Advances in Mathematical Fluid Mechanics

Hamid Bellout
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Incompressible Bipolar and Non-Newtonian Viscous Fluid Flow

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This Book is Dedicated
to
Professor Bellout's Mother
Nekhla
and to our children
Adam and Sarah Bellout
and
Daniel and Amir Bloom
who, everyday, make it all worthwhile

Preface

In 1849 G.G. Stokes published a classical paper [Sto] in which he modeled the response of a viscous incompressible fluid by a constitutive equation that has, ever since, been known as the Stokes law. When combined with the results in the earlier paper of Navier [Na], the Stokes law leads to the Navier–Stokes system of nonlinear partial differential equations; this system, subject to the incompressibility constraint on admissible velocity fields, has served as the basic mathematical model for studying the motions of incompressible viscous fluids for over a century and a half. The success of the Navier–Stokes model, in both the incompressible as well as the compressible case, has been far ranging and is unquestioned, in spite of various fundamental problems which still actively engage the attention of fluid dynamicists and mathematicians all around the globe; these problems include the well-known issues of showing that turbulent flow is a consequence of the Stokes law and discovering an adequate existence and uniqueness theory for the case of three-dimensional flows. It is widely acknowledged that there also exists a large variety of common substances (such as blood, motor oils, molasses, etc.) which exhibit fluid-like behavior but can not be adequately described using the Stokes law; such fluids are termed non-Newtonian and a significant number of constitutive relations have been proposed to describe either individual non-Newtonian fluids or entire classes of them.

This is not a book about the behavior of fluids which are universally recognized as exhibiting distinctly non-Newtonian behavior. Rather, this volume addresses the following question: what kind of model results if, in the process which leads to the formulation of the Stokes constitutive relation, we do not, a priori, impose the dual restrictions (1) that the relationship between the components of the reduced stress tensor and the rate of deformation tensor is strictly linear and (2) that the reduced stress tensor depends only on the first-order gradients of the velocity field. Thus, unlike models of non-Newtonian fluid flow, in which some ad-hoc nonlinear relation is assumed between the stress and rate of deformation tensors, and unlike some of the efforts, which will be described in this book, to regularize the Navier–Stokes model in three space dimensions by adding onto the equations terms involving higher-order spatial derivatives of the velocity field, the incompressible, nonlinear, bipolar fluid flow model treated in this volume is simply a consequence of not rigidly imposing, a priori, the two key assumptions which lead to the Stokes law; the philosophy underlying this approach has been clearly spelled out in the book by Shinbrot [Sh].

The first rigorous development of the theory of viscous, multipolar fluid flow is to be found in the fundamental paper of Nečas and Šilhavý [NS1]; the follow-up paper by Bellout et al. [BBN1] focused on elaborating the model in the incompressible, nonlinear bipolar case. With respect to the work in both [NS1] and [BBN1], it is essential to note that the development of the constitutive equations proceeds in such a manner as to render the resulting theory entirely consistent with the basic principles of material frame-indifference and the second law of thermodynamics (in the form of the Clausius-Duhem inequality). Also, as concerns the higher-order boundary conditions, which must be formulated for fluid flow problems in which spatial derivatives of the velocity of order higher than two appear, these are a rigorous consequence of the principle of virtual work coupled with some fundamental results due to Heron [HB] on the traces of divergence free vector fields; as such, these boundary conditions, which are an essential part of the theory of incompressible, nonlinear, bipolar fluid flow, stand out in stark contrast to the ad hoc types of higher-order boundary conditions which have been employed by those authors who have studied the regularizing effects of adding higher-order spatial derivatives to the Navier–Stokes equations. Since the original development of the multipolar fluid models, for both compressible and incompressible flow, a number of research groups, primarily in the United States, Eastern Europe, and China, have explored the consequences of these models; their efforts, which will be described in this book, have focused on the solution of problems in the context of specific geometries, on the existence of weak and classical solutions, and on such dynamical systems aspects of the theory as the existence of compact global attractors and inertial manifolds. The present volume is devoted exclusively to the task of elucidating some of the results which have been obtained, thus far, for the case of incompressible, nonlinear, bipolar fluid flow.

We now offer a description of the contents of this volume in the order in which the material is developed. Chapter 1 develops the theory of incompressible multipolar fluid dynamics with an emphasis on the nonlinear bipolar model. We begin in Sect. 1.1 by reviewing the hypotheses which lead to the Stokes constitutive law for viscous fluid flow and the Navier–Stokes equations which are a direct consequence of that law. In Sect. 1.2 we review the development of the general multipolar fluid model as presented in the fundamental paper of Nečas and Šilhavý [NS1]; the specialization to the case of linear bipolar fluid response appears in Sect. 1.3. Section 1.4 presents the development of the system of partial differential equations governing flow of an incompressible, nonlinear, bipolar fluid and is based, primarily, on the analysis presented in [BBN1]; a key feature of this section nis the derivation of the higher-order boundary conditions from the principle of virtual work coupled with the analysis in Heron [HB]. Elementary examples of incompressible nonlinear bipolar flows, i.e., steady plane Poiseuille flow, steady Poiseuille flow in a circular cylinder, and plane Couette flow are analyzed in Sect. 1.5. In Sect. 1.6 we describe some of the other extensions and generalizations of the standard Navier–Stokes model for incompressible viscous fluid flow which have one or more attributes in common with the bipolar fluid model; these include the non-Newtonian models of Ladyzhenskaya type [La1, 2], [DuG], [Lio1], multipolar fluids of grade 3 [BNR], dipolar fluids [BG], the extended incompressible viscous flow models of Green and Naghdi [GN1, 2], and the nonlinear dispersive Navier–Stokes alpha (NS-α) model of incompressible viscous flow, also known as the viscous Camassa-Holm equations (VCHE), treated in [CFH1, 2, 3] and [FHT1, 2]. Examples of particular flows associated with the models introduced in Sect. 1.6 are then studied in Sect. 1.7. Chapter 1 concludes by presenting, as further motivation for construction of the non-Newtonian model studied in this monograph, a catalog of experimental results which are inconsistent with the Stokes' hypothesis.

Chapter 2 is devoted to the problem of plane Poiseuille flow of incompressible bipolar viscous fluids between parallel plates; general results concerning existence, uniqueness, and continuous dependence on the constitutive parameters are elaborated in Sect. 2.2. Then, in Sect. 2.3, we obtain sharp estimates for the velocity field associated with the bipolar fluid in terms of the velocity fields associated with specializations of the bipolar constitutive theory that result from setting one or more of the key constitutive parameters equal to zero. Uniqueness of the steady Poiseuille flow within a general class of equilibrium flows in the parallel-wall domain is proven in Sect. 2.4. Finally, in Sect. 2.5 we consider the problems of existence and asymptotic stability for time-dependent plane Poiseuille flow of an incompressible, nonlinear, bipolar fluid. The work in this chapter is based, primarily, on the analysis in [BBN1] and [BB1, 2, 3].

In Chap. 3 we turn to a study of a variety of incompressible bipolar flows in special geometries and types of domains. Building on the work in [BH2], we formulate the classical problem of flow between rotating concentric cylinders for an incompressible, nonlinear, bipolar fluid in Sect. 3.2 and prove results concerning the existence, uniqueness, and continuous dependence (on constitutive parameters) for such flows. Section 3.3 is devoted to an analysis of bubble stability in a non-Newtonian viscous fluid of the type that the nonlinear bipolar model reduces to when the higher-order viscosity is set equal to zero; using the analysis presented in [Bl6] we elaborate upon the dynamics of a spherical bubble cavity in such a fluid employing both linearized dynamics and Lyapunov theory to analyze the stability of the cavity. In Sect. 3.4 we examine the problem of (steady) exterior flow of an incompressible, nonlinear, bipolar viscous fluid in the plane. Following the approach employed by Bellout and Nečas [BN] we first study the exterior problem in a truncated domain containing an obstacle and, then, proceed to obtain the solution in the original unbounded domain by implementing a limit process; it is also shown that the solution predicts the existence of a drag force on the obstacle in the direction of the velocity field at infinity. Finally, in Sect. 3.4 we study, following the analysis in [BW], the problem of flow of an incompressible, nonlinear, bipolar fluid over a non-smooth boundary, focusing on flows in polygonal domains. More specifically, the work in Sect. 3.4 analyzes the stability of solutions with respect to perturbations of the boundary of the domain and examines the regularity of solutions for problems defined on polygonal domains providing, in particular, a description of the asymptotic behavior of solutions near corners on the boundary of the domain. It is shown, in contrast with similar results based on use of the Navier–Stokes system (which are at variance with experimental data) that solutions of the bipolar initial-boundary value problem are not stable with respect to perturbations of the boundary of the domain by Lipschitz curves. Indeed, as we explicitly point out in Sect. 3.5, it has been known since the work of Nikuradze in the 1930s (e.g., [ScG]) that at high Reynolds numbers the presence of even very small protrusions on the surface of a bounding wall for a viscous flow substantially affects the flow.

Chapter 4 is devoted to proving general existence and uniqueness theorems for incompressible bipolar flow as well as for those non-Newtonian flows which result from setting the higher-order viscosity equal to zero; results are established for problems in both bounded and unbounded domains as well as for problems with periodic boundary conditions. The analysis begins in Sect. 4.2, where a Galerkin argument is used to prove the existence and uniqueness of weak solutions for the initial-boundary value problem associated with incompressible, nonlinear, bipolar flow in a bounded domain of R^n, $n = 2, 3$, with sufficiently smooth boundary. We also study the regularity of the solution and prove some estimates which establish the asymptotic stability of solutions of the initial-boundary value problem; the work in this section is based, for the most part, on the results obtained in [BBN4]. Section 4.3 establishes the existence of weak and measure-valued solutions for incompressible non-Newtonian fluids which are generated as a special case of the bipolar model with vanishing higher-order viscosity. Employing an a priori restriction to consideration of the relevant space-periodic problems, the concept of a Young measure-valued solution is first defined. Then, for a certain range of the order of the nonlinearity associated with the non-Newtonian model in space dimension $n = 2$, the Young measures are shown to be Dirac measures, while for another range the Young measures are proven to be Dirac and the associated weak solutions are shown to be regular solutions; a similar set of results is generated for the case of space dimension $n = 3$. The discussion in Sect. 4.3 is based, in large measure, on the work in [BBN2], [BBN3], and [MNN]. In Sect. 4.4 we again consider the problem of flow of an incompressible, nonlinear, bipolar fluid in an unbounded, parallel-wall channel. To prove existence of solutions for this problem, we first establish the existence of approximate solutions in bounded subdomains of the channel by using a Galerkin approach; then it is shown that there exists a subsequence of such approximate solutions whose limit is a unique weak solution of the initial-boundary value problem; the bulk of the analysis in this section first appeared in [BH4]. Finally, in Sect. 4.5 we summarize some of the most important extant results on existence and uniqueness for solutions of the Navier–Stokes equations and recall a few of the unresolved problems in three space dimensions. We also discuss related work on existence and uniqueness theorems for some of the generalizations of the Navier–Stokes model that were described in Sect. 1.6, including the non-Newtonian Ladyzhenskaya type models, viscous flow models with artificial viscosity, the multipolar fluid model of grade 3, and the viscous Camassa-Holm equations.

Chapter 5 focuses on the existence of maximal compact attractors for incompressible bipolar and non-Newtonian flows in bounded domains and for space periodic problems. Proving the existence of a maximal compact attractor involves (1) proving the existence of absorbing sets in order to deduce the uniform compactness, for large time, of the relevant solution operator, (2) establishing the uniform differentiability of the solution operator on the attractor, and (3) proving the uniform boundedness of an associated linearized operator; this operator, which is associated with the linearization of the nonlinear, incompressible, bipolar equations about an equilibrium solution, is introduced in Sect. 5.2 where we also establish the linearized stability of solutions of the incompressible bipolar equations. The results in Sect. 5.2 are based on the analysis in [Bl4]. Section 5.3 presents the results obtained in [BBN5] for incompressible bipolar initial-boundary value problems (and hold for the space periodic problems, also) in dimensions $n = 2, 3$; in Sect. 5.3 the existence of a maximal compact global attractor is proven and estimates are obtained for both the Hausdorff and fractal dimensions of the attractor. For a different range of the exponent controlling the nonlinearity in the bipolar model, it is shown in Sect. 5.4 that a maximal compact attractor exists for the space periodic problem, in $\dim n = 2$, for which both the Hausdorff and fractal dimensions are independent of the higher-order viscosity; it is also shown, independently, that the corresponding non-Newtonian space periodic problem admits a maximal compact attractor in space dimension $n = 2$; the results presented are based on the work in [Bl3]. Finally, it is shown in Sect. 5.5 that, as a consequence of the analysis in [Bl2, 3], the attractor for the bipolar problem, whose existence in the space periodic case when $n = 2$ was established in Sect. 5.4, converges in the sense of semidistance to the compact attractor for the corresponding non-Newtonian problem as the higher-order viscosity converges to zero.

Finally, Chap. 6 considers (1) the problem of the existence of an inertial manifold for bipolar, incompressible, viscous flows, and the associated phenomena of orbit squeezing, and (2) the question of whether a maximal compact global attractor exists for the flow of an incompressible bipolar viscous fluid in an unbounded, parallel walled channel; existence of solutions for this latter problem was proven in Sect. 4.4. Following the analysis in [BH3] it is proven, in Sect. 6.2, that an inertial manifold exists for the incompressible, nonlinear, bipolar viscous flow problem, subject to space periodic conditions, in both dimensions $n = 2$ and 3. The work in Sect. 6.2 also establishes a squeezing property for the orbits of the associated solution operator; a more fundamental (L^2) squeezing property is shown to hold in Sect. 6.3 by using results in [BH1]. In Sect. 6.4 we employ the analysis which appeared in [BH5] to prove the existence of a global compact attractor for the equations governing nonlinear bipolar fluid flows in unbounded two-dimensional channels. Finally, in Sect. 6.5, we survey some related recent work on the asymptotic behavior of solutions to problems for incompressible bipolar and non-Newtonian flows by other authors, highlighting developments connected with proving the existence of a global attractor.

Three appendices may be found at the end of this book, the first of which, Appendix A, sets the notation we have tried to use, consistently, throughout this volume. In the few instances when the same notation has been used with different meanings in different chapters (or sections), this circumstance has been carefully pointed out. Appendix A also reviews some important basic analysis results and definitions, including embedding and interpolation theorems for Sobolev spaces and some fundamental Sobolev space estimates. Appendix B establishes several key lemmas involving the rate of deformation tensor, including two inequalities of Korn type which are used repeatedly in Chaps. 4–6. The spectral gap condition, an essential ingredient in the proof of the existence of an inertial manifold in Chap. 6 is established in Appendix C. A reasonably comprehensive bibliography accompanies this volume; however, a substantial portion of the literature on non-Newtonian flows which is unrelated to the nonlinear bipolar model has not been detailed. In addition, by the time this volume appears, it is somewhat likely that new work within the realm of incompressible, nonlinear, bipolar viscous flow will have been published; for all such unintended omissions, the authors offer an a priori apology.

DeKalb, IL H. Bellout and F. Bloom
October 2012

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This book has been written with the spirit of our colleague, the late Professor Jindrich Nečas of Charles University in the Czech Republic, ever present. It was Professor Nečas who, in a visit to NIU in 1990, first brought the multipolar viscous fluid flow model to our attention; for over a decade, during which he split his time between faculty positions at NIU and Charles University, Jindrich was a colleague, a mentor and, above all, a dear friend. One of the great European mathematical analysts of the last half century, a mathematician with deep physical insight, and a towering figure in the modern theory of partial differential equations, Professor Nečas was a man of intense personal integrity. Unwilling to sit idly by as Soviet tanks crossed the border in 1968 to crush his beloved Czechoslovakia, he left a visiting faculty position at Berkeley to return to Prague. Against the express wishes of the second author of this book, he postponed a chemotherapy session in early Spring 2002 to attend a meeting of a committee of NIU Distinguished Research Professors who were considering that author for a similar appointment. Ever the persistent mathematician, he worked tirelessly, almost to the very end of his life, trying to establish either a global existence theorem for the three-dimensional Navier–Stokes equations or, in its absence, an argument to prove that solutions do, indeed, exhibit finite-time blow-up in some appropriate sense. Professor Nečas had an abiding faith in the universal validity of the type of model of fluid behavior elaborated upon in this book and its writing has been long overdue; if the authors have, in some small way, produced a piece of work that he would have been pleased with, then that alone suffices to justify the effort which has been made in its writing.

Contents

Chapter 1
Incompressible Multipolar Fluid Dynamics

1.1 Introduction: The Stokes Constitutive Law and the Navier–Stokes Equations

The study of the motions of an incompressible viscous fluid by mathematicians, physicists, and engineers, has been an ongoing enterprise since the publication by G.G. Stokes [Sto] of his classical memoir on the internal friction of fluids in motion in 1849. Perhaps the closest one can come to something resembling a reasonable bibliography, detailing the variety of work on the subject that was initiated by Stokes' research, is the seventy-five page list of references at the end of the 8th revised edition of the classic volume [ScG] on boundary layer theory. Among the noteworthy (and still valuable) texts and treatises dealing with incompressible viscous fluid flow based on the Stokes constitutive equation are those of Lamb [Lam], Batchelor [BaG], Landau and Lifschitz [LL], Shinbrot [Sh], and Serrin [Se]. A concise and well-written modern text which covers many of the most important elementary aspects of the theory is that of Chorin and Marsden [CM]. Finally, books which are of a more distinctly mathematical flavor, and which treat the very difficult problems of existence and uniqueness for the Navier–Stokes equations, include those of Constantin and Foias [CF], Galdi [Ga1], Temam [Te1], [Te3], Ladyzhenskaya [La1], Sohr [So], and P.-L. Lions [PL]. Three recent collections of papers on contemporary problems in mathematical fluid dynamics, including those associated with the motion of viscous incompressible fluids, are [NeP], [GHR], [BMW], and [FS1, 2]. Numerical methods for treating the Navier–Stokes equations are discussed in [Te1] and [Te3]. The list of volumes delineated above does not even begin to "scratch the surface" with respect to the literature, in book form, which deals with problems for the incompressible Navier–Stokes equations, but it is more than sufficient in terms of offering the interested reader a very comprehensive overview of the subject. In this book we will assume that the reader has a basic understanding of the field of modern continuum mechanics as well as some familiarity with elements of the subject of viscous incompressible

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fluid flow; in particular we assume that the reader has been exposed to a discussion of the fundamentals of incompressible fluid flow, i.e., to the development of the theory underlying the Navier–Stokes equations, as well as to the solution of basic flow problems within the context of that theory, at the level of, say, the text by Batchelor [BaG]. Of necessity, we also must assume that the reader is acquainted with elements of the classical and modern theory of both elliptic and parabolic partial differential equations at the level of the texts [McO], [RR], or [Ev]. Some familiarity with aspects of the modern theory of dynamical systems would also be helpful but is not absolutely essential. Appendix A provides a compendium of useful notation employed in the book, as well as some basic analysis definitions and results and a survey of embedding and interpolation theory for Sobolev spaces which is used extensively for Chaps. 4–6.

For the remainder of this chapter, $\Omega \subset R^n$, $n = 2, 3$, will denote either an open bounded domain or an unbounded domain, such as the region between two flat, parallel plates in $\mathbb{R}^3$; whenever we deal with a specific problem in this and subsequent chapters, we will be careful to indicate the nature of the domain in question. In some cases (e.g., Sect. 3.4) we will have $\Omega = \mathbb{R}^n/\Omega_*$, $n = 2, 3$, with Ω_* an open bounded domain in $\mathbb{R}^n$ so that flow in Ω represents flow around the obstacle Ω_* (the exterior flow problem). By x_i, $i = 1, \ldots, n$, we denote a set of (Eulerian) rectilinear Cartesian coordinates[1] in Ω. The velocity field of an incompressible viscous fluid, which occupies the region Ω, at time t, will be denoted most often by $\boldsymbol{v}$ (although it will sometimes be useful to denote this velocity field by $\boldsymbol{u}$). In component form, for $n = 3$, we have

$$(\boldsymbol{v})_i(\boldsymbol{x}, t) = v_i(x_1, x_2, x_3, t), \quad t \geq 0.$$

When required we will replace the rectilinear components v_1, v_2, v_3 (for flow in R^3) by the cylindrical components v_r, v_θ, v_z of the velocity field in which case, of course, the cylindrical coordinates r, θ, z are used in lieu of Cartesian coordinates.

We now pose the following question (which most researchers in fluid dynamics believe they already know the answer to), namely, where does the Stokes Hypothesis (and the corresponding system of partial differential equations known as the Navier–Stokes equations) come from? In an ideal fluid, in which there are no frictional forces, neighboring parts of the fluid may move at different velocities, without one exerting a force on the other, provided these regions in the fluid are separated by a streamline. The assumption that frictional forces are not at work (in an ideal fluid) produces troubling results such as the well-known d'Alembert "paradox". Thus, because frictional forces are caused by relative motions of neighboring regions in a fluid, any model of fluid motion which incorporates such forces would have to predict the absence of internal frictional forces whenever neighboring portions of the fluid are not moving relative to each other; frictional forces appear, however, if such relative motion is present. We will return to this elementary (but fundamental)

[1] Sometimes we will write $x_1 = x$, $x_2 = y$, $x_3 = z$.

concept after a brief pause to recall the structure of the Stokes Law and the manner in which it leads to the Navier–Stokes equations. We restrict, a priori, our considerations to the incompressible case; by virtue of the equation of continuity (conservation of mass) this assumption is, of course, equivalent to the statement that $\nabla \cdot \boldsymbol{v} = 0$, everywhere in the domain currently occupied by the fluid; it is always true if the mass density ρ is constant in space and time.

The starting point for the derivation of the Navier–Stokes equations, in most standard fluid dynamics textbooks, is the differential form of the statement of conservation of momentum in Eulerian coordinates, i.e.,

$$\rho \boldsymbol{a} = \nabla \cdot \boldsymbol{t} + \rho \boldsymbol{f}. \tag{1.1}$$

In (1.1), ρ is the mass density (assumed to be a constant), $\boldsymbol{a}$ is the fluid acceleration, $\boldsymbol{t}$ is the stress tensor, and $\boldsymbol{f}$ is the body force/mass, all of which are measured at a point $(\boldsymbol{x}, t)$ in the domain occupied by the fluid at time t. Furthermore, $\boldsymbol{a}$ is given by the convective time derivative of the velocity field $\boldsymbol{v}$ (the derivative following the motion of fluid particles), i.e., $\boldsymbol{a} = D\boldsymbol{v}/Dt$ where

$$\left(\frac{D\boldsymbol{v}}{Dt}\right)_i = \frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} \tag{1.2}$$

and

$$v_j \frac{\partial v_i}{\partial x_j} = (\boldsymbol{v} \cdot \nabla \boldsymbol{v})_i, \tag{1.3}$$

it being understood that we always sum on repeated indices such as the index j in (1.2). Finally $\boldsymbol{t}$, in (1.1), is the Cauchy stress tensor which measures force/area in the fluid in the sense that if S is a surface internal to the domain occupied by the fluid at the time t, and $\boldsymbol{x} \in S$, then the traction $\boldsymbol{T}$ at (x, t), i.e., the force/area exerted at $\boldsymbol{x}$ at time t, by neighboring portions of the fluid, is given by $\boldsymbol{T} = \boldsymbol{t}\boldsymbol{n}$ where

$$(\boldsymbol{T})_i = t_{ij} n_j. \tag{1.4}$$

The full stress tensor $\boldsymbol{t}$ is now expressed as

$$\boldsymbol{t} = -p\boldsymbol{I} + \boldsymbol{\tau} \tag{1.5}$$

where the "reduced stress tensor" $\boldsymbol{\tau}$ is identically set equal to the zero tensor in an ideal fluid in which the stress tensor is generated by a hydrostatic pressure p. If we introduce the rate of deformation tensor

$$\boldsymbol{e} = \frac{1}{2}(\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T) \tag{1.6a}$$

with components

$$e_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \tag{1.6b}$$

then the Stokes Law is the *assumption* that for some constant $\mu_0 > 0$, specific to the fluid being modeled,

$$\tau_{ij} = 2\mu_0 e_{ij} \tag{1.7}$$

where μ_0 is the viscosity. A fluid which conforms to the constitutive hypothesis (1.7) is known as a Newtonian fluid. For an incompressible viscous fluid ($\partial v_i/\partial x_i = 0$), the insertion of the *ansatz* (1.5), (1.7) into the (vector) equation (1.1), expressing balance of momentum, yields the Navier–Stokes system of partial differential equations

$$\rho\left(\frac{\partial v_i}{\partial t} + v_j\frac{\partial v_i}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} + \mu_0\nabla^2 v_i + \rho F_i \tag{1.8}$$

where $\frac{\partial p}{\partial x_i} = (\nabla p)_i$. In a bounded domain, say $\Omega \subseteq R^n$, $n = 2, 3$, the system (1.8) is to be solved subject to the standard non-slip boundary conditions $\boldsymbol{v} = \boldsymbol{0}$ on $\partial\Omega$ and the specification of an initial condition of the form $\boldsymbol{v}(\boldsymbol{x}, 0) = \boldsymbol{v}_0(\boldsymbol{x})$ for $\boldsymbol{x} \in \Omega$. In Sect. 4.5 we will review, briefly, some of what is known about the existence and uniqueness of both strong and weak solutions for initial and initial-boundary value problems for the Navier–Stokes system; suffice it to say, at this junction, that difficulties abound in R^3 with respect to obtaining what would be considered to be satisfactory results relative to, e.g., the uniqueness of weak solutions, the existence of classical solutions, globally in time, or the finite-time blow-up of classical solutions, although partial results, some of a very deep and fundamental mathematical character, have been obtained. In spite of this observation we note that (as pointed out in Sect. 3.7 of Schlichting and Gersten [ScG], where we have replaced their reference to the Stokes Law by (1.7)),

> Although Eq. (1.7) must be viewed as a pure hypothesis, or even as an educated guess, the equations of motion arising from inserting Eq. (1.7) [into the equation describing balance of momentum] can be accepted, because they have been confirmed by an exceedingly large number of experiments, sometimes in extreme conditions, as will be realised by the reader on completing this book. These equations of motion are a very good description of actual physical processes.

No researcher who is familiar with the predicative success of the system (1.8) could possibly take exception with the above quoted statement from [ScG]. Yet, beyond the difficulties with the establishment of acceptable existence/uniqueness results in $\mathbb{R}^3$, which those who are unequivocal proponents of (1.8) for "ordinary" fluids (in all circumstances) would claim is just a consequence of either not having available the proper mathematical tools, or a skilled enough mathematician, there are other substantive problems; some of these, which will be discussed further on in this volume, include the Stokes paradox for flow around an obstacle in the plane, the difficulties associated with applying the Navier–Stokes equations to study the flow over rough surfaces, the need to implement a theory, the Prandtl boundary-layer theory, which lacks an entirely firm mathematical grounding, to study problems in which the Reynolds number is high (but a transition to turbulent flow has not yet occured), and the entire question of the feasibility of using the Navier–Stokes equations to describe strong turbulence in fluids. Indeed, it is well-known that even Prandtl was not happy with the essential tenets of boundary-layer theory. Moreover, all of the shortcomings associated with applying the model based on the Stokes Law (1.7) are not simply the result of trying to implement that model for fluids which are truly non-Newtonian in nature such as magma, plastic melts, polymer solutions, or suspensions such as blood; in such fluids, as pointed out, e.g., in [Oe], the frictional stresses acting on a fluid element may depend on both the instantaneous state of motion as well as on the motion of the fluid in the past. Truly non-Newtonian fluids may not only exhibit memory effects but may also require that the linear constitutive relation (1.7) be replaced by one in which $\boldsymbol{\tau}$ is a nonlinear function of $\boldsymbol{e}$ (in all ordinary motions) through the introduction of a viscosity function which depends in a nonlinear fashion on $|\boldsymbol{e}|$. In this book we shall always have in mind "ordinary" viscous fluids, those whose behavior is described very well by the Navier–Stokes equations in most, but not all, situations. We will not be introducing a nonlinear version of (1.7), as has been done, e.g., by Ladyzhenskaya [La1, 2], so as to obtain a satisfactory existence theory; nor will we be introducing, in some ad hoc fashion, higher-order derivative terms in (1.8), along with an associated ad hoc set of boundary conditions, as has been done in, e.g., [La5], [Lio1], [BdV2, 3], or [OS1, 2] so as to regularize the Navier–Stokes system. The approach followed in this volume is based on a simple yet fundamental understanding of the genesis of the Stokes Law (1.7) and follows a tradition which is firmly ingrained in the historical development of classical physics.

While the development of the equations governing incompressible fluid flow may begin, in most textbooks on fluid dynamics, with the ansatz (1.7), i.e., the Stokes Law, this does not give a complete picture of the process that is actually taking place. A good description of the basic thought process which leads to the constitutive law (1.7) is to be found in the lecture notes by M. Shinbrot [Sh]; as Shinbrot points out in [Sh], Chap. 7, "the argument is one often used in physics and engineering and is well worth appreciating. It is based on a modification of Occam's razor: do not complicate your hypotheses unnecessarily. This means that if, under the simplest hypotheses consistent with the phenomena we wish to describe, a model can be derived that is already too difficult to cope with completely, then this model should be retained unless and until paradoxes appear within it." Shinbrot then proceeds to "apply this (modification of Occam's razor) to derive the form of the reduced stress tensor, keeping in mind that there are to be no forces between portions of a fluid that are not in relative motion. This means, first, that the reduced stress tensor $\boldsymbol{\tau}$ must be zero when the velocity $\boldsymbol{v}$ is zero.[2] Moreover $\boldsymbol{\tau}$ must be zero when the velocity $\boldsymbol{v}$ is constant throughout the fluid, for then there is no relative motion between different parts of the fluid. The simplest hypothesis that assures us of this is that $\boldsymbol{\tau}$ is a function of the derivatives of $\boldsymbol{v}$ only". In our notation this last hypothesis is equivalent to the mathematical statement that

$$\boldsymbol{\tau} = \boldsymbol{\tau}(\nabla \boldsymbol{v}, \nabla\nabla \boldsymbol{v}, \ldots) \tag{1.9}$$

with $\boldsymbol{\tau} = \mathbf{0}$ when all of the arguments of $\boldsymbol{\tau}$ are $\mathbf{0}$. To this point, (1.9) holds for every viscous fluid whether it be air, water, or some polymer solution. In the overall argument, there now appears the two basic, restrictive, assumptions which lead to the Stokes Hypothesis; as Shinbrot [Sh] argues "within the framework of the hypothesis (1.9), the simplest assumption that can be made is that $\boldsymbol{\tau}$ depends only on the *first* derivatives of $\boldsymbol{v}$. Again, the simplest hypothesis about the form of this dependence is that it is *linear*". Under the auspices of these two stated assumptions, each component of the reduced stress tensor $\boldsymbol{\tau}$ has the form

$$\tau_{ij} = a_{ij}\frac{\partial v_1}{\partial x_1} + b_{ij}\frac{\partial v_1}{\partial x_2} + c_{ij}\frac{\partial v_1}{\partial x_3} + d_{ij}\frac{\partial v_2}{\partial x_1} + e_{ij}\frac{\partial v_2}{\partial x_2} + f_{ij}\frac{\partial v_2}{\partial x_3} + g_{ij}\frac{\partial v_3}{\partial x_1} + h_{ij}\frac{\partial v_3}{\partial x_2} + k_{ij}\frac{\partial v_3}{\partial x_3} \tag{1.10}$$

where the coefficients $a_{ij}, b_{ij}, \ldots, k_{ij}$ are all constants. Shinbrot next shows that (1.10) already implies a restrictive form of the Stokes hypothesis; in fact, the argument [Sh] continues by indicating "the coefficients in (1.10) still have to be adjusted so that $\boldsymbol{\tau} = \mathbf{0}$ where there is no relative motion between different portions of a fluid. Now, in a fluid rotating like a rigid body with constant angular velocity, there is no relative motion between its various parts. Consider such a fluid, with the x_3-axis the axis of rotation. In this case, it is not hard to show that the velocity of the fluid at the point $\boldsymbol{x} = (x_1, x_2, x_3)$ is

$$\boldsymbol{v} = \omega(-x_2, x_1, 0) \tag{1.11}$$

where ω is a constant. By (1.10), then,

$$\tau_{ij} = \omega(d_{ij} - b_{ij}) \tag{1.12}$$

and this must be zero, so that $d_{ij} = b_{ij}$. In a similar way, making the axis of rotation the x_1- and then the x_2-axis, we find $g_{ij} = e_{ij}$, $h_{ij} = f_{ij}$". Therefore, we obtain from (1.10)

$$\tau_{ij} = a_{ij}\frac{\partial v_1}{\partial x_1} + e_{ij}\frac{\partial v_2}{\partial x_2} + b_{ij}\left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\right) + c_{ij}\left(\frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1}\right) + f_{ij}\left(\frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2}\right). \tag{1.13}$$

[2]The variable σ is used to denote the reduced stress tensor in [Sh] and, the velocity in [Sh] is denoted by $\boldsymbol{s}$.

The constitutive relation (1.13) is equivalent to a restrictive form of the *Stokes hypothesis*; the quantities $\frac{\partial v_1}{\partial x_1}, \frac{\partial v_2}{\partial x_2}, \frac{1}{2}\left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\right)$, etc., are just the components of the rate of deformation tensor e_{ij} in (1.6b) so that $\boldsymbol{\tau} = \boldsymbol{\tau}(\boldsymbol{e})$ and is, in fact, linear in the components of $\boldsymbol{e}$ which, as Shinbrot [Sh] points out "is more than is required by the Stokes Hypothesis. The linearity of (1.13) is a consequence of our diligent application of Occam's razor: we took as the simplest hypothesis that $\boldsymbol{\tau}$ is a linear function of $\nabla \boldsymbol{v}$". Shinbrot then pointedly notes that "some mathematicians, while accepting the other hypotheses, have been offended by the assumption of linearity, and perhaps they are correct. That is where the last part of the razor plays a role. The model should be retained *until contradictions appear*. Until they do, consistency requires that the simplest hypotheses be made". Of course, two basic assumptions, not one, have been made in arriving at (1.13), namely, (1) that $\boldsymbol{\tau}$ depends only on $\nabla \boldsymbol{v}$, and not on $\nabla\nabla \boldsymbol{v}$, $\nabla\nabla\nabla \boldsymbol{v}$, etc., and (2) that $\boldsymbol{\tau}$ is linear in the components of $\nabla \boldsymbol{v}$. It is precisely the absence of a dependence of $\boldsymbol{\tau}$ on higher-order spatial gradients of $\boldsymbol{v}$ that leads to *contradictory* results when the Navier–Stokes equations are applied to the study of flow of a viscous fluid over a surface with a rough boundary.

The reduction from the linear version (1.13) of the Stokes hypothesis to the Stokes Law (1.7), for an incompressible fluid, now proceeds in a standard manner which we will not repeat here; the basic assumption which drives this reduction is that the equations which represent a physical phenomenon have to be independent of the coordinate system used to describe it. The actual reduction which takes us from (1.13) to (1.7) may be found in most elementary fluid dynamics texts, e.g., [Sh] or [CM], with the principle which dictates the process involved very often going under the heading of material frame indifference.

Shinbrot concludes his discussion ([Sh], Chap. 7) of the Stokes hypothesis, and the subsequent derivation of the Navier–Stokes equations, with the following observations about solutions in $\mathbb{R}^3$:

> It is not known whether solutions with large initial data are unique! This is related to the smoothness question. The solution is initially smooth and unique. But at the first value of t where it is not smooth (if there is one), uniqueness may also be lost. This uniqueness question is one of the outstanding questions in theoretical hydrodynamics. Instant fame awaits the person who answers it. (Especially if the answer is negative!)
>
> If it should turn out that solutions are not unique, that fact would qualify as a paradox and would justify complicating the model. There are two obvious ways in which the model can be changed. The first is to allow the reduced stress tensor to depend on higher derivatives of the velocity than the first. The second is to allow the reduced stress tensor to depend nonlinearly on the derivatives of the velocity. A piquant aspect is lent to the whole problem by the fact that in *either* case the solution can be shown to be unique!
>
> It might be argued that as a practical matter the whole question may be irrelevant since solutions are known to be unique if the data are small. It is true, after all, that for large initial data no fluid can be looked upon as incompressible. However, the values of the initial data for which the uniqueness question can be settled are *too* small. They are well below the values for which compressibility effects in water, say, come into effect or in which our hypotheses can be expected to fail. Thus, even from a "practical" point of view, the existence and uniqueness theorems require strengthening.

> In any case, the uniqueness question remains an aggravating mathematical problem which seems extremely difficult for no very good reason.

In Sect. 1.6 we will discuss various generalizations of the (incompressible) Navier–Stokes system (1.8); some of these generalizations were constructed with a view towards proving the kind of existence and uniqueness results in $\mathbb{R}^3$ that have eluded researchers (to date) within the context of the Navier–Stokes theory. However, the absence of adequate existence and uniqueness results for initial-value and initial-boundary value problems associated with (1.8) in $\mathbb{R}^3$ does not of itself represent a paradox that warrants changing the model represented by (1.7) to allow for a nonlinear viscosity which is dependent on $|\boldsymbol{e}|$, as in the work in [La1, 2], [DuG], or [Kan], or for the additions of terms involving higher-order spatial derivatives of $\boldsymbol{v}$. Rather, it is the existence of problems like the Stokes paradox for Stokes flow of a Newtonian fluid around an obstacle in the plane, or the predictions generated by using the Newtonian model, and the associated non-slip boundary condition, to study flow over a surface with a rough boundary, which warrant at least considering changes with respect to (1.7). As will be made clear in Sect. 1.2, the changes we refer to do not amount to arbitrarily introducing nonlinear viscosity into the model, or selecting a particular type of nonlinear behavior to model the kind of phenomena exhibited by true "non-Newtonian" fluids (e.g., the Weissenberg effect); nor do they involve the somewhat arbitrary addition of higher-order derivatives of the components of the velocity vector to the system (1.8), with the consequential addition of higher-order boundary conditions. In fact, as we will point out further on in this chapter, in those papers where higher-order derivatives, and associated higher-order boundary conditions, have been used to modify the Navier–Stokes system, the boundary conditions employed are *inconsistent* with the principal of virtual work and the constraint imposed by incompressibility. As detailed in Sect. 1.2, the approach which is followed, in order to obtain the modification of the Navier–Stokes system discussed in this book, consists of beginning with the fundamental ansatz (1.9) for a viscous fluid, and imposing only those constraints which are consistent with the basic principles of continuum mechanics and thermodynamics, without a priori imposing the restriction that $\boldsymbol{\tau}$, in (1.9), be a linear function or that it depend only on $\nabla \boldsymbol{v}$.

It appears to be ingrained in the thought processes of much of the fluid dynamics research community that fluids are either Newtonian, i.e., that they always conform to the constitutive relationship (1.7), or that they are non-Newtonian; this line of reasoning is not consistent with the historical development of much of classical physics and is, in fact, contradicted by the behavior of many well-known fluids. Blood flow, for example, is very well modeled by the Navier–Stokes equations for flow in the large arteries while flow in arteries with a relatively much smaller radius can only be modelled successfully by employing non-Newtonian constitutive equations [AzP], [BHTV], [OOL], [QF]. It is, therefore, reasonable to believe that a *realistic* constitutive relation for blood would involve a set of constitutive parameters whose values, under appropriate flow conditions, would be such as to render that constitutive relation (almost) indistinguishable from the Newtonian Law (1.7).

As already noted, similar situations have repeatedly appeared in the history of classical physics. In the early part of the twentieth century, Newton's Laws of motion for particles and rigid bodies were supplanted by the (special) relativistic mechanics developed by Poincaré, Lorenz, and Einstein. For more than a century it has been accepted that Newton's Laws do not hold exactly for any body in motion; rather, they are an approximation to the laws of motion which follow from Special Relativity when a body is moving at speeds which are small compared to the speed of light. For most intents and purposes, therefore, one may work with Newton's Laws without being concerned that they are only an approximation to a more encompassing theory. A similar situation occurs in classical electromagnetic theory. It is usually assumed in most elementary treatments of electromagnetic theory that the relationship between the electric displacement $\boldsymbol{D}$ and the electric field vector $\boldsymbol{E}$ is of the form ($\epsilon = \epsilon(\boldsymbol{x}, t)$)

$$\boldsymbol{D} = \epsilon \boldsymbol{E} \tag{1.14}$$

where ϵ is called the dielectric permittivity. For "air" ϵ is a constant, $\epsilon = \epsilon_0$, the dielectric permittivity of free space. The corresponding relationship between the magnetic intensity vector $\boldsymbol{H}$, and the magnetic field vector $\boldsymbol{B}$, in free space is

$$\boldsymbol{H} = \mu_0 \boldsymbol{B} \tag{1.15}$$

where μ_0 is the magnetic permeability of empty space. The use of (1.14), with $\epsilon = \epsilon_0$, and (1.15), in conjunction with Maxwell's equations, leads to the standard, linear, wave equation for $\boldsymbol{E}$ and experimental results for the propogation of electromagnetic waves in space then yields the conclusion that $\epsilon_0 \mu_0 = 1/c^2$. In the early 1960s, however, it became clear that not all electromagnetic media conformed to the linear dielectric relationship (1.14)[3] and that the constitutive relationship (1.14) had to be broadened to allow for nonlinear dielectric media satisfying

$$\boldsymbol{D} = \epsilon(||\boldsymbol{E}||)\boldsymbol{E} \tag{1.16}$$

where $||\cdot||$ is the standard Euclidean norm on $\mathbb{R}^3$. In the nonlinear optics literature, constitutive relations of the form (see [Bl1])

$$\epsilon(||\boldsymbol{E}||) = \epsilon_0 + \epsilon_1 ||\boldsymbol{E}||^2 \tag{1.17}$$

became prevalent (as well as relations involving far more complex algebraic expressions than (1.17), and polynomial relations which involve powers of $||\boldsymbol{E}||$ higher than two). It soon became clear, however, that even for media thought to be "nonlinear dielectrics", (1.16), (1.17) only supplanted (1.14) in situations where

[3]Primarily due to the advent of the laser and the development of the subject of nonlinear optics [Bl7], [Blo].

$||\boldsymbol{E}||$ is very large (as in a high intensity laser beam) because experiments have shown that, even in such media, $\epsilon_0 << \epsilon_1$. Thus, even those media which are known to exhibit nonlinear dielectric behavior in *some circumstances* conform to the linear dielectric relation (1.14) when not subjected to conditions involving the passage of a high intensity electromagnetic wave. It is not unreasonable to believe that (1.17) also applies, say, to air, albeit with a higher order permittivity ϵ_1 that is even orders of magnitude smaller than the (already) small ϵ_1 associated with distinctly "nonlinear" dielectric media.

The philosophical and historical approach to the development of constitutive theory we have described above is the one which underlies the development of the multipolar fluid dynamics equations in Sect. 1.2. The constitutive relations for bipolar viscous fluids, which are described in Sects. 1.3 and 1.4, are entirely consistent with the general theory constructed in Sect. 1.2 and, moreover, conform to the requirement that in those circumstances, when certain constitutive parameters appearing in the model are small, the results obtained are indistinguishable from those predicted by the Navier–Stokes equations. The notion that, at some point, even for "ordinary" fluids, the Stokes Law (1.7) may have to be considered as an approximation to a more general constitutive law, is not new in the literature. We note, in particular, the following observation by David Ruelle [Ru], where in the quoted remarks we have replaced his equation numbers by their equivalents in this section, and where $\nu = \mu_0/\rho$ is the kinematic viscosity:

> Let us briefly discuss the physical meaning of the Navier–Stokes equations (1.8). It expresses the acceleration $(\partial_i v_i + \sum v_j \partial_j v_i)$ in terms of three forces. There is a viscosity term $(\nu \Delta v_i)$ corresponding to self-friction, then the gradient of the pressure term, and finally the external force term (F_i). The viscosity term is obtained by expressing the self-friction forces in terms of the rate of deformation of the fluid *in the linear approximation*; this term can therefore only be trusted for small velocity gradients. Suppose in particular that a solution of the Navier–Stokes equation develops a singularity (infinite velocity gradient), then it is clear that the term $(\nu \Delta v_i)$ is no longer physically correct. It is thus tempting to consider instead of (1.8) an equation with a more general nonlinear viscosity term, but the simplicity of (1.8) makes it well worth investigating in detail before going to something more complicated.

In Sect. 1.6 we will describe some of the results for the generalization of the Navier–Stokes equations that was introduced by Ladyzhenskaya [La2]; this model is similar, in some respects, to the special case of the nonlinear bipolar model of Sect. 1.4 when the higher order viscosity vanishes. Without going into detail, in this section, on the precise structure of the Ladyzhenskaya generalization of the Navier–Stokes equations, it is worth noting the following remarks of Du and Gunzburger [DuG]:

> There are various reasons why scientists abandon the Navier–Stokes model in favor of models employing nonlinear constitutive laws. For example, for flows of polymers or of visco-elastic or visco-plastic fluids, one generally has to use nonlinear stress-rate of strain relations. However, the models introduced by Ladyzhenskaya address a different issue, namely that the *linear* constitutive law used in the derivation of the Navier–Stokes equations presumes that derivatives of the components of velocity are small.

> Here we consider one particular model introduced by Ladyzhenskaya. The study of this model may be justified through a variety of physical and mathematical arguments. In the first place, for certain values of a parameter appearing in the model, the model still conforms with the definition of a fluid as given by Stokes. For the incompressible flow of a viscous fluid, the laws of conservation of mass and momentum, which no one questions, provide an underdetermined system of partial differential equations for the velocity, pressure, and stress fields; in order to close the system a constitutive law relating the stress to the velocity must be provided. The particular form this constitutive relation takes depends on what kind of fluid one is dealing with. Stokes introduced a series of requirements which serve to define an "ordinary" fluid, e.g., water or air. The Stokes hypotheses which define our fluid lead to a specific mathematical form for the nonlinear relation between the stress and the velocity fields. If, *in addition*, one requires that the relation between the stress and velocity fields be *linear*, then one arrives at the Navier–Stokes equations. However, if one retains the Stokes hypothesis defining a fluid and then retains some of the nonlinear terms in the general constitutive relation which a Stokesian fluid must satisfy, then one arrives at the Ladyzhenskaya model considered here. In other words, the Ladyzhenskaya model is derived by combining the principles of conservation of mass and momentum with the rules which define a Stokesian fluid and then retaining some of the nonlinear terms in the resulting constitutive law. The Navier–Stokes model is derived by first invoking exactly the same assumptions *plus* the assumption that the constitutive relation is a linear one. Thus, from a *modeling* standpoint, the Navier–Stokes equations are a special case of the Ladyzhenskaya equations considered here. This leads to the obvious conclusion that any flow which can be accurately described by solution of the Navier–Stokes equations can be at least as accurately described by solutions of the Ladyzhenskaya equations. Incidentally, Ladyzhenskaya also gives a partial justification based on kinetic theory arguments, for why one should retain the nonlinear terms she chooses to include in the constitutive relation.

We should add, of course, that these observations in [DuG] do not take account of the fact that, beyond the linearization involved, the development of the Stokes Law (1.7) also entails the a priori elimination of any dependence, whatsoever, of the reduced stress tensor $\boldsymbol{\tau}$ on higher-order spatial derivatives of $\boldsymbol{v}$; in the developments to follow, we shall see that allowing for such a dependence requires enforcing a set of higher-order boundary conditions for problems posed either in a bounded domain or on the boundary of an obstacle in $\mathbb{R}^2$ or $\mathbb{R}^3$. The higher-order boundary conditions allow for more closely correlating the resulting flow, in the domain in question, with curvature variations on the boundary.

1.2 Multipolar Fluid Dynamics

1.2.1 Introduction

In the preceding section we alluded to the fact that more than 150 years after the formulation of the Stokes Law (1.7), a number of well-defined problems still exist within the context of the associated Navier–Stokes system (1.8), even for such ordinary viscous fluids as water and air; these problems, when viewed within the context of the actual mathematical process which yields the constitutive law (1.7), led Nečas and Šilhavý [NS1] to develop a thermodynamic theory of constitutive equations for multipolar viscous fluids using the framework of the theory of Green and Rivlin [GrR1, 2]. In [NS1], the authors formulated a theory of viscous fluid flow which is based on the belief that a stronger mechanism of dissipation and viscosity, namely, a dependence of the stress on higher-order gradients of velocity, must occur in flows of viscous fluids. In fluids in which higher gradients of velocity influence the response, the rate of work of the internal forces cannot be expected to be only the product of the usual second-order stress tensor with the first gradient of velocity; instead, a more general expression must be assumed containing the sum of products of higher-order multipolar stress tensors with higher gradients of velocity. Otherwise, such materials cannot be compatible with the Clausius-Duhem inequality. In [GrR1, 2] the authors considered only the constitutive equations of elastic nonviscous materials.

The purpose of this section is to develop the thermodynamic theory of constitutive equations of multipolar viscous fluids within the framework of the theory of Green and Rivlin. The postulated constitutive equations express the Helmholtz free energy, entropy, heat flux vector, and the multipolar stress tensors as functions of the following variables: the density and its gradients up to a fixed order, gradients of velocity up to a fixed order, the temperature, and the gradient of temperature. We then derive the general restrictions which the principle of material frame-indifference and the Clausius-Duhem inequality place on the constitutive functions of the fluid. In Sect. 1.3, we restrict our attention to bipolar fluids for which the constitutive quantities depend linearly on the gradients of velocity and temperature, with coefficients independent of temperature and gradients of density. Using representation theorems for isotropic linear functions, we obtain explicit forms for the viscous stresses. The corresponding scalar coefficients, in front of the gradients of velocity in these expressions, generalize (and include as special cases) the classical viscosities. As in the classical case, the Clausius-Duhem inequality yields the nonnegativity of the viscous work which, in its strengthened form, plays a crucial role in the existence theory presented in Chap. 4. The general theory of viscous multipolar fluid flow as presented in Sect. 1.2 does not, a priori, restrict itself to the incompressible case. However, after presenting the general results for linear, viscous, multipolar fluids in Sect. 1.2, the discussion is specialized to the incompressible linear bipolar case in Sect. 1.3; this, in turn, serves to motivate the example of the incompressible, nonlinear bipolar fluid in Sect. 1.4. In Sect. 1.4, following the approach in [GrR1], we use the principle of virtual work to deduce the form of the higher-order boundary conditions associated with the bipolar model; this is done first in the general (compressible) case and then, using some deep results of Heron [HB], for the case in which the velocity satisfies the constraint of incompressibility. In Sect. 1.5, we consider some elementary examples of steady flows of isothermal, incompressible, nonlinear bipolar fluids; these include plane Poiseuille flow, Couette flow between moving parallel plates, and proper Poiseuille flow in a circular pipe. In Sect. 1.6, we offer an overview of the structure of some of the other extensions and generalizations of the Navier–Stokes equations which have appeared in the literature; results concerning specific flows, within the context of some of these other extensions, are presented in Sect. 1.7.

1.2.2 Balance Equations and the Clausius-Duhem Inequality

A detailed discussion of the thermo-mechanics of multipolar continua may be found in [GrR1, 2]; in this section we will content ourselves with presenting a summary of the basic concepts, equations, and inequalities. We use, throughout, Eulerian coordinates so that all of the fields associated with a thermo-mechanical process are functions of the actual positions $\boldsymbol{x}$ of fluid particles at time t; with $\boldsymbol{x}$ we associate the (Eulerian) coordinates x_i, $i = 1, 2, 3$ assuming (without loss of generality) motion in $\mathbb{R}^3$. Let $N \geq 1$ be an integer. Then, we make the following definition:

Definition 1.1. A thermodynamic process of a multipolar fluid of grade N is a collection of $N + 8$ functions of position and time, namely, $\boldsymbol{v}$, θ, ρ, e, η, $\boldsymbol{b}$, r, $\boldsymbol{q}$, and $\boldsymbol{t}^{(k)}$, $k = 0, 1, \ldots, N-1$, whose interpretation is as follows:

$\boldsymbol{v}$, with components v_i, is the velocity field

θ is the field of positive absolute temperature

ρ is the density of the fluid

e is the specific internal energy

η is the specific entropy

$\boldsymbol{b}$, with components b_i, is the specific external body force

r is the rate at which heat is transferred to the fluid

$\boldsymbol{q}$ is the heat flux vector

$\boldsymbol{t}^{(k)}$ with components $t_{ij_1 \ldots j_k j}$, is the spatial multipolar stress tensor of order $k + 2$, $k = 0, 1, \ldots, N-1$.

Remarks. For $k = 0$, $\boldsymbol{t}^{(0)}$ with components t_{ij}, is the usual Cauchy stress tensor. We will assume that $t_{ij_1 \ldots j_k j}$ is symmetric in the indices $j_1, \ldots, j_k$, an assumption which is motivated by the fact that $\boldsymbol{t}^{(k)}$ enters the balance equations only through its products with spatial gradients of velocity, all of which possess the same symmetry. We also assume that any function which appears in this section is continuously differentiable to whatever order is required in order to render meaningful the expressions appearing in the balance equations and the Clausius-Duhem inequality.

Each thermo-mechanical process of a viscous, multipolar fluid must satisfy the equations expressing balance of mass, energy, linear and angular momentum, and the second law of thermodynamics as embodied in the Clausius-Duhem inequality.

In the order just indicated, these equations and inequalities are as follows, where we sum on repeated indices:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho v_i) = 0, \tag{1.18}$$

$$\rho \frac{D}{Dt}\left(e + \frac{1}{2}||\boldsymbol{v}||^2\right) = \frac{\partial}{\partial x_i}\left(-q_i + \sum_{K=0}^{N-1} t_{ij_1\ldots j_k j}\frac{\partial^{k+1} v_j}{\partial x_{j_1}\cdots\partial x_{j_k}\partial x_j}\right) + \rho b_i v_i + \rho r, \tag{1.19}$$

$$\rho \frac{Dv_i}{Dt} = \frac{\partial}{\partial x_j} t_{ij} + \rho b_i, \tag{1.20}$$

$$\rho \frac{D}{Dt}(\epsilon_{jkp} x_k v_p) = \frac{\partial}{\partial x_i}(\epsilon_{jkp} x_k t_{ip} + \epsilon_{jkp} t_{ipk}) + \rho \epsilon_{jkp} x_k b_p \tag{1.21}$$

with $\boldsymbol{\epsilon}$ the usual alternating tensor, and

$$\rho \frac{D\eta}{Dt} \geq -\frac{\partial}{\partial x_i}\left(\frac{q_i}{\theta}\right) + \rho\left(\frac{r}{\theta}\right) \tag{1.22}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v_j \frac{\partial}{\partial x_j} \tag{1.23}$$

denotes the convective (or material time) derivative. We note that the higher-order stresses $\boldsymbol{t}^{(k)}$, $k = 1, \ldots, N-1$, do not enter into the equation of balance of linear momentum, while only $\boldsymbol{t}^{(0)}$ and $\boldsymbol{t}^{(1)}$ enter the equation expressing balance of angular momentum, i.e., (1.21).

Remarks. Green and Rivlin [GrR1, 2] have shown that the equations of balance of linear and angular momentum follow from the equation of balance of energy and the principle of material frame-indifference.

Standard manipulations of the balance equations (see, e.g., [GrR2] or [TN]) yield the following reduced forms for the balance of energy, balance of angular momentum, and the Clausius-Duhem inequality:

$$\rho \frac{De}{Dt} = \sum_{k=0}^{N-1}(t_{ij_1\ldots j_k j} + t_{ij_1\ldots j_k jp,p}) v_{i,j_1\ldots j_k j} - q_{i,i} + \rho r, \tag{1.19$'$}$$

$$\epsilon_{ijk}(t_{jk} + t_{jpk,p}) = 0, \tag{1.21$'$}$$

$$\rho\theta \frac{D\eta}{Dt} \geq \rho \frac{De}{Dt} - \sum_{k=0}^{N-1}(t_{ij_1\ldots j_k j} + t_{ij_1\ldots j_k jp,p}) v_{i,j_1\ldots j_k j} + \frac{q_i \theta_{,i}}{\theta} \tag{1.22$'$}$$

In (1.19′), (1.21′), and (1.22′) a comma followed by an index (or indices) denotes partial differentiation with respect to the coordinate (or coordinates) corresponding to the index (or indices).

Remarks. If we introduce the Helmholtz free energy function $\psi = e - \theta\eta$, then the following dissipation inequality is easily shown to be equivalent to the reduced form (1.22′) of the Clausius-Duhem inequality:

$$\rho \frac{D\psi}{Dt} \leq \sum_{k=0}^{N-1} (t_{ij_1 \ldots j_k j} + t_{ij_1 \ldots j_k jp,p}) v_{i,j_1 \ldots j_k j} - \rho\eta \frac{D\theta}{Dt} - \frac{q_i \theta_{,i}}{\theta}. \tag{1.24}$$

1.2.3 *The General Constitutive Equation for Multipolar Viscous Fluids*

Let $M, K \geq 1$ be prescribed positive integers and let F (respectively, $\boldsymbol{F}$ in vector or tensor-valued cases) stand for any one of the functions e, η, ψ, $\boldsymbol{q}$, or $\boldsymbol{t}^{(k)}$, $k = 0, 1, \ldots, N-1$. With the convention that $\boldsymbol{t}^{(k)} = \boldsymbol{0}$, for $k \geq N$, we consider constitutive relations of the following general form

$$F = F(\rho, \nabla\rho, \ldots \nabla^{M-1}\rho, \nabla\boldsymbol{v}, \ldots, \nabla^K \boldsymbol{v}, \theta, \nabla\theta) \tag{1.25}$$

where all of the functions $(e, \eta, \psi, \text{etc.})$, on the right-hand side of (1.25), are smooth functions of their arguments. Each of the arguments of F are evaluated at $(\boldsymbol{x}, t)$ in the fluid.

Remarks. The form of the constitutive relations, as postulated in (1.25), is motivated by the general continuum mechanics definition of a "fluid" in terms of the symmetry group of the material; in this regard, see [Noll], [Cr], [Sam], and [GVW].

We now make the following definition:

Definition 1.2. The material defined by (1.25) where $F = e, \eta, \psi$, $\boldsymbol{F} = \boldsymbol{q}, \boldsymbol{t}^{(k)}$, $k = 0, \ldots, N-1$ is called a *multipolar viscous fluid* of type (N, M, K).

Remarks. If $M > 1$ or $K > 1$ the multipolar viscous fluid of type (N, M, K) is a nonsimple fluid in the sense of [TN].

Beyond the restrictions imposed by symmetry, which have already been invoked, two general principles of continuum mechanics and thermodynamics are now used to restrict the form of the constitutive equations, namely, the principle of material frame indifference and the Clausius-Duhem inequality; these two basic principles must be satisfied in every thermo-mechanical process of a multipolar viscous fluid of type (N, M, K). In applying the principle of frame indifference, we consider a change of frame of the form

$$\bar{x}_i = Q_{ij}(t)x_j + c_i(t) \tag{1.26}$$

when the $Q_{ij}(t)$ are the components of a (time-dependent) orthogonal matrix, i.e., $Q_{ij}(t)Q_{ik}(t) = \delta_{jk}$, and the $c_i(t)$ are the components of an arbitrary time-dependent vector. For the constitutive theory to be frame-indifferent, the quantities $\theta, \rho, e, \eta, \boldsymbol{q}$, and the $\boldsymbol{t}^{(k)}$ must transform as follows under a change of frame (1.26):

$$\bar{\theta} = \theta,\ \bar{\eta} = \eta,\ \bar{e} = e,\ \bar{\rho} = \rho,\ \bar{r} = r, \tag{1.27a}$$

$$\bar{q}_i = Q_{ij}q_j, \tag{1.27b}$$

$$\bar{t}_{ij_1\ldots j_k j} = Q_{il}Q_{j_1m_1}\cdots Q_{j_km_k}Q_{jm}t_{lm_1\ldots m_km}. \tag{1.27c}$$

The arguments on the left-hand side of (1.27a,b,c) are $(\bar{\boldsymbol{x}}, t)$ while those on the right-hand side are $(\boldsymbol{x}, t)$. The transformation laws for the gradients of density, temperature, and velocity, under the change of frame (1.26), are given by

$$\bar{\rho}_{,i_1\ldots i_k} = Q_{i_1j_1}\cdots Q_{i_kj_k}\rho_{,j_1\ldots j_k}, \tag{1.28a}$$

$$\bar{\theta}_{,i} = Q_{ij}\theta_{,j}, \tag{1.28b}$$

$$\bar{v}_{i,j} = Q_{il}Q_{jm}v_{l,m} + W_{ij}, \tag{1.28c}$$

$$\bar{v}_{i,j_1\ldots j_k} = Q_{il}Q_{j_1m_1}\cdots Q_{j_km_k}v_{l,m_1\ldots m_k} \quad (k \geq 2) \tag{1.28d}$$

where

$$W_{ij} = \dot{Q}_{im}Q_{jm}, \quad W_{ij} = -W_{ji}. \tag{1.28e}$$

As a consequence of (1.28c,e), the rate of deformation tensor $e_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i})$ transforms as

$$\bar{e}_{ij} = Q_{il}Q_{jm}e_{lm}. \tag{1.29}$$

The proof of the following theorem, which we omit, is a relatively simple exercise which results from combining the transformation rules (1.27a–c) and (1.28a–e), i.e.,

Theorem 1.1. *A multipolar viscous fluid of type (N, M, K) satisfies the principle of material frame-indifference if and only if*

(i) the functions $e, \eta, \psi, \boldsymbol{q}$, and $\boldsymbol{t}^{(k)}$ depend on the first spatial gradient of velocity only through its symmetric part $\boldsymbol{e}$, i.e., $F = e, \eta, \psi$, $\boldsymbol{F} = \boldsymbol{q}, \boldsymbol{t}^{(k)}$ satisfy

$$F(\rho, \nabla\rho, \ldots, \nabla^{M-1}\rho, \nabla\boldsymbol{v}, \ldots, \nabla^K\boldsymbol{v}, \theta, \nabla\theta)$$
$$= F(\rho, \nabla\rho, \ldots, \nabla^{M-1}\rho, \boldsymbol{e}, \nabla^2\boldsymbol{v}, \ldots, \nabla^K\boldsymbol{v}, \theta, \nabla\theta). \tag{1.30}$$

(ii) The constitutive functions $e, \eta, \psi, \boldsymbol{q}, \boldsymbol{t}^{(k)}$ *are isotropic scalar, vector, and tensor-valued functions of the scalar, vector, or tensor arguments* ρ, $\nabla\rho$, ..., $\nabla^{M-1}\rho$, $\boldsymbol{e}$, $\nabla^2\boldsymbol{v}$, ..., $\nabla^K\boldsymbol{v}$, θ, *and* $\nabla\theta$.

Remarks. The second part of Theorem 1.1 means, e.g., that the k-th multipolar stress tensor $\boldsymbol{t}^{(k)}$ satisfies the functional equation

$$\begin{aligned} & t_{ij_1\ldots j_k j}(\bar{\rho}, \nabla\bar{\rho}, \ldots, \nabla^{M-1}\bar{\rho}, \bar{e}, \nabla^2\bar{\boldsymbol{v}}, \ldots, \nabla^K\bar{\boldsymbol{v}}, \bar{\theta}, \nabla\bar{\theta}) \\ &= Q_{il}Q_{j_1m_1}\cdots Q_{j_km_k}Q_{jm}t_{lm_1\ldots m_km}(\rho, \nabla\rho, \ldots, \nabla^{M-1}\rho, e, \nabla^2\boldsymbol{v}, \ldots, \nabla^K\boldsymbol{v}, \theta, \nabla\theta). \end{aligned} \tag{1.31}$$

1.2.4 Consequences of the Clausius-Duhem Inequality

We now demand, following Coleman and Noll [CN], that the Clausius-Duhem inequality be satisfied in every process compatible with the constitutive equations and the equations of balance of energy and linear momentum; these contain the external sources r and $\boldsymbol{b}$, respectively, and it is essential to admit that r and $\boldsymbol{b}$ can be arbitrary functions of $\boldsymbol{x}$, t. In view of the form of the constitutive equations this means that all possible motions and all possible variations of the absolute temperature are admissible.

To facilitate the statement of the restrictions which the Clausius-Duhem inequality places on the constitutive functions, we make the following definitions:

Definition 1.3. The equilibrium part of the multipolar stress tensor $\boldsymbol{t}^{(k)}$ is defined to be

$$\boldsymbol{t}^{(k)}_{(E)}(\rho, \nabla\rho, \ldots, \nabla^{M-1}\rho, \theta) \equiv \boldsymbol{t}^{(k)}(\rho, \nabla\rho, \ldots, \nabla^{M-1}\rho, 0, \ldots, 0, \theta, 0) \tag{1.32a}$$

for $k = 0, 1, \ldots, N-1$, while the viscous part of $\boldsymbol{t}^{(k)}$ is given by

$$\begin{aligned} \boldsymbol{t}^{(k)}_{(V)}&(\rho, \nabla\rho, \ldots, \nabla^{M-1}\rho, \nabla\boldsymbol{v}, \ldots, \nabla^K\boldsymbol{v}, \theta, \nabla\theta) \\ &= \boldsymbol{t}^{(k)}(\rho, \nabla\rho, \ldots, \nabla^{M-1}\rho, \nabla\boldsymbol{v}, \ldots, \nabla^K\boldsymbol{v}, \theta, \nabla\theta) \\ &\quad - \boldsymbol{t}^{(k)}_{(E)}(\rho, \nabla\rho, \ldots, \nabla^{M-1}\rho, \theta) \end{aligned} \tag{1.32b}$$

for $k = 0, 1, 2, \ldots, N-1$.

Remarks. The viscous part, $\boldsymbol{t}^{(k)}_{(V)}$ of the multipolar stress tensor $\boldsymbol{t}^{(k)}$ is often called the residual stress tensor and is denoted from this point on by $\boldsymbol{\tau}^{(k)}$; thus,

$$\boldsymbol{t}^{(k)} = \boldsymbol{t}^{(k)}_{(E)} + \boldsymbol{\tau}^{(k)} \tag{1.33}$$

$k = 0, 1, \ldots, N-1$, where the arguments of the tensor functions in (1.33) are as indicated in (1.32b).

Remarks. Following Nečas and Šilhavý [NS1] we set

$$\sigma = \ln \rho \tag{1.34}$$

in which case the dependence of the constitutive functions on ρ, and its gradients through order $M-1$, is replaced by a dependence on σ and its gradients.

The next definition is central to the rest of the development in this subsection:

Definition 1.4. A multipolar viscous fluid satisfies the second law of thermodynamics if the Clausius-Duhem inequality (1.22) holds in every process which is compatible with the constitutive equations (1.25) and the equations of balance of mass (1.18), energy (1.19), and linear momentum (1.20).

The major result in this section is given by Theorem 1.2, below; for the proof we refer the interested reader to the original paper [NS1].

Theorem 1.2. *A multipolar viscous fluid of type* (N, M, K) *satisfies the second law of thermodynamics if and only if, in every process, the generalized Gibbs equation*

$$\begin{aligned}\rho\frac{D\psi}{Dt} = -\rho\eta\frac{D\theta}{Dt} + \sum_{k=0}^{N-1}(t_{(E)ij_1\ldots j_k j} + t_{(E)ij_1\ldots j_k jp,p})v_{i,j_1\ldots j_k j}\\ - \sum_{k=1}^{N-1}\frac{\partial}{\partial\theta}t_{(E)ji_1\ldots j_k p}\theta_{,p}v_{j,j_1\ldots j_k}\end{aligned} \tag{1.35}$$

holds, as well as the residual dissipation inequality

$$\sum_{k=0}^{N-1}(\tau_{ij_1\ldots j_k j} + \tau_{ij_1\ldots j_k jp,p})v_{i,j_1\ldots j_k j} + \sum_{k=1}^{N-1}\frac{\partial}{\partial\theta}\tau_{(E)jj_1\ldots j_k p}\theta_{,p}v_{j,j_1\ldots j_k} - q_i\theta_{,i}/\theta \geq 0. \tag{1.36}$$

In addition, F *is independent of both* $\nabla\boldsymbol{v}, \ldots, \nabla^K\boldsymbol{v}$, *and* $\nabla\theta$, *whenever* F *stands for any one of the constitutive functions* ψ, η, *or* e, *i.e., for* $F = \psi$, η, *or* e,

$$F(\rho, \nabla\rho, \ldots, \nabla^{M-1}\rho, \nabla\boldsymbol{v}, \ldots, \nabla^K\boldsymbol{v}, \theta, \nabla\theta) = F(\rho, \nabla\rho, \ldots, \nabla^{M-1}\rho, \theta) \tag{1.37}$$

throughout the domain of the constitutive functions. Finally, both the entropy relation

$$\eta = -\frac{\partial\psi}{\partial\theta} \tag{1.38}$$

and the generalized stress relations

$$\mathrm{Sym}(t_{(E)ij_1\ldots j_k j} + t_{(E)ij_1\ldots j_k jp,p} - \frac{\partial}{\partial\theta} t_{(E)ij_1\ldots j_k jp}\theta_{,p}) = -\rho\,\mathrm{Sym}\left(\frac{\partial\psi}{\partial\sigma_{,j_1\ldots j_k}}\delta_{ij}\right) \quad (1.39)$$

for $k = 0, 1, \ldots, N-1$, *hold throughout the domain of the constitutive functions, where* Sym *denotes symmetrization with respect to the indices* $i, j_1, \ldots, j_k$.

Remarks. In the proof of Theorem 1.2, the various constitutive functions are expressed in terms of $\sigma = \ln\rho$ and its gradients, e.g.,

$$\psi = \psi(\sigma, \nabla\sigma, \ldots, \nabla^{M-1}\sigma, \nabla\boldsymbol{v}, \ldots, \nabla^K\boldsymbol{v}, \theta, \nabla\theta). \quad (1.40)$$

Our final result in this subsection serves to place upper bounds on the order of polarity N in terms of the orders M, K of the gradients of density and velocity, i.e., we have the following result, a proof of which may be found in [NS1]:

Theorem 1.3. *If a multipolar viscous fluid of type* (N, M, K) *satisfies the second law of thermodynamics and* $k \geq \max\{M, K+1\}$ *then in every process*

$$\mathrm{Sym}(t_{ij_1\ldots j_k j} + t_{ij_1\ldots j_k jp,p}) = 0. \quad (1.41)$$

Furthermore, if $t_{ij_1\ldots j_k j}$ *is symmetric in* $i, j_1, j_2, \ldots, j_k$ *for* $k \geq \max\{M, K+1\}$ *then* $N \leq K+1$ *and* $N \leq M$.

1.2.5 Linear Multipolar Viscous Fluids

For the balance of this section we will confine the discussion to the structure of constitutive functions for linear multipolar viscous fluids. We assume that only the density ρ, and not its gradients, enter the constitutive equations so that $M = 1$. Our basic assumption in this subsection is that the viscous stresses and the heat flux vector depend linearly on the gradients of the velocity field and the gradient of the temperature field. Our first result is the following:

Theorem 1.4. *If a multipolar viscous fluid of type* $(N, 1, K)$ *satisfies the second law of thermodynamics, and the principle of material frame-indifference, then*

$$t^{(0)}_{E\,ij} = -p\delta_{ij} \quad (1.42)$$

where

$$p = \rho^2\frac{\partial\psi}{\partial\rho}. \quad (1.43)$$

Furthermore, if k *is odd, then*

$$\boldsymbol{t}_E^{(k)} = \mathbf{0} \tag{1.44}$$

while if $k > 0$ *and even*

$$\boldsymbol{t}_E^{(k)} = \boldsymbol{t}_E^{(k)}(\theta) \tag{1.45}$$

and

$$\operatorname{Sym} t_{E\, ij_1 \ldots j_k j} = 0. \tag{1.46}$$

Proof. We observe that the transformation law (1.31) for the total multipolar stress tensor implies analogous laws for the equilibrium and viscous parts of the stress. Therefore,

$$t_{E\, j_1 \ldots j_{k+2}}^{(k)}(\rho, \theta) = Q_{j_1 m_1} \cdots Q_{j_{k+2} m_{k+2}} t_{E\, m_1 \ldots m_{k+2}}^{(k)}(\rho, \theta) \tag{1.47}$$

for every $\boldsymbol{Q}$ which is orthogonal. Consequencely $\boldsymbol{t}_E^{(k)}$ is an isotropic tensor of order $k+2$. If k is odd, then the only isotropic tensor of order $k+2$ is $\mathbf{0}$ (it is sufficient to set $Q_{ij} = -\delta_{ij}$ in (1.47)); this establishes (1.44). Formulas (1.42), (1.43) result from combining the generalized stress relation (1.39) with the fact that $\boldsymbol{t}_E^{(1)} = \mathbf{0}$. If k is even, and $k > 0$, then (1.39) with k replaced by $k-1$ yields

$$\operatorname{Sym}\left(t_{E\, ij_1 \ldots j_{k-1} j}^{(k-1)} + \frac{\partial}{\partial \rho} t_{E\, ij_1 \ldots j_k j p}^{(k)} \rho_{,p} \right) = 0. \tag{1.48}$$

However, as $k-1$ is odd, the first term in (1.48) vanishes by virtue of (1.44); moreover, the symmetrization is irrelevant as $t_{ij_1 \cdots j_{k-1} jp}^{(k)}$ is symmetric in $j_1 \ldots j_{k-1} j$. Thus,

$$\frac{\partial}{\partial \rho} t_{E\, ij_1 \ldots j_{k-1} jp}^{(k)} \rho_{,p} = 0. \tag{1.49}$$

This last result implies that

$$\frac{\partial}{\partial \rho} t_{E\, ij_1 \ldots j_{k-1} jp}^{(k)} = 0 \tag{1.50}$$

from which (1.45) follows. Finally, by combining (1.39) with the fact that $\boldsymbol{t}_E^{(k+1)} = \mathbf{0}$, for k even, we obtain (1.46). □

We now have the following definition of a linear, viscous multipolar fluid:

Definition 1.5. A multipolar viscous fluid of type (N, M, K) is said to be linear if $M = 1$ and for every ρ, θ the quantities

$$\boldsymbol{\tau}^{(k)} = \boldsymbol{\tau}^{(k)}(\rho, \nabla \boldsymbol{v}, \ldots, \nabla^K \boldsymbol{v}, \theta, \nabla\theta) \tag{1.51a}$$

and

$$\boldsymbol{q} = \boldsymbol{q}(\rho, \nabla \boldsymbol{v}, \ldots, \nabla^K \boldsymbol{v}, \theta, \nabla\theta) \tag{1.51b}$$

depend linearly on $\nabla \boldsymbol{v}, \ldots, \nabla^K \boldsymbol{v}$, and $\nabla\theta$.

Our first characterization of a linear, viscous multipolar fluid is

Theorem 1.5. *If a linear viscous multipolar fluid of type $(N, 1, K)$ satisfies the principle of material frame-indifference, then $\boldsymbol{\tau}^{(k)}$, with k even, depends on ρ, θ, and the odd-order spatial gradients of $\boldsymbol{v}$, i.e.,*

$$\boldsymbol{\tau}^{(k)} = \boldsymbol{\tau}^{(k)}(\rho, \theta, \nabla \boldsymbol{v}, \nabla^3 \boldsymbol{v}, \ldots, \nabla^L \boldsymbol{v}), \ k \text{ even} \tag{1.52a}$$

while $\boldsymbol{q}$ and $\boldsymbol{\tau}^{(k)}$, with k odd, depend on ρ, θ, and the even-order spatial gradients of $\boldsymbol{v}$, i.e.,

$$\boldsymbol{q} = \boldsymbol{q}(\rho, \theta, \nabla^2 \boldsymbol{v}, \ldots, \nabla^{L-1} \boldsymbol{v}, \nabla\theta) \tag{1.52b}$$

and

$$\boldsymbol{\tau}^{(k)} = \boldsymbol{\tau}^{(k)}(\rho, \theta, \nabla^2 \boldsymbol{v}, \ldots, \nabla^{L-1} \boldsymbol{v}, \nabla\theta), \ k \text{ odd} \tag{1.52c}$$

where L is an odd positive integer which depends on K.

Proof. We offer a sketch of the proof only. By linearity, the expressions for $\boldsymbol{q}$ and $\boldsymbol{\tau}^{(k)}$ are sums of a number of terms depending linearly on the gradients of velocity of different orders and of a term depending linearly on $\nabla\theta$. The transformation laws for the stresses and the heat flux vector under changes of frame must be satisfied by each of these linear terms separately. Using these transformation laws, with $Q_{ij} = -\delta_{ij}$, and counting the number of rotations in these transformation laws for each of the linear terms, one readily sees that only the terms with the orders of the gradients of velocity indicated in assertions (1.52a,b,c) can be nonzero. The details are omitted. □

The representation theorem, below, is essential for the discussion of linear bipolar fluids in Sect. 1.3, which, in turn, serves to motivate the form of the constitutive theory for the incompressible bipolar fluid in Sect. 1.4.

Theorem 1.6. *In a linear, viscous multipolar fluid satisfying the principle of material frame-indifference* $\boldsymbol{\tau}^{(0)}$, $\boldsymbol{\tau}^{(k)}$, *and* $\boldsymbol{q}$ *have the form*

$$\begin{aligned}\tau_{ij} = \lambda v_{k,k}\delta_{ij} + \mu(v_{i,j} + v_{j,i}) + \sum_{r=0}^{N-2}(\alpha^{(r)}\delta_{ij}\Delta^{r+1}v_{k,k} + \beta_1^{(r)}\Delta^{r+1}v_{i,j} \\ + \beta_2^{(r)}\Delta^{r+1}v_{j,i} + \gamma^{(r)}\Delta^r v_{k,kij}),\end{aligned} \tag{1.53}$$

$$\begin{aligned}\tau_{ijk} = \sum_{r=0}^{N-2}\Big(c_1^{(r)}\delta_{ijk}\Delta^{r+1}v_k + c_2^{(r)}\delta_{ijk}\Delta^r v_{n,nk} + c_3^{(r)}\delta_{ik}\Delta^{r+1}v_j + c_4^{(r)}\delta_{ik}\Delta^r v_{m,mj} \\ + c_5^{(r)}\delta_{jk}\Delta^{r+1}v_i + c_6^{(r)}\delta_{jk}\Delta^r v_{m,mi} + c_7^{(r)}\Delta^r v_{i,jk} + c_8^{(r)}\Delta^r v_{k,ij} \\ + c_9^{(r)}\Delta^r v_{j,ki} + c_{10}^{(r)}\Delta^{r-1}v_{m,mijk}\Big) \\ + c_{11}\delta_{ij}\theta_{,k} + c_{12}\delta_{ik}\theta_{,j} + c_{13}\delta_{jk}\theta_{,i}\end{aligned} \tag{1.54}$$

and

$$q_i = \sum_{r=0}^{N-2}\Big(d_1^{(r)}\Delta^r v_{m,mi} + d_2^{(r)}\Delta^{r+1}v_i\Big) - k\theta_{,i}. \tag{1.55}$$

Here $\lambda, \mu, \alpha^{(r)}, \beta_1^{(r)}, \beta_2^{(r)}, \gamma^{(r)}, c_1^{(r)}, \ldots, c_{13}, d_1^{(r)}, d_2^{(r)}$ *and* k *are all scalar functions of* ρ, θ *such that* $c_{10}^{(r)}$ *satisfies* $c_{10}^0 = 0$. *If the body satisfies the reduced equation of balance of angular momentum, then*

$$\beta_1^{(r)} + c_5^{(r)} + c_7^{(r)} = \beta_2^{(r)} + c_3^{(r)} + c_9^{(r)} \tag{1.56}$$

for $r = 0, 1, \ldots, N-2$. *Similar, but more complicated expressions can be obtained also for* $\boldsymbol{\tau}^{(k)}$ *with* $k \geq 2$.

Proof. In view of the transformation laws for $\boldsymbol{\tau}^{(k)}$ and $\boldsymbol{q}$ under changes of frame, the coefficients in the expressions for $\boldsymbol{q}$ and $\boldsymbol{\tau}^{(k)}$, in front of the gradients of velocity and the gradient of temperature, must be isotropic tensors. Using the general forms for isotropic tensors (see [Sp]) and the symmetry of the gradients of $\boldsymbol{v}$, one eventually arrives at (1.54)–(1.56). The details are omitted. □

The analysis presented in [NS1] offers several other characterizations of the constitutive theory for linear, multipolar viscous fluids; however, as none of these characterizations are essential for our discussion of the linear, incompressible bipolar fluid in Sect. 1.3, we will simply refer the interested reader to Sect. 5 of [NS1] for further details. A survey of the basic concepts associated with the theory of viscous multipolar fluids, in both the compressible and incompressible cases, may be found in the paper of Novotný [Nov], as well as [N3, 4]. For an analysis of the compressible bipolar problem we refer the reader to the series of papers [NeN], [NN], [NNS1, 2, 3] and [NS2].

1.3 The Linear Bipolar Fluid

In this section our goal is to specialize the discussion in Sect. 1.2.5 to account for the situation in which the fluid is bipolar (i.e., $N = 2$) as well as isothermal and incompressible. Consistent with the analysis in Sect. 1.2 we set $t_{ij_1\ldots j_N j} \equiv 0$ and suppose that $t_{E\,ij_1\ldots j_k j} \equiv 0$, for $k \geq 1$. We also postulate that $t_{ij} = t_{ji}$ and that $\frac{\partial}{\partial x_i} t_{pki} = \frac{\partial}{\partial x_i} t_{kpi}$; these two latter conditions, together with (1.20), imply that (1.21) is satisfied. In addition, we assume that for $k = 0, 1, \ldots, N-1$,

$$\tau_{ij_1\ldots j_k j} = \tau_{ij_1\ldots j_k j}(\nabla \boldsymbol{v}, \ldots, \nabla^K \boldsymbol{v}) \tag{1.57a}$$

and

$$\boldsymbol{q} = -\kappa \nabla \theta, \quad \kappa > 0 \text{ (Fourier Law of Heat Conduction).} \tag{1.57b}$$

It then follows, as a consequence of (1.57a) and the residual dissipation inequality (1.36), that

$$\sum_{k=0}^{N-1} (\tau_{ij_1\ldots j_k j} + \tau_{ij_1\ldots j_k jp,p}) \frac{\partial^{k+1} v_j}{\partial x_{j_1} \cdots \partial x_{j_k} \partial x_j} \geq 0 \tag{1.58}$$

and

$$t_{E\,ij} \equiv t^{(0)}_{E\,ij} = -p(\rho, \theta)\delta_{ij} \tag{1.59}$$

where, as in (1.43), $p = \rho^2 \frac{\partial \psi}{\partial \theta}$. In Sect. 1.2.5 we showed (Theorem 1.5) that for a linear, viscous multipolar fluid the principle of material frame-indifference implies that the $\tau_{ij_1\ldots j_k j}$ with k even, $k = 0, \ldots, N-1$, depend only on odd-order gradients of $\boldsymbol{v}$ while the $\tau_{ij_1\ldots j_k j}$ with k odd depend only on even-order gradients of $\boldsymbol{v}$. Also, material frame-indifference implies the general representation for τ_{ij} displayed in (1.53). Therefore, for a bipolar linear viscous fluid, it follows from (1.53), with $N = 2$, that

$$\tau_{ij} = \lambda v_{k,k} + \mu(v_{i,j} + v_{j,i}) + \alpha \delta_{ij} \nabla v_{k,k} + \beta_1 \Delta v_{i,j} + \beta_2 \Delta v_{j,i} + \gamma v_{k,kij}. \tag{1.60}$$

In a similar manner, with $N = 2$, (1.54) reduces to

$$\begin{aligned} \tau_{ijk} = {} & c_1 \delta_{ij} \Delta v_k + c_2 \delta_{ij} v_{m,mk} + c_3 \delta_{ij} \Delta v_j + c_4 \delta_{ik} v_{m,mj} \\ & + c_5 \delta_{jk} \Delta v_i + c_6 \delta_{jk} v_{m,m} + c_7 v_{i,jk} + c_8 v_{k,ij} + c_9 v_{j,ki}. \end{aligned} \tag{1.61}$$

In (1.60), (1.61) we have dropped the superscripts on the coefficients, i.e., $c_1 = c_1^{(0)}$, etc. We have also used the assumpion that the process is isothermal and applied the restriction $c_{10}^{(0)} = 0$.

Remarks. For $N \geq 2$ an example of a set of constitutive equations, for a linear multipolar viscous fluid, which satisfies both the principle of material frame-indifference and the Clausius-Duhem inequality (1.22) is given by the following expression:

$$\tau_{ij_1\cdots j_m j} = \sum_{l=m}^{N-1} (-1)^{l+m} \Delta^{l-m} \frac{\partial^m g_{ij}^l}{\partial x_{j_1} \cdots \partial x_{j_m}} \tag{1.62}$$

with

$$g_{ij}^l = \lambda_l v_{k,k} \delta_{ij} + 2\mu_l e_{ij}(\boldsymbol{v}), \quad \mu_l \geq 0,\ \lambda_l \geq \frac{-2\mu_l}{3} \tag{1.63a}$$

and

$$\tau_{ij_1\cdots j_m j} \equiv 0,\ \text{for } 1 \leq m \leq N-1. \tag{1.63b}$$

We now turn to the special case of an incompressible, linear, multipolar fluid; this case also serves to highlight the basic viewpoint of multipolarity. In order to be somewhat specific we will concentrate on the problem of steady flow between parallel plates at $x_2 = \pm 1$. We suppose that $p = p(x_1, x_2)$ while

$$\frac{\partial v_i}{\partial x_i} = 0, \quad \rho = \text{const.}, \tag{1.64a}$$

$$\rho \frac{\partial v_i}{\partial x_j} v_j + \frac{\partial p}{\partial x_i} - \frac{\partial \tau_{ij}}{\partial x_j} = 0. \tag{1.64b}$$

With the classical Stokes Law

$$\tau_{ij} = \lambda_0 \operatorname{div} \boldsymbol{v} \delta_{ij} + 2\mu_0 e_{ij}(\boldsymbol{v}), \tag{1.64c}$$

$\lambda_0 \geq -\frac{2}{3}\mu_0$, $\mu_0 > 0$, we obtain the usual steady-state Navier–Stokes equations. Assuming that the flow has the form $v_1 = v_1(x_2)$, $v_2 = 0$, $v_3 = 0$ (plane Poiseuille flow between fixed parallel plates), (1.64b,c), with $\operatorname{div} \boldsymbol{v} = 0$, reduce to

$$\frac{\partial p}{\partial x_1} - \mu_0 v_1'' = 0, \quad \frac{\partial p}{\partial x_2} = 0 \tag{1.65}$$

so that $p = p_0 + p_1 x_1$. Thus

$$p_1 - \mu_0 v_1'' = 0 \tag{1.66}$$

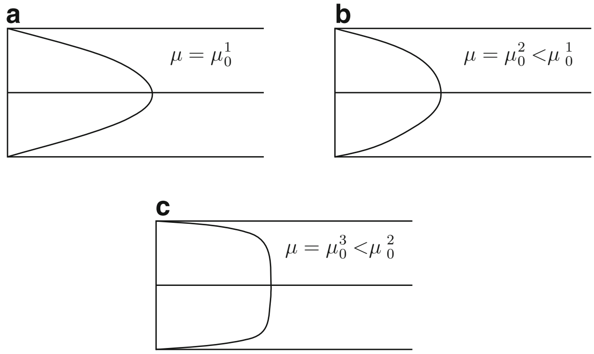

Fig. 1.1 Sequence of linear bipolar viscous profiles

and as $\boldsymbol{v} = \mathbf{0}$, for $x_2 = \pm 1$, we obtain the well-known result [Sh]

$$v_1(x_2) = -\frac{p_1}{2\mu_0}(1 - x_2^2), \quad -1 \le x_2 \le 1. \tag{1.67}$$

If we set $P = \int_{-1}^{1} v_1(x_2)\, dx_2$, then $v_1 = 3P(1 - x_2^2)/4$. A major problem with (1.67) is the fact that the form of the profile (1.67) remains parabolic as $\mu_0 \to 0^+$. From experimental observations, and the Prandtl boundary-layer theory (which is in reasonable accord with such observations) we know that as $\mu_0 \to 0^+$ the resulting profiles must flatten out with respect to the axis $x_2 = 0$ and must approach the boundaries at $x_2 = \pm 1$ in an ever increasing tangential fashion. More precisely, experimental observations and the Prandtl boundary-layer equations lead us to expect, for a sequence of viscosities $\{\mu_0^n\}$ with $\mu_0^n \to 0$, as $n \to \infty$, a sequence of progressively flattened profiles such as those depicted in Fig. 1.1a–c.

In order to gauge the effect of multipolarity on the velocity profile (1.67) we will consider special cases of (1.60), (1.61); to this end we set

$$\begin{cases} \tau_{ij}^0 = \lambda_0 \delta_{ij} \operatorname{div} \boldsymbol{v} + 2\mu_0 e_{ij} \quad (\lambda_0 \ge 0,\ \mu_0 \ge 0), \\ \tau_{ij}^1 = \lambda_1 \delta_{ij} \operatorname{div} \boldsymbol{v} + 2\mu_1 e_{ij} \quad (\lambda_1 \ge 0,\ \mu_1 \ge 0), \end{cases} \tag{1.68}$$

$$\tau_{E\,ij} = -p\delta_{ij}, \quad \tau_{E\,ijk} = 0, \tag{1.69}$$

$$\tau_{ij} = \tau_{ij}^0 - \Delta\tau_{ij}^1, \quad \tau_{E\,ijk} = \frac{\partial \tau_{ij}^1}{\partial x_k}, \tag{1.70}$$

then we easily find that

$$\tau_{ij} = -p\delta_{ij} + \lambda_0\delta_{ij} \operatorname{div} \boldsymbol{v} + 2\mu_0 e_{ij} - \lambda_1 \delta_{ij} \Delta \operatorname{div} \boldsymbol{v} - 2\mu_1 \Delta e_{ij}, \tag{1.71}$$

$$\tau_{ijk} = \lambda_0 \delta_{ij} v_{l,lk} + 2\mu_1 \frac{\partial e_{ij}}{\partial x_k}. \tag{1.72}$$

Clearly, (1.71), (1.72) are special cases of (1.60), (1.61); with the assumption of incompressibility, the full set of constitutive relations for an isothermal, linear, bipolar fluid reduce to

$$t_{ij} = -p\delta_{ij} + 2\mu_0 e_{ij} - 2\mu_1 \Delta e_{ij}, \tag{1.73}$$

$$\tau_{ijk} = 2\mu_1 \frac{\partial e_{ij}}{\partial x_k}. \tag{1.74}$$

Assume, again, that the flow has the form $v_1 = v_1(x_2)$, $v_2 = 0$, $v_3 = 0$, with $p = p(x_1, x_2)$; substituting this information into (1.73), and then the resultant form of τ_{ij} into (1.64b), we find that

$$\mu_0 v_1''(x_2) - \mu_1 v_1''''(x_2) = p_1(\text{const.}). \tag{1.75}$$

Besides the usual non-slip boundary conditions which must be satisfied, i.e., $v_1(\pm 1) = 0$, it will be shown, in Sect. 1.4, that on the boundary $\partial\Omega$ of an open, bounded domain Ω in R^n, $n = 2, 3$ (provided the boundary is sufficiently smooth) we must also satisfy

$$\tau_{ijk}\nu_j\nu_k - \tau_{jkl}\nu_j\nu_k\nu_l\nu_i = M_i, \tag{1.76}$$

$i = 1, \ldots, n$. In (1.76), $\boldsymbol{\nu}$ is the exterior unit normal to $\partial\Omega$ while $\boldsymbol{M}$ is the prescribed moment on the boundary. If we take $\boldsymbol{M} = \mathbf{0}$, then for flow between parallel plates at $x_2 = \pm 1$, (1.76) implies that $v_1''(\pm 1) = 0$. A simple calculation now shows that the solution of the boundary-value problem for (1.75) is given by

$$v_1(x_2) = \frac{P}{\frac{4}{3} - \frac{4}{\eta^2}\left(1 - \frac{1}{\eta}\frac{\sinh\eta}{\cosh\eta}\right)} \left(1 - x_2^2 + \frac{2}{\eta^2}\frac{\cosh\eta x_2}{\cosh\eta} - \frac{2}{\eta^2}\right) \tag{1.77}$$

where $\eta = \sqrt{\mu_0/\mu_1}$ and, again, $P = \int_{-1}^{1} v_1(x_2)\,dx$. As $\mu_1 \to 0^+$, it is not difficult to show that the profile given by (1.77) converges to the profile predicted by the classical Navier–Stokes relation (1.64c), with div $\boldsymbol{v} = 0$, i.e., to (1.67). We also note that it follows from (1.77) that for $\eta = \infty$, $v_1(0) = 3P/4$, while for $\eta \to 0^+$, $v_1(0) = 25P/32$; in fact for $\mu_0 = 0$, $\mu_1 > 0$ we calculate that

$$v_1(x_2) = \frac{5}{32}P\left(5 - 6x_2^2 + x_2^4\right). \tag{1.78}$$

Thus, *linear multipolarity has only a minor perturbative effect on the velocity profile for this particular steady flow*; the profiles are still distinctly parabolic in character and *do not exhibit the "flattening out" phenomenon which is predicted by the boundary layer theory for the classical Navier–Stokes equations in the case of vanishing kinematic viscosity*. A similar situation occurs within the context of the linear dipolar fluid model of Bleustein and Green [BG]; this will be demonstrated in Sect. 1.7 when we review the consequences of applying their model to the problem of Poiseuille flow in a circular cylinder.

An excellent picture, for the case of Poiseuille flow between parallel plates, of the actual velocity profiles at large Reynolds numbers, may be gleaned from Fig. 82 of [GO]; in this case flattened profiles are observed at the inlet to the channel (the Reynolds number being large, not because of the small magnitude of the kinematic viscosity, but, rather, because of the small distance traversed by the fluid); these profiles correspond precisely to those depicted in our Fig. 1.1 for $\mu < \mu_0^3$. The remedy to the problems described above is to move away from the linearity built into the constitutive relations (1.71), (1.72), for the bipolar fluid, and toward a slightly nonlinear version of these constitutive relations, one which is completely admissible within the context of the original Nečas-Šilhavý [NS1] formulation of the basic theory.

1.4 The Nonlinear Bipolar Fluid

1.4.1 Introduction

In this section we will formulate the constitutive theory for incompressible, nonlinear, viscous bipolar fluids that will be used throughout the remainder of the monograph. The constitutive relation we construct here in Sect. 1.4.2 is entirely consistent with the general development in Sect. 1.2, is in accord, therefore, with both the principle of material frame-indifference and the second law of thermodynamics for isothermal continua, and serves as a natural nonlinear extension of the linear theory discussed in Sect. 1.3. When the constitutive parameter α in the theory vanishes we recover the linear, viscous, incompressible bipolar model of Sect. 1.3; if only the higher-order viscosity μ_1 vanishes, then we recover, in essence, the nonlinear generalization of the Navier–Stokes theory studied by Ladyzhenskaya [La2] and [DuG] for α in a certain range. For $\alpha = 0$ and $\mu_1 = 0$ the model reduces to the one based on Stokes Law (1.7) and, thus, yields the incompressible Navier–Stokes equations. When $\mu_1 > 0$ the system of partial differential equations governing the velocity in the bipolar model is of fourth order and the simple non-slip boundary condition $\boldsymbol{v} = \mathbf{0}$ on $\partial\Omega$ no longer suffices to produce a well-posed problem, where $\partial\Omega$, assumed to be sufficiently smooth, is the boundary of an open bounded domain in R^n, $n = 2, 3$. In Sects. 1.4.3 and 1.4.4, following the analysis outlined by Toupin [To] for elastic media, we apply the principle of virtual work to deduce the appropriate form of the higher-order boundary conditions that must be appended to the non-slip condition on the boundary of a bounded domain; this is worked out in Sect. 1.4.3 without imposing the constraint of incompressibility on the virtual velocities and velocity gradients. Then, in Sect. 1.4.4, we apply the theory developed by Heron [HB] so as to obtain the appropriate form of the higher-order boundary conditions for incompressible bipolar fluid flow. We note that the conditions derived in Sects. 1.4.3 and 1.4.4 are very different from the (artificial) assumption that the normal derivative of the velocity field vanish in $\partial\Omega$; such an assumption has been empoyed, e.g., in [Lio1] and [OS1, 2] where higher-order derivatives of the velocity field were added onto the Navier–Stokes system of partial differential equations in order to regularize the associated initial-boundary value problem. We shall have more to say about this particular issue in Sects. 1.6 and 4.5.

1.4.2 The Constitutive Relation for a Viscous, Incompressible Nonlinear Bipolar Fluid

The general theory of multipolar fluids, as constructed in Sect. 1.2, allows for very broad types of nonlinearity in the relationships between the stress tensors and velocity gradients. In early work on the phenomenological behavior of viscous fluids, Prandtl suggested replacing the constant viscosity μ_0, in the classical Stokes Law (1.64c), with $\operatorname{div} \boldsymbol{v} = 0$, by a viscosity function μ which was defined to be directly proportional to the first spatial velocity gradient; for the plane Poiseuille flow problem (between parallel plates), which was considered in the last section, this constitutive hypothesis is embodied in the relation $\mu = \mu_0 \left| v_1' \right|$, in which case (1.66) becomes

$$p_1 - \mu(|v_1'|v_1')' = 0. \tag{1.79}$$

Solving (1.79) subject to the boundary conditions $v_1(\pm 1) = 0$ we obtain

$$v_1(x_2) = \frac{5}{6} P(1 - |x_2|^{3/2}), \tag{1.80}$$

a profile which is even further removed from the actual profile that one expects to see in a steady flow situation, as the viscosity grows smaller than the parabolic profile $v_1 = 3P(1 - x_2^2)/4$ predicted by the classical Stokes Law, or the profile (1.77) predicted by the linear theory of the bipolar fluid. The next natural step in this process would be to modify the Prandtl ansatz embodied in (1.79) and allow the viscosity μ to be an arbitrary function of $\left| v_1'(x_2) \right|$ for the case of steady plane Poiseuille flow between parallel plates at $x_2 = \pm 1$. With $\tau_{ij} = 2\mu(|\boldsymbol{e}|)e_{ij}$, (1.79) now becomes

$$p_1 - (\mu(|v_1'|)v_1')' = 0 \tag{1.81}$$

so that

$$p_1 x_2 - \mu(|v_1'|)v_1' = 0. \tag{1.82}$$

In principle, the constant pressure gradient p_1 and the velocity profile $v_1 = v_1(x_2)$ may be identified by experimental procedures; working under the auspices of that assumption, and ignoring for now the possible presence of higher-order derivatives of the velocity field (as (1.81) does), the functional form of μ may be obtained from (1.82) as follows: suppose that $v_1'(x_2)$ is monotone in $(-1, 1)$, odd, and nonnegative on $[-1, 0]$. Writing, in (1.82), $x_2 = v_1'^{-1}(v_1'(x_2))$ we find that, on the range of $\left|v_1'\right|$,

$$p_1 \frac{(v_1')^{-1}(v_1')}{v_1'} = \mu(|v_1'|). \tag{1.83}$$

Let us suppose that our measurements give us a more realistic profile than those produced in Sect. 1.3 (such profiles result, for example, from applications of boundary-layer theory for the steady-state Navier–Stokes equations). More precisely, suppose we observe a profile of the form

$$v_1(x_2) = \frac{P}{2}\frac{1+\delta}{\delta}(1 - |x_2|^{\delta}), \quad \delta \geq 2 \tag{1.84}$$

where, again, $P > 0$ is given by $\int_{-1}^{1} v_1(x_2)\, dx_2$. We look for a solution μ of (1.83) of the form

$$\mu(|v_1'|) = \beta |v_1'|^{(2-\delta)/(\delta-1)} \tag{1.85}$$

with $\beta > 0$ constant. Substituting (1.84) and (1.85) into (1.82) we find that the constitutive hypothesis (1.85) is admissible provided the constant β is given by

$$\beta = |p_1| \left(\frac{2}{P(1+\delta)}\right)^{1/(\delta-1)}. \tag{1.86}$$

For the plane Poiseuille flow governed by the classical Stokes relation we have, from the analysis in Sect. 1.3, that the viscosity μ_0 satisfies

$$\mu_0 = \frac{2|p_1|}{3P} \tag{1.87}$$

and (1.86) should be viewed in the same light. If we set $\alpha = (\delta - 2)/(\delta - 1)$, $\delta \geq 2$, then $0 < \alpha < 1$.

The formal considerations described above provide the motivation to study incompressible, isothermal, bipolar fluids in which the stress tensors t_{ij} and τ_{ijk} are given, respectively, by the following nonlinear modification of (1.73), (1.74): For some $\mu_1 > 0$,

$$t_{ij} = -p\delta_{ij} + 2\mu(|\boldsymbol{e}|)e_{ij} - 2\mu_1 \Delta e_{ij}, \tag{1.88a}$$

$$\tau_{ijk} = 2\mu_1 \frac{\partial e_{ij}}{\partial x_k} \tag{1.88b}$$

where we have replicated (1.74) as (1.88b) and where the nonlinear, lower-order viscosity μ has the form

$$\mu = \mu_0(\epsilon + e_{ij}e_{ij})^{-\alpha/2} \tag{1.89}$$

with $\epsilon \geq 0$ and μ_0 the classical viscosity associated with the Stokes Law (1.7). The heuristic argument presented, above, indicates that the constitutive theory specified by (1.88a,b), (1.89) should be investigated with α in the range $0 < \alpha < 1$. Throughout the course of this monograph, however, we will expand the analysis to include situations where $\alpha < 0$; such an assumption, relative to α, will encompass the Ladyzhenskaya modification of the Navier–Stokes system if we also set $\mu_1 = 0$ in (1.88a,b). Moreover, with $\alpha = 0$, (1.88a,b) reduces to the constitutive theory (1.73), (1.74) for the incompressible, isothermal, linear bipolar fluid, and with $\alpha = \mu_1 = 0$ we recover the constitutive theory which leads to the incompressible Navier–Stokes equations.

Remarks. In many places in this work we will set $\alpha = 2 - p$, in which case (1.89) assumes the form

$$\mu = \mu_0(\epsilon + e_{ij}e_{ij})^{\frac{p-2}{2}}. \tag{1.90}$$

A non-Newtonian fluid with a nonlinear viscosity which conforms to (1.90) is called shear-thinning if $p < 2$ and shear-thickening if $p \geq 2$.

Irrespective of how we express the nonlinear viscosity μ, the most significant point to be made about the constitutive theory embodied in (1.88a,b) is that *it is a nonlinear, isothermal constitutive theory that takes into account the presence of higher-order velocity gradients and is compatible with the basic principles of material frame-indifference and the Clausius-Duhem inequality.* By inserting (1.88a) into the differential form (1.1) of the equation expressing balance of linear momentum, we obtain the system of partial differential equations governing the velocity field in an isothermal, incompressible bipolar fluid, namely,

$$\rho\left(\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} + 2\frac{\partial}{\partial x_j}[\mu(|\boldsymbol{e}|)e_{ij}] - 2\mu_1 \frac{\partial}{\partial x_j}(\Delta e_{ij}) + \rho f_i. \tag{1.91}$$

The system (1.91), and the incompressibility condition $\nabla \cdot \boldsymbol{v} = 0$, are to hold either in all of R^n ($n = 2, 3$), in which case we consider a pure initial-value problem, or in all of R^n with periodic boundary conditions and a base cell of the form $\Omega = [0, L]^n$, $n = 2, 3$, or in the exterior of a bounded domain $\Omega \subseteq R^n$, $n = 2, 3$, or in an open bounded $\Omega \subseteq R^n$, with smooth boundary $\partial\Omega$. In the two latter situations having a well-posed boundary-value problem (for the steady flow situation) or initial-boundary value problem (for an unsteady flow) requires the specification, not only of the usual non-slip condition on $\partial\Omega$, which is common in the Newtonian case, but also of higher-order boundary conditions; this is, of course, because the system (1.91) is of fourth order in the spatial derivatives of $\boldsymbol{v}$. The multipolar stress tensor τ_{ijk} specified in (1.88b) enters into the higher-order boundary conditions as will be seen in the next two subsections. A discussion of the appropriate structure for periodic boundary conditions appears in conjunction with the general existence theory for incompressible bipolar fluids which is presented in Chap. 4.

1.4.3 The Structure of the Higher-Order Boundary Conditions 1: Virtual Work Without the Incompressibility Constraint

We now turn to the issue of the structure of the higher-order boundary conditions associated with the constitutive equations (1.88a,b), (1.90) defining an incompressible nonlinear bipolar fluid under isothermal conditions. Following the similar analysis in [To] for elastic materials, we obtain an appropriate set of boundary conditions by applying the principle of virtual work. Our analysis, in this subsection, will be carried out without imposing the constraint of incompressibility on the virtual velocities and velocity gradients; this constraint is imposed in Sect. 1.4.4 and the resulting modification of the computations presented, below, will then yield the final form of the higher-order boundary conditions associated with the multipolar stress tensor τ_{ijk}. The analysis presented here is based on the recent work in [BB4].

We begin by rewriting the constitutive equations in the form

$$t_{ij} = -p\delta_{ij} + \tau_{ij}^{(0)} - \frac{\partial}{\partial x_k}\tau_{ijk} \tag{1.92}$$

where τ_{ijk} is given by (1.88b) and

$$\tau_{ij}^{(0)} = 2\mu_0(\epsilon + e_{ij}e_{ij})^{\frac{p-2}{2}} e_{ij}. \tag{1.93}$$

Next, we define

$$\Gamma(e_{ij}) = \mu_0 \int_0^{e_{ij}e_{ij}} (\epsilon + s)^{\frac{p-2}{2}} ds, \tag{1.94a}$$

$$\Phi\left(\frac{\partial e_{ij}}{\partial x_k}\right) = \mu_1 \frac{\partial e_{ij}}{\partial x_k}\frac{\partial e_{ij}}{\partial x_k}, \tag{1.94b}$$

and

$$W\left(e_{ij}, \frac{\partial e_{ij}}{\partial x_k}\right) = \Gamma(e_{ij}) + \Phi\left(\frac{\partial e_{ij}}{\partial x_k}\right). \tag{1.95}$$

Then, clearly,

$$\tau_{ij}^{(0)} = \frac{\partial \Gamma}{\partial e_{ij}} = \frac{\partial W}{\partial e_{ij}}, \tag{1.96a}$$

$$\tau_{ijk} = \frac{\partial \Phi}{\partial \left(\frac{\partial e_{ij}}{\partial x_k}\right)} = \frac{\partial W}{\partial \left(\frac{\partial e_{ij}}{\partial x_k}\right)} \tag{1.96b}$$

so that the complete residual stress tensor $\tau_{ij} = \tau_{ij}^{(0)} - \frac{\partial}{\partial x_k}\tau_{ijk}$ is given by

$$\tau_{ij} = \frac{\partial W}{\partial e_{ij}} - \frac{\partial}{\partial x_k}\left(\frac{\partial W}{\partial \left(\frac{\partial e_{ij}}{\partial x_k}\right)}\right). \tag{1.97}$$

The potential energy of the fluid in a fixed, bounded domain $\Omega \subseteq R^n$, $n = 2, 3$, with smooth boundary $\partial\Omega$, and exterior unit normal $\boldsymbol{\nu}$ at $\boldsymbol{x} \in \partial\Omega$, is now defined by

$$E(\Omega) = \int_\Omega W\left(e_{ij}, \frac{\partial e_{ij}}{\partial x_k}\right) d\boldsymbol{x}. \tag{1.98}$$

The principle of virtual work, therefore, assumes the form

$$\delta E(\Omega) = \int_\Omega f_i \delta v_i \, d\boldsymbol{x} + \oint_{\partial\Omega} (T_i \delta v_i + M_i D\delta v_i)\, dS_{\boldsymbol{x}} \tag{1.99}$$

for arbitrary values of the virtual velocity variations δv_i, subject only to the constraint that the $\delta v_i = 0$ on $\partial\Omega$, and arbitrary variations of the virtual velocity gradients $\delta v_{i,j}$, where $D\delta v_i = \delta v_{i,j}\nu_j$ are the normal derivatives of the virtual velocity components on $\partial\Omega$, $f_i = \rho F_i - \frac{\partial p}{\partial x_i}$, T_i are the tractions/area on $\partial\Omega$, and M_i are the hypertractions (or moments)/area on $\partial\Omega$. From (1.98) we have

$$\delta E(\Omega) = \int_\Omega \left[\frac{\partial W}{\partial e_{ij}}\delta e_{ij} + \frac{\partial W}{\partial \left(\frac{\partial e_{ij}}{\partial x_k}\right)}\delta\left(\frac{\partial e_{ij}}{\partial x_k}\right)\right] d\boldsymbol{x}. \tag{1.100}$$

Now,

$$\delta e_{ij} = \frac{1}{2}(\delta v_{i,j} + \delta v_{j,i})$$

so

$$\frac{\partial W}{\partial e_{ij}}\delta e_{ij} = \frac{1}{2}\frac{\partial W}{\partial e_{ij}}(\delta v_{i,j} + \delta v_{j,i}) = \frac{\partial W}{\partial e_{ij}}\delta v_{i,j} \tag{1.101}$$

by the symmetry of e_{ij}. In a similar manner,

$$\delta\left(\frac{\partial e_{ij}}{\partial x_k}\right) = \frac{1}{2}\left(\delta v_{i,jk} + \delta v_{j,ik}\right)$$

and

$$\frac{\partial W}{\partial\left(\frac{\partial e_{ij}}{\partial x_k}\right)}\delta\left(\frac{\partial e_{ij}}{\partial x_k}\right) = \frac{\partial W}{\partial\left(\frac{\partial e_{ij}}{\partial x_k}\right)}\delta v_{i,jk}. \tag{1.102}$$

Employing (1.101), (1.102) in (1.100) we obtain

$$\delta E(\Omega) = \int_\Omega \left[\frac{\partial W}{\partial e_{ij}}\delta v_{i,j} + \frac{\partial W}{\partial\left(\frac{\partial e_{ij}}{\partial x_k}\right)}\delta v_{i,jk}\right] d\boldsymbol{x}. \tag{1.103}$$

However, by integration by parts, and the divergence theorem, we obtain for the first integral in (1.103)

$$\begin{aligned}\int_\Omega \frac{\partial W}{\partial e_{ij}}\delta v_{i,j}\, d\boldsymbol{x} &= \int_\Omega \frac{\partial}{\partial x_j}\left(\frac{\partial W}{\partial e_{ij}}\delta v_i\right) d\boldsymbol{x} - \int_\Omega \frac{\partial}{\partial x_j}\left(\frac{\partial W}{\partial e_{ij}}\right)\delta v_i\, d\boldsymbol{x} \\ &= -\int_\Omega \frac{\partial}{\partial x_j}\left(\frac{\partial W}{\partial e_{ij}}\right)\delta v_i\, d\boldsymbol{x}\end{aligned} \tag{1.104}$$

as $\delta v_i = 0$ on $\partial\Omega$. In an analogous fashion, two consecutive integrations by parts, applied to the second integral in (1.100), yields

$$\begin{aligned}\int_\Omega \frac{\partial W}{\partial\left(\frac{\partial e_{ij}}{\partial x_k}\right)}\delta v_{i,jk}\, d\boldsymbol{x} &= \int_\Omega \frac{\partial}{\partial x_k}\left(\frac{\partial W}{\partial\left(\frac{\partial e_{ij}}{\partial x_k}\right)}\delta v_{i,j}\right) d\boldsymbol{x} \\ &\quad - \int_\Omega \frac{\partial}{\partial x_k}\left(\frac{\partial W}{\partial\left(\frac{\partial e_{ij}}{\partial x_k}\right)}\right)\delta v_{i,j}\, d\boldsymbol{x} \\ &= \oint_{\partial\Omega} \frac{\partial W}{\partial\left(\frac{\partial e_{ij}}{\partial x_k}\right)}\delta v_{i,j}\, \nu_k\, dS_{\boldsymbol{x}} \\ &\quad - \int_\Omega \frac{\partial}{\partial x_j}\left[\frac{\partial}{\partial x_k}\left(\frac{\partial W}{\partial\left(\frac{\partial e_{ij}}{\partial x_k}\right)}\right)\delta v_i\right] d\boldsymbol{x} \\ &\quad + \int_\Omega \frac{\partial^2}{\partial x_j\,\partial x_k}\left(\frac{\partial W}{\partial\left(\frac{\partial e_{ij}}{\partial x_k}\right)}\right)\delta v_i\, d\boldsymbol{x}\end{aligned}$$

or

$$\int_{\Omega} \frac{\partial W}{\partial \left(\frac{\partial e_{ij}}{\partial x_k}\right)} \delta v_{i,jk}\, d\boldsymbol{x} = \oint_{\partial\Omega} \frac{\partial W}{\partial \left(\frac{\partial e_{ij}}{\partial x_k}\right)} \delta v_{i,j}\, \nu_k\, dS_{\boldsymbol{x}} + \int_{\Omega} \frac{\partial^2}{\partial x_j \partial x_k} \left(\frac{\partial W}{\partial \left(\frac{\partial e_{ij}}{\partial x_k}\right)} \right) \delta v_i\, d\boldsymbol{x} \tag{1.105}$$

where we have, again, used the fact that $\delta v_i = 0$ on $\partial\Omega$. Employing (1.104) and (1.105) in (1.103), and taking note of (1.96a,b), we find that

$$\delta E(\Omega) = -\int_{\Omega} \frac{\partial}{\partial x_j} \tau_{ij}^{(0)} \delta v_i\, d\boldsymbol{x} + \oint_{\partial\Omega} \tau_{ijk} \delta v_{i,j}\, \nu_k\, dS_{\boldsymbol{x}} + \int_{\Omega} \frac{\partial}{\partial x_j} \left(\frac{\partial}{\partial x_k} \tau_{ijk} \right) \delta v_i\, d\boldsymbol{x}$$

or, by combining like integrals,

$$\delta E(\Omega) = \int_{\Omega} \frac{\partial}{\partial x_j} \left\{ -\tau_{ij}^{0} + \frac{\partial}{\partial x_k} \tau_{ijk} \right\} \delta v_i\, d\boldsymbol{x} + \oint_{\partial\Omega} \tau_{ijk} \delta v_{i,j}\, \nu_k\, dS_{\boldsymbol{x}}. \tag{1.106}$$

If we now combine (1.99) and (1.106), and recall that $\tau_{ij} = \tau_{ij}^{(0)} - \dfrac{\partial}{\partial x_k} \tau_{ijk}$, we obtain the equation

$$\int_{\Omega} \left(f_i + \frac{\partial}{\partial x_j} \tau_{ij} \right) \delta v_i\, d\boldsymbol{x} + \oint_{\partial\Omega} (M_i \nu_j - \tau_{ijk} \nu_k) \delta v_{i,j}\, dS_{\boldsymbol{x}} = 0. \tag{1.107}$$

By virtue of the independence of the variations of the virtual velocity components (in Ω) and their gradients (on $\partial\Omega$), and in the absence, to this point, of the incompressibility constraint, we obtain from (1.107)

$$\frac{\partial}{\partial x_j} \tau_{ij} = -f_i = -\rho F_i + \frac{\partial p}{\partial x_i}, \text{ in } \Omega \tag{1.108}$$

or

$$-\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j} \tau_{ij}^{(0)} - \frac{\partial}{\partial x_j} \left(\frac{\partial}{\partial x_k} \tau_{ijk} \right) + \rho F_i = 0, \text{ in } \Omega \tag{1.109}$$

as well as

$$\tau_{ijk} \nu_k = M_i \nu_j, \text{ on } \partial\Omega. \tag{1.110}$$

If we multiply (1.110) through by ν_j, sum on j, and use the fact that $\nu_j \nu_j = 1$, it follows that

$$\tau_{ijk} \nu_j \nu_k = M_i, \text{ on } \partial\Omega \tag{1.111}$$

for $i = 1, 2, 3$ (in R^3), which is the form the higher-order boundary conditions assume, at each fixed time t, if we do not impose the incompressibility constraint on the velocity variations; in the next subsection we will modify the calculations presented here so as to take into account this constraint.

1.4.4 Structure of the Higher-Order Boundary Conditions 2: Virtual Work Subject to the Incompressibility Constraint

If we now restrict ourselves in (1.107) to smooth, divergence free virtual velocity fields with compact support in Ω, then (1.107) reduces to

$$\int_\Omega \left(f_i + \frac{\partial}{\partial x_j} \tau_{ij} \right) \delta v_i \, d\boldsymbol{x} = 0 \tag{1.112}$$

for all δv_i which are smooth and divergence free. It then follows (see, e.g., [GRa] Theorem 2.3) that (1.108) holds. Using (1.108) it is then a direct consequence of (1.107) that the boundary integrals

$$\oint_{\partial\Omega} \left(M_i \nu_j - \tau_{ijk} \nu_k \right) \delta v_{i,j} \, ds = 0 \tag{1.113}$$

for all δv_i that are divergence free and zero on $\partial\Omega$. From (1.113) it follows that for any tensor $\boldsymbol{w}$ which is the restriction of the gradient of a divergence free, smooth, vector function $\delta\boldsymbol{v}$ which vanishes on the boundary, we have

$$\oint_{\partial\Omega} \left(M_i \nu_j - \tau_{ijk} \nu_k \right) w_{ij} \, ds = 0. \tag{1.114}$$

As δv_i is zero on $\partial\Omega$ it follows that all tangential derivatives of δv_i are zero on the boundary. Thus, in (1.113), the only non-zero part of $\delta v_{i,j}$ is that component which corresponds to the normal derivative of δv_i. By the normal derivative of the vector $\delta\boldsymbol{v}$ with components δv_i we mean, of course, the vector whose ith component is $\dfrac{\partial \delta v_i}{\partial x_j} \nu_j$ and by Theorem 3.1 of [HB] all such vectors are tangential to $\partial\Omega$. Furthermore, for any smooth vector $\boldsymbol{g}$ tangential to $\partial\Omega$ there exists a function δv_i such that

$$\begin{aligned} \frac{\partial \delta v_i}{\partial \boldsymbol{\nu}} &= g_i, \text{ on } \partial\Omega, \\ \delta v_i &= 0, \text{ on } \partial\Omega, \\ \operatorname{div} \delta v_i &= 0, \text{ in } \Omega. \end{aligned} \tag{1.115}$$

Remarks. Heron, in [HB], studied higher-order traces of divergence free fields. We use only a particular form of his Theorem 3.1 and for the convenience of the reader we state below this simplified version of Heron's theorem.

Theorem (Heron [HB]). *Let Ω be a connected, bounded subset of R^n, with C^3 boundary $\partial\Omega = \Gamma$. Given a vector $\boldsymbol{g}_1 \in \boldsymbol{H}^{1/2}(\Gamma)$ there exists $\boldsymbol{u} \in \boldsymbol{W}^{2,2}(\Omega)$, with $\operatorname{div} \boldsymbol{u} = 0$ in Ω, $\boldsymbol{u} = \boldsymbol{0}$ on Γ, and $\dfrac{\partial \boldsymbol{u}}{\partial \boldsymbol{\nu}} = \boldsymbol{g}_1$ on Γ if and only if $\boldsymbol{g}_1$ is tangential to Γ, i.e., if and only if $\boldsymbol{g}_1 \cdot \boldsymbol{\nu} = 0$ where $\boldsymbol{\nu}$ is the unit normal to Γ.*

It now follows from (1.113) that the tangential component of $M_i \nu_j - \tau_{ijk}\nu_k$ is zero on $\partial\Omega$ and we may state the following

Lemma 1.1. *For an incompressible, nonlinear, bipolar viscous fluid defined by the constitutive relations (1.88a,b), (1.90), in an open bounded domain $\Omega \subseteq R^n$, $n = 2, 3$, with smooth boundary $\partial\Omega$, the higher-order boundary condition*

$$\tau_{ijk}\nu_j \nu_k \tau_i = M_i \tau_i, \text{ on } \partial\Omega \tag{1.116}$$

must be satisfied, where for $n = 2$, $\boldsymbol{\nu}$ is the exterior unit normal to the smooth curve $\partial\Omega$ at $\boldsymbol{x} \in \partial\Omega$ and $\boldsymbol{\tau}$ is the unit tangent vector, while for $n = 3$, $\boldsymbol{\nu}$ is the exterior unit normal to the surface $\partial\Omega$ at $\boldsymbol{x} \in \partial\Omega$ and $\boldsymbol{\tau}$ is any unit vector in the tangent plane to $\partial\Omega$ at $\boldsymbol{x}$.

Remarks. Throughout most of this book we will assume that $\boldsymbol{M} = \boldsymbol{0}$ on $\partial\Omega$ so that (1.116) reduces to

$$\tau_{ijk}\nu_j \nu_k \tau_i = 0, \text{ on } \partial\Omega. \tag{1.117}$$

An alternative form of the higher-order boundary conditions is then given by

Lemma 1.2. *Let Ω be an open bounded domain in R^n, $n = 2, 3$ with smooth boundary $\partial\Omega$ and let $\boldsymbol{x} \in \partial\Omega$. Let $\boldsymbol{\nu}$ be the exterior unit normal to $\partial\Omega$ at $\boldsymbol{x}$, and let $\boldsymbol{\tau}$ denote any unit vector in the tangent space to $\partial\Omega$ at $\boldsymbol{x}$. Then, at $\boldsymbol{x}$,*

$$\tau_{ijk}\nu_j \nu_k \tau_i = 0 \Leftrightarrow \tau_{ijk}\nu_j \nu_k - \tau_{jkl}\nu_j \nu_k \nu_l \nu_i = 0, \tag{1.118}$$

where $i = 1, 2$ if $n = 2$, and $i = 1, 2, 3$ if $n = 3$.

Proof. Suppose $\tau_{ijk}\nu_j \nu_k - \tau_{jkl}\nu_j \nu_k \nu_l \nu_i = 0$, $i = 1, 2, 3$. Multiplying through by τ_i, summing on i, and using the fact that $\nu_i \tau_i = 0$ we get

$$\tau_{ijk}\nu_j \nu_k \tau_i = 0, \quad i = 1, 2, 3.$$

Now, let $\tau_{ijk}\nu_j \nu_k \tau_i = 0$ and let $\hat{\boldsymbol{\tau}}$ be the vector at $\boldsymbol{x}$ with i-th component $(\hat{\boldsymbol{\tau}})_i = \tau_{ijk}\nu_j \nu_k$. Also, let $\beta\boldsymbol{\tau}$ be the projection of $\hat{\boldsymbol{\tau}}$ onto the tangent plane to $\partial\Omega$ at $\boldsymbol{x}$, where $\boldsymbol{\tau}$ is a unit vector in the tangent space, so that (see Fig. 1.2)

$$\hat{\boldsymbol{\tau}} = \alpha\boldsymbol{\nu} + \beta\boldsymbol{\tau} \tag{1.119}$$

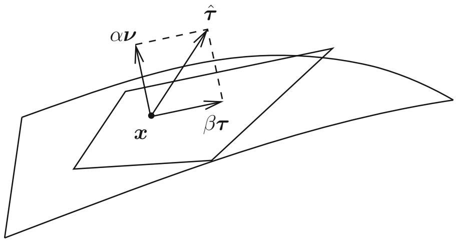

Fig. 1.2 Tangent plane to $\partial\Omega$ at x

where $\alpha\boldsymbol{\nu}$ is the projection of $\hat{\boldsymbol{\tau}}$ onto the normal direction to $\partial\Omega$ at $\boldsymbol{x}$. Therefore,

$$\hat{\tau}_i = \tau_{ijk}\nu_j\nu_k = \alpha\nu_i + \beta\tau_i \tag{1.120}$$

which, as $\nu_i\tau_i = 0$, $\nu_i\nu_i = 1$, implies that

$$\alpha = \tau_{ijk}\nu_j\nu_k\nu_i \equiv \tau_{jkl}\nu_j\nu_k\nu_l. \tag{1.121}$$

As $\tau_i\tau_i = 1$, by virtue of our hypothesis, and the definition of $\hat{\boldsymbol{\tau}}$,

$$\tau_{ijk}\nu_j\nu_k\tau_i = \beta = 0 \tag{1.122}$$

in which case

$$\tau_{ijk}\nu_j\nu_k = (\tau_{jkl}\nu_j\nu_k\nu_l)\nu_i, \quad i = 1, 2, 3 \tag{1.123}$$

so that $\tau_{ijk}\nu_j\nu_k - (\tau_{jkl}\nu_j\nu_k\nu_l)\nu_i = 0$, $i = 1, 2, 3$ whenever $\tau_{ijk}\nu_j\nu_k\tau_i = 0$. □

Remarks. The proof of Lemma 1.2 has been constructed for the case $n = 3$; the proof for $n = 2$ is a trivial modification of the proof given above.

Remarks. In Appendix B it is shown, in Lemma B.3, that the second of the conditions in (1.118) implies that

$$\tau_{ijk}e_{ij}\nu_k = 0, \text{ on } \partial\Omega. \tag{1.124}$$

By virtue of Lemma 1.2 it now follows that (1.124) is also a consequence of the condition (1.117).

Remarks. Suppose that $\tau_{ijk}\nu_j\nu_k\tau_i = M_k\tau_k \neq 0$ on $\partial\Omega$, where we sum on all repeated indices. Then, by virtue of (1.120) and (1.121) we have

$$\tau_{ijk}\nu_j\nu_k - (\tau_{jkl}\nu_j\nu_k\nu_l)\nu_i = \beta\tau_i. \tag{1.125}$$

Multiplying (1.125) by τ_i, summing on i, and using the facts that $\nu_i\tau_i = 0$, $\tau_i\tau_i = 1$, we obtain

$$\beta = \tau_{ijk}\nu_j\nu_k\tau_i = M_k\tau_k \tag{1.126}$$

the last result being valid on $\partial\Omega$ by virtue of the higher-order boundary condition (1.116). Substituting (1.126) back into (1.125) we obtain

$$\tau_{ijk}\nu_j\nu_k - (\tau_{jkl}\nu_j\nu_k\nu_l)\nu_i = (M_k\tau_k)\tau_i\,, \text{ on } \partial\Omega \tag{1.127}$$

as the form of the higher-order boundary conditions on $\partial\Omega$ which are equivalent to (1.116) whenever $M_i\tau_i \neq 0$ on $\partial\Omega$. The term on the right-hand side of (1.127) is the projection of the vector $\boldsymbol{M}$ onto the tangent plane to the surface $\partial\Omega$ at a point $\boldsymbol{x} \in \partial\Omega$ if we are working in space dimension $n = 3$; if $n = 2$, it is the projection of $\boldsymbol{M}$ onto the direction of the tangent vector to the curve $\partial\Omega$ at a point $\boldsymbol{x} \in \partial\Omega$.

1.5 Elementary Examples of Incompressible Nonlinear Bipolar Fluid Flow

1.5.1 Introduction

In the previous section we formulated the constitutive theory for an incompressible, viscous, bipolar fluid; this constitutive theory is completely defined by the relations (1.88a), (1.89), for the stress tensor t_{ij} and (1.88b) for the first multipolar stress tensor τ_{ijk}. The constitutive parameters μ_0, μ_1 are assumed to be positive, while $\epsilon \geq 0$; in this section we will look at some of the consequences of taking $0 < \alpha < 1$, the motivation for restricting α to this range having been given in Sect. 1.4.2. By collecting all of the results in Sect. 1.4, we may write the full initial-boundary value problem, for the components v_i of the velocity field of a nonlinear, incompressible bipolar viscous fluid, in the following form, where Ω is a bounded domain in R^n, $n = 2, 3$ with smooth boundary $\partial\Omega$, and the x_i are (Eulerian) Cartesian coordinates: for $i = 1, \ldots, n$ $(n = 2, 3)$,

$$\rho\left(\frac{\partial v_i}{\partial t} + v_j\frac{\partial v_j}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} + 2\mu_0\frac{\partial}{\partial x_j}\left[(\epsilon + e_{ij}e_{ij})^{-\alpha/2}e_{ij}\right] - 2\mu_1\frac{\partial}{\partial x_j}(\Delta e_{ij}) + \rho f_i \tag{1.128}$$

in $\Omega\times[0,T)$, $T > 0$, with ρ the fluid density, p the pressure, and f_i the components of the external body force/mass,

$$\nabla\cdot\boldsymbol{v} = 0, \text{ in } \Omega\times[0,T) \tag{1.129}$$

$$\boldsymbol{v} = \boldsymbol{0},\ \tau_{ijk}\nu_j\nu_k - \tau_{jkl}\nu_j\nu_k\nu_l\nu_i = (M_k\tau_k)\tau_i\,,\ i = 1,2,3, \text{ on } \partial\Omega\times[0,T) \tag{1.130}$$

on $\partial\Omega\times[0,T)$, where the M_i are the components of the hypertraction (moments) on $\partial\Omega$ at time $t \in [0,T)$, and $\boldsymbol{\nu}$ is the exterior unit normal at $\boldsymbol{x} \in \partial\Omega$, and

$$\boldsymbol{v}(\boldsymbol{x},0) = \boldsymbol{v}_0(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega. \tag{1.131}$$

For our work in this section we will assume that the $M_i = 0$ and that $f_i = 0$ as well. We want to examine the consequences of applying the theory delineated in (1.128)–(1.131) to three, standard, elementary examples of steady flow: plane Poiseuille flow between fixed parallel plates, proper Poiseuille flow in a circular pipe, and plane Couette flow over a plate which moves with constant velocity. As all of our examples involve steady flow, for each case we have $\frac{\partial v_i}{\partial t} = 0$; moreover, the geometries of the three flows selected are such that, for each i, we will also have $v_j \frac{\partial v_i}{\partial x_j} = 0$. Therefore, on the left-hand side of (1.128), $\frac{D v_i}{D t} = 0$ for each case considered, in which case the steady flow equations become

$$2\mu_0 \frac{\partial}{\partial x_j}\left[(\epsilon + e_{ij}e_{ij})^{-\alpha/2} e_{ij}\right] - 2\mu_1 \frac{\partial}{\partial x_j}(\Delta e_{ij}) = \frac{\partial p}{\partial x_i} \tag{1.128$'$}$$

in Ω, for $i = 1, \ldots, n$, $n = 2, 3$. The remaining conditions are the incompressibility constraint

$$\nabla \cdot \boldsymbol{v} = 0, \text{ in } \Omega \tag{1.129$'$}$$

and the boundary conditions

$$\boldsymbol{v} = \mathbf{0}, \ \tau_{ijk}\nu_j\nu_k - \tau_{jkl}\nu_j\nu_k\nu_l\nu_i = 0, \text{ on } \partial\Omega \tag{1.130$'$}$$

for $i = 1, \ldots, n$, $n = 2, 3$. In order to obtain closed form solutions for the relevant boundary-value problem in this section, we will set $\epsilon = \mu_1 = 0$ in (1.128$'$). However, in Chap. 2, which is devoted entirely to steady plane Poiseuille flow between parallel plates, we will prove that there exists a unique solution of the boundary-value problem (1.128$'$), (1.129$'$), (1.130$'$) which depends continuously on ϵ and μ_1 as $\epsilon \to 0^+$ and $\mu_1 \to 0^+$; for $\mu_1 \to 0^+$, continuous dependence will be proven to hold in the norm of $C^{1+\delta}$, for $0 < \delta < \frac{1}{2}$, so that, in particular, the approximation corresponding to $\epsilon = \mu_1 = 0$ is a reasonably accurate one, for small ϵ and μ_1, in the C^0 norm. An analogous continuous dependence result for the example of (steady) proper Poiseuille flow in a circular pipe will appear in the Ph.D. thesis of A. Montz [Mon].

1.5.2 Steady Plane Poiseuille Flow

We assume a flow of the form

$$v_1 = v_1(x_2), \ v_2 = 0, \ v_3 = 0 \tag{1.132}$$

between fixed parallel plates at $x_2 = \pm a$, for some $a > 0$. It is easily seen that in this case (1.128′) reduces to the fourth-order nonlinear ordinary differential equation

$$\mu_0 \left[\left(\epsilon + \frac{1}{2} v_1'^2(x_2) \right)^{-\alpha/2} v_1'(x_2) \right]' - \mu_1 v_1''''(x_2) = p_1 \tag{1.133}$$

where $p_1 = \dfrac{\partial p}{\partial x_1} = \text{const}$. The divergence free condition (1.129′) is, of course, automatically satisfied, while the boundary conditions reduce to

$$v_1(\pm a) = 0, \quad v_1''(\pm a) = 0. \tag{1.134}$$

If we set $\mu_1 = 0$ in (1.133), then only the first set of boundary conditions in (1.134) is relevant and (1.133) becomes

$$\left[\left(\epsilon + \frac{1}{2} v_1'^2(x_2) \right)^{-\alpha/2} v_1'(x_2) \right]' = \frac{p_1}{\mu_0} \tag{1.135}$$

so that for some real γ,

$$\left(\epsilon + \frac{1}{2} v_1'^2(x_2) \right)^{-\alpha/2} v_1'(x_2) = \left(\frac{p_1}{\mu_0} \right) x_2 + \gamma \equiv g(x_2). \tag{1.136}$$

If we set $w_\epsilon = \epsilon + v_1'^2(x_2)/2$ then it follows from (1.136) that w_ϵ satisfies the transcendental algebraic equation

$$w_\epsilon^{1-\alpha} - \epsilon w_\epsilon^{-\alpha} = \frac{1}{2} g^2; \quad \epsilon > 0,\ 0 < \alpha < 1 \tag{1.137}$$

whose solutions are easily seen to depend continuously on ϵ as $\epsilon \to 0^+$; we, therefore, turn our attention to (1.136) with $\epsilon = 0$, or

$$2^{\alpha/2} \left| v_1'(x_2) \right|^{-\alpha} v_1'(x_2) = g(x_2) \tag{1.138}$$

from which it follows that $\operatorname{sgn} v_1'(x_2) = \operatorname{sgn} g(x_2)$. Thus, (1.138) yields

$$2^{\alpha/2} \left| v_1'(x_2) \right|^{1-\alpha} = \operatorname{sgn} g(x_2) \cdot g(x_2) \tag{1.139}$$

or

$$\left(\sqrt{2} \right)^{\alpha/(1-\alpha)} v_1'(x_2) = \pm \left[\operatorname{sgn} g(x_2) \cdot g(x_2) \right]^{1/(1-\alpha)} \tag{1.140}$$

where we choose the plus sign in (1.140) for $v_1'(x_2) > 0$ (equivalently, $g(x_2) > 0$) and the minus sign for $v_1'(x_2) < 0$ (equivalently, $g(x_2) < 0$). In view of the viscous behavior of the fluid we know that, besides $v_1(\pm a) = 0$, we must have $v_1'(-a) > 0$ and $v_1'(a) < 0$. Although (1.140) may be integrated for arbitrary α, $0 < \alpha < 1$, it is instructive to proceed as follows: Consider the sequence $\{\alpha_n\}$, $0 < \alpha_n < 1$, for each positive integer n, given by $\alpha_n = (n-1)/n$; then $\alpha_n \to 1^-$ as $n \to \infty$ while $\alpha_1 = 0$. Setting $\alpha = \alpha_n$ in (1.140), and denoting the solution of the corresponding equation by $u_n(x_2)$, we have

$$2^{(n-1)/2} u_n'(x_2) = \pm(\operatorname{sgn} g(x_2))^n g^n(x_2) \tag{1.141}$$

so that

$$\begin{cases} 2^{(n-1)/2} u_n'(x_2) = \pm g^n(x_2), & n \text{ even}, \\ 2^{(n-1)/2} u_n'(x_2) = g^n(x_2), & n \text{ odd} \end{cases} \tag{1.142}$$

where we employ the convention that the plus sign in (1.141) corresponds to $g(x_2) > 0$ while the minus sign corresponds to $g(x_2) < 0$. We will consider two special cases of (1.142):

(a) $n = 2$ ($\alpha_n = 1/2$). In this case our differential equation reads

$$\sqrt{2} u_2'(x_2) = \pm \left[\left(\frac{p_1}{\mu_0} \right)^2 x_2^2 + \frac{2\gamma p_1}{\mu_0} x_2 + \gamma^2 \right] \tag{1.143}$$

so that for some constant $\tilde{\gamma}$

$$\sqrt{2} u_2(x_2) = \pm \left[\left(\frac{p_1}{\mu_0} \right)^2 \frac{x_2^3}{3} + \frac{\gamma p_1}{\mu_0} x_2^2 + \gamma^2 x_2 \right] + \tilde{\gamma}. \tag{1.144}$$

We now apply the boundary conditions $u_2(\pm a) = 0$, choosing the plus sign in (1.144) at $x_2 = -a$ and the minus sign at $x_2 = +a$; this follows from the fact that $u_2'(-a) > 0$ while $u_2'(a) < 0$. We obtain

$$\begin{cases} -\frac{1}{3} \left(\frac{p_1}{\mu_0} \right)^2 a^3 + \frac{\gamma p_1}{\mu_0} a^2 - \gamma^2 a + \tilde{\gamma} = 0, \\ -\left(\frac{1}{3} \left(\frac{p_1}{\mu_0} \right)^2 a^3 + \frac{\gamma p_1}{\mu_0} a^2 + \gamma^2 a \right) + \tilde{\gamma} = 0 \end{cases} \tag{1.145}$$

from which it follows that $\gamma = 0$ while $\tilde{\gamma} = (p_1/\mu_0) a^3/3$. As $\gamma = 0$, for $p_1 < 0$,

$$g(x_2) = \left(\frac{p_1}{\mu_0} \right) x_2 \begin{cases} > 0, & x_2 < 0, \\ < 0, & x_2 > 0. \end{cases} \tag{1.146}$$

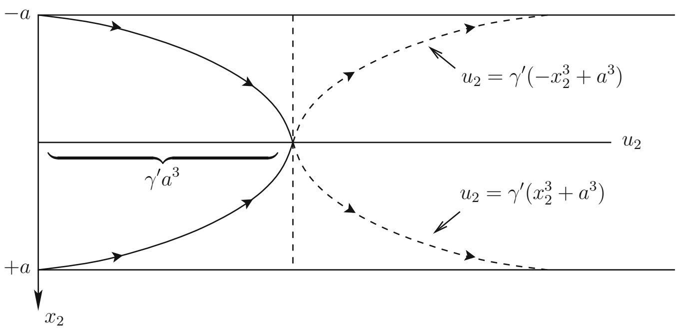

Fig. 1.3 Velocity profile for steady plane Poiseuille flow

Thus

$$u_2(x_2) = \frac{2^{-1/2}}{3}\left(\frac{p_1}{\mu_0}\right)^2 \begin{cases} x_2^3 + a^3, & x < 0, \\ -x_2^3 + a^3, & x_2 > 0. \end{cases} \tag{1.147}$$

A sketch of the velocity profile is depicted in Fig. 1.3, where we have set $\gamma' = \frac{2^{-1/2}}{3}(p_1/\mu_0)^2$.

For the velocity profile given by (1.147) we easily compute that

$$P = \int_{-a}^{a} u_2(x_2)\, dx_2 = \frac{3}{4}a^4\gamma' \equiv \frac{a^4}{2\sqrt{2}}\left(\frac{p_1}{\mu_0}\right)^2. \tag{1.148}$$

(b) $n = 3$ ($\alpha_n = 2/3$). In this case, by virtue of (1.142), the differential equation is

$$2\mu_3'(x_2) = \left(\frac{p_1}{\mu_0}\right)^3 x_2^3 + 3\left(\frac{p_1}{\mu_0}\right)^2 \gamma x_2^2 + 3\frac{p_1}{\mu_0}\gamma^2 x_2 + \gamma^3 \tag{1.149}$$

so that

$$2u_3(x_2) = \left(\frac{p_1}{\mu_0}\right)^3 \frac{x_2^4}{4} + \left(\frac{p_1}{\mu_0}\right)^2 \gamma x_2^3 + \frac{3}{2}\cdot\frac{p_1}{\mu_0}\gamma^2 x_2^2 + \gamma^3 x_2 + \bar{\gamma}. \tag{1.150}$$

Applying the boundary conditions at $x_2 = \pm a$ then yields, after some algebraic manipulation, the relations

$$\begin{cases} \frac{1}{2}\left(\frac{p_1}{\mu_0}\right)^2 a^4 + \frac{3}{2}p_1\left(\frac{\gamma}{\mu_0}\right)^2 a^2 + 2\bar{\gamma} = 0, \\ \gamma\left(2\left(\frac{p_1}{\mu_0}\right)^2 + 2\gamma^2 a\right) = 0 \end{cases} \tag{1.151}$$

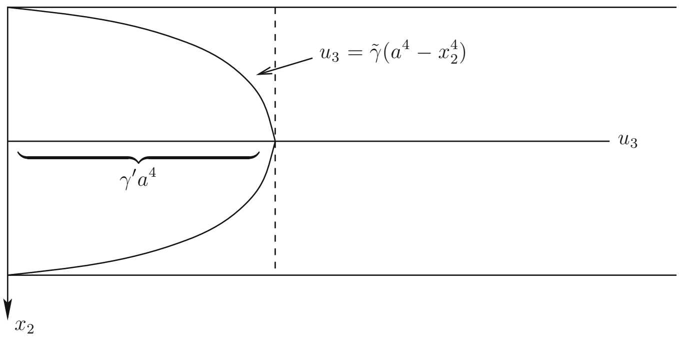

Fig. 1.4 Velocity profile for $n = 3, \alpha_n = 2/3$

from which it is immediate that $\gamma = 0$ and $\bar{\gamma} = -(1/4)(p_1/\mu_0)^3 a^4$. Substituting these results into (1.150), we find as the explicit expression for the velocity profile in this case (again, with the assumption that $p_1 < 0$):

$$u_3(x_2) = \frac{1}{8}\left(\frac{|p_1|}{\mu_0}\right)^3 (a^4 - x_2^4) \tag{1.152}$$

a sketch of which is depicted in Fig. 1.4, where we have set $\tilde{\gamma} = (1/8)(|p_1|/\mu_0^0)^3$. In this case we compute that

$$P = \int_{-a}^{a} u_3(x_2)\, dx_2 = \frac{8}{5} a^5 \tilde{\gamma} \equiv \frac{a^5}{5}\left(\frac{|p_1|}{\mu_0}\right)^3. \tag{1.153}$$

1.5.3 Steady (Proper) Poiseuille Flow in a Circular Cylinder

The physical problem treated in this subsection is genuinely two-dimensional in nature, although when formulated in terms of polar coordinates it appears to be a one-dimensional problem. We begin by looking at steady flow in a cylinder of arbitrary cross section and take the x_1-axis parallel to the generators of the cylinder (which we will, henceforth, call a pipe). This time we ask if there exists a steady flow of the form

$$v_1 = v_1(x_2, x_3), \quad v_2 = 0, \quad v_3 = 0. \tag{1.154}$$

For the flow in (1.154) all components e_{ij} of the rate of deformation tensor vanish except for

$$e_{12} = e_{21} = \frac{1}{2}\frac{\partial v_1}{\partial x_2}, \quad e_{13} = e_{31} = \frac{1}{2}\frac{\partial v_1}{\partial x_3}. \tag{1.155}$$

Then by virtue of (1.88a), (1.89)

$$t_{11} = t_{22} = t_{33} = -p, \quad t_{23} = t_{32} = 0 \tag{1.156}$$

while

$$t_{12} = \mu_0 \left(\epsilon + \frac{1}{2} \left[\left(\frac{\partial v_1}{\partial x_2} \right)^2 + \left(\frac{\partial v_1}{\partial x_3} \right)^2 \right] \right)^{-\alpha/2} \frac{\partial v_1}{\partial x_2} - \mu_1 \left(\frac{\partial^3 v_1}{\partial x_2^3} + \frac{\partial^3 v_1}{\partial x_3^2 \partial x_2} \right) \tag{1.157}$$

and

$$t_{13} = \mu_0 \left(\epsilon + \frac{1}{2} \left[\left(\frac{\partial v_1}{\partial x_2} \right)^2 + \left(\frac{\partial v_1}{\partial x_3} \right)^2 \right] \right)^{-\alpha/2} \frac{\partial v_1}{\partial x_3} - \mu_1 \left(\frac{\partial^3 v_1}{\partial x_3 \partial x_2^2} + \frac{\partial^3 v_1}{\partial x_3^3} \right). \tag{1.158}$$

The equilibrium equations (1.128′), when coupled with (1.154)–(1.158), yield

$$\frac{\partial p}{\partial x_2} = \frac{\partial p}{\partial x_3} = 0 \tag{1.159}$$

and

$$\begin{aligned} -\frac{\partial p}{\partial x_1} + \mu_0 \Bigg\{ \frac{\partial}{\partial x_2} \left[\left(\epsilon + \frac{1}{2} \left[\left(\frac{\partial v_1}{\partial x_2} \right)^2 + \left(\frac{\partial v_1}{\partial x_3} \right)^2 \right] \right)^{-\alpha/2} \frac{\partial v_1}{\partial x_2} \right] \\ + \frac{\partial}{\partial x_3} \left[\left(\epsilon + \frac{1}{2} \left[\left(\frac{\partial v_1}{\partial x_2} \right)^2 + \left(\frac{\partial v_1}{\partial x_3} \right)^2 \right] \right)^{-\alpha/2} \frac{\partial v_1}{\partial x_3} \right] \Bigg\} \\ - \mu_1 \left(\frac{\partial^4 v_1}{\partial x_2^4} + 2 \frac{\partial^4 v_1}{\partial x_2^2 \partial x_3^2} + \frac{\partial^4 v_1}{\partial x_3^4} \right) = 0. \end{aligned} \tag{1.160}$$

From (1.159), (1.160) it follows that $p = p(x_1)$ with $p'(x_1) = p_1 = \text{const}$. If, in (1.160), we set $\epsilon = \alpha = \mu_1 = 0$, we recover the Poiseuille flow predicted by the Navier–Stokes equations and governed by the Poisson equation

$$\frac{\partial^4 v_1}{\partial x_2^2} + \frac{\partial^2 v_1}{\partial x_3^2} = \frac{p_1}{\mu_0}. \tag{1.161}$$

With $r = \sqrt{x_2^2 + x_3^2}$, and $v_1(x_2, x_3) = u(r)$, the case of (proper) Poiseuille flow in a pipe of circular cross section, (1.161) becomes

$$u''(r) + \frac{1}{r} u'(r) = \frac{p_1}{\mu_0} \tag{1.162}$$

and, if the radius of a cross section is $R > 0$, then by virtue of the viscous nature of the fluid $u(R) = 0$; the well-known solution of this problem (e.g. Shinbrot [Sh]) for which $u(0) < \infty$ is given by

$$u(r) = \frac{-p_1}{4\mu_0}(R^2 - r^2) \tag{1.163}$$

so that the speed varies parabolically across the pipe with a maximum at $r = 0$.

We now consider (1.160) within the context of (proper) Poiseuille flow in a circular pipe whose cross section has radius R; i.e., we assume $v_1(x_2, x_3) = u(r)$ and introduce polar coordinates $x_2 = r\cos\theta$, $x_3 = r\sin\theta$ so that

$$\left(\frac{\partial v_1}{\partial x_2}\right)^2 + \left(\frac{\partial v_1}{\partial x_3}\right)^2 = u'^2(r). \tag{1.164}$$

A lengthy but straightforward calculation yields

$$\begin{aligned}\frac{\partial}{\partial x_2}&\left[\left(2\epsilon + \left(\frac{\partial v_1}{\partial x_2}\right)^2 + \left(\frac{\partial v_1}{\partial x_3}\right)^2\right)^{-\alpha/2}\frac{\partial v_1}{\partial x_2}\right]\\ &= \Gamma_\epsilon(r)^{-\alpha/2}\left[u''(r)\frac{x_2^2}{r^2} + \frac{u'(r)}{r}\left\{1 - \frac{x_2^2}{r^2}\right\}\right]\\ &\quad - \alpha\Gamma_\epsilon(r)^{-(\alpha/2+1)}u'^2(r)u''(r)\left(\frac{x_2^2}{r^2}\right)\end{aligned} \tag{1.165}$$

with $\Gamma_\epsilon(r) = 2\epsilon + u'^2(r)$; an entirely analogous expression is obtained for

$$\frac{\partial}{\partial x_3}\left[\left(2\epsilon + \left(\frac{\partial v_1}{\partial x_2}\right)^2 + \left(\frac{\partial v_1}{\partial x_3}\right)^2\right)^{-\alpha/2}\frac{\partial v_1}{\partial x_3}\right]$$

and addition of these expressions readily yields

$$\begin{aligned}2^{-\alpha/2}\frac{\partial}{\partial x_2}&\left[\left(\epsilon + \frac{1}{2}\left[\left(\frac{\partial v_1}{\partial x_2}\right)^2 + \left(\frac{\partial v_1}{\partial x_3}\right)^2\right]\right)^{-\alpha/2}\frac{\partial v_1}{\partial x_2}\right]\\ &+ 2^{-\alpha/2}\frac{\partial}{\partial x_3}\left[\left(\epsilon + \frac{1}{2}\left[\left(\frac{\partial v_1}{\partial x_2}\right)^2 + \left(\frac{\partial v_1}{\partial x_3}\right)^2\right]\right)^{-\alpha/2}\frac{\partial v_1}{\partial x_3}\right]\\ &= \Gamma_\epsilon(r)^{-\alpha/2}\left[u'' + \frac{1}{r}u'(r)\right] - \alpha\Gamma_\epsilon(r)^{-(\alpha/2-1)}u'^2(r)u''(r)\end{aligned} \tag{1.166}$$

so that for steady flow in a circular pipe (1.160) assumes, for $0 < r \le R$, the form

$$2^{\alpha/2}\mu_0 \left\{ \Gamma_\epsilon(r)^{-\alpha/2}\left[u''(r) + \frac{1}{r}u'(r)\right] - \alpha\Gamma_\epsilon(r)^{-(\alpha/2+1)}u'^2(r)u''(r)\right\}$$
$$- \mu_1 \left\{ u''''(r) + \frac{2}{r}u'''(r) - \frac{1}{r^2}u''(r) + \frac{1}{r^3}u'(r)\right\} = p_1. \quad (1.167)$$

By virtue of (1.130′), (1.167) is to be analyzed subject to the boundary conditions

$$\lim_{r\to 0} u(r) < \infty, \quad u(R) = 0, \quad u''(R) = 0. \quad (1.168)$$

A careful study of the full nonlinear boundary-value problem (1.167), (1.168), which includes proofs of existence, uniqueness, and continuous dependence of $u(r)$ on both ϵ and μ_1, as $\epsilon \to 0^+$, $\mu_1 \to 0^+$, may be found in the Ph.D. thesis of A. Montz [Mon]. For the purposes of the present exposition we will content ourselves with an examination of the profiles predicted by (1.167), with $\epsilon = \mu_1 = 0$, and the boundary condition $u(R) = 0$. If we set $\mu_1 = 0$ in (1.167), and also set

$$z(r) = u'(r), \quad c_1^\alpha = \frac{p_1}{\mu_0} \cdot 2^{\alpha/2} \quad (1.169)$$

we easily find that

$$z'(r)\left[1 - \frac{\alpha z^2(r)}{2\epsilon + z^2(r)}\right] = -\frac{1}{r}z(r) + c_1^\alpha(2\epsilon + z^2(r))^{\alpha/2} \quad (1.170)$$

which, for $\epsilon = 0$, reduces to the Bernoulli equation

$$(1-\alpha)z'(r) = -\frac{1}{r}z(r) + c_1^\alpha(z^2(r))^{\alpha/2}. \quad (1.171)$$

The solutions of (1.171) are given, for arbitrary real c_2, by

$$z(r) \equiv u'(r) = \pm\left(\frac{c_1^\alpha r}{2} + \frac{c_2}{r}\right)^{1/(1-\alpha)}, \quad 0 < r \le R. \quad (1.172)$$

For $p_1 > 0$ we must have $u'(R) < 0$, which dictates the choice of the minus sign in (1.172) on $(0, R]$; also, if $\lim_{r\to 0} u(r) < \infty$ then, clearly, we must also set $c_2 = 0$ in (1.172) so that, for $0 < \alpha < 1$,

$$u'(r) = -k_\alpha r^{1/(1-\alpha)}, \quad 0 < r \le R \quad (1.173)$$

with

$$k_\alpha \equiv \left(\frac{c_1^\alpha}{2}\right)^{1/(1-\alpha)} = \left(\frac{p_1}{\mu_0 2^{\frac{\alpha}{2}+1}}\right)^{1/(1-\alpha)}. \tag{1.174}$$

For $\alpha = 0$, integration of (1.173), subject to $u(R) = 0$, yields the familiar result (1.163). For arbitrary α, $0 < \alpha < 1$, integration of (1.173) yields, for some real c_3,

$$u(r) = -k_\alpha \left(\frac{1-\alpha}{2-\alpha}\right) r^{(2-\alpha)/(1-\alpha)} + c_3 \tag{1.175}$$

and then imposition of the boundary condition at $r = R$ yields the velocity profiles

$$u(r) = k_\alpha \left(\frac{1-\alpha}{2-\alpha}\right) \left[R^{(2-\alpha)/(1-\alpha)} - r^{(2-\alpha)/(1-\alpha)}\right], \quad 0 \le r \le R. \tag{1.176}$$

To obtain a clearer picture of the profiles (1.176), we again consider the sequence $\{\alpha_n\}$, $0 < \alpha_n < 1$, $\alpha_n \to 1^-$ as $n \to \infty$, given by $\alpha_n = (n-1)/n$. Setting $\alpha = \alpha_n$ in (1.176), and denoting the resulting profile by $u_n(r)$, we easily compute that

$$u_n(r) = K_n \left[R^{n+1} - r^{n+1}\right], \quad 0 \le r \le R \tag{1.177}$$

with

$$K_n = \left(p_1/\mu_0 2^{(3n-1)/2n}\right)^n \cdot \frac{1}{n+1}. \tag{1.178}$$

For $n = 2$ ($\alpha = 1/2$) we obtain

$$u_2(r) = \frac{1}{3 \cdot 2^{5/2}} \left(\frac{p_1}{\mu_0}\right)^2 \left[R^3 - r^3\right], \quad 0 \le r \le R \tag{1.179}$$

while for $n = 3$ ($\alpha = 2/3$) we get

$$u_3(r) = \frac{1}{4 \cdot 2^4} \left(\frac{p_2}{\mu_0}\right)^3 \left[R^4 - r^4\right], \quad 0 \le r \le R. \tag{1.180}$$

It is clear that the profiles given explicitly by (1.177) exhibit the "flattening out" one sees in Poiseuille flow, for small values of the standard kinematic viscosity, prior to the breakdown of laminar flow and the onset of turbulence. The persistence of the profiles (1.176) for $\epsilon \neq 0$, $\mu_1 \neq 0$ (but small) can be demonstrated by means of appropriate continuous dependence theorems for the behavior of the solutions of (1.167), (1.168) as ϵ and $\mu_1 \to 0^+$ (e.g., [Mon]); an example of just such a continuous dependence result, as has already been indicated, is proven in Chap. 2 for the problem of plane Poiseuille flow between parallel plates.

1.5.4 Plane Couette Flow

For our last example we consider the problem of steady flow between parallel plates at $x_2 = \pm a$, but now assume that the bottom plate at $x_2 = a$ is moving with constant velocity $\bar{u}$. The flow still has the form (1.132) but the boundary conditions are now

$$v_1(-a) = 0, \quad v_1(a) = \bar{u}. \tag{1.181}$$

The solution of this problem, within the framekwork of the steady Navier–Stokes equations, is well known and is given by

$$v_1(x_2) = \frac{p_1}{2\mu_0}x_2^2 + \frac{\bar{u}}{2a}x_2 + \frac{1}{2}\left(\bar{u} - \frac{p_1}{\mu_0}a^2\right) \tag{1.182}$$

which, for $\bar{u} = 0$, reduces to the solution of the plane Poiseuille flow problem between fixed parallel plates. In order to examine an example of this flow within the framework of the nonlinear theory for a bipolar fluid with $\epsilon = \mu_1 = 0$, we consider the differential equation (1.142) for $n = 2$ ($\alpha = 1/2$), i.e.,

$$\sqrt{2}u_2'(x_2) = \pm\left(\frac{p_1}{\mu_0}x_2 + \gamma\right)^2 \tag{1.183}$$

subject to the boundary conditions $u_2(-a) = 0$, $u_2(a) = \bar{u}$. Integration of (1.183) again yields (1.144), and as we must still have $u_2'(-a) > 0$, $u_2'(a) < 0$, imposition of the boundary conditions and some elementary algebraic manipulations yield

$$\gamma = -\frac{\mu_0\bar{u}}{\sqrt{2}p_1a^2}, \quad \tilde{\gamma} = \frac{1}{3}\left(\frac{p_1}{\mu_0}\right)^2 a^3 + \frac{\bar{u}}{\sqrt{2}} + \frac{1}{2}\left(\frac{\mu_0}{p_1}\right)^2\frac{\bar{u}^2}{a^3}. \tag{1.184}$$

Inserting the constants $\gamma, \tilde{\gamma}$ into (1.144) and simplifying, we find for the profile $u_2(x_2)$ the following explicit form:

$$\begin{aligned} u_2(x_2) = 2^{-1/2}\Bigg\{&\frac{1}{3}\left(\frac{p_1}{\mu_0}\right)^2(x_2^3 + a^3) + \frac{\bar{u}}{\sqrt{2}}\left(1 - \frac{x_2^2}{a^2}\right) \\ &+ \frac{1}{2}\left(\frac{\mu_0}{p_1}\right)^2\frac{\bar{u}^2}{a^4}(x_2 + a)\Bigg\}, \quad \text{for } x_2 \geq \frac{1}{\sqrt{2}}\left(\frac{\mu_0}{p_1}\right)^2\frac{\bar{u}}{a^2}, \end{aligned} \tag{1.185a}$$

$$\begin{aligned} u_2(x_2) = 2^{-1/2}\Bigg\{&\frac{1}{3}\left(\frac{p_1}{\mu_0}\right)^2(x_2^3 + a^3) + \frac{\bar{u}}{\sqrt{2}}\left(1 - \frac{x_2^2}{a^2}\right) \\ &+ \frac{1}{2}\left(\frac{\mu_0}{p_1}\right)^2\frac{\bar{u}^2}{a^4}(-x_2 + a)\Bigg\}, \quad \text{for } x_2 \leq \frac{1}{\sqrt{2}}\left(\frac{\mu_0}{p_1}\right)^2\frac{\bar{u}}{a^2}. \end{aligned} \tag{1.185b}$$

A simple comparison shows that (1.185a,b) reduce to (1.147) if $\bar{u} = 0$.

1.6 Other Extensions and Generalizations of the Navier–Stokes Model

There is, by now, a very large literature on non-Newtonian fluid dynamics as well as a bewildering array of different models that have been proposed to study non-Newtonian fluid behavior. We have no intention in this section of attempting anything approaching a comprehensive survey of non-Newtonian fluid phenomena and, indeed, such an overview of the field is available in many other excellent sources. We recommend, in particular, the following: (1) the encyclopedic volume by Joseph [Jo2] on visoelastic fluid behavior, (2) Chap. III, Sect. 2 of [GRRT] by Galdi which surveys problems in non-Newtonian fluid mechanics, (3) the article by Malek and Rajagopal [MR], (4) Rajagopal's article [Raj] on the mechanics of non-Newtonian fluids, (5) the comprehensive survey of the area of polar fluid dynamics by Cowin [Cow], and (6) the forthcoming monograph by Galdi and Robertson [GRo]; this latter volume introduces some of the most important types of non-Newtonian fluids, including Reiner-Rivlin fluids, power law models, simple fluids, and Oldroyd-B models, and presents a rigorous mathematical analysis of some of the corresponding boundary-value and initial-boundary value problems. Our limited goal, in this section, is to describe the structure of several non-Newtonian fluid models which exhibit one or more similarities to the constitutive model (1.88a,b) for the bipolar, viscous fluid; the properties we have in mind include, in particular, a (lower-order) nonlinear viscosity which resembles (1.90), or the presence of higher-order spatial derivatives of the velocity components and an associated set of higher-order boundary conditions. For each of the models considered in this section, which exhibit a dependence on velocity gradients of order greater than one, we also present a survey in Sect. 1.7 of some of the specific problems, such as plane Poiseuille flow between parallel plates, or proper Poiseuille flow in a circular pipe, which have been solved within the framework of these theories; the reader should then be able to compare several of these results with the solutions obtained, in Chaps. 2 and 3, for similar types of flows of a bipolar viscous fluid.

The paper of Friedlander and Pavlović [FPa] offers an excellent survey of the particular class of non-Newtonian fluids with a shear dependent viscosity that was studied, rigorously, for the first time by Ladyzhenskaya [La1, 2]; the authors indicate that the basic model was essentially introduced by Smagorinsky in connection with his studies of meteorlogical phenomena. Smagorinsky [Sma] has discussed the utility of employing fluid mechanics models exhibiting a nonlinear viscosity function in the context of the theory of rapidly rotating fluids and argues that the chosen form of the nonlinearity may be justified on the basis of the Heisenberg-Kolmogorov similarity theory for three-dimensional isotropic theory; the particular form chosen for the nonlinear viscosity $\mu(|\boldsymbol{e}|)$ in his studies was

$$\mu(|\boldsymbol{e}|) = \mu_*|\boldsymbol{e}|, \quad \mu_* > 0 \tag{1.186}$$

so that the stress tensor assumes the form

$$t_{ij} = -p\delta_{ij} + \mu_* |\boldsymbol{e}| e_{ij}. \tag{1.187}$$

We note, in passing, the similarity of (1.187) to the hypothesis of Prandtl, which is embodied in (1.79), for the problem of plane Poiseuille flow. The more general constitutive theory that was introduced by Ladyzhenskaya, in [La1, 2], is of the form $\boldsymbol{t} = -p\boldsymbol{I} + \boldsymbol{\tau}(\boldsymbol{e})$, where it is assumed that for some positive constants c_1, $\bar{\nu}_0$, $\bar{\nu}_1$, $\bar{\nu}_2$ and q, $\boldsymbol{\tau}(\boldsymbol{e})$ satisfies the following three conditions:

$$\left|\tau_{ij}(\boldsymbol{e})\right| \leq c_1(1 + |\boldsymbol{e}|^{2q})|\boldsymbol{e}| \tag{1.188a}$$

$$\tau_{ij}(\boldsymbol{e})\frac{\partial v_i}{\partial x_j} \geq \bar{\nu}_0|\boldsymbol{e}|^2 + \bar{\nu}_1|\boldsymbol{e}|^{2(1+q)} \tag{1.188b}$$

and, for arbitrary smooth divergence free vectors $\boldsymbol{v}$, $\boldsymbol{v}'$ which are such that $\boldsymbol{v}|_{\partial\Omega} = \boldsymbol{v}'|_{\partial\Omega}$, $\partial\Omega$ the (smooth) boundary of an open bounded domain Ω in R^n, $n = 2, 3$

$$\int_\Omega (\tau_{ij}(\boldsymbol{e}) - \tau_{ij}(\boldsymbol{e}'))\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_i'}{\partial x_j}\right) d\boldsymbol{x} \geq \bar{\nu}_2 \int_\Omega \left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_i'}{\partial x_j}\right)\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_i'}{\partial x_j}\right) d\boldsymbol{x}. \tag{1.188c}$$

The model defined by the viscosity function (1.186) conforms to (1.188a,b,c) if we take $q = 1$. Other examples of constitutive hypotheses, relative to the reduced stress tensor τ_{ij}, which satisfy (1.188a,b,c) are as follows:

$$\tau_{ij}(\boldsymbol{e}) = \bar{\nu}|\boldsymbol{e}|^{2q} e_{ij}, \tag{1.189a}$$

$$\tau_{ij}(\boldsymbol{e}) = \bar{\nu}(1 + |\boldsymbol{e}|^2)^q e_{ij}, \tag{1.189b}$$

$$\tau_{ij}(\boldsymbol{e}) = \bar{\nu}(1 + |\boldsymbol{e}|)^{2q} e_{ij}, \tag{1.189c}$$

and

$$\tau_{ij}(\boldsymbol{e}) = \bar{\nu}(1 + |\boldsymbol{e}|^{2q}) e_{ij}. \tag{1.189d}$$

The nonlinear viscosity $\mu(|\boldsymbol{e}|) = \bar{\nu}(1 + |\boldsymbol{e}|^2)^q$ is, of course, in full agreement with (1.90) if we make the identification $\bar{\nu} = \mu_0$ and $q = (p-2)/2$. In Sect. 4.5 we will recall some of Ladyzhenskaya's results for those non-Newtonian fluid models which conform to the behavior delineated in (1.188a,b,c). The Ladyzhenskaya modification of the Navier–Stokes system has also been analyzed by other authors. In [DuG] the authors study the initial-boundary value problem for an incompressible ($\nabla \cdot \boldsymbol{v} = 0$) viscous fluid satisfying the system of partial differential equations

$$\frac{\partial v_i}{\partial t} + v_j\frac{\partial v_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\mathcal{A}(\boldsymbol{v})\frac{\partial v_i}{\partial x_j}\right) + f_i \tag{1.190}$$

in a bounded domain $\Omega \subseteq R^n$, $n = 2, 3$, where we have set the density $\rho \equiv 1$ and where

$$\mathcal{A}(\boldsymbol{v}) \equiv \nu_0 + \nu_1 |\nabla \boldsymbol{v}|^{\hat{q}}, \quad \hat{q} > 0. \tag{1.191}$$

The non-Newtonian model defined by (1.190), (1.191) is essentially equivalent to the one defined by the reduced stress tensor (1.189d). We will have more to say about the results of [DuG] in Sect. 4.5. In [Lio1] Lions studies several problems which are related to the ones analyzed by Ladyzhenskaya [La1, 2] and Du and Gunzburger [DuG]; defining

$$[\boldsymbol{A}(\boldsymbol{v})]_{ij} = \frac{\partial}{\partial x_j}\left(|\nabla \boldsymbol{v}|^{p-2}\frac{\partial \boldsymbol{v}}{\partial x_j}\right) \tag{1.192}$$

and assuming that the velocity field is divergence free, the equations associated with these problems assume the following (vector) forms for some positive $\bar{\nu}$, $\bar{\nu}_0$, $\bar{\nu}_1$:

$$\frac{\partial \boldsymbol{v}}{\partial t} + v_j \frac{\partial \boldsymbol{v}}{\partial x_j} = -\nabla p + \bar{\nu}\boldsymbol{A}(\boldsymbol{v}) + \boldsymbol{f} \tag{1.193a}$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + v_j \frac{\partial \boldsymbol{v}}{\partial x_j} = -\nabla p + \bar{\nu}_0 \Delta \boldsymbol{v} + \bar{\nu}\boldsymbol{A}(\boldsymbol{v}) + \boldsymbol{f} \tag{1.193b}$$

and

$$\frac{\partial \boldsymbol{v}}{\partial t} + v_j \frac{\partial \boldsymbol{v}}{\partial x_j} = -\nabla p + \left(\bar{\nu}_0 + \bar{\nu}_1 ||\boldsymbol{v}||^2\right)\Delta \boldsymbol{v} + \boldsymbol{f}. \tag{1.193c}$$

Some of the results associated with (1.193a,b,c) will also be summarized in Sect. 4.5. To the class of Ladyzhenskaya type non-Newtonian problems described, above, we can also add the Carreau model [YKRA] for a shear-thinning fluid, namely, for some $\mu_0 > 0$, $\mu_\infty > 0$, $\gamma > 0$, and q real,

$$\mu(\boldsymbol{e}) = \mu_\infty + (\mu_0 - \mu_\infty)(1 + |\gamma \boldsymbol{e}|^2)^q. \tag{1.194}$$

The non-Newtonian model defined by (1.194) has had some success in modeling blood flow in arteries, i.e., [YKRA], where q is taken to be $q = -0.322$.

The second of the two key elements present in (1.88a,b), which distinguishes the constitutive theory from the one associated with the Stokes Law (1.7) for the reduced stress tensor, is the presence of the higher-order velocity gradients in (1.88a) and the concurrent appearance of the first multipolar stress tensor in (1.88b). Other authors have introduced, albeit in an ad hoc manner, higher-order spatial derivatives of the velocity into the fluid dynamics equations in an attempt to regularize the Navier–Stokes system. Often, the goal of such a regularization is to be able to go to the limit, as the parameter controlling the additional terms goes to zero, so as to deduce the existence of a weak solution to the original Navier–Stokes system. J.-L. Lions [Lio1, 2] perturbed the Navier–Stokes system for incompressible flow by adding to the Navier–Stokes equations the artificial viscosity term $-\epsilon(-\Delta)^{\beta}\boldsymbol{v}$ with $\epsilon > 0$; this yields the following equation for $\boldsymbol{v}$ if we set $\rho = 1$:

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\nabla p + \nu \Delta \boldsymbol{v} - \epsilon(-\Delta)^{\beta} \boldsymbol{v} + \boldsymbol{f}. \tag{1.195}$$

Friedlander and Pavlović [FPa] point out that, although the modification of Navier–Stokes reflected in (1.195) is not well-motivated, from a physical standpoint, the artificial (higher-order) viscosity term in (1.195) has proven useful in obtaining numerical approximations for incompressible viscous flow problems. Other studies which treat regularizations of Navier–Stokes that are in the same spirit as (1.195) include the papers of Beirão da Veiga [BdV2, 3] and Ladyzhenskaya [La5]. The special case of (1.195) for which $\beta = 2$ has been treated, in considerable detail, by Ou and Sritharan [OS1, 2]. In the two studies [Lio1, 2], besides the usual non-slip boundary condition, which applies for flow in a bounded domain, or in the exterior of such a domain, additional Neumann type boundary conditions are specified. With $\beta = 2$ in (1.195), this means that the full-initial boundary-value problem considered in [OS1, 2] assumes the form

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\nabla p + \nu \Delta \boldsymbol{v} - \epsilon \Delta^2 \boldsymbol{v} + \boldsymbol{f}, \text{ in } \Omega \times [0, T), \tag{1.196a}$$

$$\nabla \cdot \boldsymbol{v} = 0, \text{ in } \Omega \times [0, T), \tag{1.196b}$$

$$\boldsymbol{v} = \boldsymbol{0}, \ \frac{\partial \boldsymbol{v}}{\partial n} = 0, \text{ on } \partial\Omega \times [0, T), \tag{1.196c}$$

$$\boldsymbol{v}(\boldsymbol{x}, 0) = \boldsymbol{v}_0(\boldsymbol{x}), \text{ in } \Omega. \tag{1.196d}$$

As we have seen in Sect. 1.4, specifying the vanishing of $\partial \boldsymbol{v}/\partial n$ on $\partial\Omega \times [0, T)$ is not in accord with the (natural) boundary conditions that are derivable from the principle of virtual work when the reduced stress tensor depends on third-order spatial gradients of the velocity field; furthermore, the enforcement of the second boundary condition in (1.196c) can lead to serious inconsistencies, as has already been pointed out in Sect. 1.4.4, and as will be pointed out again in 3.5. Some of the existence and uniqueness results obtained in [Lio1, 2] and [OS1, 2] will be elaborated in Sect. 4.5.

Several other authors have considered models for incompressible, viscous fluid flow which involve spatial derivatives of the velocity of order two or higher. Because of the proliferation of partial derivatives that appear in the work to be described, we will, for the remainder of this subsection, denote such derivatives by using the standard notation $v_{i,j} = \dfrac{\partial v_i}{\partial x_j}$, etc. In [BNR] Bellout, Nečas, and Rajagopal have studied flows of multipolar fluids of grade 3; the model in question combines the theory of multipolar fluids with the theory of fluids of differential type, specifically with the subclass of such fluids that can be characterized as fluids of grade n (see, e.g., [TN]). Using a fluid flow model of differential type allows for taking into account the history of velocity gradients while using elements of multipolar theory enables one to blend non-locality into the model. The fluid of grade three conforms to the specification of a Cauchy stress tensor $\boldsymbol{t}$ of the form

$$\boldsymbol{t} = -pI + \mu_0 \boldsymbol{A}_1 + \alpha_1 \boldsymbol{A}_2 + \alpha_2 \boldsymbol{A}_1^2 + \beta_1 \boldsymbol{A}_3 + \beta_2[\boldsymbol{A}_1\boldsymbol{A}_2 + \boldsymbol{A}_2\boldsymbol{A}_1] + \beta_3(\mathrm{tr}\,\boldsymbol{A}_1^2)\boldsymbol{A}_1 \tag{1.197}$$

where $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$, the first and second Rivlin-Ericksen tensors [RE], are given by

$$\begin{cases} \boldsymbol{A}_1 = \nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^t, \\ \boldsymbol{A}_2 = \dfrac{d}{dt}\boldsymbol{A}_1 + \boldsymbol{A}_1(\nabla \boldsymbol{v}) + (\nabla \boldsymbol{v})^t \boldsymbol{A}_1. \end{cases} \tag{1.198}$$

Restrictions must be placed on the constants appearing in (1.197) so that the resulting model will be compatible with the Clausius-Duhem inequality; we will not specify those restrictions here but, rather, refer the reader to [FR]. In [BNR] the constitutive equation (1.197) is modified so as to obtain the following constitutive relations for a tripolar fluid of grade three: Let μ_0 be the classical viscosity in the Stokes law and α_1, α_2, β_3, γ, μ_1, and μ_2 (constant) material moduli. Set

$$B_{ijkm} = \mu_2(\boldsymbol{A}_1)_{ij,km}, \tag{1.199a}$$

$$S_{ijk} = \mu_1(\boldsymbol{A}_1)_{ij,k} + \gamma M_{ijk} - B_{ijkm,m}, \tag{1.199b}$$

with

$$M_{ijk} = \frac{d}{dt} e_{ij,k} + W_{mi} e_{mj,k} + W_{mj} e_{im,k} + W_{mk} e_{ij,m} \tag{1.199c}$$

where $\boldsymbol{W} = \frac{1}{2}(\nabla \boldsymbol{v} - (\nabla v)^t)$ is the spin tensor; then,

$$t_{ij} = -p\delta_{ij} + \mu_0(\boldsymbol{A}_1)_{ij} + \alpha_1(\boldsymbol{A}_2)_{ij} + \alpha_2(\boldsymbol{A}_1^2)_{ij} + \beta_3(\boldsymbol{A}_1^2)_{mm}(\boldsymbol{A}_1)_{ij} - S_{ijk,k}. \tag{1.199d}$$

In (1.199a,b) the μ_1 and μ_2 are higher-order viscosity coefficients. The associated free energy function ψ has the form

$$\rho\psi = \frac{1}{4}\alpha_1(\boldsymbol{A}_1)_{ij}(\boldsymbol{A}_1)_{ij} + \frac{1}{8}\gamma(\boldsymbol{A}_1)_{ij,k}(\boldsymbol{A}_1)_{ij,k}. \tag{1.200}$$

Some of the results on existence, uniqueness, and stability of solutions for the model proposed in [BNR] will be reviewed in Sect. 4.5. We note that in order to have a well-posed initial-boundary value problem, for the constitutive theory delineated in (1.199a–d), for the tripolar fluid of grade three, one needs to formulate a set of higher-order boundary conditions to complement the usual non-slip condition $v_i = 0$ on $\partial\Omega \times [0, T)$, with Ω an open bounded domain $\mathbb{R}^n$, $n = 2, 3$. It is shown in [BNR] that such boundary conditions assume the following formal pointwise form: Let $\boldsymbol{\nu}$ be the exterior normal to $\partial\Omega$ at $\boldsymbol{x} \in \partial\Omega$. The mean curvature of $\partial\Omega$ is then given by $2\chi = -\nu_{i;i}$ where for any function f defined on $\partial\Omega$

$$f_{;j} = \tilde{f}_{,j} - \nu_j \nu_m \tilde{f}_{,m}$$

with $\tilde{f}$ any extension of f from $\partial\Omega$ into a neighborhood of $\partial\Omega$. Let $\boldsymbol{\tau}^1$ and $\boldsymbol{\tau}^2$ be linearly independent tangent vectors to $\partial\Omega$ at $\boldsymbol{x} \in \partial\Omega$. Then,

$$B_{ijkm}\nu_j\nu_k\nu_m\tau_i^l = 0, \quad l = 1, 2, \text{ on } \partial\Omega \times [0, T) \tag{1.201a}$$

and

$$\begin{aligned}[S_{ijk}\nu_j\nu_k + 4\chi B_{ijkm}\nu_j\nu_k\nu_m - (B_{ijkm}\nu_m)_{;k}\nu_j \\ - (B_{ijkm}\nu_m\nu_k)_{;j} + (B_{ljkm}\nu_l\nu_j\nu_m\nu_k)_{;i}]\tau_i^l = 0, \\ l = 1, 2, \text{ on } \partial\Omega \times [0, T).\end{aligned} \tag{1.201b}$$

The derivation of (1.201a,b) may be found in Sect. 4.5.4.

Three models of incompressible viscous fluid flow, which involve higher-order spatial derivatives of the velocity field, as well as time derivatives of velocity gradients, may be found in the work of Bleustein and Green [BG], Green and Nagdi [GN1, 2], Chen, et. al. [CFH1, 2, 3], and Foias, et. al. [FHT1, 2]; these models have certain aspects in common with one another as well as with the model for a linear viscous, incompressible bipolar fluid constructed in Sect. 1.3. A concise summary of the equations and associated boundary conditions for these three classes of viscous flow models can be found in [QS]. Bleustein and Green [BG] were among the first to argue that it is "consistent to include multipolar stresses when considering fluids for which velocity gradients of various orders are present in the constitutive equations" due to "the problem of the formulation of boundary conditions ... in theories of the multipolar type, boundary conditions follow in a natural way"; this last point has been highlighted in Sects. 1.4.3 and 1.4.4 where it was demonstrated that the appropriate form of the higher-order boundary conditions follows directly from the principle of virtual work. In [BG] the type of multipolar fluid for which the constitutive theory is formulated is termed a "dipolar" fluid; these constitutive equations assume the form (see [BG], as well as Jordan and Puri [JP5] and Akyildiz and Bellout [AB]).

$$\tau_{ij}^* + \Phi\delta_{ij} = 2\mu e_{ij}, \tag{1.202a}$$

$$\tau_{(ij)k} + \Psi_i\delta_{jk} + \Psi_j\delta_{ik} = h_1\delta_{ij}A_{kmm} + h_2(A_{ijk} + A_{jik}) + h_3A_{kji} \tag{1.202b}$$

where τ^*_{ij} is the (total) stress tensor, the $\tau_{(ij)k}$ denote the symmetrization with respect to the indices i and j of the dipolar stress tensor τ_{ijk}, the arbitrary functions Φ and Ψ_i govern the pressure (and appear as a consequence of the fact that the velocity field $\boldsymbol{v}$ is divergence free), e_{ij} is the usual rate of deformation tensor, and $A_{ijk} = v_{i,jk} = A_{ikj}$. As a consequence of the Clausius-Duhem inequality, the shear viscosity μ and the material constants h_i, $i = 1, 2, 3$, satisfy the restrictions $\mu \geq 0$ and

$$2h_1 + h_3 \geq 0, \quad 2h_2 + h_3 \geq 0, \quad h_3 - h_2 \geq 0, \quad 5h_1 - h_2 + 2h_3 \geq 0. \quad (1.203)$$

In (1.202a) the (total) stress tensor τ^*_{ij} is related to the (nonsymmetric) monopolar stress tensor t_{ij} by

$$\tau^*_{ij} = t_{ij} + \tau_{kij,k} + \rho(F_{ij} - \Gamma_{ij}) = \tau^*_{ji} \quad (1.204)$$

where ρ is the density, which we assume to be constant in the incompressible case, F_{ij} is the dipolar (microscopic) body force/mass, and Γ_{ij} is the dipolar inertia. Two forms for the dipolar inertia have appeared in the literature, namely, the original form due to Bleustein and Green [BG]

$$\Gamma_{ij} = d^2 \left(\frac{D}{Dt} \nu_{j,i} - \nu_{j,k} \nu_{k,i} \right) \quad (1.205a)$$

and the alternative form, given by Green and Naghdi in [GN3], as

$$\Gamma_{ij} = d^2 \left(\frac{D}{Dt} \nu_{j,i} - \nu_{j,k} \nu_{k,i} - \nu_{j,k} \nu_{i,k} + \nu_{k,i} \nu_{k,j} \right). \quad (1.205b)$$

In both (1.205a) and (1.205b), $d \geq 0$ is a constant representing the micro-inertia coefficient, which has units of length. By employing the constitutive theory delineated, above, in conjunction with the usual balance of momentum equation,

$$\rho \frac{Dv_i}{Dt} = t_{ji,j} + \rho f_i,$$

f_i being the ith component of the body force/mass, we obtain the following form of the dipolar fluid dynamics equations which incorporate the dipolar inertia tensor of Green and Naghdi [GN3], and in which we have set $F_{ij} \equiv 0$:

$$(1 - d^2 \Delta) \frac{Dv_i}{Dt} + d^2 (v_{i,k} v_{k,j} + v_{i,k} v_{j,k} - v_{k,j} v_{k,i})_{,j} - \frac{1}{\rho} p_{,i} + \nu(\Delta - l^2 \Delta^2) v_i + f_i. \quad (1.206)$$

In (1.206) the pressure p is given by

$$p \equiv \Phi - 2\Psi_{i,i}, \quad (1.207)$$

also, $l^2 \equiv \dfrac{h_1 + h_3}{\mu} \geq 0$, with $\nu = \mu/\rho$ the kinematic viscosity. The boundary conditions specified in [BG], for an open bounded domain Ω, with smooth boundary $\partial\Omega$, and exterior unit normal $\boldsymbol{\nu}$ at $\boldsymbol{x} \in \partial\Omega$, consist of prescribing the monopolar traction components $T_i = t_{ki}\nu_k$ and the components $T_{ij} = \tau_{kij}\nu_k$ of the dipolar tractions; the latter condition is shown in [BG] to be equivalent to prescribing

$$\tau_{(ki)j}\nu_k\nu_i = T_{ij}\nu_i \equiv M_i, \text{ on } \partial\Omega \times [0, T). \tag{1.208}$$

The higher-order boundary conditions in (1.208) are what take the place of the analogous conditions in the bipolar model introduced in Sect. 1.4. The constitutive theory formulated in [BG], while not involving a nonlinear viscosity, is more complex than that put forth in Sect. 1.4 for the bipolar fluid. In this regard, it is important to note that like the Camassa-Holm equations in [CFH1, 2, 3], [FHT1, 2], and the isothermal viscous flow model formulated in [GN1, 2], the model constructed in [BG], unlike the bipolar model of Sect. 1.4, was envisioned as applying to more complex fluids than those which exhibit Newtonian behavior in ordinary circumstances; as pointed out in [QS], the dipolar flow theory in [BG] is believed to be capable of describing fluid motion when the fluid contains long chain molecules. Indeed, in the dipolar theory, the parameters d and l are thought of as representing the effects of fluid microstructure. To facilitate a comparison of the dipolar model in [BG], and the resultant evolution equations (1.206) for the components of the velocity field, with the analogous equations in [GN1, 2] and [CFH1, 2,3], [FHT1, 2], we follow the analysis in [QS] and rewrite those terms in (1.206) which are multiplied by d^2, but which do not involve partial derivatives with respect to time, such as

$$-v_j\Delta v_{i,j} - v_{j,i}\Delta v_j$$

in which case (1.206) assumes the form

$$\frac{Dv_i}{Dt} - d^2(\Delta v_{i,t} + v_j\Delta v_{i,j} + v_{j,i}\Delta v_j) = -p_{,i} + \mu(\Delta - l^2\Delta^2)v_i. \tag{1.209}$$

In (1.209) we have set $\rho = 1$, and $f_i = 0$, so that $\nu \equiv \mu$.

The nonlinearly dispersive Navier–Stokes alpha (NS-α) model of incompressible fluid flow, also known as the viscous Camassa-Holm equations (VCHE), were proposed as a closure approximation for the Reynolds averaged equations appearing in Navier–Stokes modeling of turbulent flow; these equations, and their consequences for particular types of turbulent flow, have been extensively explored in [CFH1, 2, 3] and [FHT1, 2]. Indeed, steady solutions of VCHE have been identified with the mean flow appearing in the Reynolds equations and the results which ensue have been compared with empirical data for turbulent flows in channels and pipes; for an overview of the statistical approach to turbulence modeling based on the Reynolds (averaged) equations, the reader is referred to part IV of [ScG]. With $\boldsymbol{v}$ denoting the usual velocity field in the fluid, and

$$\boldsymbol{u} = [\boldsymbol{I} - (\nabla \cdot \langle \boldsymbol{\sigma} \rangle]\boldsymbol{v} - \frac{\partial}{\partial x_i}\left[\langle \sigma_i \sigma_j \rangle \frac{\partial \boldsymbol{v}}{\partial x_j}\right] \tag{1.210}$$

where $\langle \quad \rangle$ denotes the usual ensemble average, and $\boldsymbol{\sigma}$ is the velocity fluctuation with components σ_i, the VCHE are, in the absence of an external bodyforce

$$\frac{d\boldsymbol{u}}{dt} + u_j \nabla v_j + \nabla \pi = \nu \nabla^2 \boldsymbol{u}, \tag{1.211a}$$

$$\nabla \cdot \boldsymbol{v} = 0 \tag{1.211b}$$

where, in (1.211a),

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla \tag{1.212a}$$

and

$$\pi = p - \frac{1}{2}||\boldsymbol{v}||^2 - \frac{1}{2}\langle \sigma_i \sigma_j \rangle \frac{\partial \boldsymbol{v}}{\partial x_i} \cdot \frac{\partial \boldsymbol{v}}{\partial x_j} \tag{1.212b}$$

is the usual pressure with p the modified pressure. If one assumes isotropy and homogeneity for the fluctuations $\boldsymbol{\sigma}$, then $\langle \boldsymbol{\sigma} \rangle = \boldsymbol{0}$ and $\langle \sigma_i \sigma_j \rangle = \alpha^2 \delta_{ij}$, with α a local length scale; by virtue of the assumption of homogeneity α is a constant. Then, under this dual assumption of homogeneity and isotropy for the σ_i, (1.211a) can be rewritten in the form

$$\frac{d\boldsymbol{v}}{dt} = \operatorname{div} \boldsymbol{t} \tag{1.213}$$

with the stress tensor $\boldsymbol{t}$ expressed as

$$\boldsymbol{t} = -p\boldsymbol{I} + 2\nu(\boldsymbol{I} - \alpha^2 \nabla^2)\boldsymbol{e} + 2\alpha^2 \dot{\boldsymbol{e}} \tag{1.214}$$

where $\boldsymbol{e} = \frac{1}{2}(\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^t)$ and the corotational (or Jaumann) derivative of $\boldsymbol{e}$ is given by

$$\dot{\boldsymbol{e}} = \frac{d\boldsymbol{e}}{dt} + \boldsymbol{e}\boldsymbol{W} - \boldsymbol{W}\boldsymbol{e} \tag{1.215}$$

with $\boldsymbol{W} = \frac{1}{2}(\nabla \boldsymbol{v} - (\nabla \boldsymbol{v})^t)$ the spin tensor introduced, previously, in (1.199c). As observed in [CFH1], in the form delineated in (1.214), (1.215), the constitutive theory underlying VCHE represents that modification of the constitutive theory defining a rate dependent incompressible homogeneous fluid of grade 2 which is obtained by modifying the viscous dissipation term through use of the Helmholtz operator $(\boldsymbol{I} - \alpha^2\nabla^2)$. Therefore, the case where the fluctuation $\boldsymbol{\sigma}$ satisfies both isotropy and inhomogeneity yields a model consistent with the basic continuum mechanics principle of material frame-indifference. In this case, where both $\langle\boldsymbol{\sigma}\rangle = 0$, and $\langle\sigma_i\sigma_j\rangle = \alpha^2\delta_{ij}$, with α constant, (1.211a), when coupled with (1.212a,b), and the fact that (1.210) reduces to

$$\boldsymbol{u} = [\boldsymbol{I} - \alpha^2\nabla^2]\boldsymbol{v} \tag{1.216}$$

assumes the explicit form

$$v_{i,t} + v_j v_{i,j} - \alpha^2(\Delta v_{i,t} + v_j \Delta v_{i,j} + v_{j,i}\Delta v_j) - p_{,i} + \nu\Delta v_i - \nu\alpha^2\Delta^2 v_i. \tag{1.217}$$

As observed in [QS], the two sets of equations (1.209) and (1.216) are essentially the same if we make the obvious identifications $d^2 = l^2 = \alpha^2$. As the parameters d^2 and l^2 appearing in (1.209) are independent of one another, the argument could be made that (1.209) is more general than (1.217); this would be misleading, of course, as the various parameters have different interpretations and the models in question were conceived to treat differental physical flow problems. With $\alpha = 0$ in (1.217) one recovers the standard form of the incompressible Navier–Stokes equations with $\rho = 1$ and $f_i = 0$, the same being true for the dipolar equations (1.209) with $l = d = 0$. Although the system of equations (1.217) is of fourth order, unlike the case with the Bleustein-Green model for the dipolar viscous fluid, there does not seem to be a well-motivated set of higher-order boundary conditions associated with VCHE. Both [CFH1] and [CFH2] use VCHE to treat the problem of turbulent channel flow in a parallel-walled channel whose half-width is a. The usual non-slip boundary condition, as applied to a steady flow solution of (1.217) of the form

$$\boldsymbol{v} \equiv \boldsymbol{U} = (U(y), 0, 0) \tag{1.218}$$

yields $U(\pm a) = 0$ in [CFH1] and [CFH2]. However, in [CFH1] the additional condition (symmetry) $U(y) = U(-y)$, $-a \le y \le a$, is used while, in [CFH2], the solution U is subject to the additional set of boundary conditions $\nu U'(\pm a) = \mp\tau_0$, with τ_0 the boundary shear stress, as well as to the assumption that U is symmetric across the channel. The condition that $\nu U'(\pm a) = \mp\tau_0$ also appears in [CFH3]. *We have been unable to discern, in the papers* [CFH1], [CFH2], [CFH3] *or* [FHT1, 2]*, a consistent set of higher-order boundary conditions that are associated with the viscous Camassa-Holm model.*

The last of the models, which incorporate higher-order spatial velocity derivatives, that we consider in this section is due to Green and Naghdi [GN1]; it is instructive to quote from their introduction:

> It is clear from extensive available experimental results, and also to some extent from existing numerical simulations, that not only the vorticity (or spin) but also the rate of change of vorticity (or "spin of the spin") affect the structure of turbulent flows. Given this premise it is natural to see if a satisfactory theory can be constructed with a vorticity vector

> $\boldsymbol{w}$ and a spin of vorticity vector $\boldsymbol{u}$ explicitly displayed in the theory, instead of just being evaluated at the end of the solution of problems from the velocity vector $\boldsymbol{v}$. Moreover, such a theory, if properly based on continuum thermodynamic equations, would have interest in its own right, apart from its possible application to turbulent flow.

In the discussion which follows, we will summarize the basic elements of only the restricted theory of incompressible viscous flow, with vorticity and spin of vorticity, that is found in [GN1]; the constitutive relations for this model have the form

$$\boldsymbol{t} = -p\boldsymbol{I} + 2\mu_0\boldsymbol{e} + 2\mu_1\boldsymbol{P}, \tag{1.219a}$$

$$\boldsymbol{M}_1 = -p_1\boldsymbol{I} + \mu_1\boldsymbol{N}, \tag{1.219b}$$

$$\boldsymbol{M}_2 = -p_2\boldsymbol{I} + 2\mu_1\boldsymbol{N} + 2\left(\frac{\mu_1^2}{\mu_0}\right)\boldsymbol{P}, \tag{1.219c}$$

where p, p_1, p_2 are arbitrary scalar functions of $\boldsymbol{x}$, t, μ_0 is the classical viscosity coefficient, and μ_1 is a second (higher-order) viscosity coefficient. In (1.219a,b,c), the tensors $\boldsymbol{P}$ and $\boldsymbol{N}$ are defined as follows: We begin by defining a spin vector $\boldsymbol{w}$ by $\boldsymbol{w} = \nabla \times \boldsymbol{v}$ so that with $\boldsymbol{W}$ the spin (or vorticity tensor) we have $\boldsymbol{W}\hat{\boldsymbol{a}} = \frac{1}{2}\boldsymbol{w} \times \hat{\boldsymbol{a}}$ for every vector $\hat{\boldsymbol{a}}$. Next, we define the vorticity (or "spin of spin") vector $\boldsymbol{u}$ to be $\boldsymbol{u} = \nabla \times \boldsymbol{w} = \nabla \times \nabla \times \boldsymbol{v} = \nabla(\nabla \cdot \boldsymbol{v}) - \nabla^2\boldsymbol{v}$ so that, for incompressible flow, $\boldsymbol{u} = -\nabla^2\boldsymbol{v}$. Finally, we set

$$\boldsymbol{N} = \nabla\boldsymbol{w} \text{ and } \boldsymbol{P} = \nabla\boldsymbol{u}. \tag{1.220}$$

It can be shown that for every vector $\hat{\boldsymbol{a}}$, $(\boldsymbol{N} - \boldsymbol{N}^t)\hat{\boldsymbol{a}} = \boldsymbol{u} \times \hat{\boldsymbol{a}}$. There remains the task of defining the tensors $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ in (1.219b) and (1.219c), respectively. If $\Omega_t \subseteq \mathbb{R}^n$, $n = 2, 3$ is a bounded open domain, at time $t > 0$, with smooth boundary $\partial\Omega_t$, then the term representing the external rate of (the supply of) work $R(\Omega_t)$, to Ω_t, has the following form in [GN1]:

$$R(\Omega_t) = \int_{\Omega_t} \rho(\boldsymbol{f}\cdot\boldsymbol{v}+\boldsymbol{c}_1\cdot\boldsymbol{w}+\boldsymbol{c}_2\cdot\boldsymbol{u})\,d\boldsymbol{x} + \oint_{\partial\Omega_t} (\boldsymbol{T}\cdot\boldsymbol{v}+\boldsymbol{m}_1\cdot\boldsymbol{w}+\boldsymbol{m}_2\cdot\boldsymbol{u})\,dS_{\boldsymbol{x}}. \tag{1.221}$$

In (1.221), $\boldsymbol{f}$ (the body force/mass), $\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{T}$ (the traction/area), $\boldsymbol{m}_1$, and $\boldsymbol{m}_2$ are all vector-valued functions of $\boldsymbol{x}$, t, either in Ω_t or on $\partial\Omega_t$. A standard "tetrahedron" argument (due to Euler) shows that the traction $\boldsymbol{T}$, at $\boldsymbol{x} \in \partial\Omega_t$ is related to the stress tensor $\boldsymbol{t}$ by the usual rule, i.e., $\boldsymbol{T} = \boldsymbol{t}\boldsymbol{\nu}$, with $\boldsymbol{\nu}$ the exterior unit normal to $\partial\Omega_t$ at $\boldsymbol{x}$; similar arguments establish the existence of second-order tensor functions $\boldsymbol{M}_1$, $\boldsymbol{M}_2$ such that

$$\boldsymbol{m}_1 = \boldsymbol{M}_1\boldsymbol{\nu} \text{ and } \boldsymbol{m}_2 = \boldsymbol{M}_2\boldsymbol{\nu}, \text{ at } (\boldsymbol{x}, t) \in \partial\Omega_t. \tag{1.222}$$

It is shown, (Sect. 4) in [GN1], that balance of mass, momentum, and energy, together with the admissible simplifying assumptions $\boldsymbol{c}_2 = \left(\frac{\mu_1}{\mu_0}\right)\boldsymbol{f}$, $\beta_2 = \left(\frac{\mu_1}{\mu_0}\right)\boldsymbol{P}$, $\boldsymbol{c}_1 = \boldsymbol{0}$, and $p_1 =$ constant, enable one to deduce from the constitutive theory (1.219a,b,c) the equation of motion

$$\rho\left(\frac{D\boldsymbol{v}}{Dt} + \left(\frac{\mu_1}{\mu_0}\right)\frac{D\boldsymbol{u}}{Dt}\right) = -\nabla p + \mu_0\nabla^2\boldsymbol{v} - 2\mu_1\nabla^4\boldsymbol{v} + \rho\boldsymbol{f}. \tag{1.223}$$

In component form the vector equation has the form (see [QS])

$$v_{i,t} + v_j v_{i,j} - \frac{\mu_1}{\rho\mu_0}(\Delta v_{i,t} + v_j\Delta v_{i,j}) = -p_{,i} + \nu\Delta v_i - \frac{2\mu_1}{\rho}\Delta^2 v_i \tag{1.224}$$

with $\nu = \frac{\mu_0}{\rho}$ again the usual kinematic viscosity. If standard reference values l_0, v_0, and ρ_0 of length, velocity, and density are introduced, and we define Reynolds numbers

$$\mathrm{Re} = \frac{\rho_0 l_0 v_0}{\mu_0} \text{ and } \mathrm{Re}^* = \frac{\rho_0 l_0^3 v_0}{\mu_1} \tag{1.225}$$

as well as the non-dimensional variables

$$\left.\begin{aligned} &\hat{\boldsymbol{x}} = \boldsymbol{x}/l_0, \quad \hat{\boldsymbol{v}} = \boldsymbol{v}/v_0, \quad \hat{t} = \frac{v_0}{l_0}t \\ &\hat{\rho} = \rho/\rho_0, \quad \hat{\boldsymbol{u}} = \frac{l_0^2}{v_0}\boldsymbol{u} \end{aligned}\right\},$$

then (1.223) can be put in the form

$$\rho\,\mathrm{Re}\left(\frac{D\boldsymbol{v}}{Dt} + \frac{1}{2\beta^2}\frac{D\boldsymbol{u}}{Dt}\right) = -\nabla p + \nabla^2\boldsymbol{v} - \frac{1}{\beta^2}\nabla^4\boldsymbol{v} + \rho\boldsymbol{f} \tag{1.226}$$

where the circumflexes have been dropped on all the non-dimensional quantities, $\boldsymbol{f}$ is non-dimensional, and $\beta^2 = \mathrm{Re}^*/2\,\mathrm{Re}$. The Green and Naghdi model for incompressible viscous fluid flow, like the other models presented in this section, is compatible with the basic principles of continuum mechanics and thermodynamics. For a problem posed in a bounded domain Ω, with smooth boundary $\partial\Omega_t$, the complete set of boundary conditions associated with the system (1.224) assumes the form

$$v_i = 0, \quad (\boldsymbol{m}_1)_i t_i^{(\alpha)} = 0, \quad \alpha = 1, 2; \text{ on } \partial\Omega \times [0, T) \tag{1.227}$$

where $(\boldsymbol{m}_1)_i = (\boldsymbol{M}_1)_{ij}\nu_j$, $(\boldsymbol{M}_1)_{ij} = -p_1\delta_{ij} + \mu_1 w_{i,j}$ (by (1.222) and (1.220)) and the $\boldsymbol{t}^{(\alpha)}$ are linearly independent vectors spanning the tangent space at $\boldsymbol{x} \in \partial\Omega$.

1.7 Some Examples of Viscous Incompressible Flow Described by the Dipolar, Camassa-Holm, and Green and Naghdi Models

1.7.1 Introduction

In this final section of Chap. 1 we will review some of the explicit results obtained, within the context of the three models described in Sect. 1.6, which incorporate higher order gradients of the velocity field; these models are the dipolar fluid of Bleustein and Green [BG], the Camassa-Holm (VCHE) model elaborated in [CFH1, 2, 3] and [FHT1, 2], and the extended theory for incompressible viscous fluid flow due to Green and Naghdi [GN1]. The problem of Poiseuille flow in a circular pipe will be considered within the context of each of the sets of model equations in this section; steady channel flow between parallel plates will be considered using the framework of VCHE, while non-steady flow for the same geometry will be treated using the Green and Naghdi model. For the dipolar model of Bleustein and Green we will also review the results obtained for the unsteady plane Couette flow, generated by fluid motion between parallel plates, which is initiated by the sudden acceleration of the upper plate.

1.7.2 Examples of Viscous Incompressible Flow for Dipolar Equations

Isothermal Steady Poiseuille Flow of a Homogeneous Isotropic Dipolar Fluid in a Cylinder

For this case, assuming $\rho = 1$ and $f_i = 0$, the dipolar system of equations (1.209) reduces to

$$v_j \frac{\partial v_i}{\partial x_j} - d^2(v_j \Delta v_{i,j} + v_{j,i}\Delta v_j) = -p_{,i} + \mu(\Delta - l^2\Delta^2)v_i. \tag{1.228}$$

We begin by assuming flow in a cylinder of arbitrary cross-section, with generators parallel to the z-axis of a rectilinear Cartesian coordinate system, and look for solutions of (1.228), subject to $v_{i,i} = 0$, of the form

$$\boldsymbol{v} = v(r)\boldsymbol{e}_z,\ p = p(r,z);\quad r = \sqrt{x^2 + y^2}. \tag{1.229}$$

Furthermore, it is assumed in [BG] that in (1.207)

$$\Phi = \Phi(r, z), \quad \Psi = \Psi(r, z)\boldsymbol{e}_z \tag{1.230}$$

so that

$$p(r, z) = \Phi(r, z) - 2\frac{\partial \Psi(r, z)}{\partial z}. \tag{1.231}$$

For the non-zero physical components of the dipolar stress tensor τ_{ijk} in cylindrical coordinates we compute, by setting the indeterminate stresses $\tau_{(ij)k} = 0$ (without loss of generality)

$$\begin{cases} \tau_{rrz} = (h_1 + h_3)v'' + h_1 r^{-1} v', \\ \tau_{\theta\theta z} = h_1 v'' + (h_1 + h_3) r^{-1} v', \\ \tau_{zzz} = -2\Psi + h_1(v'' + r^{-1} v'), \\ \tau_{zrr} = \tau_{rzr} = -\Psi + h_2 v'', \\ \tau_{z\theta\theta} = \tau_{\theta z\theta} = -\Psi + h_2 r^{-1} v' \end{cases} \tag{1.232}$$

where $' = \dfrac{d}{dr}$. From (1.202a), (1.204) with $\rho = 1$ and $F_{ij} = 0$, (1.205b) with $v_{j,it} = 0$, and (1.232) we compute that, in cylindrical coordinates, the non-zero components of the monopolar stress tensor t_{ij} are

$$\begin{cases} t_{rr} = t_{\theta\theta} = -\Phi + \dfrac{\partial \Psi}{\partial z}, \\ t_{zz} = -p, \\ t_{rz} = \mu v' - (h_1 + h_3)[v''' + r^{-1} v'' + r^{-2} v'], \\ t_{zr} = \mu v' + \Psi' - h_2[v''' + r^{-1} v'' - r^{-2} v']. \end{cases} \tag{1.233}$$

By virtue of the ansatz (1.229), the incompressibility constraint $v_{i,i} = 0$ is satisfied identically, while (1.228) reduces to the system

$$\begin{aligned} \frac{\partial p}{\partial r} &= 0 \\ (1 - l^2 D_r^2) D_r^2 v &= \frac{1}{\mu}\frac{\partial p}{\partial z} \end{aligned} \tag{1.234}$$

with $D_r^2 \equiv \dfrac{d^2}{dr^2} + \dfrac{1}{r}\dfrac{d}{dr}$. From the first equation in (1.234) it follows that $p = p(z)$, while the second equation implies that $\dfrac{dp}{dz} = \text{const.} \equiv p_1$; thus, from (1.234) we obtain

$$(1 - l^2 D_r^2) D_r^2 v = \frac{1}{\mu} p_1 \tag{1.235}$$

whose general solution, as given in [BG], is

$$v = \frac{p_1 r^2}{4\mu} + c_1 \ln r + c_2 + c_3 I_0\left(\frac{r}{l}\right) + c_4 K_0\left(\frac{r}{l}\right) \tag{1.236}$$

with I_0 and K_0 being the modified Bessel functions of the first and second kind, respectively, and the c_i, $i = 1, \ldots, 4$, being arbitrary constants.

Remarks. Equation (1.235) corresponds exactly to the bipolar equation (1.167) in Sect. 1.5 if we set $\alpha = 0$, identify μ in (1.235) with μ_0, and set $l = \sqrt{\mu_1/\mu_0}$. We now specialize, as we did in 1.5, to Poiseuille flow in a circular cylinder of radius R whose axis is coincident with the z-axis of the cylindrical coordinate system. From the requirement that $v(0) < \infty$ we obtain, from (1.236), the conclusions $c_1 = c_4 = 0$, so that (1.236) reduces to

$$v = \frac{p_1 r^2}{4\mu} + c_2 + c_3 I_0\left(\frac{r}{l}\right). \tag{1.237}$$

Along the wall of the circular tube, at $r = R$, the boundary conditions in the dipolar theory require that $v(R) = 0$ and (see (1.208))

$$\begin{cases} \tau_{rrr}(R) = M_r, \\ \tau_{rr\theta}(R) = M_\theta, \\ \tau_{rrz}(R) = M_z. \end{cases} \tag{1.238}$$

By comparing (1.238) with (1.232) one easily sees that the assumptions (1.229), (1.230) are compatible with (1.238) if $M_r = M_\theta = 0$. Although, as pointed out in [BG], the value of M_z depends on the interaction of the fluid with the wall of the circular cylinder, we only consider here the case where $M_z = 0$. From (1.237), the boundary conditions $v(R) = 0$ and $\tau_{rrz}(R) = 0$, and the expression for τ_{rrz} in (1.232), it then follows that

$$\begin{aligned} v(r) = \frac{-p_1 R^2}{4\mu}\left[1 - \left(\frac{r}{R}\right)^2\right] \\ + \frac{p_1 R^2}{2\mu}\left\{\frac{\left(\frac{l}{R}\right)\left[I_0\left(\frac{R}{l}\right) - I_0\left(\frac{r}{l}\right)\right]}{\left(\frac{R}{l}\right)\left(\frac{h_1+h_3}{2h_1+h_3}\right)\Phi_0\left(\frac{R}{l}\right) - \left(\frac{h_3}{2h_1+h_3}\right) I_0\left(\frac{R}{l}\right)}\right\}. \end{aligned} \tag{1.239}$$

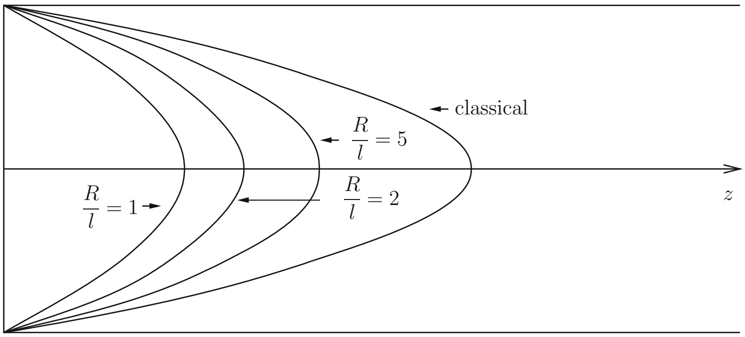

Fig. 1.5 Parabolic velocity profiles for the dipolar fluid

It is clear, that for fixed $\frac{R}{l}$, the velocity profile specified by (1.239) depends on the material parameters h_1 and h_3 only through the ratio h_1/h_3. In Fig. 1.5 the velocity profiles corresponding to (1.239) are plotted for different values of the ratio $\frac{R}{l}$ with $\frac{h_1}{h_3} = 0.3$.

As $l \to 0^+$ the system (1.234) formally converges to the one governed by the Stokes Law for Poiseuille flow in a circular pipe and the profiles in Fig. 1.5 converge, as well, to the parabolic profile predicted by the Stokes law. For small viscosity μ, the profiles predicted by (1.239) deviate more significantly than those observed in practice, and computed through use of the Prandtl boundary-layer theory, than that of the classical parabolic profile for small viscosity. The treatment we have presented here follows the original solution as constructed in [BG].

Unsteady Plane Couette Flow of a Dipolar Fluid

Jordan and Puri, [JP1]–[JP5], have analyzed, in considerable detail, a number of special types of flows of a viscous incompressible fluid within the framework of the theory of dipolar fluids as presented in [BG]; we will content ourselves here with a summary of their solution of the problem of unsteady plane Couette flow, between parallel plates, which is set in motion by the sudden acceleration of the top plate. Solutions of the aforementioned problem are found in [JP4] for arbitrary, nonnegative values of the dipolar constants d and l in (1.206); for $d = l \to 0^+$, the known solution of this problem for the viscous Newtonian fluid (see [ScG]) is recovered, as will be seen below.

In [JP4], the incompressible dipolar fluid occupies the domain $0 < y < h$ between two infinite ($-\infty < x < \infty$, $-\infty < z < \infty$) parallel plates at $y = 0$ and $y = h$. The fluid and the plates are initially at rest but, at time $t = 0^+$, the fluid is set in motion as a consequence of the sudden acceleration of the upper plate at $y = 0$, along the x-axis, to a constant velocity V_1. Thus the velocity of the upper plate can be expressed in the form $\boldsymbol{v}(h,t) = (V_1 H(t), 0, 0)$ where $H(t)$ is the usual Heaviside function. We seek a solution of (1.206) with $f_i = 0$ (we have already assumed that $F_{ij} \equiv 0$) in the form

$$\boldsymbol{v}(y,t) = (u(y,t), 0, 0); \quad 0 < y < h,\ t > 0 \tag{1.240}$$

in which case $\nabla \cdot \boldsymbol{v} = 0$ is satisfied identically, the only surviving equation in (1.206) is the one with $i = 1$, and, in addition, all the nonlinear terms in that equation vanish as well. Furthermore, it is easily shown that the pressure p must be of the form

$$p = \rho d^2 u_{,y}^2 + \chi(t) \tag{1.241}$$

where χ is an arbitrary function. For $i = 1$ in (1.206), with $f_1 = 0$, the ansatz (1.240) yields the equation of motion

$$u_{,t} - \nu u_{,yy} - d^2 u_{,yyt} + \nu l^2 u_{,yyyy} = 0, \quad 0 < y < h,\ t > 0. \tag{1.242}$$

Associated with (1.242) are the boundary conditions

$$u(0,t) = 0,\ u(h,t) = V_1 H(t), \quad t > 0, \tag{1.243a}$$

$$\mu l^2 u_{,yy}(0,t) = M_0,\ \mu l^2 u_{,yy}(h,t) = M_1, \quad t > 0, \tag{1.243b}$$

the latter set being the form assumed by (1.208) for the given geometry and the ansatz (1.240). The constants M_0 and M_1 represent the prescribed values of τ_{221} at $y = 0, h$. Finally, we prescribe the initial condition

$$u(y,0) = 0, \quad 0 < y < h \tag{1.244}$$

The complete initial-boundary value problem now consists of (1.242), (1.243a,b), and (1.244). If, as in [JP4], we employ the non-dimensional variables

$$y' = \frac{y}{h}, \quad u' = \frac{u}{V_1}, \quad t' = \frac{\nu t}{h^2}, \quad l_1 = \frac{d}{h}, \quad l_2 = \frac{l}{h}, \tag{1.245}$$

then, with the primes in (1.245) dropped, the initial-boundary value problem (1.242), (1.243a,b), (1.244), in terms of the new variables and parameters, becomes

$$u_t - u_{yy} - l_1^2 u_{yyt} + l_2^2 u_{yyyy} = 0, \quad (y,t) \in (0,1) \times (0,\infty), \tag{1.246a}$$

$$u(0,t) = 0,\ u(1,t) = H(t),\ u_{yy}(0,t) = \tilde{M}_0,\ u_{yy}(1,t) = \tilde{M}_1,\ \ t > 0, \tag{1.246b}$$

$$u(y,0) = 0, \quad 0 < y < 1 \tag{1.246c}$$

with $\tilde{M}_0$ and $\tilde{M}_1$ the non-dimensional forms of M_0 and M_1, respectively. Now, let $\bar{u}$ be the Laplace transform of u in the temporal variable, i.e., $\bar{u}(y,s) = \mathcal{L}[u(y,t)]$; applying the Laplace transform to (1.246a,b,c) produces the following boundary-value problem for $\bar{u}$:

$$\left.\begin{aligned} l_2^2\frac{d^4\bar{u}}{dy^4} - (sl_1^2+1)\frac{d^2\bar{u}}{dy^2} + s\bar{u} = 0 \\ \bar{u}(0,s)=0, \quad \bar{u}(1,s)=\frac{1}{s}, \quad \frac{d^2\bar{u}}{dy^2}(0,s)=\frac{\tilde{M}_0}{s}, \quad \frac{d^2\bar{u}}{dy^2}(1,s)=\frac{\tilde{M}_1}{s} \end{aligned}\right\}. \tag{1.247}$$

The solution of (1.247) may be obtained, by elementary techniques for linear, constant coefficient, ordinary differential equations; for $l_2 > 0$ it has the explicit form

$$\begin{aligned} \bar{u}(y,s) = {} & \frac{\tilde{M}_0}{s(r_2^2-r_1^2)}\left\{\frac{\sinh(r_2(1-y))}{\sinh(r_2)} - \frac{\sinh(r_1(1-y))}{\sinh(r_1)}\right\} \\ & + \frac{\tilde{M}_1}{s(r_2^2-r_1^2)}\left\{\frac{\sinh(r_2 y)}{\sinh(r_2)} - \frac{\sinh(r_1 y)}{\sinh(r_1)}\right\} \\ & + \frac{r_2^2}{s(r_2^2-r_1^2)}\left\{\frac{\sinh(r_1 y)}{\sinh(r_1)}\right\} - \frac{r_1^2}{s(r_2^2-r_1^2)}\left\{\frac{\sinh(r_2 y)}{\sinh(r_2)}\right\} \end{aligned} \tag{1.248}$$

where

$$r_{1,2} = \frac{1}{l_2}\sqrt{\frac{sl_1^2+1 \mp \sqrt{(sl_1^2+1)^2-4sl_2^2}}{2}}. \tag{1.249}$$

The singular points of $\bar{u}(y,s)$ are simple poles at

$$s = 0 \text{ and } s_n = -\frac{n^2\pi^2(1+n^2\pi^2 l_2^2)}{1+n^2\pi^2 l_1^2}, \quad n = 1,2,3,\ldots. \tag{1.250}$$

The expression for $\bar{u}(y,s)$ in (1.248) may, therefore, be inverted by adding up the residues at the poles delineated in (1.250); this process yields the following solution recorded in [JP4]:

$$\begin{aligned} u(y,t) = H(t)\Bigg\{ & u_\infty(y,l_2) \\ & - \frac{2l_2^2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n\exp[-\alpha_n^2(l_1,l_2)t]}{n(1+n^2\pi^2l_2^2)}[\tilde{M}_0\sin[n\pi(1-y)] + \tilde{M}_1\sin[n\pi y]] \end{aligned}$$

$$+ \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n \exp[-\alpha_n^2(l_1, l_2)t] \sin[n\pi y]}{n(1 + n^2\pi^2 l_1^2)} \Bigg\} \tag{1.251}$$

where $\alpha_n^2(l_1, l_2) = |s_n|$ and $u_\infty(y, l_2)$ is the steady-state part of u, which is given by

$$u_\infty(y, l_2) = -\tilde{M}_0 l_2^2 + y\{l + l_2^2(\tilde{M}_0 - \tilde{M}_1)\} + \frac{l_2^2\{\tilde{M}_1 \sinh[y/l_2] + \tilde{M}_0 \sinh[(1 - y)/l_2]\}}{\sinh[1/l_2]}. \tag{1.252}$$

As pointed out by Jordan and Puri, if one sets $l_1 = l_2 = L > 0$ in (1.251), (1.252), and then takes the limit as $L \to 0^+$, the solution converges to the non-dimensional form of the velocity field for unsteady Couette flow of a viscous, incompressible Newtonian fluid; this velocity field, which corresponds to setting $l_1 = l_2 = 0$ in (1.246a), and solving the resulting equation subject to

$$\left.\begin{aligned} u(0, t) = 0,\ u(1, t) = H(t), \quad t > 0 \\ u(y, 0) = 0, \quad 0 < y < h \end{aligned}\right\} \tag{1.253}$$

is given by

$$u(y, t) = H(t) \left\{ y + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n e^{-n^2\pi^2 t} \sin(n\pi y)}{n} \right\}. \tag{1.254}$$

The solution (1.254) for the Newtonian fluid may be expressed in the form (see [ScG], Chap. 5)

$$u(y, t) = H(t) \sum_{n=0}^{\infty} \left[\operatorname{erfc}\left(\frac{2n + 1 - y}{2\sqrt{t}} \right) - \operatorname{erfc}\left(\frac{2n + 1 + y}{2\sqrt{t}} \right) \right]. \tag{1.255}$$

An extensive analysis of the solution (1.251), (1.252) for this dipolar flow problem may be found in [JP4].

1.7.3 Some Applications of the Viscous Camassa-Holm Equations

Steady Flow in a Channel

Following the analysis in [CFH1], we consider the application of the viscous Camassa-Holm equations (VCHE) given by (1.210), (1.211a,b), and (1.212a,b) to the problem of turbulent flow in a channel of width $2a$ with walls at $y = \pm a$. The velocity field is assumed to be steady and of the form

$$\boldsymbol{v} = (U(y), 0, 0) \tag{1.256}$$

and we look for a solution which satisfies the boundary conditions $U(\pm a) = 0$ as well as the symmetry condition $U(-y) = U(y)$, for $-a < y < a$. With the ansatz (1.256), the system VCHE reduces to

$$-\nu[(1-\beta')U]'' + \nu(\alpha^2 U')''' = -\frac{\partial \tilde{\pi}}{\partial x}, \tag{1.257a}$$

$$\frac{\partial \tilde{\pi}}{\partial y} = \frac{\partial \tilde{\pi}}{\partial z} = 0 \tag{1.257b}$$

where $' = \dfrac{d}{dy}$ and, employing the same basic notation as in [CFH1],

$$\alpha^2 = \langle \sigma_2^2 \rangle, \quad \beta = \langle \sigma_2 \rangle, \tag{1.258a}$$

$$\tilde{\pi} = \pi + \int [U(y) - \beta' U(y) - (\alpha^2 U'(y))']U'(y)\, dy. \tag{1.258b}$$

Now, in a turbulent channel flow, the mean velocity appearing in the Reynolds averaged Navier–Stokes equations has the form $\langle \boldsymbol{v} \rangle = (\bar{U}(y), 0, 0)$ and the mean pressure the form $\langle p \rangle = \bar{P}(x, y, z)$. In this situation (see, [ScG]) the Reynolds equations reduce to the system

$$-\nu \bar{U}''(y) + \frac{\partial}{\partial y}\langle wu \rangle = -\frac{\partial}{\partial x}\bar{P}, \tag{1.259a}$$

$$\frac{\partial}{\partial y}\langle wv \rangle = -\frac{\partial}{\partial z}\bar{P}, \quad \frac{\partial}{\partial y}\langle w^2 \rangle = -\frac{\partial}{\partial y}\bar{P} \tag{1.259b}$$

where (u, v, w) is the fluctuation velocity in the infinite channel given by $-a \le y \le a$, $-\infty < x, z < \infty$. By comparing (1.257a,b) and (1.259a,b) the authors [CFH1] deduce that $\bar{U} = U$, $-a \le y \le a$, and

$$\nu[(\alpha^2 U')''' - (\beta' U)''] + p_0 = \frac{\partial}{\partial y}\langle wu \rangle \tag{1.260a}$$

and

$$\nabla(\bar{P} + \langle w^2 \rangle) = \nabla(\tilde{\pi} - p_0 x), \quad \frac{\partial}{\partial y}\langle wv \rangle = 0 \tag{1.260b}$$

for some constant p_0. From (1.260a,b) it follows that $\langle wv \rangle(y) = 0$ and

$$-p_0 y - \nu[(\alpha^2 U')''(y) - (\beta' U)'(y)] = -\langle wu \rangle(y). \tag{1.261}$$

If one assumes isotropy and homogeneity throughout the channel, then with α constant, and $\beta = \langle \sigma_2 \rangle = 0$, the general solution of (1.257a), subject to $U(\pm a) = 0$ and $U(y) = U(-y)$, $-a < y < a$, is given by

$$U(y) = \lambda \left[1 - \frac{\cosh(y/\alpha)}{\cosh(a/\alpha)} \right] + \gamma \left[1 - \frac{y^2}{a^2} \right] \tag{1.262}$$

for some constants λ, γ. The analysis in [CFH1] continues by making the assumption that isotropy and homogeneity hold away from the walls at $y = \pm a$ so that $\alpha(y) = \alpha_0 =$ const. and $\beta(y) \equiv 0$ for $|y| \leq a_0$ (for some a_0, $0 < a_0 < a$); we refer the interested reader to the original paper [CFH1] for details of the analysis that follows from this assumption, as well as for a comparison of the mean-velocity profiles in the channel predicted by the constant-α VCHE with experimental data for turbulent channel flow. *It is important to emphasize that, unlike the bipolar model of Sects. 1.3 and 1.4, the Bleustein and Green dipolar fluid model [BG], or the Green and Naghdi model [GN1] for viscous flow, the Camassa-Holm model is thought of as applying to turbulent flow; more specifically, the ansatz has been made, for the channel flow problem, that U in the VCHE corresponds to the average velocity $\bar{U}$ in the Reynolds equations.* The channel flow problem for the VCHE has also been discussed in [CFH2].

The Camassa-Holm Equations and Turbulent Flow in a Circular Pipe

The problem of turbulent flow in a circular pipe of radius a, whose axis is coincident with the x-axis, has been studied in [CFH2] using the machinery of the Camassa-Holm theory. We assume a steady state situation with the fluid flowing, on average, only in the x direction; as in [CFH2], we denote by U the mean velocity of the fluid in that direction. Because of symmetry, U and the averages of the fluctuations $\langle \sigma_r \rangle$ depend only on r, the radial distance from the x-axis. From the stated hypothesis, we infer from (1.210), (1.211a,b), (1.212a,b) the system of equations

$$V \frac{\partial U}{\partial r} = -\frac{\partial \pi}{\partial r}, \quad 0 = -\frac{1}{r}\frac{\partial \pi}{\partial \theta}, \quad \text{and} \quad -\nu \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial V}{\partial r} \right) = -\frac{\partial \pi}{\partial x} \tag{1.263}$$

where

$$V = U - \left\{ \left(\frac{1}{r} + \frac{\partial}{\partial r} \right) \langle \sigma_r \rangle \right\} U - \left(\frac{1}{r} + \frac{\partial}{\partial r} \right) \left\{ \langle \sigma_r^2 \rangle \frac{\partial U}{\partial r} \right\}. \tag{1.264}$$

From the second of the equations in (1.263) we infer that π is dependent on θ. Integration of the last of the equations in (1.263) with respect to x then yields

$$-x \frac{\nu}{r} \frac{\partial}{\partial r} \left(r \frac{\partial V}{\partial r} \right) + c(r) = -\pi \tag{1.265}$$

for some arbitrary function $c(r)$. We now differentiate (1.265) through with respect to r and use the first equation in (1.263) so as to obtain

$$-x\frac{\partial}{\partial r}\left\{\frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right)\right\}+c'(r)=-\frac{\partial \pi}{\partial r}=V\frac{\partial U}{\partial r}. \tag{1.266}$$

As the right-hand side of (1.266) depends only on r, so must the left-hand side; this, however, then implies that

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right)\right)=0, \quad 0<r<a. \tag{1.267}$$

If we solve (1.267) under the assumption that $V(0)<\infty$ we find that

$$V(r)=k_1\left(\frac{r}{a}\right)^2+k_2, \quad 0<r<a \tag{1.268}$$

for some real constants k_1 and k_2. If, in addition to our previous hypotheses, we now assume that the velocity fluctuations are isotropic and homogeneous (away from the wall of the pipe) then $\langle\sigma_r\rangle=0$, and $\alpha^2=\langle\sigma_r^2\rangle$ is independent of r; in these circumstances (1.264) reduces to

$$V(r)=U(r)-\alpha^2\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right), \quad 0<r<a. \tag{1.269}$$

Following the analysis in [CFH2] we now introduce the following non-dimensional quantities: τ_0 is the boundary shear stress, i.e., the shear stress at $r=a$, $u_*^2=\tau_0$, $\phi(\eta)=U(r)/u_*$, and $\eta=R_0\left(\frac{r}{a}\right)$, where $R_0=\dfrac{u_*a}{\nu}$ is the so-called skin friction Reynolds number; in terms of these variables and parameters (1.269), with V given by (1.268), becomes

$$\frac{R_0^2}{\xi^2}\left(1-\frac{\eta}{R_0}\right)^{-1}\frac{\partial}{\partial\eta}\left\{\left(1-\frac{\eta}{R_0}\right)\frac{\partial\phi}{\partial\eta}\right\}-\phi=-f_0-2f_1\left(1-\frac{\eta}{R_0}\right)^2$$

where $f_0=k_2/u_*$, $f_1=k_1/(2u_*)$, and $\xi=a/\alpha$. Solving this equation yields

$$\phi(\eta)=CI_0\left(\xi\left(1-\frac{\eta}{R_0}\right)\right)+2f_1\left(1-\frac{\eta}{R_0}\right)^2+8\frac{f_1}{\xi^2}+f_0 \tag{1.270}$$

where

$$I_0(\xi)=\sum_{n=0}^{\infty}\frac{1}{(n!)^2}\left(\frac{r^2}{4}\right)^n \tag{1.271}$$

is the modified Bessel function of the first kind. The second term in (1.270) is the classical Hagen-Poiseuille solution for laminar flow in a pipe which is generated by the Navier–Stokes system for this same problem. A detailed analysis of the solution (1.270), (1.271) may be found in [CFH2].

1.7.4 Fluid Dynamics Problems for the Extended Green-Naghdi Viscous Flow Model

Unsteady Plane Poiseuille Flow in a Parallel-Wall Channel

We consider, within the context of the non-dimensional form (1.226) of the Green-Naghdi theory [GN1], the problem of a time-dependent plane Poiseuille flow in the channel $-1 \le y \le 1$, $-\infty < x, z < \infty$. In (1.226) we recall that $\boldsymbol{v}, \boldsymbol{u}$ are non-dimensional versions, respectively, of the fluid velocity and the curl of the spin vector $\boldsymbol{w} = \nabla \times \boldsymbol{v}$, i.e., in the incompressible situation $\boldsymbol{u} = -\nabla^2 \boldsymbol{v} \equiv \nabla \times \nabla \times \boldsymbol{v}$. If we denote by $\boldsymbol{e}_i$ the (orthonormal) basis vectors along the x, y, and z axes, then the flow we are interested in has a velocity field of the form

$$\boldsymbol{v} = v(y,t)\boldsymbol{e}_1 \tag{1.272}$$

in which case

$$\boldsymbol{w} = -\frac{\partial v}{\partial y}\boldsymbol{e}_3, \quad \boldsymbol{u} = -\frac{\partial^2 v}{\partial y^2}\boldsymbol{e}_1. \tag{1.273}$$

If $\boldsymbol{f} = \boldsymbol{0}$ in (1.226) and we use (1.272), (1.273) then

$$\begin{cases} \mathrm{Re}\left(\dfrac{\partial v}{\partial t} - \dfrac{1}{2\beta^2}\dfrac{\partial^3 v}{\partial t \partial y^2}\right) = -\dfrac{\partial p}{\partial x} + \dfrac{\partial^2 v}{\partial y^2} - \dfrac{1}{\beta^4}\dfrac{\partial^4 v}{\partial y^4} \\ \dfrac{\partial p}{\partial y} = \dfrac{\partial p}{\partial t} = 0. \end{cases} \tag{1.274}$$

In [GN1], solutions of (1.274) are sought which are even in y and which include the classical steady-state Poiseuille flow at $t = 0$ when β is large; this leads to

$$\begin{aligned} v(y,t) &= L + \frac{1}{2}A(1-y^2) + Be^{\lambda t}\cosh(\alpha\beta y) \\ p &= -Ax \end{aligned} \tag{1.275}$$

with α, L, A, B constants and

$$\mathrm{Re}\,\lambda\left(1 - \frac{1}{2}\alpha^2\right) = \alpha^2\beta^2(1-\alpha^2). \tag{1.276}$$

If it is assumed that the walls at $y = \pm 1$ are both moving with velocity V, then we obtain from (1.275)

$$V = L + B\cosh(\alpha\beta) \tag{1.277}$$

Additionally, it is assumed in [GN1] that the component of the couple $\boldsymbol{m}_1$ at the walls at $y = \pm 1$, in the direction $\boldsymbol{e}_3$ normal to the channel, is zero when $t = 0$. As $\boldsymbol{m}_1 = \boldsymbol{M}_1\boldsymbol{v}$ and $\boldsymbol{M}_1 = -p_1\boldsymbol{I} + \mu_1\boldsymbol{N}$, i.e., (1.219b), (1.222), it follows that

$$\boldsymbol{m}_1 \cdot \boldsymbol{e}_3 = (\boldsymbol{M}_1\boldsymbol{e}_2)\cdot\boldsymbol{e}_3 = \mu N_{32} = \mu\frac{\partial w}{\partial y} = 0 \tag{1.278}$$

so that $\left.\dfrac{\partial^2 V}{\partial y^2}\right|_{y=\pm 1} = 0$, in which case, by (1.275)

$$-A + \alpha^2\beta^2 B\cosh(\alpha\beta) = 0. \tag{1.279}$$

Then

$$v = V + A\left\{\frac{1}{2}(1-y^2) + e^{\lambda t}\frac{\cosh(\alpha\beta y) - \cosh(\alpha\beta)}{\alpha^2\beta^2\cosh(\alpha\beta)}\right\}. \tag{1.280}$$

When β is large, this solution reduces to the classical Poiseuille flow

$$v = V + \frac{1}{2}A(1-y^2). \tag{1.281}$$

If either $\alpha^2 > 2$, or $\alpha^2 < 1$, then by (1.276) $\lambda > 0$ and the solution (1.280) increases with increasing t. However, if $1 < \alpha^2 < 2$, then $\lambda < 0$ and the solution in (1.280) are asymptotic to that in (1.281) as $t \to \infty$. For $\alpha = 1$, $\lambda = 0$, and (1.280) becomes the steady-state solution

$$v = V + A\left\{\frac{1}{2}(1-y^2) + \frac{cosh(\beta y) - \cosh\beta}{\beta^2\cosh\beta}\right\}. \tag{1.282}$$

From (1.282) we infer that the maximum value $v_{\max}$ of v occurs at the center of the channel, i.e., at $y = 0$, so that

$$v_{\max} = V + A\left\{\frac{1}{2} + \frac{1-\cosh\beta}{\beta^2\cosh\beta}\right\}. \tag{1.283}$$

If we set $(V/v_{\max}) = k$ then from (1.283) we deduce that

$$\frac{A}{V}\left\{\frac{1}{2} + \frac{1-\cosh\beta}{\beta^2\cosh\beta}\right\} = \frac{1-k}{k} \tag{1.284}$$

and

$$\frac{v}{v_{\max}} = k + \frac{(1-k)\left\{\frac{1}{2}(1-y^2) + \dfrac{\cosh(\beta y) - \cosh\beta}{\beta^2\cosh\beta}\right\}}{\dfrac{1}{2} + \dfrac{1-\cosh\beta}{\beta^2\cosh\beta}}. \tag{1.285}$$

As $\beta \to \infty$ in (1.285),

$$\frac{v}{v_{\max}} \to 1 - (1-k)y^2 \tag{1.286}$$

which is the classical result obtainable from the Navier–Stokes equations, while for $\beta \to 0^+$,

$$\frac{v}{v_{\max}} \to k + \frac{1-k}{5}(1-y^2)(5-y^2). \tag{1.287}$$

It is observed in [GN1] that neither of the results in (1.286) or (1.287) depend on the Reynolds numbers Re, Re* defined in (1.225).

Poiseuille Flow of a Green-Naghdi Fluid in a Circular Pipe

For this example we take the z-axis to be the axis of a circular cylinder of radius one and consider a steady flow solution of (1.226), in cylindrical coordinates, of the form ($r = \sqrt{x^2+y^2}$),

$$\boldsymbol{v} = v(r)\boldsymbol{e}_3, \quad (r = \sqrt{x^2+y^2}) \tag{1.288}$$

in which case it is easily computed that

$$\boldsymbol{w} = \frac{yv'(r)}{r}\boldsymbol{e}_1 - \frac{xv'(r)}{r}\boldsymbol{e}_2, \tag{1.289a}$$

$$\boldsymbol{u} = -\left(v''(r) + \frac{v'(r)}{r}\right)\boldsymbol{e}_3 \tag{1.289b}$$

with $' = \dfrac{d}{dr}$. Taking $\boldsymbol{f} = \boldsymbol{0}$ again in (1.226) we obtain the system

$$\begin{cases} -\dfrac{\partial p}{\partial z} + v'' + \dfrac{v'}{r} - \dfrac{1}{\beta^2}\left(\dfrac{d^2}{dr^2} + \dfrac{1}{r}\dfrac{d}{dr}\right)\left(v'' + \dfrac{v'}{r}\right) = 0 \\ \dfrac{\partial p}{\partial x} = 0, \quad \dfrac{\partial p}{\partial y} = 0 \end{cases} \tag{1.290}$$

for which a solution of the form

$$v = L - \frac{Ar^2}{4} + BI_0(\beta r) \tag{1.291}$$

is easily obtained, where A, B, L are constants and I_0 is, once again, the modified Bessel function of the first kind. With $v(1) = V$ we obtain from (1.291)

$$V = L - \frac{1}{4}A + BI_0(\beta)$$

and (1.291) becomes

$$v = V + \frac{1}{4}A(1 - r^2) + B(I_0(\beta r)). \tag{1.292}$$

It is assumed in [GN1] that the $\boldsymbol{e}_\theta$ component of the couple $\boldsymbol{m}$ acting across the surface $r = 1$ is zero; this condition is equivalent to $v''(1) = 0$ which is, of course, the constraint that follows from the higher-order boundary conditions, for this same problem, for the bipolar fluid in Sect. 1.5.3. Applying the condition $v''(1) = 0$ to (1.292) yields

$$-\frac{1}{2}A + \beta^2 BI_0''(\beta) = 0$$

in which case (1.292) becomes

$$v = V + \frac{1}{2}A(1 - r^2) + \frac{A(I_0(\beta r) - I_0(\beta))}{2\beta^2 I_0''(\beta)}. \tag{1.293}$$

From (1.293) we now obtain

$$\lim_{\beta\to\infty} v(r) = V + \frac{1}{4}A(1 - r^2) \tag{1.294}$$

which is the classical solution. Also

$$\lim_{\beta\to 0+} v(r) = V + \frac{A\beta^2}{64}(1 - r^2)(5 - r^2). \tag{1.295}$$

Finally $v_{\max}$ occurs along the axis of the cylinder at $r = 0$ in which case (1.293) yields the result

$$v_{\max} = V + \frac{1}{4}A + \frac{A(1 - I_0(\beta))}{2\beta^2 I_0''(\beta)}. \tag{1.296}$$

As in the previous example we set $\dfrac{V}{v_{\max}} = k$ so that, by virtue of (1.296),

$$\frac{1}{k} = 1 + \frac{A}{V}\left\{\frac{1}{4} + \frac{1 - I_0(\beta)}{2\beta^2 I_0''(\beta)}\right\}. \tag{1.297}$$

Employing (1.296) and (1.297) in (1.293), we compute that

$$\frac{v}{v_{\max}} = k + \frac{(1-k)\left\{\frac{1}{4}(1-r^2) + \frac{I_0(\beta r) - I_0(\beta)}{2\beta^2 I_0''(\beta)}\right\}}{\frac{1}{4} + \frac{1 - I_0(\beta)}{2\beta^2 I_0''(\beta)}}. \tag{1.298}$$

From (1.298), we obtain the classical result associated with the Navier–Stokes model if we extract the limit as $\beta \to \infty$, i.e.,

$$\lim_{\beta\to\infty} \frac{v(r)}{v_{\max}} = 1 - (1-k)r^2. \tag{1.299}$$

On the other hand, (1.298) also yields

$$\lim_{\beta\to 0} \frac{v(r)}{v_{\max}} = k + \frac{(1-k)}{5}(1-r^2)(5-r^2). \tag{1.300}$$

Remarks. The three models of viscous fluid flow considered in this section, and Sect. 1.6, involve higher-order spatial velocity gradients as well as nonlinearity; the nonlinear aspects of these models do not come into play in any of the specific examples looked at in this section, because of the specific types of solution sought and the geometry of the domain; this is why explicit solutions were obtained for these problems. The most feasible way in which to compare the predictions of the models, involving higher order spatial velocity gradients that were introduced in Sect. 1.6, with the bipolar model of Sect. 1.4, would be to look at a specific type of flow for the linear bipolar fluid; this is, of course, because the presence of the nonlinear viscosity in the general bipolar model means that the governing equations of motion remain nonlinear even in the context of the elementary flows discussed in this section. Thus, even though the genesis and interpretation of the various models are different, one could, in principle, point to the similarities between the result in (1.77) for plane Poiseuille flow, in the context of the linear bipolar theory, and the analogous result (1.262) for the same kind of flow when treated using the viscous Camassa-Holm equations.

We conclude this chapter by presenting, below, a brief review of some of the better known experimental results which are at variance with predictions based on the standard model of incompressible viscous fluid flow generated by the Stokes' hypothesis.

1.8 A Catalog of Experimental Results Which Are Inconsistent with the Stokes' Hypothesis

1.8.1 Couette Flow

Consider two long coaxial vertical cylinders. The inner cylinder, with radius $r = R_1$, and the external cylinder having radius $r = R_2$. Naturally $R_1 < R_2$ and the fluid occupies the toroidal region (r, z); $r \in (R_1, R_2)$, $z \in (0, L)$. There is no vertical motion of the cylinders but the cylinders are rotating around their common axis ($r = 0$) with respective angular velocities ω_1, and ω_2. For the incompressible bipolar fluid this problem is treated in Sect. 3.2.

Wall Pressure

The Navier–Stokes equations predict, in this situation, that the pressure on the inner cylindrical wall will be lower than the pressure exerted by the fluid on the outer cylindrical wall. This is indeed the case for many fluids. But there are well documented experiments of fluids behaving in exactly the opposite way. Figure 1.3 on page 4 of [CMN] displays a striking example of a fluid[4] which produces exactly the opposite of the Navier–Stokes prediction, i.e., the pressure of the fluid on the wall of the inner cylinder is higher than the pressure on the outer cylinder wall. In this regard, see also the movie [Mar1] and the web site [Mar2].

Rod Climbing

When the two cylinders described above are at rest, the fluid occupies the region (r, z), $R_1 \le r \le R_2$, $0 \le z \le L$. The horizontal surface $z = L$ separates the liquid from the ambient atmosphere. As the cylinders rotate, the surface separating the liquid from the ambient air deforms from this planar region and is now described by a function $Z(r)$; $r \in (R_1, R_2)$. The Navier–Stokes equations predict that the function $Z(r)$ will be an increasing function of r. However this is not always true for all fluids. There are known examples (see again Fig. 1.3 on page 4 of [CMN]) where it can be clearly seen that the fluid has climbed along the inner rod; this is commonly referred to in the literature as the Weissenberg effect. In the example referred to here it can be easily seen that the function $Z(r)$ is a monotone decreasing function of r.

[4] An 8.5 % solution of polyisobutylene in decalin.

Angular Velocity as a Function of Torque

Once again, within the context of Couette flow, suppose we now assume that the outer cylinder is not rotating while the inner cylinder is rotating with constant angular velocity ω. Navier–Stokes predicts that the torque T per unit of height which must be applied to the inner cylinder to maintain the constant angular velocity is given by

$$T = c\frac{R_1^2 R_2^2}{R_2^2 - R_1^2}\omega$$

where c is a constant. The formula above states that T depends linearly on ω. Experiments going as far back as 1954 clearly show this is not always the case. The experiment reported in [KM] shows that a fluid consisting of rubber latex containing 62.2 % of solids, with $R_1 = 2.357$ cm and $R_2 = 2.508$ cm, exhibits a torque T which depends on ω in a superlinear way. In this regard see also [CMN], Fig. 1.2, page 3.

1.8.2 Poiseuille Flow

Consider a vertical pipe with a circular cross section of constant radius R_p. The pipe is long enough for one to assume it occupies the region $0 \leq r \leq R_p$, $z \geq 0$. At the bottom end, i.e., at $z = 0$, the pipe is open and fluid is coming out of the pipe at a constant steady rate into the surrounding ambient atmosphere which is at constant pressure A. As the fluid exits the cylinder it will now occupy the region $r \in (0, R(z))$, $z \leq 0$. The Navier–Stokes equations predict that $R(z) \leq R_p$ for all $z \leq 0$. But this is not always so. There are fluids which exhibit swelling upon leaving the pipe, i.e., there are fluids for which $R(z) > R_p$ for $z \leq 0$. In [Mar2] it may be clearly seen that a solution of glycerine behaves as predicted by Navier–Stokes while a 2 % solution of poly-ethylene oxide in water swells as it leaves the orifice of the pipe. See also [CMN], Fig. (27.4), page 72.

Chapter 2
Plane Poiseuille Flow of Incompressible Bipolar Viscous Fluids

2.1 Introduction

In Sect. 1.4 we introduced the model of an incompressible, nonlinear, bipolar viscous fluid; this model, which is consistent with the basic principles of continuum mechanics and thermodynamics, as delineated in Sect. 1.4, is based on the following constitutive hypotheses for the Cauchy stress tensor τ_{ij} and the first multipolar stress tensor τ_{ijk}:

$$\tau_{ij} = -p\delta_{ij} + 2\mu_0(\epsilon + e_{ij}e_{ij})^{-\alpha/2} - 2\mu_1 \Delta e_{ij} \tag{2.1a}$$

and

$$\tau_{ijk} = 2\mu_1 \frac{\partial e_{ij}}{\partial x_k} \tag{2.1b}$$

where p is the pressure, $e_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$ is the rate of deformation tensor, $\boldsymbol{v}$ is the fluid velocity field, and μ_0, μ_1, ϵ, and α are the constitutive constants, the first three of which are positive while α, in this chapter, will be assumed to satisfy $0 < \alpha < 1$. In a bounded domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, with smooth boundary $\partial\Omega$, the constitutive hypotheses (2.1a,b) yield (see Sect. 1.4) the following initial-boundary value problem (take the density $\rho \equiv 1$):

$$\begin{aligned}\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\nabla p + 2\nabla \cdot (\mu(|\boldsymbol{e}|)\boldsymbol{e}) \\ - 2\mu_1 \nabla \cdot (\Delta \boldsymbol{e}) + \boldsymbol{f}\,,\end{aligned} \tag{2.2a}$$

H. Bellout and F. Bloom, *Incompressible Bipolar and Non-Newtonian Viscous Fluid Flow*, Advances in Mathematical Fluid Mechanics, DOI 10.1007/978-3-319-00891-2_2,

$$\nabla \cdot \boldsymbol{v} = 0, \tag{2.2b}$$

$$\boldsymbol{v} = \mathbf{0}, \quad \tau_{ijk}\nu_j\nu_k - \tau_{jkl}\nu_j\nu_k\nu_l\nu_i = 0, \tag{2.2c}$$

$$\boldsymbol{v}(\boldsymbol{x}, 0) = \boldsymbol{v}_0(\boldsymbol{x}). \tag{2.2d}$$

The system of partial differential equations (2.2a), (2.2b) holds in $\Omega \times [0, T)$, $T > 0$, while the initial condition (2.2d) is assumed to hold in Ω, at $t = 0$, and the boundary data (2.2c) is specified for $(\boldsymbol{x}, t) \in \partial\Omega \times [0, T)$, with $\boldsymbol{\nu}$ the exterior unit normal to $\partial\Omega$ at time t. In (2.2a), $\boldsymbol{f}$ specifies the external body force/volume in Ω while

$$\mu(|\boldsymbol{e}|) = 2\mu_0(\epsilon + |\boldsymbol{e}|^2)^{-\alpha/2} \tag{2.3}$$

is the nonlinear viscosity function. For $\alpha = 0$, $\mu_1 = 0$, and in the absence of the second set of boundary conditions in (2.2c), the system (2.2a)–(2.2d) reduces to the specification of the standard initial-boundary value problem for the (incompressible) Navier–Stokes equations.

In Sect. 1.5.2 we considered, for the bipolar model (2.2a)–(2.2d), the most standard of all classical problems in fluid dynamics, namely, the problem of Poiseuille flow between parallel plates whose location, in the Cartesian coordinate system (x_1, x_2, x_3), is at $x_2 = \pm a$, with $a > 0$. As is customary in considering plane Poiseuille flow between parallel plates we assumed, in Sect. 1.6, a velocity field (for steady flow) of the form

$$v_1 = v_1(x_2), \quad v_2 = 0, \quad v_3 = 0, \tag{2.4}$$

In this case, with $\boldsymbol{f} = \mathbf{0}$, the steady plane Poiseuille flow of an incompressible, bipolar, viscous fluid satisfies (see Sect. 1.5.2) the following boundary-value problem, where $p_1 = \dfrac{\partial p}{\partial x_1}$ is a constant:

$$\mu_0\left[\left(\epsilon + \frac{1}{2}v_1'^2(x_2)\right)^{-\alpha/2} v_1'(x_2)\right]' - \mu_1 v_1''''(x_2) = p_1 \tag{2.5a}$$

$$v_1(\pm a) = 0, \quad v_1''(\pm a) = 0. \tag{2.5b}$$

Explicit solutions for the non-Newtonian problem derived from (2.5a,b) by setting $\epsilon = 0$, $\mu_1 = 0$, and deleting the second set of boundary conditions in (2.5b), were obtained in (1.5.2) and compared with the standard solution obtained for the Navier–Stokes model (i.e., with the case $\alpha = 0$).

In this chapter we will treat, in depth, the behavior of both steady and time-dependent plane Poiseuille flow solutions for the incompressible, bipolar, fluid flow model. We begin in Sect. 2.2 by considering the problems of existence, uniqueness, and continuous dependence for the generalization of the nonlinear boundary-value problem (2.5a,b) in which the constant p_1 is replaced by a function $f \in L^2(-a, a)$.

In (2.5a), $\mu_1 > 0$ and we are interested in the behavior of solutions of (2.5a,b), not only as $\epsilon \to 0^+$, but also as $\mu_1 \to 0^+$; any continuous dependence result, in this case, can not hold in the C^2 sense (as boundary-layer theory comes into effect at that level of smoothness) but will be shown to hold in the norm of $C^{1+\delta}$ for $0 < \delta < \frac{1}{2}$. More explicit results for the boundary-value problem are delineated in Sect. 2.3. Suppose we display the dependence of the solution of (2.5a,b) on ϵ and μ_1, for a fixed $\alpha \in (0, 1)$, by writing $v_1 = u(x_2; \epsilon, \mu_1)$ and, for the same fixed α, write $u(x_2; 0, 0) = u_0(x_2)$; then, it will be shown in Sect. 2.3 that with $y = x_2$,

$$u(y; \epsilon, \mu_1) > 0, \quad -a \le y < 0; \qquad u'(y; \epsilon, \mu_1) < 0, \quad 0 < y \le a \tag{2.6a}$$

with $u''(y; \epsilon, \mu_1) \le 0, -a < y < a$, for all $\epsilon, \mu_1 \ge 0$, Also

$$u'''(-a; \epsilon, \mu_1) < 0, \tag{2.6b}$$

$$u'(-a; \epsilon, \mu_1) = -u'(a; \epsilon, \mu_1) \tag{2.6c}$$

for all $\epsilon, \mu_1 \ge 0$ and

$$\left|u'(y; \epsilon, 0) - u_0'(y)\right| < \left(1 + \frac{1}{\sqrt{1-\alpha}}\right)\sqrt{\epsilon} \tag{2.6d}$$

for $y \in [-a, a]$, $\epsilon > 0$. It is also proven in 2.3 that $\exists C_+$, C_1, C_2, positive and independent of both ϵ and μ_1, such that

$$\begin{aligned} -\left(1 + \frac{1}{\sqrt{1-\alpha}}\right) a\sqrt{\epsilon} - \frac{\sqrt{aC_2}}{1-\alpha}\left(\sqrt{\epsilon} + C_1\right)^{\alpha} \mu_1^{*1/2} &\le u(y; \epsilon, \mu_1) - u_0(y) \\ &\le \left(1 + \frac{1}{\sqrt{1-\alpha}}\right) a\sqrt{\epsilon} + \frac{aC_+}{1-\alpha}\left(\sqrt{\epsilon} + C_1\right)^{\alpha} \mu_1^* \end{aligned} \tag{2.7}$$

with $\mu_1^* = \mu_1/\mu_0$. In Sect. 2.4 we reconsider the problem of uniqueness in relation to the nonlinear boundary-value problem (2.5a,b). If we denote the region between the parallel plates at $y = \pm a$ by

$$\Omega_a = \{(x, y, z) \mid y \in [-a, a],\ a > 0,\ -\infty < x, z < \infty\} \tag{2.8}$$

and the uniquely determined vector field corresponding to the solution of (2.5a,b) by

$$\boldsymbol{v}^P = (u(y; \epsilon, \mu_1), 0, 0) \tag{2.9}$$

where we again set $v_1 = u$, $x_2 = y$, then it will be proven in Sect. 2.4 that for μ_1 sufficiently large $\boldsymbol{v}^P$ is, in fact, the unique equilibrium solution $\boldsymbol{v}$ of (2.2a)–(2.2d) in Ω_a which satisfies $\boldsymbol{v} - \boldsymbol{v}^P \in \boldsymbol{H}^4(\Omega_a)$. The existence of other equilibrium solutions of (2.2a)–(2.2d) in Ω_a, beyond the Poiseuille vector field (2.9), when μ_1 is not sufficiently large, is an open problem. In Sect. 2.5 we take up the problem of existence and asymptotic stability of time-dependent Poiseuille flow in the domain Ω_a; specifically we ask whether or not there exists, globally in time, smooth Poiseuille solutions of (2.2a)–(2.2d) in $\Omega_a \times [0,T)$, $T > 0$, of the form

$$\boldsymbol{v}^P(\boldsymbol{x},t) = (v(y,t;\epsilon,\mu_1),0,0). \tag{2.10}$$

We show, in Sect. 2.5 that there exists a unique weak solution to the corresponding initial-boundary value problem which is of class $C^{4,1}(y,t)$ on $(-a,a)\times[0,T)$, for any $T > 0$, in which case the weak solution is actually a classical solution of the problem. Finally, it is also demonstrated in Sect. 2.5 that the unique steady Poiseuille flow solution (2.9) is linearly asymptotically stable, as well as asymptotically stable, within the class of all flows in $\Omega_a \times [0,T)$, $T > 0$, of the Poiseuille type (2.10).

2.2 Existence, Uniqueness, and Continuous Dependence for Steady Poiseuille Flow

We consider, in this section, a slight generalization of the boundary-value problem (2.5a,b) associated with the steady flow of an incompressible, bipolar, viscous fluid in a parallel-wall channel, namely,

$$-\left[\frac{u'(y)}{(\epsilon + u'^2(y))^{\alpha/2}}\right]' + \mu_1 u''''(y) = f(y), \quad -a < y < a, \tag{2.11a}$$

$$u(\pm a) = u''(\pm a) = 0 \tag{2.11b}$$

where we have written $y = x_2$, $u = v_1$, and where $f \in L^2(-a,a)$. Our first basic result is the following existence and uniqueness theorem:

Theorem 2.1. *Let* $V = H_0^{\frac{3}{2}+\delta}(-a,a)$*, with* $0 < \delta < \frac{1}{2}$*, let* $B_M(0)$ *be the ball of radius* $M > 0$ *in* V*, and set*

$$W_M = B_M(0) \cap H^2(-a,a). \tag{2.12}$$

Then, for M *sufficiently large, there exists a unique solution* $u \in W_M$ *of the boundary-value problem* (2.11a,b).

Proof. By virtue of standard embedding results (Appendix A) W_M is compact in V for any $\delta < 1/2$. For $v \in V$ we define

$$L_v u = -\left[\frac{u'}{(\epsilon + v'^2)^{\alpha/2}}\right]' + \mu_1 u''''. \tag{2.13}$$

Then, for fixed $v \in V$, the linear boundary-value problem

$$\begin{cases} L_v u = f, \quad -a < y < a, & (2.14a) \\ u(\pm a) = u''(\pm a) = 0 & (2.14b) \end{cases}$$

has, as a consequence of the Lax-Milgram Lemma (see Appendix A), a unique solution $u \in H^2(-a,a)$ for which

$$||u||_{H^2(-a,a)} \leq c||f||_{L^2(-a,a)} \tag{2.15}$$

with $c > 0$ independent of u. Let $T : v \to u$ where u is the unique solution of (2.14a,b). For any given $f \in L^2(-a,a)$ it is a direct consequence of (2.15) that $\exists M > 0$ sufficiently large such that $T : W_M \to W_M$. We want to show that T is a continuous map. For $v, w \in W_M$, let $u_1 = Tv$, $u_2 = Tw$; then

$$-\left[\frac{u_1'}{(\epsilon + v'^2)^{\alpha/2}}\right]' + \left[\frac{u_2'}{(\epsilon + w'^2)^{\alpha/2}}\right]' + \mu_1[u_1 - u_2]'''' = 0. \tag{2.16}$$

Multiplying (2.16) by $u_1 - u_2$, integrating over $(-a, a)$, and then integrating by parts we obtain, in view of (2.14b),

$$\mu_1 \int_{-a}^{a} [(u_1 - u_2)''(y)]^2 \, dy + \int_{-a}^{a} \frac{u_1'(y)(u_1 - u_2)'(y)}{(\epsilon + v'^2(y))^{\alpha/2}} \, dy - \int_{-a}^{a} \frac{u_2'(y)(u_1 - u_2)'(y)}{(\epsilon + w'^2(y))^{\alpha/2}} \, dy = 0 \tag{2.17}$$

or

$$\mu_1 ||u_1 - u_2||^2_{H^2(-a,a)} + \int_{-a}^{a} \frac{(u_1 - u_2)'(y)u_1'(y)[(\epsilon + w'^2(y))^{\alpha/2} - (\epsilon + v'^2(y))^{\alpha/2}]}{(\epsilon + v'^2(y))^{\alpha/2}(\epsilon + w'^2(y))^{\alpha/2}} \, dy + \int_{-a}^{a} \frac{[(u_1 - u_2)'(y)]^2}{(\epsilon + w'^2(y))^{\alpha/2}} \, dy = 0. \tag{2.18}$$

As $u_1, u_2 \in H^2(-a,a)$, and $u_1', u_2' \in L^\infty(-a,a)$, we may estimate the first integral in (2.18) from above, and drop the (nonnegative) second integral, so as to obtain an estimate of the form

$$||u_1 - u_2||_{H^2(-a,a)} \leq c_1 \left[\int_{-a}^{a} \left(\left|w'(y)\right|^\alpha - \left|v'(y)\right|^\alpha\right)^2 \, dy\right]^{1/2} \tag{2.19}$$

for some $c_1 > 0$; in obtaining (2.19) we have also employed the mean value theorem in the integrand of the first integral in (2.18). The continuity of T follows directly from the estimate (2.19). By the Schauder fixed-point theorem it now follows that there exists, for $M > 0$ sufficiently large, a unique $u \in W_M$ such that $u = Tu$ and, thus, we have established the existence of a unique solution of (2.11a,b) for arbitrary $\mu_1 > 0$. ☐

For the second of the fundamental results relative to the boundary-value problem (2.11a,b) we assume that $f(y)$ is a constant, say, $f(y) = K$, $-a \le y \le a$; this clearly covers the case of the constant pressure gradient p_1 in (2.5a). Also, we define $\bar{u} = \bar{u}(y)$ to be the unique solution of (2.11a) for $\mu_1 = 0$, and $f(y) = K$, which is subject to the boundary conditions $\bar{u}(\pm a) = 0$. We state the following result, which highlights the continuous dependence of the solution of (2.1a,b), established in Theorem 2.1, on the positive constitutive parameter μ_1:

Theorem 2.2. *For fixed* $\epsilon > 0$*, and* $\alpha \in (0, 1)$*, denote by* $u_{\mu_1}(x)$ *the unique solution of* (2.11a,b) *with* $f(x) = K$*. Then, as* $\mu_1 \to 0^+$*,*

$$u_{\mu_1} \to \bar{u}, \text{ in } C^{1+\delta} \tag{2.20}$$

for $0 < \delta < \dfrac{1}{2}$.

Proof. From Theorem 2.1, we infer, for $M > 0$ sufficiently large, the existence of a unique solution $u_{\mu_1} \in W_M$ of the boundary-value problem

$$-\left[\frac{u'_{\mu_1}(y)}{(\epsilon + u'^2_{\mu_1}(y))^{\alpha/2}}\right]' + \mu_1 u''''_{\mu}(y) = K, \quad -a < y < a, \tag{2.21a}$$

$$u_{\mu_1}(\pm a) = 0, \quad u''_{\mu_1}(\pm a) = 0. \tag{2.21b}$$

We now set $v_{\mu_1} = u'_{\mu_1}$ in (2.21a) and then integrate the resulting equation over $(-a, x)$, $x < a$, so as to obtain

$$-\frac{v_{\mu_1}(y)}{(\epsilon + v^2_{\mu_1}(y))^{\alpha/2}} + \mu v''_{\mu_1}(y) = K(y + a) - A_{\mu_1} \tag{2.22}$$

with

$$A_{\mu_1} = \frac{v_{\mu_1}(-a)}{(\epsilon + v^2_{\mu_1}(-a))^{\alpha/2}} - \mu_1 v''_{\mu_1}(-a). \tag{2.23}$$

In view of the boundary conditions (2.21b), $v'_{\mu_1}(-a) = v'_{\mu_1}(a) = 0$ and

$$\int_{-a}^{a} v_{\mu_1}(y)\, dy = u_{\mu_1}(0) - u_{\mu_1}(-a) = 0. \tag{2.24}$$

We now multiply (2.22) by $v_{\mu_1}(y)$, integrate over $(-a,a)$, and then integrate by parts to obtain

$$\int_{-a}^{a} \frac{v_{\mu_1}^2(y)}{(\epsilon + v_{\mu_1}^2(y))^{\alpha/2}}\,dx + \mu_1 \int_{-a}^{a} v_{\mu_1}'^2(y)\,dx = -K\int_{-a}^{a} y v_{\mu_1}(y)\,dy \tag{2.25}$$

where we have used (2.24). Now, let

$$E_\epsilon = \left\{ y \mid v_{\mu_1}^2(y) > \epsilon \right\}. \tag{2.26}$$

Then $\forall y \in E_\epsilon$, $v_{\mu_1}^2/(\epsilon + v_{\mu_1}^2)^{\alpha/2} > \beta_1 v_{\mu_1}^{2-\alpha}$, $\beta_1 = 2^{-\alpha/2} > 0$; similarly, as $u'_{\mu_1} \in L^\infty(-a,a)$, on $E_\epsilon^c = [-a,a]/E_\epsilon$, $\exists \beta_2, \rho > 0$ such that $v_{\mu_1}^{2-\alpha} \le \beta_2 \epsilon^\rho$. Therefore

$$\begin{aligned}\int_{-a}^{a} v_{\mu_1}^{2-\alpha}(y)\,dy &= \int_{E_\epsilon} v_{\mu_1}^{2-\alpha} dy + \int_{E_\epsilon^c} v_{\mu_1}^{2-\alpha}\,dy \\ &\le \frac{1}{\beta_1}\int_{E_\epsilon} \frac{v_{\mu_1}^2}{(\epsilon + v_{\mu_1}^2)^{\alpha/2}} dy + \beta_2 \epsilon^\rho \ \text{meas}(E_\epsilon^c) \\ &\le \frac{1}{\beta_1}\int_{-a}^{a} \frac{v_{\mu_1}^2}{(\epsilon + v_{\mu_1}^2)^{\alpha/2}} dy + \beta_3\end{aligned} \tag{2.27}$$

Using the last estimate in (2.27) in (2.25) we have

$$\int_{-a}^{a} v_{\mu_1}^{2-\alpha}(y)\,dy + \frac{\mu}{\beta_1}\int_{-a}^{a} v_{\mu_1}'^2(y)\,dy \le \frac{K}{\beta_1}\int_{-a}^{a} |y|\left|v_{\mu_1}(y)\right|dy + \beta_3. \tag{2.28}$$

By virtue of the Hölder Inequality, (2.28) yields

$$\int_{-a}^{a} \left|v_{\mu_1}\right|^{2-a} dy \le \frac{K}{\beta_1}\left[\int_{-a}^{a} |y|^{(2-\alpha)/(1-\alpha)} dy\right]^{(1-\alpha)/(2-\alpha)} \left[\int_{-a}^{a} \left|v_{\mu_1}\right|^{2-\alpha} dy\right]^{1/(2-\alpha)} + \beta_3. \tag{2.29}$$

For arbitrary $\delta > 0$, we now use Young's inequality (see appendix A)

$$|a|\cdot|b| \le \delta |a|^p + \delta^{-1/(p-1)}|b|^{p'}, \qquad \frac{1}{p} + \frac{1}{p'} = 1$$

on the right-hand side of (2.29) with $p = 2-\alpha$; for δ chosen sufficiently small we obtain from (2.29) an estimate of the form

$$\int_{-a}^{a} v_{\mu_1}^{2-\alpha}(y)\,dy \le a_1 \int_{-a}^{a} |y|^{(2-\alpha)/(1-\alpha)} dy + a_2 \tag{2.30}$$

with $a_1, a_2 > 0$. To obtain our next set of estimates we multipy (2.22) by $v''_{\mu_1}(y)$, integrate over $(-a, a)$, and integrate by parts; inasmuch as $v'_{\mu_1}(-a) = v'_{\mu_1}(a) = 0$ we easily find that

$$\int_{-a}^{a} \left[\frac{v_{\mu_1}(y)}{(\epsilon + v_{\mu_1}^2(y))^{\alpha/2}} \right]' v'_{\mu_1}(y)\, dy + \mu_1 \int_{-a}^{a} (v''_{\mu_1}(y))^2\, dy = -K \int_{-a}^{a} v'_{\mu_1}(y)\, dy$$
$$= K[v_{\mu_1}(-a) - v_{\mu_1}(a)]$$

or

$$\int_{-a}^{a} \frac{v'^2_{\mu_1}(y)}{(\epsilon + v_{\mu_1}^2(y))^{\alpha/2}} \cdot \left[\frac{\epsilon + (1-\alpha) v_{\mu_1}^2(y)}{\epsilon + v_{\mu_1}^2(y)} \right] dy + \mu_1 \int_{-a}^{a} (v''_{\mu_1}(y))^2 dy = K[v_{\mu_1}(-a) - v_{\mu_1}(a)]. \tag{2.31}$$

Now, $\forall \alpha$ with $0 < \alpha < 1$, $\exists \bar{c}_1, \bar{c}_2 > 0$ such that

$$\bar{c}_1 \le \frac{\epsilon + (1-\alpha)\eta}{\epsilon + \eta} \le \bar{c}_2, \quad \forall \eta \ge 0 \tag{2.32}$$

where the $\bar{c}_i$, $i = 1, 2$, depend on α but not on η. Applying (2.32) to (2.31), with $\eta = v_{\mu_1}^2$, we get the estimate

$$\begin{aligned} \int_{-a}^{a} \frac{v'^2_{\mu_1}(y)}{(\epsilon + v_{\mu_1}^2(y))^{\alpha/2}} dy + \frac{\mu_1}{\bar{c}_1} \int_{-a}^{a} (v''_{\mu_1}(y))^2 dy &\le \frac{K}{\bar{c}_1} [v_{\mu_1}(-a) - v_{\mu_1}(a)] \\ &\le \bar{c}_3 \max_{[-a,a]} \left| v_{\mu_1}(y) \right|. \end{aligned} \tag{2.33}$$

We now set

$$\Psi(v_{\mu_1}) = \int_0^{v_{\mu_1}} \frac{ds}{(\epsilon + s^2)^{\alpha/4}}. \tag{2.34}$$

Then it follows directly from (2.34) that

$$\int_{-a}^{a} \left(\frac{d}{dx} \Psi(v_{\mu_1}(y)) \right)^2 dx \le \bar{c}_3 \max_{[-a,a]} \left| v_{\mu_1}(y) \right|. \tag{2.35}$$

As $\Psi(v_{\mu_1})$ is an even function, and $1/(\epsilon + s^2)^{\alpha/4} \le 1/(s^2)^{\alpha/4}$, we have that

$$\left| \Psi(v_{\mu_1}) \right| = \Psi(\left| v_{\mu_1} \right|) \le \left(1 - \frac{\alpha}{2} \right)^{-1} \left| v_{\mu_1} \right|^{1-\alpha/2} \le 4 \left| v_{\mu_1} \right|^{1-\alpha/2}. \tag{2.36}$$

Therefore, by virtue of our previous estimate (2.30), $\exists \Psi_0 > 0$ (const.) such that

$$\int_{-a}^{a} \Psi^2(v_{\mu_1}(y))dx \le \Psi_0. \tag{2.37}$$

Now, $\forall w \in H^1[-a,a]$, and $\forall \delta > 0$, $\exists c_\delta > 0$ such that

$$\max_{[-a,a]} |w| \le \delta \left(\int_{-a}^{a} w'^2(y)\, dy \right)^{1/2} + c_\delta \left(\int_{-a}^{a} w^2(y)\, dy \right)^{1/2} \tag{2.38}$$

(see, e.g., [Lio1] Lemma 5.1); applying (2.38) with $w = \Psi(v_{\mu_1})$, and making use of both (2.35) and (2.37), we find that for some $d_\delta > 0$,

$$\max_{[-a,a]} \left|\Psi(v_{\mu_1})\right| \le \delta \left[\max_{[-a,a]} \left|v_{\mu_1}(y)\right| \right]^{1/2} + d_\delta. \tag{2.39}$$

Our goal now is to show that for some $c > 0$,

$$\max_{[-a,a]} \left|v_{\mu_1}(y)\right| \le c \left[\max \left\{ 1, \max_{[-a,a]} \left|\Psi(v_{\mu_1})\right|^{\beta} \right\} \right] \tag{2.40}$$

with c independent of v_{μ_1}, and $\beta = 1/(1-\alpha/2)$. To this end we define, for $s \in R^1$,

$$F(s) = \Psi(s) - ks^{1-\alpha/2} \tag{2.41}$$

where k is chosen so that $\Psi(1) > k$. Thus, $F(1) \ge 0$, while

$$F'(s) = \frac{1}{(\epsilon + s)^{\alpha/4}} - k\left(1 - \frac{\alpha}{2}\right) s^{-\alpha/2}$$

so that, for k chosen sufficiently small, $F'(s) \ge 0$, $\forall s \in R^1$. Consequently, $F(s) \ge 0$, $\forall s \ge 1$ so that

$$ks^{1/\beta} \le \Psi(s), \quad \forall s \ge 1 \tag{2.42}$$

with $\beta = 1/(1-\alpha/2)$. Employing (2.42) in (2.39) we obtain

$$\max_{[-a,a]} \left|\Psi(v_{\mu_1})\right| \le \delta' \left[\max \left\{ 1, \max_{[-a,a]} \left|\Psi(v_{\mu_1})\right|^{\beta/2} \right\} \right] + d_\delta \tag{2.43}$$

with $\delta' = \delta k^{-\beta/2}$. If $\max_{[-a,a]} \left|\Psi(v_{\mu_1})\right|^{\beta/2} > 1$, then

$$\max_{[-a,a]} \left|\Psi(v_{\mu_1})\right| - \delta' \max_{[-a,a]} \left|\Psi(v_{\mu_1})\right|^{\beta/2} \le d_\delta$$

or, as $\beta/2 = 1/(2-\alpha) < 1$, for $0 < \alpha < 1$,

$$(1-\delta') \max_{[-a,a]} \left|\Psi(v_{\mu_1})\right| \le d_\delta. \tag{2.44}$$

Therefore, for δ chosen sufficiently small, it follows that $\exists C > 0$ such that (recall that $\left|\Psi(v_{\mu_1})\right| = \Psi(\left|v_{\mu_1}\right|)$):

$$\max_{[-a,a]} \left[\Psi(\left|v_{\mu_1}(y)\right|)\right] \le C. \tag{2.45}$$

Clearly, an estimate of the form (2.45) also follows from (2.43) if $\max\limits_{[-a,a]} \left|\Psi(v_{\mu_1})\right|^{\beta/2} \ge 1$. Now, the estimate (2.40) is a direct consequence of (2.42), and the use of (2.45) in (2.42) then produces a bound of the form

$$\max_{[-a,a]} \left|v_{\mu_1}(y)\right| \le C'$$

for some $C' > 0$. Thus, by virtue of (2.33), we have, for some $C > 0$,

$$\int_{-a}^{a} \left[\frac{d}{dx}\Psi(v_{\mu_1}(y))\right]^2 dx + \mu \int_{-a}^{a} (v''_{\mu_1}(y))^2 dy \le C. \tag{2.46}$$

Combining (2.46) with (2.37), it follows that $\exists \tilde{C}$, independent of μ_1, such that

$$\left|\left|\Psi(v_{\mu_1})\right|\right|_{H^1(-a,a)} \le \tilde{C}. \tag{2.47}$$

Therefore, $\exists \Psi^0 \in H^1(-a,a)$ such that

$$\Psi(v_{\mu_1}) \to \Psi^0, \text{ in } H^1(-a,a), \text{ as } \mu_1 \to 0^+ \tag{2.48}$$

and, by virtue of (2.46), we also note that

$$\mu v''_{\mu_1} \to 0, \text{ in } L^2(-a,a), \text{ as } \mu_1 \to 0^+. \tag{2.49}$$

In view of (2.48), and Theorem 2.1, for some $\bar{\Psi}$ we have

$$\Psi(v_{\mu_1}) \to \bar{\Psi}, \text{ in } C^{0,\delta}, \text{ for } 0 < \delta < 1/2, \text{ as } \mu_1 \to 0^+. \tag{2.50}$$

But Ψ, being monotone, is invertible, and as $\Psi^{-1} \in C^1(R^1)$ we find that

$$v_{\mu_1} \to \bar{u}, \text{ in } C^{0,\delta}, \text{ for } 0 < \delta < 1/2, \text{ as } \mu_1 \to 0^+ \tag{2.51}$$

Finally, as $u'_{\mu_1} = v_{\mu_1}$ we have, for the unique solution $u_{\mu_1}(y)$ of (2.21a,b), that

$$v_{\mu_1} \to \bar{u}, \text{ in } C^{1+\delta}, \text{ for } 0 < \delta < 1/2, \text{ as } \mu_1 \to 0^+ \tag{2.52}$$

with $\bar{u}$ the unique solution of (2.21a), with $\mu = 0$, subject to the boundary conditions $\bar{u}(\pm a) = 0$; this concludes the demonstration of the continuous dependence of u_{μ_1} on μ_1, as $\mu_1 \to 0^+$, in the $C^{1+\delta}$ norm, for $0 < \delta < 1/2$. □

Remarks. If $\mu_1 = 0$, then (2.5a) reduces to

$$\left[\left(\epsilon + \frac{1}{2}v_1'^2(x_2)\right)^{-\alpha/2} v_1'(x_2)\right]' = p_1/\mu_0$$

so that for some real constant γ,

$$\left(\epsilon + \frac{1}{2}v_1'^2(x_2)\right)^{-\alpha/2} v_1'(x_2) = g_\gamma(x_2) \tag{2.53}$$

where

$$g_\gamma(x_2) = \left(\frac{p_1}{\mu_0}\right) x_2 + \gamma.$$

If we now set $W_\epsilon(x_2) = \epsilon + \frac{1}{2}v_1'^2(x_2)$, then it follows from (2.53) that W_ϵ satisfies the transcendental algebraic equation

$$W_\epsilon^{1-\alpha} - \epsilon W_\epsilon^{\alpha} = \frac{1}{2}g_\gamma^2; \quad \epsilon > 0,\ 0 < \alpha < 1 \tag{2.54}$$

whose solutions are easily seen to be dependent continuously on ϵ as $\epsilon \to 0^+$. Thus, solutions of (2.11a), with $f(y) = p_1$ and $\mu_1 = 0$, subject to $u(\pm a) = 0$, depend continuously on ϵ; that the same continuous dependence with respect to ϵ holds for the full boundary-value problem (2.11a,b), with $\mu_1 > 0$ and $f(y) = p_1$, follows from the detailed estimates in Sect. 2.3.

2.3 Estimates and Generalized Reynolds Numbers for Steady Plane Poiseuille Flow

We will continue here the practice of using the notation $u = v_1$, and $y = x_2$, in which case (2.5a,b) assumes the equivalent form

$$\mu_0\left[\left(\epsilon + \frac{1}{2}u'^2(y)\right)^{-\alpha/2} u'(y)\right]' - \mu_1 u''''(y) = p_1, \quad -a < y < a \tag{2.55a}$$

$$u(\pm a) = 0, \quad u''(\pm a) = 0 \tag{2.55b}$$

where we have written $u(y)$ in lieu of $u(y;\epsilon,\mu_1)$, while for $\epsilon = \mu_1 = 0$ the above fourth-order boundary-value problem reduces to

$$\mu_0 \left[\left(\frac{1}{2} u_0'^2(y) \right)^{-\alpha/2} u_0'(y) \right]' = p_1, \tag{2.56a}$$

$$u_0(\pm a) = 0 \tag{2.56b}$$

where $u_0(y) = u(y;0,0)$. Whenever it is deemed important to avoid any possible confusion, we will make explicit the dependence of the solution u of (2.55a,b) on ϵ and μ_1.

In this section we continue the study of plane equilibrium Poiseuille flows of incompressible, isothermal, bipolar fluids initiated in Sect. 2.2. Through the use of dimensional analysis applied to (2.55a) we isolate the natural counterparts of the Reynolds number associated with the Navier–Stokes theory. We then investigate, in greater detail than was done in Sect. 2.2, properties of the solutions $u_0(\cdot)$ of equations (2.56a,b) and use the solutions to compute the associated mean velocity, maximum velocity, volume flow, and pressure drop. Finally, although a continuous dependence result for solutions of (2.55a,b), (in $C^{1+\delta}$, $0 < \delta < 1/2$) as $\epsilon, \mu_1 \to 0^+$, was established in Sect. 2.2 precise estimates of the errors incurred by setting $\epsilon = \mu_1 = 0$ and using, in place of $u(\cdot;\epsilon,\mu_1)$ the solutions $u_0(\cdot)$ of equations (2.56a,b) were not presented there; such estimates are derived here and are subsequently employed to establish the related estimates for the volume flow, etc. It is hoped (and expected) that such estimates will eventually serve as a guide for the formulation of experiments directed at the determination of the constitutive constants in the model.

2.3.1 *Generalized Reynolds Numbers for Plane Poiseuille Flow of a Bipolar Fluid*

In this subsection we indicate the appropriate form which a dimensionless version of the evolution equation associated with (2.55a) assumes and, in the process, are led to the definition of generalized Reynolds numbers that are connected with plane Poiseuille flows of an incompressible bipolar fluid. Employing a standard approach (and not taking, a priori, the density $\rho = 1$) we set

$$\bar{y} = \frac{y}{a}, \quad \bar{t} = \frac{V}{a}t, \quad \bar{u} = \frac{u}{V}, \quad \bar{p} = \frac{p}{\rho V^2} \tag{2.57}$$

in the evolution equation for plane Poiseuille flow of an incompressible bipolar fluid,

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial y} + \mu_0 \frac{\partial}{\partial y}\left\{\left[\epsilon + \left(\frac{\partial u}{\partial y}\right)^2\right]^{-\alpha/2} \frac{\partial u}{\partial y}\right\} - \mu_1 \frac{\partial^4 u}{\partial y^4}. \tag{2.58}$$

Here V is a measure of the mean- or far-field velocity associated with the flow and $u = u(y,t)$, where we have supressed the dependence of u on ϵ and μ_1. With $\bar{u}(\bar{y},\bar{t}) = u(a\bar{y}, a\bar{t}/V)$, $\bar{p}(\bar{y}) = p(a\bar{y})$, elementary calculations yield

$$\frac{\partial u}{\partial t} = \frac{V^2}{a}\frac{\partial \bar{u}}{\partial \bar{t}}, \tag{2.59a}$$

$$\frac{\partial p}{\partial y} = \frac{\rho V^2}{a}\frac{\partial \bar{p}}{\partial \bar{y}}, \tag{2.59b}$$

$$\frac{\partial u}{\partial y} = \frac{V}{a}\frac{\partial \bar{u}}{\partial \bar{y}}, \tag{2.59c}$$

$$\frac{\partial^4 u}{\partial y^4} = \frac{V}{a^4}\frac{\partial^4 \bar{u}}{\partial \bar{y}^4}, \tag{2.59d}$$

the substitution of which into (2.58) produces

$$\left(\frac{\rho V^2}{a}\right)\frac{\partial \bar{u}}{\partial \bar{t}} = -\left(\frac{\rho V^2}{a}\right)\frac{\partial \bar{p}}{\partial \bar{y}} + \frac{\mu_0}{a}\frac{\partial}{\partial \bar{y}}\left\{\left[\epsilon + \frac{V^2}{a^2}\left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)^2\right]^{-\frac{\alpha}{2}}\left(\frac{V}{a}\right)\frac{\partial \bar{u}}{\partial \bar{y}}\right\} - \frac{\mu_1 V}{a^4}\frac{\partial^4 \bar{u}}{\partial \bar{y}^4}. \tag{2.60}$$

After multiplying (2.60) by $\dfrac{a}{\rho V^2}$, and setting

$$\nu_0 = \frac{\mu_0}{\rho}, \quad \nu_1 = \frac{\mu_1}{\rho} \tag{2.61}$$

(2.60) becomes

$$\frac{\partial \bar{u}}{\partial \bar{t}} = -\frac{\partial \bar{p}}{\partial \bar{y}} + \frac{\nu_0}{aV}\frac{\partial}{\partial \bar{y}}\left\{\left[\epsilon + \frac{V^2}{a^2}\left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)^2\right]^{-\frac{\alpha}{2}}\frac{\partial \bar{u}}{\partial \bar{y}}\right\} - \frac{\nu_1}{a^3 V}\frac{\partial^4 \bar{u}}{\partial \bar{y}^4}. \tag{2.62}$$

In as much as ϵ must have the dimension of a velocity gradient squared,

$$\bar{\epsilon} = \epsilon a^2/V^2 \tag{2.63}$$

is dimensionless; using the definition (2.63) of $\bar{\epsilon}$ we may now rewrite (2.62) as

$$\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial y} + \frac{\nu_0}{aV}\left(\frac{a}{V}\right)^{\alpha}\frac{\partial}{\partial y}\left\{\left[\epsilon + \left(\frac{\partial u}{\partial y}\right)^2\right]^{-\frac{\alpha}{2}}\frac{\partial u}{\partial y}\right\} - \frac{\nu_1}{a^3 V}\frac{\partial^4 u}{\partial y^4} \tag{2.64}$$

where we have dropped the superposed bars from y, t, p, u, and ϵ. The dimensionless version (2.64) of (2.58) leads naturally to the definition of two generalized Reynolds numbers that are associated with plane Poiseuille flow of a bipolar viscous fluid, namely,

$$R_0^{(\alpha)} = \frac{V^{\alpha+1}}{\nu_0 a^{\alpha-1}}, \quad R_1 = \frac{a^3 V}{\nu_1}. \tag{2.65}$$

Using the definitions (2.65), the evolution equation for $u(y,t)$ assumes the form

$$\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial y} + \frac{1}{R_0^{(\alpha)}} \frac{\partial}{\partial y} \left\{ \left[\epsilon + \left(\frac{\partial u}{\partial y} \right)^2 \right]^{-\frac{\alpha}{2}} \frac{\partial u}{\partial y} \right\} - \frac{1}{R_1} \frac{\partial^4 u}{\partial y^4}. \tag{2.66}$$

For $\alpha = \mu_1 = 0$, clearly $R_0^{(0)} = Va/\nu_0$ and $R_1^{-1} = 0$, so that (2.66) reduces to the standard dimensionless form for plane Poiseuille flow within the context of the Navier–Stokes formulation, with $R_0^{(0)}$ being the usual Reynolds number.

2.3.2 The Poiseuille Flow for $\epsilon = \mu_1 = 0$

In this subsection we will look, in greater detail than that which was done in Sect. 1.7.1, at the behavior of the solution of (2.56a,b). In the next subsection our interest will be in obtaining estimates which relate the behavior of the solution of the boundary-value problem (2.56a,b) to that of the solution $u(y;\epsilon,\mu_1)$ of (2.55a,b); the quantities of particular interest to us will be the volume flow

$$Q_{\epsilon,\mu_1} = \int_{-a}^{a} u(y;\epsilon,\mu_1)\,dy \tag{2.67a}$$

which for $\epsilon = \mu_1 = 0$ has the form

$$Q_{0,0} \equiv Q_0 = \int_{-a}^{a} u_0(y)\,dy \tag{2.67b}$$

the mean velocity

$$\bar{u}_{\epsilon,\mu_1} = \frac{1}{2a} Q_{\epsilon,\mu_1} \tag{2.68a}$$

and its counterpart for $\epsilon = \mu_1 = 0$, i.e.,

$$\bar{u}_0 = \frac{1}{2a} Q_0 \tag{2.68b}$$

and the friction factors

$$f_{\epsilon,\mu_1} = \frac{4\tau_{12}(\pm a, \epsilon, \mu_1)}{\frac{1}{2}\rho \bar{u}^2_{\epsilon,\mu_1}} \tag{2.69a}$$

$$f_0 = \frac{4\tau_{12}(\pm a, 0, 0)}{\frac{1}{2}\rho \bar{u}^2_0} \tag{2.69b}$$

where $\tau_{12}(\pm a, \epsilon, \mu_1)$ is the shear stress at the walls located at $y = \pm a$, i.e.,

$$\tau_{12}(\pm a, \epsilon, \mu_1) = \mu_0 \left[\epsilon + \frac{1}{2}u'^2(\pm a, \epsilon, \mu_1)\right]^{-\frac{\alpha}{2}} u'(\pm a, \epsilon, \mu_1) - \mu_1 u''''(\pm a, \epsilon, \mu_1). \tag{2.70}$$

We begin by noting that if $u_0(y)$ is a solution of the boundary-value problem (2.56a,b) then so is $u_0(-y)$ and, thus, by uniqueness of solutions $u_0(y) = u_0(-y)$, $-a \le y \le a$; from this result it follows that $u_0'(y) = -u_0'(-y)$, so that $u_0'(0) = 0$. Moreover, with p_1, the constant pressure gradient, negative, a first integration of equation (2.56a) yields

$$\mu_0 \left[\frac{1}{2}u_0'^2(y)\right]^{-\frac{\alpha}{2}} u_0'(y) = -|p_1| y \tag{2.71}$$

where the constant of integration vanishes in view of the fact that $0 < a < 1$ and $u_0'(0) = 0$. From (2.71) it is immediate that

$$\begin{aligned} u_0'(y) &> 0, \quad y \in (-a, 0), \\ u_0'(y) &< 0, \quad y \in (0, a). \end{aligned} \tag{2.72}$$

In as much as $u_0'(y) \le 0$, for $y \in (0, a)$, we have $u_0'(y) = -\left|u_0'(y)\right|$, $0 \le y \le a$; therefore, if we set $C_\alpha = \mu_0 2^{\alpha/2}$, equation (2.71) becomes, on $(0, a)$,

$$\left|u_0'(y)\right|^{1-\alpha} = \frac{|p_1|}{C_\alpha} y, \quad 0 < y < a$$

or

$$\left|u_0'(y)\right| = C_\alpha y^{1/(1-\alpha)}, \quad 0 < y < a \tag{2.73}$$

with $C_\alpha = \left(\frac{|p_1|}{C_\alpha}\right)^{1/(1-\alpha)}$. We rewrite (2.73) as

$$u_0'(y) = -C_\alpha y^{1/(1-\alpha)}, \quad 0 < y < a \tag{2.74}$$

and integrate from a to y obtaining

$$u_0(y) = \frac{C_\alpha}{\bar{\gamma}+1}\left[a^{\bar{\gamma}+1} - y^{\bar{\gamma}+1}\right]; \quad \bar{\gamma} = \frac{1}{1-\alpha}. \tag{2.75}$$

In obtaining (2.75) we have used, of course, the boundary condition $u_0(a) = 0$. Substituting for $\bar{\gamma}$ in (2.75), and noting that $u_0(y) = u_0(-y)$, we find that

$$u_0(y) = d_\alpha \left[1 - \left(\frac{|y|}{a}\right)^{(2-\alpha)/(1-\alpha)}\right], \quad -a \le y \le a \tag{2.76}$$

where

$$\begin{aligned} d_\alpha &= C_\alpha \left(\frac{1-\alpha}{2-\alpha}\right) a^{(2-\alpha)/(1-\alpha)} \\ &= \left(\frac{|p_1|}{C_\alpha}\right)^{1/(1-\alpha)} a^{(2-\alpha)/(1-\alpha)} \left(\frac{1-\alpha}{2-\alpha}\right) \\ &= \left(\frac{1-\alpha}{2-\alpha}\right)\left(\frac{|p_1| a^{2-\alpha}}{\mu_0 2^{\alpha/2}}\right)^{1/(1-\alpha)}. \end{aligned} \tag{2.77}$$

It is clear, from (2.76), that

$$\max_{[-a,a]} u_0(y) \equiv u_0^{\max} = u_0(0) = d_\alpha. \tag{2.78}$$

By direct calculation, the mean velocity $\bar{u}_0$ associated with $u_0(y)$ is

$$\begin{aligned} \bar{u}_0 &= \frac{1}{2a}\int_{-a}^{a} u_0(y)\, dy \\ &= \frac{d_\alpha}{a}\int_0^a \left[1 - \left(\frac{\lambda}{a}\right)^\delta\right] d\lambda; \quad \delta = \frac{2-\alpha}{1-\alpha}. \end{aligned} \tag{2.79}$$

Carrying out the integration in (2.79), we are led to

$$\bar{u}_0 - \left(\frac{2-\alpha}{3-2\alpha}\right) d_\alpha \equiv \left(\frac{2-\alpha}{3-2\alpha}\right) u_0^{\max} \tag{2.80}$$

or, in view of (2.77),

$$\bar{u}_0 = \left(\frac{1-\alpha}{3-2\alpha}\right)\left(\frac{|p_1| a^{2-\alpha}}{\mu_0 2^{\alpha/2}}\right)^{1/(1-\alpha)}. \tag{2.81}$$

We note that

$$\bar{u}_0\Big|_{\alpha=0} = \frac{1}{3}\frac{|p_1|a^2}{\mu_0} \tag{2.82}$$

which is the classical result associated with Navier–Stokes. From (2.78) and (2.80) it follows that

$$\lim_{\alpha\to 1-} u_0^{\max} = \lim_{\alpha\to 1-} d_\alpha = \lim_{\alpha\to 1-} \bar{u}_0. \tag{2.83}$$

However, in view of (2.77),

$$\lim_{\alpha\to 1^-} d_\alpha = \lim_{\alpha\to 1^-}\left[\left(\frac{1-\alpha}{2-\alpha}\right)\left\{\frac{|p_1|a^{2-\alpha}}{\mu_0 2^{\alpha/2}}\right\}^{1/(1-\alpha)}\right]$$

from which it is clear that the critical quantity in computing $\lim_{\alpha\to 1-} u_0^{\max}$ is

$$e_\alpha = \frac{|p_1|a^{2-\alpha}}{\mu_0 2^{\alpha/2}}. \tag{2.84}$$

Specifically, if $e_\alpha > 1$, for α sufficiently close to 1, then $u_0^{\max} \to \infty$ as $\alpha \to 1^-$. Suppose that we set $V = \bar{u}_0|_{\alpha=0}$ in (2.65), so that $|p_1| = \rho V^2/a$, then

$$e_\alpha^{1/(1-\alpha)} = \left[\frac{V^2}{\nu_0 2^{\alpha/2}}\right]^{1/(1-\alpha)} > \left[\frac{V^2}{\nu_0\sqrt{2}}\right]^{1/(1-\alpha)} \tag{2.85}$$

while

$$\lim_{\alpha\to 1^-} R_0^{(\alpha)} = \frac{V^2}{\nu_0}.$$

Thus,

$$u_0^{\max} \to \infty, \text{ as } \alpha \to 1^- \tag{2.86a}$$

provided

$$\lim_{\alpha\to 1^-} R_0^{(\alpha)} > \sqrt{2}. \tag{2.86b}$$

To emphasize further the role of the criteria (2.86b) and its connection with the status of the parameter α in situations involving small physical viscosity (i.e., α close to 1), we compute the $\lim_{\alpha\to 1^-} u_0'(a)$. From (2.76) and (2.80), we have

$$u_0(y) = \left(\frac{3-2\alpha}{2-\alpha}\right)\bar{u}_0\left[1-\left(\frac{y}{a}\right)^{(2-\alpha)/(1-\alpha)}\right], \quad 0 \le y \le a \tag{2.87}$$

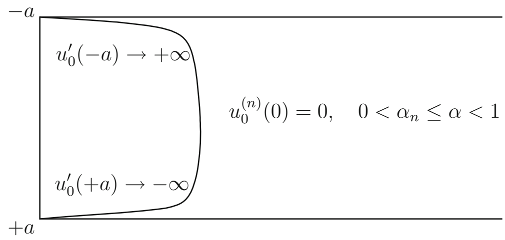

Fig. 2.1 Non-Newtonian velocity profile, $0 < \alpha < 1$

so that

$$u_0'(y) = -\left(\frac{3-2\alpha}{1-\alpha}\right)\bar{u}_0\left(\frac{y}{a}\right)^{1/(1-\alpha)} \tag{2.88}$$

and, thus,

$$\frac{1}{(3-2\alpha)}u_0'(a) = \frac{1}{(1-\alpha)}\bar{u}_0. \tag{2.89}$$

Therefore, by virtue of (2.81) and (2.84),

$$\begin{aligned}\lim_{\alpha\to 1^-} u_0'(a) &= -\lim_{\alpha\to 1^-}\left[\frac{1}{(1-\alpha)}\bar{u}_0\right]\\ &= -\lim_{\alpha\to 1^-}\left[\frac{1}{(3-2\alpha)}e_\alpha^{1/(1-\alpha)}\right]\\ &= -\lim_{\alpha\to 1^-} e_\alpha^{1/(1-\alpha)} = -\infty\end{aligned} \tag{2.90}$$

if $\lim_{\alpha\to 1^-} R_0^{(\alpha)} > \sqrt{2}$; under these same conditions

$$\lim_{\alpha\to 1^-} u_0'(-a) = \lim_{\alpha\to 1^-}\left[-u_0'(a)\right] = +\infty \tag{2.91}$$

so that if $\lim_{\alpha\to 1^-} R_0^{(\alpha)} > \sqrt{2}$, and α is close to 1, the velocity profile assumes the form indicated in Fig. 2.1, above.

In fact, not only is $u_0'(0) = 0$, but the rapid flattening of the profile, depicted in Fig. 2.1, with respect to the axis $y = 0$, as $\alpha \to 1^-$, is easily demonstrated as follows: from (2.88), and the companion result for $-a \le y \le 0$,

$$u_0''(y) = \frac{(3-2\alpha)}{(1-\alpha)^2}\bar{u}_0\left(\frac{|y|}{a}\right)^{\alpha/(1-\alpha)} \tag{2.92}$$

so that $u_0''(0) = 0$, for all α, $0 < \alpha < 1$. Then

$$u_0'''(y) = -\alpha \frac{(3-2\alpha)}{(1-\alpha)^3} \bar{u}_0 \left(\frac{|y|}{a} \right)^{(2\alpha-1)/(1-\alpha)} \tag{2.93}$$

so that $u_0'''(0) = 0$, for $\alpha > \frac{1}{2}$. A further computation shows that $u_0''''(0) = 0$, for $\alpha > \frac{2}{3}$, and it is clear that by an induction argument we may show that $u_0^{(n)}(0) = 0$, $\alpha > \alpha_n$, for some α_n sufficiently close to 1.

From (2.81) it follows directly that the volume flow Q_0, as given by (2.67b), is

$$Q_0 = 2a \left(\frac{1-\alpha}{3-2\alpha} \right) \left[\frac{|p_1| a^{2-\alpha}}{\mu_0 2^{\alpha/2}} \right]^{1/(1-\alpha)}. \tag{2.94}$$

Also, from (2.81), it is a simple matter to compute that

$$|p_1| = \frac{\mu_0 2^{\alpha/2}}{a^{2-\alpha}} \left(\frac{3-2\alpha}{1-\alpha} \right) \bar{u}_0^{1-\alpha}. \tag{2.95}$$

Now, by (2.70), $\tau_{12}(-a, 0, 0) = |p_1|a$, or

$$\tau_{12}(-a, 0, 0) = \frac{\mu_0 2^{\alpha/2}}{a^{1-\alpha}} \left(\frac{3-2\alpha}{1-\alpha} \right) \bar{u}_0^{1-\alpha} \tag{2.96}$$

so that, by (2.69b), the friction factor is

$$f_0 = \frac{\nu_0 2^{3+(\alpha/2)}}{a^{1-\alpha}} \left(\frac{3-2\alpha}{1-\alpha} \right) \bar{u}_0^{-(\alpha+1)}. \tag{2.97}$$

Also, in view of (2.95), and the fact that $a > 1$,

$$\frac{\mu_0 \bar{u}_0^{1-\alpha}}{a^2} < |p_1| < \frac{3\sqrt{2}}{a(1-\alpha)} \mu_0 \bar{u}_0^{1-\alpha} \tag{2.98}$$

an estimate which should be useful in experimentally approximating the values of the constitutive parameters μ_0, α based on careful measurements of the magnitude of the pressure gradient p_1 and the mean velocity $\bar{u}_0$.

2.3.3 Estimates for the Poiseuille Flow $u(y; \epsilon, \mu_1)$ when $\epsilon, \mu_1 \neq 0$

In this section we provide those precise qualitative estimates, for the unique solution $u(y; \epsilon, \mu_1)$ of the nonlinear boundary-value problem (2.55a,b), which allow us to compare $u(y;\epsilon,\mu_1)$ with the solution $u_0(y)$ of (2.56a,b) and which are missing from the analysis in Sect. 2.2; the results obtained in this subsection render meaningful the content of results such as (2.97), (2.98) for the flow $u_0(y)$. We begin by introducing the following notation: we set

$$w(y;\epsilon,\mu_1) = u'(y;\epsilon,\mu_1), \tag{2.99a}$$

$$\hat{w}(y;\epsilon) = w(y;\epsilon,0), \tag{2.99b}$$

$$z(y;\epsilon,\mu_1) = \epsilon + |w(y;\epsilon,\mu_1)|^2, \tag{2.99c}$$

$$\hat{z}(y;\epsilon) = z(y;\epsilon,0) = \epsilon + |\hat{w}(y;\epsilon)|^2 \tag{2.99d}$$

and

$$\gamma(w) = (\epsilon + |w|^2)^{-\alpha/2} w = z^{-\alpha/2} w. \tag{2.100}$$

When the interpretation is obvious, we will supress the explicit dependence on y and write, e.g., $\hat{z}(\epsilon) = \hat{z}(y;\epsilon)$. We also note that $\hat{w}(y;0) = u'(y;0,0) \equiv u_0'(y)$ which is given explicitly by (2.88). Our goal is to estimate the difference $u(y;\epsilon,\mu_1) - u(y;0,0) = u(y;\epsilon,\mu_1) - u_0(y)$, from both below and above and to then use the resulting bounds to estimate quantities such as Q_{ϵ,μ_1}, the volume flow associated with the unique solution of the boundary-value problem (2.55a,b). The bounds for $u(y;\epsilon,\mu) - u_0(y)$ on $[-a,a]$ will be obtained from a series of lemmas which culminate in Theorem 2.3; the first of these results is as follows:

Lemma 2.1. *Let $u(y;\epsilon,\mu_1)$ be the unique classical solution of* (2.55a,b) *and $u_0(y)$ the unique solution of* (2.56a,b). *Then $\exists K_\alpha > 0$, depending only on α, such that on* $[-a,a]$

$$\left|u'(y;\epsilon,0) - u_0'(y)\right| < (1 + K_\alpha)\sqrt{\epsilon}. \tag{2.101}$$

Proof. We set $\mu_1 = 0$ in (2.55a,b), divide through by μ_0 and set $p_1^* = p_1/\mu_0$; then $u(y;\epsilon,0)$ is the solution of the boundary-value problem

$$\left[\left(\epsilon + |\hat{w}(y;\epsilon)|^2\right)^{-\alpha/2} \hat{w}(y;\epsilon)\right]' = p_1^*, \quad -a < y < a, \tag{2.102a}$$

$$u(\pm a;\epsilon,0) = 0 \tag{2.102b}$$

where we have used the definitions in (2.99a,b). Now, if $u(y;\epsilon,\mu_1)$ is a solution of (2.55a,b), then so is $u(-y;\epsilon,\mu_1)$ for any $\epsilon,\mu_1 \geq 0$; by uniqueness of solutions to the boundary-value problem we must have $u(y;\epsilon,\mu_1) = u(-y;\epsilon,\mu_1)$ from which it follows that

$$u'(y;\epsilon,\mu_1) = -u'(-y;\epsilon,\mu_1), \quad -a < y < a \tag{2.103}$$

and, therefore, $u'(0;\epsilon,\mu_1) = 0$, for all $\epsilon, \mu_1 \geq 0$. Integration of equation (2.102a) leads, therefore, to

$$\left(\epsilon + |\hat{w}(y;\epsilon)|^2\right)^{-\alpha/2} \hat{w}(y;\epsilon) = p_1^* y, \quad -a < y < a \tag{2.104}$$

as $\hat{w}(0;\epsilon) = u'(0;\epsilon,0) = 0$. It then follows from (2.104) that

$$\hat{w}(y;\epsilon) \neq 0, \quad \forall \epsilon \geq 0, \quad y \neq 0 \tag{2.105}$$

as $0 < \alpha < 1$. Squaring both sides of (2.104) and using the definition (2.99d), we obtain

$$\hat{z}(y;\epsilon)^{-\alpha} \hat{w}^2(y;\epsilon) = p_1^{*2} y^2, \quad -a < y < a. \tag{2.106}$$

We rewrite (2.106) in the form [recall that $\hat{z}(\epsilon) = \hat{z}(y;\epsilon)$, $-a \leq y \leq a$]

$$\hat{z}(\epsilon)^{-\alpha}[\hat{z}(\epsilon) - \epsilon] = p_1^{*2} y^2$$

or

$$\hat{z}(\epsilon)^{1-\alpha} - \epsilon \hat{z}(\epsilon)^{-\alpha} = p_1^{*2} y^2, \quad -a < y < a. \tag{2.107}$$

If we now differentiate (2.107) with respect to ϵ we obtain, after a simple calculation,

$$\hat{z}(\epsilon)^{-\alpha}\left[(1-\alpha)\hat{z}_\epsilon - 1 + \alpha\epsilon\hat{z}(\epsilon)^{-1}\hat{z}_\epsilon\right] = 0 \tag{2.108}$$

where $\hat{z}_\epsilon = \dfrac{\partial}{\partial\epsilon}\hat{z}(y;\epsilon)$. We now restrict our attention to the set of all $y \in (-a, 0)$. As

$$\hat{z}(\epsilon) = \epsilon + \left|u'(y;\epsilon,0)\right|^2 \neq 0, \quad \forall \epsilon \geq 0, \quad y \in (-a, 0) \tag{2.109}$$

it follows from (2.108) that

$$(1-\alpha)\hat{z}_\epsilon - 1 + \alpha\epsilon\hat{z}(\epsilon)^{-1}\hat{z}_\epsilon = 0, \quad \epsilon \geq 0, \quad y \in (-a, 0) \tag{2.110}$$

in which case we find that

$$\hat{z}_\epsilon = -\frac{\hat{z}(\epsilon)}{(1-\alpha)\hat{z}(\epsilon) + \alpha\epsilon}, \quad \epsilon \geq 0, \quad y \in (-a, 0). \tag{2.111}$$

As a direct consequence of (2.111) we see that

$$0 \le \hat{z}_\epsilon(y;\epsilon,0) \le \frac{1}{1-\alpha}, \quad \epsilon \ge 0, \quad y \in (-a,0). \tag{2.112}$$

Now, for $y \in (-a,0)$ we may write that

$$\hat{z}(y;\epsilon) = \hat{z}(y;0) + \int_0^\epsilon \hat{z}_\lambda(y;\lambda)\,d\lambda. \tag{2.113}$$

Combining (2.112) and (2.113), we then have

$$0 \le \hat{z}(y;\epsilon) - \hat{z}(y;0) \le \frac{\epsilon}{1-\alpha}; \quad \epsilon \ge 0,\ y \in [-a,0) \tag{2.114}$$

where we have used the continuity of $u'(y;\epsilon,\mu)$ to extend the result to $y = -a$. However,

$$\hat{z}(0;0) = |\hat{w}(0;0)|^2 = u'^2(0;0,0) = 0 \tag{2.115a}$$

and

$$\begin{aligned} \hat{z}(0,\epsilon) &= \epsilon + |\hat{w}(0;\epsilon)|^2 \\ &= \epsilon + \left|u'(0;\epsilon,0)\right|^2 \\ &= \epsilon \end{aligned} \tag{2.115b}$$

so $\hat{z}(0;\epsilon) - \hat{z}(0;0) = \epsilon < \dfrac{\epsilon}{1-\alpha}$, for $0 < \alpha < 1$, and, thus, equation (2.114) also holds at $y = 0$. Now, (2.114) is equivalent to

$$0 \le \left[\epsilon + \hat{w}^2(y;\epsilon)\right] - \hat{w}^2(y;0) \le \frac{\epsilon}{1-\alpha}, \quad y \in [-a,0]$$

or

$$-\epsilon \le \hat{w}^2(y;\epsilon) - \hat{w}^2(y;0) \le \frac{\alpha\epsilon}{1-\alpha}, \quad y \in [-a,0] \tag{2.116}$$

which, in turn, yields the two estimates

$$\hat{w}^2(y;0) - \epsilon \le \hat{w}^2(y;\epsilon), \tag{2.117a}$$

$$\hat{w}^2(y;\epsilon) \le \hat{w}^2(y;0) + \frac{\alpha\epsilon}{1-\alpha} \tag{2.117b}$$

on $[-a,0]$. Consider the set of all $y \in [-a,0]$ such that $\hat{w}(y;0) \ge \sqrt{\epsilon}$ for fixed $\epsilon > 0$; for y in this set it follows from (2.117b) that

$$0 \le \hat{w}^2(y;\epsilon) \le \hat{w}^2(y;0) + \frac{\alpha\epsilon}{1-\alpha}, \quad \begin{cases} y \in [-a,0] \\ \hat{w}(y;0) \ge \sqrt{\epsilon} \end{cases} \tag{2.118}$$

with the upper bound holding, of course, on all of $[-a,0]$. Therefore, for all $y \in [-a,0]$, such that $\hat{w}(y;0) \ge \sqrt{\epsilon}$,

$$0 \le \hat{w}(y;\epsilon) \le \hat{w}(y;0) + \sqrt{\frac{\alpha\epsilon}{1-\alpha}} \tag{2.119}$$

and we have used the fact that (2.104), with $p_1^* < 0$, implies that $\hat{w}(y;\epsilon) > 0$, $\forall\epsilon \ne 0$, $y \in [-a,0)$. Now, suppose that $y \in [-a,0]$ but $\hat{w}(y;0) < \sqrt{\epsilon}$; then by (2.117b) we have

$$\hat{w}^2(y;\epsilon) < \epsilon + \frac{\alpha\epsilon}{1-\alpha} = K_\alpha^2\epsilon \tag{2.120}$$

with $K_\alpha^2 = 1/(1-\alpha)$. Thus, if $y \in [-a,0]$ and $\hat{w}(y;0) < \sqrt{\epsilon}$ then

$$\hat{w}(y;\epsilon) < K_\alpha\sqrt{\epsilon} \tag{2.121}$$

and

$$\begin{aligned} |\hat{w}(y;\epsilon) - \hat{w}(y;0)| &\le \hat{w}(y;\epsilon) + \hat{w}(y;0) \\ &< (1+K_\alpha)\sqrt{\epsilon} \end{aligned}$$

or

$$\hat{w}(y;0) - (1+K_\alpha)\sqrt{\epsilon} < \hat{w}(y;\epsilon) < \hat{w}(y;0) + (1+K_\alpha)\sqrt{\epsilon} \tag{2.122}$$

for all $y \in [-a,0]$ such that $\hat{w}(y;0) < \sqrt{\epsilon}$. However,

$$K_\alpha = \sqrt{1 + \frac{\alpha}{1-\alpha}} > \sqrt{\frac{\alpha}{1-\alpha}}$$

so a comparison of (2.119) and (2.122) shows that (2.122) holds for all $y \in [-a,0]$. Using the definitions of $\hat{w}(y;\epsilon)$, $\hat{w}(y;0)$ we may rewrite (2.122) as

$$\left|u'(y;\epsilon,0) - u_0'(y)\right| < (1+K_\alpha)\sqrt{\epsilon}, \quad y \in [-a,0] \tag{2.123}$$

where $K_\alpha = \dfrac{1}{\sqrt{1-\alpha}}$. Replacing, y by $-y$, for $y \in [0,a]$, we see that (2.123) holds for all y, $-a \le y \le a$, as both $u'(y;\epsilon,0)$ and $u_0'(y)$ are odd functions on $[-a,a]$; this establishes the validity of the estimate (2.101) and concludes the proof of the lemma. □

Lemma 2.1 enables us to compare $u(y;\epsilon,0)$ with $u_0(y)$ on $[-a,a]$; our next set of lemmas are aimed at enabling us to compare $u(y;\epsilon,\mu_1)$ with $u(y;\epsilon,0)$, the first of these being stated as follows:

Lemma 2.2. *Let $u(y;\epsilon,\mu_1)$ be the unique classical solution of* (2.55a,b) *and set $t(y;\epsilon,\mu_1) = u'''(y;\epsilon,\mu_1)$. Then $\exists C_+, C_- > 0$, both independent of ϵ and μ, such that*

$$t(y;\epsilon,\mu_1) \leq C_+, \quad y \in [-a,0], \tag{2.124a}$$

$$t(y;\epsilon,\mu_1) \geq -C_-, \quad y \in [0,a]. \tag{2.124b}$$

Proof. We will establish only (2.124a), which is all that is needed in the sequel: the proof of (2.124b) follows in an entirely analogous fashion. We begin by recalling that $u(y;\epsilon,\mu)$ and $u(y;\epsilon,0)$ are, respectively, the solutions of the nonlinear ordinary differential equations

$$\left\{\left[\epsilon + w^2(y;\epsilon,\mu_1)\right]^{-\alpha/2} w(y;\epsilon,\mu_1)\right\}' - \mu_1^* w'''(y;\epsilon,\mu_1) = p_1^*, \tag{2.125a}$$

$$\left\{\left[\epsilon + \hat{w}^2(y;\epsilon)\right]^{-\alpha/2} \hat{w}(y;\epsilon)\right\}' = p_1^* \tag{2.125b}$$

subject to $u(\pm a;\epsilon,\mu_1) = u''(\pm a;\epsilon,\mu_1) = 0$ and $u(\pm a;\epsilon,0) = 0$, where $\mu_1^* = \mu_1/\mu_0$. Subtracting (2.125a) from (2.125b), and integrating with respect to y, we obtain

$$\frac{w(y;\epsilon,\mu)}{[\epsilon + w^2(y;\epsilon,\mu_1)]^{\alpha/2}} - \frac{\hat{w}(y;\epsilon)}{[\epsilon + \hat{w}^2(y;\epsilon)]^{\alpha/2}} = \mu_1^* w''(y;\epsilon,\mu_1). \tag{2.126}$$

For future reference we record here the following: first of all, as

$$u(-y;\epsilon,\mu_1) = u(y;\epsilon,\mu_1), \quad y \in [-a,a]$$

not only is $u'(y;\epsilon,\mu_1)$ an odd function on $[-a,a]$ but so is $u'''(y;\epsilon,\mu_1)$, while $u''(y;\epsilon,\mu_1)$ is an even function: in particular, $\forall \epsilon, \mu_1 \geq 0$, $u'''(0;\epsilon,\mu_1) = 0$. Next we observe that the use of definition (2.100) enables us to write (2.125a) in the form

$$\gamma\left[u'(y;\epsilon,\mu_1)\right]' - \mu_1^* u''''(y;\epsilon,\mu_1) = p_1^* \tag{2.127}$$

and that

$$\gamma'(w) = (\epsilon + w^2)^{-\alpha/2}\left[1 - \alpha w^2(\epsilon + w^2)^{-1}\right] \tag{2.128a}$$

$$\gamma''(w) = -w\left\{\frac{\alpha}{(\epsilon + w^2)^{(\alpha/2)+1}}\left[\frac{\epsilon + (1-\alpha)w^2}{\epsilon + w^2}\right] + \frac{2\alpha\epsilon}{(\epsilon + w^2)^{2+(\alpha/2)}}\right\} \tag{2.128b}$$

from which it follows, as $0 < \alpha < 1$, that $\gamma'(w) > 0$, $\forall \epsilon > 0$, while $\operatorname{sgn} \gamma''(w) = -\operatorname{sgn} w$. Now, from (2.127) with

$$s(y; \epsilon, \mu_1) = u''(y; \epsilon, \mu_1) \tag{2.129}$$

we have

$$\gamma'(u'(y; \epsilon, \mu_1))s(y; \epsilon, \mu_1) - \mu_1^* s''(y; \epsilon, \mu_1) = p_1^* < 0, \tag{2.130a}$$

$$s(-a; \epsilon, \mu_1) = s(a; \epsilon, \mu_1) = 0. \tag{2.130b}$$

Suppose that $s(y; \epsilon, \mu_1)$ takes a positive maximum at some $y_0 \in (-a, a)$, so that $s(y_0; \epsilon, \mu_1) > 0$. From (2.130a) we have

$$\gamma' \left[u'(y_0; \epsilon, \mu_1)\right] s(y_0; \epsilon, \mu_1) + |p_1^*| = \mu_1^* s''(y_0; \epsilon, \mu_1). \tag{2.131}$$

But $\gamma'(u'(y_0; \epsilon, \mu_1)) > 0$, while $s''(y_0; \epsilon, \mu_1) \leq 0$, if y_0 is interior to $[-a, a]$ and s has a maximum there. Thus, $s(y; \epsilon, \mu_1)$ cannot achieve a positive maximum at a point $y_0 \in (-a, a)$ and any positive maximum of $s(y; \epsilon, \mu_1)$ must, therefore, occur at $y = \pm a$. In view of the boundary conditions (2.130b), it follows that there is no positive maximum for $s(y; \epsilon, \mu_1)$ anywhere on $[-a, a]$; thus, if the maximum of $s(y; \epsilon, \mu_1)$ occurs at an interior point $y_0 \in (-a, a)$ we must have $s(y_0; \epsilon, \mu_1) < 0$ in which case

$$s(y; \epsilon, \mu_1) \leq s(y_0; \epsilon, \mu_1) < 0, \quad y \in (-a, a) \tag{2.132}$$

and the same result holds if the maximum occurs at $y = \pm a$, where s vanishes. By (2.131), $s(y; \epsilon, \mu_1)$ cannot have a zero maximum at interior point $y_0 \in (-a, a)$. Thus,

$$u''(y; \epsilon, \mu_1) < 0; \quad y \in (-a, a), \quad \epsilon, \mu_1 > 0 \tag{2.133}$$

which shows that the graph of $u(y; \epsilon, \mu_1)$ is concave (down) on $(-a, a)$. Now let $y \in (-a, \delta)$ for any $\delta \leq a$. Then

$$\int_{-a}^{y} u'''(\lambda; \epsilon, \mu_1)\, d\lambda = u''(y; \epsilon, \mu_1) < 0 \tag{2.134}$$

and as y may be chosen arbitrarily close to $-a$ (and $u'''(y; \epsilon, \mu_1)$ is continuous in y on $(-a, a)$) it follows that

$$u'''(-a; \epsilon, \mu_1) < 0 \tag{2.135a}$$

$$u'''(a; \epsilon, \mu_1) > 0 \tag{2.135b}$$

since $u'''(y;\epsilon,\mu_1)$ is an odd function of y on $(-a,a)$. From the definitions of $s(y;\epsilon,\mu_1)$ and $t(y;\epsilon,\mu_1)$,

$$t(y;\epsilon,\mu_1) = s'(y;\epsilon,\mu_1), \quad y \in (-a,a). \tag{2.136}$$

Therefore, if we differentiate (2.131) with respect to y we readily obtain

$$\gamma'[u'(y;\epsilon,\mu_1)]t(y;\epsilon,\mu_1) + \gamma''[u'(y;\epsilon,\mu_1)][u''(y;\epsilon,\mu_1)]^2 - \mu_1^* t''(y;\epsilon,\mu_1) = 0. \tag{2.137}$$

The calculation leading to (2.137) may be validated by the following elementary argument: in Sect. 2.2 it was demonstrated that the boundary-value problem given by (2.130a,b), subject to the additional constraint $u(\pm a,\epsilon,\mu_1) = 0$, has a unique classical solution, i.e. a solution in $C^4(-a,a)$; in light of this observation, and the definition of γ,

$$s''(y;\epsilon,\mu_1) = \frac{1}{\mu_1^*}\left\{|p_1^*| + \gamma'\left[u'(y;\epsilon,\mu_1)\right]s(y;\epsilon,\mu_1)\right\}$$

is continuously differentiable in y on $(-a,a)$ and (2.137) holds. Repetition of this argument shows that the unique classical solution of the boundary-value problem is, in fact, in $C^\alpha(-a,a)$.

We now return to (2.137) and assume that $t(y;\epsilon,\mu)$ achieves a positive maximum at $y_0 \in (-a,a)$ so that $t''(y_0;\epsilon,\mu_1) \le 0$; then, by (2.137) it must be true that

$$\gamma'[u'(y_0;\epsilon,\mu_1)]t(y_0;\epsilon,\mu_1) + \gamma''[u'(y_0;\epsilon,\mu_1)][u''[y_0;\epsilon,\mu_1)]^2 \le 0. \tag{2.138}$$

However, by (2.128b)

$$\operatorname{sgn}\gamma''[u'(y_0;\epsilon,\mu_1)] = -\operatorname{sgn}u'(y_0;\epsilon,\mu_1). \tag{2.139}$$

But $u''(y;\epsilon,\mu_1) < 0$, $y \in (-a,a)$, while $u'(0;\epsilon,\mu_1) = 0$, so $u'(y;\epsilon,\mu_1) < 0$ for $y \in (0,a)$. Thus, if $y_0 \in (0,a)$, then by (2.139) we must have $\gamma''[u'(y_0;\epsilon,\mu_1)] > 0$, contradicting (2.138). This means, of course, that if $t(y;\epsilon,\mu_1)$ achieves a positive maximum at $y_0 \in (-a,a)$ then, in fact, $y_0 \in (-a,0)$; note that $t(0;\epsilon,\mu) = 0$ as $u'''(y;\epsilon,\mu_1)$ is odd on $(-a,a)$. At such a $y_0 \in (-a,0)$ we will have, by virtue of (2.138),

$$t(y_0;\epsilon,\mu_1) \le \frac{-\gamma''[u'(y_0;\epsilon,\mu_1)][u''(y_0;\epsilon,\mu_1)]^2}{\gamma'[u'(y_0;\epsilon,\mu_1)]}. \tag{2.140}$$

Now, at y_0, $t'(y_0,;\epsilon,\mu_1) = s''(y_0;\epsilon,\mu_1) = 0$, in which case it follows from (2.130a) that

$$\gamma[u'(y_0;\epsilon,\mu_1)]u''(y_0;\epsilon,\mu_1) = p_1^* < 0 \tag{2.141}$$

so that

$$u''(y_0;\epsilon,\mu_1) = p_1^*/\gamma'[u'(y_0;\epsilon,\mu_1)]. \tag{2.142}$$

Substituting from (2.144) into (2.140), we obtain

$$t(y_0;\epsilon,\mu_1) \le \frac{-\gamma''[u'(y_0;\epsilon,\mu_1)]}{\gamma'^3[u'(y_0;\epsilon,\mu_1)]} p_1^{*2}. \tag{2.143}$$

However, by the pointwise bound established in Sect. 2.2, which precedes (2.46), we have – with $v_{\mu_1}(y) = u'(y;\epsilon,\mu_1)$ – the existence of $C_1 > 0$ such that

$$\max_{[-a,a]} \left|u'(y;\epsilon,\mu)\right| \le C_1. \tag{2.144}$$

It now follows from (2.143) and (2.135a) that $\exists C_+ > 0$, independent of both ϵ and μ_1, such that

$$t(y;\epsilon,\mu_1) \le t(y_0;\epsilon,\mu_1) \le C_+, \quad y \in [-a,0]. \tag{2.145}$$

If $t(y;\epsilon,\mu_1) \le 0$ on $(-a,0)$, so that no positive maximum exists on $[-a,0]$, then certainly (2.145) holds $\forall C_+ > 0$. An analogous argument, which begins with the assumption that $t(y;\epsilon,\mu_1)$ has a negative minimum on $(-a,a)$ can be used, as above, to establish the existence of a $C_- > 0$, independent of both ϵ and μ, such that

$$t(y;\epsilon,\mu_1) \ge -C_-, \quad y \in [0,a] \tag{2.146}$$

but we omit the details. □

The next lemma provides us with an upper bound for $u(y;\epsilon,\mu_1) - u(y;\epsilon,0)$; we have, specifically, the following result:

Lemma 2.3. *Let $u(y;\epsilon,\mu_1)$ be the unique classical solution of* (2.55a,b). *Then for all $\epsilon, \mu_1 > 0$, and all $y \in [-a,a]$,*

$$u(y;\epsilon,\mu_1) - u(y;\epsilon,0) \le \frac{aC_+}{1-\alpha}\left(\sqrt{\epsilon} + C_1\right)^{\alpha} \mu_1^* \tag{2.147}$$

where C_+, C_1, independent of both ϵ and μ, are the positive constants appearing, respectively, in (2.124a) *and* (2.144).

Proof. From (2.126) and the definitions of $\gamma(\cdot)$ and $t(y;\epsilon,\mu)$ we have

$$\gamma[w(y;\epsilon,\mu_1)] - \gamma[\hat{w}(y;\epsilon)] = \mu_1^* t(y;\epsilon,\mu_1). \tag{2.148}$$

However,

$$\gamma(w) - \gamma(\hat{w}) = \gamma'(\bar{w})(w - \hat{w}) \tag{2.149}$$

with

$$\hat{w}(y;\epsilon) \le \bar{w}(y;\epsilon,\mu_1) \le w(y;\epsilon,\mu_1)$$

for each fixed $y \in [-a, a]$, $\epsilon, \mu_1 > 0$. Thus,

$$w(y;\epsilon,\mu_1) - \hat{w}(y;\epsilon) = \frac{\mu_1^* t(y;\epsilon,\mu_1)}{\gamma'[\bar{w}(y;\epsilon,\mu_1)]} \tag{2.150}$$

From (2.128a),

$$\gamma'(w) = \frac{\epsilon + (1-\alpha)w^2}{(\epsilon + w^2)^{1+(\alpha/2)}}$$

so

$$\frac{1}{\gamma'(\bar{w})} = \frac{(\epsilon + \bar{w}^2)^{1+(\alpha/2)}}{\epsilon + (1-\alpha)\bar{w}^2} < \frac{(\epsilon + \bar{w}^2)^{1+(\alpha/2)}}{(1-\alpha)\epsilon + (1-\alpha)\bar{w}^2} = \frac{1}{1-\alpha}(\epsilon + \bar{w}^2)^{\alpha/2}$$
$$\le \frac{1}{1-\alpha}\left(\sqrt{\epsilon} + |\bar{w}|\right)^{\alpha} \le \frac{1}{1-\alpha}\left(\sqrt{\epsilon} + C_1\right)^{\alpha}$$

by virtue of (2.144) and (2.149). Employing this last estimate for $[\gamma'(\bar{w})]^{-1}$, as well as (2.124a), in (2.150) we are led to the upper bound

$$w(y;\epsilon,\mu_1) - \hat{w}(y;\epsilon) \le \frac{\mu_1^* C_+}{1-\alpha}\left(\sqrt{\epsilon} + C_1\right)^{\alpha}, \quad y \in [-a, 0). \tag{2.151}$$

Choosing $y \in (-a, 0]$, and integrating both sides of (2.151) from $-a$ to y, we have, in view of the definitions of $w(y;\epsilon,\mu_1)$ and $\hat{w}(y;\epsilon)$, and the fact that $u(-a;\epsilon,\mu_1) = 0$, for $\epsilon > 0$ and $\mu_1 \ge 0$,

$$u(y;\epsilon,\mu_1) - u(y;\epsilon,0) \le \frac{aC_+}{1-\alpha}\left(\sqrt{\epsilon} + C_1\right)^{\alpha}\mu_1^*, \quad y \in [-a, 0]. \tag{2.152}$$

However, $u(y;\epsilon,\mu_1) = u(-y;\epsilon,\mu_1)$, for $y \in [-a, a]$, so we see that the upper bound in (2.152) is, in fact, valid for all y, $-a \le y \le a$. □

The next to the last lemma in this sequence is

Lemma 2.4. *Let $u(y;\epsilon,\mu_1)$ be the unique classical solution of* (2.55a,b). *Then for $-a \le y \le a$,*

$$|u(y;\epsilon,\mu_1) - u(y;\epsilon,0)| \le \frac{\sqrt{aC_2}}{1-\alpha}\left(\sqrt{\epsilon}+C_1\right)^{\alpha} \mu_1^{*1/2} \tag{2.153}$$

with $C_1 > 0$ *the constant appearing in* (2.144)*, and* $C_2 > 0$ *the constant appearing in the estimate* (2.46)*.*

Proof. By virtue of (2.150), it follows that for $y \in [-a, 0]$,

$$\left|\int_{-a}^{y} [w(\lambda;\epsilon,\mu_1) - \hat{w}(\lambda;\epsilon)]\, d\lambda\right| = \mu_1^* \left|\int_{-a}^{y} \frac{t(\lambda;\epsilon,\mu_1)}{\gamma'(\bar{w}(\lambda;\epsilon,\mu_1)]}\, d\lambda\right| \tag{2.154}$$

from which we obtain

$$|u(y;\epsilon,\mu_1) - u(y;\epsilon,0)| \le \frac{\mu_1^*}{1-\alpha}\left(\sqrt{\epsilon}+C_1\right)^{\alpha} \int_{-a}^{y} |t(\lambda;\epsilon,\mu_1)|\, d\lambda \tag{2.155}$$

by again bounding $[\gamma'(\bar{w})]^{-1}$ from above and using the definitions of w and $\hat{w}$. However,

$$\begin{aligned}
\mu_1^* \int_{-a}^{y} |t|\, d\lambda &= \mu_1^{*1/2} \int_{-a}^{y} \mu_1^{*1/2} |t|\, d\lambda \\
&\le \mu_1^{*1/2} \left(\int_{-a}^{y} d\lambda\right)^{1/2} \left(\int_{-a}^{y} \mu_1^* |t|^2 d\lambda\right)^{1/2} \\
&\le \sqrt{a\mu_1^*} \left(\int_{-a}^{a} \mu_1^* |t|^2\, d\lambda\right)^{1/2}.
\end{aligned}$$

However, by virtue of the estimate (2.46) of Sect. 2.2, there exists a positive constant, independent of both ϵ and μ_1, which we will denote by C_2, such that

$$\int_{-a}^{a} \mu_1^* |t|^2\, d\lambda \le C_2$$

and we are led to the bound

$$\mu_1^* \int_{-a}^{y} |t|\, d\lambda \le (a\mu_1^* C_2)^{1/2}. \tag{2.156}$$

Use of the bound (2.156) in the estimate (2.155) now yields the estimate (2.153) for $y \in [-a, 0]$ and the fact that $u(y;\epsilon,\mu_1)$ is an even function of y on $[-a, a]$, for all $\epsilon, \mu_1 \ge 0$, then establishes the validity of the estimate in (2.153) for all y, $-a \le y \le a$. □

Our final lemma in this section is merely a synthesis of Lemmas 2.1–2.4, namely,

Lemma 2.5. *Let* $u(y;\epsilon,\mu_1)$ *be the unique classical solution of the problem* (2.55a,b)*. Then,* $\exists C_+, C_1, C_2$*, all positive and independent of both* ϵ *and* μ_1*, such* *that for all* y, $-a \le y \le a$,

$$\frac{-\sqrt{aC_2}}{1-\alpha}\left(\sqrt{\epsilon}+C_1\right)^{\alpha}\mu_1^{*1/2} \le u(y;\epsilon,\mu_1)-u(y;\epsilon,0) \le \frac{aC_+}{1-\alpha}\left(\sqrt{\epsilon}+C_1\right)^{\alpha}\mu_1^{*}. \tag{2.157}$$

We are now in a position to state and prove the basic result of this section, i.e., we have

Theorem 2.3. *If* $u(y;\epsilon,\mu_1)$ *is the unique classical solution of the boundary-value problem* (2.55a,b)*, while* $\mu_0(y)$ *is the corresponding solution of the boundary-value problem* (2.56a,b)*, then* $\exists C_+, C_1, C_2$*, all positive and independent of* ϵ *and* μ_1*, such that for all* y, $-a \le y \le a$*, we have, with* $K_\alpha = (1-\alpha)^{-1/2}$, $0 < \alpha < 1$,

$$-(1+K_\alpha)a\sqrt{\epsilon}-\frac{\sqrt{aC_2}}{1-\alpha}\left(\sqrt{\epsilon}+C_1\right)^{\alpha}\mu_1^{*1/2} \le u(y;\epsilon,\mu_1)-\mu_0(y) \le (1+K_\alpha)a\sqrt{\epsilon}+\frac{aC_+}{1-\alpha}\left(\sqrt{\epsilon}+C_1\right)^{\alpha}\mu_1^{*}. \tag{2.158}$$

Proof. By virtue of Lemma 2.1, we have

$$-(1+K_\alpha)\sqrt{\epsilon} < u'(y;\epsilon,0)-u_0'(y) < (1+K_\alpha)\sqrt{\epsilon} \tag{2.159}$$

for all $y \in [-a,a]$, where $K_\alpha = (1-\alpha)^{-1/2}$. By integrating (2.159) from $-a$ to y, for $y \in [-a,0)$, we find that

$$-(1+K_\alpha)a\sqrt{\epsilon} < u(y;\epsilon,0)-u_0(y) < (1+K_\alpha)a\sqrt{\epsilon} \tag{2.160}$$

with this last result holding for all $y \in [-a,a]$, as $u(y;\epsilon,0)$, $u_0(y)$ are both even functions of y. Since

$$u(y;\epsilon,\mu_1)-u_0(y) = [u(y;\epsilon,\mu_1)-u(y;\epsilon,0)]+[u(y;\epsilon,0)-u_0(y)]$$

the Theorem 2.3 now follows by combining (2.157) and (2.159). ☐

As a direct consequence of the estimates in Theorem 2.3, we have the following bounds for the difference of the net volume flows Q_{ϵ,μ_1} and Q_0, and the mean velocities $\bar{u}_{\epsilon,\mu_1}$ and $\bar{u}_0$:

Theorem 2.4. *Under the same conditions as those which prevail in Theorem 2.3 the difference* $\bar{u}_{\epsilon,\mu_1}-\bar{u}_0$ *of the mean velocities also satisfies the estimate* (2.158)*, while the difference* $Q_{\epsilon,\mu_1}-Q_0$ *of the volume flows satisfies*

$$-2(1+K_\alpha)a^2\sqrt{\epsilon}-\frac{2a^{3/2}C_2^{1/2}}{1-\alpha}\left(\sqrt{\epsilon}+C_1\right)^{\alpha}\mu_1^{*1/2} \le Q_{\epsilon,\mu_1}-Q_0 \le 2(1+K_\alpha)a^2\sqrt{\epsilon}+\frac{2a^2C_+}{1-\alpha}\left(\sqrt{\epsilon}+C_1\right)^{\alpha}\mu_1^{*}. \tag{2.161}$$

Remarks. The proof of Theorem 2.4 is a direct consequence of the estimates in (2.158) and the definitions of the mean velocities and volume flows. Moreover, in view of (2.94) and (2.161) we may exhibit the explicit bounds

$$Q_{\epsilon,\mu_1} \geq 2a\left(\frac{1-\alpha}{3-2\alpha}\right)\left[\frac{|p_1|a^{2-\alpha}}{\mu_0 2^{\alpha/2}}\right]^{1/(1-\alpha)} - 2(1+K_\alpha)a^2\sqrt{\epsilon} - \frac{2a^{3/2}C_2^{1/2}}{1-\alpha}\left(\sqrt{\epsilon}+C_1\right)^\alpha\left(\frac{\mu_1}{\mu_0}\right)^{1/2} \tag{2.162a}$$

and

$$Q_{\epsilon,\mu} \leq 2a\left(\frac{1-\alpha}{3-2\alpha}\right)\left[\frac{|p_1|a^{2-\alpha}}{\mu_0 2^{\alpha/2}}\right]^{1/(1-\alpha)} + 2(1+K_\alpha)a^2\sqrt{\epsilon} - \frac{2a^2C_+}{1-\alpha}\left(\sqrt{\epsilon}+C_1\right)^\alpha\left(\frac{\mu_1}{\mu_0}\right). \tag{2.162b}$$

Similar estimates may be developed for the friction factor f_{ϵ,μ_1} in (2.69a), by employing the bounds for $\bar{u}_{\epsilon,\mu_1}$, if we observe that (2.55a) is equivalent to

$$\frac{\partial}{\partial y}\tau_{12}(y;\epsilon,\mu_1) = p_1 \tag{2.163}$$

so that for all $\epsilon, \mu_1 \geq 0$, $\tau_{12}(\pm a;\epsilon,\mu_1) = \pm|p_1|a$.

2.4 Uniqueness of Steady Poiseuille Flow in the Class of Equilibrium Flows Between Parallel Plates

2.4.1 Introduction

In Sect. 2.2 we considered the problem of existence and uniqueness for steady Poiseuille flow of an incompressible, bipolar, viscous fluid in a parallel-wall channel. In rectangular Cartesian coordinates the flow assumes the form $\boldsymbol{v}^p = (u(y), 0, 0)$ and satisfies the nonlinear boundary-value problem (2.55a,b) where the channel walls are located at $y = \pm a$. The existence of a unique solution u of (2.55a,b) was established in the set W_M, for $M > 0$ sufficiently large, W_M as given by (2.12), with $B_M(0)$ the ball of radius M in $H_0^{\frac{3}{2}+\delta}(-a,a)$, $0 < \delta < \frac{1}{2}$. In this section we will consider the broader problem of uniqueness for steady, bipolar, viscous flows in the domain Ω_a specified by (2.8). It will be more convenient in this section to return to the subscript notation for coordinates and vector components. Therefore, we will write for Ω_a,

$$\Omega_a = \{(x_1,x_2,x_3) \mid x_2 \in [-a,a],\ -\infty < x_1, x_3 < \infty\}. \tag{2.164}$$

In the domain Ω_a, a steady, bipolar, viscous flow $\boldsymbol{v}$ (without an external forcing function) will satisfy

$$\boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\nabla p + \nabla \cdot (2\mu \boldsymbol{e}) - 2\mu_1 \nabla \cdot (\Delta \boldsymbol{e}) \tag{2.165a}$$

$$\operatorname{div} \boldsymbol{v} = \mathbf{0} \tag{2.165b}$$

where $\mu = \mu(|\boldsymbol{e}|)$ represents the nonlinear viscosity

$$\mu = \mu_0(\epsilon + e_{kl}e_{kl})^{-\alpha/2}. \tag{2.165c}$$

In addition to (2.165a,b), $\boldsymbol{v}$ must satisfy, on $\partial\Omega_a$,

$$\boldsymbol{v}(x_1, \pm a, x_3) = \mathbf{0}, \quad \tau_{i22}(\boldsymbol{v})\big|_{x_2=\pm a} = 0, \quad i = 1, 2, 3 \tag{2.166}$$

where the higher-order boundary conditions in (2.166) follow from the stipulation that

$$\tau_{ijk}\nu_j\nu_k - \tau_{jkl}\nu_j\nu_k\nu_l\nu_i = 0, \quad i = 1, 2, 3$$

on $\partial\Omega_a$, coupled with the fact that the exterior unit normal is given by $\boldsymbol{\nu} = (0, \pm 1, 0)$. For the steady Poiseuille velocity field in Ω_a we will write

$$\boldsymbol{v}^p = (u(x_2; \epsilon, \mu_1), 0, 0), \quad -a \le x_2 \le a \tag{2.167}$$

where $u(x_2; \epsilon, \mu_1) \equiv u(x_2)$ satisfies

$$\mu_0 \left[\left(\epsilon + \frac{1}{2} u'^2(x_2) \right)^{-\alpha/2} u'(x_2) \right]' - \mu_1 u''''(x_2) = p_1, \quad -a < x_2 < a, \tag{2.168a}$$

$$u(\pm a) = u''(\pm a) = 0 \tag{2.168b}$$

with $p_1 = \dfrac{\partial p}{\partial x_2}$ the constant pressure gradient. At this point, we know that there exists at least one solution of the nonlinear boundary-value problem (2.165a,b,c), (2.166), such that

$$\boldsymbol{v} - \boldsymbol{v}^p \in \boldsymbol{H}^4(\Omega_a) \tag{2.169}$$

namely, $\boldsymbol{v} = \boldsymbol{v}^p$.

Our goal in this section will be to show that, under specific restrictions on the constitutive parameters ϵ, μ_0, μ_1, and α, the plate separation $2a$, and the constant pressure gradient p_1 associated with the problem (2.168a,b), $\boldsymbol{v} = \boldsymbol{v}^p$ is the unique solution of the boundary-value problem (2.165a,b,c), (2.166) in the domain Ω_a

which satisfies the regularity condition (2.169); to this end we set $\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{v}^p$ and examine some of the consequences of the condition $\boldsymbol{w} \in \boldsymbol{H}^4(\Omega_a)$ in the next subsection.

2.4.2 The Condition $\boldsymbol{w} \in \boldsymbol{H}^4(\boldsymbol{\Omega}_a)$

In Sect. 2.4.1 we set $\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{v}^p$ where $\boldsymbol{v}$ and $\boldsymbol{v}^p$ are, respectively, (1) any solution of (2.165a,b,c), (2.166) and (2) the vector field (2.167) which is determined by the unique, classical, solution of (2.168a,b). In this subsection we use the conventional notation

$$\boldsymbol{D}^{\boldsymbol{\beta}}\boldsymbol{w} = \frac{\partial^{|\boldsymbol{\beta}|}\boldsymbol{w}}{\partial^{\beta_1}x_1\partial^{\beta_2}x_2\partial^{\beta_3}x_3}$$

where $\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3)$, $\beta_i \geq 0$, $i = 1, 2, 3$, and $|\boldsymbol{\beta}| = \beta_1 + \beta_2 + \beta_3$. The condition $\boldsymbol{w} \in \boldsymbol{H}^4(\Omega_a)$ then reads

$$\sum_{|\boldsymbol{\beta}|\leq 4} \int_{\Omega_a} \left|\left|\boldsymbol{D}^{\boldsymbol{\beta}}\boldsymbol{w}\right|\right|^2 d\boldsymbol{x} < \infty$$

which is equivalent to

$$\sum_{i=1}^{3}\sum_{|\boldsymbol{\beta}|\leq 4} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\int_{-a}^{a}\left|\boldsymbol{D}^{\boldsymbol{\beta}}w_i(x_1, x_2, x_3)\right|^2 dx_2\right) dx_1\, dx_3 < \infty \tag{2.170}$$

Setting

$$W_i^{\boldsymbol{\beta}}(x_1, x_3) = \int_{-a}^{a}\left|\boldsymbol{D}^{\boldsymbol{\beta}}w_i(x_1, x_2, x_3)\right|^2 dx_2 \tag{2.171}$$

we may rewrite (2.170) in the form

$$\sum_{i=1}^{3}\sum_{|\boldsymbol{\beta}|\leq 4} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_i^{\boldsymbol{\beta}}(x_1, x_3)dx_1\, dx_3 < \infty. \tag{2.172}$$

The finiteness of the integrals displayed in (2.172) now implies (see Fig. 2.2) that there exists a sequence $\{S_n\}$ of smooth surfaces in x_1, x_2, x_3 space intersecting, for each n, the planes at $x_2 = \pm a$, in a sufficiently smooth fashion, along which $x_1^2 + x_3^2 \to \infty$, as $n \to \infty$, and such that

$$W_i^{\boldsymbol{\beta}}(x_1, x_3)\big|_{\bar{S}_n} \to 0, \text{ as } n \to \infty \tag{2.173}$$

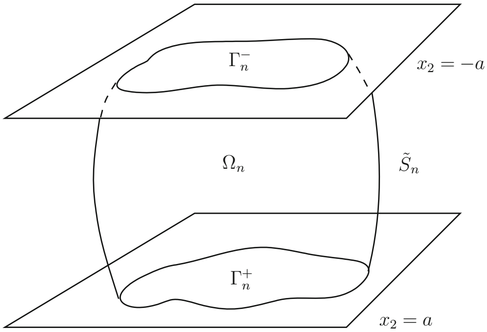

Fig. 2.2 The domain Ω_n in Lemma 2.6

for $|\boldsymbol{\beta}| \leq 4$, $1 \leq i \leq 3$, where $\bar{S}_n$ is the set of all points (x_1, x_2, x_3) lying on S_n such that $-a < x_2 < a$. If we define $\Gamma_n^{\pm}$ to be the sets of points lying on the planes $x_2 = \pm a$ which are bounded, respectively, by the curves of intersection of S_n with the planes at $x_2 = \pm a$, then by the terminology "sufficiently smooth", in the definition of S_n, we mean simply that the bounded domain $\Omega_n \subseteq R^3$ with boundary

$$\partial\Omega_n = \bar{S}_n \cup \Gamma_n^+ \cup \Gamma_n^- \tag{2.174}$$

admits of application of the divergence theorem. By virtue of the definition (2.171) of $W_i^{\boldsymbol{\beta}}$, the criterion (2.173), and the Sobolev embedding lemma (see Appendix A), it follows that for $|\boldsymbol{\beta}| \leq 3$,

$$\max_{[-a,a]} \left| \boldsymbol{D}^{\boldsymbol{\beta}} w_i(x_1, x_2, x_3) \right|_{\bar{S}_n} \to 0, \text{ as } n \to \infty \tag{2.175}$$

for each $i = 1, 2, 3$. The radiation conditions expressed by (2.175) are the principal consequences of the restriction $\boldsymbol{w} \in \boldsymbol{H}^4(\Omega_a)$, which will be of use to us in the following subsections; among the results which follow from this restriction is the following lemma of Poincaré type:

Lemma 2.6. *Let* $\boldsymbol{w} \in \boldsymbol{H}^4(\Omega_a)$ *with* $\boldsymbol{w}(\pm a) = 0$; *then,*

$$\int_{\Omega_a} ||\boldsymbol{w}||^2 d\boldsymbol{x} \leq (2a^2 + \theta)^2 \int_{\Omega_a} \left|\left|\nabla^2 \boldsymbol{w}\right|\right|^2 d\boldsymbol{x} \tag{2.176}$$

for any $\theta > 0$, *where* $||\cdot||$ *is the Euclidean norm on* $\mathbb{R}^3$.

Proof. We set $\boldsymbol{w} = (w_1, w_2, w_3)$ and let w represent any of the w_i, $i = 1, 2$, or 3. As

$$w(x_1, x_2, x_3) = \int_{-a}^{x_2} \frac{\partial w}{\partial x_2}(x_1, \tau, x_3)\, d\tau$$

we have

$$w^2(x_1, x_2, x_3) \leq 2a \int_{-a}^{a} \left(\frac{\partial w}{\partial x_2} \right)^2 dx_2$$

and, therefore,

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-a}^{a} w^2(x_1, x_2, x_3) dx_2\, dx_1\, dx_2 &\leq 4a^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-a}^{a} \left(\frac{\partial w}{\partial x_2} \right)^2 dx_2\, dx_1\, dx_3 \\ &\leq 4a^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-a}^{a} ||\nabla w||^2 dx_2\, dx_1\, dx_3. \end{aligned} \tag{2.177}$$

Now we consider $\int_{\Omega_a} ||\nabla w||^2 \, d\boldsymbol{x}$ where Ω_n is the bounded domain in $\mathbb{R}^3$ which is bounded by the smooth surface $\partial\Omega_n$ of (2.174); we compute that

$$\begin{aligned} \int_{\Omega_n} ||\nabla w||^2 \, d\boldsymbol{x} &= \int_{\Omega_n} \frac{\partial w}{\partial x_i} \frac{\partial w}{\partial x_i} \, d\boldsymbol{x} = \int_{\Omega_n} \frac{\partial}{\partial x_i} \left(w \frac{\partial w}{\partial x_i} \right) d\boldsymbol{x} - \int_{\Omega_n} w \nabla^2 w \, d\boldsymbol{x} \\ &= \int_{\partial\Omega_n} w \frac{\partial w}{\partial x_i} \nu_{i(n)} d\sigma_n - \int_{\Omega_n} w \nabla^2 w \, d\boldsymbol{x} \\ &= - \int_{\Gamma_n^+} w \frac{\partial w}{\partial x_1} d\boldsymbol{a}^+ + \int_{\Gamma_n^-} w \frac{\partial w}{\partial x_1} d\boldsymbol{a}^- \\ &\quad + \int_{\tilde{S}_n} w \frac{\partial w}{\partial x_i} \tilde{\nu}_{i(n)} d\tilde{\sigma}_n - \int_{\Omega_n} w \nabla^2 w \, d\boldsymbol{x} \end{aligned} \tag{2.178}$$

where $\boldsymbol{\nu}(n)$ is the exterior unit normal to $\partial\Omega_n$, $\tilde{\boldsymbol{\nu}}_{(n)}$ the exterior unit normal to $\tilde{S}_n$, $d\boldsymbol{a}^{\pm}$ the infinitesimal surface elements in the domains $\Gamma_n^{\pm}$, located in the planes at $x_2 = \pm a$, $d\sigma_n$ the infinitesimal surface element on $\partial\Omega_n$, and $d\tilde{\sigma}_n$ the infinitesimal surface element on $\tilde{S}_n$. In as much as $w(\pm a) = 0$, w vanishes on $\Gamma_n^{\pm}$, for each n, and (2.178) reduces to

$$\int_{\Omega_n} ||\nabla w||^2 d\boldsymbol{x} = \int_{\tilde{S}_n} w \frac{\partial w}{\partial x_i} \tilde{\nu}_{i(n)} d\tilde{\sigma}_n - \int_{\Omega_n} w \nabla^2 w \, d\boldsymbol{x}. \tag{2.179}$$

Letting $n \to \infty$ in (2.179), and employing the radiation condition (2.175), we obtain

$$\int_{\Omega_a} ||\nabla w||^2 \, d\boldsymbol{x} = \int_{\Omega_a} (-w) \nabla^2 w \, d\boldsymbol{x} \leq \frac{1}{2\lambda} \int_{\Omega_a} w^2 d\boldsymbol{x} + \frac{\lambda}{2} \int_{\Omega_a} (\nabla^2 w)^2 \, d\boldsymbol{x} \tag{2.180}$$

for any $\lambda > 0$. Combining (2.180) with (2.177) we have, therefore, the estimate

$$\frac{1}{4a^2} \int_{\Omega_a} w^2 \, d\boldsymbol{x} \leq \frac{1}{2\lambda} \int_{\Omega_a} w^2 \, d\boldsymbol{x} + \frac{\lambda}{2} \int_{\Omega} (\nabla^2 w)^2 \, d\boldsymbol{x}$$

or, for any $\lambda > 2a^2$,

$$\int_{\Omega_a} w^2\, d\boldsymbol{x} \le \frac{\lambda}{2\left(\frac{1}{4a^2} - \frac{1}{2\lambda}\right)} \int_{\Omega_a} (\nabla^2 w)^2\, d\boldsymbol{x}. \tag{2.181}$$

The lemma now follows if we write down (2.181) for each of w_1, w_2, and w_3, sum the resulting three estimates, and take $\lambda = 2a^2 + \theta$, with θ any positive constant.

□

2.4.3 Key Lemmas for Nonlinear Viscosity and the Poiseuille Flow in Ω_a

Prior to stating and proving the uniqueness theorem for steady channel flow of a bipolar, viscous fluid in Sect. 2.4.4, we will establish, in this subsection, two key lemmas; one of these relates directly to the structure of the nonlinear viscosity (2.165c) while the other is based on the structure of the Poiseuille flow field in Ω_a. The first result is the following inequality for vectors in $\mathbb{R}^n$:

Lemma 2.7. *Let* $\boldsymbol{u} = (u_1, \ldots, u_n)$*,* $\boldsymbol{v} = (v_1, \ldots, v_n)$*, and suppose that* $\epsilon > 0$ *and* $0 < \alpha < 1$*; then*

$$\sum_{i=1}^{n} \left(\frac{u_i}{\left(\epsilon + ||\boldsymbol{u}||^2\right)^{\alpha/2}} - \frac{v_i}{\left(\epsilon + ||\boldsymbol{v}||^2\right)^{\alpha/2}} \right) (u_i - v_i) \ge 0 \tag{2.182}$$

Proof. We let σ stand for the sum on the left-hand side of the inequality in (2.182); then,

$$\begin{aligned}
\sigma &= \frac{||\boldsymbol{u}||^2}{\left(\epsilon + ||\boldsymbol{u}||^2\right)^{\alpha/2}} + \frac{||\boldsymbol{v}||^2}{\left(\epsilon + ||\boldsymbol{v}||^2\right)^{\alpha/2}} - \boldsymbol{u}\cdot\boldsymbol{v} \left[\frac{1}{\left(\epsilon + ||\boldsymbol{u}||^2\right)^{\alpha/2}} + \frac{1}{\left(\epsilon + ||\boldsymbol{v}||^2\right)^{\alpha/2}} \right] \\
&\ge \frac{||\boldsymbol{u}||^2}{\left(\epsilon + ||\boldsymbol{u}||^2\right)^{\alpha/2}} + \frac{||\boldsymbol{v}||^2}{\left(\epsilon + ||\boldsymbol{v}||^2\right)^{\alpha/2}} - ||\boldsymbol{u}||||\boldsymbol{v}|| \left[\frac{1}{\left(\epsilon + ||\boldsymbol{u}||^2\right)^{\alpha/2}} + \frac{1}{\left(\epsilon + ||\boldsymbol{v}||^2\right)^{\alpha/2}} \right] \\
&= \left(\frac{||\boldsymbol{u}||}{\left(\epsilon + ||\boldsymbol{u}||^2\right)^{\alpha/2}} - \frac{||\boldsymbol{v}||}{\left(\epsilon + ||\boldsymbol{v}||^2\right)^{\alpha/2}} \right) (||\boldsymbol{u}|| - ||\boldsymbol{v}||)\,.
\end{aligned} \tag{2.183}$$

We now set, for $s \in R^1$,

$$\gamma(s) = s(\epsilon + s^2)^{-\alpha/2}, \quad 0 < \alpha < 1 \tag{2.184a}$$

then

$$\gamma'(s) = (\epsilon + s^2)^{-(\alpha/2+1)}(\epsilon + (1-\alpha)s^2) \tag{2.184b}$$

so that $\gamma'(s) > 0$, in which case for $s_1 \geq s_2$,

$$(\gamma(s_1) - \gamma(s)2))(s_1 - s_2) \geq 0$$

and the Lemma 2.7 follows from the last inequality in (2.183). □

The second lemma in this subsection depends on the qualitative behavior of solutions to the nonlinear boundary-value problem (2.168a,b) as well as on the structure of the solutions to the associated non-Newtonian problem (2.56a,b), which we rewrite here as

$$\mu_0 \left[\left(\frac{1}{2} u_0'(x_2) \right)^{-\alpha/2} u_0'(x_2) \right]' = p_1, \quad -a < x_2 < a \tag{2.185a}$$

$$u_0(\pm a) = 0. \tag{2.185b}$$

Lemma 2.8. *Let $\boldsymbol{v}^p$ be defined by (2.167) with $u(x_2; \epsilon, \mu_1)$, $-a \leq x_2 \leq a$, the unique solution of (2.168a,b). Let $\boldsymbol{e}^p = \frac{1}{2}(\nabla \boldsymbol{v}^p + (\nabla \boldsymbol{v}^p)^t)$ be the associated rate of deformation tensor. Then, for any vector field $\boldsymbol{w}(\cdot) \in \boldsymbol{L}^2(\Omega_a)$,*

$$-\int_{\Omega_a} \boldsymbol{w} \cdot \boldsymbol{e}^p \cdot \boldsymbol{w}\, d\boldsymbol{x} \leq \Gamma \int_{\Omega_a} ||\boldsymbol{w}||^2\, d\boldsymbol{x} \tag{2.186}$$

where

$$\Gamma = \Gamma(a, p_1; \mu_0, \mu_1, \alpha, \epsilon) \equiv a \left(\frac{|p_1|a}{\mu_0 2^{\alpha/2}} \right)^{1/(1-\alpha)} + \frac{1}{2}\left(1 + \frac{1}{\sqrt{1-\alpha}}\right)\sqrt{\epsilon}. \tag{2.187}$$

Proof. We let $\boldsymbol{x} \in \Omega_a$ and refer the rate of deformation tensor $\boldsymbol{e}^p$ to its principal axes at $\boldsymbol{x}$. Then at $\boldsymbol{x}$

$$\boldsymbol{w} \cdot \boldsymbol{e}^p \cdot \boldsymbol{w} = \boldsymbol{e}^p_{ij} w_i w_j \geq ||\boldsymbol{w}||^2 \cdot \min[e^p_{11}, e^p_{22}, e^p_{33}] \tag{2.188}$$

where the e^p_{ii}, $i = 1, 2, 3$, are the eigenvalues of $\boldsymbol{e}^p$. As

$$\operatorname{div} \boldsymbol{v}^p = \operatorname{tr} \boldsymbol{e}^p = e^p_{11} + e^p_{22} + e^p_{33} = 0 \tag{2.189}$$

at least one of the eigenvalues of $\boldsymbol{e}^p$ must be negative at $\boldsymbol{x}$. We denote the largest negative eigenvalue of $\boldsymbol{e}^p$ at $\boldsymbol{x} \in \Omega_a$ by $-|\lambda_{\boldsymbol{e}^p}(\boldsymbol{x})|$ so that, at $\boldsymbol{x}$,

$$\boldsymbol{w} \cdot \boldsymbol{e}^p \cdot \boldsymbol{w} \geq -|\lambda_{\boldsymbol{e}^p}(\boldsymbol{x})| ||\boldsymbol{w}||^2, \quad \boldsymbol{x} \in \Omega_a. \tag{2.190}$$

However, $\boldsymbol{v}^p$ and, hence, $\boldsymbol{e}^p$ and $\lambda_{\boldsymbol{e}^p}$ depend only on x_2 (and not on either x_1 or x_3) so $\lambda_{\boldsymbol{e}^p}(\boldsymbol{x}) = \bar{\lambda}_{\boldsymbol{e}^p}(x_2)$, $-a \le x_2 \le a$. Setting, therefore

$$|\lambda| = \max_{-a \le x_2 \le a} \left|\bar{\lambda}_{\boldsymbol{e}^p}(x_2)\right| \tag{2.191}$$

we have

$$\boldsymbol{w}(\boldsymbol{x}) \cdot \boldsymbol{e}^p(\boldsymbol{x}) \cdot \boldsymbol{w}(\boldsymbol{x}) \ge -|\lambda| ||\boldsymbol{w}(\boldsymbol{x})||^2, \quad \boldsymbol{x} \in \Omega_a \tag{2.192}$$

in which case

$$-\int_{\Omega_a} \boldsymbol{w} \cdot \boldsymbol{e}^p \cdot \boldsymbol{w}\, d\boldsymbol{x} \le |\lambda| \int_{\Omega_a} ||\boldsymbol{w}(x)||^2\, d\boldsymbol{x}. \tag{2.193}$$

Now, consider the tensor field

$$\nabla \boldsymbol{v}^p = \begin{pmatrix} 0 & u'(x_2;\epsilon,\mu_1) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which yields the rate of deformation tensor

$$\boldsymbol{e}^p = \frac{1}{2} \begin{pmatrix} 0 & u'(x_2;\epsilon,\mu_1) & 0 \\ u'(x_2;\epsilon,\mu_1) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.194}$$

Therefore,

$$\det(\boldsymbol{e}^p - \eta \boldsymbol{I}) = -\eta \left(\eta^2 - \frac{1}{4} u'^2(x_2;\epsilon,\mu_1) \right) \tag{2.195}$$

so that the eigenvalues of $\boldsymbol{e}^p$ at any $\boldsymbol{x} \in \Omega_a$ are given by $\eta_1 = 0$, $\eta_{2,3} = \pm u'(x_2;\epsilon,\mu_1)$. We now avail ourselves of the results in Sect. 2.3.3. As $u'(x_2;\epsilon,\mu_1) > 0$, $-a < x_2 < 0$, and $u'(x_2;\epsilon,\mu_1) < 0$, $0 < x_2 < a$,

$$\begin{aligned} |\lambda| &= \frac{1}{2} \max_{[-a,a]} \left|u'(x_2;\epsilon,\mu_1)\right| \\ &= \frac{1}{2} \max_{[-a,a]} u'(x_2,\epsilon,\mu_1). \end{aligned} \tag{2.196}$$

But, $u''(x_2;\epsilon,\mu_1) < 0$, for $x_2 \in (-a,a)$, $\epsilon,\mu_1 > 0$ so, in fact,

$$|\lambda| = \frac{1}{2} u'(-a;\epsilon,\mu_1). \tag{2.197}$$

Now, if we define, once again, $\gamma(s)$, $s \in R^1$, by (2.184a), then (2.168a) may be written in the form

$$\mu_0 \gamma(u')' - \mu_1 u'''' = p_1. \tag{2.198}$$

Writing (2.198) down for the cases $\mu_1 > 0$, and $\mu_1 = 0$, subtracting the resulting equations, and then integrating with respect to x_2, we find that

$$\gamma(u'(x_2;\epsilon,\mu_1)) - \gamma(u'(x_2;\epsilon,0)) = \mu_1^* u'''(x_2;\epsilon,\mu_1) \tag{2.199}$$

where $\mu_1^* = (\mu_1/\mu_0)$. Setting $x_2 = -a$ in (2.199), and using the fact that $u'''(-a;\epsilon,\mu_1) < 0$, (all derivatives at $x_2 = -a$ are, of course, the usual right-handed derivatives) we find that

$$\gamma(u'(-a;\epsilon,\mu_1)) < \gamma(u'(-a;\epsilon,0)); \quad \epsilon,\mu_1 > 0. \tag{2.200}$$

In view of (2.184b), $\gamma'(s) > 0$, $\forall s \in R^1$, so it follows from (2.200) that

$$u'(-a;\epsilon,\mu_1) < u'(-a;\epsilon,0), \quad \epsilon,\mu_1 > 0 \tag{2.201}$$

and, thus, by (2.197)

$$|\lambda| < \frac{1}{2}u'(-a;\epsilon,0), \quad \epsilon,\mu_1 > 0. \tag{2.202}$$

But, in light of (2.123), $\forall x_2 \in [-a,0]$ and $\epsilon > 0$,

$$u'(x_2;\epsilon,0) < u_0'(x_2) + \left(1 + \frac{1}{\sqrt{1-\alpha}}\right)\sqrt{\epsilon}. \tag{2.203}$$

A direct calculation based on (2.76) and (2.77), with y replaced by x_2, produces

$$u_0'(-a) = a\left(\frac{|p_1|a}{u_0 2^{\alpha/2}}\right)^{1/(1-\alpha)}, \quad 0 < \alpha < 1 \tag{2.204}$$

so that, by virtue of (2.202)–(2.204),

$$|\lambda| < \frac{a}{2}\left(\frac{|p_1|a}{\mu_0^0 2^{\alpha/2}}\right)^{1/(1-\alpha)} + \frac{1}{2}\left(1 + \frac{1}{\sqrt{1-\alpha}}\right)\sqrt{\epsilon}. \tag{2.205}$$

The desired conclusion of Lemma 2.8, namely, the estimate (2.186), with Γ given by (2.181), now follows directly from (2.193) and (2.205). □

2.4.4 Uniqueness of Solutions

Having established Lemmas 2.6–2.8, in the previous subsections, we are now in a position to state and prove the main result of this section, namely,

Theorem 2.5. *For $\mu_1 > 0$ sufficiently large $\boldsymbol{v}^p$, as defined by* (2.167)*,* (2.168a,b)*, is the unique solution of the nonlinear boundary-value problem* (2.165a,b)*,* (2.166) *satisfying* (2.169)*.*

Proof. For simplicity we will begin by replacing 2μ and $2\mu_1$, respectively, in (2.165a) by μ and μ_1 where $\mu = \mu(\boldsymbol{v})$ is given by (2.165c). We already know that $\boldsymbol{v}^p$ is a solution of (2.165a,b), (2.166) which obviously, satisfies (2.169); suppose that $\boldsymbol{v}(\boldsymbol{x})$ is any other solution. Then

$$\begin{aligned} \boldsymbol{v}\cdot\nabla\boldsymbol{v} &= -\nabla p + \nabla\cdot(\mu\boldsymbol{e}) - \mu_1\nabla\cdot(\Delta\boldsymbol{e}),\\ \boldsymbol{v}^p\cdot\nabla\boldsymbol{v}^p &= -\nabla p^p + \nabla\cdot(\mu_p\boldsymbol{e}^p) - \mu_1\nabla\cdot(\Delta\boldsymbol{e}^p) \end{aligned} \tag{2.206}$$

where $\boldsymbol{e}^p = \dfrac{1}{2}(\nabla\boldsymbol{v}^p + (\nabla\boldsymbol{v}^p)^t)$, and $\mu_p = \mu(\boldsymbol{e}^p)$, so that $\mu_p = \mu_0(\epsilon + e^p_{kl}e^p_{kl})^{-\alpha/2}$; also $\nabla p^p = (p_1, 0, 0)$. Of course $\boldsymbol{v}^p\cdot\nabla\boldsymbol{v}^p = 0$, and the second equation in (2.206) just reduces to (2.168a), but we shall find it convenient for our present purposes to leave it in the form in which we have written it. If we subtract the second equation in (2.206) from the first, and set

$$\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{v}^p \text{ and } P = p - p^p \tag{2.207}$$

then we obtain

$$(\boldsymbol{v}^p+\boldsymbol{w})\cdot\nabla(\boldsymbol{v}^p+\boldsymbol{w}) - \boldsymbol{v}^p\cdot\nabla\boldsymbol{v}^p = -\nabla P + \nabla\cdot(\mu\boldsymbol{e} - \mu_p\boldsymbol{e}^p) - \mu_1\nabla\cdot\Delta(\boldsymbol{e} - \boldsymbol{e}^p) \tag{2.208}$$

the above result holding throughout Ω_a. Expanding the left-hand side of (2.208), and then integrating the resulting expression over Ω_n (refer to Fig. 2.2), we obtain

$$\begin{aligned} \int_{\Omega_n} \boldsymbol{v}^p\cdot\nabla\boldsymbol{w}\cdot\boldsymbol{w}\,d\boldsymbol{x} + \int_{\Omega_n} \boldsymbol{w}\cdot\nabla\boldsymbol{w}\cdot\boldsymbol{w}\,d\boldsymbol{x} + \int_{\Omega_n} \boldsymbol{w}\cdot\nabla\boldsymbol{v}^p\cdot\boldsymbol{w}\,d\boldsymbol{x} &= -\int_{\Omega_n} \nabla p\cdot\boldsymbol{w}\,d\boldsymbol{x}\\ + \int_{\Omega_n} \nabla\cdot(\mu\boldsymbol{e} - \mu_0\boldsymbol{e}^p)\cdot\boldsymbol{w}\,d\boldsymbol{x} - \mu_1\int_{\Omega_n} \nabla\cdot\Delta(\boldsymbol{e} - \boldsymbol{e}^p)\cdot\boldsymbol{w}\,d\boldsymbol{x}. \end{aligned} \tag{2.209}$$

We now proceed to study each of the integrals over Ω_n, which are displayed in (2.209), in the limit as $n \to \infty$. First of all,

$$\begin{aligned} \int_{\Omega_n} \boldsymbol{v}^p\cdot\nabla\boldsymbol{w}\cdot\boldsymbol{w}\,d\boldsymbol{x} &= \int_{\Omega_n} v^p_j \frac{\partial w_i}{\partial x_j} w_i\,d\boldsymbol{x} = \frac{1}{2}\int_{\Omega_n} v^p_i \frac{\partial}{\partial x_j}(w_i\,w_i)\,d\boldsymbol{x}\\ &= \frac{1}{2}\int_{\partial\Omega_n} v^p_j w_i w_i \nu_{j(n)} d\sigma_n - \frac{1}{2}\int_{\Omega_n} w_i w_i \frac{\partial v^p_j}{\partial x_j}\,d\boldsymbol{x}. \end{aligned}$$

But div $\boldsymbol{v}^p = 0$, while $\boldsymbol{w}$ vanishes on both $\Gamma_n^\pm$, so

$$\int_{\Omega_n} \boldsymbol{v}^p \cdot \nabla \boldsymbol{w} \cdot \boldsymbol{w}\, d\boldsymbol{x} = \frac{1}{2}\int_{\tilde{S}_n} v_j^p w_i w_i \tilde{\nu}_{j(n)}\, d\tilde{\sigma}_n. \tag{2.210}$$

As $\boldsymbol{w} \in \boldsymbol{H}^4(\Omega_a)$ by (2.169), it follows from (2.175) that $w_i\big|_{\tilde{S}_n} \to 0$, as $n \to \infty$, uniformly for $-a \le x_2 \le a$. Thus

$$\int_{\Omega_n} \boldsymbol{v}^p \cdot \nabla \boldsymbol{w} \cdot \boldsymbol{w}\, d\boldsymbol{x} \to 0, \text{ as } n \to \infty. \tag{2.211}$$

Next,

$$\begin{aligned}\int_{\Omega_n} \boldsymbol{w} \cdot \nabla \boldsymbol{w} \cdot \boldsymbol{w}\, d\boldsymbol{x} &= \int_{\Omega_n} w_i \frac{\partial w_i}{\partial x_j} w_j\, d\boldsymbol{x} = \frac{1}{2}\int_{\Omega_n} w_j \frac{\partial}{\partial x_j}(w_i w_i)\, d\boldsymbol{x} \\ &= \frac{1}{2}\int_{\partial\Omega_n} w_i w_i w_j \nu_{j(n)} d\sigma_n - \frac{1}{2}\int_{\Omega_n} w_i w_i \frac{\partial w_j}{\partial x_j}\, d\boldsymbol{x}.\end{aligned}$$

But $\nabla \cdot \boldsymbol{w} = 0$, with $\boldsymbol{w}$ vanishing on $\Gamma_n^\pm$, so

$$\int_{\Omega_n} \boldsymbol{w} \cdot \nabla \boldsymbol{w} \cdot \boldsymbol{w}\, d\boldsymbol{x} = \frac{1}{2}\int_{\tilde{S}_n} w_i w_i w_j \tilde{\nu}_{j(n)} d\tilde{\sigma}_n \tag{2.212}$$

in which case, just as in (2.210),

$$\int_{\Omega_n} \boldsymbol{w} \cdot \nabla \boldsymbol{w} \cdot \boldsymbol{w}\, d\boldsymbol{x} \to 0, \text{ as } n \to \infty. \tag{2.213}$$

Continuing, we have,

$$\begin{aligned}\int_{\Omega_n} \nabla P \cdot \boldsymbol{w}\, d\boldsymbol{x} &= \int_{\Omega_n} \nabla \cdot (P\boldsymbol{w})\, d\boldsymbol{x} \text{ (as div } \boldsymbol{w} = 0) \\ &= \int_{\partial\Omega_n} P\boldsymbol{w} \cdot \boldsymbol{\nu}_{(n)}\, d\sigma_n \\ &= \int_{\tilde{S}_n} P\boldsymbol{w} \cdot \tilde{\boldsymbol{\nu}}_{(n)}\, d\tilde{\sigma}_n.\end{aligned}$$

Thus, if P is bounded on Ω_a, then by (2.175)

$$\int_{\Omega_n} \nabla P \cdot \boldsymbol{w}\, d\boldsymbol{x} \to 0, \text{ as } n \to \infty \tag{2.214}$$

We now turn our attention to those integrals over Ω_n in (2.209) which do not vanish, in the limit, as $n \to \infty$. By Lemma 2.8

$$\lim_{n\to\infty} \int_{\Omega_n} \boldsymbol{w} \cdot \nabla \boldsymbol{v}^p \cdot \boldsymbol{w}\, d\boldsymbol{x} = \int_{\Omega_a} \boldsymbol{w} \cdot \nabla \boldsymbol{v}^p \cdot \boldsymbol{w}\, d\boldsymbol{x} \geq -\Gamma \int_{\Omega_a} ||\boldsymbol{w}||^2\, d\boldsymbol{x} \tag{2.215}$$

with Γ given by (2.187). Also,

$$\begin{aligned}
\int_{\Omega_n} \nabla \cdot (\mu \boldsymbol{e} - \mu_p \boldsymbol{e}^p) \cdot \boldsymbol{w}\, d\boldsymbol{x} &= \int_{\Omega_n} \frac{\partial}{\partial x_j}(\mu e_{ij} - \mu_p e_{ij}^p)(v_i - v_i^p)\, d\boldsymbol{x} \\
&= \int_{\partial\Omega_n} (\mu e_{ij} - \mu_p e_{ij}^p)(v_i - v_i^p)\nu_{j(n)}\, d\sigma_n \\
&\quad - \int_{\Omega_n} (\mu e_{ij} - \mu_0 e_{ij}^p)\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_i^p}{\partial x_j}\right) d\boldsymbol{x} \\
&= \int_{\tilde{S}_n} (\mu e_{ij} - \mu_0 e_{ij}^p) w_i \tilde{\nu}_{j(n)}\, d\tilde{\sigma}_n \\
&\quad - \int_{\Omega_n} (\mu e_{ij} - \mu_0 e_{ij}^0)\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_i^p}{\partial x_j}\right) d\boldsymbol{x}
\end{aligned}$$

as $w_i = v_i - v_i^p$ vanishes on both $\Gamma_n^{\pm}$. The regularity requirement (2.169) now implies that

$$\mu_{ij} = \mu(\boldsymbol{v}) e_{ij}(\boldsymbol{v}) - \mu_p(\boldsymbol{v}^p) e_{ij}^p(\boldsymbol{v}^p) \tag{2.216}$$

is, for each pair i, j, $1 \leq i, j \leq 3$, bounded on Ω_a while, as previously, $w_i\big|_{\tilde{S}_n} \to 0$ as $n \to \infty$. Thus, letting $n \to \infty$ in the last of the equalities preceding (2.216) we find that

$$\begin{aligned}
\int_{\Omega_a} \nabla \cdot (\mu \boldsymbol{e} - \mu_p \boldsymbol{e}^p) \cdot \boldsymbol{w}\, d\boldsymbol{x} &= - \int_{\Omega_a} (\mu e_{ij} - \mu_0 e_{ij}^p)\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_i^p}{\partial x_j}\right) d\boldsymbol{x} \\
&= - \int_{\Omega_a} (\mu e_{ij} - \mu_p e_{ij}^p)(e_{ij} - e_{ij}^p)\, d\boldsymbol{x}
\end{aligned} \tag{2.217}$$

where the last result follows from the symmetry of e_{ij}, e_{ij}^p in their indices. Finally,

$$\begin{aligned}
-\mu_1 \int_{\Omega_n} \nabla \cdot \Delta(\boldsymbol{e} - \boldsymbol{e}^p) \cdot \boldsymbol{w}\, d\boldsymbol{x} &= -\mu_1 \int_{\Omega_n} \frac{\partial}{\partial x_j}\left[\frac{\partial^2}{\partial x_k^2}(e_{ij} - e_{ij}^p)\right] w_i\, d\boldsymbol{x} \\
&= -\mu_1 \int_{\partial\Omega_n} \frac{\partial^2}{\partial x_k^2}(e_{ij} - e_{ij}^p) w_i \nu_{j(n)}\, d\sigma_n
\end{aligned}$$

$$
\begin{aligned}
&+ \mu_1 \int_{\Omega_n} \frac{\partial w_i}{\partial x_j} \frac{\partial^2}{\partial x_k^2}(e_{ij} - e_{ij}^p)\, d\boldsymbol{x} \\
&= -\mu_1 \int_{\partial\Omega_n} \nabla^2(e_{ij} - e_{ij}^p) w_i \nu_{j(n)}\, d\sigma_n \\
&\quad + \mu_1 \int_{\partial\Omega_n} \frac{\partial w_i}{\partial x_j} \frac{\partial}{\partial x_k}(e_{ij} - e_{ij}^p)\nu_{k(n)} d\sigma_n \\
&\quad + \mu_1 \int_{\Omega_n} \frac{\partial^2 w_i}{\partial x_j \partial x_k} \frac{\partial}{\partial x_k}(e_{ij} - e_{ij}^p)\, d\boldsymbol{x}.
\end{aligned}
$$

For the first boundary integral in this last set of identities, i.e., for

$$
\int_{\partial\Omega_n} \nabla^2(e_{ij} - e_{ij}^p) w_i \nu_{j(n)}\, d\sigma_n
$$

we have $w_i\big|_{\Gamma_n} = 0$, $i = 1, 2, 3$ while $\nabla^2(e_{ij} - e_{ij}^p)$ is bounded on Ω_a, $1 \le i, j \le 3$ and $w_i\big|_{\tilde{S}_n} \to 0$, as $n \to \infty$, $i = 1, 2, 3$, so

$$
\int_{\partial\Omega_n} \nabla^2(e_{ij} - e_{ij}^p) w_i \nu_{j(n)} d\sigma_n \to 0, \text{ as } n \to \infty. \tag{2.218}
$$

For the second boundary integral in the last identity following (2.217) we have

$$
\begin{aligned}
\int_{\partial\Omega_n} \frac{\partial w_i}{\partial x_j} \frac{\partial}{\partial x_k}(e_{ij} - e_{ij}^p)\nu_{k(n)}\, d\sigma_n &= \int_{\Gamma_n^+} \frac{\partial w_i}{\partial x_j} \frac{\partial}{\partial x_k}(e_{ij} - e_{ij}^p)\nu_{k(n)}\, da^+ \\
&\quad + \int_{\Gamma_n^-} \frac{\partial w_i}{\partial x_j} \frac{\partial}{\partial x_k}(e_{ij} - e_{ij}^0)\nu_{k(n)}\, da^- \qquad (2.219) \\
&\quad + \int_{\tilde{S}_n} \frac{\partial w_i}{\partial x_j} \frac{\partial}{\partial x_k}(e_{ij} - e_{ij}^p)\tilde{\nu}_{k(n)}\, d\tilde{\sigma}_n.
\end{aligned}
$$

Although $\nu_{k(n)}\big|_{\Gamma_n}\pm = \pm\delta_{k2}$ for all n, we need not avail ourselves of that fact here; instead we note that we may write

$$
\begin{aligned}
\int_{\Gamma_n^\pm} \frac{\partial w_i}{\partial x_j} \frac{\partial}{\partial x_k}(e_{ij} - e_{ij}^p)\nu_{k(n)} da^\pm &= \int_{\Gamma_n^\pm} \frac{\partial}{\partial x_k}(e_{ij}(\boldsymbol{v}) - e_{ij}^p(\boldsymbol{v}^p)) e_{ij}(\boldsymbol{w})\nu_{k(n)} da^\pm \\
&= \int_{\Gamma_n^\pm} \frac{\partial}{\partial x_k}(e_{ij}(\boldsymbol{w})) e_{ij}(\boldsymbol{w})\nu_{k(n)} da^\pm \\
&= \int_{\Gamma_n^\pm} \tau_{ijk}(\boldsymbol{w}) e_{ij}(\boldsymbol{w})\nu_{k(n)} da^\pm = 0
\end{aligned} \tag{2.220}
$$

by virtue of Lemma B.3 and the second set of boundary conditions in (2.166). The result of the calculation in (2.220) is to reduce the boundary integral in (2.219) to

$$\int_{\partial\Omega_n} \frac{\partial w_i}{\partial x_j}\frac{\partial}{\partial x_k}(e_{ij} - e^p_{ij})\nu_{k(n)}d\sigma_n = \int_{\tilde{S}_n} \frac{\partial w_i}{\partial x_j}\frac{\partial}{\partial x_k}(e_{ij} - e^p_{ij})\tilde{\nu}_{k(n)}d\tilde{\sigma}_n. \quad (2.221)$$

However, $\frac{\partial}{\partial x_k}(e_{ij} - e^p_{ij})$ is bounded on Ω_a, for $1 \le i, j, k \le 3$, while $\left.\frac{\partial w_i}{\partial x_j}\right|_{\tilde{S}_n} \to 0$, as $n \to \infty$, for $1 \le i, j \le 3$ (all by virtue of the regularity condition (2.169). Therefore,

$$\int_{\partial\Omega_n} \frac{\partial w_i}{\partial x_j}\frac{\partial}{\partial x_k}(e_{ij} - e^p_{ij})\nu_{k(n)}d\sigma_n \to 0, \text{ as } n \to \infty. \quad (2.222)$$

Employing (2.217) and (2.218) in the last of the equations following (2.217), and letting $n \to \infty$, there results:

$$\begin{aligned} -\mu_1 \int_{\Omega_a} \nabla \cdot \Delta(\boldsymbol{e} - \boldsymbol{e}^p) \cdot \boldsymbol{w}\, d\boldsymbol{x} &= -\mu_1 \int_{\Omega_a} \frac{\partial^2 w_i}{\partial x_j \partial x_k}\frac{\partial}{\partial x_k}(e_{ij} - e^p_{ij})\, d\boldsymbol{x} \\ &= -\mu_1 \int_{\Omega_a} \frac{\partial^2 w_i}{\partial x_j \partial x_k}\frac{\partial}{\partial x_k} e_{ij}(\boldsymbol{w})\, d\boldsymbol{x} \\ &= -\mu_1 \int_{\Omega_a} \frac{\partial e_{ij}(\boldsymbol{w})}{\partial x_k}\frac{\partial e_{ij}(\boldsymbol{w})}{\partial x_k}\, d\boldsymbol{x}. \end{aligned} \quad (2.223)$$

If we now let $n \to \infty$ in (2.209), and make use of the results in (2.211), (2.213), (2.214), (2.215), (2.217) and (2.223), we obtain the estimate

$$-\Gamma \int_{\Omega_a} ||\boldsymbol{w}||^2\, d\boldsymbol{x} + \int_{\Omega_a} (\mu e_{ij} - \mu_0 e^p_{ij})(e_{ij} - e^p_{ij})\, d\boldsymbol{x} \le -\mu_1 \int_{\Omega_a} \frac{\partial e_{ij}(\boldsymbol{w})}{\partial x_k}\frac{\partial e_{ij}(\boldsymbol{w})}{\partial x_k}\, d\boldsymbol{x}. \quad (2.224)$$

It requires, however, only a trivial extension of Lemma 2.7 to conclude that at each fixed $\boldsymbol{x} \in \Omega_a$, for $0 < \alpha < 1$,

$$\left[\frac{e_{ij}(\boldsymbol{v})}{(\epsilon + e_{kl}(\boldsymbol{v})e_{kl}(\boldsymbol{v}))^{\alpha/2}} - \frac{e_{ij}(\boldsymbol{v}^p)}{(\epsilon + e_{kl}(\boldsymbol{v}^p)e_{kl}(\boldsymbol{v}^p))^{\alpha/2}}\right] \cdot (e_{ij}(\boldsymbol{v}) - e_{ij}(\boldsymbol{v}^p)) \ge 0 \quad (2.225)$$

and, of course, this leads us from (2.224) to

$$\mu_1 \int_{\Omega_a} \frac{\partial e_{ij}(\boldsymbol{w})}{\partial x_k}\frac{\partial e_{ij}(\boldsymbol{w})}{\partial x_k}\, d\boldsymbol{x} \le \Gamma \int_{\Omega_a} ||\boldsymbol{w}||^2\, d\boldsymbol{x}. \quad (2.226)$$

However, for $\boldsymbol{w} \in \boldsymbol{H}^4(\Omega_a)$ satisfying the boundary conditions (2.166),

$$\int_{\Omega_a} \frac{\partial e_{ij}(\boldsymbol{w})}{\partial x_k} \frac{\partial e_{ij}(\boldsymbol{x})}{\partial x_k}\, d\boldsymbol{x} \geq \frac{1}{2} \int_{\Omega_a} \frac{\partial^2 w_i}{\partial x_j \partial x_k} \frac{\partial^2 w_i}{\partial x_j \partial x_k}\, d\boldsymbol{x} \tag{2.227a}$$

while

$$\int_{\Omega_a} \frac{\partial^2 w_i}{\partial x_j \partial x_k} \frac{\partial^2 w_i}{\partial x_j \partial x_k}\, d\boldsymbol{x} \geq \frac{1}{(2a^2+\theta)^2} \int_{\Omega_a} ||\boldsymbol{w}||^2\, d\boldsymbol{x} \tag{2.227b}$$

for any $\theta > 0$, the last inequality following by the Poincaré-type estimate (2.176) of Lemma 2.6. Combining (2.226) with (2.227a,b) now produces

$$\left(\frac{\mu_1}{2(2a^2+\theta)^2} - \Gamma \right) \int_{\Omega_a} ||\boldsymbol{w}||^2\, d\boldsymbol{x} \leq 0 \tag{2.228}$$

from which it is immediate that $\boldsymbol{w} = 0$, a.e. in $\boldsymbol{L}^2(\Omega_a)$, if $\mu_1 > 2(2a^2+\theta)^2\Gamma$. □

2.5 Existence and Asymptotic Stability of Time-Dependent Poiseuille Flows

2.5.1 Introduction

In this section our attention will be focused on the natural counterpart to the boundary-value problem (2.55a,b), i.e., we consider time-dependent Poiseuille flow in the domain $\Omega_a \times [0, T)$, $T > 0$, of the form

$$\boldsymbol{v} = (u(y, t; \epsilon, \mu_1), 0, 0) \tag{2.229}$$

which satisfies the initial-boundary value problem

$$\rho \dot{u} = -p' + \mu_0 \left[(\epsilon + u'^2)^{-\alpha/2} u' \right]' - \mu_1 u'''', \tag{2.230a}$$

$$u(\pm a, t; \epsilon, \mu_1) = u''(\pm a, t; \epsilon, \mu_1) = 0, \quad t \in [0, T), \tag{2.230b}$$

$$u(y, 0; \epsilon, \mu_1) = u_0(y; \epsilon, \mu_1) \tag{2.230c}$$

where we have once again written $v_1 = u$ and $y = x_2$, with $\dot{u} = \frac{\partial u}{\partial t}$, $u' = \frac{\partial u}{\partial y}$, etc. In certain places we may write u_t for $\frac{\partial u}{\partial t}$ and, without loss of generality, we will set $\rho = 1$. The initial data function u_0 is assumed to be of class $C^2(-a, a)$ while the pressure distribution $p(y, t)$ is taken as being prescribed, with $p \in C^{1,0}(y, t)$ for $y \in (-a, a)$, $t > 0$.

Our goals in this section are twofold: first of all, to establish the existence of a unique solution to the initial-boundary value problem (2.230a,b,c) which is of class $C^{4,1}$ on $(-a,a) \times [0,T)$, for any $T > 0$, and, secondly, to prove that the unique equilibrium solution of the boundary-value problem (2.55a,b) is stable (in fact, asymptotically stable) in a sense which will be made precise in Sect. 2.5.4.

2.5.2 *Some Preliminary Estimates for an Associated Parabolic Problem*

We begin with some considerations related to the solutions of the following linear parabolic initial-boundary value problem for $w = w(y,t)$ on $(-a,a) \times [0,T)$:

$$\dot{w} = -\mu_1 w^{(4)} + f(y,t), \tag{2.231a}$$

$$w(\pm a,t) = w_{xx}(\pm a,t) = 0, \quad t > 0, \tag{2.231b}$$

$$w(y,0) = w_0(y), \quad y \in [-a,a] \tag{2.231c}$$

where $w^{(4)} \equiv w''''$, $w_0(\cdot) \in C^2(-a,a)$, $f \in C^{0,0}(y,t)$ for $(y,t) \in (-a,a) \times [0,T)$, and $\mu_1 > 0$. It is a direct consequence of standard results for linear parabolic initial boundary-value problems (specifically, e.g., Sects. 9, 10 of part 2 in [Fr]) that under the hypotheses delineated above, there exists a unique solution $w(y,t)$ of (2.231a,b,c) on $[-a,a] \times [0,T)$, for any $T > 0$, which is of class $C^{4,1}(y,t)$; in order to carry out the analysis in Sect. 2.5.3, we are interested in deriving certain *a priori* estimates which are satisfied by the unique solution $w(x,t)$ of (2.231a,b,c). Our first step consists of multiplying (2.231a) through by $w(y,\tau)$, $(y,\tau) \in (-a,a) \times [0,T)$, $t < T$, and integrating over $(-a,a) \times [0,t)$; we obtain

$$\frac{1}{2}\int_{-a}^{a} w^2(x,t)\,dx + \mu_1 \int_0^t \int_{-a}^{a} w^{(4)} w\,dx\,d\tau = \frac{1}{2}\int_{-a}^{a} w^2(x,0)\,dx + \int_0^t \int_{-a}^{a} w \cdot f\,dx\,d\tau \tag{2.232}$$

However, in view of the boundary conditions (2.231b), two successive integrations by parts yield

$$\int_0^t \int_{-a}^{a} w^{(4)} w\,dx\,d\tau = \int_0^t \int_{-a}^{a} w_{xx}^2\,dx\,d\tau \tag{2.233}$$

and, therefore,

$$\int_{-a}^{a} w^2(x,t)\,dx + 2\mu_1 \int_0^t \int_{-a}^{a} w_{xx}^2 dx\,d\tau = 2\int_0^t \int_{-a}^{a} w \cdot f\,dx\,d\tau + \int_{-a}^{a} w^2(x,0)\,dx. \tag{2.234}$$

Next, we multiply (2.231a) by $\dot{w}(y,\tau)$ and, again, integrate over $[-a,a]\times[0,t)$ so as to obtain

$$\int_0^t \int_{-a}^a \dot{w}^2\,dx\,d\tau = \int_0^t \int_{-a}^a f\cdot\dot{w}\,dx\,d\tau - \mu_1 \int_0^t \int_{-a}^a w^{(4)}\dot{w}\,dx\,d\tau. \tag{2.235}$$

Applying two successive integrations by parts to the second integral on the right-hand side of (2.235) yields, in view of (2.231b),

$$\begin{aligned}\int_0^t \int_{-a}^a w^{(4)}\dot{w}\,dx\,d\tau &= \int_0^t \int_{-a}^a w_{xx}\dot{w}_{xx}\,dx\,d\tau \\ &= \frac{1}{2}\int_{-a}^a w_{xx}^2(x,t)\,dx - \frac{1}{2}\int_{-a}^a w_{xx}^2(x,0)\,dx\end{aligned} \tag{2.236}$$

so that

$$\begin{aligned}&2\int_0^t \int_{-a}^a \dot{w}^2\,dx\,d\tau + \mu_1 \int_{-a}^a w_{xx}^2(x,t)\,dx \\ &\qquad = 2\int_0^t \int_{-a}^a f\cdot\dot{w}\,dx\,d\tau + \mu_1\int_{-a}^a w_{xx}^2(x,0)\,dx.\end{aligned} \tag{2.237}$$

Now, if $w(y,t)$ is a solution of (2.231a,b,c), then so is $w(-y,t)$; by uniqueness of solutions, therefore, $w(y,t) = w(-y,t)$, for $y\in(-a,a)$, $t>0$. It then follows that $w_y(y,t) = -w_y(-y,t)$ for $y\in(-a,a)$, $t>0$, in which case $w_y(0,t)=0$ for $t>0$. In view of this observation, elementary calculations show that $\exists k(a)>0$ (in fact, is is a simple exercise to show that we may take $k(a)=a^4/4$) such that

$$\int_{-a}^a w^2(x,t)\,dx \le k(a)\int_{-a}^a w_{xx}^2(x,t)\,dx, \quad t>0, \tag{2.238a}$$

$$\int_0^t \int_{-a}^a w^2(x,\tau)\,dx\,d\tau \le k(a)\int_0^t \int_{-a}^a w_{xx}^2(x,\tau)\,dx\,d\tau. \tag{2.238b}$$

Returning to (2.234) we have, by (2.238b), for any $\delta>0$,

$$\begin{aligned}2\int_0^t \int_{-a}^a f\cdot w\,dx\,d\tau &\le \delta\int_0^t f^2 dx\,d\tau + \frac{1}{\delta}\int_0^t \int_{-a}^a w^2\,dx\,d\tau \\ &\le \delta\int_0^t \int_{-a}^a f^2\,dx\,d\tau + \frac{k(a)}{\delta}\int_0^t \int_{-a}^a w_{xx}^2\,dx\,d\tau\end{aligned} \tag{2.239}$$

so employing this last estimate in (2.234) yields

$$\begin{aligned}&\int_{-a}^a w^2(x,t)\,dx + \left(2\mu_1 - \frac{k(a)}{\delta}\right)\int_0^t \int_{-a}^a w_{xx}^2\,dx\,d\tau \\ &\qquad \le \delta\int_0^t \int_{-a}^a f^2\,dx\,d\tau + \int_{-a}^a w^2(x,0)\,dx\end{aligned} \tag{2.240}$$

where we assume, of course, that δ has been chosen so large that $\delta > k(a)/2\mu_1$. In a similar fashion

$$2\int_0^t \int_{-a}^a f \cdot \dot{w}\,dx\,d\tau \le \int_0^t \int_{-a}^a f^2\,dx\,d\tau + \int_0^t \int_{-a}^a \dot{w}^2\,dx\,d\tau \tag{2.241}$$

and use of this estimate in (2.237) leads us to

$$\begin{aligned}\int_0^t \int_{-a}^a \dot{w}^2\,dx\,d\tau + \mu_1 \int_{-a}^a w_{xx}^2(x,t)\,dx \\ \le \int_0^t \int_{-a}^a f^2\,dx\,d\tau + \mu_1 \int_{-a}^a w_{xx}^2(x,0)\,dx.\end{aligned} \tag{2.242}$$

By virtue of the estimates (2.240), (2.242) we see that the unique solution of (2.231a,b,c) satisfies

$$w(\cdot,t) \in W^{2,2}(-a,a), \quad t > 0, \tag{2.243a}$$

$$\dot{w} \in L^2((-a,a)\times[0,t)), \quad t > 0, \tag{2.243b}$$

$$w \in H^2((-a,a)\times[0,t)), \quad t > 0. \tag{2.243c}$$

2.5.3 Existence of Weak Solutions

In this section we will employ an iteration scheme to establish the existence of an appropriately defined weak solution to the following initial-boundary value problem on $[-a,a]\times[0,T)$, $T > 0$, for the function $v = v(y,t)$:

$$\dot{v} = -\mu_1 v^{(4)} + \bar{\gamma}(v')' + g(y,t), \quad (y,t) \in (-a,a)\times[0,T), \tag{2.244a}$$

$$v(\pm a,t) = v_{xx}(\pm a,t) = 0, \quad t > 0, \tag{2.244b}$$

$$v(y,0) = v_0(y). \tag{2.244c}$$

In (2.244a,b,c), for $s \in R^1$

$$\bar{\gamma}(s) = \mu_0(\epsilon + s^2)^{-\alpha/2} s \tag{2.245}$$

and we take $g \in C^{0,0}(y,t)$ for $(y,t) \in [-a,a]\times[0,T)$. The initial-boundary value problem (2.230a,b,c) may be directly identified with (2.244a,b,c) if we fix $\epsilon > 0$, $\mu_1 > 0$, take $\rho = 1$, $u(y,t;\epsilon,\mu) = v(y,t)$, and $\dfrac{\partial p}{\partial y}(y,t) = g(y,t)$. We make the following definition rrelative to the problem (2.244a,b,c):

Definition 2.1. A function $v(y,t)$ defined on $Q_T = [-a,a] \times [0,T)$, is a weak solution of the initial-boundary value problem (2.244a,b,c) provided that $\dot{v} \in L^2(Q_T)$, $v'' \in L^2(Q_T)$, while v satisfies (2.244b,c) and, for every test function $\phi \in C_0^\infty(Q_T)$,

$$\iint_{Q_T} \phi \dot{v}\, dx\, dt + \mu_1 \iint_{Q_T} \phi'' v''\, dx\, dt = \iint_{Q_T} \phi \bar{\gamma}'(w') w''\, dx\, dt + \iint_{Q_T} \phi g\, dx\, dt. \tag{2.246}$$

Prior to introducing the iteration scheme associated with (2.244a,b,c) we again note that (as a direct consequence of Sects. 9, 10, part 2, of [Fr]) the linear parabolic equation

$$\dot{v} = -\mu_1 v^{(4)} + h(y,t) v_{yy} + g(y,t) \tag{2.247}$$

subject to the initial and boundary data (2.244b,c), possesses, for $h \in C^{0,0}(y,t)$, on $[-a,a] \times [0,T)$, a unique classical solution $v \in C^{4,1}(y,t)$ on $(-a,a) \times [0,T)$.

Consider now the iteration scheme defined by solutions of the initial-boundary value problem

$$\dot{w}_n = -\mu_1 w_n^{(4)} + \bar{\gamma}'(w'_{n-1}) w''_n + g(y,t), \tag{2.248a}$$

$$w_n(\pm a, t) = w''_n(\pm a, t) = 0, \tag{2.248b}$$

$$w_n(y,0) = v_0(y) \tag{2.248c}$$

for $(y,t) \in [-a,a] \times [0,T)$, and each integer $n \geq 1$, where

$$w_0(y,t) = j(y,t), \quad (x,t) \in [-a,a] \times [0,T) \tag{2.248d}$$

is given, with $j \in C^{1,0}(y,t)$ for $(y,t) \in (-a,a) \times [0,T)$. Clearly, for each $n \geq 1$, $h_n(y,t) \equiv \bar{\gamma}'(w'_{n-1}(y,t))$ is continuous on $[-a,a] \times [0,T)$ so that (2.248a–d) possesses a unique solution $w_n(y,t)$ on $[-a,a] \times [0,T)$, with $w_n \in C^{4,1}(y,t)$ for $(y,t) \in (-a,a) \times [0,T)$; for this solution $w_n(y,t)$ we may state the following:

Lemma 2.9. *Let $w_n(y,t)$, for $n \geq 1$, be the unique classical solution of* (2.248a,b,c) *on $[-a,a] \times [0,T)$, $T > 0$, subject to* (2.248d)*. Then $\exists k(T) > 0$ such that, for $t \leq T$,*

$$||w''_n||_{L^\infty([0,t);L^2(-a,a))} \leq k^{1/2}(T), \tag{2.249a}$$

$$||\dot{w}_n||_{L^2([0,t)\times(-a,a))} \leq \mu_1^{1/2} k^{1/2}(T), \tag{2.249b}$$

$$||w''_n||_{L^2([0,t)\times(-a,a))} \leq \frac{1}{\sqrt{2}} \mu_1^{1/2} \epsilon^{\alpha/2} k^{1/2}(T). \tag{2.249c}$$

Proof. If we set

$$f_n(y,t) = g(y,t) + \bar{\gamma}'(w'_{n-1})w''_n \tag{2.250}$$

then (2.248a) assumes the form

$$\dot{w}_n = -\mu_1 w_n^{(4)} + f_n(y,t), \quad n \geq 1 \tag{2.251}$$

and the *a priori* estimates associated with the parabolic initial-boundary value problem (2.231a,b,c) apply to the unique solution $w_n(y,t)$, $n \geq 1$ of (2.251), (2.248b,c) subject to the choice of the initial iterate as per (2.248d). In order to implement the aforementioned estimates we note that by virtue of (2.245), for $s \in R^1$,

$$\bar{\gamma}'(s) = \frac{\epsilon + (1-\alpha)s^2}{(\epsilon + s^2)^{1+(\alpha/2)}} > 0 \tag{2.252}$$

and that $\bar{\gamma}'$ is an even function of $s \in R^1$. Furthermore, a straightforward calculation yields

$$\bar{\gamma}''(s) = -s\left\{\frac{\alpha}{(\epsilon + s^2)^{1+(\alpha/2)}}\left[\frac{\epsilon + (1-\alpha)s^2}{\epsilon + s^2}\right] + \frac{2\alpha\epsilon}{(\epsilon + s^2)^{2+(\alpha/2)}}\right\} \tag{2.253}$$

so that $\bar{\gamma}''(s) < 0$, for $s > 0$. It thus follows that

$$\max_{s \in R^1} \bar{\gamma}'(s) = \bar{\gamma}'(0) = \epsilon^{-\alpha/2}. \tag{2.254}$$

Now, in view of (2.250) and (2.254),

$$\begin{aligned}\int_0^t \int_{-a}^a f_n^2\, dx\, d\tau &\leq 2\int_0^t \int_{-a}^a g^2(x,\tau)\, dx\, d\tau \\ &\quad + 2\int_0^t \int_{-a}^a \left(\bar{\gamma}'(w'_{n-1})\right)^2 (w''_n)^2\, dx\, d\tau \\ &\leq 2\int_0^t \int_{-a}^a g^2(x,\tau)\, dx\, d\tau + 2\epsilon^{-\alpha}\int_0^t \int_{-a}^a (w''_n)^2\, dx\, d\tau.\end{aligned} \tag{2.255}$$

We set, for $n \geq 1$,

$$b_n(t) = \int_0^t \int_{-a}^a (w''_n(x,\tau))^2\, dx\, d\tau \tag{2.256}$$

so that $b_n(0) = 0$, while

$$b_n(t) = \int_{-a}^a w''^2_n(x,t)\, dx \tag{2.257}$$

and by (2.248c)

$$\dot{b}_n(0) = \int_{-a}^{a} w_n''^2(x,0)\,dx = \int_{-a}^{a} v_0''^2(x)\,dx. \tag{2.258}$$

Applying the estimate (2.242) with $w \to w_n$ and $f \to f_n$ we have

$$\int_0^t \int_{-a}^{a} \dot{w}_n^2\,dx\,d\tau + \mu_1 \int_{-a}^{a} w_n''^2(x,t)\,dx \leq \int_0^t \int_{-a}^{a} f_n^2\,dx\,d\tau + \mu_1 \int_{-a}^{a} w_n''^2(x,0)\,dx. \tag{2.259}$$

If we drop the first integral on the left-hand side of (2.259), and employ (2.255)–(2.258), we are led to the differential inequality

$$\mu\dot{b}_n(t) \leq C(T) + 2\epsilon^{-\alpha} b_n(t) + \mu_1 \dot{b}_n(0) \tag{2.260}$$

for $t \leq T$, where

$$C(T) \equiv 2\int_0^T \int_{-a}^{a} g^2(x,\tau)\,dx\,d\tau. \tag{2.261}$$

Setting, for $n \geq 1$,

$$\begin{aligned} d_n(T) &= \frac{1}{\mu_1} C(T) + \dot{b}_n(0) \\ &= \frac{1}{\mu_1} C(T) + \int_{-a}^{a} v_0''^2(x)\,dx \equiv d(T) \end{aligned} \tag{2.262}$$

we see that (2.260) can be rewritten in the form

$$\dot{b}_n(t) \leq d(T) + \left(\frac{2}{\mu_1 \epsilon^\alpha}\right) b_n(t) \tag{2.263}$$

which, by use of the integrating factor $\exp\left(-\frac{2}{\mu_1\epsilon^\alpha} t\right)$, yields the estimate

$$b_n(t) \leq \frac{\mu_1 \epsilon^\alpha}{2} d(T) \left[\exp\left(\frac{2t}{\mu_1 \epsilon^\alpha}\right) - 1\right]. \tag{2.264}$$

However, substitution of (2.264) back into (2.263) then produces the bound

$$\dot{b}_n(t) \leq d(T) \exp\left[\frac{2}{\mu_1 \epsilon^\alpha} t\right], \quad t \leq T. \tag{2.265}$$

We now return to the estimate (2.259) which, when coupled with (2.255)–(2.258) and (2.261), we can use to deduce the inequality

$$\int_0^t \int_{-a}^a \dot{w}_n^2 \, dx \, d\tau + \mu_1 \dot{b}_n(t) \leq C(T) + 2\epsilon^{-\alpha} b_n(t) + \mu_1 \dot{b}_n(0). \tag{2.266}$$

Employing the estimate (2.264) for $b_n(t)$ on the right-hand side of (2.266) we now find, in succession, that for $t \geq T$,

$$\begin{aligned}\int_0^t \int_{-a}^a \dot{w}_n^2 \, dx \, d\tau + \mu_1 \dot{b}_n(t) &\leq C(T) + \mu_1 \dot{b}_n(0) \\ &\quad + \mu d(T) \left[\exp\left(\frac{2t}{\mu_1 \epsilon^\alpha} \right) - 1 \right] \\ &= (C(T) + \mu \dot{b}_n(0)) \exp\left(\frac{2t}{\mu_1 \epsilon^\alpha} \right) \\ &= \mu d(T) \exp\left(\frac{2t}{\mu_1 \epsilon^\alpha} \right).\end{aligned} \tag{2.267}$$

If we set

$$k(T) \equiv d(T) \exp\{(2/\mu_1 \epsilon^\alpha) T\} \tag{2.268}$$

then directly from (2.265) and (2.267) we have, for $t \leq T$, and $n \geq 1$,

$$\int_{-a}^a w_n''^2(x,t) \, dx \leq k(T), \tag{2.269a}$$

$$\int_0^t \int_{-a}^a \dot{w}_n^2(x,\tau) \, dx \, d\tau \leq \mu_1 k(T) \tag{2.269b}$$

while, in view of (2.264)

$$\int_0^t \int_{-a}^a w_n''^2(x,\tau) \, dx \, d\tau \leq \frac{1}{2} \mu_1 \epsilon^\alpha k(T) \tag{2.269c}$$

for $t \leq T$ and $n \geq 1$. The statements embodied in (2.269a,b,c) are, of course, equivalent to (2.249a,b,c). □

We are now ready to state and prove the main result in this section, namely,

Theorem 2.6. *Let $w_n(y,t)$ be the unique classical solution, for $n \geq 1$, of* (2.248a,b,c) *on $[-a,a] \times [0,T)$, $T > 0$, which is subject to* (2.248d)*. Then, as $n \to \infty$, $\{w_n\}$ converges to a weak solution of the initial-boundary value problem* (2.244a,b,c).

Proof. As a consequence of (2.249b,c), the boundary conditions (2.248d), and a straightforward interpolation argument, it follows that $\exists C > 0$ such that for $t \leq T$, and $n \geq 1$,

$$\begin{aligned} \|w_n'\|_{H^{1/2}([0,t)\times(-a,a))} &\leq C\left(\|\dot{w}\|_{L^2([0,t)\times(-a,a))} + \|w_n''\|_{L^2([0,t)\times(-a,a))}\right) \\ &\leq \left(1+\frac{1}{\sqrt{2}}\epsilon^{\alpha/2}\right) C\mu_1^{1/2}k^{1/2}(T). \end{aligned} \tag{2.270}$$

We now set $Q_T = [-a,a]\times[0,T)$. As a direct result of the bounds (2.249b,c) and (2.270) it follows that (by picking, if necessary, subsequences of previously chosen subsequences) we may single out a subsequence of $\{w_n\}$, which we will also denote by $\{w_n\}$, such that $w_n \to w$ weakly in $L^2(Q_T)$ and, hence, also in the sense of distributions, while

$$w_n'' \to w'', \text{ weakly in } L^2(Q_T), \tag{2.271a}$$

$$\dot{w}_n \to \dot{w}, \text{ weakly in } L^2(Q_T), \tag{2.271b}$$

$$w_n' \to w', \text{ weakly in } H^{1/2}(Q_T). \tag{2.271c}$$

However, by the compactness of the embedding $H^{1/2}(Q_T) \hookrightarrow L^2(Q_T)$ (see Appendix A) it follows from (2.271c) that

$$\{w_n' \to w' \text{ strongly in } L^2(Q_T)\} \Rightarrow \{w_n' \to w' \text{ a.e. in } Q_T\}. \tag{2.272}$$

Now, let $\phi(y,t)$ be a test function, i.e., suppose that $\phi \in C_0^\infty(Q_T)$; we multiply (2.248a) through by ϕ and integrate over $[-a,a]\times[0,T)$ obtaining, after two integrations by parts,

$$\begin{aligned} &\iint_{Q_T} \phi\dot{w}_n\,dx\,dt + \mu_1\iint_{Q_T}\phi'' w_n''\,dx\,dt \\ &\qquad = \iint_{Q_T}\phi\bar{\gamma}'(w_{n-1}')w_n''\,dx\,dt + \iint_{Q_T}\phi g\,dx\,dt. \end{aligned} \tag{2.273}$$

As $\bar{\gamma}'$ is clearly continuous, by (2.272) we have that $\bar{\gamma}'(w_{n-1}') \to \bar{\gamma}'(w')$, a.e. in Q_T, while $\bar{\gamma}' > 0$ with $\bar{\gamma}'(w_{n-1}') \leq \bar{\gamma}'(0)$, $\forall n \geq 1$. Thus,

$$\bar{\gamma}'(w_{n-1}') \to \bar{\gamma}'(w'), \text{ strongly in } L^2(Q_T). \tag{2.274}$$

Therefore, by the dominated convergence theorem,

$$\iint_{Q_T}\phi\bar{\gamma}'(w_{n-1}')w_n''dx\,dt \to \iint_{Q_T}\phi\bar{\gamma}'(w')w''\,dx\,dt \tag{2.275}$$

as $n \to \infty$. Extracting the limit in (2.273) as $n \to \infty$, we find that, as a consequence of (2.271a,b) and (2.275), $w(y,t)$ satisfies

$$\iint_{Q_T} \phi \dot{w}\, dx\, dt + \mu_1 \iint_{Q_T} \phi'' w''\, dx\, dt \\ = \iint_{Q_T} \phi \bar{\gamma}'(w') w''\, dx\, dt + \iint_{Q_T} \phi g\, dx\, dt. \tag{2.276}$$

Clearly w satisfies, as well, the boundary conditions (2.244b) and the initial condition (2.244c), which completes the proof of Theorem 2.6. □

Remarks. The regularity of the weak solution to (2.244a,b,c) now follows by standard arguments (e.g., see [Fr]). It is, in fact, easily established that $v \in C^4(y,t)$ on $(-a,a) \times [0,T)$ for any $T > 0$.

2.5.4 Uniqueness and Stability of Solutions

In the last subsection we established the existence of a weak solution $v(y,t)$ to the initial-boundary value problem (2.244a,b,c), indicating that, in fact, $v \in C^{4,1}(y,t)$ on $(-a,a) \times [0,T)$, $T > 0$, so that the weak solution is actually a classical solution of our problem. In this section we prove that any classical solution of (2.244a,b,c) is unique and that the (unique) equilibrium Poiseuille flow (2.9), with $u(y;\epsilon,\mu_1)$ being the solution of the boundary-value problem (2.55a,b), is linearly asympotitically stable as well as asymptotically stable, within the class of all flows in $\Omega_a \times [0,T)$, $T > 0$, of the Poiseuille type (2.10). We begin with the following result:

Theorem 2.7. *If $u(y,t)$, $v(y,t)$ are any two classical solutions of the initial boundary-value problem (2.244a,b,c), i.e., two solutions in $C^{4,1}(y,t)$, and $w(y,t) = v(y,t) - u(y,t)$, then for any $t > 0$, $||w(\cdot,t)||_{L^2(-a,a)} = 0$.*

Proof. As in the hypothesis of the theorem we set $w = v - u$ in which case $w(y,t)$ clearly satisfies

$$\dot{w} + \mu_1 w^{(4)} = \bar{\gamma}(w' + u')' - \bar{\gamma}(u')'. \tag{2.277}$$

Multiplying (2.277) through by $w(y,t)$, integrating over $[-a,a]$, and then integrating by parts, twice in succession, the resulting integral $\int_{-a}^{a} w w^{(4)}\, dx$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{-a}^{a} w^2\, dx + \mu_1 \int_{-a}^{a} w_{xx}^2\, dx = \int_{-a}^{a} \left(\bar{\gamma}(w' + u') - \bar{\gamma}(u')\right)' w\, dx \tag{2.278}$$

where we have used the fact that $\forall t > 0$, $w(\pm a, t) = w_{xx}(\pm a, t) = 0$. Integrating the integral on the right-hand side of (2.278) by parts, we are led from (2.278) to

$$\frac{1}{2}\frac{d}{dt}\int_{-a}^{a} w^2\,dx + \mu_1 \int_{-a}^{a} w_{xx}^2\,dx = -\int_{-a}^{a} \left(\bar{\gamma}(w' + u') - \bar{\gamma}(u')\right) w'\,dx$$

or

$$\frac{1}{2}\frac{d}{dt}\int_{-a}^{a} w^2\,dx + \mu_1 \int_{-a}^{a} w_{xx}^2\,dx = -\int_{-a}^{a} \left[\bar{\gamma}(v') - \bar{\gamma}(u')\right](v' - u')\,dx. \quad (2.279)$$

However,

$$(\bar{\gamma}(p) - \bar{\gamma}(q))(p - q) \geq 0, \quad \forall p, q \in R^1$$

by virtue of the monotonicity of $\bar{\gamma}$, i.e. (2.252). From (2.279), therefore, we obtain

$$\int_{-a}^{a} w^2(x,t)\,dx \leq \int_{-a}^{a} w^2(x,0)\,dx = 0 \quad (2.280)$$

as $u(y,0) = v(y,0) \equiv v_0(y)$, $\forall y \in [-a, a]$. □

Now, if we make use of (2.252), we see that (2.244a) has the equivalent form

$$\dot{v} = -\mu_1 v^{(4)} + \left[\frac{\epsilon + (1-\alpha)v'^2}{(\epsilon + v'^2)^{1+(\alpha/2)}}\right] v'' + g(y,t) \quad (2.281)$$

from which it is apparent that if $v(y,t)$ is a classical solution of (2.244a,b,c) on $[-a, a] \times [0, T)$, $T > 0$, then so is $v(-y, t)$. In view of the uniqueness theorem proven above we then have $v(y,t) = v(-y,t)$, $y \in [-a, a]$, $t > 0$, from which it follows that $v'(0,t) = 0$, $t > 0$. The Poincaré-type estimate of (2.238a) with $k(a) = a^4/4$, applies, therefore, to v as does the usual estimate

$$\int_{-a}^{a} v^2(x,t)\,dx \leq 4a^2 \int_{-a}^{a} v_x^2(x,t)\,dx. \quad (2.282)$$

From here on we will again denote the unique solution of (2.55a,b) by $u(y)$. Thus, $u(y)$ satisfies, for $y \in (-a, a)$,

$$\bar{\gamma}(u'(y))' - \mu_1 u''''(y) = p_1 \quad (2.283a)$$

with

$$u(\pm a) = u''(\pm a) = 0. \quad (2.283b)$$

Also, let $v_\delta(y,t)$ be the unique classical solution of (2.244a,b,c) with $g(y,t) = -\dfrac{\partial p(y,t)}{\partial y} \equiv -p_1$, $\forall t > 0$, and $v_0(y) = u(y) + \delta f(y)$, with $\delta > 0$ and $||f||_{L^\infty(-a,a)} \le K$, for some $K > 0$, i.e., $v_\delta(y,t)$ satisfies

$$\dot{v}_\delta = -\mu_1 v_\delta^{(4)} + \bar{\gamma}(v'_\delta)' - p_1, \quad y \in (-a,a),\ t > 0, \tag{2.284a}$$

$$v_\delta(\pm a, t) = v''_\delta(\pm a, t) = 0, \quad t > 0, \tag{2.284b}$$

$$v_\delta(y,0) = u(y) + \delta f(y), \quad y \in [-a,a]. \tag{2.284c}$$

Now, if $\delta = 0$, then the unique solution of (2.284a,b,c) is clearly $v_0(y,t) = u(y)$, $y \in (-a,a)$, $\forall t > 0$. Our goal is to study the behavior of the unique classical solution $v_\delta(y,t)$, $y \in (-a,a)$, as $t \to \infty$. To this end we set

$$w_\delta(y,t) = v_\delta(y,t) - u(y); \quad y \in [-a,a],\ t > 0 \tag{2.285}$$

in which case w_δ is easily seen to satisfy

$$\dot{w}_\delta + \mu_1 w_\delta^{(4)} = \bar{\gamma}(v'_\delta)' - \bar{\gamma}(u')', \quad y \in (-a,a),\ t > 0, \tag{2.286a}$$

$$w_\delta(\pm a, t) = w''_\delta(\pm a, t) = 0, \quad t > 0, \tag{2.286b}$$

$$w_\delta(y,0) = \delta f(y), \quad y \in [-a,a]. \tag{2.286c}$$

We first look at the problem of linear asymptotic stability of the equilibrium solution u. We write that, to within terms of order $O(\left|\left|w'_\delta\right|\right|^2_{L^\infty(-a,a)})$,

$$\bar{\gamma}(v'_\delta) \equiv \bar{\gamma}(w'_\delta + u') = \bar{\gamma}(u') + \bar{\gamma}'(u')w'_\delta \tag{2.287}$$

in which case the linearized equation associated with (2.286a) is just

$$\dot{w}_\delta + \mu_1 w_\delta^{(4)} = \left(\bar{\gamma}'(u')w'_\delta\right)' \tag{2.288}$$

where

$$\bar{\gamma}'(u') = \frac{\epsilon + (1-\alpha)u'^2}{(\epsilon + u'^2)^{1+(\alpha/2)}}. \tag{2.289}$$

We then have the following result concerning the linearized asymptotic stability of the (equilibrium) plane Poiseuille solution $u(y)$:

Theorem 2.8. *Let u be the unique solution of the boundary-value problem* (2.283a,b) *and w_δ the unique solution of the linearized initial-boundary value problem* (2.288), (2.286b,c). *Then for any $\delta > 0$, and $f \in L^2(-a,a)$, $||w_\delta(\cdot,t)||_{L^2(-a,a)}$ decays to zero, exponentially, as $t \to \infty$.*

Proof. Multiplying (2.288) through by $w_\delta(y,t)$, integrating over $(-a,a)$, and then integrating the term involving μ_1 by parts, in the familiar fashion, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{-a}^{a} w_\delta(x,t)\,dx + \mu_1\int_{-a}^{a}(w_\delta''(x,t))^2\,dx &= \int_{-a}^{a}(\bar{\gamma}'(u')w_\delta')'w_\delta\,dx \\ &= -\int_{-a}^{a}\bar{\gamma}'(u')w_\delta'^2\,dx.\end{aligned} \tag{2.290}$$

From (2.289)

$$\begin{aligned}\bar{\gamma}'(u') &= (\epsilon + u'^2)^{-\alpha/2}\left[1-\alpha\frac{u'^2}{\epsilon+u'^2}\right] \\ &\geq (1-\alpha)/(\epsilon+u'^2)^{\alpha/2}.\end{aligned} \tag{2.291}$$

But, from the analysis in Sect. 2.2, we know that $\exists C' > 0$, independent of ϵ and μ_1, such that $\left|\left|u'\right|\right|_{L^\infty(-a,a)} \leq C'$, in which case (2.291) implies that

$$\bar{\gamma}'(u'(y)) \geq (1-\alpha)/(\epsilon + C'^2)^{\alpha/2}, \quad y \in [-a,a]. \tag{2.292}$$

Employing the lower bound (2.292) in (2.290) now yields the estimate

$$\frac{1}{2}\frac{d}{dt}\int_{-a}^{a} w_\delta^2(x,t)\,dx \leq \frac{-(1-\alpha)}{(\epsilon+C'^2)^{\alpha/2}}\int_{-a}^{a} w_\delta'^2(x,t)\,dx \tag{2.293}$$

which is valid for all $\mu_1 > 0$, any $\epsilon > 0$, and $t > 0$. However, by virtue of (2.282), which holds if $v(\pm a) = 0$ (without regard to whether the condition $u'(0) = 0$ is satisfied),

$$\int_{-a}^{a} w_\delta^2(x,t)\,dx \leq 4a^2\int_{-a}^{a} w_\delta'^2(x,t)\,dx, \quad \forall t > 0. \tag{2.294}$$

Combining (2.293) and (2.294), we see that, for $t > 0$,

$$\frac{d}{dt}\int_{-a}^{a} w_\delta^2(x,t)\,dx \leq \frac{-(1-\alpha)}{2a^2(\epsilon+C'^2)^{\alpha/2}}\int_{-a}^{a} w_\delta^2(x,t)\,dx \tag{2.295}$$

from which it follows by (2.286c) that, for $t > 0$,

$$\int_{-a}^{a} w_\delta^2(x,t)\,dx \leq \delta^2\int_{-a}^{a} f^2(x)\,dx \cdot \exp(-\eta t) \tag{2.296}$$

where $\eta > 0$ is given by

$$\eta = (1-\alpha)/2a^2(\epsilon + C'^2)^{\alpha/2}. \tag{2.297}$$

Irrespective of the magnitude of the perturbation δf, therefore, $||w_\delta(\cdot,t)||_{L^2(-a,a)} \to 0$, as $t \to \infty$, and the proof of Theorem 2.8 is complete. □

Because of the monotonicity property associated with $\bar{\gamma}(\cdot)$, and the Poincaré type inequality (2.283a), we can actually do much better than the linearized asymptotic stability result expressed by Theorem 2.8 above; in fact we may state the following:

Theorem 2.9. *Let u be the unique solution of* (2.283a,b)*, v_δ the unique solution of* (2.284a,b,c)*, and set $w_\delta = v_\delta - u$, so that w_δ satisfies* (2.286a,b,c)*. Then, for any $\delta > 0$, and $f \in L^2(-a,a)$, $||w_\delta(\cdot,t)||_{L^2(-a,a)}$ decays to zero, exponentially, as $t \to \infty$.*

Proof. We multiply (2.286a) by w_δ, integrate over $[-a,a]$, and effectuate the required integrations by parts employing (2.286b), so as to obtain

$$\frac{1}{2}\frac{d}{dt}\int_{-a}^{a} w_\delta^2(x,t)\,dx + \mu_1 \int_{-a}^{a} w_\delta''^2(x,t)\,dt = -\int_{-a}^{a} \left[\bar{\gamma}(v_\delta') - \bar{\gamma}(u')\right][v_\delta' - u']\,dx. \tag{2.298}$$

Using the monotonicity property of $\bar{\gamma}$, i.e., $\forall\, p,q \in R^1$, $(\bar{\gamma}(p) - \bar{\gamma}(q))(p-q) \geq 0$, we obtain from (2.298)

$$\frac{d}{dt}\int_{-a}^{a} w_\delta^2(x,t)\,dx \leq 2\mu_1 \int_{-a}^{a} w_\delta''^2(x,t)\,dx. \tag{2.299}$$

However, $v_\delta(y,t) = v_\delta(-y,t)$, and $u(y) = u(-y)$, for $t > 0$, $y \in [-a,a]$, so $w_\delta(y,t) = w_\delta(-y,t)$, for $t > 0$, $y \in [-a,a]$. Thus, $w_\delta'(0,t) = 0$, $\forall t \geq 0$, while $w_\delta(\pm a,t) = 0$, $\forall t > 0$. The Poincaré type inequality (2.238a) applies, therefore, to w_δ and its use in (2.299) produces

$$\frac{d}{dt}\int_{-a}^{a} w_\delta^2(x,t)\,dx \leq -\frac{\mu_1}{2a^4}\int_{-a}^{a} w_\delta^2(x,t)\,dx, \quad t > 0. \tag{2.300}$$

Integration of (2.300) yields

$$\int_{-a}^{a} w_\delta^2(x,t)\,dx \leq \delta^2 \left(\int_{-a}^{a} f^2(x)\,dx\right) \exp\left[\frac{-\mu_1 t}{2a^4}\right], \quad t > 0 \tag{2.301}$$

and Theorem 2.9 has been established. □

Chapter 3
Incompressible Bipolar Fluid Dynamics: Examples of Other Flows and Geometries

3.1 Introduction

The mathematical model of a nonlinear, incompressible, bipolar viscous fluid was introduced in Sect. 1.6 and conforms to the constitutive hypotheses for the Cauchy stress tensor τ_{ij} and the first multipolar stress tensor τ_{ijk} which are given, respectively, by (2.1a,b). In (2.1a,b), p is the pressure, $e_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$ is the usual rate of deformation tensor, $\boldsymbol{v}$ is the fluid velocity (which satisfies $\nabla \cdot \boldsymbol{v} = 0$ in the relevant domain), and the constitutive parameters μ_0, μ_1, and ϵ are positive real numbers, while α (in this chapter) again satisfies $0 < \alpha < 1$. When $\alpha = \mu_1 = 0$, the bipolar model reduces to the standard one associated with the Stokes constitutive hypothesis, as delineated in Sect. 1.1, and the associated boundary-value, space-periodic, or pure initial value problems are those associated with the Navier–Stokes equations.

In Sect. 1.7, three standard problems for incompressible fluid flow were analyzed within the context of the bipolar model with $\epsilon = \mu_1 = 0$, namely, plane Poiseuille flow between parallel plates, proper Poiseuille flow in a circular pipe, and Couette flow over a plate moving with constant velocity. Then, in Chap. 2, a deeper study of the plane Poiseuille flow of an incompressible, nonlinear, bipolar viscous fluid was presented; existence, uniqueness, and continuous dependence (on the parameters ϵ and μ_1, as $\epsilon \to 0$, $\mu_1 \to 0$) were established in Sect. 2.2, for this problem, the continuous dependence with respect to μ_1 holding in the norm of $C^{1+\delta}$ for $0 < \delta < 1/2$. Appropriate Reynolds numbers for steady Poiseuille flow were generated in Sect. 2.3, as well as explicit estimates for the solution of the problem with $\epsilon \neq 0$, $\mu_1 \neq 0$ in terms of the solution of the associated problem corresponding to $\epsilon = \mu_1 = 0$. In Sect. 2.4, the solution $\boldsymbol{v}^p$ of the boundary-value problem for steady plane Poiseuille flow of an incompressible bipolar fluid in the parallel-walled channel given, e.g., by (2.164) was proven to be unique within the class of all steady flows of a viscous bipolar fluid in Ω_a whose associated velocity fields $\boldsymbol{v}$ satisfy $\boldsymbol{v} - \boldsymbol{v}^p \in \boldsymbol{H}^4(\Omega_a)$. Finally, existence of a unique solution $\boldsymbol{v}$ of the time-dependent

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plane Poiseuille problem given by (2.229), (2.230a,b,c), where $u = v_1$ and $y = x_2$, was established in Sect. 2.5 using an iteration scheme in conjunction with estimates for an associated parabolic initial boundary-value problem; in addition it was shown that the unique, steady plane Poiseuille flow $\boldsymbol{v}^p$ is both linearly asymptotically stable, as well as asymptotically stable, within the class of all flows in $\Omega_a \times [0, T)$, $T > 0$, of the Poiseuille type (2.229), (2.230a,b,c).

In this chapter, we move beyond the elementary examples given in Sect. 1.7 and the detailed description and analysis of plane Poiseuille flow of an incompressible, nonlinear, viscous bipolar fluid presented in Chap. 2. We consider four distinct problems involving the motion of a nonlinear, incompressible, bipolar fluid: (1) the classical problem of flow between concentric rotating cylinders, (2) the problem of the stability of a bubble immersed in an incompressible bipolar fluid, albeit with $\epsilon = \mu_1 = 0$, (3) the exterior flow of a viscous, bipolar fluid around an obstacle in the plane, and (4) the problem of flow over a non-smooth boundary, specifically flow of an incompressible, bipolar, viscous fluid in a polygonal domain.

The steady flow of an incompressible, bipolar, viscous fluid between rotating concentric circular cylinders, i.e., proper Couette flow, is studied in Sect. 3.2. The boundary-value problem governing this steady, proper, Couette flow is first formulated and then solved for degenerate values of the constitutive parameters. For the general situation in which the constitutive parameters satisfy $\epsilon > 0$, $\mu_1 > 0$, existence and uniqueness of the solution to the boundary-value problem is established. Continuous dependence of the solution, in appropriate norms, is also established with respect to the parameters governing the nonlinearity and multipolarity of the model as those constitutive parameters converge to zero. In Sect. 3.3 we consider the dynamical behavior of a spherical cavity, with a fixed center, in an unbounded, incompressible, viscous non-Newtonian fluid whose Cauchy stress tensor $\boldsymbol{\tau}$ is of the form (2.1a) with $\epsilon = \mu_1 = 0$. Using elementary dynamical systems theory we delineate the locally asymptotically stable, stable, and unstable equilibrium states of the spherical vapor bubble; stability results for the equilibrium states of bubbles immersed in a Newtonian fluid are obtained as a special case. The problem of steady flow of an incompressible fluid past a fixed body $\Omega' \subset \mathbb{R}^2$, as modeled by the Navier–Stokes system, has been studied extensively but is, for the most part, unresolved. Leray in 1933 [Le1] proved the existence of a velocity field which satisfies, in a weak sense, the relevant boundary-value problem, but it is still unknown as to whether or not this solution also satisfies the associated radiation condition at infinity in $\mathbb{R}^2$; this particular problem was resolved in [FiS] but only under the condition that the velocity at infinity is sufficiently small in magnitude. However, for the problem of stationary flow of an incompressible, bipolar, viscous fluid past a bounded domain in $\mathbb{R}^2$ we are able, in Sect. 3.4, to prove the existence of a unique solution satisfying prescribed conditions at infinity. It is also shown in Sect. 3.4 that the bipolar model of fluid flow predicts the existence of a drag on the immersed body and, thus, does not lead to a d'Alembert type paradox. The last problem investigated in this chapter is that of the stability of solutions to the incompressible bipolar equations with respect to perturbations of the boundary of the domain; unlike the situation with respect to the Navier–Stokes equations, it is demonstrated in Sect. 3.5 that, in general, solutions for the bipolar model are not stable with respect to perturbations of the boundary by Lipschitz curves. We also study, in Sect. 3.5, the regularity of solutions to the bipolar equations in a polygonal domain Ω; it is shown that near a corner in Ω, if the forcing function for the system, $\boldsymbol{F} \in \boldsymbol{L}^2(\Omega)$, then any weak solution in $\boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}_0^1(\Omega)$ may be written as the sum of two vector fields, a regular field in $\boldsymbol{H}_{loc}^4(\Omega)$, and a singular field which is not in $\boldsymbol{H}_{loc}^4(\Omega)$ and whose precise behavior depends on the interior angle of the corner. Finally, we provide in Sect. 3.5 an explicit characterization of the local singularities at a corner on $\partial\Omega$ in terms of the interior angle of the corner; the behavior of these local singularities are computed using MAPLE and sharp a priori estimates are used to show that there are no other singularities.

3.2 Flow Between Rotating Cylinders

3.2.1 Introduction

A classical problem in the study of motions of an incompressible viscous fluid is that of steady (or equilibrium) flow between rotating concentric circular cylinders, i.e., proper Couette flow. Under the assumption that the fluid is governed by the classical Stokes Law, the solution of the problem of proper Couette flow may be found in most classical texts on fluid dynamics (e.g., [CM, BaG], or [LL]). In fact, if $v(r)$ denotes the tangential velocity of the fluid, which is supposed to lie between rigid cylinders of radii r_1 and r_2 ($> r_1$) that rotate with constant angular velocities Ω_1 and Ω_2, r being the radial distance from the center line of the inner cylinder to a point in the fluid, then

$$v(r) = \frac{1}{r}\left(\frac{\Omega_1 - \Omega_2}{r_1^{-2} - r_2^{-2}}\right) + r\left(\frac{\Omega_1 r_1^2 - \Omega_2 r_2^2}{r_1^2 - r_2^2}\right) \tag{3.1}$$

is the resulting steady flow determined by Stokes Law and the non-slip boundary conditions

$$\Omega(r_1) \equiv \frac{v(r_1)}{r_1} = \Omega_1, \quad \Omega(r_2) \equiv \frac{v(r_2)}{r_2} = \Omega_2. \tag{3.2}$$

Also, the pressure distribution required to maintain the flow (3.1) is given by

$$\rho^{-1} p(r) = \frac{1}{2}A^2 r^2 - \frac{1}{2}B^2 r^{-2} + 2AB \ln r \tag{3.3}$$

where ρ is the (constant) fluid density and

$$A = \frac{r_2^2\Omega_2 - r_1^2\Omega_1}{r_2^2 - r_1^2}, \quad B = \frac{\Omega_2 - \Omega_1}{r_2^{-2} - r_1^{-2}}. \tag{3.4}$$

Finally, the frictional couple exerted, per unit length, across a cylindrical surface in the fluid of radius r, $r_1 < r < r_2$, is independent of r and is given by

$$2\pi r^2 \tau_{r\theta} = -r\pi\mu_0 \left(\frac{\Omega_1 - \Omega_2}{r_1^{-2} - r_2^{-2}} \right) \tag{3.5}$$

where $\tau_{r\theta}$ is the tangential stress. In deriving the relations (3.1), (3.3), (3.4), and (3.5), one writes the equilibrium Navier–Stokes equations in cylindrical coordinates (r, θ, z) and looks for solutions, subject to (3.2), of the form

$$v_r = \dot{r} = 0, \quad v_\theta = r\dot{\theta}(r), \quad v_z = \dot{z} = 0 \tag{3.6}$$

which can be supported by a pressure distribution $p = p(r)$; in such a situation (3.1) reduces to

$$\tau_{r\theta} = 2\mu_0 e_{r\theta} \tag{3.7}$$

where

$$e_{r\theta} = \frac{1}{2}\left(\frac{\partial v}{\partial r} - \frac{v}{r} \right). \tag{3.8}$$

In this section we will be concerned with the existence (and structure) of equilibrium solutions of the form (3.6) for the case in which (1) the domain Ω is the region bounded by two concentric circular cylinders of radii r_1 and r_2 that rotate with constant angular velocities Ω_1 and Ω_2 and (2) the fluid conforms to the constitutive hypotheses (2.1a,b), with $0 \le \alpha < 1$ and $\mu_1 > 0$; we recall that, if we set $\alpha = 2 - p$, then our assumption relative to α implies that $1 < p \le 2$ in the nonlinear viscosity (2.3), when we rewrite it as

$$\mu(|\boldsymbol{e}|) = \mu_0(\epsilon + |\boldsymbol{e}|^2)^{\frac{p-2}{2}}. \tag{3.9}$$

In the present work we do not set ourselves the task of establishing, within the context of the bipolar model (2.1a,b), with $0 \le \alpha < 1$, as broad a range of results for the problem of proper Couette flow as has been established, to date, for the problem of plane Poiseuille flow; rather we shall content ourselves with deriving the relevant nonlinear boundary-value problem, with solving (in closed form) that problem for the case in which both ϵ and μ_1 are zero, and then proving the existence of a unique solution that depends continuously on ϵ and μ_1 as these constitutive parameters tend to zero. Along the way we will compare our results to those predicted by the classical solution, as given by (3.1), will study the limit of the tangential velocity field as $\alpha \to 1^-$, and will compute relevant quantities such as the frictional couple exerted on the fluid inside a cylindrical surface of radius r by the fluid exterior to that surface. Our results are expected to be of some utility to experimental fluid dynamicists.

3.2.2 *The Nonlinear Boundary-Value Problem for Proper Couette Flow of an Incompressible Bipolar Fluid*

One way to proceed, in this section, would be to rewrite the viscous, incompressible, bipolar flow problem (2.2a–d) in cylindrical coordinates and then look for solutions of the form (3.6); indeed, the transformation of this problem into cylindrical coordinates may be found in the Ph.D. thesis of A. Montz [Mon]. However, it is somewhat instructive to transform the bipolar equations in Cartesian form

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + 2\frac{\partial}{\partial x_j}(\mu(|\boldsymbol{e}|)e_{ij}) - 2\mu_1 \frac{\partial}{\partial x_j}(\Delta e_{ij}) + f_i, \text{ in } \Omega \times [0, T) \tag{3.10}$$

to cylindrical coordinates by starting with the ansatz

$$\begin{aligned} u_1 &= -v(r)\sin\theta, \\ u_2 &= v(r)\cos\theta, \\ u_3 &= 0 \end{aligned} \tag{3.11}$$

or, $\boldsymbol{v} = (-v(r)\sin\theta, v(r)\cos\theta, 0)$. Equations (3.11) result, of course, from the relations

$$\begin{aligned} u_1 &= v_r \cos\theta - v_\theta \sin\theta, \\ u_2 &= v_r \sin\theta + v_\theta \cos\theta, \\ u_3 &= v_z \end{aligned} \tag{3.12}$$

and the hypothesis (3.6). Throughout this section we will use the notation $x_1 = x$, $x_2 = y$, $x_3 = z$.

Remarks. Since the angular velocity $\dot{\theta} = \Omega(r) = \dfrac{v(r)}{r}$, we may also write

$$\boldsymbol{v} = (-\Omega(r)y, \Omega(r)x, 0).$$

Our point of departure is the set of standard relations

$$\frac{\partial r}{\partial x} = \cos\theta, \quad \frac{\partial r}{\partial y} = \sin\theta, \quad \frac{\partial \theta}{\partial x} = \frac{-\sin\theta}{r}, \quad \frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}. \tag{3.13}$$

Our task now is to express the quantities e_{ij}, $\dfrac{\partial}{\partial x_k}e_{ij}$, and Δe_{ij} in terms of r and θ; we begin by noting that, as a consequence of (3.11), (3.13),

$$
\begin{aligned}
\frac{\partial u_1}{\partial x} &= -\left(v'(r) - \frac{v(r)}{r}\right)\sin\theta\cos\theta, \\
\frac{\partial u_1}{\partial y} &= -v'(r)\sin^2\theta - \frac{v(r)}{r}\cos^2\theta, \\
\frac{\partial u_2}{\partial x} &= v'(r)\cos^2\theta + \frac{v(r)}{r}\sin^2\theta, \\
\frac{\partial u_2}{\partial y} &= \left(v'(r) - \frac{v(r)}{r}\right)\sin\theta\cos\theta = -\frac{\partial u_1}{\partial x}.
\end{aligned}
\tag{3.14}
$$

For the sake of convenience we set

$$
f(r) = v'(r) - \frac{v(r)}{r}. \tag{3.15}
$$

Now, for $i = 1, 2$, we compute for the convective derivatives appearing on the left-hand side of (3.10):

$$
\begin{aligned}
\frac{\partial u_1}{\partial t} + u_j\frac{\partial u_1}{\partial x_j} &= -\frac{v^2(r)}{r}\cos\theta, \\
\frac{\partial u_2}{\partial t} + u_j\frac{\partial u_2}{\partial x_j} &= -\frac{v^2(r)}{r}\sin\theta.
\end{aligned}
\tag{3.16}
$$

Also,

$$
\begin{aligned}
e_{11} &= \frac{\partial u_1}{\partial x} = -f(r)\sin\theta\cos\theta = -\frac{1}{2}f(r)\sin 2\theta, \\
e_{22} &= \frac{\partial u_2}{\partial y} = f(r)\sin\theta\cos\theta = \frac{1}{2}f(r)\sin 2\theta, \\
e_{12} &= e_{21} = \frac{1}{2}\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right) = \frac{1}{2}f(r)\cos 2\theta,
\end{aligned}
\tag{3.17}
$$

and $e_{ij} = 0$, otherwise. From (3.16) it follows that

$$
|\boldsymbol{e}|^2 = e_{ij}e_{ij} = \frac{1}{2}f^2(r). \tag{3.18}
$$

We now set

$$
h(r) = \left(\epsilon + \frac{1}{2}f^2(r)\right)^{-\alpha/2} f(r) \tag{3.19}
$$

and compute that

$$\begin{aligned}\frac{\partial}{\partial x}\left[\left(\epsilon+|\boldsymbol{e}|^2\right)^{-\alpha/2}e_{11}\right] &= -\frac{1}{2}\frac{\partial}{\partial x}\left[\left(\epsilon+|\boldsymbol{e}|^2\right)^{-\alpha/2}f(r)\sin 2\theta\right]\\ &= -\frac{1}{2}\frac{\partial}{\partial x}\left[h(r)\sin 2\theta\right]\\ &= -\frac{1}{2}\left(h'(r)\sin 2\theta\cos\theta-\frac{2h(r)}{r}\cos 2\theta\sin\theta\right).\end{aligned}\tag{3.20a}$$

An analogous computation produces

$$\frac{\partial}{\partial y}\left[\left(\epsilon+|\boldsymbol{e}|^2\right)^{-\alpha/2}e_{12}\right]=\frac{1}{2}\left(h'(r)\cos 2\theta\sin\theta-\frac{2h(r)}{r}\sin 2\theta\cos\theta\right)\tag{3.20b}$$

and, therefore

$$\frac{\partial}{\partial x}\left[\left(\epsilon+|\boldsymbol{e}|^2\right)^{-\alpha/2}e_{11}\right]+\frac{\partial}{\partial y}\left[\left(\epsilon+|\boldsymbol{e}|^2\right)^{-\alpha/2}e_{12}\right]=-\left(\frac{1}{2}h'(r)+\frac{h(r)}{r}\right)\sin\theta.\tag{3.21}$$

In a similar manner we have

$$\frac{\partial}{\partial x}\left[\left(\epsilon+|\boldsymbol{e}|^2\right)^{-\alpha/2}e_{21}\right]=\frac{1}{2}\left(h'(r)\cos 2\theta\cos\theta+\frac{2h(r)}{r}\sin 2\theta\sin\theta\right)\tag{3.22a}$$

and

$$\frac{\partial}{\partial y}\left[\left(\epsilon+|\boldsymbol{e}|^2\right)^{-\alpha/2}e_{22}\right]=\frac{1}{2}\left(h'(r)\sin 2\theta\sin\theta+\frac{2h(r)}{r}\cos 2\theta\cos\theta\right)\tag{3.22b}$$

so that

$$\frac{\partial}{\partial x}\left[\left(\epsilon+|\boldsymbol{e}|^2\right)^{-\alpha/2}e_{21}\right]+\frac{\partial}{\partial y}\left[\left(\epsilon+|\boldsymbol{e}|^2\right)^{-\alpha/2}e_{22}\right]=\left(\frac{h'(r)}{2}+\frac{h(r)}{r}\right)\cos\theta.\tag{3.23}$$

Our next set of computations is directed at producing the components of the tensor

$$\tau_{ijk}=\frac{\partial}{\partial x_k}(e_{ij}).$$

Note that, by virtue of (2.1b), $\tau_{ijk} = 2\mu_1 \dfrac{\partial e_{ij}}{\partial x_k}$. We then have, as a consequence of (3.17),

$$\tau_{111} = \frac{\partial}{\partial x} e_{11} = -\frac{1}{2} f'(r) \frac{\partial r}{\partial x} \sin 2\theta - f(r) \cos 2\theta \frac{\partial \theta}{\partial x}$$

so that

$$\tau_{111} = -\frac{1}{2} f'(r) \sin 2\theta \cos\theta + \frac{f(r)}{r} \cos 2\theta \sin\theta \tag{3.24a}$$

and, in a similar manner,

$$\tau_{112} = -\frac{1}{2} f'(r) \sin 2\theta \sin\theta - \frac{f(r)}{r} \cos 2\theta \cos\theta, \tag{3.24b}$$

$$\tau_{221} = -\tau_{111}, \quad \tau_{222} = -\tau_{112}, \tag{3.24c}$$

$$\tau_{121} = \frac{1}{2} f'(r) \cos 2\theta \cos\theta + \frac{f(r)}{r} \sin 2\theta \sin\theta, \tag{3.24d}$$

$$\tau_{122} = \frac{1}{2} f'(r) \cos 2\theta \sin\theta - \frac{f(r)}{r} \sin 2\theta \cos\theta, \tag{3.24e}$$

$$\tau_{221} = \tau_{121}, \quad \tau_{212} = \tau_{122}, \tag{3.24f}$$

and $\tau_{ijk} = 0$, otherwise. Since the fluid is assumed to be confined between rigid cylinders of radii r_1 and r_2 ($> r_1$),

$$\begin{cases} \text{for } r = r_2 : \nu_1 = \cos\theta,\ \nu_2 = \sin\theta,\ \nu_3 = 0, \\ \text{for } r = r_1 : \nu_1 = -\cos\theta,\ \nu_2 = -\sin\theta,\ \nu_3 = 0. \end{cases} \tag{3.25}$$

Now, the first set of boundary conditions in (2.2c) is identical with (3.2). By combining (3.24a–f) and (3.25), a straightforward but tedious computation shows that the second set of boundary conditions in (2.2c) is satisfied if and only if, for $\mu_1 \neq 0$,

$$f'(r_i) = 0, \quad i = 1, 2. \tag{3.26}$$

If $\mu_1 = 0$, the boundary conditions in (3.26) are not applicable.

Our next task is to express the terms Δe_{ij}, in cylindrical coordinates, for the special steady motion defined by (3.6). We begin by recalling a series of trigonometric identities, namely

$$\sin 2\theta \cos\theta + \cos 2\theta \sin\theta = \sin 3\theta,$$

$$\sin 2\theta \cos\theta - \cos 2\theta \sin\theta = \sin\theta,$$

$$\frac{d}{d\theta}(\sin 2\theta \cos \theta) = \frac{1}{2}(3\cos 3\theta + \cos \theta),$$
$$\frac{d}{d\theta}(\cos 2\theta \sin \theta) = \frac{1}{2}(3\cos 3\theta - \cos \theta),$$
$$\cos 2\theta \cos \theta - \sin 2\theta \sin \theta = \cos 3\theta,$$
$$\cos 2\theta \cos \theta + \sin 2\theta \sin \theta = \cos \theta,$$

and

$$\frac{d}{d\theta}(\cos 2\theta \cos \theta) = -\frac{1}{2}(3\sin 3\theta + \sin \theta),$$
$$\frac{d}{d\theta}(\sin 2\theta \sin \theta) = \frac{1}{2}(3\sin 3\theta - \sin \theta).$$

We now compute, with the aid of these identities, that

$$\begin{aligned}\frac{\partial^2}{\partial x^2}e_{11} = &-\frac{1}{2}f''r_x \sin 2\theta \cos \theta - \frac{1}{4}f'(3\cos 3\theta + \cos \theta)\theta_x \\ &+ \left(\frac{f}{r}\right)' r_x \cos 2\theta \sin \theta + \frac{1}{2}\frac{f}{r}(3\cos 3\theta - \cos \theta)\theta_x\end{aligned}$$

or

$$\begin{aligned}\frac{\partial^2}{\partial x^2}e_{11} = &-\frac{1}{2}f'' \sin 2\theta \cos^2 \theta = \frac{1}{4}\frac{f}{r}(3\cos 3\theta + \cos \theta)\sin \theta \\ &+ \left(\frac{f}{r}\right)' \cos 2\theta \sin \theta \cos \theta - \frac{1}{2}\frac{f}{r^2}(3\cos 3\theta - \cos \theta)\sin \theta\end{aligned} \tag{3.27a}$$

and, in a like fashion,

$$\begin{aligned}\frac{\partial^2}{\partial y^2}e_{11} = &-\frac{1}{2}f'' \sin 2\theta \sin^2 \theta - \frac{1}{4}\frac{f'}{r}(3\sin 3\theta - \sin \theta)\cos \theta \\ &- \left(\frac{f}{r}\right)' \cos 2\theta \sin \theta \cos \theta + \frac{1}{2}\frac{f}{r^2}(3\sin 3\theta + \sin \theta)\cos \theta\end{aligned} \tag{3.27b}$$

so that

$$\Delta e_{11} = \left(-\frac{1}{2}f'' - \frac{1}{2}\frac{f'}{r} + \frac{2f}{r^2}\right)\sin 2\theta. \tag{3.28a}$$

As is easily verified, a series of calculations entirely analogous to those that led to (3.28a) now yields the following results:

$$\Delta e_{22} = -\Delta e_{11}, \tag{3.28b}$$

$$\Delta e_{12} = \left(\frac{1}{2} f'' + \frac{1}{2}\frac{f'}{r} - \frac{2f}{r^2}\right)\cos\theta, \tag{3.28c}$$

$$\Delta e_{21} = \Delta e_{12}, \tag{3.28d}$$

and $\Delta e_{ij} = 0$, otherwise. If we set

$$g(r) = -\frac{1}{2} f'' - \frac{1}{2}\frac{f'}{r} - \frac{2f}{r^2} \tag{3.29}$$

then (3.28a–d) can be expressed as

$$\begin{aligned} \Delta e_{11} &= g(r)\sin 2\theta = -\Delta e_{22}, \\ \Delta e_{12} &= -g(r)\cos 2\theta = \Delta e_{21}, \\ \Delta e_{ij} &= 0 \text{ (otherwise)}. \end{aligned} \tag{3.30}$$

Progressing in a more or less methodical fashion with the computation of the quantities $\frac{\partial}{\partial x_j}(\Delta e_{ij})$ we have, by virtue of (3.30), and (3.11):

$$\frac{\partial}{\partial x}(\Delta e_{11}) = g' \sin 2\theta \cos\theta - \frac{2g}{r}\cos 2\theta \sin\theta, \tag{3.31a}$$

$$\frac{\partial}{\partial y}(\Delta e_{11}) = g' \sin 2\theta \sin\theta + \frac{2g}{r}\cos 2\theta \cos\theta, \tag{3.31b}$$

$$\frac{\partial}{\partial x}(\Delta e_{12}) = -g' \cos 2\theta \cos\theta - \frac{2g}{r}\sin 2\theta \sin\theta, \tag{3.31c}$$

and

$$\frac{\partial}{\partial y}(\Delta e_{12}) = -g' \cos 2\theta \sin\theta + \frac{2g}{r}\sin 2\theta \cos\theta. \tag{3.31d}$$

Therefore, for $i = 1$:

$$\begin{aligned} \frac{\partial}{\partial x_j}(\Delta e_{1j}) &= \frac{\partial}{\partial x}\Delta e_{11} + \frac{\partial}{\partial y}\Delta e_{12} \\ &= \left(g' + \frac{2g}{r}\right)\sin\theta \end{aligned} \tag{3.32a}$$

while for $i = 2$:

$$\frac{\partial}{\partial x_j}(\Delta e_{2j}) = \frac{\partial}{\partial x}\Delta e_{21} + \frac{\partial}{\partial y}\Delta e_{22} = -\left(g' + \frac{2g}{r}\right)\cos\theta. \tag{3.32b}$$

Finally, with $p = p(r, \theta)$:

$$\begin{aligned}\frac{\partial p}{\partial x} &= \frac{\partial p}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial p}{\partial \theta}\sin\theta,\\ \frac{\partial p}{\partial y} &= \frac{\partial p}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial p}{\partial \theta}\cos\theta.\end{aligned} \tag{3.33}$$

In order to synthesize the set of equations for the proper Couette flow of a bipolar viscous fluid, we now combine (3.16), (3.33), (3.27a,b), (3.29), and (3.32a,b), where $f(r)$, $h(r)$, and $g(r)$ are given, respectively, by (3.15), (3.19), and (3.29). In this manner, we obtain the steady flow equations

$$\frac{\partial p}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial p}{\partial \theta}\sin\theta - \rho\frac{v^2(r)}{r}\cos\theta = -\mu_0\left(\frac{h'}{2} + \frac{h}{r}\right)\sin\theta + 2\mu_1\left(g' + \frac{2g}{r}\right)\sin\theta, \tag{3.34}$$

$$\frac{\partial p}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial p}{\partial \theta}\cos\theta - \rho\frac{v^2(r)}{r}\sin\theta = 2\mu_0\left(\frac{h'}{2} + \frac{h}{r}\right)\cos\theta - 2\mu_1\left(g' + \frac{2g}{r}\right)\cos\theta. \tag{3.35}$$

Multiplying (3.34) by $\cos\theta$, (3.35) by $\sin\theta$ and adding, we find that

$$\frac{\partial p}{\partial r} - \rho\frac{v^2(r)}{r} = 0. \tag{3.36}$$

Multiplying (3.34) by $\cos\theta$, and (3.35) by $\sin\theta$, and subtracting the resulting equations, we obtain

$$\frac{1}{r}\frac{\partial p}{\partial \theta} = 2\mu_0\left(\frac{h'}{2} + \frac{h}{r}\right) + 2\mu_1\left(g' + \frac{2g}{r}\right) \tag{3.37}$$

where $p = p(r, \theta)$. Associated with (3.36), (3.37) are the boundary conditions

$$\frac{v(r_i)}{r_i} \equiv \Omega(r_i) = \Omega_i, \quad i = 1, 2 \tag{3.38a}$$

and

$$f'(r_i) = 0, \quad i = 1, 2. \tag{3.38b}$$

It is easily seen that (3.37) may be written in the form

$$r\frac{\partial p}{\partial \theta} = \mu_0(r^2h)' + 2\mu_1(r^2g)'. \tag{3.39}$$

If $\mu_1 > 0$ and, as in the classical situation, we look for a solution for which $\dfrac{\partial p}{\partial \theta} = 0$, then our boundary-value problem may be summarized as follows: find $v = v(r)$ such that

$$\mu_0(r^2h)' + 2\mu_1(r^2g)' = 0, \quad 0 < r_1 \le r \le r_2, \tag{3.40a}$$

$$v(r_i) = \Omega_i r_i, \quad i = 1, 2, \tag{3.40b}$$

$$f'(r_i) = 0, \quad i = 1, 2, \tag{3.40c}$$

where $f(r) = v'(r) - \dfrac{v(r)}{r}$, $h(r) = \left(\epsilon + \dfrac{1}{2}f^2(r)\right)^{-\alpha/2} f(r)$, $g(r) = -\dfrac{1}{2}f''(r) - \dfrac{1}{2}\dfrac{f'(r)}{r} + \dfrac{2f(r)}{r^2}$, and $0 \le \alpha < 1$. Once the tangential velocity distribution $v(r)$ has been determined by (3.40a,b,c), the pressure distribution $p = p(r)$ may be deduced by integration of (3.36). It will turn out to be useful, with respect to the analysis that follows, to rewrite the boundary-value problem (3.40a,b,c) in terms of the angular velocity $\Omega(r) = v(r)/r$; to this end we note the series of identities

$$f(r) = r\Omega'(r), \tag{3.41a}$$

$$h(r) = \left(\epsilon + \frac{1}{2}(r\Omega'(r))^2\right)^{-\alpha/2} r\Omega'(r), \tag{3.41b}$$

$$g(r) = -\frac{1}{2}\left(r\Omega'''(r) + 3\Omega''(r) - \frac{3\Omega'(r)}{r}\right), \tag{3.41c}$$

from which it follows that (3.40a,b,c) is equivalent to

$$-\left[\frac{r^3\Omega'(r)}{\left(\epsilon + \frac{1}{2}(r\Omega'(r)^2)\right)^{\alpha/2}}\right]' + \mu_1^*(r^3\Omega'''(r) + 3r^2\Omega''(r) - 3r\Omega'(r))' = 0, \tag{3.42a}$$

$$\boldsymbol{\Omega}(r_i) = \Omega_i, \quad i = 1, 2, \tag{3.42b}$$

$$(r\Omega'(r))'(r_i) = 0, \quad i = 1, 2 \tag{3.42c}$$

with $\mu_1^* = \mu_1/\mu_0$, and $p(r)$ determined by (3.36).

3.2.3 The Proper Couette Flow: $\epsilon = \mu_1 = 0$

In the special case in which $\epsilon = \mu_1 = 0$, the boundary-value problem (3.40a,b,c)—equivalently, (3.42a,b,c)—may be integrated in closed form; it is instructive to compute this solution when $0 \le \alpha < 1$. For the mathematical analysis of existence, uniqueness, and continuous dependence, which is presented in the next section, it is more advantageous to work with (3.42a,b,c); in the present situation it is better to work with (3.40a,b,c). Thus, with $\epsilon = \mu_1 = 0$ in (3.40a), we have

$$h'_\alpha(r) + \frac{2}{r} h_\alpha(r) = 0 \tag{3.43}$$

with

$$h_\alpha(r) = 2^{\alpha/2} \cdot \frac{f_\alpha(r)}{|f_\alpha(r)|^\alpha} \tag{3.44}$$

and $f_\alpha(r) = v'_\alpha(r) - \dfrac{v_\alpha(r)}{r}$. Integrating (3.43), and noting (3.44), we have for some c_α

$$\frac{f_\alpha(r)}{|f_\alpha(r)|^\alpha} = \frac{c_\alpha}{r^2}. \tag{3.45}$$

We now make the assumption that for $0 < r_1 < r < r_2$,

$$f_\alpha(r) > 0 \Leftrightarrow r\Omega'_\alpha(r) > 0 \tag{3.46}$$

in which case $c_\alpha > 0$, and for $0 < \alpha < 1$,

$$f_\alpha(r) = c_\alpha^{1/(1-\alpha)} r^{2/(\alpha-1)} \equiv \lambda_\alpha r^{2/(\alpha-1)}, \quad \lambda_\alpha > 0. \tag{3.47}$$

From the definition of $f(r)$ it follows easily from (3.47) that, for $0 < r_1 < r < r_2$,

$$\left(\frac{v_\alpha(r)}{r}\right)' = \lambda_\alpha r^{(3-\alpha)/(\alpha-1)}, \quad 0 \le \alpha < 1. \tag{3.48}$$

Integration of (3.48) now yields (for some constant δ_α, and with $\eta_\alpha = \dfrac{1}{2}\lambda_\alpha(\alpha - 1)$),

$$v_\alpha(r) = \eta_\alpha r^{(\alpha+1)/(\alpha-1)} + \delta_\alpha r, \quad 0 < \alpha < 1 \tag{3.49}$$

for $0 < r_1 < r < r_2$, as the expression for the tangential velocity field, while for the angular velocity we have, of course,

$$\Omega_\alpha(r) = \eta_\alpha r^{2/(\alpha-1)} + \delta_\alpha, \quad 0 \le \alpha < 1. \tag{3.50}$$

Applying the boundary conditions (3.40b,c) we easily find that

$$\eta_\alpha = \frac{\Omega_1 - \Omega_2}{r_1^{2/(\alpha-1)} - r_2^{2/(\alpha-1)}}, \quad 0 \le \alpha < 1, \tag{3.51a}$$

$$\delta_\alpha = \frac{\Omega_1 r_2^{2/(\alpha-1)} - \Omega_2 r_1^{2/(\alpha-1)}}{r_2^{2/(\alpha-1)} - r_1^{2/(\alpha-1)}}, \quad 0 \le \alpha < 1 \tag{3.51b}$$

and, therefore, for $0 \le \alpha < 1$,

$$v_\alpha(r) = \left(\frac{\Omega_1 r_2^{2/(\alpha-1)} - \Omega_2 r_1^{2/(\alpha-1)}}{r_2^{2/(\alpha-1)} - r_1^{2/(\alpha-1)}}\right) r + \left(\frac{\Omega_1 - \Omega_2}{r_1^{2/(\alpha-1)} - r_2^{2/(\alpha-1)}}\right) r^{(\alpha+1)/(\alpha-1)}. \tag{3.52}$$

Remarks. (i) For $\alpha = 0$ it is easy to check that (3.52) reduces to the classical result as described by (3.1).

(ii) It is interesting to note the result for $\lim_{\alpha\to 1^-} v_\alpha(r)$ as given by (3.52). First of all we may write (since $\alpha - 1 = -|\alpha - 1|$, $0 \le \alpha < 1$) that

$$\eta_\alpha r^{(\alpha+1)/(\alpha-1)} = \frac{r_1(\Omega_1 - \Omega_2)}{\left(\frac{r}{r_1}\right)^{(\alpha+1)/|\alpha-1|}\left(1 - \left(\frac{r_1}{r_2}\right)^{(\alpha+1)/|\alpha-1|}\frac{r_1}{r_2}\right)}$$

where $r_2 > r > r_1$; therefore $\eta_\alpha^{(\alpha+1)/(\alpha-1)} \to 0$, as $\alpha \to 1^-$. Also,

$$\delta_\alpha = \Omega_2 \left[\frac{\left(\frac{\Omega_1}{\Omega_2}\right)\left(\frac{r_1}{r_2}\right)^{2/|\alpha-1|} - 1}{\left(\frac{r_1}{r_2}\right)^{2/|\alpha-1|} - 1}\right]$$

so that $\delta_\alpha \to \Omega_2$ as $\alpha \to 1^-$. Thus, for $\Omega_2 > \Omega_1$, $v_\alpha(r) \to \Omega_2 r$ as $\alpha \to 1^-$, which is a rigid-body rotation in which the tangential stresses are everywhere zero.

(iii) It is a relatively easy task to compute the tangential stress on an element of the surface of a cylinder of radius r, $r_1 \le r \le r_2$, if the tangential velocity distribution is prescribed by (3.52). We note that, as a consequence of (3.8) and (3.52),

$$e_{r\theta} = \frac{1}{\alpha - 1}\eta_\alpha r^{2/(\alpha-1)}, \quad r_1 \le r \le r_2 \tag{3.53}$$

and

$$|\boldsymbol{e}|^2 = \frac{1}{2}\left(v'(r) - \frac{v(r)}{r}\right)^2 = \frac{1}{2}(2e_{r\theta})^2 = 2e_{r\theta}^2. \tag{3.54}$$

Therefore, with $\epsilon = \mu_1 = 0, 0 \leq \alpha < 1$,

$$\begin{aligned}\tau_{r\theta} &= 2\mu_0 \left[2e_{r\theta}^2\right]^{-\alpha/2} e_{r\theta} \\ &= 2^{1-\alpha/2}\mu_0 \left(\frac{\eta_\alpha}{\alpha - 1}\right)^{1-\alpha} r^{-2} \\ &= 2^{1-\alpha/2}\mu_0 \left(\frac{|\eta_\alpha|}{1 - \alpha}\right)^{1-\alpha} r^{-2}\end{aligned}$$

since $\eta_\alpha < 0$ for $\Omega_2 > \Omega_1$ and $r_2 > r_1$. Substituting for η_α in the above expression for $\tau_{r\theta}$ and simplifying, we obtain, for $0 \leq \alpha < 1$,

$$\tau_{r\theta} = 2^{1-\alpha/2}\mu_0 \left[\frac{\Omega_2 - \Omega_1}{(1-\alpha)\left(r_1^{2/(\alpha-1)} - r_2^{2/(\alpha-1)}\right)}\right]^{1-\alpha} r^{-2}. \tag{3.55}$$

For the frictional couple exerted on the fluid inside a cylindrical surface of radius r by the fluid outside, $r_1 < r < r_2$, we then have

$$2\pi r^2 \tau_{r\theta} = 2^{1-\alpha/2}\pi\mu_0 \left[\frac{\Omega_2 - \Omega_1}{(1-\alpha)\left(r_1^{2/(\alpha-1)} - r_2^{2/(\alpha-1)}\right)}\right]^{1-\alpha} \tag{3.56}$$

which is, of course, in agreement with the classical result for $\alpha = 0$.

3.2.4 Existence, Uniqueness, and Continuous Dependence

In this section we will establish the existence of a unique solution to the boundary-value problem (3.42a,b,c) in an appropriate class of functions, and then show that the solution depends continuously on the parameters ϵ and μ_1 as both $\epsilon \to 0^+$ and $\mu_1 \to 0^+$; in a sense that will be made precise, below, this analysis will justify our study of the case in which $\epsilon = \mu_1 = 0$, i.e., the situation in which (3.42a,b,c) reduces to the boundary-value problem

$$\left[\frac{r^3\Omega'(r)}{(r\Omega'(r))^\alpha}\right]' = 0, \quad r_1 < r < r_2, \tag{3.57a}$$

$$\Omega(r_i) = \Omega_i \tag{3.57b}$$

whose solution is

$$\Omega_\alpha(r) = \left(\frac{\Omega_1 - \Omega_2}{r_1^{2/(\alpha-1)} - r_2^{2/(\alpha-1)}}\right) r^{2/(\alpha-1)} + \left(\frac{\Omega_1 r_2^{2/(\alpha-1)} - \Omega_2 r_1^{2/(\alpha-1)}}{r_2^{2/(\alpha-1)} - r_1^{2/(\alpha-1)}}\right) \tag{3.58}$$

for $r_1 \le r \le r_2$, $0 \le \alpha < 1$. When $\alpha = \mu_1 = 0$, (3.42a,b,c) reduces to the boundary-value problem based on the Stokes' constitutive law, namely,

$$(r^3\Omega'(r))' = 0, \quad r_1 < r < r_2, \tag{3.59a}$$

$$\Omega(r_i) = \Omega_i, \quad i = 1, 2 \tag{3.59b}$$

whose unique solution

$$\Omega_0(r) = \frac{1}{r^2}\left(\frac{\Omega_1 - \Omega_2}{r_1^{-2} - r_2^{-2}}\right) + \left(\frac{\Omega_1 r_1^2 - \Omega_2 r_2^2}{r_1^2 - r_2^2}\right) \tag{3.60}$$

may be obtained either from (3.1) or from (3.58), upon setting $\alpha = 0$. We note that $\Omega_0(r)$ satisfies

$$(r^3\Omega_0'''(r) + 3r^2\Omega_0''(r) - 3r\Omega_0'(r))' \equiv 0 \tag{3.61}$$

on (r_1, r_2) but that

$$(r\Omega_0'(r))' = \frac{4}{r^3}\left(\frac{\Omega_1 - \Omega_2}{r_1^{-2} - r_2^{-2}}\right) \ne 0. \tag{3.62}$$

Finally, when $\mu_1 = 0$, the boundary-value problem (3.42a,b,c) reduces to

$$\left[\frac{r^3\Omega'(r)}{(\epsilon + \frac{1}{2}(r\Omega'(r))^2)^{\alpha/2}}\right]' = 0, \quad r_1 < r < r_2, \tag{3.63a}$$

$$\Omega(r_i) = \Omega_i, \quad i = 1, 2 \tag{3.63b}$$

whose solution (granted that one exists and is uniquely determined) will be denoted as $\Omega_{\alpha,\epsilon}(r)$. In fact, if we denote the solution of the boundary-value problem (3.42a,b,c) by $\Omega(r; \epsilon, \mu_1, \alpha)$—again, granted that one exists and is uniquely determined—then we clearly have the identifications:

$$\begin{aligned} \Omega_{\alpha,\epsilon}(r) &= \Omega(r; \epsilon, 0, \alpha), \\ \Omega_0(r) &= \Omega(r; \epsilon, 0, 0), \\ \Omega_\alpha(r) &= \Omega(r; 0, 0, \alpha) \equiv \Omega_{\alpha,0}(r). \end{aligned} \tag{3.64}$$

Our first theorem is the existence and uniqueness result for the system (3.42a,b,c):

Theorem 3.1. *For* $\mu_1 > 0$ *the boundary-value problem* (3.42a,b,c) *has a unique solution* $\Omega(r; \epsilon, \mu, \alpha)$*, in* $H_0^2(r_1, r_2)$*, for all* α *such that* $0 \le \alpha < 1$*, and for all* $\epsilon \ge 0$.

Proof. The case $\alpha = 0$ is trivial; when $\alpha = 0$, (3.42a,b,c) reduces to the linear boundary-value problem

$$-(r^3\Omega'(r))' + \mu_1^*(r^3\Omega'''(r) + 3r^2\Omega''(r) - 3r\Omega'(r))' = 0 \tag{3.65a}$$

for $r_1 < r < r_2$, with

$$\Omega(r_i) = \Omega_i, \quad i = 1, 2, \tag{3.65b}$$

$$(r\Omega'(r))'(r_i) = 0, \quad i = 1, 2 \tag{3.65c}$$

which has, by the standard theory for linear boundary-value problems [Ev], a unique solution in $H^2(r_1, r_2)$; actually, it is easy to show that the unique solution of (3.65a,b,c) lies in $C^\infty(r_1, r_2)$.

Now, consider the case where $0 < \alpha < 1$. We denote by $\tilde{\Omega}(r)$ the unique classical solution of the boundary-value problem

$$(r^3\tilde{\Omega}'''(r) + 3r^2\tilde{\Omega}''(r) - 3r\tilde{\Omega}'(r))' = 0 \tag{3.66a}$$

for $r_1 < r < r_2$, with, once again

$$\tilde{\Omega}(r_i) = \Omega_i, \quad i = 1, 2, \tag{3.66b}$$

$$(r\tilde{\Omega}'(r))'(r_i) = 0, \quad i = 1, 2. \tag{3.66c}$$

We note that $\tilde{\Omega}(r) \in C^\infty(r_1, r_2)$ and that although, as a consequence of (3.61), the classical solution $\Omega_0(r)$ of (3.59a,b), satisfies (3.66a), $\tilde{\Omega}(r) \neq \Omega_0(r)$ by virtue of (3.62). If we set

$$u(r) = \Omega(r; \epsilon, \mu_1, \alpha) - \tilde{\Omega}(r) \tag{3.67}$$

then, as a consequence of (3.42a,b,c), coupled with (3.66a,b,c), we easily find that for $r_1 < r < r_2$, $u(r)$ satisfies ($\mu_1^* = \mu_1/\mu_0$),

$$-\left[\frac{r^3(u'(r) + \tilde{\Omega}'(r))}{(\epsilon + \frac{1}{2}(ru'(r) + r\tilde{\Omega}'(r))^2)^{\alpha/2}}\right]' + \mu_1^*(r^3u'''(r) + 3r^2u''(r) - 3ru'(r))' = 0 \tag{3.68a}$$

and

$$u(r_i) = 0, \quad i = 1, 2, \tag{3.68b}$$

$$(ru'(r))'(r_i) = 0, \quad i = 1, 2. \tag{3.68c}$$

Thus, in order to show that (3.42a,b,c) possesses a unique solution in $H_0^2(r_1, r_2)$, it is sufficient to prove that (3.68a,b,c) has a unique solution in $H_0^2(r_1, r_2)$.

Let $H = H_0^{3/2+\sigma}(r_1, r_2)$ with $0 < \sigma < 1/2$ and denote by W_M the closed ball of radius $M > 0$ in $H_0^2(r_1, r_2)$. As a consequence of standard embedding results (e.g., [Ev]) we know that W_M is compactly embedded in H for any $\sigma < 1/2$. For the sake of convenience, we define the linear map L by

$$(Lu)(r) = \mu_1^*(r^3u'''(r) + 3r^2u''(r) - 3ru'(r))' \tag{3.69}$$

and for a fixed, but arbitrary, $h \in H$ we consider the linear boundary-value problem

$$(Lu)(r) = \left[\frac{r^3(h'(r) + \tilde{\Omega}'(r))}{(\epsilon + \frac{1}{2}(rh'(r) + r\tilde{\Omega}'(r))^2)^{\alpha/2}}\right]', \quad r_1 < r < r_2, \tag{3.70a}$$

$$u(r_i) = (ru'(r))'(r_i) = 0, \quad i = 1, 2. \tag{3.70b}$$

With

$$a(p, q) \equiv \int_{r_1}^{r_2} p(r)(Lq)(r)\, dr \tag{3.71}$$

we have

$$\begin{aligned} a(u, u) &= \mu_1^* \int_{r_1}^{r_2} (r^3u'''(r) + 3r^2u''(r) - 3ru'(r))'u(r)\, dr \\ &= -\mu_1^* \int_{r_1}^{r_2} (r^3u'''(r) + 3r^2u''(r) - 3ru'(r))u'(r)\, dr \\ &= -\mu_1^* \int_{r_1}^{r_2} [r(r(ru')')' - 4ru'(r)]u'(r)\, dr \\ &= \mu_1^* \int_{r_1}^{r_2} r[(ru')'^2 + 4u'^2(r)]\, dr \\ &= \mu_1^* \int_{r_1}^{r_2} r[r^2u''(r)^2 + 2ru''(r)u'(r) + 5u'^2(r)]\, dr \end{aligned}$$

and, therefore,

$$\begin{aligned} a(u, u) &\geq \frac{3}{4}\mu_1^* \int_{r_1}^{r_2} [r^3u''(r)^2 + ru'^2(r)]\, dr \\ &\geq \alpha_0 ||u||^2_{H_0^2(r_1, r_2)} \end{aligned} \tag{3.72}$$

where $\alpha_0 = \dfrac{3}{4}\mu_1^* \min(r_1^3, r_1)$. By the Lax-Milgram Lemma (Appendix A) we may conclude that the boundary-value problem (3.70a,b) has a unique solution $u \in H_0^2(r_1, r_2)$ that satisfies

$$
\begin{aligned}
||u||_{H_0^2(r_1,r_2)} &\le \frac{1}{\alpha_0}\left|\left|\frac{r^3(h'(r)+\tilde{\Omega}'(r))}{(\epsilon+\frac{1}{2}(rh'(r)+r\tilde{\Omega}'(r))^2)^{\alpha/2}}\right|\right|_{L^2(r_1,r_2)} \\
&\le \frac{1}{\alpha_0}\left|\left|2^{\alpha/2} r^{3-\alpha}\left|h'(r)+\tilde{\Omega}'(r)\right|^{1-\alpha}\right|\right|_{L^2(r_1,r_2)} \\
&= \frac{1}{\alpha_0}\cdot 2^{\alpha/2}\left[\int_{r_1}^{r_2} r^{2(3-\alpha)/\alpha}\, dr\right]^{\alpha/2} \\
&\qquad \times \left|\left|h'(r)+\tilde{\Omega}'(r)\right|\right|_{L^2(r_1,r_2)}^{1-\alpha}
\end{aligned}
$$

where we have used the Hölder Inequality. Thus,

$$
||u||_{H_0^2(r_1,r_2)} \le C\left|\left|h'(r)+\tilde{\Omega}'(r)\right|\right|_{L^2(r_1,r_2)}^{1-\alpha} \tag{3.73}
$$

with $C = \dfrac{1}{\alpha_0} 2^{\alpha/2}\left[\displaystyle\int_{r_1}^{r_2} r^{\frac{2(3-\alpha)}{\alpha}}\, dr\right]^{\alpha/2}$. We now apply Young's inequality in the form

$$
|a|\cdot|b| \le \sigma|a|^p + \sigma^{-\frac{1}{p-1}}|b|^{p'}, \quad \frac{1}{p}+\frac{1}{p'}=1
$$

to (3.73), with $p = 1/(1-\alpha)$ and $\sigma = \dfrac{1}{4}$, and we obtain the estimate

$$
\begin{aligned}
||u||_{H_0^2(r_1,r_2)} &\le \frac{1}{4}\left|\left|h'(r)+\tilde{\Omega}'(r)\right|\right|_{L^2(r_1,r_2)} + \tilde{C} \\
&\le \frac{1}{4}\left(||h||_{H_0^1(r_1,r_2)} + ||\tilde{\Omega}||_{H_0^1(r_1,r_2)}\right) + \tilde{C}
\end{aligned} \tag{3.74}
$$

with $\tilde{C} = \tilde{C}(\alpha; r_1, r_2)$ independent of u.

We now define the mapping $T : h \to u$ where for each fixed $h \in H$, u is the unique solution of the boundary-value problem (3.70a,b). By virtue of the estimate (3.74), $\exists M > 0$, sufficiently large, such that $T : W_M \to W_M$; we want to show that the map T is continuous. For $h_1, h_2 \in W_M$, we set $u_1 = Th_1$ and $u_2 = Th_2$. Then

$$
\begin{aligned}
&\mu_1^*(r^3(u_1-u_2)''' + 3r^2(u_1-u_2)'' - 3r(u_1-u_2)')' \\
&\qquad = \left[\frac{r^3(h_1'(r)+\tilde{\Omega}'(r))}{z_1(r)}\right]' - \left[\frac{r^3(h_2'(r)+\tilde{\Omega}'(r))}{z_2(r)}\right]'
\end{aligned} \tag{3.75}
$$

with

$$z_i(r) = \left(\epsilon + \frac{1}{2}(rh_i'(r) + r\tilde{\Omega}'(r))^2\right)^{\alpha/2}, \quad i = 1, 2. \tag{3.76}$$

Multiplying (3.75) by $u_1 - u_2$, integrating the result over (r_1, r_2), and then integrating by parts, we obtain

$$\begin{aligned} &\mu_1^* \int_{r_1}^{r_2} [r^3(u_1 - u_2)''' + 3r^2(u_1 - u_2)'' - 3r(u_1 - u_2)']'(u_1 - u_2)\, dr \\ &= -\int_{r_1}^{r_2} \frac{r^3(h_1'(r) + \tilde{\Omega}'(r))}{z_1(r)}(u_1 - u_2)'\, dr + \int_{r_1}^{r_2} \frac{r^3(h_2'(r) + \tilde{\Omega}'(r))}{z_2(r)}(u_1 - u_2)'\, dr. \end{aligned} \tag{3.77}$$

Now, from (3.77), the definition of the bilinear form a in (3.71), and the estimate (3.72), we have

$$\begin{aligned} \alpha_0 ||u_1 - u_2||^2_{H_0^2(r_1, r_2)} \leq &\int_{r_1}^{r_2} \frac{r^3(h_2' - h_1')}{z_2(r)}(u_1 - u_2)'\, dr \\ &+ \int_{r_1}^{r_2} \frac{r^3(u_1 - u_2)'(h_1' + \tilde{\Omega}')[z_1(r) - z_2(r)]}{z_1(r) z_2(r)}\, dr \end{aligned} \tag{3.78}$$

where we have added (and subtracted) the integral $\int_{r_1}^{r_2} \frac{r^2(h_1' + \tilde{\Omega}')}{z_2}(u_2 - u_1)'\, dr$. For the case where $\epsilon > 0$ we have, clearly, that $z_i(r) \geq \epsilon^{\alpha/2}$ for $i = 1, 2$ and, since $h_1, h_2, u_1, u_2 \in W_M$, $\exists N > 0$ such that

$$|u_i| \leq N,\ |h_i| \leq N,\ \left|u_i'\right| \leq N,\ \left|h_i'\right| \leq N, \quad i = 1, 2 \tag{3.79}$$

Using this information now in (3.78) we find the estimate

$$\begin{aligned} \alpha_0 ||u_1 - u_2||^2_{H_0^2(r_1, r_2)} \leq &\frac{2N}{\epsilon^{\alpha/2}} r_2^3 \int_{r_1}^{r_2} \left|h_2'(r) - h_1'(r)\right| dr \\ &+ \frac{4N^2 r_2^3}{\epsilon^{\alpha}} \int_{r_1}^{r_2} |z_1(r) - z_2(r)|\, dr. \end{aligned} \tag{3.80}$$

Noting that with $\eta(\xi) = \left(\epsilon + \frac{1}{2}\xi^2\right)^{\alpha/2}$, for $\xi \in R^1$, the derivative $\eta'(\xi) = \frac{\alpha}{2}\xi\left(\epsilon + \frac{1}{2}\xi^2\right)^{-1+(\alpha/2)}$ is bounded on compact subsets of R^1, and taking account of (3.79), and the definition of $z_i(r)$, $i = 1, 2$, i.e., (3.76), we have, for some $C_1 > 0$,

$$|z_1(r) - z_2(r)| \leq C_1 \left|(rh_1'(r) + r\tilde{\Omega}'(r)) - (rh_2'(r) + r\tilde{\Omega}'(r))\right|$$

or

$$|z_1(r) - z_2(r)| \le C_1 r \left| h_1'(r) - h_2'(r) \right|. \tag{3.81}$$

Combining (3.81) with (3.80) we, therefore, deduce the existence of a $\tilde{C} > 0$, independent of u_i, $i = 1, 2$, such that

$$||u_1 - u_2||_{H_0^2(r_1,r_2)} \le \tilde{C} ||h_1 - h_2||_{H_0^1(r_1,r_2)} \tag{3.82}$$

thus establishing the continuity of the map $T : W_M \to W_M$ for the case in which $\epsilon > 0$. On the other hand, if $\epsilon = 0$ then by (3.72) and (3.77) we have

$$\alpha_0 ||u_1 - u_2||_{H_0^2(r_1,r_2)} \le 2^{\alpha/2} \int_{r_1}^{r_2} r^{3-\alpha} (u_1' - u_2') \left[\frac{h_2' + \tilde{\Omega}'}{\left| h_2' + \tilde{\Omega}' \right|^{\alpha}} - \frac{h_1' + \tilde{\Omega}'}{\left| h_1' + \tilde{\Omega}' \right|^{\alpha}} \right] dr. \tag{3.83}$$

However, for arbitrary $a, b \in R^1$, and $0 \le \alpha < 1$, it is an easy exercise to verify the elementary inequality

$$\left| \frac{a}{|a|^{\alpha}} - \frac{b}{|b|^{\alpha}} \right| \le 2^{\alpha} |a - b|^{1-\alpha}. \tag{3.84}$$

Combining, in this case, (3.79), (3.83), and (3.84), we find, with the help of the Hölder Inequality, that

$$\begin{aligned} \alpha_0 ||u_1 - u_2||^2_{H_0^2(r_1,r_2)} &\le C_2 \int_{r_1}^{r_2} \left| h_1' - h_2' \right|^{1-\alpha} dr \\ &\le C_3 \left[\int_{r_1}^{r_2} \left| h_1' - h_2' \right|^2 dr \right]^{(1-\alpha)/2} \end{aligned}$$

for some $C_2, C_3 > 0$. Therefore, with $C_4 = C_3/\alpha_0$

$$||u_1 - u_2||_{H_0^2(r_1,r_2)} \le C_4 ||h_1 - h_2||^{(1-\alpha)/2}_{H_0^1(r_1,r_2)} \tag{3.85}$$

and the continuity of $T : W_M \to W_M$ again follows, this time for the case $\epsilon = 0$.

As a direct consequence of the Schauder fixed-point theorem we may now conclude that there exists, for $M > 0$ sufficiently large, a unique $u \in W_M$ such that $Tu = u$; this establishes, of course, the existence of a unique solution u of the boundary-value problem (3.68a,b,c), for $\mu_1 > 0$, and, hence, for the original boundary-value problem (3.42a,b,c). □

We now turn our attention to the boundary-value problem (3.63a,b), which is, of course, what the boundary-value problem (3.42a,b,c) formally reduces to if we set $\mu_1 = 0$; our basic result may be stated as follows:

Theorem 3.2. *For $\epsilon \geq 0$, if a solution of the boundary-value problem* (3.63a,b) *exists in $H^1(r_1, r_2)$, then the solution is unique.*

Proof. For $\epsilon = 0$ the unique solution of (3.63a,b), $\Omega_\alpha(r) = \Omega_{\alpha,0}(r)$, is given explicitly by (3.58); so we turn to the case in which $\epsilon > 0$. We will establish the uniqueness of solutions to (3.63a,b) under the assumption that solutions do exist in $H^1(r_1, r_2)$. So, suppose that $\omega_1, \omega_2 \in H^1(r_1, r_2)$ are solutions of (3.63a,b). Then

$$-\left[\frac{r^3\omega_1'(r)}{\left(\epsilon + \frac{1}{2}(r\omega_1'(r))^2\right)^{\alpha/2}}\right]' + \left[\frac{r^3\omega_2'(r)}{\left(\epsilon + \frac{1}{2}(r\omega_2'(r))^2\right)^{\alpha/2}}\right]' = 0. \tag{3.86}$$

We note that $\omega_1(r_i) = \Omega_i$, $\omega_2(r_i) = \Omega_i$, $i = 1, 2$. Multiplying (3.86) by $\omega_1 - \omega_2$, and integrating by parts, we obtain

$$\int_{r_1}^{r_2} r^2(\omega_1' - \omega_2') \left[\frac{r\omega_1'(r)}{\left(\epsilon + \frac{1}{2}(r\omega_1'(r))^2\right)^{\alpha/2}} - \frac{r\omega_2'(r)}{\left(\epsilon + \frac{1}{2}(r\omega_2'(r))^2\right)^{\alpha/2}}\right] dr = 0. \tag{3.87}$$

Now for any $a, b \in R^1$, $\epsilon \geq 0$, and $0 \leq \alpha < 1$, we have the elementary inequality

$$(a - b)\left(\frac{a}{\left(\epsilon + \frac{1}{2}a^2\right)^{\alpha/2}} - \frac{b}{\left(\epsilon + \frac{1}{2}b^2\right)^{\alpha/2}}\right) \geq 0 \tag{3.88}$$

and, therefore, as a direct consequence of (3.87) we have $\omega_1' = \omega_2'$ a.e. on $[r_1, r_2]$. However, $\omega_1(r_i) = \omega_2(r_i)$, $i = 1, 2$; so $\omega_1 = \omega_2$ a.e. on (r_1, r_2) and it follows that solutions of (3.63a,b) in $H^1(r_1, r_2)$ are unique if they exist. □

There remains the task of showing that, for $\epsilon > 0$, there exist solutions of (3.63a,b) in $H^1(r_1, r_2)$; to do this we will show that the unique solution of (3.42a,b,c) converges, as $\mu_1 \to 0^+$, to the (unique) solution of (3.63a,b); the convergence will be in the norm of $C^{1+\sigma}$, for $0 < \sigma < 1/2$, and will also establish the continuous dependence of the solutions of (3.42a,b,c) on μ_1 as $\mu_1 \to 0^+$. The precise result is the following:

Theorem 3.3. *As $\mu_1 \to 0^+$, the unique solution of the boundary-value problem* (3.42a,b,c)*, $\Omega(\cdot; \epsilon, \mu_1, \alpha)$, converges in $C^{1,\sigma}(r_1, r_2)$, $0 < \sigma < 1/2$, to the (unique) solution $\Omega_{\epsilon,\alpha}(\cdot)$ of* (3.63a,b) *and, in fact, $\Omega_{\epsilon,\alpha}(\cdot) = \hat{\Omega}(\cdot)$, where $\hat{\Omega}(\cdot)$ is given by* (3.130)*.*

Proof. As in the proof of Theorem 3.1, we set $u(r) = \Omega(r;\epsilon,\mu_1,\alpha) - \tilde{\Omega}(r)$ with $\Omega(r;\epsilon,\mu_1,\alpha)$ the unique solution of (3.42a,b,c), for $\mu_1 > 0$, and $\tilde{\Omega}(r)$ the unique solution of (3.66a,b,c). We also set

$$\begin{aligned} s(r) &= u'(r), \\ W_0(r) &= r\tilde{\Omega}'(r), \\ W(r) &= r\Omega'(r;\epsilon,\mu_1,\alpha), \\ Z(r) &= \epsilon + \frac{1}{2}W^2(r). \end{aligned} \tag{3.89}$$

Clearly, both $u(r)$ and $s(r)$ depend on μ_1 but we will refrain, for the time being, from writing $u_{\mu_1}(r)$ or $s_{\mu_1}(r)$. Using the notation in (3.67), (3.89), and the fact that $\tilde{\Omega}(r)$ satisfies (3.66a), we may rewrite (3.42a) in the form

$$-\left[\frac{r^2W(r)}{Z(r)^{\alpha/2}}\right]' + \mu_1^*(r^3s''(r) + 3r^2s'(r) - 3rs(r))' = 0. \tag{3.90}$$

Integrating (3.90) over (r_1, r), for $r \le r_2$, we find that

$$-\frac{r^2W(r)}{Z(r)^{\alpha/2}} + \mu_1^*(r^3s''(r) + 3r^2s'(r) - 3rs(r)) = A_{\mu_1} \tag{3.91}$$

where

$$A_{\mu_1} = \frac{r_1^2W(r_1)}{Z(r_1)^{\alpha/2}} + \mu_1^*(r_1^3s''(r_1) + 3r_1^2s'(r_1) - 3r_1s(r_1)). \tag{3.92}$$

By virtue of (3.42b) and (3.66b),

$$\begin{aligned} \int_{r_1}^{r_2} s(r)\,dr &= u(r_2) - u(r_1) \\ &= (\Omega(r_2) - \tilde{\Omega}(r_2)) - (\Omega(r_1) - \tilde{\Omega}(r_1)) \\ &= 0. \end{aligned} \tag{3.93}$$

Therefore, if we multiply (3.91) by $s(r)$, integrate over (r_1, r_2), and then integrate by parts, we obtain

$$\int_{r_1}^{r_2} \frac{rW^2(r)}{Z(r)^{\alpha/2}}\,dr - \int_{r_1}^{r_2} \frac{rW(r)W_0(r)}{Z(r)^{\alpha/2}}\,dr + \mu_1^* \int_{r_1}^{r_2} r\left[(rs(r))'^2 + 4s^2(r)\right]dr = 0 \tag{3.94}$$

where we have used (3.93), the obvious relation $rs(r) = W(r) - W_0(r)$, and the fact that as $(rs(r))'(r_i) = 0$, for $i = 1, 2$, so that

$$\int_{r_1}^{r_2} (r^3 s''(r) + 3r^2 s'(r) - 3rs(r))s(r)\,dr = \int_{r_1}^{r_2} [r(r(rs(r))')' - 4rs(r)]s(r)\,dr$$
$$= -\int_{r_1}^{r_2} r[(rs(r))'^2 + 4s^2(r)]\,dr. \tag{3.95}$$

Now, for $\epsilon > 0$, we set

$$E_\epsilon = \{r \mid W^2(r) > 2\epsilon\}. \tag{3.96}$$

Then $\forall r \in E_\epsilon$,

$$\frac{rW^2(r)}{Z(r)^{\alpha/2}} = \frac{rW^2(r)}{\left(\epsilon + \frac{1}{2}W^2(r)\right)^{\alpha/2}} > r_1|W(r)|^{2-\alpha} \tag{3.97}$$

while $\forall r \in \{[r_1, r_2]/E_\epsilon\} \equiv E_\epsilon^c$, we have $|W(r)|^{2-\alpha} \le (2\epsilon)^{(2-\alpha)/2}$. Therefore,

$$\begin{aligned}\int_{r_1}^{r_2} |W(r)|^{2-\alpha} dr &= \int_{E_\epsilon} |W(r)|^{2-\alpha}\,dr + \int_{E_\epsilon^c} |W(r)|^{2-\alpha}\,dr \\ &\le \frac{1}{r_1}\int_{r_1}^{r_2} \frac{rW^2(r)}{Z(r)^{\alpha/2}}\,dr + (2\epsilon)^{(2-\alpha)/2}\,\text{meas}(E_\epsilon^c)\end{aligned} \tag{3.98}$$

where $\text{meas}(E_\epsilon^c) \le r_2 - r_1$. Combining (3.94) with (3.98) we obtain the estimate

$$\int_{r_1}^{r_2} |W(r)|^{2-\alpha} dr + \frac{\mu_1^*}{r_1}\int_{r_1}^{r_2} r[rs(r))'^2 + 4s^2(r)]\,dr \le \frac{1}{r_1}\int_{r_1}^{r_2} \frac{rW(r)W_0(r)}{Z(r)^{\alpha/2}} dr + (2\epsilon)^{(2-\alpha)/2}(r_2 - r_1). \tag{3.99}$$

By virtue of the Hölder Inequality, and the definition of $Z(r)$, i.e., (3.89), we have the following estimate for the integral on the right-hand side of (3.99):

$$\left|\int_{r_1}^{r_2} \frac{rW(r)W_0(r)}{Z(r)^{\alpha/2}}\,dr\right| \le 2^{\alpha/2}\int_{r_1}^{r_2} r|W(r)|^{1-\alpha}|W_0(r)|\,dr$$
$$\le 2^{\alpha/2}\left[\int_{r_1}^{r_2} |W(r)|^{2-\alpha}\,dr\right]^{(1-\alpha)/(2-\alpha)} \times \left[\int_{r_1}^{r_2} |rW_0(r)|^{2-\alpha}\,dr\right]^{1/(2-\alpha)}. \tag{3.100}$$

Applying Young's inequality to (3.100), with $p = \dfrac{2-\alpha}{1-\alpha}$ and $\sigma = \dfrac{r_1}{2^{1+\alpha/2}}$, we now find that

$$\frac{1}{r_1}\left|\int_{r_1}^{r_2} \frac{rW(r)W_0(r)}{Z(r)^{\alpha/2}}\,dr\right| \le \frac{1}{2}\int_{r_1}^{r_2} |W(r)|^{2-\alpha}\,dr + C_* \tag{3.101}$$

with C_* independent of both μ_1 and ϵ, where we have used the fact that $W_0(\cdot) \in L^\infty(r_1, r_2)$. Combining (3.99) with (3.101) we are led to the estimate

$$\begin{aligned}\int_{r_1}^{r_2} |W(r)|^{2-\alpha}\,dr + \frac{2\mu_1^*}{r_1}\int_{r_1}^{r_2} r[rs(r))'^2 + 4s^2(r)]\,dr \\ \le 2(2\epsilon)^{(2-\alpha)/2}(r_2 - r_1) + 2C_*.\end{aligned} \tag{3.102}$$

For our next set of estimates, we multiply (3.91) by $(rs(r))''$, integrate over (r_1, r_2), and then integrate by parts; since

$$A_{\mu_1}\int_{r_1}^{r_2}(rs(r))''\,dr = A_{\mu_1}(rs(r))'\Big|_{r_1}^{r_2} = 0 \tag{3.103}$$

and

$$\begin{aligned}\int_{r_1}^{r_2}(r^3 s''(r) + 3r^2 s'(r) - 3rs(r))(rs(r))''\,dr \\ = \int_{r_1}^{r_2}[r^2(rs(r))''^2 + 3(rs(r))'^2 + r^2 s'(r)(rs(r))'']\,dr\end{aligned} \tag{3.104}$$

we obtain

$$\begin{aligned}\int_{r_1}^{r_2}\left[\frac{r^2 W(r)}{Z(r)^{\alpha/2}}\right]'(rs(r))'\,dr + \mu_1^*\int_{r_1}^{r_2} r^2(rs(r))''^2\,dr \\ + 3\mu_1^*\int_{r_1}^{r_2}(rs(r))'^2\,dr + \mu_1^*\int_{r_1}^{r_2} r^2 s'(r)(rs(r))''\,dr = 0.\end{aligned} \tag{3.105}$$

We now note that

$$\begin{aligned}\int_{r_1}^{r_2}\left[\frac{r^2 W(r)}{Z(r)^{\alpha/2}}\right]'(rs(r))'\,dr &= \int_{r_1}^{r_2}\left(\frac{2rW(r)}{Z(r)^{\alpha/2}} + r^2\left[\frac{W(r)}{Z(r)^{\alpha/2}}\right]'\right)(rs(r))'\,dr \\ &= \int_{r_1}^{r_2} r^2\left[\frac{W(r)}{Z(r)^{\alpha/2}}\right]' W'(r)\,dr - \int_{r_1}^{r_2} r^2\left[\frac{W(r)}{Z(r)^{\alpha/2}}\right]' W_0'(r)\,dr \\ &\quad + \int_{r_1}^{r_2}\frac{2rW(r)}{Z(r)^{\alpha/2}}(rs(r))'\,dr \\ &= \int_{r_1}^{r_2} r^2\left[\frac{W(r)}{Z(r)^{\alpha/2}}\right]' W'(r)\,dr + \int_{r_1}^{r_2}\frac{W(r)}{Z(r)^{\alpha/2}}(r^2 W_0'(r))'\,dr \\ &\quad + \int_{r_1}^{r_2}\frac{2rW(r)}{Z(r)^{\alpha/2}}(rs(r))'\,dr\end{aligned} \tag{3.106}$$

where we have used the fact that $rs(r) = W(r) - W_0(r)$ as well as the boundary conditions $W_0'(r_i) = (r\tilde{\Omega}'(r))'(r_i) = 0$, $i = 1, 2$. For the first integral on the right-hand side of the last equation in (3.106), we compute that

$$\int_{r_1}^{r_2} r^2 \left[\frac{W(r)}{Z(r)^{\alpha/2}}\right]' W'(r)\, dr = \int_{r_1}^{r_2} r^2 \frac{W'^2(r)}{Z(r)^{\alpha/2}} \left\{\frac{\epsilon + \frac{1}{2}(1-\alpha)W^2(r)}{\epsilon + \frac{1}{2}W^2(r)}\right\} dr. \tag{3.107}$$

However, $\forall \alpha$ such that $0 \leq \alpha < 1$,

$$1 - \alpha \leq \frac{\epsilon + \frac{1}{2}(1-\alpha)\eta}{\epsilon + \frac{1}{2}\eta} \leq 1, \quad \forall \eta \geq 0. \tag{3.108}$$

Therefore, combining (3.105)–(3.108) we easily obtain the estimate

$$\begin{aligned}
(1-\alpha) \int_{r_1}^{r_2} r^2 \frac{W'^2(r)}{Z(r)^{\alpha/2}}\, dr + \mu_1^* \int_{r_1}^{r_2} r^2 (rs(r))''^2\, dr \\
3\mu_1^* \int_{r_1}^{r_2} (rs(r))'^2\, dr \leq -\mu_1^* \int_{r_1}^{r_2} r^2 s'(r)(rs(r))''\, dr \\
- \int_{r_1}^{r_2} \frac{W(r)}{Z(r)^{\alpha/2}} (r^2 W_0'(r))'\, dr - \int_{r_1}^{r_2} \frac{2rW(r)}{Z(r)^{\alpha/2}} (rs(r))'\, dr.
\end{aligned} \tag{3.109}$$

Using Young's inequality, and the estimate (3.102), we now note the following series of estimates for the first integral on the right-hand side of (3.109):

$$\begin{aligned}
&\left| -\mu_1^* \int_{r_1}^{r_2} r^2 s'(r)(rs(r))''\, dr \right| \\
&\leq \mu_1^* \int_{r_1}^{r_2} \left| rs'(r) \right| \cdot \left| r(rs(r))'' \right| dr \\
&\leq \mu_1^* \left[\int_{r_1}^{r_2} r^2 s'^2(r)\, dr\right]^{1/2} \left[\int_{r_1}^{r_2} r^2 (rs(r))''^2\, dr\right]^{1/2} \\
&\leq \frac{1}{2}\mu_1^* \int_{r_1}^{r_2} r^2 (rs(r))''^2\, dr + 2\mu_1^* \int_{r_1}^{r_2} r^2 s'^2(r)\, dr \\
&\leq \frac{1}{2}\mu_1^* \int_{r_1}^{r_2} r^2 (rs(r))''^2\, dr + 2\mu_1^* \int_{r_1}^{r_2} [(rs(r))' - s(r)]^2\, dr \\
&\leq \frac{1}{2}\mu_1^* \int_{r_1}^{r_2} r^2 (rs(r))''^2\, dr + 4\mu_1^* \int_{r_1}^{r_2} ((rs(r))'^2 + s^2(r))\, dr \\
&\leq \frac{1}{2}\mu_1^* \int_{r_1}^{r_2} r^2 (rs(r))''^2\, dr + \frac{4\mu_1^*}{r_1} \int_{r_1}^{r_2} r[(rs(r))'^2 + s^2(r)]\, dr.
\end{aligned}$$

Therefore,

$$\left|-\mu_1^* \int_{r_1}^{r_2} r^2 s'(r)(rs(r))''\, dr\right| \le \frac{1}{2}\mu_1^* \int_{r_1}^{r_2} r^2 (rs(r))''^2\, dr + 4(r_2-r_1)(2\epsilon)^{(2-\alpha)/2} + 4C_*. \tag{3.110}$$

For the second integral on the right-hand side of (3.109) we have the estimates

$$\begin{aligned} \left|\int_{r_1}^{r_2} \frac{W(r)}{Z(r)^{\alpha/2}} (r^2 W_0'(r))'\, dr\right| &\le 2^{\alpha/2} \int_{r_1}^{r_2} |W(r)|^{1-\alpha} \left|r^2 W_0'(r))'\right| dr \\ &\le \frac{1}{2}\int_{r_1}^{r_2} |W(r)|^{2-\alpha}\, dr + C_1 \end{aligned} \tag{3.111}$$

where we have, once again, used the Hölder and Young inequalities, and C_1 is independent of both μ_1 and ϵ. Finally, for the last integral on the right-hand side of (3.109), we compute that

$$\left|\int_{r_1}^{r_2} \frac{2rW(r)}{Z(r)^{\alpha/2}} \cdot (rs(r))'\, dr\right| \le \left|\int_{r_1}^{r_2} \frac{2rW(r)}{Z(r)^{\alpha/2}} \cdot W'(r)\, dr\right| + \left|\int_{r_1}^{r_2} \frac{2rW(r)}{Z(r)^{\alpha/2}} \cdot W_0'(r)\, dr\right| \tag{3.112}$$

since $rs(r) = W(r) - W_0(r)$. However,

$$\begin{aligned} \left|\int_{r_1}^{r_2} \frac{2rW(r)}{Z(r)^{\alpha/2}} W_0'(r)\, dr\right| &\le 2^{\frac{\alpha}{2}+1} \int_{r_1}^{r_2} |W(r)|^{1-\alpha} \left|rW_0'(r)\right| dr \\ &\le \frac{1}{2}\int_{r_1}^{r_2} |W(r)|^{2-\alpha}\, dr + C_2 \end{aligned} \tag{3.113}$$

with C_2 independent of μ_1 and ϵ, while

$$\begin{aligned} \left|\int_{r_1}^{r_2} \frac{2rW(r)}{Z(r)^{\alpha/2}} W'(r)\, dr\right| &\le 2\int_{r_1}^{r_2} \left|\frac{W(r)}{Z(r)^{\alpha/4}}\right| \cdot \left|\frac{rW'(r)}{Z(r)^{\alpha/4}}\right| dr \\ &\le 2\left[\int_{r_1}^{r_2} \frac{W^2(r)}{Z(r)^{\alpha/2}}\, dr\right]^{1/2} \left[\int_{r_1}^{r_2} \frac{r^2 W'^2(r)}{Z(r)^{\alpha/2}}\, dr\right]^{1/2} \\ &\le \frac{1}{2}(1-\alpha)\int_{r_1}^{r_2} \frac{r^2 W'^2(r)}{Z(r)^{\alpha/2}}\, dr + \frac{8}{1-\alpha}\int_{r_1}^{r_2} \frac{W^2(r)}{Z(r)^{\alpha/2}}\, dr \\ &\le \frac{1}{2}(1-\alpha)\int_{r_1}^{r_2} \frac{r^2 W'^2(r)}{Z(r)^{\alpha/2}}\, dr + \frac{8}{1-\alpha}\int_{r_1}^{r_2} |W|^{2-\alpha}\, dr. \end{aligned} \tag{3.114}$$

Combining (3.109)–(3.114) with (3.102) we obtain an estimate of the form

$$\int_{r_1}^{r_2} \frac{r^2 W'^2(r)}{Z(r)^{\alpha/2}}\,dr + \frac{\mu_1^*}{1-\alpha}\int_{r_1}^{r_2} r^2 (rs(r))''^2\,dr + \frac{6\mu_1^*}{1-\alpha}\int_{r_1}^{r_2} (rs(r))'^2\,dr \le C_3\left(1 + (r_2 - r_1)(2\epsilon)^{(2-\alpha)/2}\right) \tag{3.115}$$

with C_3 independent of both μ_1 and ϵ. We now set

$$\psi(W) = \int_0^W \frac{d\xi}{\left(\epsilon + \frac{1}{2}\xi^2\right)^{\alpha/4}}. \tag{3.116}$$

Then, as a consequence of the estimate (3.115), we obtain

$$\begin{aligned}\int_{r_1}^{r_2} r^2\left[\frac{d}{dr}\psi(W(r))\right]^2 dr &= \int_{r_1}^{r_2} \frac{r^2 W'^2(r)}{\left(\epsilon + \frac{1}{2}W^2(r)\right)^{\alpha/2}}\,dr \\ &\le C_3\left(1 + (r_2 - r_1)(2\epsilon)^{(2-\alpha)/2}\right).\end{aligned} \tag{3.117}$$

Since

$$\frac{1}{\left(\epsilon + \frac{1}{2}\xi^2\right)^{\alpha/4}} \le 2^{\alpha/4}|\xi|^{-\alpha/2}, \quad \forall \xi \in R^1$$

we have, for $0 \le \alpha < 1$,

$$\begin{aligned}|\psi(W)| &\le \psi(|W|) \\ &\le 2^{\alpha/4}(1-\alpha/2)^{-1}|W|^{1-\alpha/2} \\ &\le 4|W|^{1-\alpha/2}.\end{aligned} \tag{3.118}$$

Employing the estimate (3.102), in conjunction with the bound (3.118), we see that $\exists \psi_0 > 0$ (const.) such that

$$\int_{r_1}^{r_2} \psi^2(W(r))\,dr < \psi_0 \tag{3.119}$$

with ψ_0 independent of μ_1. Now $\forall f(\cdot) \in H^1(r_1, r_2)$, and $\forall \sigma > 0$, $\exists C_\sigma > 0$ such that

$$\max_{[r_1,r_2]} |f(r)| \le \sigma\left(\int_{r_1}^{r_2} f'^2(r)\,dr\right)^{1/2} + C_\sigma\left(\int_{r_1}^{r_2} f^2(r)\,dr\right)^{1/2} \tag{3.120}$$

(see, e.g., [Lio1, Lemma 5.1]); applying (3.120) with $f(r) = \psi(W(r))$, and employing both (3.117) and (3.119), we determine that for some $C_4 > 0$, C_4 independent of μ_1, we have

$$\max_{[r_1,r_2]} |\psi(W(r))| \le C_4. \tag{3.121}$$

In view of the definition (3.116), for $|\rho| > (2\epsilon)^{1/2}$,

$$|\psi(\rho)| = \int_0^{|\rho|} \frac{d\xi}{\left(\epsilon + \frac{1}{2}\xi^2\right)^{\alpha/4}} \ge (1-\alpha/2)^{-1}\left[|\rho|^{1-\alpha/2} - \epsilon^{1-\alpha/2}\right]$$

or

$$|\rho|^{1-\alpha/2} \le (1-\alpha/2)|\psi(\rho)| + \epsilon^{1-\alpha/2}, \text{ for } |\rho| > (2\epsilon)^{1/2}. \tag{3.122}$$

Combining (3.122) with (3.121), we easily deduce the existence of a constant $C > 0$, independent of μ_1, such that

$$\max_{[r_1,r_2]} |W(r)| \le C. \tag{3.123}$$

If we now use (3.117) in conjunction with (3.119), and no longer suppress the dependence of W, u, or s on μ_1, it follows that $\exists \tilde{C} > 0$, independent of μ_1, such that

$$\left|\left|\psi(W_{\mu_1}(\cdot)\right|\right|_{H^1(r_1,r_2)} \le \tilde{C}. \tag{3.124}$$

As a consequence of the uniform bound (3.124), if $\{\mu_n\}$, $\mu_n > 0$ for each integer n, is a sequence such that $\mu_n \to 0^+$, as $n \to \infty$, then there exists a subsequence $\{\mu_{n_k}\}$, and a function $\psi^0 \in H^1(r_1, r_2)$, such that as $n_k \to \infty$,

$$\psi(W_{\mu_{n_k}}) \to \psi^0, \text{ in } H^1(r_1, r_2). \tag{3.125}$$

From standard embedding results (Appendix A) we deduce from (3.125) that we also have, as $n_k \to \infty$,

$$\psi(W_{\mu_{n_k}}) \to \psi^0, \text{ in } C^{0,\sigma}, \quad 0 < \sigma < \frac{1}{2}. \tag{3.126}$$

Also, as ψ is monotone, ψ is invertible; thus, by (3.126), for $n_k \to \infty$,

$$W_{\mu_{n_k}} \to \psi^{-1}(\psi^0), \text{ in } C^{0,\sigma}(r_1, r_2), \quad 0 < \sigma < 1/2. \tag{3.127}$$

Using the definition of $W(r)$, i.e. (3.89), and (3.127), we now find that, as $n_k \to \infty$,

$$\Omega'(\cdot; \epsilon, \mu_{n_k}, \alpha) \to \frac{1}{r}\psi^{-1}(\psi^0), \text{ in } C^{0,\sigma}(r_1, r_2) \tag{3.128}$$

and

$$\Omega(\cdot;\epsilon,\mu_{n_k},\alpha)\to\hat{\Omega}(\cdot),\ \text{in } C^{1,\sigma}(r_1,r_2) \tag{3.129}$$

for $0<\sigma<1/2$, where for $r\in[r_1,r_2]$,

$$\hat{\Omega}(r)=\Omega_1+\int_{r_1}^{r}\frac{1}{\xi}\psi^{-1}(\psi_0(\xi))\,d\xi. \tag{3.130}$$

By virtue of (3.102) and (3.115), we see that for some constant $C>0$, independent of μ_1,

$$\mu_{n_k}\left|\left|s_{\mu_{n_k}}(\cdot)\right|\right|^2_{H^2(r_1,r_2)}\le C \tag{3.131}$$

so that

$$\begin{cases}\mu_{n_k}s_{\mu_{n_k}}\to 0,\\ \mu_{n_k}s'_{\mu_{n_k}}\to 0,\\ \mu_{n_k}s''_{\mu_{n_k}}\to 0\end{cases} \tag{3.132}$$

in $L^2(r_1,r_2)$ as $n_k\to\infty$. Employing (3.129) and (3.132) it is easy to show that $\hat{\Omega}(\cdot)$ is a solution of (3.63a,b); however, we have already shown that (Theorem 3.2) the solution of the boundary-value problem (3.63a,b) is, for $\epsilon\ge 0$, uniquely defined in $H^1(r_1,r_2)$ if it exists. The proof of the Theorem 3.3 is now complete. □

Remarks. By examining the estimates that led us to the uniform bound (3.131), i.e., (3.102) and (3.105), it is an easy exercise to show that the solutions of (3.42a,b,c) also depend continuously on ϵ, as $\epsilon\to 0^+$, in the norm of $C^{2+\sigma}(r_1,r_2)$, for $0<\sigma<1/2$.

3.3 Bubble Stability in an Incompressible Non-Newtonian Viscous Fluid

3.3.1 Introduction

In this section we consider the dynamical behavior of a spherical cavity which is immersed in an unbounded, incompressible, viscous, non-Newtonian fluid; we assume that the associated Cauchy stress tensor for the fluid is of the form (2.1a), with $\epsilon=\mu_1=0$. This subsection will begin by briefly reviewing some of the literature on bubble dynamics in Newtonian and non-Newtonian fluids. Then, in Sect. 3.3.2 a non-Newtonian version of the basic Rayleigh-Plesset equation is derived; some examples of linearized dynamics for the associated model are worked out in Sect. 3.3.3. Finally, a theorem concerning the nonlinear stability of the equilibrium states of spherical vapor cavities is proven by establishing the existence of a suitable Liapunov function.

The basic equation governing the dynamical behavior of a spherical cavity of radius $R(t)$, at time $t > 0$, in an unbounded Newtonian fluid, under the assumption that there is negligible mass transfer at the cavity/fluid interface, is the well-known Rayleigh-Plesset equation, i.e.,

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = \frac{1}{\rho_\ell}\left(p_v - p_\infty - \frac{2\sigma}{R} - 4\mu_0\frac{\dot{R}}{R}\right) \tag{3.133}$$

where p_v is the (vapor) pressure acting on the inner surface of the cavity/fluid interface, p_∞ is the pressure at a very large (infinite) distance from the cavity, μ_0 and ρ_ℓ are the (assumed) constant viscosity and density of the fluid, and σ is the interfacial tension. Although a general time-dependent boundary condition of the form $p_\infty = p_\infty(t)$ may be considered, for our purposes in this section, we shall assume that p_∞ = const. In (3.133) the superposed dot denotes time differentiation. There are many excellent discussions in the literature of bubble dynamics in Newtonian fluids and, thus, we will forgo here surveying the various consequences of (3.133); for an extensive review of the literature related to the Rayleigh-Plesset equation one may consult the survey in [PP]. An elementary treatment of various solutions which have been obtained for the coupled systems consisting of (3.133) and associated energy or diffusion equations may be found in [PP] as well as in the tutorial [Th]. The stability of spherical vapor cavities in an unbounded viscous Newtonian fluid has been studied by Plesset and Mitchell [PM].

In the recent literature there have been several efforts directed at studying the dynamical behavior of spherical vapor cavities in non-Newtonian or viscoelastic fluids; most notable among these are the papers [FG, Pr], and [BK]; in what follows, below, we will briefly summarize the derivation of the model equations which are to be found in these papers and will attempt to indicate the relations which exist between them, the Rayleigh-Plesset model, and the non-Newtonian model equations that will be delineated in Sect. 3.3.2. If, following Prosperetti [Pr] we denote by $U_\ell(t)$ the radial liquid velocity at the cavity/fluid interface then, irrespective of the fluid rheology, it follows from mass conservation (as will be shown in Sect. 3.3.2) that

$$u(r,t) = \left(\frac{R^2}{r^2}\right)U_\ell(t) \tag{3.134}$$

where $u(r,t)$ is the radial liquid velocity at a distance r from the center of the cavity. The radial component of the momentum equation may be easily shown [Pr] to have the form

$$\rho_\ell\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r}\right) = -\frac{\partial p}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{rr}) - \left(\frac{\tau_{\theta\theta} + \tau_{\phi\phi}}{r}\right) \tag{3.135}$$

in spherical coordinates, where $p = p(r,t)$ is the pressure while τ_{rr}, $\tau_{\theta\theta}$, and $\tau_{\phi\phi}$ are the diagonal components of the Cauchy stress tensor in the fluid. As the fluid is assumed to be incompressible, $\nabla \cdot \boldsymbol{\tau} = 0$, or

$$\tau_{\theta\theta} + \tau_{\phi\phi} = -\tau_{rr}. \tag{3.136}$$

Using (3.136) in (3.135), introducing the expression (3.134) for $u(r,t)$, and then integrating the resulting equation from $r = R(t)$ to $r = \infty$ yields

$$R\dot{U}_\ell + 2\dot{R}U_\ell - \frac{1}{2}U_\ell^2 = \frac{1}{\rho_\ell}\left(p(R,t) - p_\infty - \tau_{rr}(R,t) + 3\int_R^\infty r^{-1}\tau_{rr}\,dr\right). \tag{3.137}$$

As a consequence of conservation of mass and momentum across the cavity/fluid interface, it follows from the work in [Pr] that

$$J \equiv \rho_\ell(U_\ell - \dot{R}) = \rho_\nu(U_\nu - \dot{R}), \tag{3.138a}$$

$$J^2\left(\frac{1}{\rho_\nu} - \frac{1}{\rho_\ell}\right) + p_\nu - p(R,t) + \tau_{rr}(R,t) = \frac{2\sigma}{R} \tag{3.138b}$$

where J is the mass flux across the interface while ρ_ν and U_ν are, respectively, the density and velocity of the vapor at the inner surface of the cavity/fluid interface. If $J = 0$ (as will be assumed in this paper) then, by virtue of (3.138a), $U_\ell = U_\nu = \dot{R}$ in which case (3.137) becomes

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = \frac{1}{\rho_\ell}\left(p(R,t) - p_\infty - \tau_{rr}(R,t) + 3\int_R^\infty r^{-1}\tau_{rr}(r,t)\,dr\right) \tag{3.139}$$

while (3.138b) reduces to

$$p(R,t) = p_\nu + \tau_{rr}(R,t) - \frac{2\sigma}{R}. \tag{3.140}$$

Now, in an incompressible Newtonian fluid

$$\tau_{rr} = 2\mu_0\frac{\partial u}{\partial r} \tag{3.141}$$

while, as a consequence of (3.134), and the assumption that $J = 0$,

$$u(r,t) = \frac{\dot{R}R^2}{r^2}. \tag{3.142}$$

Combining (3.141) and (3.142) we have

$$\tau_{rr} = -4\mu_0\dot{R}R^2r^{-3} \tag{3.143}$$

for the Newtonian fluid, in which case

$$\tau_{rr}(R,t) = -4\mu_0 \left(\frac{\dot{R}}{R}\right). \tag{3.144}$$

Also

$$3\int_R^\infty r^{-1}\tau_{rr}\,dr = -12\mu_0 \dot{R}R^2 \int_R^\infty r^{-4}\,dr = -4\mu_0\left(\frac{\dot{R}}{R}\right) \tag{3.145}$$

so that for the incompressible Newtonian fluid

$$-\tau_{rr}(R,t) + 3\int_R^\infty r^{-1}\tau_{rr}\,dr = 0. \tag{3.146}$$

Thus, (3.139) reduces to

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = \frac{1}{\rho_\ell}(p(R,t) - p_\infty) \tag{3.147}$$

while (3.140) becomes

$$p(R,t) = p_v - \frac{2\sigma}{R} - 4\mu_0\left(\frac{\dot{R}}{R}\right). \tag{3.148}$$

The classical Rayleigh-Plesset equation (3.133) now follows by combining (3.147) and (3.148).

Remarks. By combining (3.139) and (3.140) one obtains

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = \frac{1}{\rho_\ell}\left(p_v - p_\infty - \frac{2\sigma}{R} + 3\int_R^\infty r^{-1}\tau_{rr}\,dr\right) \tag{3.149}$$

which is, in essence, equation (8) of [FG]. In [FG], (3.149) is coupled to the linear viscoelastic fluid constitutive relation

$$\tau_{rr}(r,t) = -2\int_0^t N(t-\tau)e_{rr}(\tau)\,d\tau \tag{3.150}$$

where $e_{rr} = \frac{\partial u}{\partial r}$ is the (radial) strain rate while $N(t)$ is the memory function (or relaxation modulus); the dynamical behavior of the vapor cavity is then governed by the nonlinear integrodifferential equation which results by combining (3.149) and (3.150).

Remarks. In [Pr] no specific non-Newtonian fluid constitutive relation is introduced; rather, the focus is on the generalization of (3.149) which results from combining (3.137), (3.138a,b) under the assumption that $J \neq 0$, i.e.,

$$R\ddot{U}_\ell + \frac{3}{2}U_\ell^2 - \frac{J}{\rho_\ell}\left[2U_\ell + J\left(\frac{1}{\rho_\nu} - \frac{1}{\rho_\ell}\right)\right] = \frac{1}{\rho_\ell}\left(p_\nu - p_\infty - \frac{2\sigma}{R} + 3\int_R^\infty r^{-1}\tau_{rr}\,dr\right). \tag{3.151}$$

As has been indicated in [Pr], (3.151) is particularly useful in those cases where the mass flux can be computed independently of conditions within the bubble, e.g., in vapor bubble growth which is heat transfer controlled, liquid vaporization at the interface serves as a source of new vapor within the cavity (with the rate of vapor production limited by the rate at which heat can be conducted through the bubble wall in order to satisfy latent heat requirements). In this heat transfer controlled situation one may use, as an approximation, $J = \frac{1}{\lambda}q(R,t)$, where λ is the latent heat/mass and $q(R,t)$ is the radial heat flux in the liquid at the cavity/liquid interface.

A recent work which considers vapor bubble growth within the context of a non-Newtonian situation is that of [BK]; in this paper the authors take, for $\alpha < 1$,

$$\mu = \mu_0\left(\frac{\dot{R}R^2}{r^3}\right)^{-\alpha}. \tag{3.152}$$

The analysis in [BK] is limited by the assumption that $\dot{R}R^2/r^3 >> 1$, as the authors are only interested in the final stage of the collapse of a spherical cavity. It is further assumed in [BK] that the normal tension at the bubble surface is zero, i.e., that

$$\tau_{rr}(R,t) = p(R,t), \quad t > 0 \tag{3.153}$$

so that, by virtue of (3.140), $p_\nu = \frac{2\sigma}{R}$ while (3.139) reduces to

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = \frac{1}{\rho_\ell}\left(-p_\infty + 3\int_R^\infty r^{-1}\tau_{rr}\,dr\right). \tag{3.154}$$

Setting

$$\Lambda(r,t) = \frac{\dot{R}(t)R^2(t)}{r^3} \tag{3.155}$$

we find, as a direct consequence of (3.142) and (3.152), that

$$\tau_{rr} = 2\mu_0\Lambda^{-\alpha}\frac{\partial u}{\partial r} = -4\mu_0\dot{R}^{1-\alpha}R^{2-2\alpha}r^{3\alpha-3} \tag{3.156}$$

so that

$$\tau_{rr}(R,t) = -4\mu_0 \left(\frac{\dot{R}}{R}\right)^{1-\alpha}. \tag{3.157}$$

Thus, the assumption, which is made in [BK] of zero normal tension at the bubble surface (i.e., (3.153)) is equivalent to

$$p(R,t) = -4\mu_0 \left(\frac{\dot{R}}{R}\right)^{1-\alpha}. \tag{3.158}$$

For the similar case of zero normal tension at the bubble surface in the Newtonian case one would have, as a consequence of (3.148) and the fact that $p_\nu = \dfrac{2\sigma}{R}$,

$$p(R,t) = -4\mu_0 \left(\frac{\dot{R}}{R}\right)$$

which is, of course, the special case of (3.158) that corresponds to the choice $\alpha = 0$. Using (3.156) one now easily computes that

$$3\int_R^\infty r^{-1}\tau_{rr}\,dr = -12\mu_0 \dot{R}^{1-\alpha} R^{2-2\alpha} \int_R^\infty r^{3\alpha-4}\,dr. \tag{3.159}$$

For most rheological models of the type (3.152) it is generally accepted that $\alpha < 1$, in which case the integral on the right-hand side of (3.159) is convergent and, in fact

$$3\int_R^\infty r^{-1}\tau_{rr}\,dr = \frac{-4\mu_0}{(1-\alpha)} \left(\frac{\dot{R}}{R}\right)^{1-\alpha}, \quad \alpha < 1 \tag{3.160}$$

for the model considered in [BK]. Combining (3.154) with (3.160) we find that

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = \frac{1}{\rho_\ell}\left(-p_\infty - \frac{4\mu_0}{1-\alpha}\left(\frac{\dot{R}}{R}\right)^{1-\alpha}\right), \quad \alpha < 1. \tag{3.161}$$

As has already been indicated, the analysis in [BK] is predicated upon the dual hypotheses that $\dfrac{\dot{R}R^2}{r^3} >> 1$ and that $p_\nu = 2\sigma/R$; if one does not enforce the latter assumption in [BK] then, as a consequence of (3.157) and (3.160), for $\alpha < 1$,

$$-\tau_{rr}(R,t) + 3\int_R^\infty r^{-1}\tau_{rr}\,dr = \frac{-4\alpha\mu_0}{1-\alpha}\left(\frac{\dot{R}}{R}\right)^{1-\alpha} \tag{3.162}$$

and, in lieu of (3.161), we would obtain for the evolution of $R(t)$

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = \frac{1}{\rho_\ell}\left(p(R,t) - p_\infty - \frac{4\alpha\mu_0}{1-\alpha}\left(\frac{\dot{R}}{R}\right)^{1-\alpha}\right). \tag{3.163}$$

In lieu of (3.163) we may first combine (3.140) with (3.157) so as to obtain

$$p(R,t) = p_v - \frac{2\sigma}{R} - 4\mu_0\left(\frac{\dot{R}}{R}\right)^{1-\alpha}. \tag{3.164}$$

Employing (3.164) in (3.163) then results in the evolution equation

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = \frac{1}{\rho_\ell}\left(p_v - p_\infty - \frac{2\sigma}{R} - \frac{4\mu_0}{1-\alpha}\left(\frac{\dot{R}}{R}\right)^{1-\alpha}\right) \tag{3.165}$$

which is, of course, the generalization of the Brutyan and Krapivsky model equation (3.161) for the case $\alpha < 1$ when $p_v \neq \frac{2\sigma}{R}$. For $\alpha = 0$, (3.165) yields the Rayleigh-Plesset equation (3.133).

3.3.2 Dynamics of a Spherical Vapor Cavity in an Incompressible Bipolar, Viscous Fluid: $\epsilon = \mu_1 = 0$

Even within the context of the restrictive hypotheses imposed in [BK], one of the difficulties inherent in this work is the choice of a constitutive assumption, (3.152), which reflects a dependence of the viscosity on the components of the rate of strain tensor instead of upon their magnitudes. For the incompressible, bipolar, viscous fluid with $\epsilon = \mu_1 = 0$, the resulting non-Newtonian model conforms to the nonlinear viscosity $\mu = \mu(|\boldsymbol{e}|)$ with

$$\mu = \mu_0(|\boldsymbol{e}|^2)^{-\alpha/2} \tag{3.166}$$

where it is only assumed, in this section, that $\alpha < 1$. The model which results from the constitutive hypothesis ((3.166) will generate a bubble dynamics which is similar to the one considered by [BK] except with respect to one very important qualitative difference, which will be highlighted in this section. As in previous sections, for the incompressible bipolar viscous fluid, we will set the fluid density $\rho_\ell = 1$. Also, to conform to the bulk of the literature dealing with the growth of a spherical vapor cavity in a viscous fluid, we will denote the fluid velocity vector by $\boldsymbol{u}$ (instead of $\boldsymbol{v}$), so that in a spherical coordinate system (r, θ, ϕ), which is related to the Cartesian coordinates (x_1, x_2, x_3) in the standard fashion, $\boldsymbol{u}$ is given by $\boldsymbol{u} = (v_r, v_\theta, v_\phi)$. With μ given by (3.166), the equations for the motion of the fluid, in Cartesian coordinates, have the form

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + 2\frac{\partial}{\partial x_j}\left\{\mu(|\boldsymbol{e}|^2)e_{ij}\right\} \tag{3.167}$$

with the u_i, $i = 1, 2, 3$, the Cartesian components of $\boldsymbol{u}$. We will denote the radial component of the velocity by $v_r = u(r,t)$. In fact, as we consider only radial motions of a spherical bubble surface, within the non-Newtonian fluid, we will postulate a spherically symmetric fluid velocity field, i.e.,

$$v_r = u(r,t), \quad v_\theta = 0, \quad v_\phi = 0. \tag{3.168}$$

With

$$v(r,t) \equiv \frac{1}{r}u(r,t) = \frac{1}{r}v_r(r,t) \tag{3.169}$$

we have

$$\begin{cases} v_1 = \dfrac{x_1}{r}u(r,t) \equiv x_1 v(r,t), \\ v_2 = \dfrac{x_2}{r}u(r,t) \equiv x_2 v(r,t), \\ u_3 = \dfrac{x_3}{r}u(r,t) \equiv x_3 v(r,t). \end{cases} \tag{3.170}$$

From (3.170) we obtain

$$|\boldsymbol{e}|^2 = r^2\left(\frac{\partial v}{\partial r}\right)^2 + rv\frac{\partial v}{\partial r}. \tag{3.171}$$

The incompressibility condition $\operatorname{div}\boldsymbol{u} = 0$ yields

$$r\frac{\partial v}{\partial r} + 3v = 0 \tag{3.172}$$

and leads to the conclusion that

$$v(r,t) = \frac{C(t)}{r^2}. \tag{3.173}$$

As $R(t)$ denotes the bubble radius at time t, so that $\dot{R}(t)$ is the outward normal speed of the spherical bubble surface, we have, by virtue of continuity across the bubble/fluid interface

$$u(R(t),t) = \dot{R}(t) \tag{3.174}$$

in which case (3.173) yields the expression (3.142) for $u(r,t)$. From (3.142), (3.169), and (3.171) we obtain

$$|\boldsymbol{e}|^2 = \frac{6}{r^6}\dot{R}^2(t)R^4(t) \tag{3.175}$$

in which case (3.166) becomes

$$\mu = \mu_0 6^{-\frac{\alpha}{2}}\left(\frac{|\dot{R}|R^2}{r^3}\right)^{-\alpha} \tag{3.176}$$

thus highlighting the difference between the present non-Newtonian structure and the model used in [BK], i.e., the one based on (3.152) for motions of the form (3.142).

Using incompressibility, i.e., (3.172), the convective terms $u_j\dfrac{\partial u_i}{\partial x_j}$ appearing in (3.167) are easily computed to be

$$\begin{cases} u_j\dfrac{\partial u_1}{\partial x_j} = -2x_1 v^2(r,t), \\ u_j\dfrac{\partial u_2}{\partial x_j} = -2x_2 v^2(r,t), \\ u_j\dfrac{\partial u_3}{\partial x_j} = -2x_3 v^2(r,t). \end{cases} \tag{3.177}$$

We now set

$$g(r,t) = \left(|\boldsymbol{e}(r,t)|^2\right)^{-\alpha/2}. \tag{3.178}$$

From (3.169), (3.171), and (3.172) we have, in fact,

$$g(r,t) = \left(\frac{6}{r^2}u^2(r,t)\right)^{-\alpha/2}. \tag{3.179}$$

Then (3.167) becomes

$$\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + 2\mu_0\frac{\partial}{\partial x_j}(g(r,t)e_{ij}). \tag{3.180}$$

Using (3.170) and (3.177), it is easy to show that, for $i = 1$, (3.180) is equivalent to

$$x_1\frac{\partial v}{\partial t} - 2x_1 v^2 = \frac{-x_1}{r}\frac{\partial p}{\partial r} - \frac{4\mu_0 x_1}{r}\frac{\partial g}{\partial r}v \tag{3.181}$$

while the results for $i = 2, 3$ may be obtained from (3.181) by replacing, in turn, $x_1 \to x_2, x_3$. Thus, for $i = 1, 2, 3$, each of the equations of motion reduce to

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} = -\frac{\partial p}{\partial r} - \frac{4\mu_0}{\partial r}\frac{\partial g}{\partial r}u. \tag{3.182}$$

However, a straightforward computation based on (3.142) and (3.179) produces

$$\frac{\partial g}{\partial r}(r,t) = 18\alpha\left\{\frac{6\dot{R}^2(t)R^4(t)}{r^6}\right\}^{-\left(1+\frac{\alpha}{2}\right)}\dot{R}^2(t)R^4(t)r^{-7}$$

or

$$\frac{\partial g}{\partial r} = (3\alpha)6^{-\frac{\alpha}{2}}\left|\dot{R}(t)\right|^{-\alpha}R^{-2\alpha}(t)r^{3\alpha-1} \tag{3.183}$$

while

$$\begin{cases} \dfrac{\partial u}{\partial r} = -2\dfrac{R^2(t)\dot{R}(t)}{r^3}, \\ \dfrac{\partial u}{\partial t} = (\ddot{R}(t)R^2(t) + 2R(t)\dot{R}^2(t))r^{-2}, \\ u\dfrac{\partial u}{\partial r} = -2R^4(t)\dot{R}^2(t)r^{-5}. \end{cases} \tag{3.184}$$

Finally,

$$u\frac{\partial g}{\partial r} - (3\alpha)6^{-\frac{\alpha}{2}}\left|\dot{R}\right|R^{2-2\alpha}r^{3\alpha-3}. \tag{3.185}$$

Substituting (3.184), and (3.185), into (3.182) we obtain

$$(\ddot{R}R^2 + 2R\dot{R}^2)r^{-2} - 2R^4\dot{R}^2r^{-5} = -\frac{\partial p}{\partial r} - 12\alpha\mu_0 6^{-\frac{\alpha}{2}}\left|\dot{R}\right|^{-\alpha}\dot{R}R^{2-2\alpha}r^{3\alpha-4}. \tag{3.186}$$

Remarks. Let

$$h(\alpha) = \left(\frac{6\dot{R}^2R^4}{r^6}\right)^{-\frac{\alpha}{2}} \tag{3.187}$$

so that

$$h'(\alpha) = -\frac{1}{2}h(\alpha)\ln\left[\frac{6\dot{R}^2R^4}{r^6}\right].$$

Thus,

$$h'(0) = -\frac{1}{2}\ln\left[\frac{6\dot{R}^2R^4}{r^6}\right].$$

However, for $|\alpha| \approx 0$, $h(\alpha) = h(0) + h'(0)\alpha + \mathcal{O}(|\alpha|^2)$, so that

$$h(\alpha) \approx 1 - \frac{1}{2}\left[\ln\left(\frac{6\dot{R}^2R^4}{r^6}\right)\right]\alpha. \tag{3.188}$$

From (3.183) and (3.188) we have, for $|\alpha| \approx 0$,

$$\begin{aligned}\frac{\partial g}{\partial r} &= 3\alpha r^{-1}h(\alpha)\\ &\approx 3\alpha r^{-1}\left(1 - \frac{1}{2}\left[\ln\left(\frac{6\dot{R}^2R^4}{r^6}\right)\right]\alpha\right)\\ &= \frac{3\alpha}{r} + \mathcal{O}(|\alpha|^2).\end{aligned}$$

Therefore, for $|\alpha| \approx 0$, we have to within terms of order $\mathcal{O}(|\alpha|^2)$

$$(\ddot{R}R^2 + 2R\dot{R}^2)r^{-2} - 2R^4\dot{R}^2r^{-5} = -\frac{\partial p}{\partial r} - 12\alpha\mu_0R^2\dot{R}r^{-4} \tag{3.189}$$

which is the form of (3.186) which is appropriate for small α. As the fluid velocity field has the structure

$$u(r,t) = \frac{1}{r^2}\left\{R^2(t)\dot{R}(t)\right\} \equiv \mathcal{U}(r; R(t)) \tag{3.190}$$

it is quite natural to assume an analogous form for the pressure field, i.e.,

$$p(r,t) = \mathcal{P}(r; R(t)). \tag{3.191}$$

We now integrate over (3.186) with respect to r from $r = R(t)$ to $r = \infty$ and use the fact that

$$-\frac{\partial\mathcal{P}(r; R(t))}{\partial r}\bigg|_{R(t)}^{\infty} = \hat{p}(R(t)) - p_\infty \tag{3.192}$$

where $\hat{p}(R(t)) \equiv \mathcal{P}(R(t); R(t))$ and $p_\infty \equiv \mathcal{P}(\infty; R(t))$. In this manner, we obtain the nonlinear ordinary differential equation which is the non-Newtonian version of the Rayleigh-Plesset equation (3.144), with fluid density $\rho_\ell = 1$, i.e.,

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = (\hat{p}(R(t)) - p_\infty - \frac{12\alpha\mu_0}{6^{\alpha/2}} R^{2(1-\alpha)} \left|\dot{R}\right|^{-\alpha} \dot{R} \left. \frac{r^{3\alpha-3}}{(3\alpha-3)} \right|_{R(t)}^{\infty} . \tag{3.193}$$

The last term on the right-hand side of (3.193) is clearly convergent, as $\alpha < 1$, in which case

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = \hat{p}(R) - p_\infty - \frac{4\alpha\mu_0}{(1-\alpha)6^{\alpha/2}} R^{\alpha-1} \dot{R}\left|\dot{R}\right|^{-\alpha}. \tag{3.194}$$

The evolution equation for $R(t)$, (3.194), will be studied in Sects. 3.3.3 and 3.3.4; it clearly reduces to (3.147), if $\rho_\ell = 1$ and $\alpha = 0$, provided we identify $p(R,t) = \hat{p}(R(t))$. The process described above, when applied to the approximate relation (3.189) for $|\alpha| \approx 0$, yields

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = (\hat{p}(R) - p_\infty) - 4\alpha\mu_0 \left(\frac{\dot{R}}{R}\right). \tag{3.195}$$

Remarks. The classical Rayleigh-Plesset equation for the dynamics of a spherical cavity of radius $R(t)$ in an unbounded, incompressible, Newtonian viscous fluid (neglecting mass transfer at the cavity/fluid interface) may be put in the form (3.133) by combining (3.147) with (3.148). In general, as a consequence of (3.140), (3.191), and the definitions of $\hat{p}$ and p_∞, we have

$$\hat{p}(R(t)) = p_v(t) - \frac{2\sigma}{R(t)} + \tau_{rr}(R(t)). \tag{3.196}$$

However, for the non-Newtonian model defined by (3.166) we have

$$\tau_{rr} = 2\mu(|\boldsymbol{e}|^2)e_{rr}, \quad e_{rr} = \frac{\partial u}{\partial r}$$

and by (3.142),

$$e_{rr} = -2r^{-3}\dot{R}(t)R^2(t)$$

in which case,

$$\tau_{rr}(R(t)) = -4\mu(|\boldsymbol{e}(R(t))|^2)\frac{\dot{R}(t)}{R(t)}. \tag{3.197}$$

Thus,

$$\hat{p}(R(t)) = p_v(t) - \frac{2\sigma}{R(t)} - 4\mu(|\boldsymbol{e}(R(t))|^2)\frac{\dot{R}(t)}{R(t)}. \tag{3.198}$$

Using (3.176) we then obtain from (3.198)

$$\hat{p}(R(t)) = p_v(t) - \frac{2\sigma}{R(t)} - \frac{4\mu_0}{6^{\alpha/2}} \frac{\left|\dot{R}\right|^{-\alpha} \dot{R}}{R^{1-\alpha}}. \tag{3.199}$$

Clearly, if we again identify $p(R,t) \equiv \hat{p}(R(t))$, (3.199) reduces to (3.148) when $\alpha = 0$. Using (3.199) in conjunction with (3.194) we obtain

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = (p_v(t) - p_\infty) - \frac{2\sigma}{R} - \frac{4\mu_0}{(1-\alpha)6^{\alpha/2}} \left\{ \frac{|\dot{R}|^{-\alpha}\dot{R}}{R^{1-\alpha}} \right\} \tag{3.200}$$

and we note that (3.200) reduces to (3.133), for $\alpha = 0$, when $\rho_\ell = 1$. On the other hand, if we employ (3.199) in the approximate relation (3.195), for $|\alpha| \approx 0$ we obtain

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = (p_v(t) - p_\infty) - \frac{2\sigma}{R} - 4\mu_0 \left(\frac{\dot{R}}{R}\right) \left\{ \alpha + 6^{-\alpha/2} \left(\frac{|\dot{R}|}{R}\right)^{-\alpha} \right\} \tag{3.201}$$

which also reduces to (3.133) when $\alpha = 0$ and $\rho_\ell = 1$.

3.3.3 Linearized Dynamics for a Spherical Cavity in a Non-Newtonian Fluid

For spherical cavity growth in an unbounded non-Newtonian fluid, which conforms to the constitutive hypothesis (2.1) with $\epsilon = \mu_1 = 0$, we have shown that the equations governing the evolution of the cavity of radius $R(t)$, in the absence of mass transfer across the cavity/fluid interface, are (3.194) for $\alpha < 1$ and (3.195) for $|\alpha| \approx 0$.

We now set $S(t) = \dot{R}(t)$; then, in terms of R and S, (3.194) may be rewritten as the system

$$\begin{cases} R = S, \\ \dot{S} = (\hat{p}(R) - p_\infty)R^{-1} - \dfrac{3}{2}R^{-1}S^2 - k(\alpha;\mu_0)R^{\alpha-2}S|S|^{-\alpha}, \end{cases} \tag{3.202}$$

for $\alpha < 1$, where

$$k(\alpha;\mu_0) \equiv \frac{4\alpha\mu_0}{(1-\alpha)6^{\alpha/2}} \tag{3.203}$$

while (3.195) becomes

$$\begin{cases} \dot{R} = S, \\ \dot{S} = (\hat{p}(R) - p_\infty)R^{-1} - \dfrac{3}{2}R^{-1}S^2 - 4\alpha\mu_0 R^{-2}S, \end{cases} \tag{3.204}$$

for $|\alpha| \approx 0$. Both of the systems (3.202) and (3.204) are planar nonlinear systems of the form

$$\begin{cases} \dot{R} = F(R, S), \\ \dot{S} = G(R, S) \end{cases}$$

and it is easily seen that for each of these systems (R_0, S_0) is an equilibrium state, i.e., $F(R_0, S_0) = G(R_0, S_0) = 0$, if and only if

$$S_0 = 0 \text{ and } \hat{p}(R_0) = p_\infty. \tag{3.205}$$

In this subsection we will look at some aspects of the linearized stability of the equilibrium state $(R_0, 0)$, $\hat{p}(R_0) = p_\infty$, for $|\alpha| \approx 0$; the nonlinear stability of such equilibrium states with respect to the nonlinear system (3.202), for $\alpha < 1$, will be examined in Sect. 3.3.4 using Liapunov theory.

For the system (3.204) we have

$$\begin{cases} F(R, S) = S, \\ G(R, S) = (\hat{p}(R) - p_\infty)R^{-1} - \dfrac{3}{2}R^{-1}S^2 - 4\alpha\mu_0 R^{-2}S. \end{cases} \tag{3.206}$$

Setting $\bar{R} = R - R_0$, $\bar{S} = S$, and linearizing about $(R_0, 0)$, we obtain a system of the form

$$\dot{\boldsymbol{u}} = \boldsymbol{\Lambda}(\boldsymbol{u}_0)\boldsymbol{u} \tag{3.207}$$

with

$$\boldsymbol{u} = \begin{pmatrix} \bar{R} \\ \bar{S} \end{pmatrix}, \quad \boldsymbol{u}_0 = \begin{pmatrix} R_0 \\ 0 \end{pmatrix} \tag{3.208a}$$

and

$$\boldsymbol{\Lambda}(\boldsymbol{u}_0) = \begin{pmatrix} \dfrac{\partial F}{\partial R} & \dfrac{\partial F}{\partial S} \\ \dfrac{\partial G}{\partial R} & \dfrac{\partial G}{\partial s} \end{pmatrix}_{(R,S)=(R_0,0)}. \tag{3.208b}$$

Using (3.206) and the fact that $\hat{p}(R_0) = p_\infty$, one easily computes that

$$\begin{cases} \dfrac{\partial F}{\partial R}(R_0, 0) = 0, \quad \dfrac{\partial F}{\partial S}(R_0, 0) = 1, \\ \dfrac{\partial G}{\partial R}(R_0, 0) = \dfrac{1}{R_0}(\hat{p}'(R_0)), \\ \dfrac{\partial G}{\partial S}(R_0, 0) = -4\alpha\mu_0 R_0^{-2}. \end{cases} \tag{3.209}$$

If we define

$$\psi(R) \equiv \hat{p}(R) - p_\infty \tag{3.210}$$

then

$$\mathbf{\Lambda}(\boldsymbol{u}_0) = \begin{pmatrix} 0 & 1 \\ \dfrac{\psi'(R_0)}{R_0} & \dfrac{-4\alpha\mu_0}{R_0^2} \end{pmatrix} \tag{3.211}$$

in which case $\det(\mathbf{\Lambda}(\boldsymbol{u}_0) - \lambda \boldsymbol{I}) = 0$ if and only if

$$\lambda^2 + \gamma(R_0)\lambda - \frac{1}{R}\psi'(R_0) = 0 \tag{3.212}$$

with

$$\gamma(R_0) = 4\alpha\mu_0 R_0^{-2}. \tag{3.213}$$

From (3.212) we obtain for the eigenvalues of $\mathbf{\Lambda}(\boldsymbol{u}_0)$

$$\lambda = -2\alpha\mu_0 R_0^{-2} \pm \sqrt{4\alpha^2\mu_0^2 R_0^{-4} + \psi'(R_0)R_0^{-1}}. \tag{3.214}$$

In (3.214) $\alpha \approx 0$, but we may have $\alpha > 0$, $\alpha < 0$, or even $\alpha = 0$. The conclusions which follow from the relation (3.214), concerning the linearized stability of the equilibrium state $(R_0, 0)$, and their consequences for the nonlinear stability of $(R_0, 0)$ with respect to the system (3.204), for $|\alpha| \approx 0$, are a direct result of standard elementary theorems for planar dynamical systems (e.g., [Pe] or [HK]) given that the vector field $(F(R, S), G(R, S))$ is of class C^2 in a neighborhood of $R_0 \neq 0$, $S_0 = 0$.

Case I:

$\alpha \approx 0, \alpha > 0$

(a) $\psi'(R_0) > 0 \Leftrightarrow \hat{p}'(R_0) > 0$
In this subcase it follows from (3.214) that the eigenvalues λ_1, λ_2 satisfy $\lambda_1 < 0 < \lambda_2$; thus $(R_0, 0)$ is an unstable saddle point for the linearized problem (3.208) and, also, an *(unstable) saddle point* for the (approximate) nonlinear system (3.204).

(b) $\psi'(R_0) < 0 \Leftrightarrow \hat{p}'(R_0) < 0$
For this subcase (3.214) becomes

$$\lambda = -2\alpha\mu_0 R_0^{-2} \pm \sqrt{4\alpha^2\mu_0 R_0^{-4} - |\psi'(R_0)| R_0^{-1}} \tag{3.215}$$

so that three situations are possible:

(i) $|\psi'(R_0)| < 4\alpha^2\mu_0^2 R_0^{-3}$, in which case $\lambda_1 < 0$, $\lambda_2 < 0$ and $(R_0, 0)$ is an *asymptotically stable node* both for the linearized problem and the nonlinear (approximate) system

(ii) $|\psi'(R_0)| > 4\alpha^2\mu_0^2 R_0^{-3}$, in which case λ is of the form $\lambda = a \pm ib$, with $a < 0$. In this subcase, $(R_0, 0)$ is an *asymptotically stable focus* for both the linearized problem and the nonlinear (approximate) system

(iii) $|\psi'(R_0)| = 4\alpha^2\mu_0^2 R_0^{-3}$, in which case $\lambda_1 = \lambda_2 = -2\alpha\mu_0 R_0^{-2} < 0$ and $(R_0, 0)$ is again an *asymptotically stable node* for both the linearized problem and the nonlinear (approximate) system

Case II:

$\alpha \approx 0, \alpha < 0$
For all of the subcases considered below

$$\lambda = 2|\alpha|\mu_0 R_0^{-2} \pm \sqrt{4\alpha^2\mu_0 R_0^{-4} + \psi'(R_0) R_0^{-1}} \tag{3.216}$$

(a) $\psi'(R_0) > 0 \Leftrightarrow \tilde{p}'(R_0) > 0$
In this subcase it easily follows from (3.216) that $\lambda_1 < 0 < \lambda_2$ so that $(R_0, 0)$ is an *(unstable) saddle point* for both the linearized and nonlinear (approximate) problems

(b) $\psi'(R_0) < 0 \Leftrightarrow \hat{p}'(R_0) < 0$
In this scenario,

$$\lambda = 2|\alpha|\mu_0 R_0^{-2} \pm \sqrt{4\alpha^2\mu_0^2 R_0^{-4} - |\psi'(R_0)| R_0^{-1}} \tag{3.217}$$

and, as in case Ib, three situations are possible:

(i) $|\psi'(R_0)| < 4\alpha^2\mu_0^2 R_0^{-3}$, in which case $\lambda_1 > 0$, $\lambda_2 > 0$ and $(R_0, 0)$ is an *unstable node* both for the linearized problem and the nonlinear (approximate) problem

(ii) $\left|\psi'(R_0)\right| > 4\alpha^2\mu_0^2 R_0^{-3}$, in which case $\lambda = a \pm ib$ with $a > 0$ so that $(R_0, 0)$ is an *unstable focus* for both the linearized problem as well as the nonlinear (approximate) problem

(iii) $\left|\psi'(R_0)\right| = 4\alpha^2\mu_0^2 R_0^{-3}$, in which case $\lambda_1 = \lambda_2 = 2|\alpha|\mu_0 R_0^{-2} > 0$ so that $(R_0, 0)$ is an *unstable node* with respect to both the linearized and nonlinear (approximate) problems

Case III:

$\alpha = 0$

This case corresponds, of course, to the Newtonian model, i.e., $\mu = \mu_0$ in (3.167), for which both systems (3.202) and (3.204), reduce to

$$\begin{cases} \dot{R} = S, \\ \dot{S} = \psi(R)R^{-1} - \dfrac{3}{2}R^{-1}S^2. \end{cases} \tag{3.218}$$

From (3.214) we obtain

$$\lambda = \pm\sqrt{\psi'(R_0)R_0^{-1}} \tag{3.219}$$

in which case for

(a) $\psi'(R_0) > 0 \Leftrightarrow \hat{p}'(R_0) > 0$

We have $\lambda_1 < 0 < \lambda_2$ and, consequently, $(R_0, 0)$ is an *unstable saddle point* both for the linearized system

$$\begin{cases} \dot{R} = S \\ \dot{S} = \dfrac{1}{R_0}\psi'(R_0)(R - R_0) \end{cases} \tag{3.220}$$

as well as for the nonlinear system (3.218)

(b) $\psi'(R_0) < 0 \Leftrightarrow \hat{p}'(R_0) < 0$

We have $\lambda = \pm i\sqrt{|\psi'(R_0)|R_0^{-1}}$, in which case the linearized problem (3.220) exhibits a *stable center* at $(R_0, 0)$ but no definitive conclusions concerning the nature of the equilibrium at $(R_0, 0)$ for the nonlinear problem (3.218) may be drawn from this result. A conclusion of stability, however, for the equilibrium at $(R_0, 0)$, with respect to the nonlinear system (3.218) will follow, in Sect. 3.3.4, from Liapunov theory when $\psi'(R_0) = 0$ and $\psi''(R_0) < 0$, i.e., when $\hat{p}'(R_0) = 0$ and $\hat{p}''(R_0) < 0$.

Remarks. In each of the cases treated above we have avoided the possible situation in which $\psi'(R_0) = 0$. In this case λ, as given by (3.214), reduces to

$$\lambda = -2\alpha\mu_0 R_0^{-2} \pm \sqrt{4\alpha^2\mu_0^2 R_0^{-4}} \tag{3.221}$$

so

$$\begin{aligned}
&\text{for } \alpha > 0 : \lambda_1 = 0,\ \lambda_2 < 0,\\
&\text{for } \alpha < 0 : \lambda_1 = 0,\ \lambda_2 > 0,\\
&\text{for } \alpha = 0 : \lambda_1 = \lambda_2 = 0.
\end{aligned} \tag{3.222}$$

Therefore, if $(R_0, 0)$ is an equilibrium state with respect to the nonlinear system (3.204) for $|\alpha| \approx 0$, for the linear problem (3.207), (3.208a), (3.211), $(R_0, 0)$ is a (degenerate) nonhyperbolic equilibrium when $\psi'(R_0) = 0$. Various results concerning the nature of the stability of degenerate equilibria corresponding to the possibilities exhibited in (3.222) follow from the work of Poincaré, Bendixson, Andronov, et. al. (see, e.g., [Pe]) but we will not pursue these issues here; rather, the stability of $(R_0, 0)$ in this situation may be handled within the context of the Liapunov theory introduced in the next section under the additional hypothesis that $\psi''(R_0) < 0$, i.e., that ψ (equivalently, $\hat{p}$) has a relative maximum at $R = R_0$.

3.3.4 Liapunov Theory and Nonlinear Stability

In this section we consider the nonlinear system (3.202) where $k(\alpha; \mu_0)$ is given by (3.203) and $\alpha < 1$. Setting, once again, $\bar{R} = R - R_0$ and $\bar{S} = S$ we rewrite (3.202) in the form

$$\begin{cases}
\dot{\bar{R}} = \bar{S},\\
\dot{\bar{S}} = \bar{\psi}(\bar{R}; R_0)(\bar{R} + R_0)^{-1} - \dfrac{3}{2}(\bar{R} + R_0)^{-1}\bar{S}^2\\
\qquad\qquad + k(\alpha; \mu_0)(\bar{R} + R_0)^{\alpha-2}\bar{S}|\bar{S}|^{-\alpha},
\end{cases} \tag{3.223}$$

where

$$\bar{\psi}(\bar{R}; R_0) \equiv \psi(\bar{R} + R_0) = \hat{p}(\bar{R} + R_0) - p_\infty. \tag{3.224}$$

As $(R_0, 0)$ is an equilibrium point of (3.202), $\bar{\psi}(0; R_0) = 0$. The additional assumption that will be made in this subsection is the following: $\exists \delta > 0$ such that

$$\bar{\psi}(\lambda; R_0) \le 0, \quad |\lambda| < \delta. \tag{3.225}$$

As $\bar{\psi}(0; R_0) = 0$, (3.225) is satisfied if $\bar{\psi}(\lambda; R_0)$ has a relative maximum at $\lambda = 0$, i.e., if

$$\bar{\psi}'(0; R_0) = 0 \text{ and } \bar{\psi}''(0; R_0) < 0. \tag{3.226}$$

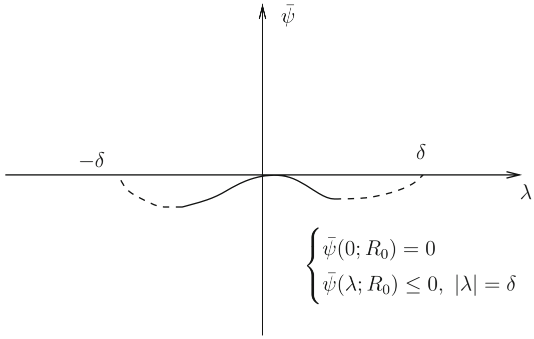

Fig. 3.1 Sketch of the function $\bar{\psi}(\lambda; R_0)$, $|\lambda| < \delta$

Clearly, (3.226) is equivalent to

$$\hat{p}'(R_0) = 0 \text{ and } \hat{p}''(R_0) < 0. \tag{3.227}$$

The situation delineated above is sketched above (note that $\bar{\psi}(0; R_0) = 0 \Leftrightarrow \hat{p}(R_0) = p_\infty$) in Fig. 3.1. We now consider the function

$$\mathcal{V}(\bar{R}, \bar{S}) \equiv (\bar{R} + R_0)^3 \bar{S}^2 - 2 \int_0^{\bar{R}} (\lambda + R_0)^2 \bar{\psi}(\lambda; R_0)\, d\lambda. \tag{3.228}$$

Clearly, $\mathcal{V}(0, 0) = 0$. We define a neighborhood

$$\mathcal{N}(0, 0) \equiv \{(\bar{R}, \bar{S}) \mid \bar{R}^2 + \bar{S}^2 \leq \delta^2\} \tag{3.229}$$

of the origin in the $\bar{R}, \bar{S}$ plane and note that for any $\bar{R}$ such that $(\bar{R}, \bar{S}) \in \mathcal{N}(0, 0)$, $\bar{\psi}(\lambda; R_0) \leq 0$ if $\lambda \in [0, \bar{R}]$, as a consequence of (3.225) and the definition (3.229). Thus, for $(\bar{R}, \bar{S}) \in \mathcal{N}(0, 0)$ (as $\bar{R} + R_0 \equiv R \geq 0$) it follows from (3.228) that $\mathcal{V}(\bar{R}, \bar{S}) \geq 0$. Furthermore, $\mathcal{V}(\bar{R}, \bar{S}) = 0$ implies that $(\bar{R}, \bar{S}) = (0, 0)$ as

$$\begin{cases} \mathcal{V}_1(\bar{R}, \bar{S}) \equiv (\bar{R} + R_0)^3 \bar{S}^2 \geq 0, \\ \mathcal{V}_2(\bar{R}, \bar{S}) \equiv -2 \int_0^{\bar{R}} (\lambda + R_0)^2 \bar{\psi}(\lambda; R_0)\, d\lambda \geq 0 \end{cases} \tag{3.230}$$

for $(\bar{R}, \bar{S}) \in \mathcal{N}(0, 0)$. Therefore, $\mathcal{V}(\bar{R}, \bar{S}) > 0$ for $(\bar{R}, \bar{S}) \in \mathcal{N}(0, 0)$ with $(\bar{R}, \bar{S}) \neq (0, 0)$. For $\bar{\psi}(\lambda; R_0)$ continuous in λ, $\mathcal{V}(\bar{R}, \bar{S})$ is clearly of class C^2 on $\mathcal{N}(0, 0)$ in which case $\mathcal{V}(\bar{R}, \bar{S})$ constitutes a Liapunov function.

We now consider the derivative of $\mathcal{V}(\bar{R}, \bar{S})$ along the trajectories of the system (3.223), i.e., we want to compute

$$\begin{aligned} L_t \mathcal{V}(\bar{R}, \bar{S}) &= \nabla \mathcal{V} \cdot (\dot{\bar{R}}, \dot{\bar{S}}) \\ &\equiv \frac{\partial \mathcal{V}}{\partial \bar{R}} \dot{\bar{R}} + \frac{\partial \mathcal{V}}{\partial \bar{S}} \dot{\bar{S}}. \end{aligned} \tag{3.231}$$

By virtue of the definition of $\mathcal{V}$, i.e., (3.228),

$$\begin{cases} \dfrac{\partial \mathcal{V}}{\partial \bar{R}} = 3(\bar{R} + R_0)^3 \bar{S}^2 - 2(\bar{R} + R_0)^2 \bar{\psi}(\bar{R}; R_0), \\ \dfrac{\partial \mathcal{V}}{\partial \bar{S}} = 2(\bar{R} + R_0)^3 \bar{S}, \end{cases} \tag{3.232}$$

so that

$$L_t \mathcal{V} = \{3(\bar{R} + R_0)^3 \bar{S}^2 - 2(\bar{R} + R_0)^2 \bar{\psi}(\bar{R}; R_0)\}\dot{R} + 2(\bar{R} + R_0)^3 \bar{S}\dot{\bar{S}} \tag{3.233}$$

or, employing (3.223),

$$\begin{aligned} L_t \mathcal{V} = {} & \{3(\bar{R} + R_0)^3 \bar{S}^2 - 2(\bar{R} + R_0)^2 \bar{\psi}(\bar{R}; R_0)\} S \\ & + 2(\bar{R} + R_0)^3 \bar{S}\{\bar{\psi}(\bar{R}; R_0)(\bar{R} + R_0)^{-1} \\ & - \frac{3}{2}(\bar{R} + R_0)^{-1} \bar{S}^2 - k(\alpha; \mu_0)(\bar{R} + R_0)^{\alpha - 2} \bar{S}|\bar{S}|^{-\alpha}\}. \end{aligned} \tag{3.234}$$

Upon simplifying (3.234) we obtain

$$L_t \mathcal{V} = -2k(\alpha; \mu_0)(\bar{R} + R_0)^{1+\alpha} \bar{S}^2 \left|\bar{S}\right|^{-\alpha} \tag{3.235}$$

and, as $\bar{R} + R_0 = R > 0$, for $\alpha < 1$ we obtain from (3.235) and the definition, (3.203), of $k(\alpha; \mu_0)$ the following conclusions

$$\begin{cases} L_t \mathcal{V} < 0, & \alpha > 0, \\ L_t \mathcal{V} = 0, & \alpha = 0, \\ L_t \mathcal{V} > 0, & \alpha < 0. \end{cases} \tag{3.236}$$

We may summarize our results in the following

Theorem 3.4. *Consider a spherical vapor cavity (bubble), whose radius at time $t > 0$ is $R(t)$, immersed in an unbounded non-Newtonian fluid defined by the constitutive law (2.1a), with $\epsilon = \mu_1 = 0$, and $\alpha < 1$. Let the fluid pressure be of the form (3.191) and define $\hat{p}(R) = \mathcal{P}(R; R)$ where it is assumed that $\hat{p}$ is differentiable on $\mathbb{R}^+$. Then*

(i) All equilibrium states of the bubble are of the form $R = R_0$, where the constant R_0 satisfies $\hat{p}(R_0) = p_\infty$
(ii) If we define, for $\lambda \in (-\infty, \infty)$,

$$\bar{\psi}(\lambda; R_0) = \hat{p}(\lambda + R_0) - p_\infty \tag{3.237}$$

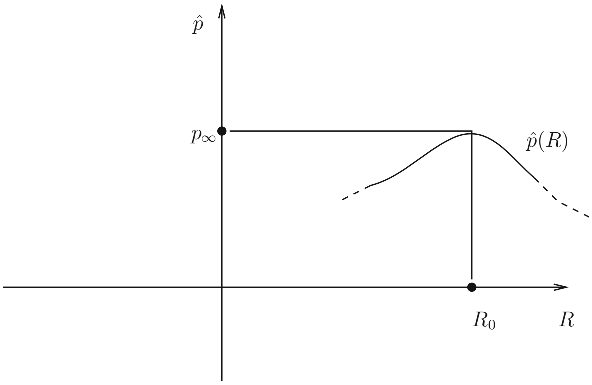

Fig. 3.2 The function $\bar{\psi}(\lambda; R_0)$ in Theorem 3.4

and $\bar{\psi}$ *satisfies* (3.225)*, for some* $\delta > 0$*, the equilibrium state* $(R_0, 0)$ *is*

(a) (locally) asymptotically stable for $\alpha > 0$
(b) stable for $\alpha = 0$
(c) unstable for $\alpha < 0$

An example which corresponds to the conditions of Theorem 3.4 is sketched above in Fig. 3.2.

Remarks. Using the Liapunov function $\mathcal{V}(\bar{R}, \bar{S})$ in (3.228) we may reconsider the nonlinear (approximate) system, corresponding to $|\alpha| \approx 0$, for the previously indeterminate case in which $\psi'(R_0) = 0$. In terms of $\bar{R} = R - R_0$, and $\bar{S} = S$, (3.204) becomes

$$\begin{cases} \dot{\bar{R}} = \bar{S}, \\ \dot{\bar{S}} = (\hat{p}(\bar{R} + R_0) - p_\infty)(\bar{R} + R_0)^{-1} \\ \qquad - \dfrac{3}{2}(\bar{R} + R_0)^{-1}\bar{S}^2 - 4\alpha\mu_0(\bar{R} + R_0)^{-2}\bar{S}. \end{cases} \tag{3.238}$$

Under the same conditions which prevailed earlier, i.e., the definition of $\mathcal{N}(0,0)$ in (3.229), and the hypothesis relative to $\bar{\psi}(\lambda; R_0) = \hat{p}(\lambda + R_0) - p_\infty$, we find that

$$L_t\mathcal{V} = -8\alpha\mu_0(\bar{R} + R_0)\bar{S}^2. \tag{3.239}$$

In particular, (3.239) holds if (see (3.226)) $\psi'(R_0) = 0$ and, in addition, $\psi''(R_0) < 0$; thus for $\alpha < 0$ the equilibrium state $(R_0, 0)$ is unstable while it is stable in the Newtonian case in which $\alpha = 0$ and (locally) asymptotically stable for $\alpha > 0$.

3.4 Exterior Flow of a Bipolar Viscous Fluid in the Plane

3.4.1 Introduction

We continue in this section the process of looking at the motion of bipolar viscous fluids in specific domains; in this case the problem in question is that of steady flow of an incompressible bipolar viscous fluid past a fixed body $\Omega' \subset R^2$. The hypothesis of steady flow conditions for Navier–Stokes, when coupled with the assumption that the external body force/mass $\boldsymbol{f} = \mathbf{0}$, and the normalizations $\rho = 1$, $\mu_0 = 1$, yields the following exterior flow problem:

$$\Delta \boldsymbol{v} - \boldsymbol{v} \cdot \nabla \boldsymbol{v} - \nabla p = 0, \text{ in } R^2/\bar{\Omega}', \tag{3.240a}$$

$$\nabla \cdot \boldsymbol{v} = 0, \text{ in } R^2/\bar{\Omega}', \tag{3.240b}$$

$$\boldsymbol{v} = \mathbf{0}, \text{ on } \partial\Omega', \tag{3.240c}$$

$$\lim_{||\boldsymbol{x}|| \to \infty} \boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{v}_\infty. \tag{3.240d}$$

In 1933, Leray [Le1] proved the existence of a function $\boldsymbol{v}$ which satisfies (3.240a,b,c) in a weak sense. It is not, however, known yet whether or not the weak solution constructed in [Le1] satisfies the radiation condition (3.240d). In [FiS], Finn and Smith provided an affirmative answer to the problem under the restriction that $||\boldsymbol{v}_\infty||$ be sufficiently small; the general existence problem for (3.240a–d) remains open. In this section we will consider the same classical problem (of steady flow past a bounded set Ω' in the plane) within the framework of the theory of the incompressible bipolar viscous fluid; we will show that the problem which replaces (3.240a–d), in the case of a bipolar fluid, has a solution and will use that solution to compute the drag on the body Ω'.

3.4.2 Formulation of the Exterior Flow Problem for a Bipolar Fluid

Let $\Omega' \subset R^2$ be a simply connected bounded domain with smooth boundary $\partial\Omega$ and set $\Omega = R^2/\bar{\Omega}'$. With the nonlinear viscosity μ as given by (1.89), where μ_0, μ_1 and ϵ are all positive constants, and $0 < \alpha < 1$, our problem assumes the form

$$(\boldsymbol{L}(\boldsymbol{v}))_i \equiv \mu_1 \frac{\partial}{\partial x_j} \Delta e_{ij} - \frac{\partial}{\partial x_j}(\mu(|\boldsymbol{e}|)e_{ij}) + v_j \frac{\partial v_i}{\partial x_j} = -\frac{\partial p}{\partial x_i}, \text{ in } \Omega, \tag{3.241a}$$

$$\nabla \cdot \boldsymbol{v} = 0, \text{ in } \Omega, \tag{3.241b}$$

$$\boldsymbol{v} = \mathbf{0}, \text{ on } \partial\Omega', \tag{3.241c}$$

$$\left(\frac{\partial}{\partial x_k} e_{ij}\right) \nu_j \nu_k \tau_i = 0, \text{ on } \partial\Omega', \tag{3.241d}$$

$$\lim_{||\boldsymbol{x}|| \to \infty} \boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{v}_\infty \equiv \begin{pmatrix} \lambda \\ 0 \end{pmatrix}, \tag{3.241e}$$

$$\lim_{||\boldsymbol{x}|| \to \infty} \boldsymbol{D}^2 \boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{0}. \tag{3.241f}$$

In (3.241d), $\boldsymbol{\nu}$ is the exterior unit normal to $\partial\Omega'$ (and, thus, lies in the interior of the domain Ω) while $\boldsymbol{\tau}$ is the unit tangent vector to $\partial\Omega'$. In what follows we will use the notation

$$\left.\begin{aligned} ||\boldsymbol{D}^2\boldsymbol{u}||^2_{\boldsymbol{L}^2(\Omega)} &\equiv \sum_{i,j,k} \iint \left(\frac{\partial^2 u_i}{\partial x_j \partial x_k}\right)^2 d\boldsymbol{x} \\ ||\boldsymbol{D}\boldsymbol{u}||^s_{\boldsymbol{L}^s} &\equiv \sum_{i,j} \iint \left|\frac{\partial u_i}{\partial x_j}\right|^s d\boldsymbol{x} \end{aligned}\right\}. \tag{3.242}$$

Elementary algebraic estimates (see (B.14) for (3.243a)) yield that there exists a positive constant c such that for all $\boldsymbol{u}$,

$$\frac{\partial}{\partial x_k} e_{ij}(\boldsymbol{u}) \frac{\partial}{\partial x_k} e_{ij}(\boldsymbol{u}) \geq c \sum_{i,j,k} \left(\frac{\partial^2 u_i}{\partial x_j \partial x_k}\right)^2, \tag{3.243a}$$

$$\mu(|\boldsymbol{e}|) e_{ij}(\boldsymbol{u}) e_{ij}(\boldsymbol{u}) \geq c\mu_0 \sum_{i,j} \left|e_{ij}(\boldsymbol{u})\right|^{2-\alpha}. \tag{3.243b}$$

Next, we introduce the function spaces

$$\boldsymbol{H}_\alpha(\Omega) = \{\boldsymbol{u} \mid \boldsymbol{u} = \boldsymbol{0} \text{ on } \partial\Omega',\ \nabla \cdot \boldsymbol{v} = 0 \text{ in } \Omega, \text{ and } \nabla\boldsymbol{u} \in \boldsymbol{L}^{2-\alpha}(\Omega)\}, \tag{3.244a}$$

$$\boldsymbol{V}_\alpha(\Omega) = \{\boldsymbol{u} \mid \boldsymbol{u} \in \boldsymbol{H}_\alpha(\Omega) \text{ and } \boldsymbol{D}^2\boldsymbol{u} \in \boldsymbol{L}^2(\Omega)\} \tag{3.244b}$$

and

$$\boldsymbol{V}^b_\alpha = \{\boldsymbol{u} \mid \boldsymbol{u} \in \boldsymbol{V}_\alpha(\Omega) \text{ such that } \operatorname{supp} \boldsymbol{u} \text{ is bounded}\}. \tag{3.244c}$$

We will need the following result from Appendix B which is proven in [HN]:

$$\iint_\Omega \sum_{i,j} \left|e_{ij}(\boldsymbol{u})\right|^{2-\alpha} d\boldsymbol{x} \geq c\, ||\boldsymbol{D}\boldsymbol{u}||^{2-\alpha}_{\boldsymbol{L}^{2-\alpha}}. \tag{3.245}$$

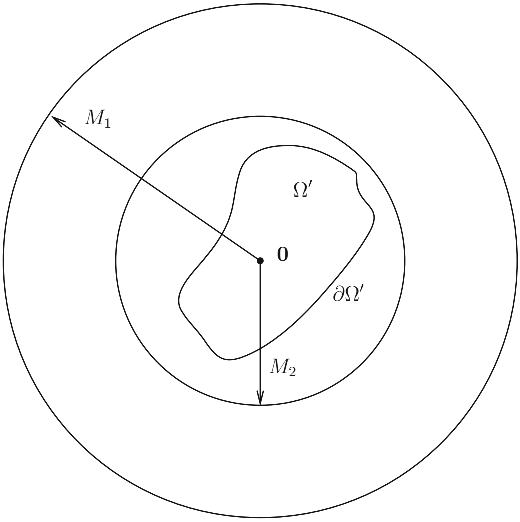

Fig. 3.3 The domain Ω on which $w(x)$ is defined

In order to proceed, we first transform the problem (3.241a–f) into an equivalent problem with homogeneous boundary conditions; to this end, we use some results of Heywood (see [He1, 2]) to infer the existence of a function $\boldsymbol{w}(\boldsymbol{x})$, which is defined for $\boldsymbol{x} \in \Omega$, and which satisfies $\boldsymbol{w} \in \boldsymbol{C}^{\infty}(\Omega)$ and

$$\boldsymbol{w}(\boldsymbol{x}) \equiv \begin{pmatrix} -\lambda \\ 0 \end{pmatrix}, \text{ for } |\boldsymbol{x}| < M_2, \tag{3.246a}$$

$$\boldsymbol{w}(\boldsymbol{x}) \equiv \boldsymbol{0}, \text{ for } |x| > M_1, \tag{3.246b}$$

$$\operatorname{div} \boldsymbol{w} = 0, \text{ in } \Omega \tag{3.246c}$$

where $M_2 > 0$ is such that $\Omega' \subset B_{M_2}(\boldsymbol{0})$ and $M_1 > M_2$ (see Fig. 3.3). Furthermore, it can be shown (see, e.g., [Lio1]) that, for all $\epsilon > 0$, we can choose $\boldsymbol{w}$ such that, with $\Omega_{M_1} = \Omega \cap B_{M_1}(\boldsymbol{0})$,

$$\sum_{i,j} \iint_{\Omega_{M_1}} \left(w_j, u_i\right)^2_{\boldsymbol{L}^2(\Omega)} \, d\boldsymbol{x} \le \epsilon ||\boldsymbol{D}\boldsymbol{u}||^2_{\boldsymbol{L}^2(\Omega_{M_1})}, \ \forall \boldsymbol{u} \in \boldsymbol{V}_{\alpha}(\Omega_{M_1}). \tag{3.247}$$

Setting $\boldsymbol{u} = \boldsymbol{v} - \boldsymbol{v}_\infty - \boldsymbol{w}$ we then have that $\boldsymbol{u}$ satisfies:

$$\boldsymbol{L}(\boldsymbol{u} + \boldsymbol{v}_\infty + \boldsymbol{w}) = -\nabla p, \text{ in } \Omega, \tag{3.248a}$$

$$\boldsymbol{u} = \boldsymbol{0}, \text{ on } \partial\Omega', \tag{3.248b}$$

$$\left(\frac{\partial}{\partial x_k} e_{ij}\right) \nu_j \nu_k \tau_i = 0, \text{ on } \partial\Omega', \tag{3.248c}$$

$$\lim_{||\boldsymbol{x}||\to\infty} \boldsymbol{u} = \boldsymbol{0}, \tag{3.248d}$$

$$\lim_{||\boldsymbol{x}||\to\infty} \boldsymbol{D}^2\boldsymbol{u} = \boldsymbol{0}, \tag{3.248e}$$

$$\nabla \cdot \boldsymbol{u} = 0, \text{ in } \Omega. \tag{3.248f}$$

Our goal in the next two subsections will be to prove the existence of a weak solution for the problem (3.248a–f); to that end we make the following

Definition 3.1. A function $\boldsymbol{u}$ is said to be a weak solution of (3.248a–f) if $\boldsymbol{u} \in \boldsymbol{V}_\alpha(\Omega)$, $\boldsymbol{u}$ satisfies (3.248d,e,f) and, for all $\boldsymbol{v} \in \boldsymbol{V}_\alpha^b(\Omega)$,

$$\begin{aligned}\iint_\Omega & \mu(|\boldsymbol{e}(\boldsymbol{u}+\boldsymbol{w})|)e_{ij}(\boldsymbol{u}+\boldsymbol{w})e_{ij}(\boldsymbol{v})\,d\boldsymbol{x} \\ &+ \mu_1 \iint_\Omega \frac{\partial}{\partial x_k}e_{ij}(\boldsymbol{u}+\boldsymbol{w})\frac{\partial}{\partial x_k}e_{ij}(\boldsymbol{v})\,d\boldsymbol{x} \\ &+ \iint_\Omega (\boldsymbol{u}+\boldsymbol{v}_\infty+\boldsymbol{w})_j \frac{\partial(\boldsymbol{u}+\boldsymbol{w})_i}{\partial x_j} v_i\,d\boldsymbol{x} = 0. \end{aligned} \tag{3.249}$$

3.4.3 Solution of the Exterior Problem in a Truncated Domain

To prove the existence of a weak solution for (3.248a–f) we will first establish the existence of a solution $\boldsymbol{u}^N$ in the truncated domain $\Omega_N = \Omega \cap B_N(\boldsymbol{0})$; then, in Sect. 3.4.4, we will let $N \to \infty$ and prove that $\lim_{N\to\infty} \boldsymbol{u}^N \equiv \boldsymbol{u}$ is a weak solution to (3.248a–f) in the sense of Definition 3.1. Therefore, let

$$B_N(\boldsymbol{0}) = \{\boldsymbol{x} \in R^2 \mid ||\boldsymbol{x}|| < N\}.$$

We assume that $N > 0$ is so large that $N > M_1 > M_2 > \operatorname{diam}\Omega'$. With $\Omega_N = \Omega \cap B_N(\boldsymbol{0})$, we want to prove the existence of a function $\boldsymbol{u}^N$ which satisfies $\nabla \cdot \boldsymbol{u}^N = 0$ in Ω_N with

$$\boldsymbol{u}^N \in \boldsymbol{W}^{2,2}(\Omega_N) \cap \boldsymbol{W}_0^{1,2}(\Omega_N), \tag{3.250a}$$

$$\left(\frac{\partial}{\partial x_k}e_{ij}(\boldsymbol{u}^N)\right)\nu_j\nu_k\tau_i = 0, \text{ on } \partial\Omega_N \tag{3.250b}$$

and for all $\boldsymbol{v} \in \boldsymbol{W}^{2,2}(\Omega_N) \cap \boldsymbol{W}_0^{1,2}(\Omega_N)$ such that $\nabla \cdot \boldsymbol{v} = 0$ in Ω_N,

$$\iint_{\Omega_N} \mu\left(\left|\boldsymbol{e}(\boldsymbol{u}^N + \boldsymbol{w} + \boldsymbol{v}_\infty)\right|\right) e_{ij}(\boldsymbol{u}^N + \boldsymbol{w} + \boldsymbol{v}_\infty) e_{ij}(\boldsymbol{v})\, d\boldsymbol{x}$$
$$+ \mu_1 \iint_{\Omega_N} \frac{\partial}{\partial x_k} e_{ij}(\boldsymbol{u}^N + \boldsymbol{w} + \boldsymbol{v}_\infty) \frac{\partial}{\partial x_k} e_{ij}(\boldsymbol{v})\, d\boldsymbol{x}$$
$$+ \iint_{\Omega_N} (\boldsymbol{u}^N + \boldsymbol{w} + \boldsymbol{v}_\infty)_j \frac{\partial(\boldsymbol{u}^N + \boldsymbol{w})_i}{\partial x_j} v_i\, d\boldsymbol{x} = 0. \tag{3.251}$$

Our first step in this direction consists of proving the following a priori estimate:

Theorem 3.5. *There exists a positive constant C, independent of N, such that any solution $\boldsymbol{u}^N$ of* (3.251) *satisfies:*

$$\iint_{\Omega_N} (\boldsymbol{D}^2 \boldsymbol{u}^N)^2\, d\boldsymbol{x} + \iint_{\Omega} (\boldsymbol{D}\boldsymbol{u}^N)^{2-\alpha}\, d\boldsymbol{x} \le C. \tag{3.252}$$

Proof. Setting $\boldsymbol{v} = \boldsymbol{u}^N$ in (3.251), it follows that

$$\iint_{\Omega_N} \mu(\left|\boldsymbol{e}(\boldsymbol{u}^N + \boldsymbol{w})\right|) e_{ij}(\boldsymbol{u}^N + \boldsymbol{w}) e_{ij}(\boldsymbol{u}^N + \boldsymbol{w})\, d\boldsymbol{x} \tag{3.253}$$
$$+ \mu_1 \iint_{\Omega_N} \frac{\partial}{\partial x_k} e_{ij}(\boldsymbol{u}^N + \boldsymbol{w}) \frac{\partial}{\partial x_k} (e_{ij}(\boldsymbol{u}^N + \boldsymbol{w}))\, d\boldsymbol{x}$$
$$\le \left| \iint_{\Omega_N} \mu(\boldsymbol{u}^N + \boldsymbol{w}) e_{ij}(\boldsymbol{u}^N + \boldsymbol{w}) e_{ij}(\boldsymbol{w})\, d\boldsymbol{x} \right|$$
$$+ \mu_1 \left| \iint_{\Omega_N} \frac{\partial}{\partial x_k} e_k(\boldsymbol{u}^N + \boldsymbol{w}) \frac{\partial}{\partial x_k} e_{ij}(\boldsymbol{w})\, d\boldsymbol{x} \right|$$
$$+ \left| \iint_{\Omega_N} (\boldsymbol{u}^N + \boldsymbol{v}_\infty + \boldsymbol{w})_j \frac{\partial(\boldsymbol{u}^N + \boldsymbol{w})_i}{\partial x_j} \boldsymbol{u}_i^N\, d\boldsymbol{x} \right|.$$

Also (with no summation over the repeated indices i, j, k) we have, for any $\delta > 0$,

$$\left| \frac{\partial}{\partial x_k} e_{ij}(\boldsymbol{u}^N + \boldsymbol{w}) \frac{\partial}{\partial x_k} e_{ij}(\boldsymbol{w}) \right| \le \frac{\delta}{2} \left(\frac{\partial}{\partial x_k} e_{ij}(\boldsymbol{u}^N + \boldsymbol{w}) \right)^2$$
$$+ \frac{1}{2\delta} \left(\frac{\partial}{\partial x_k} e_{ij}(\boldsymbol{w}) \right)^2 \tag{3.254}$$

and, for some $c_1 > 0$,

$$\left| \iint_{\Omega_N} \mu\left(\left|\boldsymbol{e}(\boldsymbol{u}^N + \boldsymbol{w})\right|\right) e_{ij}(\boldsymbol{u}^N + \boldsymbol{w}) e_{ij}(\boldsymbol{w})\, d\boldsymbol{x} \right| \\ \leq \sum_{ij} \left(c_1 \delta \left|\left| e_{ij}(\boldsymbol{u}^N + \boldsymbol{w}) \right|\right|^{2-\alpha}_{L^{2-\alpha}(\Omega_N)} + c_1 \delta^{\alpha-1} \left|\left| e_{ij}(\boldsymbol{w}) \right|\right|^{2-\alpha}_{L^{2-\alpha}(\Omega_N)} \right). \tag{3.255}$$

By (3.243a,b), (3.245), and (3.253), it follows that $\exists c_1 > 0$, $c_2 > 0$ such that for any $\delta > 0$,

$$c_2 \iint_{\Omega_N} (\boldsymbol{D}^2(\boldsymbol{u}^N + \boldsymbol{w}))^2\, d\boldsymbol{x} + c_2 \iint_{\Omega_N} (\boldsymbol{D}(\boldsymbol{u}^N + \boldsymbol{w}))^{2-\alpha}\, d\boldsymbol{x} \tag{3.256}$$

$$\leq (1 - \delta c_1) \iint_{\Omega_N} \mu(\left|\boldsymbol{e}(\boldsymbol{u}^N + \boldsymbol{w})\right|) e_{ij}(\boldsymbol{u}^N + \boldsymbol{w}) e_{ij}(\boldsymbol{u}^N + \boldsymbol{w})\, d\boldsymbol{x} \\ + \mu_1 \left(1 - \frac{\delta}{2}\right) \iint_{\Omega_N} \frac{\partial}{\partial x_k} e_{ij}(\boldsymbol{u}^N + \boldsymbol{w}) \frac{\partial}{\partial x_k} e_{ij}(\boldsymbol{u}^N + \boldsymbol{w})\, d\boldsymbol{x} \\ \leq \mu_1 \frac{1}{2\delta} \sum_{i,j} \left|\left| \frac{\partial}{\partial x_k} e_{ij}(\boldsymbol{w}) \right|\right|^2_{L^2(\Omega_N)} + c_1 \delta^{\alpha-1} \sum_{i,j} \left|\left| e_{ij}(\boldsymbol{w}) \right|\right|^{2-\alpha}_{L^{2-\alpha}(\Omega_N)} \\ + \left| b(\boldsymbol{u}^N + \boldsymbol{w} + \boldsymbol{v}_\infty, \boldsymbol{u}^N + \boldsymbol{w}, \boldsymbol{u}^N) \right|$$

where for any domain $\Omega \subseteq R^n$,

$$b(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) = \iint_{\Omega} f_j \frac{\partial g_i}{\partial x_j} h_i\, d\boldsymbol{x}.$$

Integration by parts yields $b(\boldsymbol{u}^N + \boldsymbol{w} + \boldsymbol{v}_\infty, \boldsymbol{u}^N, \boldsymbol{u}^N) = 0$, hence

$$b(\boldsymbol{u}^N + \boldsymbol{w} + \boldsymbol{v}_\infty, \boldsymbol{u}^N + \boldsymbol{w}, \boldsymbol{u}^N) = b(\boldsymbol{w} + \boldsymbol{v}_\infty, \boldsymbol{w}, \boldsymbol{u}^N) + b(\boldsymbol{u}^N, \boldsymbol{w}, \boldsymbol{u}^N).$$

Since $\partial w_i / \partial x_j$ has fixed support (independent of N) we have that

$$\left| b(\boldsymbol{w} + \boldsymbol{v}_\infty, \boldsymbol{w}, \boldsymbol{u}^N) \right| \leq \delta \left|\left| \boldsymbol{u}^N \right|\right|^2_{\boldsymbol{L}^\infty(\Omega_{M_2})} + \frac{1}{\delta} ||\boldsymbol{w} + \boldsymbol{v}_\infty||^2_{\boldsymbol{L}^\infty(\Omega_{M_2})} \left|\left| \frac{\partial w_i}{\partial x_j} \right|\right|^2_{L^2(\Omega_{M_2})} \\ \leq c_3 \delta \left|\left| \boldsymbol{D}^2 \boldsymbol{u}^N \right|\right|^2_{\boldsymbol{L}^2(\Omega_{M_2})} + \frac{1}{\delta} ||\boldsymbol{w} + \boldsymbol{v}_\infty||^2_{\boldsymbol{L}^\infty(\Omega_{M_2})} \left|\left| \frac{\partial w_i}{\partial x_j} \right|\right|^2_{L^2(\Omega_{M_2})} \tag{3.257}$$

for any $\delta > 0$, where c_3 depends on Γ, M_2 but not on N.

From (3.247), we deduce that

$$\left|b(\boldsymbol{u}^N, \boldsymbol{w}, \boldsymbol{u}^N)\right| \leq \epsilon \left|\left|\boldsymbol{D}\boldsymbol{u}^N\right|\right|^2_{\boldsymbol{L}^2(\Omega_{M_2})} \leq \epsilon c_7 \left|\left|\boldsymbol{D}^2\boldsymbol{u}^N\right|\right|^2_{\boldsymbol{L}^2(\Omega_{M_2})} \tag{3.258}$$

where $c_7 > 0$ is independent of N. Choosing ϵ and δ small enough, it then follows from (3.253) that there exists a positive $C(\epsilon, \delta, \mu_0, \mu_1)$, independent of N, $\boldsymbol{u}^N$ such that

$$\left|\left|\boldsymbol{D}^2\boldsymbol{u}^N\right|\right|^2_{\boldsymbol{L}^2(\Omega_N)} + \left|\left|\boldsymbol{D}\boldsymbol{u}^N\right|\right|^{2-\alpha}_{\boldsymbol{L}^{2-\alpha}(\Omega_N)} \leq C. \qquad \square$$

Our goal now is to employ the Galerkin method to prove the existence of a solution $\boldsymbol{u}^N$ to (3.251) which satisfies (3.250a,b) and $\nabla \cdot \boldsymbol{u}^N = 0$ in Ω_N. In order to introduce the required basis we define the space

$$\boldsymbol{\mathcal{H}} = \{\boldsymbol{u} \mid \boldsymbol{u} \in \boldsymbol{W}^{2,2}(\Omega_N) \cap \boldsymbol{W}^{1,2}_0(\Omega_N);\ \nabla \cdot \boldsymbol{u} = 0\}. \tag{3.259}$$

In $\boldsymbol{\mathcal{H}}$ the scalar product is taken to be

$$((\boldsymbol{w}, \boldsymbol{\varphi})) = \int_{\Omega_N} \frac{\partial}{\partial x_k} e_{ij}(\boldsymbol{w}) \frac{\partial}{\partial x_k} e_{ij}(\boldsymbol{\varphi})\, d\boldsymbol{x}. \tag{3.260}$$

We now have the following

Lemma 3.1. *The eigenvalue problem*

$$((\boldsymbol{w}, \boldsymbol{\varphi})) = \lambda\, (\boldsymbol{w}, \boldsymbol{\varphi})_{\boldsymbol{L}^2(\Omega)}, \quad \forall \boldsymbol{\varphi} \in \boldsymbol{\mathcal{H}} \tag{3.261}$$

has a sequence of solutions $\boldsymbol{\mathcal{W}}^i \in \boldsymbol{\mathcal{H}} \cap \boldsymbol{C}^\infty(\Omega_N)$ *corresponding to a sequence of positive eigenvalues* λ_l. *Furthermore,*

1. $((\partial/\partial x_k)e_{ij}(\boldsymbol{\mathcal{W}}^l))\nu_j \nu_k \tau_i = 0$, *on* $\partial\Omega_N$, *for all* $l = 1, 2, \ldots$.
2. *The sequence* $\boldsymbol{\mathcal{W}}^l$ *is a basis for the closure of* $\boldsymbol{\mathcal{H}}$ *under the* $\boldsymbol{L}^2$ *norm.*
3. *The sequence* $\boldsymbol{\mathcal{W}}^l$ *is a basis of* $\boldsymbol{\mathcal{H}}$.
4. $(\boldsymbol{\mathcal{W}}^l, \boldsymbol{\mathcal{W}}^k) = \delta_{lk}$.

Proof. This is a standard consequence of the estimate (3.243a). $\square$

Now, for K fixed, let $\boldsymbol{u}^{N,K} \in \boldsymbol{E}_K = \text{span}\{\boldsymbol{\mathcal{W}}^1 \cdots \boldsymbol{\mathcal{W}}^K\}$, with $\boldsymbol{u}^{N,K}(\boldsymbol{x}) = \sum_{l=1}^{K} \boldsymbol{A}_l \boldsymbol{\mathcal{W}}^l(\boldsymbol{x})$ the solution of

$$\iint_{\Omega_N} \mu(|\boldsymbol{e}(\boldsymbol{u}^{N,K} + \boldsymbol{w} + \boldsymbol{v}_\infty)|) e_{ij}(\boldsymbol{u}^{N,K} + \boldsymbol{w} + \boldsymbol{v}_\infty) e_{ij}(\boldsymbol{v})\, d\boldsymbol{x}$$
$$+ \iint_{\Omega_N} \frac{\partial}{\partial x_k} e_{ij}(\boldsymbol{u}^{N,K} + \boldsymbol{w} + \boldsymbol{v}_\infty) \frac{\partial}{\partial x_k} e_{ij}(\boldsymbol{v})\, d\boldsymbol{x}$$
$$+ \iint_{\Omega_N} (\boldsymbol{u}^{N,K} + \boldsymbol{w} + \boldsymbol{v}_\infty)_j \frac{\partial(\boldsymbol{u}^{N,K} + \boldsymbol{w})_i}{\partial x_j} v_i\, d\boldsymbol{x} = 0 \quad (3.262)$$

$\forall \boldsymbol{w} \in \boldsymbol{E}_K$. Then,

Lemma 3.2. *The problem* (3.262) *has a solution* $\boldsymbol{u}^{N,K}$.

In order to establish the validity of Lemma 3.2 we recall the following result in [Te1]:

Lemma 3.3. *Let* $\boldsymbol{X}$ *be a finite-dimensional Hilbert space with scalar product* $[\cdot,\cdot]$ *and norm* $[\cdot]$, *and let* $\boldsymbol{P}$ *be a continuous mapping from* $\boldsymbol{X}$ *into itself such that*

$$[\boldsymbol{P}(\boldsymbol{\xi}), \boldsymbol{\xi}] > 0 \, for \, [\boldsymbol{\xi}] = A > 0. \quad (3.263)$$

Then there exists $\boldsymbol{\xi} \in \boldsymbol{X}$, $[\boldsymbol{\xi}] \le A$, *such that* $\boldsymbol{P}(\boldsymbol{\xi}) = \boldsymbol{0}$.

Proof (Lemma 3.2). We set $\boldsymbol{X} = \boldsymbol{E}_K$ and define $\boldsymbol{P}$ by

$$[\boldsymbol{P}(\boldsymbol{u}), \boldsymbol{v}] = \mu_1 \iint_{\Omega_N} \frac{\partial}{\partial x_k} e_{ij}(\boldsymbol{u} + \boldsymbol{w}) \frac{\partial}{\partial x_k} e_{ij}(\boldsymbol{v})\, d\boldsymbol{x}$$
$$+ \iint_{\Omega_N} \mu(|\boldsymbol{e}(\boldsymbol{u} + \boldsymbol{w})|) e_{ij}(\boldsymbol{u} + \boldsymbol{w}) e_{ij}(\boldsymbol{v})\, d\boldsymbol{x}$$
$$+ b(\boldsymbol{u} + \boldsymbol{w} + \boldsymbol{v}_\infty, \boldsymbol{u} + \boldsymbol{w}, \boldsymbol{v}) \quad (3.264)$$

$\forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{E}_K$. The continuity of $\boldsymbol{P}$ is clear; also, proceeding as in the proof of the a priori estimate (3.252) it can be shown that there exists a positive c (independent of K and N) such that if $[\boldsymbol{P}(\boldsymbol{u}), \boldsymbol{u}] \le 0$, then $||\boldsymbol{u}|| \le c$. Therefore, if $A > c$, condition (3.263) of the lemma is satisfied. Hence, there exists a solution $\boldsymbol{u}^{N,K}$ to the problem (3.262). □

Since (3.262) holds for $\boldsymbol{v} = \boldsymbol{u}^{N,K}$, the solution $\boldsymbol{u}^{N,K}$ satisfies the a priori estimate of Theorem 3.5. In particular, there exists a positive c independent of K such that

$$||\boldsymbol{D}^2 \boldsymbol{u}^{N,K}||_{\boldsymbol{L}^2(\Omega_N)} \le c.$$

Hence, for fixed N the sequence $\boldsymbol{u}^{N,K}$ is weakly compact in $\boldsymbol{W}^{2,2}(\Omega_N) \cap \boldsymbol{W}_0^{1,2}(\Omega_N)$; from this fact the following existence theorem can be deduced by letting $K \to \infty$ in (3.262) and using standard arguments:

Theorem 3.6. *The problem (3.251) has a (weak) solution* $\boldsymbol{u}^N \in \boldsymbol{\mathcal{H}}$ *for all* $\boldsymbol{v} \in \boldsymbol{W}^{2,2}(\Omega_N) \cap \boldsymbol{W}_0^{1,2}(\Omega_N)$ *such that* $\nabla \cdot \boldsymbol{v} = 0$ *in* Ω_N.

Our next theorem concerns a regularity result for the solution $\boldsymbol{u}^N$ of Theorem 3.6; it shows, in particular, that $\boldsymbol{u}^N$ satisfies the higher-order boundary condition (3.250b).

Theorem 3.7. *Let* $\boldsymbol{u}^N \in \boldsymbol{\mathcal{H}}$ *be the weak solution of Theorem 3.6. Then* $\boldsymbol{u}^N \in \boldsymbol{W}^{3,2}(\Omega_N)$ *and satisfies the boundary condition*

$$\frac{\partial}{\partial x_k} e_{ij}(\boldsymbol{u}^N)\nu_j \nu_k \tau_i = 0.$$

Moreover, there exists $\boldsymbol{p}^N \in \boldsymbol{L}^2(\Omega_N)$ *such that* $\boldsymbol{u}^N$ *satisfies*

$$\boldsymbol{L}(\boldsymbol{u}^N + \boldsymbol{v}_\infty + \boldsymbol{w}) = -\nabla \boldsymbol{p}^N, \ \text{in } \Omega_N. \tag{3.265}$$

Proof. The fact that $\boldsymbol{u}^N$ satisfies (3.265) is an immediate consequence of (3.251) and the de Rham Theorem (see, e.g., [GRa]). Since $\boldsymbol{u}^N \in \boldsymbol{W}_0^{1,2}(\Omega_N) \cap \boldsymbol{W}^{2,2}(\Omega_N)$ and is divergence free, there exists a unique function $\Psi^N(x, y) \in \boldsymbol{W}^{3,2}(\Omega_N)$ such that

$$\boldsymbol{u}^N = (-\Psi_y^N, \Psi_x^N), \tag{3.266a}$$

$$\Psi^N\big|_{\partial\Omega'} = c^N, \quad \Psi^N\big|_{||\boldsymbol{x}||=N} = 0, \tag{3.266b}$$

$$\frac{\partial \Psi^N}{\partial n}\bigg|_{\partial\Omega_N} = 0. \tag{3.266c}$$

Substituting $(-\Psi_y^N, \Psi_x^N)$ for $\boldsymbol{u}^N$ in the partial differential equation (3.265) and taking the curl, we find that

$$\begin{aligned} -\mu_1 \Delta^3 \Psi^N = {} & \frac{\partial^2}{\partial x_1 \partial x_j} \mu(|\boldsymbol{e}(\boldsymbol{u}^N + \boldsymbol{w} + \boldsymbol{v}_\infty)|) e_{2j}(\boldsymbol{u}^N + \boldsymbol{w} + \boldsymbol{v}_\infty) \\ & - \frac{\partial^2}{\partial x_2 \partial x_j} \mu(|\boldsymbol{e}(\boldsymbol{u}^N + \boldsymbol{w} + \boldsymbol{v}_\infty)|) e_{1j}(\boldsymbol{u}^N + \boldsymbol{w} + \boldsymbol{v}_\infty) \\ & + \frac{\partial}{\partial x_1}\left((\boldsymbol{u}^N + \boldsymbol{w} + \boldsymbol{v}_\infty)_j \frac{\partial(\boldsymbol{u}^N + \boldsymbol{w})_2}{\partial x_j} \right) \\ & - \frac{\partial}{\partial x_2}\left((\boldsymbol{u}^N + \boldsymbol{w} + \boldsymbol{v}_\infty)_j \frac{\partial(\boldsymbol{u}^N + \boldsymbol{w})_1}{\partial x_j} \right) \\ & + \mu_1 \frac{\partial^2}{\partial x_1 \partial x_j} \Delta e_{2j}(\boldsymbol{w}) - \mu_1 \frac{\partial^2}{\partial x_2 \partial x_j} \Delta e_{1j}(\boldsymbol{w}). \end{aligned} \tag{3.267}$$

From the regularity of $\boldsymbol{u}^N$ and $\boldsymbol{w}$ the right-hand side of (3.267) is in $\boldsymbol{W}^{-2,2}(\Omega_N)$ so that $\Delta^3\Psi^N \in \boldsymbol{W}^{-2,3}(\Omega_N)$. Using that $(\Psi^N, \Delta^3\Psi^N) \in \boldsymbol{W}^{3,2}(\Omega_N) \times \boldsymbol{W}^{-2,3}(\Omega_N)$ we deduce via duality, in the usual fashion, that we can define $\partial^2\Psi^N/\partial\nu^3 \in \boldsymbol{W}^{-1/2,2}(\partial\Omega_N)$. Therefore, the traces of all third-order derivatives of Ψ^N are defined, and the traces of all second-order derivatives of $\boldsymbol{u}^N$ are defined. As a consequence of (3.251) and (3.265) we have that

$$\int_{\partial\Omega_N} \frac{\partial}{\partial x_k} e_{ij}(\boldsymbol{u}^N)\nu_j\nu_k \frac{\partial v_i}{\partial\nu}\, dS = 0, \quad \forall \boldsymbol{v} \in \mathcal{H} \tag{3.268}$$

from which it follows, by virtue of the analysis in Sect. 1.4.4 and Theorem 3.1 of Heron [HB], that the tangential component of $\left(\frac{\partial}{\partial x_k}\right) e_{ij}(\boldsymbol{u}^N)\nu_j\nu_k$ vanishes on $\partial\Omega_N$, i.e.,

$$\left.\frac{\partial}{\partial x_k} e_{ij}(\boldsymbol{u}^N)\nu_j\nu_k\tau_i\right|_{\partial\Omega_N} = 0. \tag{3.269}$$

Therefore, Ψ^N satisfies

$$\left.\frac{\partial}{\partial\nu}\Delta\Psi^N\right|_{\partial\Omega_N} = 0. \tag{3.270}$$

From the regularity theory of elliptic partial differential equations (see, e.g., [Ev] or [LM]), it follows that $\Psi^N \in W^{4,2}(\Omega_N)$, which then yields that $\boldsymbol{u}^N \in \boldsymbol{W}^{3,2}(\Omega_N)$ and $\boldsymbol{p}^N \in \boldsymbol{L}^2(\Omega_N)$. □

3.4.4 Solution of the Exterior Problem in an Unbounded Domain

We will now use the weak solutions $\boldsymbol{u}^N$ for the truncated domains Ω_N to construct a solution of the exterior problem in $\Omega = R^2/\bar{\Omega}'$. Our first result in this direction is the following

Lemma 3.4. *There exists a constant $c > 0$ independent of N such that*

$$||\boldsymbol{u}^N||_{L^{4/\alpha-2}(\Omega_N)} \leq c. \tag{3.271}$$

Proof. This is a direct consequence of estimate (3.252) and the estimate

$$||\boldsymbol{u}||_{L^q(\Omega)} \leq c||\nabla\boldsymbol{u}||_{L^p(\Omega)} \tag{3.272}$$

with $1/p - 1/2 = 1/q$ (see, e.g., [Te1], page 158). □

Remarks. From (3.271) it follows that the sequence $\boldsymbol{u}^N$ has a subsequence, denoted also by $\boldsymbol{u}^N$, which is weakly convergent in $\boldsymbol{L}^{4/\alpha-2}(\Omega)$ to a function $\boldsymbol{u}$. By the estimate (3.252), and a diagonal process, there exists a subsequence, which will again be denoted by $\boldsymbol{u}^N$, such that $\boldsymbol{u}^N$ converges strongly to $\boldsymbol{u}$ in $\boldsymbol{W}^{1,2-\alpha}_{loc}(\Omega)$ and weakly in $\boldsymbol{W}^{2,2}_{loc}(\Omega)$. From (3.252) it follows that

$$\iint_\Omega (\boldsymbol{D}^2\boldsymbol{u})^2\,d\boldsymbol{x} + \iint_\Omega |\boldsymbol{D}\boldsymbol{u}|^{2-\alpha}\,d\boldsymbol{x} \le c. \tag{3.273}$$

and, from (3.271), that

$$\iint_\Omega |\boldsymbol{u}|^{4/\alpha-2}\,d\boldsymbol{x} \le c. \tag{3.274}$$

Going to the limit in (3.251), we then find that

$$\begin{aligned}\iint_\Omega &\mu(|\boldsymbol{e}(\boldsymbol{u}+\boldsymbol{w}+\boldsymbol{v}_\infty)|)e_{ij}(\boldsymbol{u}+\boldsymbol{w}+\boldsymbol{v}_\infty)e_{ij}(\boldsymbol{v})\,d\boldsymbol{x}\\ &+\mu_1\iint_\Omega \frac{\partial}{\partial x_k}e_{ij}(\boldsymbol{u}+\boldsymbol{w}+\boldsymbol{v}_\infty)\frac{\partial e_{ij}}{\partial x_k}(\boldsymbol{v})\,d\boldsymbol{x}\\ &+\iint_\Omega (\boldsymbol{u}+\boldsymbol{w}+\boldsymbol{v}_\infty)_j\frac{\partial(\boldsymbol{u}+\boldsymbol{w}+\boldsymbol{v}_\infty)_i}{\partial x_j}v_i\,d\boldsymbol{x} = 0\end{aligned} \tag{3.275}$$

$\forall \boldsymbol{v} \in \boldsymbol{V}^b_\alpha(\Omega)$. It now follows from (3.275) that $\boldsymbol{u}$ satisfies, in Ω,

$$\begin{aligned}\mu_1\frac{\partial}{\partial x_j}\Delta e_{ij}(\boldsymbol{u}+\boldsymbol{w}+\boldsymbol{v}_\infty) &- \frac{\partial}{\partial x_j}\left(\mu(|\boldsymbol{e}(\boldsymbol{u}+\boldsymbol{w}+\boldsymbol{v}_\infty)|)e_{ij}(\boldsymbol{u}+\boldsymbol{w}+\boldsymbol{v}_\infty)\right)\\ &+(\boldsymbol{u}+\boldsymbol{w}+\boldsymbol{v}_\infty)_j\frac{\partial}{\partial x_j}(\boldsymbol{u}+\boldsymbol{w}+\boldsymbol{v}_\infty)_i = -\frac{\partial p}{\partial x_i}\end{aligned} \tag{3.276}$$

where $p \in W^{-1,2}(\Omega)$. As the right-hand side of (3.267) is uniformly bounded in $W^{-2,2}(\Omega)$, it follows from local regularity results for elliptic equations that $\Psi^N - c^N$ (see (3.266b)) is uniformly bounded in $\boldsymbol{W}^{4,2}_{loc}(\Omega)$. Therefore, $\boldsymbol{u}^N$ converges in $\boldsymbol{W}^{3,2}_{loc}(\Omega)$. It also follows from (3.269) that

$$\frac{\partial}{\partial x_k}e_{ij}(\boldsymbol{u})\nu_j\nu_k\tau_i\bigg|_{\partial\Omega'} = 0.$$

Our last lemma, which leads up to the statement of the existence theorem for the exterior problem in Ω is

Lemma 3.5. *The function* $\boldsymbol{u}$ *which has been obtained as the limit, as* $N \to \infty$*, of a subsequence of the weak solutions* $\boldsymbol{u}^N$*, and which satisfies* (3.276)*,* $\forall \boldsymbol{v} \in \boldsymbol{V}_\alpha^b(\Omega)$*, also satisfies*

(i) $\boldsymbol{u} \in \boldsymbol{L}^\infty(\Omega)$,
(ii) $\lim_{||\boldsymbol{x}|| \to \infty} \boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{0}$.

Proof. Since $\boldsymbol{u} = \boldsymbol{0}$ on $\partial\Omega'$, it follows from (3.273) that $\boldsymbol{u} \in \boldsymbol{L}^\infty_{loc}(\Omega)$. Assume that $\boldsymbol{u} \notin \boldsymbol{L}^\infty(\Omega)$. Then there exists $\boldsymbol{x}_n \to \infty$ such that $||\boldsymbol{u}(\boldsymbol{x}_n)|| \to \infty$. But, since for some $c > 0$,

$$\left|\left|\boldsymbol{D}^2\boldsymbol{u}\right|\right|_{L^2(B_1(\boldsymbol{x}_n))} \leq c, \quad ||\boldsymbol{u}||_{L^{4/\alpha-2}(B_1(\boldsymbol{x}_n))} \leq c \tag{3.277a}$$

(c independent of n) it follows from the estimate

$$||\boldsymbol{u}||_{\boldsymbol{L}^\infty(B_1(\boldsymbol{x}_n))} \leq c\left(\left|\left|\boldsymbol{D}^2\boldsymbol{u}\right|\right|_{\boldsymbol{L}^2(B_1(\boldsymbol{x}_n))} + ||\boldsymbol{u}||_{\boldsymbol{L}^{4/\alpha-2}(B_1(\boldsymbol{x}_n))}\right) \tag{3.277b}$$

that

$$||\boldsymbol{u}||_{\boldsymbol{L}^\infty(B_1(\boldsymbol{x}_n))} \leq c \tag{3.277c}$$

where c is independent of n; this establishes part (i) of the Lemma. To establish part (ii) of the lemma we only have to observe that, by (3.273) and (3.274), the right-hand side of (3.277b) goes to zero as $n \to \infty$. □

By combining the discussion which follows Lemma 3.4 with the results of Lemma 3.5, we observe that we have established the following

Theorem 3.8. *The problem* (3.248a–f) *has a weak solution in the sense of Definition 3.1.*

Remarks. The condition (3.248d) is satisfied in the sense of pointwise limit, while (3.248c) is satisfied in the sense that $\boldsymbol{D}^2\boldsymbol{u} \in \boldsymbol{L}^2(\Omega)$.

3.4.5 Existence of a Drag Force

In this brief subsection we show that the solution of problem (3.248a–f), equivalently, the solution of (3.241a–f), predicts the existence of a drag force on the body Ω' in the direction of $\boldsymbol{v}_\infty$. We denote by $\boldsymbol{\mathcal{F}}(\boldsymbol{v}_\infty)$ the force exerted on the body Ω' as a consequence of the motion of the fluid so that

$$\boldsymbol{\mathcal{F}}(\boldsymbol{v}_\infty) = \int_{\partial\Omega'} t_{ij}\nu_j \, dS \tag{3.278}$$

with $t_{ij} = -p\delta_{ij} + \mu(|\boldsymbol{e}(\boldsymbol{v})|)e_{ij} - 2\mu_1\Delta e_{ij}$ for the bipolar viscous fluid in $\Omega = R^2/\bar{\Omega}'$. We have the following result

Theorem 3.9. *The solution* $\boldsymbol{v}$ *of* (3.241a–f) *exhibits a drag force in the direction of* $\boldsymbol{v}_\infty$*, i.e.,* $\mathcal{F}(\boldsymbol{v}_\infty) \cdot \boldsymbol{v}_\infty > 0$.

Proof. By (3.278) and (3.241c)

$$\mathcal{F}(\boldsymbol{v}_\infty) \cdot \boldsymbol{v}_\infty = \lambda \int_{\partial\Omega'} \tau_{1,j}(\boldsymbol{v})\nu_j \, dS.$$

Setting $\boldsymbol{v}^N = \boldsymbol{u}^N + \boldsymbol{w} + \boldsymbol{v}_\infty$, multiplying (2.266) by $(\boldsymbol{v}^N - \boldsymbol{v}_\infty)$, integrating by parts, and using the fact that $v_j^N(\boldsymbol{v}^N - \boldsymbol{v}_\infty)_i = 0$, on $\partial\Omega_N$, we find that

$$\lambda \int_{\partial\Omega_N} \tau_{1,j}(\boldsymbol{v}^N)\nu_j \, dS = \iint_{\Omega_N} \mu(\boldsymbol{v}^N) e_{ij}(\boldsymbol{v}^N) \, d\boldsymbol{x} + \mu_1 \iint_{\Omega_N} \frac{\partial}{\partial x_k} e_{ij}(\boldsymbol{v}^N) \frac{\partial}{\partial x_k} e_{ij}(\boldsymbol{v}^N) \, d\boldsymbol{x} > 0. \tag{3.279}$$

Letting $N \to \infty$ in (3.279) then yields

$$\lambda \int_{\partial\Omega'} \tau_{1,j}(\boldsymbol{v})\nu_j \, dS = \iint_{\Omega} \mu(\boldsymbol{v}) e_{ij}(\boldsymbol{v}) e_{ij}(\boldsymbol{v}) \, d\boldsymbol{x} + \mu_1 \iint_{\Omega} \frac{\partial}{\partial x_k} e_{ij}(\boldsymbol{v}) \frac{\partial}{\partial x_k} e_{ij}(\boldsymbol{v}) \, d\boldsymbol{x} > 0. \tag{3.280}$$

□

3.5 Flow Over Non-smooth Boundaries

3.5.1 Introduction

The initial-boundary value problem for a nonlinear bipolar fluid in a domain $\Omega \subseteq \mathbb{R}^n$, $n = 2, 3$ is given by (1.128)–(1.131). In this section we will consider such problems for $\Omega \subset \mathbb{R}^2$ and, to emphasize this point, we will write the velocity vector as $\boldsymbol{w} = (w_1(x_1, x_2), w_2(x_1, x_2))$. We also assume that in the higher-order boundary conditions (1.116) $M_i = 0$, $i = 1, 2$. With the aforementioned assumptions, the nonlinear, bipolar, initial-boundary value problem in this section takes on the form

$$\rho \frac{\partial w_i}{\partial t} = -\rho w_j \frac{\partial w_i}{\partial x_j} - \frac{\partial p}{\partial x_i} - 2\mu_1 \frac{\partial \Delta e_{ij}(\boldsymbol{w})}{\partial x_j} + 2\frac{\partial(\mu(\boldsymbol{w}) e_{ij}(\boldsymbol{w}))}{\partial x_j} + \rho f_i, \text{ in } \Omega \times (0, T), \tag{3.281a}$$

$$\nabla \cdot \boldsymbol{w} = 0, \text{ in } \Omega \times (0, T), \tag{3.281b}$$

$$\boldsymbol{w}(\boldsymbol{x}, 0) = \boldsymbol{h}(\boldsymbol{x}), \text{ in } \Omega, \tag{3.281c}$$

$$\boldsymbol{w} = \boldsymbol{0}, \text{ on } \partial\Omega \times (0, T) \tag{3.281d}$$

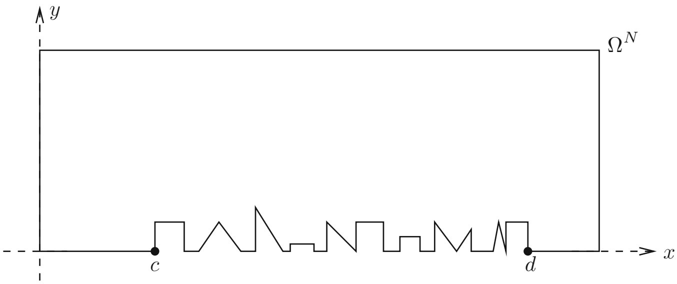

Fig. 3.4 A sequence of domains $\Omega^N \to \Omega$

and

$$\frac{\partial e_{ij}(\boldsymbol{w})}{\partial x_k} \nu_j \nu_k \tau_i = 0, \text{ on } \partial\Omega \times (0, T) \tag{3.281e}$$

with $\boldsymbol{f} = (f_1(x_1, x_2), f_2(x_1, x_2))$, $\boldsymbol{h}(x_1, x_2)$ given functions, $\boldsymbol{\nu}$ the exterior unit normal to $\partial\Omega$, $\boldsymbol{\tau}$ the unit tangent vector to $\partial\Omega$, $e_{ij}(\boldsymbol{w}) = \frac{1}{2}\left(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i}\right)$, and $\mu(\boldsymbol{w}) \equiv \mu(|\boldsymbol{e}(\boldsymbol{w})|) = \mu_0(\epsilon + e_{ij}(\boldsymbol{w})e_{ij}(\boldsymbol{w}))^{-\alpha/2}$. As in previous sections of this volume, the parameters in (3.281a) satisfy $\epsilon, \mu_0, \mu_1 > 0$ and $0 \le \alpha < 1$.

We wish to investigate the dependence of the solutions to problem (3.281a–e) on the domain Ω. More specifically, we are interested in the stability of solutions with respect to perturbations of the boundary of the domain Ω (i.e. If a domain Ω^N is close to a domain Ω, does it follow that the corresponding solution $\boldsymbol{w}^N$ to problem (3.281a–e) in Ω^N is close to the solution $\boldsymbol{w}$ to problem (3.281a–e) in Ω?). We also study the regularity of the solution to problem (3.281a–e) defined on a polygonal domain.

In Sect. 3.5.2 we state the existence and uniqueness results for solutions of problem (3.281a–e). In Sect. 3.5.3 we investigate the question of stability of the solutions with respect to perturbations of the boundary of the domain Ω and prove that instability does occur; specifically, we take Ω to be a rectangular region and consider a family of domains Ω^N which converge to Ω, as $N \to \infty$, where the sequence of domains Ω^N are rectangular in shape except for N indentations on the bottom side, (see Fig. 3.4).

Thus, we will let $\boldsymbol{w}^N$ be the solution of problem (3.281a–e) in the domain Ω^N, and show that as $N \to \infty$, the functions $\boldsymbol{w}^N$ approach the function $\boldsymbol{w}$ which satisfies, under certain assumptions on the asymptotic behavior of the ratio of the heights and widths of the indentations, equations (3.281a–d) and

$$\frac{\partial e_{ij}(\boldsymbol{w})}{\partial x_k} \nu_j \nu_k \tau_i = 0, \text{ on } (\partial\Omega \setminus \Lambda) \times (0, T), \tag{3.282a}$$

$$\frac{\partial \boldsymbol{w}}{\partial \nu} = 0, \text{ on } \Lambda \times (0, T) \tag{3.282b}$$

instead of the boundary condition (3.281e), where $\Lambda = (c,d) \times \{0\}$ in Fig. 3.4. Thus, the boundary conditions are not preserved in the limit; this result illustrates that, in general, the solutions of problem (3.281a–e) are not stable with respect to perturbations of the boundary of Ω by Lipschitz curves. This type of stability question has been considered by several authors (see, e.g., Babuška [BaI], Keldysh [Ke], Maz'ya and Nazarov [MaN], Sapondzhyan [Sap], and the references which are cited therein) and has yielded surprising instability results. The best known of these results is the Sapondzhyan-Babuška paradox (or polygon to circle paradox) in the theory of thin plates; this paradox consists in the fact that, when a thin circular plate is approximated by regular polygons with freely supported edges, the limit solution does not satisfy the conditions of free support on the circle. In Sect. 3.5.4 we study the regularity of the solution to the steady state problem associated with problem (3.281a–e) in a polygonal domain Ω^N. We show that if $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega^N)$, then any weak solution $\boldsymbol{w}^N \in \boldsymbol{H}^2(\Omega^N) \cap \boldsymbol{H}^1_0(\Omega^N)$ satisfies $\boldsymbol{w}^N \in \boldsymbol{H}^4_{loc}$ away from the vertices of Λ^N, where Λ^N is the polygonal line joining the points $(c,0)$ and $(d,0)$ in Fig. 3.4. Moreover, near the vertices of Λ^N, we show that the solution $\boldsymbol{w}^N$ may be written as the sum $\boldsymbol{w}^N = \boldsymbol{w}^N_{reg} + \boldsymbol{w}^N_{sing}$, where $\boldsymbol{w}^N_{reg} \in \boldsymbol{H}^4_{loc}$ is the regular part whose behavior is not affected by the presence of corners and $\boldsymbol{w}^N_{sing}$ is the singular part which is not in $\boldsymbol{H}^4_{loc}$ and whose precise behavior depends on the interior angle at the vertex of each corner. We also provide an explicit characterization of the local singularities in terms of the interior angle at the vertex. To obtain this kind of characterization, we introduce a stream function ψ such that $\boldsymbol{w}^N = \operatorname{curl}\psi$, which reduces the problem to the study of the regularity of the solution ψ to the partial differential equation $\Delta^3\psi = f \in H^{-1}$ near each of the vertices. The regularity of ψ is obtained by using the general theory of Kondratiev [Ko]. The corresponding problem for the Navier–Stokes equations, as well as for second- and fourth-order equations, have been extensively studied (see, for example, Kondratiev [Ko], Grisvard [Gr1, 2], Kellogg and Osborn [KO], Osborn [OS], Moffatt [Mo], Blum and Rannacher [BR], and the literature therein).

Essential to the analysis presented in this section is the role of the higher-order boundary condition $\tau_{ijk}\nu_j\nu_k\tau_i = 0$ on $\partial\Omega$; this boundary condition was derived for a "smooth" boundary in Sect. 1.4.4. When used by mathematicians, the meaning of "smooth" boundary changes with the context and, when needed, a definition is usually provided. We will use smooth here to mean that the boundary is a C^2 submanifold of $\mathbb{R}^2$. On the other hand, the concept of smooth is used in engineering as an idealization for a boundary, where the imperfections of the surface (such as machining marks, etc.) are negligible with respect to other variables. In this context, the concept of "smooth" boundary thus represents an idealization of a surface whenever the size of the imperfections is deemed small enough. This idealization provides the advantage of working in a domain with a simple geometry and is highly desirable. For example, in the case of flow between two plates, it is very advantageous to think of the two plates as just two parallel planes rather than attempting to incorporate the precise geometry and then thinking of flow between two curved surfaces, each of which departs slightly and irregularly from being a plane. Alternately, one can think of Ω^N as being a perturbation of the region Ω and the question then arises as to whether or not the flow in Ω^N is a stable perturbation of the flow in Ω. Whenever the problem is studied in Ω there is a tacit and de facto assumption that the flow is stable under such perturbations of the domain. It has been known for a very long time that for high Reynolds numbers even the presence of very small protrusions on the surface of the wall substantially affects the flow. Nikuradze has already observed and reported such findings in the 1930's (see [ScG] for example). This is consistent with our instability results for the velocity $\boldsymbol{w}$ with respect to perturbations of the boundary and is an indication that, at least qualitatively, the fluid flow model (3.281a–e) is sensitive to wall roughness. Moreover, the result in (3.282b) indicates that, for large N, the gradient of the velocity $\boldsymbol{w}^N$ is small near the rough part of the boundary, which is consistent with experimental data for fluid flow in domains with rough walls, (see the measurements of Nikuradze in [ScG]). For other relevant experimental results see also [Wa, Ya, Ni] and the references therein.

3.5.2 Existence and Uniqueness of Solutions

We want to study the following initial-boundary value problem for an isothermal, incompressible, bipolar viscous fluid in a planar domain of the type depicted in Fig. 3.4:

$$\rho\frac{\partial w_i}{\partial t} = -\rho w_j\frac{\partial w_i}{\partial x_j} - \frac{\partial p}{\partial x_i} - 2\mu_1\frac{\partial \Delta e_{ij}(\boldsymbol{w})}{\partial x_j} + 2\frac{\partial(\mu(\boldsymbol{w})e_{ij}(\boldsymbol{w}))}{\partial x_j} + \rho f_i,\ \text{in } \Omega\times(0,T), \tag{3.283a}$$

$$\nabla\cdot\boldsymbol{w} = 0,\ \text{in } \Omega\times(0,T), \tag{3.283b}$$

$$\boldsymbol{w}(\boldsymbol{x},0) = \boldsymbol{h}(\boldsymbol{x}),\ \text{in } \Omega, \tag{3.283c}$$

$$\boldsymbol{\gamma}^l\boldsymbol{w} = \boldsymbol{0},\ \text{on } \Gamma_l\times(0,T), \tag{3.283d}$$

$$\boldsymbol{\gamma}^l\left[\frac{\partial e_{ij}(\boldsymbol{w})}{\partial x_k}\nu_j\nu_k\tau_i\right] = 0,\ \text{on } \Gamma_l\times(0,T) \tag{3.283e}$$

for $i, j = 1, 2$, where the domain $\Omega\subset R^2$ is rectangular in shape except on one side where the points c and d (see Fig. 3.4) are joined together by polygonal lines which produce rectangular and (acute and right) triangular indentations, and the boundary is composed of a union of a finite number of linear segments, denoted by $\bar{\Gamma}_l$.

In (3.283a–e), $\boldsymbol{\gamma}^l$ denotes the trace operator on each side of the polygonal domain, $\boldsymbol{f}\in L^2_{loc}(0,\infty;\boldsymbol{L}^2(\Omega))$, and $\boldsymbol{h}\in\boldsymbol{H}^2(\Omega)$, with

$$\int_0^t ||\boldsymbol{f}||^2_{\boldsymbol{L}^2(\Omega)}\,ds \le c_1 e^{c_1\lambda t} \text{ and } ||\boldsymbol{h}||^2_{\boldsymbol{H}^2(\Omega)} \le c_2 \tag{3.284}$$

where the positive constants c_2 and c_2 depend on the area of the domain. We set

$$\boldsymbol{H} = \{\boldsymbol{w} \in \boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}_0^1(\Omega) \mid \nabla \cdot \boldsymbol{w} = 0, \text{ in } \Omega\} \tag{3.285}$$

and define a weak solution to the initial-boundary value problem (3.283a–e) to be a function $\boldsymbol{w} \in \boldsymbol{H}$ which satisfies, a.e. on $(0, T)$,

$$\int_{\Omega_t} \rho \frac{\partial w_i}{\partial t} \psi_i \, d\boldsymbol{x} + 2\mu_1 \int_{\Omega_t} \frac{\partial e_{ij}(\boldsymbol{w})}{\partial x_k} \frac{\partial e_{ij}(\boldsymbol{\psi})}{\partial x_k} \, d\boldsymbol{x}$$
$$= -\int_{\Omega_t} \rho w_j \frac{\partial w_i}{\partial x_j} \psi_i \, d\boldsymbol{x} - 2 \int_{\Omega_t} \mu(\boldsymbol{w}) e_{ij}(\boldsymbol{w}) \frac{\partial \psi_i}{\partial x_j} \, d\boldsymbol{x} + \int_{\Omega_t} \rho f_i \psi_i \, d\boldsymbol{x} \tag{3.286}$$

for all $\boldsymbol{\psi} \in \boldsymbol{H}$. We then have the following existence and uniqueness result:

Theorem 3.10. *Suppose* $\boldsymbol{f} \in \boldsymbol{L}^2_{loc}(0, \infty; \boldsymbol{L}^2(\Omega))$, $\boldsymbol{h} \in \boldsymbol{H}$, *and* (3.284) *holds. Then problem* (3.283a–e) *has a unique weak solution* $\boldsymbol{w} \in \boldsymbol{H}$, *in the sense of* (3.286), *which satisfies*

$$\sup_{t \in (0,T)} ||\boldsymbol{w}||^2_{\boldsymbol{L}^2(\Omega_t)} + \int_0^T ||\boldsymbol{w}||^2_{\boldsymbol{H}^2(\Omega_t)} \, dt \le c \tag{3.287}$$

$$\sup_{t \in (0,T)} ||\boldsymbol{w}||^2_{\boldsymbol{H}^2(\Omega_t)} + \int_0^T ||\boldsymbol{w}_t||^2_{\boldsymbol{L}^2(\Omega_t)} \, dt \le c \tag{3.288}$$

where $c > 0$ *depends on* T *and on the domain through the Lipschitz norm of the boundary of* Ω *and the area of the domain* Ω.

Remarks. In the case where Ω is a smooth (C^2) domain, this result has been proved in [BBN4]. The proof is similar in the case where Ω is a polygonal domain and we will omit it.

3.5.3 Perturbation of Domain Results

In this subsection, we study whether or not the solution $\boldsymbol{w}^N$ of the bipolar fluid flow equations (3.283a–e) in a rectangular domain Ω^N with small rectangular or triangular perturbations on one side of the boundary is close to the solution $\boldsymbol{w}$ of the bipolar fluid flow equations in a domain Ω, where Ω is rectangular in shape and has no perturbations. It will be shown, under some assumptions on the asymptotic behavior of the ratio of the heights and the widths of the rectangular and triangular perturbations, that the solution $\boldsymbol{w}^N$ is *not* close to the solution $\boldsymbol{w}$.

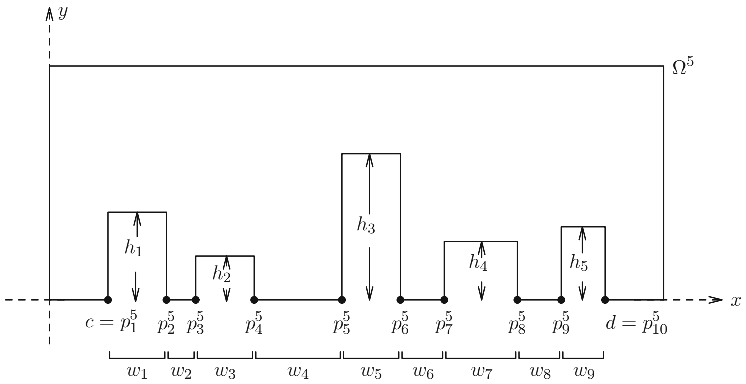

Fig. 3.5 Sequence of domains with rectangular indentations

Rectangular Perturbations

We begin by considering a sequence of domains Ω^N which are rectangular in shape except each have N rectangular indentations on the bottom side, (see Fig. 3.5).

A more precise definition of the domains Ω^N is as follows: Let $P^N = \{p_j^N\}_{j=1}^{2N}$ be a set of points on the x-axis such that $c = p_1^N < p_2^N < \cdots < p_{2N}^N = d$. Let the set of points $(x, 0)$ for $x \in (c, d)$ be denoted by Λ. Construct a rectangular indentation with height h_j^N between the points p_{2j-1}^N and p_{2j}^N for $j = 1, 2, \ldots, N$. Let $w_{2j-1}^N = \left|p_{2j}^N - p_{2j-1}^N\right|$ for $j = 1, 2, \ldots, N$ and $w_{2j}^N = \left|p_{2j+1}^N - p_{2j}^N\right|$ for $j = 1, 2, \ldots, N-1$. Thus, the jth rectangle has height h_j^N and width w_{2j-1}^N, where $j = 1, 2, \ldots, N$, and the jth and $(j+1)$st rectangles are a distance w_{2j}^N apart, where $j = 1, 2, \ldots, N-1$. Let $(w_j^N)^* = \max\{w_{2j-2}^N, w_{2j-1}^N\}$ for $j = 1, 2, \ldots, N$ where $w_0^N = w_2^N$. Let

$$s_N = \max_{0 \le j \le N-1} \left\{ \frac{((w_{j+1}^N)^*)^2}{h_{j+1}^N} \right\} \tag{3.289}$$

and

$$r_N = \max_{0 \le j \le N-1} \left\{ \frac{w_{2j+1}^N}{w_{2j}^N} \right\}. \tag{3.290}$$

Using the notation just described, we can now state the main result in this subsection.

Theorem 3.11. *Let Ω^N for $N = 1, 2, \ldots$, be a sequence of domains described as in Fig. 3.5. Suppose that, as $N \to \infty$, we have $h_j^N \to 0$, $s_N \to 0$, and $r_N \to c_1$, where c_1 is a positive constant bounded away from zero. Let $\boldsymbol{w}^N = (u^N, v^N)$ be a solution to problem (3.283a–e) on the domain $\Omega^N \times (0, T)$. Then, as $N \to \infty$, the domain Ω^N approaches a rectangular domain Ω, and the sequence of solutions $\{\boldsymbol{w}^N\}$ converges in $L^2((0,T); \boldsymbol{H}_0^1(\Omega))$ to a solution $\boldsymbol{w} = (u, v) \in L^2((0,T); \boldsymbol{H}^2_{loc}(\bar{\Omega}\backslash\Lambda))$ which satisfies equations (3.283a–d), equation (3.283e) on $\Gamma_l \setminus \Lambda \times (0,T)$ and $\dfrac{\partial \boldsymbol{w}}{\partial \boldsymbol{\nu}} = 0$ on $\Lambda \times (0, T)$.*

Remarks. If we consider the specific case where the domains Ω^N have N uniform rectangular indentations each with height h^N and width w^N, where the rectangular indentations are a distance d^N apart, then the conclusion of Theorem 3.10 is valid provided that as $N \to \infty$, we have $h^N \to 0$, $\dfrac{\max\{w^N, d^N\}}{h^N} \to 0$, and $\dfrac{w^N}{d^N} \to c_1$, where c_1 is a positive constant bounded away from zero.

Proof (Theorem 3.11). By Theorem 3.10, for each domain Ω^N, there exists a solution $\boldsymbol{w}^N = (u^N, v^N)$ which satisfies the inequality

$$\int_0^T ||\boldsymbol{w}_t^N||^2_{\boldsymbol{L}^2(\Omega_t^N)}\, dt + \int_0^T ||\boldsymbol{w}^N||^2_{\boldsymbol{H}^2(\Omega_t^N)}\, dt \le c \tag{3.291}$$

where the dependence of c on the domain with a Lipschitz boundary is through the Lipschitz norm and the area of Ω^N. Because of the nature of the domains Ω^N, the Lipschitz norm can be bounded by two for every Ω^N and we can find a uniform bound on the area for every Ω^N. Therefore, the sequence $\{\boldsymbol{w}^N\}$ is bounded in $L^2((0,T); \boldsymbol{H}^2(\Omega^N))$ independently of N. Since $\boldsymbol{w}^N \in L^2((0,T); \boldsymbol{H}^2(\Omega^N))$ and $\dfrac{d\boldsymbol{w}^N}{dt} \in L^2((0,T); \boldsymbol{L}^2(\Omega^N))$, we can deduce that the sequence $\{\boldsymbol{w}^N\}$ converges strongly in $L^2((0,T); \boldsymbol{H}^1_{loc}(\Omega \setminus \Lambda))$. Since $\{\boldsymbol{w}^N\}$ is in $L^2((0,T); \boldsymbol{H}_0^1(\Omega^N))$, we can extend $\{\boldsymbol{w}^N\}$ by $\boldsymbol{0}$ to all of Ω to obtain a new sequence, denoted again by $\{\boldsymbol{w}^N\}$, which is strongly convergent in $L^2((0,T); \boldsymbol{H}_0^1(\Omega))$ to a function $\boldsymbol{w}$. These results are sufficient to pass to the limit in (3.286) and recover that the limit function $\boldsymbol{w}$ satisfies (3.283a–d) and (3.283e) in $L^2((0,T); \boldsymbol{H}^{-1/2}(\Gamma_l \setminus \Lambda))$ by using the usual duality argument. To finish the proof of Theorem 3.11 we will need the following lemma, which we will prove later:

Lemma 3.6. *Under the assumptions of Theorem 3.11, there exists a positive constant c such that for all sufficiently small $\delta > 0$,*

$$\int_0^T \int_c^d u_y^2(x, \delta)\, dx\, dt \le c\delta. \tag{3.292}$$

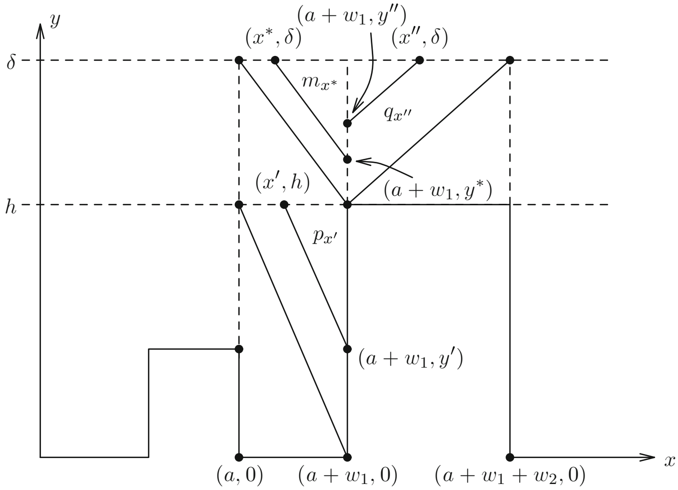

Fig. 3.6 Domain in the proof of Lemma 3.7

To finish the proof of Theorem 3.11, if we let δ go to 0 in (3.292), then

$$\int_0^T \int_c^d u_y^2(x,0)\,dx\,dt = 0. \tag{3.293}$$

Therefore, $u_y(x,0) = 0$ on $\Lambda \times (0,T)$, and the theorem is proved. □

We now must prove Lemma 3.6. First, however, we will need to obtain estimates on some norms of u_y^N. These estimates will be established with the help of the next five lemmas. Fix a domain Ω^N, let the solution $\boldsymbol{w}^N$ on Ω^N be denoted simply by $\boldsymbol{w}$, and consider two of the rectangular indentations with heights and widths as shown in Fig. 3.6, where δ is a positive number chosen such that $\delta > \max\limits_{1\le j\le N}\{h_j^N\}$.

Lemma 3.7. *With the same notation as in Fig. 3.6, we have*

$$\int_a^{a+w_1} u_y^2(x,h)\,dx \le 2\left(\frac{w_1^2}{h} + h\right)\int_D\int (u_{xy}^2 + u_{yy}^2)\,dx\,dy \tag{3.294}$$

where D is the triangular region with vertices (a,h), $(a+w_1,0)$, and $(a+w_1,h)$.

Proof. Recall that a solution $\boldsymbol{w} = (u,v)$ of the boundary-value problem (3.283a–e) is zero on the boundary of the domain. So, $u_y = 0$ on the vertical boundary lines. From Fig. 3.6, a parameterization of line $p_{x'}$, where $(a+w_1,y')$ is at $s = 0$ and (x',h) is at $s = 1$ is given by:

$$\begin{cases} x(s) = (x' - a - w_1)s + a + w_1, \\ y(s) = \dfrac{h}{w_1}(a + w_1 - x')(s - 1_+ h, \quad \text{for } 0 \le s \le 1. \end{cases} \tag{3.295}$$

Let $F(s) = u_y(x(s), y(s))$, for $0 \leq s \leq 1$. Then, using Taylor's Formula $F(1) = F(0) + \int_0^1 F'(s)\, ds$, we have

$$u_y(x', h) = \int_0^1 F'(s)\, ds \tag{3.296}$$

since $F(0) = u_y(a + w_1, y') = 0$. By squaring both sides of (3.296), applying the Cauchy-Schwarz inequality, and integrating with respect to x' from a to $a + w_1$, we obtain

$$\int_a^{a+w_1} u_y^2(x', h)\, dx' \leq 2\left(1 + \frac{h^2}{w_1^2}\right) \int_a^{a+w_1} \int_0^1 (x' - a - w_1)^2 (u_{xy}^2 + u_{yy}^2)\, ds\, dx'. \tag{3.297}$$

Note that

$$\int_a^{a+w_1} \int_0^1 (x' - a - w_1)^2 (u_{xy}^2 + u_{yy}^2)\, ds\, dx' \leq \frac{w_1^2}{h} \int_D \int (u_{xy}^2 + u_{yy}^2)\, dx\, dy. \tag{3.298}$$

The lemma follows from the inequalities (3.294) and (3.298). □

Lemma 3.8. *With the same notation as in Fig. 3.6, we have the inequality*

$$\int_a^{a+w_1} u_y^2(x, \delta)\, dx \leq 4 \max\left(\frac{w_1^2}{h} + h, \delta - h\right) \int_K \int (u_{xy}^2 + u_{yy}^2)\, dx\, dy \tag{3.299}$$

where K is the quadrilateral with vertices $(a+w_1, 0)$, $(a+w_1, \delta)$, (a, δ), and (a, h).

Proof. Since for x fixed

$$u_y(x, \delta) = u_y(x, h) + \int_h^\delta u_{yy}(x, s)\, ds,$$

it easily follows by the Cauchy-Schwarz inequality that

$$\int_a^{a+w_1} u_y^2(x, \delta)\, dx \leq 2 \int_a^{a+w_1} u_y^2(x, h)\, dx + 2(\delta - h) \int_a^{a+w_1} \int_h^\delta u_{yy}^2\, dy\, dx. \tag{3.300}$$

Applying Lemma 3.7 to the first term on the right-hand side of (3.300), we obtain the stated result. □

Lemma 3.9. *With the same notation as in Fig. 3.6, we have the inequality*

$$\frac{w_1}{\delta - h}\int_h^\delta u_y^2(a+w_1,y)\,dy \leq 16\max\left(\frac{w_1^2}{h}+h,\delta-h,\frac{w_1^2}{\delta-h}+\delta-h\right)\int_K\int(u_{xy}^2+u_{yy}^2)\,dx\,dy \tag{3.301}$$

where K is the region defined in Lemma 3.8.

Proof. Using Taylor's Formula again, we find that

$$u_y(a+w_1,y^*) = u_y(x^*,\delta) + \int_0^1 F'(s)\,ds \tag{3.302}$$

where $F(s) = u_y((a+w_1-x^*)s+x^*, y^*s+\delta)$ and $y^* = \frac{h-\delta}{w_1}(a+w_1-x^*)+\delta$. Squaring both sides of the above equality, using the Cauchy-Schwarz inequality, and integrating with respect to x^* from a to $a+w_1$, we get

$$\frac{w_1}{\delta - h}\int_h^\delta u_y^2(a+w_1,y^*)\,dy^* \leq 2\int_a^{a+w_1} u_y^2(x^*,\delta)\,dx^* + 4\left(\frac{w_1^2}{\delta-h}+\delta-h\right)\int_T\int(u_{xy}^2+u_{yy}^2)\,dx\,dy \tag{3.303}$$

where T is the triangle with vertices (a,δ), $(a+w_1,\delta)$, and $(a+w_1,h)$. Using Lemma 3.8 to estimate the first term on the right-hand side of (3.303) and noting that $T \subseteq K$, we obtain the result. □

Lemma 3.10. *Using the same notation as in Fig. 3.6, we have*

$$\int_{a+w_1}^{a+w_1+w_2} u_y^2(x'',\delta)\,dx'' \leq 32\max\left(\frac{w_1w_2}{h}+\frac{w_2h}{w_1},\frac{w_2}{w_1}(\delta-h),\frac{w_1w_2}{\delta-h}+\frac{w_2}{w_1}(\delta-h),\frac{w_2^2}{\delta-h}+\delta-h\right)\int_P\int(u_{xy}^2+u_{yy}^2)\,dx\,dy \tag{3.304}$$

where P is the region with vertices $(a+w_1,0)$, $(a+w_1,h)$, $(a+w_1+w_2,\delta)$, (a,δ), and (a,h).

Proof. If we parameterize line $q_{x''}$ in Fig. 3.6, apply similar techniques as in Lemma 3.9, and use the inequality (3.301), we obtain the result. □

Lemma 3.11. *Using the same notation as in Fig. 3.6, we have the estimate*

$$\begin{aligned}\int_a^{a+w_1+w_2} & u_y^2(x,\delta)\,dx \\ & \leq 64\max\left(\frac{w_1^2}{h}+h,\delta-h,\frac{w_1w_2}{h}+\frac{w_2h}{w_1},\frac{w_2^2}{\delta-h}+\delta-h,\right. \\ & \left.\frac{w_1w_2}{\delta-h}+\frac{w_2}{w_1}(\delta-h),\frac{w_2}{w_1}(\delta-h)\right)\int_P\int(u_{xy}^2+u_{yy}^2)\,dx\,dy \end{aligned} \tag{3.305}$$

where the region P is defined in Lemma 3.10.

Proof. The proof follows from Lemmas 3.8 and 3.10. □

Proof (Lemma 3.6). For a fixed N, we can apply Lemma 3.11 to each of the N rectangular regions (as in Fig. 3.5) to obtain

$$\begin{aligned}\int_c^d & (u_y^N)^2(x,\delta)\,dx \\ & \leq 128\max\left(s_N+a_N,\delta-b_N,s_N+r_Na_N,\frac{c_N^2}{\delta-a_N}+\delta-b_N,\right. \\ & \left.\frac{c_N^2}{\delta-a_N}+r_N(\delta-b_N),r_N(\delta-b_N)\right)\int_{\sum_{j=1}^N P_j}\int(D^2u^N)^2\,dx\,dy \end{aligned} \tag{3.306}$$

where

$$a_N=\max_{1\leq j\leq N}\{h_j^N\},\quad b_N=\min_{1\leq j\leq N}\{h_j^N\},\quad \text{and } c_N=\max_{1\leq j\leq N}\{(w_j^N)^*\}.$$

But,

$$\begin{aligned}\int_0^T\int_{\sum_{j=1}^N P_j^t}\int(D^2u^N)^2\,dx\,dy\,dt & \leq \int_0^T\int_{\Omega_t^N}\int(D^2u^N)^2\,dx\,dy\,dt \\ & \leq \int_0^T ||u^N||^2_{\boldsymbol{H}^2(\Omega_t^N)}\,dt\leq c\end{aligned} \tag{3.307}$$

where c is independent of N. Since $u^N\to u$ in $L^2((0,T);\boldsymbol{H}^2_{loc}(\bar\Omega\setminus\Lambda)$, we have $u^N\to u$ in $L^2((0,T);\boldsymbol{H}^2_{loc}(\bar\Omega\setminus\Lambda)$ for ϵ a fixed, small, positive number. Thus,

$$u_y^N(x,\delta)\to u^y(x,\delta),\ \text{ in } L^2((0,T);\boldsymbol{L}^2_{loc}(\bar\Omega\setminus\Lambda) \tag{3.308}$$

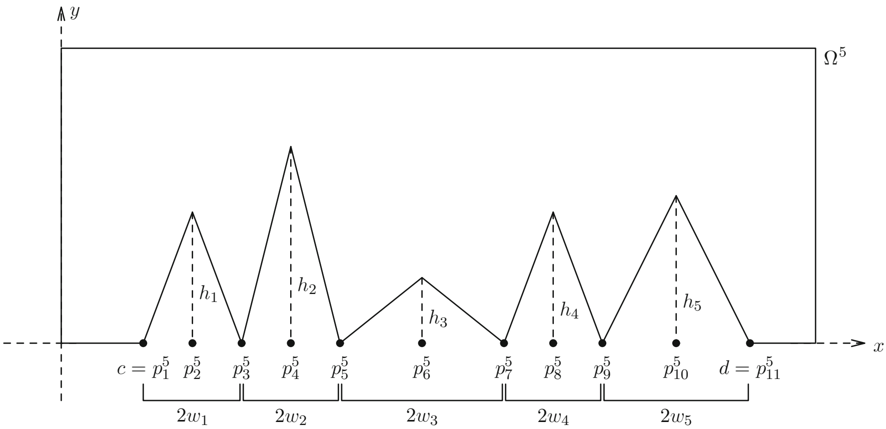

Fig. 3.7 Sequence of domains with triangular indentations

as $N \to \infty$. From (3.307), (3.308), and the assumptions of Theorem 3.11, inequality (3.306) becomes

$$\int_0^T \int_c^d u_y^2(x,\delta)\,dx\,dt \leq 128c\max(\delta, c_1\delta) \tag{3.309}$$

as $N \to \infty$. Therefore, as $N \to \infty$, (3.309) is true for all sufficiently small $\delta > 0$. □

Triangular Indentations

Our interest in triangular protrusions stems from the fact that in experimental work, the type of protrusions considered are often of this type, (see [LRRH, Wa], and the references therein). In this part of Sect. 3.5.3, we consider a sequence of domains Ω^N which are rectangular in shape except each have N triangular indentations on the bottom side, where the indentations are in the shape of isosceles triangles (see Fig. 3.7).

A more precise definition of the domains Ω^N is as follows: Let $P^N = \{p_j^N\}_{j=1}^{2N+1}$ be a set of points on the x-axis such that $c = p_1^N < p_2^N < \cdots < p_{2N+1}^N = d$. Let the set of points $(x, 0)$ for $x \in (c, d)$ be denoted by Λ. Construct an isosceles triangle with height h_j^N between the points p_{2j-1}^N and P_{2j+1}^N for $j = 1, 2, \ldots, N$. Let $w_j^N = \dfrac{1}{2}|p_{2j+1}^N - p_{2j-1}^N|$ for $j = 1, 2, \ldots, N$. Thus, the jth triangle has height h_j^N and width $2w_j^N$, where $j = 1, 2, \ldots, N$. Let

$$\hat{t}_N = \max_{1 \le j \le N} \left\{ \frac{w_j^N}{h_j^N} \right\}. \tag{3.310}$$

Using the notation just described, we can now state the following theorem.

Theorem 3.12. *Let Ω^N for $N = 1, 2, \ldots$, be a sequence of domains described as in Fig. 3.7. Suppose that, as $N \to \infty$, we have $h_j^N \to 0$ and $\hat{t}_N \to c_2$, where c_2 is a positive constant bounded way from zero. Let $\boldsymbol{w}^N = (u^N, v^N)$ be a solution to problem (3.283a–e) on the domain $\Omega^N \times (0, T)$. Then, as $N \to \infty$, the domain Ω^N approaches a rectangular domain Ω, and the sequence of solutions $\{\boldsymbol{w}^N\}$ converges in $L^2((0,T); \boldsymbol{H}_0^1(\Omega))$ to a solution $\boldsymbol{w} = (u, v) \in L^2((0,T); \boldsymbol{H}_{loc}^2(\bar{\Omega} \setminus \Lambda))$ which satisfies (3.283a–d), equation (3.283e) on $\Gamma_1 \setminus \Lambda \times (0, T)$, and $\dfrac{\partial \boldsymbol{w}}{\partial \boldsymbol{\nu}} = 0$ on $\Lambda \times (0, T)$.*

Remarks. The condition that $\hat{t}_N \to c_2$ as $N \to \infty$, where c_2 is a positive constant bounded away from zero, means that as $N \to \infty$, the triangles shrink in size but do not flatten.

The proof of Theorem 3.12 is based on the same ideas as the proof of Theorem 3.11 and the following

Lemma 3.12. *Under the assumptions of Theorem 3.12, there exists a positivecon-stant c' such that for all sufficiently small $\delta > 0$,*

$$\int_0^T \int_c^d u_y^2(x, \delta)\, dx\, dt \le c'\delta. \tag{3.311}$$

Given that (3.12) is valid, the proof of Theorem 3.12 follows in the same manner that Theorem 3.11 follows from (3.292). Thus, we can omit the proof of Theorem 3.12. However, some of the technical details are different enough to make it necessary to provide at least an outline of the proof of Lemma 3.12. In order to prove Lemma 3.12, we will need to obtain estimates on some norms of u_y^N. We cannot directly obtain estimates on u_y^N, as in the proof of Lemma 3.7, since $u_y^N \neq 0$ on the nonvertical boundary lines of the triangular indentations. But, as $u^N = 0$ on the boundary, the directional derivatives of u^N in the direction of the sides of the triangular indentations will be zero. As u_y^N can be written as a linear combination of certain directional derivatives of u^N, we can estimate certain norms of u_y^N by calculating upper bounds for these directional derivatives of u^N. These estimates will be established with the help of the next six lemmas.

For the following lemmas, fix a domain Ω^N, let the solution $\boldsymbol{w}^N$ on Ω^N be denoted simply by $\boldsymbol{w}$, and consider an isosceles triangular indentation with height h and width $2w$ as shown in Fig. 3.8. Also, let δ be a positive number chosen such that $\delta > \max_{1 \le j \le N} \{h_j^N\}$.

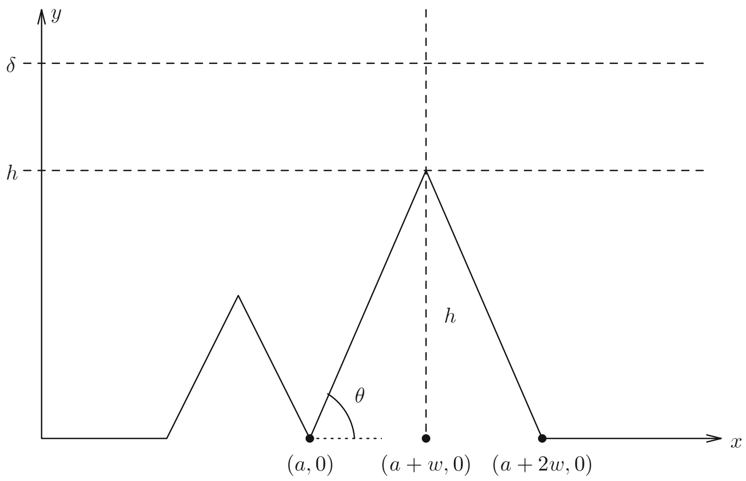

Fig. 3.8 Domain in the proof of Lemma 3.13

Lemma 3.13. *With the same notation as in Fig. 3.8, we have*

$$\int_a^{a+w} [\nabla_{\alpha_1} u(x,\delta)]^2 \, dx \le \delta \int_{\bar{D}} \int (u_{xy}^2 + u_{yy}^2)\, dx\, dy \tag{3.312}$$

where ∇_{α_1} *is the derivative taken in the direction of the vector* $(\cos\alpha_1, \sin\alpha_1)$ *and* $\bar{D}$ *is the region with vertices* $(a,0)$*,* (a,δ)*,* $(a+w,\delta)$*, and* $(a+w,h)$*.*

Proof. For x fixed in $(a, a+w)$, by Taylor's Formula, we have

$$\nabla_{\alpha_1} u(x,\delta) = \nabla_{\alpha_1} u\left(x, \frac{h}{w}(x-a)\right) + \int_{\frac{h}{w}(x-a)}^{\delta} \frac{\partial}{\partial y} (\nabla_{\alpha_1} u(x,s))\, ds. \tag{3.313}$$

Since $u = 0$ on the boundary of the domain, we have $\nabla_{\alpha_1} u\left(x, \frac{h}{w}(x-a)\right) = 0$ on the line passing through the points $(a,0)$ and $(a+w,h)$. Squaring both sides of (3.313), applying the Cauchy-Schwarz inequality, and integrating with respect to x from a to $a+w$, we obtain

$$\int_a^{a+w} [\nabla_{\alpha_1} u(x,\delta)]^2 \, dx \le \left(\delta - \frac{h}{w}(x-a)\right) \int_a^{a+w} \int_{\frac{h}{w}(x-a)}^{\delta} \left[\frac{\partial}{\partial y} (\nabla_{\alpha_1} u(x,y))\right]^2 dy\, dx. \tag{3.314}$$

But,

$$\left[\frac{\partial}{\partial y} (\nabla_{\alpha_1} u(x,y))\right]^2 = \left[\frac{\partial}{\partial y} \left(\frac{w}{\sqrt{w^2+h^2}} u_x + \frac{h}{\sqrt{w^2+h^2}} u_y\right)\right]^2$$

$$\leq \frac{w^2}{w^2+h^2}u_{xy}^2 + \frac{h^2}{w^2+h^2}u_{yy}^2 + \frac{h^2}{w^2+h^2}u_{xy}^2 + \frac{w^2}{w^2+h^2}u_{yy}^2$$
$$= u_{xy}^2 + u_{yy}^2. \tag{3.315}$$

Substituting (3.315) into the right-hand side of (3.314), and noting that $\delta - \dfrac{h}{w}(x-a) \leq \delta$ for $x \in (a, a+w)$, we obtain the result. □

Lemma 3.14. *With the same notation as Fig. 3.8, we have*

$$\frac{w}{\delta-h}\int_h^\delta \left[\nabla_{\alpha_1}u(a+w,y)\right]^2\,dy \leq 4\max\left(\delta, \frac{w^2}{\delta-h}+\delta-h\right)\int_{\bar{D}}\int (u_{xx}^2+2u_{xy}^2+u_{yy}^2)\,dx\,dy \tag{3.316}$$

where ∇_{α_1} *and the region* $\bar{D}$ *are defined in Lemma 3.13.*

Proof. Setting

$$F(s) = \nabla_{\alpha_1}u\left((a+w)s + x''(1-s), \left[\frac{-(\delta-h)}{w}(a+w-x'')+\delta\right]s + \delta(1-s)\right) \tag{3.317}$$

for $0 \leq s \leq 1$, where $x'' \in [a, a+w]$ and $y'' \in [h, \delta]$, and using the formula $F(1) = F(0) + \displaystyle\int_0^1 F'(s)\,ds$, we find that

$$\nabla_{\alpha_1}u(a+w, y'') = \nabla_{\alpha_1}u(x'', \delta) + \int_0^1 F'(s)\,ds \tag{3.318}$$

where $y'' = \dfrac{-(\delta-h)}{w}(a+w-x'')+\delta$. Squaring both sides of (3.318), using the Cauchy-Schwarz inequality, and integrating with respect to x'' from a to $a+w$, we obtain

$$\int_a^{a+w}\left[\nabla_{\alpha_1}u(a+w,y'')\right]^2\,dx'' \leq 2\int_a^{a+w}\left[\nabla_{\alpha_1}u(x'',\delta)\right]^2\,dx'' + 2\int_a^{a+w}\int_0^1 (F'(s))^2\,ds\,dx''. \tag{3.319}$$

If we let $a_1 = a+w-x''$ and $a_2 = \dfrac{-(\delta-h)}{w}(a+w-x'')$, then

$$(F'(s))^2 = \left[\frac{w}{\sqrt{w^2+h^2}}a_1u_{xx} + \frac{h}{\sqrt{w^2+h^2}}a_2u_{yy} + \left(\frac{w}{\sqrt{w^2+h^2}}a_2 + \frac{h}{\sqrt{w^2+h^2}}a_1\right)u_{xy}\right]^2$$
$$\leq \left(\frac{w^2a_1^2+h^2a_2^2+w^2a_2^2+h^2a_1^2}{w^2+h^2}\right)(u_{xx}^2+2u_{xy}^2+u_{yy}^2)$$
$$= (a_1^2+a_2^2)(u_{xx}^2+2u_{xy}^2+u_{yy}^2). \tag{3.320}$$

Also, it is easy to see that

$$\int_a^{a+w} \left[\nabla_{\alpha_1} u(a_+ w, y'')\right]^2 dx'' = \frac{w}{\delta - h} \int_h^{\delta} \left[\nabla_{\alpha_1} u(a + w, y'')\right]^2 dy'' \tag{3.321}$$

and

$$\int_a^{a+w} \int_0^1 (a + w - x'')^2 (u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2)\, ds\, dx'' \le \frac{w^2}{\delta - h} \int_{\bar{T}} \int (u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2)\, dx\, dy \tag{3.322}$$

where $\bar{T}$ is the triangle with vertices (a, δ), $(a + w, \delta)$, and $(a + w, h)$. The lemma follows from substituting (3.320)–(3.322) into (3.319), using Lemma 3.13, and noting that $\bar{T} \subset \bar{D}$. □

Lemma 3.15. *With the same notation as in Fig. 3.8, we have the inequality*

$$\int_{a+w}^{a+2w} [\nabla_{\alpha_1} u(x, \delta)]^2\, dx \le 4 \max\left(\delta, \frac{w^2}{\delta - h} + \delta - h\right) \int_{\tilde{P}} \int (u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2)\, dx\, dy \tag{3.323}$$

where ∇_{α_1} *is defined in Lemma 3.13 and* $\tilde{P}$ *is the region with vertices* $(a, 0)$*,* (a, δ)*,* $(a + 2w, \delta)$*, and* $(a + w, h)$*.*

Proof. Lemma 3.15 can be deduced from Lemma 3.14 in the same way that Lemma 3.10 was deduced from Lemma 3.9. □

Lemma 3.16. *Using the same notation as in Fig. 3.8, we have the inequality*

$$\int_a^{a+2w} [\nabla_{\alpha_1} u(x, \delta)]^2\, dx \le 8 \max\left(\delta, \frac{w^2}{\delta - h} + \delta - h\right) \int_{TP} \int (u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2)\, dx\, dy \tag{3.324}$$

where ∇_{α_1} *is defined in Lemma 3.13 and the region* $\tilde{P}$ *is defined in Lemma 3.15.*

Proof. The proof follows from Lemmas 3.13 and 3.15. □

Lemma 3.16 gives an upper bound for a certain norm of the directional derivative $\nabla_{\alpha_1} u$. An upper bound for a particular norm of $\nabla_{\alpha_2} u$, where ∇_{α_2} is the directional derivative in the direction of the line passing through the points $(a + w, h)$ and $(a+2w, 0)$ in Fig. 3.8, can be derived by using techniques similar to those employed in Lemma 3.13 to Lemma 3.16. In fact, it can be shown that we have the following result:

Lemma 3.17. *Using the same notation as in Fig. 3.8, we have the inequality*

$$\int_a^{a+2w} [\nabla_{\alpha_2} u(x,\delta)]^2 \, dx \leq 8 \max\left(\delta, \frac{w^2}{\delta - h} + \delta - h\right) \int_{\tilde{A}}\int (u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2)\, dx\, dy \tag{3.325}$$

where ∇_{α_2} *is the derivative taken in the direction of the line passing through the points* $(a + w, h)$ *and* $(a_{+}2w, 0)$*, and the region* $\tilde{A}$ *has vertices* $(a + 2w, 0)$*,* $(a + 2w, \delta)$*,* (a, δ)*, and* $(a + w, h)$*.*

Since we now have upper bounds for certain norms of the directional derivatives $\nabla_{\alpha_1} u$ and $\nabla_{\alpha_2} u$, we can find upper bounds for some norms of u_y as follows:

Lemma 3.18. *Using the same notation as in Fig. 3.8, we have the inequality*

$$\int_a^{a+2w} u_y^2(x,\delta)\, dx \leq 8 \max\left[\left(\frac{w^2}{h^2} + 1\right)\delta, \left(\frac{w^2}{h^2} + 1\right)\left(\frac{w^2}{\delta - h} + \delta - h\right)\right] \int_{\hat{P}}\int (u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2)\, dx\, dy \tag{3.326}$$

where $\hat{P}$ *is the region with vertices* $(a, 0)$*,* (a, δ)*,* $(a + 2w, \delta)$*,* $(a + 2w, 0)$*, and* $(a + w, h)$*.*

Proof. Recall that

$$\nabla_{\alpha_1} u = \frac{w}{\sqrt{w^2 + h^2}} u_x + \frac{h}{\sqrt{w^2 + h^2}} u_y \text{ and } \nabla_{\alpha_2} u = \frac{-w}{\sqrt{w^2 + h^2}} u_x + \frac{h}{\sqrt{w^2 + h^2}} u_y. \tag{3.327}$$

Adding the above two equations together, solving for u_y, and then squaring both sides of the resulting equation and applying the Cauchy-Schwarz inequality gives

$$u_y^2 \leq \frac{w^2 + h^2}{2h^2} (\nabla_{\alpha_1} u)^2 + \frac{w^2 + h^2}{2h^2} (\nabla_{\alpha_2} u)^2. \tag{3.328}$$

Using (3.328), we obtain the estimate

$$\int_a^{a+2w} u_y^2(x,\delta)\, dx \leq \frac{1}{2}\left(\frac{w^2}{h^2} + 1\right) \int_a^{a+2w} [\nabla_{\alpha_1} u(x,\delta)]^2 \, dx + \frac{1}{2}\left(\frac{w^2}{h^2} + 1\right) \int_a^{a+2w} [\nabla_{\alpha_2} u(x,\delta)]^2 \, dx \tag{3.329}$$

and the result follows from Lemmas 3.16 and 3.17. □

Using the results in Lemmas 3.13–3.18 we may finally construct the following

Proof (Lemma 3.12). For a fixed N, we can apply Lemma 3.18 to each of the N triangular regions (in Fig. 3.7) to obtain

$$\int_c^d (u_y^N)^2(x,\delta)\,dx \le 16\max\left[(\tilde{t}_N^2+1)\delta, (\tilde{t}_N^2+1)\left(\frac{\hat{c}_N^2}{\delta-\hat{b}_N}+\delta-\hat{a}_N\right)\right] \int_{\sum_{j=1}^N \hat{P}_j}\int (D^2u^N)^2\,dx\,dy \tag{3.330}$$

where

$$\hat{a}_N = \min_{1\le j\le N}\{h_j^N\},\ \hat{b}_N = \max_{1\le j\le N}\{h_j^N\},\ \text{and}\ \hat{c}_N = \max_{1\le j\le N}\{w_j^N\}. \tag{3.331}$$

Then, by a procedure similar to that used in the proof of Lemma 3.6, and the assumptions of Theorem 3.12, inequality (3.330) becomes

$$\int_0^T\int_c^d u_y^2(x,\delta)\,dx\,dt \le 16c'(c_2^2+1)\delta \tag{3.332}$$

as $N\to\infty$. Thus, as $N\to\infty$, (3.332) is true for all sufficiently small $\delta>0$. □

Remarks. We note that a solution of the incompressible bipolar equations (3.283a–e) in a rectangular domain having rectangular or triangular indentations on one of its sides (as in Theorems 3.11 or 3.12) can not be approximated by a solution $\boldsymbol{w}^* = (u^*, v^*)$ of the bipolar equations (3.283a–e) in a rectangular domain with no indentations, since $u_{yy}^* = 0$ on Λ, while $u_y = 0$ on Λ.

3.5.4 Regularity Results in Polygonal Domains

In this subsection, we shall study the regularity in space of the weak solution $\boldsymbol{w}(x,t)$ for $t>0$ of (3.283a–e) in a plane domain with a polygonal boundary, where the boundary is composed of a union of a finite number of linear segments, denoted by $\bar{\Gamma}_l$ for $1\le l\le J$. For this purpose, we fix $t=t_0$ where t_0 is such that $\boldsymbol{w}(x,t_0)\in \boldsymbol{H}^2(\Omega)\cap\boldsymbol{H}_0^1(\Omega)$ and $\boldsymbol{w}_t(x,t_0)\in\boldsymbol{L}^2(\Omega)$. By (3.287) and (3.288), this holds for almost every $t>0$, so that the existence of such a t_0 is not in question. Thus, we consider (3.283a–e) at time t_0. Since all the lower-order terms appearing in (3.283a) are in $\boldsymbol{L}^2(\Omega)$, we can then, without loss of generality, incorporate them with the forcing term $\boldsymbol{f}$ and confine our study of the regularity to that of the study of the linearized steady-state problem

$$\frac{\partial p}{\partial x_i} + 2\mu_1 \frac{\partial}{\partial x_j} \Delta e_{ij}(\boldsymbol{w}) = \rho f_i, \text{ in } \Omega, \quad i, j = 1, 2, \tag{3.333a}$$

$$\operatorname{div} \boldsymbol{w} = 0, \text{ in } \Omega, \tag{3.333b}$$

$$\boldsymbol{\gamma}^l \boldsymbol{w} = \boldsymbol{0}, \text{ on } \Gamma_l, \tag{3.333c}$$

$$\boldsymbol{\gamma}^l \left[\frac{\partial e_{ij}(\boldsymbol{w})}{\partial x_k} \nu_j \nu_k \tau_i \right] = 0, \text{ on } \Gamma_l \tag{3.333d}$$

where $\boldsymbol{f} = (f_1, f_2) \in \boldsymbol{L}^2(\Omega)$. The regularity of p and $\boldsymbol{w}$ away from the vertices of $\partial\Omega$ can easily be deduced from the usual regularity theory of elliptic equations and the de Rham Theorem. It is well-known that solutions of elliptic equations fail to be regular near corners of the boundary. We will provide a description of the asymptotic behavior of the solution near corners. To study the regularity, we use the vorticity-stream formulation instead of the velocity-pressure formulation. To this end, we first introduce a stream function ψ such that $\boldsymbol{w} = (u, v) = (-\psi_y, \psi_x)$, establish the regularity of ψ, and deduce from that the regularity results for $\boldsymbol{w}$. By the use of the stream function ψ, we will reduce problem (3.333a–d) to a problem of the type $\Delta^3 \psi = f$. More specifically, we have

Lemma 3.19. *Let* $\boldsymbol{w} = (u, v) \in \boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}_0^1(\Omega)$ *be the solution of problem* (3.333a–d). *Then there exists* $\psi \in H^3(\Omega) \cap H_0^2(\Omega)$ *such that*

$$(u, v) = (-\psi_y, \psi_x) \tag{3.334}$$

with ψ *a solution to the problem*

$$\Delta^3 \psi = \hat{f}, \textit{ in } \Omega, \tag{3.335a}$$

$$\gamma^l \psi = \gamma^l \left(\frac{\partial \psi}{\partial \boldsymbol{\nu}} \right) = 0, \textit{ on } \Gamma_l, \tag{3.335b}$$

$$\gamma^l \left(\frac{\partial \Delta \psi}{\partial \boldsymbol{\nu}} \right) = 0, \textit{ on } \Gamma_l \tag{3.335c}$$

where $\hat{f} = \dfrac{\rho}{\mu_1} \left(\dfrac{\partial f_2}{\partial x} - \dfrac{\partial f}{\partial y} \right) \in H^{-1}(\Omega)$.

The last boundary condition is satisfied in the sense of $H^{-1/2}(\Gamma_l)$ since $(\Delta^3 \psi, \psi) \in \boldsymbol{H}^{-1}(\Omega) \times \boldsymbol{H}^3(\Omega)$, so by a standard duality argument, $\dfrac{\partial \Delta \psi}{\partial \boldsymbol{\nu}} = H^{-1/2}(\Gamma_l)$. In the following theorem, we assume that the Γ_l for $l = 1, 2, \ldots, J$, are numbered in the counterclockwise direction around the domain. We denote the terminal point of Γ_l by S_l and denote a neighborhood of the corner S_l by V_l. Our first regularity result is given by

Theorem 3.13. *Let* $\boldsymbol{w} \in \boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}_0^1(\Omega)$ *and* $p \in H^{-1}(\Omega)$ *be solutions to* (3.333a–d). *Then for* $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, *we have* $\boldsymbol{w} \in \boldsymbol{H}^4_{loc}(\bar{\Omega} \setminus (\cup_{l=1}^J V_l))$ *and* $p \in H^1_{loc}(\bar{\Omega} \setminus (\cup_{l=1}^J V_l))$.

Proof. Using techniques similar to those employed in the general regularity results for elliptic equations in [ADN], it follows that the solution ψ of problem (3.335a,b,c) is in $H^5_{loc}(\bar{\Omega} \setminus (\cup_{l=1}^J V_l))$ when the right-hand side $\hat{f}$ is in $H^{-1}(\Omega)$. From the definition of ψ, we then deduce the regularity result for $\boldsymbol{w}$. The result for p then follows from the partial differential equation (3.333a) and the de Rham Theorem. □

Next we wish to describe the regularity of $\boldsymbol{w}$ near the corners of the domain and, also, provide an asymptotic expansion valid near the corners. Thus, let S be a vertex of the polygonal boundary and denote by Γ_1 and Γ_2 the two sides of the polygonal boundary around S. We choose polar coordinates with origin at S such that $\theta = 0$ on Γ_1 and $\theta = \omega$ on Γ_2. To give the flavor of the type of results we are seeking, let us state our results for the particular case where the interior angle $\omega = \frac{3\pi}{2}$.

Theorem 3.14. *Let* $\boldsymbol{w} \in \boldsymbol{H}^2(\Omega)$ *be a weak solution of problem* (3.333a–d). *Assume that at the origin there is a vertex with interior angle* $\omega = \frac{3\pi}{2}$; *then, there exists* $\delta > 0$ *such that*

$$\boldsymbol{w}(\boldsymbol{x}) = \boldsymbol{w}_{reg}(\boldsymbol{x}) + \boldsymbol{w}_{sing}(\boldsymbol{x}) \tag{3.336}$$

where $\boldsymbol{w}_{reg} \in \boldsymbol{H}^4(B(0,\delta))$ *and*

$$\begin{aligned} \boldsymbol{w}_{sing}(x) &= \boldsymbol{w}_1(\theta) r^{1.27} + \boldsymbol{w}_2(\theta) r^{1.39} + \boldsymbol{w}_3(\theta) r^{5/3} + \boldsymbol{w}_4(\theta) r^{7/3} + \boldsymbol{w}_5(\theta) r^{2.56} \\ &\quad + \boldsymbol{w}_6(\theta) r^{2.88} + \boldsymbol{w}_7(\theta) r^3 \operatorname{Si}(\ln r) + \boldsymbol{w}_8(\theta) r^3 \frac{\sin(\ln r)}{\ln r} \end{aligned} \tag{3.337}$$

where the $\boldsymbol{w}_i(\theta)$, *for* $i = 1, 2, \ldots, 8$, *are infinitely differentiable functions of* θ.

Remarks. We will present the proof of this theorem in Sect. 3.5.5. The singular behavior of $\boldsymbol{w}$ near a vertex depends on the interior angle ω. We want to describe that behavior for a general angle ω; this will be done first for the stream function ψ, and then the result for $\boldsymbol{w}$ will easily follow.

To study the regularity of ψ near the corners, we will localize the problem near a corner and apply the method introduced in [Ko] to study the behavior of the solution. For this purpose, we let η be a radial cut-off function, i.e. $\frac{\partial \eta}{\partial \theta} = 0$, which is equal to one near S and has bounded support which does not intersect any of the $\bar{\Gamma}_j$ except Γ_1 and Γ_2. Let G denote the infinite sector whose vertex is at S and whose sides are the extension to infinity of the two lines Γ_1 and Γ_2, so $G = \{re^{i\theta} \mid r > 0,\ 0 \le \theta \le \omega\}$. Let $\psi^* = \widetilde{\eta\psi}$ where the tilde represents extension by zero. Then, equations (3.335a,b,c) become

$$\Delta^3 \psi^* = f^*, \text{ in } G, \tag{3.338a}$$

$$\psi^* = 0, \text{ on } \Gamma_1, \Gamma_2, \tag{3.338b}$$

$$\frac{\partial \psi^*}{\partial \boldsymbol{\nu}} = 0, \text{ on } \Gamma_1, \Gamma_2, \tag{3.338c}$$

$$\frac{\partial \Delta \psi^*}{\partial \boldsymbol{\nu}} = 0, \text{ on } \Gamma_1, \Gamma_2 \tag{3.338d}$$

where $\psi^* \in H^3(G) \cap H_0^2(G)$ has bounded support (i.e. ψ^* vanishes for $r \geq R$), and $f^* \in H^{-1}(G)$). The method of Kondratiev [Ko] consists of performing a change of variables that replaces the problem in an infinite sector by a similar problem in an infinite strip. Then, by using partial Fourier transforms, the problem in an infinite strip is reduced to a two-point boundary-value problem for a sixth-order differential equation which depends on a parameter. The singularities of the solution to this differential equation then can be described by using the Residue Theorem from complex analysis. Therefore, we must carefully study the analyticity of the solution, calculate precise growth conditions for the solution, and find Laurent expansions of the solution near singular points. By changing to polar coordinates, setting $r = e^t$, and letting $z(t, \theta) = e^{-t}\psi^*(e^t \cos\theta, e^t \sin\theta)$, it can be shown that $z \in H^3(B) \cap H_0^2(B)$, where $B = R \times [0, \omega]$, (see Chap. 7 [Gr1]). Also, equations (3.338a–d) are transformed into

$$\begin{aligned} &\frac{\partial^6 z}{\partial t^6} - 6\frac{\partial^5 z}{\partial t^5} + 7\frac{\partial^4 z}{\partial t^4} + 12\frac{\partial^3 z}{\partial t^3} - 17\frac{\partial^2 z}{\partial t^2} - 6\frac{\partial z}{\partial t} + 9z \\ &\qquad + 3\frac{\partial^6 z}{\partial \theta^2 \partial t^4} - 12\frac{\partial^5 z}{\partial \theta^2 \partial t^3} + 18\frac{\partial^4 z}{\partial \theta^2 \partial t^2} - 12\frac{\partial^3 z}{\partial \theta^2 \partial t} + 19\frac{\partial^2 z}{\partial \theta^2} \\ &\qquad + 3\frac{\partial^6 z}{\partial \theta^4 \partial t^2} - 6\frac{\partial^5 z}{\partial \theta^4 \partial t} + 11\frac{\partial^4 z}{\partial \theta^4 \partial t} + 11\frac{\partial^4 z}{\partial \theta^4} + \frac{\partial^6 z}{\partial \theta^6} = g, \end{aligned} \tag{3.339a}$$

$$z(t,0) = \frac{\partial z}{\partial \theta}(t,0) = z(t,\omega) = \frac{\partial z}{\partial \theta}(t,\omega) = \frac{\partial^3 z}{\partial \theta^3}(t,0) = \frac{\partial^3 z}{\partial \theta^3}(t,\omega) = 0 \tag{3.339b}$$

where the first equation holds in B and $g(t, \theta) = e^{5t} f^*(e^5 \cos\theta, e^5 \sin\theta)$. Using the fact that $f^* \in H^{-1}(G)$, it can be shown that $e^{-3t} g(t, \theta) \in H^{-1}(B)$, (Chap. 7 of [Gr1]).

We will study solutions to the boundary-value problem (3.339a,b). For this purpose, we recall that the partial Fourier transform of z with respect to t is defined by

$$\hat{z}(\tau, \theta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-it\tau} z(t, \theta)\, dt \tag{3.340}$$

where τ is the dual variable of t and τ can be a complex number. In the next lemma, we collect some well-known results about the properties of the partial Fourier transform.

Lemma 3.20. *If* $z \in H^3(B) \cap H_0^2(B)$ *and* $z(t,\theta) = 0$ *for* $t \geq \ln R$, *then* $\hat{z}(\tau,\theta)$ *is defined for* $\operatorname{Im}\tau \geq 0$, *the mapping* $\tau \mapsto \hat{z}(\tau,\theta)$ *from* C *into* $L^2([0,\omega])$ *is analytic in* τ *for* $\operatorname{Im}\tau > 0$, *and*

$$\sum_{j=0}^{3} \int_{-\infty}^{\infty} |\tau_1 + i\tau_2|^{6-2j} ||\hat{z}(\tau_1 + i\tau_2, \theta)||^2_{H^j([0,\omega])}\, d\tau_1 \leq cR^{2\tau_2}||z||_{H^3(B)} \tag{3.341}$$

for every $\tau_2 \geq 0$, *where* $\tau = \tau_1 + i\tau_2$ *and* c *is a constant.*

Proof. This is a direct consequence of Plancherel's theorem and the definition of the partial Fourier transform. □

It now follows that the function $\theta \mapsto \hat{z}(\tau,\theta)$ is, for $\tau_2 \geq 0$ and almost every $\tau_1 \in R$, an element of the space $H^3([0,\omega])$. Taking the partial Fourier transform of (3.339a,b) we get

$$\frac{\partial^6 \hat{z}}{\partial\theta^6} + (-3\tau^2 - 6\tau i + 11)\frac{\partial^4 \hat{z}}{\partial\theta^4} + (3\tau^4 + 12\tau^3 i - 18\tau^2 - 12\tau i + 19)\frac{\partial^2 \hat{z}}{\partial\theta^2}$$
$$+ (-\tau^6 - 6\tau^5 i + 7\tau^4 - 12\tau^3 i + 17\tau^2 - 16\tau i + 0)\hat{z} = \hat{g}, \tag{3.342a}$$

$$\hat{z}(\tau,0) = \frac{\partial\hat{z}}{\partial\theta}(\tau,0) = \frac{\partial^3\hat{z}}{\partial\theta^3}(\tau,0) = \hat{z}(\tau,\omega) = \frac{\partial\hat{z}}{\partial\theta}(\tau,\omega) = \frac{\partial^3\hat{z}}{\partial\theta^3}(\tau,\omega) = 0. \tag{3.342b}$$

For each τ, equations (3.342a,b) define a two-point boundary-value problem for a sixth-order differential equation in $[0,\omega]$. The characteristic equation for the differential equation (3.342a) has roots $\pm(\tau + i)$, $\pm(\tau - i)$, and $\pm(\tau + 3i)$. Recall that, for each τ, problem (3.342a,b) has a unique solution for a given $\hat{g}$ if and only if the corresponding homogeneous problem has only the zero solution.

Lemma 3.21. *The homogeneous problem* (3.342a,b) *has only the zero solution in the following cases:*

1. $\tau \neq 0, \pm i, -2i, -3i$ *and* τ *is not a root of the equation:*

$$[\sin\omega(-40\tau^6 i + 192\tau^5 + 256\tau^4 i - 96\tau^3 + 8\tau^2 i)$$
$$+ \sin 3\omega(20\tau^6 i - 144\tau^5 - 392\tau^4 i + 432\tau^3 + 20\tau^2 i + 288\tau + 144i)$$
$$+ \sin 5\omega(-4\tau^6 i + 48\tau^5 + 184\tau^4 i - 240\tau^3 - 100\tau^2 i)]\cosh\omega\tau$$
$$+ [\cos\omega(-8\tau^6 - 96\tau^5 i + 320\tau^4 + 288\tau^3 i + 40\tau^2 + 96\tau i)$$

$$+ \cos 3\omega(12\tau^6 + 144\tau^5 i - 504\tau^4 - 528\tau^3 i - 372\tau^2 - 960\tau i + 432)$$
$$+ \cos 5\omega(-4\tau^4 - 48\tau^5 i + 184\tau^4 + 240\tau^3 i - 100\tau^2)] \sinh \omega\tau$$
$$+ [\cos 3\omega(144\tau^2 + 288\tau i - 144)] \sinh 3\omega\tau$$
$$+ [\sin 3\omega(144\tau^2 i - 288\tau - 144i)] \cosh 3\omega\tau = 0. \tag{3.343}$$

2. $\tau = 0, 2i$, if the determinant

$$\sin\omega(6\omega\cos\omega + 2\sin\omega + 5\cos\omega\sin\omega + 3\omega)(6\omega\cos\omega + 2\sin\omega - 5\cos\omega\sin\omega - 3\omega) \tag{3.344}$$

is nonzero.

3. $\tau = i, -3i$, if the determinant

$$\cos\omega(\cos\omega - 1)(\cos\omega + 1)(6\omega\cos^3\omega - 7\cos^2\omega\sin\omega - 3\omega\cos\omega + 4\sin\omega) \tag{3.345}$$

is nonzero.

4. $\tau = -i$, if the determinant

$$-128(\cos\omega - 1)(\cos\omega + 1)(2\omega\cos^2\omega - 3\cos\omega\sin\omega + \omega) \tag{3.346}$$

is nonzero.

Proof. When $\tau \neq 0, \pm i, -2i, -3i$, the characteristic equation has six distinct roots and the solution of the homogeneous problem (3.342a,b) is of the form

$$\begin{aligned}\hat{z}(\theta) = &A\cos\theta\cosh\tau\theta + B\cos\theta\sinh\tau\theta + C\sin\theta\cosh\tau\theta \\ &+ D\sin\theta\sinh\tau\theta + E\cosh(\tau + 3i)\theta + F\sinh(\tau + 3i)\theta\end{aligned} \tag{3.347}$$

where A, B, C, D, E, F are complex numbers which can depend on τ. Substituting $\hat{z}$ into the boundary conditions (3.342b), we obtain a homogeneous system of six linear equations in the six unknowns A, B, C, D, E, F. By calculating the corresponding determinant, (using the symbolic manipulator Maple), we obtain the left-hand side of (3.343). When $\tau = 0$ or $-2i$, the characteristic equation has double roots of $\pm i$ and single roots of $\pm 3i$. The solution of the homogeneous problem (3.342a,b) now assumes the form

$$\hat{z}(\theta) = A\cos\theta + B\sin\theta + C\theta\cos\theta + D\theta\sin\theta + E\cos 3\theta + F\sin 3\theta$$

and the corresponding determinant is given by (3.344). When $\tau = i$ or $-3i$, the characteristic equation has 0 as a double root and $\pm i, \pm 4i$ as single roots. The solution of the homogeneous problem (3.342a,b) has the form

$$\hat{z}(\theta) = A + B\theta + C\cos 2\theta + D\sin 2\theta + E\cos 4\theta + F\sin 4\theta \tag{3.348}$$

and the corresponding determinant is (3.345). Finally, when $\tau = -i$, the characteristic equation has double roots 0 and $\pm 2i$. The solution of the homogeneous problem (3.342a,b) is of the form

$$\hat{z}(\theta) = A + B\theta + C\cos 2\theta + D\sin 2\theta + E\theta\cos 2\theta + F\theta\sin 2\theta \tag{3.349}$$

and the corresponding determinant is (3.346), which completes the proof of the lemma. □

Remarks. By using the symbolic manipulation software package Maple, it can be seen that the determinant (3.344) is zero when $\omega = 180°, 360°$ and $\omega \approx 118.3°, 238.6°, 299.2°$. Likewise, the determinant (3.345) is zero when $\omega = 90°, 180°, 270°, 360°$ and $\omega \approx 136.3°$, $226.9°$, $252.8°$, $315.7°$. Also, the determinant (3.346) is zero when $\omega = 180°, 360°$.

In what follows, we let D denote the set of noncharacteristic values for the problem (3.342a,b), (i.e. the values of τ for which the homogeneous problem (3.342a,b) has only the zero solution). Our next lemma follows from general results about two-point boundary-value problems.

Lemma 3.22. *For $\tau \in D$, the problem* (3.342a,b) *has a unique solution $\hat{z} \in H^3([0,\omega]) \cap H_0^2([0,\omega])$ provided $\hat{g} \in H^{-3}([0,\omega])$.*

In order to study the analyticity of $\hat{z}$, we must construct a fundamental system of solutions which is analytic in τ, even when $\tau = 0, \pm i, -2i, -3i$. Since any solution of the homogeneous problem (3.342a,b) is a linear combination of fundamental solutions, (see the proof of Lemma 3.21), consider the linear combination

$$z = Az_1 + Bz_2 + Cz_3 + Dz_4 + Ez_5 + Fz_6 \tag{3.350}$$

where

$$\begin{aligned}
z_1(\tau,\theta) &= \cos\theta\cosh\tau\theta,\\
z_2(\tau,\theta) &= \frac{3\tau^2-1}{2\tau(\tau^2+1)}\cos\theta\sinh\tau\theta + \frac{-\tau^3+3\tau}{2\tau(\tau^2+1)}\sin\theta\cosh\tau\theta,\ \text{for } \tau \neq -i, 0, i,\\
z_2(\pm i,\theta) &= \theta,\\
z_2(0,\theta) &= \frac{-1}{2}\theta\cos\theta + \frac{3}{2}\sin\theta,\\
z_3(\tau,\theta) &= \frac{1}{2(\tau^2+1)}\sin\theta\cosh\tau\theta - \frac{1}{2\tau(\tau^2+1)}\cos\theta\sinh\tau\theta,\ \text{for } \tau \neq -i, 0, i,\\
z_3(\pm i,\theta) &= \frac{-1}{8}\sin 2\theta + \frac{1}{4}\theta,\\
z_3(0,\theta) &= \frac{1}{2}\sin\theta - \frac{1}{2}\theta\cos\theta,\\
z_4(\tau,\theta) &= \frac{\sin\theta\sinh\tau\theta}{\tau},\ \text{for } \tau \neq 0,
\end{aligned}$$

$$z_4(0,\theta) = \theta \sin\theta,$$

$$z_5(\tau,\theta) = -\cos\theta \cosh\tau\theta + \cosh(\tau+3i)\theta,$$

$$z_6(\tau,\theta) = \frac{1}{\tau^3+6\tau^2 i-11\tau-6i}\left(\frac{-\tau^3-13\tau-12i}{\tau^3+\tau}\cos\theta\sinh\tau\theta \right.$$
$$\left. + \frac{-3\tau^2 i+12\tau+9i}{\tau^2+1}\sin\theta\cosh\tau\theta + \sinh(\tau+3i)\theta\right), \text{ for } \tau \neq -3i,-2i,-i,0,i,$$

$$z_6(i,\theta) = \frac{1}{2}i\theta - \frac{1}{3}i\sin 2\theta + \frac{1}{24}i\sin 4\theta,$$

$$z_6(0,\theta) = -2i\theta\cos\theta + \frac{3}{2}i\sin\theta + \frac{1}{6}i\sin 3\theta,$$

$$z_6(-i,\theta) = -2\theta - \theta\cos 2\theta + \frac{3}{2}\sin 2\theta,$$

$$z_6(-2i,\theta) = 2\theta\cos\theta - \frac{3}{2}\sin\theta - \frac{1}{6}\sin 3\theta,$$

$$z_6(-3i,\theta) = \frac{-1}{2}\theta + \frac{1}{3}\sin 2\theta - \frac{1}{24}\sin 4\theta.$$

It is easy to show that the z_i, $i = 1,2,\ldots,6$, are analytic functions of τ, $z_1(\tau,0) = 1$, $z_i(\tau,0) = 0$ for $i = 2,3,4,5,6$, $\frac{\partial z_2}{\partial\theta}(\tau,0) = 1$, $\frac{\partial z_i}{\partial\theta}(\tau,0) = 0$ for $i = 1,3,4,5,6$, $\frac{\partial^3 z_3}{\partial\theta^3}(\tau,0) = 1$, and $\frac{\partial^3 z_i}{\partial\theta^3}(\tau,0) = 0$ for $i = 1,2,4,5,6$. It then follows that the solution $\hat{z}$ of problem (3.342a,b) is such that $d(\tau)\hat{z}$ is an entire analytic function of τ (provided that $\hat{g}$ is an entire analytic function of τ) where

$$d(\tau) = \begin{vmatrix} z_4(\tau,\omega) & z_5(\tau,\omega) & z_6(\tau,\omega) \\ \frac{\partial z_4}{\partial\theta}(\tau,\omega) & \frac{\partial z_5}{\partial\theta}(\tau,\omega) & \frac{\partial z_6}{\partial\theta}(\tau,\omega) \\ \frac{\partial^3 z_4}{\partial\theta^3}(\tau,\omega) & \frac{\partial^3 z_5}{\partial\theta^3}(\tau,\omega) & \frac{\partial^3 z_6}{\partial\theta^3}(\tau,\omega) \end{vmatrix}. \tag{3.351}$$

Therefore, $\hat{z}$ is analytic in τ except for the values of τ where $d(\tau) = 0$. Easy calculations show that

$$d(\tau) = \frac{h(\tau)}{(\tau-i)\tau^2(\tau+i)^2(\tau+2i)(\tau+3i)} \tag{3.352}$$

where $h(\tau)$ is the left-hand side of (3.343) when $\tau \neq 0, \pm i, -2i, -3i$. Let D'' denote the values of τ such that $d(\tau) \neq 0$; it then follows from perturbation theory (see Lemma 13, page 592 in [DS]) that the following result holds

Lemma 3.23. *If E is an open subset of the complex plane such that $\tau \mapsto \hat{g}$ is analytic from E into $H^{-3}([0,\omega])$, then $\tau \mapsto \hat{z}$ is analytic from $D'' \cap E$ into $H^3([0,\omega]) \cap H_0^2([0,\omega])$.*

We now seek a subset E such that $\tau \mapsto \hat{g}$ is analytic from E into $H^{-3}([0,\omega])$. The essential elements of the proof of the following lemma can be found in [Gr1].

Lemma 3.24. *Suppose $e^{-3t}g \in H^{-1}(B)$ and that g vanishes for $t \geq \ln R$. Then $\hat{g}$ is defined for $\operatorname{Im}\tau \geq -3$ and the mapping $\tau \mapsto \hat{g}(\tau,\theta)$ is analytic for $\operatorname{Im}\tau > -3$ with values in $H^{-1}([0,\omega])$. Furthermore, for each $R' > R$, there exists $\hat{g}_1$ and $\hat{g}_2$ such that $\hat{g} = \hat{g}_1 + \hat{g}_2$, where $\hat{g}_1$ and $\hat{g}_2$ are defined for $\operatorname{Im}\tau \geq -3$, the mappings $\tau \mapsto \hat{g}_i(\tau,\theta)$, $i = 1$ and 2, are analytic for $\operatorname{Im}\tau > -3$, $\hat{g}_1$ has values in $L^2([0,\omega])$, and $\hat{g}_2$ has values in $H^{-1}([0,\omega])$. In addition, there exists a constant C such that*

$$\int_{-\infty}^{\infty} \left[|\tau^*|^{-2} ||\hat{g}_1(\tau,\theta)||^2_{L^2([0,\omega])} + ||\hat{g}_2(\tau,\theta)||^2_{H^{-1}([0,\omega])} \right] d\tau_1 \leq C(R')^{2(\tau_2+3)} \tag{3.353}$$

for every $\tau_2 \geq -3$, where $\tau = \tau_1 + i\tau_2$ and $\tau^* = \tau_1 + i(\tau_2 + 3)$.

Since the mapping $\tau \mapsto \hat{z}(\tau,\theta)$ is analytic for $\operatorname{Im}\tau > 0$ and the mapping $\tau \mapsto \hat{g}(\tau,\theta)$ is analytic for $\operatorname{Im}\tau > -3$, we have, by Lemma 3.23, that the mapping $\tau \mapsto \hat{z}(\tau,\theta)$ has an analytic continuation to the domain $\{\operatorname{Im}\tau > 0\} \cup [\{\operatorname{Im}\tau > -3\} \cap D'']$. We proceed by deriving some growth conditions on $\hat{z}$, beginning with

Lemma 3.25. *Suppose that for a given function $\theta \mapsto \hat{g}(\tau,\theta)$ in the space $H^{-1}([0,\omega])$, the function $\theta \mapsto \hat{z}(\tau,\theta)$ satisfies (3.342a,b), where the function $\theta \mapsto \hat{z}(\tau,\theta)$ is an element of the space $H^3([0,\omega]) \cap H_0^2([0,\omega])$ for $\tau_2 \geq -3$ and almost every $\tau_1 \in R$. Then, there exist constants $c > 0$ and $M > 0$ such that*

$$\sum_{j=0}^{5} |\tau_1|^{5-j} ||\hat{z}(\tau,\theta)||_{H^j([0,\omega])} \leq c||\hat{g}(\tau,\theta)||_{H^{-1}([0,\omega])} \tag{3.354}$$

for $|\tau_1| \geq M$ and $-3 \leq \tau_2 \leq 0$, where $\tau = \tau_1 + i\tau_2$.

Proof. Multiplying (3.342a) by $-\bar{\tau}^4\bar{\hat{z}}$, integrating by parts from 0 to ω, and performing algebraic manipulations, we obtain

$$\begin{aligned} |\tau|^4 \left|\left| \frac{\partial^3\hat{z}}{\partial\theta^3} \right|\right|^2 + 3|\tau|^6 \left|\left| \frac{\partial^2\hat{z}}{\partial\theta^2} \right|\right|^2 + 3|\tau|^8 \left|\left| \frac{\partial\hat{z}}{\partial\theta} \right|\right|^2 + |\tau|^{10}||\hat{z}||^2 \\ = |\tau|^4 A_1 \left|\left| \frac{\partial^3\hat{z}}{\partial\theta^3} \right|\right|^2 + |\tau|^6 A_2 \left|\left| \frac{\partial^2\hat{z}}{\partial\theta^2} \right|\right|^2 \\ + |\tau|^8 A_3 \left|\left| \frac{\partial\hat{z}}{\partial\theta} \right|\right|^2 + |\tau|^{10} A_4 ||\hat{z}||^2 - \int_0^\omega \bar{\tau}^4 \hat{g}\bar{\hat{z}}\, d\theta \end{aligned} \tag{3.355}$$

where $||\quad|| = ||\quad||_{L^2([0,\omega])}$, and

$$A_1 = \left(1 - \frac{\bar{\tau}^4}{|\tau|^4}\right), \tag{3.356a}$$

$$A_2 = \left[3\left(1 - \frac{\bar{\tau}^2}{|\tau|^2}\right) - \frac{6\bar{\tau}^3 i}{|\tau|^4} + \frac{11\bar{\tau}^4}{|\tau|^6}\right], \tag{3.356b}$$

$$A_3 = \left(\frac{-12\bar{\tau}i}{|\tau|^2} + \frac{18\bar{\tau}^2}{|\tau|^4} + \frac{12\bar{\tau}^3 i}{|\tau|^6} - \frac{19\bar{\tau}^4}{|\tau|^8}\right), \tag{3.356c}$$

$$A_4 = \left[\left(1 - \frac{\tau^2}{|\tau|^2}\right) - \frac{6\tau i}{|\tau|^2} + \frac{7}{|\tau|^2} - \frac{12\bar{\tau}i}{|\tau|^4} + \frac{17\bar{\tau}^2}{|\tau|^6} - \frac{6\bar{\tau}^3 i}{|\tau|^8} + \frac{9\bar{\tau}^4}{|\tau|^{10}}\right]. \tag{3.356d}$$

By taking the modulus of both sides of (3.355), noting the fact that

$$\left|\int_0^\omega \bar{\tau}^4 \hat{g}\bar{\hat{z}}\, d\theta\right| \leq \frac{1}{2}|\tau|^8 ||\hat{z}||^2_{H_0^1([0,\omega])} + \frac{1}{2}||\hat{g}||^2_{H^{-1}([0,\omega])} \tag{3.357}$$

and using well-known interpolation results (as in [Gr1] or [LM], we obtain the result. □

We now state an important growth condition for $\hat{z}$, whose proof is similar to a corresponding result found in [Gr1].

Lemma 3.26. *Under the same assumptions as in Lemma 3.25, we have*

$$\sup_{-3\leq\tau_2\leq 0}\left\{\int_{|\tau_1\geq M|}\sum_{j=0}^{5}|\tau_1|^{2(5-j)}||\hat{z}(\tau_1 + i\tau_2, \theta)||^2_{H^j([0,\omega])}\, d\tau_1\right\} < \infty. \tag{3.358}$$

Since the solution $\hat{z}$ to (3.342a,b) is such that the mapping $\tau \mapsto \hat{z}$ is analytic for $E' = \{\mathrm{Im}\,\tau > 0\} \cup [\{\mathrm{Im}\,\tau > -3\} \cap D'']$, the mapping $\tau \mapsto \hat{z}$ may have poles at the set of points $E^* = \{\mathrm{Im}\,\tau > -3\} \setminus E'$. We now elaborate the behavior of the solution near a point in E^*; to this end we have

Lemma 3.27. *Let $\tau_m \in E^*$. Let $\tau \mapsto \hat{g}(\tau, \theta)$ be any analytic function in a neighborhood of τ_m, with values in $H^{-3}([0, \omega])$. Then, the corresponding solution of* (3.342a,b) *has the following Laurent expansion near τ_m:*

1. *If τ_m is a simple zero of* (3.352)*, then*

$$\hat{z}(\tau, \theta) = \frac{\psi_m(\theta)}{\tau - \tau_m} + \hat{w}_m(\tau, \theta) \tag{3.359}$$

where ψ_m is a solution of the homogeneous problem (3.342a,b) *with $\tau = \tau_m$, and $\tau \mapsto \hat{w}_m(\tau, \theta)$ is an analytic function near τ_m with values in $H^3([0, \omega])$.*

2. *If τ_m is a double zero of* (3.352)*, then*

$$\hat{z}(\tau, \theta) = \frac{\psi_m(\theta)}{(\tau - \tau_m)^2} + \frac{\varphi_m(\theta)}{\tau - \tau_m} + \hat{w}_m(\tau, \theta) \tag{3.360}$$

where ψ_m and $\hat{w}_m$ are defined as in (3.359), φ_m is a solution of

$$L(\tau_m, D_6)\varphi_m = -L'_\tau(\tau_m, D_\theta)\psi_m \tag{3.361}$$

in $[0, \omega]$ with the boundary conditions in (3.342b), and

$$L(\tau, D_\theta) = \frac{\partial^6}{\partial\theta^6} + (-3\tau^2 - 6\tau i + 11)\frac{\partial^4}{\partial\theta^4} + (3\tau^4 + 12\tau^3 i - 18\tau^2 - 12\tau i + 19)\frac{\partial^2}{\partial\theta^2} + (-\tau^6 - 6\tau^5 i + 7\tau^4 - 12\tau^3 i + 17\tau^2 - 6\tau i + 9). \tag{3.362}$$

3. *If τ_m is a triple zero of (3.352), then*

$$\hat{z}(\tau, \theta) = \frac{\psi_m(\theta)}{(\tau - \tau_m)^3} + \frac{\varphi_m(\theta)}{(\tau - \tau_m)^2} + \frac{\Phi_M(\theta)}{\tau - \tau_m)} + \hat{w}_m(\tau, \theta) \tag{3.363}$$

where ψ_m, φ_m, and $\hat{w}_m$ are defined as in (3.359) and (3.360), and Φ_m is a solution of

$$L(\tau_m, D_\theta)\Phi_m = -\frac{1}{2}L''_\tau(\tau_m, D_\theta)\psi_m - L'_\tau(\tau_m, D_\theta)\varphi_m \tag{3.364}$$

in $[0, \omega]$ with the boundary conditions in (3.342b).

Proof. The ideas for the proof of (3.359) and (3.360) can be found in [Gr1]. Hence, we will only prove (3.363). If $d(\tau)$ has a triple zero at τ_m, then $\hat{z}$ has a triple pole at τ_m, and its Laurent expansion is given by

$$\hat{z}(\tau, \theta) = \frac{\psi_m(\theta)}{(\tau - \tau_m)^3} + \frac{\varphi_m(\theta)}{(\tau - \tau_m)^2} + \frac{\Phi_m(\theta)}{\tau - \tau_m} + \hat{w}_m(\tau, \theta) \tag{3.365}$$

where $\tau \mapsto \hat{w}_m(\tau, \theta)$ is an analytic function near τ_m with values in $H^3([0, \omega])$. Multiplying (3.365) by $(\tau - \tau_m)^3$ and applying the differential operator $L(\tau, D_\theta)$ gives $L(\tau_m, D_\theta)\psi_m = 0$. Also, multiplying (3.365) by $(\tau - \tau_m)^2$, applying the differential operator $L(\tau, D_\theta)$, and using the fact that $L(\tau_m, D_\theta)\psi_m = 0$, we get $L(\tau_m, D_\theta)\varphi_m = -L'_\tau(\tau_m, D_\theta)\psi_m$. It is easy to show that ψ_m and φ_m satisfy the boundary conditions in (3.342b). Now, multiplying (3.365) by $(\tau - \tau_m)$ and applying the differential operator $L(\tau, D_\theta)$ gives

$$L(\tau, D_\theta)\Phi_m = (\tau - \tau_m)\hat{g} - \frac{L(\tau, D_\theta)\psi_m + (\tau - \tau_m)L(\tau, D_\theta)\varphi_m}{(\tau - \tau_m)^2} - (\tau - \tau_m)L(\tau, D_\theta)\hat{w}_m. \tag{3.366}$$

Letting $\tau \to \tau_m$, and applying L'Hôpital's rule twice, we get

$$L(\tau_m, D_\theta)\Psi_m = -\frac{1}{2}L''_\tau(\tau_m, D_\theta)\psi_m - L'_\tau(\tau_m, D_\theta)\varphi_m. \tag{3.367}$$

It is now a straightforward calculation to show that Φ_m satisfies the boundary conditions in (3.342b). □

The existence of φ_m and Φ_m in Lemma 3.27 is not obvious since τ_m is a characteristic value. Thus, we need to derive explicit formulas for ψ_m, φ_m, and Φ_m when the solution space to the homogeneous problem in (3.342a,b) at $\tau = \tau_m$ is one-dimensional; for this purpose we will prove

Lemma 3.28. *Suppose $\tau = \tau_m$ is a characteristic value of the homogeneous problem* (3.342a,b) *and the solutions ψ_m span a one-dimensional space. Then there exist constants c_m, d_m, and f_m such that*

1. $\psi_m(\theta) = c_m B(\tau_m, \theta)$ *when τ_m is a simple zero of* (3.352)
2. $\varphi_m(\theta) = c_m B'_\tau(\tau_m, \theta) + d_m B(\tau_m, \theta)$ *when τ_m is a double zero of* (3.352)
3. $\Phi_m(\theta) = f_m B(\tau_m, \theta) + d_m B'_\tau(\tau_m, \theta) + \frac{1}{2} c_m B''_\tau(\tau_m, \theta)$, *when τ_m is a triple zero of* (3.352), *where*

$$
\begin{aligned}
B(\tau, \theta) = {} & \left[z_6(\tau, \omega) \frac{\partial^3 z_5}{\partial \theta^3}(\tau, \omega) - z_5(\tau, \omega) \frac{\partial^3 z_6}{\partial \theta^3}(\tau, \omega) \right] z_4(\tau, \theta) \\
& + \left[z_4(\tau, \omega) \frac{\partial^3 z_6}{\partial \theta^3}(\tau, \omega) - z_6(\tau, \omega) \frac{\partial^3 z_4}{\partial \theta^3}(\tau, \omega) \right] z_5(\tau, \theta) \\
& + \left[z_5(\tau, \omega) \frac{\partial^3 z_4}{\partial \theta^3}(\tau, \omega) - z_4(\tau, \omega) \frac{\partial^3 z_5}{\partial \theta^3}(\tau, \omega) \right] z_6(\tau, \theta)
\end{aligned} \tag{3.368}
$$

and z_4, z_5, and z_6 are defined following (3.350).

Proof. The function B is entire in τ and is a solution of

$$L(\tau, D_\theta) B(\tau, \theta) = 0, \tag{3.369a}$$

$$B(\tau, 0) = \frac{\partial B}{\partial \theta}(\tau, 0) = \frac{\partial^3 B}{\partial \theta^3}(\tau, 0) = B(\tau, \omega) = \frac{\partial^3 B}{\partial \theta^3}(\tau, \omega) = 0, \tag{3.369b}$$

$$\frac{\partial B}{\partial \theta}(\tau, \omega) = d(\tau). \tag{3.369c}$$

When τ_m is a zero of $d(\tau)$, $\dfrac{\partial B}{\partial \theta}(\tau_m, \omega) = 0$, and $B(\tau_m, \theta)$ is a solution to the homogeneous problem (3.342a,b). Since the solution space is one-dimensional and $B(\tau_m, \theta)$ does not vanish everywhere, we have $\psi_m(\theta) = c_m B(\tau_m, \theta)$ for some constant c_m. Now, differentiating $L(\tau, D_\theta) B(\tau, \theta) = 0$ with respect to τ, we obtain

$$L'_\tau(\tau, D_\theta) B(\tau, \theta) = -L(\tau, D_\theta) B'_\tau(\tau, \theta) \tag{3.370a}$$

for every τ. Differentiating the boundary conditions (3.369b,c) with respect to τ gives

$$B'_\tau(\tau,0) = \frac{\partial}{\partial\theta}[B'_\tau(\tau,0)] = \frac{\partial^3}{\partial\theta^3}[B'_\tau(\tau,0)] = B'_\tau(\tau,\omega) = \frac{\partial^3}{\partial\theta^3}[B'_\tau(\tau,\omega)] = 0 \tag{3.370b}$$

and

$$\frac{\partial}{\partial\theta}[B'_\tau(\tau,\omega)] = d'(\tau) \tag{3.370c}$$

for every τ. When τ_m is a double zero of $d(\tau)$, $B'_\tau(\tau_m,\theta)$ satisfies the boundary conditions in (3.342b). It follows that $c_m B'_\tau(\tau_m,\theta)$ solves the same problem as $\varphi_m(\theta)$ in Part 2 of Lemma 3.27 and so their differences must be a solution of the homogeneous problem (3.342a,b) at $\tau = \tau_m$. Therefore, $\varphi_m(\theta) = c_m B'_\tau(\tau_m,\theta) + d_m B(\tau_m,\theta)$ for some constant d_m. Now, let $a(\tau) = d_m(\tau-\tau_m)+c_m$. Differentiating $L(\tau, D_\theta)a(\tau)B(\tau,\theta) = 0$ twice with respect to τ gives

$$\begin{aligned} L''_\tau(\tau, D_\theta)a(\tau)B(\tau,\theta) + 2L'_\tau(\tau, D_\theta)a'(\tau)B(\tau,\theta) \\ + 2L'_\tau(\tau, D_\theta)a(\tau)B'_\tau(\tau,\theta) + 2L(\tau, D_\theta)a'(\tau)B'_\tau(\tau,\theta) \\ + L(\tau, D_\theta)a''(\tau)B(\tau,\theta) + L(\tau, D_\theta)a(\tau)B''_\tau(\tau,\theta) = 0 \end{aligned} \tag{3.371}$$

for every τ. Letting $\tau = \tau_m$ in (3.371), we get

$$L(\tau_m, D_\theta)\left[d_m B'_\tau(\tau_m,\theta) + \frac{1}{2}c_m B''_\tau(\tau_m,\theta)\right] = -\frac{1}{2}L''_\tau(\tau_m, D_\theta)\psi_m(\theta) - L'_\tau(\tau_m, D_\theta)\varphi_m(\theta) \tag{3.372}$$

Set $C(\tau_m,\theta) = d_m B'_\tau(\tau_m,\theta) + \frac{1}{2}c_m B''_\tau(\tau_m,\theta)$. Then, it can be shown that

$$C(\tau_m,0) = \frac{\partial C}{\partial\theta}(\tau_m,0) = \frac{\partial^3 C}{\partial\theta^3}(\tau_m,0) = C(\tau_m,\omega) = \frac{\partial^3 C}{\partial\theta^3}(\tau_m,\omega) = 0 \tag{3.373a}$$

$$\frac{\partial C}{\partial\theta}(\tau_m,\omega) = d_m d'(\tau_m) + \frac{1}{2}c_m d''(\tau_m). \tag{3.373b}$$

When τ_m is a triple zero of $d(\tau)$, $C(\tau_m,\theta)$ solves the same problem as $\Phi_m(\theta)$ in Part 3 of Lemma 3.27 and so their difference must be a solution of the homogeneous problem (3.342a,b). Therefore, $\Phi_m(\theta) = f_m B(\tau_m,\theta) + d_m B'_\tau(\tau_m,\theta) + \frac{1}{2}c_m B''_\tau(\tau_m,\theta)$ for some constant f_m.

Remarks. When the solution space to the homogeneous problem in (3.342a,b) is two-dimensional at $\tau = \tau_m$, the function ψ_m will be a linear combination of two functions, say ψ_m^1 and ψ_m^2. In most cases, when τ_m is a double zero of (3.352), ψ_m must vanish in order to satisfy the solvability condition for (3.361), and so φ_m will be a linear combination of ψ_m^1 and ψ_m^2. Similar results occur for solution spaces with dimension greater than two and for triple zeros. The basis functions for the solution space and the solvability condition requirements for specific angles ω can be calculated (for example, by using the symbolic manipulation software package Maple) in order to determine the structure of φ_m and Φ_m.

Remarks. In order to describe the singularities of the solution $z(t, \theta)$ to problem (3.339a,b), we will use the Residue Theorem, where the path chosen includes the line $\operatorname{Im}\tau = -3$. It is therefore necessary to study the zeros of (3.352) on the line $\operatorname{Im}\tau = -3$. On the line $\operatorname{Im}\tau = -3$, (i) if $\operatorname{Re}\tau = 0$, then for the zeros of (3.345), ($\omega = 90°, 270°$ and $\omega \approx 136.3°, 226.9°, 252.8°, 315.7°$) (3.352) has a simple zero at $\tau = -3i$; (ii) if $\operatorname{Re}\tau \neq 0$, then there is no angle such that τ is a zero of (3.352). Part (i) follows from Lemma 3.21 and the remarks following (3.349). Part (ii) follows from easy but tedious calculations.

We can now describe the singularities of the solution $z(t, \theta)$ to (3.339a,b) as follows:

Lemma 3.29. *The solution* $z(t, \theta) \in H^3(B) \cap H_0^2(B)$ *of* (3.339a,b) *with* $e^{-3t} g(t, \theta) \in H^{-1}(B)$ *and* $g(t, \theta) = e^{5t} f^*(e^t \cos\theta, e^t \sin\theta)$ *has the following form:*

1. *If* $\omega \in (0, 2\pi)$ *and* ω *is not a zero of* (3.345)*, then*

$$z(t, \theta) = e^{3t} w(t, \theta) + \sum_{-3<\operatorname{Im}\tau_m<0} S_m(t, \theta). \tag{3.374}$$

2. *If* $\omega \in (0, 2\pi)$ *and* ω *is a zero of* (3.345)*, then*

$$z(t, \theta) = e^{3t} w(t, \theta) + \frac{2i\psi_{-3i}(\theta)}{\sqrt{2\pi}} e^{3t}\operatorname{Si}(t) - i\sqrt{\frac{\pi}{2}}\psi_{-3i}(\theta) e^{3t} + \sum_{-3<\operatorname{Im}\tau_m<0} S_m(t, \theta). \tag{3.375}$$

In (3.374)*,* (3.375)*,* $w(t, \theta) \in H^5(B)$*,* τ_m *denotes the sequence of characteristic values (i.e. the zeros of* (3.352)*),* $S_m(t, \theta)$ *is the residue of* $\tau \mapsto i\sqrt{2\pi} e^{it\tau} \hat{z}(\tau, \theta)$ *at* $\tau = \tau_m$*,* $\psi_{-3i}(\theta)$ *is a solution of the homogeneous problem* (3.342a,b) *with* $\tau = -3i$*, and* $\operatorname{Si}(t) = \int_0^t \frac{\sin\tau^*}{\tau^*}\, d\tau^*$*.*

Proof. If $\omega \in (0, 2\pi)$ and ω is not a zero of (3.345), then (3.374) easily follows from the Residue Theorem by integrating the function $\frac{1}{\sqrt{2\pi}} e^{it\tau} \hat{z}(\tau, \theta)$ around the

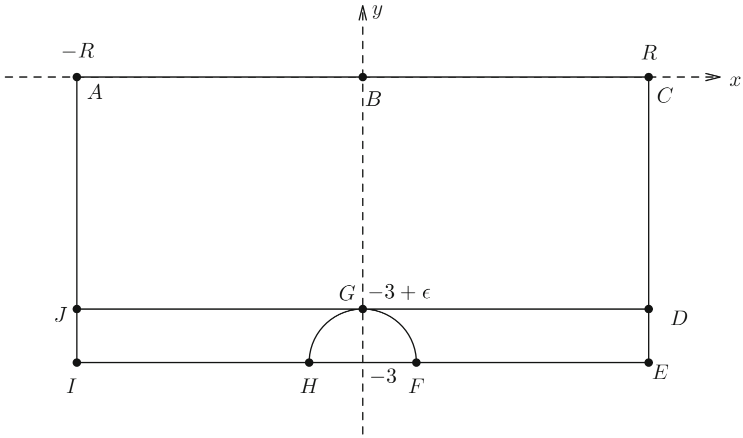

Fig. 3.9 Domain in the proof of Lemma 3.29

path $ACEIA$, (see Fig. 3.9). It follows from Lemma 3.26 that $w(t,\theta) \in H^5(B)$ where

$$w(t,\theta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{it\tau_1} \hat{z}(\tau_1 - 3i, \theta)\, d\tau_1.$$

Now, suppose $\omega \in (0, 2\pi)$ and ω is a zero of (3.345). Since $\tau = -3i$ is a simple zero of (3.352), by the remarks preceding Lemma 3.29 $\hat{z}(\tau,\theta)$ is not well-defined at $\tau = -3i$; thus, we will integrate along the path $ABCEFGHIA$, (Fig. 3.9). But, by Cauchy's Theorem, it suffices to integrate along the path $ABCDGJA$, which we will denote by $\mathcal{C}$. The function $\frac{1}{\sqrt{2\pi}} e^{it\tau} \hat{z}(\tau,\theta)$ is single-valued and analytic inside and on the closed curve $\mathcal{C}$ except at the singularities inside the curve (i.e. at the zeros of (3.352) where $-3+\epsilon < \operatorname{Im}\tau_m < 0$). Applying the Residue Theorem and using the fact that the Fourier transform of an L^1 function is a continuous function with zero limit at infinity, we obtain

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{it\tau_1} \hat{z}(\tau_1,\theta)\, d\tau_1 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{it(\tau_1 - 3i + \epsilon i)} \hat{z}(\tau_1 - 3i + \epsilon i, \theta)\, d\tau_1 + \sum_{-3+\epsilon < \operatorname{Im}\tau_m < 0} S_m(t,\theta) \tag{3.376}$$

as $R \to \infty$. Now, let

$$\begin{aligned} \hat{R}(\tau_1 - 3i + \epsilon i, \theta) &= \hat{z}(\tau_1 - 3i + \epsilon i, \theta) - \frac{\psi_{-3i}(\theta)\chi_{(-1,1)}}{\tau_1 + \epsilon i} \\ &= \begin{cases} \hat{z}(\tau_1 - 3i + \epsilon i, \theta), & \text{for } \tau_1 \le 1 \text{ or } \tau_1 \ge 1, \\ \hat{w}_{-3i}(\tau_1 - 3i + \epsilon i, \theta), & \text{for } -1 < \tau_1 < 1 \end{cases} \end{aligned} \tag{3.377}$$

where χ denotes the characteristic function on τ_1, and set

$$w(t,\theta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \epsilon^{it\tau_1} \hat{R}(\tau_1 - 3i, \theta)\, d\tau_1 \tag{3.378}$$

From Lemma 3.26, the fact that $\hat{z}$ is well-defined for $\tau = \tau_1 - 3i$, where $\tau_1 \leq -1$ or $\tau_1 \geq 1$, and the fact $\hat{w}_{-3i}(\tau,\theta)$ is analytic for $\tau = \tau_1 - 3i$, where $-1 \leq \tau_1 \leq 1$, we have $w(t,\theta) \in H^5(B)$. Also,

$$\begin{aligned}\int_{-\infty}^{\infty} e^{it(\tau_1 - 3i + \epsilon i)} \frac{\psi_{-3i}(\theta)\chi_{(-1,1)}}{\tau_1 + \epsilon i}\, d\tau_1 &= 2i\psi_{-3i}(\theta) e^{3t-\epsilon t} \int_0^1 \sin t\tau_1 \frac{\tau_1}{\tau_1^2 + \epsilon^2}\, d\tau_1 \\ &\quad - 2i\psi_{-3i}(\theta) e^{3t-\epsilon t} \int_0^1 \cos t\tau_1 \frac{\epsilon}{\tau_1^2 + \epsilon^2}\, d\tau_1.\end{aligned} \tag{3.379}$$

The first term on the right-hand side of (3.379) approaches $2i\psi_{-3i}(\theta)e^{3t}\,\mathrm{Si}(t)$ as $\epsilon \to 0$. Also, by the Lebesgue Convergence Theorem (see Lemma 3.30 for a proof),

$$\lim_{\epsilon\to 0} \int_0^1 \cos t\tau_1 \frac{\epsilon}{\tau_1^2 + \epsilon^2}\, d\tau_1 = \lim_{\epsilon\to 0} \int_0^{\infty} \cos t\tau_1 \frac{\epsilon}{\tau_1^2 + \epsilon^2}\, d\tau_1 = \frac{\pi}{2}. \tag{3.380}$$

Letting $\epsilon \to 0$ in (3.376), we obtain (3.375). □

We will now prove (3.380).

Lemma 3.30. *For any real number t,*

$$\lim_{\epsilon\to 0} \int_0^1 \cos t\tau_1 \frac{\epsilon}{\tau_1^2 + \epsilon^2}\, d\tau_1 = \frac{\pi}{2}.$$

Proof. We first consider the expression

$$\lim_{\epsilon\to 0} \int_1^{\infty} \cos t\tau_r \frac{\epsilon}{\tau_1^2 + \epsilon^2}\, d\tau_1$$

If we let $f_\epsilon(\tau_1) = \dfrac{\epsilon}{\tau_1^2 + \epsilon^2}$, then for a fixed $\tau_1 \in [1,\infty)$, $f_\epsilon(\tau_1) \to 0$ as $\epsilon \to 0$. Thus, for each sequence $\{\epsilon_n\}_{n=1}^{\infty}$ such that $\epsilon_n \to 0$, as $n \to \infty$, and $\epsilon_n \neq 0$ for all n, the sequence $\{f_{\epsilon_n}(\tau_1)\}_{n=1}^{\infty}$ approaches 0 as $n \to \infty$ for a fixed $\tau_1 \in [1,\infty)$. Since, for a fixed $\tau_1 \in [1,\infty)$,

$$\frac{\partial f_\epsilon}{\partial \epsilon}(\tau_1) = \frac{\tau_1^2 - \epsilon^2}{(\tau_1^2 + \epsilon^2)} > 0$$

for ϵ small (i.e., $0 < \epsilon < \frac{1}{2}$), f_ϵ is an increasing function of ϵ. So, for a fixed $\tau_1 \in [1, \infty)$,

$$|f_{\epsilon_n}(\tau_1)| = \left| \frac{\epsilon_n}{\tau_1^2 + \epsilon_n^2} \right| \leq \frac{\frac{1}{2}}{\tau_1^2 + \left(\frac{1}{2}\right)^2} = \frac{2}{4\tau_1^2 + 1} \tag{3.381}$$

as f_{ϵ_n} is increasing, and $0 < \epsilon_n < \frac{1}{2}$. Also,

$$\begin{aligned} \int_1^\infty \frac{2}{4\tau_1^2 + 1}\, d\tau_1 &= \frac{1}{2} \int_1^\infty \frac{1}{\tau_1^2 + \left(\frac{1}{2}\right)^2}\, d\tau_1 = \tan^{-1} 2\tau_1 \Big|_1^\infty \\ &= \frac{\pi}{2} - \tan^{-1} 2 < \infty \end{aligned} \tag{3.382}$$

since $\int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1} \frac{u}{a} + C$ for $a > 0$. Therefore, $g(\tau_1) = \frac{2}{4\tau_1^2 + 1}$ is integrable over $[1, \infty)$, $|f_{\epsilon_n}(\tau_1)| \leq g(\tau_1)$ on $[1, \infty)$, and

$$\lim_{\epsilon \to 0} f_\epsilon(\tau_1) = \lim_{n \to \infty} f_{\epsilon_n}(\tau_1) = \lim_{n \to \infty} \frac{\epsilon_n}{\tau_1^2 + \epsilon_n^2} = 0.$$

Thus, by the Lebesgue Convergence Theorem,

$$\lim_{\epsilon \to 0} \int_1^\infty \frac{\epsilon}{\tau_1^2 + \epsilon^2}\, d\tau_1 = \lim_{n \to \infty} \int_1^\infty \frac{\epsilon_n}{\tau_1^2 + \epsilon_n^2}\, d\tau_1 = 0. \tag{3.383}$$

From (3.383) and the fact that

$$\left| \int_1^\infty \cos t\tau_1 \frac{\epsilon}{\tau_1^2 + \epsilon^2}\, d\tau_1 \right| \leq \int_1^\infty \frac{\epsilon}{\tau_1^2 + \epsilon^2}\, d\tau_1$$

we get

$$\lim_{\epsilon \to 0} \int_1^\infty \cos t\tau_1 \frac{\epsilon}{\tau_1^2 + \epsilon^2}\, d\tau_1 = 0. \tag{3.384}$$

Also, from integration tables, we find that for positive constants a and m,

$$\int_0^\infty \frac{\cos mx}{x^2 + a^2}\, dx = \frac{\pi}{2a} e^{-ma}. \tag{3.385}$$

So, from (3.385),

$$\lim_{\epsilon\to 0}\int_0^\infty \cos t\tau_1 \frac{\epsilon}{\tau_1^2+\epsilon^2}\,d\tau_1 = \lim_{\epsilon\to 0}\epsilon\int_0^\infty \frac{\cos t\tau_1}{\tau_1^2+\epsilon^2}\,d\tau_1 = \lim_{\epsilon\to 0}\epsilon\frac{\pi}{2\epsilon}e^{-|t|\epsilon} = \lim_{\epsilon\to 0}\frac{\pi}{2}e^{-|t|\epsilon} = \frac{\pi}{2}. \tag{3.386}$$

The result follows from (3.384) and (3.386). □

We can now describe the behavior of the solution ψ^* to problem (3.338a–d), i.e., we have

Theorem 3.15. *Suppose* $\psi^* \in H^3(G) \cap H_0^2(G)$ *is a solution, with bounded suppport, of the system* (3.338a–d). *If* $f^* \in H^{-1}(G)$, *then* $\psi^* = \psi^*_{reg} + \psi^*_{sing}$, *where* $\psi^*_{reg} \in H^5(G)$ *and*

$$\begin{aligned}\psi^*_{sing}(r,\theta) = &\sum_{-3<\operatorname{Im}\tau'_m<0} i\sqrt{2\pi}r^{1+i\tau'_m}\psi_m(\theta) + \sum_{-3<\operatorname{Im}\tau''_m<0} i\sqrt{2\pi}r^{1+i\tau''_m}\{\varphi_m(\theta)+i(\ln r)\psi_m(\theta)\}\\ &+\sum_{-3<\operatorname{Im}\tau'''_m<0} i\sqrt{2\pi}r^{1+i\tau'''_m}\left\{\Phi_m(\theta)+i(\ln r)\varphi_m(\theta)-\frac{(\ln r)^2}{2}\psi_m(\theta)\right\}\end{aligned} \tag{3.387}$$

if $\omega \in (0,2\pi)$ *and* ω *is not a zero of* (3.345), *while*

$$\begin{aligned}\psi^*_{sing}(r,\theta) = &\sum_{-3<\operatorname{Im}\tau'_m<0} i\sqrt{2\pi}r^{1+i\tau'_m}\psi_m(\theta) + \sum_{-3<\operatorname{Im}\tau''_m<0} i\sqrt{2\pi}r^{1+i\tau''_m}\{\varphi_m(\theta)+i(\ln r)\psi_m(\theta)\}\\ &+\sum_{-3<\operatorname{Im}\tau'''_m<0} i\sqrt{2\pi}r^{1+i\tau'''_m}\left\{\Phi_m(\theta)+i(\ln r)\varphi_m(\theta)-\frac{(\ln r)^2}{2}\psi_m(\theta)\right\}\\ &+i\sqrt{\frac{2}{\pi}}\psi_{-3i}(\theta)r^4\operatorname{Si}(\ln r)\end{aligned} \tag{3.388}$$

if $\omega \in (0,2\pi)$ *and* ω *is a zero of* (3.345).

Remarks. In (3.387) and (3.388), the real parts of ψ^*_{sing} are considered relevant, $\psi_{-3i}(\theta)$ and $\operatorname{Si}(\ln r)$ are defined in Lemma 3.29, and the functions $\psi_m(\theta)$, $\varphi_m(\theta)$, and $\Phi_m(\theta)$ are defined in Lemma 3.27. Also, τ'_m denotes the sequence of simple zeros of (3.352), τ''_m denotes the sequence of double zeros of (3.352), and τ'''_m denotes the sequence of triple zeros of (3.352).

Proof (Theorem 3.15). By (3.376), if $\tau = \tau_m$ is a simple pole of $\hat{z}$, then the residue of $\tau \mapsto i\sqrt{2\pi}e^{it\tau}\hat{z}(\tau,\theta)$ at $\tau = \tau_m$ is given by $i\sqrt{2\pi}e^{it\tau_m}\psi_m(\theta)$, where $\psi_m(\theta)$ is defined in Part 1 of Lemma 3.27. if $\tau = \tau_m$ is a double pole of $\hat{z}$, then the residue of $\tau \mapsto i\sqrt{2\pi}e^{it\tau}\hat{z}(\tau,\theta)$ at $\tau = \tau_m$ is given by $i\sqrt{2\pi}e^{it\tau_m}\{\varphi_m(\theta)+it\psi_m(\theta)\}$, where $\psi_m(\theta)$ and $\varphi_m(\theta)$ are defined in Part 2 of Lemma 3.27. Similarly, if $\tau = \tau_m$ is a triple pole of $\hat{z}$, then the residue of $\tau \mapsto i\sqrt{2\pi}e^{it\tau}\hat{z}(\tau,\theta)$ at $\tau = \tau_m$ is given by $i\sqrt{2\pi}e^{it\tau_m}\left\{\Phi_m(\theta)+it\varphi_m(\theta)-\frac{t^2}{2}\psi_m(\theta)\right\}$, where $\psi_m(\theta)$, $\varphi_m(\theta)$, and $\Phi_m(\theta)$ are defined in Part 3 of Lemma 3.27. If we perform the change of variable $r = e^t$ in (3.374) and (3.375), and notice that $\psi^*_{reg}(x, y) = e^{4t} w(t, \theta) \in H^5(G)$ (by Lemma 7.2.1.3 in [Gr1]) then the result follows. □

As $u = -\dfrac{\partial \psi}{\partial y}$ and $v = \dfrac{\partial \psi}{\partial x}$, we also obtain the following theorem:

Theorem 3.16. *Suppose* $\boldsymbol{w} = (u, v) \in \boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}^1_0(\Omega)$ *is a solution to* (3.333a–d). *If* $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, *then in a neighborhood of the corner* S_l, *the solution* $\boldsymbol{w} = (u, v)$ *of* (3.333a–d) *has the form:*

$$u = u_{reg} - \sum_{-3<\mathrm{Im}\,\tau'_m<0} A_{1,m}(r,\theta) - \sum_{-3<\mathrm{Im}\,\tau''_m<0} B_{1,m}(r,\theta) - \sum_{-3<\mathrm{Im}\,\tau'''_m<0} C_{1,m}(r,\theta) \tag{3.389a}$$

and

$$v = v_{reg} + \sum_{-3<\mathrm{Im}\,\tau'_m<0} A_{2,m}(r,\theta) + \sum_{-3<\mathrm{Im}\,\tau''_m<0} B_{2,m}(r,\theta) + \sum_{-3<\mathrm{Im}\,\tau'''_m<0} C_{2,m}(r,\theta) \tag{3.389b}$$

if $\omega \in (0, 2\pi)$ *and* ω *is not a zero of* (3.345), *while*

$$u = u_{reg} - \sum_{-3<\mathrm{Im}\,\tau'_m<0} A_{1,m}(r,\theta) - \sum_{-3<\mathrm{Im}\,\tau''_m<0} B_{1,m}(r,\theta) - \sum_{-3<\mathrm{Im}\,\tau'''_m<0} C_{1,m}(r,\theta) + D_1(r,\theta) \tag{3.390a}$$

and

$$v = v_{reg} + \sum_{-3<\mathrm{Im}\,\tau'_m<0} A_{2,m}(r,\theta) + \sum_{-3<\mathrm{Im}\,\tau''_m<0} B_{2,m}(r,\theta) + \sum_{-3<\mathrm{Im}\,\tau'''_m<0} C_{2,m}(r,\theta) + D_2(r,\theta) \tag{3.390b}$$

if $\omega \in (0, 2\pi)$ *and* ω *is a zero of* (3.345). *In* (3.389a,b), (3.390a,b),

$$\begin{aligned}
A_{1,m}(r,\theta) &= i\sqrt{2\pi}\, r^{i\tau'_m}[(1 + i\tau'_m)\sin\theta\,\psi_m(\theta) + \cos\theta\,\psi'_m(\theta)],\\
A_{2,m}(r,\theta) &= i\sqrt{2\pi}\, r^{i\tau'_m}[(1 + i\tau'_m)\cos\theta\,\psi_m(\theta) - \sin\theta\,\psi'_m(\theta)],\\
B_{1,m}(r,\theta) &= i\sqrt{2\pi}\, r^{i\tau''_m}[(1 + i\tau''_m)(\sin\theta)(\varphi_m(\theta) + i\ln r\,\psi_m(\theta))\\
&\qquad + i\sin\theta\,\psi_m(\theta) + \cos\theta\,\varphi'_m(\theta) + i\ln r\cos\theta\,\psi'_m(\theta)],\\
B_{2,m}(r,\theta) &= i\sqrt{2\pi}\, r^{i\tau''_m}[(1 + i\tau''_m)(\cos\theta)(\varphi_m(\theta) + i\ln r\,\psi_m(\theta))\\
&\qquad + i\cos\theta\,\psi_m(\theta) - \sin\theta\,\varphi'_m(\theta) - i\ln r\sin\theta\,\psi'_m(\theta)],
\end{aligned}$$

$$C_{1,m}(r,\theta) = i\sqrt{2\pi} r^{i\tau_m'''}[(1 + i\tau_m''')(\sin\theta)(\Phi_m(\theta) + i\ln r\varphi_m(\theta) - \frac{(\ln r)^2}{2}\psi_m(\theta)) + i\sin\theta\varphi_m(\theta) - \ln r\sin\theta\psi_m(\theta) + \cos\theta\Phi_m'(\theta) + i\ln r\cos\theta\varphi_m'(\theta) - \frac{(\ln r)^2}{2}\cos\theta\psi_m'(\theta)],$$

$$C_{2,m}(r,\theta) = i\sqrt{2\pi} r^{i\tau_m'''}[(1 + i\tau_m''')(\cos\theta)(\Phi_m(\theta) + i\ln r\varphi_m(\theta) - \frac{(\ln r)^2}{2}\psi_m(\theta)) + i\cos\theta\varphi_m(\theta) - \ln r\cos\theta\psi_m(\theta) - \sin\theta\Phi_m'(\theta) - i\ln r\sin\theta\varphi_m'(\theta) + \frac{(\ln r)^2}{2}\sin\theta\psi_m'(\theta)],$$

$$D_1(r,\theta) = i\sqrt{\frac{2}{\pi}} r^3[4\psi_{-3i}(\theta)\sin\theta\,\mathrm{Si}(\ln r) + \psi_{-3i}(\theta)\sin\theta\frac{\sin(\ln r)}{\ln r} + \psi_{-3i}'(\theta)\cos\theta\,\mathrm{Si}(\ln r)],$$

$$D_2(r,\theta) = i\sqrt{\frac{2}{\pi}} r^3[4\psi_{-3i}(\theta)\cos\theta\,\mathrm{Si}(\ln r) + \psi_{-3i}(\theta)\cos\theta\frac{\sin(\ln r)}{\ln r} - \psi_{-3i}'(\theta)\sin\theta\,\mathrm{Si}(\ln r)]$$

and the real parts of u and v are considered relevant. Also, $u_{reg} \in H^4$ *and* $v_{reg} \in H^4$ *in a neighborhood of the corner* S_l*,* $\psi_{-3i}(\theta)$ *and* $\mathrm{Si}(\ln r)$ *are defined in Lemma 3.29, the functions* $\psi_m(\theta)$*,* $\varphi_m(\theta)$*, and* $\Phi_m(\theta)$ *are given in Lemma 3.28, and* τ_m'*,* τ_m''*, and* τ_m''' *are defined in Theorem 3.15.*

3.5.5 Some Specific Regularity Results

In this subsection, we will calculate the local singularities of the solution to the bipolar equations around a corner with angle measurement $\omega = \frac{3\pi}{2}$ and prove Theorem 3.14; this angle is of particular importance in many research areas such as the study of fluid flow around a corner (see [Mo]), the study of drag reduction by using rectangular riblets (see [Wa]), and the study of the effect of rectangular riblets on the laminar to turbulent transition of flow over a flat plate in a water tunnel (see [LRRH]).

From Theorem 3.16, we can calculate the local singularities of the solution to the bipolar equations for a specific interior angle measurement by determining the zeros of (3.352), calculating the multiplicity of these zeros, finding the functions

$\psi_m(\theta)$, $\varphi_m(\theta)$, and $\Phi_m(\theta)$ defined in Lemma 3.27, and calculating the dimension of the solution space spanned by these functions; such computations are most easily performed by using a symbolic manipulation package such as Maple. For example, if we consider a domain which has a corner with an angle measurement of $\frac{3\pi}{2}$, and localize the problem near this corner, then letting $\omega = \frac{3\pi}{2}$ in Lemma 3.21 we obtain the following theorem:

Theorem 3.17. *For* $\omega = \frac{3\pi}{2}$ *the homogeneous problem* (3.342a,b) *has only the zero solution in the following cases: (i)* $\tau \neq 0, \pm i, -2i, -3i$ *and* τ *is not a root of the equation*

$$(64\tau^6 i - 384\tau^5 - 832\tau^4 i + 768\tau^3 + 112\tau^2 i + 288\tau + 144i)\cosh\frac{3\pi\tau}{2} + (144\tau^2 i - 288\tau - 144i)\cosh\frac{9\pi\tau}{2} = 0, \tag{3.391}$$

(ii) $\tau = 0$*, (iii)* $\tau = -i$*, and (iv)* $\tau = -2i$*. The roots of* (3.391) *are described by the following lemma:*

Lemma 3.31. *In the region* $-3 \leq \operatorname{Im}\tau \leq 0$*, equation* (3.391) *has the simple roots* $\tau = -\frac{7}{3}i, -\frac{5}{3}i, -i, -\frac{1}{3}i$ *and* $\tau \approx -2.88i, -2.56i, -1.39i, -1.27i, -.73i, -.61i,$ *and the double roots* $\tau = -3i, -2i, -i, 0$*; also, this equation has no root with multiplicity larger than two.*

It is now possible to calculate the values of τ for which $d(\tau) = 0$, where $d(\tau)$ is defined in (3.352) and, thus, establish the representation (3.336), (3.337) in Theorem 3.14. In fact, from Lemma 3.31, we obtain the following result:

Lemma 3.32. *In the region* $-3 \leq \operatorname{Im}\tau \leq 0$*,* $d(\tau)$ *has simple zeros at* $\tau = -3i$*,* $-\frac{7}{3}i$*,* $-2i$*,* $-\frac{5}{3}i$*,* $-i$*,* $-\frac{1}{3}i$ *and* $\tau \approx -2.88i$*,* $-2.56i$*,* $-1.39i$*,* $-1.27i$*,* $-.73i$*,* $-.61i$*, and no zeros with multiplicity larger than one.*

Proof (Theorem 3.14). As a consequence of Lemmas 3.31 and 3.32, we obtain the representation

$$\psi^* = \psi^*_{reg} + i\sqrt{2\pi}\big(r^{3.88}\psi_{-2.88i}(\theta) + r^{3.56}\psi_{-2.56i}(\theta) + r^{\frac{10}{3}}\psi_{-\frac{7}{3}i}(\theta) + r^{\frac{8}{3}}\psi_{-\frac{5}{3}i}(\theta) + r^{2.39}\psi_{-1.39i}(\theta) + r^{2.27}\psi_{-1.27i}(\theta) + r^{1.73}\psi_{-.73i}(\theta) + r^{1.61}\psi_{-.61i}(\theta) + r^{\frac{4}{3}}\psi_{-\frac{1}{3}i}(\theta)\big) + i\sqrt{\frac{2}{\pi}}\psi_{-3i}(\theta)r^4\,\mathrm{Si}(\ln r) \tag{3.392}$$

where $\psi^*_{reg} \in H^5(G)$. It can be verified that the solutions $\psi_\tau(\theta)$ where $\tau = -3i$, $-\frac{7}{3}i$, $-\frac{5}{3}i$, $-\frac{1}{3}i$, and $\tau \approx -2.88i, -2.56i, -1.39i, -1.27i, -.73i, -.61i$, span a one-dimensional space. For example, $\psi_{-3i}(\theta) = A_{-3i}i(1 - \cos\theta\cos 3\theta)$ and $\psi_{-\frac{7}{3}i}(\theta) = A_{-\frac{7}{3}i}\left(-\cos\frac{10\theta}{3} + \cos\frac{2\theta}{3}\right)$, where A_{-3i} and $A_{-\frac{7}{3}i}$ are real constants. Since we are assuming that $\psi^* \in H^3(G)$, we must have $A_{-\frac{1}{3}i} = A_{-.61i} = A_{-.73i} = 0$. Therefore,

$$\psi^* = \psi^*_{reg} + i\sqrt{2\pi}(r^{3.88}\psi_{-2.88i}(\theta) + r^{3.56}\psi_{-2.56i}(\theta) + r^{\frac{10}{3}}\psi_{-\frac{7}{3}i}(\theta) + r^{\frac{8}{3}}\psi_{-\frac{5}{3}i}(\theta)$$
$$+ r^{2.39}\psi_{-1.39i}(\theta) + r^{2.27}\psi_{-1.27i}(\theta)) + i\sqrt{\frac{2}{\pi}}\psi_{-3i}(\theta)r^4\,\mathrm{Si}(\ln r). \tag{3.393}$$

From (3.390a,b), in a neighborhood of a corner with interior angle $\omega = \frac{3\pi}{2}$, the solution $\boldsymbol{w} = (u, v)$ of (3.333a–d) has the form

$$u = u_{reg} - A_{1,-2.88i}(r,\theta) - A_{1,-2.56i}(r,\theta) - A_{1,-\frac{7}{3}i}(r,\theta) \tag{3.394a}$$
$$- A_{1,-\frac{5}{3}i}(r,\theta) - A_{1,-1.39i}(r,\theta) - A_{1,-1.27i}(r,\theta) + D_1(r,\theta),$$
$$v = v_{reg} + A_{2,-2.88i}(r,\theta) + A_{2,-2.56i}(r,\theta) + A_{2,-\frac{7}{3}i}(r,\theta) \tag{3.394b}$$
$$+ A_{2,-\frac{5}{3}i}(r,\theta) + A_{2,-1.39i}(r,\theta) + A_{2,-1.27i}(r,\theta) + D_2(r,\theta)$$

where the notation is the same as in Theorem 3.16. The conclusions in Theorem 3.14, i.e. (3.336), (3.337), now follow directly from (3.394a,b). □

For a further discussion of the influence of the rough boundaries on fluid flow see also [Sj] and the references therein.

Chapter 4
General Existence and Uniqueness Theorems for Incompressible Bipolar and Non-Newtonian Fluid Flow

4.1 Introduction

In Sect. 1.4 we introduced the equations which govern the motion of a nonlinear, incompressible, bipolar fluid. For a bounded domain in $\mathbb{R}^n$, $n = 2, 3$ the appropriate boundary conditions were set forth in Sect. 1.4 and, for flows in all of $\mathbb{R}^n$, the relevant periodic (boundary) conditions were also delineated. In both of the two previous chapters, various theorems pertaining to existence, uniqueness, and continuous dependence on the constitutive parameters were formulated and established, albeit for specific types of flows and geometries. Thus, in Chap. 2, we proved (i) existence, uniqueness, and continuous dependence results for steady plane Poiseuille flows of a viscous, incompressible, nonlinear bipolar fluid, (ii) uniqueness of the steady Poiseuille flow within a broad class of equilibrium flows between parallel plates, and (iii) existence, uniqueness, and asymptotic stability theorems for time-dependent Poiseuille flows of a bipolar fluid. In Chap. 3, theorems establishing existence and uniqueness of solutions for problems governing the motion of an incompressible, nonlinear, bipolar fluid were established for the cases of (i) flow between rotating cylinders, (ii) exterior flow around an obstacle in the plane with a smooth boundary, and (iii) flow over non-smooth boundaries (in particular, flow in polygonal domains).

In this chapter, existence and uniqueness results of a more general character will be developed. First, in Sect. 4.2 we obtain, via a Galerkin argument, the existence of a unique weak solution to the initial-boundary value problem for an incompressible bipolar viscous fluid satisfying nonhomogeneous boundary conditions. The domain in this section is a general open bounded domain $\Omega \subseteq \mathbb{R}^n$, $n = 2, 3$, with smooth boundary $\partial\Omega$. Regularity results are also established in Sect. 4.2 and the solution is proven to be asymptotically stable when the forcing function and the initial and boundary data decay in an appropriate sense. Section 4.3 is both the heart and soul of this chapter. In Sect. 4.3 we reconsider the problem of existence and uniqueness of solutions for the incompressible bipolar fluid for flows in $\mathbb{R}^n$, $n = 2, 3$, with associated space-periodic conditions; the appropriate space-periodic conditions are

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introduced at the end of Sect. 4.2. The proof is constructed in such a manner as to allow for the conclusion that for $p > 1$, when $n = 2$, and for $p > 6/5$, when $n = 3$, the weak limit of a sequence $\{\boldsymbol{v}^N\}$ of unique weak solutions, corresponding to a sequence $\{\mu_1^N\}$ of higher-order viscosities, is a measure-valued solution of the initial-boundary value problem with $\mu_1 = 0$. In the formulation of the flow problem in Sect. 4.3 we weaken the constitutive hypothesis (2.1a), where $\alpha = 2 - p$, assuming only that for $\mu_1 = 0$, the viscous part of the stress tensor satisfies a growth condition of the form $\left|\tau_{ij}(\boldsymbol{e})\right| \leq C(1 + |\boldsymbol{e}|)^{p-1}$, for some $C > 0$. Then, we prove that the measure-valued solutions are weak solutions, when $\mu_1 = 0$, for $3/2 < p < 2$ (in $\dim n = 2$) and for $9/5 < p < 11/5$ (in $\dim n = 3$). Finally, it is shown that the measure-valued solutions are, in fact, unique regular weak solutions for $p \geq 2$ (in $\dim n = 2$) and for $p \geq 11/5$ (in $\dim n = 3$).

In Sect. 4.4 we return to the problem of flow of an incompressible, nonlinear, bipolar fluid in an unbounded, parallel-wall channel, i.e., in the domain Ω_a defined by (2.164). Existence of solutions for this problem is established by first considering a sequence of approximate solutions in bounded subdomains of Ω_a; we then prove that there exists a subsequence of such approximate solutions, whose limit is a weak solution of the initial-boundary value problem, and that the solution is unique. In the last section of this chapter, Sect. 4.5, we recall the results in some of the work on existence and uniqueness for the Navier–Stokes equations that has appeared in the literature, as well as the related work for some of the generalizations of the Navier–Stokes model which have been described in Sect. 1.6.

4.2 Existence, Uniqueness, and Stability of Solutions to the Initial-Boundary Value Problem for Bipolar Viscous Fluids

4.2.1 Introduction

In this section, we consider the most general initial-boundary value problem associated with a bipolar viscous fluid; we will study the problem of existence, uniquenss and stability, and will allow for general boundary conditions, i.e., we will not simply restrict our analysis to the non-slip boundary condition and will also allow for non-zero moments of the tractions on the boundary of the domain (see also [NR] and [Po]).

Let $\Omega \subseteq R^n$, $n = 2, 3$, be an open bounded domain with smooth boundary $\partial\Omega$. We recall here the form of the initial-boundary value for the bipolar viscous fluid as previously delineated in Sect. 1.4:

$$\rho\frac{\partial v_i}{\partial t} + \rho v_j\frac{\partial v_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} - 2\mu_1\frac{\partial}{\partial x_j}\Delta e_{ij}(\boldsymbol{v}) + \frac{\partial}{\partial x_j}(\gamma(\boldsymbol{v})e_{ij}(\boldsymbol{v})) + \rho F_i, \text{ in } \Omega \times (0, T), \tag{4.1}$$

$$\operatorname{div} \boldsymbol{v} = 0, \text{ in } \Omega \times (0, T), \tag{4.2}$$

$$\tau_{ijk}(\boldsymbol{v})\nu_j \nu_k - \tau_{jkl}(\boldsymbol{v})\nu_j \nu_k \nu_l \nu_i = (M_k \tau_k)\tau_i, \text{ on } \partial\Omega \times (0, T), \tag{4.3}$$

$$\boldsymbol{v} = \boldsymbol{g}, \text{ on } \partial\Omega \times (0, T), \tag{4.4}$$

$$\boldsymbol{v}(\boldsymbol{x}, 0) = \boldsymbol{v}_0(\boldsymbol{x}), \text{ in } \Omega \tag{4.5}$$

where $\boldsymbol{v}$ is the velocity field associated with the flow of an incompressible bipolar fluid, p is the pressure, F_i, $\boldsymbol{g}$, M_i and $\boldsymbol{v}_0$ are given functions, the smoothness of which will be specified later, and e_{ij} is the rate of deformation tensor, i.e.,

$$e_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \tag{4.6}$$

Also, in (4.1), ρ is the density (which we set equal to one, without loss of generality) and $\gamma(\boldsymbol{v})$ is given by

$$\gamma(\boldsymbol{v}) = 2\mu_0(\epsilon + e_{ij}(\boldsymbol{v})e_{ij}(\boldsymbol{v}))^{-\alpha/2} = 2\mu(|\boldsymbol{e}(\boldsymbol{v})|) \tag{4.7}$$

while μ_0, μ_1, and ϵ are positive constants and $\alpha \in (0, 1)$. Finally, in (4.3) the τ_{ijk} are the components of the first multipolar stress tensor

$$\tau_{ijk}(\boldsymbol{v}) = 2\mu_1 \frac{\partial e_{ij}(\boldsymbol{v})}{\partial x_k}. \tag{4.8}$$

We will assume in our work that the field $\boldsymbol{g}$ given in (4.4) satisfies

$$\int_{\partial\Omega} \boldsymbol{g}(\cdot, t) ds = 0 \tag{4.9}$$

for every t. This is a necessary and sufficient condition [HB] to ensure the existence of a vector field $\tilde{\boldsymbol{v}}$ such that

$$\tilde{\boldsymbol{v}} = \boldsymbol{g} \text{ on } \partial\Omega \times (0, T), \tag{4.10}$$

$$\operatorname{div} \tilde{\boldsymbol{v}} = 0 \text{ in } \Omega \times (0, T). \tag{4.11}$$

This section is organized as follows: in Sect. 4.2.2 we define the notion of weak solution for the initial-boundary value problem (4.1)–(4.5); we then implement the Galerkin method to prove the existence of a weak solution. In 4.2.3 two regularity results are proven for the weak solution whose existence was established in Sect. 4.2.2. The uniqueness of the weak solution to (4.1)–(4.5) is proven in Sect. 4.2.4. Finally, with appropriate assumptions relative to the data, the asymptotic stability of the solution to the initial-boundary value problem is demonstrated in Sect. 4.2.5.

4.2.2 Existence of a Weak Solution

To prove the existence of a solution to the problem (4.1)–(4.5), we will start by formulating the equations for the corresponding weak solution. The existence of a weak solution will be established using the Galerkin method. We begin by introducing the space

$$\boldsymbol{H} = \{\boldsymbol{u} \in \boldsymbol{W}^{2,2}(\Omega) \cap \boldsymbol{W}_0^{1,2}(\Omega) \mid \operatorname{div} \boldsymbol{u} = 0\} \tag{4.12}$$

Next, we define a weak solution to our problem to be a function $\boldsymbol{v}$ such that $\boldsymbol{v} - \tilde{\boldsymbol{v}} \in L^{\infty}_{loc}(0,\infty; \boldsymbol{L}^2(\Omega)) \cap L^2_{loc}(0,\infty; \boldsymbol{H}) \cap W^{-1,2}_{loc}(0,\infty; \boldsymbol{W}^{4,2}(\Omega))$, satisfies

$$\begin{aligned}
-\int_0^\infty \int_{\Omega_t} \frac{\partial \psi_i}{\partial t} v_i \, d\boldsymbol{x}\, dt + \int_\Omega \psi_i(\boldsymbol{x},0) v_i(\boldsymbol{x},0)\, d\boldsymbol{x} + 2\mu_1 \int_0^\infty \int_{\Omega_t} \frac{\partial e_{ij}(\boldsymbol{v})}{\partial x_k} \frac{\partial e_{ij}(\boldsymbol{\psi})}{\partial x_k} d\boldsymbol{x}\, dt \\
= \int_0^\infty \int_{\Omega_t} v_j \frac{\partial \psi_i}{\partial x_j} v_i \, d\boldsymbol{x}\, dt - \int_0^\infty \int_{\Omega_t} \gamma(\boldsymbol{v}) e_{ij}(\boldsymbol{v}) \frac{\partial \psi_i}{\partial x_j} d\boldsymbol{x}\, dt \\
+ \int_0^\infty \int_{\Omega_t} F_i \psi_i \, d\boldsymbol{x}\, dt + \int_0^\infty \int_{\partial\Omega_t} (M_k \tau_k)\tau_i \frac{\partial \psi_i}{\partial \nu} d\boldsymbol{s}\, dt,
\end{aligned} \tag{4.13}$$

$\forall \boldsymbol{\psi} \in W^{1,2}(0,\infty; \boldsymbol{H})$, with compact support in $[0,\infty)$. The form of the boundary integral in (4.13) follows from the discussion in Appendix A.II which leads up to (B.33). For the remainder of our work in this section we will set $\tilde{M}_i = (M_k \tau_k)\tau_i$.

In order to use the Galerkin method to prove the existence of the weak solution $\boldsymbol{w}$, we need to introduce a suitable basis; we define in $\boldsymbol{H}$ the scalar product

$$((\boldsymbol{w}, \boldsymbol{\psi})) = \int_\Omega \frac{\partial}{\partial x_k} e_{ij}(\boldsymbol{w}) \frac{\partial}{\partial x_k} e_{ij}(\boldsymbol{\psi})\, d\boldsymbol{x}$$

and denote by $(\boldsymbol{w}, \boldsymbol{\psi})_{\boldsymbol{L}^2(\Omega)}$ the usual $\boldsymbol{L}^2$ scalar product. We begin with

Lemma 4.1. *The eigenvalue problem*

$$((\boldsymbol{w}, \boldsymbol{\psi})) = \lambda (\boldsymbol{w}, \boldsymbol{\psi})_{\boldsymbol{L}^2(\Omega)}, \quad \forall \boldsymbol{\psi} \in \boldsymbol{H} \tag{4.14}$$

has a sequence of solutions $\mathcal{W}^l \in \boldsymbol{H} \cap \boldsymbol{C}^\infty(\Omega)$ corresponding to a sequence of positive eigenvalues λ_l. Furthermore

(i) the sequence $\mathcal{W}^l$ is a basis for the closure of $\boldsymbol{H}$ under the $\boldsymbol{L}^2$ norm;
(ii) the sequence $\mathcal{W}^l$ is a basis of $\boldsymbol{H}$;
(iii) $\left(\mathcal{W}^l, \mathcal{W}^k\right)_{\boldsymbol{L}^2(\Omega)} = \delta_{lk}$.

Proof. This is a standard consequence of Lemma B.2 in Appendix B and classical results in the spectral theory of operators in Hilbert spaces (see [Bre, Yos] or [Kat] for example). □

As we have indicated we will prove the existence of a weak solution using the Galerkin technique. We start by assembling some technical results. For l fixed, let $\boldsymbol{w}^l \in \boldsymbol{E}_l = \text{span}\{\boldsymbol{\mathcal{W}}^l \cdots \boldsymbol{\mathcal{W}}^l\}$ and let

$$\boldsymbol{w}^l(\boldsymbol{x},t) = \sum_{k=1}^{l} C_{l,k}(t)\boldsymbol{\mathcal{W}}^k(\boldsymbol{x})$$

be the solution of

$$\int_{\Omega_t} \frac{\partial w_i^l}{\partial t}\psi_i\,d\boldsymbol{x} + 2\mu_1 \int_{\Omega_t} \frac{\partial e_{ij}(\boldsymbol{w}^l)}{\partial x_k}\frac{\partial e_{ij}(\boldsymbol{\psi})}{\partial x_k}d\boldsymbol{x} = \int_{\Omega_t} v_j^l \frac{\partial v_i^l}{\partial x_j}\psi_i\,d\boldsymbol{x} - \int_{\Omega_t} \gamma(\boldsymbol{v}^l)e_{ij}(\boldsymbol{v}^l)\frac{\partial \psi_i}{\partial x_j}d\boldsymbol{x}$$
$$+ \int_{\Omega_t} F_i\psi_i\,d\boldsymbol{x} - \int_{\Omega_t}\frac{\partial \tilde{v}_i}{\partial t}\psi_i\,d\boldsymbol{x} + \int_{\Omega_t}\frac{\partial e_{ij}(\tilde{\boldsymbol{v}})}{\partial x_k}\frac{\partial e_{ij}(\boldsymbol{\psi})}{\partial x_k}d\boldsymbol{x} + \int_{\partial\Omega_t}\tilde{M}_i\frac{\partial \psi_i}{\partial \nu}d\boldsymbol{s}, \tag{4.15}$$

$\forall \boldsymbol{\psi} \in \boldsymbol{E}_l$, where $\boldsymbol{v}^l = \boldsymbol{w}^l + \tilde{\boldsymbol{v}}$.

The nonlinear system of ordinary differential equations for the coefficients $C_{l,k}(t)$ generated by (4.15) satisfies the conditions of Picard's theorem because of the regularity of $\gamma(\boldsymbol{v})$; hence, this system, along with the initial conditions

$$C_{l,k}(0) = \int_{\Omega_t}(\boldsymbol{v}_0 - \tilde{\boldsymbol{v}})\cdot\boldsymbol{\mathcal{W}}^l\,d\boldsymbol{x}, \quad \forall(l,k) \tag{4.16}$$

has a unique local solution on an interval $[0, T_l]$.

We will now proceed with proving some a priori estimates for $\boldsymbol{w}^l$. In order to state our first a priori estimate we introduce the energy $E^l(t)$ defined by

$$E^l(t) = \frac{1}{2}\left|\left|\boldsymbol{w}^l\right|\right|^2_{\boldsymbol{L}^2(\Omega_t)} + \sigma\int_0^t \left|\left|\boldsymbol{w}^l\right|\right|^2_{\boldsymbol{W}^{2,2}(\Omega_t)}dt, \quad \sigma > 0. \tag{4.17}$$

For the energy $E^l(t)$ we now establish the following

Lemma 4.2. *There exists $c > 0$, and $\hat{c} > 0$, such that, for all l,*

$$E^l(t) \le \hat{c}e^{ct}, \quad \forall t \ge 0. \tag{4.18}$$

Proof. Set $\boldsymbol{\psi} = \boldsymbol{w}^l$ in (4.15) and sum over i; it then follows that

$$\frac{1}{2}\frac{\partial}{\partial t}\int_{\Omega_t}\left|\left|\boldsymbol{w}^l\right|\right|^2 d\boldsymbol{x} + 2\mu_1\int_{\Omega_t}\frac{\partial e_{ij}(\boldsymbol{w}^l)}{\partial x_k}\frac{\partial e_{ij}(\boldsymbol{w}^l)}{\partial x_k}d\boldsymbol{x} = -b(\boldsymbol{v}^l,\boldsymbol{v}^l,\boldsymbol{w}^l)$$
$$- \int_{\Omega_t}\gamma(\boldsymbol{v}^l)e_{ij}(\boldsymbol{v}^l)e_{ij}(\boldsymbol{w}^l)d\boldsymbol{x} + \int_{\Omega_t}\boldsymbol{F}\cdot\boldsymbol{w}^l\,d\boldsymbol{x} - \int_{\Omega_t}\frac{\partial\tilde{\boldsymbol{v}}}{\partial t}\cdot\boldsymbol{w}^l\,d\boldsymbol{x}$$
$$- \mu_1\int_{\Omega_t}\frac{\partial e_{ij}(\tilde{\boldsymbol{v}})}{\partial x_k}\frac{\partial e_{ij}(\boldsymbol{w}^l)}{\partial x_k}d\boldsymbol{x} + \int_{\partial\Omega_t}\tilde{M}_i\frac{\partial w_i^l}{\partial\nu}d\boldsymbol{s}, \tag{4.19}$$

where $b(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v}) = \int_{\Omega_t} u_j \frac{\partial w_i}{\partial x_j} v_i \, d\boldsymbol{x}$. Since $\boldsymbol{v}^l = \boldsymbol{w}^l + \tilde{\boldsymbol{v}}$,

$$b(\boldsymbol{v}^l, \boldsymbol{v}^l, \boldsymbol{w}^l) = b(\boldsymbol{w}^l, \boldsymbol{w}^l, \boldsymbol{w}^l) + b(\tilde{\boldsymbol{v}}, \boldsymbol{w}^l, \boldsymbol{w}^l) + b(\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{v}}, \boldsymbol{w}^l) + b(\boldsymbol{w}^l, \tilde{\boldsymbol{v}}, \boldsymbol{w}^l). \tag{4.20}$$

However, $\boldsymbol{w}^l$ is divergence free, and $\boldsymbol{w}^l = 0$ on $\partial\Omega_t = \partial\Omega \times [0, t)$, so an easy calculation shows that $b(\boldsymbol{w}^l, \boldsymbol{w}^l, \boldsymbol{w}^l) = 0$. Similarly, $b(\tilde{\boldsymbol{v}}, \boldsymbol{w}^l, \boldsymbol{w}^l) = 0$. Hence,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Omega_t} \left|\left|\boldsymbol{w}^l\right|\right|^2 d\boldsymbol{x} + 2\mu_1 \int_{\Omega_t} \frac{\partial e_{ij}(\boldsymbol{w}^l)}{\partial x_k}\frac{\partial e_{ij}(\boldsymbol{w}^l)}{\partial x_k} d\boldsymbol{x} + \int_{\Omega_t} \gamma(\boldsymbol{v}^l) e_{ij}(\boldsymbol{w}^l) e_{ij}(\boldsymbol{w}^l) d\boldsymbol{x} \\
&\quad = -\int_{\Omega_t} \gamma(\boldsymbol{v}^l) e_{ij}(\tilde{\boldsymbol{v}}) e_{ij}(\boldsymbol{w}^l) d\boldsymbol{x} + \int_{\Omega_t} \boldsymbol{F} \cdot \boldsymbol{w}^l d\boldsymbol{x} - b(\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{v}}, \boldsymbol{w}^l) - b(\boldsymbol{w}^l, \tilde{\boldsymbol{v}}, \boldsymbol{w}^l) \\
&\qquad - \int_{\Omega_t} \frac{\partial \tilde{\boldsymbol{v}}}{\partial t} \cdot \boldsymbol{w}^l d\boldsymbol{x} - \mu_1 \int_{\Omega_t} \frac{\partial e_{ij}(\tilde{\boldsymbol{v}})}{\partial x_k}\frac{\partial e_{ij}(\boldsymbol{w}^l)}{\partial x_k} d\boldsymbol{x} + \int_{\partial\Omega_t} \tilde{M}_i \frac{\partial w_i^l}{\partial \nu} d\boldsymbol{s}.
\end{aligned} \tag{4.21}$$

Thus, for some generic $c > 0$,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Omega_t} \left|\left|\boldsymbol{w}^l\right|\right|^2 d\boldsymbol{x} + 2\mu_1 \int_{\Omega_t} \frac{\partial e_{ij}(\boldsymbol{w}^l)}{\partial x_k}\frac{\partial e_{ij}(\boldsymbol{w}^l)}{\partial x_k} d\boldsymbol{x} + \int_{\Omega_t} \gamma(\boldsymbol{v}^l) e_{ij}(\boldsymbol{w}^l) e_{ij}(\boldsymbol{w}^l) d\boldsymbol{x} \\
&\quad \le (c\,||\tilde{\boldsymbol{v}}||_{\boldsymbol{W}^{1,2}(\Omega_t)} + ||\boldsymbol{F}||_{\boldsymbol{W}^{-1,2}(\Omega_t)} + ||\tilde{\boldsymbol{v}}_t||_{\boldsymbol{W}^{1,2}(\Omega_t)}) \left|\left|\boldsymbol{w}^l\right|\right|_{\boldsymbol{W}^{1,2}(\Omega_t)} \\
&\qquad + (c\,||\tilde{\boldsymbol{v}}||_{\boldsymbol{W}^{2,2}(\Omega_t)} + \left|\left|\tilde{M}_i\right|\right|_{W^{-1/2,2}(\partial\Omega_t)}) \left|\left|\boldsymbol{w}^l\right|\right|_{\boldsymbol{W}^{2,2}(\Omega_t)} \\
&\qquad + \left|b(\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{v}}, \boldsymbol{w}^l)\right| + \left|b(\boldsymbol{w}^l, \tilde{\boldsymbol{v}}, \boldsymbol{w}^l)\right|.
\end{aligned} \tag{4.22}$$

Using the fact that $||\tilde{\boldsymbol{v}}||_{\boldsymbol{L}^\infty(\Omega_t)} \le C\,||\tilde{\boldsymbol{v}}||_{\boldsymbol{W}^{2,2}(\Omega_t)}$, for some $C > 0$, we find that

$$\left|b(\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{v}}, \boldsymbol{w}^l)\right| \le C\,||\tilde{\boldsymbol{v}}||^2_{\boldsymbol{W}^{2,2}(\Omega_t)} \cdot \left|\left|\boldsymbol{v}^l\right|\right|_{\boldsymbol{W}^{1,2}(\Omega_t)}.$$

Also

$$\begin{aligned}
&\left|b(\boldsymbol{w}^l, \tilde{\boldsymbol{v}}, \boldsymbol{w}^l)\right| ||\tilde{\boldsymbol{v}}||_{\boldsymbol{L}^\infty(\Omega_t)} \left[\int_{\Omega_t} \left|\boldsymbol{w}^l\right| \cdot \left|\nabla \boldsymbol{w}^l\right| d\boldsymbol{x}\right] \\
&\qquad \le C\,||\tilde{\boldsymbol{v}}||_{\boldsymbol{W}^{2,2}(\Omega_t)} \left|\left|\boldsymbol{w}^l\right|\right|_{\boldsymbol{L}^2(\Omega_t)} \left|\left|\boldsymbol{w}^l\right|\right|_{\boldsymbol{W}^{2,2}(\Omega_t)}.
\end{aligned} \tag{4.23}$$

The inequality in (4.22) then yields, for some $\theta > 0$,

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_t} \left|\left|\boldsymbol{w}^l\right|\right|^2 d\boldsymbol{x} + 2\mu_1 \int_{\Omega_t} \frac{\partial e_{ij}(\boldsymbol{w}^l)}{\partial x_k}\frac{\partial e_{ij}(\boldsymbol{w}^l)}{\partial x_k} d\boldsymbol{x} + \int_{\Omega_t} \gamma(\boldsymbol{v}^l) e_{ij}(\boldsymbol{w}^l) e_{ij}(\boldsymbol{w}^l) d\boldsymbol{x}$$
$$\leq \Big(c\,||\tilde{\boldsymbol{v}}||_{\boldsymbol{W}^{1,2}(\Omega_t)} + ||\boldsymbol{F}||_{\boldsymbol{W}^{-1,2}(\Omega_t)} + ||\tilde{\boldsymbol{v}}_t||_{\boldsymbol{W}^{1,2}(\Omega_t)} + c\,||\tilde{\boldsymbol{v}}||_{\boldsymbol{W}^{2,2}(\Omega_t)}$$
$$+ c\,||\tilde{\boldsymbol{v}}||_{\boldsymbol{W}^{2,2}(\Omega_t)} \left|\left|\tilde{M}_i\right|\right|_{W^{-1/2,2}(\partial\Omega_t)}\Big)^2 + 2\theta^{-1}\left(C\,||\tilde{\boldsymbol{v}}||_{\boldsymbol{W}^{2,2}(\Omega_t)}\left|\left|\boldsymbol{w}^l\right|\right|_{\boldsymbol{L}^2}\right)^2. \tag{4.24}$$

Using the inequality $|xy| \leq 1/2(x^2/\theta + y^2\theta)$ on the right-hand side of (4.24), as well as Lemma B.2 to estimate the second term on the left-hand side, and then dropping the non-negative third term on the left-hand side, we find that (4.24) yields, for some $\sigma > 0$

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_t} \left|\left|\boldsymbol{w}^l\right|\right|^2 d\boldsymbol{x} + \sigma\left|\left|\boldsymbol{w}^l\right|\right|^2_{\boldsymbol{W}^{2,2}(\Omega_t)}$$
$$\leq \theta^{-1}(c\,||\tilde{\boldsymbol{v}}||_{\boldsymbol{W}^{1,2}(\Omega_t)} + ||\boldsymbol{F}||_{\boldsymbol{W}^{-1,2}(\Omega_t)} + ||\tilde{\boldsymbol{v}}_t||_{\boldsymbol{W}^{1,2}(\Omega_t)} + c\,||\tilde{\boldsymbol{v}}||_{\boldsymbol{W}^{2,2}(\Omega_t)}$$
$$+ ||M_i||_{W^{-1/2,2}(\partial\Omega_t)} + C\,||\tilde{\boldsymbol{v}}||_{\boldsymbol{W}^{2,2}(\Omega_t)}\left|\left|\boldsymbol{w}^l\right|\right|_{\boldsymbol{L}^2})^2$$
$$\leq 2\theta^{-1}(c\,||\tilde{\boldsymbol{v}}||_{\boldsymbol{W}^{1,2}(\Omega_t)} + ||\boldsymbol{F}||_{\boldsymbol{W}^{-1,2}(\Omega_t)} + ||\tilde{\boldsymbol{v}}_t||_{\boldsymbol{W}^{1,2}(\Omega_t)}$$
$$+ c\,||\tilde{\boldsymbol{v}}||_{\boldsymbol{W}^{2,2}(\Omega_t)}\left|\left|\tilde{M}_i\right|\right|_{W^{-1/2,2}(\partial\Omega_t)})^2 + 2\theta^{-1}(C\,||\tilde{\boldsymbol{v}}||_{\boldsymbol{W}^{2,2}(\Omega_t)}\left|\left|\boldsymbol{w}^l\right|\right|_{\boldsymbol{L}^2(\Omega_t)})^2. \tag{4.25}$$

Therefore,

$$\frac{dE^l(t)}{dt} \leq aE^l(t) + b \tag{4.26}$$

where

$$a = \frac{C}{\sigma}\sup_{t\geq 0}\left(||\tilde{\boldsymbol{v}}||^2_{\boldsymbol{W}^{2,2}(\Omega_t)}\right) \tag{4.27}$$

and

$$b = \frac{C}{\sigma}\sup_{t\geq 0}\Big(c\,||\tilde{\boldsymbol{v}}||_{\boldsymbol{W}^{1,2}(\Omega_t)} + ||\boldsymbol{F}||_{\boldsymbol{W}^{-1,2}(\Omega_t)} + ||\tilde{\boldsymbol{v}}_t||_{\boldsymbol{W}^{1,2}(\Omega_t)} \tag{4.28}$$
$$+ c\,||\tilde{\boldsymbol{v}}||_{\boldsymbol{W}^{2,2}(\Omega_t)} + \left|\left|\tilde{M}_i\right|\right|_{W^{-1/2,2}(\partial\Omega_t)}\Big)^2.$$

The lemma is then a direct consequence of Gronwall's inequality. (See [Hen].) □

The purpose of the next lemma is to be able to use Theorem 5.1 of [Lio1] to obtain strong convergence of $\boldsymbol{w}^l$ in a suitable space.

Lemma 4.3. *The norm of* $\dfrac{d\boldsymbol{w}^l}{dt}$ *in* $L^{3/2}(0,T;\boldsymbol{W}^{-2,2})$ *is bounded independently of* l.

Proof. Let $\boldsymbol{u} \in L^3(0,T;\boldsymbol{W}_0^{2,2}(\Omega))$, choose $s > 2$ (s will be specified later), and set $\boldsymbol{u}^l = \boldsymbol{P}_l(\boldsymbol{u})$ where $\boldsymbol{P}_l$ is the projection operator onto the space $E_l = \text{span}\{\mathcal{W}^l \cdots \mathcal{W}^l\}$. We then have that

$$\begin{aligned}
\int_0^T \int_{\Omega_t} \frac{d\boldsymbol{w}^l}{dt} \cdot \boldsymbol{u}\, d\boldsymbol{x}\, dt &= \int_0^T \int_{\Omega_t} \frac{d\boldsymbol{w}^l}{dt} \cdot \boldsymbol{u}^l\, d\boldsymbol{x}\, dt \\
&\equiv -\int_0^T \int_{\Omega_t} \frac{d\tilde{\boldsymbol{v}}}{dt} \cdot \boldsymbol{u}^l\, d\boldsymbol{x}\, dt + \int_0^T \int_{\Omega_t} \boldsymbol{v}_j^l \frac{\partial \boldsymbol{v}_i^l}{\partial x_j} \boldsymbol{u}_i^l\, d\boldsymbol{x}\, dt \\
&\quad - 2\mu_1 \int_0^T \int_{\Omega_t} \frac{\partial e_{ij}(\boldsymbol{v}^l)}{\partial x_k} \frac{\partial e_{ij}(\boldsymbol{u}^l)}{\partial x_k}\, d\boldsymbol{x}\, dt - \int_0^T \int_{\Omega_t} \gamma(\boldsymbol{v}) e_{ij}(\boldsymbol{v}^l) \frac{\partial u_i^l}{\partial x_j}\, d\boldsymbol{x}\, dt \\
&\quad + \int_0^T \int_{\Omega_t} F_i u_i^l\, d\boldsymbol{x}\, dt \equiv I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned} \tag{4.29}$$

Using Lemma 4.2 it is easy to see that for some $\bar{C} > 0$,

$$|I_1 + I_3 + I_4 + I_5| \le \bar{C} \left|\left|\boldsymbol{u}^l\right|\right|_{L^2(0,T;\boldsymbol{W}^{2,2}(\Omega))} \le \bar{C} \left|\left|\boldsymbol{u}^l\right|\right|_{L^p(0,T;\boldsymbol{W}_0^{2,2}(\Omega))}. \tag{4.30}$$

Let p' be the conjugate of p; then by virtue of the Hölder Inequality

$$|I_2| = \left| -\int_0^T \int_{\Omega_t} \boldsymbol{v}_j^l \boldsymbol{v}_i^l \frac{\partial u_i^l}{\partial x_j}\, d\boldsymbol{x}\, dt \right| \le \int_0^t \left(\int_{\Omega_t} \left|\left|\boldsymbol{v}^l\right|\right|^{2p'} d\boldsymbol{x} \right)^{\frac{1}{2p'}} \left[\int_{\Omega_t} \left|\left|\nabla \boldsymbol{u}^l\right|\right|^p d\boldsymbol{x} \right]^{\frac{1}{p}} dt. \tag{4.31}$$

Recall that $\left|\left|\boldsymbol{D}\boldsymbol{u}^l\right|\right|_{\boldsymbol{L}^p(\Omega)} \le \bar{c}\, ||\boldsymbol{u}||_{W^{2,2}(\Omega)}$, for $p \le \dfrac{2n}{n-2}$, and some $\bar{c} > 0$. Also

$$\left(\int_{\Omega_t} \left|\left|\boldsymbol{v}^l\right|\right|^{2p'} d\boldsymbol{x} \right) \le \left(\int_{\Omega_t} \left|\left|\boldsymbol{v}^l\right|\right|^2 d\boldsymbol{x} \right)^{\frac{1}{2p'}} \cdot \left(\int_{\Omega_t} \left|\left|\boldsymbol{v}^l\right|\right|^{4p'-2} d\boldsymbol{x} \right)^{\frac{1}{2p'}}. \tag{4.32}$$

By Lemma 4.2 we have that $\int_{\Omega_t} \left|\left|\boldsymbol{v}^l\right|\right|^2 d\boldsymbol{x}$ is uniformly bounded in $L^\infty(0,T)$; hence, for some $\hat{C}(T) > 0$,

$$|I_2| \le \hat{C} \int_0^t \left(\int_{\Omega_t} \left| \left|\left|\boldsymbol{v}^l\right|\right|^{4p'-2} \right| d\boldsymbol{x} \right)^{\frac{1}{2p'}} \cdot ||\boldsymbol{u}||_{\boldsymbol{W}^{2,2}(\Omega)}\, dt, \quad 0 < t < T.$$

We are now interested in estimating

$$\int_0^t \left(\int_{\Omega_t} \left|\left| \boldsymbol{v}^l \right|\right|^{4p'-2} d\boldsymbol{x} \right)^{\frac{s'}{2p'}} dt. \tag{4.33}$$

For this purpose we use the inequality $\left(\int_{\Omega_t} \left|\left| \boldsymbol{v}^l \right|\right|^{4p'-2} d\boldsymbol{x} \right)^{\frac{1}{4p'-2}} \leq \hat{c} \left|\left| (\boldsymbol{v}^l) \right|\right|_{\boldsymbol{W}^{2,2}(\Omega)}$, for some $\hat{c} > 0$, which is valid provided $p > n/2 - 1$, from Lemma 4.2 it then follows that

$$\int_0^t \left(\int_{\Omega_t} \left|\left| \boldsymbol{v}^l \right|\right|^{4p'-2} d\boldsymbol{x} \right)^{\frac{2}{4p'-2}} dt \leq \hat{c} e^{cT}. \tag{4.34}$$

Therefore, for $\dfrac{s'}{2p'} = \dfrac{2}{4p'-2}$, we have that

$$\int_0^t \left(\int_{\Omega_t} \left|\left| \boldsymbol{v}^l \right|\right|^{4p'-2} d\boldsymbol{x} \right)^{\frac{s'}{2p'}} dt \leq \hat{c} e^{cT}. \tag{4.35}$$

For $n \leq 3$, the choice $p = 3$ satisfies all the restrictions cited above and yields $s' = 3/2$, $s = 3$. Hence, for $\tilde{c} > 0$, $\tilde{C} > 0$,

$$\int_0^T \int_{\Omega_t} \frac{d\boldsymbol{w}^l}{dt} \cdot \boldsymbol{u}\, d\boldsymbol{x}\, dt \leq \tilde{C}\, ||\boldsymbol{u}||_{L^3(0,T,\boldsymbol{W}_0^{2,3}(\Omega))} \tag{4.36}$$

and

$$\left|\left| \frac{d\boldsymbol{w}^l}{dt} \right|\right|_{L^{3/2}(0,T;\boldsymbol{W}^{-2,3}(\Omega))} \leq \tilde{c}. \tag{4.37}$$

□

We are now in a position to state and prove the main result of this subsection, namely,

Theorem 4.1. *Assume that*

(i) $\tilde{\boldsymbol{v}}$ *is in* $L^\infty((0,\infty); \boldsymbol{W}^{2,2}(\Omega))$,
(ii) $\tilde{\boldsymbol{v}}_t$ *is in* $L^2_{loc}([0,\infty); \boldsymbol{W}^{-1,2}(\Omega))$,
(iii) $\boldsymbol{F}$ *is in* $L^2_{loc}([0,\infty); \boldsymbol{W}^{1,2}(\Omega))$,
(iv) $\tilde{M}_i$ *is in* $L^2_{loc}([0,\infty); \boldsymbol{W}^{-1/2,2}(\partial\Omega))$,
(v) $\boldsymbol{h}$ *is in* $\boldsymbol{L}^2(\Omega)$,
(vi) there exists $c > 0$ *and* $\hat{c} > 0$ *such that*

$$\int_0^t \left(||\boldsymbol{F}||_{\boldsymbol{W}^{-1,2}(\Omega_t)} + ||\boldsymbol{v}_t||_{W^{-1,2}(\Omega_t)} + \left|\left| \tilde{M}_i \right|\right|_{W^{-1/2,2}(\partial\Omega_t)} \right)^2 dt \leq \hat{c} e^{ct}. \tag{4.38}$$

Then the problem (4.1)–(4.5) *has a weak solution* $\boldsymbol{v}$ *in the sense of* (4.13) *which satisfies*

$$\frac{1}{2}||\boldsymbol{v}||_{L^2(\Omega_t)} + \sigma \int_0^t ||\boldsymbol{v}||^2_{\boldsymbol{W}^{2,2}(\Omega_t)} \, dt \leq \hat{c} e^{ct}. \tag{4.39}$$

Proof. From Lemma 4.2 it follows that the sequence $\boldsymbol{w}^l$ has a convergent subsequence, denoted again by $\boldsymbol{w}^l$, which is convergent in $L^2_{loc}([0,\infty); \boldsymbol{W}^{2,2}(\Omega))$ to a function $\boldsymbol{w}$. Furthermore,

$$\frac{1}{2}\,||\boldsymbol{w}||_{L^2(\Omega_t)} + \sigma \int_0^t ||\boldsymbol{w}||^2_{\boldsymbol{W}^{2,2}(\Omega_t)} \, dt \leq \hat{c} e^{ct}. \tag{4.40}$$

Using Lemmas 4.2, 4.3 and Theorem 5.1 of [Lio1] we then have that the sequence $\boldsymbol{w}^l$ is compact in $L^2((0,T); \boldsymbol{W}^{s,2}(\Omega))$, $\forall s < 2$. Hence there exists a subsequence, denoted again by $\boldsymbol{w}^l$, which converges, strongly, to $\boldsymbol{w}$ in $L^2((0,T); W^{s,2}(\Omega))$. From the compactness of the embedding of Sobolev spaces (see Appendix A) it then follows that $(\boldsymbol{w}^l + \tilde{\boldsymbol{v}})_j(\boldsymbol{w}^l + \tilde{\boldsymbol{v}})_i$ converges weakly in $L^2_{loc}([0,\infty); \boldsymbol{L}^2(\Omega))$ and strongly in $L^1_{loc}([0,\infty); \boldsymbol{L}^2(\Omega))$; thus its limit is $(\boldsymbol{w} + \tilde{\boldsymbol{v}})_j(\boldsymbol{w} + \tilde{\boldsymbol{v}})_i$. Similarly, $\gamma(\boldsymbol{v}^l)e_{ij}(\boldsymbol{v}^l)$ converges strongly in $L^2((0,T); \boldsymbol{L}^2(\Omega))$ to $\gamma(\boldsymbol{v})e_{ij}(\boldsymbol{v})$.

Letting l go to infinity in (4.15) it then follows that $\forall \boldsymbol{\psi} \in W^{1,2}(0,\infty; \boldsymbol{H})$, with compact support in $[0,\infty)$,

$$\begin{aligned}
-\int_0^\infty \int_{\Omega_t} \frac{\partial \psi_i}{\partial t} v_i \, d\boldsymbol{x}\, dt + \int_\Omega \psi_i(x,0) h_i(x)\, d\boldsymbol{x} + 2\mu_1 \int_0^\infty \int_{\Omega_t} \frac{\partial e_{ij}(\boldsymbol{v})}{\partial x_k} \frac{\partial e_{ij}(\psi)}{\partial x_k} d\boldsymbol{x}\, dt \\
= \int_0^\infty \int_{\Omega_t} v_j \frac{\partial \psi_i}{\partial x_j} v_i \, d\boldsymbol{x}\, dt + \int_0^\infty \int_{\Omega_t} \gamma(\boldsymbol{v}) e_{ij}(\boldsymbol{v}) \frac{\partial \psi_i}{\partial x_j} d\boldsymbol{x}\, dt \\
+ \int_0^\infty \int_{\Omega_t} F_i \psi_i \, d\boldsymbol{x}\, dt + \int_0^\infty \int_{\partial\Omega_t} \tilde{M}_i \frac{\partial \psi_i}{\partial \nu} d\boldsymbol{s}\, dt
\end{aligned}$$

where $\boldsymbol{v} = \boldsymbol{w} + \tilde{\boldsymbol{v}}$. □

Remarks. The above result is still valid, and the proof is essentially unchanged, if instead of assumption (4.7) one requires that γ be a positive decreasing function and that $\lim_{s \to 0} s\gamma(s) < \infty$. In particular, one can set $\epsilon = 0$ in (4.7).

4.2.3 Regularity of the Solution

In this section we will establish two regularity results for the solution whose existence was proven in Sect. 4.2.2; we will also make precise the sense in which the solution of the variational problem satisfies the boundary-value problem. Our first result shows that if we assume additional regularity of the data, for $t > 0$, we do get additional regularity of the solution for $t > 0$. This is made precise by the following theorem:

Theorem 4.2. *Assume that*

(i) $\tilde{\boldsymbol{v}}$ *is in* $L^\infty((0,\infty); \boldsymbol{W}^{2,2}(\Omega)) \cap W^{-1,2}((0,\infty); \boldsymbol{W}^{4,2}(\Omega))$,
(ii) $\tilde{\boldsymbol{v}}_t$ *is in* $L^2_{loc}(([0,\infty)); \boldsymbol{L}^2(\Omega))$,
(iii) $\boldsymbol{F}$ *is in* $L^2_{loc}([0,\infty); \boldsymbol{L}^2(\Omega))$,
(iv) $\tilde{M}_i$ *is in* $W^{1,2}_{loc}([0,\infty); W^{-1/2,2}(\partial\Omega))$.

Then Problem (4.1)–(4.5) *has a weak solution* $\boldsymbol{v}$ *in the sense of* (4.13) *which satisfies*

(i) $\boldsymbol{v} \in W^{1,2}_{loc}((0,\infty); \boldsymbol{L}^2(\Omega))$,
(ii) $\boldsymbol{v} \in L^\infty_{loc}((0,\infty); \boldsymbol{H})$.

Proof. Set $\boldsymbol{\psi} = \boldsymbol{w}^l_t$ in (4.13). After integration by parts over (t_1, T) and summing over i, it follows that

$$\int_{t_1}^{T}\int_{\Omega_t}\left|\left|\frac{\partial \boldsymbol{v}^l}{\partial t}\right|\right|^2 d\boldsymbol{x}\,dt + \int_{\Omega_t}\gamma(\boldsymbol{v}^l)\,d\boldsymbol{x} + \mu_1\int_{\Omega_t}\frac{\partial e_{ij}(\boldsymbol{v}^l)}{\partial x_k}\frac{\partial e_{ij}(\boldsymbol{v}^l)}{\partial x_k}d\boldsymbol{x}$$
$$= -\int_{t_1}^{T}\int_{\Omega_t}\boldsymbol{v}^l_j\frac{\partial \boldsymbol{v}^l_i}{\partial x_j}(\boldsymbol{v}^l_i)d\boldsymbol{x}\,dt + \int_{\Omega_{t_1}}\bar{\gamma}(\boldsymbol{v}^l)\,d\boldsymbol{x} + \mu_1\int_{\Omega_{t_1}}\frac{\partial e_{ij}(\boldsymbol{v}^l)}{\partial x_k}\frac{\partial e_{ij}(\boldsymbol{v}^l)}{\partial x_k}\,d\boldsymbol{x}$$
$$+\int_{t_1}^{T}\int_{\Omega_t}\boldsymbol{F}\cdot\boldsymbol{v}^l_t\,d\boldsymbol{x}\,dt - \int_{t_1}^{T}\int_{\partial\Omega_t}\frac{\partial\tilde{M}_i}{\partial t}\frac{\partial\boldsymbol{v}^l_i}{\partial\nu}d\boldsymbol{s}\,dt + \int_{\partial\Omega_\tau}\tilde{M}_i\frac{\partial\boldsymbol{v}^l_i}{\partial\nu}\,d\boldsymbol{s} - \int_{\partial\Omega_{t_1}}\tilde{M}_i\frac{\partial\boldsymbol{v}^l_i}{\partial\nu}\,d\boldsymbol{s} \tag{4.41}$$

where $\bar{\gamma}(\boldsymbol{v}) = \dfrac{1}{2-\alpha}(\epsilon + e_{ij}e_{ij})^{(2-\alpha)/2}$. Hence, for some $C > 0$,

$$\int_{t_1}^{T}||\boldsymbol{v}^l_t||^2_{\boldsymbol{L}^2(\Omega_t)}dt + ||\boldsymbol{v}^l||^2_{\boldsymbol{W}^{2,2}(\Omega_t)}$$
$$\leq C\left(||\boldsymbol{v}^l||^2_{\boldsymbol{W}^{2,2}(\Omega_{t_1})} + \int_{t_1}^{t}||\boldsymbol{F}||^2_{\boldsymbol{L}^2(\Omega_t)}\,d\tau + \int_{t_1}^{t}||\boldsymbol{v}^l\,\boldsymbol{D}\boldsymbol{v}^l||^2_{\boldsymbol{L}^2(\Omega_t)}d\tau\right.$$
$$\left. + \int_{t_1}^{t}\left|\left|\frac{\partial M_i}{\partial t}\right|\right|^2_{W^{-1/2,2}(\partial\Omega_t)}d\tau + ||\tilde{M}_i||^2_{W^{-1/2,2}(\partial\Omega_t)} + ||\tilde{M}_i||^2_{W^{-1/2,2}(\partial\Omega_{t_1})}\right)$$
$$\leq A(T) + C\,||\boldsymbol{v}^l||^2_{\boldsymbol{W}^{2,2}(\Omega_{t_1})} + C\int_{t_1}^{t}||\boldsymbol{v}^l\,\boldsymbol{D}\boldsymbol{v}^l||^2_{\boldsymbol{L}^2(\Omega_t)}\,d\tau \tag{4.42}$$

where

$$A(T) = C\left(||\boldsymbol{F}||^2_{L^2((0,T);\boldsymbol{L}^2(\Omega))} + \int_0^T \left|\left|\frac{\partial \tilde{M}_i}{\partial t}\right|\right|_{W^{-1/2,2}(\partial\Omega_t)} d\tau \right.$$
$$\left. + \ ||\tilde{M}_i||^2_{W^{1,2}((0,T);W^{-1/2,2}(\partial\Omega))}\right). \tag{4.43}$$

Next, we estimate the last term on the right-hand side of (4.42) as follows:

$$\int_{t_1}^t \left|\left|\boldsymbol{v}^l \boldsymbol{D}\boldsymbol{v}^l\right|\right|^2_{\boldsymbol{L}^2(\Omega_t)} d\tau \le C\int_{t_1}^t \left|\left|\boldsymbol{v}^l\right|\right|^2_{\boldsymbol{W}^{2,2}(\Omega_t)} \left|\left|\boldsymbol{D}\boldsymbol{v}^l\right|\right|^2_{\boldsymbol{L}^2(\Omega_t)} d\tau$$
$$\le \epsilon C\int_{t_1}^t \left|\left|\boldsymbol{v}^l\right|\right|^4_{\boldsymbol{W}^{2,2}(\Omega_t)} d\tau + C\int_{t_1}^t \left|\left|\boldsymbol{v}^l\right|\right|^2_{\boldsymbol{W}^{2,2}(\Omega_t)} \left|\left|\boldsymbol{v}^l\right|\right|^2_{\boldsymbol{L}^2(\Omega_t)} d\tau$$

where standard embedding results (see Appendix A) and the estimate

$$||\boldsymbol{D}\boldsymbol{v}^l||^2_{\boldsymbol{L}^2(\Omega_t)} \le \epsilon\, ||\boldsymbol{v}^l||^2_{\boldsymbol{W}^{2,2}(\Omega_t)} + C(\epsilon)\, ||\boldsymbol{v}^l||^2_{\boldsymbol{L}^2(\Omega_t)} \tag{4.44}$$

were used. Hence, $\forall\epsilon > 0$, we have

$$\int_{t_1}^t ||\boldsymbol{v}^l \boldsymbol{D}\boldsymbol{v}^l||^2_{\boldsymbol{L}^2(\Omega_t)} d\tau \le \epsilon\left(\sup_{t_1\le t\le T} ||\boldsymbol{v}^l||^2_{\boldsymbol{W}^{2,2}(\Omega_t)}\right)\int_{t_1}^t ||\boldsymbol{v}^l||^2_{\boldsymbol{W}^{2,2}(\Omega_t)}$$
$$+ C(\epsilon)\left(\sup_{t_1\le t\le T} ||\boldsymbol{v}^l||^2_{\boldsymbol{L}^2(\Omega_t)}\right)\int_{t_1}^t ||\boldsymbol{v}^l||^2_{\boldsymbol{W}^{2,2}(\Omega_t)} d\tau. \tag{4.45}$$

Using Lemma 4.2, (4.45) with a small enough ϵ, and (4.42), we find that

$$\int_0^T ||\boldsymbol{v}^l_t||^2_{\boldsymbol{L}^2(\Omega_t)} dt + ||\boldsymbol{v}^l||^2_{\boldsymbol{W}^{2,2}(\Omega_t)}$$
$$\le \frac{1}{2}\sup_{t_1\le t\le T} ||\boldsymbol{v}^l||^2_{\boldsymbol{W}^{2,2}(\Omega_t)} + A(T) + C\, ||\boldsymbol{v}^l||^2_{\boldsymbol{W}^{2,2}(\Omega_{t_1})} \tag{4.46}$$

where $A(T)$ is given by (4.43) and depends only on $\boldsymbol{F}$, $\tilde{M}_i$, $\boldsymbol{g}$. Taking the $\sup_t$ in the above inequality we then find that

$$\int_{t_1}^T ||\boldsymbol{v}^l_t||^2_{\boldsymbol{L}^2(\Omega_t)} dt + \sup_{t_1\le t\le T} ||\boldsymbol{v}^l||^2_{\boldsymbol{W}^{2,2}(\Omega_t)} \le 2A(T) + C\, ||\boldsymbol{v}^l||^2_{\boldsymbol{W}^{2,2}(\Omega_{t_1})}. \tag{4.47}$$

After integration with respect to t_1, over the interval $(s, 2s)$, we then have

$$\int_{2s}^{T} ||\boldsymbol{v}_t^l||^2_{L^2(\Omega_t)} dt + \sup_{2s \le t \le T} ||\boldsymbol{v}^l||^2_{\boldsymbol{W}^{2,2}(\Omega_t)} \le 2A(T) + \frac{C}{s} \int_0^T ||\boldsymbol{v}^l||^2_{\boldsymbol{W}^{2,2}(\Omega_t)} dt. \tag{4.48}$$

Using Lemma 4.2, again, and letting l go to ∞ completes the proof. □

Next we show that if the additional regularity of the data extends to $t = 0$, so does the regularity of the solution. The content of this second regularity result is expressed by the following:

Theorem 4.3. *Assume that*

(i) $\tilde{\boldsymbol{v}}$ *is in* $L^\infty((0,\infty); \boldsymbol{W}^{2,2}(\Omega)) \cap W^{-1,2}((0,\infty); \boldsymbol{W}^{4,2}(\Omega))$,
(ii) $\tilde{\boldsymbol{v}}_t$ *is in* $L^2_{loc}([0,\infty); \boldsymbol{L}^2(\Omega))$,
(iii) $\boldsymbol{F}$ *is in* $L^2_{loc}([0,\infty); \boldsymbol{L}^2(\Omega))$,
(iv) $\tilde{M}_i$ *is in* $W^{1,2}_{loc}([0,\infty); W^{-1/2,2}(\Omega))$,
(v) $\boldsymbol{v}_0$ *is in* $\boldsymbol{H}$ *and* $\boldsymbol{v}_0 - \tilde{\boldsymbol{v}}(\boldsymbol{x}, 0)$ *is in* $\boldsymbol{W}^{1,2}_0(\Omega)$.

Then problem (4.1)–(4.5) *has a weak solution* $\boldsymbol{v}$ *in the sense of* (4.13) *which satisfies*

(i) $\boldsymbol{v} \in W^{1,2}_{loc}([0,\infty); \boldsymbol{L}^2(\Omega))$,
(ii) $\boldsymbol{v} \in L^\infty_{loc}([0,\infty); \boldsymbol{H})$.

Proof. Set $t_1 = 0$ in the proof of Theorem 4.2. □

Finally, we can relate the solution of the variational problem to that of the boundary-value problem. Specifically, we need to account for the fact that the test functions used are divergence free.

Theorem 4.4. *Under the assumptions of Theorem 4.3, the weak solution* $\boldsymbol{v}$ *of problem* (4.1)–(4.5) *that was constructed in Theorem 4.1 satisfies*

(i) $\boldsymbol{v} \in W^{1,2}_{loc}([0,\infty); \boldsymbol{H}')$,
(ii) there exists $p(x,t) \in L^2_{loc}((0,\infty); W^{-1,2}(\Omega))$ *such that* $(\boldsymbol{v}, p)$ *is a solution of* (4.1)–(4.5).

Proof. These results are direct consequences of (4.41), Theorems 4.1 and 4.2, and results in [Te1]. □

4.2.4 Uniqueness of the Weak Solution

We establish in this section the uniqueness of the weak solution constructed in Sect. 4.2.2; specifically, we will prove

Theorem 4.5. *Under the assumptions in Theorem 4.3, the weak solution of the problem (4.1)–(4.5) is unique.*

Proof. Assume that we have two distinct weak solutions $\boldsymbol{v}, \boldsymbol{u}$ of our problem and let $\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{u}$. Taking the difference of the equations satisfied by $\boldsymbol{v}$ and $\boldsymbol{u}$, multiplying the resulting equation by $\boldsymbol{w}$ and integrating by parts over Ω_t we find that

$$\int_{\Omega_t} \frac{\partial \boldsymbol{w}}{\partial t} \cdot \boldsymbol{w}\, d\boldsymbol{x} + \int_{\Omega_t} (\gamma(\boldsymbol{v}) e_{ij}(\boldsymbol{v}) - \gamma(\boldsymbol{u}) e_{ij}(\boldsymbol{u}))((e_{ij})(\boldsymbol{v}) - (e_{ij})(\boldsymbol{u}))\, d\boldsymbol{x}$$
$$+ 2\mu_1 \int_{\Omega_t} \frac{\partial e_{ij}(\boldsymbol{w})}{\partial x_k} \frac{\partial e_{ij}(\boldsymbol{w})}{\partial x_k}\, d\boldsymbol{x} = \int_{\Omega_t} \left(v_j \frac{\partial v_i}{\partial x_j} - u_j \frac{\partial u_i}{\partial x_j} \right) w_i\, d\boldsymbol{x}. \quad (4.49)$$

From the monotonicity of γ it follows that the second term on the left-hand side of (4.49) is positive and can be dropped. A simple calculation shows that

$$\int_{\Omega_t} \left(v_j \frac{\partial v_i}{\partial x_j} - u_j \frac{\partial u_i}{\partial x_j} \right) w_i\, d\boldsymbol{x} = \int_{\Omega_t} w_j \frac{\partial w_i}{\partial x_j} v_i\, d\boldsymbol{x}. \quad (4.50)$$

However, for some $C > 0$, and any $\epsilon > 0$,

$$\left| \int_{\Omega_t} w_j \frac{\partial w_i}{\partial x_j} v_i\, d\boldsymbol{x} \right| \le ||\boldsymbol{v}||_{L^\infty} \left(\int_{\Omega_t} \left(\frac{\partial w_i}{\partial x_j} \right)^2 d\boldsymbol{x} \right)^{1/2} \left(\int_{\Omega_t} (w_j)^2\, d\boldsymbol{x} \right)^{1/2}$$
$$\le \epsilon C\, ||\boldsymbol{v}||^2_{\boldsymbol{W}^{2,2}}\, ||\boldsymbol{w}||_{\boldsymbol{W}^{2,2}} + \frac{1}{\epsilon}\, ||\boldsymbol{w}||_{\boldsymbol{L}^2} \quad (4.51)$$

so choosing ϵ small enough it then follows from (4.49), and the Korn-type inequality of Lemma B.2, that

$$\frac{d}{dt}\, ||\boldsymbol{w}||_{\boldsymbol{L}^2(\Omega_t)} \le ||\boldsymbol{w}||_{\boldsymbol{L}^2(\Omega_t)}\,.$$

As $\boldsymbol{w} = \boldsymbol{0}$ at $t = 0$, it now follows that $\boldsymbol{w}(\boldsymbol{x}, t) = 0$, $\forall t > 0$. □

4.2.5 Stability of the Solution

Finally, in this last subsection, we will prove some estimates which establish the asymptotic stability of the solution $\boldsymbol{v}$ of (4.1)–(4.5) under an appropriate set of conditions on $\boldsymbol{F}$, $\boldsymbol{g}$, $\boldsymbol{M}$, and $\boldsymbol{v}_0$. We begin with

Lemma 4.4. *There exists, $\forall \delta > 0$, a function $\boldsymbol{G}_\delta$ such that*

(i) $\boldsymbol{G}_\delta = \boldsymbol{g}$ *on* $\partial\Omega \times (0, \infty)$,
(ii) $|b(\boldsymbol{v}, \boldsymbol{G}_\delta, \boldsymbol{v})| \le \delta\, ||\boldsymbol{v}||^2_{\boldsymbol{W}_0^{1,2}(\Omega)}$, $\forall \boldsymbol{v} \in \boldsymbol{W}_0^{1,2}(\Omega)$,
(iii) $\operatorname{div} \boldsymbol{G}_\delta = 0$ *and* $||\boldsymbol{G}_\delta||_{\boldsymbol{W}^{2,2}(\Omega)} \le C\, ||\boldsymbol{g}||_{\boldsymbol{W}^{2,2}(\Omega)}$, *for some* $C > 0$.

Proof. The lemma is a direct consequence of the results of Hopf in [Ho1]. □

Now, let $\boldsymbol{v}$ be the unique solution of problem (4.1)–(4.5) and set $\boldsymbol{w}_\delta = \boldsymbol{v} - \boldsymbol{G}_\delta$. Taking $\boldsymbol{w} = \boldsymbol{\psi}$ in (4.13), and for the sake of convenience, dropping the δ subscript on $\boldsymbol{G}_\delta$, we find that

$$\int_{\Omega_t} \frac{\partial w_i}{\partial t} w_i \, d\boldsymbol{x} + 2\mu_1 \int_{\Omega_t} \frac{\partial e_{ij}(\boldsymbol{w})}{\partial x_k} \frac{\partial e_{ij}(\boldsymbol{w})}{\partial x_k} d\boldsymbol{x} = \int_{\Omega_t} v_j \frac{\partial v_i}{\partial x_j} w_i \, d\boldsymbol{x} - \int_{\Omega_t} \gamma(\boldsymbol{v}) e_{ij}(\boldsymbol{v}) \frac{\partial w_i}{\partial x_j} \, d\boldsymbol{x}$$
$$+ \int_{\Omega_t} F_i w_i \, d\boldsymbol{x} - \int_{\Omega_t} \frac{\partial \tilde{v}_i}{\partial t} w_i \, d\boldsymbol{x} + 2\mu_1 \int_{\Omega_t} \frac{\partial e_{ij}(\boldsymbol{G})}{\partial x_k} \frac{\partial e_{ij}(\boldsymbol{w})}{\partial x_k} d\boldsymbol{x} + \int_{\partial\Omega_t} \tilde{M}_i \frac{\partial w_i}{\partial \nu} d\boldsymbol{s}.$$

Hence, for some $c > 0$,

$$\frac{1}{2}\frac{d}{dt} \int_{\Omega_t} ||\boldsymbol{w}||^2 \, d\boldsymbol{x} + 2\mu_1 \int_{\Omega_t} \frac{\partial e_{ij}(\boldsymbol{w})}{\partial x_k} \frac{\partial e_{ij}(\boldsymbol{w})}{\partial x_k} d\boldsymbol{x} + \int_{\Omega_t} \gamma(\boldsymbol{v}) e_{ij}(\boldsymbol{w}) e_{ij}(\boldsymbol{w}) \, d\boldsymbol{x}$$
$$\le (c\, ||\boldsymbol{G}||_{\boldsymbol{W}^{1,2}(\Omega_t)} + ||\boldsymbol{F}||_{\boldsymbol{W}^{-1,2}(\Omega_t)} + ||\boldsymbol{G}_t||_{\boldsymbol{W}^{1,2}(\Omega_t)}) \, ||\boldsymbol{w}||_{\boldsymbol{W}^{1,2}(\Omega_t)}$$
$$+ (c\, ||\boldsymbol{G}||_{\boldsymbol{W}^{2,2}(\Omega_t)} + \left|\left|\tilde{M}_i\right|\right|_{W^{-1/2,2}(\partial\Omega_t)}) \, ||\boldsymbol{w}||_{\boldsymbol{W}^{2,2}(\Omega_t)} + |b(\boldsymbol{v}, \boldsymbol{v}, \boldsymbol{w})|. \tag{4.52}$$

Since $b(\boldsymbol{v}, \boldsymbol{v}, \boldsymbol{w}) = b(\boldsymbol{w}, \boldsymbol{G}, \boldsymbol{w}) + b(\boldsymbol{G}, \boldsymbol{G}, \boldsymbol{w})$ and $G_i G_j \in L^2(\Omega_t)$, we have that

$$\begin{aligned} |b(\boldsymbol{v}, \boldsymbol{v}, \boldsymbol{w})| &\le |b(\boldsymbol{w}, \boldsymbol{G}, \boldsymbol{w})| + |b(\boldsymbol{G}, \boldsymbol{G}, \boldsymbol{w})| \\ &\le \delta\, ||\boldsymbol{v}||^2_{\boldsymbol{W}_0^{1,2}(\Omega)} + c \left|\left|G_i G_j\right|\right|_{L^2(\Omega)} ||\boldsymbol{v}||_{\boldsymbol{W}_0^{1,2}(\Omega)} \end{aligned} \tag{4.53}$$

where Lemma 4.4 has been used. It now follows from (4.52) that

$$\frac{1}{2}\frac{d}{dt} \int_{\Omega_t} ||\boldsymbol{w}||^2 \, d\boldsymbol{x} + 2\mu_1 \int_{\Omega_t} \frac{\partial e_{ij}(\boldsymbol{w})}{\partial x_k} \frac{\partial e_{ij}(\boldsymbol{w})}{\partial x_k} d\boldsymbol{x} + \int_{\Omega_t} \gamma(\boldsymbol{v}) e_{ij}(\boldsymbol{w}) e_{ij}(\boldsymbol{w}) \, d\boldsymbol{x}$$
$$\le \frac{1}{2\delta}(c\, ||\boldsymbol{G}||_{\boldsymbol{W}^{1,2}(\Omega_t)} + ||\boldsymbol{F}||_{\boldsymbol{W}^{-1,2}(\Omega_t)} + ||\boldsymbol{G}_t||_{\boldsymbol{W}^{1,2}(\Omega_t)}$$
$$+ c\, ||\boldsymbol{G}||_{\boldsymbol{W}^{2,2}(\Omega_t)} + \left|\left|\tilde{M}_i\right|\right|_{W^{-1/2,2}(\partial\Omega_t)} + c \left|\left|G_i G_j\right|\right|_{L^2(\Omega)})^2 + 2\delta\, ||\boldsymbol{w}||^2_{\boldsymbol{W}^{2,2}(\Omega_t)}. \tag{4.54}$$

Using Lemma 4.4, and dropping the positive term $\int_{\Omega_t} \gamma(\boldsymbol{v}) e_{ij}(\boldsymbol{w}) e_{ij}(\boldsymbol{w}) d\boldsymbol{x}$ on the left-hand side of (5.54), we find that, for δ small enough,

$$\frac{d}{dt} ||\boldsymbol{w}||^2_{\boldsymbol{L}^2(\Omega_t)} + \frac{\sigma}{2} ||\boldsymbol{w}||^2_{\boldsymbol{W}^{2,2}(\Omega_t)} \le c B(t) \tag{4.55}$$

where $\sigma > 0$ and

$$B(t) = \left(||\boldsymbol{F}||_{W^{-1,2}(\Omega_t)} + ||\boldsymbol{g}_t||_{W^{1,2}(\Omega_t)} + ||\boldsymbol{g}||_{W^{2,2}(\Omega_t)} + \left|\left|\tilde{M}_i\right|\right|_{W^{-1/2,2}(\partial\Omega_t)} \right)^2 . \tag{4.56}$$

As an immediate consequence of (4.55) we have the following:

Theorem 4.6. *Assume that $B(t) \in L^\infty(0,\infty)$; then the solution $\boldsymbol{v}$ of problem (4.1)–(4.5) is in $L^\infty((0,\infty);\boldsymbol{L}^2(\Omega))$; furthermore, if $B(t)$ decays exponentially to 0, then $||\boldsymbol{v}||_{L^2(\Omega)}$ also decays exponentially to 0.*

Remarks. In the next section we will consider the existence of weak and measure-valued solutions for incompressible bipolar fluids, with $\mu_1 = 0$, in the presence of periodic boundary conditions. For the bipolar problem, posed in the domain $\Omega = [0, L]^n$, $n = 2, 3$, $L > 0$, and satisfying space-periodic conditions, the appropriate conditions are as follows: Let $\boldsymbol{e}_j$ be the unit vector in the jth coordinate direction. Then, if $\mu_1 = 0$,

$$\begin{aligned} \boldsymbol{v}(\boldsymbol{0},t) &= \boldsymbol{v}(L\boldsymbol{e}_j,t), \quad t \geq 0, \\ \int_\Omega \boldsymbol{v}(\boldsymbol{x},t)\,d\boldsymbol{x} &= \boldsymbol{0}, \quad t \geq 0 \end{aligned} \tag{4.57a}$$

while, for $\mu_1 > 0$, we require that (4.57a) be satisfied and, in addition, for any vector $\boldsymbol{\tau}$ in the tangent space to $\partial\Omega$

$$\tau_{ijk}(\boldsymbol{v}(0,t))\nu_j\nu_k\tau_i = \tau_{ijk}(\boldsymbol{v}(L\boldsymbol{e}_j,t))\nu_j\nu_k\tau_i . \tag{4.57b}$$

4.3 Weak and Measure-Valued Solutions for Incompressible Bipolar Fluids with $\mu_1 = 0$

4.3.1 Introduction

In this section we consider the existence problem for the incompressible, bipolar viscous fluid with higher-order viscosity $\mu_1 = 0$; for ease of exposition, we focus on the space-periodic problem in a domain $\Omega = [0, L]^n$, $L > 0$, $n = 2, 3$. Our results are obtained by examining the limits of solutions of the corresponding bipolar fluid problems, with $\mu_1 > 0$, as $\mu_1 \to 0^+$. In lieu of tying the analysis to the specific form (1.90) of the nonlinear viscosity $\mu(|\boldsymbol{e}|)$, and the resultant ansatz that the lower-order residual stress tensor $\boldsymbol{\tau}_0^v$ satisfy

$$(\boldsymbol{\tau}_0^v)_{ij} = 2\mu(|\boldsymbol{e}|)e_{ij} = 2\mu_0(\epsilon + |\boldsymbol{e}|^2)^{p-2}e_{ij},$$

we broaden the discussion to include those $\boldsymbol{\tau}_0^v$ which have continuous components $(\boldsymbol{\tau}_0^v)_{ij}$ that satisfy the polynomial growth condition

$$\left|(\boldsymbol{\tau}_0^v)_{ij}(\boldsymbol{e})\right| \leq C(1+|\boldsymbol{e}|)^{p-1}$$

for some $C > 0$, with $1 < p < \infty$ in dimension $n = 2$, and $1 < p < 6$ in dimension $n = 3$. A lower bound of the form $(\boldsymbol{\tau}_0^v)_{ij}e_{ij} \geq c_1 |\boldsymbol{e}|^{p-1}$ is also needed in order to establish the existence of measure-valued solutions; such solutions are defined in Sect. 4.3.2; the existence of such Young measure-valued solutions is established in this section for $p > 1$ if $n = 2$ and for $p > 6/5$ if $n = 3$. The existence of weak solutions is then proven for $p > 3/2$ if $n = 2$ and for $p > 9/5$ when $n = 3$. Finally, the existence of unique regular weak solutions is established, in space dimension $n = 2$ for $p \geq 2$, and in space dimension $n = 3$ for $p \geq 11/5$ (see, also, [BdV1]).

4.3.2 Young Measure-Valued Solutions

Measure-valued solutions are a generalization of the concept of weak solutions. Measure-valued solutions rely on the existence of solutions associated with some probability measure; the particular case where the associated probability measure is the Dirac measure yields the classical concept of weak solution (see, e.g., [Ta1]).

We will begin by giving a brief heuristic introduction to the concept of Young measures. Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 1$, be an open bounded domain with smooth boundary (or, all of $\mathbb{R}^n$). Suppose we have a sequence of functions $u_n(\boldsymbol{x})$ which converge in the weak $*$ topology of $L^\infty(\Omega)$ to a function $u \in L^\infty(\Omega)$, and let f be a smooth bounded function defined on $\mathbb{R}^1$ (say, f is C^∞ and with compact support, for example). It is well-known that $v_n = f(u_n)$ will not necessarily converge to $f(u)$ in any sense. However, it turns out that there is a family of probability measures $\nu(\boldsymbol{x}, \lambda)$; $\boldsymbol{x} \in \Omega$, $\lambda \in \mathbb{R}^1$ such that:

$$f(u_n) \overset{*}{\rightharpoonup} \langle f(\lambda); \nu(\boldsymbol{x}, \lambda)\rangle$$

for a.e. $\boldsymbol{x} \in \Omega$, where $\langle\ ,\ \rangle$ denotes the duality between the space of continuous functions $C(\mathbb{R}^1)$ and the space of measures. We will clarify this, below, using more precise statements.

We now introduce the probability measure of L.C. Young [You] in the form proposed by J. Ball [BaJ]. To this end, we consider the space $C_0(\mathbb{R}^{n^2})$ of continuous functions f from $\mathbb{R}^{n^2} \to R^1$ which satisfy $\lim\limits_{|\boldsymbol{\lambda}|\to 0} f(\boldsymbol{\lambda}) = 0$. By the Riesz representation theorem we have that $(C_0(\mathbb{R}^{n^2}))^* = M(\mathbb{R}^{n^2})$, $M(\mathbb{R}^{n^2})$ being the space of Radon measures. Now, consider the Bochner space $L^1(Q_T; C_0(\mathbb{R}^{n^2}))$ where $Q_T = \Omega \times [0, T)$. It is proven in [Ed] that

$$(L^1(Q_T; C_0(\mathbb{R}^{n^2})))^* = L_w^\infty(Q_T; M(\mathbb{R}^{n^2}))$$

where $g \in L^{\infty}_{w}(Q_T; M(\mathbb{R}^{n^2}))$ means that $\forall f \in L^1(Q_T; C_0(\mathbb{R}^{n^2}))$ the function $\langle f(\boldsymbol{x},t), g(\boldsymbol{x},t)\rangle$ is measurable in Q_T and essentially bounded, with $\langle\, ,\, \rangle$ denoting the duality between $C_0(\mathbb{R}^{n^2})$ and $M(\mathbb{R}^{n^2})$. We have, in fact, the following:

Lemma 4.5. *Let $G \in (L^1(Q_T; C_0(\mathbb{R}^{n^2})))^*$ be a bounded, positive, linear function. Then there exists a unique $g \in L^{\infty}_{w}(Q_T; M(\mathbb{R}^{n^2}))$, with $g(\boldsymbol{x},t)$ a positive Radon measure, such that $\forall f \in L^1(Q_T; C_0(\mathbb{R}^{n^2}))$,*

$$\langle G, f\rangle = \int_{Q_T} \langle f(\boldsymbol{x},t), g(\boldsymbol{x},t)\rangle \, d\boldsymbol{x}\, dt$$

and

$$||G|| = \operatorname*{ess\,sup}_{Q_T} ||g(\boldsymbol{x},t)||_{M(\mathbb{R}^{n^2})}.$$

We now return to the weak solution of the initial-boundary value problem for the incompressible bipolar fluid, i.e., to the weak solution of (4.1)–(4.5). In (4.1)–(4.5) we will take $\rho = 1$ and, without loss of generality, set $\boldsymbol{g} = \boldsymbol{0}$, $M_i = 0$. For the corresponding problem for the non-Newtonian fluid, with $\mu_1 = 0$, the condition in (4.3) does not, of course, apply. Moreover, it is important to note that in [BBN2, BBN3], existence of a weak solution for the incompressible, bipolar initial-boundary value problem was established for a larger class of nonlinear viscosities than that which is specified by, say, (2.3), i.e.,

$$\mu(|\boldsymbol{e}|) = 2\mu_0(\epsilon + |\boldsymbol{e}|^2)^{\frac{p-2}{2}} \tag{4.58a}$$

for which the lower-order part of the fluid stress tensor is given by

$$\boldsymbol{\tau}_0^v = \mu(|\boldsymbol{e}|)\boldsymbol{e}. \tag{4.58b}$$

In fact, it was assumed in [BBN2, BBN3], that $\boldsymbol{\tau}_0^v$ has continuous components $(\boldsymbol{\tau}_0^v)_{ij}$ that satisfy the polynomial growth condition

$$\left|(\boldsymbol{\tau}_0^v)_{ij}(\boldsymbol{e})\right| \le C(1 + |\boldsymbol{e}|)^{p-1} \tag{4.59}$$

for some $C > 0$, with $1 < p < \infty$, in space dimension $n = 2$, and $1 < p < 6$ in space dimension $n = 3$. Relative to (4.58b) it is certainly true that

$$(\boldsymbol{\tau}_0^v)_{ij}(\boldsymbol{e})e_{ij} \ge 0. \tag{4.60}$$

However, in considering measure-valued solutions, in this subsection, we will require the stronger condition

$$(\boldsymbol{\tau}_0^v)_{ij}(\boldsymbol{e})e_{ij} \ge c_1\,|\boldsymbol{e}|^p \tag{4.61}$$

for some $c_1 > 0$. To further simplify the computations which follow we assume the existence of a potential function $\Gamma(\boldsymbol{e})$ for which

$$(\boldsymbol{\tau}_0^v)_{ij} = \frac{\partial \Gamma}{\partial e_{ij}}. \tag{4.62}$$

We assume that $\Gamma(\cdot)$ is once continuously differentiable in R^{n^2}, $\Gamma > 0$, $\Gamma(0) = 0$ and, for some $C > 0$,

$$\Gamma(\boldsymbol{e}) \le C(1 + |\boldsymbol{e}|)^p. \tag{4.63}$$

Clearly, Γ exists if we use the constitutive hypothesis (4.58a) and, in fact,

$$\Gamma(\boldsymbol{e}) = \mu_0 \int_0^{e_{ij}e_{ij}} (\epsilon + s)^{\frac{p-2}{2}}\, ds.$$

The structure of the potential (4.63) will figure prominently in our work in Chaps. 5 and 6.

We intend to show that the non-Newtonian ($p > 1$), monopolar ($\mu_1 = 0$) model has a measure-valued solution. We will focus here on the case of the periodic boundary conditions introduced in the remarks concluding Sect. 4.2. To obtain solutions for the case $\mu_1 = 0$, we intend to first set $\mu_1 = \mu_1^N$, prove the existence of a corresponding solution $\boldsymbol{v}^N$, and by an appropriate limit procedure, show that we recover a function $\boldsymbol{v}$ which is a solution to the problem with $\mu_1 = 0$. For the remainder of this section we will assume that

$$\Omega = [0, L]^n, \quad L > 0$$

and denote by $L^2_{per}(\Omega)$, $\boldsymbol{W}^{k,p}_{per}(\Omega)$ the indicated spaces of L-periodic functions in any space variable. Now, let $\boldsymbol{v}^N$ be the unique weak solution to the problem consisting of (4.1), (4.2), (4.5) and the periodic boundary conditions

$$\boldsymbol{v}^N(\boldsymbol{0}, t) = \boldsymbol{v}^N(L\boldsymbol{e}_j, t), \quad t \ge 0, \tag{4.64a}$$

$$\tau_{ijk}(\boldsymbol{v}^N(0,t))\nu_j\nu_k\tau_i = \tau_{ijk}(\boldsymbol{v}^N(L,t))\nu_j\nu_k\tau_i, \quad t \ge 0, \tag{4.64b}$$

$$\int_\Omega \boldsymbol{v}(\boldsymbol{x}, t)\, d\boldsymbol{x} = \boldsymbol{0}, \quad t \ge 0 \tag{4.64c}$$

where $\boldsymbol{e}_j$ is the unit vector in the jth coordinate direction, and $\boldsymbol{\tau}$ is any vector in the tangent space to $\partial\Omega$. Even though the proof of existence and uniqueness in Sect. 4.2 was carried out for Dirichlet-type boundary conditions, the proof for the space-periodic problem is very similar and we will not repeat it here. We will use the following notation in this section:

$$\boldsymbol{V} = \{\boldsymbol{v} \in \boldsymbol{W}^{1,p}_{per}(\Omega) \mid \operatorname{div} \boldsymbol{v} = 0 \text{ and } \boldsymbol{v} \text{ satisfies (4.64a,c)}\}.$$

We note that for every $N > 0$, $\boldsymbol{v}^N(\boldsymbol{x}, t)$ is a weak solution of (4.1), (4.2), (4.5), and (4.64a,b,c) if

$$\boldsymbol{v}^N(x,t) \in L^2((0,T); \boldsymbol{W}^{2,2}_{per}(\Omega)) \cap L^2((0,T); \boldsymbol{V})$$

and $\forall \boldsymbol{w} \in \boldsymbol{V} \cap \boldsymbol{W}^{2,2}_{per}(\Omega)$, a.e. on $(0,T)$, we have

$$\int_\Omega \dot{v}_i^N w_i \, d\boldsymbol{x} + \int_\Omega v_j^N \frac{\partial v_i^N}{\partial x_j} w_i \, d\boldsymbol{x} + \frac{1}{\rho}\int_\Omega (\tau_0^v)_{ij}(\boldsymbol{v}^N) e_{ij}(\boldsymbol{w}) \, d\boldsymbol{x} + \frac{2\mu_1^N}{\rho}\int_\Omega \frac{\partial e_{ij}(\boldsymbol{v}^N)}{\partial x_k}\frac{\partial e_{ij}(\boldsymbol{w})}{\partial x_k} \, d\boldsymbol{x} - \int_\Omega F_i w_i \, d\boldsymbol{x} = 0. \quad (4.65)$$

We will now obtain some estimates for $\boldsymbol{v}^N$, in the process making explicit the dependence of $\boldsymbol{v}^N$ on μ_1^N. The first two estimates are fairly standard and so we will only sketch the proofs.

Lemma 4.6. *There exists $c_1, c_2 > 0$, independent of μ_1^N, such that*

$$\int_{\Omega_t} ||\boldsymbol{v}^N||^2 \, d\boldsymbol{x} + c_1 \int_0^t ||\boldsymbol{v}^N||^p_{\boldsymbol{W}^{1,p}(\Omega)} \, d\tau + \mu_1^N \int_0^t ||\boldsymbol{v}^N||^2_{\boldsymbol{W}^{2,2}(\Omega)} \, d\tau \leq \int_{\Omega_0} ||\boldsymbol{v}^N||^2 \, d\boldsymbol{x} + c_2 \int_0^t ||\boldsymbol{F}||^2_{\boldsymbol{L}^2(\Omega)} \, d\tau. \quad (4.66)$$

Proof. This is obtained by setting $\boldsymbol{w} = \boldsymbol{v}^N$ in (4.65), integrating by parts in space, and finally integrating in time over $(0,t)$. □

Lemma 4.7. *There exists $c > 0$, independent of N, such that the weak solution $\boldsymbol{v}^N$ satisfies*

$$\left|\left|\frac{\partial \boldsymbol{v}^N}{\partial t}\right|\right|_{L^2((0,T);\boldsymbol{B}')} \leq c \quad (4.67)$$

where $\boldsymbol{B}'$ is the dual of $\boldsymbol{B} = \boldsymbol{V} \cap \boldsymbol{W}^{3,2}_{per}(\Omega)$.

Proof. From (4.65) we have

$$\int_\Omega \frac{\partial v_i^N}{\partial t} w_i \, d\boldsymbol{x} = -\int_\Omega v_j^N \frac{\partial v_i^N}{\partial x_j} w_i \, d\boldsymbol{x} + \int_\Omega (\tau_0^v)_{ij}(\boldsymbol{v}^N) e_{ij}(\boldsymbol{w}) \, d\boldsymbol{x} + \mu_1 \int_\Omega \frac{\partial e_{ij}(\boldsymbol{v}^N)}{\partial x_k} \cdot \frac{\partial e_{ij}(\boldsymbol{w})}{\partial x_k} \, d\boldsymbol{x} - \int_\Omega F_i w_i \, d\boldsymbol{x}. \quad (4.68)$$

We start by noticing that, by virtue of Sobolev embeddings, there exists a positive constant c such that

$$\sup_{\boldsymbol{x}\in\Omega}\left|\left|\frac{\partial \boldsymbol{w}}{\partial x_j}(\boldsymbol{x},t)\right|\right| \leq c\,||\boldsymbol{w}||_{\boldsymbol{W}^{3,2}(\Omega_t)}\,.$$

Hence

$$\begin{aligned}\left|\int_\Omega v_j^N\,\frac{\partial v_i^N}{\partial x_j} w_i\,d\boldsymbol{x}\right| &= \left|\int v_j\,v_i\,\frac{\partial w_i}{\partial x_j}\,d\boldsymbol{x}\right|\\ &\leq \sup_{\boldsymbol{x}}\left|\frac{\partial w_i}{\partial x_j}(\boldsymbol{x},t)\right|\int_\Omega ||\boldsymbol{v}||^2\,d\boldsymbol{x}\\ &\leq ||\boldsymbol{w}||_{W^{3,2}(\Omega_t)}\int_\Omega ||\boldsymbol{v}||^2\,d\boldsymbol{x}. \qquad (4.69)\end{aligned}$$

Also,

$$\begin{aligned}\left|\int_{\Omega_t}(\tau_0^v)_{ij}(\boldsymbol{v}^N)e_{ij}(\boldsymbol{w})\,d\boldsymbol{x}\right| &\leq \sup_{\boldsymbol{x}}\left|\frac{\partial w_i}{\partial x_j}(\boldsymbol{x},t)\right|\int_{\Omega_t}(\tau_0^v)_{ij}(\boldsymbol{v}^N)\,d\boldsymbol{x}\\ &\leq c||\boldsymbol{w}(\boldsymbol{x},t)||_{\boldsymbol{W}^{3,2}_{per}(\Omega_t)}\int_{\Omega_t}(\tau_0^v)(\boldsymbol{v}^N)\,d\boldsymbol{x}\\ &\leq c||\boldsymbol{w}(\boldsymbol{x},t)||_{\boldsymbol{W}^{3,2}_{per}(\Omega_t)}\int_{\Omega_t}||\nabla v||^{p-1}\,d\boldsymbol{x}\\ &\leq c||\boldsymbol{w}(\boldsymbol{x},t)||_{\boldsymbol{W}^{3,2}_{per}(\Omega_t)}\left(\int_{\Omega_t}||\nabla v||^p\,d\boldsymbol{x}\right)^{\frac{p-1}{p}}. \qquad (4.70)\end{aligned}$$

It then follows from (4.68), (4.69) and (4.70), after integration in time over $(0,t)$, that:

$$\begin{aligned}\left|\iint\frac{\partial v_i^N}{\partial t}w_i\,d\boldsymbol{x}\right| &\leq c\,||\boldsymbol{w}||_{L^2((0,t);\boldsymbol{W}^{3,2}_{per}(\Omega))}\left(\int_0^t\left(\int_\Omega ||\boldsymbol{v}||^2\,d\boldsymbol{x}\right)^2 dt\right)^{1/2}\\ &+ c\,||\boldsymbol{w}||_{L^p((0,t);\boldsymbol{W}^{3,2}_{per}(\Omega))}\int_0^t\int\left|\left|\nabla v^N\right|\right|^p d\boldsymbol{x}\,dt\\ &+ \mu_1\,||\boldsymbol{w}||_{L^2((0,t);\boldsymbol{W}^{2,2}(\Omega))}\left|\left|\boldsymbol{v}^N\right|\right|_{L^2((0,t);\boldsymbol{W}^{2,2}(\Omega))}\\ &+ c\,||\boldsymbol{F}||_{L^2((0,t);\boldsymbol{L}^2(\Omega))}\,||\boldsymbol{w}||_{L^2((0,t);\boldsymbol{L}^2(\Omega))}\,.\end{aligned}$$

Therefore, there exists $c > 0$ such that

$$\left|\left|\iint \frac{\partial v_i^N}{\partial t} w_i \, dx \, dt\right|\right| \le c \left[||\boldsymbol{w}||_{L^2((0,t);\boldsymbol{W}^{3,2}(\Omega))} + ||\boldsymbol{w}||_{L^p((0,t);\boldsymbol{W}^{3,2}(\Omega))}\right]$$
$$\le c \, ||\boldsymbol{w}||_{L^s((0,t);\boldsymbol{W}^{3,2}(\Omega))}$$

where $s = \max(2, p)$. Thus, $\left|\left|\frac{\partial v_i^N}{\partial t}\right|\right|_{L^{s'}((0,T);\boldsymbol{B}')}$ is bounded independently of μ_1^N, where s' is the conjugate of $s = \max(2, p)$. □

For what follows in this subsection we will need two lemmas from [BBN3], namely,

Lemma 4.8. *Let $\boldsymbol{B}_2$ be a Banach space and $\boldsymbol{B}_i$, $i = 0, 1$, separable, reflexive Banach spaces. Suppose $\boldsymbol{B}_0 \hookrightarrow\hookrightarrow \boldsymbol{B}_2 \hookrightarrow \boldsymbol{B}_1$ (where $\hookrightarrow$ denotes continuous embedding and $\hookrightarrow\hookrightarrow$ compact embedding). Let*

$$\boldsymbol{W} = \left\{ \boldsymbol{v} \in \boldsymbol{L}^{p_0}(I; \boldsymbol{B}_0) \mid \frac{d\boldsymbol{v}}{dt} \in \boldsymbol{L}^{p_1}(I; \boldsymbol{B}_1) \right\} \tag{4.71}$$

with $I \subset R^1$ a bounded interval, and where p_i, $i = 0, 1$ satisfies $1 < p_i < \infty$. Then $\boldsymbol{W} \hookrightarrow\hookrightarrow \boldsymbol{L}^{p_0}(I; \boldsymbol{B}_2)$.

Proof. See [Lio1], Lemma 5.2. □

Lemma 4.9. *Suppose that, for some $c > 0$,*

$$\begin{cases} ||\boldsymbol{v}^N||_{L^\infty(I;\boldsymbol{L}^2(\Omega))} \le c < \infty, \\ ||\boldsymbol{v}^N||_{L^{p_1}(I;\boldsymbol{W}^{1,p_1}(\Omega))} \le c < \infty, \ p_1 > \dfrac{2n}{n+2} \end{cases} \tag{4.72a}$$

and

$$\left|\left|\frac{d}{dt}\boldsymbol{v}^N\right|\right|_{L^{s'}(I,B')} \le c \tag{4.72b}$$

where $s' = \min\left\{2, \dfrac{p}{p-1}\right\}$. If $\boldsymbol{v}^N \rightharpoonup \boldsymbol{v}$ in $\boldsymbol{L}^2(Q_T)$, then $\boldsymbol{v}^N \to \boldsymbol{v}$ in $\boldsymbol{L}^2(Q_T)$, i.e., if $\boldsymbol{v}^N$ converges weakly to $\boldsymbol{v}$ in $\boldsymbol{L}^2(Q_T)$ it also converges strongly in $\boldsymbol{L}^2(Q_T)$.

Proof. We will use Lemma 4.8 with $\boldsymbol{B}_0 = \boldsymbol{W}^{1,p_1}(\Omega)$, $\boldsymbol{B}_1 = \boldsymbol{B}'$ and $\boldsymbol{B}_2 = \boldsymbol{L}^2(\Omega)$. It then follows that $\boldsymbol{v}^N$ converges strongly in $L^{p_1}(0, T; L^2(\Omega))$. Here we made use of the fact that for $p_1 \ge \dfrac{2n}{n+2}$, $\boldsymbol{W}^{1,p_1} \hookrightarrow\hookrightarrow \boldsymbol{L}^2(\Omega)$. If $p_1 > 2$ then it immediately follows that $\boldsymbol{v}^N$ converges strongly in $L^2(0, T; \boldsymbol{L}^2(\Omega))$. If $p_1 < 2$ then we use the convergence of $\boldsymbol{v}^N$ in $L^{p_1}(0,T;\boldsymbol{L}^2(\Omega))$, the estimate $\left|\left|\boldsymbol{v}^N\right|\right|_{L^\infty(0,T,\boldsymbol{L}^2(\Omega))} \le c$ together, with the inequality

$$\int_{Q_T} \left|\boldsymbol{v}^N - \boldsymbol{v}\right|^2 d\boldsymbol{x}\,dt \le \int_0^T \left(\int_\Omega \left|\boldsymbol{v}^N - \boldsymbol{v}\right|^2 d\boldsymbol{x}\right)^a \cdot \left(\int_\Omega \left|\boldsymbol{v}^N - \boldsymbol{v}\right|^2 d\boldsymbol{x}\right)^{p_1/2} dt$$

where $a = 1 - p_1/2$, to deduce that $\boldsymbol{v}^N$ does converge strongly in $L^2(0,T,\boldsymbol{L}^2(\Omega))$. □

For our purposes here, we will suppose that

$$F_i \in L^{p_1'}((0,T);\boldsymbol{L}^2(\Omega)), \qquad \frac{1}{p_1} + \frac{1}{p_1'} = 1. \tag{4.73}$$

It follows from (4.61) and (4.13) with $M_i = 0$, that if we denote by $\boldsymbol{v}^N$ the unique weak solution of the initial-boundary value problem corresponding to $\mu_1 = \mu_1^N$, the conditions of Lemma 4.9 are satisfied[1]; we can, therefore, suppose that $\boldsymbol{v}^N \overset{*}{\rightharpoonup} \boldsymbol{v}$ in $L^\infty(I;\boldsymbol{L}^2(\Omega))$, as well as $\boldsymbol{v}^N \to \boldsymbol{v}$ in $L^{p_1}((0,T);\boldsymbol{W}^{1,p_1}(\Omega))$.

Let $\Psi(\boldsymbol{x},t,\boldsymbol{\lambda}) \in L^1(Q_T;C_0(\mathbb{R}^{n^2}))$, so that a.e., for $(\boldsymbol{x},t) \in Q_T$, we have that $\Psi(\boldsymbol{x},t,\cdot) \in C_0(\mathbb{R}^{n^2})$. We now set

$$\int_{Q_T} \left\langle \Psi(\boldsymbol{x},t,\cdot), \nu_{\boldsymbol{x},t}^N(\cdot)\right\rangle d\boldsymbol{x}\,dt = \int_{Q_T} \Psi(\boldsymbol{x},t,\nabla\boldsymbol{v}^N(\boldsymbol{x},t))\,d\boldsymbol{x}\,dt. \tag{4.74}$$

This serves to define the measures $\nu^N \in L_w^\infty(Q_T;M(\mathbb{R}^{n^2}))$ and we have a.e., in Q_T, that

$$\nu_{\boldsymbol{x},t}^N \ge 0, \quad \left|\left|\nu_{\boldsymbol{x},t}^N\right|\right|_{M(\mathbb{R}^{n^2})} = 1$$

and

$$\left|\int_{Q_T} \Psi(\boldsymbol{x},t,\nabla\boldsymbol{v}^N(\boldsymbol{x},t))\,d\boldsymbol{x}\,dt\right| \le ||\Psi||_{L^1(Q_T;C_0(\mathbb{R}^{n^2}))}.$$

Therefore, we can suppose that for $\nu^N \in L_w^\infty(Q_T;M(\mathbb{R}^{n^2}))$ we have $\nu^N \overset{*}{\rightharpoonup} \nu$ (ν^N a subsequence of the original sequence, if necessary) as a consequence of the separability of the space $L^1(Q_T;C_0(\mathbb{R}^{n^2}))$. Hence, by Lemma 4.5,

$$\nu_{\boldsymbol{x},t} \ge 0, \quad ||\nu_{\boldsymbol{x},t}||_{M(\mathbb{R}^{n^2})} \le 1. \tag{4.75}$$

---

[1]The sequence $\{\mu_1^N\}$ is chosen so that $\mu_1^N \to 0^+$, as $N \to \infty$.

In fact, the following is true:

Theorem 4.7. *The measure* $\nu_{\boldsymbol{x},t}$ *satisfies* $||\nu_{\boldsymbol{x},t}||_{M(\mathbb{R}^{n^2})} = 1$, *a.e. in* Q_T.

Proof. We sketch the proof (which may be found in [BaJ]). Set

$$\vartheta(\boldsymbol{\lambda}) = \begin{cases} 1, & |\boldsymbol{\lambda}| \le k, \\ 1 + k - |\boldsymbol{\lambda}|, & k \le |\boldsymbol{\lambda}| \le 1 + k, \\ 0, & |\boldsymbol{\lambda}| \ge 1 + k. \end{cases}$$

Then, for any measurable $E \subset Q_T$,

$$\begin{aligned} \lim_{N\to\infty} \frac{1}{\text{meas } E} \int\int_E \vartheta^k(\nabla \boldsymbol{v}^N(\boldsymbol{x},t))\, d\boldsymbol{x}\, dt &= \frac{1}{\text{meas } E} \int\int_E \left\langle \vartheta^k, \nu_{\boldsymbol{x},t} \right\rangle d\boldsymbol{x}\, dt \\ &\le \frac{1}{\text{meas } E} \int\int_E ||\nu_{\boldsymbol{x},t}||_{M(\mathbb{R}^{n^2})}\, d\boldsymbol{x}\, dt. \end{aligned} \tag{4.76}$$

On the other hand, for some $c' > 0$,

$$\begin{aligned} &\frac{1}{\text{meas } E} \int_E \left(1 - \left\langle \vartheta^k, \nu^N_{\boldsymbol{x},t} \right\rangle\right) d\boldsymbol{x}\, dt \\ &\qquad \le \frac{1}{\text{meas } E} \left[\text{meas}\left\{(\boldsymbol{x},t) \in E \mid \left|\left|\nabla \boldsymbol{v}^N(\boldsymbol{x},t)\right|\right| \ge k\right\}\right] \\ &\qquad \le \frac{c'}{\text{meas } E} \cdot k^{-1} [\text{meas } E]^{1/p_1'}. \end{aligned} \tag{4.77}$$

Therefore,

$$1 - k^{-1} [\text{meas } E]^{\frac{1}{p_1'} - 1} \cdot c' \le \frac{1}{\text{meas } E} \int_E \left\langle \vartheta^k, \nu^N_{\boldsymbol{x},t} \right\rangle d\boldsymbol{x}\, dt \tag{4.78}$$

so that

$$\begin{aligned} 1 - k^{-1} [\text{meas } E]^{\frac{1}{p_1'} - 1} \cdot c' &\le \frac{1}{\text{meas } E} \int_E \left\langle \vartheta^k, \nu_{\boldsymbol{x},t} \right\rangle d\boldsymbol{x}\, dt \\ &\le \frac{1}{\text{meas } E} \int_E ||\nu_{\boldsymbol{x},t}||_{M(\mathbb{R}^{n^2})}\, d\boldsymbol{x}\, dt. \end{aligned} \tag{4.79}$$

Letting $k \to \infty$ yields the required result. □

We now state:

Theorem 4.8. *Let* $h(\cdot) \in C(\mathbb{R}^{n^2})$ *be such that* $|h(\boldsymbol{\lambda})| \leq C(1 + |\boldsymbol{\lambda}|)^{p-1}$ *and choose* $\psi \in L^q(Q_T)$ *with* $q \geq p_1/(p-1)$*. Then the duality*

$$\int_{Q_T} \langle \Psi(\boldsymbol{x}, t, \cdot), \nu_{\boldsymbol{x},t}(\cdot)\rangle \, d\boldsymbol{x} \, dt$$

can be continuously extended to $\Psi(\boldsymbol{x}, t, \boldsymbol{\lambda}) = \psi(\boldsymbol{x}, t)h(\boldsymbol{\lambda})$ *as the limit of the functions* $\psi(\boldsymbol{x}, t)b_m(\boldsymbol{\lambda})$*, where* $b_m \in C_0(\mathbb{R}^{n^2})$ *satisfies* $|b_m(\boldsymbol{\lambda})| \leq C(1 + |\boldsymbol{\lambda}|)^{p-1}$ *and* $b_m(\cdot) \to h(\cdot)$*, locally in* $\mathbb{R}^{n^2}$*, as* $m \to \infty$*.*

Proof. We have, for some $C > 0$,

$$||\langle b_m(\cdot), \nu_{\boldsymbol{x},t}\rangle||_{L^{\frac{p_1}{p-1}}(Q_T)} \leq C. \tag{4.80}$$

Let $1 < p_2 < \dfrac{p_1}{p-1}$; then, we claim that $\exists \chi \in L^{p_2}(Q_T)$ such that

$$\langle b_m(\cdot), \nu_{\boldsymbol{x},t}(\cdot)\rangle \to \chi(\boldsymbol{x}, t), \text{ in } L^{p_2}(Q_T). \tag{4.81}$$

In order to verify (4.81) we compute that

$$\begin{aligned}
&\int_{Q_T} |\langle b_{m_1}(\cdot), \nu_{\boldsymbol{x},t}(\cdot)\rangle - \langle b_{m_2}(\cdot), \nu_{\boldsymbol{x},t}(\cdot)\rangle|^{p_2} \, d\boldsymbol{x} \, dt \\
&\quad = \lim_{N\to\infty} \int_{Q_T} \left| b_{m_1}(\nabla \boldsymbol{v}^N(\boldsymbol{x}, t)) - b_{m_2}(\nabla \boldsymbol{v}^N(\boldsymbol{x}, t))\right|^{p_2} d\boldsymbol{x} \, dt \\
&\quad = \lim_{N\to\infty} \int_{Q_{T,k,N}} \left| b_{m_1}(\nabla \boldsymbol{v}^N(\boldsymbol{x}, t)) - b_{m_2}(\nabla \boldsymbol{v}^N(\boldsymbol{x}, t))\right|^{p_2} d\boldsymbol{x} \, dt \\
&\qquad + \overline{\lim_{N\to\infty}} \int_{Q_T/Q_{T,k,N}} \left| b_{m_1}(\nabla \boldsymbol{v}^N(\boldsymbol{x}, t)) - b_{m_2}(\nabla \boldsymbol{v}^N(\boldsymbol{x}, t))\right|^{p_2} d\boldsymbol{x} \, dt
\end{aligned}$$

where

$$Q_{T,k,N} = \{(\boldsymbol{x}, t) \in Q_T \mid \left|\left|\nabla \boldsymbol{v}^N(\boldsymbol{x}, t)\right|\right| \geq k\}. \tag{4.82}$$

However,

$$\text{meas } Q_{T,k,N} \leq ck^{-1} \tag{4.83}$$

so, by virtue of the growth assumption for $|b_m(\cdot)|$,

$$\begin{aligned}\int_{Q_{T,k,N}} &\left|b_{m_1}(\nabla \boldsymbol{v}^N(\boldsymbol{x},t)) - b_{m_2}(\nabla \boldsymbol{v}^N(\boldsymbol{x},t))\right| d\boldsymbol{x}\, dt \\ &\le C \int_{Q_{T,k,N}} \left(1 + \left|\nabla \boldsymbol{v}^N(\boldsymbol{x},t)\right|\right)^{(p-1)p_2}\, d\boldsymbol{x}\, dt \\ &\le C \left(\int_{Q_{T,k,N}} \left(1 + \left|\nabla \boldsymbol{v}^N\right|\right)^{p_1}\, d\boldsymbol{x}\, dt\right)^{\frac{(p-1)p_2}{p_1}} \cdot [\text{meas } Q_{T,k,N}]^{\frac{1-(p-1)p_2}{p-1}}.\end{aligned} \tag{4.84}$$

Therefore, for any $\epsilon > 0$ we may choose k so large that

$$\int_{Q_{T,k,N}} \left|b_{m_1}(\nabla \boldsymbol{v}^N(\boldsymbol{x},t)) - b_{m_2}(\nabla \boldsymbol{v}^N(\boldsymbol{x},t))\right| d\boldsymbol{x}\, dt \le \frac{\epsilon}{2}. \tag{4.85}$$

Fixing k so large that (4.85) holds, we now have for m_1, m_2 sufficiently large, say, $m_1, m_2 \ge m$,

$$\overline{\lim_{N\to\infty}} \int_{Q_T/Q_{T,k,N}} \left|b_{m_1}(\nabla \boldsymbol{v}^N(\boldsymbol{x},t)) - b_{m_2}(\nabla \boldsymbol{v}^N(\boldsymbol{x},t))\right|^{p_2} d\boldsymbol{x}\, dt \le \frac{\epsilon}{2} \tag{4.86}$$

and the desired result follows. □

Once again let $\boldsymbol{v}^N$ be the unique weak solution to the space-periodic problem for the incompressible bipolar fluid corresponding to the choice of the higher-order viscosity $\mu_1 = \mu_1^N$. As above we take $\nu_{\boldsymbol{x},t}$ to be the Young measure constructed from the sequence of solutions $\{\boldsymbol{v}^N\}$ as $N \to \infty$ (in which case, of course, we have $\mu_1^N \to 0^+$). Then the results of this subsection easily imply the following theorem on the existence of measure-valued solutions:

Theorem 4.9. *Let $\boldsymbol{v}$ denote the weak limit of the sequence $\{\boldsymbol{v}^N\}$ of unique weak solutions to the space-periodic problem* (4.1)*,* (4.2)*,* (4.5)*, and* (4.64a,b,c)*, with $\rho = 1$, for the incompressible bipolar fluid. Then, provided $p > 1$, when $n = 2$, and $p > 6/5$, when $n = 3$, $\boldsymbol{v}$ satisfies* (4.1)*,* (4.2)*,* (4.5)*, and* (4.64a,c)*, with $\mu_1 = 0$, in the following sense: $\forall \boldsymbol{\phi} \in \boldsymbol{V} \cap \boldsymbol{W}^{2,2}_{per}(\Omega)$, a.e. on $(0,T)$, $\Omega = [0,L]^n$, $L > 0$, $n = 2, 3$, where*

$$\boldsymbol{V} = \{\boldsymbol{\phi} \in \boldsymbol{W}^{1,p}_{per}(\Omega) \mid \nabla \cdot \boldsymbol{\phi} = \boldsymbol{0} \text{ and } \boldsymbol{\phi} \text{ satisfies } (4.64\text{a,c})\}$$

$$\int_{Q_T} \left[-v_i \frac{\partial \phi_i}{\partial t} - v_i v_j \frac{\partial \phi_i}{\partial x_j} + e_{ij}(\boldsymbol{\phi}) \int_{\mathbb{R}^{n^2}} (\boldsymbol{\tau}_0^v)_{ij}(e_{ij}(\boldsymbol{\lambda}))\, d\nu_{\boldsymbol{x},t}(\boldsymbol{\lambda}) - F_i \phi_i\right] d\boldsymbol{x}\, dt = 0. \tag{4.87}$$

4.3.3 Young Measures Are Dirac and the Weak Solutions are Regular for $p \geq 2$, $n = 2$

In order to prove the basic results in this and succeeding subsections, we need to establish some additional estimates for the velocity that are independent of $\mu_1 > 0$. These estimates will allow us to prove regularity results for the solution of some problems associated with non-Newtonian monopolar fluids and are the sharpest of their kind that are known. For this purpose we will introduce and use a specific test function. We recall that for a space-periodic function with zero average we can use the definition for the $\boldsymbol{W}^{1,2}(\Omega)$ norm given by

$$||\boldsymbol{v}||_{1,2}^2 = \int_\Omega |\nabla \boldsymbol{v}|^2 \, dx$$

which is easily seen to follow from Friedrich's inequality (see Appendix A). We need to treat separately the various cases depending on the space dimension n and the parameter p. We begin with the case in which $p \geq 2$. By virtue of the embedding $\boldsymbol{W}^{2,2}(\Omega) \hookrightarrow \boldsymbol{L}^\infty(\Omega)$ and the standard (interpolation) inequality (e.g., [Lio1]):

$$||\boldsymbol{v}||_{\boldsymbol{W}^{1,2}(\Omega)}^2 \leq \delta \, ||\boldsymbol{v}||_{\boldsymbol{W}^{2,2}(\Omega)}^2 + \lambda(\delta) \, ||\boldsymbol{v}||_{\boldsymbol{L}^2(\Omega)}^2$$

for any $\delta > 0$, and some $\lambda(\delta) > 0$, we have for some $c > 0$,

$$\int_{Q_T} ||\boldsymbol{v}||^2 \, |\nabla \boldsymbol{v}|^2 \, d\boldsymbol{x} \, dt \leq c \int_0^T ||\boldsymbol{v}||_{\boldsymbol{W}^{2,2}(\Omega_t)}^2 \left[\delta \int_{Q_t} ||\boldsymbol{v}||_{\boldsymbol{W}^{2,2}(\Omega_t)}^2 + \lambda(\delta) \, ||\boldsymbol{v}||_{\boldsymbol{L}^2(\Omega_t)}^2 \right] dt.$$

The standard method of differences then yields for the weak solution $\boldsymbol{v}$ and positive constants c, c_1, and c_2, dependent on γ,

$$\sup_{(0,T)} ||\boldsymbol{v}||_{1,2} \leq c(\gamma), \tag{4.88a}$$

$$\int_0^T \int_\Omega (1 + |\boldsymbol{e}|)^{p-2} e_{ij} \left(\frac{\partial \boldsymbol{v}}{\partial x_l} \right) e_{ij} \left(\frac{\partial \boldsymbol{v}}{\partial x_l} \right) d\boldsymbol{x} \, dt \leq c_1(\gamma), \tag{4.88b}$$

$$\int_0^T \int_\Omega \sum_{|\alpha|=2} \left(D^\alpha \frac{\partial \boldsymbol{v}}{\partial x_l} \cdot D^\alpha \frac{\partial \boldsymbol{v}}{\partial x_l} \right) d\boldsymbol{x} \, dt \leq c_2(\gamma) \tag{4.88c}$$

where $||\cdot||_{1,2} \equiv ||\cdot||_{\boldsymbol{W}^{1,2}(\Omega)}$ (and we will continue to employ this more compact notation in the sequel). Our goal, however, is to obtain some estimates which are independent of $\gamma \equiv 2\mu_1/\rho$.

For the sake of simplicity of notation, we assume that the initial function, as well as the forcing term $\boldsymbol{F}$, have zero spatial average. From this it easily follows that the solution also has zero spatial average and, therefore, the Poincaré inquality is valid (see Appendix A); these results can be easily extended to the general case. We will also assume that

$$\dot{F}_i \in L^2((0,T); L^2_{per}(\Omega)). \tag{4.89}$$

Employing, once again, the difference method, or directly from the Galerkin approximations, one can then prove that, for some positive c, c_3, c_4,

$$\sup_{(0,T)} ||\dot{\boldsymbol{v}}||_{0,2} \leq c(\gamma), \tag{4.90a}$$

$$\int_0^T \int_\Omega (1 + |\boldsymbol{e}|)^{p-2} e_{ij}(\dot{\boldsymbol{v}}) e_{ij}(\dot{\boldsymbol{v}})\, d\boldsymbol{x}\, dt \leq c_3(\gamma), \tag{4.90b}$$

$$\int_0^T \int_\Omega \sum_{|\boldsymbol{\alpha}|=2} |D^{\boldsymbol{\alpha}} \dot{\boldsymbol{v}}|^2\, d\boldsymbol{x}\, dt \leq c_4(\gamma). \tag{4.90c}$$

Remarks. In the above regularity results we made use of the smoothness of the decomposition result, Corollary 3.7 of [GRa]. In [GRa] the result is stated for smooth (C^2) bounded domains. However this result extends easily to the case under consideration here, i.e., the space-periodic case.

Now, let us take $\lambda \geq 0$ and denote, for any fixed subscript l, $\boldsymbol{v}' = \dfrac{\partial}{\partial x_l}\boldsymbol{v}$. As in Sect. 4.3.2, we again set

$$\boldsymbol{V} = \{\boldsymbol{v} \in \boldsymbol{W}^{1,p}_{per}(\Omega) \mid \nabla \cdot \boldsymbol{v} = 0 \text{ and } \boldsymbol{v} \text{ satisfies (4.64a,c)}\}$$

and make use of the following definition of a weak solution $\boldsymbol{v}$ of the problem (4.1), (4.2), (4.5), (4.64a,b,c) which is embodied in (4.65):

Definition 4.1. A weak solution of the problem (4.1), (4.2), (4.5), (4.64a,b,c) is a function

$$\boldsymbol{v} \in L^2((0,T); \boldsymbol{W}^{2,2}_{per}(\Omega)) \cap L^2((0,T); \boldsymbol{V})$$

which satisfies

$$\int_\Omega \dot{v}_i w_i\, d\boldsymbol{x} + \int_\Omega v_j \frac{\partial v_i}{\partial x_j} w_i\, d\boldsymbol{x} + \frac{1}{\rho}\int_\Omega (\boldsymbol{\tau}_0^v)_{ij}(\boldsymbol{v}) e_{ij}(\boldsymbol{w})\, d\boldsymbol{x}$$
$$+ \gamma \int_\Omega \frac{\partial e_{ij}}{\partial x_k}(\boldsymbol{v}) \frac{\partial e_{ij}}{\partial x_k}(\boldsymbol{w})\, d\boldsymbol{x} - \int_\Omega F_i w_i\, d\boldsymbol{x} = 0, \tag{4.91}$$

$\forall \boldsymbol{w} \in \boldsymbol{V} \cap \boldsymbol{W}^{2,2}_{per}(\Omega)$, where $\gamma = 2\mu_1/\rho$.

In (4.91) we now take as the test function $\boldsymbol{w} \in \boldsymbol{V} \cap \boldsymbol{W}^{2,2}_{per}(\Omega)$ with components

$$w_i = \frac{v_i''}{(1 + |\boldsymbol{v}|^2_{1,2})^\lambda}$$

and obtain, for $\lambda \neq 1$ and almost all t:

$$\begin{aligned}\frac{1}{2(1-\lambda)} \frac{d}{dt}(1 + ||\boldsymbol{v}||^2_{1,2})^{1-\lambda} + \int_\Omega v_j \frac{\partial v_i}{\partial x_j} \frac{v_i''}{(1 + ||\boldsymbol{v}||^2_{1,2})^\lambda}\, d\boldsymbol{x} \\ + \frac{\beta_2}{\rho} \int_\Omega (1 + |\boldsymbol{e}|)^{p-2} e_{ij}(\boldsymbol{v}') e_{ij}(\boldsymbol{v}') \cdot (1 + ||\boldsymbol{v}||^2_{1,2})^{-\lambda}\, d\boldsymbol{x} \\ + \gamma (1 + ||\boldsymbol{v}||^2_{1,2})^{-\lambda} \int_\Omega \frac{\partial e_{ij}(\boldsymbol{v}')}{\partial x_k} \frac{\partial e_{ij}(\boldsymbol{v}')}{\partial x_k}\, d\boldsymbol{x} \\ - (1 + ||\boldsymbol{v}||^2_{1,2})^{-\lambda} \int_\Omega F_i v_i''\, d\boldsymbol{x} \le 0. \quad (4.92)\end{aligned}$$

For the case in which $\lambda = 1$, the first term in (4.92) must be replaced by $\frac{d}{dt}\log(1+||\boldsymbol{v}||^2_{1,2})$. Now, via integration by parts, and our assumption of spatial periodicity,

$$-\int_\Omega v_j \frac{\partial v_i}{\partial x_j} v_i''\, d\boldsymbol{x} = \int_\Omega v_j' \frac{\partial v_i}{\partial x_j} v_i'\, d\boldsymbol{x}$$

thus,

$$\left| \int_\Omega v_j \frac{\partial v_i}{\partial x_j} \frac{v_i''}{(1 + ||\boldsymbol{v}||^2_{1,2})^\lambda}\, d\boldsymbol{x} \right| \le c(1 + ||\boldsymbol{v}||^2_{1,2})^{-\lambda}\, ||\boldsymbol{v}||^3_{1,3}\,. \quad (4.93)$$

Also, using that $p \ge 2$, after summing over l we have

$$\begin{aligned}(1 + ||\boldsymbol{v}||^2_{1,2})^{-\lambda} \left| \int_\Omega b_i v_i''\, d\boldsymbol{x} \right| \le c \left(\int_\Omega (1 + |\boldsymbol{e}|)^{p-2} e_{ij}(\boldsymbol{v}') e_{ij}(\boldsymbol{v}') \cdot (1 + ||\boldsymbol{v}||^2_{1,2})^{-\lambda}\, d\boldsymbol{x} \right)^{1/2} \\ \times \left(||\boldsymbol{F}||^2_{0,2}\, (1 + ||\boldsymbol{v}||^2_{1,2})^{-\lambda} \right)^{1/2}. \quad (4.94)\end{aligned}$$

After integration in time it then follows from (4.92) that

$$\begin{aligned}\frac{1}{2(1-\lambda)} (1 + ||\boldsymbol{v}(T)||^2_{1,2})^{1-\lambda} \\ + \frac{1}{2\rho} \int_0^T \int_\Omega (1 + |\boldsymbol{e}|)^{p-2} e_{ij}(\boldsymbol{v}') e_{ij}(\boldsymbol{v}') \cdot (1 + ||\boldsymbol{v}||^2_{1,2})^{-\lambda}\, d\boldsymbol{x}\, dt\end{aligned}$$

$$+ \gamma \int_0^T (1 + ||\boldsymbol{v}||_{1,2}^2)^{-\lambda} \int_\Omega \frac{\partial e_{ij}(\boldsymbol{v}')}{\partial x_k} \frac{\partial e_{ij}(\boldsymbol{v}')}{\partial x_k} \, d\boldsymbol{x} \, dt.$$

$$\leq \frac{1}{2(1-\lambda)} (1 + ||\boldsymbol{v}(0)||_{1,2}^2)^{1-\lambda} + c \int_0^T ||\boldsymbol{F}||_{0,2}^2 \, dt + c \int_0^T (1 + ||\boldsymbol{v}||_{1,2}^2)^{-\lambda} \, ||\boldsymbol{v}||_{1,3}^3 \, dt. \tag{4.95}$$

Thus, we want to estimate the integral:

$$\int_0^T (1 + ||\boldsymbol{v}||_{1,2}^2)^{-\lambda} \, ||\boldsymbol{v}||_{1,3}^3 \, dt.$$

We will make repeated use of the usual interpolation estimate (see [Tr] p. 186, for example)

$$||\boldsymbol{v}||_{s,r} \leq c \, ||\boldsymbol{v}||_{s_1,r_1}^{\theta} \cdot ||\boldsymbol{v}||_{s_2,r_2}^{1-\theta} \tag{4.96}$$

where $\frac{1}{r} = \frac{\theta}{r_1} + \frac{1-\theta}{r_2}$, $s = \theta s_1 + (1-\theta) s_2$ and $\theta \in [0, 1]$, as well as of the general embedding estimate (see [Tr] p. 328)

$$||\boldsymbol{v}||_{s,r} \leq c \, ||\boldsymbol{v}||_{s_1,r_1}$$

where $\frac{1}{r} = \frac{1}{r_1} - \frac{s_1 - s}{n}$. Now by virtue of the embedding $\boldsymbol{W}^{4/3,2}(\Omega) \hookrightarrow \boldsymbol{W}^{1,3}(\Omega)$, for $n = 2$, and the interpolation estimate

$$||\boldsymbol{v}||_{4/3,2} \leq c \, ||\boldsymbol{v}||_{1,2}^{2/3} \cdot ||\boldsymbol{v}||_{2,2}^{1/3}$$

we have

$$||\boldsymbol{v}||_{1,3} \leq c \, ||\boldsymbol{v}||_{1,2}^{2/3} \cdot ||\boldsymbol{v}||_{2,2}^{1/3} \, . \tag{4.97}$$

Thus, it follows that

$$\begin{aligned}
&\int_0^T ||\boldsymbol{v}||_{1,3}^3 (1 + ||\boldsymbol{v}||_{1,2}^2)^{-\lambda} \, dt \\
&\quad \leq c \int_0^T (1 + ||\boldsymbol{v}||_{1,2}^2)^{1-\lambda} \, ||\boldsymbol{v}||_{2,2} \, dt \\
&\quad \leq c \int_0^T (1 + ||\boldsymbol{v}||_{1,2}^2)^{1-\lambda/2} \, ||\boldsymbol{v}||_{2,2} \, (1 + ||\boldsymbol{v}||_{1,2}^2)^{-\lambda/2} \, dt \\
&\quad \leq c \left(\int_0^T (1 + ||\boldsymbol{v}||_{1,2}^2)^{2-\lambda} dt \right)^{1/2} \left(\int_0^T ||\boldsymbol{v}||_{2,2}^2 \, (1 + ||\boldsymbol{v}||_{1,2}^2)^{-\lambda} \, dt \right)^{1/2} .
\end{aligned} \tag{4.98}$$

Employing (4.98) in (4.95), with $\lambda = 1$, we obtain

$$\log(1 + ||\boldsymbol{v}(t)||_{1,2}^2) + \tilde{c}_1 \int_0^T ||\boldsymbol{v}(t)||_{2,2}^2 (1 + ||\boldsymbol{v}(t)||_{1,2}^2)^{-1}\, dt \le \tilde{c}_2 \tag{4.99}$$

for some positive constants $\tilde{c}_1, \tilde{c}_2$; we have, therefore, established the following: Let $\tilde{\boldsymbol{V}} = \{\boldsymbol{v} \in \boldsymbol{W}_{per}^{2,2}(\Omega),\, \Omega = [0, L]^2 \mid \nabla \cdot \boldsymbol{v} = 0\}$ and assume that $\boldsymbol{\tau}_0^v$ satisfies a strong monotonicity condition of the form

$$\frac{\partial(\boldsymbol{\tau}_0^v)_{ij}}{\partial e_{kl}} \xi_{ij}\xi_{kl} \ge \beta_{\boldsymbol{v}}(1 + |\boldsymbol{e}|^{p-2})||\boldsymbol{\xi}||^2, \quad \xi_{ij} = \xi_{ji} \tag{4.100}$$

for some $\beta_{\boldsymbol{v}} > 0$; then

Lemma 4.10. *The unique weak solution for the incompressible bipolar fluid*[2] $\boldsymbol{v} \in L^\infty((0,T); \tilde{\boldsymbol{V}})$*, in the case where* $p \ge 2$*, and satisfies, for any* $\gamma > 0$*,*

$$\begin{cases} \sup\limits_{(0,T)} ||\boldsymbol{v}(t)||_{1,2} \le \bar{c}, \\ \displaystyle\int_0^T ||\boldsymbol{v}(t)||_{2,2}^2\, dt \le \bar{c} \end{cases} \tag{4.101}$$

where $\bar{c}$, $0 < \bar{c} < \infty$, *is independent of* γ.

We are now in a position to prove the following:

Theorem 4.10. *For the space-periodic problem with* $n = 2$ *and* $p \ge 2$*, under the assumptions* (4.101)*,* (4.59)*,* (4.63)*, and* $F_i, \dot{F}_i \in L^\infty((0,T); \boldsymbol{L}^2(\Omega))$*, if* $\boldsymbol{v}_{\gamma_k} \rightharpoonup \boldsymbol{v}$ *in* $L^p((0,T); \boldsymbol{W}^{1,p}(\Omega))$ *then*

$$\boldsymbol{v}_{\gamma_k} \to \boldsymbol{v}, \text{ in } L^2((0,T); \boldsymbol{W}^{1,q}(\Omega)), \quad q > 1, \tag{4.102a}$$

$$\boldsymbol{v}_{\gamma_k} \to \boldsymbol{v}, \text{ in } L^{\tilde{p}}((0,T); \boldsymbol{W}^{1,p}(\Omega)), \quad \tilde{p} < p \tag{4.102b}$$

where $\boldsymbol{v}_{\nu_k}$ *is the solution of the space-periodic problem corresponding to* $\gamma_k = 2\mu_1^k/\rho$ *with* $\mu_1^k \to 0^+$ *as* $k \to \infty$*. Furthermore, the bounds* (4.101) *apply to the limiting function* $\boldsymbol{v}$*.*

Remarks. We may conclude from (4.97), (4.98) that the Young measure $\nu_{\boldsymbol{x},t}$ generated by $\{\boldsymbol{v}_{\gamma_k}\}$ is Dirac, i.e. $\nu_{\boldsymbol{x},t} = \delta(\boldsymbol{\lambda} - \nabla \boldsymbol{v}(\boldsymbol{x}, t))$.

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[2] We should write $\boldsymbol{v}_\gamma$ in (4.101), and similar estimates in the sequel, but will decline from doing so if the meaning is clear.

Proof (Theorem 4.10). We set $I = (0, T)$. Then by (4.101) and Lemma 4.9 with $p_1 = 2$,

$$||\boldsymbol{v}_{\gamma_j}||_{L^p(I;W^{1,p}(\Omega))} \le c < \infty, \tag{4.103}$$

$$\left|\left|\frac{d}{dt}\boldsymbol{v}_{\gamma_j}\right|\right|_{L^2(I;W^{-2,2}(\Omega))} \le c < \infty, \tag{4.104}$$

the conclusions of the theorem, i.e. (4.102a,b) and (4.98) now follow as a direct consequence of (4.103), (4.104), and Lemma 4.8. □

4.3.4 Young Measures Are Dirac for $3/2 < p < 2$, $n = 2$

We continue here the considerations of Sect. 4.3.3 but will have to resort to more delicate estimates than those previously employed. As was the case in Sect. 4.3.3, the estimates (4.88a,b,c), (4.90a,b,c) remain valid here. Also, we may take $\lambda > 1$ and use the same test function $w_i = v_i''(1 + ||\boldsymbol{v}||_{1,2}^2)^{-\lambda}$ so as to obtain (4.92). Since $p < 2$, instead of (4.94), we will use the estimate

$$(1 + ||\boldsymbol{v}||_{1,2}^2)^{-\lambda}\left|\int_\Omega b_i v_i''\, d\boldsymbol{x}\right| \le c\,||\boldsymbol{v}||_{2,p}\,||\boldsymbol{b}||_{0,p'}\,(1 + ||\boldsymbol{v}||_{1,2}^2)^{-\lambda}. \tag{4.105}$$

We will also need the estimate

$$\begin{aligned} ||\boldsymbol{v}||_{2,p} \le c &\left(\int_\Omega (1 + |\boldsymbol{e}|)^{p-2} e_{ij}\left(\frac{\partial \boldsymbol{v}}{\partial x_k}\right) e_{ij}\left(\frac{\partial \boldsymbol{v}}{\partial x_k}\right) d\boldsymbol{x}\right)^{1/2} \\ &\times \left(\int_\Omega (1 + |\boldsymbol{e}|)^p\, d\boldsymbol{x}\right)^{\frac{1}{p}-\frac{1}{2}} \end{aligned} \tag{4.106}$$

which is a direct consequence of the Hölder Inequality and the estimate (B.14). Hence,

$$\begin{aligned} (1 + ||\boldsymbol{v}||_{1,2}^2)^{-\lambda}\left|\int_\Omega b_i v_i''\, d\boldsymbol{x}\right| &\le c\,||\boldsymbol{b}||_{0,p'}\,(1 + ||\boldsymbol{v}||_{1,2}^2)^{-\lambda/2}\,||\boldsymbol{v}||_{1,p}^{\frac{2-p}{2}} \\ \times \left(\int_\Omega (1 + |\boldsymbol{e}|)^{p-2} e_{ij}\left(\frac{\partial \boldsymbol{v}}{\partial x_k}\right) e_{ij}\left(\frac{\partial \boldsymbol{v}}{\partial x_k}\right) (1 + ||\boldsymbol{v}||_{1,2}^2)^{-\lambda}\, d\boldsymbol{x}\right)^{1/2}&. \end{aligned} \tag{4.107}$$

Therefore we again obtain inequality (4.95) except that the term $\int_0^T ||\boldsymbol{F}||_{0,2}^2\, dt$ on the right-hand side is now replaced by the term $\sup_{(0,T)} ||\boldsymbol{F}||_{0,p'}^2$. We again seek to estimate the integral $\int_0^T (1 + ||\boldsymbol{v}||_{1,2}^2)^{-\lambda} \, ||\boldsymbol{v}||_{1,3}^3 \, dt$. However, with $\delta = 2 - \dfrac{2}{p}$, $\boldsymbol{W}^{2,p}(\Omega) \hookrightarrow \boldsymbol{W}^{1+\delta,2}(\Omega)$ and, thus, as $\boldsymbol{W}^{4/3,2}(\Omega) \hookrightarrow \boldsymbol{W}^{1,3}(\Omega)$ we have

$$\begin{aligned} ||\boldsymbol{v}||_{1,3} &\le \bar{c}_1 \, ||\boldsymbol{v}||_{4/3,2} \\ &\le \bar{c}_2 \, ||\boldsymbol{v}||_{1,2}^{1-\frac{p}{6(p-1)}} \, ||\boldsymbol{v}||_{1+\delta,2}^{\frac{p}{6(p-1)}} \,. \end{aligned} \tag{4.108}$$

For (4.108) to be meaningful we must have $1 - p/6(p-1) > 0$, or $p \ge 6/5$. Because of the embedding $\boldsymbol{W}^{2,p}(\Omega) \hookrightarrow \boldsymbol{W}^{1,q}(\Omega)$, $\dfrac{1}{q} = \dfrac{1}{p} - \dfrac{1}{2}$, we also have, for some $\tilde{c} > 0$,

$$||\boldsymbol{v}||_{1,3} \le \tilde{c} \, ||\boldsymbol{v}||_{1,p}^{\frac{5p-6}{3p}} \, ||\boldsymbol{v}||_{2,p}^{\frac{6-2p}{3p}} \,. \tag{4.109}$$

Therefore, for any β, $0 < \beta < 1$, it follows that,

$$\begin{aligned} &\int_0^T ||\boldsymbol{v}||_{1,3}^3 \, (1 + ||\boldsymbol{v}||_{1,2}^2)^{-\lambda} \, dt \\ &\quad \le \hat{c} \int_0^T \Bigg(||\boldsymbol{v}||_{1,2}^{\left(3-\frac{p}{2(p-1)}\right)\beta} \, ||\boldsymbol{v}||_{1,p}^{\frac{5p-6}{p}(1-\beta)} \, ||\boldsymbol{v}||_{2,p}^{\frac{p\beta}{2(p-1)}} \, ||\boldsymbol{v}||_{2,p}^{\frac{6-2p}{p}(1-\beta)} \\ &\qquad \times (1 + ||\boldsymbol{v}||_{1,2})^{-\lambda} \Bigg) \, dt. \end{aligned} \tag{4.110}$$

Setting $\beta = \dfrac{2(p-1)(3-p)}{(5p-6)}$, and $\lambda = \dfrac{3-p}{p-1}$ in (4.110), using the Hölder Inequality, and the estimate (4.106), we obtain for $3/2 < p < 2$, uniformly with respect to γ and p, the estimate

$$\begin{aligned} &\int_0^T ||\boldsymbol{v}||_{1,3}^3 \, (1 + ||\boldsymbol{v}||_{1,2}^2)^{-\lambda} \, dt \\ &\quad \le c \left(\int_0^T \left[\int_\Omega (1 + |\boldsymbol{e}|)^{p-2} \frac{\partial e_{ij}}{\partial x_k} \frac{\partial e_{ij}}{\partial x_k} \, d\boldsymbol{x} \right] \left(1 + ||\boldsymbol{v}||_{1,2}^2\right)^{-\lambda} \, dt \right)^{\frac{3-p}{2}} \\ &\qquad \times \left(\int_0^T (1 + ||\boldsymbol{v}||_{1,p})^p \, dt \right)^{\frac{p-1}{2}} \,. \end{aligned} \tag{4.111}$$

By virtue of (4.66), the weak solution $\boldsymbol{v}$ of (4.1), (4.2), (4.5), and (4.64a,b,c) satisfies, for all $T > 0$,

$$\sup_{[0,T)} ||\boldsymbol{v}||_{\boldsymbol{L}^2(\Omega_t)}^2 \le c, \quad \int_0^T ||\boldsymbol{v}||_{\boldsymbol{W}^{1,p}(\Omega_t)}^p \, dt \le c$$

for some $c > 0$ which is independent of μ_1. Since $\dfrac{3-p}{2} < 1$ using the last two estimates, above, and (4.95) we also find that

$$\int_0^T \left(\int_\Omega (1+|\boldsymbol{e}|)^{p-2} \frac{\partial e_{ij}}{\partial x_k} \frac{\partial e_{ij}}{\partial x_k}\, d\boldsymbol{x} \right) (1+||\boldsymbol{v}||_{1,2}^2)^{-\lambda}\, dt \le c < \infty \tag{4.112}$$

and, by (4.106)

$$\int_0^T ||\boldsymbol{v}||_{2,p}^2 \,(1+||\boldsymbol{v}||_{1,p})^{p-2}(1+||\boldsymbol{v}||_{1,2}^2)^{-\lambda}\, dt \le c < \infty. \tag{4.113}$$

Also,

$$||\boldsymbol{v}||_{1,2} \le \bar{c}_1\, ||\boldsymbol{v}||_{2/p,p} \le \bar{c}_2\, ||\boldsymbol{v}||_{1,p}^{2-\frac{2}{p}}\, ||\boldsymbol{v}||_{2,p}^{\frac{2}{p}-1}. \tag{4.114}$$

It then follows from (4.113) that

$$\int_0^T ||\boldsymbol{v}||_{2,p}^2 \left(1+||\boldsymbol{v}||_{1,p}^{2-p+2\lambda\left(2-\frac{2}{p}\right)}\right)^{-1} \left(1+||\boldsymbol{v}||_{2,p}^{2\lambda\left(\frac{2}{p}-1\right)}\right)^{-1} dt \le c < \infty. \tag{4.115}$$

Now let $M = \{t \in (0,T) \mid ||\boldsymbol{v}||_{2,p} > 1\}$ and recall that $\lambda = \dfrac{3-p}{p-1}$; then as a direct consequence of (4.115) we find that

$$\int_M ||\boldsymbol{v}||_{2,p}^{\frac{4(2p-3)}{p(p-1)}} \left(1+||\boldsymbol{v}||_{1,p}^{\frac{12-2p-p^2}{p}}\right)^{-1} dt \le c < \infty. \tag{4.116}$$

Setting $P = \dfrac{12-2p}{p^2}$, $\theta(p) = \dfrac{4(2p-3)p}{(12-2p)(p-1)}$ and using the Hölder Inequality we obtain

$$\int_M ||\boldsymbol{v}||_{2,p}^{\theta(p)}\, dt \le \left(\int_M ||\boldsymbol{v}||_{2,p}^{\frac{4(2p-3)}{p(p-1)}} \left(1+||\boldsymbol{v}||_{1,p}^{\frac{12-2p-p^2}{p}}\right)^{-1} dt \right)^{\frac{1}{P}} \times \left(\int_M (1+||\boldsymbol{v}||_{1,p})^p\, dt \right)^{\frac{P-1}{P}}$$

so that for $p \in (3/2, 2)$ we have, uniformly with respect to $\gamma > 0$,

$$\int_0^T ||\boldsymbol{v}||_{2,p}^{\theta(p)}\, dt \le c < \infty. \tag{4.117}$$

Now, we choose q such that $\theta(p) < q < p$. As

$$\int_0^T ||\boldsymbol{v}||_{1+\sigma,p}^q \, dt \leq c^* \int_0^T ||\boldsymbol{v}||_{1,p}^{(1-\sigma)q} \, ||\boldsymbol{v}||_{2,p}^{\sigma q} \, dt \tag{4.118}$$

with $\sigma = \theta(p)(p-q)/q(p-\theta(p))$, by the estimates which follow (4.111) and (4.117) we get, uniformly with respect to $\gamma > 0$ and p (in $(3/2, 2)$), the bound

$$\int_0^T ||\boldsymbol{v}||_{1+\sigma,p}^q \, dt \leq c < \infty \tag{4.119}$$

and as a consequence the following theorem:

Theorem 4.11. *For the space-periodic problem, in dimension $n = 2$ with $3/2 < p < 2$ (under the same assumptions which apply in Theorem 4.10) if $\boldsymbol{v}_{\gamma_k} \rightharpoonup \boldsymbol{v}$ in $L^p((0,T); \boldsymbol{W}^{1,p}(\Omega))$ then, for any $\tilde{p} < p$, $\boldsymbol{v}_{\gamma_k} \to \boldsymbol{v}$ in $L^{\tilde{p}}((0,T); \boldsymbol{W}^{1,p}(\Omega))$ and $\boldsymbol{v}_{\gamma_k} \to \boldsymbol{v}$ in $L^q((0,T); \boldsymbol{W}^{1+\tilde{\sigma},p}(\Omega))$, for $\tilde{\sigma} < \sigma$. Furthermore the limit function $\boldsymbol{v}$ satisfies the bound (4.119) and the Young measure $\nu_{\boldsymbol{x},t}(\boldsymbol{\lambda}) = \delta(\boldsymbol{\lambda} - \nabla \boldsymbol{v}(\boldsymbol{x},t))$, a. e. in $\Omega \times (0,T)$.*

4.3.5 Young Measures Are Dirac and the Weak Solutions Are Regular for $7/3 \leq p < 6$, $n = 3$

In this subsection we will show that for $7/3 \leq p < 6$, and space dimension $n = 3$, the sequence $\boldsymbol{v}_{\gamma_k} \to \boldsymbol{v}$ (in a sense to be made precise below) and that $\boldsymbol{v}$ is, for $7/3 \leq p < 6$, the unique weak solution of (4.1), with $\mu_1 = 0$, (4.2), (4.5), and (4.64a,c). We indicate here that the constraint $p < 6$ is rather artificial and results from our use of the embedding (for $n = 3$) $\boldsymbol{W}^{1,2}(\Omega) \hookrightarrow \boldsymbol{L}^6(\Omega)$; the constraint may, in fact, be easily removed and our efforts are directed principally at identifying the smallest value of p, in dimension $n = 3$, for which $\boldsymbol{v}$ is a unique weak solution.

We begin this time by choosing as the test function $w_i = v_i''(1 + ||\boldsymbol{v}||_{1,2}^2)^{-\lambda}$; following the pattern of Sects. 4.3.3 and 4.3.4 we seek to estimate

$$\int_0^T \left(\int_\Omega v_j' \frac{\partial v_i}{\partial x_j} v_i' \, d\boldsymbol{x} \right) (1 + ||\boldsymbol{v}||_{1,2}^2)^{-\lambda} \, dt.$$

In dimension $n = 3$, the interpolation-embedding estimates

$$||\boldsymbol{v}||_{1,3} \leq \bar{c}_1 \, ||\boldsymbol{v}||_{1,2}^{1/2} \, ||\boldsymbol{v}||_{2,2}^{1/2} \tag{4.120a}$$

and

$$||\boldsymbol{v}||_{1,3} \leq \bar{c}_2 \, ||\boldsymbol{v}||_{1,p}^{p/2-1/2} \, ||\boldsymbol{v}||_{1,3p}^{3/2-p/2} \tag{4.120b}$$

hold, for some $\bar{c}_1, \bar{c}_2 > 0$. We now set

$$\mathcal{I} = \int_\Omega (1 + |\boldsymbol{e}|)^{p-2} \frac{\partial e_{ij}}{\partial x_k} \frac{\partial e_{ij}}{\partial x_k}\, d\boldsymbol{x}. \tag{4.121}$$

We will need the estimates

$$||\boldsymbol{v}||_{2,2} \le \bar{c}_3 \mathcal{I}^{1/2} \tag{4.122a}$$

and

$$||\boldsymbol{v}||_{1,3p} \le \bar{c}_4 \mathcal{I}^{1/p}. \tag{4.122b}$$

Estimate (4.122a) follows from (B.14). Estimate (4.122b) follows from the Friedrich's inequality

$$\left|\left|(e_{ij}e_{ij})^{p/2}\right|\right|_{1,2} \le c\mathcal{I}^{1/2} \tag{4.123}$$

the embedding of $\boldsymbol{W}^{1,2}$ into $\boldsymbol{L}^6$, and the usual $\boldsymbol{L}^r$ (with $r = 3p$) Korn inequality (as proven, e.g., in [N1]). For $p \ge 3$ the estimation of the key integral is trivial, i.e., $\exists \bar{c}_5 > 0$ such that

$$\left| \int_0^T \left(\int_\Omega v_j' \frac{\partial v_i}{\partial x_j} v_i'\, d\boldsymbol{x} \right) \left(1 + ||\boldsymbol{v}||_{1,2}^2 \right)^{-\lambda} dt \right| \le \bar{c}_5 \int_0^T ||\boldsymbol{v}||_{1,p}^3\, dt. \tag{4.124}$$

Therefore, we will turn our attention to the case in which $7/3 \le p < 3$. By virtue of (4.120a,b), for any β such that $0 < \beta < 1$, and some $\bar{c}_6 > 0$,

$$||\boldsymbol{v}||_{1,3} \le \bar{c}_6\, ||\boldsymbol{v}||_{1,2}^{\beta/2}\, ||\boldsymbol{v}||_{2,2}^{\beta/2}\, ||\boldsymbol{v}||_{1,p}^{(1-\beta)\frac{(p-1)}{2}}\, ||\boldsymbol{v}||_{1,3p}^{(1-\beta)\frac{(3-p)}{2}}. \tag{4.125}$$

Then, with $\beta = \dfrac{4(3-p)}{3(4-p)}$ and $\lambda = 2\left(\dfrac{3-p}{p-1}\right)$ we have

$$\begin{aligned} &\int_0^T ||\boldsymbol{v}||_{1,3}^3 (1 + ||\boldsymbol{v}||_{1,2}^2)^{-\lambda}\, dt \\ &\quad \le \bar{c}_7 \int_0^T \left\{ ||\boldsymbol{v}||_{1,2}^{3\beta/2}\, ||\boldsymbol{v}||_{1,p}^{\frac{3(1-\beta)(p-1)}{2}}\, \mathcal{I}^{(\frac{3\beta}{4} + \frac{3(3-p)(1-\beta)}{2p})} (1 + ||\boldsymbol{v}||_{1,2}^2)^{-\lambda} \right\} dt \\ &\quad \le \bar{c}_8 \left(\int_0^T \mathcal{I}(1 + ||\boldsymbol{v}||_{1,2}^2)^{-\lambda}\, dt \right)^{\frac{3(3-p)}{2(4-p)}} \times \left(\int_0^T ||\boldsymbol{v}||_{1,p}^p\, dt \right)^{\frac{p-1}{2(4-p)}}. \end{aligned} \tag{4.126}$$

Combining our estimates, we are led to the following two results:

$$\int_0^T \mathcal{I}(1+||\boldsymbol{v}||_{1,2}^2)^{-\lambda}\,dt \le c \tag{4.127a}$$

$$(1+||\boldsymbol{v}||_{1,2}^2)^{1-\lambda} \le c \tag{4.127b}$$

where c is independent of γ.

The estimates (4.127a,b) hold for $p \geq 2$; however, λ is clearly decreasing in p and for $p = 7/3$, $\lambda = 1$. We may, therefore, state

Lemma 4.11. *For $7/3 \le p < 6$, independently of γ, the unique regular weak solution $\boldsymbol{v} \in L^\infty((0,T);\hat{\boldsymbol{V}})$, $\hat{\boldsymbol{V}} = \boldsymbol{V} \cap \boldsymbol{W}^{2,2}_{per}(\Omega)$, for the incompressible bipolar fluid satisfies, for some $c > 0$,*

$$\sup_{(0,T)} ||\boldsymbol{v}||_{1,2} \le c < \infty, \tag{4.128a}$$

$$\int_0^T \int_\Omega (1+|\boldsymbol{e}|)^{p-2}\frac{\partial e_{ij}}{\partial x_k}\frac{\partial e_{ij}}{\partial x_k}\,d\boldsymbol{x} \le c < \infty. \tag{4.128b}$$

As a direct consequence of Lemma 4.11 we have the following

Theorem 4.12. *For $p \ge 7/3$ and $n = 3$, in the case of the space-periodic problem (and under the same assumptions as apply in Theorem 4.10) if $\boldsymbol{v}_{\gamma_k} \rightharpoonup \boldsymbol{v}$ in $L^p((0,T);\boldsymbol{W}^{1,p}(\Omega))$ then $\boldsymbol{v}_{\gamma_k} \to \boldsymbol{v}$ in $L^{\tilde{p}}((0,T);\boldsymbol{W}^{1,3\tilde{p}}(\Omega))$, for any $\tilde{p} < p$, and the bounds (4.128a,b) remain valid for the limit function $\boldsymbol{v}$. Therefore, $\nu_{\boldsymbol{x},t}(\boldsymbol{\lambda}) = \delta(\boldsymbol{\lambda} - \nabla\boldsymbol{v}(\boldsymbol{x},t))$, a.e. in Q_T and, in addition, we have for the limit function $\boldsymbol{v}$,*

$$\int_0^T ||\boldsymbol{v}||_{2,2}^2\,dt \le c' < \infty, \tag{4.129a}$$

$$\int_0^T ||\boldsymbol{v}||_{1,3p}^p\,dt \le c' < \infty, \tag{4.129b}$$

for some $c' > 0$.

We now make the following definition:

Definition 4.2. A regular weak solution to the equations (4.1), with $\gamma = 0$, (4.2), (4.5), and (4.64a,c) is a function $\boldsymbol{v} \in L^\infty((0,T);\boldsymbol{W}^{1,2}_{per}(\Omega)) \cap L^p((0,T);\boldsymbol{W}^{1,p}_{per}(\Omega)) \cap L^2((0,T);\boldsymbol{W}^{2,2}_{per}(\Omega))$, with $\operatorname{div}\boldsymbol{v} = 0$, which satisfies (4.91), with $\gamma = 0$, and for which $\dot{\boldsymbol{v}} \in L^p((0,T);\boldsymbol{W}^{-1,p}(\Omega))$.

Theorem 4.13. *The weak solution to (4.1), with $\gamma = 0$, (4.2), (4.5), and (4.64a,c), which is given by Theorem 4.12 for the space-periodic problem in space dimension $n = 3$, is unique provided $p \ge 7/3$.*

Proof. We begin by noting that for $\boldsymbol{\phi} \in C_0^\infty(\Omega)$,

$$\begin{aligned}\int_0^T \sup_{||\boldsymbol{\phi}||_{1,p}\le 1} \left| \int_\Omega v_j \frac{\partial v_i}{\partial x_j} \phi_i \, d\boldsymbol{x} \right|^p dt &= \int_0^T \sup_{||\boldsymbol{\phi}||_{1,p}\le 1} \left| \int_\Omega v_j v_i \frac{\partial \phi_i}{\partial x_j} \, d\boldsymbol{x} \right|^p dt \\ &\le c \int_0^T \left(\int_\Omega |\boldsymbol{v}|^{2\frac{p}{p-1}} \, d\boldsymbol{x} \right)^{p-1} dt.\end{aligned} \tag{4.130}$$

However, $\dfrac{2p}{p-1} \le 6$, and $\sup\limits_{(0,T)} ||\boldsymbol{v}||_{1,2} < \infty$, so for some $\tilde{c} > 0$,

$$\int_0^T \sup_{||\boldsymbol{\phi}||_{1,p}\le 1} \left| \int_\Omega v_j \frac{\partial v_i}{\partial x_j} \phi_i \, d\boldsymbol{x} \right|^p dt \le \tilde{c} < \infty. \tag{4.131}$$

Now, let $\boldsymbol{v}^1, \boldsymbol{v}^2$ be two solutions of (4.91), with $\gamma = 0$, such that $\operatorname{div} \boldsymbol{v}^j = 0$, $\boldsymbol{v}^j \in L^\infty((0,T); \boldsymbol{W}^{1,2}(\Omega))$, $j = 1, 2$. If we take the difference of (4.91) for $\boldsymbol{v} = \boldsymbol{v}^1$ and $\boldsymbol{v} = \boldsymbol{v}^2$, and integrate over $(0, T)$, it is clear that the troublesome term which must be dealt with is

$$\begin{aligned}&\int_0^T \left| \int_\Omega (v_j^2 - v_j^1) \frac{\partial v_i^1}{\partial x_j} (v_i^2 - v_i^1) \, d\boldsymbol{x} \right| dt \\ &\quad\le \int_0^T \left(\int_\Omega \left| \boldsymbol{v}^2 - \boldsymbol{v}^1 \right|^4 d\boldsymbol{x} \right)^{1/2} \left|\left| \boldsymbol{v}^1 \right|\right|_{1,2} dt \\ &\quad\le c_1 \int_0^T \left(\int_\Omega \left| \boldsymbol{v}^2 - \boldsymbol{v}^1 \right|^2 d\boldsymbol{x} \right)^{1/4} \left(\int_\Omega \left| \boldsymbol{v}^2 - \boldsymbol{v}^1 \right|^6 d\boldsymbol{x} \right)^{1/4} dt\end{aligned} \tag{4.132}$$

where we have used the Hölder Inequality and the fact that $\sup\limits_{(0,T)} \left|\left| \boldsymbol{v}^1 \right|\right|_{1,2} \le c_1 < \infty$.

Therefore, by Young's inequality (Appendix A) and the embedding $\boldsymbol{W}^{1,2}(\Omega) \hookrightarrow \boldsymbol{L}^6(\Omega)$, we have for any $\delta > 0$, and some $\lambda(\delta) > 0$,

$$\begin{aligned}&\int_0^T \left| \int_\Omega (v_j^2 - v_j^1) \frac{\partial v_i^1}{\partial x_j} (v_i^2 - v_i^1) \, d\boldsymbol{x} \right| dt \\ &\quad\le c_1 \delta \int_0^T \left(\int_\Omega \left| \boldsymbol{v}^2 - \boldsymbol{v}^1 \right|^6 d\boldsymbol{x} \right)^{1/3} dt + c_2 \lambda(\delta) \int_0^T \left(\int_\Omega \left| \boldsymbol{v}^2 - \boldsymbol{v}^1 \right|^2 d\boldsymbol{x} \right) dt \\ &\quad\le c_3 \delta \int_0^T \left|\left| \boldsymbol{v}^2 - \boldsymbol{v}^1 \right|\right|_{1,2}^2 dt + c_2 \lambda(\delta) \int_0^T \int_\Omega \left| \boldsymbol{v}^2 - \boldsymbol{v}^1 \right|^2 d\boldsymbol{x} dt.\end{aligned} \tag{4.133}$$

Thus, if we difference (4.91) for $\boldsymbol{v} = \boldsymbol{v}^1$, $\boldsymbol{v} = \boldsymbol{v}^2$, and integrate over $(0, T)$, for δ sufficiently small the term

$$\delta \int_0^T \left\|\boldsymbol{v}^2 - \boldsymbol{v}^1\right\|_{1,2}^2 dt$$

in the estimate (4.133), above, can be absorbed, by virtue of (4.61), by the integral involving the nonlinear viscosity in (4.91); the uniqueness theorem then follows by a standard application of the usual Gronwall lemma. □

4.3.6 *Young Measures Are Dirac and the Weak Solutions Are Regular for* $11/5 \leq p < 7/3$, $n = 3$

In this subsection we will show that the $L^\infty((0, T); \boldsymbol{W}^{1,2})$ estimate for $\boldsymbol{v}$ holds also for $p \in (11/5, 7/3)$. We first note that the following interpolation estimate holds for some $\bar{c}_2 > 0$,

$$||\boldsymbol{v}||_{1,3} \leq \bar{c}_2 \, ||\boldsymbol{v}||_{1,2}^{\frac{2(p-1)}{3p-2}} \, ||\boldsymbol{v}||_{1,3p}^{\frac{p}{3p-2}} . \tag{4.134}$$

By virtue of (4.120b) and (4.134), for $\beta \in [0, 1]$, and some $\bar{c}_6 > 0$,

$$||\boldsymbol{v}||_{1,3} \leq \bar{c}_6 \, ||\boldsymbol{v}||_{1,2}^{\beta \frac{2(p-1)}{3p-2}} \, ||\boldsymbol{v}||_{1,p}^{(1-\beta)\frac{(p-1)}{2}} \, ||\boldsymbol{v}||_{1,3p}^{(1-\beta)\frac{(3-p)}{2} + \beta \frac{p}{3p-2}} . \tag{4.135}$$

Then, with $\beta = \dfrac{(3p-2)(3-p)}{6(p-1)}$, λ_1 given by

$$\lambda_1 = \frac{(3-p)}{2}\left(1 + \frac{3\lambda}{2}\right) \tag{4.136}$$

and λ as in the previous section, i.e., $\lambda = 2\left(\dfrac{3-p}{p-1}\right)$, we have

$$\begin{aligned}
&\int_0^T ||\boldsymbol{v}||_{1,3}^3 \left(1 + ||\boldsymbol{v}||_{1,2}^2\right)^{-\lambda_1} dt \\
&\quad \leq \bar{c}_7 \int_0^T \left\{ ||\boldsymbol{v}||_{1,2}^{\beta \frac{6(p-1)}{3p-2}} \, ||\boldsymbol{v}||_{1,p}^{(1-\beta)\frac{3(p-1)}{2}} \, ||\boldsymbol{v}||_{1,3p}^{(1-\beta)\frac{3(3-p)}{2} + \beta \frac{3p}{3p-2}} \right. \\
&\qquad \left. \times \left(1 + ||\boldsymbol{v}||_{1,2}^2\right)^{-\lambda_1} \right\} dt \\
&\quad \leq \bar{c}_8 \left(\int_0^T ||\boldsymbol{v}||_{1,3p}^p \left(1 + ||\boldsymbol{v}||_{1,2}^2\right)^{-\lambda} dt \right)^{\frac{3(3-p)}{4}} \times \left(\int_0^T ||\boldsymbol{v}||_{1,p}^p \, dt \right)^{\frac{3p-5}{4}} .
\end{aligned} \tag{4.137}$$

Now, by (4.122b) and (4.122a), the term $\left(\int_0^T ||\boldsymbol{v}||_{1,3p}^p \left(1 + ||\boldsymbol{v}||_{1,2}^2\right)^{-\lambda} dt\right)$ is bounded independently of γ; therefore, we have for some $c > 0$, independent of γ,

$$\int_0^T \left(\int_\Omega (1 + |\boldsymbol{e}|)^{p-2} \frac{\partial e_{ij}}{\partial x_k} \frac{\partial e_{ij}}{\partial x_k} \, d\boldsymbol{x}\right) \left(1 + ||\boldsymbol{v}||_{1,2}^2\right)^{-\lambda_1} dt \leq c \tag{4.138a}$$

and

$$\sup_{(0,T)} \left(1 + ||\boldsymbol{v}(t)||_{1,2}^2\right)^{1-\lambda_1} \leq c. \tag{4.138b}$$

We now set

$$\lambda_k = \frac{3-p}{2}\left(1 + \frac{3\lambda_{k-1}}{2}\right) \tag{4.139}$$

and assume that we have proved, for some c independent of γ, that

$$\int_0^T \left(\int_\Omega (1 + |\boldsymbol{e}|)^{p-2} \frac{\partial e_{ij}}{\partial x_k} \frac{\partial e_{ij}}{\partial x_k} \, d\boldsymbol{x}\right) \left(1 + ||\boldsymbol{v}||_{1,2}^2\right)^{-\lambda_{k-1}} dt \leq c. \tag{4.140}$$

By repeating the above procedure we find that

$$\begin{aligned} &\int_0^T ||\boldsymbol{v}||_{1,3}^3 \, (1 + ||\boldsymbol{v}||_{1,2}^2)^{-\lambda_k} \, dt \\ &\quad \leq \bar{c}_8 \left(\int_0^T ||\boldsymbol{v}||_{1,3p}^p \, (1 + ||\boldsymbol{v}||_{1,2}^2)^{-\lambda_{k-1}} \, dt\right)^{\frac{3(3-p)}{4}} \times \left(\int_0^T ||\boldsymbol{v}||_{1,p}^p \, dt\right)^{\frac{3p-5}{4}} \end{aligned} \tag{4.141}$$

from which it then follows that, for some $C_k > 0$, which is independent of γ, we have

$$\int_0^T \left(\int_\Omega (1 + |\boldsymbol{e}|)^{p-2} \frac{\partial e_{ij}}{\partial x_k} \frac{\partial e_{ij}}{\partial x_k} \, d\boldsymbol{x}\right) (1 + ||\boldsymbol{v}||_{1,2}^2)^{-\lambda_k} \, dt \equiv A_k \leq C_k. \tag{4.142a}$$

and

$$\sup_{(0,T)} (1 + ||\boldsymbol{v}(t)||_{1,2}^2)^{1-\lambda_k} \equiv B_k \leq C_k. \tag{4.142b}$$

The sequence λ_k is decreasing and convergent, i.e.,

$$\lim_{k\to\infty} \lambda_k = \lambda_\infty = 2\left(\frac{3-p}{3p-5}\right). \tag{4.143}$$

Also, we have that $\lambda_\infty < 1$ for $p > 11/5$; therefore, $\forall p > 11/5$, $\exists k(p)$ such that $\lambda_{k(p)} < 1$. Thus, after finitely many iterations of the above scheme we find that Lemma 4.11 also holds for $p > 11/5$. The case $p = 11/5$ requires infinitely many iterations and, thus, we need to look at the limit of the sequence A_k. We rewrite (4.95) in the form

$$\begin{aligned}
&\frac{1}{2\rho}\int_0^T \int_\Omega (1+|\boldsymbol{e}|)^{p-2} e_{ij}(\boldsymbol{v}') e_{ij}(\boldsymbol{v}') \left(1+||\boldsymbol{v}||_{1,2}^2\right)^{-\lambda} d\boldsymbol{x}\, dt \\
&\qquad + \gamma \int_0^T \left(1+||\boldsymbol{v}||_{1,2}^2\right)^{-\lambda} \int_\Omega \frac{\partial e_{ij}(\boldsymbol{v}')}{\partial x_k}\frac{\partial e_{ij}(\boldsymbol{v}')}{\partial x_k}\, d\boldsymbol{x}\, dt \\
&\quad \leq \frac{1}{2(1-\lambda)}\left((1+||\boldsymbol{v}(0)||_{1,2}^2)^{1-\lambda} - (1+||\boldsymbol{v}(T)||_{1,2}^2)^{1-\lambda}\right) \\
&\qquad + c\int_0^T ||\boldsymbol{F}||_{0,2}^2\, dt + c\int_0^T (1+||\boldsymbol{v}||_{1,2}^2)^{-\lambda}\, ||\boldsymbol{v}||_{1,3}^3\, dt
\end{aligned} \tag{4.144}$$

and note that either $\frac{1}{2(1-\lambda)}\left((1+||\boldsymbol{v}(0)||_{1,2}^2)^{1-\lambda} - (1+||\boldsymbol{v}(T)||_{1,2}^2)^{1-\lambda}\right)$ is non-positive or else

$$\frac{1}{2(1-\lambda)}\left((1+||\boldsymbol{v}(0)||_{1,2}^2)^{1-\lambda} - \left(1+||\boldsymbol{v}(T)||_{1,2}^2\right)^{1-\lambda}\right) \leq \frac{1}{2}\log\left(1+||\boldsymbol{v}(0)||_{1,2}^2\right).$$

Using (4.141) and noting that $3(3-p)/4 = 0.6$, for $p = 11/5$, we then have that for some c_1, c_2, independent of both k and γ,

$$A_{k+1} \leq c_1 + c_2 \cdot A_k^{0.6} \tag{4.145}$$

from which it can be easily seen that the sequence A_k is bounded independently of γ. Letting k go to infinity in (4.142a) we find that, for some $c > 0$, independent of γ,

$$\int_0^T \left(\int_\Omega (1+|\boldsymbol{e}|)^{p-2}\frac{\partial e_{ij}}{\partial x_k}\frac{\partial e_{ij}}{\partial x_k}\, d\boldsymbol{x}\right)(1+||\boldsymbol{v}||_{1,2}^2)^{-1}\, dt \leq c. \tag{4.146}$$

from which we then deduce via (4.95) and (4.141) that

$$\sup_{(0,T)}\left(1+||\boldsymbol{v}(t)||_{1,2}^2\right) \leq c. \tag{4.147}$$

We can, therefore, state

Theorem 4.14. *Lemma 4.11, Theorems 4.12 and 4.13 all hold for* $11/5 \leq p < 6$.

4.3.7 Young Measures Are Dirac for $2 \leq p < 11/5$, $n = 3$

In this subsection we again choose as the test function $w_i = v_i''\left(1 + ||\boldsymbol{v}||_{1,2}^2\right)^{-\lambda}$; proceeding as we did for the case $p = \dfrac{11}{5}$ we get

$$A_{k+1} \leq c_1 + c_2 \cdot A_k^{\frac{3(3-p)}{4}} \tag{4.148}$$

instead of (4.145). Since $\dfrac{3(3-p)}{4} < 1$ in this range of p, it then follows that for $\lambda = \lambda_\infty = 2\dfrac{3-p}{3p-5}$ there exists $c > 0$, independent of γ, such that

$$\int_0^T \left(\int_\Omega (1 + |\boldsymbol{e}|)^{p-2} \frac{\partial e_{ij}}{\partial x_k}\frac{\partial e_{ij}}{\partial x_k}\, d\boldsymbol{x}\right)(1 + ||\boldsymbol{v}||_{1,2}^2)^{-\lambda}\, dt \leq c. \tag{4.149}$$

Using the Hölder Inequality we then have that for $\theta = \dfrac{p}{2\lambda + p}$, $0 < \theta < 1$, and

$$\begin{aligned}\int_0^T \mathcal{I}^\theta\, dt &= \int_0^T \mathcal{I}^\theta \left(1 + ||\boldsymbol{v}||_{1,2}^2\right)^{-\lambda\theta}\left(1 + ||\boldsymbol{v}||_{1,2}^2\right)^{\lambda\theta} dt\\ &\leq \left(\int_0^T \mathcal{I}\left(1 + ||\boldsymbol{v}||_{1,2}^2\right)^{-\lambda} dt\right)^\theta \times \left(\int_0^T \left(1 + ||\boldsymbol{v}||_{1,2}^2\right)^{\frac{p}{2}} dt\right)^{1-\theta}\end{aligned} \tag{4.150}$$

where $\mathcal{I}$ is given by (4.121). By (4.150) and (4.122b) we have

$$\int_0^T ||\boldsymbol{v}||_{2,2}^{2\theta}\, dt \leq c < \infty; \quad \theta = \frac{p(3p-5)}{3(p^2-3p+4)}. \tag{4.151}$$

We now choose $q > 1$ so that $2\theta < q < p$. Because $\boldsymbol{W}^{2,2}(\Omega) \hookrightarrow \boldsymbol{W}^{1+s,p}(\Omega)$, with $s = (6-p)/2p$, we get, with $\Theta = \dfrac{qp - 2\theta p}{qp - 2\theta q}$, and $\sigma = (1-\Theta)s$,

$$\begin{aligned}\int_0^T ||\boldsymbol{v}||_{1+\sigma,p}^q\, dt &\leq c' \int_0^T ||\boldsymbol{v}||_{1,p}^{\Theta q}\, ||\boldsymbol{v}||_{1+s,p}^{(1-\Theta)q}\, dt\\ &\leq \bar{c}\left(\int_0^T ||\boldsymbol{v}||_{2,2}^{2\theta}\, dt\right)^{\frac{(1-\Theta)q}{2\theta}}\left(\int_0^T ||\boldsymbol{v}||_{1,p}^p\, dt\right)^{\frac{2\theta - q + \Theta q}{2\theta}}\\ &\leq c^* < \infty.\end{aligned} \tag{4.152}$$

We can, therefore, state the following:

Theorem 4.15. *For $2 \leq p < 11/5$ and $n = 3$, in the case of the space-periodic problem (and under the same assumptions as apply in Theorem 4.10) the weak convergence $\boldsymbol{v}_{\gamma_k} \rightharpoonup \boldsymbol{v}$ in $L^p((0,T);\boldsymbol{W}^{1,p}(\Omega))$ implies the strong convergence $\boldsymbol{v}_{\gamma_k} \to \boldsymbol{v}$, in $L^{\tilde{p}}((0,T);\boldsymbol{W}^{1,p}(\Omega))$ for any $\tilde{p} < p$, and in $L^q((0,T);\boldsymbol{W}^{1+\tilde{\sigma},p}(\Omega))$ for $\tilde{\sigma} < \sigma$. Thus, in this situation $\nu_{\boldsymbol{x},t}(\boldsymbol{\lambda}) = \delta(\boldsymbol{\lambda} - \nabla\boldsymbol{v}(\boldsymbol{x},t))$ and the bound represented by (4.152) applies to the limit function $\boldsymbol{v}$.*

Remarks. For the particular case of the standard Navier–Stokes equations $p = 2$, ($2\theta = \frac{2}{3}$), estimate (4.151) is already known. More precisely, in 1981 Foias et al. [FGT] proved it for the space periodic case and in 1990 Duff, [Duff] established it for a general bounded domain.

4.3.8 Young Measures Are Dirac for $9/5 < p < 2$*,* $n = 3$

For $9/5 < p < 2$, $n = 3$, we have $\boldsymbol{W}^{2,p}(\Omega) \hookrightarrow \boldsymbol{W}^{1,q}(\Omega)$ with $\frac{1}{q} = \frac{1}{p} - \frac{1}{3} = (3-p)/3p$; so, for some $c > 0$,

$$||\boldsymbol{v}||_{1,3} \leq c\,||\boldsymbol{v}||_{1,p}^{\frac{2p-3}{p}}\,||\boldsymbol{v}||_{1,q}^{\frac{3-p}{p}} \tag{4.153}$$

which is, in fact, valid for $p > 3/2$. Furthermore, we have $\boldsymbol{W}^{2,p}(\Omega) \hookrightarrow \boldsymbol{W}^{1+\sigma,2}(\Omega)$, for $p > 6/5$, with $\sigma = \frac{5}{2} - \frac{3}{p}$ and, as $\boldsymbol{W}^{1+\delta,2}(\Omega) \hookrightarrow \boldsymbol{W}^{1,3}(\Omega)$, with $\delta = 1/2$ we get

$$||\boldsymbol{v}||_{1,3} \leq c\,||\boldsymbol{v}||_{1,2}^{\frac{4p-6}{5p-6}} \cdot ||\boldsymbol{v}||_{2,p}^{\frac{p}{5p-6}}\,. \tag{4.154}$$

Thus, the relevant integral, generated by the convective term, may be estimated by

$$\begin{aligned}\int_0^T ||\boldsymbol{v}||_{1,3}^3\,(1+||\boldsymbol{v}||_{1,2}^2)^{-\lambda}\,dt \leq c' \int_0^T \Bigg[||\boldsymbol{v}||_{1,2}^{3\left(\frac{4p-6}{5p-6}\right)\beta}\,||\boldsymbol{v}||_{2,p}^{\frac{3p\beta}{5p-6}}\,dt \\ \times ||\boldsymbol{v}||_{1,p}^{3\left(\frac{2p-3}{p}\right)(1-\beta)}\,||\boldsymbol{v}||_{2,p}^{3(\frac{3-p}{p})(1-\beta)}\,(1+||\boldsymbol{v}||_{1,2}^2)^{-\lambda}\Bigg]\,dt\end{aligned} \tag{4.155}$$

for $0 < \beta < 1$. We now set

$$\beta = -\frac{(5p-6)(3-p)}{6(3-2p)}, \quad \lambda = \frac{2(3-p)}{3p-5} \tag{4.156}$$

and note that for $p > 9/5$, $\beta < 1$. By employing (4.106), we see that (4.155) yields the estimate

$$\begin{aligned}\int_0^T & ||\boldsymbol{v}||_{1,3}^3 \,(1 + ||\boldsymbol{v}||_{1,2}^2)^{-\lambda}\, dt \\ &\le \bar{c} \left(\int_0^T \left(\int_\Omega (1 + |\boldsymbol{e}|)^{p-2} \frac{\partial e_{ij}}{\partial x_k} \frac{\partial e_{ij}}{\partial x_k}\, d\boldsymbol{x} \right) \cdot \left(1 + ||\boldsymbol{v}||_{1,2}^2\right)^{-\lambda}\, dt \right)^{\frac{3(3-p)}{4}} \\ &\quad \times \left(\int_0^T (1 + ||\boldsymbol{v}||_{1,p})^p\, dt \right)^{\frac{3p-5}{4}}.\end{aligned} \tag{4.157}$$

Therefore, for some $\hat{c} > 0$, $\hat{c}$ independent of γ,

$$\int_0^T \left(\int_\Omega (1 + |\boldsymbol{e}|)^{p-2} \frac{\partial e_{ij}}{\partial x_k} \frac{\partial e_{ij}}{\partial x_k}\, d\boldsymbol{x} \right) \cdot \left(1 + ||\boldsymbol{v}||_{1,2}^2\right)^{-\lambda}\, dt \le \hat{c} < \infty \tag{4.158}$$

in which case, we also have for $c^* > 0$ independent of γ, that

$$\int_0^T ||\boldsymbol{v}||_{2,p}^2 \left(\int_\Omega (1 + |\boldsymbol{e}|^p\, d\boldsymbol{x} \right)^{1-\frac{2}{p}} (1 + ||\boldsymbol{v}||_{1,2}^2)^{-\lambda}\, dt \le c^* < \infty. \tag{4.159}$$

As

$$||\boldsymbol{v}||_{1,2} \le \tilde{c}\, ||\boldsymbol{v}||_{1,p}^{\frac{5p-6}{2p}} \cdot ||\boldsymbol{v}||_{2,p}^{\frac{6-3p}{2p}} \tag{4.160}$$

we find that, for c independent of γ,

$$\int_0^T ||\boldsymbol{v}||_{2,p}^{\gamma_1} \left[||\boldsymbol{v}||_{2,p}^{\gamma_2} + ||\boldsymbol{v}||_{1,p}^{\gamma_3} \right]^{-1} \cdot (1 + ||\boldsymbol{v}||_{1,p})^{p-2}\, dt \le c < \infty \tag{4.161}$$

with

$$\gamma_1 = 2 - \frac{2(6-3p)(3-p)}{p(3p-5)}, \tag{4.162a}$$

$$\gamma_2 = -\frac{2(6-3p)(3-p)}{p(3p-5)}, \tag{4.162b}$$

and

$$\gamma_3 = \frac{2(3-p)(5p-6)}{p(3p-5)}. \tag{4.162c}$$

Integrating (4.91) over $M = \{t \in (0,T) \mid ||\boldsymbol{v}||_{2,p} > 1\}$, we then find that, for c' independent of γ,

$$\int_M ||\boldsymbol{v}||_{2,p}^{\frac{4(5p-9)}{p(3p-5)}} \cdot \left(1 + ||\boldsymbol{v}||_{1,p}\right)^{\gamma_4} \, dt \le c' < \infty \tag{4.163a}$$

with

$$\gamma_4 = -\frac{2(3-p)(5p-6)}{p(3p-5)} + p - 2. \tag{4.163b}$$

However, it is clear that with $I = (0,T)$ we have, for some c'' independent of γ,

$$\int_{I\backslash M} ||\boldsymbol{v}||_{2,p}^{\frac{4(5p-9)}{p(3p-5)}} \cdot \left(1 + ||\boldsymbol{v}||_{1,p}\right)^{\gamma_4} \, dt \le c'' < \infty. \tag{4.164}$$

Setting $P = \dfrac{4(9-2p-p^2)}{p(3p-5)}$, $\theta = \dfrac{4(5p-9)}{p(3p-5)}\dfrac{1}{P}$, $\gamma_5 = \dfrac{\gamma_4}{P}$, $s = p\left(1 - \dfrac{1}{P}\right)$, and using the Hölder Inequality (Appendix A) we find

$$\begin{aligned}
&\int_0^T ||\boldsymbol{v}||_{2,p}^{\theta} \cdot \left(1 + ||\boldsymbol{v}||_{1,p}\right)^{\gamma_5} \left(1 + ||\boldsymbol{v}||_{1,p}\right)^{s} \, dt \\
&\qquad \le \left(\int_0^T ||\boldsymbol{v}||_{2,p}^{\frac{4(5p-9)}{p(3p-5)}} \cdot (1 + ||\boldsymbol{v}||_{1,p})^{\gamma_4} \, dt\right)^{1/P} \\
&\qquad\qquad \times \left(\int_0^T \left(1 + ||\boldsymbol{v}||_{1,p}\right)^{p} \, dt\right)^{1-1/P} \le c' < \infty.
\end{aligned} \tag{4.165}$$

Noting that

$$||\boldsymbol{v}||_{1,p} \le \tilde{c}\, ||\boldsymbol{v}||_{0,p}^{\frac{1}{2}} \cdot ||\boldsymbol{v}||_{2,p}^{\frac{1}{2}} \le c\, ||\boldsymbol{v}||_{2,p}^{\frac{1}{2}} \tag{4.166}$$

and letting $\eta = \min\left\{\theta + \dfrac{1}{2}(\gamma_5 + s); \theta\right\} = \min\left\{\theta; \dfrac{p-1}{2}\right\} > 0$ we obtain, for $\bar{c}$ independent of γ, the estimate

$$\int_0^T ||\boldsymbol{v}||_{2,p}^{\eta} \, dt \le \bar{c} < \infty. \tag{4.167}$$

Finally, choosing q such that $\eta < q < p$ we get, for $9/5 < p < 2$, with $\sigma = \dfrac{\eta(p-q)}{q(p-\eta)}$,

$$\int_0^T ||\boldsymbol{v}||^2_{1+\sigma,p}\, dt \leq c_1 \int_0^T ||\boldsymbol{v}||^{(1-\sigma)q}_{1,p} \cdot ||\boldsymbol{v}||^{\sigma q}_{2,p}\, dt \leq c_2 < \infty \tag{4.168}$$

with c_1, c_2 both independent of γ and we have proved the following result:

Theorem 4.16. *The conclusions of Theorem 4.12 hold for $n = 3$, when $9/5 < p < 2$, including the estimate (4.168) for the limit function $\mathbf{v}$, provided we take $\sigma = \eta(p-q)/q(p-\eta)$, and $2\eta < q < p$.*

4.4 Existence and Uniqueness for Incompressible Flow in an Unbounded Channel

4.4.1 Introduction

In Sects. 4.2 and 4.3, existence and uniqueness theorems were established for the bipolar flow problem (4.1), (4.2), (4.5) subject to both the boundary conditions (4.3), (4.4), for the case of a problem posed on a bounded domain $\Omega \subseteq \mathbb{R}^n$, $n = 2, 3$, as well as for the space-periodic problem posed in $\Omega = [0, L]^n$, $n = 2, 3$, $L > 0$, for which the conditions (4.57a,b) are assumed to hold; analogous results were also proven for both the boundary-value problem (for a bounded domain) and the space-periodic problem, in dim $n = 2, 3$, for the non-Newtonian flow problem (4.1), (4.2), (4.5) with $\mu_1 = 0$. If $\mu_1 = 0$ in (4.1) then, for the boundary-value problem, only the condition (4.4) is applied while for the space-periodic case only (4.57a) applies. In this section we again consider the problem of existence and uniqueness for solutions of the initial-boundary value problem, this time in the unbounded channel Ω_a defined by (2.164), i.e.,

$$\Omega_a = \{(x_1, x_2, x_3) \mid x_2 \in [-a, a],\ -\infty < x_1, x_3 < \infty\}.$$

Most of the results which are presented in this section first appeared in the Ph.D. thesis [Hao]. To establish existence and uniqueness of solutions to the initial-boundary value problem, with $\Omega = \Omega_a$, we begin by formulating the problem in a Hilbert space setting and then prove some preliminary results about functions in the spaces in which the solutions will be established; this is accomplished in Sect. 4.4.2. Then in Sect. 4.4.3 we establish the existence of solutions by considering a sequence of approximate solutions in bounded subdomains of Ω_a; we show, in Sect. 4.4.3, that there exists a subsequence of such approximate solutions whose limit is a weak solution of the initial-boundary value problem with $\Omega = \Omega_a$. Finally, the uniqueness of the solution is proven in Sect. 4.4.4. For the remainder of this section, $\boldsymbol{\nu}$ will denote the exterior unit normal to $\partial\Omega$ and it will be understood that $\Omega \equiv \Omega_a$.

4.4.2 Formulation of the Problem in Hilbert Space and Some Preliminary Lemmas

We begin by introducing the spaces

$$\bar{\boldsymbol{V}} \equiv \text{ the closure of } \boldsymbol{J}(\Omega) \text{ in } \boldsymbol{H}^2(\Omega) \tag{4.169}$$

and

$$\bar{\boldsymbol{H}} \equiv \text{ the closure of } \boldsymbol{J}(\Omega) \text{ in } \boldsymbol{L}^2(\Omega) \tag{4.170}$$

where $\Omega = \Omega_a$, and

$$\boldsymbol{J}(\Omega) = \{\boldsymbol{\varphi} \in \boldsymbol{C}_0^\infty(\bar{\Omega}) \mid \boldsymbol{\varphi} = \boldsymbol{0} \text{ on } \partial\Omega \ \& \ \operatorname{div}\boldsymbol{\varphi} = 0 \text{ in } \Omega\}. \tag{4.171}$$

We also let $\bar{\boldsymbol{V}}'$ and $\bar{\boldsymbol{H}}'$ be the dual spaces of $\bar{\boldsymbol{V}}$ and $\bar{\boldsymbol{H}}$ respectively. It is clear that $\boldsymbol{V} \subset \boldsymbol{H}$ and $\bar{\boldsymbol{V}}$ is dense in $\bar{\boldsymbol{H}}$, the injection being continuous. The scalar product and the norm in $\bar{\boldsymbol{H}}$ are given by $(\boldsymbol{u}, \boldsymbol{v})_{L^2(\Omega)}$ and $||\boldsymbol{u}||_{L^2(\Omega)}$ respectively. By duality, if $\bar{\boldsymbol{H}}'$ is the dual of $\bar{\boldsymbol{H}}$, then the adjoint $\boldsymbol{i}^*$ of the identity is injective, $\boldsymbol{i}^*(\bar{\boldsymbol{H}}')$ is dense in $\bar{\boldsymbol{V}}'$, and we can identify $\bar{\boldsymbol{H}}'$ with a dense subspace of $\bar{\boldsymbol{V}}'$. If we identify $\bar{\boldsymbol{H}}$ with its dual $\bar{\boldsymbol{H}}'$, we obtain

$$\bar{\boldsymbol{V}} \subset \bar{\boldsymbol{H}} \equiv \bar{\boldsymbol{H}}' \subset \bar{\boldsymbol{V}}' \tag{4.172}$$

where each space is dense in the following, the injection being continuous. We now introduce the linear operator $\boldsymbol{A}$ as follows: consider the positive definite $\boldsymbol{V}$-elliptic symmetric bilinear form $\bar{a}(\cdot,\cdot) : \boldsymbol{V} \times \boldsymbol{V} \to R^1$ given by

$$\bar{a}(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \frac{\partial e_{ij}}{\partial x_j}(\boldsymbol{u}) \frac{\partial e_{ij}}{\partial x_j}(\boldsymbol{v})\, d\boldsymbol{x}. \tag{4.173}$$

As a consequence of the Lax-Milgram Lemma of Appendix A, we obtain an isometry $\bar{\boldsymbol{A}} \in \mathcal{L}(\bar{\boldsymbol{V}}; \bar{\boldsymbol{V}}')$, via

$$\langle \boldsymbol{A}\boldsymbol{u}, \boldsymbol{v}\rangle_{\bar{V}' \times \bar{V}} = \bar{a}(\boldsymbol{u}, \boldsymbol{v}) = \langle \boldsymbol{f}, \boldsymbol{v}\rangle_{\bar{V}' \times \bar{V}'}, \quad \forall \boldsymbol{v} \in \bar{\boldsymbol{V}} \tag{4.174}$$

with $\boldsymbol{f} \in \bar{\boldsymbol{V}}'$, where the domain of $\bar{\boldsymbol{A}}$ is

$$\boldsymbol{D}(\bar{\boldsymbol{A}}) = \{\boldsymbol{u} \in \bar{\boldsymbol{V}} \mid \bar{a}(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v})_{L^2(\Omega)},\ \boldsymbol{f} \in \bar{\boldsymbol{H}} \subset \bar{\boldsymbol{V}}',\ \forall \boldsymbol{v} \in \bar{\boldsymbol{V}}\}. \tag{4.175}$$

Thus, $\bar{\boldsymbol{A}} \in \mathcal{L}(\boldsymbol{D}(\bar{\boldsymbol{A}}); \bar{\boldsymbol{H}}) \cap \mathcal{L}(\bar{\boldsymbol{V}}, \bar{\boldsymbol{V}}')$.

We now have the following series of lemmas which will be used later to establish the existence and uniqueness of solutions of the initial-boundary value problem:

Lemma 4.12. *If* $\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)$ *then*

$$||\boldsymbol{v}||_{\boldsymbol{L}^4(\Omega)} \leq 2^{\frac{1}{4}}\, ||\boldsymbol{v}||^{\frac{1}{2}}_{\boldsymbol{L}^2(\Omega)}\, ||\nabla \boldsymbol{v}||^{\frac{1}{2}}_{\boldsymbol{L}^2(\Omega)}. \tag{4.176}$$

Proof. (See [Te4].) □

Lemma 4.13. *If* $\boldsymbol{v} \in \bar{\boldsymbol{V}}$, *then there exists a positive constant* c_1, *depending only on* a, *such that*

$$||\nabla \boldsymbol{v}||_{\boldsymbol{L}^4(\Omega)} \leq c_1\, ||\boldsymbol{v}||^{\frac{1}{2}}_{\boldsymbol{H}^1(\Omega)}\, ||\boldsymbol{v}||^{\frac{1}{2}}_{\boldsymbol{H}^2(\Omega)}\,. \tag{4.177}$$

Proof. In view of the definition of $\bar{\boldsymbol{V}}$, it suffices to show that if $f \in C^1(\bar{\Omega})$, and has compact support, then

$$||f||_{L^2(\Omega)} \leq c_1 ||f||^{\frac{1}{2}}_{L^2(\Omega)} ||f||^{\frac{1}{2}}_{H^1(\Omega)}. \tag{4.178}$$

Let $h(x_1, x_2) = \dfrac{x_2 + a}{2a}$ and $g(x_1, x_2) = h(x_1, x_2) \cdot f(x_1, x_2)$, for all $(x_1, x_2) \in \bar{\Omega}$, i.e., $-\infty < x_1 < +\infty$ and $-a \leq x_2 \leq a$. It is obvious that $g \in C^1(\bar{\Omega})$, $g(x_1, -a) = 0$, $-\infty < x_1 < +\infty$, and that the support of g is compact because the support of f is compact. We write

$$|g(x_1, x_2)|^2 = 2 \int_{-\infty}^{x_1} g(\xi, x_2) \frac{\partial g}{\partial x_1}(\xi, x_2)\, d\xi$$

so that

$$|g(x_1, x_2)|^2 = 2 g_1(x_2)$$

where

$$g_1(x_2) = \int_{-\infty}^{+\infty} |g(\xi, x_2)| \left| \frac{\partial g}{\partial x_1}(\xi, x_2) \right| d\xi.$$

Similarly, if we write

$$|g(x_1, x_2)|^2 = 2\int_{-a}^{x_2} g(x_1, \eta)\frac{\partial g}{\partial x_2}(x_1, \eta)\,d\eta$$

then we have

$$|g(x_1, x_2)|^2 = 2g_2(x_1)$$

where

$$g_2(x_1) = \int_{-a}^{a} |g(x_1, \eta)|\left|\frac{\partial g}{\partial x_2}(x_1, \eta)\right| d\eta.$$

Therefore,

$$\begin{aligned}\int_\Omega |g(x_1, x_2)|^4\,dx_1\,dx_2 &\le 4\left(\int_{-a}^{a} g_1(x_2)\,dx_2\right)\cdot\left(\int_{-\infty}^{\infty} g_2(x_1)\,dx_1\right)\\ &\le 4||g||^2_{L^2(\Omega)}\left|\left|\frac{\partial g}{\partial x_1}\right|\right|_{L^2(\Omega)}\left|\left|\frac{\partial g}{\partial x_2}\right|\right|_{L^2(\Omega)}.\end{aligned} \tag{4.179}$$

Since $0 \le h \le 1$, and $\dfrac{\partial h}{\partial x_2} = \dfrac{1}{2a}$, we have

$$\left|\frac{\partial g}{\partial x_1}\right|^2 = \left|h\frac{\partial f}{\partial x_1}\right|^2 \le \left|\frac{\partial f}{\partial x_1}\right|^2 \tag{4.180}$$

and

$$\begin{aligned}\left|\frac{\partial g}{\partial x_2}\right|^2 &= \left|\frac{\partial h}{\partial x_2}f + h\frac{\partial f}{\partial x_2}\right|^2\\ &\le 2\left(\left|\frac{\partial h}{\partial x_2}f\right|^2 + \left|h\frac{\partial f}{\partial x_2}\right|^2\right)\\ &\le 2\left(\frac{1}{4a^2}f^2 + \left|\frac{\partial f}{\partial x_2}\right|^2\right)\\ &\le 2\max\left(\frac{1}{4a^2}, 1\right)\left(f^2 + \left|\frac{\partial f}{\partial x_2}\right|^2\right).\end{aligned} \tag{4.181}$$

Employing (4.179)–(4.181) yields

$$\int_\Omega |g(x_1, x_2)|^4\,dx_1\,dx_2 \le 4\sqrt{2}\max\left(\frac{1}{2a}, 1\right)||f||^2_{L^2(\Omega)}||f||^2_{H^1(\Omega)} \tag{4.182}$$

where we have used the fact that $|g| \leq |f|$ on Ω. Similarly, we can also show that

$$\int_\Omega |(1-h)f|^4\, dx_1 dx_2 \leq 4\sqrt{2}\max\left(\frac{1}{2a}, 1\right) ||f||^2_{L^2(\Omega)} ||f||^2_{H^1(\Omega)}. \tag{4.183}$$

Combining (4.182) and (4.183), we obtain the estimate

$$\begin{aligned} \int_\Omega |f|^4\, dx_1\, dx_2 &\leq \int_\Omega |(h + (1-h))f|^4\, dx_1\, dx_2 \\ &\leq 4\left\{\int_\Omega |hf|^4\, dx_1\, dx_2 + \int_\Omega |(1-h)f|^4\, dx_1\, dx_2\right\} \\ &= 4\left\{\int_\Omega |g|^4\, dx_1\, dx_2 + \int_\Omega |(1-h)f|^4\, dx_1\, dx_2\right\} \\ &\leq 32\sqrt{2}\max\left(\frac{1}{2a}, 1\right) ||f||^2_{L^2(\Omega)} ||f||^2_{H^1(\Omega)} \end{aligned} \tag{4.184}$$

so that (4.177) is a direct consequence of (4.178). □

The lower bound in the next lemma may be inferred directly from Lemma B.2 by approximating $\Omega = \Omega_a$ by a sequence of bounded domains $\{\Omega_n\} \subset \Omega$ and, then, going to the limit as $n \to \infty$. It is instructive, however, to give a direct (and straightforward) proof for the two-dimensional unbounded case $\Omega = \Omega_a$.

Lemma 4.14. *There exist positive constants c_2 and c_3 such that, $\forall \boldsymbol{v} \in \bar{\boldsymbol{V}}$,*

$$c_2 ||\boldsymbol{v}||_{\boldsymbol{H}^2(\Omega)} \leq \left(\frac{\partial e_{ij}(\boldsymbol{v})}{\partial x_k}, \frac{\partial e_{ij}(\boldsymbol{v})}{\partial x_k}\right)_{L^2(\Omega)} \leq c_3 ||\boldsymbol{v}||_{\boldsymbol{H}^2(\Omega)}. \tag{4.185}$$

Proof. Recall that $e_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i})$, for $i, j = 1, 2$. Employing a direct computation of the derivatives $\partial e_{ij}/\partial x_k$ we obtain

$$\begin{aligned} \left(\frac{\partial e_{ij}(\boldsymbol{v})}{\partial x_k}, \frac{\partial e_{ij}(\boldsymbol{v})}{\partial x_k}\right)_{L^2(\Omega)} &= \left(\frac{\partial^2 v_1}{\partial x_1^2}, \frac{\partial^2 v_1}{\partial x_1^2}\right)_{L^2(\Omega)} \\ &+ \frac{1}{2}\left(\frac{\partial^2 v_1}{\partial x_1 \partial x_2} + \frac{\partial^2 v_2}{\partial x_1^2}, \frac{\partial^2 v_1}{\partial x_1 \partial x_2} + \frac{\partial^2 v_2}{\partial x_1^2}\right)_{L^2(\Omega)} \\ &+ \left(\frac{\partial^2 v_2}{\partial x_1 \partial x_2}, \frac{\partial^2 v_2}{\partial x_1 \partial x_2}\right)_{L^2(\Omega)} + \left(\frac{\partial^2 v_1}{\partial x_1 \partial x_2}, \frac{\partial^2 v_1}{\partial x_1 \partial x_2}\right)_{L^2(\Omega)} \\ &+ \frac{1}{2}\left(\frac{\partial^2 v_1}{\partial x_2^2} + \frac{\partial^2 v_2}{\partial x_1 \partial x_2}, \frac{\partial^2 v_1}{\partial x_2^2} + \frac{\partial^2 v_2}{\partial x_1 \partial x_2}\right)_{L^2(\Omega)} + \left(\frac{\partial^2 v_2}{\partial x_2^2}, \frac{\partial^2 v_2}{\partial x_2^2}\right)_{L^2(\Omega)} \end{aligned}$$

in which case

$$
\begin{aligned}
&\left(\frac{\partial e_{ij}(\boldsymbol{v})}{\partial x_k}, \frac{\partial e_{ij}(\boldsymbol{v})}{\partial x_k}\right)_{L^2(\Omega)} \\
&\quad \leq \left|\left|\frac{\partial^2 v_1}{\partial x_1^2}\right|\right|^2_{L^2(\Omega)} + \frac{1}{2}\left(\left|\left|\frac{\partial^2 v_1}{\partial x_1 \partial x_2}\right|\right|_{L^2(\Omega)} + \left|\left|\frac{\partial^2 v_2}{\partial x_1^2}\right|\right|_{L^2(\Omega)}\right)^2 \\
&\qquad + \left|\left|\frac{\partial^2 v_2}{\partial x_1 \partial x_2}\right|\right|^2_{L^2(\Omega)} + \left|\left|\frac{\partial^2 v_1}{\partial x_1 \partial x_2}\right|\right|^2_{L^2(\Omega)} \\
&\qquad + \frac{1}{2}\left(\left|\left|\frac{\partial^2 v_1}{\partial x_2^2}\right|\right|_{L^2(\Omega)} + \left|\left|\frac{\partial^2 v_2}{\partial x_1 \partial x_2}\right|\right|_{L^2(\Omega)}\right)^2 + \left|\left|\frac{\partial^2 v_2}{\partial x_2^2}\right|\right|^2_{L^2(\Omega)} \\
&\quad \leq \left|\left|\frac{\partial^2 v_1}{\partial x_1^2}\right|\right|^2_{L^2(\Omega)} + \left|\left|\frac{\partial^2 v_1}{\partial x_1 \partial x_2}\right|\right|^2_{L^2(\Omega)} + \left|\left|\frac{\partial^2 v_2}{\partial x_1^2}\right|\right|^2_{L^2(\Omega)} + \left|\left|\frac{\partial^2 v_2}{\partial x_1 \partial x_2}\right|\right|^2_{L^2(\Omega)} \\
&\qquad + \left|\left|\frac{\partial^2 v_1}{\partial x_1 \partial x_2}\right|\right|^2_{L^2(\Omega)} + \left|\left|\frac{\partial^2 v_1}{\partial x_2^2}\right|\right|^2_{L^2(\Omega)} + \left|\left|\frac{\partial^2 v_2}{\partial x_1 \partial x_2}\right|\right|^2_{L^2(\Omega)} + \left|\left|\frac{\partial^2 v_2}{\partial x_2^2}\right|\right|^2_{L^2(\Omega)} \\
&\quad \leq ||\boldsymbol{v}||^2_{\boldsymbol{H}^2(\Omega)}\,.
\end{aligned}
\tag{4.186}
$$

Therefore, the second inequality in (4.185) follows with $c_3 = 1$. To establish the first inequality in (4.185), we note that by the virtue of the elementary lower bound

$$
||f + g||^2_{L^2(\Omega)} \geq \left(||f||_{L^2(\Omega)} - ||g||_{L^2(\Omega)}\right)^2 \tag{4.187}
$$

(4.184) yields

$$
\begin{aligned}
\left(\frac{\partial e_{ij}(\boldsymbol{v})}{\partial x_k}, \frac{\partial e_{ij}(\boldsymbol{v})}{\partial x_k}\right)_{L^2(\Omega)} &\geq \left|\left|\frac{\partial^2 v_1}{\partial x_1^2}\right|\right|^2_{L^2(\Omega)} + \frac{1}{2}\left(\left|\left|\frac{\partial^2 v_1}{\partial x_1 \partial x_2}\right|\right|_{L^2(\Omega)} - \left|\left|\frac{\partial^2 v_2}{\partial x_1^2}\right|\right|_{L^2(\Omega)}\right)^2 \\
&\quad + \left|\left|\frac{\partial^2 v_2}{\partial x_1 \partial x_2}\right|\right|^2_{L^2(\Omega)} + \left|\left|\frac{\partial^2 v_1}{\partial x_1 \partial x_2}\right|\right|^2_{L^2(\Omega)} \\
&\quad + \frac{1}{2}\left(\left|\left|\frac{\partial^2 v_1}{\partial x_2^2}\right|\right|_{L^2(\Omega)} - \left|\left|\frac{\partial^2 v_2}{\partial x_1 \partial x_2}\right|\right|_{L^2(\Omega)}\right)^2 + \left|\left|\frac{\partial^2 v_2}{\partial x_2^2}\right|\right|^2_{L^2(\Omega)} \\
&\geq \left|\left|\frac{\partial^2 v_1}{\partial x_1^2}\right|\right|^2_{L^2(\Omega)} + \frac{1}{2}\left|\left|\frac{\partial^2 v_1}{\partial x_1 \partial x_2}\right|\right|^2_{L^2(\Omega)}
\end{aligned}
$$

$$+\left(\frac{1}{4}\left\|\frac{\partial^2 v_2}{\partial x_1^2}\right\|^2_{L^2(\Omega)}-\left\|\frac{\partial^2 v_1}{\partial x_1 \partial x_2}\right\|_{L^2(\Omega)}\left\|\frac{\partial^2 v_2}{\partial x_1^2}\right\|_{L^2(\Omega)}+\left\|\frac{\partial^2 v_1}{\partial x_1 \partial x_2}\right\|^2_{L^2(\Omega)}\right)$$
$$+\frac{1}{4}\left\|\frac{\partial^2 v_2}{\partial x_1^2}\right\|^2_{L^2(\Omega)}+\frac{1}{4}\left\|\frac{\partial^2 v_1}{\partial x_2^2}\right\|^2_{L^2(\Omega)}+\frac{1}{2}\left\|\frac{\partial^2 v_2}{\partial x_1 \partial x_2}\right\|^2_{L^2(\Omega)}+\left\|\frac{\partial^2 v_2}{\partial x_2^2}\right\|^2_{L^2(\Omega)}$$
$$+\left(\frac{1}{4}\left\|\frac{\partial^2 v_1}{\partial x_2^2}\right\|^2_{L^2(\Omega)}-\left\|\frac{\partial^2 v_1}{\partial x_2^2}\right\|_{L^2(\Omega)}\left\|\frac{\partial^2 v_2}{\partial x_1 \partial x_2}\right\|_{L^2(\Omega)}+\left\|\frac{\partial^2 v_2}{\partial x_1 \partial x_2}\right\|^2_{L^2(\Omega)}\right)$$
$$\geq \frac{1}{4}\left(\frac{\partial^2 v_i}{\partial x_j \partial x_k}, \frac{\partial^2 v_i}{\partial x_j \partial x_k}\right)_{L^2(\Omega)}. \tag{4.188}$$

To prove the first inequality in (4.185), it suffices to establish the following generalized Poincaré type inequalities:

$$\begin{cases} \int_\Omega ||\boldsymbol{v}||^2 \, d\boldsymbol{x} \leq 4a^2 \int_\Omega ||\nabla \boldsymbol{v}||^2 \, d\boldsymbol{x}, \quad \forall \boldsymbol{v} \in \bar{\boldsymbol{V}}, & (4.189a) \\ \int_\Omega ||\nabla \boldsymbol{v}||^2 \, d\boldsymbol{x} \leq 4a^2 \int_\Omega ||\Delta \boldsymbol{v}||^2 \, d\boldsymbol{x}, \quad \forall \boldsymbol{v} \in \bar{\boldsymbol{V}} & (4.189b) \end{cases}$$

where $||\cdot||$ denotes the standard Euclidean norm on $\mathbb{R}^2$. Using Hölder's inequality, we have

$$\begin{aligned} \int_\Omega ||\boldsymbol{v}||^2 \, d\boldsymbol{x} &= \int_\Omega |v_1|^2 \, d\boldsymbol{x} + \int_\Omega |v_2|^2 \, d\boldsymbol{x} \\ &\leq \int_\Omega \left| \int_{-a}^{x_2} \frac{\partial v_1}{\partial x_2} \, dx_2 \right|^2 d\boldsymbol{x} + \int_\Omega \left| \int_{-a}^{x_2} \frac{\partial v_2}{\partial x_2} \, dx_2 \right|^2 d\boldsymbol{x} \\ &\leq 2a \int_\Omega \int_{-a}^{x_2} \left| \frac{\partial v_1}{\partial x_2} \right|^2 dx_2 \, d\boldsymbol{x} + 2a \int_\Omega \int_{-a}^{x_2} \left| \frac{\partial v_2}{\partial x_2} \right|^2 dx_2 \, d\boldsymbol{x} \qquad (4.190) \\ &= 4a^2 \left[\int_\Omega \left| \frac{\partial v_1}{\partial x_2} \right|^2 d\boldsymbol{x} + \int_\Omega \left| \frac{\partial v_2}{\partial x_2} \right|^2 d\boldsymbol{x} \right] \\ &\leq 4a^2 \int_\Omega ||\nabla \boldsymbol{v}||^2 \, d\boldsymbol{x} \end{aligned}$$

which establishes (4.189a), while integration by parts yields

$$
\begin{aligned}
\int_\Omega ||\nabla \boldsymbol{v}||^2 \, d\boldsymbol{x} &= -\int_\Omega \boldsymbol{v} \cdot \Delta \boldsymbol{v} \, d\boldsymbol{x} \\
&\le \left(\int_\Omega ||\boldsymbol{v}||^2 \; d\boldsymbol{x} \right)^{\frac{1}{2}} \cdot \left(\int_\Omega ||\boldsymbol{v}||^2 \; d\boldsymbol{x} \right)^{\frac{1}{2}} \\
&\le 2a \left(\int_\Omega ||\nabla \boldsymbol{v}||^2 \, d\boldsymbol{x} \right)^{\frac{1}{2}} \left(\int_\Omega ||\Delta \boldsymbol{v}||^2 \, d\boldsymbol{x} \right)^{\frac{1}{2}} \\
&\le \frac{1}{2} \int_\Omega ||\nabla \boldsymbol{v}||^2 \, d\boldsymbol{x} + 2a^2 \int_\Omega ||\Delta \boldsymbol{v}||^2 \, d\boldsymbol{x}
\end{aligned}
\tag{4.191}
$$

which serves to establish (4.189b). Combining (4.189a) and (4.189b) now produces

$$
\int_\Omega ||\boldsymbol{v}||^2 \; d\boldsymbol{x} \le 4a^2 \int_\Omega ||\nabla \boldsymbol{v}||^2 \, d\boldsymbol{x} \le 16a^4 \int_\Omega ||\Delta \boldsymbol{v}||^2 \, d\boldsymbol{x}. \tag{4.192}
$$

Also, by virtue of (4.189a,b) we have the estimates

$$
\begin{aligned}
||\boldsymbol{v}||^2_{\boldsymbol{H}^2(\Omega)} &= \int_\Omega ||\boldsymbol{v}||^2 \; d\boldsymbol{x} + \int_\Omega ||\nabla \boldsymbol{v}||^2 \, d\boldsymbol{x} + \left(\frac{\partial^2 v_i}{\partial x_j \partial x_k}, \frac{\partial^2 v_i}{\partial x_j \partial x_k} \right)_{L^2(\Omega)} \\
&\le (16a^4 + 4a^2) \int_\Omega ||\Delta \boldsymbol{v}||^2 \, d\boldsymbol{x} + \left(\frac{\partial^2 v_i}{\partial x_j \partial x_k}, \frac{\partial^2 v_i}{\partial x_j \partial x_k} \right)_{L^2(\Omega)} \\
&\le (32a^4 + 8a^2 + 1) \left(\frac{\partial^2 v_i}{\partial x_j \partial x_k}, \frac{\partial^2 v_i}{\partial x_j \partial x_k} \right)_{L^2(\Omega)} \\
&\le 4(32a^4 + 8a^2 + 1) \left(\frac{\partial e_{ij}}{\partial x_k}, \frac{\partial e_{ij}}{\partial x_k} \right)_{L^2(\Omega)}
\end{aligned}
\tag{4.193}
$$

where we have used (4.188). Thus the first inequality in (4.185), i.e., the lower bound, has been established with $c_2 = 1/(144a^4 + 32a^2 + 4)$. □

Remarks. Aside from the three lemmas proven, thus far, in this subsection the analysis in the remainder of this section will make essential use of Lemmas A.8 and A.9 of Appendix A.

We now reformulate our problem in a Hilbert space setting. We begin by defining on $\boldsymbol{H}^1_0(\Omega)$ and, thus, on $\bar{\boldsymbol{V}}$, a trilinear continuous form $b(\cdot, \cdot, \cdot)$ by setting

$$
b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j \, d\boldsymbol{x}, \text{ for } \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{H}^1_0(\Omega). \tag{4.194}
$$

We note that as $\nabla \cdot \boldsymbol{u} = 0$,

$$\left.\begin{aligned} b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{v}) &= 0 \\ b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) &= -b(\boldsymbol{u},\boldsymbol{w},\boldsymbol{v}) \end{aligned}\right\} \forall \boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega). \tag{4.195}$$

For $\boldsymbol{u}, \boldsymbol{v} \in \bar{\boldsymbol{V}}$, we denote by $\boldsymbol{B}(\boldsymbol{u},\boldsymbol{v})$ the element of $\bar{\boldsymbol{V}}'$ defined by

$$\langle \boldsymbol{B}(\boldsymbol{u},\boldsymbol{v}), \boldsymbol{w}\rangle = b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}), \quad \forall \boldsymbol{w} \in \bar{\boldsymbol{V}} \tag{4.196}$$

and set

$$\boldsymbol{B}(\boldsymbol{u}) = \boldsymbol{B}(\boldsymbol{u},\boldsymbol{u}) \in \bar{\boldsymbol{V}}', \quad \forall \boldsymbol{u} \in \bar{\boldsymbol{V}}. \tag{4.197}$$

For $\boldsymbol{u} \in \bar{\boldsymbol{V}}$, we also denote by $\boldsymbol{N}(\boldsymbol{u})$ the element of $\bar{\boldsymbol{V}}'$ defined by

$$\langle \boldsymbol{N}(\boldsymbol{u}), \boldsymbol{v}\rangle = 2\int_\Omega \mu(\boldsymbol{u}) e_{ij}(\boldsymbol{u}) e_{ij}(\boldsymbol{v})\, d\boldsymbol{x}, \quad \forall \boldsymbol{v} \in \bar{\boldsymbol{V}} \tag{4.198}$$

where $\mu(\boldsymbol{u})$ is the nonlinear viscosity given, e.g., by (2.3) with $\boldsymbol{e} = \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T)$. We assume that $(\boldsymbol{v}, p)$ is a classical solution of (2.2a–d) such that $\boldsymbol{v}(\boldsymbol{x},t)$ and its derivatives of order less than or equal to four tend to zero as $|\boldsymbol{x}| \to +\infty$. If $\boldsymbol{\phi} \in \boldsymbol{J}(\Omega)$, it is easy to see that

$$\begin{aligned} \left(\frac{\partial \boldsymbol{v}}{\partial t}, \boldsymbol{\phi}\right)_{\boldsymbol{L}^2(\Omega)} &+ 2\mu_1 \left(\frac{\partial e_{ij}(\boldsymbol{v})}{\partial x_k}, \frac{\partial e_{ij}(\boldsymbol{\phi})}{\partial x_k}\right)_{\boldsymbol{L}^2(\Omega)} \\ &+ \langle \boldsymbol{N}(\boldsymbol{v}), \boldsymbol{\phi}\rangle + b(\boldsymbol{v},\boldsymbol{v},\boldsymbol{\phi}) = (\boldsymbol{f}, \boldsymbol{\phi})_{\boldsymbol{L}^2(\Omega)}. \end{aligned} \tag{4.199}$$

By continuity, (4.199) holds for each $\boldsymbol{\phi} \in \bar{\boldsymbol{V}}$. This suggests the following weak formulation of the problem (4.1)–(4.5), with $M_i = 0$, $\boldsymbol{g} = \boldsymbol{0}$:

Definition 4.3. Let $\boldsymbol{f}$ and $\boldsymbol{v}_0$ be given with $\boldsymbol{f} \in L^2((0,T);\bar{\boldsymbol{H}})$ and $\boldsymbol{v}_0 \in \bar{\boldsymbol{H}}$. Then $\boldsymbol{v}$ is a weak solution of (4.1)–(4.5) with $M_i = 0$, $\boldsymbol{g} = \boldsymbol{0}$, and $\Omega = \Omega_a$, if $\boldsymbol{v} \in L^2((0,T);\bar{\boldsymbol{V}}) \cap L^\infty((0,T);\bar{\boldsymbol{H}})$, $\boldsymbol{v}(0) = \boldsymbol{v}_0$, and (4.199) is satisfied $\forall \boldsymbol{\phi} \in \bar{\boldsymbol{V}}$.

Remarks. If $\boldsymbol{v}$ only belongs to $L^2((0,T);\bar{\boldsymbol{V}}) \cap L^\infty((0,T);\bar{\boldsymbol{H}})$, then the initial condition need not make sense. However, if $\boldsymbol{v} \in L^2((0,T);\bar{\boldsymbol{V}}) \cap L^\infty((0,T);\bar{\boldsymbol{H}})$ and also satisfies (4.199), $\forall \boldsymbol{\phi} \in \bar{\boldsymbol{V}}$, the following lemma guarantees that $\boldsymbol{v}' \in L^2((0,T);\bar{\boldsymbol{V}}')$; then, by virtue of Lemma A.9, we have that $\boldsymbol{v}$ is equal, a.e., to some continuous function from $[0,T]$ into $\bar{\boldsymbol{H}}$ so that the initial condition $\boldsymbol{v}(0) = \boldsymbol{v}_0$ is meaningful.

Lemma 4.15. *Assume that $\boldsymbol{v}$ belongs to $L^2((0,T);\bar{\boldsymbol{V}}) \cap L^\infty((0,T);\bar{\boldsymbol{H}})$. Then the function $\boldsymbol{B}(\boldsymbol{v}(t))$ defined by*

$$\langle \boldsymbol{B}(\boldsymbol{v}(t)), \boldsymbol{\varphi}\rangle = b(\boldsymbol{v}(t), \boldsymbol{v}(t), \boldsymbol{\varphi}), \quad \forall \boldsymbol{\varphi} \in \bar{\boldsymbol{V}}, \text{ a.e. for } t \in [0,T]$$

belongs to $L^2((0,T);\bar{\boldsymbol{V}}')$. Also, the function $\boldsymbol{N}(\boldsymbol{v}(t))$ given by

$$\langle \boldsymbol{N}(\boldsymbol{v}(t)), \boldsymbol{\varphi}\rangle = 2\int_\Omega \mu(\boldsymbol{e}(\boldsymbol{v}))e_{ij}(\boldsymbol{v})e_{ij}(\boldsymbol{\varphi})\, d\boldsymbol{x}$$

belongs to $L^2((0,T);\bar{\boldsymbol{V}}')$.

Proof. For almost all t, $\boldsymbol{B}(\boldsymbol{v}(t))$ and $\boldsymbol{N}(\boldsymbol{v}(t))$ are elements of $\bar{\boldsymbol{V}}'$, and the measurability of the functions

$$t \in [0,T] \to \boldsymbol{B}(\boldsymbol{v}(t)) \in \bar{\boldsymbol{V}}'$$

and

$$t \in [0,T] \to \boldsymbol{N}(\boldsymbol{v}(t)) \in \bar{\boldsymbol{V}}'$$

is easy to check. Moreover, employing (4.195), the Hölder Inequality, the embedding theorems, and Lemma 4.12, we have for some $c_1 > 0$, $c_2 > 0$,

$$\begin{aligned} |\langle \boldsymbol{B}(\boldsymbol{v}(t)), \boldsymbol{\varphi}\rangle| &= |b(\boldsymbol{v}(t), \boldsymbol{v}(t), \boldsymbol{\varphi})| \\ &= |-b(\boldsymbol{v}(t), \boldsymbol{\varphi}, \boldsymbol{v}(t))| \\ &= \left|\int_\Omega v_i \frac{\partial \phi_j}{\partial x_i} v_j \, d\boldsymbol{x}\right| \\ &\le c_1 \,||\boldsymbol{v}||^2_{\boldsymbol{L}^4(\Omega)}\, ||\boldsymbol{\varphi}||_{\boldsymbol{H}^1_0(\Omega)} \\ &\le c_2 ||\boldsymbol{v}||_{\boldsymbol{L}^2(\Omega)}\, ||\boldsymbol{v}||_{\boldsymbol{H}^1_0(\Omega)}\, ||\boldsymbol{\varphi}||_{\boldsymbol{H}^1_0(\Omega)}. \end{aligned} \tag{4.200}$$

Using the fact that $\mu(\boldsymbol{v}) = \mu_0(\epsilon + e_{ij}e_{ij})^{-\alpha/2} \le \mu_0\epsilon^{-\alpha/2}$, we also have for some $c_3 > 0$, $c_4 > 0$

$$\begin{aligned} |\langle \boldsymbol{N}(\boldsymbol{v}(t), \boldsymbol{\varphi}\rangle| &= \left|2\int_\Omega \mu(\boldsymbol{e}(\boldsymbol{v}))e_{ij}(\boldsymbol{v})e_{ij}(\boldsymbol{\varphi})\, d\boldsymbol{x}\right| \\ &\le \frac{2\mu_0}{\epsilon^{-\alpha/2}} \int_\Omega \left|e_{ij}(\boldsymbol{v})e_{ij}(\boldsymbol{\varphi})\right| d\boldsymbol{x} \\ &\le c_3 \,||\boldsymbol{v}||_{\boldsymbol{H}^1_0(\Omega)}\, ||\boldsymbol{\varphi}||_{\boldsymbol{H}^1_0(\Omega)} \\ &\le c_4 \,||\boldsymbol{v}||_V \, ||\boldsymbol{\varphi}||_V . \end{aligned} \tag{4.201}$$

As a consequence of (4.200) and (4.201), we obtain, therefore, the estimates

$$||\boldsymbol{B}(\boldsymbol{v}(t))||_{\bar{\boldsymbol{V}}'} \le c_5 ||\boldsymbol{v}||_{\boldsymbol{L}^2(\Omega)} \, ||\boldsymbol{v}||_{\bar{\boldsymbol{V}}} \tag{4.202}$$

and

$$||\boldsymbol{N}(\boldsymbol{v}(t))||_{\bar{\boldsymbol{V}}'} \le c_6 \, ||\boldsymbol{v}||_{\bar{\boldsymbol{V}}} \tag{4.203}$$

for some $c_5 > 0$, $c_6 > 0$. Therefore,

$$\begin{aligned}\int_0^T ||\boldsymbol{B}(\boldsymbol{v}(t))||^2_{\bar{\boldsymbol{V}}'} \, dt &\le c_5^2 \int_0^T ||\boldsymbol{v}||^2_{\boldsymbol{L}^2(\Omega)} \, ||\boldsymbol{v}||^2_{\bar{\boldsymbol{V}}} \, dt \\ &\le c_5^2 \, ||\boldsymbol{v}||^2_{L^\infty((0,T);\bar{\boldsymbol{H}})} \int_0^T ||\boldsymbol{v}||^2_{\bar{\boldsymbol{V}}} \, dt < \infty\end{aligned} \tag{4.204}$$

and

$$\int_0^T ||\boldsymbol{N}(\boldsymbol{v}(t))||^2_{\bar{\boldsymbol{V}}'} \, dt \le c_6^2 \int_0^T ||\boldsymbol{v}||^2_{\bar{\boldsymbol{V}}} \, dt \tag{4.205}$$

and the Lemma 4.15 has been proved. □

Remarks. If $\boldsymbol{v}$ satisfies (4.199), $\forall \boldsymbol{\phi} \in \bar{\boldsymbol{V}}$, and $\boldsymbol{v}(0) = \boldsymbol{v}_0$, then by (4.173), (4.174), and Lemma 4.15 one can rewrite (4.199) in the form

$$\frac{d}{dt}\langle \boldsymbol{v}, \boldsymbol{\varphi}\rangle = \big\langle \boldsymbol{f} - 2\mu_1 \bar{\boldsymbol{A}}\,\boldsymbol{v} - \boldsymbol{N}(\boldsymbol{v}) - \boldsymbol{B}(\boldsymbol{v}), \boldsymbol{\varphi}\big\rangle, \quad \forall \boldsymbol{\varphi} \in \bar{\boldsymbol{V}} \tag{4.206}$$

Because $\bar{\boldsymbol{A}}$ is linear, and continuous from $\bar{\boldsymbol{V}}$ into $\bar{\boldsymbol{V}}'$, and $\boldsymbol{v} \in L^2((0,T);\bar{\boldsymbol{V}}')$, the function $\bar{\boldsymbol{A}}\,\boldsymbol{v} \in L^2((0,T);\bar{\boldsymbol{V}}')$; hence $\boldsymbol{f} - 2\mu_1 \bar{\boldsymbol{A}}\,\boldsymbol{v} - \boldsymbol{N}(\boldsymbol{v}) - \boldsymbol{B}(\boldsymbol{v}) \in L^2((0,T);\bar{\boldsymbol{V}}')$. Therefore, (4.205) and Lemma A.8 show that $\boldsymbol{v}' \in L^2((0,T);\bar{\boldsymbol{V}}')$.

Now suppose that $\boldsymbol{f} \in L^2((0,T);\bar{\boldsymbol{H}})$. If $\boldsymbol{v} \in L^2((0,T);\bar{\boldsymbol{V}}) \cap L^\infty((0,T);\bar{\boldsymbol{H}})$ then, as shown above, $\boldsymbol{v}$ satisfies (4.205) and (4.206). By Lemma A.8, (4.199) is equivalent to

$$\boldsymbol{v}' + 2\mu_1 \bar{\boldsymbol{A}}\,\boldsymbol{v} + \boldsymbol{N}(\boldsymbol{v}) + \boldsymbol{B}(\boldsymbol{v}) = \boldsymbol{f}. \tag{4.207}$$

Conversely if $\boldsymbol{v} \in L^2((0,T);\bar{\boldsymbol{V}}) \cap L^\infty((0,T);\bar{\boldsymbol{H}})$, and $\boldsymbol{v}' \in L^2((0,T);\bar{\boldsymbol{V}}')$, then $\boldsymbol{v}$ satisfies (4.199), for all $\boldsymbol{v} \in \bar{\boldsymbol{V}}$. Therefore, an alternative weak formulation of the problem under consideration is the following:

Given $\boldsymbol{f}$ and $\boldsymbol{v}_0$ satisfying $\boldsymbol{f} \in L^2((0,T);\bar{\boldsymbol{H}})$ and $\boldsymbol{v}_0 \in \bar{\boldsymbol{H}}$, find $\boldsymbol{v}$ satisfying

$$\boldsymbol{v} \in L^2((0,T);\bar{\boldsymbol{V}}) \cap L^\infty((0,T);\bar{\boldsymbol{H}}), \quad \boldsymbol{v}' \in L^2((0,T);\bar{\boldsymbol{V}}'), \tag{4.208}$$

$$\boldsymbol{v}' + 2\mu_1 \bar{\boldsymbol{A}}\boldsymbol{v} + \bar{\boldsymbol{N}}(\boldsymbol{v}) + \boldsymbol{B}(\boldsymbol{v}) = \boldsymbol{f}, \tag{4.209}$$

$$\boldsymbol{v}(0) = \boldsymbol{v}_0. \tag{4.210}$$

Any solution of (4.208), (4.209), (4.210) with $\boldsymbol{v}' \in L^2((0,T);\bar{\boldsymbol{V}}')$ is a solution of (4.209) satisfying $\boldsymbol{v}(0) = \boldsymbol{v}_0$ with $\boldsymbol{v} \in L^2((0,T);\bar{\boldsymbol{V}}) \cap L^\infty((0,T);\bar{\boldsymbol{H}})$, and conversely. Because Ω is unbounded, the embeddings $\boldsymbol{H}^2(\Omega) \hookrightarrow \boldsymbol{L}^2(\Omega)$ and $\boldsymbol{H}^1(\Omega) \hookrightarrow \boldsymbol{L}^2(\Omega)$ are not compact so the existence of solutions to (4.208)–(4.210) can not be established directly by using the Galerkin method. Instead, we will first establish the existence of approximate solutions. Then, we will show that there exists a convergent subsequence of approximate solutions whose limit is a solution of our problem. This procedure necessitates the introduction of some notation as follows: using the ideas in [BB2], as delineated in Sect. 2.4, we let $\{\Omega_N\}$, $N = 1, 2, \ldots$ be an expanding sequence of simply connected, bounded subdomains of Ω such that $\Omega_N \to \Omega$, as $N \to \infty$, and $\partial\Omega_N$ of class $\boldsymbol{C}^\infty$. We set

$$\begin{cases} \Gamma_N^+ = \{(x_1, a) \mid (x_1, a) \in \bar{\Omega}_N\}, \\ \Gamma_N^- = \{(x_1, -a) \mid (x_1, -a) \in \bar{\Omega}_N\}, \end{cases} \tag{4.211}$$

$$\boldsymbol{J}(\Omega_N) = \{\boldsymbol{\varphi} \in \boldsymbol{J}(\Omega) \mid \boldsymbol{\varphi} \in (\boldsymbol{\mathcal{D}}(\Omega_N) \cup \Gamma_N^+ \cup \Gamma_N^-)\}, \tag{4.212}$$

$$\begin{cases} \bar{\boldsymbol{V}}_N = \text{ the closure of } \boldsymbol{J}(\Omega_N) \text{ in } \boldsymbol{H}^2(\Omega_N), \\ \bar{\boldsymbol{H}}_N = \text{ the closure of } \boldsymbol{J}(\Omega_N) \text{ in } \boldsymbol{L}^2(\Omega_N), \end{cases} \tag{4.213}$$

and we denote by $\bar{\boldsymbol{V}}_N'$ and $\bar{\boldsymbol{H}}_N'$, respectively, the dual of $\bar{\boldsymbol{V}}_N$ and $\bar{\boldsymbol{H}}_N$. We also define $\bar{\boldsymbol{A}}_N$ by

$$\langle \bar{\boldsymbol{A}}_N \boldsymbol{v}, \boldsymbol{\varphi} \rangle = \left(\frac{\partial e_{ij}(\boldsymbol{v})}{\partial x_k}, \frac{\partial e_{ij}(\boldsymbol{\varphi})}{\partial x_k} \right)_{L^2(\Omega_N)}, \quad \forall \boldsymbol{v}, \boldsymbol{\varphi} \in \bar{\boldsymbol{V}}_N. \tag{4.214}$$

Remarks. (i) It is obvious that $\boldsymbol{J}(\Omega_1) \subset \boldsymbol{J}(\Omega_2) \subset \cdots \subset \boldsymbol{J}(\Omega)$.
(ii) $\forall \boldsymbol{v} \in \bar{\boldsymbol{V}}_N$ ($\boldsymbol{v} \in \bar{\boldsymbol{H}}_N$), if we extend $\boldsymbol{v}$ by setting $\boldsymbol{v} = \boldsymbol{0}$ outside Ω_N, then $\boldsymbol{v} \in \bar{\boldsymbol{V}}_{N+j} \subset \bar{\boldsymbol{V}}$, $j = 1, 2, \ldots$ ($\boldsymbol{v} \in \bar{\boldsymbol{H}}_{N+j} \subset \bar{\boldsymbol{H}}$, $j = 1, 2, \ldots$). i.e. $\bar{\boldsymbol{V}}_1 \subset \bar{\boldsymbol{V}}_2 \subset \cdots \subset \bar{\boldsymbol{V}}$ and $\bar{\boldsymbol{H}}_1 \subset \bar{\boldsymbol{H}}_2 \subset \cdots \subset \bar{\boldsymbol{H}}$.

4.4.3 The Existence Problem for a Sequence of Approximations

In this subsection we will prove an existence theorem for a sequence of approximating problems which are naturally associated with (4.208)–(4.210); these problems may be stated as follows: For $\boldsymbol{f}^N$ and $\boldsymbol{v}_0^N$ satisfying

$$\begin{cases} \boldsymbol{f}^N \in L^2((0,T);\bar{\boldsymbol{H}}_N), \\ \boldsymbol{v}_0^N \in \bar{\boldsymbol{H}}_N, \end{cases} \tag{4.215}$$

find $\boldsymbol{v}^N$ such that

$$\boldsymbol{v}^N \in L^2((0,T);\bar{\boldsymbol{V}}_N) \cap L^\infty((0,T);\bar{\boldsymbol{H}}_N), \quad (\boldsymbol{v}^N)' \in L^2((0,T);\bar{\boldsymbol{V}}'_N) \tag{4.216}$$

$$(\boldsymbol{v}^N)' + 2\mu_1 \bar{\boldsymbol{A}}\boldsymbol{v}^N + \boldsymbol{N}(\boldsymbol{v}^N) + \boldsymbol{B}(\boldsymbol{v}^N) = \boldsymbol{f}^N \tag{4.217}$$

$$\boldsymbol{v}^N(0) = \boldsymbol{v}_0^N. \tag{4.218}$$

Before stating and proving the relevant existence theorem for the system (4.216)–(4.218) we will need to establish the following key lemma:

Lemma 4.16. *Let $\{\boldsymbol{v}^{N,K}\}$ be a sequence of elements in $L^2((0,T);\bar{\boldsymbol{V}}_N) \cap L^\infty((0,T);\bar{\boldsymbol{H}}_N)$ such that, as $K \to \infty$, $\boldsymbol{v}^{N,K} \to \boldsymbol{v}^N$ weakly in $L^2((0,T);\bar{\boldsymbol{V}}_N)$, and weak * in $L^\infty((0,T);\bar{\boldsymbol{H}}_N)$, as well as strongly in $L^2((0,T);\boldsymbol{H}_0^1(\Omega_N))$. Then for any $\boldsymbol{\varphi} \in \mathcal{Y} = \{\boldsymbol{\varphi} \in C((0,T);\bar{\boldsymbol{V}}_N) \mid \boldsymbol{\varphi}' \in L^2((0,T);\bar{\boldsymbol{H}}_N)\}$, we have the following:*

$$\lim_{K\to\infty} \int_0^T \left(\boldsymbol{v}^{N,K}, \boldsymbol{\varphi}'(t)\right)_{\boldsymbol{L}^2(\Omega_N)} dt = \int_0^T \left(\boldsymbol{v}^N(t), \boldsymbol{\varphi}'(t)\right)_{\boldsymbol{L}^2(\Omega_N)} dt, \tag{4.219a}$$

$$\lim_{K\to\infty} \int_0^T \left(\frac{\partial e_{ij}(\boldsymbol{v}^{N,K})}{\partial x_k}, \frac{\partial e_{ij}(\boldsymbol{\varphi}(t))}{\partial x_k}\right)_{L^2(\Omega_N)} dt = \int_0^T \left(\frac{\partial e_{ij}(\boldsymbol{v}^N)}{\partial x_k}, \frac{\partial e_{ij}(\boldsymbol{\varphi}(t))}{\partial x_k}\right)_{L^2(\Omega_N)} dt, \tag{4.219b}$$

$$\lim_{K\to\infty} \int_0^T \left\langle \boldsymbol{N}(\boldsymbol{v}^{N,K}), \boldsymbol{\varphi}(t)\right\rangle dt = \int_0^T \left\langle \boldsymbol{N}(\boldsymbol{v}^N), \boldsymbol{\varphi}(t)\right\rangle dt, \tag{4.219c}$$

$$\lim_{K\to\infty} \int_0^T b\left(\boldsymbol{v}^{N,K}, \boldsymbol{v}^{N,K}, \boldsymbol{\varphi}(t)\right) dt = \int_0^T b\left(\boldsymbol{v}^N, \boldsymbol{v}^N, \boldsymbol{\varphi}(t)\right) dt. \tag{4.219d}$$

Proof. The result in (4.219a) follows directly from the fact that $\boldsymbol{v}^{N,K} \to \boldsymbol{v}^N$ strongly in $L^2((0,T);\boldsymbol{H}_0^1(\Omega_N))$ and $\boldsymbol{\phi} \in L^2((0,T);\bar{\boldsymbol{H}}_N)$. As far as (4.219b) is concerned, we note that the relation $\langle \boldsymbol{v}, \bar{\boldsymbol{A}}\boldsymbol{\phi}\rangle_{\bar{V}_N \times \bar{V}'_N} = \left(\frac{\partial e_{ij}(\boldsymbol{v})}{\partial x_k}, \frac{\partial e_{ij}(\boldsymbol{\varphi})}{\partial x_k}\right)_{L^2(\Omega_N)}$, $\forall \boldsymbol{v}, \boldsymbol{\varphi} \in \bar{\boldsymbol{V}}_N$, and the fact that $\boldsymbol{v}^{N,K} \rightharpoonup \boldsymbol{v}^N$, weakly in $L^2((0,T);\bar{\boldsymbol{V}}_N)$, imply that

$$\begin{aligned}
&\lim_{K\to\infty}\int_0^T \left(\frac{\partial e_{ij}(\boldsymbol{v}^{N,K})}{\partial x_k}, \frac{\partial e_{ij}(\boldsymbol{\varphi}(t))}{\partial x_k}\right)_{L^2(\Omega_N)} dt \\
&\qquad = \lim_{K\to\infty}\int_0^T \left\langle \boldsymbol{v}^{N,K}, \bar{\boldsymbol{A}}\boldsymbol{\varphi}(t)\right\rangle dt \\
&\qquad = \int_0^T \left\langle \boldsymbol{v}^{N}, \bar{\boldsymbol{A}}\boldsymbol{\varphi}(t)\right\rangle dt \\
&\qquad = \int_0^T \left(\frac{\partial e_{ij}(\boldsymbol{v}^{N})}{\partial x_k}, \frac{\partial e_{ij}(\boldsymbol{\varphi}(t))}{\partial x_k}\right)_{L^2(\Omega_N)} dt.
\end{aligned} \tag{4.220}$$

For the result in (4.219c) we remark that a straightforward calculation exhibits the existence of a $c > 0$ for which

$$\left|\left\langle \boldsymbol{N}(\boldsymbol{v}^{N,K}) - \boldsymbol{N}(\boldsymbol{v}^N), \boldsymbol{\varphi}(t)\right\rangle\right| \leq c\left|\left|\boldsymbol{v}^{N,K} - \boldsymbol{v}^N\right|\right|_{\boldsymbol{H}_0^1(\Omega_N)} ||\boldsymbol{\varphi}(t)||_{\boldsymbol{H}_0^1(\Omega_N)}. \tag{4.221}$$

This result, the fact that $\boldsymbol{v}^{N,K} \to \boldsymbol{v}^N$ strongly in $L^2((0,T); \boldsymbol{H}_0^1(\Omega_N))$, and the condition $\boldsymbol{\varphi} \in C((0,T); \bar{\boldsymbol{V}}_N)$, imply that

$$\lim_{K\to\infty}\int_0^T \left\langle \boldsymbol{N}(\boldsymbol{v}^{N,K}), \boldsymbol{\varphi}(t)\right\rangle dt = \int_0^T \left\langle \boldsymbol{N}(\boldsymbol{v}^{N}), \boldsymbol{\varphi}(t)\right\rangle dt. \tag{4.222}$$

Now, because for some $c_1 > 0$, $c_2 > 0$

$$\begin{aligned}
&\left|b\left(\boldsymbol{v}^{N,K}, \boldsymbol{v}^{N,K}, \boldsymbol{\varphi}(t)\right) - b\left(\boldsymbol{v}^{N}, \boldsymbol{v}^{N}, \boldsymbol{\varphi}(t)\right)\right| \\
&\quad = \left|b\left(\boldsymbol{v}^{N,K} - \boldsymbol{v}^N, \boldsymbol{v}^{N,K}, \boldsymbol{\varphi}(t)\right) - b\left(\boldsymbol{v}^{N}, \boldsymbol{v}^{N,K} - \boldsymbol{v}^N, \boldsymbol{\varphi}(t)\right)\right| \\
&\quad = \left|-b\left(\boldsymbol{v}^{N,K} - \boldsymbol{v}^N, \boldsymbol{\varphi}(t), \boldsymbol{v}^{N,K}\right) - b\left(\boldsymbol{v}^{N}, \boldsymbol{\varphi}(t), \boldsymbol{v}^{N,K} - \boldsymbol{v}^N\right)\right| \\
&\quad \leq c_1\left|\left|\boldsymbol{v}^{N,K} - \boldsymbol{v}^N\right|\right|_{\boldsymbol{L}^4(\Omega_N)} ||\boldsymbol{\varphi}(t)||_{\boldsymbol{H}_0^1(\Omega_N)} \left|\left|\boldsymbol{v}^{N,K}\right|\right|_{\boldsymbol{L}^4(\Omega_N)} \\
&\qquad + c_1\left|\left|\boldsymbol{v}^{N}\right|\right|_{\boldsymbol{L}^4(\Omega_N)} ||\boldsymbol{\varphi}(t)||_{\boldsymbol{H}_0^1(\Omega_N)} \left|\left|\boldsymbol{v}^{N,K} - \boldsymbol{v}^N\right|\right|_{\boldsymbol{L}^4(\Omega_N)} \\
&\quad \leq \Bigg(c_2\left|\left|\boldsymbol{v}^{N,K} - \boldsymbol{v}^N\right|\right|^{\frac{1}{2}}_{\boldsymbol{L}^2(\Omega_N)} \left|\left|\boldsymbol{v}^{N,K} - \boldsymbol{v}^N\right|\right|^{\frac{1}{2}}_{\boldsymbol{H}_0^1(\Omega_N)} ||\boldsymbol{\varphi}(t)||_{\boldsymbol{H}_0^1(\Omega_N)} \\
&\qquad\quad \times\left|\left|\boldsymbol{v}^{N,K}\right|\right|^{\frac{1}{2}}_{\boldsymbol{L}^2(\Omega_N)} \left|\left|\boldsymbol{v}^{N,K}\right|\right|^{\frac{1}{2}}_{\boldsymbol{H}_0^1(\Omega_N)}\Bigg) \\
&\qquad + \Bigg(c_2\left|\left|\boldsymbol{v}^{N}\right|\right|^{\frac{1}{2}}_{\boldsymbol{L}^2(\Omega_N)} \left|\left|\boldsymbol{v}^{N}\right|\right|^{\frac{1}{2}}_{\boldsymbol{H}_0^1(\Omega_N)} ||\boldsymbol{\varphi}(t)||_{\boldsymbol{H}_0^1(\Omega_N)} \\
&\qquad\quad \times\left|\left|\boldsymbol{v}^{N,K} - \boldsymbol{v}^N\right|\right|^{\frac{1}{2}}_{\boldsymbol{L}^2(\Omega_N)} \left|\left|\boldsymbol{v}^{N,K} - \boldsymbol{v}^N\right|\right|^{\frac{1}{2}}_{\boldsymbol{H}_0^1(\Omega_N)}\Bigg),
\end{aligned}$$

we have, for some $c_3 > 0$,

$$\begin{aligned}
&\left|\int_0^T \left[b\left(\boldsymbol{v}^{N,K}, \boldsymbol{v}^{N,K}, \boldsymbol{\varphi}(t)\right) - b\left(\boldsymbol{v}^N, \boldsymbol{v}^N, \boldsymbol{\varphi}(t)\right)\right] dt\right| \\
&\quad \le \Bigg(c_3 \left|\left|\boldsymbol{v}^{N,K} - \boldsymbol{v}^N\right|\right|^{\frac{1}{4}}_{L^2((0,T);\bar{\boldsymbol{H}}_N)} \left|\left|\boldsymbol{v}^{N,K} - \boldsymbol{v}^N\right|\right|^{\frac{1}{4}}_{L^2((0,T);\boldsymbol{H}^1_0(\Omega_N))} \\
&\qquad \times \left|\left|\boldsymbol{v}^N\right|\right|^{\frac{1}{4}}_{L^2((0,T);\bar{\boldsymbol{H}}_N)} \left|\left|\boldsymbol{v}^{N,K}\right|\right|^{\frac{1}{4}}_{L^2((0,T);\boldsymbol{H}^1_0(\Omega_N))}\Bigg) \\
&\quad + \Bigg(c_3 \left|\left|\boldsymbol{v}^N\right|\right|^{\frac{1}{4}}_{L^2((0,T);\bar{\boldsymbol{H}}_N)} \left|\left|\boldsymbol{v}^N\right|\right|^{\frac{1}{4}}_{L^2((0,T);\boldsymbol{H}^1_0(\Omega_N))} \\
&\qquad \times \left|\left|\boldsymbol{v}^{N,K} - \boldsymbol{v}^N\right|\right|^{\frac{1}{4}}_{L^2((0,T);\bar{\boldsymbol{H}}_N)} \left|\left|\boldsymbol{v}^{N,K} - \boldsymbol{v}^N\right|\right|^{\frac{1}{4}}_{L^2((0,T);\boldsymbol{H}^1_0(\Omega_N))}\Bigg) \to 0
\end{aligned} \tag{4.223}$$

as $\boldsymbol{v}^{N,K} \to \boldsymbol{v}^N$, strongly in $L^2((0,T); \boldsymbol{H}^1_0(\Omega_N))$, which serves to complete the proof of (4.219d). □

With Lemma 4.16 in hand, we are now in a position to state and prove the following result for the sequence of approximating problems (4.217), (4.218):

Theorem 4.17. *Let* $\boldsymbol{f}^N$ *and* $\boldsymbol{v}_0^N$ *satisfy* (4.215)*. Then there exists a function* $\boldsymbol{v}^N$ *which satisfies* (4.216)*,* (4.217) *and* (4.218)*. Moreover, for any* $t \in (0,T)$*,* $\boldsymbol{v}^N$ *satisfies the energy identity*

$$\begin{aligned}
\left|\left|\boldsymbol{v}^N\right|\right|^2_{\boldsymbol{L}^2(\Omega_N)} + 4\mu_1 \int_0^t \int_{\Omega_N} \frac{\partial e_{ij}(\boldsymbol{v}^N)}{\partial x_k} \cdot \frac{\partial e_{ij}(\boldsymbol{v}^N)}{\partial x_k}\, d\boldsymbol{x}\, d\tau + 2\int_0^t \left\langle \mu(\boldsymbol{v}^N), \boldsymbol{v}^N\right\rangle d\tau \\
= \left|\left|\boldsymbol{v}_0^N\right|\right|^2_{\boldsymbol{L}^2(\Omega_N)} + 2\int_0^t \left(\boldsymbol{f}^N(\tau), \boldsymbol{v}^N\right) d\tau.
\end{aligned} \tag{4.224}$$

Proof. To prove Theorem 4.17 we will use the Galerkin method. Since $\bar{\boldsymbol{V}}_N$ is separable and $\boldsymbol{J}(\Omega_N)$ is dense in $\bar{\boldsymbol{V}}_N$, there exists a sequence $\boldsymbol{w}^1, \boldsymbol{w}^2, \ldots$ of elements of $\boldsymbol{J}(\Omega_N)$ which form a basis of $\bar{\boldsymbol{V}}_N$, with $(\boldsymbol{w}^i, \boldsymbol{w}^j) = 0$, for $i \neq j$, and $\left(\boldsymbol{w}^i, \boldsymbol{w}^j\right)_{\boldsymbol{L}^2(\Omega_N)} =$, for $i = j$. For each fixed K, we define an approximate solution $\boldsymbol{v}^{N,K}$ of (4.217), (4.218) as follows: set

$$\boldsymbol{v}^{N,K} = \sum_{l=1}^{K} g_{kl}(t)\boldsymbol{w}^l. \tag{4.225}$$

Then, $\boldsymbol{v}^{N,K}$ is to satisfy

$$\int_{\Omega_N} \frac{d v^{N,K}{}_i}{dt} w_i^m \, d\boldsymbol{x} + 2\mu_1 \int_{\Omega_N} \frac{\partial e_{ij}(\boldsymbol{v}^{N,K})}{\partial x_k} \frac{\partial e_{ij}(\boldsymbol{w}^m)}{\partial x_k} \, d\boldsymbol{x} + \int_{\Omega_N} 2\mu(\boldsymbol{v}^{N,K}) e_{ij}(\boldsymbol{v}^{N,K}) e_{ij}(\boldsymbol{w}^m) \, d\boldsymbol{x} + \int_{\Omega_N} v^{N,K}{}_j \frac{\partial v^{N,K}{}_i}{\partial x_j} w_i^m \, d\boldsymbol{x} - \int_{\Omega_N} f_i^N w_i^m \, d\boldsymbol{x} = 0 \tag{4.226}$$

for $m = 1, 2, \ldots, K$ and

$$\boldsymbol{v}^{N,K}(0) = \boldsymbol{v}^{N,K}{}_0 \tag{4.227}$$

where $\boldsymbol{v}^{N,K}(0)$ is the orthogonal projection in $\bar{\boldsymbol{H}}_N$ of $\boldsymbol{v}_0^N$ onto the space spanned by $\boldsymbol{w}^1, \boldsymbol{w}^2, \ldots, \boldsymbol{w}^k$.

Equations (4.225), (4.226), (4.227) yield a nonlinear differential system for the functions $g_{1K}, g_{2K}, \ldots, g_{KK}$, namely,

$$g'_{lK} = \left(\boldsymbol{f}^N(t), \boldsymbol{w}^l\right)_{\boldsymbol{L}^2(\Omega_N)} + \phi_l(g_{1K}, g_{2K}, \ldots, g_{KK}) \tag{4.228}$$

$$g_{lK}(0) = \text{ the } l\text{-th component of } \boldsymbol{v}^{N,K}{}_0 \tag{4.229}$$

where ϕ_l, $l = 1, \ldots, K$ is a Lipschitz function of $g_{1K}, g_{2K}, \ldots, g_{KK}$ on any bounded domain of R^K because of the regularity of $\mu(\boldsymbol{v})$. Since the $F_l(t) = \left(\boldsymbol{f}^N(t), \boldsymbol{w}^l\right)_{\boldsymbol{L}^2(\Omega_N)}$ are square integrable, there exists a maximal solution of (4.228), (4.229) on $[0, t_K]$. If $t_K < T$, then $\left|\left|\boldsymbol{v}^{N,K}\right|\right|_{\boldsymbol{L}^2(\Omega_N)}$ must tend to $+\infty$ as $t \to t_K$; the a priori estimates we shall prove show that this does not happen and therefore $t_K = T$.

In order to proceed we multiply (4.226) by $g_{lK}(t)$ and sum the resulting equations for $l = 1, \ldots, K$. Using the fact that $b(\boldsymbol{v}^{N,K}, \boldsymbol{v}^{N,K}, \boldsymbol{v}^{N,K}) = 0$, we obtain

$$\int_{\Omega_N} \frac{d v^{N,K}{}_i}{dt} v^{N,K}{}_i \, d\boldsymbol{x} + 2\mu_1 \int_{\Omega_N} \frac{\partial e_{ij}(\boldsymbol{v}^{N,K})}{\partial x_k} \frac{\partial e_{ij}(\boldsymbol{v}^{N,K})}{\partial x_k} \, d\boldsymbol{x} + \int_{\Omega_N} 2\mu(\boldsymbol{v}^{N,K}) e_{ij}(\boldsymbol{v}^{N,K}) e_{ij}(\boldsymbol{v}^{N,K}) \, d\boldsymbol{x} - \int_{\Omega_N} f_i^N v^{N,K}{}_i \, d\boldsymbol{x} = 0. \tag{4.230}$$

Dropping the positive term $\int_{\Omega_N} 2\mu(\boldsymbol{v}^{N,K})e_{ij}(\boldsymbol{v}^{N,K})e_{ij}(\boldsymbol{v}^{N,K})\,d\boldsymbol{x}$ in (4.230), and applying Lemma 4.14, Young's inequality (see Appendix A), and the embedding $\boldsymbol{H}^2(\Omega_N) \hookrightarrow \boldsymbol{L}^2(\Omega_N)$, we find that there exists a constant $c_1 > 0$, depending on μ_1 and a, such that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||\boldsymbol{v}^{N,K}||^2_{\boldsymbol{L}^2(\Omega_N)}+c_1||\boldsymbol{v}^{N,K}||^2_{\boldsymbol{H}^2(\Omega_N)}&\\ &\leq ||\boldsymbol{f}^N||_{\boldsymbol{L}^2(\Omega_N)}||\boldsymbol{v}^{N,K}||_{\boldsymbol{L}^2(\Omega_N)}\\ &\leq \frac{1}{2c_1}||\boldsymbol{f}^N||_{\boldsymbol{L}^2(\Omega_N)} + \frac{1}{2}c_1||\boldsymbol{v}^{N,K}||^2_{\boldsymbol{L}^2(\Omega_N)}\\ &\leq \frac{1}{2c_1}||\boldsymbol{f}^N||^2_{\boldsymbol{L}^2(\Omega_N)} + \frac{1}{2}c_1||\boldsymbol{v}^{N,K}||^2_{\boldsymbol{H}^2(\Omega_N)}.\end{aligned} \tag{4.231}$$

Therefore, for some $c_2 > 0$, depending on μ_1 and a, it follows that

$$\frac{1}{2}\frac{d}{dt}||\boldsymbol{v}^{N,K}||^2_{\boldsymbol{L}^2(\Omega_N)} + c_1||\boldsymbol{v}^{N,K}||_{\boldsymbol{H}^2(\Omega_N)} \leq c_2||\boldsymbol{f}^N||^2_{\boldsymbol{L}^2(\Omega_N)}. \tag{4.232}$$

Integrating (4.232) from 0 to $s \leq T$ now yields the estimate

$$\begin{aligned}||\boldsymbol{v}^{N,K}(s)||^2_{\boldsymbol{L}^2(\Omega_N)}+c_1\int_0^t ||\boldsymbol{v}^{N,K}(\tau)||^2_{\boldsymbol{H}^2(\Omega_N)}\,d\tau&\\ &\leq ||\boldsymbol{v}^{N,K}{}_0(s)||_{\boldsymbol{L}^2(\Omega_N)} + c_2\int_0^t ||\boldsymbol{f}^N||^2_{\boldsymbol{L}^2(\Omega_N)}\,d\tau\\ &\leq ||\boldsymbol{v}_0(s)||^2_{\boldsymbol{L}^2(\Omega_N)} + c_2\int_0^t ||\boldsymbol{f}^N||^2_{\boldsymbol{L}^2(\Omega_N)}\,d\tau.\end{aligned} \tag{4.233}$$

As a consequence of (4.233), we have

$$\sup_{s\in[0,T]} ||\boldsymbol{v}^{N,K}(s)||^2_{\boldsymbol{L}^2(\Omega_N)} \leq ||\boldsymbol{v}_0(s)||^2_{\boldsymbol{L}^2(\Omega_N)} + c_2\int_0^t ||\boldsymbol{f}^N||^2_{\boldsymbol{L}^2(\Omega_N)}\,d\tau \tag{4.234}$$

and

$$\int_0^t ||\boldsymbol{v}^{N,K}(\tau)||^2_{\boldsymbol{H}^2(\Omega_N)}\,d\tau \leq \frac{1}{c_1}||\boldsymbol{v}_0(s)||^2_{\boldsymbol{L}^2(\Omega_N)} + \frac{c_2}{c_1}\int_0^t ||\boldsymbol{f}^N||^2_{\boldsymbol{L}^2(\Omega_N)}\,d\tau. \tag{4.235}$$

Now, let $\boldsymbol{\varphi} \in C^1([0,T],\bar{\boldsymbol{V}}_N)$ and let $\boldsymbol{\varphi}^K$ be the projection of $\boldsymbol{\varphi}$, in $\bar{\boldsymbol{V}}_N$, onto the space generated by $\boldsymbol{w}^1, \boldsymbol{w}^2, \ldots, \boldsymbol{w}^K$. By (4.226), we have

$$\begin{aligned}\int_{\Omega_N} \frac{d v^{N,K}{}_i}{dt} w_i^m \, d\boldsymbol{x} &= -2\mu_1 \int_{\Omega_N} \frac{\partial e_{ij}(\boldsymbol{v}^{N,K})}{\partial x_k} \frac{\partial e_{ij}(\boldsymbol{w}^m)}{\partial x_k} \, d\boldsymbol{x} \\ &\quad - \int_{\Omega_N} 2\mu(\boldsymbol{v}^{N,K}) e_{ij}(\boldsymbol{v}^{N,K}) e_{ij}(\boldsymbol{w}^m) \, d\boldsymbol{x} \\ &\quad - \int_{\Omega_N} v^{N,K}{}_j \frac{\partial v^{N,K}{}_i}{\partial x_j} w_i^m \, d\boldsymbol{x} \\ &\quad + \int_{\Omega_N} f_i^N w_i^m \, d\boldsymbol{x}, \quad m = 1, 2, \ldots, K.\end{aligned} \tag{4.236}$$

Using (4.236), and the definition of $\boldsymbol{\varphi}^K$, we obtain

$$\begin{aligned}\int_0^T \int_{\Omega_N} \frac{d v^{N,K}{}_i}{dt} \boldsymbol{\varphi}^K \, d\boldsymbol{x} \, dt &= 2\mu_1 \int_0^T \int_{\Omega_N} \frac{\partial e_{ij}(\boldsymbol{v}^{N,K})}{\partial x_k} \frac{\partial e_{ij}(\boldsymbol{\varphi}^K)}{\partial x_k} \, d\boldsymbol{x} \, dt \\ &\quad - \int_0^T \int_{\Omega_N} 2\mu(\boldsymbol{v}^{N,K}) e_{ij}(\boldsymbol{v}^{N,K}) e_{ij}(\boldsymbol{\varphi}^K) \, d\boldsymbol{x} \, dt \\ &\quad - \int_0^T \int_{\Omega_N} v^{N,K}{}_j \frac{\partial v^{N,K}{}_i}{\partial x_j} \varphi_i^K \, d\boldsymbol{x} \, dt \\ &\quad + \int_0^T \int_{\Omega_N} \boldsymbol{f}^N \boldsymbol{\varphi}^K \, d\boldsymbol{x} \, dt \\ &= I_1 + I_2 + I_3 + I_4.\end{aligned} \tag{4.237}$$

However, by virtue of (4.235), for some $C_1 > 0$,

$$|I_1 + I_4| \leq C_1 \big\| \boldsymbol{\varphi}^K \big\|_{L^2((0,T);\bar{V}_N)}. \tag{4.238}$$

Also, as

$$\mu(\boldsymbol{v}^{N,K}) \leq \mu_0 \epsilon^{-\frac{\alpha}{2}}$$

we have, for some $C_2 > 0$,

$$\begin{aligned}|I_2| &\leq 2\mu_0 \epsilon^{-\frac{\alpha}{2}} \int_0^T \int_{\Omega_N} \big| e_{ij}(\boldsymbol{v}^{N,K}) e_{ij}(\boldsymbol{\varphi}^K) \big| \, d\boldsymbol{x} \, dt \\ &\leq C_2 \big\| \boldsymbol{\varphi}^K \big\|_{\boldsymbol{L}^2((0,T);\bar{V}_N)}.\end{aligned} \tag{4.239}$$

Employing integration by parts in the integral I_3, and using the fact that $\operatorname{div} \boldsymbol{v}^{N,K} = 0$, as well as Lemma 4.12 yields, for some $C_3 > 0$,

$$\begin{aligned}
|I_3| &= \left| -\int_0^T \int_{\Omega_N} v^{N,K}{}_j \frac{\partial v^{N,K}{}_i}{\partial x_j} \varphi_i^K \, d\boldsymbol{x}\, dt \right| \\
&\le \left| \int_0^T \int_{\Omega_N} v^{N,K}{}_j v^{N,K}{}_i \frac{\varphi_i^K}{\partial x_j} \, d\boldsymbol{x}\, dt \right| \\
&\le \int_0^T \left(\int_{\Omega_N} \left| v^{N,K}{}_j \right|^4 \right)^{\frac{1}{4}} \left(\int_{\Omega_N} \left| v^{N,K}{}_i \right|^4 \right)^{\frac{1}{4}} \left(\int_{\Omega_N} \left| \frac{\partial \varphi_i^K}{\partial x_j} \right|^2 \right)^{\frac{1}{2}} dt \\
&\le \int_0^T \Big\{ ||\boldsymbol{v}^{N,K}||^{\frac{1}{2}}_{\boldsymbol{L}^2(\Omega_N)} ||\boldsymbol{v}^{N,K}||^{\frac{1}{2}}_{\boldsymbol{H}^1(\Omega_N)} ||\boldsymbol{v}^{N,K}||^{\frac{1}{2}}_{\boldsymbol{L}^2(\Omega_N)} \times \\
&\qquad ||\boldsymbol{v}^{N,K}||^{\frac{1}{2}}_{\boldsymbol{H}^1(\Omega_N)} ||\boldsymbol{\varphi}^K||_{\boldsymbol{H}^1(\Omega_N)} \Big\} \, dt \\
&\le C_3 \int_0^T ||\boldsymbol{v}^{N,K}||_{\boldsymbol{L}^2(\Omega_N)} ||\boldsymbol{v}^{N,K}||_{\boldsymbol{H}^1(\Omega_N)} ||\boldsymbol{\varphi}^K||_{\boldsymbol{H}^1(\Omega_N)} \, dt
\end{aligned} \tag{4.240}$$

and, thus, by virtue of (4.234), we obtain for some $C_4 > 0$, $C_5 > 0$, the bounds

$$\begin{aligned}
|I_3| &\le C_4 \int_0^T ||\boldsymbol{v}^{N,K}||_{\boldsymbol{H}^1(\Omega_N)} ||\boldsymbol{\varphi}^K||_{\boldsymbol{H}^1(\Omega_N)} \, dt \\
&\le C_5 ||\boldsymbol{\varphi}^K||_{L^2((0,T);\bar{\boldsymbol{V}}_N)}.
\end{aligned} \tag{4.241}$$

Combining (4.237) with (4.238), (4.239) and (4.241) we may conclude that, for some $C > 0$,

$$\begin{aligned}
\left| \int_0^T \int_{\Omega_N} \frac{d v^{N,K}{}_i}{dt} \boldsymbol{\varphi} \, d\boldsymbol{x}\, dt \right| &\le C ||\boldsymbol{\varphi}^K||_{L^2((0,T);\bar{\boldsymbol{V}}_N)} \\
&\le C ||\boldsymbol{\varphi}||_{L^2((0,T);\bar{\boldsymbol{V}}_N)}
\end{aligned} \tag{4.242}$$

as well as

$$\left|\left| \frac{d v^{N,K}{}_i}{dt} \right|\right|_{L^2((0,T);\bar{\boldsymbol{V}}'_N)} \le C. \tag{4.243}$$

However, (4.234) implies that the sequence $\boldsymbol{v}^{N,K}$ remains in a bounded set of $L^\infty((0,T); \bar{\boldsymbol{H}}_N)$; therefore, (4.235) and (4.243) imply that the sequence $\boldsymbol{v}^{N,K}$ remains in a bounded set of the Banach space $\mathcal{Y} = \{\boldsymbol{v} \in L^2((0,T); \bar{\boldsymbol{V}}_N) \mid \boldsymbol{v}' \in L^2((0,T); \bar{\boldsymbol{V}}'_N)\}$. By Lemma A.9, $\mathcal{Y} \subset L^2((0,T); \boldsymbol{H}_0^1(\Omega_N))$ is compact; hence, we can extract a subsequence $\boldsymbol{v}^{N,K'}$ of $\boldsymbol{v}^{N,K}$ such that

$$\boldsymbol{v}^{N,K'} \to (\boldsymbol{v}^N)^*, \text{ in the weak * topology of } L^\infty((0,T);\bar{\boldsymbol{H}}_N),$$
$$\boldsymbol{v}^{N,K'} \to \boldsymbol{v}^N, \text{ in the weak topology of } L^2((0,T);\bar{\boldsymbol{V}}_N),$$

and

$$\boldsymbol{v}^{N,K'} \to \boldsymbol{v}^N, \text{ in the strong topology of } L^2((0,T);\boldsymbol{H}_0^1(\Omega_N)).$$

Thus, as $K' \to +\infty$,

$$\int_0^T \left(\boldsymbol{v}^{N,K'} - (\boldsymbol{v}^N), \boldsymbol{\varphi}\right)_{\boldsymbol{L}^2(\Omega_N)} dt \to 0, \quad \forall \boldsymbol{\varphi} \in L^1((0,T);\bar{\boldsymbol{H}}_N) \tag{4.244}$$

and

$$\int_0^T \left\langle \boldsymbol{v}^{N,K'} - \boldsymbol{v}^N, \boldsymbol{\varphi}\right\rangle_{\bar{V}\times\bar{V}'} dt \to 0, \quad \forall \boldsymbol{\varphi} \in L^2((0,T);\bar{\boldsymbol{V}}'_N). \tag{4.245}$$

If we now use the fact that

$$L^2((0,T);\bar{\boldsymbol{H}}_N) \hookrightarrow L^1((0,T);\bar{\boldsymbol{H}}_N)$$

as well as the embedding

$$L^2((0,T);\bar{\boldsymbol{H}}_N) \hookrightarrow L^2((0,T);\bar{\boldsymbol{V}}'_N)$$

then (4.244) and (4.245) yield

$$\lim_{K'\to\infty} \int_0^T \left(\boldsymbol{v}^{N,K'} - (\boldsymbol{v}^N)^*, \boldsymbol{\varphi}\right) dt \to 0, \quad \forall \boldsymbol{\varphi} \in L^2((0,T);\bar{\boldsymbol{H}}_N) \tag{4.246}$$

and

$$\lim_{K'\to\infty} \int_0^T \left\langle \boldsymbol{v}^{N,K'} - \boldsymbol{v}^N, \boldsymbol{\varphi}\right\rangle_{\bar{V}\times\bar{V}'} dt \to 0, \quad \forall \boldsymbol{\varphi} \in L^2((0,T);\bar{\boldsymbol{H}}_N). \tag{4.247}$$

By subtracting (4.247) from (4.246), we find that $\boldsymbol{v}^N = (\boldsymbol{v}^N)^* \in L^2((0,T);\bar{\boldsymbol{V}}_N) \cap L^\infty((0,T);\bar{\boldsymbol{H}}_N)$; thus, we can extract a subsequence $\boldsymbol{v}^{N,K'}$ from $\boldsymbol{v}^{N,K}$ such that, as $K' \to \infty$,

$$\begin{cases} \boldsymbol{v}^{N,K'} \to \boldsymbol{v}^N, \text{ in } L^2((0,T);\bar{\boldsymbol{H}}_N), \text{ weak } *, \\ \boldsymbol{v}^{N,K'} \rightharpoonup \boldsymbol{v}^N, \text{ in } L^2((0,T);\bar{\boldsymbol{V}}_N), \text{ weakly}, \\ \boldsymbol{v}^{N,K'} \to \boldsymbol{v}^N, \text{ in } L^2((0,T);\bar{\boldsymbol{H}}_N), \text{ strongly}. \end{cases} \tag{4.248}$$

Now, let $\phi \in C^1[0, T]$, with $\phi(0) = 0$. We multiply (4.226) by ϕ, and then integrate by parts, so as to obtain

$$
\begin{aligned}
-\int_0^T \left(\boldsymbol{v}^{N,K'}, \phi'(t)\boldsymbol{w}^m\right)_{\boldsymbol{L}^2(\Omega_N)} dt + 2\mu_1 \int_0^T \left(\frac{\partial e_{ij}(\boldsymbol{v}^{N,K'})}{\partial x_k}, \frac{\partial e_{ij}(\phi(t)\boldsymbol{w}^m)}{\partial x_k}\right)_{L^2(\Omega_N)} dt \\
+ \int_0^T \left\langle \boldsymbol{N}(\boldsymbol{v}^{N,K'}), \phi(t)\boldsymbol{w}^m \right\rangle dt \\
+ \int_0^T b\left(\boldsymbol{v}^{N,K'}, \boldsymbol{v}^{N,K'}, \phi(t)\boldsymbol{w}^m\right) dt \\
= \int_0^T \left(\boldsymbol{f}^N, \phi(t)\boldsymbol{w}^m\right)_{\boldsymbol{L}^2(\Omega_N)} dt + \left(\boldsymbol{v}^{N,K'}{}_0, \boldsymbol{w}^m\right)_{\boldsymbol{L}^2(\Omega_N)} \phi(0).
\end{aligned}
\tag{4.249}
$$

Using Lemma 4.16, and taking the limit as $m \to \infty$ in (4.249), yields

$$
\begin{aligned}
-\int_0^T \left(\boldsymbol{v}^{N,K'}, \phi'(t)\boldsymbol{\varphi}\right)_{\boldsymbol{L}^2(\Omega_N)} dt + 2\mu_1 \int_0^T \left(\frac{\partial e_{ij}(\boldsymbol{v}^{N,K'})}{\partial x_k}, \frac{\partial e_{ij}(\phi(t)\boldsymbol{\varphi})}{\partial x_k}\right)_{L^2(\Omega_N)} dt \\
+ \int_0^T \left\langle \boldsymbol{N}(\boldsymbol{v}^{N,K'}), \phi(t)\boldsymbol{\varphi} \right\rangle dt \\
+ \int_0^T b\left(\boldsymbol{v}^{N,K'}, \boldsymbol{v}^{N,K'}, \phi(t)\boldsymbol{\varphi}\right) dt \\
= \int_0^T (\boldsymbol{f}^N, \phi(t)\boldsymbol{\varphi})_{\boldsymbol{L}^2(\Omega_N)} dt + \left(\boldsymbol{v}^{N,K'}{}_0, \boldsymbol{\varphi}\right)_{\boldsymbol{L}^2(\Omega_N)} \phi(0).
\end{aligned}
\tag{4.250}
$$

Equation (4.250) holds for any $\boldsymbol{\varphi}$ expressible as a linear combination of $\boldsymbol{w}^1, \boldsymbol{w}^2, \ldots$; also, it holds, by continuity, for any $\boldsymbol{\varphi} \in \bar{\boldsymbol{V}}_N$. Now let $\phi \in \mathcal{D}(0, T)$; then

$$
\begin{aligned}
-\int_0^T \left(\boldsymbol{v}^{N,K'}, \boldsymbol{\varphi}\right) \phi'(t)\, dt \\
= \int_0^T \Bigg\{ 2\mu_1 \left(\frac{\partial e_{ij}(\boldsymbol{v}^{N,K'})}{\partial x_k}, \frac{\partial e_{ij}(\boldsymbol{\varphi})}{\partial x_k}\right)_{\boldsymbol{L}^2(\Omega_N)} - \left\langle \boldsymbol{N}(\boldsymbol{v}^{N,K'}), \boldsymbol{\varphi} \right\rangle \\
- b\left(\boldsymbol{v}^{N,K'}, \boldsymbol{v}^{N,K'}, \boldsymbol{\varphi}\right) + (\boldsymbol{f}^N, \boldsymbol{\varphi})_{\boldsymbol{L}^2(\Omega)} \Bigg\} \phi(t)\, dt.
\end{aligned}
\tag{4.251}
$$

Therefore, $\forall \boldsymbol{\varphi} \in \bar{\boldsymbol{V}}_N$,

$$\begin{aligned}\frac{d}{dt}\left(\boldsymbol{v}^{N,K'}, \boldsymbol{\varphi}\right)_{\boldsymbol{L}^2(\Omega_N)} &+ 2\mu_1\left(\frac{\partial e_{ij}(\boldsymbol{v}^{N,K'})}{\partial x_k}, \frac{\partial e_{ij}(\boldsymbol{\varphi})}{\partial x_k}\right)_{\boldsymbol{L}^2(\Omega_N)} \\ &+ \left\langle \boldsymbol{N}(\boldsymbol{v}^{N,K'}), \boldsymbol{\varphi}\right\rangle + b\left(\boldsymbol{v}^{N,K'}, \boldsymbol{v}^{N,K'}, \boldsymbol{\varphi}\right) \\ &= \left(\boldsymbol{f}^N \boldsymbol{\varphi}\right)_{\boldsymbol{L}^2(\Omega)}.\end{aligned} \tag{4.252}$$

in the distribution sense. By Lemma A.8, $\boldsymbol{v}^N$ satisfies (4.217) and $(\boldsymbol{v}^N)' \in L^2((0,T); \bar{\boldsymbol{V}}'_N)$. We want to show next that $\boldsymbol{v}^N$ satisfies (4.218); to do this we multiply (4.252) by $\phi(t)$ and integrate from 0 to T so as to obtain

$$\begin{aligned}-\int_0^T \left(\boldsymbol{v}^{N,K'}, \boldsymbol{\varphi}\right)_{\boldsymbol{L}^2(\Omega)} \phi'(t)\, dt + \int_0^T \left\{-2\mu_1\left(\frac{\partial e_{ij}(\boldsymbol{v}^{N,K'})}{\partial x_k}, \frac{\partial e_{ij}(\phi(t)\boldsymbol{\varphi})}{\partial x_k}\right)_{L^2(\Omega_N)}\right\} dt \\ + \int_0^T \left\langle \boldsymbol{N}(\boldsymbol{v}^{N,K'}), \phi(t)\boldsymbol{\varphi}\right\rangle dt + \int_0^T b\left(\boldsymbol{v}^{N,K'}, \boldsymbol{v}^{N,K'}, \phi(t)\boldsymbol{\varphi}\right) dt \\ = \int_0^T \left(\boldsymbol{f}^N, \phi(t)\boldsymbol{\varphi}\right)_{\boldsymbol{L}^2(\Omega_N)} dt + \left(\boldsymbol{v}^N(0), \boldsymbol{\varphi}\right)_{\boldsymbol{L}^2(\Omega_N)} \phi(0).\end{aligned} \tag{4.253}$$

By comparison with (4.250), we may conclude that

$$\left(\boldsymbol{v}^N(0) - \boldsymbol{v}_0^N, \boldsymbol{v}\right)_{\boldsymbol{L}^2(\Omega_N)} \phi(0) = 0, \quad \forall \boldsymbol{v} \in \bar{\boldsymbol{V}}_N \tag{4.254}$$

and, therefore, if we choose $\phi(0) \neq 0$, we obtain

$$\left(\boldsymbol{v}^N(0) - \boldsymbol{v}_0^N, \boldsymbol{v}\right)_{\boldsymbol{L}^2(\Omega_N)} = 0, \quad \forall \boldsymbol{v} \in \bar{\boldsymbol{V}}_N \tag{4.255}$$

which implies that $\boldsymbol{v}^N(0) = \boldsymbol{v}_0^N$. Our last task is to establish the energy inequality (4.224); we begin by noting that as $\boldsymbol{v}' \in L^2((0,T); \bar{\boldsymbol{V}}'_N)$ and $\boldsymbol{v} \in L^2((0,T); \bar{\boldsymbol{V}}_N)$, Lemma A.9 implies that $\boldsymbol{v} \in C([0,T]; \bar{\boldsymbol{H}}_N)$ as well as

$$\frac{d}{dt}||\boldsymbol{v}||^2_{\boldsymbol{L}^2(\Omega_N)} = 2\left(\boldsymbol{v}', \boldsymbol{v}\right)_{\boldsymbol{L}^2(\Omega_N)} \tag{4.256}$$

in the distribution sense. Taking the inner product of (4.217) with $\boldsymbol{v}^N$, and using the fact that $b(\boldsymbol{v}^N, \boldsymbol{v}^N, \boldsymbol{v}^N) = 0$, as well as (4.256), we obtain

$$\begin{aligned}\frac{d}{dt}||\boldsymbol{v}^N||^2_{\boldsymbol{L}^2(\Omega_N)} + 4\mu_1 \int_{\Omega_N} \frac{\partial e_{ij}(\boldsymbol{v}^N)}{\partial x_k}\frac{\partial e_{ij}(\boldsymbol{v}^N)}{\partial x_k}\, d\boldsymbol{x} + 2\left\langle \boldsymbol{N}(\boldsymbol{v}^N), \boldsymbol{v}^N\right\rangle \\ = 2\left(\boldsymbol{f}^N, \boldsymbol{v}^N\right)_{\boldsymbol{L}^2(\Omega_N)}.\end{aligned} \tag{4.257}$$

By integrating this last equation from 0 to t, we find that

$$\begin{aligned} ||\boldsymbol{v}^N||^2_{\boldsymbol{L}^2(\Omega_N)} + 4\mu_1 \int_0^t \int_{\Omega_N} \frac{\partial e_{ij}(\boldsymbol{v}^N)}{\partial x_k} \frac{\partial e_{ij}(\boldsymbol{v}^N)}{\partial x_k}\, d\boldsymbol{x} + 2\int_0^t \langle \boldsymbol{N}(\boldsymbol{v}^N), \boldsymbol{v}^N \rangle \, d\tau \\ = ||\boldsymbol{v}_0^N||^2_{\boldsymbol{L}^2(\Omega_N)} + 2\int_0^t \left(\boldsymbol{f}^N, \boldsymbol{v}^N\right)_{\boldsymbol{L}^2(\Omega_N)} d\tau \end{aligned} \tag{4.258}$$

which completes the proof of Theorem 4.17. □

4.4.4 Existence of Incompressible, Bipolar, Viscous Flow in Ω_a

With the existence result of Theorem 4.17 for the approximation (4.216)–(4.218) in hand, we are now in a position to establish both existence and uniqueness for (4.208)–(4.210), given that $\boldsymbol{f} \in L^2((0,T); \bar{\boldsymbol{H}})$ and $\boldsymbol{v}_0 \in \bar{\boldsymbol{H}}$. Existence of solutions for the problem (4.208)–(4.210) will be established below in Theorem 4.18, while the uniqueness result will follow in Sect. 4.4.5, i.e., Theorem 4.19. The precise existence result is the following:

Theorem 4.18. *Given $\boldsymbol{f} \in L^2((0,T); \bar{\boldsymbol{H}})$ and $\boldsymbol{v}_0 \in \bar{\boldsymbol{H}}$, there exists a solution $\boldsymbol{v}$ of* (4.209)*,* (4.210) *which satisfies the regularity conditions* (4.208)*.*

Proof. Let $\bar{\boldsymbol{V}}$ and $\bar{\boldsymbol{H}}$ be defined by (4.169), (4.170) and $\bar{\boldsymbol{V}}_N$, $\bar{\boldsymbol{H}}_N$ by (4.213) so that $\bar{\boldsymbol{V}}_1 \subset \bar{\boldsymbol{V}}_2 \subset \cdots \subset \bar{\boldsymbol{V}}$ and $\bar{\boldsymbol{H}}_1 \subset \bar{\boldsymbol{H}}_2 \subset \cdots \subset \bar{\boldsymbol{H}}$. For $\boldsymbol{f} \in L((0,T); \bar{\boldsymbol{H}})$, let $\boldsymbol{f}^N$ be the projection of $\boldsymbol{f}(t) \in \bar{\boldsymbol{H}}$ onto $\bar{\boldsymbol{H}}_N$, $t \in (0,T)$. Similarly, for $\boldsymbol{v}_0 \in \bar{\boldsymbol{H}}$ let $\boldsymbol{v}_0^N$ be the projection of $\boldsymbol{v}_0$ onto $\bar{\boldsymbol{H}}_N$. Then we have $\boldsymbol{f}^N \in L((0,T); \bar{\boldsymbol{H}}_N)$ and $\boldsymbol{v}_0^N \in \bar{\boldsymbol{H}}_N$. Furthermore, it is obvious that

$$||\boldsymbol{f}^N||_{L^2((0,T);\bar{\boldsymbol{H}}_N)} \le ||\boldsymbol{f}||_{L^2((0,T);\bar{\boldsymbol{H}})}, \tag{4.259a}$$

$$||\boldsymbol{v}_0^N||_{\boldsymbol{L}^2(\Omega_N)} \le ||\boldsymbol{v}_0||_{\boldsymbol{L}^2(\Omega)}. \tag{4.259b}$$

Now, let $\boldsymbol{v}^N$ be a solution of (4.216)–(4.218) corresponding to $\boldsymbol{f}^N$ and $\boldsymbol{v}_0^N$; the existence of such a solution is guaranteed by Theorem 4.17. By Theorem 4.17, and an application of Hölder's inequality, (4.224) yields

$$\begin{aligned} ||\boldsymbol{v}^N||^2_{\boldsymbol{L}^2(\Omega_N)} + 4\mu_1 \int_0^t \int_{\Omega_N} \frac{\partial e_{ij}(\boldsymbol{v}^N)}{\partial x_k} \frac{\partial e_{ij}(\boldsymbol{v}^N)}{\partial x_k}\, d\boldsymbol{x}\, d\tau \\ + 2\int_0^t \langle \mu(\boldsymbol{v}^N), \boldsymbol{v}^N \rangle \, d\tau \\ = ||\boldsymbol{v}_0^N||^2_{\boldsymbol{L}^2(\Omega_N)} + 2\int_0^t ||\boldsymbol{f}^N(\tau)||_{\boldsymbol{L}^2(\Omega_N)} \cdot ||\boldsymbol{v}^N||_{\boldsymbol{L}^2(\Omega_N)}\, d\tau. \end{aligned} \tag{4.260}$$

Thus, if we use Lemma 4.14, and Young's inequality, we find that there exist constants $c_1, c_2 > 0$, independent of N, such that

$$\begin{aligned} &||\boldsymbol{v}^N||^2_{\boldsymbol{L}^2(\Omega_N)} + 2c_1 \int_0^t ||\boldsymbol{v}^N||^2_{\boldsymbol{H}^2(\Omega_N)}\, d\tau \\ &\quad + 2\int_0^t \langle \mu(\boldsymbol{v}^N), \boldsymbol{v}^N\rangle\, d\tau \\ &\quad \le ||\boldsymbol{v}_0^N||^2_{\boldsymbol{L}^2(\Omega_N)} + c_2 \int_0^t ||\boldsymbol{f}^N(\tau)||^2_{\boldsymbol{L}^2(\Omega_N)} + c_1 \int_0^t ||\boldsymbol{v}^N||^2_{\boldsymbol{L}^2(\Omega_N)}\, d\tau \\ &\quad \le ||\boldsymbol{v}_0^N||^2_{\boldsymbol{L}^2(\Omega_N)} + c_2 \int_0^t ||\boldsymbol{f}(\tau)||^2_{\boldsymbol{L}^2(\Omega)} + c_1 \int_0^t ||\boldsymbol{v}||^2_{\boldsymbol{L}^2(\Omega)}\, d\tau. \end{aligned} \tag{4.261}$$

Dropping the positive term $2\int_0^t \langle \mu(\boldsymbol{v}^N), \boldsymbol{v}^N\rangle\, d\tau$ in (4.261), and simplifying, we then obtain the estimate

$$||\boldsymbol{v}^N||^2_{\boldsymbol{L}^2(\Omega_N)} + c_1 \int_0^t ||\boldsymbol{v}^N||^2_{\boldsymbol{H}^2(\Omega_N)}\, d\tau \le ||\boldsymbol{v}_0^N||^2_{\boldsymbol{L}^2(\Omega_N)} + c_2 \int_0^t ||\boldsymbol{f}(\tau)||^2_{\boldsymbol{L}^2(\Omega)} \tag{4.262}$$

in which case

$$\sup_{s\in[0,T]} ||\boldsymbol{v}^N(s)||^2_{\boldsymbol{L}^2(\Omega_N)} \le ||\boldsymbol{v}_0^N||^2_{\boldsymbol{L}^2(\Omega_N)} + c_2 \int_0^t ||\boldsymbol{f}(\tau)||^2_{\boldsymbol{L}^2(\Omega)} \tag{4.263a}$$

and

$$\int_0^t ||\boldsymbol{v}^N||^2_{\boldsymbol{H}^2(\Omega_N)} d\tau \le \frac{1}{c_1}||\boldsymbol{v}_0^N||^2_{\boldsymbol{L}^2(\Omega_N)} + \frac{c_2}{c_1}\int_0^t ||\boldsymbol{f}(\tau)||^2_{\boldsymbol{L}^2(\Omega)}. \tag{4.263b}$$

We also have the a priori estimate

$$\left|\left|\frac{d\boldsymbol{v}^N}{dt}\right|\right|_{L^2((0,T);\bar{\boldsymbol{V}}')} \le c_3, \quad N = 1, 2, \ldots \tag{4.264}$$

for some constant $c_3 > 0$, independent of N. The proof of (4.264) is similar to that of (4.242), the only difference being that, in (4.247), we must take $\boldsymbol{\varphi} \in L^2((0,T); \bar{\boldsymbol{V}})$ and consider $\boldsymbol{\varphi}^N$ as the projection of $\boldsymbol{\varphi}$ in $\bar{\boldsymbol{V}}$ onto $\bar{\boldsymbol{V}}_N$.

Now, because $\bar{\boldsymbol{V}}_1 \subset \bar{\boldsymbol{V}}_2 \subset \cdots \subset \bar{\boldsymbol{V}}$, we have $\bar{\boldsymbol{V}}'_1 \supset \bar{\boldsymbol{V}}'_2 \supset \cdots \supset \bar{\boldsymbol{V}}'$. As a conequence of (4.264) the following estimate is, therefore, valid:

$$\left|\left|\frac{d\boldsymbol{v}^N}{dt}\right|\right|_{L^2((0,T);\bar{\boldsymbol{V}}'_M)} \le c_3, \quad N = 1, 2, \ldots \text{ and } M = 1, 2, \ldots. \tag{4.265}$$

Also, we know that

$$\bar{\boldsymbol{V}}_N \hookrightarrow \boldsymbol{W}^{2,2}(\Omega_N) \hookrightarrow \boldsymbol{W}^{1,2}(\Omega_N) \hookrightarrow \bar{\boldsymbol{V}}'_N \tag{4.266}$$

with $\boldsymbol{W}^{2,2}(\Omega_N) \hookrightarrow \boldsymbol{W}^{1,2}(\Omega_N)$ being compact. Therefore, employing (4.263a,b), (4.264), and (4.265), and using a standard diagonal process, it is easy to show that there exists a subsequence $\boldsymbol{v}^{N'}$ which converges to $\boldsymbol{v}$, weak * in $L^\infty((0,T);\bar{\boldsymbol{H}})$, and weakly in $L^2((0,T);\bar{\boldsymbol{V}})$, as well as strongly in $L^2((0,T);\boldsymbol{H}_0^1(\Omega_N))$, for each $N = 1, 2, \ldots$. We next show that $\boldsymbol{v}$ satisfies (4.208) and (4.209); we begin by noting that for $\boldsymbol{v} \in \boldsymbol{J}(\Omega)$, there exists M such that the support of $\boldsymbol{v}$ is in $\Omega_{N'} \cup \Gamma^+_{N'} \cup \Gamma^-_{N'}$, for $N' \geq M$. Since $\boldsymbol{v} \in \bar{\boldsymbol{V}}_{N'}$, by (4.249), for $N' \geq M$, we have $\forall \phi \in C[0,T]$, with $\phi(1) = 0$,

$$-\int_0^T \left(\boldsymbol{v}^N, \phi'(t)\boldsymbol{\varphi}\right) dt + 2\mu_1 \int_0^T \left(\frac{\partial e_{ij}(\boldsymbol{v}^{N'})}{\partial x_k}, \frac{\partial e_{ij}(\phi(t)\boldsymbol{\varphi})}{\partial x_k}\right)_{L^2(\Omega_N)} dt$$
$$+ \int_0^T \left\langle \boldsymbol{N}(\boldsymbol{v}^{N'}), \phi(t)\boldsymbol{\varphi}\right\rangle dt + \int_0^T b\left(\boldsymbol{v}^{N'}, \boldsymbol{v}^{N'}, \phi(t)\boldsymbol{\varphi}\right) dt$$
$$= \int_0^T \left(\boldsymbol{f}^{N'}, \phi(t)\boldsymbol{\varphi}\right)_{\boldsymbol{L}^2(\Omega)} dt + \left(\boldsymbol{v}_0^{N'}, \boldsymbol{\varphi}\right)_{\boldsymbol{L}^2(\Omega)} \phi(0). \tag{4.267}$$

As $\boldsymbol{v}^{N'} \to \boldsymbol{v}$ strongly in $L^2((0,T);\boldsymbol{H}_0^1(\Omega_N))$, so that $\boldsymbol{v}$ is the strong limit of a sequence of functions with compact support in $\Omega_{N'} \cup \Gamma^+_{N'} \cup \Gamma^-_{N'}$, it is easy to establish the following results:

$$\lim_{N'\to\infty} \int_0^T \left(\boldsymbol{v}^{N'}, \phi'(t)\boldsymbol{\varphi}\right)_{\boldsymbol{L}^2(\Omega)} dt = \int_0^T \left(\boldsymbol{v}, \phi'(t)\boldsymbol{\varphi}\right)_{\boldsymbol{L}^2(\Omega)} dt, \tag{4.268a}$$

$$\lim_{N'\to\infty} \int_0^T \left\langle \boldsymbol{N}(\boldsymbol{v}^{N'}), \phi(t)\boldsymbol{\varphi}\right\rangle dt = \int_0^T \langle \boldsymbol{N}(\boldsymbol{v}, \phi(t)\boldsymbol{\varphi}\rangle\, dt, \tag{4.268b}$$

and

$$\lim_{N'\to\infty} \int_0^T b\left(\boldsymbol{v}^{N'}, \boldsymbol{v}^{N'}, \phi(t)\boldsymbol{\varphi}\right) dt = \int_0^T b(\boldsymbol{v}, \boldsymbol{v}, \phi(t)\boldsymbol{\varphi})\, dt. \tag{4.268c}$$

Furthermore, the fact that $\boldsymbol{v}^{N'} \rightharpoonup \boldsymbol{v}$ weakly in $L^2((0,T);\bar{\boldsymbol{V}}')$, yields

$$\lim_{N'\to\infty} \int_0^T \left(\frac{\partial e_{ij}(\boldsymbol{v}^{N'})}{\partial x_k}, \frac{\partial e_{ij}(\phi(t)\boldsymbol{\varphi})}{\partial x_k}\right)_{\boldsymbol{L}^2(\Omega_N)} dt$$
$$= \int_0^T \left(\frac{\partial e_{ij}(\boldsymbol{v})}{\partial x_k}, \frac{\partial e_{ij}(\phi(t)\boldsymbol{\varphi})}{\partial x_k}\right)_{L^2(\Omega)} dt. \tag{4.269}$$

Using (4.268a,b,c), and taking the limit as $N' \to \infty$ in (4.267), we obtain

$$-\int_0^T \big(\boldsymbol{v}, \phi'(t)\boldsymbol{\varphi}\big)_{\boldsymbol{L}^2(\Omega)}\, dt + 2\mu_1 \int_0^T \left(\frac{\partial e_{ij}(\boldsymbol{v})}{\partial x_k}, \frac{\partial e_{ij}(\phi(t)\boldsymbol{\varphi})}{\partial x_k}\right)_{L^2(\Omega)} dt$$
$$+ \int_0^T \langle \boldsymbol{N}(\boldsymbol{v}), \phi(t)\boldsymbol{\varphi}\rangle\, dt + \int_0^T b(\boldsymbol{v}, \boldsymbol{v}, \phi(t)\boldsymbol{\varphi})\, dt$$
$$= \int_0^T (\boldsymbol{f}, \phi(t)\boldsymbol{\varphi})_{\boldsymbol{L}^2(\Omega)}\, dt + (\boldsymbol{v}_0, \boldsymbol{\varphi})_{\boldsymbol{L}^2(\Omega)}\phi(0), \quad \forall \boldsymbol{\varphi} \in \boldsymbol{J}(\Omega). \tag{4.270}$$

By continuity, (4.270) holds for all $\boldsymbol{v} \in \bar{\boldsymbol{V}}$; if we take $\phi \in \mathcal{D}(0, T)$ in (4.270), we obtain

$$-\int_0^T \big(\boldsymbol{v}, \phi'(t)\boldsymbol{\varphi}\big)_{\boldsymbol{L}^2(\Omega)}\, dt + 2\mu_1 \int_0^T \left(\frac{\partial e_{ij}(\boldsymbol{v})}{\partial x_k}, \frac{\partial e_{ij}(\phi(t)\boldsymbol{\varphi})}{\partial x_k}\right)_{L^2(\Omega_N)} dt$$
$$+ \int_0^T \langle \boldsymbol{N}(\boldsymbol{v}), \phi(t)\boldsymbol{\varphi}\rangle\, dt + \int_0^T b(\boldsymbol{v}, \boldsymbol{v}, \phi(t)\boldsymbol{\varphi})\, dt$$
$$= \int_0^T (\boldsymbol{f}, \phi(t)\boldsymbol{\varphi})_{\boldsymbol{L}^2(\Omega)}\, dt \tag{4.271}$$

which implies that, in the sense of distributions,

$$\frac{d}{dt}(\boldsymbol{v}, \boldsymbol{\varphi})_{\boldsymbol{L}^2(\Omega)} + 2\mu_1 \left(\frac{\partial e_{ij}(\boldsymbol{v})}{\partial x_k}, \frac{\partial e_{ij}(\phi(t)\boldsymbol{\varphi})}{\partial x_k}\right)_{L^2(\Omega_N)}$$
$$+ \langle \boldsymbol{N}(\boldsymbol{v}), \phi(t)\boldsymbol{\varphi}\rangle + b(\boldsymbol{v}, \boldsymbol{v}, \phi(t)\boldsymbol{\varphi})$$
$$= (\boldsymbol{f}, \phi(t)\boldsymbol{\varphi})_{\boldsymbol{L}^2(\Omega)}, \quad \forall \boldsymbol{v} \in \bar{\boldsymbol{V}}. \tag{4.272}$$

As a direct consequence of (4.272) we have, therefore,

$$\frac{d\boldsymbol{v}}{dt} + 2\mu_1 A\boldsymbol{v} + \boldsymbol{N}(\boldsymbol{v}) + \boldsymbol{B}(\boldsymbol{v}, \boldsymbol{v}) = \boldsymbol{f} \tag{4.273}$$

and

$$\frac{d\boldsymbol{v}}{dt} \in L^2((0, T); \bar{\boldsymbol{V}}'). \tag{4.274}$$

The proof will be complete if we can show that $\boldsymbol{v}$ satisfies the initial condition; to this end, we take the inner product, in $\boldsymbol{L}^2(\Omega)$, of (4.273) with $\phi(t)\boldsymbol{\varphi}$, where $\boldsymbol{\varphi} \in \bar{\boldsymbol{V}}$; we then integrate by parts so as to obtain

$$-\int_0^T \big(\boldsymbol{v}, \phi'(t)\boldsymbol{\varphi}\big)_{\boldsymbol{L}^2(\Omega)}\, dt + 2\mu_1 \int_0^T \left(\frac{\partial e_{ij}(\boldsymbol{v})}{\partial x_k}, \frac{\partial e_{ij}(\phi(t)\boldsymbol{\varphi})}{\partial x_k}\right)_{L^2(\Omega_N)} dt$$
$$+ \int_0^T \langle \boldsymbol{N}(\boldsymbol{v}), \phi(t)\boldsymbol{\varphi}\rangle\, dt + \int_0^T b(\boldsymbol{v}, \boldsymbol{v}, \phi(t)\boldsymbol{\varphi})\, dt$$
$$= \int_0^T (\boldsymbol{f}, \phi(t)\boldsymbol{\varphi})_{\boldsymbol{L}^2(\Omega)}\, dt + (\boldsymbol{v}(0), \boldsymbol{\varphi})_{\boldsymbol{L}^2(\Omega)}\phi(0). \quad (4.275)$$

Comparing (4.270) with (4.275), we obtain

$$(\boldsymbol{v}(0) - \boldsymbol{v}_0, \boldsymbol{\varphi})_{\boldsymbol{L}^2(\Omega)}\phi(0) = 0, \quad \forall \phi \in C[0,1] \text{ and } \forall \boldsymbol{\varphi} \in \bar{\boldsymbol{V}} \quad (4.276)$$

which implies that $\boldsymbol{v}(0) = \boldsymbol{v}_0$. □

Remarks. By virtue of Theorem 4.18, and Lemma A.9, it follows that the solution of (4.209), (4.210), which satisfies (4.208), actually belongs to $C([0,T]; \bar{\boldsymbol{H}})$.

4.4.5 Uniqueness of Bipolar Flow in $\boldsymbol{\Omega_a}$

Theorem 4.18 of Sect. 4.4.4 established the existence of a solution of (4.209), (4.210), satisfying (4.208); in the subsection we will prove that the solution is, in fact, unique. Thus, suppose that $\boldsymbol{u}$ and $\boldsymbol{v}$ are two solutions of (4.209), (4.210), which satisfy (4.208), and let $T < \infty$ be fixed. Then,

$$\begin{cases} \dfrac{d\boldsymbol{u}}{dt} + 2\mu_1 \bar{\boldsymbol{A}}\boldsymbol{u} + \boldsymbol{N}(\boldsymbol{u}) + \boldsymbol{B}(\boldsymbol{u}, \boldsymbol{u}) = \boldsymbol{f}, \\ \boldsymbol{u}(0) = \boldsymbol{v}_0, \end{cases} \quad (4.277)$$

and

$$\begin{cases} \dfrac{d\boldsymbol{v}}{dt} + 2\mu_1 \bar{\boldsymbol{A}}\boldsymbol{v} + \boldsymbol{N}(\boldsymbol{v}) + \boldsymbol{B}(\boldsymbol{v}, \boldsymbol{v}) = \boldsymbol{f}, \\ \boldsymbol{v}(0) = \boldsymbol{v}_0. \end{cases} \quad (4.278)$$

Let $\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{v}$, so that $\boldsymbol{w}(0) = \boldsymbol{0}$. Subtracting (4.278) from (4.277) we obtain the equation

$$\boldsymbol{w}' + 2\mu_1 \bar{\boldsymbol{A}}\boldsymbol{w} + \boldsymbol{N}(\boldsymbol{u}) - \boldsymbol{N}(\boldsymbol{v}) + \boldsymbol{B}(\boldsymbol{u}, \boldsymbol{u}) - \boldsymbol{B}(\boldsymbol{v}, \boldsymbol{v}) = \boldsymbol{0}. \quad (4.279)$$

Taking the inner product of (4.279) with $\boldsymbol{w}$ then yields

$$\frac{1}{2}\frac{d}{dt}||\boldsymbol{w}^2||_{\boldsymbol{L}^2(\Omega)} + 2\mu_1\left(\frac{\partial e_{ij}(\boldsymbol{w})}{\partial x_k}, \frac{\partial e_{ij}(\boldsymbol{w})}{\partial x_k}\right)_{L^2(\Omega)} + \langle \boldsymbol{N}(\boldsymbol{u}) - \boldsymbol{N}(\boldsymbol{v}), \boldsymbol{w}\rangle + b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{w}) - b(\boldsymbol{v},\boldsymbol{v},\boldsymbol{w}) = 0. \quad (4.280)$$

From the monotonicity of $\mu(\boldsymbol{u})$ it follows that

$$\begin{aligned}\langle \boldsymbol{N}(\boldsymbol{u}) - \boldsymbol{N}(\boldsymbol{v}), \boldsymbol{w}\rangle &= 2\int_\Omega \left[\mu(\boldsymbol{u})e_{ij}(\boldsymbol{u}) - \mu(\boldsymbol{v})e_{ij}(\boldsymbol{v})\right] e_{ij}(\boldsymbol{w})\, d\boldsymbol{x} \\ &\geq 0\end{aligned} \quad (4.281)$$

and we also have

$$\begin{aligned}b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{w}) - b(\boldsymbol{v},\boldsymbol{v},\boldsymbol{w}) &= b(\boldsymbol{w},\boldsymbol{u},\boldsymbol{w}) - b(\boldsymbol{v},\boldsymbol{w},\boldsymbol{w}) \\ &= b(\boldsymbol{w},\boldsymbol{u},\boldsymbol{w})\end{aligned} \quad (4.282)$$

so that for some $c_i > 0$, $i = 1, 2, 3$,

$$\begin{aligned}|b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{w}) - b(\boldsymbol{v},\boldsymbol{v},\boldsymbol{w})| &\leq \left|\int_\Omega w_j \frac{\partial u_i}{\partial x_j} w_i\, d\boldsymbol{x}\right| \\ &\leq c_1\, ||\boldsymbol{w}||_{\boldsymbol{L}^4(\Omega)}\, ||\boldsymbol{w}||_{\boldsymbol{L}^4(\Omega)}\, |\boldsymbol{u}|_{\boldsymbol{H}^1(\Omega)} \\ &\leq c_2 ||\boldsymbol{w}||_{\boldsymbol{L}^2(\Omega)} ||\boldsymbol{w}||_{\boldsymbol{H}^1(\Omega)} |\boldsymbol{u}|_{\boldsymbol{H}^1(\Omega)} \\ &\leq c_3 ||\boldsymbol{w}||_{\boldsymbol{L}^2(\Omega)} ||\boldsymbol{w}||_{\boldsymbol{H}^2(\Omega)} ||\boldsymbol{u}||_{\boldsymbol{H}^2(\Omega)}.\end{aligned} \quad (4.283)$$

Applying Lemma 4.14, in conjunction with (4.280)–(4.283), we find that, for some $c_4 > 0$,

$$\begin{aligned}\frac{d}{dt}\, ||\boldsymbol{w}||^2_{\boldsymbol{L}^2(\Omega)} + 2c_4\, ||\boldsymbol{w}||^2_{\boldsymbol{H}^2(\Omega)} &\leq 2c_3 ||\boldsymbol{w}||_{\boldsymbol{L}^2(\Omega)} ||\boldsymbol{w}||_{\boldsymbol{H}^2(\Omega)} ||\boldsymbol{u}||_{\boldsymbol{H}^2(\Omega)} \\ &\leq \frac{c_3^2}{2c_4}\, ||\boldsymbol{w}||^2_{\boldsymbol{L}^2(\Omega)}\, ||\boldsymbol{u}||^2_{\boldsymbol{H}^2(\Omega)} + 2c_4\, ||\boldsymbol{w}||^2_{\boldsymbol{H}^2(\Omega)}\end{aligned} \quad (4.284)$$

or, for some $c_5 > 0$,

$$\frac{d}{dt}\, ||\boldsymbol{w}||^2_{\boldsymbol{L}^2(\Omega)} - c_5\, ||\boldsymbol{u}||^2_{\boldsymbol{H}^2(\Omega)}\, ||\boldsymbol{w}||^2_{\boldsymbol{L}^2(\Omega)} < 0. \quad (4.285)$$

We now multiply (4.285) by $e^{-c_5 \int_0^t ||\boldsymbol{u}||^2_{\boldsymbol{H}^2(\Omega)}\, d\tau}$ so as to obtain

$$\frac{d}{dt}\left\{||\boldsymbol{w}||^2_{\boldsymbol{L}^2(\Omega)}\, e^{-c_5 \int_0^t ||\boldsymbol{u}||^2_{\boldsymbol{H}^2(\Omega)}\, d\tau}\right\} \leq 0. \quad (4.286)$$

Finally, integrating (4.286) from 0 to t, and using the fact that $||\boldsymbol{w}(0)||^2_{\boldsymbol{L}^2(\Omega)} = 0$, we have

$$||\boldsymbol{w}||^2_{\boldsymbol{L}^2(\Omega)} \, e^{-c_5 \int_0^t ||\boldsymbol{u}||^2_{\boldsymbol{H}^2(\Omega)} \, d\tau} \leq 0 \tag{4.287}$$

so that $\boldsymbol{w} = \boldsymbol{0}$, $0 \leq t \leq T$; this serves to establish the following result:

Theorem 4.19. *Given that* $\boldsymbol{f} \in L^2((0,T); \bar{\boldsymbol{H}})$ *and* $\boldsymbol{v}_0 \in \bar{\boldsymbol{H}}$*, the solution of* $\boldsymbol{v}$ *of* (4.209)*,* (4.210)*, which satisfies the regularity conditions* (4.208)*, is uniquely determined.*

4.5 Related Work on Existence and Uniqueness for Navier–Stokes and Some Generalizations

4.5.1 Introduction

In this section we take up the problems of existence, uniqueness, and stability for some of the non-Newtonian fluid dynamics models which were introduced in Sect. 1.6 and which are related, in various ways, to the model of the nonlinear bipolar fluid. We also present a brief review of existence and uniqueness results for the incompressible Navier–Stokes equations, to which the bipolar equations reduce when $\alpha = \mu_1 = 0$; this review is offered in Sect. 4.5.2, where we discuss the history of the outstanding problems in space dimension $n = 3$. We also provide both general and technical versions of these problems as they appear on the Clay Institute's website in connection with the formulation of the corresponding millennium prize problem. Some of the classical results on partial regularity for the Navier–Stokes in R^3, which were obtained in [Sch,CKN], and [Lin], are also presented. Existence and uniqueness results for solutions of the Ladyzhenskaya generalization of the Navier–Stokes equations, which was introduced in Sect. 1.6, are presented in Sect. 4.5.3; this model is, in many respects, similar to the one the nonlinear bipolar model reduces to when the higher-order viscosity $\mu_1 = 0$. For the problems discussed in Sect. 4.5.3 we present, not only the conclusions obtained in the original work of Ladyzhenskaya [La6], but also the results obtained in [Lio2] and, more recently, those claimed by Du and Gunzburger [DuG]. In Sect. 4.5.4 we introduce the existence and uniqueness theorems obtained by Lions [Lio1, 2], Ladyzhenskaya [La5], and Beirão da Veiga [BdV2, 3] for the regularization of the Navier–Stokes equations obtained by adding artificial viscosity to the model. In somewhat greater detail, we present the more recent results which are to be found in [OS1, 2] for these models; they are mathematically similar to the one for the linear bipolar fluid except for the fact that the work referenced in Sect. 4.5.4 employs Neumann type boundary conditions in addition to the usual non-slip condition (as opposed to the type of higher-order, physically motivated, boundary conditions derived in Sect. 1.4.4 for the incompressible bipolar model). Also discussed, in Sect. 4.5.4

is the issue of convergence of the solutions of the regularized problem to that of the corresponding Navier–Stokes system as the regularization parameter tends to zero. In Sect. 4.5.5 we outline, in some considerable detail, the results on existence, uniqueness, and stability for multipolar fluids of grade 3 obtained in [BNR]. After first defining the notion of weak solution, such a solution is obtained as the limit of a suitable sequence of Galerkin approximations; the solutions obtained satisfy specific regularity conditions and a unique weak solution is then obtained under different assumptions relative to the initial data with the concurrent loss of some regularity. It is then shown, in Sect. 4.5.5, how the higher-order boundary conditions for multipolar fluids of grade 3, which were specified in Sect. 1.6, follow from the definition of a weak solution; this subsection concludes with the presentation of an estimate which establishes the asymptotic stability of the rest state of a multipolar fluid of grade 3, and which is based on the introduction and properties of a suitably defined total energy functional. Finally, in Sect. 4.5.6 we briefly review some of the results obtained in [FHT2] concerning the global, in time, regularity of the three-dimensional viscous Camassa-Holm equations that were introduced in Sect. 1.6; these global regularity results are proven in [FHT2] for the case of periodic boundary conditions. Among the results established in [FHT2], and discussed in Sect. 4.5.5, is the convergence, as a key length scale in the model tends to zero, of a subsequence of solutions of the VCHE to a weak solution of the three-dimensional Navier–Stokes system.

4.5.2 Existence and Uniqueness for the Incompressible Navier–Stokes Equations

The Navier–Stokes system consists of the nonlinear partial differential equations (1.8) subject to the constraint of incompressibility; we rewrite the problem here as follows: find $v_i(\boldsymbol{x}, t)$, $i = 1, \ldots, n$, and $p(\boldsymbol{x}, t)$ such that

$$\rho\left(\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} + \mu_0 \nabla^2 v_i + \rho F_i, \tag{4.288a}$$

$$\frac{\partial v_i}{\partial x_i} = 0 \tag{4.288b}$$

where $n = 2, 3$, we sum on repeated indices, and all the quantities in (4.288a,b) have previously been defined. Several different types of problems can be associated with the system (4.288a,b) in both $\mathbb{R}^2$ and $\mathbb{R}^3$, namely, (i) the pure initial-value problem posed in all of $\mathbb{R}^n$, $n = 2, 3$, for which an initial condition $\boldsymbol{v}(\boldsymbol{x}, 0) = \boldsymbol{v}_0(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^n$, is appended to (4.288a,b), (ii) the initial-boundary value problem posed on an open, bounded domain $\Omega \subseteq \mathbb{R}^n$ with, say, smooth boundary $\partial\Omega$ for which (4.288a,b) is to hold in $\Omega \times [0, T)$, $T > 0$, with $\boldsymbol{v}$ a prescribed function on $\partial\Omega \times [0, T)$, $T > 0$, and $\boldsymbol{v}(x, 0) = \boldsymbol{v}_0(x)$, for $\boldsymbol{x} \in \Omega$, (iii) the exterior problem, which is similar to

(ii) except it is posed in $\Omega \times [0, T)$ with $\Omega = \mathbb{R}^n/\bar{\Omega}'$, Ω' an open bounded set in $\mathbb{R}^n$, and we also specify a radiation condition of the form $\lim_{|\boldsymbol{x}|\to\infty} \boldsymbol{v}(\boldsymbol{x}, t) = \boldsymbol{v}_\infty$, and (iv) the space-periodic problem posed on $\Omega \times [0, T)$, with $\Omega = [0, L]^n$, $L > 0$ and periodic conditions imposed on both $\boldsymbol{v}_0$ and $\boldsymbol{F}$.

The most sweeping generalization that one can make about all the problems posed, above, for (4.288a,b) is that in $\mathbb{R}^2$, as concerns both the existence of unique weak solutions and the existence and uniqueness of classical solutions starting from sufficiently smooth initial data, almost everything is settled in the affirmative (see, however, the discussion of the current state of affairs concerning existence of solutions for the steady state exterior problem in two dimensions in Sect. 3.4); for existence and uniqueness for Navier–Stokes in dim $n = 2$ we refer the reader to the original works of Kiselev and Ladyzhenskaya [KL] and Lions and Prodi [LP] as well as to the monograph [La1] and the references listed in Sect. 1.1, namely, [CF, Ga1], [Te1, 3], and [PL]. A very good (and relatively recent) survey article covering the state of affairs with respect to open problems for the Navier–Stokes system in dimensions $n = 2$ and 3 is that of Heywood [He2]. At the other end of the spectrum, the most sweeping generalization that one can make about existence and uniqueness for the various problems associated with (4.288a,b) in space dim $n = 3$ is that almost all the problems remain unresolved; these problems have proven to be so notoriously difficult, even though many partial results abound (e.g., [CKN] for some of the most notable of them), that their resolution constitutes one of the Millennium Prize Problems (see [Dek] or the Clay Mathematics Institute web site at www.claymath.org/millennium).

It is not the purpose of this subsection to provide an account of the difficulties that are associated with proving the existence and uniqueness of sufficiently smooth (global in time) solutions for the various types of problems associated with the system (4.288a,b); there would be little to be gained from the exertion of such an effort here for the following reasons: (1) the rationale for the introduction of the viscous, incompressible bipolar fluid model in Sect. 1.4 was not motivated, as has been the case with some of the other modifications of Navier–Stokes that have appeared in the literature, by a desire to resolve the outstanding existence and uniqueness issues associated with Navier–Stokes in space dim $n = 3$, and (2) so many talented mathematicians have spent a considerable portion of their careers addressing the problems involved, including the delineation of the many technical reasons why the outstanding issues have not been settled, that there exists a wealth of information available which is very easy to access. For example, on the Clay Institute website itself there is a very nice concise article by C. Fefferman [Fe]; other discussions well worth looking at include the remarks in [Tao], the survey by Galdi [Ga2], and the brief article by Constantin [Con2]. We will, therefore, exercise some restraint and offer a few (mostly) non-technical remarks concerning the initial-value problem in $\mathbb{R}^3$ and some (slightly more technical) details relative to the initial-boundary value problem posed on an open bounded domain $\Omega \subset \mathbb{R}^3$.

The actual statement of the Navier–Stokes millennium prize problem on the Clay Institute's website is of a distinctly non-technical nature, as follows:

> Prove or give a counter-example of the following statement: In three space dimensions and time, given an initial velocity field, there exists a vector velocity and a scalar pressure field, which are both smooth and globally defined, that solve the Navier–Stokes equations.

One technical version of this problem as stated in [Fe] would be the following: For the Navier–Stokes system (4.288a,b) in space dim $n = 3$, with $\boldsymbol{F} = \boldsymbol{0}$ and an associated initial data function $\boldsymbol{v}_0(\cdot)$ that is smooth, divergence free, and satisfies the growth condition at infinity

$$||D^{\boldsymbol{\alpha}} \boldsymbol{v}_0(\boldsymbol{x})|| \leq C_{\boldsymbol{\alpha} K}(1 + ||\boldsymbol{x}||)^{-K}, \text{ on } R^3 \tag{4.289}$$

for any positive integer K, and any multi-index $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$, prove that there exist smooth functions $p(\boldsymbol{x}, t)$, $v_i(\boldsymbol{x}, t)$, $i = 1, 2, 3$ on $R^3 \times [0, \infty)$, i.e., $p, v_i \in C^\infty(R^3 \times [0, \infty))$, that satisfy the initial-value problem and the energy bound,

$$\int_{R^3} ||\boldsymbol{v}(\boldsymbol{x}, t)||^2 \, d\boldsymbol{x} \leq C, \quad \forall t \geq 0 \tag{4.290}$$

for some $C > 0$. Three other versions of the millennium prize problem, as it relates just to the pure initial-value problem in $\mathbb{R}^3$, may also be found in [Fe]. It is known that the problem posed above has an affirmative answer provided $\boldsymbol{v}_0$ is sufficiently small and if $\boldsymbol{v}_0$ is not assumed to be "small" then the result holds but only on some "small" time interval $[0, t_1]$, with t_1 depending on the size of the initial data; the largest such t_1, for given initial data, is known as the "blow-up" time. For the Navier–Stokes system (4.288a,b), should it ever be proven that a finite "blow-up" time t_1 exists it would imply that one or more of the velocity components $v_i(\boldsymbol{x}, t)$ would become unbounded as $t \to t_1^-$.

The earliest (serious) work on the issues of existence and uniqueness for the Navier–Stokes system is usually credited to Leráy [Le1, 2] who showed that a weak solution of the initial-value problem in R^3, $(p, \boldsymbol{v})$, exists with appropriate growth properties; to this day it is not known whether the weak solutions constructed by Leray (see, also, Hopf [Ho2]) are uniquely defined. Weak solutions for the initial-value problem associated with (4.288a,b) are, of course, defined in a manner entirely analogous to the way in which weak solutions for the incompressible bipolar model were introduced in Sect. 4.2, i.e., integrate (4.288a) against a test function and then integrate by parts to pull all the derivatives off the v_i. For the various types of problems associated with the system (4.288a,b) there are partial regularity results available as well as a wealth of theorems asserting that, provided certain other quantities associated with $\boldsymbol{v}(\boldsymbol{x}, t)$ can be proven to be globally controlled, global existence and uniqueness for the associated Navier–Stokes system in three space dimensions follows; details may be found in the many references cited at the beginning of this subsection. For example, for the bounded domain case, with $\Omega \subset R^3$, the total kinetic energy of the fluid is given by

$$\text{KE}(t) = \frac{1}{2} \int_\Omega ||\boldsymbol{v}(\boldsymbol{x}, t)||^2 \, d\boldsymbol{x}$$

and, if there is no external source of energy, then there we have dissipation of the kinetic energy because it is easily shown that

$$\frac{1}{2}\int_\Omega ||\boldsymbol{v}(\boldsymbol{x},t)||^2\,d\boldsymbol{x} + \nu\int_0^t\int_\Omega |\nabla\boldsymbol{v}(\boldsymbol{x},s)|^2\,d\boldsymbol{x}\,ds \le \frac{1}{2}\int_\Omega ||\boldsymbol{v}(\boldsymbol{x},0)||^2\,d\boldsymbol{x}.$$

As noted, e.g., in [Con2], "the dissipation of kinetic energy is the strongest quantitative information about the Navier–Stokes equations that is presently known for general solutions. In his classical work [Le2] Leráy used this dissipation to construct weak solutions with finite kinetic energy that exist for all time.... The solutions have partial regularity [CKN] but are not known to be smooth.... The uniqueness of the Leráy weak solutions is not known." Many sufficient conditions have been identified that would guarantee the global (in time) existence of smooth solutions, e.g., the condition that, for arbitrary $T > 0$,

$$\int_0^T \left(\int_\Omega |\nabla\boldsymbol{v}(\boldsymbol{x},t)|^2\,dx\right)^2 dt < \infty$$

guarantees that the velocity fields $\boldsymbol{v} \in C^\infty(\Omega)$ for all $t > 0$. We note that the best partial regularity results appear to be those attributed to [Sch, CKN], and [Lin] which characterize the singular set of a weak solution $\boldsymbol{v}(\boldsymbol{x},t)$, namely, the set of all points $(\boldsymbol{x},t) \in R^3 \times R^1$ such that $\boldsymbol{v}$ is unbounded in every neighborhood of $(\boldsymbol{x},t)$, as belonging to a set of Hausdorff dimension of measure at most equal to one (actually, a parabolic analogue of the usual concept of Hausdorff dimension which is defined in Sect. 5.3.5 in connection with our discussion of attractors for the bipolar model); this partial regularity result implies, in particular, that the singular set does not contain a smooth curve in $R^3 \times R^1$. Because of its importance in the development of efforts to either prove (or disprove) regularity for the solutions of the Navier–Stokes equations in space dim $n = 3$, we will expand a little on these remarks. A class of weak solutions of the Navier–Stokes system is introduced in [CKN] which the authors refer to as the class of suitable weak solutions; these solutions are functions $\boldsymbol{v}$ which satisfy

$$\begin{aligned} &2\int_0^\infty\int_\Omega |\nabla\boldsymbol{v}|^2\phi\,d\boldsymbol{x}\,dt \\ &\quad \le \int_0^\infty\int_\Omega \left\{||\boldsymbol{v}||^2(\phi_{,t} + \Delta\phi) + (||\boldsymbol{v}||^2 + 2p)\boldsymbol{v}\cdot\nabla\phi + 2(\boldsymbol{v}\cdot\boldsymbol{F})\phi\right\} d\boldsymbol{x}\,dt \end{aligned}$$

for all (bump) functions $\phi(\boldsymbol{x},t)$ which have compact support in $\Omega \times [0,\infty)$. An important step in the analysis in [CKN] is the demonstration that the term

$$\int_0^\infty\int_\Omega ||\boldsymbol{v}||^2(\phi_{,t} + \Delta\phi)\,d\boldsymbol{x}\,dt$$

can be made arbitrarily small by choosing ϕ to satisfy the backwards heat equation. To define the parabolic Hausdorff dimension one uses parabolic cylinders S_λ instead of spheres in $\mathbb{R}^n$. The S_λ have the form, for $\lambda > 0$,

$$S_\lambda(\boldsymbol{x}, t) = \{(\boldsymbol{y}, \tau) \mid ||\boldsymbol{y} - \boldsymbol{x}|| < \lambda,\ t - \lambda^2 < \tau < t\}.$$

It is first proven in [CKN] that, if $\boldsymbol{v}$, p, and $\boldsymbol{F}$ are sufficiently small on S_λ, then $\boldsymbol{v}$ is regular on $S_{\lambda/2}$; this result is then shown to imply an estimate for the minimum rate at which the development of a singularity could occur which, in turn, yields a sufficient condition for a point $(\boldsymbol{x}, t)$ to be a regular point of $\boldsymbol{v}$. Finally, using a covering of the singular set, the authors [CKN] show that any suitable weak solution has the property that its singular set has a one-dimensional parabolic Hausdorff measure equal to zero.

We will close this subsection by offering a few remarks about what is known concerning existence and uniqueness of solutions for the initial-boundary value problem associated with (4.288a,b) on an open bounded domain $\Omega \subset R^3$, with smooth boundary $\partial\Omega$, initial condition $\boldsymbol{v}(\boldsymbol{x}, 0) = \boldsymbol{v}_0(\boldsymbol{x})$, $\boldsymbol{x} \in \Omega$, and the (usual) non-slip boundary condition $\boldsymbol{v}(\boldsymbol{x}, t) = 0$, for $(\boldsymbol{x}, t) \in \partial\Omega \times [0, T)$, $T > 0$. We employ the following standard (i.e., [Te1]) notation employed when dealing with the Navier–Stokes system (4.288a,b):

$$\boldsymbol{H} = \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \mid \nabla \cdot \boldsymbol{v} = 0,\ \text{in}\ \Omega,\ \text{and}\ \boldsymbol{v} \cdot \boldsymbol{\nu} = 0,\ \text{on}\ \partial\Omega\}, \tag{4.291a}$$

$$\boldsymbol{V} = \{\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \mid \nabla \cdot \boldsymbol{v} = 0,\ \text{in}\ \Omega\}, \tag{4.291b}$$

$$\boldsymbol{D}(\boldsymbol{A}) = \boldsymbol{H}_0^1(\Omega) \cap \boldsymbol{H}^2(\Omega), \tag{4.291c}$$

and

$$\boldsymbol{A}\boldsymbol{v} = -\boldsymbol{P}\Delta\boldsymbol{v} \tag{4.291d}$$

with $\boldsymbol{P}$ the orthogonal projection from $\boldsymbol{L}^2(\Omega)$ onto $\boldsymbol{H}$. We rewrite the Navier–Stokes initial-boundary value problem as an initial-value problem for a nonlinear evolution equation in the Hilbert space $\boldsymbol{H}$, i.e., with

$$\boldsymbol{B}(\boldsymbol{v}) = \boldsymbol{B}(\boldsymbol{v}, \boldsymbol{v}), \quad \boldsymbol{B}(\boldsymbol{u}, \boldsymbol{v}) = \boldsymbol{P}(\boldsymbol{u} \cdot \boldsymbol{v}),\ \text{and}\ \boldsymbol{f} \to \boldsymbol{P}\boldsymbol{f}$$

we have

$$\frac{d\boldsymbol{v}}{dt} + \mu_0 \boldsymbol{A}\boldsymbol{v} + \boldsymbol{B}(\boldsymbol{v}) = \boldsymbol{f} \tag{4.292a}$$

$$\boldsymbol{v}(0) = \boldsymbol{v}_0. \tag{4.292b}$$

For $\boldsymbol{v}_0 \in \boldsymbol{V}$ and $\boldsymbol{f} \in L^\infty((0,\infty);\boldsymbol{H})$ a strong solution of (4.292a,b) on some interval $[0,T)$, $T > 0$, is defined to be a function $\boldsymbol{v} \in L^\infty([0,T);\boldsymbol{V}) \cap L^2([0,T);\boldsymbol{D}(\boldsymbol{A}))$, while a weak solution on $[0,T)$ is a function $\boldsymbol{v} \in L^\infty([0,T);\boldsymbol{H}) \cap L^2([0,T);\boldsymbol{V})$. For space dimension $n = 2$, it is known that a strong (and, thus, a weak) solution exists, is unique, and we may take $T = \infty$; however, if $n = 3$, then a strong solution is known to exist and be unique only on some interval $[0,t_1]$, with t_1 having the form

$$t_1 = t_1(|\boldsymbol{v}_0|_{\boldsymbol{L}^2(\Omega)}) = \frac{c_1}{(1+|\boldsymbol{v}_0|^2_{\boldsymbol{L}^2(\Omega)})^2}, \tag{4.293}$$

the constant c_1 depending only on $|\boldsymbol{f}|_{L^\infty((0,\infty);\boldsymbol{H})}$. A weak solution of (4.292a,b), when $n = 3$, exists for all $T > 0$ and agrees with the strong solution on the interval $[0,t_1]$ but it is still not known if this weak solution is unique.

Remarks. For $\boldsymbol{v}_0 \in \boldsymbol{H}$ and $\boldsymbol{f} \in L^\infty((0,\infty);\boldsymbol{H})$ it is known that there exists, in space dimension $n = 2$, a unique solution $\boldsymbol{v}$ of (4.292a,b) satisfying $\boldsymbol{v} \in L^\infty([0,\infty);\boldsymbol{H}) \cap L^2([0,\infty);\boldsymbol{V})$ such that $\boldsymbol{v}$ is analytic in t with values in $\boldsymbol{D}(\boldsymbol{A})$ for $t > 0$; furthermore, the mapping $\boldsymbol{v}_0 \mapsto \boldsymbol{v}(t)$ is continuous from $\boldsymbol{H}$ into $\boldsymbol{D}(\boldsymbol{A})$, $\forall t > 0$. Such a result, for $\boldsymbol{v}_0 \in \boldsymbol{H}$, enables us to define the (nonlinear semigroup of) operators $\boldsymbol{S}(t) : \boldsymbol{v}_0 \mapsto \boldsymbol{v}(t)$ which enjoy the properties

$$\boldsymbol{S}(t+s) = \boldsymbol{S}(t)\cdot\boldsymbol{S}(s), \quad \forall s,t \geq 0 \tag{4.294a}$$

$$\boldsymbol{S}(0) = \boldsymbol{I} \quad \text{(identity in } \boldsymbol{H}) \tag{4.294b}$$

with

$$\boldsymbol{S}(t) \text{ a continuous nonlinear operator from } \boldsymbol{H} \text{ into itself, } \forall t \geq 0. \tag{4.294c}$$

In fact, for $n = 2$, and $\boldsymbol{v}_0 \in \boldsymbol{H}$, the $\boldsymbol{S}(t)$ are continuous, $\forall t \geq 0$ from $\boldsymbol{H}$ into $\boldsymbol{D}(\boldsymbol{A})$. However, for $n = 3$ it is not known if solutions $\boldsymbol{v}$ of (4.292a,b), with $\boldsymbol{v}_0 \in \boldsymbol{H}$, are uniquely defined in $L^\infty([0,T);\boldsymbol{H})$, $\forall T > 0$, thus making impossible the definition of a corresponding nonlinear semigroup of continuous operators $\boldsymbol{S}(t)$. For the initial-boundary value problem for the nonlinear bipolar viscous fluid, as we have seen in Sect. 4.2, we have an entirely different and more satisfactory situation; this fact (something which is also valid in reference to the existence and uniqueness theory for the other sets of fluid dynamics equations which were introduced in Sect. 1.6) does nothing to detract from the importance of resolving the outstanding issues related to the solutions of the Navier–Stokes equations in three space dimensions.

4.5.3 Existence and Uniqueness for Solutions of the Ladyzhenskaya Model Equations

In Sect. 1.6 we pointed out that each of the forms of the reduced stress tensor $\boldsymbol{\tau}$ in (1.189a–d) satisfy the three conditions (1.188a,b,c) which characterize the Ladyzhenskaya generalization of the Stokes Law (1.7). In order to be somewhat more specific in this subsection, we will focus either on the particular model indicated in (1.189c), or one of several equivalent forms, i.e., we can assume, for example, that for some $\bar{\nu} > 0$, $q > 0$,

$$\tau_{ij}(\boldsymbol{e}) = \bar{\nu}(1 + |\boldsymbol{e}|^2)^q e_{ij}. \tag{4.295a}$$

This ansatz places the resultant non-Newtonian fluid model in accord with the non-Newtonian reduction of the bipolar viscous model of Sect. 1.4 when $\mu_1 = 0$, more precisely, the reduced stress tensor in the bipolar model with $\mu_1 = 0$ takes on the form

$$\tau_{ij}(\boldsymbol{e}) = 2\mu_0(\epsilon + |\boldsymbol{e}|^2)^{\frac{p-2}{2}} e_{ij} \tag{4.295b}$$

which is equivalent to (4.295a) if we identify q with $(p-2)/2$ and rescale the other constitutive parameters. Another form of $\boldsymbol{\tau}$ which is equivalent to (4.295a) is the one which appears in [DuG], namely, for ν_0, ν_1, and r all positive

$$\tau_{ij}(\boldsymbol{e}) = (\nu_0 + \nu_1|\nabla \boldsymbol{v}|^r)\frac{\partial v_i}{\partial x_j} \tag{4.295c}$$

provided we identify r with $2q$. The special case of (4.295c) with $r = 2$ also appears in the monograph [La1] of Ladyzhenskaya. Another example of a reduced stress tensor which satisfies the Ladyzhenskaya condition is (see [FPa])

$$\tau_{ij}(\boldsymbol{e}) = \bar{\nu}\,|\boldsymbol{e}|^{2q}\, e_{ij}. \tag{4.295d}$$

The form of $\boldsymbol{\tau}$ expressed in (4.295d) is but a special case of (4.295b), which holds in the bipolar model (for $\mu_1 = 0$) if we set $\epsilon = 0$, $q = p-2$, and identify $\bar{\nu} = 2\mu_0$. An equivalent form of (4.295d) has also appeared in the work of Lions [Lio1], i.e.,

$$\tau_{ij}(\nabla \boldsymbol{v}) = \nu|\nabla \boldsymbol{v}|^{p-2}\frac{\partial v_i}{\partial x_j}. \tag{4.295e}$$

We now present some of the highlights of the existence and uniqueness results for these models, which generalize Navier–Stokes, involve shear dependent viscosities, and have appeared in [La6, DuG], and [Lio1].

Under the hypothesis that $q \geq 1/4$ in (4.295a) it is proven in [La6] that there exist globally (in time) unique weak solutions of the initial-boundary value problem on an open bounded domain $\Omega \subset R^3$, satisfying $\boldsymbol{v} = \boldsymbol{0}$ on $\partial\Omega \times [0, T)$, and $\nabla \cdot \boldsymbol{v} = 0$ in $\Omega \times [0, T)$, for any $T > 0$, if the initial data function $\boldsymbol{v}_0$ belongs to the closure of the set of functions which are divergence free, of class C^∞ on Ω, and have compact support in Ω. Ladyzhenskaya has also indicated [La2] that the boundary-value problem for the corresponding steady state equations, with $\boldsymbol{v} = \boldsymbol{0}$ on $\partial\Omega$ has at least one solution for any $\nu_0 > 0$, $\nu_1 \geq 0$, and $r > 0$, irregardless of the size of the body force term. In [DuG] the authors establish some new a priori estimates for weak solutions of

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j} \tau_{ij}(\boldsymbol{e}) + F_i, \text{ in } \Omega \times [0, T), \tag{4.296a}$$

$$\nabla \cdot \boldsymbol{v} = 0, \text{ in } \Omega \times [0, T), \tag{4.296b}$$

$$\boldsymbol{v}(\boldsymbol{x}, t) = \boldsymbol{0}, \text{ on } \partial\Omega \times [0, T), \tag{4.296c}$$

$$\boldsymbol{v}(\boldsymbol{x}, 0) = \boldsymbol{v}_0(\boldsymbol{x}), \text{ in } \Omega, \tag{4.296d}$$

with $\Omega \subset R^3$ an open bounded domain with smooth $\partial\Omega$ and where $\tau_{ij}(\boldsymbol{e})$ has the form specified in (4.295c); they claim to show that (i) weak solutions of (4.296a–d) are globally defined (in time) and unique provided $r \geq 1/5$ in (4.295c). For the non-Newtonian reduction of the bipolar model defined by (4.295b), the restriction $r \geq 1/5$ is equivalent to having $p \geq 11/5$; thus the result claimed in [DuG], for the problem (4.296a–d), is equivalent to the similar result obtained in [BBN2, 3] for the space-periodic problem which was presented in Sect. 4.3. For the initial-boundary value problem (4.296a–d) on $\Omega \subset R^3$, with a reduced stress tensor $\boldsymbol{\tau}(\boldsymbol{e})$ satisfying growth and coercivity conditions, which we specify below, and which include the models (4.295a,b,c), Málek et al. [MNR1, 2] prove the existence of a globally (in time) defined unique regular weak solution which, for the model defined by (4.295b), holds for $p \geq 9/4$; for the initial-boundary value problem, this is a result which is a little weaker than that established in Sect. 4.3 for the space-periodic problem and, also, that claimed in [DuG] for the problem (4.296a–d). The specific assumptions in [MNR2], which hold for each of the models (4.295a,b,c), are as follows: Let $R^{3\times 3}_{sym} = \{\boldsymbol{e} \in R^{3\times 3} \mid e_{ij} = e_{ji};\ i, j = 1, 2, 3\}$. Then it is assumed that there exists a potential $\Gamma : R^+ \to R^+$, and constants $c_1, c_2 > 0$, such that for some $p > 1$, all $i, j, k, l = 1, 2, 3$, and $\boldsymbol{a}, \boldsymbol{e} \in R^{3\times 3}_{sym}$,

$$\tau_{ij}(\boldsymbol{e}) = \frac{\partial}{\partial e_{ij}} \Gamma(|\boldsymbol{e}|), \tag{4.297a}$$

$$\Gamma(0) = \frac{\partial}{\partial e_{ij}} \Phi|_{\boldsymbol{x}=\boldsymbol{0}} = 0, \tag{4.297b}$$

$$\frac{\partial}{\partial e_{ij}} \frac{\partial}{\partial e_{kl}} \Gamma(|\boldsymbol{e}|) a_{ij} a_{kl} > c_1 (1 + |\boldsymbol{e}|)^{p-2} |\boldsymbol{e}|^2, \tag{4.297c}$$

and

$$\left|\frac{\partial}{\partial e_{ij}}\frac{\partial}{\partial e_{kl}}\Gamma(|\boldsymbol{e}|)\right| \le c_2(1+|\boldsymbol{e}|)^{p-2}. \tag{4.297d}$$

We note that the work in [MNR2] and, in particular, the existence of a potential Γ satisfying (4.297a–d), does not hold for (4.295d) or its equivalent (4.295e); however, the situation in which the reduced stress tensor $\boldsymbol{\tau}(\boldsymbol{e})$ is given by (4.295e), for some $\nu > 0$, has been treated by Lions in Chap. 2 of [Lio1] as follows: Consider, for $\Omega \subset R^n$ with smooth boundary $\partial\Omega$, the initial-boundary value problem (4.296a–d) and let the $\tau_{ij}(\boldsymbol{e})$ be specified as in (4.295e). Let

$$\mathcal{V} = \{\boldsymbol{\varphi} \mid \boldsymbol{\varphi} = \{\varphi_1, \ldots, \varphi_n\},\ \varphi_i \in C_0^\infty(\Omega) \text{ and } \nabla\cdot\boldsymbol{\varphi} = 0\} \tag{4.298a}$$

and

$$\bar{\mathcal{V}} \equiv \text{ the completion of } \mathcal{V} \text{ in } \boldsymbol{W}^{1,p}(\Omega). \tag{4.298b}$$

It can be shown that

$$\bar{\mathcal{V}} = \{\boldsymbol{v} \mid \boldsymbol{v} \in \boldsymbol{W}_0^{1,p}(\Omega) \text{ with } \nabla\cdot\boldsymbol{v} = 0\}. \tag{4.298c}$$

Finally, let

$$\bar{\mathcal{H}} \equiv \text{ the completion of } \mathcal{V} \text{ in } \boldsymbol{L}^2(\Omega). \tag{4.298d}$$

Then, it is proven in [Lio1] that, if $\boldsymbol{F} \in L^{p'}((0,T); \bar{\mathcal{V}}')$ and $\boldsymbol{v}_0 \in \bar{\mathcal{H}}$, for

$$p \ge 1 + \frac{2n}{n+2} \tag{4.299}$$

there exists a solution $(\boldsymbol{v}, p)$ of (4.296a–d) such that

$$\boldsymbol{v} \in L^p((0,T); \bar{\mathcal{V}}) \cap L^\infty((0,T); \bar{\mathcal{H}}). \tag{4.300}$$

The problem of the uniqueness of this solution appears to be an open one (see Remark 5.3 in [Lio1]). For $n = 3$ in (4.299) we recover the criterion that $p \ge 11/5$ (although, without the corresponding uniqueness of the solution established in Sect. 4.3 for the space-periodic problem and claimed in [DuG] for the boundary-value problem). Finally, with a view towards applications to non-Newtonian blood flow in mind, the existence and uniqueness of weak solutions for the problem of flow in a compliant vessel is treated in [LuZ1, 2]; in this case the domain is time-dependent and the fluid dynamics equations are coupled to a generalized string equation which ties the motion of the boundary of the domain to the influence of the pressure and shear stress exerted by the fluid. Admissible forms of the reduced stress tensor $\boldsymbol{\tau}(\boldsymbol{e})$ in [LuZ2] are assumed to conform to the growth and coercivity conditions (4.297a–d) in [MNR2].

Remarks (On the results claimed in [DuG]:). In [DuG] it is claimed that the existence of a weak solution proved in [La6] by Ladyzenskaya for $r \geq 1/2$ (4.295c) is being extended to $r \geq 1/5$. However, as indicated in [BBN3], several points in the proof of [DuG] were not clear to us. Ladyzenskaya in a footnote in [La7] commented on the claims in [DuG] saying: *But in [DuG] there are no explanation*(s) *as to what is possible* (*i.e.,* needed) *for the reduction of the requirement* $r \geq 1/2$ *to the requirement* $r \geq 1/5$.

4.5.4 Existence and Uniqueness Results for Viscous Flow Models with Artificial Viscosity

As we have previously indicated in Sect. 1.6, dissipative modifications of the Navier–Stokes equations which lead to systems of the type (1.195), for the velocity components v_i of a viscous fluid, subject to the incompressibility constraint $\nabla \cdot \boldsymbol{v} = 0$, appear in the work of Lions [Lio1, 2], Ladyzhenskaya [La5], and Beirão da Veiga [BdV2, 3]; the latter author studied this type of regularization for problems posed in all of $\mathbb{R}^n$ and was able to prove convergence theorems for the situation in which the regularization parameter $\epsilon \to 0^+$. The regularization in [Lio1, 2] has the form in (1.195) with $\beta = 2m$ and, for problems in a bounded domain $\Omega \subset \mathbb{R}^n$, with smooth boundary $\partial\Omega$, m additional boundary conditions of Neumann type are appended to the non-slip condition $\boldsymbol{v} = \boldsymbol{0}$ on $\partial\Omega \times [0, T)$; for such systems Lions was able to prove the global existence of weak solutions.

In this subsection we will describe the existence and uniqueness results obtained in [OS1], for the regularization employed by Lions, in that special case where $m = 1$; in a follow-up paper [OS2] the authors establish the existence of a compact global attractor, as well as invariant manifolds, for the same modification of the incompressible Navier–Stokes equations. The problem in [OS1] is posed in a bounded open domain $\Omega \subset \mathbb{R}^n$, with $n \leq 6$, whose boundary $\partial\Omega$ is of class C^r, $r \geq 4$; with the velocity field $\boldsymbol{v}$ given by $\boldsymbol{v} = (v_1, \ldots, v_n)$ the precise form of the initial-boundary value problem in [OS1] is as follows:

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\nabla p + \nu \Delta \boldsymbol{v} - \epsilon \Delta^2 \boldsymbol{v} + \boldsymbol{F}, \text{ in } \Omega \times [0, T), \tag{4.301a}$$

$$\nabla \cdot \boldsymbol{v} = 0, \text{ in } \Omega \times [0, T), \tag{4.301b}$$

$$\boldsymbol{v} = \boldsymbol{0}, \quad \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{\nu}} = \nabla \boldsymbol{v} \cdot \boldsymbol{\nu} = \boldsymbol{0}, \text{ on } \partial\Omega \times [0, T), \tag{4.301c}$$

$$\boldsymbol{v}(\boldsymbol{x}, 0) = \boldsymbol{v}_0(\boldsymbol{x}), \text{ in } \Omega \tag{4.301d}$$

where $\nu = \mu_0/\rho$ is the usual kinematic viscosity and $\boldsymbol{\nu}$ is the unit outward normal on $\partial\Omega$. Results similar to those described, below, are also established in [OS1] for problems posed on $\Omega = [0, L]^n$ with periodic boundary conditions replacing those in (4.301c). *We emphasize here, as we did in Sect. 1.6, that the addition of the higher-order derivatives in* (4.301a) *has been made on an ad hoc basis and is not supported by a physical argument of the kind which forms the basis for the viscous bipolar model introduced in Sect. 1.4. Moreover, the Neumann boundary condition in* (4.301c) *is inconsistent with any form of the principle of virtual work which can be associated with a constitutive theory that would lead to the (vector) evolution equation* (4.301a). In spite of this observation, results obtained for systems of the form (4.301a–d) are of interest for several reasons. As pointed out in [OS1] numerical approximations to solutions of the Navier–Stokes equations produce truncation errors which depend on the chosen mesh size; the system (4.301a–d) can be considered as a model for a modification of the Navier–Stokes equations where the artificial viscosity depends on that mesh size. Additionally, it is of some mathematical interest to prove, as has been done by Ladyzhenskaya [La5] that strong (regular) solutions of a system like (4.301a–d) converge to a solution of the Navier–Stokes initial-boundary value problem as $\epsilon \to 0^+$. Indeed, one of the basic results obtained in [OS1], which is highlighted in this subsection, is that weak solutions of (4.301a–d) converge, for sufficiently small Reynolds numbers, to a weak solution of the corresponding Navier–Stokes problem. The existence theorem for weak solutions for the regularized system (4.301a–d) was first established by Lions in [Lio1].

With $\boldsymbol{H}$ defined as in (5.5a) and

$$\hat{\boldsymbol{V}} = \{\boldsymbol{v} \in \boldsymbol{H}_0^2(\Omega) \mid \nabla \cdot \boldsymbol{v} = 0\}, \tag{4.302}$$

the weak formulation of (4.301a–d) in [OS1] is the following: Suppose that $\boldsymbol{F} \in L^2((0,T), \hat{\boldsymbol{V}}')$. Find

$$\boldsymbol{v} \in L^2((0,T); \hat{\boldsymbol{V}}) \cap L^\infty((0,T); \boldsymbol{H})$$

such that $\boldsymbol{v}(0) = \boldsymbol{v}_0 \in \boldsymbol{H}$ and, $\forall \boldsymbol{u} \in \hat{\boldsymbol{V}}$,

$$\frac{d}{dt}(\boldsymbol{v}, \boldsymbol{u})_{L^2(\Omega)} + \epsilon(\Delta \boldsymbol{v}, \Delta \boldsymbol{u})_{L^2(\Omega)} + \nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v})_{L^2(\Omega)} + b(\boldsymbol{v}, \boldsymbol{v}, \boldsymbol{u}) = \langle \boldsymbol{F}, \boldsymbol{u} \rangle, \text{ in } \mathcal{D}'(0,T) \tag{4.303}$$

where $\langle , \rangle$ is the usual duality pairing between $\hat{\boldsymbol{V}}$ and $\hat{\boldsymbol{V}}'$, $\mathcal{D}'(0,T)$ is the dual of the space of test functions $\mathcal{D}(0,T)$ on $(0,T)$, and $b(\cdot,\cdot,\cdot)$ is the standard trilinear form, $b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \int_\Omega u_i \frac{\partial v_i}{\partial x_j} w_j \, d\boldsymbol{x}$. An alternative but (as shown in [OS1]) equivalent weak formulation of (4.301a–d) may be stated as follows: given $\boldsymbol{F} \in L^2((0,T); \hat{\boldsymbol{V}}')$, find $\boldsymbol{v} \in L^2((0,T); \hat{\boldsymbol{V}})$ with $\boldsymbol{v}' \in L^2((0,T); \hat{\boldsymbol{V}}')$ such that

$$\boldsymbol{v}' + \epsilon \boldsymbol{A}\boldsymbol{u} + \nu \boldsymbol{A}_1 \boldsymbol{v} + \boldsymbol{B}\boldsymbol{v} = \boldsymbol{F}, \text{ in } \mathcal{D}'((0,T); \hat{\boldsymbol{V}}') \tag{4.304}$$

with $\boldsymbol{v}(0) = \boldsymbol{v}_0 \in H$. To define the operators $\boldsymbol{A}$, $\boldsymbol{A}_1$, and $\boldsymbol{B}$ in (4.304) we first introduce the bilinear form $a(\boldsymbol{u}, \boldsymbol{v}) = (\Delta \boldsymbol{u}, \Delta \boldsymbol{v})_{\boldsymbol{L}^2(\Omega)}$, observe that the conditions of the Lax-Milgram Lemma (Appendix A) apply, so that for $\boldsymbol{F} \in \hat{\boldsymbol{V}}'$ there exists a unique $\boldsymbol{v} \in \hat{\boldsymbol{V}}$ such that $a(\boldsymbol{v}, \boldsymbol{u}) = \langle \boldsymbol{F}, \boldsymbol{u} \rangle$, $\forall \boldsymbol{u} \in \boldsymbol{V}$, and then write the correspondence $\boldsymbol{F} \rightarrow \boldsymbol{v}$ as $\boldsymbol{A}\boldsymbol{v} = \boldsymbol{F}$. Continuity of the bilinear form $a(\cdot, \cdot)$ implies the continuity of $\boldsymbol{A}$ while the coercivity of $a(\cdot, \cdot)$ implies that $\boldsymbol{A}$ is injective. In fact $\boldsymbol{A} \in \mathcal{L}(\hat{\boldsymbol{V}}; \hat{\boldsymbol{V}}')$ is an isomorphism. With $\boldsymbol{P}$ the orthogonal projection of $\boldsymbol{L}^2(\Omega)$ onto $\boldsymbol{H}$, $\boldsymbol{A}_1$ is given by $\boldsymbol{A}_1 \boldsymbol{v} = -\boldsymbol{P}\Delta \boldsymbol{v}$, $\forall \boldsymbol{v} \in \boldsymbol{H}^2(\Omega) \cap \boldsymbol{V}^*$ where

$$\boldsymbol{V}^* = \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega) \mid \boldsymbol{v}|_{\partial\Omega} = \boldsymbol{0} \text{ and } \nabla \cdot \boldsymbol{v} = 0, \text{ in } \Omega\}. \tag{4.305}$$

The following results on the existence, uniqueness, and regularity of weak solutions to the initial-boundary value problem (4.301a–d) have been established in [OS1], with a Galerkin approximation method employed for the existence part:

Theorem 4.20. *Let $n \leq 6$, let $\boldsymbol{F} \in L^2((0, T); \hat{\boldsymbol{V}}')$, and let $\boldsymbol{v}_0 \in \boldsymbol{H}$. Then there exists a unique solution $\boldsymbol{v} \in L^2((0, T); \hat{\boldsymbol{V}}) \cap C((0, T); \boldsymbol{H})$ to the regularized Navier–Stokes system* (4.301a–d) *which also satisfies $\boldsymbol{v}' \in L^2((0, T); \hat{\boldsymbol{V}}')$. The solution $\boldsymbol{v}$ also satisfies, for any $t \in (0, T)$, the energy identity*

$$||\boldsymbol{v}(t)||^2 + 2\epsilon \int_0^t ||\boldsymbol{v}(\tau)||^2_{\boldsymbol{L}^2(\Omega)}\, d\tau + 2\nu \int_0^t |\nabla \boldsymbol{v}|^2(\tau)\, d\tau = ||\boldsymbol{v}_0||^2 + 2\int_0^t \langle \boldsymbol{F}, \boldsymbol{v}\rangle\, d\tau. \tag{4.306}$$

Using Theorem 4.20, Ou and Sritharan also prove that the weak solution of (4.301a–d) satisfies, uniformly in the artificial viscosity ϵ, the estimates in the following theorem:

Theorem 4.21. *Let $\boldsymbol{F} \in L^2((0, T); \boldsymbol{H})$. Then weak solutions $\boldsymbol{v}$ of* (4.301a–d) *satisfy, for some $\lambda_1 > 0$, $\lambda_2 > 0$,*

$$||\boldsymbol{v}(t)||^2 \leq ||\boldsymbol{v}_0||^2 e^{-(2\epsilon\lambda_1 + \nu\lambda_2)t} + \frac{1}{\nu\lambda_2}\int_0^t ||\boldsymbol{F}(\tau)||^2_{\boldsymbol{L}^2(\Omega)}\, d\tau \tag{4.307a}$$

and

$$\int_0^T |\nabla \boldsymbol{v}|^2\, dt \leq \frac{1}{\nu}\left[||\boldsymbol{v}_0||^2 + \frac{1}{\nu\lambda_2}\int_0^T ||\boldsymbol{F}(t)||^2_{\boldsymbol{L}^2(\Omega)}\, dt\right]. \tag{4.307b}$$

As is common, of course, for such problems, with more regularity for the initial data $\boldsymbol{v}_0(\cdot)$ comes better regularity results for the solution; in fact we may state

Theorem 4.22. *If* $\Omega \subset R^2$, $\boldsymbol{v}_0 \in \hat{V}$, *and* $\boldsymbol{F} \in L^2((0,T);\boldsymbol{H})$, *then the weak solution of* (4.301a–d) *satisfies* $\boldsymbol{v} \in L^2((0,T),T);\mathcal{D}(\boldsymbol{A})) \cap C((0,T);\boldsymbol{V})$ *and* $\boldsymbol{v}' \in L^2((0,T);\boldsymbol{H})$. *Moreover, these (strong) solutions also satisfy, for some* $K > 0$, *which is independent of both* ϵ *and* t, $|\nabla \boldsymbol{v}(t)| \le K$, $\forall t \in [0,T]$.

The final result in this sequence is a result which shows that weak solutions of (4.301a–d) converge, as $\epsilon \to 0$, to a corresponding weak solution of the Navier–Stokes initial-boundary value problem; in stating this result we explicitly display the dependence of the solutions of (4.301a–d) on the artificial viscosity parameter ϵ:

Theorem 4.23. *Let* $\Omega \subset R^3$, *with* $\partial\Omega$ *of class* C^4, *and let* $\boldsymbol{v}_\epsilon$ *be the unique weak solution of the modified Navier–Stokes system* (4.301a–d) *with* $\boldsymbol{F} \in L^2((0,T);\boldsymbol{H})$. *Then for* $\nu > 0$ *sufficiently large,* $\boldsymbol{v}_\epsilon$ *converges to a unique limit* $\boldsymbol{v}^*$, *which is a weak solution of* (4.301a–d) *with* $\epsilon = 0$ *satisfying the first boundary condition (only) in* (4.301c). *More precisely, it follows that, uniformly in* ϵ,

$$\boldsymbol{v}_\epsilon \to \boldsymbol{v}^*, \ \textit{in } L^2((0,T);\boldsymbol{V}^*), \ \textit{weakly as } \epsilon \to 0^+,$$

$$\boldsymbol{v}_\epsilon \to \boldsymbol{v}^*, \ \textit{in } L^\infty((0,T);\boldsymbol{H}), \ \textit{weak } * \textit{ as } \epsilon \to 0^+,$$

and

$$\boldsymbol{v}_\epsilon \to \boldsymbol{v}^*, \ \textit{in } L^2((0,T);\boldsymbol{L}^2(\Omega)), \textit{ strongly as } \epsilon \to 0^+.$$

Other results which are established in [OS1] for the system (4.301a–d) include a proof of the existence of at least one time-periodic solution (with the same period) which corresponds to the prescription of a time-periodic forcing function $\boldsymbol{F}$ and the demonstration that a squeezing property holds for orbits of (4.301a–d) emanating from neighboring initial values in $\hat{\boldsymbol{V}}$. (See Sects. 6.2.4 and 6.2.5 for the analogous property for the viscous bipolar fluid). We leave it to the interested reader to consult the original paper [OS1] for the details as well as for the proofs of Theorems 4.20–4.23 of this subsection, each of which represents an amalgamation of a set of results in [OS1]. Those readers interested in the subject matter of Chaps. 5 and 6 may also wish to consult [OS2].

4.5.5 Existence, Uniqueness, and Stability Theorems for Multipolar Fluids of Grade 3

The stress tensor associated with the response of a multipolar fluid of grade 3 is defined in [BNR] by the constitutive law (1.199d), where the first and second Rivlin-Ericksen tensors $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are given by (1.198) and the tensor with components S_{ijk} in (1.199d) is defined by (1.199a,b,c). For the balance of this subsection, the components of the velocity gradient will be denoted as L_{ij} $(= v_{i,j})$. Also, $\Omega \subset R^3$ will be an open bounded domain, with smooth boundary $\partial\Omega$, $I = [0,T)$, and for

$T > 0$ we set $Q_T = \Omega \times [0, T)$. We begin by defining what is meant by a weak solution of the problem

$$\rho \dot{v}_i = t_{ij,j} + \rho f_i, \text{ in } Q_T, \tag{4.308a}$$

$$\nabla \cdot \boldsymbol{v} = 0, \text{ in } Q_T, \tag{4.308b}$$

$$\boldsymbol{v}(\boldsymbol{x}, 0) = \boldsymbol{v}_0(x), \text{ in } \Omega, \tag{4.308c}$$

and we note that the formulation presented, below, is broad enough to incorporate both the usual non-slip boundary condition $\boldsymbol{v} = \boldsymbol{0}$ on $\partial\Omega \times [0, T)$, as well as the relevant set (1.201a,b) of higher-order boundary conditions.

Definition 4.4. Let $\alpha_1 > 0$, $\alpha_2 \in (-\infty, \infty)$, $\beta_3 > 0$, $\mu_0 \geq 0$, $\mu_1 \geq 0$, $\mu_2 > 0$, and $\gamma \geq 0$. A function $\boldsymbol{v} \in L^4(I; \boldsymbol{W}_0^{1,4}(\Omega))$ which also satisfies $\boldsymbol{v} \in L^2(I; \boldsymbol{W}^{3,2}(\Omega))$ and $\boldsymbol{v}_{,t} \in L^2(I; \boldsymbol{W}^{2,2}(\Omega))$ is a weak solution of (4.308a,b,c) if a.e., $\forall t \geq 0$, and all $\boldsymbol{\phi} \in \boldsymbol{W}_0^{1,2}(\Omega) \cap \boldsymbol{W}^{3,2}(\Omega)$ such that $\nabla \cdot \boldsymbol{\phi} = 0$,

$$\begin{aligned}
&\int_\Omega \frac{\partial v_i}{\partial t} \phi_i \, d\boldsymbol{x} + \alpha_1 \int_\Omega \frac{\partial (\boldsymbol{A}_1)_{ij}}{\partial t} \phi_{i,j} \, d\boldsymbol{x} + \gamma \int_\Omega \frac{\partial D_{ij,k}}{\partial t} \phi_{i,jk} \, d\boldsymbol{x} + \mu_0 \int_\Omega (\boldsymbol{A}_1)_{ij} \phi_{i,j} \, d\boldsymbol{x} \\
&\quad + \alpha_1 \int_\Omega \left[\frac{\partial (\boldsymbol{A}_1)_{ij}}{\partial x_l} v_l + L_{mi} (\boldsymbol{A}_1)_{mj} + L_{mj} (\boldsymbol{A}_1)_{im} \right] \phi_{i,j} \, d\boldsymbol{x} + \alpha_2 \int_\Omega (\boldsymbol{A}_1^2)_{ij} \phi_{i,j} \, d\boldsymbol{x} \\
&\quad + \beta_3 \int_\Omega (\boldsymbol{A}_1^2)_{mm} (\boldsymbol{A}_1)_{ij} \phi_{i,j} \, d\boldsymbol{x} + \mu_1 \int_\Omega (\boldsymbol{A}_1)_{ij,k} \phi_{i,jk} \, d\boldsymbol{x} \\
&\quad + \int_\Omega v_j \frac{v_i}{\partial x_j} \phi_i \, d\boldsymbol{x} + \mu_2 \int_\Omega (\boldsymbol{A}_1)_{ij,km} \phi_{i,jkm} \, d\boldsymbol{x} \\
&\quad - \int_\Omega f_i \phi_i \, d\boldsymbol{x} + \gamma \int_\Omega (W_{mi} D_{mj,k} + W_{mj} D_{im,k} + W_{mk} D_{ij,m}) \phi_{i,jk} \, d\boldsymbol{x} = 0.
\end{aligned} \tag{4.309}$$

Prior to stating our first existence theorem for the model in [BNR] two lemmas need to be established; these lemmas, as well as the proof of the existence theorem itself, are based on the introduction of the framework for a Galerkin argument as follows: Consider the sequence $\{\boldsymbol{w}^l\}$ of eigenfunctions for the Stokes problem in Ω with homogeneous Dirichlet boundary conditions. It is well known that $\{\boldsymbol{w}^l\}$ constitutes a Schauder basis for the space

$$\boldsymbol{W} = \{\boldsymbol{w} \mid \boldsymbol{w} \in \boldsymbol{W}^{3,2}(\Omega) \cap \boldsymbol{W}_0^{1,2}(\Omega);\ \nabla \cdot \boldsymbol{w} = 0\}.$$

We set $\boldsymbol{E}_m = \text{span}\{\boldsymbol{w}^l;\ l = 1, 2, \ldots, m\}$ and let

$$\boldsymbol{v}^m = \sum_{l \leq m} c^l(t) \boldsymbol{w}^l$$

be such that if we set $\boldsymbol{v} = \boldsymbol{v}^m$ in (4.309), then (4.309) is satisfied for all test functions $\boldsymbol{\phi} \in \boldsymbol{E}_m$ provided we choose $c^l(0) = \int_\Omega \boldsymbol{v}_0 \cdot \boldsymbol{w}^l \, d\boldsymbol{x}$, $\forall l$. The existence and uniqueness of $\boldsymbol{v}^m$ follows from the Picard theorem. Our first lemma assumes the following form:

Lemma 4.17. *There exists $C > 0$, independent of m, such that for each $i = 1, 2, 3$,*

$$||\boldsymbol{v}^m||_{L^\infty((0,T);\boldsymbol{W}^{1,2}(\Omega))} \leq C\left(||\boldsymbol{v}_0||^2_{\boldsymbol{W}^{2,2}(\Omega)} + ||f_i||^2_{L^2(Q_T)} + 1\right), \tag{4.310a}$$

$$||\boldsymbol{v}^m||_{L^2((0,T);\boldsymbol{W}^{3,2}(\Omega))} \leq C\left(||\boldsymbol{v}_0||^2_{\boldsymbol{W}^{2,2}(\Omega)} + ||f_i||^2_{L^2(Q_T)} + 1\right), \tag{4.310b}$$

$$||\boldsymbol{v}^m||_{L^4((0,T);\boldsymbol{W}^{2,4}(\Omega))} \leq C\left(||\boldsymbol{v}_0||^2_{\boldsymbol{W}^{2,2}(\Omega)} + ||f_i||^2_{L^2(Q_T)} + 1\right). \tag{4.310c}$$

Proof. We take $\boldsymbol{v} = \boldsymbol{v}^m$ in (4.309) and, also, use $\boldsymbol{\phi} = \boldsymbol{v}^m$ as a test function. With Ω_t denoting the configuration occupied by the fluid at time t, we obtain after integration in t the following estimate, where for the sake of not having to deal with an overcumbersome notation we have deleted the m superscript on all relevant quantities:

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega_t} v_i v_i \, d\boldsymbol{x} + \alpha_1 \int_{\Omega_t} D_{ij} D_{ij} \, d\boldsymbol{x} + \frac{1}{2}\gamma \int_{\Omega_t} D_{ij,k} D_{ij,k} \, d\boldsymbol{x} + \mu_0 \int_0^T \int_{\Omega_t} (\boldsymbol{A}_1)_{ij} (\boldsymbol{A}_1)_{ij} \, d\boldsymbol{x} \, dt \\
&\quad + \mu_1 \int_0^T \int_{\Omega_t} (\boldsymbol{A}_1)_{ij,k} (\boldsymbol{A}_1)_{ij,k} \, d\boldsymbol{x} \, dt + \mu_2 \int_0^T \int_\Omega (\boldsymbol{A}_1)_{ij,km} (\boldsymbol{A}_1)_{ij,km} \, d\boldsymbol{x} \, dt \\
&\quad + \beta_3 \int_0^T \int_\Omega (\boldsymbol{A}_1^2)_{mm} (\boldsymbol{A}_1)_{ij} (\boldsymbol{A}_1)_{ij} (\boldsymbol{A}_1)_{ij} \, d\boldsymbol{x} \, dt \\
&\leq \frac{1}{2}\int_\Omega v_i v_i \, d\boldsymbol{x} + \alpha_1 \int_\Omega D_{ij} D_{ij} \, d\boldsymbol{x} + \frac{1}{2}\gamma \int_\Omega D_{ij,k} D_{ij,k} \, d\boldsymbol{x} + \left| \int_0^T \int_\Omega b_i v_i \, d\boldsymbol{x} \, dt \right| \\
&\quad + \left| \alpha_1 \int_0^T \int_\Omega [L_{mi} (\boldsymbol{A}_1)_{mj} + L_{mj} (\boldsymbol{A}_1)_{im}] v_{i,j} \, d\boldsymbol{x} \, dt \right| + \left| \alpha_2 \int_0^T (\boldsymbol{A}_1^2)_{ij} v_{i,j} \, d\boldsymbol{x} \, dt \right|.
\end{aligned} \tag{4.311}$$

In arriving at (4.311) we have made use of the identities

$$\alpha_1 \int_0^T \int_\Omega \frac{\partial (\boldsymbol{A}_1)_{ij}}{\partial x_l} v_l v_{i,j} \, d\boldsymbol{x} \, dt = 0, \tag{4.312a}$$

$$\int_0^T \int_\Omega (W_{mi} D_{mj,k} + W_{mj} D_{im,k} + W_{mk} D_{ij,m}) v_{i,jk} \, d\boldsymbol{x} \, dt = 0, \tag{4.312b}$$

$$\int_\Omega v_j \frac{\partial v_i}{\partial x_j} v_i \, d\boldsymbol{x} = 0. \tag{4.312c}$$

It then follows from (4.311), upon reinstating the m superscript in the Galerkin approximation $\boldsymbol{v}^m$, that

$$||\boldsymbol{v}^m||^2_{\boldsymbol{W}^{2,2}(\Omega_T)} + ||\boldsymbol{v}^m||^2_{L^2((0,T);\boldsymbol{W}^{3,2}(\Omega))} + ||\boldsymbol{v}^m||^4_{L^4((0,T);\boldsymbol{W}^{2,4}(\Omega))}$$
$$\leq c\left(||\boldsymbol{v}^m||^2_{\boldsymbol{W}^{2,2}(\Omega)} + ||v_i^m||_{L^2(Q_T)}||f_i||_{L^2(Q_T)} + ||Dv_i^m||^3_{L^3(Q_T)}\right) \quad (4.313)$$

for some $c > 0$. Using standard techniques it now follows from (4.313) that, for each $i = 1, 2, 3$,

$$||\boldsymbol{v}^m||_{\boldsymbol{W}^{2,2}(\Omega_T)} + ||\boldsymbol{v}_0||^2_{L^2((0,T);\boldsymbol{W}^{3,2}(\Omega))} + ||\boldsymbol{v}^m||^4_{L^4((0,T);\boldsymbol{W}^{2,4}(\Omega))}$$
$$\leq c\left(||\boldsymbol{v}^m||^2_{\boldsymbol{W}^{2,2}(\Omega)} + ||f_i||^2_{L^2(Q_T)} + 1\right) \quad (4.314)$$

which completes the proof of Lemma 4.17. □

The second lemma needed for the proof of the basic existence theorem for multipolar fluids of grade 3 is

Lemma 4.18. *For the Galerkin approximations* $\boldsymbol{v}^m$, $\exists C > 0$, *independent of m, such that*

$$\left|\left|\frac{\partial \boldsymbol{v}^m}{\partial t}\right|\right|_{L^2((0,T);\boldsymbol{W}^{2,2}(\Omega))} \leq C\left(||\boldsymbol{v}_0||^2_{\boldsymbol{W}^{3,2}(\Omega)} + ||f_i||^2_{L^2(Q_T)} + 1\right). \quad (4.315)$$

Proof. As $\frac{\partial \boldsymbol{v}^m}{\partial t} \in E_m$, we may use $\boldsymbol{\phi} = \frac{\partial \boldsymbol{v}^m}{\partial t}$ in (4.309). Deleting, as in the proof of Lemma 4.17, the superscript m we find after integration in time

$$\int_0^T \int_\Omega \frac{\partial v_i}{\partial t}\frac{\partial v_i}{\partial t}\, d\boldsymbol{x}\, dt + \alpha_1 \int_0^T \int_\Omega \frac{\partial (\boldsymbol{A}_1)_{ij}}{\partial t}\frac{\partial (\boldsymbol{A}_1)_{ij}}{\partial t}\, d\boldsymbol{x}\, dt \quad (4.316)$$
$$+ \gamma \int_0^T \int_\Omega \frac{\partial D_{ij,k}}{\partial t}\frac{\partial D_{ij,k}}{\partial t}\, d\boldsymbol{x}\, dt + \frac{1}{2}\mu_0 \int_{\Omega_T} (\boldsymbol{A}_1)_{ij}(\boldsymbol{A}_1)_{ij}\, d\boldsymbol{x}$$
$$+ \frac{1}{2}\mu_1 \int_{\Omega_T} (\boldsymbol{A}_1)_{ij,k}(\boldsymbol{A}_1)_{ij,k}\, d\boldsymbol{x} + \frac{1}{2}\mu_2 \int_{\Omega_T} (\boldsymbol{A}_1)_{ij,km}(\boldsymbol{A}_1)_{ij,km}\, d\boldsymbol{x}$$
$$\leq \frac{1}{2}\mu_0 \int_\Omega (\boldsymbol{A}_1)_{ij}(\boldsymbol{A}_1)_{ij}\, d\boldsymbol{x} + \frac{1}{2}\mu_1 \int_\Omega (\boldsymbol{A}_1)_{ij,k}(\boldsymbol{A}_1)_{ij,k}\, d\boldsymbol{x}$$
$$+ \frac{1}{2}\mu_2 \int_\Omega (\boldsymbol{A}_1)_{ij,km}(\boldsymbol{A}_1)_{ij,km}\, d\boldsymbol{x}$$
$$+ \left|\alpha_1 \int_0^T \int_\Omega \left[\frac{\partial (\boldsymbol{A}_1)_{ij}}{\partial x_l} v_l + L_{mi}(\boldsymbol{A}_1)_{mj} + L_{mj}(\boldsymbol{A}_1)_{im}\right] \frac{\partial v_{i,j}}{\partial t}\, d\boldsymbol{x}\, dt\right|$$
$$+ \left|\alpha_2 \int_0^T \int_\Omega (\boldsymbol{A}_1^2)_{ij} \frac{\partial v_{i,j}}{\partial t}\, d\boldsymbol{x}\, dt\right| + \left|\beta_3 \int_0^T \int_\Omega (\boldsymbol{A}_1^2)_{mm}(\boldsymbol{A}_1)_{ij} \frac{\partial v_{i,j}}{\partial t}\, d\boldsymbol{x}\, dt\right|$$

$$+ \left| \int_0^T \int_\Omega v_j \frac{\partial v_i}{\partial x_j} \frac{\partial v_i}{\partial t} \, d\boldsymbol{x} \, dt \right| + \left| \int_0^T \int_\Omega b_i \frac{\partial v_i}{\partial t} \, d\boldsymbol{x} \, dt \right|$$
$$+ \left| \gamma \int_0^T \int_\Omega (W_{mi} D_{mj,k} + W_{mj} D_{im,k} + W_{mk} D_{ij,m}) \frac{\partial v_{i,jk}}{\partial t} \, d\boldsymbol{x} \, dt \right|$$
$$\equiv I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8 + I_9.$$

Now $I_1 + I_2 + I_3$ can be bounded by using the norm of the initial condition in $\boldsymbol{W}^{3,2}(\Omega)$ while I_7 can be easily estimated by using the Cauchy-Schwarz inequality, i.e., for any $a > 0$,

$$I_7 \le \frac{1}{a} ||f_i||^2_{L^2(Q_T)} + a \left|\left| \frac{\partial v_i}{\partial t} \right|\right|^2_{L^2(Q_T)}. \tag{4.317}$$

By choosing $a > 0$ small enough, we can absorb the term $a \left|\left| \frac{\partial v_i}{\partial t} \right|\right|^2_{L^2(Q_T)}$ in the left-hand side of (4.316). In a similar manner, by using embedding theorems (see Appendix A) and the Hölder Inequality, the remaining terms on the right-hand side of (4.316) satisfy the bound

$$I_3 + I_4 + I_5 + I_6 + I_8 \le \frac{1}{2} ||v_i||^2_{L^2((0,T);\boldsymbol{W}^{3,2}(\Omega))} + a \left| \frac{\partial v_i}{\partial t} \right|^2_{L^2((0,T);\boldsymbol{W}^{3,2}(\Omega))}. \tag{4.318}$$

Again, choosing a sufficiently small we may absorb the second term on the right-hand side of (4.318) in the left-hand side of (4.316); the required result, i.e., (4.315) now follows from Lemma 4.17 upon reinstatement of the superscript m. □

We can now state and prove the following existence theorem for multipolar fluids of grade 3:

Theorem 4.24. *Assume that $\Omega \subset R^3$ is an open bounded domain with $\partial\Omega$ sufficiently smooth. Then for any $f_i \in L^2(Q_T)$ and any $\boldsymbol{v}_0 \in \boldsymbol{W}^{3,2}(\Omega) \cap \boldsymbol{W}_0^{1,2}(\Omega)$ such that $\nabla \cdot \boldsymbol{v}_0 = 0$, there exists a weak solution to the system* (4.308a,b,c) *satisfying the regularity conditions specified in Definition 4.4. Furthermore, the weak solution $\boldsymbol{v}$ satisfies $\frac{\partial \boldsymbol{v}}{\partial t} \in L^2((0,T); \boldsymbol{W}^{2,2}(\Omega))$.*

Proof. From Lemma 4.17 we deduce that there exists a subsequence, denoted again by $\boldsymbol{v}^m$, which converges, as $m \to \infty$, weakly in $L^2((0,T); \boldsymbol{W}^{3,2}(\Omega) \cap \boldsymbol{W}_0^{1,2}(\Omega))$ to a function $\boldsymbol{v}$. Using the estimate (4.315) and Aubin's Lemma (see Appendix A) we conclude that there exists a subsequence, again denoted by $\boldsymbol{v}^m$, which converges in $L^2((0,T); \boldsymbol{W}^{2,2}(\Omega) \cap \boldsymbol{W}_0^{1,2}(\Omega))$. Both types of convergence referenced above are sufficient to allow for passing to the limit in each of the terms in (4.309), with $v_i = v_i^m$, as $m \to \infty$. □

The main existence and uniqueness result for the model analyzed in [BNR] has weaker assumptions relative to the initial data $\boldsymbol{v}_0$ than those in Theorem 4.24 but yields a weak solution which does not necessarily satisfy the condition that $\frac{\partial \boldsymbol{v}}{\partial t} \in L^2((0,T); \boldsymbol{W}^{2,2}(\Omega))$; we state this result as

Theorem 4.25. *Let $\Omega \subset R^3$ be an open bounded domain with $\partial\Omega$ sufficiently smooth. Then for any $f_i \in L^2(Q_T)$, $i = 1, 2, 3$ and any $\boldsymbol{v}_0 \in \boldsymbol{W}^{2,2}(\Omega) \cap \boldsymbol{W}_0^{1,2}(\Omega)$, such that $\nabla \cdot \boldsymbol{v}_0 = 0$, there exists a unique weak solution of the system satisfying the regularity condition of Definition 4.4.*

Proof. The proof is based on a density argument. As Ω is bounded and $\partial\Omega$ is (sufficiently) smooth the set

$$\boldsymbol{W} = \{\boldsymbol{w} \mid \boldsymbol{w} \in \boldsymbol{W}^{3,2}(\Omega) \cap \boldsymbol{W}_0^{1,2}(\Omega);\ \nabla \cdot \boldsymbol{w} = 0\}$$

is dense in the set

$$\bar{\boldsymbol{W}} = \{\boldsymbol{w} \mid \boldsymbol{w} \in \boldsymbol{W}^{2,2}(\Omega) \cap \boldsymbol{W}_0^{1,2}(\Omega);\ \nabla \cdot \boldsymbol{w} = 0\}.$$

An easy way to see this is to recall that the eigenfunctions of the Stokes problem in Ω are smooth and constitute a basis of the space $\bar{\boldsymbol{W}}$. Now, let $\boldsymbol{v}_0^k$ be a sequence of divergence free functions in $\boldsymbol{W}^{3,2}(\Omega) \cap \boldsymbol{W}_0^{1,2}(\Omega)$ such that $\boldsymbol{v}_0^k$ converges strongly to $\boldsymbol{v}_0$ in $W^{2,2}(\Omega)$ as $k \to \infty$. By Theorem 4.24, for every fixed k the problem (4.308a,b,c) with initial condition $\boldsymbol{v}_0^k$ has a weak solution satisfying the regularity criteria of Definition 4.4. As $\boldsymbol{v}_0^k$ is bounded $\boldsymbol{W}^{2,2}(\Omega)$, independent of k, it follows that $\exists C > 0$, independent of k, such that for each i, $i = 1, 2, 3$,

$$||\boldsymbol{v}^k||_{L^\infty((0,T);\boldsymbol{W}^{2,2}(\Omega))} \leq C\left(||\boldsymbol{v}_0||^2_{\boldsymbol{W}^{2,2}(\Omega)} + ||f_i||^2_{L^2(Q_T)} + 1\right), \tag{4.319a}$$

$$||\boldsymbol{v}^k||_{L^2((0,T);\boldsymbol{W}^{3,2}(\Omega))} \leq C\left(||\boldsymbol{v}_0||^2_{\boldsymbol{W}^{2,2}(\Omega)} + ||f_i||^2_{L^2(Q_T)} + 1\right), \tag{4.319b}$$

$$||\boldsymbol{v}^k||_{L^4((0,T);\boldsymbol{W}^{2,4}(\Omega))} \leq C\left(||\boldsymbol{v}_0||^2_{\boldsymbol{W}^{2,2}(\Omega)} + ||f_i||^2_{L^2(Q_T)} + 1\right). \tag{4.319c}$$

Next, for $\boldsymbol{\phi} \in L^2((0,T); \boldsymbol{W}_0^{3,2}(\Omega))$, with $\nabla \cdot \boldsymbol{\phi} = 0$, it follows from the definition of a weak solution that for every positive integer k,

$$\begin{aligned}
&\int_\Omega \frac{\partial v_i^k}{\partial t}(\phi_i - \alpha_1 \phi_{i,jj} + \gamma \phi_{i,jkjk})\, d\boldsymbol{x} \\
&= -\mu_0 \int_\Omega (\boldsymbol{A}_1)_{ij} \phi_{i,j}\, d\boldsymbol{x} \\
&\quad - \alpha_1 \int_\Omega \left[\frac{\partial (\boldsymbol{A}_1^k)_{ij}}{\partial x_l} v_l^k + L_{mi}^k (\boldsymbol{A}^k)_{mj} + L_{mj}^k (\boldsymbol{A}_1^k)_{im}\right] \phi_{i,j}\, d\boldsymbol{x} - \alpha_2 \int_\Omega (\boldsymbol{A}_1^2)_{ij}^k \phi_{i,j}\, d\boldsymbol{x} \\
&\quad - \beta_3 \int_\Omega (\boldsymbol{A}_1^2)_{mm}^k (\boldsymbol{A}_1)_{ij}^k \phi_{i,j}\, d\boldsymbol{x} - \mu_1 \int_\Omega (\boldsymbol{A}_1)_{ij}^k \phi_{i,jl}\, d\boldsymbol{x}
\end{aligned}$$

$$- \int_\Omega v_j^k \frac{\partial v_i^k}{\partial x_j} \phi_i \, d\boldsymbol{x} - \mu_2 \int_\Omega (\boldsymbol{A}_1)_{ij,lm}^k \phi_{i,jlm} \, d\boldsymbol{x}$$
$$+ \int_\Omega f_i \phi_i \, d\boldsymbol{x} - \gamma \int_\Omega (W_{mi}^k D_{mj,l}^k + W_{im,l}^k + W_{ml}^k D_{ij,m}^k) \phi_{i,jk} \, d\boldsymbol{x} \tag{4.320}$$

so that, by virtue of (4.319a,b,c), we can conclude that for some $c > 0$, independent of k,

$$\left| \int_\Omega \frac{\partial v_i^k}{\partial t} (\phi_i - \alpha_1 \phi_{i,jj} + \gamma \phi_{i,jkjk}) \, d\boldsymbol{x} \right| \le c ||\boldsymbol{\phi}||_{\boldsymbol{W}_0^{3,2}(\Omega)}. \tag{4.321}$$

After an integration by parts we then obtain, for some $C > 0$, which is also independent of k,

$$\left|\left| \sum_i \frac{\partial}{\partial t} (v_i^k - \alpha_1 v_{i,jj}^k + \gamma v_{i,jkjk}^k) \right|\right|_{L^2((0,T);\boldsymbol{W}^{3,2}(\Omega))} \le C. \tag{4.322}$$

Also, by virtue of (4.319a,b,c),

$$\left|\left| \sum_i (v_i^k - \alpha_1 v_{i,jj}^k + \gamma v_{i,jkjk}^k) \right|\right|_{L^2((0,T);\boldsymbol{W}^{-1,2}(\Omega))}. \tag{4.323}$$

It now follows from Aubin's Lemma that the sequence $\sum_i (v_i^k - \alpha_1 v_{i,jj}^k + \gamma v_{i,jkjk}^k)$ is in a compact subset, with respect to the strong topology of $L^2((0,T);\boldsymbol{W}^{-s,2}(\Omega))$, for all $s > 1$. Thus, as in the proof of Theorem 4.24, we may, after choosing $v_i = v_i^k$ in (4.309), pass to the limit to conclude that $\boldsymbol{v} = \lim_{k\to\infty} \boldsymbol{v}^k$ is a weak solution. To establish uniqueness of the weak solution, assume that $\boldsymbol{v}^{(1)}$ and $\boldsymbol{v}^{(2)}$ are distinct weak solutions; we set $\boldsymbol{v} = \boldsymbol{v}^{(1)} - \boldsymbol{v}^{(2)}$ and use $\boldsymbol{v}$ as a test function. As the linear terms in the partial differential equation (4.308a) do not present any difficulty, we need only note the following estimates, which result from taking the difference of the equations satisfied by $\boldsymbol{v}^{(1)}$ and $\boldsymbol{v}^{(2)}$: for some $C > 0$,

$$\int_{\Omega_t} (D_{ij,k}^{(1)} - D_{ij,k}^{(2)})(D_{ij,k}^{(1)} - D_{ij,k}^{(2)}) ||\boldsymbol{v}^{(2)}||^2 \, d\boldsymbol{x} \le C ||\boldsymbol{v}^{(1)} - \boldsymbol{v}^{(2)}||_{\boldsymbol{W}^{2,2}(\Omega)}^2, \tag{4.324a}$$

$$\int_{\Omega_t} (D_{ij,k}^{(1)} - D_{ij,k}^{(2)})(D_{ij,k}^{(1)} - D_{ij,k}^{(2)}) \left| D\boldsymbol{v}^{(2)} \right|^2 d\boldsymbol{x} \le C \left|\left| \boldsymbol{v}^{(1)} - \boldsymbol{v}^{(2)} \right|\right|_{\boldsymbol{W}^{2,2}(\Omega)}^2 \left|\left| \boldsymbol{v}^{(2)} \right|\right|_{\boldsymbol{W}^{3,2}(\Omega)}^2, \tag{4.324b}$$

$$\int_{\Omega_t} \left| D\boldsymbol{v}^{(1)} - D\boldsymbol{v}^{(2)} \right|^2 \left| D_{ij,k}^{(2)}) \right|^2 d\boldsymbol{x} \le C ||\boldsymbol{v}^{(1)} - \boldsymbol{v}^{(2)}||_{\boldsymbol{W}^{2,2}(\Omega)}^2 ||\boldsymbol{v}^{(2)}||_{\boldsymbol{W}^{3,2}(\Omega)}^2. \tag{4.324c}$$

The estimates (4.324a,b,c) now yield a Gronwall inequality (see Appendix A) for the quantity

$$\chi(t) = \frac{1}{2}\int_{\Omega_t} \left|\left|\boldsymbol{v}^{(1)} - \boldsymbol{v}^{(2)}\right|\right|^2 + \alpha_1\left[e_{ij}(\boldsymbol{v}^{(1)} - \boldsymbol{v}^{(2)})e_{ij}(\boldsymbol{v}^{(1)} - \boldsymbol{v}^{(2)})\right]\gamma(D^{(1)}_{ij,k} - D^{(2)}_{ij,k})(D^{(1)}_{ij,k} - D^{(2)}_{ij,k})\, d\boldsymbol{x} \quad (4.325)$$

from which the uniqueness of the weak solution follows immediately. □

Remarks (Higher-Order Boundary Conditions for Multipolar Fluids of Grade 3*).* The conditions specified in Definition 4.4 allow, in principle, for a broad range of higher-order conditions to be satisfied by a weak solution of (4.308a,b,c); in point of fact, however, it is not difficult to see that these higher-order conditions must assume the form (1.201a,b) where $\boldsymbol{\tau}^l$, $l = 1, 2$ are linearly independent tangent vectors to $\partial\Omega$ at $\boldsymbol{x} \in \partial\Omega$, $\boldsymbol{\nu}$ is the exterior unit normal to $\partial\Omega$ at $\boldsymbol{x} \in \partial\Omega$, and $B_{ijkm}(\boldsymbol{v})$, $S_{ijk}(\boldsymbol{v})$ are defined by (1.199a,b,c). To begin with, it follows from the definition of a weak solution that for every test function $\boldsymbol{\phi}(\boldsymbol{x}, t)$ defined on $\Omega \times [0, T)$, and vanishing on $\partial\Omega \times [0, T)$,

$$\int_{\partial\Omega} (S_{ijk}(\boldsymbol{v})\phi_{i,k} + B_{ikmj}(\boldsymbol{v})\phi_{i,km})\nu_j\, dA = 0, \quad t \in [0, T) \quad (4.326)$$

where dA denotes the surface measure on $\partial\Omega$. The demonstration that (4.326), for sufficiently smooth fields S_{ijk}, B_{ikmj}, implies (1.201a,b) depends on the following surface divergence theorem which asserts that, for any smooth vector field $\boldsymbol{w}$ defined on $\partial\Omega$,

$$\int_{\partial\Omega} w_{i;i}\, dA = \int_{\Omega} 2\chi w_i \nu_i\, dA \quad (4.327)$$

where ; followed by an index denotes the surface gradient (see 1.6) and χ is the mean curvature of $\partial\Omega$, $2\chi = -\nu_{i;i}$. The result (4.327) can be obtained by applying Stokes formula to the vector field $\epsilon_{ijk} w_j \nu_k$. We now note that $\phi_{i,j} = \frac{\partial\phi_i}{\partial\nu}\nu_j$, as ϕ_i vanishes on $\partial\Omega$, and further denote by $\frac{\partial^2\phi_i}{\partial\nu^2} = \frac{\partial^2\phi_i}{\partial\nu_k\partial\nu_l}\nu_k\nu_l$ the second normal derivative of ϕ_i. The second term in the integrand in (4.326) may be transformed as follows:

$$\begin{aligned} B_{ijkm}\nu_m\phi_{i,jk} &= B_{ijkm}\nu_m\delta_{kp}\phi_{i,jp} \\ &= B_{ijkm}\nu_m(\delta_{kp} - \nu_k\nu_p)\phi_{i,jp} + B_{ijkm}\nu_m\nu_k\nu_p\phi_{i,jp} \\ &= B_{ijkm}\nu_m\phi_{i,j;k} + B_{ijkm}\nu_m\nu_k\nu_p\delta_{jp}\phi_{i,qp} \end{aligned}$$

$$
\begin{aligned}
&= B_{ijkm}\nu_m\phi_{i,j;k} + B_{ijkm}\nu_m\nu_k\nu_p(\delta_{jq} - \nu_j\nu_q)\phi_{i,qp} \\
&\quad + B_{ijkm}\nu_m\nu_k\nu_p\nu_j\nu_q\phi_{i,qp} \\
&= B_{ijkm}\nu_m\phi_{i,j;k} + B_{ijkm}\nu_m\nu_k\nu_p\phi_{i,p;j} + B_{ijkm}\nu_j\nu_k\nu_m\frac{\partial^2\phi_i}{\partial\nu^2} \\
&= (B_{ijkm}\nu_m\phi_{i,j})_{;k} - (B_{ijkm}\nu_m)_{;k}\phi_{i,j} + (B_{ijkm}\nu_m\nu_k\nu_p\phi_{i,p})_{;j} \\
&\quad - (B_{ijkm}\nu_m\nu_k\nu_p)_{;j}\phi_{i,p} + B_{ijkm}\nu_j\nu_k\nu_m\frac{\partial^2\phi_i}{\partial\nu^2}\, dA.
\end{aligned}
$$

Therefore, if we apply (4.327) twice in succession, we obtain

$$
\begin{aligned}
&\int_{\partial\Omega} B_{ijkm}\phi_{i,jk}\nu_m\, dA \qquad (4.328)\\
&= \int_{\partial\Omega}(B_{ijkm}\nu_m\phi_{i,j})_{;k}\, dA - \int_{\partial\Omega}(B_{ijkm}\nu_m)_{;k}\phi_{i,j}\, dA \\
&\quad + \int_{\partial\Omega}(B_{ijkm}\nu_m\nu_k\nu_p\phi_{i,p})_{;j} - \int_{\partial\Omega}(B_{ijkm}\nu_m\nu_k\nu_p)_{;j}\phi_{i,p}\, dA \\
&\quad + \int_{\partial\Omega} B_{ijkm}\nu_j\nu_k\nu_m\frac{\partial^2\phi_i}{\partial\nu^2}\, dA \\
&= \int_{\partial\Omega} 2\chi B_{ijkm}\nu_m\nu_k\phi_{i,j}\, dA - \int_{\partial\Omega}(B_{ijkm}\nu_m)_{;k}\phi_{i,j}\, dA \\
&\quad + \int_{\partial\Omega} 2\chi B_{ijkm}\nu_m\nu_k\nu_p\phi_{i,p}\nu_j\, dA - \int_{\partial\Omega}(B_{ijkm}\nu_m\nu_k\nu_p)_{;j}\phi_{i,p}\, dA \\
&\quad + \int_{\partial\Omega} B_{ijkm}\nu_j\nu_k\nu_m\frac{\partial^2\phi_i}{\partial\nu^2}\, dA.
\end{aligned}
$$

Furthermore, we have $\nu_p\nu_p = 1$; therefore, $\nu_{p;k}\nu_p = 0$ and, thus, one can rewrite the term $(B_{ijkm}\nu_m\nu_k\nu_p)_{;}\phi_{i,p}$ on the right-hand side of (4.328) as

$$
\begin{aligned}
(B_{ijkm}\nu_m\nu_k\nu_p)_{;j}\phi_{i,p} &= (B_{ijkm}\nu_m\nu_k\nu_p)_{;j}\nu_p\frac{\partial\phi_i}{\partial\boldsymbol{\nu}} \\
&= [(B_{ijkm}\nu_m\nu_k)_{;j}\nu_p + B_{ijkm}\nu_m\nu_k\nu_{p;j}]\nu_p\frac{\partial\phi_i}{\partial\boldsymbol{\nu}} \qquad (4.329)\\
&= (B_{ijkm}\nu_m\nu_k)_{;j}\frac{\partial\phi_i}{\partial\boldsymbol{\nu}}
\end{aligned}
$$

because $\dfrac{\partial\phi_i}{\partial\boldsymbol{\nu}} = \dfrac{\partial\phi_i}{\partial x_p}\nu_p$. By using (4.329) again in conjunction with (4.328) one finally obtains

$$\int_{\partial\Omega} B_{ijkm}\phi_{i;jk}\nu_m \, dA = \int_{\partial\Omega} [2\chi B_{ijkm}\nu_m\nu_k\nu_j - (B_{ijkm}\nu_m)_{;k}\nu_j + 2\chi B_{ijkm}\nu_k\nu_k\nu_j - (B_{ijkm}\nu_m\nu_k)_{;j}] \frac{\partial\phi_i}{\partial\boldsymbol{\nu}} \, dA + \int_{\partial\Omega} B_{ijkm}\nu_j\nu_k\nu_m \frac{\partial^2\phi_i}{\partial\nu^2} \, dA. \quad (4.330)$$

The boundary condition (4.326) now can be written in the form

$$\int_{\partial\Omega} [S_{ijk}\nu_j\nu_k + 2\chi B_{ijkm}\nu_m\nu_k\nu_j - (B_{ijkm}\nu_m)_{;k}\nu_j + 2\chi B_{ijkm}\nu_m\nu_k\nu_j - (B_{ijkm}\nu_m\nu_k)_{;j}] \frac{\partial\phi_i}{\partial\boldsymbol{\nu}} \, dA + \int_{\partial\Omega} B_{ijkm}\nu_j\nu_m \frac{\partial^2\phi_i}{\partial\boldsymbol{\nu}^2} \, dA = 0. \quad (4.331)$$

As ϕ_i is a divergence free vector field, the functions $\partial\phi_i/\partial\boldsymbol{\nu}$, $\partial^2\phi_i/\partial\boldsymbol{\nu}^2$ are not completely arbitrary and we cannot conclude that each of the integrands in (4.331) is zero; however, for a divergence free vector field ϕ_i, which vanishes on the boundary of Ω, the most general forms of the normal derivatives on the boundary are given by:

$$\frac{\partial\phi_i}{\partial\boldsymbol{\nu}} = g_i, \text{ on } \partial\Omega, \quad (4.332a)$$

$$\frac{\partial^2\phi_i}{\partial\boldsymbol{\nu}^2} = h_i - \nu_i \operatorname{div}_s(\boldsymbol{g}), \text{ on } \partial\Omega \quad (4.332b)$$

where $\boldsymbol{h}$ and $\boldsymbol{g}$ are arbitrary vectors tangent to the surface $\partial\Omega$ and div_s is the surface divergence operator defined by $\operatorname{div}_s(\boldsymbol{v}) := v_{i;i}$. The last term in (4.331) now becomes

$$\int_{\partial\Omega} B_{ijkm}\nu_j\nu_k\nu_m \frac{\partial^2\phi_i}{\partial\boldsymbol{\nu}^2} \, dA = \int_{\partial\Omega} B_{ijkm}\nu_j\nu_k\nu_m h_i \, dA - \int_{\partial\Omega} B_{ijkm}\nu_j\nu_k\nu_m \operatorname{div}_s(g_i)\nu_i \, dA. \quad (4.333)$$

Since the vector $\boldsymbol{g}$ is tangential to $\partial\Omega$ we have, by integration by parts,

$$-\int_{\partial\Omega} B_{ijkm}\nu_j\nu_k\nu_m\nu_i \operatorname{div}_s(\boldsymbol{g}) \, dA = \int_{\partial\Omega} g_l (B_{ijkm}\nu_j\nu_k\nu_m\nu_i)_{;l} \, dA. \quad (4.334)$$

Equation (4.321) then yields

$$\int_{\partial\Omega} [S_{ijk}\nu_j\nu_k + 4\chi B_{ijkm}\nu_m\nu_k\nu_j - (B_{ijkm}\nu_m)_{;k}\nu_j - (B_{ijkm}\nu_m\nu_k)_{;j} + (B_{ijkm}\nu_l\nu_j\nu_k)_{;i}] g_i \, dA + \int_{\partial\Omega} B_{ijkm}\nu_j\nu_k\nu_m h_i \, dA = 0. \quad (4.335)$$

Since $\boldsymbol{h}$ and $\boldsymbol{g}$ are arbitrary tangent vector fields, it follows that if $\boldsymbol{\tau}^1$ and $\boldsymbol{\tau}^2$ are linearly independent tangent to $\partial\Omega$, then

$$B_{ijkm}\nu_j\nu_k\nu_m\tau_i^l = 0, \quad l = 1, 2$$

and

$$[S_{ijk}\nu_j\nu_k + 4\chi B_{ijkm}\nu_m\nu_k\nu_j - (B_{ijkm}\nu_m)_{;k}\nu_j - (B_{ijkm}\nu_m\nu_k)_{;j} + (B_{ijkm}\nu_l\nu_j\nu_m\nu_k)_{;i}]\tau_i^p = 0, \quad p = 1, 2$$

which are, of course, (1.201a,b), respectively.

Remarks (An Energy Inequality and Stability of the Rest State). If we integrate the inequality $\mu(\boldsymbol{A}_1)_{ij,k}(\boldsymbol{A}_1)_{ij,k} \geq 0$ over Ω and use the divergence theorem in conjunction with (4.325), after setting $\phi_i \equiv v_i$, we obtain the identity

$$\frac{d}{dt}\int_\Omega \rho\left(\frac{1}{2}v_i v_i + \psi\right) dx + \frac{1}{2}\int_\Omega [\mu_0(\boldsymbol{A}_1^2)_{mm} + (\alpha_1 + \alpha_2)(\boldsymbol{A}_1^3)_{mm} + \beta_3(\boldsymbol{A}_1^2)_{mm}(\boldsymbol{A}_1^2)_{mm} + \mu_1(\boldsymbol{A}_1)_{ij,k}(\boldsymbol{A}_1)_{ij,k} + \mu_2(\boldsymbol{A}_1)_{ij,km}(\boldsymbol{A}_1)_{ij,km}]\, d\boldsymbol{x} = 0 \tag{4.336}$$

where ψ is the free energy function as given by (1.200). The quantity

$$E(t) = \int_\Omega \rho\left(\frac{1}{2}v_i v_i + \psi\right) d\boldsymbol{x} = \int_\Omega \left(\frac{1}{2}\rho v_i v_i + \alpha_1 D_{ij}D_{ij} + \frac{1}{2}\gamma D_{ij,k}D_{ij,k}\right) d\boldsymbol{x} \tag{4.337}$$

is the total energy of the fluid at time t. Since the second integral is non-negative as a consequence of the second law of thermodynamics (see [BNR] for the details) it follows that $\dot{E}(t) \leq 0$ for every process in the fluid. We now assume that the rest state $v_i \equiv 0$ is stable in the sense that every perturbation of the rest state is eventually damped out by the dissipative mechanisms of the fluid. It is natural to assume that during a process the energy of the fluid tends to the zero energy of the rest state, i.e., $E(t) \to 0$ as $t \to \infty$; from this result, coupled with $\dot{E}(t) \leq 0$, we obtain $E(t) \geq 0$ and, as this must hold for every initial perturbation of the rest state, one finds that a necessary condition for (formal) stability is

$$E[\boldsymbol{v}(\cdot)] = \int_\Omega \left(\frac{1}{2}\rho v_i v_i + \alpha_1 D_{ij}D_{ij} + \frac{1}{2}\gamma D_{ij,k}D_{ij,k}\right) d\boldsymbol{x} \geq 0 \tag{4.338}$$

for every velocity field v_i with $v_i = 0$ on $\partial\Omega$ and $v_{i,i} = 0$. Clearly a necessary condition for (4.338) to hold for every velocity field is that $\gamma \geq 0$. The following theorem gives a sufficient condition for the validity of (4.338):

Theorem 4.26. *Let $\gamma > 0$ and suppose that either (i) $\alpha_1 \geq 0$ or (ii) $\alpha_1 \leq 0$ and $\gamma > \frac{3}{\rho}\alpha_1^2$. Then $\exists c > 0$ such that $\forall \boldsymbol{v} \in \boldsymbol{W}_0^{1,2}(\Omega) \cap \boldsymbol{W}^{2,2}(\Omega)$, with $v_{i,i} = 0$ in Ω, we have*

$$E(\boldsymbol{v}(\cdot)) \geq c\,||\boldsymbol{v}||_{\boldsymbol{W}^{2,2}(\Omega)}\,. \tag{4.339}$$

Proof. If $\alpha_1 \geq 0$ the proof is immediate from (4.338). In order to establish the theorem under the conditions in part (ii) we need to establish a pair of lemmas, the first of which is

Lemma 4.19. *For every $\phi \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$ one has*

$$||\text{grad}\,\phi||^2_{L^2(\Omega)} \leq ||\phi||_{L^2(\Omega)}||\Delta\phi||_{L^2(\Omega)}. \tag{4.340}$$

Proof.

$$\begin{aligned}
||\text{grad}\,\phi||^2_{L^2(\Omega)} &= \int_\Omega \frac{\partial\phi}{\partial x_i}\frac{\partial\phi}{\partial x_i}\,d\boldsymbol{x} \\
&= \int_\Omega \left[\frac{\partial}{\partial x_i}\left(\frac{\partial\phi}{\partial x_i}\phi\right) - \phi\Delta\phi\right] d\boldsymbol{x} \\
&= \int_{\partial\Omega} \frac{\partial\phi}{\partial x_i}\phi\nu_i\,dA - \int_\Omega \phi\Delta\phi\,d\boldsymbol{x} \\
&\leq ||\phi||_{L^2(\Omega)}||\Delta\phi||_{L^2(\Omega)}.
\end{aligned}$$

□

Remarks. It is worth noting that the constant $c = 1$ in the inequality

$$||\text{grad}\,\phi||^2_{L^2(\Omega)} \leq c||\phi||_{L^2(\Omega)}||\Delta\phi||_{L^2(\Omega)}$$

is optimum. Indeed for eigenfunctions of the Laplacian (4.340) becomes an equality.

The second result which is needed in order to complete the proof of Theorem 4.26 is

Lemma 4.20. *For every vector-valued function $\boldsymbol{v} \in \boldsymbol{W}_0^{1,2}(\Omega) \cap \boldsymbol{W}^{2,2}(\Omega)$, with $v_{i,i} = 0$ in Ω, one has*

$$\int_\Omega D_{ij}D_{ij}\,dx \leq \sqrt{3}||\boldsymbol{v}||_{\boldsymbol{L}^2(\Omega)}\int_\Omega D_{ij,k}D_{ij,k}\,d\boldsymbol{x}. \tag{4.341}$$

Proof. The proof follows from the elementary results

$$(\Delta v_i)^2 \le 12(D_{i1,1}^2 + \cdots + D_{i3,3}^2), \tag{4.342a}$$

$$D_{ij}D_{ij} = \frac{1}{2}|\nabla \boldsymbol{v}|^2 + \frac{1}{2}v_{i,j}v_{j,i}, \tag{4.342b}$$

the boundary conditions satisfied by $\boldsymbol{v}$, and the incompressibility constraint; details may be found in [BNR]. □

Proof (Continuation of Theorem 4.26). By (4.342a), as $\alpha_1 \le 0$,

$$E[\boldsymbol{v}(\cdot)] \ge \frac{1}{2}\rho\,||\boldsymbol{v}||_{\boldsymbol{L}^2}^2 + \sqrt{3}\alpha_1\,||\boldsymbol{v}||_{\boldsymbol{L}^2}^2\,|\text{grad}\,\boldsymbol{D}|_{\boldsymbol{L}^2} + \frac{1}{2}\gamma|\text{grad}\,\boldsymbol{D}|_{\boldsymbol{L}^2} \tag{4.343}$$

where $|\text{grad}\,\boldsymbol{D}|^2 = D_{ij,k}D_{ij,k}$. The right-hand side of (4.343) is a quadratic form in $||\boldsymbol{v}||_{\boldsymbol{L}^2}$ and $|\text{grad}\,\boldsymbol{D}|_{\boldsymbol{L}^2}$ and its discriminant is negative if and only if $\alpha_1 \le 0$ and $\gamma > \dfrac{3}{\rho}\alpha_1^2$; these conditions imply, therefore, that the right-hand side of (4.343) is positive definite and, hence, for some $c > 0$,

$$E[\boldsymbol{v}(\cdot)] \ge c|\text{grad}\,\boldsymbol{D}|_{\boldsymbol{L}^2}^2. \tag{4.344}$$

On the other hand, for some other constant $\hat{c} > 0$, one has that

$$|\text{grad}\,\boldsymbol{D}|_{\boldsymbol{L}^2} \ge \hat{c}\left|\text{grad}^2\,\boldsymbol{v}\right|_{\boldsymbol{L}^2} \tag{4.345}$$

and (4.339) now follows by combining (4.344) and (4.345) with the Poincaré inequality (see Appendix A). □

We conclude this subsection by noting that it has also been shown in [BNR] that for $\mu_1 \ge 0$, $\mu_2 \ge 0$, with $\mu_1 + \mu_2 > 0$, and either $\alpha_1 \ge 0$, or $\alpha_1 \le 0$ and $\gamma > 3\alpha_1^2/\rho$, $\exists c > 0$, $c_1 > 0$ such that weak solutions $\boldsymbol{v}$ which satisfy the boundary conditions on $\partial\Omega \times [0, \infty)$ also satisfy

$$||\boldsymbol{v}(\cdot, t)||_{W^{2,2}(\Omega)} \le c_1||\boldsymbol{v}(\cdot, 0)||_{W^{2,2}(\Omega)}e^{-ct} \tag{4.346}$$

for every $t \ge 0$; this result establishes, of course, the asymptotic stability of the rest state for the multipolar fluid of grade 3.

4.5.6 Global Regularity of Solutions to the Viscous Camassa-Holm Equations

The Navier–Stokes alpha (NS-α) model of incompressible fluid flow, known also as the viscous Camassa-Holm equations (VCHE), was introduced in Sect. 1.6; some applications of the model were then presented in Sect. 1.7.3. In this final subsection of Sect. 4.5 we will briefly summarize some of the results obtained in [FHT2] concerning the global, in time, regularity of solutions of VCHE in three space dimensions; we also discuss the relation of these equations to the Navier–Stokes equations by delineating the result in [FHT2] which says, in essence, that as a certain length scale in the NS-α model tends to zero, a subsequence of solutions of the NS-α equations converges to a weak solution of the three dimensional Navier–Stokes equations. In this subsection we assume that Ω is the periodic box, $\Omega = [0, L]^3$, $L > 0$; the associated problem for the VCHE (1.217) with an external body force $\boldsymbol{f}$ and constant density ρ_0 can then be written in the form

$$\frac{\partial}{\partial t}(\alpha_0^2 \boldsymbol{v} - \alpha_1^2 \boldsymbol{v}) - \nu\Delta(\alpha_0^2 \boldsymbol{v} - \alpha_1^2 \boldsymbol{v}) - \boldsymbol{v} \times (\nabla \times (\alpha_0^2 \boldsymbol{v} - \alpha_1^2 \Delta \boldsymbol{v})) + \frac{1}{\rho_0}\nabla p = \boldsymbol{f}, \tag{4.347a}$$

$$\nabla \cdot \boldsymbol{v} = 0, \tag{4.347b}$$

$$\boldsymbol{v}(\boldsymbol{x}, 0) = \boldsymbol{v}_0(\boldsymbol{x}) \tag{4.347c}$$

where isotropy and homogeneity of the velocity fluctions $\boldsymbol{\sigma}$ has been assumed (see Sect. 1.6). Also, by (1.212b), with the aforementioned isotropy hypothesis, and the inclusion of the constant density ρ_0, the modified pressure p and the usual pressure π are related by

$$\frac{p}{\rho_0} = \frac{\pi}{\rho_0} + \alpha_0^2 \,||\boldsymbol{v}||^2 - \alpha_1^2(\boldsymbol{v} \cdot \Delta \boldsymbol{v}). \tag{4.348}$$

We recall, from the discussion in Sect. 1.6, that in (4.347a) $\nu > 0$ is the constant viscosity while $\alpha_0 > 0$ and $\alpha_1 \geq 0$ are scale parameters (in (1.217), $\alpha_0 = 1$ and $\alpha_1 = \nu\alpha^2$). In the limiting case where $\alpha_0 = 1$ and $\alpha_1 = 0$, one obtains the three-dimensional Navier–Stokes equations with periodic boundary conditions. In [FHT2] it is assumed that $\boldsymbol{f}(\boldsymbol{x}, t) \equiv \boldsymbol{f}(\boldsymbol{x})$. As a consequence of (4.347a,b), and integration by parts,

$$\frac{d}{dt}\int_\Omega (\alpha_0^2 \boldsymbol{v} - \alpha_1^2 \Delta \boldsymbol{v})\, d\boldsymbol{x} = \int_\Omega \boldsymbol{f}\, d\boldsymbol{x} \tag{4.349}$$

while, because of the spatial periodicity of the solution of (4.347a,b,c), $\int_\Omega \Delta \boldsymbol{v}\, d\boldsymbol{x} = \boldsymbol{0}$. Therefore, $\frac{d}{dt}\int_\Omega \boldsymbol{v}\, d\boldsymbol{x} = \int_\Omega \boldsymbol{f}\, d\boldsymbol{x}$ so that the mean of the solution is invariant if the mean of the forcing term is zero. In [FHT2] forcing terms and initial conditions are considered which satisfy $\int_\Omega \boldsymbol{v}_0(\boldsymbol{x})\, d\boldsymbol{x} = \int_\Omega \boldsymbol{f}(\boldsymbol{x})\, d\boldsymbol{x} = \boldsymbol{0}$ so that $\int_\Omega \boldsymbol{v}(\boldsymbol{x}, t)\, d\boldsymbol{x} = \boldsymbol{0}$.

In order to delineate the results obtained in [FHT2] we need to introduce the following notation:

(i) $\boldsymbol{X}$ is the linear subspace of integrable functions on Ω with $\dot{\boldsymbol{X}} = \{\boldsymbol{\varphi} \in \boldsymbol{X} \mid \int_\Omega \boldsymbol{\varphi}(\boldsymbol{x})\, d\boldsymbol{x} = \boldsymbol{0}\}$.

(ii) $\boldsymbol{\mathcal{V}} = \{\boldsymbol{\varphi} \mid \boldsymbol{\varphi}$ is a (vector-valued) trigonometric polynomial defined on Ω, $\nabla \cdot \boldsymbol{\varphi} = 0$, and $\int_\Omega \boldsymbol{\varphi}(\boldsymbol{x})\, d\boldsymbol{x} = \boldsymbol{0}\}$.

(iii) $\boldsymbol{H}$ and $\boldsymbol{V}$ are, respectively, the closures of $\boldsymbol{\mathcal{V}}$ in $\boldsymbol{L}^2(\Omega)$ and $\boldsymbol{H}^1(\Omega)$.

(iv) $\boldsymbol{P}_\sigma : \boldsymbol{L}^2(\Omega) \to \boldsymbol{H}$ is the Leray projector while $\boldsymbol{A} = -\boldsymbol{P}_\sigma \boldsymbol{\Delta}$ is the Stokes operator with $D(\boldsymbol{A}) = \boldsymbol{H}^2(\Omega) \cap \boldsymbol{V}$. (In the case of periodic boundary conditions, $\boldsymbol{A} = -\boldsymbol{\Delta}|_{D(\boldsymbol{A})}$ is a positive self-adjoint operator, with compact inverse $\boldsymbol{A}^{-1}$, so that–in the usual manner–$\boldsymbol{H}$ has an orthonormal basis $\{\boldsymbol{w}_j\}_{j=1}^\infty$ of eigenfunctions of $\boldsymbol{A}$, $\boldsymbol{A}\boldsymbol{w}_j = \lambda_j \boldsymbol{w}_j$, with $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_j \to \infty$; for the nonlinear bipolar fluid, analogous considerations will be central to the analysis in Chap. 6.)

With the same notation as commonly applied in studies of the Navier–Stokes equations (e.g., [Te1]), and which we will employ again in Chaps. 5 and 6, we set

$$\boldsymbol{B}(\boldsymbol{u},\boldsymbol{v}) = \boldsymbol{P}_\sigma(\boldsymbol{u} \cdot \nabla \boldsymbol{v}) \text{ and } \boldsymbol{B}(\boldsymbol{v})\boldsymbol{u} = \boldsymbol{B}(\boldsymbol{u},\boldsymbol{v}), \quad \forall \boldsymbol{u},\boldsymbol{v} \in \boldsymbol{V}, \tag{4.350a}$$

$$\tilde{\boldsymbol{B}}(\boldsymbol{u},\boldsymbol{v}) = -\boldsymbol{P}_\sigma(\boldsymbol{u} \times (\nabla \times \boldsymbol{v})), \quad \forall \boldsymbol{u},\boldsymbol{v} \in \boldsymbol{V}, \tag{4.350b}$$

from which it follows that

$$(\boldsymbol{B}(\boldsymbol{u},\boldsymbol{v}),\boldsymbol{w})_{\boldsymbol{L}^2(\Omega)} = -(\boldsymbol{B}(\boldsymbol{u},\boldsymbol{w}),\boldsymbol{v})_{\boldsymbol{L}^2(\Omega)}, \quad \forall \boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in \boldsymbol{V}, \tag{4.351a}$$

$$\left(\tilde{\boldsymbol{B}}(\boldsymbol{u},\boldsymbol{v}),\boldsymbol{w}\right)_{\boldsymbol{L}^2(\Omega)} = (\boldsymbol{B}(\boldsymbol{u},\boldsymbol{v}),\boldsymbol{w})_{\boldsymbol{L}^2(\Omega)} - (\boldsymbol{B}(\boldsymbol{w},\boldsymbol{v}),\boldsymbol{u})_{\boldsymbol{L}^2(\Omega)}, \quad \forall \boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in \boldsymbol{V}, \tag{4.351b}$$

and

$$\tilde{\boldsymbol{B}}(\boldsymbol{u},\boldsymbol{v}) = (\boldsymbol{B}(\boldsymbol{v}) - \boldsymbol{B}^*(\boldsymbol{v}))\boldsymbol{u}, \quad \forall \boldsymbol{u},\boldsymbol{v} \in \boldsymbol{V} \tag{4.351c}$$

where $\boldsymbol{B}^*$ denotes the adjoint operator.

It is a direct consequence of the Poincaré inequality (see Appendix A) that the Stokes operator $\boldsymbol{A} = -\boldsymbol{P}_\sigma \boldsymbol{\Delta}$ satisfies the following estimates: for some $c > 0$,

$$c||\boldsymbol{A}\boldsymbol{w}||_{\boldsymbol{L}^2(\Omega)} \le ||\boldsymbol{w}||_{\boldsymbol{H}^2(\Omega)} \le c^{-1}||\boldsymbol{A}\boldsymbol{w}||_{\boldsymbol{L}^2(\Omega)}, \quad \forall \boldsymbol{w} \in D(\boldsymbol{A}) \tag{4.352a}$$

and

$$c\left|\left|\boldsymbol{A}^{1/2}\boldsymbol{w}\right|\right|_{\boldsymbol{L}^2(\Omega)} \le ||\boldsymbol{w}||_{\boldsymbol{H}^1(\Omega)} \le c^{-1}\left|\left|\boldsymbol{A}^{1/2}\boldsymbol{w}\right|\right|_{\boldsymbol{L}^2(\Omega)}, \quad \forall \boldsymbol{w} \in \boldsymbol{V} \tag{4.352b}$$

where we have used the fact that $\boldsymbol{V} = D(\boldsymbol{A}^{1/2})$, e.g., see [Te1]. The inner product and norm on $\boldsymbol{V}$ are then given, respectively, by

$$(\boldsymbol{u}, \boldsymbol{v})_V = \left(\boldsymbol{A}^{1/2}\boldsymbol{u}, \boldsymbol{A}^{1/2}\boldsymbol{v}\right)_{\boldsymbol{L}^2(\Omega)} \text{ and } ||\boldsymbol{u}||_V = \left|\left|\boldsymbol{A}^{1/2}\boldsymbol{u}\right|\right|_{\boldsymbol{L}^2(\Omega)}, \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V} \tag{4.353}$$

and it is easily shown that the inner product $(\cdot, \cdot)_V$ on $\boldsymbol{V}$ is equivalent to the $\boldsymbol{H}^1$ inner product

$$[\boldsymbol{u}, \boldsymbol{v}]_{\boldsymbol{H}^1} = \alpha_0^2(\boldsymbol{u}, \boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} + \alpha_1^2(\boldsymbol{u}, \boldsymbol{v})_V, \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V} \tag{4.354}$$

if $\alpha_1 > 0$. The technical machinery needed for the proof of the global existence and uniqueness theorem in [FHT2] is embodied in the following lemma:

Lemma 4.21. *(i) The operator $\boldsymbol{A}$ can be extended continuously so as to be defined on $\boldsymbol{V} = D(\boldsymbol{A}^{1/2})$, with values in $\boldsymbol{V}' \subset \boldsymbol{H}^{-1}$, such that*

$$\langle \boldsymbol{A}\boldsymbol{u}, \boldsymbol{v}\rangle_{V'} = \left(\boldsymbol{A}^{1/2}\boldsymbol{u}, \boldsymbol{A}^{1/2}\boldsymbol{v}\right)_{\boldsymbol{L}^2(\Omega)} = \int_\Omega (\nabla \boldsymbol{u} : \nabla \boldsymbol{v})\, d\boldsymbol{x}$$

for every $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}$.

(ii) Similarly, the operator $\boldsymbol{A}^2$ can be extended continuously so as to be defined on $D(\boldsymbol{A})$ with values in $D(\boldsymbol{A})'$ such that

$$\left\langle \boldsymbol{A}^2\boldsymbol{u}, \boldsymbol{v}\right\rangle_{D(\boldsymbol{A})'} = (\boldsymbol{A}\boldsymbol{u}, \boldsymbol{A}\boldsymbol{v})_{\boldsymbol{L}^2(\Omega)}, \text{ for every } \boldsymbol{u}, \boldsymbol{v} \in D(\boldsymbol{A}).$$

(iii) The operator $\tilde{\boldsymbol{B}}$ can be extended continuously from $\boldsymbol{V} \times \boldsymbol{V}$ with values in $\boldsymbol{V}'$ and satisfies, for some $c > 0$,

$$\left|\left\langle \tilde{\boldsymbol{B}}(\boldsymbol{u}, \boldsymbol{v}), \boldsymbol{w}\right\rangle_{V'}\right| \le c\, ||\boldsymbol{u}||^{1/2}_{\boldsymbol{L}^2(\Omega)}\, ||\boldsymbol{u}||^{1/2}_V\, ||\boldsymbol{v}||_V ||\boldsymbol{w}||_V,$$
$$\left|\left\langle \tilde{\boldsymbol{B}}(\boldsymbol{u}, \boldsymbol{v}), \boldsymbol{w}\right\rangle_{V'}\right| \le c||\boldsymbol{u}||_V ||\boldsymbol{v}||_V\, ||\boldsymbol{w}||^{1/2}_{\boldsymbol{L}^2(\Omega)}\, ||\boldsymbol{w}||^{1/2}_V$$

for all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{V}$. Also,

$$\left\langle \tilde{\boldsymbol{B}}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\right\rangle_{V'} = -\left\langle \tilde{\boldsymbol{B}}(\boldsymbol{w}, \boldsymbol{v}), \boldsymbol{u}\right\rangle_{V'}, \quad \forall \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{V}$$

so that

$$\left\langle \tilde{\boldsymbol{B}}(\boldsymbol{u}, \boldsymbol{v}), \boldsymbol{u}\right\rangle_{V'} \equiv 0, \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}.$$

(iv) For every $\boldsymbol{u} \in \boldsymbol{H}$, $\boldsymbol{v} \in \boldsymbol{V}$, *and* $\boldsymbol{w} \in D(\boldsymbol{A})$,

$$\left|\left\langle \tilde{\boldsymbol{B}}(\boldsymbol{u}, \boldsymbol{v}), \boldsymbol{w}\right\rangle_{D(\boldsymbol{A})'}\right| \leq c||\boldsymbol{u}||_{\boldsymbol{L}^2(\Omega)}||\boldsymbol{v}||_{\boldsymbol{V}}\, ||\boldsymbol{w}||_{\boldsymbol{V}}^{1/2}\, ||\boldsymbol{A}\boldsymbol{w}||_{\boldsymbol{L}^2(\Omega)}^{1/2}$$

and, by symmetry,

$$\left|\left(\tilde{\boldsymbol{B}}(\boldsymbol{u}, \boldsymbol{v}), \boldsymbol{w}\right|\right)_{\boldsymbol{L}^2(\Omega)} \leq c\, ||\boldsymbol{u}||_{\boldsymbol{V}}^{1/2}\, ||\boldsymbol{A}\boldsymbol{u}||_{\boldsymbol{L}^2(\Omega)}^{1/2}||\boldsymbol{v}||_{\boldsymbol{V}}||\boldsymbol{w}||_{\boldsymbol{L}^2(\Omega)}$$

for every $\boldsymbol{u} \in D(\boldsymbol{A})$, $\boldsymbol{v} \in \boldsymbol{V}$, *and* $\boldsymbol{w} \in \boldsymbol{H}$.

(v) For every $\boldsymbol{u} \in \boldsymbol{V}$, $\boldsymbol{v} \in \boldsymbol{H}$, *and* $\boldsymbol{w} \in D(\boldsymbol{A})$, *we have for some* $c > 0$,

$$\left|\left\langle \tilde{\boldsymbol{B}}(\boldsymbol{u}, \boldsymbol{v}), \boldsymbol{w}\right\rangle_{D(\boldsymbol{A})'}\right| \leq c\left(||\boldsymbol{u}||_{\boldsymbol{L}^2(\Omega)}^{1/2}\, ||\boldsymbol{u}||_{\boldsymbol{V}}^{1/2}\, ||\boldsymbol{v}||_{\boldsymbol{L}^2(\Omega)}||\boldsymbol{A}\boldsymbol{w}||_{\boldsymbol{L}^2(\Omega)} + ||\boldsymbol{v}||_{\boldsymbol{L}^2(\Omega)}||\boldsymbol{u}||_{\boldsymbol{V}}\, ||\boldsymbol{w}||_{\boldsymbol{V}}^{1/2}\, ||\boldsymbol{A}\boldsymbol{w}||_{\boldsymbol{L}^2(\Omega)}^{1/2}\right)$$

and, additionally, for every $\boldsymbol{u} \in D(\boldsymbol{A})$, $\boldsymbol{v} \in \boldsymbol{H}$, *and* $\boldsymbol{w} \in \boldsymbol{V}$, *and some* $c > 0$,

$$\left|\left\langle \tilde{\boldsymbol{B}}(\boldsymbol{u}, \boldsymbol{v}), \boldsymbol{w}\right\rangle_{\boldsymbol{V}'}\right| \leq c\left(||\boldsymbol{u}||_{\boldsymbol{V}}^{1/2}\, ||\boldsymbol{A}\boldsymbol{u}||_{\boldsymbol{V}}^{1/2}||\boldsymbol{v}||_{\boldsymbol{L}^2(\Omega)}||\boldsymbol{w}||_{\boldsymbol{V}} + ||\boldsymbol{A}\boldsymbol{u}||_{\boldsymbol{L}^2(\Omega)}||\boldsymbol{v}||_{\boldsymbol{L}^2(\Omega)}\, ||\boldsymbol{w}||_{\boldsymbol{L}^2(\Omega)}^{1/2}\, ||\boldsymbol{w}||_{\boldsymbol{V}}^{1/2}\right).$$

Proof. For the proof of this technical lemma we refer the interested reader to [FHT2]. □

If we apply $\boldsymbol{P}_\sigma$ to (4.347a,b,c), using the notation developed above, we obtain the following equivalent system of equations:

$$\frac{d}{dt}(\alpha_0^2\boldsymbol{v} + \alpha_1^2\boldsymbol{A}\boldsymbol{v}) + \nu\boldsymbol{A}(\alpha_0^2 + \alpha_1^2\boldsymbol{A})\boldsymbol{v} + \tilde{\boldsymbol{B}}(\boldsymbol{v}, \alpha_0^2\boldsymbol{v} + \alpha_1^2\boldsymbol{A}\boldsymbol{v}) = \boldsymbol{P}_\sigma\boldsymbol{f} \tag{4.355a}$$

$$\boldsymbol{v}(0) = \boldsymbol{v}_0 \tag{4.355b}$$

or, if we set

$$\boldsymbol{u} = \alpha_0^2\boldsymbol{v} + \alpha_1^2\boldsymbol{A}\boldsymbol{v} \tag{4.356}$$

$$\frac{d\boldsymbol{u}}{dt} + \nu\boldsymbol{A}\boldsymbol{u} + \boldsymbol{B}(\boldsymbol{u})\boldsymbol{v} - \boldsymbol{B}^*(\boldsymbol{u})\boldsymbol{v} = \boldsymbol{P}_\sigma\boldsymbol{f}, \tag{4.357a}$$

$$\boldsymbol{v}(0) = \boldsymbol{v}_0. \tag{4.357b}$$

As in [FHT2], we assume that $\boldsymbol{P}_\sigma \boldsymbol{f} = \boldsymbol{f}$; if not, then we may add the gradient part of $\boldsymbol{f}$ to the modified pressure and relabel $\boldsymbol{P}_\sigma \boldsymbol{f}$ as $\boldsymbol{f}$. We now have the following definition of a regular solution:

Definition 4.5. Let $\boldsymbol{f} \in \boldsymbol{H}$ and let $T > 0$. A function $\boldsymbol{v} \in C([0,T);\boldsymbol{V}) \cap L^2([0,T);D(\boldsymbol{A}))$ with $\dfrac{d\boldsymbol{v}}{dt} \in L^2([0,T);\boldsymbol{H})$ is said to be a regular solution of (4.355a,b) in the interval $[0,T)$ if it satisfies

$$\left\langle \frac{d}{dt}(\alpha_0^2 \boldsymbol{v} + \alpha_1^2 \boldsymbol{A}\boldsymbol{v}), \boldsymbol{w} \right\rangle_{D(\boldsymbol{A})'} + \nu \langle \boldsymbol{A}(\alpha_0^2 \boldsymbol{v} + \alpha_1^2 \boldsymbol{A}\boldsymbol{v}), \boldsymbol{w}\rangle_{D(\boldsymbol{A})'}$$
$$+ \left\langle \tilde{\boldsymbol{B}}(\boldsymbol{v}, \alpha_0^2 \boldsymbol{v} + \alpha_1^2 \boldsymbol{A}\boldsymbol{v}), \boldsymbol{w} \right\rangle_{D(\boldsymbol{A})'} = (\boldsymbol{f}, \boldsymbol{w})_{\boldsymbol{L}^2(\Omega)} \quad (4.358)$$

for every $\boldsymbol{w} \in D(\boldsymbol{A})$ and almost every $t \in [0,T)$. Also, $\boldsymbol{v}(0) = \boldsymbol{v}_0$ in $\boldsymbol{V}$.

Remarks. Equation (4.358) is to be understood in the following sense: For every $t_0, t \in [0,T)$ we have

$$\left(\alpha_0^2 \boldsymbol{v}(t) + \alpha_1^2 \boldsymbol{A}\boldsymbol{v}(t), \boldsymbol{w}\right)_{\boldsymbol{L}^2(\Omega)} + \nu \int_{t_0}^{t} \left(\alpha_0^2 \boldsymbol{v}(s) + \alpha_1^2 \boldsymbol{A}\boldsymbol{v}(s), \boldsymbol{w}\right)_{\boldsymbol{L}^2(\Omega)} ds$$
$$+ \int_{t_0}^{t} \left\langle \tilde{\boldsymbol{B}}\left(\boldsymbol{v}(s), \alpha_0^2 \boldsymbol{v}(s) + \alpha_1^2 \boldsymbol{A}\boldsymbol{v}(s)\right), \boldsymbol{w} \right\rangle_{D(\boldsymbol{A})'} ds = \int_{t_0}^{t} (\boldsymbol{f}, \boldsymbol{w})_{\boldsymbol{L}^2(\Omega)}\, ds. \quad (4.359)$$

We are now in a position to state the central result in [FHT2] concerning global existence and uniqueness of regular solutions to (4.355a,b).

Theorem 4.27. *Let* $\boldsymbol{f} \in \boldsymbol{H}$ *and* $\boldsymbol{v}_0 \in \boldsymbol{V}$. *Then for any* $T > 0$, *the system* (4.355a,b) *has a unique regular solution* $\boldsymbol{v}$ *on* $[0,T)$. *Moreover, this solution satisfies*

(i) $\boldsymbol{v} \in L^\infty_{loc}((0,T];\boldsymbol{H}^3(\Omega))$
(ii) *There are constants* R_k, *for* $k = 0,1,2,3$, *which depend only on* ν, α_0, α_1, *and* $\boldsymbol{f}$, *but not on* $\boldsymbol{v}_0$, *such that*

$$\limsup_{t\to\infty} \left(\alpha_0^2 \left|\left| \boldsymbol{A}^{\frac{k}{2}} \boldsymbol{v} \right|\right|^2_{\boldsymbol{L}^2(\Omega)} + \alpha_1^2 \left|\left| \boldsymbol{A}^{\frac{k+1}{2}} \boldsymbol{v} \right|\right|^2_{\boldsymbol{L}^2(\Omega)} \right) = R_k^2$$

for $k = 0,1,2,3$. *In particular, we have*

$$R_0^2 = \frac{1}{\nu\lambda_1} \min\left\{ \frac{\left|\left|\boldsymbol{A}^{-1/2}\boldsymbol{f}\right|\right|^2_{\boldsymbol{L}^2(\Omega)}}{\nu\alpha_0^2}, \frac{\left|\left|\boldsymbol{A}^{-1/2}\boldsymbol{f}\right|\right|_{\boldsymbol{L}^2(\Omega)}}{\nu\alpha_1^2} \right\} \quad (4.360)$$
$$\le \min\left\{ \frac{||\boldsymbol{f}||_{\boldsymbol{L}^2(\Omega)}}{\nu^2\lambda_1^2\alpha_0^2}, \frac{||\boldsymbol{f}||^2_{\boldsymbol{L}^2(\Omega)}}{\nu^2\lambda_1^3\alpha_1^2} \right\}$$

so that

$$R_0^2 \leq \frac{G^2\nu^2}{\lambda_1^{1/2}} \min\left\{\frac{1}{\alpha_0^2}, \frac{1}{\alpha_1^2\lambda_1}\right\} = \frac{G^2\nu^2}{\gamma\lambda_1^{1/2}}$$

where $G = \dfrac{||\boldsymbol{f}||_{L^2(\Omega)}}{\nu^2\lambda_1^{3/4}}$ *is the Grashoff number and* $\gamma^{-1} = \min\left\{\dfrac{1}{\alpha_0^2}, \dfrac{1}{\alpha_1^2\lambda_1}\right\}$.
Finally, for all $t \geq 0$*,*

$$\limsup_{T\to\infty} \frac{\nu}{T}\int_t^{t+T} \left(\alpha_0^2||\boldsymbol{v}(s)||_{\boldsymbol{V}}^2 + \alpha_1^2||\boldsymbol{A}\boldsymbol{v}(s)||_{\boldsymbol{L}^2(\Omega)}^2\right) ds$$
$$\leq \nu\lambda_1 R_0^2 \leq \frac{G^2\nu^3\lambda_1^{1/2}}{\gamma}. \quad (4.361)$$

Proof. The existence proof is based on a Galerkin procedure to establish the required ($\boldsymbol{H}^1$, $\boldsymbol{H}^2$, and $\boldsymbol{H}^3$) a priori estimates and the Aubin Compactness Theorem (see, e.g., [CF]); for the details of the rather lengthy proof of the existence part of Theorem 4.27 we refer the reader to the original paper [FHT2]. Uniqueness of regular solutions will follow if we establish continuous dependence of those solutions on the initial data. To this end, let $\boldsymbol{v}$ and $\bar{\boldsymbol{v}}$ be any two solutions of (4.355a,b) on $[0, T]$ with initial data $\boldsymbol{v}(0) = \boldsymbol{v}_0$ and $\bar{\boldsymbol{v}}(0) = \bar{\boldsymbol{v}}_0$, respectively. We set

$$\boldsymbol{u} = \alpha_0^2\boldsymbol{v} + \alpha_1^2\boldsymbol{A}\boldsymbol{v} \text{ and } \bar{\boldsymbol{u}} = \alpha_0^2\bar{\boldsymbol{v}} + \alpha_1^2\boldsymbol{A}\bar{\boldsymbol{v}}$$

and let $\delta\boldsymbol{v} = \boldsymbol{v} - \bar{\boldsymbol{v}}$ and $\delta\boldsymbol{u} = \boldsymbol{u} - \bar{\boldsymbol{u}}$. Then by (4.355a)

$$\frac{d}{dt}\boldsymbol{u} + \nu\boldsymbol{A}\boldsymbol{u} + \tilde{\boldsymbol{B}}(\delta\boldsymbol{v}, \boldsymbol{u}) + \tilde{\boldsymbol{B}}(\bar{\boldsymbol{v}}, \delta\boldsymbol{u}) = \boldsymbol{0}. \quad (4.362)$$

Equation (4.362) holds in $L^2([0, T], D(\boldsymbol{A})')$; as $\delta\boldsymbol{v} \in L^2([0, T]; D(\boldsymbol{A}))$, which is the dual space of $L^2([0, T]; D(\boldsymbol{A})')$, we infer from Lemma 4.21 that

$$\left\langle \frac{d}{dt}\boldsymbol{u}, \delta\boldsymbol{v}\right\rangle_{D(\boldsymbol{A})'} + \nu\left(\alpha_0^2||\delta\boldsymbol{v}||_{\boldsymbol{V}}^2 + \alpha_1^2||\boldsymbol{A}\delta\boldsymbol{v}||_{\boldsymbol{L}^2(\Omega)}^2\right) + \left\langle\tilde{\boldsymbol{B}}(\bar{\boldsymbol{v}}, \delta\boldsymbol{u}), \delta\boldsymbol{v}\right\rangle_{D(\boldsymbol{A})'} = 0. \quad (4.363)$$

As (see [Te1], Chap. III, Lemma 1.2)

$$\left\langle \frac{d\boldsymbol{u}}{dt}, \delta\boldsymbol{v}\right\rangle_{D(\boldsymbol{A})'} = \frac{1}{2}\frac{d}{dt}\left(\alpha_0^2||\delta\boldsymbol{v}||_{\boldsymbol{L}^2(\Omega)}^2 + \alpha_1^2||\delta\boldsymbol{v}||_{\boldsymbol{V}}^2\right)$$

it follows from (4.363) that

$$\frac{1}{2}\frac{d}{dt}\left(\alpha_0^2||\delta \boldsymbol{v}||^2_{\boldsymbol{L}^2(\Omega)} + \alpha_1^2||\delta \boldsymbol{v}||^2_{\boldsymbol{V}}\right) + \nu\left(\alpha_0^2||\delta \boldsymbol{v}||^2_{\boldsymbol{V}} + \alpha_1^2||\boldsymbol{A}\,\delta \boldsymbol{v}||^2_{\boldsymbol{L}^2(\Omega)}\right) + \left\langle \tilde{\boldsymbol{B}}(\bar{\boldsymbol{v}}, \delta \boldsymbol{u}), \delta \boldsymbol{v}\right\rangle_{D(\boldsymbol{A})'} = 0. \quad (4.364)$$

By the second of the two results in part (v) of Lemma 4.21,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\left(\alpha_0^2||\delta \boldsymbol{v}||^2_{\boldsymbol{L}^2(\Omega)} + \alpha_1^2||\delta \boldsymbol{v}||^2_{\boldsymbol{V}}\right) + \nu\left(\alpha_0^2||\delta \boldsymbol{v}||^2_{\boldsymbol{V}} + \alpha_1^2||\boldsymbol{A}\,\delta \boldsymbol{v}||^2_{\boldsymbol{L}^2(\Omega)}\right)\\ &\le c\left(||\bar{\boldsymbol{v}}||^{1/2}_{\boldsymbol{V}}||\boldsymbol{A}\,\bar{\boldsymbol{v}}||^{1/2}_{\boldsymbol{L}^2(\Omega)}||\delta \boldsymbol{u}||_{\boldsymbol{L}^2(\Omega)}||\delta \boldsymbol{v}||_{\boldsymbol{V}}\right.\\ &\quad\left. + ||\boldsymbol{A}\,\bar{\boldsymbol{v}}||_{\boldsymbol{L}^2(\Omega)}||\delta \boldsymbol{u}||_{\boldsymbol{L}^2(\Omega)}||\delta \boldsymbol{v}||^{1/2}_{\boldsymbol{L}^2(\Omega)}||\delta \boldsymbol{v}||^{1/2}_{\boldsymbol{V}}\right) \quad (4.365)\end{aligned}$$

for some $c > 0$ and, by Young's inequality (Appendix A)

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\left(\alpha_0^2||\delta \boldsymbol{v}||^2_{\boldsymbol{L}^2(\Omega)} + \alpha_1^2||\delta \boldsymbol{v}||^2_{\boldsymbol{V}}\right) + \nu\left(\alpha_0^2||\delta \boldsymbol{v}||^2_{\boldsymbol{V}} + \alpha_1^2||\boldsymbol{A}\,\delta \boldsymbol{v}||^2_{\boldsymbol{L}^2(\Omega)}\right)\\ &\le \frac{c}{\nu}\left(||\bar{\boldsymbol{v}}||_{\boldsymbol{V}}||\boldsymbol{A}\,\bar{\boldsymbol{v}}||_{\boldsymbol{L}^2(\Omega)}||\delta \boldsymbol{v}||^2_{\boldsymbol{V}} + ||\boldsymbol{A}\,\bar{\boldsymbol{v}}||^2_{\boldsymbol{L}^2(\Omega)}||\delta \boldsymbol{v}||_{\boldsymbol{L}^2(\Omega)}||\delta \boldsymbol{v}||_{\boldsymbol{V}}\right)\\ &\quad + \frac{\nu}{2}\left(\alpha_0^2||\delta \boldsymbol{v}||^2_{\boldsymbol{V}} + \alpha_1^2||\boldsymbol{A}\,\delta \boldsymbol{v}||^2_{\boldsymbol{L}^2(\Omega)}\right) \quad (4.366)\\ &\le \frac{c}{2\nu\alpha_1^2\lambda_1^{1/2}}||\boldsymbol{A}\,\bar{\boldsymbol{v}}||^2_{\boldsymbol{L}^2(\Omega)}\left(\alpha_0^2||\delta \boldsymbol{v}||^2_{\boldsymbol{L}^2(\Omega)} + \alpha_1^2||\delta \boldsymbol{v}||^2_{\boldsymbol{V}}\right)\\ &\quad + \frac{\nu}{2}\left(\alpha_0^2||\delta \boldsymbol{v}||^2_{\boldsymbol{V}} + \alpha_1^2||\boldsymbol{A}\,\delta \boldsymbol{v}||^2_{\boldsymbol{L}^2(\Omega)}\right).\end{aligned}$$

Therefore,

$$\begin{aligned}&\left(\alpha_0^2||\delta \boldsymbol{v}(t)||^2_{\boldsymbol{L}^2(\Omega)} + \alpha_1^2||\delta \boldsymbol{v}(t)||^2_{\boldsymbol{V}}\right)\\ &\quad\le \left(\alpha_0^2||\delta \boldsymbol{v}(0)||^2_{\boldsymbol{L}^2(\Omega)} + \alpha_1^2||\delta \boldsymbol{v}(0)||^2_{\boldsymbol{V}}\right)\exp\left[\int_0^t \frac{c||\boldsymbol{A}\,\bar{\boldsymbol{v}}(s)||^2_{\boldsymbol{L}^2(\Omega)}}{\nu\alpha_1^2\lambda_1^{1/2}}\,ds\right]. \quad (4.367)\end{aligned}$$

As $\bar{\boldsymbol{v}} \in L^2([0,T]; D(\boldsymbol{A}))$ we conclude that solutions of (4.355a,b) depend continuously on the initial data on any bounded interval $[0,T]$; in particular, regular solutions of (4.355a,b) are uniquely defined. □

We conclude this subsection by setting $\alpha_0 = 1$ and stating the following result from [FHT2] concerning the convergence of the solutions of the system (4.355a,b) as $\alpha_1 \to 0^+$:

Theorem 4.28. *Let* $\boldsymbol{f} \in \boldsymbol{H}$, $\boldsymbol{v}_0 \in \boldsymbol{V}$, *and set* $\alpha_0 = 1$. *Also, let* $\boldsymbol{v}_{\alpha_1}$ *denote the unique regular solution of* (4.355a,b) *and set* $\boldsymbol{u}_{\alpha_1} = \boldsymbol{v}_{\alpha_1} + \alpha_1^2 \boldsymbol{A} \boldsymbol{v}_{\alpha_1}$. *Then there are subsequences* $\boldsymbol{v}_{\alpha_1^j}$ *and* $\boldsymbol{u}_{\alpha_1^j}$, *and a function* $\boldsymbol{v}$ *such that as* $\alpha_1^j \to 0^+$ *we have*

(i) $\boldsymbol{v}_{\alpha_1^j} \to \boldsymbol{v}$, *strongly in* $L^2_{loc}([0,\infty); \boldsymbol{H})$,

(ii) $\boldsymbol{v}_{\alpha_1^j} \to \boldsymbol{v}$, *weakly in* $L^2_{loc}([0,\infty); \boldsymbol{V})$,

(iii) For every $T \in (0,\infty)$, *and every* $\boldsymbol{w} \in \boldsymbol{H}$, *we have* $\left(\boldsymbol{v}_{\alpha_1^j}(t), \boldsymbol{w}\right)_{\boldsymbol{L}^2(\Omega)} \to (\boldsymbol{v}(t), \boldsymbol{w})_{\boldsymbol{L}^2(\Omega)}$, *uniformly on* $[0,T]$,

(iv) $\boldsymbol{u}_{\alpha_1^j} \to \boldsymbol{v}$, *weakly in* $L^2_{loc}([0,\infty); \boldsymbol{H})$ *and strongly in* $L^2_{loc}([0,\infty); \boldsymbol{V}')$.

Furthermore, $\boldsymbol{v}$ *is a weak solution of the three-dimensional Navier–Stokes equations with the initial data* $\boldsymbol{v}(0) = \boldsymbol{v}_0$ *and space-periodic boundary conditions.*

Remarks. The analysis in [FHT2] also provides upper bounds for the Hausdorff and fractal dimensions $d_H(\boldsymbol{\mathcal{A}})$ and $d_F(\boldsymbol{\mathcal{A}})$, respectively, of the global attractor $\boldsymbol{\mathcal{A}}$ for the NS-α equations of the form

$$d_H(\boldsymbol{\mathcal{A}}) \le d_F(\boldsymbol{\mathcal{A}}) \le c \max\left\{ G^{4/3}\left(\frac{1}{\gamma\alpha_1^2\lambda_1}\right)^{2/3}, G^{3/2}\left(\frac{1}{\alpha_0^4\gamma^2\lambda_1\alpha_1^2}\right)^{3/8} \right\} \tag{4.368}$$

where all the relevant parameters have been defined in the statement of Theorem 4.27; the estimates in (4.368) may be compared to the analogous results in Chap. 5 for the bipolar viscous fluid with $\alpha = 0$.

Chapter 5
Attractors for Incompressible Bipolar and Non-Newtonian Flows: Bounded Domains and Space Periodic Problems

5.1 Introduction

From the existence and uniqueness theorems established in Chap. 4, both for the initial-boundary value problems, as well as for the space-periodic problems associated with nonlinear, incompressible, bipolar ($\mu_1 > 0$) and non-Newtonian flow ($\mu_1 = 0$), it follows that under appropriate sets of conditions one may show that the solution operator $\boldsymbol{S}(t)$ yields a nonlinear semigroup; in this chapter we examine the behavior of the orbits of such semigroups as $t \to \infty$. Our interest here is focused on the existence of maximal compact global attractors for bounded domains and space periodic problems.

Given a nonlinear semigroup of solution operators $\boldsymbol{S}(t)$, $t \geq 0$, associated with a well-posed boundary-value problem (or space-periodic problem), for some nonlinear system of evolutionary partial differential equations, the general definition of an attractor $\mathcal{A}$ is that it is a set which is invariant, i.e., $\boldsymbol{S}(t)\mathcal{A} = \mathcal{A}$, $\forall t \geq 0$, and which satisfies $d(\boldsymbol{S}(t)\boldsymbol{v}_0, \mathcal{A}) \to 0$, as $t \to \infty$, for all $\boldsymbol{v}_0$ in some neighborhood of $\mathcal{A}$. Here d is the standard distance measure given by Definition 5.3. Establishing the existence of a maximal compact attractor is a multi-step process which involves proving the existence of appropriate absorbing sets (in order to deduce the uniform compactness of the semigroup $\boldsymbol{S}(t)$, for large t), the uniform differentiability of $\boldsymbol{S}(t)$ on the attractor, and the uniform boundedness of related linearized operators $\mathcal{L}(t; \boldsymbol{u}_0)$ for $\boldsymbol{u}_0 \in \mathcal{A}$. Once the existence of a maximal compact attractor has been established, it is then often possible to use the framework established, e.g., by Constatin, Foias, and Temam (see [Te4, CFT1]) in order to deduce upper bounds for both the Hausdorff and fractal dimensions $d_H(\mathcal{A})$ and $d_F(\mathcal{A})$, respectively; these concepts will be carefully defined in this chapter. Besides the two works cited above, the following books and papers contain extensive analyses of problems involving the existence of attractors for dissipative partial differential equations and systems: [BV1, 3, 4, 5], [CV1, CVW], [EZ1, 2, 3], [GT, Hal, La4, Ro], and [SY2]. Additionally, there is a well-developed literature which is focused, specifically, on the problem of the existence of attractors for the Navier–Stokes equations;

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prominent among such works are the following references: [BaA, BV2, CF, CFT2, CV2, FP, FT, Gu, La3, LWZ], and [Ru].

We now offer a synposis of the work which is described in this chapter; in order to keep the chapter reasonably self-contained, we first review the basic equations and Hilbert spaces associated with the modeling and analysis of viscous, incompressible, bipolar fluid flow and then recall the various existence and uniqueness results of Chap. 4 which are so essential to our work in this chapter. The constitutive equations relating the stress and multipolar stress tensors to the rate of deformation tensor, which were introduced in Chap. 1, have the form

$$\tau_{ij} = -p\delta_{ij} + 2\mu_0(\epsilon + |\boldsymbol{e}|^2)^{-\frac{\alpha}{2}} e_{ij} - 2\mu_1 \Delta e_{ij}, \tag{5.1a}$$

$$\tau_{ijk} = 2\mu_1 \frac{\partial e_{ij}}{\partial x_\kappa} \tag{5.1b}$$

with $\epsilon, \mu_0 > 0$, $\mu_1 \geq 0$. The nonlinear viscosity μ is, therefore, given as

$$\mu(|\boldsymbol{e}|) = \mu_0(\epsilon + |\boldsymbol{e}|^2)^{-\frac{\alpha}{2}} \tag{5.1c}$$

and the main cases of interest are either $0 \leq \alpha < 1$ or $\alpha < 0$. We will, as in Chap. 4, often set $\alpha = 2 - p$ so that

$$\mu(|\boldsymbol{e}|) = \mu_0(\epsilon + |\boldsymbol{e}|^2)^{\frac{p-2}{2}} \tag{5.1d}$$

in which case

$$0 \leq \alpha < 1 \Leftrightarrow 1 < p \leq 2,$$
$$\alpha < 0 \Leftrightarrow p > 2.$$

The velocity vector $\boldsymbol{v}$ satisfies, for an incompressible flow,

$$\rho\left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla v\right) = -\nabla p + 2\nabla \cdot (\mu(|\boldsymbol{e}|)\boldsymbol{e}) - 2\mu_1 \nabla \cdot (\Delta \boldsymbol{e}) + \rho \boldsymbol{f} \tag{5.2a}$$

$$\nabla \cdot \boldsymbol{v} = 0 \tag{5.2b}$$

where $\boldsymbol{f}$ is the body force/mass. Equations (5.2a,b) are to hold in $\Omega \times [0, T)$, $T > 0$ where, for the boundary-value problem, $\Omega \subseteq \mathbb{R}^n$, $n = 2, 3$, is a bounded domain, while for the case of the space-periodic problem $\Omega = [0, L]^n$, $L > 0$, $n = 2, 3$. If $\Omega \subseteq \mathbb{R}^n$ is a bounded domain then to (5.2a,b) we append the boundary conditions (Sect. 1.4)

$$\boldsymbol{v} = \boldsymbol{0},\ \tau_{ijk}\nu_j\nu_k - \tau_{jkl}\nu_j\nu_k\nu_l\nu_i = 0,\ i = 1, 2, 3, \text{ on } \partial\Omega \times [0, T) \tag{5.3a}$$

where $\boldsymbol{\nu}$ is the exterior unit normal to $\partial\Omega$, while for the space-periodic problem, in dimensions $n = 2$ or 3, we require that

$$\left.\begin{aligned} &v_i(\mathbf{0},t) = v_i(L\boldsymbol{e}_j,t), && t \geq 0\\ &\int_\Omega \boldsymbol{v}(\boldsymbol{x},t)\,d\boldsymbol{x} = 0, && t \geq 0\\ &\tau_{ijk}(\boldsymbol{v}(0,t))\nu_j\nu_k\tau_i = \tau_{ijk}(\boldsymbol{v}(L\boldsymbol{e}_j,t))\nu_j\nu_k\tau_i, && t \geq 0 \end{aligned}\right\} \tag{5.3b}$$

with $\boldsymbol{e}_j$ the unit vector in the jth coordinate direction and $\boldsymbol{\tau}$ any vector in the tangent space to $\partial\Omega$. Associated with either (5.2a,b), (5.3a) or (5.2a,b), (5.3b) is the specification of an initial condition

$$\boldsymbol{v}(\boldsymbol{x},0) = \boldsymbol{v}_0(\boldsymbol{x}),\ \boldsymbol{x} \in \Omega. \tag{5.4}$$

If $\alpha = 0$, $\mu_1 = 0$ (or $p = 2$, $\mu_1 = 0$) we are dealing with the Navier–Stokes equations and the associated boundary-value or space-periodic problems where, of course, only the first set of boundary conditions in (5.3a) are relevant for the boundary-value problem. For $\alpha \neq 0$, but $\mu_1 = 0$, we are working with a shear-thinning ($p < 2$ or $\alpha > 0$) non-Newtonian fluid or a shear-thickening ($p > 2$ or $\alpha < 0$) non-Newtonian fluid; in either case, only the first set of boundary conditions in (5.3a) is, again, relevant for the associated boundary-value problem. Finally, the fundamental Hilbert spaces for the two types of problems cited, above, are

$$\boldsymbol{H} = \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \mid \nabla\cdot\boldsymbol{v} = 0, \text{ in } \Omega,\ \boldsymbol{v}\cdot\boldsymbol{\nu} = 0, \text{ on } \partial\Omega\} \tag{5.5a}$$

for the case of a bounded domain, Ω, in dimension $n = 3$, and

$$\begin{aligned} \boldsymbol{H}_{per} = \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)\ |\nabla\cdot\boldsymbol{v} = 0, \text{ in } \Omega = [0,L]^3, \text{ with}\\ v_i(\mathbf{0}) = v_i(L\boldsymbol{e}_j) \text{ and } \int_\Omega \boldsymbol{v}(\boldsymbol{x})d\boldsymbol{x} = \mathbf{0}\} \end{aligned} \tag{5.5b}$$

for the space-periodic problem in dimension $n = 3$ with $\mu_1 = 0$. For the space-periodic problem with $\mu_1 > 0$ we add the last condition in (5.3b) to those present in the definition of $\boldsymbol{H}$ and denote this new Hilbert space as $\boldsymbol{H}^2_{per}$.

We now recall the basic existence and uniqueness results that were described in Chap. 4 for the bipolar and non-Newtonian problems.

I. [BBN4]: Suppose that $\mu_1 > 0$. Then, for the boundary-value problem, with $\boldsymbol{H}$ given by (5.5a), or for the space-periodic problem where $\boldsymbol{H}$ is replaced by $\boldsymbol{H}^2_{per}$, if $p > 1$, $f_i \in L^\infty([0,\infty); \boldsymbol{H}_{per})$, $i = 1,2,3$, and $\boldsymbol{v}_0 \in \boldsymbol{H}_{per}$, $\exists$ a unique solution of either (5.2a,b), (5.3a,b), (5.4) or (5.2a,b), (5.3b), (5.4) which satisfies, $\forall t_0 > 0$,

$$\boldsymbol{v} \in L^\infty([0,\infty); \boldsymbol{H}) \cap L^\infty((t_0,\infty), \boldsymbol{H}^2(\Omega)).$$

Moreover, the unique solution $\boldsymbol{v} \in C([0,T]; H)$, $\forall T > 0$, so that the solution operator $\boldsymbol{S}_{\mu_1}(t) : \boldsymbol{v}_0 \to \boldsymbol{v}(t)$ constitutes a nonlinear semigroup of operators.

Remarks. While the results in (I) have been cited for the spaces in (5.5a,b), corresponding to dimension $n = 3$, they also hold, of course, in dimension $n = 2$; as pointed out in Chap. 4, in dimension $n = 3$ these results contrast sharply with what is known about existence and uniqueness for the Navier–Stokes equations, for which there exists a unique strong solution only on some interval $[0, t_1]$ with $t_1 = t_1(||\boldsymbol{v}_0||_{\boldsymbol{L}^2})$ and a weak solution $\forall t > 0$ (which agrees with the strong solution on $[0, t_1]$ but which may not be unique).

II. [BBN2, 3]: Suppose that $\mu_1 = 0$. For the space-periodic problem with $\Omega = [0, L]^n$, $n = 2, 3$, $L > 0$, and

$$\boldsymbol{v}_0 \in \boldsymbol{W}_0^{1,2}(\Omega) \cap \boldsymbol{W}_{per}^{2,2}(\Omega)$$

we have the following results:

(i) $\exists$ a (possible non-unique) weak solution for $n = 2$ if $\frac{3}{2} < p < 2$ ($\Leftrightarrow 0 < \alpha < \frac{1}{2}$) and for $n = 3$ if $\frac{9}{5} < p < \frac{11}{5}$ ($\Leftrightarrow -\frac{1}{5} < \alpha < \frac{1}{5}$).

(ii) $\exists$ a unique, regular, weak solution in $L^p([0,T); \boldsymbol{W}_{per}^{1,p}(\Omega)) \cap L^\infty([0,T); \boldsymbol{W}_{per}^{1,2}(\Omega))$ for $n = 2$ if $p \geq 2$ ($\Leftrightarrow \alpha \leq 0$) and for $n = 3$ if $p \geq \frac{11}{5}$ ($\Leftrightarrow \alpha \leq -\frac{1}{5}$).

(iii) $\exists$ a unique Young measure-valued solution for $n = 2$ if $p > 1$ ($\Leftrightarrow \alpha < 1$) and for $n = 3$ if $\frac{6}{5} < p \leq \frac{9}{5}$ ($\Leftrightarrow \frac{1}{5} \leq \alpha < \frac{4}{5}$); this latter result in dimension $n = 3$ holds for the boundary-value problem as well.

Related results for both the boundary-value problem, and the space-periodic problem, when $\mu_1 = 0$, appear, e.g., in [La1, 2], [MN, DuG, MNR2, GZ] and [BaH].

Having reviewed the basic equations associated with the modeling of viscous, incompressible, bipolar fluid flow, which were introduced in Chap. 2, and summarized the existence and uniqueness results presented in Chap. 4, we now offer an overview of those results on the existence of global attractors which will be proven in this chapter:

In Sect. 5.2 we introduce the linear operator $\boldsymbol{L}(t;\boldsymbol{U})$ associated with the linearization of the nonlinear, incompressible, bipolar equations about an equilibrium solution $\boldsymbol{U}$; the operator $\boldsymbol{L}$ will occupy a central role in the analysis presented in the balance of the present chapter. Based on results obtained in [Bl4], we also establish in Sect. 5.2 the linearized stability of solutions of the incompressible bipolar flow equations.

In Sect. 5.3 we present the results obtained in [BBN5] for the initial-boundary value problem associated with incompressible, bipolar flow, i.e., for (5.2a,b), (5.3), (5.4) with $\mu_1 > 0$ and $0 \le \alpha < 1$ $(1 < p \le 2)$; the results hold in dimensions $n = 2$ or 3 and are valid for the space-periodic problem as well. It is demonstrated that a maximal compact global attractor $\boldsymbol{\mathcal{A}}_{\mu_1} \subset \boldsymbol{W}^{2,2}(\Omega)$ exists and that the Hausdorf and Fractal dimensions of $\boldsymbol{\mathcal{A}}_{\mu_1}$ satisfy $d_H(\boldsymbol{\mathcal{A}}_{\mu_1}) \le k^*$, $d_F(\boldsymbol{\mathcal{A}}_{\mu_1}) \le 2k^*$ where $k^* < 1 + g(\mu_1;\Omega)|\boldsymbol{f}|_\infty^3$ with $|\boldsymbol{f}|_\infty = ||\boldsymbol{f}||_{L^\infty([0,\infty;\boldsymbol{H})}$ and $g \sim \mu_1^{-6}$ as $\mu_1 \to 0$.

For $\mu_1 > 0$ it is first proven in Sect. 5.4 that, as a consequence of the results obtained in [Bl3], a maximal compact attractor, $\boldsymbol{\mathcal{A}}_{\mu_1} \subset \boldsymbol{W}^{2,2}(\Omega)$, exists for the space-periodic problem when $n = 2$ and $p > 2$ $(\alpha < 0)$. More importantly, however, is the fact that now, for $p > 2$, we are able to show that $d_H(\boldsymbol{\mathcal{A}}_{\mu_1})$ and $d_F(\boldsymbol{\mathcal{A}}_{\mu_1})$ are both independent of μ_1. Indeed, for $\mu_1 = 0$, the corresponding non-Newtonian, space-periodic problem, for $p > 2$ and $n = 2$ is shown to admit a maximal compact global attractor $\boldsymbol{\mathcal{A}}_0 \subset \boldsymbol{W}^{2,2}(\Omega)$ provided $\boldsymbol{v}_0(\cdot) \subset \boldsymbol{L}^2(\Omega)$, $\Omega = [0, L]^2$, $L > 0$.

Finally, in Sect. 5.5 we show that, as a consequence of the analysis presented in [Bl2] and [Bl3], the attractors $\boldsymbol{\mathcal{A}}_{\mu_1}$, whose existence has been established for the space-periodic problem in the case $p > 2$ and $n = 2$, converge (in the sense of semidistance) to the global compact attractor $\boldsymbol{\mathcal{A}}_0$ for the same case, with $\mu_1 = 0$, as $\mu_1 \to 0$. The fact that convergence holds only in the sense of semidistance is, as noted in [Bl3], a consequence of the failure, to this point, of establishing uniform differentiability for the nonlinear semigroup $\boldsymbol{S}_0(t)$ associated with (5.2a,b), with $\mu_1 = 0$, (5.3b), (5.4) for $n = 2$ and $p > 2$.

5.2 Linearized Stability of Viscous Incompressible Bipolar Equations

In this section we will establish a sufficient condition for the linearized stability of equilibrium solutions to the boundary-value problem for the bipolar viscous flow equations in bounded subdomains of $\mathbb{R}^3$; the basic tools to be employed in the analysis are interpolation and Sobolev space estimates and embeddings, and both $\boldsymbol{H}^2$ and $\boldsymbol{L}^p$, $1 < p < 2$, versions of the Korn inequality. Related results for solutions of the Navier–Stokes equations may be found in many sources with the treatise by Joseph [Jo1] being a standard reference.

5.2.1 Linearized Bipolar Equations

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth C^1 boundary $\partial\Omega$. Let $\boldsymbol{U}$ be an equilibrium solution of (5.2a,b), (5.3a), (5.4), where the nonlinear viscosity $\mu(|\boldsymbol{e}|)$ is given by (5.1c),[1] with $\mu_0 > 0$, $\epsilon > 0$, and where, in this section, we specify that $0 < \alpha < 1$. Alternatively, $\mu(|\boldsymbol{e}|)$ is defined by (5.1d) and we assume that $1 < p < 2$. Also, for ease of exposition in this section we will set $\rho = 1$ and $\boldsymbol{f} = \boldsymbol{0}$. Thus, $\boldsymbol{U}$ is a solution of the following boundary-value problem with pressure $\tilde{p}$:

$$\boldsymbol{U} \cdot \nabla \boldsymbol{U} = -\nabla \tilde{p} + \nabla \cdot (2\mu \boldsymbol{e}(\boldsymbol{U})) - 2\mu_1 \nabla \cdot (\Delta \boldsymbol{e}(\boldsymbol{U})) \tag{5.6}$$

on Ω,

$$\mu = \mu(\boldsymbol{e}(\boldsymbol{U})) \equiv \mu_0(\epsilon + |\boldsymbol{e}(\boldsymbol{U})|^2)^{-\alpha/2} \tag{5.7}$$

$0 < \alpha < 1$, with

$$\boldsymbol{U} = 0,\ \tau_{ijk}(\boldsymbol{U})\nu_j \nu_k \tau_i = 0, , \text{ on } \partial\Omega. \tag{5.8}$$

If $\boldsymbol{u}$ is any solution of (5.2a,b), (5.3a), and (5.4) and we set

$$\boldsymbol{v}(\boldsymbol{x}, t) \equiv \boldsymbol{u}(\boldsymbol{x}, t) - \boldsymbol{U}(\boldsymbol{x}), \tag{5.9}$$

then $\boldsymbol{v}$ satisfies

$$\begin{aligned} &\frac{\partial v_i}{\partial t} + (U_j + v_j)\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial v_i}{\partial x_j}\right) = -\frac{\partial}{\partial x_i}(p - \tilde{p}) \\ &+ \frac{\partial}{\partial x_j}[\mu(\boldsymbol{e}(\boldsymbol{U} + \boldsymbol{v}))(e_{ij}(\boldsymbol{U}) + e_{ij}(\boldsymbol{v}))] - 2\mu_1 \frac{\partial}{\partial x_j}[\Delta e_{ij}(\boldsymbol{U}) + \Delta e_{ij}(\boldsymbol{v})]. \end{aligned} \tag{5.10}$$

On the assumption that the "perturbation" $\boldsymbol{v}$ is small (say, in the $\boldsymbol{C}^1(\Omega)$ norm) we now expand the products in (5.10), dropping all terms which are quadratic in $v_i(\boldsymbol{x}, t)$ and its spatial derivatives. We note, first of all that, with the usual summation convention for repeated indices, if we drop terms quadratic in the derivatives of v_i, we obtain

$$\begin{aligned} \mu(\boldsymbol{e}(\boldsymbol{U} + \boldsymbol{v})) &= \mu(\boldsymbol{e}(\boldsymbol{U}) + \boldsymbol{e}(\boldsymbol{v})) \\ &= 2\mu_0(\epsilon + (e_{kl}(\boldsymbol{U}) + e_{kl}(\boldsymbol{v}))(e_{kl}(\boldsymbol{U}) + e_{kl}(\boldsymbol{v}))]^{-\frac{\alpha}{2}} \\ &\simeq 2\mu_0[\epsilon + e_{kl}(\boldsymbol{U})e_{kl}(\boldsymbol{U}) + 2e_{kl}(\boldsymbol{U})e_{kl}(\boldsymbol{v})]^{-\frac{\alpha}{2}} \end{aligned}$$

[1] When there is no possibility of confusion we will often write $\mu(\boldsymbol{e})$ in lieu of $\mu(|\boldsymbol{e}|)$.

or

$$\frac{1}{2\mu_0}\cdot\mu(\boldsymbol{e}(\boldsymbol{U}+\boldsymbol{v})\simeq(\epsilon+|\boldsymbol{e}(\boldsymbol{U})|^2)^{-\alpha/2}-\alpha(\epsilon+|\boldsymbol{e}(\boldsymbol{U})|^2)^{-(1+\frac{\alpha}{2})}e_{ij}(\boldsymbol{U})e_{ij}(\boldsymbol{v}) \tag{5.11}$$

so that

$$\begin{aligned}\mu(\boldsymbol{e}(\boldsymbol{U}+\boldsymbol{v}))&\simeq\mu(\boldsymbol{e}(\boldsymbol{U}))-2\alpha\mu_0(\epsilon+|\boldsymbol{e}(\boldsymbol{U})|^2)^{-(1+\frac{\alpha}{2})}e_{ij}(\boldsymbol{U})e_{ij}(\boldsymbol{v})\\&=\mu(\boldsymbol{e}(\boldsymbol{U}))-A_{ij}(\boldsymbol{e}(\boldsymbol{U}))e_{ij}(\boldsymbol{v})\end{aligned} \tag{5.12}$$

where

$$A_{ij}(\boldsymbol{e}(\boldsymbol{w}))\equiv2\alpha\mu_0(\epsilon+|\boldsymbol{e}(\boldsymbol{w})|^2)^{-(1+\frac{\alpha}{2})}e_{ij}(\boldsymbol{w}). \tag{5.13}$$

Using (5.12) in (5.10) and subsequently dropping those terms which are quadratic in v_i, and its spatial dervatives, we are led to the equation

$$\begin{aligned}\frac{\partial v_i}{\partial t}+U_j\frac{\partial U_i}{\partial x_j}+U_j\frac{\partial v_i}{\partial x_j}+v_j\frac{\partial U_i}{\partial x_j}&=-\frac{\partial}{\partial x_i}(p-\tilde{p})\\&+\frac{\partial}{\partial x_j}[\mu(\boldsymbol{e}(\boldsymbol{U}))e_{ij}(\boldsymbol{U})-A_{kl}(\boldsymbol{e}(\boldsymbol{U}))e_{ij}(\boldsymbol{U})e_{kl}(\boldsymbol{v})]\\&+\frac{\partial}{\partial x_j}[\mu(\boldsymbol{e}(\boldsymbol{U}))e_{ij}(\boldsymbol{v})]-2\mu_1\frac{\partial}{\partial x_j}[\Delta e_{ij}(\boldsymbol{U})]-2\mu_1\frac{\partial}{\partial x_j}[\Delta e_{ij}(\boldsymbol{v})]\end{aligned}$$

which, after taking into account (5.6), and setting $P=p-\tilde{p}$, reduces to

$$\begin{aligned}\frac{\partial v_i}{\partial t}+U_j\frac{\partial v_i}{\partial x_j}+v_j\frac{\partial U_i}{\partial x_j}=-\frac{\partial}{\partial x_i}P+\frac{\partial}{\partial x_j}[\mu(\boldsymbol{e}(\boldsymbol{U}))e_{ij}(\boldsymbol{v})-A_{kl}(\boldsymbol{e}(\boldsymbol{U}))e_{ij}(\boldsymbol{U})e_{kl}(\boldsymbol{v})]\\-2\mu_1\frac{\partial}{\partial x_j}[\Delta e_{ij}(\boldsymbol{v})].\end{aligned} \tag{5.14}$$

Noting that $\frac{\partial}{\partial x_j}(\Delta e_{ij}(\boldsymbol{v}))=\frac{1}{2}\Delta^2 v_i$, as a consequence of the fact that $\nabla\cdot\boldsymbol{v}=0$, and setting

$$B_{ijkl}(\boldsymbol{w})=\frac{1}{\alpha}e_{ij}(\boldsymbol{w})A_{kl}(\boldsymbol{e}(\boldsymbol{w}))\equiv2\mu_0(\epsilon+|\boldsymbol{e}(\boldsymbol{w})|^2)^{-(1-\frac{\alpha}{2})}e_{ij}(\boldsymbol{w})e_{kl}(\boldsymbol{w}) \tag{5.15}$$

we may write (5.14) in the final form

$$\frac{\partial v_i}{\partial t}+U_i\frac{\partial v_i}{\partial x_j}+v_j\frac{\partial U_i}{\partial x_j}=-\frac{\partial P}{\partial x_i}+\frac{\partial}{\partial x_j}[\mu(\boldsymbol{e}(\boldsymbol{U}))e_{ij}(\boldsymbol{v})-\alpha B_{ijkl}(\boldsymbol{U})e_{kl}(\boldsymbol{v})]-\mu_1\Delta^2 v_i \tag{5.16}$$

and we make the following:

Definition 5.1. The system of equations (5.16), $i = 1, 2, 3$, on $\Omega \times [0, T]$, $\Omega \subseteq \mathbb{R}^3$, $T > 0$, represents the linearization of the system of nonlinear incompressible bipolar equations about an equilibrium solution $\boldsymbol{U}$.

Remarks. The existence of a unique solution $\boldsymbol{v}$ for the system (5.16), subject to $\nabla \cdot \boldsymbol{v} = 0$ in Ω, an initial condition $\boldsymbol{v}(\boldsymbol{x}, 0) = \boldsymbol{v}_0(\boldsymbol{x})$ in Ω, and the boundary conditions $\boldsymbol{v} = 0$, $\tau_{ijk}\nu_j\nu_k\tau_i = 0$, on $\partial\Omega \times [0, T)$, $\boldsymbol{\nu}$ being the exterior unit normal to $\partial\Omega$ and $\boldsymbol{\tau}$ any tangent vector to $\partial\Omega$, follows from the work in [BBN4] as described in Chap. 4.

In order to proceed with the analysis we formally define operators $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ as follows:

$$(\boldsymbol{A}_1\boldsymbol{v})_i = \Delta\Delta v_i \equiv 2\frac{\partial}{\partial x_j}(\Delta e_{ij}), \tag{5.17a}$$

$$(\boldsymbol{A}_2\boldsymbol{v})_i = -\frac{\partial}{\partial x_j}[\mu(\boldsymbol{e}(\boldsymbol{U}))e_{ij}(\boldsymbol{v}) - \alpha B_{ijkl}(\boldsymbol{U})e_{kl}(\boldsymbol{v})]. \tag{5.17b}$$

For $\boldsymbol{v}_0 \in \boldsymbol{W}^{1,2}(\Omega)$, with $\nabla \cdot \boldsymbol{v}_0 = 0$, we know from our work in Chap. 4 that the solution of (5.16), subject to the condition $\nabla \cdot \boldsymbol{v} = 0$ and the boundary conditions (5.3a), satisfies

$$\boldsymbol{v} \in L^\infty([0,\infty); \boldsymbol{W}^{1,2}(\Omega)) \cap L^\infty([0,\infty); \boldsymbol{W}^{1,p}(\Omega)) \cap L^\infty((t_1,\infty), \boldsymbol{W}^{2,2}(\Omega))$$

$\forall t_1 > 0$, so that $\boldsymbol{A}_1$, $\boldsymbol{A}_2$ are well-defined. If we also set

$$\boldsymbol{R}(\boldsymbol{U}, \boldsymbol{v}) = \boldsymbol{U} \cdot \nabla\boldsymbol{v} + \boldsymbol{v} \cdot \nabla\boldsymbol{U} \tag{5.18}$$

we see that (5.16) may be rewritten in the form

$$\frac{\partial \boldsymbol{v}}{\partial t} + \mu_1\boldsymbol{A}_1\boldsymbol{v} + \boldsymbol{A}_2\boldsymbol{v} + \boldsymbol{R}(\boldsymbol{U}, \boldsymbol{v}) = -\nabla P,$$

or, if we employ the standard device of projecting each term in this last equation onto the subspace of $\boldsymbol{L}^2(\Omega)$ consisting of the solenoidal vector fields, as

$$\frac{\partial \boldsymbol{v}}{\partial t} + \mu_1\boldsymbol{A}_1\boldsymbol{v} + \boldsymbol{A}_2\boldsymbol{v} + \boldsymbol{R}(\boldsymbol{U}, \boldsymbol{v}) = \boldsymbol{0}. \tag{5.19}$$

Remarks. As is customary, e.g., in work on the Navier–Stokes equations, we have not displayed the relevant projection operator $\boldsymbol{\mathcal{P}}$ in (5.19), i.e., $\boldsymbol{R}(\boldsymbol{U}, \boldsymbol{v})$ is actually to be replaced by $\mathcal{P}\boldsymbol{R}(\boldsymbol{U}, \boldsymbol{v}) \equiv \mathcal{P}(\boldsymbol{U} \cdot \nabla\boldsymbol{v}) + \mathcal{P}(\boldsymbol{v} \cdot \nabla\boldsymbol{U})$.

Remarks. It is a simple matter to show that $\boldsymbol{A}_1$, as defined by (5.17a) is a symmetric operator; that $\boldsymbol{A}_2$ is also symmetric follows from the fact that, for $\boldsymbol{w} \in \boldsymbol{W}_0^{1,2}(\Omega)$,

$$\begin{aligned}\int_\Omega (\boldsymbol{A}_2\boldsymbol{v})_i w_i\, d\boldsymbol{x} &= -\int_\Omega \frac{\partial}{\partial x_j}[\mu(\boldsymbol{e}(\boldsymbol{U}))e_{ij}(\boldsymbol{v}) - \alpha B_{ijkl}(\boldsymbol{U})e_{kl}(\boldsymbol{v})]w_i\, d\boldsymbol{x}\\ &= \int_\Omega [\mu(\boldsymbol{e}(\boldsymbol{U}))e_{ij}(\boldsymbol{v}) - \alpha B_{ijkl}(\boldsymbol{U})e_{kl}(\boldsymbol{v})]\frac{\partial w_i}{\partial x_j}\, d\boldsymbol{x}\\ &= \int_\Omega [\mu(\boldsymbol{e}(\boldsymbol{U}))e_{ij}(\boldsymbol{v})e_{ij}(\boldsymbol{w}) - \alpha B_{ijkl}(\boldsymbol{U})e_{ij}(\boldsymbol{w})e_{kl}(\boldsymbol{v})]\, d\boldsymbol{x}\\ &= \int_\Omega (\boldsymbol{A}_2\boldsymbol{w})_i v_i\, d\boldsymbol{x}\end{aligned}$$

because of the symmetries $B_{ijkl} = B_{jikl} = B_{ijlk} = B_{klij}$. However, in analogy with the situation for the Navier–Stokes operator, the operator

$$\hat{\boldsymbol{A}} \equiv \mu_1 \boldsymbol{A}_1 + \boldsymbol{A}_2 + \boldsymbol{R}(\boldsymbol{U}, \cdot)$$

is not symmetric owing to the character of the linear mapping $\boldsymbol{R}$ generated by the convective term; in spite of this fact, we will be able to use (5.19) to establish a sufficient condition for the linearized stability of equilibrium solutions of the nonlinear, incompressible, bipolar equations.

5.2.2 Basic Estimates for the Rate of Deformation

In this section we will elucidate five lemmas which apply to the rate of deformation tensor $\boldsymbol{e}$ under specific assumptions relative to the velocity field $\boldsymbol{v}$, where $\boldsymbol{e} = \frac{1}{2}(\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^t)$. While $\boldsymbol{v}$ and $\boldsymbol{e}$ have the specific interpretations that are germane to the present work, these lemmas are, of course, valid for any vector field $\boldsymbol{v}$ satisfying the hypotheses of the individual results, below, and any tensor $\boldsymbol{e}$ obtained as the symmetric part of the gradient of a sufficiently smooth vector field. We state only the lemmas in this subsection; a restatement of the results, with proofs, may be found in Appendix B. Our first two results are the following $\boldsymbol{L}^p$ and $\boldsymbol{H}^2$ versions, respectively, of the standard Korn inequality:

Lemma 5.1. *For* $\boldsymbol{v} \in \boldsymbol{W}_0^{1,p}(\Omega)$*,* $p > 1$*, and* Ω *a bounded domain in* $\mathbb{R}^n$*,* $n = 2, 3$*, with smooth boundary,* $\exists c_1 = c_1(p; \Omega) > 0$ *such that*

$$\int_\Omega [e_{ij}(\boldsymbol{v})e_{ij}(\boldsymbol{v})]^{p/2} d\boldsymbol{x} \geq c_1 \,||\boldsymbol{v}||^p_{\boldsymbol{W}^{1,p}(\Omega)}\,. \tag{5.20}$$

Remarks. Lemma 5.1 is a well-known result of Nečas [N1] which holds also in the space periodic case where $\Omega = [0, L]^n$, $L > 0$, $n = 2, 3$; it is identical with Lemma B.1.

Lemma 5.2. *Let $\boldsymbol{v} \in \boldsymbol{W}^{2,2}(\Omega) \cap \boldsymbol{W}_0^{1,2}(\Omega)$, $n = 2, 3$, $\Omega \subseteq \mathbb{R}^n$ a bounded domain with smooth boundary. Then $\exists c_2 = c_2(\Omega) > 0$ such that*

$$\int_\Omega \frac{\partial e_{ij}(\boldsymbol{v})}{\partial x_k} \frac{\partial e_{ij}(\boldsymbol{v})}{\partial x_k} d\boldsymbol{x} \geq c_2 \, ||\boldsymbol{v}||^2_{\boldsymbol{W}^{2,2}(\Omega)} \, . \tag{5.21}$$

Remarks. This result is identical with Lemma B.2; the proof in Appendix B establishes the existence of $c_3(\Omega) > 0$ such that for $\boldsymbol{v} \in \boldsymbol{W}^{2,2}(\Omega) \cap \boldsymbol{W}_0^{1,2}(\Omega)$, $\Omega \subseteq \mathbb{R}^n$, $n = 2, 3$,

$$\int_\Omega \frac{\partial e_{ij}(\boldsymbol{v})}{\partial x_k} \frac{\partial e_{ij}(\boldsymbol{v})}{\partial x_k} d\boldsymbol{x} \geq c_3 \sum_{k=1}^{n} \left|\left| \frac{\partial \boldsymbol{v}}{\partial x_k} \right|\right|^2_{\boldsymbol{L}^2(\Omega)} . \tag{5.22}$$

If (5.22) is valid, then (5.21) is a direct consequence of standard regularity results for elliptic equations.

Our third lemma in this section provides an alternative characterization of the higher-order boundary conditions in (5.3a), one that is better suited to some of the integration by parts computations in Sect. 5.2.3.

Lemma 5.3. *Let $S \subseteq \mathbb{R}^3$ be a smooth surface and $\boldsymbol{v}(\cdot)$ a divergence free C^2 vector field defined on a neighborhood of S, with $\boldsymbol{v} = \boldsymbol{0}$ on S. If $\tau_{ijk}(\boldsymbol{v})\nu_j\nu_k - \tau_{jkl}(\boldsymbol{v})\nu_j\nu_k\nu_l\nu_i|_S = 0$, for $i = 1, 2, 3$, where $\boldsymbol{\nu}$ is the exterior unit normal on S, then $\tau_{ijk}(\boldsymbol{v})e_{ij}(\boldsymbol{v})\nu_k|_S = 0$.*

Remarks. This lemma is the same as Lemma B.3.

Our next lemma in this section provides an elementary lower bound for the integral

$$\int_\Omega \boldsymbol{w} \cdot \boldsymbol{e}(\boldsymbol{U}) \cdot \boldsymbol{w} d\boldsymbol{x}, \ \boldsymbol{w} \in \boldsymbol{L}^2(\Omega) \tag{5.23}$$

and figures prominently in the analysis presented in the next subsection; it is identical with Lemma B.4.

Lemma 5.4. *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$. For any $\boldsymbol{w} \in \boldsymbol{L}^2(\Omega)$, and $\boldsymbol{U} \in \boldsymbol{W}^{1,2}(\Omega)$, $\exists \Lambda(\boldsymbol{U}) > 0$ such that*

$$||\boldsymbol{w}||^2_{\boldsymbol{L}^2(\Omega)} \geq -\left(\frac{1}{\Lambda}\right) \int_\Omega \boldsymbol{w} \cdot \boldsymbol{e}(\boldsymbol{U}) \cdot \boldsymbol{w} d\boldsymbol{x}. \tag{5.24}$$

Remarks. We note that all of the results in this section retain their validity, with obvious cosmetic modifications, if we replace the relevant real-valued $\boldsymbol{L}^p$ and Sobolev spaces by their corresponding complex-valued spaces, i.e., if for $w_i, U_i \in \mathbb{C}$ in (5.24) we write

$$\begin{cases} ||\boldsymbol{w}||^2_{\boldsymbol{L}^2(\Omega)} = \int_\Omega w_i w_i^* d\boldsymbol{x}, \\ \boldsymbol{w} \cdot \boldsymbol{e}(\boldsymbol{U}) \cdot \boldsymbol{w} = e_{ij}(\boldsymbol{U}) w_i w_j^* \end{cases}$$

where $e_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j^*}{\partial x_i}\right) = e_{ji}^*$, the asterisk denoting complex conjugation.

Our last lemma in the current sequence has been restated and proven as Lemma B.5.

Lemma 5.5. *Let $\boldsymbol{u}(t)$, $\boldsymbol{v}(t)$ be the unique solutions of* (5.2a,b)*,* (5.3a)*,* (5.4) *which correspond, respectively, to initial data $\boldsymbol{u}(0) = \boldsymbol{u}_0$ and $\boldsymbol{v}(0) = \boldsymbol{v}_0$. Then, for $1 < p \le 2$,*

$$\int_\Omega \left[\gamma(\boldsymbol{v})e_{ij}(\boldsymbol{v}) - \gamma(\boldsymbol{u})e_{ij}(\boldsymbol{u})\right]\left[e_{ij}(\boldsymbol{v}) - e_{ij}(\boldsymbol{u})\right] d\boldsymbol{x} \ge 0 \tag{5.25}$$

where $\gamma(\boldsymbol{v}) = \mu(|\boldsymbol{e}(\boldsymbol{v})|)$.

5.2.3 A Sufficient Condition for Linearized Stability

We are now in a position to establish the linearized stability of the solution $\boldsymbol{U}$ of the boundary-value problem (5.6)–(5.8). We begin with the following

Definition 5.2. An equilibrium solution $\boldsymbol{U}$ of the bipolar initial-boundary value problem is said to exhibit linearized stability if every solution $\boldsymbol{v}$ of the linearized problem (5.19), (5.3a), (5.4) which has the form

$$\boldsymbol{v}(\boldsymbol{x}, t) = \tilde{\boldsymbol{v}}(\boldsymbol{x})e^{-\sigma t}, \ \sigma \in \mathbb{C} \tag{5.26}$$

satisfies $\int_\Omega v_i(\boldsymbol{x}, t) v_i^*(\boldsymbol{x}, t) d\boldsymbol{x} \to 0$, as $t \to +\infty$.

We observe that (5.19) was obtained by projection onto the space of divergence free vector fields so any solution of (5.19) satisfies $\nabla \cdot \boldsymbol{v} = 0$.

If in (5.19) we set

$$\hat{\boldsymbol{A}} = \mu_1 \boldsymbol{A}_1 + \boldsymbol{A}_2 + \boldsymbol{R}(\boldsymbol{U}, \cdot) \tag{5.27}$$

so that (5.19) assumes the form

$$\frac{\partial \boldsymbol{v}}{\partial t} + \hat{\boldsymbol{A}}\boldsymbol{v} = \boldsymbol{0}, \quad t > 0 \tag{5.28}$$

then, in the usual manner, $\boldsymbol{v}$ (of the form (5.26)) is a solution of (5.28) if and only if $\tilde{\boldsymbol{v}}$ is an eigenvector of $\boldsymbol{A}$ with corresponding eigenvalue σ. As

$$\sigma = \sigma_r + i\sigma_c,\ \sigma_r, \sigma_c \in \mathbb{R} \tag{5.29}$$

it then follows that $\boldsymbol{U}$ exhibits linearized stability if and only if $\sigma_r = \operatorname{Re}\sigma > 0$ for σ any eigenvalue of $\hat{\boldsymbol{A}}$ with corresponding eigenvector $\tilde{\boldsymbol{v}}$. Our task, therefore, is to characterize the eigenvalues of $\hat{\boldsymbol{A}}$, i.e., to obtain a lower bound for $\operatorname{Re}\sigma$, the positivity of which will ensure the linearized stability of the equilibrium solution $\boldsymbol{U}$. For the remainder of this section our computations will be carried out using the inner-product $\langle\ ,\ \rangle$ in the complex-valued Hilbert space $\boldsymbol{L}_c^2(\Omega)$, where for $v_i, w_i \in \mathbb{C}$, $\langle \boldsymbol{v}, \boldsymbol{w}\rangle = \int_\Omega v_i w_i^* d\boldsymbol{x}$, so that $\langle \boldsymbol{v}, \boldsymbol{w}\rangle = (\tilde{\boldsymbol{v}}, \boldsymbol{w})_{\boldsymbol{L}_c^2(\Omega)}$ and $\sqrt{\langle \boldsymbol{v}, \boldsymbol{v}\rangle}$ will simply be denoted as $||\boldsymbol{v}||$. We are now in a position to state the following result concerning the linearized stability of equilibrium solutions of the incompressible bipolar equations:

Theorem 5.1. *Let $\boldsymbol{U}$ be an equilibrium solution of the incompressible bipolar boundary-value problem on $\Omega \subseteq \mathbb{R}^3$. Then $\exists \Sigma = \Sigma(\mu_0, \mu_1, \alpha; \Omega)$ such that for $\Sigma > 0$ any eigenvalue σ of $\hat{\boldsymbol{A}}$, as given by (5.27), satisfies*

$$\operatorname{Re}\sigma \geq \Sigma > 0 \tag{5.30}$$

and $\boldsymbol{U}$ is a linearly stable solution.

The proof of Theorem 5.1 will be preceded by a series of lemmas delineating the structure of the operators $\boldsymbol{A}_1$, $\boldsymbol{A}_2$ and $\boldsymbol{R}(\boldsymbol{U}, \cdot)$ in (5.27).

Lemma 5.6. *$\boldsymbol{A}_1$, as given by (5.17a) satisfies*

$$\langle \boldsymbol{A}_1\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{v}}\rangle = 2\int_\Omega \frac{\partial e_{ij}}{\partial x_k}\frac{\partial e_{ij}^*}{\partial x_k} d\boldsymbol{x}. \tag{5.31}$$

Proof. Substituting (5.26) into (5.28), and using the definition of $\hat{\boldsymbol{A}}$ in (5.27) we obtain

$$\mu_1 \boldsymbol{A}_1\tilde{\boldsymbol{v}} + \boldsymbol{A}_2\tilde{\boldsymbol{v}} + \boldsymbol{R}(\boldsymbol{U}, \tilde{\boldsymbol{v}}) = \sigma\tilde{\boldsymbol{v}}. \tag{5.32}$$

By taking the inner-product in $\boldsymbol{H}$ of (5.32) with $\tilde{\boldsymbol{v}}$ we find that

$$\mu_1 \langle \boldsymbol{A}_1\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{v}}\rangle + \langle \boldsymbol{A}_2\tilde{\boldsymbol{v}}, \boldsymbol{v}\rangle + \langle \boldsymbol{R}(\boldsymbol{U}, \tilde{\boldsymbol{v}}), \tilde{\boldsymbol{v}}\rangle = \sigma\, ||\boldsymbol{v}||^2. \tag{5.33}$$

Therefore,

$$\mu_1 \langle \boldsymbol{A}_1\tilde{\boldsymbol{v}}, \boldsymbol{v}\rangle + \langle \boldsymbol{A}_2\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{v}}\rangle + Re\,\langle \boldsymbol{R}(\boldsymbol{U}, \tilde{\boldsymbol{v}}), \tilde{\boldsymbol{v}}\rangle = \sigma_r\, ||\tilde{\boldsymbol{v}}||^2 \tag{5.34}$$

if we extract the real parts on both sides of (5.33). We now compute as follows: As $\boldsymbol{v} = \boldsymbol{0}$, on $\partial\Omega$,

$$\begin{aligned}
\langle \boldsymbol{A}_1\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{v}}\rangle &= 2\int_\Omega \frac{\partial}{\partial x_j}(\Delta e_{ij})\tilde{v}_i^* d\boldsymbol{x} \\
&= -2\int_\Omega \Delta e_{ij}\frac{\partial \tilde{v}_i^*}{\partial x_j} d\boldsymbol{x} \\
&= -2\int_\Omega \frac{\partial}{\partial x_k}\left(\frac{\partial e_{ij}}{\partial x_k}\frac{\partial \tilde{v}_i^*}{\partial x_j}\right) d\boldsymbol{x} + 2\int_\Omega \frac{\partial e_{ij}}{\partial x_k}\frac{\partial^2 \tilde{v}_i^*}{\partial x_k \partial x_j} d\boldsymbol{x} \\
&= -2\oint_{\partial\Omega} \tau_{ijk} e_{ij}^* \nu_k dS + 2\int_\Omega \frac{\partial e_{ij}}{\partial x_k}\frac{\partial e_{ij}^*}{\partial x_k} d\boldsymbol{x}
\end{aligned} \tag{5.35}$$

from which (5.31) follows if we apply Lemma 5.3 and the second boundary condition in (5.3a). □

Lemma 5.7. *The linear operator* $\boldsymbol{A}_2$*, as given by* (5.17b)*, satisfies*

$$\begin{aligned}
\langle \boldsymbol{A}_2\tilde{\boldsymbol{v}}, \boldsymbol{v}\rangle = &\int_\Omega \mu(\boldsymbol{e}(\boldsymbol{U}))e_{ij}(\tilde{\boldsymbol{v}})e_{ij}^*(\tilde{\boldsymbol{v}}) d\boldsymbol{x} \\
&- \alpha\int_\Omega \Gamma(\boldsymbol{e}(\boldsymbol{U}))e_{ij}(\boldsymbol{U})e_{kl}^*(\boldsymbol{U})e_{ij}(\tilde{\boldsymbol{v}}) d\boldsymbol{x}
\end{aligned} \tag{5.36}$$

where

$$\Gamma(e(\boldsymbol{U})) = 2\mu_0(\epsilon + |\boldsymbol{e}(\boldsymbol{U})|^2)^{-(1+\frac{\alpha}{2})}. \tag{5.37}$$

Proof. We compute, using (5.17b) that

$$\begin{aligned}
\langle \boldsymbol{A}_2\tilde{\boldsymbol{v}}, \boldsymbol{v}\rangle &= -\int_\Omega \frac{\partial}{\partial x_j}[\mu(\boldsymbol{e}(\boldsymbol{U}))e_{ij}(\tilde{\boldsymbol{v}}) - \alpha B_{ijkl}(\boldsymbol{U})e_{kl}(\tilde{\boldsymbol{v}})]\tilde{v}_i^* d\boldsymbol{x} \\
&= \int_\Omega \left[\mu(\boldsymbol{e}(\boldsymbol{U}))e_{ij}(\tilde{\boldsymbol{v}})\frac{\partial v_i^*}{\partial x_j} - \alpha B_{ijkl}(\boldsymbol{U})e_{kl}(\tilde{\boldsymbol{v}})\frac{\partial \tilde{v}_i^*}{\partial x_j}\right] d\boldsymbol{x}
\end{aligned} \tag{5.38}$$

so that

$$\langle \boldsymbol{A}_2\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{v}}\rangle = \int_\Omega [\mu(\boldsymbol{e}(\boldsymbol{U}))e_{ij}(\tilde{\boldsymbol{v}})e_{ij}^*(\tilde{\boldsymbol{v}}) - \alpha B_{ijkl}(\boldsymbol{U})e_{ij}(\tilde{\boldsymbol{v}})e_{kl}(\tilde{\boldsymbol{v}}) d\boldsymbol{x} \tag{5.39}$$

and (5.36) now follows from (5.39) by using the obvious modification of (5.15) for the complex-valued situation. □

Lemma 5.8. *Define* $\boldsymbol{R}(\boldsymbol{U},\cdot)$ *as in* (5.18)*; then*

$$\langle \boldsymbol{R}(\boldsymbol{U},\tilde{\boldsymbol{v}}),\tilde{\boldsymbol{v}}\rangle = \int_\Omega \tilde{\boldsymbol{v}}\cdot\nabla\boldsymbol{U}\cdot\tilde{\boldsymbol{v}}^*\,d\boldsymbol{x}. \tag{5.40}$$

Proof. Using (5.18) we compute, directly, that

$$\begin{aligned}\langle \boldsymbol{R}(\boldsymbol{U},\tilde{\boldsymbol{v}}),\tilde{\boldsymbol{v}}\rangle &= \int_\Omega\left(U_j\frac{\partial\tilde{v}_i}{\partial x_j}+\tilde{v}_j\frac{\partial U_i}{\partial x_j}\right)\tilde{v}_i^*\,d\boldsymbol{x}\\ &= \int_\Omega \tilde{\boldsymbol{v}}\cdot\nabla\boldsymbol{U}\cdot\tilde{\boldsymbol{v}}^*\,d\boldsymbol{x}+\int_\Omega U_j\frac{\partial\tilde{v}_i}{\partial x_j}\tilde{v}_i^*\,d\boldsymbol{x}\\ &= \int_\Omega \tilde{\boldsymbol{v}}\cdot\nabla\boldsymbol{U}\cdot\tilde{\boldsymbol{v}}^*\,d\boldsymbol{x}+\frac{1}{2}\int_\Omega U_j\frac{\partial}{\partial x_j}\,||\tilde{\boldsymbol{v}}||^2\,d\boldsymbol{x}.\end{aligned} \tag{5.41}$$

However,

$$\int_\Omega U_j\frac{\partial}{\partial x_j}\,||\tilde{\boldsymbol{v}}||^2\,d\boldsymbol{x} = \frac{1}{2}\oint_{\partial\Omega}U_j\,||\boldsymbol{v}||^2\,\nu_j\,dS-\frac{1}{2}\int_\Omega||\tilde{\boldsymbol{v}}||^2\,\nabla\cdot\boldsymbol{U}\,d\boldsymbol{x}=0$$

as $\tilde{\boldsymbol{v}}=0$ on $\partial\Omega$ and $\nabla\cdot\boldsymbol{U}=0$ on Ω. □

We are now in a position to establish (5.30).

Proof (Theorem 5.1). We set

$$I = \int_\Omega \tilde{v}_i\frac{\partial U_i}{\partial x_j}\tilde{v}_j^*\,d\boldsymbol{x} \tag{5.42}$$

in which case

$$I^* = \int_\Omega \tilde{v}_i^*\frac{\partial U_i^*}{\partial x_j}\tilde{v}_j\,d\boldsymbol{x} = \int_\Omega \tilde{v}_j^*\frac{\partial U_j^*}{\partial x_i}\tilde{v}_i\,d\boldsymbol{x} = \int_\Omega \tilde{v}_i\frac{\partial U_j^*}{\partial x_i}\tilde{v}_j^*\,d\boldsymbol{x}$$

so that

$$\operatorname{Re} I = \frac{1}{2}(I+I^*) \equiv \int_\Omega e_{ij}(\boldsymbol{U})\tilde{v}_i\tilde{v}_j^*\,d\boldsymbol{x}. \tag{5.43}$$

By virtue of (5.40) and (5.43) we have

$$\operatorname{Re}\langle \boldsymbol{R}(\boldsymbol{U},\tilde{\boldsymbol{v}}),\tilde{\boldsymbol{v}}\rangle = \int_\Omega e_{ij}(\boldsymbol{U})\tilde{v}_i\tilde{v}_j^*\,d\boldsymbol{x}. \tag{5.44}$$

Combining (5.31), (5.36), and (5.44) with (5.27), we see that

$$
\begin{aligned}
\left\langle \hat{\boldsymbol{A}}\,\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{v}} \right\rangle = 2\mu_1 \int_\Omega \frac{\partial e_{ij}}{\partial x_k} \frac{\partial e_{ij}^*}{\partial x_k} d\boldsymbol{x} \\
+ \int_\Omega \mu(\boldsymbol{e}(\boldsymbol{U})) e_{ij}(\tilde{\boldsymbol{v}}) e_{ij}^*(\tilde{\boldsymbol{v}}) d\boldsymbol{x} \\
- \alpha \int_\Omega \Gamma(\boldsymbol{e}(\boldsymbol{U})) e_{ij}(\boldsymbol{U}) e_{kl}^*(\boldsymbol{U}) e_{ij}(\tilde{\boldsymbol{v}}) e_{kl}^*(\tilde{\boldsymbol{v}}) d\boldsymbol{x} + \int_\Omega e_{ij}(\boldsymbol{U}) \tilde{v}_i \tilde{v}_j^* d\boldsymbol{x}.
\end{aligned}
\tag{5.45}
$$

Next, we apply the obvious complex-valued modifications of (5.21), in $\dim n = 3$, with $\boldsymbol{v} = \tilde{\boldsymbol{v}}$, and of (5.24), with $\boldsymbol{w} = \tilde{\boldsymbol{v}}$, to (5.45) so as to conclude that

$$
\begin{aligned}
\left\langle \hat{\boldsymbol{A}}\,\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{v}} \right\rangle \geq 2\mu_1 c_2(\Omega)\, ||\tilde{\boldsymbol{v}}||_{\boldsymbol{H}^2(\Omega)} - \Lambda\, ||\tilde{\boldsymbol{v}}||^2 + \int_\Omega \mu(\boldsymbol{e}(\boldsymbol{U})) e_{ij}(\tilde{\boldsymbol{v}}) e_{ij}^*(\tilde{\boldsymbol{v}}) d\boldsymbol{x} \\
- \alpha \int_\Omega \Gamma(\boldsymbol{e}(\boldsymbol{U})) e_{ij}(\boldsymbol{U}) e_{kl}^*(\boldsymbol{U}) e_{ij}(\tilde{\boldsymbol{v}}) e_{kl}^*(\tilde{\boldsymbol{v}}) d\boldsymbol{x}
\end{aligned}
\tag{5.46}
$$

with $c_2(\Omega) > 0$, $\Lambda(\boldsymbol{U}) > 0$. From (5.34) and (5.46) we obtain the estimate

$$
(\sigma_r + \Lambda)\, ||\tilde{\boldsymbol{v}}||^2 \geq 2\mu_1 c_2(\Omega)\, ||\tilde{\boldsymbol{v}}||^2_{\boldsymbol{H}^2(\Omega)} + J \tag{5.47}
$$

with

$$
J = \int_\Omega [\mu(\boldsymbol{e}(\boldsymbol{U})) e_{ij}(\tilde{\boldsymbol{v}}) e_{ij}^*(\tilde{\boldsymbol{v}}) - \alpha \Gamma(\boldsymbol{e}(\boldsymbol{U})) e_{ij}(\boldsymbol{U}) e_{kl}^*(\boldsymbol{U}) e_{ij}(\tilde{\boldsymbol{v}}) e_{kl}^*(\tilde{\boldsymbol{v}}) d\boldsymbol{x}. \tag{5.48}
$$

The basic task now is to obtain a reasonable lower bound for J; to this end, we first substitute for $\mu(\boldsymbol{e}(\boldsymbol{U}))$ and $\Gamma(\boldsymbol{e}(\boldsymbol{U}))$ in (5.48) so that

$$
J = \int_\Omega \left[\frac{2\mu_0 e_{ij}(\tilde{\boldsymbol{v}}) e_{ij}^*(\tilde{\boldsymbol{v}})}{(\epsilon + |\boldsymbol{e}(\boldsymbol{U})|^2)^{\frac{\alpha}{2}}} - \frac{2\alpha\mu_0 e_{ij}(\boldsymbol{U}) e_{kl}^*(\boldsymbol{U}) e_{ij}(\tilde{\boldsymbol{v}}) e_{kl}^*(\tilde{\boldsymbol{v}})}{(\epsilon + |\boldsymbol{e}(\boldsymbol{U})|^2)^{1+\frac{\alpha}{2}}} \right] d\boldsymbol{x} \tag{5.49}
$$

where $\alpha = 2 - p$ satisfies $0 < \alpha < 1$ so that $1 < p < 2$. We now note that

$$
\begin{aligned}
K &= \int_\Omega \left(\frac{e_{ij}(\boldsymbol{U}) e_{kl}^*(\boldsymbol{U}) e_{ij}(\tilde{\boldsymbol{v}}) e_{kl}^*(\tilde{\boldsymbol{v}})}{(\epsilon + |\boldsymbol{e}(\boldsymbol{U})|^2)^{1+\frac{\alpha}{2}}} \right) d\boldsymbol{x} \\
&= \int_\Omega \left[\frac{e_{ij}(\boldsymbol{U}) e_{kl}^*(\tilde{\boldsymbol{v}})}{(\epsilon + |\boldsymbol{e}(\boldsymbol{U})|^2)^{\frac{1}{2}+\frac{\alpha}{4}}} \right] \left[\frac{e_{ij}(\tilde{\boldsymbol{v}}) e_{kl}^*(\boldsymbol{U})}{(\epsilon + |\boldsymbol{e}(\boldsymbol{U})|^2)^{\frac{1}{2}+\frac{\alpha}{4}}} \right] d\boldsymbol{x} \\
&\leq \left(\int_\Omega \left[\frac{e_{ij}(\boldsymbol{U}) e_{kl}^*(\tilde{\boldsymbol{v}}) e_{ij}^*(\boldsymbol{U}) e_{kl}(\tilde{\boldsymbol{v}})}{(\epsilon + |\boldsymbol{e}(\boldsymbol{U})|^2)^{1+\frac{\alpha}{2}}} \right] d\boldsymbol{x} \right)^{\frac{1}{2}} \times \left(\int_\Omega \left[\frac{e_{ij}(\tilde{\boldsymbol{v}}) e_{kl}^*(\boldsymbol{U}) e_{ij}^*(\tilde{\boldsymbol{v}}) e_{kl}(\boldsymbol{U})}{(\epsilon + |\boldsymbol{e}(\boldsymbol{U})|^2)^{1+\frac{\alpha}{2}}} \right] d\boldsymbol{x} \right)^{\frac{1}{2}} \\
&= \int_\Omega \left[\frac{e_{ij}(\boldsymbol{U}) e_{ij}^*(\boldsymbol{U}) e_{kl}(\tilde{\boldsymbol{v}}) e_{kl}^*(\tilde{\boldsymbol{v}})}{(\epsilon + |\boldsymbol{e}(\boldsymbol{U})|^2)^{1+\frac{\alpha}{2}}} \right] d\boldsymbol{x}
\end{aligned}
$$

or

$$K \leq \int_{\Omega} \frac{|\boldsymbol{e}(\boldsymbol{U})|^2 |\boldsymbol{e}(\tilde{\boldsymbol{v}})|^2}{(\epsilon + |\boldsymbol{e}(\boldsymbol{U})|^2)^{1+\frac{\alpha}{2}}} d\boldsymbol{x}. \tag{5.50}$$

Employing the estimate (5.50) in (5.49) we find that

$$\begin{aligned} J &\geq \int_{\Omega} \left[\frac{2\mu_0(\epsilon + |\boldsymbol{e}(\boldsymbol{U})|^2)|\boldsymbol{e}(\tilde{\boldsymbol{v}})|^2 - 2\alpha\mu_0 |\boldsymbol{e}(\boldsymbol{U})|^2 |\boldsymbol{e}(\tilde{\boldsymbol{v}})|^2}{(\epsilon + |\boldsymbol{e}(\boldsymbol{U})|^2)^{1+\frac{\alpha}{2}}} \right] d\boldsymbol{x} \\ &= 2\epsilon\mu_0 \int_{\Omega} \frac{|\boldsymbol{e}(\tilde{\boldsymbol{v}})|^2}{(\epsilon + |\boldsymbol{e}(\boldsymbol{U})|^2)^{1+\frac{\alpha}{2}}} d\boldsymbol{x} \\ &\quad + 2(1-\alpha)\mu_0 \int_{\Omega} \frac{|\boldsymbol{e}(\boldsymbol{U})|^2 |\boldsymbol{e}(\tilde{\boldsymbol{v}})|^2}{(\epsilon + |\boldsymbol{e}(\boldsymbol{U})|^2)^{1+\frac{\alpha}{2}}} d\boldsymbol{x}. \end{aligned} \tag{5.51}$$

The use of (5.51) in (5.47) now produces the lower bound

$$\begin{aligned} (\sigma_r + \Lambda)\,||\tilde{\boldsymbol{v}}||^2 \geq 2\mu_1 c_2(\Omega)\,||\tilde{\boldsymbol{v}}||^2_{\boldsymbol{H}^2(\Omega)} \\ + 2\epsilon\mu_0 \int_{\Omega} \frac{|\boldsymbol{e}(\tilde{\boldsymbol{v}})|^2}{(\epsilon + |\boldsymbol{e}(\boldsymbol{U})|^2)^{1+\frac{\alpha}{2}}} d\boldsymbol{x} \\ + 2(1-\alpha)\mu_0 \int_{\Omega} \frac{|\boldsymbol{e}(\boldsymbol{U})|^2 |\boldsymbol{e}(\tilde{\boldsymbol{v}})|^2}{(\epsilon + |\boldsymbol{e}(\boldsymbol{U})|^2)^{1+\frac{\alpha}{2}}} d\boldsymbol{x}. \end{aligned} \tag{5.52}$$

We now add and subtract the expression

$$2(1-\alpha)\epsilon\mu_0 \int_{\Omega} \frac{|\boldsymbol{e}(\tilde{\boldsymbol{v}})|^2}{(\epsilon + |\boldsymbol{e}(\boldsymbol{U})|^2)^{1+\frac{\alpha}{2}}} d\boldsymbol{x}$$

on the right-hand side of (5.52), rearrange terms, and obtain the estimate

$$\begin{aligned} (\sigma_r + \Lambda)\,||\tilde{\boldsymbol{v}}||^2 \geq 2\mu_1 c_2(\Omega)\,||\tilde{\boldsymbol{v}}||^2_{\boldsymbol{H}^2(\Omega)} \\ + 2\epsilon\mu_0[1-(1-\alpha)] \int_{\Omega} \frac{|\boldsymbol{e}(\tilde{\boldsymbol{v}})|^2}{(\epsilon + |\boldsymbol{e}(\boldsymbol{U})|^2)^{1+\frac{\alpha}{2}}} d\boldsymbol{x} \\ + 2(1-\alpha)\mu_0 \int_{\Omega} \frac{(\epsilon + |\boldsymbol{e}(\boldsymbol{U})|^2)|\boldsymbol{e}(\tilde{\boldsymbol{v}})|^2}{(\epsilon + |\boldsymbol{e}(\boldsymbol{U})|^2)^{1+\frac{\alpha}{2}}} d\boldsymbol{x} \end{aligned}$$

or, as $0 < \alpha < 1$,

$$(\sigma_r + \Lambda)\,||\tilde{\boldsymbol{v}}||^2 \geq 2\mu_1 c_2(\Omega)\,||\tilde{\boldsymbol{v}}||^2_{\boldsymbol{H}^2(\Omega)} + 2(1-\alpha)\mu_0 \int_{\Omega} \frac{|\boldsymbol{e}(\tilde{\boldsymbol{v}})|^2}{(\epsilon + |\boldsymbol{e}(\boldsymbol{U})|^2)^{\alpha/2}} d\boldsymbol{x}. \tag{5.53}$$

By Lemma 5.14 (5.3.5), with $\boldsymbol{\phi} = \tilde{\boldsymbol{v}}$ and $\boldsymbol{u} = \boldsymbol{U}$, we obtain from (5.53) the lower bound

$$(\sigma_r + \Lambda)\,||\tilde{\boldsymbol{v}}||^2 \geq 2\mu_1 c_2(\Omega)\,||\tilde{\boldsymbol{v}}||^2_{\boldsymbol{H}^2(\Omega)} + 2(1-\alpha)\mu_0 c_4(\Omega)\,||\boldsymbol{v}||^2_{\boldsymbol{W}^{1,p}(\Omega)}\,. \tag{5.54}$$

However, by Lemma A.6 with $\delta' = 2\left(\frac{2-\alpha}{4+\alpha}\right)$, we have the existence of $d_{\delta'}(\Omega) > 0$ such that, for any $\zeta > 0$,

$$||\boldsymbol{v}||^2_{\boldsymbol{W}^{1,p}(\Omega)} \geq \frac{\zeta^{1/\delta'}}{\delta' d_{\delta'}}\,||\boldsymbol{v}||^2_{\boldsymbol{H}^1(\Omega)} - \zeta^{\frac{1}{\delta'(1-\delta')}}\left(\frac{1-\delta'}{\delta'}\right)||\boldsymbol{v}||^2_{\boldsymbol{H}^2(\Omega)}$$

and, thus

$$(\sigma_r + \Lambda)\,||\boldsymbol{v}||^2 \geq 2\left[\mu_1 c_2 - (1-\alpha)\mu_0 c_4 \zeta^{\frac{1}{\delta'(1-\delta')}}\left(\frac{1-\delta'}{\delta'}\right)\right]||\boldsymbol{v}||^2_{\boldsymbol{H}^2(\Omega)} + \left(\frac{2(1-\alpha)\mu_0 c_4 \zeta^{1/\delta'}}{\delta' d_{\delta'}}\right)||\boldsymbol{v}||^2_{\boldsymbol{H}^1(\Omega)}\,.$$

Therefore, for $\zeta > 0$ chosen sufficiently small,

$$(\sigma_r + \Lambda)\,||\boldsymbol{v}||^2 \geq \left(\frac{2(1-\alpha)\mu_0 c_4 \zeta^{1/\delta'}}{\delta' d_{\delta'}}\right)||\boldsymbol{v}||^2_{\boldsymbol{H}^1(\Omega)}\,. \tag{5.55}$$

We note that, in general, $\zeta = \zeta(\mu_1, \mu_1\alpha; \Omega)$. At this point we appeal to the equivalence of the $\boldsymbol{H}^1$ seminorm and the $\boldsymbol{W}^{1,2}$ norm, for $\boldsymbol{v} \in \boldsymbol{W}^{1,2}_0(\Omega)$, and then apply the Poincaré inequality so as to deduce the existence of $\kappa(\Omega) > 0$ such that

$$||\tilde{\boldsymbol{v}}||^2 \leq \kappa(\Omega)\,||\tilde{\boldsymbol{v}}||^2_{\boldsymbol{H}^1(\Omega)}\,. \tag{5.56}$$

From (5.55), (5.56) we have

$$\left(\sigma_r + \Lambda - \frac{2(1-\alpha)\mu_0 c_4 \kappa \zeta^{1/\delta'}}{\delta' d_{\delta'}}\right)||\tilde{\boldsymbol{v}}||^2 \geq 0 \tag{5.57}$$

or

$$\sigma_r \geq \frac{2(1-\alpha)\mu_0 c_4 \kappa \zeta^{1/\delta'}}{\delta' d_{\delta'}} - \Lambda. \tag{5.58}$$

The theorem now follows with $\Sigma(\mu_0, \mu_1, \alpha_1; \Omega)$ given by

$$\Sigma = \frac{2(1-\alpha)\mu_0 c_4 \kappa \zeta^{1/\delta'}}{\delta' d_{\delta'}} - \Lambda, \tag{5.59}$$

i.e., if $\Sigma > 0$, then $\sigma_r > 0$, and all solutions of the linearized bipolar equations of the form (5.26) decay to zero, in the norm on $\boldsymbol{L}_c^2(\Omega)$, as $t \to +\infty$. □

5.3 Bounds for the Dimensions of Attractors for Nonlinear Bipolar Fluid Flow ($0 \leq \alpha < 1$)

In this section, we begin our study of the problem of the existence of maximal compact global attractors for the dynamical systems generated by the initial-boundary value and space-period problems associated with the motion of nonlinear bipolar and non-Newtonian fluids; we begin with the nonlinear bipolar fluid ($\mu_1 > 0$) for which the initial-boundary value problem is given by (5.2a,b), (5.3a), (5.4) while the space periodic problem is defined by (5.2a,b), (5.3b), (5.4). It will be seen that the maximal global attractor has the form $\boldsymbol{\mathcal{A}}_{\mu_1} = \bigcap_{t \geq 0} \boldsymbol{S}_{\mu_1}(t) B^{\rho'}_{\boldsymbol{H}^2(\Omega)}$, where $\boldsymbol{S}_{\mu_1}(t)$ is the relevant nonlinear semigroup and $B^{\rho'}_{\boldsymbol{H}^2(\Omega)}$ (the ball of radius $\rho' > 0$ in $\boldsymbol{H}^2(\Omega)$) is, for $\rho' > 0$ sufficiently large, what will be defined to be an absorbing set in $\boldsymbol{H}^2(\Omega)$. One of the highlights of this section will be the derivation of upper bounds for the Hausdorff and Fractal dimensions of the attractor $\boldsymbol{\mathcal{A}}_{\mu_1}$. Throughout this section we will assume that $0 \leq \alpha < 1$ so that, with $p = 2 - \alpha$, $1 < p \leq 2$. We also set $\rho = 1$ in the bipolar equations.

5.3.1 Some Preliminary Concepts

Throughout the remainder of this section the Hilbert spaces $\boldsymbol{H}$ and $\boldsymbol{H}^2_{per}$ will be as defined in Sect. 5.1. Furthermore for $\Omega \subseteq \mathbb{R}^n$, $n = 2, 3$ we will set

$$\boldsymbol{V} = \{\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \big| \nabla \cdot \boldsymbol{v} = 0\} \tag{5.60}$$

and

$$\boldsymbol{A}\boldsymbol{v} = -\boldsymbol{\mathcal{P}}\Delta \boldsymbol{v}, \;\; \boldsymbol{v} \in \boldsymbol{H}^2(\Omega) \tag{5.61}$$

where $\boldsymbol{\mathcal{P}}$ is the orthogonal projection operator from $\boldsymbol{L}^2(\Omega)$ onto $\boldsymbol{H}$. In addition, we have

$$\boldsymbol{B}(\boldsymbol{u}, \boldsymbol{v}) = \boldsymbol{\mathcal{P}}(\boldsymbol{u} \cdot \nabla \boldsymbol{v}) \text{ and } \boldsymbol{B}(\boldsymbol{v}) = \boldsymbol{B}(\boldsymbol{v}, \boldsymbol{v}). \tag{5.62}$$

For the initial-boundary value problem (5.2a,b), (5.3a), (5.4), we have already noted, in Sect. 5.1, that as a consequence of the results in [BBN4] a unique solution

$$\boldsymbol{v} \in L^\infty([0, \infty; \boldsymbol{H}) \cap L^\infty((t_0, \infty); \boldsymbol{H}^2(\Omega))$$

exists. Thus, the solution operators $\boldsymbol{S}_{\mu_1}(t) : \boldsymbol{v}_0 \to \boldsymbol{v}(t)$ yield a nonlinear semigroup of operators which enjoys the properties

$$\boldsymbol{S}_{\mu_1}(t+s) = \boldsymbol{S}_{\mu_1}(t) \cdot \boldsymbol{S}_{\mu_1}(s), \ \forall s, t \geq 0, \tag{5.63a}$$

$$\boldsymbol{S}_{\mu_1}(0) = \boldsymbol{I} \tag{5.63b}$$

with

$$\begin{gathered} \boldsymbol{S}_{\mu_1}(t) \text{ a continuous nonlinear operator} \\ \text{from } \boldsymbol{H} \text{ into itself, for any } \mu_1 > 0, \forall t \geq 0. \end{gathered} \tag{5.63c}$$

Our basic goal in this section is to study the behavior, as $t \to +\infty$, of the orbits of the semigroups $\boldsymbol{S}_{\mu_1}(\cdot)$.

In Sect. 5.2 we studied the linearized stability of an equilibrium solution $\boldsymbol{U}$ of the bipolar initial-boundary value problem, with zero body force, where $\boldsymbol{U}$ satisfies (5.6)–(5.8). By letting $\boldsymbol{v}(\boldsymbol{x}, t) = \boldsymbol{u}(\boldsymbol{x}, t) - \boldsymbol{U}(\boldsymbol{x})$, with $\boldsymbol{u}$ any solution of (5.2a,b), (5.3a), (5.4), we found, as the linearized equations for the v_i (5.16), where $P = p - \tilde{p}$ is the difference of the pressures associated with $\boldsymbol{u}$ and $\boldsymbol{U}$, and $B_{ijkl}(\boldsymbol{U})$ is given by (5.15) with $\boldsymbol{w} \to \boldsymbol{U}$.

In the present section we will let $(\boldsymbol{v}, p)$ be a solution of the initial-boundary value problem (5.2a,b), (5.3a), (5.4) and $(\boldsymbol{v}^*, p^*)$ any other solution corresponding to the same body force density $\boldsymbol{f}$. Now, we set

$$\boldsymbol{U}(\boldsymbol{x}, t) = \boldsymbol{v}^*(\boldsymbol{x}, t) - \boldsymbol{v}(\boldsymbol{x}, t), \tag{5.64a}$$

$$\boldsymbol{P}(\boldsymbol{x}, t) = p^*(\boldsymbol{x}, t) - p(\boldsymbol{x}, t). \tag{5.64b}$$

Using the abbreviated notation

$$\gamma(\boldsymbol{v}) = \mu(\boldsymbol{e}(\boldsymbol{v})) \tag{5.65}$$

we obtain for the linearization of the system of bipolar fluid equations, about the solution $(\boldsymbol{v}, p)$, the system

$$\begin{aligned} \frac{\partial U_i}{\partial t} + U_j \frac{\partial v_i}{\partial x_j} + v_j \frac{\partial U_i}{\partial x_j} = -\frac{\partial p_U}{\partial x_i} + \frac{\partial}{\partial x_j}[\gamma(\boldsymbol{v}) e_{ij}(\boldsymbol{U}) - \alpha B_{ijkl}(\boldsymbol{v}) e_{kl}(\boldsymbol{U})] \\ - 2\mu_1 \frac{\partial}{\partial x_j}(\Delta e_{ij}(\boldsymbol{U})) \end{aligned} \tag{5.66a}$$

on $\partial\Omega \times [0, T)$, with

$$\nabla \cdot \boldsymbol{U} = 0, \text{ in } \Omega \times [0, T), \tag{5.66b}$$

$$\boldsymbol{U}(\boldsymbol{x}, 0) = \boldsymbol{v}^*(\boldsymbol{x}, 0) - \boldsymbol{v}(\boldsymbol{x}, 0) \equiv \boldsymbol{U}_0(\boldsymbol{x}), \text{ in } \Omega \tag{5.66c}$$

and

$$U_i = 0,\ \tau_{ijk}(\boldsymbol{U})\nu_j\nu_k\tau_i = 0, \text{ on } \partial\Omega \times [0, T). \tag{5.66d}$$

Remarks. We note that B_{ijkl}, as defined by (5.15) results from the computation of the Fréchet derivative of the nonlinear viscosity. Additionally, if we assume that $e_{ij} = e^*_{ij}$, then by a judicious application of the Cauchy-Schwarz inequality to the integral

$$\hat{I} \equiv \int_\Omega \frac{e_{ij}(\boldsymbol{U})e_{kl}(\boldsymbol{U})e_{ij}(\boldsymbol{v})e_{kl}(\boldsymbol{v})}{(\epsilon + e_{mn}(\boldsymbol{v})e_{mn}(\boldsymbol{v}))^{1+\alpha/2}} d\boldsymbol{x} \tag{5.67}$$

we obtain, in a manner entirely similar to that which yielded the lower bound (5.51) for the integral J in (5.48), the estimate

$$\begin{aligned}\int_\Omega [\gamma(\boldsymbol{v})e_{ij}(\boldsymbol{U})e_{ij}(\boldsymbol{U}) - \alpha B_{ijkl}(\boldsymbol{v})e_{ij}(\boldsymbol{U})e_{kl}(\boldsymbol{U})]d\boldsymbol{x} \\ \geq 2\epsilon\alpha\mu_0 \int_\Omega \frac{e_{ij}(\boldsymbol{U})e_{ij}(\boldsymbol{U})}{(\epsilon + e_{kl}(\boldsymbol{v})e_{kl}(\boldsymbol{v}))^{1+\alpha/2}} d\boldsymbol{x} \\ + 2(1-\alpha)\mu_0 \int_\Omega \frac{e_{ij}(\boldsymbol{U})e_{ij}(\boldsymbol{U})}{(\epsilon + e_{kl}(\boldsymbol{v})e_{kl}(\boldsymbol{v}))^{\alpha/2}} d\boldsymbol{x}\end{aligned} \tag{5.68}$$

for all $\epsilon, \mu_0 \geq 0$, $0 \leq \alpha < 1$. Finally, we note that, when it exists, the Fréchet differential of the semigroup of solution operators $\boldsymbol{S}_{\mu_1}(t) : \boldsymbol{v}_0 \to \boldsymbol{v}(t)$, generated by (5.2a,b), (5.3a), (5.4), at $\boldsymbol{v}_0$ is the mapping

$$\boldsymbol{L}_{\mu_1}(t, \boldsymbol{v}_0) : \boldsymbol{\xi} \to \boldsymbol{U}(t) \tag{5.69}$$

where $\boldsymbol{U}$ is the solution of the linearized problem (5.66a–d) with $\boldsymbol{U}(0) = \boldsymbol{\xi}$, and with $\boldsymbol{v}$ in (5.66a) given by $\boldsymbol{S}_{\mu_1}(t)\boldsymbol{v}_0$.

Concerning the existence and uniqueness of solutions for the linearized initial-boundary value problem (5.66a–d), we have the following theorem which may be inferred from the results in Sect. 4.2:

Theorem 5.2. *Choose $\boldsymbol{v}_0$ and $\boldsymbol{U}_0$ to be in $\boldsymbol{L}^2(\Omega)$ and divergence free; then there exists a unique solution $\boldsymbol{U}(t)$ of* (5.66a–d) *such that,* $\forall t_1 > 0$,

$$\boldsymbol{U} \in L^\infty([t_1, \infty); \boldsymbol{H}) \cap L^\infty([t_1, \infty); \boldsymbol{H}^2(\Omega)).$$

In what follows, we will need to talk of solutions $\boldsymbol{U}$ to the linearized system only for the situation where we linearize about a solution $\boldsymbol{v}(t)$ of (5.2a,b), (5.3a), (5.4) for which the initial data $\boldsymbol{v}_0$ is chosen in a compact subset $X \subset \boldsymbol{H}$ which is invariant with respect to the nonlinear semigroup $\boldsymbol{S}_{\mu_1}(t)$, $\forall t \geq 0$; in this case it is easily shown that the unique solution of (5.66a–d), corresponding to a choice of $\boldsymbol{U}_0 \in H$, satisfies the stronger result

$$\boldsymbol{U}(t) \in L^\infty([0,\infty); \boldsymbol{H}) \cap L^\infty([0,\infty); \boldsymbol{H}^2(\Omega)).$$

We conclude this section with a few well-known definitions (see, e.g., [Te4]).

Definition 5.3. An attractor for the semigroup $\boldsymbol{S}_{\mu_1}(t)$ is a set $\boldsymbol{\mathcal{A}}_{\mu_1} \subset \boldsymbol{H}$ such that

(i) $\boldsymbol{\mathcal{A}}_{\mu_1}$ is a functionally invariant set of the semigroup $\boldsymbol{S}_{\mu_1}(t)$, i.e., $\boldsymbol{S}_{\mu_1}(t)\boldsymbol{\mathcal{A}}_{\mu_1} = \boldsymbol{\mathcal{A}}_{\mu_1}$, $\forall t \geq 0$.
(ii) $\boldsymbol{\mathcal{A}}_{\mu_1}$ has an open neighborhood Ω such that for every $\boldsymbol{v}_0 \in \Omega$, $\boldsymbol{S}_{\mu_1}(t)\boldsymbol{v}_0$ converges to $\boldsymbol{\mathcal{A}}_{\mu_1}$ as $t \to +\infty$, i.e., $\text{dist}(\boldsymbol{S}_{\mu_1}(t)\boldsymbol{v}_0, \boldsymbol{\mathcal{A}}_{\mu_1}) \to 0$ as $t \to +\infty$ where

$$d(\boldsymbol{x}, \boldsymbol{\mathcal{A}}_{\mu_1}) = \inf_{\boldsymbol{y} \in \boldsymbol{\mathcal{A}}_{\mu_1}} d(\boldsymbol{x}, \boldsymbol{y}),$$

$d(\boldsymbol{x}, \boldsymbol{y})$ being the distance from $\boldsymbol{x}$ to $\boldsymbol{y}$ in $\boldsymbol{H}$.

Definition 5.4. We call $\boldsymbol{\mathcal{A}}_{\mu_1} \subset \boldsymbol{H}$ the global (or universal) attractor for the semigroup $\boldsymbol{S}_{\mu_1}(t)$ if $\boldsymbol{\mathcal{A}}_{\mu_1}$ is a compact attractor that attracts the bounded sets of $\boldsymbol{H}$; the basin of attraction of $\boldsymbol{\mathcal{A}}_{\mu_1}$ is then said to be all of $\boldsymbol{H}$.

Definition 5.5. Let B be a subset of $\boldsymbol{H}$ and Ω an open set containing B. Then B is said to be absorbing in Ω if for every bounded set $B_0 \subset \Omega$, $\exists t_0(B_0)$ such that $\boldsymbol{S}_{\mu_1}(t)B_0 \subset B$, $\forall t \geq t_0(B_0)$.

In order to prove the existence of the global attractor for the semigroup $\boldsymbol{S}_{\mu_1}(t)$ generated by (5.66a–d) we will establish in the next section the existence of an absorbing set $B^\rho_{\boldsymbol{H}}$ in all of $\boldsymbol{H}$ and then use that result to establish the existence of an absorbing set $B^{\rho'}_{\boldsymbol{H}^2(\Omega)}$, where $B^\rho_{\boldsymbol{H}}$ is the (closed) ball in $\boldsymbol{H}$ of radius $\rho > 0$ with a similar interpretation for $B^{\rho'}_{\boldsymbol{H}^2(\Omega)}$; the existence of the global attractor $\boldsymbol{\mathcal{A}}_{\mu_1}$ in the form

$$\boldsymbol{\mathcal{A}}_{\mu_1} = \bigcap_{t>0} \boldsymbol{S}_{\mu_1}(t) B^{\rho'}_{\boldsymbol{H}^2(\Omega)} \tag{5.70}$$

will then follow from a theorem in [CF]. Alternatively, we could take $\boldsymbol{\mathcal{A}}_{\mu_1}$ to be the ω-limit set of the absorbing set $B^\rho_{\boldsymbol{H}}$ as in [Te4]; in this latter approach we view the existence of the absorbing set $B^{\rho'}_{\boldsymbol{H}^2(\Omega)}$ as establishing the uniform compactness of the operators $\boldsymbol{S}_{\mu_1}(t)$ for t large. We remark that all of the calculations below may be made entirely rigorous by making use of the existence and uniqueness results of Sect. 4.2 and standard density arguments.

5.3.2 Absorbing Sets in $\boldsymbol{H}$ and $H^2(\Omega)$ and Existence of a Global Compact Attractor $\mathcal{A}_{\mu_1}$

The existence of an absorbing set in $\boldsymbol{H}$ for solutions of (5.66a–d) is the essential content of

Lemma 5.9. *For $\mu_1 > 0$, and $0 \le \alpha < 1$, let $\boldsymbol{v}(t)$ be the unique solution of the initial-boundary value problem* (5.2a,b), (5.3a), (5.4) *where, without loss of generality, we set $\rho = 1$. Then, $\exists t_0' > 0$, $t_0' = t_0'(||\boldsymbol{v}_0||_{L^2(\Omega)})$, $\beta > 0$, and $\lambda > 0$, such that for $t \ge t_0'$,*

$$||\boldsymbol{v}(t)||_{L^2(\Omega)} \le \frac{2\beta}{\lambda}\,||\boldsymbol{f}||_\infty \equiv \hat{\rho} \tag{5.71}$$

where $|\boldsymbol{f}|_\infty = \sup_{[0,\infty)} ||\boldsymbol{f}||_{L^2(\Omega)} \equiv ||\boldsymbol{f}||_{L^\infty([0,\infty);L^2(\Omega))}$.

Proof. We write (5.2a) in component form, with $\rho = 1$, as

$$\frac{\partial v_i}{\partial t} + v_j\frac{\partial v_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}(\gamma(\boldsymbol{v})e_{ij}) - 2\mu_1\frac{\partial}{\partial x_j}(\Delta e_{ij}) + f_i \tag{5.72}$$

where $\gamma(\boldsymbol{v}) = \mu(|\boldsymbol{e}(v)|)$. Our first step is to multiply (5.72) through by $v_i(\boldsymbol{x}, t)$ and integrate over Ω so as to obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega v_i v_i\,d\boldsymbol{x} = \int_\Omega v_i\frac{\partial}{\partial x_j}(\gamma(\boldsymbol{v})e_{ij})\,d\boldsymbol{x} - 2\mu_1\int_\Omega v_i\frac{\partial}{\partial x_j}(\Delta e_{ij})\,d\boldsymbol{x} + \int_\Omega f_i v_i\,d\boldsymbol{x}. \tag{5.73}$$

In obtaining (5.73) we use the fact that

$$\int_\Omega v_j\frac{\partial v_i}{\partial x_j}v_i\,d\boldsymbol{x} = \int_\Omega v_i\frac{\partial p}{\partial x_i}\,d\boldsymbol{x} = 0 \tag{5.74}$$

both of which follow by integration by parts coupled with the fact that $v_i = 0$ on $\partial\Omega \times [0, T)$, as well as div $\boldsymbol{v} = 0$ in $\Omega \times [0, T)$. Now

$$\int_\Omega v_i\frac{\partial}{\partial x_j}(\gamma(\boldsymbol{v})e_{ij})\,d\boldsymbol{x} = -\int_\Omega \gamma(\boldsymbol{v})e_{ij}e_{ij}\,d\boldsymbol{x} \tag{5.75}$$

while two successive integrations by parts applied to the second integral on the right-hand side of (5.73) yields

$$\int_\Omega v_i\frac{\partial}{\partial x_j}(\Delta e_{ij})\,d\boldsymbol{x} = \int_\Omega \frac{\partial e_{ij}}{\partial x_k}\frac{\partial e_{ij}}{\partial x_k}\,d\boldsymbol{x}. \tag{5.76}$$

In obtaining (5.76), we actually make use of the fact that

$$\int_{\partial\Omega} \tau_{ijk} e_{ij} v_k \, dS = 0 \tag{5.77}$$

which is a direct consequence of Lemma 5.3. Combining (5.72), (5.74) and (5.76) we find that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\,||\boldsymbol{v}||^2_{\boldsymbol{L}^2(\Omega)} + \int_\Omega \gamma(\boldsymbol{v}) e_{ij} e_{ij}\, d\boldsymbol{x} + 2\mu_1 \int_\Omega \frac{\partial e_{ij}}{\partial x_k}\frac{\partial e_{ij}}{\partial x_k}\, d\boldsymbol{x} \\ &\quad = \int_\Omega f_i v_i \, d\boldsymbol{x} \le ||\boldsymbol{f}||_{\boldsymbol{L}^2(\Omega)}\, ||\boldsymbol{v}||_{\boldsymbol{L}^2(\Omega)} \le |\boldsymbol{f}|_\infty \, ||\boldsymbol{v}||_{\boldsymbol{L}^2(\Omega)} \end{aligned} \tag{5.78}$$

with

$$|\boldsymbol{f}|_\infty = \sup_{[0,\infty)} ||\boldsymbol{f}||_{\boldsymbol{L}^2(\Omega)} \equiv ||\boldsymbol{f}||_{L^\infty([0,\infty);\boldsymbol{L}^2(\Omega))}\,. \tag{5.79}$$

Now for $\boldsymbol{v}_0 \in \boldsymbol{H}$, our results in Sect. 4.2 guarantee the existence of a $t_0 = t_0(||\boldsymbol{v}_0||_{\boldsymbol{L}^2(\Omega)})$ such that $\boldsymbol{v} \in L^\infty([t_0,\infty);\boldsymbol{H}^2(\Omega))$; for $t \ge t_0$, therefore, we may apply the extension of the Korn inequality expressed by Lemma 5.2 so as to conclude that for $t \ge t_0(||\boldsymbol{v}_0||_{L^2(\Omega)})$

$$\frac{1}{2}\frac{d}{dt}\,||\boldsymbol{v}||^2_{\boldsymbol{L}^2(\Omega)} + 2\mu_1 k(\Omega)\, ||\boldsymbol{v}||^2_{\boldsymbol{H}^2(\Omega)} \le |\boldsymbol{f}|_\infty \, ||\boldsymbol{v}||_{\boldsymbol{L}^2(\Omega)}$$

in which case, for any $\beta > 0$,

$$\frac{1}{2}\frac{d}{dt}\,||\boldsymbol{v}||^2_{L^2(\Omega)} + 2\mu_1 k(\Omega)\, ||\boldsymbol{v}||^2_{\boldsymbol{L}^2(\Omega)} \le \frac{\beta}{2}|\boldsymbol{f}|^2_\infty + \frac{2}{\beta}\, ||\boldsymbol{v}||^2_{\boldsymbol{L}^2(\Omega)}\,. \tag{5.80}$$

Therefore, for $\beta > 0$ sufficiently large, we see that there exists $\lambda(\mu_1, k(\Omega)) > 0$ such that

$$\frac{d}{dt}\,||\boldsymbol{v}||^2_{\boldsymbol{L}^2(\Omega)} + \lambda\, ||\boldsymbol{v}||^2_{\boldsymbol{L}^2(\Omega)} \le \beta|\boldsymbol{f}|^2_\infty \tag{5.81}$$

for $t \ge t_0(||\boldsymbol{v}_0||_{\boldsymbol{L}^2(\Omega)})$. In the standard manner (5.81) leads to the estimate

$$||\boldsymbol{v}(t)||^2_{\boldsymbol{L}^2(\Omega)} \le e^{-\lambda(t-t_0)}\, ||\boldsymbol{v}(t_0)||^2_{\boldsymbol{L}^2(\Omega)} + \frac{\beta|\boldsymbol{f}|^2_\infty}{\lambda}[1 - e^{-\lambda(t-t_0)}]. \tag{5.82}$$

But, by virtue of the fact that $\boldsymbol{v} \in L^\infty([0,\infty);\boldsymbol{H})$, $\exists C = C(||\boldsymbol{v}_0||_{\boldsymbol{L}^2(\Omega)})$ such that $||\boldsymbol{v}(t_0)||_{\boldsymbol{L}^2(\Omega)} \le C$. Thus,

$$||\boldsymbol{v}(t)||^2_{\boldsymbol{L}^2(\Omega)} \le \boldsymbol{e}^{-\lambda(t-t_0)} C + \frac{\beta|\boldsymbol{f}|^2_\infty}{\lambda} \tag{5.83}$$

and it follows that $\exists t_0' = t_0'(||\boldsymbol{v}_0||_{\boldsymbol{L}^2(\Omega)}) \geq t_0$ such that

$$||\boldsymbol{v}(t)||_{\boldsymbol{L}^2(\Omega)} \leq \frac{2\beta}{\lambda}|\boldsymbol{f}|_\infty^2 \equiv \hat{\rho} \tag{5.84}$$

for $t \geq t_0'(||\boldsymbol{v}_0||_{\boldsymbol{L}^2(\Omega)})$; the estimate (5.84) establishes the existence of an absorbing set in $\boldsymbol{H}$, i.e., if $\boldsymbol{v}_0$ is in a bounded set $B_0 \subset \boldsymbol{H}$, then $\exists R_0 > 0$ s.t. $||\boldsymbol{v}_0||_{\boldsymbol{L}^2(\Omega)} \leq R_0$, $\forall \boldsymbol{v}_0 \in B_0$ and $\boldsymbol{S}_{\mu_1}(t)\boldsymbol{v}_0 \in B_H^\rho$, for $t \geq t_0'(R_0)$. □

Now, as a consequence of (5.80) and (5.84), we also have

$$\frac{1}{2}\frac{d}{dt}||\boldsymbol{v}||_{\boldsymbol{L}^2(\Omega)}^2 + 2\mu_1 k(\Omega)|\boldsymbol{v}|_{\boldsymbol{H}^1(\Omega)}^2 \leq \rho|\boldsymbol{f}|_\infty \tag{5.85}$$

for $t \geq t_0'(|\boldsymbol{v}|_{L^2(\Omega)})$ where $|\boldsymbol{v}|_{\boldsymbol{H}^1(\Omega)}$ is the seminorm

$$|\boldsymbol{v}|_{\boldsymbol{H}^1(\Omega)} = \left(\int_\Omega \frac{\partial v_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}\,d\boldsymbol{x}\right)^{1/2}. \tag{5.86}$$

Let $r > 0$ and $t \geq t_0'$; integrating (5.85) we find that

$$\begin{aligned} ||\boldsymbol{v}(t+r)||_{\boldsymbol{L}^2(\Omega)}^2 + 2\mu_1 k(\Omega)\int_t^{t+r} |\boldsymbol{v}|_{\boldsymbol{H}^1(\Omega)}^2\,d\tau &\leq \rho r|\boldsymbol{f}|_\infty + ||\boldsymbol{v}(t)||_{\boldsymbol{L}^2(\Omega)}^2 \\ &\leq \rho r|\boldsymbol{f}|_\infty + \hat{\rho}^2. \end{aligned} \tag{5.87}$$

Therefore, for $t \geq t_0'(||\boldsymbol{v}_0||_{\boldsymbol{L}^2(\Omega)})$

$$\int_t^{t+r} |\boldsymbol{v}|_{\boldsymbol{H}^1(\Omega)}^2\,d\tau \leq \frac{\rho r|\boldsymbol{f}|_\infty + \hat{\rho}^2}{2\mu_1 k(\Omega)}. \tag{5.88}$$

By virtue of the inequality immediately preceding (5.80) it is clear that (5.88) also holds with $|\boldsymbol{v}|_{\boldsymbol{H}^1(\Omega)}^2$ replaced by $||\boldsymbol{v}||_{\boldsymbol{H}^2(\Omega)}^2$. The estimate (5.88) will be useful in helping us to establish the existence of an absorbing set in $\boldsymbol{H}^2(\Omega)$. We may, in fact, state the following result:

Lemma 5.10. *For $\mu_1 > 0$, and $0 \leq \alpha < 1$, let $\boldsymbol{v}(t)$ be the unique solution of the initial-boundary value problem* (5.2a,b)*,* (5.3a)*,* (5.4) *with $\rho = 1$. Then, for any $r > 0$, $\exists K(r) > 0$ such that for $t_0'(||\boldsymbol{v}_0||_{\boldsymbol{L}^2(\Omega)})$, as given in Lemma 5.9,*

$$||\boldsymbol{v}(t+r)||_{\boldsymbol{H}^2(\Omega)} \leq K(r),\ \forall t \geq t_0'. \tag{5.89}$$

Proof. We begin by multiplying (5.72) through by $\partial v_i/\partial t$ and integrating over Ω, i.e.,

$$\int_\Omega \frac{\partial v_i}{\partial t}\frac{\partial v_i}{\partial t}\,d\boldsymbol{x} + \int_\Omega v_j \frac{\partial v_i}{\partial x_j}\frac{\partial v_i}{\partial t}\,d\boldsymbol{x}$$
$$= \int_\Omega \frac{\partial}{\partial x_j}(\gamma(\boldsymbol{v})e_{ij})\frac{\partial v_i}{\partial t}d\boldsymbol{x} - 2\mu_1 \int_\Omega \frac{\partial}{\partial x_j}(\Delta e_{ij})\frac{\partial v_i}{\partial t}d\boldsymbol{x} + \int_\Omega f_i \frac{\partial v_i}{\partial t}\,d\boldsymbol{x} \tag{5.90}$$

where we have used the fact that

$$\int_\Omega \frac{\partial p}{\partial x_i}\frac{\partial v_i}{\partial t}\,d\boldsymbol{x} = 0.$$

Treating the first two integrals on the right-hand side of (5.90) in a manner similar to that employed in the proof of Lemma 5.9, we find that

$$\left\|\frac{\partial \boldsymbol{v}}{\partial t}\right\|^2_{\boldsymbol{L}^2(\Omega)} + \int_\Omega \gamma(\boldsymbol{v})e_{ij}\cdot\frac{\partial e_{ij}}{\partial t}\,d\boldsymbol{x} + \mu_1\frac{d}{dt}\int_\Omega \frac{\partial e_{ij}}{\partial x_k}\frac{\partial e_{ij}}{\partial x_k}\,d\boldsymbol{x}$$
$$= -\int_\Omega v_j\frac{\partial v_i}{\partial x_j}\frac{\partial v_i}{\partial t}\,d\boldsymbol{x} + \int_\Omega f_i\frac{\partial v_i}{\partial t}\,d\boldsymbol{x}. \tag{5.91}$$

We now introduce the potential

$$\Gamma(e_{ij}e_{ij}) = \int_0^{e_{ij}e_{ij}} \mu_0(\epsilon + s)^{-\alpha/2}\,ds \tag{5.92}$$

so that

$$\frac{d\Gamma}{dt} = \gamma(\boldsymbol{v})e_{ij}\frac{\partial e_{ij}}{\partial t}, \tag{5.93}$$

then we obtain from (5.91)

$$\left\|\frac{\partial \boldsymbol{v}}{\partial t}\right\|^2_{L^2(\Omega)} + \frac{d}{dt}\left\{\int_\Omega \Gamma(e_{ij}e_{ij})\,d\boldsymbol{x} + \mu_1\int_\Omega \frac{\partial e_{ij}}{\partial x_k}\frac{\partial e_{ij}}{\partial x_k}\,d\boldsymbol{x}\right\}$$
$$\le -\int_\Omega v_j\frac{\partial v_i}{\partial x_j}\frac{\partial v_i}{\partial t}\,d\boldsymbol{x} + |\boldsymbol{f}|_\infty\left\|\frac{\partial \boldsymbol{v}}{\partial t}\right\|_{L^2(\Omega)}. \tag{5.94}$$

From (5.94) we have, immediately, that

$$\frac{1}{2}\left\|\frac{\partial \boldsymbol{v}}{\partial t}\right\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{d}{dt}\left\{\int_\Omega \Gamma(e_{ij}e_{ij}\,d\boldsymbol{x} + \mu_1\int_\Omega \frac{\partial e_{ij}}{\partial x_k}\frac{\partial e_{ij}}{\partial x_k}\,d\boldsymbol{x}\right\}$$
$$\le \frac{1}{2}|\boldsymbol{f}|^2_\infty + \left|\int_\Omega v_j\frac{\partial v_i}{\partial x_j}\frac{\partial v_i}{\partial t}\,d\boldsymbol{x}\right|. \tag{5.95}$$

Now,

$$
\begin{aligned}
\left|\int_\Omega \left(v_j \frac{\partial v_i}{\partial x_j}\right) \frac{\partial v_i}{\partial t}\, d\boldsymbol{x}\right| &\le \left(\int_\Omega ||\boldsymbol{v}\cdot\nabla\boldsymbol{v}||^2\, d\boldsymbol{x}\right)^{1/2} \left|\left|\frac{\partial \boldsymbol{v}}{\partial t}\right|\right|_{\boldsymbol{L}^2(\Omega)} \\
&\le \frac{\delta}{2}\int_\Omega ||\boldsymbol{v}\cdot\nabla\boldsymbol{v}||^2\, d\boldsymbol{x} + \frac{1}{2\delta}\left|\left|\frac{\partial \boldsymbol{v}}{\partial t}\right|\right|^2_{\boldsymbol{L}^2(\Omega)}
\end{aligned} \tag{5.96}
$$

for any $\delta > 0$. Choosing $\delta > 1$ in (5.96), and then combining this result with (5.95), we are led to the estimate

$$
\frac{d}{dt}\left\{\int_\Omega \Gamma(e_{ij}e_{ij})\, d\boldsymbol{x} + \mu_1 \int_\Omega \frac{\partial e_{ij}}{\partial x_k}\frac{\partial e_{ij}}{\partial x_k}\, d\boldsymbol{x}\right\} \le \frac{1}{2}|\boldsymbol{f}|^2_\infty + \frac{\delta}{2}\int_\Omega ||\boldsymbol{v}\cdot\nabla\boldsymbol{v}||^2\, d\boldsymbol{x}. \tag{5.97}
$$

Next,

$$
\begin{aligned}
\int_\Omega ||\boldsymbol{v}\cdot\nabla\boldsymbol{v}||^2\, d\boldsymbol{x} &= \int_\Omega v_j \frac{\partial v_i}{\partial x_j} v_k \frac{\partial v_i}{\partial x_k}\, d\boldsymbol{x} \le \sup_\Omega (v_i v_i) \int_\Omega \frac{\partial v_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}\, d\boldsymbol{x} \\
&= ||\boldsymbol{v}||^2_{\boldsymbol{L}^2(\Omega)}\, ||\boldsymbol{v}||^2_{\boldsymbol{L}^2(\Omega)} \le c(\Omega)\, ||\boldsymbol{v}||^2_{\boldsymbol{H}^2(\Omega)}\, ||\boldsymbol{v}||^2_{\boldsymbol{H}^1(\Omega)}
\end{aligned} \tag{5.98}
$$

for some $c(\Omega) > 0$, by virtue of the embedding of $\boldsymbol{W}^{2,2}(\Omega)$ into $\boldsymbol{C}(\Omega)$ in dimension $n = 3$. Therefore, from (5.97) we now obtain

$$
\begin{aligned}
&\frac{d}{dt}\left\{\int_\Omega \Gamma(e_{ij}e_{ij})\, d\boldsymbol{x} + \mu_1 \int_\Omega \frac{\partial e_{ij}}{\partial x_k}\frac{\partial e_{ij}}{\partial x_k}\, d\boldsymbol{x}\right\} \\
&\quad \le \left\{\frac{\delta c(\Omega)}{2\mu_1}|\boldsymbol{v}|^2_{\boldsymbol{H}^1(\Omega)}\right\} \mu_1\, ||\boldsymbol{v}||^2_{\boldsymbol{H}^2(\Omega)} + \frac{1}{2}|\boldsymbol{f}|^2_\infty \\
&\quad \le \left[\frac{\delta \tilde{c}(\Omega)}{2\mu_1}|\boldsymbol{v}|^2_{\boldsymbol{H}^1(\Omega)}\right]\left\{\int_\Omega \Gamma(e_{ij}e_{ij})\, d\boldsymbol{x} + \mu_1 \int_\Omega \frac{\partial e_{ij}}{\partial x_k}\frac{\partial e_{ij}}{\partial x_k}\, d\boldsymbol{x}\right\} + \frac{1}{2}|\boldsymbol{f}|^2_\infty
\end{aligned} \tag{5.99}
$$

with $\tilde{c}(\Omega) = c(\Omega)/k(\Omega)$ where we have again used Lemma 5.2. If we define

$$
y(t) = \int_\Omega \Gamma(e_{ij}e_{ij})\, d\boldsymbol{x} + \mu_1 \int_\Omega \frac{\partial e_{ij}}{\partial x_k}\frac{\partial e_{ij}}{\partial x_k}\, d\boldsymbol{x} \tag{5.100}
$$

then, by virtue of (5.99), we see that we have arrived at a differential inequality of the form

$$
\frac{dy}{dt} \le a(t)y(t) + b(t) \tag{5.101}
$$

with

$$a(t) = \frac{\delta \tilde{c}(\Omega)}{2\mu_1} |\boldsymbol{v}(t)|^2_{\boldsymbol{H}^1(\Omega)}, \quad b(t) = \frac{1}{2}|\boldsymbol{f}|^2_\infty. \tag{5.102}$$

At this point we apply the Uniform Gronwall Lemma (see Appendix A or [FP,Te4]); this result requires, for its application here that, for all $t \geq t_0' = t_0'(||\boldsymbol{v}_0||_{\boldsymbol{L}^2(\Omega)})$, $\exists k_1, k_2, k_3$, and r, all positive constants, such that

$$\int_t^{t+r} a(s)\,ds \leq k_1, \quad \int_t^{t+r} b(s)\,ds \leq k_2, \quad \int_t^{t+r} y(s)\,ds \leq k_3. \tag{5.103}$$

We now show that the conditions of (5.103) are satisfied for appropriate $k_1, k_2, k_3 > 0$. First of all, from the definition of $a(t)$, i.e., (5.102) and (5.88), which is valid for any $r > 0$, if $t \geq t_0'(||\boldsymbol{v}_0||_{\boldsymbol{L}^2(\Omega)})$, we have

$$\int_t^{t+r} a(s)\,ds = \frac{\delta \tilde{c}(\Omega)}{2\mu_1} \int_t^{t+r} |\boldsymbol{v}(s)|^2_{\boldsymbol{H}^1(\Omega)}\,ds \leq \frac{\delta \tilde{c}(\Omega)\rho(r|\boldsymbol{f}|_\infty + \hat{\rho})}{4\mu_1^2 k(\Omega)} \equiv k_1. \tag{5.104}$$

Also

$$\int_t^{t+r} b(s)\,ds = \frac{r}{2}|\boldsymbol{f}|^2_\infty \equiv k_2. \tag{5.105}$$

It remains, therefore, to show that for $t \geq t_0'(||\boldsymbol{v}_0||_{\boldsymbol{L}^2(\Omega)})$

$$\int_t^{t+r} \int_\Omega \left\{ \Gamma(e_{ij}e_{ij}) + \mu_1 \frac{\partial e_{ij}}{\partial x_k} \frac{\partial e_{ij}}{\partial x_k} \right\} d\boldsymbol{x}\,ds \leq k_3 \tag{5.106}$$

for some $k_3 > 0$. Now, directly from (5.78) we infer that

$$\frac{1}{2}\frac{d}{dt} ||\boldsymbol{v}||^2_{\boldsymbol{L}^2(\Omega)} + 2\mu_1 \int_\Omega \frac{\partial e_{ij}}{\partial x_k} \frac{\partial e_{ij}}{\partial x_k}\,d\boldsymbol{x} \leq |\boldsymbol{f}|_\infty ||\boldsymbol{v}||_{\boldsymbol{L}^2(\Omega)} \leq \hat{\rho}|\boldsymbol{f}|_\infty \tag{5.107}$$

for $t \geq t_0'(||\boldsymbol{v}_0||_{\boldsymbol{L}^2(\Omega)})$, where we have used (5.84), i.e., the existence of the absorbing set in $\boldsymbol{H}$. Integrating (5.107) from t to $t + r$, for $t \geq t_0'$, yields

$$\frac{1}{2} ||\boldsymbol{v}(t+r)||^2_{\boldsymbol{L}^2(\Omega)} + 2\mu_1 \int_t^{t+r} \int_\Omega \frac{\partial e_{ij}}{\partial x_k} \frac{\partial e_{ij}}{\partial x_k}\,d\boldsymbol{x}\,ds \leq \hat{\rho} r|\boldsymbol{f}|_\infty + \frac{1}{2} ||\boldsymbol{v}(t)||^2_{\boldsymbol{L}^2(\Omega)}. \tag{5.108}$$

Therefore, if we again make use of (5.84) we are led directly from (5.108) to the estimate

$$\int_t^{t+r} \int_\Omega \frac{\partial e_{ij}}{\partial x_k} \frac{\partial e_{ij}}{\partial x_k}\,d\boldsymbol{x}\,ds \leq \frac{\rho r|\boldsymbol{f}|_\infty + \hat{\rho}^2/2}{2\mu_1} \tag{5.109}$$

which is valid for $t \geq t_0'(||\boldsymbol{v}_0||_{\boldsymbol{L}^2(\Omega)})$. We now consider the function $\Gamma(e_{ij}e_{ij})$ which is defined by (5.92); for $s \geq 0$ the integrand

$$g(s) = \mu_0(\epsilon + s)^{-\alpha/2} \tag{5.110}$$

satisfies

$$g'(s) = -\frac{\alpha\mu_0}{2}(\epsilon + s)^{-(\frac{\alpha}{2}+1)} < 0, \quad g(0) = \mu_0\epsilon^{-\alpha/2} > 0.$$

Therefore,

$$\Gamma(e_{ij}e_{ij}) \leq \int_0^{e_{ij}e_{ij}} \left[\sup_{s\geq 0} g(s)\right] ds \equiv \frac{\mu_0}{\epsilon^{\alpha/2}} e_{ij}e_{ij} \tag{5.111}$$

so that for $t \geq t_0'(||\boldsymbol{v}_0||_{\boldsymbol{L}^2(\Omega)})$ we have, by virtue of (5.88),

$$\begin{aligned} \int_0^{t+r}\int_\Omega \Gamma(e_{ij}e_{ij})\, d\boldsymbol{x}\, ds &\leq \frac{\mu_0}{\epsilon^{\alpha/2}} \int_t^{t+r} \left(\int_\Omega e_{ij}e_{ij}\, d\boldsymbol{x}\right) ds \\ &\leq \frac{\mu_0 c'(\Omega)}{\epsilon^{\alpha/2}} \int_t^{t+r} |\boldsymbol{v}(s)|^2_{\boldsymbol{H}^1(\Omega)}\, ds \\ &\leq \frac{\mu_0 c'(\Omega)}{2\mu_1\epsilon^{\alpha/2}k(\Omega)}\{\rho r|\boldsymbol{f}|_\infty + \hat{\rho}^2\}. \end{aligned} \tag{5.112}$$

By combining (5.109) with (5.112) we now see that (5.106) is valid, for $t \geq t_0'(||\boldsymbol{v}_0||_{\boldsymbol{L}^2(\Omega)})$, with

$$k_3 = \left\{1 + \frac{\mu_0 c'(\Omega)}{\mu_1\epsilon^{\alpha/2}k(\Omega)}\right\}(\rho r|\boldsymbol{f}|_\infty + \hat{\rho}^2). \tag{5.113}$$

According to the Uniform Gronwall Lemma, for locally integrable positive functions $y(t)$, $a(t)$, $b(t)$ on $[t_0', \infty)$, satisfying (5.101) for $t \geq t_0'$, and the conditions (5.103), we have

$$y(t + r) \leq \left(\frac{k_3}{r} + k_2\right)\exp(k_1),\ \forall t \geq t_0'. \tag{5.114}$$

Defining, therefore, k_1, k_2, and k_3 as in (5.104), (5.105), and (5.113), we find that

$$\int_\Omega \left\{\Gamma(e_{ij}e_{ij}) + \mu_1\frac{\partial e_{ij}}{\partial x_k}\frac{\partial e_{ij}}{\partial x_k}\right\} d\boldsymbol{x}\bigg|_{t+r} \leq \left(\frac{k_3}{r} + k_2\right)\exp(k_1),\ \forall t \geq t_0'. \tag{5.115}$$

Applying the extension of Korn's lemma, i.e., Lemma 5.2, to (5.115) we infer the existence of $K = K(r)$ such that

$$||\boldsymbol{v}(t+r)||_{\boldsymbol{H}^2(\Omega)} \le K(r), \; \forall t \ge t_0'(||\boldsymbol{v}_0||_{\boldsymbol{L}^2(\Omega)}). \qquad \square$$

Remarks. It follows directly from (5.89) that

$$||\boldsymbol{v}(t)||_{\boldsymbol{H}^2(\Omega)} \le K(r), \; \forall t \ge t_0' + r. \tag{5.116}$$

Of course, $K(r)$ depends, as well, on μ_0, μ_1, ϵ, Ω, and $|\boldsymbol{f}|_\infty$ but not on $||\boldsymbol{v}_0||_{\boldsymbol{L}^2(\Omega)}$. The estimate (5.116) shows that, for $\boldsymbol{v}_0$ in a bounded set, $B_0 \subset H$, so that $||\boldsymbol{v}_0||_{\boldsymbol{L}^2(\Omega)} \le R_0$, for some $R_0 > 0$ and all $\boldsymbol{v}_0 \in B_0$, $\boldsymbol{S}_{\mu_1}(t)\boldsymbol{v}_0 \in B^{\rho'}_{\boldsymbol{H}^2}$, for $t \ge t_0'(R_0)$, where $\rho' = K(r)$.

We may combine Lemmas 5.9 and 5.10 into the following

Theorem 5.3. *For the nonlinear semigroup $\boldsymbol{S}_{\mu_1}(t)$ defined by the solution of the initial-boundary value problem* (5.2a,b), (5.3a), (5.4)*, with initial data $\boldsymbol{v}_0 \in H$, there exist absorbing sets $B^{\hat{\rho}}_{\boldsymbol{H}}$, $B^{\rho'}_{\boldsymbol{H}^2(\Omega)}$, respectively, in $\boldsymbol{H}$ and $\boldsymbol{H}^2(\Omega)$; furthermore, for $\boldsymbol{v}_0 \in B_0 \subset \boldsymbol{H}$, a bounded set contained in the ball in $\boldsymbol{H}$ of radius R_0 centered at the zero element in $\boldsymbol{L}^2(\Omega)$, $\exists t_0' = t_0'(R_0)$ such that $\boldsymbol{S}_{\mu_1}(t)\boldsymbol{v}_0 \in B^{\hat{\rho}}_{\boldsymbol{H}} \cap B^{\rho'}_{\boldsymbol{H}^2(\Omega)}$ for $t \ge t_0'$.*

The existence of a (global) compact attractor $\boldsymbol{\mathcal{A}}_{\mu_1}$, which attracts all the bounded sets of $\boldsymbol{H}$, can now be inferred from results in [Te4] if we take $\boldsymbol{\mathcal{A}}_{\mu_1} = \omega(B^{\hat{\rho}}_{\boldsymbol{H}})$, the ω-limit set of $B^{\hat{\rho}}_{\boldsymbol{H}}$, because the existence of the absorbing set $B^{\rho'}_{\boldsymbol{H}^2(\Omega)}$ in $\boldsymbol{H}^2(\Omega)$ enables us to conclude that

$$\bigcup_{t \ge t_0'} \boldsymbol{S}_{\mu_1}(t) B_0$$

is relatively compact in $\boldsymbol{H}$ for every bounded set $B_0 \subset \boldsymbol{H}$; in other words the operators $\boldsymbol{S}_{\mu_1}(t)$ are uniformly compact for t sufficiently large. Alternatively, we may follow the approach in [CF] so as to conclude that

$$\boldsymbol{\mathcal{A}}_{\mu_1} = \bigcap_{t>0} \boldsymbol{S}_{\mu_1}(t) B^{\rho'}_{\boldsymbol{H}^2(\Omega)} \tag{5.117}$$

in which case the properties already established for the semigroup $\boldsymbol{S}_{\mu_1}(t)$ generated by (5.2a,b), (5.3a), (5.4), and the fact that $B^{\rho'}_{\boldsymbol{H}^2(\Omega)}$ is an absorbing set in $\boldsymbol{H}^2(\Omega)$, permit us to follow, almost verbatim, the proof of Proposition 14.1 of [CF] so as to conclude the following:

Theorem 5.4. *Let* $\mathcal{A}_{\mu_1}$ *be defined as in* (5.117) *where* $S_{\mu_1}(t)$ *is the nonlinear semigroup of (solution) operators generated by the bipolar initial-boundary value problem* (5.2a,b)*,* (5.3a)*,* (5.4)*, with* $v_0 \in H$*, and* $B^{\rho'}_{H^2(\Omega)}$ *is the absorbing set whose existence was established in the proof of Lemma* 5.10*. Then*

(i) $\mathcal{A}_{\mu_1}$ *is compact in* H*.*
(ii) $S_{\mu_1}(t)\mathcal{A}_{\mu_1} = \mathcal{A}_{\mu_1}$, $\forall t \geq 0$.
(iii) If $B \subset H$ *is bounded and satisfies* $S_{\mu_1}(t)B = B$, $\forall t \geq 0$, *then* $B \subset \mathcal{A}_{\mu_1}$.
(iv) $\mathcal{A}_{\mu_1}$ *is a connected set.*
(v) For every $v_0 \in H$, $\lim_{t\to+\infty} d(S_{\mu_1}(t)v_0, \mathcal{A}_{\mu_1}) = 0$.

Theorem 5.4 establishes $\mathcal{A}_{\mu_1}$, as defined by (5.117), as the desired global attractor for the semigroup $S_{\mu_1}(t)$; our next goal is to establish upper bounds for both the Hausdorff and Fractal dimensions of $\mathcal{A}_{\mu_1}$. We choose to establish the upper bounds for the Hausdorff and Fractal dimensions of $\mathcal{A}_{\mu_1}$ by using the general framework in [Te4, CF], and [CFT1]; to this end we must establish two basic facts: (i) that the operators $S_{\mu_1}(t)$ are uniformly differentiable on $\mathcal{A}_{\mu_1}$ for $t \geq 0$ and (ii) that the Fréchet differential $L_{\mu_1}(t; v_0)$ of $S_{\mu_1}(t)$, at $v_0 \in \mathcal{A}_{\mu_1}$, is uniformly bounded, $\forall t \geq 0$, on $\mathcal{A}_{\mu_1}$ in the strong operator norm of $\mathcal{L}(H; H)$. We begin by considering the first of these two problems in the next section.

5.3.3 The Uniform Differentiability of $S_{\mu_1}(t)$

Let $S_{\mu_1}(t)$ be the nonlinear semigroup generated by the solution of (5.2a,b), (5.3a), (5.4), corresponding to initial datum $v_0 \in H$ and let $\mathcal{A}_{\mu_1}$ be the global attractor for $S_{\mu_1}(t)$ defined by (5.117). We then have the following

Definition 5.6. Let $t > 0$ be given. Then $S_{\mu_1}(t)$ is uniformly differentiable on $\mathcal{A}_{\mu_1}$ if for every $v_0 \in \mathcal{A}_{\mu_1}$ there exists a linear operator $L_{\mu_1}(t; u_0) \in \mathcal{L}(H; H)$ such that, as $\epsilon \to 0$,

$$\sup_{\substack{v_0, u_0 \in \mathcal{A}_{\mu_1} \\ 0<|v_0-u_0|_{L^2(\Omega)}\leq\epsilon}} \frac{S_{\mu_1}(t)v_0 - S_{\mu_1}(t)u_0 - L_{\mu_1}(t; u_0)(v_0 - u_0)|_{L^2(\Omega)}}{|v_0 - u_0|_{L^2(\Omega)}} \to 0.$$

Remarks.

(i) When $L_{\mu_1}(t; u_0)$ exists, $L_{\mu_1}(t; u_0)(v_0 - u_0)$ will be the solution $U(t)$ of the linearized initial-boundary value problem (5.66a–d), (about $u(t) = S_{\mu_1}(t)u_0$), with $U_0(x) = v_0(x) - u_0(x)$ in Ω.

(ii) If $v_0 \in \mathcal{A}_{\mu_1}$ then $v(t) = S_{\mu_1}(t)v_0 \in \mathcal{A}_{\mu_1}$, $\forall t \geq 0$. As $S_{\mu_1}(t)B^{\rho'}_{H^2(\Omega)} \subset B^{\rho'}_{H^2(\Omega)}$, for t sufficiently large, we will have $v(t) \in B^{\rho'}_{H^2(\Omega)}$, $\forall t \geq 0$, inasmuch as $\boldsymbol{v}(t) \in \boldsymbol{\mathcal{A}}_{\mu_1}$, $\forall t \geq 0 \Rightarrow \boldsymbol{v}(t) \in \bigcap_{s>0} \boldsymbol{S}_{\mu_1}(s) B^{\rho'}_{\boldsymbol{H}^2(\Omega)} \Rightarrow \boldsymbol{v}(t) \in \boldsymbol{S}_{\mu_1}(\tau) B^{\rho'}_{\boldsymbol{H}^2(\Omega)} \subset B^{\rho'}_{\boldsymbol{H}^2(\Omega)}$, for τ sufficiently large, and all $t \geq 0$.

(iii) By virtue of (ii), above, $\boldsymbol{U}(t) = \boldsymbol{L}_{\mu_1}(t;\boldsymbol{u}_0)(\boldsymbol{v}_0 - \boldsymbol{u}_0)$ will be the solution of the linearized initial-boundary value problem, with the linearization about $\boldsymbol{u}(t) \in B^{\rho'}_{\boldsymbol{H}^2(\Omega)}$, $\forall t \geq 0$. In this case Theorem 5.2 may be easily strengthened so as to imply the existence of a unique solution $\boldsymbol{U}(t)$ of (5.66a–d) which is in $L^\infty([0,t);\boldsymbol{H}) \cap L^\infty([0,t);\boldsymbol{H}^2(\Omega))$, $\forall t > 0$, as $\boldsymbol{U}(0) \in \boldsymbol{H}^2(\Omega)$.

We now have the following result:

Theorem 5.5. *The semigroup $\boldsymbol{S}_{\mu_1}(t)$ defined by the solution of (5.2a,b), (5.3a), (5.4), with $\boldsymbol{v}_0 \in \boldsymbol{\mathcal{A}}_{\mu_1}$, is uniformly differentiable on the global attractor $\boldsymbol{\mathcal{A}}_{\mu_1}$ defined by (5.117). In particular, if we set*

$$\boldsymbol{\Theta}(t) = \boldsymbol{S}_{\mu_1}(t)\boldsymbol{v}_0 - \boldsymbol{S}_{\mu_1}(t)\boldsymbol{u}_0 - \boldsymbol{L}_{\mu_1}(t;\boldsymbol{u}_0)(\boldsymbol{v}_0 - \boldsymbol{u}_0)$$

with

(i) $\boldsymbol{v}(t) = \boldsymbol{S}_{\mu_1}(t)\boldsymbol{v}_0$ *the unique solution of* (5.2a,b), (5.3a), (5.4) *with* $\boldsymbol{v}(0) = \boldsymbol{v}_0 \in \boldsymbol{\mathcal{A}}_{\mu_1}$,

(ii) $\boldsymbol{u}(t) = \boldsymbol{S}_{\mu_1}(t)\boldsymbol{u}_0$ *the unique solution of* (5.2a,b), (5.3a), (5.4) *with* $\boldsymbol{u}(0) = \boldsymbol{u}_0 \in \boldsymbol{\mathcal{A}}_{\mu_1}$,

(iii) $\boldsymbol{U}(t) = \boldsymbol{L}_{\mu_1}(t;\boldsymbol{u}_0)(\boldsymbol{v}_0 - \boldsymbol{u}_0)$ *the unique solution of* (5.66a–d) *satisfying* $\boldsymbol{U}(0) = \boldsymbol{v}_0 - \boldsymbol{u}_0$, *then* $\exists j(t) < \infty$, *for each* $t > 0$, *such that*

$$\frac{||\boldsymbol{\Theta}(t)||_{\boldsymbol{L}^2(\Omega)}}{||\boldsymbol{v}_0 - \boldsymbol{u}_0||_{\boldsymbol{L}^2(\Omega)}} \leq j(t)\epsilon^{1/5} \to 0, \text{ as } \epsilon \to 0. \tag{5.118}$$

Prior to proceeding with the proof of Theorem 5.5 we will first pause to establish the following:

Lemma 5.11. *Under the conditions stated in Theorem 5.5, $\boldsymbol{\Theta}(t)$ satisfies, $\forall t \geq 0$,*

$$\sup_{\substack{\boldsymbol{v}_0,\boldsymbol{u}_0 \in \boldsymbol{\mathcal{A}}_{\mu_1} \\ 0<||\boldsymbol{v}_0-\boldsymbol{u}_0||_{\boldsymbol{L}^2(\Omega)}<\epsilon}} ||\boldsymbol{\Theta}||_{\boldsymbol{L}^2(\Omega)} \leq c_1 e^{c_2 t} \cdot \epsilon \tag{5.119}$$

for some $c_1 > 0$, $c_2 > 0$ which depend (at most) on μ_1, ρ', and Ω.

Proof. By the remarks above, and the assumption that both $\boldsymbol{v}_0, \boldsymbol{u}_0 \in \boldsymbol{\mathcal{A}}_{\mu_1}$, we have, in fact, that $\boldsymbol{v}$, $\boldsymbol{u}$, and $\boldsymbol{U}$ are all in $L^\infty([0,\infty);\boldsymbol{H}) \cap L^\infty([0,\infty);\boldsymbol{H}^2(\Omega))$. We set $\boldsymbol{w}(t) = \boldsymbol{v}(t) - \boldsymbol{u}(t)$ in which case $\boldsymbol{\Theta}(t) = \boldsymbol{w}(t) - \boldsymbol{U}(t)$. It is easy to deduce that $\boldsymbol{w}(t)$ satisfies, in $\Omega \times [0,T)$,

$$\begin{aligned}\frac{\partial w_i}{\partial t} &+ w_j\frac{\partial u_i}{\partial x_j} + u_j\frac{\partial w_i}{\partial x_j} + w_j\frac{\partial w_i}{\partial x_j}\\ &= -\frac{\partial p_w}{\partial x_i} + \frac{\partial}{\partial x_j}[\gamma(\boldsymbol{v})e_{ij}(\boldsymbol{v}) - \gamma(\boldsymbol{u})e_{ij}(\boldsymbol{u})]\\ &- 2\mu_1\frac{\partial}{\partial x_j}(\Delta e_{ij}(\boldsymbol{w}))\end{aligned} \tag{5.120}$$

with p_w the difference of the pressures corresponding to the two velocity fields $\boldsymbol{v}$ and $\boldsymbol{u}$, and

$$\operatorname{div}\boldsymbol{w} = 0, \text{ in } \Omega \times [0, T), \tag{5.121a}$$

$$\boldsymbol{w} = \boldsymbol{0}, \quad \tau_{ijk}(\boldsymbol{w})\nu_j\nu_k\tau_i = 0, \text{ on } \partial\Omega \times [0, T), \tag{5.121b}$$

$$\boldsymbol{w}(0) = \boldsymbol{v}_0 - \boldsymbol{u}_0, \text{ in } \Omega. \tag{5.121c}$$

We now multiply (5.120) through by w_i, integrate over Ω, and sum over $i = 1, 2, 3$ so as to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||\boldsymbol{w}||^2_{\boldsymbol{L}^2(\Omega)} &+ \int_\Omega w_j\frac{\partial u_i}{\partial x_j}w_i\,d\boldsymbol{x}\\ &+ \int_\Omega [\gamma(\boldsymbol{v})e_{ij}(\boldsymbol{v}) - \gamma(\boldsymbol{u})e_{ij}(\boldsymbol{u})][e_{ij}(\boldsymbol{v}) - e_{ij}(\boldsymbol{u})]d\boldsymbol{x}\\ &+ 2\mu_1\int_\Omega \frac{\partial e_{ij}(\boldsymbol{w})}{\partial x_k}\frac{\partial e_{ij}(\boldsymbol{w})}{\partial x_k}d\boldsymbol{x} = 0.\end{aligned} \tag{5.122}$$

However, as demonstrated in Lemma B.5,

$$\int_\Omega [\gamma(\boldsymbol{v})e_{ij}(\boldsymbol{v}) - \gamma(\boldsymbol{u})e_{ij}(\boldsymbol{u})][e_{ij}(\boldsymbol{v}) - e_{ij}(\boldsymbol{u})]d\boldsymbol{x} \geq 0 \tag{5.123}$$

so that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||\boldsymbol{w}||^2_{\boldsymbol{L}^2(\Omega)} &+ 2\mu_1\int_\Omega \frac{\partial e_{ij}(\boldsymbol{w})}{\partial x_k}\frac{\partial e_{ij}(\boldsymbol{w})}{\partial x_k}d\boldsymbol{x}\\ &\leq -\int_\Omega w_j\frac{\partial u_i}{\partial x_j}w_i\,d\boldsymbol{x} = \int_\Omega u_i\frac{\partial w_i}{\partial x_j}w_j\,d\boldsymbol{x}.\end{aligned} \tag{5.124}$$

Moreover, for any $\delta > 0$,

$$\begin{aligned}\left|\int_\Omega u_i \frac{\partial w_i}{\partial x_j} w_j \, d\boldsymbol{x}\right| &\le ||\boldsymbol{u}||_{\boldsymbol{L}^\infty(\Omega)} \, ||\boldsymbol{w}||_{\boldsymbol{H}^1(\Omega)} \, ||\boldsymbol{w}||_{\boldsymbol{L}^2(\Omega)} \\ &\le c(\Omega) \, ||\boldsymbol{u}||_{\boldsymbol{H}^2(\Omega)} \, ||\boldsymbol{w}||_{\boldsymbol{H}^1(\Omega)} \, ||\boldsymbol{w}||_{\boldsymbol{L}^2(\Omega)} \\ &\le c(\Omega)\rho' \left(\frac{\delta}{2} \, ||\boldsymbol{w}||^2_{\boldsymbol{H}^2(\Omega)} + \frac{1}{2\delta} \, ||\boldsymbol{w}||^2_{\boldsymbol{L}^2(\Omega)} \right)\end{aligned} \tag{5.125}$$

where we have, again, used the embedding of $\boldsymbol{W}^{2,2}(\Omega)$ into $\boldsymbol{C}(\Omega)$, valid for $n = 3$, and the fact that $\forall t \ge 0, \boldsymbol{u}(t) \in B^{\rho'}_{\boldsymbol{H}^2(\Omega)}$ if $\boldsymbol{u}_0 \in \boldsymbol{\mathcal{A}}_{\mu_1}$. Applying the Korn inequality, Lemma 5.2, to (5.124), the estimate of Lemma 5.2 now being valid $\forall t \ge 0$ since $\boldsymbol{w}(t) \in \boldsymbol{H}^{\rho'}_{\boldsymbol{H}^2(\Omega)}$, $\forall t \ge 0$, and then using (5.125), we infer the existence of constants $\eta_1 = \eta_1(\Omega) > 0$ and $\eta_2(\Omega) > 0$ such that

$$\frac{d}{dt} \, ||\boldsymbol{w}||^2_{\boldsymbol{L}^2(\Omega)} + \eta_1 \, ||\boldsymbol{w}||^2_{\boldsymbol{H}^2(\Omega)} \le \eta_2 \, ||\boldsymbol{w}||^2_{\boldsymbol{L}^2(\Omega)}, \quad t \ge 0. \tag{5.126}$$

From (5.126) we easily deduce that

$$||\boldsymbol{v}(t) - \boldsymbol{u}(t)||^2_{\boldsymbol{L}^2(\Omega)} \le ||\boldsymbol{v}_0 - \boldsymbol{u}_0||^2_{\boldsymbol{L}^2(\Omega)} \exp(\eta_2 t), \quad t \ge 0 \tag{5.127}$$

and, thus,

$$\int_0^t ||\boldsymbol{w}(\tau)||^2_{\boldsymbol{H}^2(\Omega)} \, d\tau \le \left(\frac{\eta_2}{\eta_1}\right) ||\boldsymbol{v}_0 - \boldsymbol{u}_0||^2_{\boldsymbol{L}^2(\Omega)} \exp(\eta_2 t). \tag{5.128}$$

Next, from (5.120) and (5.66a) we compute that for $\boldsymbol{\Theta} = \boldsymbol{w} - \boldsymbol{U}$:

$$\begin{aligned}\frac{\partial \Theta_i}{\partial t} + \Theta_j \frac{\partial u_i}{\partial x_j} + u_j \frac{\partial \Theta_i}{\partial x_j} + w_j \frac{\partial w_i}{\partial x_j} &= -\frac{\partial P_{\boldsymbol{\Theta}}}{\partial x_i} + \frac{\partial}{\partial x_j}[\gamma(\boldsymbol{v}) e_{ij}(\boldsymbol{v}) - \gamma(\boldsymbol{u}) e_{ij}(\boldsymbol{u})] \\ -\frac{\partial}{\partial x_j}[\gamma(\boldsymbol{u}) e_{ij}(\boldsymbol{U}) &- \alpha B_{ijkl}(\boldsymbol{u}) e_{kl}(\boldsymbol{U})] - 2\mu_1 \frac{\partial}{\partial x_j}(\Delta e_{ij}(\boldsymbol{\Theta}))\end{aligned} \tag{5.129}$$

in $\Omega \times [0, T)$, with

$$\operatorname{div} \boldsymbol{\Theta} = 0, \text{ in } \Omega \times [0, T), \tag{5.130a}$$

$$\Theta_i = \tau_{ijk}(\boldsymbol{\Theta}) \nu_j \nu_k - \tau_{jkl}(\boldsymbol{\Theta}) \nu_j \nu_k \nu_l \nu_i = 0, \text{ on } \partial\Omega \times [0, T), \tag{5.130b}$$

$$\boldsymbol{\Theta}(0) = \boldsymbol{0}, \text{ in } \Omega. \tag{5.130c}$$

In (5.129), $P_{\boldsymbol{\Theta}}$ represents, of course, the difference of the pressures $p_{\boldsymbol{w}}$ and $p_{\boldsymbol{U}}$ corresponding to the solutions of (5.120), (5.121a,b,c), and (5.66a–d), respectively, with the linearization in (5.66a) about $\boldsymbol{u}(t) = \boldsymbol{S}_{\mu_1}(t)\boldsymbol{u}_0$. If we multiply (5.129) through by Θ_i, integrate over Ω, and sum on $i = 1, 2, 3$, then the usual integration by parts scheme, coupled with (5.130a,b), leads us to

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\,||\boldsymbol{\Theta}||^2_{\boldsymbol{L}^2(\Omega)} &+ \int_\Omega \Theta_i \frac{\partial u_i}{\partial x_j}\Theta_j\,d\boldsymbol{x} + \int_\Omega w_j\,\frac{\partial w_i}{\partial x_j}\Theta_i\,d\boldsymbol{x}\\ &+ 2\mu_1 \int_\Omega \frac{\partial e_{ij}(\boldsymbol{\Theta})}{\partial x_k}\frac{\partial e_{ij}(\boldsymbol{\Theta})}{\partial x_k}d\boldsymbol{x}\\ &+ \int_\Omega [\gamma(\boldsymbol{v})e_{ij}(\boldsymbol{v}) - \gamma(\boldsymbol{u})e_{ij}(\boldsymbol{u})]e_{ij}(\boldsymbol{\Theta})d\boldsymbol{x}\\ &- \int_{\Omega)} [\gamma(\boldsymbol{u})e_{ij}(\boldsymbol{U}) - \alpha B_{ijkl}(\boldsymbol{u})e_{kl}(\boldsymbol{U})]e_{ij}(\boldsymbol{\Theta})d\boldsymbol{x} = 0.\end{aligned} \tag{5.131}$$

From the definition of the potential $\Gamma(e_{ij}e_{ij})$ in (5.92) we obtain

$$\frac{\partial \Gamma}{\partial e_{ij}}(\boldsymbol{e}(\boldsymbol{v})) = \gamma(\boldsymbol{v})e_{ij}(\boldsymbol{v}). \tag{5.132}$$

Using (5.132) we can now estimate the sum of the last two terms on the left-hand side of (5.131) as follows:

$$\begin{aligned}&\int_\Omega [\gamma(\boldsymbol{v})e_{ij}(\boldsymbol{v}) - \gamma(\boldsymbol{u})e_{ij}(\boldsymbol{u})]e_{ij}(\boldsymbol{\Theta})d\boldsymbol{x}\\ &\quad - \int_\Omega [\gamma(\boldsymbol{u})e_{ij}(\boldsymbol{U}) - \alpha B_{ijkl}(\boldsymbol{U})e_{kl}(\boldsymbol{U})]e_{ij}(\boldsymbol{\Theta})d\boldsymbol{x}\\ &= \int_\Omega \left[\frac{\partial \Gamma}{\partial e_{ij}}(\boldsymbol{e}(\boldsymbol{v}(t))) - \frac{\partial \Gamma}{\partial e_{ij}}(\boldsymbol{e}(\boldsymbol{u}(t)))\right] e_{ij}(\boldsymbol{\Theta})d\boldsymbol{x}\\ &\quad - \int_{\Omega)} \frac{\partial^2 \Gamma}{\partial e_{ij}\partial e_{kl}}(\boldsymbol{e}(\boldsymbol{u}(t)))e_{kl}(\boldsymbol{U})e_{ij}(\boldsymbol{\Theta})d\boldsymbol{x}\\ &= \int_\Omega \left(\int_0^1 \frac{\partial^2 \Gamma}{\partial e_{ij}\partial e_{kl}}(\boldsymbol{e}(\boldsymbol{u}+z\boldsymbol{w}))dz\right) e_{kl}(\boldsymbol{w})e_{ij}(\boldsymbol{\Theta})d\boldsymbol{x}\\ &\quad - \int_\Omega \left(\int_0^1 \frac{\partial^2 \Gamma}{\partial e_{ij}\partial e_{kl}}(\boldsymbol{e}(\boldsymbol{u}))dz\right) e_{kl}(\boldsymbol{U})e_{ij}(\boldsymbol{\Theta})d\boldsymbol{x}\\ &\geq \int_\Omega \left(\int_0^1 \frac{\partial^2 \Gamma}{\partial e_{ij}\partial e_{kl}}(\boldsymbol{e}(\boldsymbol{u}+z\boldsymbol{w}))dz\right) e_{kl}(\boldsymbol{w})e_{ij}(\boldsymbol{\Theta})d\boldsymbol{x}\\ &\quad \int_\Omega \left(\int_0^1 \frac{\partial^2 \Gamma}{\partial e_{ij}\partial e_{kl}}(\boldsymbol{e}(\boldsymbol{u}))dz\right) e_{kl}(\boldsymbol{w})e_{ij}(\boldsymbol{\Theta})d\boldsymbol{x}\end{aligned} \tag{5.133}$$

where we have dropped the term

$$\int_\Omega \frac{\partial^2 \Gamma}{\partial e_{ij}\partial e_{kl}}(\boldsymbol{e}(\boldsymbol{u}))e_{ij}(\boldsymbol{\Theta})e_{kl}(\boldsymbol{\Theta})d\boldsymbol{x}$$
$$= \int_\Omega [\gamma(\boldsymbol{u})e_{ij}(\boldsymbol{\Theta}) - \alpha B_{ijkl}(\boldsymbol{u})e_{kl}(\boldsymbol{\Theta})]e_{ij}(\boldsymbol{\Theta})d\boldsymbol{x} \geq 0$$

by virtue of (5.68) with $\boldsymbol{v} \to \boldsymbol{u}$ and $\boldsymbol{U} \to \boldsymbol{\Theta}$. Thus

$$\begin{aligned}
&\int_\Omega [\gamma(\boldsymbol{v})e_{ij}(\boldsymbol{v}) - \gamma(\boldsymbol{u})e_{ij}(\boldsymbol{u})]e_{ij}(\boldsymbol{\Theta})d\boldsymbol{x} - \int_\Omega [\gamma(\boldsymbol{u})e_{ij}(\boldsymbol{U}) - \alpha B_{ijkl}(\boldsymbol{U})e_{kl}(\boldsymbol{U})]e_{ij}(\boldsymbol{\Theta})d\boldsymbol{x} \\
&\geq \int_\Omega \left[\int_0^1 \left(\frac{\partial^2 \Gamma}{\partial e_{ij}\partial e_{kl}}(\boldsymbol{e}(\boldsymbol{u}+\tau\boldsymbol{w}) - \frac{\partial^2 \Gamma}{\partial e_{ij}\partial e_{kl}}(\boldsymbol{e}(\boldsymbol{u}))\right) d\tau\right] e_{kl}(\boldsymbol{w})e_{ij}(\boldsymbol{\Theta})d\boldsymbol{x} \qquad (5.134) \\
&= \int_\Omega \left(\int_0^1\int_0^1 \frac{\partial^3 \Gamma}{\partial e_{ij}\partial e_{kl}\partial e_{mn}}(\boldsymbol{e}[\boldsymbol{u}+\sigma((\boldsymbol{u}+\tau\boldsymbol{w})-\boldsymbol{u})])e_{mn}(\tau\boldsymbol{w})d\tau d\sigma\right) e_{kl}(\boldsymbol{w})e_{ij}(\boldsymbol{\Theta})d\boldsymbol{x} \\
&= \int_\Omega \left(\int_0^1\int_0^1 \frac{\partial^3 \Gamma}{\partial e_{ij}\partial e_{kl}\partial e_{mn}}(\boldsymbol{e}(\boldsymbol{u}+\sigma\tau\boldsymbol{w}))\cdot\tau d\tau d\sigma\right) e_{kl}(\boldsymbol{w})e_{ij}(\boldsymbol{\Theta})d\boldsymbol{x} \\
&= \int_\Omega \Gamma_{ijklmn}e_{ij}(\boldsymbol{\Theta})e_{kl}(\boldsymbol{w})e_{mn}(\boldsymbol{w})d\boldsymbol{x}
\end{aligned}$$

with

$$\Gamma_{ijklmn} = \int_0^1\int_0^1 \frac{\partial^3 \Gamma}{\partial e_{ij}\partial e_{kl}\partial e_{mn}}(\boldsymbol{e}(\boldsymbol{u}+\sigma\tau\boldsymbol{w}))\cdot\tau d\tau d\sigma.$$

Combining the last estimate in (5.134) with (5.131) we now obtain the differential inequality

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}||\boldsymbol{\Theta}||^2_{\boldsymbol{L}^2(\Omega)} + 2\mu_1 \int_\Omega \frac{\partial e_{ij}(\boldsymbol{\Theta})}{\partial x_k}\frac{\partial e_{ij}(\boldsymbol{\Theta})}{\partial x_k}d\boldsymbol{x} \\
&\quad \leq \left|\int_\Omega \Theta_i \frac{\partial u_i}{\partial x_j}\Theta_j d\boldsymbol{x}\right| + \left|\int_\Omega w_j \frac{\partial w_i}{\partial x_j}\Theta_i d\boldsymbol{x}\right| \qquad (5.135) \\
&\quad + \left|\int_\Omega \Gamma_{ijklmn}e_{ij}(\boldsymbol{\Theta})e_{kl}(\boldsymbol{w})e_{mn}(\boldsymbol{w})d\boldsymbol{x}\right|.
\end{aligned}$$

There remains the task of estimating the three terms on the right-hand side of (5.135); for the first of these terms, we have for any $\gamma > 0$, as $\boldsymbol{u}(t) \in B^{\rho'}_{\boldsymbol{H}^2(\Omega)}$, $\forall t \geq 0$

$$
\begin{aligned}
\left| \int_\Omega \Theta_i \frac{\partial u_i}{\partial x_j} \Theta_j \, d\boldsymbol{x} \right| &\leq ||\boldsymbol{\Theta}||_{\boldsymbol{L}^\infty(\Omega)} \, |\boldsymbol{u}|_{\boldsymbol{H}^1(\Omega)} \, ||\boldsymbol{\Theta}||_{\boldsymbol{L}^2(\Omega)} \\
&\leq c(\Omega) \, ||\boldsymbol{\Theta}||_{\boldsymbol{H}^2(\Omega)} \, |\boldsymbol{u}|_{\boldsymbol{H}^1(\Omega)} \, ||\boldsymbol{\Theta}||_{\boldsymbol{L}^2(\Omega)} \\
&\leq c(\Omega)\rho' \left(\frac{\gamma}{2} \, ||\boldsymbol{\Theta}||^2_{\boldsymbol{H}^2(\Omega)} + \frac{1}{2\gamma} \, ||\boldsymbol{\Theta}||^2_{\boldsymbol{L}^2(\Omega)} \right)
\end{aligned} \tag{5.136a}
$$

for some $c(\Omega) > 0$. Next, as $\boldsymbol{w}(t) \in B^{\rho'}_{\boldsymbol{H}^2(\Omega)}$, $\forall t \geq 0$,

$$
\begin{aligned}
\left| \int_\Omega w_j \frac{\partial w_i}{\partial x_j} \Theta_i \, d\boldsymbol{x} \right| &\leq ||\boldsymbol{w}||_{\boldsymbol{L}^\infty(\Omega)} \, |\boldsymbol{w}|_{\boldsymbol{H}^1(\Omega)} \, ||\boldsymbol{\Theta}||_{\boldsymbol{L}^2(\Omega)} \\
&\leq c(\Omega) \, ||\boldsymbol{u}||_{\boldsymbol{H}^2(\Omega)} \, |\boldsymbol{w}|_{\boldsymbol{H}^1(\Omega)} \, ||\boldsymbol{\Theta}||_{\boldsymbol{L}^2(\Omega)} \\
&\leq \frac{c(\Omega)\rho'}{2} \left(||\boldsymbol{w}||^2_{\boldsymbol{H}^2(\Omega)} + ||\boldsymbol{\Theta}||^2_{\boldsymbol{L}^2(\Omega)} \right).
\end{aligned} \tag{5.136b}
$$

A straightforward computation, based on the definitions of Γ and Γ_{ijklmn}, shows that for all possible combinations of the tensor indices, and all values of the arguments, $|\Gamma_{ijklmn}| \leq \epsilon^{-(1+\alpha)/2}$. By virtue of the Hölder Inequality we have, therefore, for some $c^{\#} = c^{\#}(\epsilon, \alpha, \Omega) > 0$,

$$
\int_\Omega \Gamma_{ijklmn} e_{ij}(\boldsymbol{\Theta}) e_{kl}(\boldsymbol{w}) e_{mn}(\boldsymbol{w}) \, d\boldsymbol{x} \leq c^{\#} \, ||\boldsymbol{w}||^2_{W^{1,3}(\Omega)} \, ||\boldsymbol{\Theta}||_{\boldsymbol{W}^{1,3}(\Omega)} \, .
$$

Applying the continuous embedding of $\boldsymbol{H}^2(\Omega)$ into $\boldsymbol{W}^{1,3}(\Omega)$ (a direct consequence of the continuous embedding of $\boldsymbol{W}^{1,2}(\Omega)$ into $\boldsymbol{L}^6(\Omega)$, e.g., Appendix A) we obtain from the last estimate, for some $\hat{c}(\epsilon, \alpha, \Omega) > 0$, and any $\gamma > 0$,

$$
\begin{aligned}
&\int_\Omega \Gamma_{ijklmn} e_{ij}(\boldsymbol{\Theta}) e_{kl}(\boldsymbol{w}) e_{mn}(\boldsymbol{w}) \, d\boldsymbol{x} \\
&\qquad \leq \hat{c} \, ||\boldsymbol{w}||^2_{\boldsymbol{H}^2(\Omega)} \, ||\boldsymbol{\Theta}||_{\boldsymbol{H}^2(\Omega)} \\
&\qquad \leq \hat{c}\rho' \, ||\boldsymbol{w}||_{\boldsymbol{H}^2(\Omega)} \, ||\boldsymbol{\Theta}||_{\boldsymbol{H}^2(\Omega)} \\
&\qquad \leq d(\rho')(\gamma \, ||\boldsymbol{\Theta}||^2_{\boldsymbol{H}^2(\Omega)} + \frac{1}{\gamma} \, ||\boldsymbol{w}||^2_{\boldsymbol{H}^2(\Omega)})
\end{aligned} \tag{5.136c}
$$

where $d(\rho') = \frac{1}{2}\hat{c}\rho'$ and we have again used the fact that $\boldsymbol{w}(t) \in B^{\rho'}_{\boldsymbol{H}^2(\Omega)}$, $\forall t \geq 0$. By combining (5.136a,b,c) with (5.135) we now obtain an estimate of the form

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt} \, ||\boldsymbol{\Theta}||^2_{\boldsymbol{L}^2(\Omega)} + 2\mu_1 \int_\Omega \frac{\partial e_{ij}(\boldsymbol{\Theta})}{\partial x_k} \frac{\partial e_{ij}(\boldsymbol{\Theta})}{\partial x_k} d\boldsymbol{x} &\leq (c(\Omega)\rho' + d(\rho'))\frac{\gamma}{2} \, ||\boldsymbol{\Theta}||^2_{\boldsymbol{H}^2(\Omega)} \\
+ \frac{1}{2}c(\Omega)\rho' \left(1 + \frac{1}{\gamma} \right) ||\boldsymbol{\Theta}||^2_{\boldsymbol{L}^2(\Omega)} &+ \frac{1}{2} \left(c(\Omega)\rho' + \frac{d(\rho')}{\gamma} \right) ||\boldsymbol{w}||^2_{\boldsymbol{H}^2(\Omega)} \, .
\end{aligned} \tag{5.137}
$$

Because $\boldsymbol{\Theta}(t) \in \boldsymbol{W}^{2,2}(\Omega)$, $\forall t \geq 0$, we may apply Lemma 5.2 to (5.137) and conclude that for $\gamma > 0$ chosen sufficiently small, $\exists \rho_1, \rho_2 > 0$ which depend (at most) on μ_1, ρ', and Ω, for which

$$\frac{d}{dt} ||\boldsymbol{\Theta}||^2_{\boldsymbol{L}^2(\Omega)} \leq \rho_1 ||\boldsymbol{w}||^2_{\boldsymbol{H}^2(\Omega)} + \rho_2 ||\boldsymbol{\Theta}||^2_{\boldsymbol{L}^2(\Omega)}, \tag{5.138}$$

for all $t \geq 0$. Because $\boldsymbol{\Theta}(0) = \boldsymbol{0}$, (5.138) implies that

$$||\boldsymbol{\Theta}||^2_{\boldsymbol{L}^2(\Omega)} \leq \rho_1 e^{\rho_2 t} \int_0^t ||\boldsymbol{w}||^2_{\boldsymbol{H}^2(\Omega)} \, ds, \quad t \geq 0. \tag{5.139}$$

Using the bound for $\int_0^t ||\boldsymbol{w}||^2_{\boldsymbol{H}^2(\Omega)} \, d\tau$, in (5.128), in the estimate (5.139), we are now led to the bound

$$||\boldsymbol{\Theta}||^2_{\boldsymbol{L}^2(\Omega)} \leq \rho_1 \left(\frac{\eta_2}{\eta_1}\right) e^{(\rho_2+\eta_2)t} \, ||\boldsymbol{v}_0 - \boldsymbol{u}_0||^2_{\boldsymbol{L}^2(\Omega)} \tag{5.140}$$

for $t \geq 0$ which has, as an immediate consequence, the following estimate for $t \geq 0$:

$$\sup_{\substack{\boldsymbol{v}_0, \boldsymbol{u}_0 \in \boldsymbol{\mathcal{A}}_{\mu_1} \\ 0 < ||\boldsymbol{v}_0 - \boldsymbol{u}_0||_{\boldsymbol{L}^2(\Omega)} < \epsilon}} ||\boldsymbol{\Theta}||_{\boldsymbol{L}^2(\Omega)} \leq \sqrt{\rho_1 \left(\frac{\eta_2}{\eta_1}\right) e^{(\rho_2+\eta_2)t}} \cdot \epsilon. \tag{5.141}$$

We now see that (5.119) holds with $c_1 = \sqrt{\rho_1 \left(\frac{\eta_2}{\eta_1}\right)}$ and $c_2 = \rho_2 + \eta_2$, thus, establishing the validity of Lemma 5.11. □

We are now in a position to proceed with the proof of the uniform differentiability of $\boldsymbol{S}_{\mu_1}(t)$.

Proof (Theorem 5.5). We begin by recycling (5.141) through suitably modified versions of some of our previous estimates. The estimate (5.136a) is retained in its current form but (5.136b) is altered to

$$\begin{aligned} \left| \int_\Omega w_j \frac{\partial w_i}{\partial x_j} \Theta_i \, d\boldsymbol{x} \right| &\leq c(\Omega) \, ||\boldsymbol{w}||_{\boldsymbol{H}^2(\Omega)} \, ||\boldsymbol{w}||_{\boldsymbol{H}^1(\Omega)} \, ||\boldsymbol{\Theta}||_{\boldsymbol{L}^2(\Omega)} \\ &\leq c(\Omega) \sqrt{\frac{\rho_1 \eta_2}{\eta_1}} e^{(\rho_2+\eta_2)t/2} \cdot \epsilon \, ||\boldsymbol{w}||^2_{\boldsymbol{H}^2(\Omega)} \end{aligned} \tag{5.142}$$

while, by the continuous embedding of $\boldsymbol{H}^2(\Omega)$ into $\boldsymbol{W}^{1,3}(\Omega)$, the first estimate in (5.136c) yields

$$\left| \int_\Omega \Gamma_{ijklmn} e_{ij}(\boldsymbol{\Theta}) e_{kl}(\boldsymbol{w}) e_{mn}(\boldsymbol{w}) \, d\boldsymbol{x} \right| \leq c'(\Omega) \, ||\boldsymbol{w}||^2_{\boldsymbol{H}^2(\Omega)} \, ||\boldsymbol{\Theta}||_{\boldsymbol{W}^{1,3}(\Omega)}. \tag{5.143}$$

We now avail ourselves of the following results on embedding and interpolation (see, e.g., [Te4] as well as Appendix A):

(i) $\boldsymbol{W}^{3/2,2}(\Omega)$ is continuously embedded in $\boldsymbol{W}^{1,3}(\Omega)$,
(ii) $\boldsymbol{H}^{m+s}(\Omega)$, $0 < s < 1$, $m \in N$, may be interpolated between $\boldsymbol{H}^{m+1}(\Omega)$ and $\boldsymbol{L}^2(\Omega)$.

In particular, for the case where $m = 1$ and $s = 1/2$, we have, for some $c(\Omega) > 0$, that for $0 < \delta < 1$

$$||\boldsymbol{\Theta}||_{\boldsymbol{W}^{3/2,2}(\Omega)} \leq c\, ||\boldsymbol{\Theta}||^{\delta}_{\boldsymbol{L}^2(\Omega)}\, ||\boldsymbol{\Theta}||^{1-\delta}_{\boldsymbol{H}^2(\Omega)}\,.$$

Thus, if we apply this last result with $\delta = 1/4$, and use the embedding of $\boldsymbol{W}^{3/2,2}$ into $\boldsymbol{W}^{1,3}$, we find that, for some $c^*(\Omega) > 0$

$$||\boldsymbol{\Theta}||_{\boldsymbol{W}^{1,3}(\Omega)} \leq c^*\, ||\boldsymbol{\Theta}||^{1/4}_{\boldsymbol{L}^2(\Omega)}\, ||\boldsymbol{\Theta}||^{3/4}_{\boldsymbol{H}^2(\Omega)}\,. \tag{5.144}$$

However, by virtue of Young's inequality, for any $\zeta > 0$, $q > 1$, we have

$$||\boldsymbol{\Theta}||^{1/4}_{\boldsymbol{L}^2(\Omega)}\, ||\boldsymbol{\Theta}||^{3/4}_{\boldsymbol{H}^2(\Omega)} \leq \frac{1}{q}\zeta^q\, ||\boldsymbol{\Theta}||^{3q/4}_{\boldsymbol{H}^2(\Omega)} + \frac{q-1}{q\zeta^{q/q-1}}\, ||\boldsymbol{\Theta}||^{q/4(q-1)}_{\boldsymbol{L}^2(\Omega)}\,. \tag{5.145}$$

Choosing $q = 8/3$ in (5.145) we obtain, as a consequence of (5.144)

$$||\boldsymbol{\Theta}||_{\boldsymbol{W}^{1,3}(\Omega)} \leq c^* \left\{ \frac{3\zeta^{8/3}}{8}\, ||\boldsymbol{\Theta}||^2_{\boldsymbol{H}^2(\Omega)} + \frac{5}{8\zeta^{8/5}}\, ||\boldsymbol{\Theta}||^{2/5}_{\boldsymbol{L}^2(\Omega)} \right\}. \tag{5.146}$$

By combining (5.143) and (5.146) we find that

$$\begin{aligned} &\left| \int_\Omega \Gamma_{ijklmn} e_{ij}(\boldsymbol{\Theta}) e_{kl}(\boldsymbol{w}) e_{mn}(\boldsymbol{w})\, d\boldsymbol{x} \right| \\ &\qquad \leq \bar{c}\, ||\boldsymbol{w}||^2_{\boldsymbol{H}^2(\Omega)} \left\{ 3\zeta^{8/3}\, ||\boldsymbol{\Theta}||^2_{\boldsymbol{H}^2(\Omega)} + 5\zeta^{-8/5}\, ||\boldsymbol{\Theta}||^{2/5}_{\boldsymbol{L}^2(\Omega)} \right\} \end{aligned} \tag{5.147}$$

which is valid for any $\zeta > 0$ and all $t \geq 0$. We now collect the estimates (5.136a), (5.142), and (5.147) and employ them in the differential inequality (5.135) so as to obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\, ||\boldsymbol{\Theta}||^2_{\boldsymbol{L}^2(\Omega)} + 2\mu_1 \int_\Omega \frac{\partial e_{ij}(\boldsymbol{\Theta})}{\partial x_k} \frac{\partial e_{ij}(\boldsymbol{\Theta})}{\partial x_k}\, d\boldsymbol{x} \\ &\quad \leq \frac{c(\Omega)\rho'}{2} \left(\gamma\, ||\boldsymbol{\Theta}||^2_{\boldsymbol{H}^2(\Omega)} + \frac{1}{\gamma}\, ||\boldsymbol{\Theta}||^2_{\boldsymbol{L}^2(\Omega)} \right) \\ &\qquad + \sqrt{\frac{\rho_1\eta_2}{\eta_1}}\, e^{(\rho_2+\eta_2)t/2} \cdot \epsilon\, ||\boldsymbol{w}||^2_{\boldsymbol{H}^2(\Omega)} \\ &\qquad + \bar{c}\, ||\boldsymbol{w}||^2_{\boldsymbol{H}^2(\Omega)} \left\{ 3\zeta^{8/3}\, ||\boldsymbol{\Theta}||^2_{\boldsymbol{H}^2(\Omega)} + 5\zeta^{-8/5}\, ||\boldsymbol{\Theta}||^{3/5}_{\boldsymbol{L}^2(\Omega)} \right\}. \end{aligned} \tag{5.148}$$

By choosing γ, ζ both sufficiently small, and taking account of the fact that $||\boldsymbol{w}||^2_{\boldsymbol{H}^2(\Omega)} \leq \rho'^2$, $\forall t \geq 0$, we may absorb those terms on the right-hand side of (5.148) which involve $||\boldsymbol{\Theta}||^2_{\boldsymbol{H}^2(\Omega)}$ into the second term on the left-hand side of this estimate by again using Lemma 5.2. In this manner we obtain from (5.148) the estimate

$$\frac{d}{dt}\,||\boldsymbol{\Theta}||^2_{\boldsymbol{L}^2(\Omega)} \leq 2\sqrt{\frac{\rho_1\eta_2}{\eta_1}}e^{(\rho_2+\eta_2)t/2}\cdot\epsilon\,||\boldsymbol{w}||^2_{\boldsymbol{H}^2(\Omega)} + 10\bar{c}\zeta^{-8/5}\,||\boldsymbol{\Theta}||^{2/5}_{\boldsymbol{L}^2(\Omega)}\,||\boldsymbol{w}||^2_{\boldsymbol{H}^2(\Omega)} + \frac{c(\Omega)\rho'}{\gamma}\,||\boldsymbol{\Theta}||^2_{\boldsymbol{L}^2(\Omega)}\,. \tag{5.149}$$

Using (5.41) to bound the term $||\boldsymbol{\Theta}||^{2/5}_{\boldsymbol{L}^2(\Omega)}$ in (5.149), we obtain the differential inequality

$$\frac{d}{dt}\,||\boldsymbol{\Theta}||^2_{\boldsymbol{L}^2(\Omega)} \leq [\epsilon k_1(t) + \epsilon^{2/5}k_2(t)]\,||\boldsymbol{w}||^2_{\boldsymbol{H}^2(\Omega)} + \frac{c(\Omega)\rho'}{\gamma}\,||\boldsymbol{\Theta}||^2_{\boldsymbol{L}^2(\Omega)} \tag{5.150}$$

with

$$\begin{cases} k_1(t) = 2\sqrt{\dfrac{\rho_1\eta_2}{\eta_1}}e^{(\rho_2+\eta_2)t/2}, \\ k_2(t) = 10\bar{c}\tau^{-8/5}\left(\dfrac{\rho_1\eta_2}{\eta_1}\right)^{1/5} e^{(\rho_2+\eta_2)t/5}. \end{cases} \tag{5.151}$$

As $\boldsymbol{\Theta}(0) = \boldsymbol{0}$ it follows from (5.150) that, $\forall t \geq 0$,

$$||\boldsymbol{\Theta}||^2_{\boldsymbol{L}^2(\Omega)} \leq [\epsilon k_1(t) + \epsilon^{2/5}k_2(t)]\exp\left(\frac{c(\Omega)\rho'}{\gamma}t\right)\int_0^t ||\boldsymbol{w}||^2_{\boldsymbol{H}^2(\Omega)}\,d\tau. \tag{5.152}$$

Therefore, by employing (5.128) in (5.152) we may conclude that, $\forall t \geq 0$,

$$||\boldsymbol{\Theta}||^2_{\boldsymbol{L}^2(\Omega)} \leq \frac{[\epsilon k_1(t) + \epsilon^{2/5}k_2(t)]}{\eta_1}\exp\left(\left[\eta_2 + \frac{c(\Omega)\rho'}{\gamma}\right]t\right)||\boldsymbol{v}_0 - \boldsymbol{u}_0||^2_{\boldsymbol{L}^2(\Omega)} \tag{5.153}$$

in which case, $\forall \boldsymbol{u}_0, \boldsymbol{v}_0 \in \boldsymbol{\mathcal{A}}_{\mu_1}$, with $||\boldsymbol{v}_0 - \boldsymbol{u}_0||_{L^2(\Omega)} < \epsilon$ sufficiently small, $\exists j(t) > 0$, for each $t > 0$, such that (5.118) is satisfied. □

The second of the two tasks delineated at the end of Sect. 5.3.2 was to establish that the Fréchet differential $\boldsymbol{L}_{\mu_1}(t;\boldsymbol{u}_0)$ of $\boldsymbol{S}_{\mu_1}(t)$, at any $\boldsymbol{u}_0 \in \boldsymbol{\mathcal{A}}_{\mu_1}$, is uniformly bounded, $\forall t \geq 0$, in the strong operator norm of $\boldsymbol{\mathcal{L}}(\boldsymbol{H};\boldsymbol{H})$; it is to this problem that we turn in the next subsection.

5.3.4 Uniform Boundedness of $L_{\mu_1}(t; u_0)$

In this section we show that the Fréchet differential $\boldsymbol{L}(t;\boldsymbol{u}_0)$, of the nonlinear semigroup $\boldsymbol{S}_{\mu_1}(t)$, at $\boldsymbol{u}_0 \in \boldsymbol{\mathcal{A}}_{\mu_1}$, is uniformly bounded, in the strong operator norm of $\boldsymbol{\mathcal{L}}(\boldsymbol{H};\boldsymbol{H})$, on the global attractor $\boldsymbol{\mathcal{A}}_{\mu_1}$, for all $t \ge 0$, where $\boldsymbol{H} = \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \mid \nabla \cdot \boldsymbol{v} = 0\}$. In more precise terms, we will demonstrate the following:

Theorem 5.6. *Let $\boldsymbol{J} \in \boldsymbol{H}$ and let $\boldsymbol{L}_{\mu_1}(t,\boldsymbol{u}_0)$: $\boldsymbol{J} \mapsto \boldsymbol{U}(t)$, where $\boldsymbol{U}(t)$ is the unique solution of (5.66a–d) in $\boldsymbol{L}^\infty([0,\infty);\boldsymbol{H})$ satisfying $\boldsymbol{U}_0 = \boldsymbol{J}$, the linearization in (5.66a) being taken about $\boldsymbol{u}(t) = \boldsymbol{S}_{\mu_1}(t)\boldsymbol{u}_0$, with $\boldsymbol{u}_0 \in \boldsymbol{\mathcal{A}}_{\mu_1}$. Then $\exists \ell > 0$ such that for all $t \ge 0$*

$$\sup_{\boldsymbol{u}_0 \in \boldsymbol{\mathcal{A}}_{\mu_1}} |\boldsymbol{L}_{\mu_1}(t;\boldsymbol{u}_0)|_{\boldsymbol{\mathcal{L}}(\boldsymbol{H};\boldsymbol{H})} \le \ell^{[t]+1}. \tag{5.154}$$

Proof. The linearized equations are, in this case,

$$\begin{aligned}\frac{\partial U_i}{\partial t} + U_j \frac{\partial u_i}{\partial x_j} + u_j \frac{\partial U_i}{\partial x_j} &= -\frac{\partial P_{\boldsymbol{U}}}{\partial x_i} \\ &+ \frac{\partial}{\partial x_j}[\gamma(\boldsymbol{u})e_{ij}(\boldsymbol{U}) - \alpha B_{ijkl}(\boldsymbol{u})e_{kl}(\boldsymbol{U})] - 2\mu_1 \frac{\partial}{\partial x_j}(\Delta e_{ij}(\boldsymbol{U}))\end{aligned} \tag{5.155}$$

in $\Omega \times [0,T]$; the U_i also satisfy (5.66b,c,d) with $\boldsymbol{U}_0 = \boldsymbol{J} \in \boldsymbol{H}$. As a consequence of $\boldsymbol{u}_0 \in \boldsymbol{\mathcal{A}}_{\mu_1}$, $\boldsymbol{u}(t) \in B^{\rho'}_{\boldsymbol{H}^2(\Omega)}$, $\forall t \ge 0$, while for any $t > 0$ and $\boldsymbol{u}_0 \in \boldsymbol{\mathcal{A}}_{\mu_1}$,

$$\left|\left|\boldsymbol{L}_{\mu_1}(t;\boldsymbol{u}_0)\right|\right|_{\boldsymbol{\mathcal{L}}(\boldsymbol{H};\boldsymbol{H})} = \sup_{\boldsymbol{J}\in\boldsymbol{H}} \frac{\left|\left|\boldsymbol{L}_{\mu_1}(t;\boldsymbol{u}_0)\boldsymbol{J}\right|\right|_{\boldsymbol{L}^2(\Omega)}}{||\boldsymbol{J}||_{\boldsymbol{L}^2(\Omega)}}. \tag{5.156}$$

In the usual manner, we multiply (5.155) through by U_i, integrate over Ω, and sum over $i = 1, 2, 3$, obtaining,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||\boldsymbol{U}||^2_{\boldsymbol{L}^2(\Omega)} &+ 2\mu_1 \int_\Omega \frac{\partial e_{ij}(\boldsymbol{U})}{\partial x_k}\frac{\partial e_{ij}(\boldsymbol{U})}{\partial x_k}d\boldsymbol{x} \\ &\le \left|\int_\Omega U_j \frac{\partial u_i}{\partial x_j} U_i \, d\boldsymbol{x}\right| = \left|\int_\Omega U_j \frac{\partial U_i}{\partial x_j} u_i \, d\boldsymbol{x}\right|\end{aligned} \tag{5.157}$$

where, on the left-hand side of the estimate (5.157) we have dropped the non-negative integral

$$\int_\Omega \left[\gamma(\boldsymbol{u})e_{ij}(\boldsymbol{U})e_{ij}(\boldsymbol{U}) - \alpha B_{ijkl}(\boldsymbol{u})e_{ij}(\boldsymbol{U})e_{kl}(\boldsymbol{U})\right] d\boldsymbol{x}.$$

From (5.157) we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\,||\boldsymbol{U}||^2_{\boldsymbol{L}^2(\Omega)} + 2\mu_1 \int_\Omega \frac{\partial e_{ij}(\boldsymbol{U})}{\partial x_k}\frac{\partial e_{ij}(\boldsymbol{U})}{\partial x_k}d\boldsymbol{x} &\\ \le ||\boldsymbol{U}||_{\boldsymbol{L}^2(\Omega)}\,||\boldsymbol{U}||_{\boldsymbol{H}^1(\Omega)}\,||\boldsymbol{u}||_{\boldsymbol{L}^\infty(\Omega)} &\\ \le ||\boldsymbol{U}||_{\boldsymbol{L}^2(\Omega)}\,||\boldsymbol{U}||_{\boldsymbol{H}^2(\Omega)}\,||\boldsymbol{u}||_{\boldsymbol{H}^2(\Omega)} &\\ \le \rho'\,||\boldsymbol{U}||_{\boldsymbol{L}^2(\Omega)}\,||\boldsymbol{U}||_{\boldsymbol{H}^2(\Omega)} &\end{aligned} \tag{5.158}$$

as $||\boldsymbol{u}||_{\boldsymbol{H}^2(\Omega)} \le \rho'$, $\forall t \ge 0$. For any $\delta > 0$ we have, therefore,

$$\frac{1}{2}\frac{d}{dt}\,||\boldsymbol{U}||^2_{\boldsymbol{L}^2(\Omega)} + 2\mu_1 \int_\Omega \frac{\partial e_{ij}(\boldsymbol{U})}{\partial x_k}\frac{\partial e_{ij}(\boldsymbol{U})}{\partial x_k}d\boldsymbol{x} \le \frac{\rho'}{2\delta}\,||\boldsymbol{U}||^2_{\boldsymbol{L}^2(\Omega)} + \frac{\rho'\delta}{2}\,||\boldsymbol{U}||^2_{\boldsymbol{H}^2(\Omega)}\,. \tag{5.159}$$

Noting that with $\boldsymbol{u}(t) \in B^{\rho'}_{\boldsymbol{H}^2(\Omega)}$, $\forall t \ge 0$, we also have $\boldsymbol{U} \in \boldsymbol{L}^\infty((0,t);\boldsymbol{H}^2(\Omega))$, $\forall t > 0$, an application of the Korn-type estimate of Lemma 5.2 to (5.159) produces, for $\delta > 0$ chosen sufficiently small,

$$\frac{d}{dt}\,||\boldsymbol{U}||^2_{\boldsymbol{L}^2(\Omega)} + \kappa\,||\boldsymbol{U}||^2_{\boldsymbol{H}^2(\Omega)} \le \left(\frac{\rho'}{2\delta}\right)||\boldsymbol{U}||^2_{\boldsymbol{L}^2(\Omega)} \tag{5.160}$$

for some $\kappa = \kappa(\rho', \mu_1, \Omega) > 0$. From (5.160) it is immediate that

$$||\boldsymbol{U}(t)||_{\boldsymbol{L}^2(\Omega)} \le ||\boldsymbol{U}(0)||_{\boldsymbol{L}^2(\Omega)}\,e^{\frac{\rho'}{2\delta}t}, \quad \forall t \ge 0. \tag{5.161}$$

Therefore, for all $t \ge 0$, $\boldsymbol{u}_0 \in \boldsymbol{\mathcal{A}}_{\mu_1}$, and $\boldsymbol{J} \in \boldsymbol{H}$,

$$\frac{\left|\left|\boldsymbol{L}_{\mu_1}(t;\boldsymbol{u}_0)\boldsymbol{J}\right|\right|_{\boldsymbol{L}^2(\Omega)}}{||\boldsymbol{J}||_{\boldsymbol{L}^2(\Omega)}} \le e^{\frac{\rho'}{2\delta}t} \tag{5.162}$$

from which it follows, immediately, that

$$\sup_{0\le t\le 1}\ \sup_{\boldsymbol{u}_0\in\boldsymbol{\mathcal{A}}_{\mu_1}} \left|\left|\boldsymbol{L}_{\mu_1}(t,\boldsymbol{u}_0)\right|\right|_{\boldsymbol{\mathcal{L}}(\boldsymbol{H};\boldsymbol{H})} \le e^{\frac{\rho'}{2\delta}} \equiv \ell. \tag{5.163}$$

Following the analysis in [Te4], Chap. V, Sect. 2, we conclude from the relation

$$\boldsymbol{S}_{\mu_1}(t) = \boldsymbol{S}_{\mu_1}(t-[t])\boldsymbol{S}_{\mu_1}(t)^{[t]} \tag{5.164}$$

that (5.163) implies that (5.154) holds $\forall t \ge 0$. □

5.3.5 Hausdorff and Fractal Dimensions of the Global Attractor $\mathcal{A}_{\mu_1}$

Having established, in Sect. 5.3.3, the uniform differentiability of $\boldsymbol{S}_{\mu_1}(5)$ on $\mathcal{A}_{\mu_1}$, and in Sect. 5.3.4 the uniform boundedness of $\boldsymbol{L}_{\mu_1}(t;,\boldsymbol{u}_0)$, $\boldsymbol{u}_0 \in \mathcal{A}_{\mu_1}$, we are now in a position to estimate the Hausdorff and fractal dimensions of the global attractor $\mathcal{A}_{\mu_1}$. We begin by recalling some basic properties and definitions that will be employed in the computations of $d_H(\mathcal{A}_{\mu_1})$ and $d_F(\mathcal{A}_{\mu_1})$. In particular, for any $\boldsymbol{L} \in \mathcal{L}(\boldsymbol{H};\boldsymbol{H})$, and each positive integer k, we set

$$\alpha_k(\boldsymbol{L}) = \sup_{\substack{G\subset H \\ \dim G = k}} \inf_{\substack{\xi\in G \\ ||\xi||_{\boldsymbol{H}}=1}} ||\boldsymbol{L}\boldsymbol{\zeta}||_{\boldsymbol{L}^2(\Omega)} \tag{5.165}$$

and

$$\omega_k(\boldsymbol{L}) = \alpha_1(\boldsymbol{L})\cdots\alpha_k(\boldsymbol{L}). \tag{5.166}$$

The sequence $\{\alpha_k(\boldsymbol{L})\}$ is non-increasing and for $\boldsymbol{L}$ a compact self-adjoint non-negative linear operator on $\boldsymbol{H}$ the $\alpha_k(\boldsymbol{L})$ are just the eigenvalues of $(\boldsymbol{L}^*\boldsymbol{L})^{1/2}$, with $\alpha_1(\boldsymbol{L}) \geq \alpha_2(\boldsymbol{L}) \geq \cdots \geq 0$, where $\boldsymbol{L}^*$ is the adjoint operator. For $\boldsymbol{L} \in \mathcal{L}(\boldsymbol{H};\boldsymbol{H})$ and $d \in R^+$, $d = n + s$, n an integer ≥ 1 and $0 < s < 1$, we define

$$\omega_d(\boldsymbol{L}) = \omega_n(\boldsymbol{L})^{1-s}\omega_{n+1}(\boldsymbol{L})^s. \tag{5.167}$$

It is easy to see (e.g., [Te4], Chp. 5, Sect. 2) that $d \mapsto \omega_d(\boldsymbol{L})$ is a non-increasing function from $[1,\infty)$ into R^+. Let $\boldsymbol{S}_{\mu_1}(t)$ be the nonlinear semigroup generated by the solution of the initial-boundary value problem (5.2a,b), (5.3a), (5.4) and let $\boldsymbol{L}_{\mu_1}(t;\boldsymbol{u}_0)$ be the associated Fréchet differential, with $\boldsymbol{u}_0 \in \mathcal{A}_{\mu_1}$; the numbers $\omega_k(\boldsymbol{L}_{\mu_1}(t;\boldsymbol{u}_0))$ may be proven to bound the largest distortion of an infinitesimal k-dimensional volume produced by $\boldsymbol{S}_{\mu_1}(t)$ around the point μ_0. By virtue of Theorem 5.4 which yields the uniform differentiability of $\boldsymbol{S}_{\mu_1}(t)$ on $\mathcal{A}_{\mu_1}$, the numbers $\omega_k(\boldsymbol{L}_{\mu_1}(t;\boldsymbol{u}_0))$ are well-defined $\forall t \geq 0$, $k \in N$, and $\boldsymbol{u}_0 \in \mathcal{A}_{\mu_1}$. Now, we set

$$\bar{\omega}_k^{\mu_1}(t) = \sup_{\boldsymbol{u}_0\in\mathcal{A}_{\mu_1}} (\boldsymbol{L}_{\mu_1}(t;\boldsymbol{u}_0)), \quad k \in N, \quad t \geq 0. \tag{5.168}$$

By virtue of the definitions (5.167), (5.168) and the estimate (5.162), it follows that the functions $t \mapsto \bar{\omega}_k^{\mu_1}(t)$ are subexponential, i.e.,

$$\bar{\omega}_k^{\mu_1}(t+s) \leq \bar{\omega}_k^{\mu_1}(t)\bar{\omega}_k^{\mu_1}(s), \quad \forall t,s \geq 0. \tag{5.169}$$

Thus, by results in [Te4], the limit $\lim\limits_{t\to\infty}\{\bar{\omega}_k^{\mu_1}(t)\}^{1/t}$ exists and is equal to

$$\Pi_k^{\mu_1} = \inf_{t>0}\{\bar{\omega}_k^{\mu_1}(t)\}^{1/t}. \tag{5.170}$$

Next, we define, recursively, the numbers

$$\Lambda_1^{\mu_1} = \Pi_1^{\mu_1}, \quad \Lambda_1^{\mu_1}\Lambda_2^{\mu_1} = \Pi_2^{\mu_1}, \quad \dots, \quad \Lambda_1^{\mu_1}\cdots\Lambda_k^{\mu_1} = \Pi_k^{\mu_1}$$

or

$$\Lambda_1^{\mu_1} = \Pi_1^{\mu_1}, \quad \Lambda_k^{\mu_1} = \frac{\Pi_k^{\mu_1}}{\Pi_{k-1}^{\mu_1}}, \quad k \geq 2. \tag{5.171}$$

The $\Lambda_k^{\mu_1}$ are the global (or uniform) Lyapunov numbers on $\boldsymbol{\mathcal{A}}_{\mu_1}$ while the numbers

$$\lambda_k^{\mu_1} = \log \Lambda_k^{\mu_1}, \quad k \geq 1, \tag{5.172}$$

are the global (uniform) Lyapunov exponents. From (5.170)–(5.172) it follows that

$$\inf_{t>0}(\bar{\omega}_k^{\mu_1}(t))^{1/t} = \exp(\lambda_k^{\mu_1} + \cdots + \lambda_k^{\mu_1}). \tag{5.173}$$

Employing the linearized bipolar equations (5.155) we define the linear operator $\boldsymbol{\mathcal{L}}_{\mu_1}(\boldsymbol{u})$:

$$\boldsymbol{\mathcal{L}}_{\mu_1}(\boldsymbol{u}) : \tilde{\boldsymbol{H}} \to \boldsymbol{H}; \quad \boldsymbol{u} = \boldsymbol{S}_{\mu_1}(t)\boldsymbol{u}_0, \quad \boldsymbol{u}_0 \in \boldsymbol{\mathcal{A}}_{\mu_1},$$

where

$$\tilde{\boldsymbol{H}} = \{\boldsymbol{\phi} \in \boldsymbol{W}_0^{1,2}(\Omega) \cap \boldsymbol{W}^{2,2}(\Omega) \mid \operatorname{div}\boldsymbol{\phi} = 0 \text{ in } \Omega$$
$$\text{with } \tau_{ijk}(\boldsymbol{\phi})\nu_j\nu_k - \tau_{jkl}(\boldsymbol{\phi})\nu_j\nu_k\nu_l\nu_i = 0 \text{ on } \partial\Omega,\ i = 1,2,3\}$$

by

$$(\boldsymbol{\mathcal{L}}_{\mu_1}(\boldsymbol{u})\boldsymbol{\phi})_i = 2\mu_1\frac{\partial}{\partial x_j}(\Delta e_{ij}(\boldsymbol{\phi})) - \frac{\partial}{\partial x_j}[\gamma(\boldsymbol{u})e_{ij}(\boldsymbol{\phi}) - \alpha B_{ijkl}(\boldsymbol{u})e_{kl}(\boldsymbol{\phi})] + u_j\frac{\partial \phi_i}{\partial x_j} + \phi_j\frac{\partial u_i}{\partial x_j}. \tag{5.174}$$

Appealing yet one more time to the general framework in [Te4], Chap. 5, Sect. 2, it may be shown that

$$\lambda_1^{\mu_1} + \cdots + \lambda_k^{\mu_1} \leq -q_k^{\mu_1} \tag{5.175}$$

where

$$q_k^{\mu_1} = \lim_{t\to\infty}\sup_{\boldsymbol{u}_0\in\boldsymbol{\mathcal{A}}_{\mu_1}}\frac{1}{t}\int_0^t \inf_{\operatorname{rank}\boldsymbol{Q}=k}\operatorname{tr}(\boldsymbol{\mathcal{L}}_{\mu_1}(\boldsymbol{u})\circ\boldsymbol{Q})ds \tag{5.176}$$

with tr denoting the trace operation and $\boldsymbol{Q}$ an orthogonal projection on $\tilde{\boldsymbol{H}}$ of rank k. For the purposes of computation, we note here that

$$\mathrm{tr}(\boldsymbol{\mathcal{L}}_{\mu_1}(\boldsymbol{u}) \circ \boldsymbol{Q}) = \sum_{j=1}^{k} (\boldsymbol{\mathcal{L}}_{\mu_1}(\boldsymbol{u})\boldsymbol{\phi}_j, \boldsymbol{\phi}_j)_{\boldsymbol{L}^2(\Omega)} \tag{5.177}$$

where $\{\boldsymbol{\phi}_j\}_{j\in N}$ is any basis of $\tilde{\boldsymbol{H}}$, with the $\boldsymbol{\phi}_j$ orthonormal in $\boldsymbol{L}^2(\Omega)$ and such that $\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_k$ is a basis of $\boldsymbol{Q} \cdot \boldsymbol{H}$.

In order to connect the numbers $q_k^{\mu_1}$ to the computation (presented below), of the upper bounds for the Hausdorff and fractal dimensions of the global attractor $\boldsymbol{\mathcal{A}}_{\mu_1}$, we recall the precise definitions of these quantities: Let $\boldsymbol{X} \subset \boldsymbol{H}$, $d \in R^+$, and $\delta > 0$ and set

$$\mu(\boldsymbol{X}, d, \delta) = \inf \sum_j r_j^d \tag{5.178}$$

where the inf is taken over all coverings of $\boldsymbol{X}$ by a family of balls in $\boldsymbol{H}$ of radius $r_j \leq \delta$. Then $\mu(\boldsymbol{X}, d, \delta)$ is a decreasing function of δ and

$$\mu(\boldsymbol{X}, d) = \lim_{\delta \to 0} \mu(\boldsymbol{X}, d, \delta) \tag{5.179}$$

is the d-dimensional Hausdorff measure of $\boldsymbol{X}$. It follows that $\exists d_0 \in [0, \infty)$ such that $\mu(\boldsymbol{X}, d) = +\infty$ for $d < d_0$ while $\mu(\boldsymbol{X}, d) = 0$ for $d > d_0$; the number $d_0 = d_H(\boldsymbol{X})$ is called the *Hausdorff dimension of* $\boldsymbol{X}$. Next, let $n_{\boldsymbol{X}}(\delta)$ be the minimal number of balls in $\boldsymbol{H}$ of radius δ needed to cover $\boldsymbol{X}$; then the *fractal dimension* $d_F(\boldsymbol{X})$ is the number defined by

$$d_F(\boldsymbol{X}) = \limsup_{\delta \to 0} \frac{\log n_{\boldsymbol{X}}(\delta)}{\log(1/\delta)} \tag{5.180}$$

and it may be shown that

$$d_F(\boldsymbol{X}) = \inf\{d > 0 \mid \limsup_{\delta \to 0} \delta^d n_{\boldsymbol{X}}(\delta) = 0\} \tag{5.181}$$

from which it follows that $d_H(\boldsymbol{X}) \leq d_F(\boldsymbol{X})$.

We want to apply the definitions, above, of the Hausdorff and fractal dimensions to the case where $\boldsymbol{X} = \boldsymbol{\mathcal{A}}_{\mu_1}$, the global attractor for the semigroup $\boldsymbol{S}_{\mu_1}(t)$ generated by the solution of (5.2a,b), (5.3a), (5.4). The following result is a direct consequence of the existence theorem of Sect. 4.2, Theorems 5.2–5.6, the fact that the initial data $\boldsymbol{u}_0 \in \boldsymbol{\mathcal{A}}_{\mu_1}$, and Theorem 3.3 in [Te4]:

Lemma 5.12. *If for some* $k \geq 1$, $\lambda_1^{\mu_1} + \cdots + \lambda_k^{\mu_1} < 0$, *then* $\lambda_k^{\mu_1} < 0$, $(\lambda_1^{\mu_1} + \cdots + \lambda_{k-1}^{\mu_1})/|\lambda_k^{\mu_1}| < 1$ *and*

$$d_H(\mathcal{A}_{\mu_1}) \leq (k-1) + \frac{(\lambda_1^{\mu_1} + \cdots + \lambda_{k-1}^{\mu_1})_+}{|\lambda_k^{\mu_1}|} \tag{5.182a}$$

$$d_F(\mathcal{A}_{\mu_1}) \leq k \cdot \left\{ \max_{1 \leq j \leq k-1} 1 + \frac{(\lambda_1^{\mu_1} + \cdots + \lambda_j^{\mu_1})_+}{|\lambda_1^{\mu_1} + \cdots + \lambda_k^{\mu_1}|} \right\} \tag{5.182b}$$

where, for any $r \in R$, $r_+ = \max(r, 0)$.

Our main result in this subsection can now be expressed in the following form:

Theorem 5.7. *Consider the dynamical system defined by the nonlinear bipolar initial-boundary value problem* (5.2a,b), (5.3a), (5.4) *with* $\boldsymbol{v}_0 \in \boldsymbol{H}$. *Then the global attractor* $\mathcal{A}_{\mu_1}$ *defined by* (5.17) *has Hausdorff dimension less than or equal to* k, *and fractal dimension less than or equal to* $2k$, *where* $k \in N$ *satisfies*

$$k - 1 < \psi |\boldsymbol{f}|_\infty^3 (\lambda_1 \Lambda)^{-3/2} \mu_1^{-9/2} < k \tag{5.183}$$

for some $\psi = \psi(\Omega)$, $\Lambda = \Lambda(\mu_0, \mu_1, \alpha; \Omega)$ *where* λ_1 *is the smallest eigenvalue of* $-\Delta$ *on* Ω *such that the corresponding eigenvector* $\boldsymbol{w}_1$ *is in* $\boldsymbol{H}_0^1(\Omega)$ *and satisfies* $\nabla \cdot \boldsymbol{w}_1 = 0$.

In order to prove Theorem 5.7 we will proceed with first establishing a series of lemmas, beginning with

Lemma 5.13. *Let* $\boldsymbol{Q}$ *be an orthogonal projection on* $\tilde{\boldsymbol{H}}$ *of rank* k; *then for* $\boldsymbol{\phi}_j \in \tilde{\boldsymbol{H}}$, $j = 1, 2, \ldots, k$, *such that* $\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_k$ *forms a basis of* $\boldsymbol{Q} \circ \tilde{\boldsymbol{H}}$, *with the* $\boldsymbol{\phi}_j$ *orthonormal in* $\boldsymbol{L}^2(\Omega)$, $\exists c = c(\Omega)$, $\tilde{k} = \tilde{k}(\Omega)$ *such that for any* $\delta > 0$,

$$\begin{aligned}
\operatorname{tr}(\mathcal{L}_{\mu_1}(\boldsymbol{u}) \circ \boldsymbol{Q}) &= \sum_{t=1}^{k} (\mathcal{L}_{\mu_1}(\boldsymbol{u})\boldsymbol{\phi}_\ell, \boldsymbol{\phi}_\ell)_{\boldsymbol{L}^2(\Omega)} \\
&\geq 2\mu_1 \sum_{\ell=1}^{k} \int_\Omega \frac{\partial e_{ij}(\boldsymbol{\phi}_\ell)}{\partial x_k} \frac{\partial e_{ij}(\boldsymbol{\phi}_\ell)}{\partial x_k} d\boldsymbol{x} \\
&\quad + 2\tilde{k}(\Omega)(1-\alpha)\mu_0 \sum_{\ell=1}^{k} ||\boldsymbol{\phi}_\ell||_{\boldsymbol{W}^{1,p}(\Omega)}^2 \\
&\quad - \delta c(\Omega) \sum_{\ell=1}^{k} |\boldsymbol{\phi}_\ell|_{\boldsymbol{H}^1(\Omega)}^2 - \frac{c(\Omega)}{\delta} ||\boldsymbol{u}||_{\boldsymbol{H}^2(\Omega)}.
\end{aligned} \tag{5.184}$$

Proof. For any $\boldsymbol{\phi} \in \tilde{\boldsymbol{H}}$ we compute that

$$
\begin{aligned}
(\boldsymbol{\mathcal{L}}_{\mu_1}(\boldsymbol{u})\boldsymbol{\phi}, \boldsymbol{\phi})_{\boldsymbol{L}^2(\Omega)} &= 2\mu_1 \int_\Omega \frac{\partial e_{ij}(\boldsymbol{\phi})}{\partial x_k}\frac{\partial e_{ij}(\boldsymbol{\phi})}{\partial x_k} d\boldsymbol{x} \\
&+ \int_\Omega [\gamma(\boldsymbol{u})e_{ij}(\boldsymbol{\phi})e_{ij}(\boldsymbol{\phi}) - \alpha B_{ijkl}(\boldsymbol{u})e_{ij}(\boldsymbol{\phi})e_{kl}(\boldsymbol{\phi}) d\boldsymbol{x} + \int_\Omega \phi_j \frac{\partial u_i}{\partial x_j}\phi_i \, d\boldsymbol{x}
\end{aligned}
$$

in which case

$$
\begin{aligned}
(\boldsymbol{\mathcal{L}}_{\mu_1}(\boldsymbol{u})\boldsymbol{\phi}, \boldsymbol{\phi})_{\boldsymbol{L}^2(\Omega)} &\geq 2\mu_1 \int_\Omega \frac{\partial e_{ij}(\boldsymbol{\phi})}{\partial x_k}\frac{\partial e_{ij}(\boldsymbol{\phi})}{\partial x_k} d\boldsymbol{x} \\
&+ 2(1-\alpha)\mu_0 \int_\Omega \frac{e_{ij}(\boldsymbol{\phi})e_{ij}(\boldsymbol{\phi})}{(\epsilon + e_{kl}(\boldsymbol{u})e_{kl}(\boldsymbol{u}))^{\alpha/2}} d\boldsymbol{x} + \int_\Omega \phi_j \frac{\partial u_i}{\partial x_j}\phi_i \, d\boldsymbol{x}.
\end{aligned}
\tag{5.185}
$$

Now, for any $\boldsymbol{\phi} \in \boldsymbol{H}_0^1(\Omega)$ such that $||\boldsymbol{\phi}||_{\boldsymbol{L}^2(\Omega)} = 1$,

$$
\begin{aligned}
\left| \int_\Omega \phi_j \frac{\partial u_i}{\partial x_j}\phi_i \, d\boldsymbol{x} \right| &\leq ||\boldsymbol{\phi}||_{\boldsymbol{L}^2(\Omega)} \, ||\boldsymbol{\phi}||_{\boldsymbol{H}^1(\Omega)} \, ||\boldsymbol{u}||_{\boldsymbol{L}^\infty(\Omega)} \\
&\leq 2c(\Omega) \, ||\boldsymbol{\phi}||_{\boldsymbol{H}^1(\Omega)} \, ||\boldsymbol{u}||_{\boldsymbol{H}^2(\Omega)}
\end{aligned}
\tag{5.186}
$$

for some $c(\Omega) > 0$. Therefore, for any $\delta > 0$,

$$
\left| \int_\Omega \phi_j \frac{\partial u_i}{\partial x_j}\phi_i \, d\boldsymbol{x} \right| \leq c(\Omega) \left[\delta \, ||\boldsymbol{\phi}||^2_{\boldsymbol{H}^1(\Omega)} + \frac{1}{\delta} \, ||\boldsymbol{u}||^2_{\boldsymbol{H}^2(\Omega)} \right]. \tag{5.187}
$$

We now need to estimate, from below, the integral

$$
I = \int_\Omega \frac{e_{ij}(\boldsymbol{\phi})e_{ij}(\boldsymbol{\phi})}{(\epsilon + e_{kl}(\boldsymbol{u})e_{kl}(\boldsymbol{u}))^{\alpha/2}} d\boldsymbol{x}. \tag{5.188}
$$

For convenience we set $p = 2 - \alpha$ and $|\boldsymbol{e}(\boldsymbol{u})|^2 = e_{ij}(\boldsymbol{u})e_{ij}(\boldsymbol{u})$; then we have the following result which is of independent interest and is, thus, stated separately as

Lemma 5.14. *Let* $\boldsymbol{\phi} \in \tilde{\boldsymbol{H}} = \{\boldsymbol{\psi} \in \boldsymbol{W}_0^{1,2}(\Omega) \cap \boldsymbol{W}^{2,2}(\Omega) \mid \nabla \cdot \boldsymbol{\psi} = 0$ *in* Ω *with* $\tau_{ijk}(\boldsymbol{\psi})\nu_j\nu_k - \tau_{jkl}(\boldsymbol{\psi})\nu_j\nu_k\nu_l\nu_i = 0$ *on* $\partial\Omega$, $i = 1, 2, 3\}$. *Then if* $\boldsymbol{u}$ *is the solution of* (5.2a,b)*,* (5.3a)*,* (5.4) *corresponding to initial data* $\boldsymbol{u}_0 \in \boldsymbol{\mathcal{A}}_{\mu_1}$*,*

$$
\int_\Omega \frac{e_{ij}(\boldsymbol{\phi})e_{ij}(\boldsymbol{\phi})}{(\epsilon + |\boldsymbol{e}(\boldsymbol{u})|^2)^{(2-p)/2}} d\boldsymbol{x} \geq \tilde{k}(\Omega) \, ||\boldsymbol{\phi}||^2_{\boldsymbol{W}^{1,p}(\Omega)} \tag{5.189}
$$

for some $\tilde{k}(\Omega) > 0$ *and all* $t \geq 0$*.*

Proof. By the Hölder Inequality

$$\int_\Omega [e_{ij}(\boldsymbol{\phi})e_{ij}(\boldsymbol{\phi})]^{p/2} d\boldsymbol{x} = \int_\Omega \left[\frac{e_{ij}(\boldsymbol{\phi})e_{ij}(\boldsymbol{\phi})}{(\epsilon + |\boldsymbol{e}(\boldsymbol{u})|^2)^{(2-p)/2}}\right]^{p/2} \cdot (\epsilon + |\boldsymbol{e}(\boldsymbol{u})|^2)^{p(2-p)/4} d\boldsymbol{x}$$
$$\leq \left(\int_\Omega \left[\frac{e_{ij}(\boldsymbol{\phi})e_{ij}(\boldsymbol{\phi})}{(\epsilon + |\boldsymbol{e}(\boldsymbol{u})|^2)^{(2-p)/2}}\right] d\boldsymbol{x}\right)^{p/2} \times \left(\int_\Omega (\epsilon + |\boldsymbol{e}(\boldsymbol{u})|^2)^{p/2} d\boldsymbol{x}\right)^{(2-p)/2}$$

or

$$\left(\int_\Omega [e_{ij}(\boldsymbol{\phi})e_{ij}(\boldsymbol{\phi})]^{p/2} d\boldsymbol{x}\right)^{2/p} \leq \left(\int_\Omega \frac{e_{ij}(\boldsymbol{\phi})e_{ij}(\boldsymbol{\phi})}{(\epsilon + |\boldsymbol{e}(\boldsymbol{u})|^2)^{(2-p)/2}} d\boldsymbol{x}\right) \times \left(\int_\Omega (\epsilon + |\boldsymbol{e}(\boldsymbol{u})|^2)^{p/2} d\boldsymbol{x}\right)^{(2-p)/p}. \tag{5.190}$$

However, from the results in [BBN2, 3], as described in Sect. 4.3, we know that the solution of (5.2a,b), (5.3a), (5.4) is in $\boldsymbol{L}^\infty([0,\infty); \boldsymbol{W}^{1,p}(\Omega))$; therefore, $\exists k = k(\Omega)$ such that

$$\left(\int_\Omega (\epsilon + |\boldsymbol{e}(\boldsymbol{u})|^2)^{p/2} d\boldsymbol{x}\right)^{1/p} \leq k(\Omega) \tag{5.191}$$

$\forall t \geq 0$, with $k(\Omega)$ independent of $\boldsymbol{u}_0$ for $\boldsymbol{u}_0 \in \boldsymbol{\mathcal{A}}_{\mu_1}$. Also, by the L^p version of the Korn inequality (Lemma 5.1) we know that

$$\left(\int_\Omega [e_{ij}(\boldsymbol{\phi})e_{ij}(\boldsymbol{\phi})]^{p/2} d\boldsymbol{x}\right)^{2/p} \geq k'(\Omega)\, ||\boldsymbol{\phi}||^2_{\boldsymbol{W}^{1,p}(\Omega)}. \tag{5.192}$$

Combining (5.190) with (5.191) and (5.192) we find that (5.189) is satisfied for some $\tilde{k}(\Omega) > 0$ and all $t \geq 0$. □

Returning to the proof of Lemma 5.13, and combining (5.185), (5.187), and (5.189), we arrive at the indicated lower bound for $\mathrm{tr}(\boldsymbol{\mathcal{L}}_{\mu_1}(\boldsymbol{u}) \circ \boldsymbol{Q})$, i.e., (5.184), which completes the proof of Lemma 5.14. □

Now, it is well-known [CFT1] that

$$\sum_{\ell=1}^k |\boldsymbol{\phi}_\ell|^2_{\boldsymbol{H}^1(\Omega)} \geq \lambda_1 + \cdots + \lambda_k$$

with $\lambda_j \geq \tilde{c}\lambda_1 j^{2/3}$, $\forall j \geq 1$, for some $\tilde{c} > 0$ (and independent of j), where the λ_j, $j = 1, \ldots, k$, are the first k eigenvalues of $-\Delta$ on Ω such that the corresponding eigenvectors $\boldsymbol{w}_j$ are in $\boldsymbol{H}^1_0(\Omega)$ and satisfy the constraint $\Delta \cdot \boldsymbol{w}_j = 0$. In view of the above estimate, and the lower bound cited for the λ_j, $j \geq 1$ (this latter result being due to Metivier [Me]) we find that for some $c' > 0$,

$$\sum_{\ell=1}^{k} |\boldsymbol{\phi}_\ell|^2_{\boldsymbol{H}^1(\Omega)} \geq \lambda_1 c' k^{5/3}. \tag{5.193}$$

To deal with the term $\sum_{l=1}^{k} ||\boldsymbol{\phi}_l||^2_{\boldsymbol{W}^{1,p}(\Omega)}$ in (5.184) we will make use Lemma A.6 which says that

$$\begin{aligned} ||\boldsymbol{\phi}||^2_{\boldsymbol{W}^{1,p}(\Omega)} \geq \frac{\zeta^{1/\delta'}}{\delta' d_{\delta'}(\Omega)} |\boldsymbol{\phi}|^2_{\boldsymbol{H}^1(\Omega)} \\ - \zeta^{1/\delta'(1-\delta')} \cdot \frac{1-\delta'}{\delta'} ||\boldsymbol{\phi}||^2_{\boldsymbol{H}^2(\Omega)} \end{aligned} \tag{5.194}$$

for $1 < p \leq 2$, any $\zeta > 0$, and some $d_{\delta'}(\Omega) > 0$, if $\boldsymbol{\phi} \in \boldsymbol{W}^{1,p}(\Omega) \cap \boldsymbol{W}_0^{1,2}(\Omega) \cap \boldsymbol{W}^{2,2}(\Omega)$.

The last lemma which is needed before we can address the proof of Theorem 5.7 is the following:

Lemma 5.15. *Under the conditions stated in Theorem 5.7, the uniform Lyapunov exponents $\lambda_j^{\mu_1}$ on $\boldsymbol{\mathcal{A}}_{\mu_1}$, $j = 1, 2, \ldots, k$, satisfy*

$$\begin{aligned} \lambda_1^{\mu_1} + \cdots + \lambda_k^{\mu_1} \leq -\lambda_1 c' \Lambda k^{5/3} \\ + \frac{k}{\mu_1} n(\Omega) \lim_{t\to\infty} \sup_{\boldsymbol{u}_0 \in \boldsymbol{\mathcal{A}}_{\mu_1}} \int_0^t ||\boldsymbol{u}||^2_{\boldsymbol{H}^2(\Omega)} \, d\tau \end{aligned} \tag{5.195a}$$

where

$$\Lambda = \frac{1}{2}\mu_1 k(\Omega) + \frac{\mu_0(1-\alpha)}{\delta'} \bar{m}(\Omega) \left[\frac{\mu_1 \delta'}{1-\delta'} k(\Omega) \right]^{1-\delta'} \tag{5.195b}$$

with $k(\Omega)$ as in (5.191), $\delta' = 2p/6 - p$, for $1 < p < 2$, $\bar{m}(\Omega) = 2\tilde{k}(\Omega)/d(\Omega)$, $\tilde{k}(\Omega)$ the constant in the estimate (5.189), $n(\Omega) = 2c^2(\Omega)/k(\Omega)$, and $d(\Omega)$ the constant in the estimate (5.194), and the dependence of $d(\Omega)$ on δ' being understood.

Proof. By the estimate (5.184) of Lemma 5.13, combined with the lower bound (5.194) of Lemma A.6 we have

$$\mathrm{tr}(\boldsymbol{\mathcal{L}}_{\mu_1}(\boldsymbol{u}) \circ \boldsymbol{Q}) \geq 2\mu_1 \sum_{\ell=1}^{k} \int_\Omega \frac{\partial e_{ij}(\boldsymbol{\phi}_\ell)}{\partial x_k} \frac{\partial e_{ij}(\boldsymbol{\phi}_\ell)}{\partial x_k} d\boldsymbol{x}$$

$$+\left[\frac{2\tilde{k}(\Omega)(1-\alpha)\mu_0\zeta^{1/\delta}}{\delta' d(\Omega)} - \delta c(\Omega)\right]\sum_{\ell=1}^{k}|\boldsymbol{\phi}_\ell|^2_{\boldsymbol{H}^1(\Omega)}$$

$$-\zeta^{1/\delta(1-\delta)}\frac{(1-\delta')}{\delta'}\sum_{\ell=1}^{k}||\boldsymbol{\phi}_\ell||^2_{\boldsymbol{H}^2(\Omega)} - \frac{kc(\Omega)}{\delta}\,||\boldsymbol{u}||^2_{\boldsymbol{H}^2(\Omega)}$$

or

$$\operatorname{tr}(\boldsymbol{\mathcal{L}}_{\mu_1}(\boldsymbol{u})\circ\boldsymbol{Q}) \geq \left[2\mu_1 k(\Omega) - \zeta^{1/\delta'(1-\delta')}\frac{(1-\delta')}{\delta'}\right]\sum_{\ell=1}^{k}||\boldsymbol{\phi}_\ell||^2_{\boldsymbol{H}^2(\Omega)}$$
$$+\left(\left[\frac{2\mu_0(1-\alpha)\tilde{k}(\Omega)}{\delta' d(\Omega)}\zeta^{1/\delta'} - \delta c(\Omega)\right]\times\sum_{\ell}^{k}|\boldsymbol{\phi}_\ell|^2_{\boldsymbol{H}^1(\Omega)}\right) - \frac{kc(\Omega)}{\delta}\,||\boldsymbol{u}||^2_{\boldsymbol{H}^2(\Omega)}\,. \tag{5.196}$$

Choosing $\zeta > 0$ such that

$$2\mu_1 k(\Omega) - \zeta^{1/\delta'(1-\delta')}\frac{(1-\delta')}{\delta'} = \mu_1 k(\Omega)$$

i.e.,

$$\zeta = \left[\frac{\mu_1\delta'}{1-\delta'}k(\Omega)\right]^{\delta'(1-\delta')} \tag{5.197}$$

and $\delta > 0$ sufficiently small, say,

$$\delta = \mu_1 k(\Omega)/2c(\Omega) \tag{5.198}$$

we obtain from (5.196) the estimate

$$\operatorname{tr}(\boldsymbol{\mathcal{L}}_{\mu_1}(\boldsymbol{u})\circ\boldsymbol{Q}) \geq \Lambda\sum_{\ell=1}^{k}|\boldsymbol{\phi}_\ell|^2_{\boldsymbol{H}^1(\Omega)} - \frac{2kc^2(\Omega)}{\mu_1 k(\Omega)}\,||\boldsymbol{u}||^2_{\boldsymbol{H}^2(\Omega)} \tag{5.199}$$

with Λ given by (5.195b). Employing the definition of the $q_k^{\mu_1}$, i.e., (5.176) we obtain the lower bound

$$q_k^{\mu_1} \geq \Lambda\sum_{\ell=1}^{k}|\boldsymbol{\phi}_\ell|^2_{\boldsymbol{H}^1(\Omega)} - \frac{k}{\mu_1}n(\Omega)\lim_{t\to\infty}\sup_{\boldsymbol{u}_0\in\boldsymbol{\mathcal{A}}_{\mu_1}}\int_0^t ||\boldsymbol{u}||^2_{\boldsymbol{H}^2(\Omega)}\,d\tau \tag{5.200}$$

where $n(\Omega) = 2c^2(\Omega)/k(\Omega)$. In view of (5.193) we are led from (5.200) to

$$q_k^{\mu_1} \geq \lambda_1 c' \Lambda k^{5/3} - \frac{k}{\mu_1} n(\Omega) \lim_{t\to\infty} \sup_{\boldsymbol{u}_0 \in \boldsymbol{\mathcal{A}}_{\mu_1}} \int_0^t ||\boldsymbol{u}||^2_{\boldsymbol{H}^2(\Omega)}\, d\tau$$

so that the uniform Lyapunov exponents $\lambda_j^{\mu_1}$ on $\boldsymbol{\mathcal{A}}_{\mu_1}$, $j = 1, 2, \dots, k$, satisfy (5.195a). □

We are now in a position to establish (5.183).

Proof (Theorem 5.7). By Lemma 5.12 we want to determine the first positive integer k such that the right-hand side of (5.195a) is negative. To this end we return to (5.80) with $\boldsymbol{u}(t)$ in place of $\boldsymbol{v}(t)$, i.e.,

$$\frac{1}{2}\frac{d}{dt} ||\boldsymbol{u}||^2_{\boldsymbol{L}^2(\Omega)} + 2\mu_1 k(\Omega)\, ||\boldsymbol{u}||^2_{\boldsymbol{H}^2(\Omega)} \leq |\boldsymbol{f}|_\infty\, ||\boldsymbol{u}||_{\boldsymbol{L}^2(\Omega)} \tag{5.201}$$

where, unlike (5.80), (5.201) now holds $\forall t \geq 0$; this is because $\boldsymbol{u}_0 \in \boldsymbol{\mathcal{A}}_{\mu_1}$ implies that $\boldsymbol{u}(t) \in B^{\rho'}_{\boldsymbol{H}^2(\Omega)} \subset \boldsymbol{H}^2(\Omega)$, $\forall t \geq 0$, so the Korn-type estimate of Lemma 5.2 can be employed in (5.78) from time $t = 0$. For any $\gamma > 0$ we now have

$$\frac{1}{2}\frac{d}{dt} ||\boldsymbol{u}||^2_{\boldsymbol{L}^2(\Omega)} + 2\mu_1 k(\Omega)\, ||\boldsymbol{u}||^2_{\boldsymbol{H}^2(\Omega)} \leq \frac{\gamma}{2} |\boldsymbol{f}|^2_\infty + \frac{1}{2\gamma} ||\boldsymbol{u}||^2_{\boldsymbol{L}^2(\Omega)} \tag{5.202}$$

which for γ sufficiently large leads us to

$$\frac{1}{2}\frac{d}{dt} ||\boldsymbol{u}||^2_{\boldsymbol{L}^2(\Omega)} + \mu_1 k(\Omega)\, ||\boldsymbol{u}||^2_{\boldsymbol{H}^2(\Omega)} \leq \frac{\gamma}{2} |\boldsymbol{f}|^2_\infty\,. \tag{5.203}$$

From (5.203) we obtain, $\forall t > 0$,

$$\frac{1}{2} ||\boldsymbol{u}(t)||^2_{\boldsymbol{L}^2(\Omega)} + \mu_1 k(\Omega) \int_0^t ||\boldsymbol{u}(\tau)||^2_{\boldsymbol{H}^2(\Omega)}\, d\tau \leq \frac{\gamma}{2} t\, |\boldsymbol{f}|^2_\infty + \frac{1}{2} ||\boldsymbol{u}_0||^2_{\boldsymbol{L}^2(\Omega)} \tag{5.204}$$

so that

$$\mu_1 k(\Omega) \frac{1}{t} \int_0^t ||\boldsymbol{u}||^2_{\boldsymbol{H}^2(\Omega)}\, d\tau \leq \frac{\gamma}{2} |\boldsymbol{f}|^2_\infty + \frac{1}{2t} ||\boldsymbol{u}_0||^2_{\boldsymbol{L}^2(\Omega)}\,. \tag{5.205}$$

Therefore,

$$\lim_{t\to\infty} \sup_{\boldsymbol{u}_0 \in \boldsymbol{\mathcal{A}}_{\mu_1}} \frac{1}{t} \int_0^t ||\boldsymbol{u}||^2_{\boldsymbol{H}^2(\Omega)}\, d\tau \leq \frac{1}{4\mu_1^2 k^2(\Omega)} |\boldsymbol{f}|^2_\infty \tag{5.206}$$

if we choose $\gamma = 1/2\mu_1 k(\Omega)$. Using (5.206) in (5.195a) we compute that

$$\lambda_1^{\mu_1} + \cdots \lambda_k^{\mu_1} \leq -\lambda_1 c' \Lambda k^{5/3} + \frac{k n(\Omega)}{4\mu_1^3 k^2(\Omega)} |\boldsymbol{f}|^2_\infty\,. \tag{5.207}$$

From (5.207) it is immediate that $\lambda_1^{\mu_1} + \cdots + \lambda_k^{\mu_1} < 0$ if k is the smallest positive integer such that if

$$k^{2/3} > \frac{n(\Omega)\,|\boldsymbol{f}|_\infty^2}{(4\mu_1^3\lambda_1 c'\Lambda)k^2(\Omega)}$$

i.e., if $k \in N$ satisfies

$$k - 1 < \frac{n^{3/2}(\Omega)\,|\boldsymbol{f}|_\infty^3}{8\mu_1^{9/2}(\lambda_1 c'\Lambda)^{3/2}k^3(\Omega)} < k \tag{5.208}$$

where Λ is given by (5.195b) and $\delta' = 2(2-\alpha)/(4+\alpha)$. □

The result of Theorem 5.7 may be improved upon if it could be established that

$$M = \sup_{t>0}\sup_{\Omega}\left\{\sup_{\boldsymbol{u}_0\in\boldsymbol{\mathcal{A}}_{\mu_1}} (e_{ij}(\boldsymbol{u})e_{ij}(\boldsymbol{u}))\right\} < \infty. \tag{5.209}$$

It is not yet known if (5.209) is valid for the bipolar fluid model; however, (5.209) holds if it is true that $\boldsymbol{\mathcal{A}}_{\mu_1} \subset \boldsymbol{H}^3(\Omega)$ as $\boldsymbol{H}^3(\Omega) \subset C^1(\bar{\Omega})$ by the Sobolev embedding theorem in $\dim n = 3$, and this result is known to be true [Gu] for the Navier–Stokes equations when $\boldsymbol{f} \in \boldsymbol{H}^1(\Omega)$ and is independent of t. If (5.209) could be established for the solution of the bipolar initial-boundary value problem we would have the following result:

Theorem 5.8. *For the dynamical system defined by the nonlinear bipolar initial-value problem* (5.2a,b)*,* (5.3a)*,* (5.4)*, if* (5.209) *holds then* $\lambda_1^{\mu_1} + \cdots + \lambda_k^{\mu_1} < 0$ *for* $k \in N$ *satisfying*

$$k - 1 < \frac{m^{3/2}(\Omega)\,|\boldsymbol{f}|_\infty^3}{8(\lambda_1 c'\gamma)^{3/2}\mu_1^3(\mu_1 k(\Omega) + (\gamma/4))^{3/2}} < k \tag{5.210}$$

where $m(\Omega) = (c(\Omega)/k(\Omega))^2$ *and* $\gamma = \dfrac{(1-\alpha)\mu_0}{(\epsilon + M)^{\alpha/2}}$.

Proof. If (5.209) holds then, from (5.185) and (5.187), we obtain the lower bound

$$\begin{aligned}(\boldsymbol{\mathcal{L}}_{\mu_1}(\boldsymbol{u})\boldsymbol{\phi},\boldsymbol{\phi})_{\boldsymbol{L}^2(\Omega)} \geq 2\mu_1 \int_\Omega \frac{\partial e_{ij}(\boldsymbol{\phi})}{\partial x_k}\frac{\partial e_{ij}(\boldsymbol{\phi})}{\partial x_k}d\boldsymbol{x}\\ + (\gamma - \delta c(\Omega))\,|\boldsymbol{\phi}|_{\boldsymbol{H}^1(\Omega)}^2 - \frac{c(\Omega)}{\delta}\,||\boldsymbol{u}||_{\boldsymbol{H}^2(\Omega)}^2\end{aligned} \tag{5.211}$$

with

$$\gamma = (1-\alpha)\mu_0/(\epsilon + M)^{\alpha/2}. \tag{5.212}$$

In obtaining (5.211) we have also used the elementary result

$$\int_{\Omega} e_{ij}(\boldsymbol{\phi}) e_{ij}(\boldsymbol{\phi}) d\boldsymbol{x} \geq \frac{1}{2} |\boldsymbol{\phi}|^2_{\boldsymbol{H}^1(\Omega)}, \quad \forall \boldsymbol{\phi} \in \boldsymbol{H}^1_0(\Omega). \tag{5.213}$$

By (5.211) we have

$$\begin{aligned} \operatorname{tr}(\boldsymbol{\mathcal{L}}_{\mu_1}(\boldsymbol{u})\boldsymbol{\phi}, \boldsymbol{\phi})_{\boldsymbol{L}^2(\Omega)} &\geq 2\mu_1 \sum_{\ell=1}^{k} \int_{\Omega} \frac{\partial e_{ij}(\boldsymbol{\phi}_\ell)}{\partial x_k} \frac{\partial e_{ij}(\boldsymbol{\phi}_\ell)}{\partial x_k} d\boldsymbol{x} \\ &\quad + (\gamma - \delta c(\Omega)) \sum_{\ell=1}^{k} |\boldsymbol{\phi}_\ell|^2_{\boldsymbol{H}^1(\Omega)} - \frac{k c(\Omega)}{\delta} ||\boldsymbol{u}||^2_{\boldsymbol{H}^2(\Omega)} \end{aligned} \tag{5.214}$$

with the $\boldsymbol{\phi}_j$, $j = 1, 2, \ldots, k$ chosen as in the proof of Theorem 5.7. Applying (5.193), and choosing $\delta = \gamma/2c(\Omega)$, we have, by virtue of Lemma 5.2

$$\operatorname{tr}(\boldsymbol{\mathcal{L}}_{\mu_1}(\boldsymbol{u})\boldsymbol{\phi}, \boldsymbol{\phi})_{\boldsymbol{L}^2(\Omega)} \geq 2\mu_1 k(\Omega) \sum_{\ell=1}^{k} ||\boldsymbol{\phi}_\ell||^2_{\boldsymbol{H}^2(\Omega)} + \frac{\gamma \lambda_1 c' k^{5/3}}{2} - \frac{2kc^2(\Omega)}{\gamma} ||\boldsymbol{u}||^2_{\boldsymbol{H}^2(\Omega)}$$

in which case

$$\operatorname{tr}(\boldsymbol{\mathcal{L}}_{\mu_1}(\boldsymbol{u})\boldsymbol{\phi}, \boldsymbol{\phi})_{\boldsymbol{L}^2(\Omega)} \geq \left(2\mu_1 k(\Omega) + \frac{\gamma}{2}\right) \lambda_1 c' k^{5/3} - \frac{2kc^2(\Omega)}{\gamma} ||\boldsymbol{u}||^2_{\boldsymbol{H}^2(\Omega)}. \tag{5.215}$$

From (5.215) and (5.206) we now obtain an estimate analogous to (5.207) when (5.209) holds, namely,

$$\lambda_1^{\mu_1} + \cdots \lambda_k^{\mu_1} \leq -\left(2\mu_1 k(\Omega) + \frac{\gamma}{2}\right) \lambda_1 c' k^{5/3} + \frac{kc^2(\Omega)}{2\mu_1^2 \gamma k^2(\Omega)} |\boldsymbol{f}|^2_\infty. \tag{5.216}$$

Thus, $\lambda_1^{\mu_1} + \cdots + \lambda_k^{\mu_1} < 0$ for k the smallest positive integer such that

$$k^{2/3} > \frac{m(\Omega) |\boldsymbol{f}|^2_\infty}{2\lambda_1 c' \gamma \mu_1^2 (2\mu_1 k(\Omega) + (\gamma/2))} \tag{5.217}$$

from which (5.210) follows immediately. □

5.4 Attractors for the Bipolar and Non-Newtonian Problems $(-1 < \alpha < 0)$

Having established, in Sect. 5.3, the existence of a maximal compact global attractor $\boldsymbol{\mathcal{A}}_{\mu_1}$ for the bipolar initial-boundary value problem when $0 \leq \alpha < 1$, so that $1 < p \leq 2$ (a result which is also valid for the space-periodic problem in

$\dim n = 2, 3$) we now turn our attention to both the bipolar and non-Newtonian problems for the case $\alpha < 0$ ($p > 2$). In actuality our results will only apply in the range $-1 < \alpha < 0$ ($2 < p < 3$) with the problem still being open for $p \geq 3$. As in Sect. 5.3 we will be able to prove the existence of maximal compact global attractors $\boldsymbol{\mathcal{A}}_{\mu_1} \subset \boldsymbol{W}^{2,2}(\Omega)$. For technical reasons we confine our attention in this paper to space dimension $n = 2$; we intend to use the results for the situation in which $\mu_1 > 0$ in order to establish the existence of a maximal compact global attractor $\boldsymbol{\mathcal{A}}_0$ for the non-Newtonian subcase in which $\mu_1 = 0$ and $p > 2$. We note that in [MN] the authors have established, directly, the existence of a maximal compact global attractor $\boldsymbol{\mathcal{A}}_0$ for the space periodic version of the non-Newtonian problem ($\mu_1 = 0$), in $\dim n = 3$, when $p \geq 5/2$ and $\boldsymbol{v}_0 \in \boldsymbol{H}_{per}$; they were also able to prove that the fractal dimension of $\boldsymbol{\mathcal{A}}_0$ is bounded. Although for the case $p > 2$, and $\mu_1 > 0$, we may speak almost interchangeably of the space-periodic and boundary-value problems, because of the delicacy of dealing with the boundary-value problem for the non-Newtonian case in which $\mu_1 = 0$, we will restrict our attention throughout this section to the relevant space-periodic problems. *For the remainder of this chapter, as well as in Chap. 6, we will employ the following protocol: when referring to the space-periodic problem for the bipolar equations with* $\mu_1 > 0$ *we will use the full set of hypotheses in* (5.3b)*; however, for the case in which* $\mu_1 = 0$ *only the first set of conditions in* (5.3b) *will apply. This specification mirrors that for the boundary-value problem where only the first condition in* (5.3a) *holds when* $\mu_1 = 0$*, while both sets of boundary conditions in* (5.3a) *apply to the case where* $\mu_1 > 0$. Beyond establishing, for $\mu_1 \geq 0$ and $p > 2$, the existence of maximal compact global attrctors $\boldsymbol{\mathcal{A}}_{\mu_1}$, for the incompressible bipolar viscous fluid, we will also show that for $\mu_1 > 0$, and $2 < p < 3$, there exist upper bounds for the Hausdorff and fractal dimensions of these attractors. Unlike the situation we encountered in Sect. 5.3, where it was determined that $d_H(\boldsymbol{\mathcal{A}}_{\mu_1}) < k$, $d_F(\boldsymbol{\mathcal{A}}_{\mu_1}) < 2k$, with $k \in N$ satisfying (5.183), for the case $1 < p \leq 2$, the upper bounds for the $d_H(\boldsymbol{\mathcal{A}}_{\mu_1})$ and $d_F(\boldsymbol{\mathcal{A}}_{\mu_1})$, when $2 < p < 3$, will turn out to be independent of μ_1. In Sect. 5.5, under the conditions which prevail in this section, it will be shown that as $\mu_1 \to 0^+$, $\hat{d}(\boldsymbol{\mathcal{A}}_{\mu_1}, \boldsymbol{\mathcal{A}}_0) \to 0$ where $\hat{d}$ is the semidistance for sets, the underlying norm being that of $\boldsymbol{L}^2(\Omega)$. Unfortunately, the fact that (for $p > 2$) the upper bounds for $d_H(\boldsymbol{\mathcal{A}}_{\mu_1})$ and $d_F(\boldsymbol{\mathcal{A}}_{\mu_1})$ are independent of $\mu_1 > 0$, even when coupled with the result $\hat{d}(\boldsymbol{\mathcal{A}}_{\mu_1}, \boldsymbol{\mathcal{A}}_0) \to 0$, as $\mu_1 \to 0^+$, does not permit us to obtain an upper bound for either $d_H(\boldsymbol{\mathcal{A}}_0)$ or $d_F(\boldsymbol{\mathcal{A}}_0)$. As in Sect. 5.3 we will set the density $\rho = 1$ throughout this section.

5.4.1 Existence of Absorbing Sets and Maximal Compact Global Attractors for $S_{\mu_1}(t)$, $\mu_1 \geq 0$

In order to establish the existence of maximal compact global attractors for the space-periodic problem (5.2a,b), (5.3b), (5.4), in space dimension $n = 2$, when

$p > 2$, and $\mu_1 \geq 0$, we assume that $\boldsymbol{v}_0 \in \boldsymbol{L}^2(\Omega)$, $\Omega = [0, L]^2$ for some $L > 0$. As a precursor to establishing the existence of the attractors $\boldsymbol{\mathcal{A}}_{\mu_1}$, $\mu_1 \geq 0$, we will (as in Sect. 5.3) first establish the existence of suitable absorbing sets for the problems in question; we want, in fact, to prove the following result:

Theorem 5.9. *Consider the problem* (5.2a,b)*,* (5.3b)*,* (5.4)*, in* $\dim n = 2$*, with* $p > 2$*; then there exist absorbing sets in* $\boldsymbol{H}_{per}$ *and* $\boldsymbol{W}^{1,2}(\Omega)$ *for* $\boldsymbol{S}_{\mu_1}(t)$*,* $\mu_1 \geq 0$*, which are independent of* $\mu_1 > 0$*, as well as absorbing sets for* $\boldsymbol{S}_{\mu_1}(t)$*, in* $\boldsymbol{W}^{2,2}(\Omega)$*, when* $\mu_1 > 0$*.*

The proof of Theorem 5.9 will be broken up into a series of lemmas, the first of which is

Lemma 5.16. *Let* $\boldsymbol{v}(t)$ *be the unique solution of* (5.2a,b)*,* (5.3b)*,* (5.4) *in* $\dim n = 2$*, with* $p > 2$*; then* $\exists \bar{\beta} > 0$*,* $\bar{t}_0 > 0$*, such that* $\forall t \geq \bar{t}_0$*, and all* $\mu_1 \geq 0$*,*

$$||\boldsymbol{v}(t)||^2_{\boldsymbol{L}^2(\Omega)} \leq \bar{\beta}|\boldsymbol{f}|^2_\infty \tag{5.218}$$

in which case $\boldsymbol{S}_{\mu_1}(t)$*,* $\mu_1 \geq 0$*, has an absorbing set in* $\boldsymbol{H}_{per}$*.*

Proof. We again employ the notation $\gamma(\boldsymbol{v}) = \mu(|\boldsymbol{e}(\boldsymbol{v})|)$ and begin by multiplying (5.2a) by v_i and integrating by parts. In view of the space-periodicity assumption (5.3b) this produces the estimate

$$\frac{1}{2}\frac{d}{dt}||\boldsymbol{v}||^2_{\boldsymbol{L}^2(\Omega)} + \int_\Omega \gamma(\boldsymbol{v}) e_{ij} e_{ij} d\boldsymbol{x} + 2\mu_1 \int_\Omega \frac{\partial e_{ij}}{\partial x_k}\frac{\partial e_{ij}}{\partial x_k} d\boldsymbol{x} = \int_\Omega f_i v_i d\boldsymbol{x} \leq |\boldsymbol{f}|_\infty ||\boldsymbol{v}||_{\boldsymbol{L}^2(\Omega)}. \tag{5.219}$$

We note that for $p > 2$, $\epsilon > 0$, $\exists k_1(\Omega) > 0$ such that

$$\begin{aligned}\int_\Omega \gamma(\boldsymbol{v}) e_{ij} e_{ij} d\boldsymbol{x} &= \int_\Omega (\epsilon + |\boldsymbol{e}|^2)^{\frac{p-2}{2}} |\boldsymbol{e}|^2 d\boldsymbol{x} \\ &\geq \int_\Omega |\boldsymbol{e}|^p d\boldsymbol{x} \geq k_1(\Omega)\, ||\boldsymbol{v}||^p_{\boldsymbol{W}^{1,p}(\Omega)}\end{aligned} \tag{5.220a}$$

and, also, that $\exists k_2(\Omega) > 0$ such that

$$\int_\Omega \gamma(\boldsymbol{v}) e_{ij} e_{ij} d\boldsymbol{x} \geq \epsilon^{\frac{p-2}{2}} \int_\Omega |\boldsymbol{e}|^2 d\boldsymbol{x} \geq \epsilon^{\frac{p-2}{2}} k_2(\Omega)\, ||\boldsymbol{v}||^2_{\boldsymbol{W}^{1,2}(\Omega)}. \tag{5.220b}$$

If we drop the expression involving μ_1 in (5.219), and use (5.220a), we obtain

$$\frac{1}{2}\frac{d}{dt}||\boldsymbol{v}||^2_{\boldsymbol{L}^2(\Omega)} + k_1(\Omega)\, ||\boldsymbol{v}||^p_{\boldsymbol{W}^{1,p}(\Omega)} \leq |\boldsymbol{f}|_\infty ||\boldsymbol{v}||_{\boldsymbol{L}^2(\Omega)} \tag{5.221}$$

while dropping the expression involving μ_1 in (5.219) and using (5.220b) we find that for any $\beta > 0$

$$\frac{1}{2}\frac{d}{dt}\,||\boldsymbol{v}||^2_{\boldsymbol{L}^2(\Omega)} + \epsilon^{\frac{p-2}{2}} k_2(\Omega)\,||\boldsymbol{v}||^2_{\boldsymbol{W}^{1,2}(\Omega)} \le \frac{1}{2}\beta|\boldsymbol{f}|^2_\infty + \frac{1}{2\beta}\,||\boldsymbol{v}||^2_{\boldsymbol{L}^2(\Omega)}. \tag{5.222}$$

Both (5.221) and (5.222) hold $\forall \mu_1 \ge 0$. From (5.222) we have, for $\beta > 0$ sufficiently large, and some $\bar{k}(\epsilon, \rho; \Omega) > 0$, an estimate of the form

$$\frac{d}{dt}\,||\boldsymbol{v}||^2_{\boldsymbol{L}^2(\Omega)} + \bar{k}(\epsilon, \rho; \Omega)\,||\boldsymbol{v}||^2_{\boldsymbol{L}^2(\Omega)} \le \beta|\boldsymbol{f}|^2_\infty \tag{5.223}$$

from which it follows that, $\forall \mu_1 \ge 0$,

$$||\boldsymbol{v}(t)||^2_{\boldsymbol{L}^2(\Omega)} \le e^{-\bar{k}t}\,||\boldsymbol{v}_0||^2_{\boldsymbol{L}^2(\Omega)} + \frac{\beta}{\bar{k}}|\boldsymbol{f}|^2_\infty. \tag{5.224}$$

As an immediate consequence of (5.224) we obtain the existence of a $\bar{t}_0 = \bar{t}_0(||\boldsymbol{v}_0||_{\boldsymbol{L}^2(\Omega)})$ such that $\forall t \ge \bar{t}_0$, and all $\mu_1 \ge 0$,

$$||\boldsymbol{v}(t)||^2_{\boldsymbol{L}^2(\Omega)} \le \frac{2\beta}{\bar{k}}|\boldsymbol{f}|^2_\infty \tag{5.225}$$

in which case (5.218) follows with $\bar{\beta} = 2\beta/\bar{k}$.

Thus, if for some bounded set $B_0 \subset \boldsymbol{L}^2(\Omega)$, with $B_0 \subset B_{R_0}(\boldsymbol{0})$, a ball of radius R_0 centered at $\boldsymbol{0}$, we have $\boldsymbol{v}_0 \in B_0$, then $\exists t_0 = t_0(R_0)$ such that (5.225) holds $\forall t \ge t_0$, and all $\mu_1 \ge 0$; this serves to establish the existence of the absorbing set in $\boldsymbol{H}_{per}$ for $\boldsymbol{S}_{\mu_1}(t)$ when $\mu_1 \ge 0$. □

Before moving on to the problem of the existence of absorbing sets in $\boldsymbol{W}^{1,2}(\Omega)$, we make note here of a result that we will need later on. As a consequence of (5.221) and (5.225), if $\boldsymbol{v}_0 \in B_0$ then for $t \ge t_0(R_0)$, and all $\mu_1 \ge 0$

$$\frac{1}{2}\frac{d}{dt}\,||\boldsymbol{v}||^2_{\boldsymbol{L}^2(\Omega)} + k_1(\Omega)\,||\boldsymbol{v}||^p_{\boldsymbol{W}^{1,p}(\Omega)} \le \rho|\boldsymbol{f}|^2_\infty \tag{5.226}$$

where $\rho = \sqrt{\frac{2\beta}{\bar{k}}}$ (and does not stand for the density of the fluid, which we have set equal to one). Taking any $r > 0$ and integrating (5.226) from t to $t + r$, with $t \ge t_0(R_0)$, we obtain the estimates

$$\begin{aligned}\frac{1}{2}\,||\boldsymbol{v}(t+r)||^2_{\boldsymbol{L}^2(\Omega)} + k_1(\Omega)\int_t^{t+r}||\boldsymbol{v}||^p_{\boldsymbol{W}^{1,p}(\Omega)}\,d\tau &\le \frac{1}{2}\,||\boldsymbol{v}(t)||^2_{\boldsymbol{L}^2(\Omega)} + \rho r|\boldsymbol{f}|^2_\infty \\ &\le \left(\frac{1}{2}\rho^2 + \rho r\right)|\boldsymbol{f}|^2_\infty.\end{aligned} \tag{5.227}$$

Thus, if $\boldsymbol{v}_0 \in B_0$, then for $t \geq t_0(R_0)$, any $r > 0$, all $\mu_1 \geq 0$, and $p > 2$,

$$\int_t^{t+r} ||\boldsymbol{v}||^p_{\boldsymbol{W}^{1,p}(\Omega)} \, dt \leq \frac{1}{k_1(\Omega)} \left(\frac{1}{2}\rho^2 + \rho r \right) |\boldsymbol{f}|^2_\infty. \tag{5.228}$$

Lemma 5.17. *Let $\boldsymbol{v}(t)$ be the unique solution of* (5.2a,b)*,* (5.3b)*,* (5.4) *in* dim $n = 2$*, with $p > 2$; then for any $r > 0$, and all $\mu_1 \geq 0$, $\exists K = K(\epsilon, p, r, |\boldsymbol{f}|_\infty; \Omega)$ such that*

$$||\boldsymbol{v}||^2_{\boldsymbol{W}^{1,2}(\Omega)}(t) \leq K, \text{ for } t \geq t_0(R_0) + r \tag{5.229}$$

from which it follows that $\exists B^{\rho'}_{\boldsymbol{W}^{1,2}(\Omega)}$, a ball of radius ρ' in $\boldsymbol{W}^{1,2}(\Omega)$, which is an absorbing set for $\boldsymbol{S}_{\mu_1}$, $\mu_1 \geq 0$, that is independent of μ_1.

Proof. We multiply (5.2a) by $\dfrac{\partial v_i}{\partial t}$, integrate over Ω, and then integrate the resulting equation by parts; using the definition (5.92) of the potential Γ, with $\alpha = 2 - p$, so that

$$\frac{\partial \Gamma}{\partial t} = \gamma(\boldsymbol{v}) e_{ij} \frac{\partial e_{ij}}{\partial t} \tag{5.230}$$

we obtain the estimate

$$\frac{1}{2} \left\| \frac{\partial \boldsymbol{v}}{\partial t} \right\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{d}{dt} \left\{ \int_\Omega \Gamma(e_{ij}e_{ij}) d\boldsymbol{x} + \mu_1 \int_\Omega \frac{\partial e_{ij}}{\partial x_k} \frac{\partial e_{ij}}{\partial x_k} d\boldsymbol{x} \right\} \leq \left| \int_\Omega v_j \frac{\partial v_i}{\partial x_j} \frac{\partial v_i}{\partial t} d\boldsymbol{x} \right| + \frac{1}{2} |\boldsymbol{f}|^2_\infty \tag{5.231}$$

which holds $\forall \mu_1 \geq 0$ and all $t > 0$. Now, for any $\delta > 0$,

$$\left| \int_\Omega v_j \frac{\partial v_i}{\partial x_j} \frac{\partial v_i}{\partial t} d\boldsymbol{x} \right| \leq \frac{\delta}{2} ||\boldsymbol{v}||^2_{\boldsymbol{L}^\infty(\Omega)} ||\boldsymbol{v}||^2_{\boldsymbol{H}^1(\Omega)} + \frac{1}{2\delta} \left\| \frac{\partial \boldsymbol{v}}{\partial t} \right\|^2_{\boldsymbol{L}^2(\Omega)}. \tag{5.232}$$

However, $\boldsymbol{W}^{1,p}(\Omega) \hookrightarrow \boldsymbol{L}^\infty(\Omega)$, for $p > n = 2$, so $\exists k_3(\Omega) > 0$ such that

$$\left| \int_\Omega v_j \frac{\partial v_i}{\partial x_j} \frac{\partial v_i}{\partial t} d\boldsymbol{x} \right| \leq \frac{\partial \delta k_3}{2} ||\boldsymbol{v}||^2_{\boldsymbol{W}^{1,p}(\Omega)} ||\boldsymbol{v}||^2_{\boldsymbol{H}^1(\Omega)} + \frac{1}{2\delta} \left\| \frac{\partial \boldsymbol{v}}{\partial t} \right\|^2_{\boldsymbol{L}^2(\Omega)}. \tag{5.233}$$

Choosing $\delta = 2$ in (5.232), and employing the result in (5.231), we find that for $p > 2$, $t > 0$, and all $\mu_1 > 0$,

$$\frac{d}{dt}\left\{\int_\Omega \Gamma(e_{ij}e_{ij})d\boldsymbol{x} + \mu_1 \int_\Omega \frac{\partial e_{ij}}{\partial x_k}\frac{\partial e_{ij}}{\partial x_k}d\boldsymbol{x}\right\} \le k_3(\Omega)\,||\boldsymbol{v}||^2_{\boldsymbol{W}^{1,p}(\Omega)}\,||\boldsymbol{v}||^2_{\boldsymbol{H}^1(\Omega)} + \frac{1}{2}|\boldsymbol{f}|^2_\infty, \tag{5.234}$$

We now need to obtain, for $\boldsymbol{v}_0 \in B_0$, $r > 0$, and $t \ge t_0(R_0)$, estimates for the integrals

$$\int_t^{t+r}\int_\Omega \Gamma(e_{ij}e_{ij})d\boldsymbol{x}\,d\tau \text{ and } \int_t^{t+r}\int_\Omega \frac{\partial e_{ij}}{\partial x_k}\frac{\partial e_{ij}}{\partial x_k}d\boldsymbol{x}\,d\tau.$$

To begin with, we note that, for $p > 2$,

$$\begin{aligned}\Gamma(e_{ij}e_{ij}) &= \int_0^{e_{ij}e_{ij}} (\epsilon + s)^{\frac{p-2}{2}}ds \\ &= \frac{2}{p}\{(\epsilon + |\boldsymbol{e}|^2)^{\frac{p}{2}} - \epsilon^{\frac{p}{2}}\} \le \frac{2}{p}(\epsilon + |\boldsymbol{e}|^2)^{\frac{p}{2}}\end{aligned}$$

and, therefore, by the Hölder Inequality

$$\int_\Omega \Gamma(e_{ij}e_{ij})d\boldsymbol{x} \le k_p\left(\int_\Omega(\epsilon + |\boldsymbol{e}|^2)d\boldsymbol{x}\right)^{\frac{p}{2}} \tag{5.235}$$

where $k_p = \dfrac{2}{p}(\text{meas}\,\Omega)^{\frac{2-p}{2}}$. Using Hölder again we have

$$\begin{aligned}\int_t^{t+r}\left(\int_\Omega(\epsilon + |\boldsymbol{e}|^2)d\boldsymbol{x}\right)^{\frac{p}{2}}d\tau &\le r^{\frac{2-p}{2}}\left(\int_t^{t+r}\int_\Omega(\epsilon + |\boldsymbol{e}|^2)d\boldsymbol{x}\,d\tau\right)^{\frac{p}{2}} \\ &\le r^{\frac{2-p}{2}}\left(\epsilon r\,\text{meas}(\Omega) + \int_t^{t+r} e_{ij}e_{ij}d\boldsymbol{x}\,d\tau\right)^{\frac{p}{2}}\end{aligned} \tag{5.236}$$

which, when combined with (5.235), yields

$$\int_t^{t+r}\int_\Omega \Gamma(e_{ij}e_{ij})d\boldsymbol{x}\,d\tau \le k_p'(\Omega)r^{\frac{2-p}{2}}\left(\epsilon r\,\text{meas}(\Omega) + \int_t^{t+r}||\boldsymbol{v}||^2_{\boldsymbol{W}^{1,2}(\Omega)}\,d\tau\right) \tag{5.237}$$

for some $k_p'(\Omega) > 0$, when $\boldsymbol{v}_0 \in B_0$, $r > 0$, and $t \ge t_0(R_0)$. Returning to (5.222), and choosing $\beta = \beta^* > 0$ sufficiently large, we infer the existence of $k^*(\epsilon, p; \Omega) > 0$ such that

$$\frac{d}{dt}\,||\boldsymbol{v}||^2_{\boldsymbol{L}^2(\Omega)} + k^*\,||\boldsymbol{v}||^2_{\boldsymbol{W}^{1,2}(\Omega)} \le \beta^*|\boldsymbol{f}|^2_\infty \tag{5.238}$$

which, upon integration from t to $t+r$, yields

$$||\boldsymbol{v}(t+r)||^2_{\boldsymbol{L}^2(\Omega)} + k^* \int_t^{t+r} ||\boldsymbol{v}||^2_{\boldsymbol{W}^{1,2}(\Omega)}\, d\tau \le \left(\beta^* r + \frac{2\beta}{\bar{k}}\right) |\boldsymbol{f}|^2_\infty \tag{5.239}$$

where we have again used (5.225), which is valid for $\boldsymbol{v}_0 \in B_0$, $\forall \mu_1 \ge 0$, if $t \ge t_0(R_0)$. From (5.239) it is immediate that for $t \ge t_0(R_0)$, $r > 0$,

$$\int_t^{t+r} ||\boldsymbol{v}||^2_{\boldsymbol{W}^{1,2}(\Omega)}\, d\tau \le \frac{1}{k^*}\left(\beta^* r + \frac{2\beta}{\bar{k}}\right) |\boldsymbol{f}|^2_\infty \tag{5.240}$$

provided $\boldsymbol{v}_0 \in B_0$. Combining (5.237) with (5.240) we infer the existence of $C_1 = C_1(r, p, \epsilon, \Omega, |\boldsymbol{f}|_\infty) > 0$ and independent of $\mu_1 \ge 0$, such that for $\boldsymbol{v}_0 \in B_0$ and $t \ge t_0(R_0)$

$$\int_t^{t+r} \int_\Omega \Gamma(e_{ij}e_{ij})\, d\boldsymbol{x}\, d\tau \le C_1. \tag{5.241}$$

Also, as a consequence of (5.219) and (5.225), we have for $\boldsymbol{v}_0 \in B_0$, $t \ge t_0(R_0)$, and all $\mu_1 \ge 0$

$$\frac{1}{2}\frac{d}{dt} ||\boldsymbol{v}||^2_{\boldsymbol{L}^2(\Omega)} + 2\mu_1 \int_\Omega \frac{\partial e_{ij}}{\partial x_k}\frac{\partial e_{ij}}{\partial x_k}\, d\boldsymbol{x} \le \rho |\boldsymbol{f}|^2_\infty \tag{5.242}$$

with $\rho = \sqrt{\frac{2\beta}{\bar{k}}}$ as in (5.226); integrating this last estimate from t to $t+r$, and employing (5.225) again, we find that $\forall \mu_1 \ge 0$, $r > 0$,

$$\mu_1 \int_t^{t+r} \int_\Omega \frac{\partial e_{ij}}{\partial x_k}\frac{\partial e_{ij}}{\partial x_k}\, d\boldsymbol{x}\, d\tau \le (\rho^2 + 2\rho r) |\boldsymbol{f}|^2_\infty \tag{5.243}$$

provided $\boldsymbol{v}_0 \in B_0$ and $t \ge t_0(R_0)$. We now want to deal with the differential inequality (5.234); to this end we note that, in a manner similar to (5.211), the standard Korn inequality [Ev] yields

$$\begin{aligned} \int_\Omega \Gamma(e_{ij}e_{ij})\, d\boldsymbol{x} &= \int_\Omega \int_0^{e_{ij}e_{ij}} (\epsilon + s)^{\frac{p-2}{2}}\, ds\, d\boldsymbol{x} \\ &\ge \epsilon^{\frac{p-2}{2}} \int_\Omega e_{ij}e_{ij}\, d\boldsymbol{x} \ge \epsilon^{\frac{p-2}{2}} k_2(\Omega)\, ||\boldsymbol{v}||^2_{\boldsymbol{W}^{1,2}(\Omega)}\,. \end{aligned} \tag{5.244}$$

Therefore,

$$||\boldsymbol{v}||^2_{\boldsymbol{W}^{1,2}(\Omega)} \le C_2 \left(\int_\Omega \Gamma(e_{ij}e_{ij})\, d\boldsymbol{x} + \mu_1 \int_\Omega \frac{\partial e_{ij}}{\partial x_k}\frac{\partial e_{ij}}{\partial x_k}\, d\boldsymbol{x} \right) \tag{5.245}$$

for all $\mu_1 \geq 0$, where $C_2(\epsilon, p; \Omega) = (\epsilon^{\frac{p-2}{2}} k_2(\Omega))^{-1}$. Employing (5.245) in (5.234) we obtain a differential inequality of the form

$$\frac{dy}{dt} \leq a(t)y(t) + b(t), \quad t > 0 \tag{5.246}$$

where

$$y(t) = \int_\Omega \Gamma(e_{ij}e_{ij})d\boldsymbol{x} + \mu_1 \int_\Omega \frac{\partial e_{ij}}{\partial x_k}\frac{\partial e_{ij}}{\partial x_k}d\boldsymbol{x}, \tag{5.247a}$$

$$a(t) = k_3(\Omega)C_2(\epsilon, p; \Omega)\,||\boldsymbol{v}||^2_{\boldsymbol{W}^{1,p}(\Omega)}, \tag{5.247b}$$

and

$$b(t) = \frac{1}{2}|\boldsymbol{f}|^2_\infty. \tag{5.247c}$$

The differential inequality displayed in (5.246) is valid $\forall p > 2$ and $\forall \mu_1 \geq 0$. We now apply the Uniform Gronwall Lemma of Foias and Prodi [Te4] to (5.246); to begin with, we define

$$\chi(t) = \left\{\tau \in [t, t+r] \,\middle|\, ||\boldsymbol{v}(\tau)||_{\boldsymbol{W}^{1,p}(\Omega)} \leq 1\right\} \tag{5.248}$$

for any $t \geq 0$ and any fixed $r > 0$. Thus, on $[t, t+r]/\chi(t)$ we have $||\boldsymbol{v}||_{\boldsymbol{W}^{1,p}(\Omega)} > 1$ so that

$$||\boldsymbol{v}(\tau)||^2_{\boldsymbol{W}^{1,p}(\Omega)} < ||\boldsymbol{v}(\tau)||^p_{\boldsymbol{W}^{1,p}(\Omega)}, \quad p > 2$$

for $\tau \in [t, t+r]/\chi(t)$. Therefore,

$$\int_{\chi(t)} ||\boldsymbol{v}(\tau)||^2_{\boldsymbol{W}^{1,p}(\Omega)}\,d\tau \leq \text{meas}(\chi(t)) \leq r \tag{5.249}$$

while for $\boldsymbol{v}_0 \in B_0$, $t \geq t_0(R_0)$, and all $\mu_1 \geq 0$,

$$\begin{aligned}\int_{[t,t+r]/\chi(t)} ||\boldsymbol{v}(\tau)||^2_{\boldsymbol{W}^{1,p}(\Omega)}\,d\tau &\leq \int_{[t,t+r]/\chi(t)} ||\boldsymbol{v}(\tau)||^p_{\boldsymbol{W}^{1,p}(\Omega)}\,d\tau \\ &\leq \int_t^{t+r} ||\boldsymbol{v}(\tau)||^p_{\boldsymbol{W}^{1,p}(\Omega)}\,d\tau \leq \frac{1}{k_1(\Omega)}\left(\frac{1}{2}\rho^2 + \rho r\right)|\boldsymbol{f}|^2_\infty\end{aligned} \tag{5.250}$$

by virtue of (5.228). Combining (5.249) and (5.250), and using the definition (5.247b) of $a(t)$, we obtain an estimate of the form

$$\int_t^{t+r} a(\tau)d\tau \leq k_1(\epsilon, p, r, |\boldsymbol{f}|_\infty; \Omega) \tag{5.251}$$

with $k_1 > 0$, which is valid for $t \geq t_0(R_0)$, if $\boldsymbol{v}_0 \in B_0$, and all $\mu_1 \geq 0$. Also, by (5.247c)

$$\int_t^{t+r} b(\tau)d\tau \leq k_2(r; |\boldsymbol{f}|_\infty) \equiv \frac{1}{2} r |\boldsymbol{f}|_\infty^2 \tag{5.252}$$

while, in view of (5.241), (5.243), and (5.247a) we have for $\boldsymbol{v}_0 \in B_0$, $t \geq t_0(R_0)$, and all $\mu_1 \geq 0$

$$\int_t^{t+r} y(\tau)d\tau \leq k_3(\epsilon, p, r, |\boldsymbol{f}|_\infty; \Omega) \tag{5.253}$$

with $k_3 > 0$. In view of (5.251)–(5.253), a direct application of the Uniform Gronwall Lemma to (5.246) now yields the estimate

$$y(t+r) \leq \left(\frac{k_3}{r} + k_2\right) \exp(k_1)$$

or

$$\int_\Omega \left\{ \Gamma(e_{ij}e_{ij}) + \mu_1 \frac{\partial e_{ij}}{\partial x_k} \frac{\partial e_{ij}}{\partial x_k} \right\} d\boldsymbol{x} \bigg|_{t+r} \leq K \tag{5.254}$$

with $K = K(\epsilon, \rho, r, |\boldsymbol{f}|_\infty; \Omega)$; the estimate (5.254) holds $\forall p > 2$, $t \geq t_0(R_0)$, for $\boldsymbol{v}_0 \in B_0$, and all $\mu_1 \geq 0$. As a direct consequence of (5.254), and the fact that (for $p > 2$) (5.244) holds we see that $\forall p > 2$, $t \geq t_0(R_0)$, and $\mu_1 \geq 0$,

$$||\boldsymbol{v}||^2_{\boldsymbol{W}^{1,2}(\Omega)}(t+r) \leq K \tag{5.255}$$

from which (5.229) follows. We now see that if $\boldsymbol{v} \in B_0 \subset B_{R_0}(\boldsymbol{0})$, so that $||\boldsymbol{v}_0||_{\boldsymbol{L}^2(\Omega)} \leq R_0$, then $\exists \rho' > 0$, ρ' independent of $\mu_1 \geq 0$, such that for any $p > 2$,

$$\boldsymbol{S}_{\mu_1}(t)\boldsymbol{v}_0 \in B^{\rho'}_{\boldsymbol{W}^{1,2}(\Omega)}, \text{ for } t \geq t_0'(R_0) \tag{5.256}$$

so that the ball $B^{\rho'}_{\boldsymbol{W}^{1,2}(\Omega)}$ of radius ρ' in $\boldsymbol{W}^{1,2}(\Omega)$ is an absorbing set for $\boldsymbol{S}_{\mu_1}$, which is independent of μ_1 for all $\mu_1 \geq 0$. □

Proof (Theorem 5.9). Lemmas 5.16 and 5.17 have established the existence of absorbing sets in $\boldsymbol{H}_{per}$ and $\boldsymbol{W}^{1,2}(\Omega)$, for $\boldsymbol{S}_{\mu_1}(t)$, with $\mu_1 \geq 0$, which are independent of $\mu_1 > 0$. There remains the task of exhibiting, for $\mu_1 > 0$, an absorbing set for $\boldsymbol{S}_{\mu_1}(t)$ in $\boldsymbol{W}^{2,2}(\Omega)$. However, by (5.254) it follows that $\forall \mu_1 \geq 0$, all $p > 2$, and $t \geq t_0(R_0)$,

$$\mu_1 \int_\Omega \frac{\partial e_{ij}}{\partial x_k} \frac{\partial e_{ij}}{\partial x_k} d\boldsymbol{x} \bigg|_{t+r} \leq K. \tag{5.257}$$

Then, by virtue of the generalized Korn inequality of Lemma 5.2, for $\boldsymbol{w} \in \boldsymbol{W}_0^{1,2}(\Omega) \cap \boldsymbol{W}^{2,2}(\Omega)$, $\exists \hat{k}(\Omega) > 0$ such that

$$\int_\Omega \frac{\partial e_{ij}}{\partial x_k} \frac{\partial e_{ij}}{\partial x_k} d\boldsymbol{x} \geq \hat{k}(\Omega) \, ||\boldsymbol{v}||^2_{\boldsymbol{H}^2(\Omega)}$$

so that (5.257) implies that for $p > 2$, $t \geq t_0'(R_0)$, and all $\mu_1 > 0$,

$$||\boldsymbol{v}||^2_{\boldsymbol{H}^2(\Omega)}(t) \leq \frac{1}{\mu_1} \hat{K}(\epsilon, p, r, |\boldsymbol{f}|_\infty; \Omega). \tag{5.258}$$

From (5.258) we deduce the existence of absorbing sets for $\boldsymbol{S}_{\mu_1}$, for any $\mu_1 > 0$, of radii ρ_{μ_1} where, clearly, $\rho_{\mu_1} \sim \dfrac{1}{\mu_1}$ as $\mu_1 \to 0^+$; in other words for $\boldsymbol{v}_0 \in B_0 \subset B_{R_0}(\boldsymbol{0})$, $p > 2$, and any $\mu_1 > 0$,

$$\boldsymbol{S}_{\mu_1}(t)\boldsymbol{v}_0 \in B^{\rho_{\mu_1}}_{\boldsymbol{W}^{2,2}(\Omega)}, \text{ for } t \geq t_0'(R_0) \tag{5.259}$$

thus completing the proof of the theorem. □

Remarks. For fixed $\mu_1 > 0$ the existence of the absorbing set $B^{\rho_{\mu_1}}_{\boldsymbol{W}^{2,2}(\Omega)}$ yields the uniform compactness of $\boldsymbol{S}_{\mu_1}(t)$, for t large, and similar remarks apply to $\boldsymbol{S}_{\mu_1}(t)$ for $\mu_1 \geq 0$ with respect to the existence of the absorbing sets $B^{\rho'}_{\boldsymbol{W}^{1,2}(\Omega)}$.

Following the analysis in [CF] we may now define the maximal compact global attractors for $\boldsymbol{S}_{\mu_1}$, $\mu_1 \geq 0$, as follows: for $\mu_1 > 0$ we set

$$\boldsymbol{\mathcal{A}}_{\mu_1} = \bigcap_{t>0} \boldsymbol{S}_{\mu_1}(t) B^{\rho_{\mu_1}}_{\boldsymbol{W}^{2,2}(\Omega)}, \quad \mu_1 > 0 \tag{5.260}$$

while for $\mu_1 = 0$ we define

$$\boldsymbol{\mathcal{A}}_0 = \bigcap_{t>0} \boldsymbol{S}_0(t) B^{\rho'}_{\boldsymbol{W}^{1,2}(\Omega)}. \tag{5.261}$$

That $\boldsymbol{\mathcal{A}}_{\mu_1}$, $\mu_1 > 0$, and $\boldsymbol{\mathcal{A}}_0$ are, respectively, the maximal compact global attractors for $\boldsymbol{S}_{\mu_1}$, $\mu_1 > 0$, and $\boldsymbol{S}_0$ follows from an argument entirely analogous to the one presented in Sect. 5.3 and will not be repeated here. Our goal in the next subsection will be to establish (1) the uniform diferentiability of $\boldsymbol{S}_{\mu_1}(t)$, $\mu_1 > 0$, on $\boldsymbol{\mathcal{A}}_{\mu_1}$ and (2) the uniform boundedness of the Fréchet differential $\boldsymbol{L}_{\mu_1}(t; \boldsymbol{u}_0)$, $\boldsymbol{u}_0 \in \boldsymbol{\mathcal{A}}_{\mu_1}$, of $\boldsymbol{S}_{\mu_1}(t)$ in the strong operator norm of $\boldsymbol{\mathcal{L}}_{\mu_1}(\boldsymbol{H}_{per}; \boldsymbol{H}_{per})$ on the global attractor $\boldsymbol{\mathcal{A}}_{\mu_1}$, $\forall t > 0$; once these results have been established, the techniques employed in Sect. 5.3 can be used to compute upper bounds for the Hausdorff and fractal dimensions $d_H(\boldsymbol{\mathcal{A}}_{\mu_1})$ and $d_F(\boldsymbol{\mathcal{A}}_{\mu_1})$, $\mu_1 > 0$, respectively; we also explain, in Sect. 5.4.2 why similar techniques can not be used to compute upper bounds for $d_H(\boldsymbol{\mathcal{A}}_0)$ and $d_F(\boldsymbol{\mathcal{A}}_0)$.

5.4.2 Uniform Differentiability of S_{μ_1} on the Maximal Compact Global Attractor $\mathcal{A}_{\mu_1}$, $\mu_1 > 0$

As in Sect. 5.3.3, for the case $1 < p \leq 2$, we now want to prove uniform differentiability of the nonlinear semigroup $\boldsymbol{S}_{\mu_1}$ on the maximal compact global attractor $\mathcal{A}_{\mu_1}$ whose existence was established in the last section; thus far, it has only proven possible to do this for $\mu_1 > 0$ and for p in the range $2 < p < 3$. The definition of uniform differentability presented in Sect. 5.3.3 must be altered slightly to account for the fact that we are now dealing with the space-periodic problem and not the boundary-value problem; we make the

Definition 5.7. The nonlinear semigroup $\boldsymbol{S}_{\mu_1}$, $\mu_1 > 0$, is uniformly differentiable on $\mathcal{A}_{\mu_1}$ if, $\forall \boldsymbol{u}_0$, $\exists \boldsymbol{L}_{\mu_1}(t, \boldsymbol{u}_0) \in \mathcal{L}_{\mu_1}(\boldsymbol{H}_{per}, \boldsymbol{H}_{per})$ such that, as $\epsilon \to 0$, we have $\forall t > 0$

$$\sup_{\substack{\boldsymbol{u}_0, \boldsymbol{v}_0 \in \mathcal{A}_{\mu_1} \\ 0 < ||\boldsymbol{u}_0 - \boldsymbol{v}_0|| \leq \epsilon}} \frac{|\boldsymbol{S}_{\mu_1}(t)\boldsymbol{v}_0 - \boldsymbol{S}_{\mu_1}(t)\boldsymbol{u}_0 - \boldsymbol{L}_{\mu_1}(t; \boldsymbol{u}_0)(\boldsymbol{v}_0 - \boldsymbol{u}_0)|_{L^2(\Omega)}}{||\boldsymbol{v}_0 - \boldsymbol{u}_0||_{L^2(\Omega)}} \to 0. \tag{5.262}$$

Remarks. When $\boldsymbol{L}_{\mu_1}(t; \boldsymbol{u}_0)$, $\mu_1 > 0$, $t > 0$, exists $\boldsymbol{L}_{\mu_1}(t; \boldsymbol{u}_0)(\boldsymbol{v}_0 - \boldsymbol{u}_0) = \boldsymbol{U}(t; \mu_1)$ is a solution of the linearized problem

$$\begin{aligned} \frac{\partial U_i}{\partial t} + U_j \frac{\partial u_i}{\partial x_j} + u_j \frac{\partial U_i}{\partial x_j} = -\frac{\partial \tilde{p}}{\partial x_i} + \frac{\partial}{\partial x_j}[\gamma(\boldsymbol{u}) e_{ij}(\boldsymbol{U}) - \alpha B_{ijkl}(\boldsymbol{u}) e_{kl}(\boldsymbol{U})] \\ - 2\mu_1 \frac{\partial}{\partial x_j}(\Delta e_{ij}(\boldsymbol{U})), \text{ in } \Omega \times [0, T). \end{aligned} \tag{5.263}$$

In (5.263), $\boldsymbol{u}(t) = \boldsymbol{S}_{\mu_1}(t)\boldsymbol{u}_0$, $\tilde{p}$ is the difference of the pressures corresponding to $\boldsymbol{u}(t) = \boldsymbol{S}_{\mu_1}(t)\boldsymbol{u}_0$ and $\boldsymbol{v}(t) = \boldsymbol{S}_{\mu_1}(t)\boldsymbol{v}_0$, and

$$B_{ijkl}(\boldsymbol{u}) = \mu_0(\epsilon + |\boldsymbol{e}(\boldsymbol{u})|^2)^{\frac{p-4}{2}} e_{ij}(\boldsymbol{u}) e_{ij}(\boldsymbol{u}), \quad p > 2. \tag{5.264}$$

The solution of (5.263) is subject to an initial condition of the form

$$\boldsymbol{U}(\boldsymbol{x}, 0) = \boldsymbol{U}_0 \in \boldsymbol{H}_{per}(\Omega) \cap \boldsymbol{H}^2(\Omega) \tag{5.265}$$

as well as the constraint of incompressibility and the periodicity condition (5.3b), with $\boldsymbol{v}$ replaced by $\boldsymbol{U}$.

Remarks. It may be easily established that there exists a unique solution of the space-periodic problem for $\boldsymbol{U}(t)$, when $p > 2$, in both space dim $n = 2, 3$, which satisfies

$$\boldsymbol{U}(\cdot) \in L^\infty([0, T); \boldsymbol{H}_{per} \cap \boldsymbol{H}^2(\Omega)), \quad \forall T > 0. \tag{5.266}$$

We also record here the following result which is a direct consequence of the definitions of the potential Γ in (5.92) and the tensor B_{ijkl}: for $\alpha < 0$,

$$\gamma(\boldsymbol{u})e_{ij}(\boldsymbol{U}) - \alpha B_{ijkl}(\boldsymbol{u})e_{kl}(\boldsymbol{U}) = \frac{\partial^2\Gamma}{\partial e_{ij}\partial e_{kl}}(\boldsymbol{e}(\boldsymbol{u}))e_{kl}(\boldsymbol{U}). \tag{5.267}$$

To establish the uniform differentiability of $\boldsymbol{S}_{\mu_1}$ on the attractor $\boldsymbol{\mathcal{A}}_{\mu_1}$, as defined by (5.260), we need to estimate the $\boldsymbol{L}^2(\Omega)$ norm of

$$\boldsymbol{\theta}(t;\mu_1) = \boldsymbol{S}_{\mu_1}(t)\boldsymbol{v}_0 - \boldsymbol{S}_{\mu_1}(t)\boldsymbol{u}_0 - \boldsymbol{L}_{\mu_1}(t;\boldsymbol{u}_0)(\boldsymbol{v}_0 - \boldsymbol{u}_0) \tag{5.268}$$

for $\boldsymbol{u}_0, \boldsymbol{v}_0 \in \boldsymbol{\mathcal{A}}_{\mu_1}$ such that $||\boldsymbol{u}_0 - \boldsymbol{v}_0||_{\boldsymbol{L}^2(\Omega)} \le \epsilon$; to this end we set

$$\boldsymbol{w}(t;\mu_1) = \boldsymbol{v}(t;\mu_1) - \boldsymbol{u}(t;\mu_1), \tag{5.269a}$$

$$\boldsymbol{\theta}(t;\mu_1) = \boldsymbol{w}(t;\mu_1) - \boldsymbol{U}(t;\mu_1) \tag{5.269b}$$

and state the following

Lemma 5.18. *Let $\boldsymbol{w}, \boldsymbol{\theta}$ be defined as in (5.269a,b). Then $\exists$ constants $\bar{k}(\Omega)$, $\tilde{k}(\Omega)$ and $\bar{c}_p(\Omega)$, all positive, such that for all $t > 0$,*

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}||\boldsymbol{\theta}||^2_{\boldsymbol{L}^2(\Omega)} + \epsilon^{p-2}\bar{k}(\Omega)\,||\boldsymbol{\theta}||^2_{\boldsymbol{W}^{1,2}(\Omega)} + 2\mu_1\hat{k}(\Omega)\,||\boldsymbol{\theta}||^2_{\boldsymbol{H}^2(\Omega)}\\ &\le \bar{c}_p\int_\Omega |\boldsymbol{e}(\boldsymbol{\theta})||\boldsymbol{e}(\boldsymbol{w})|^2 d\boldsymbol{x} + \left|\int_\Omega \theta_i\frac{\partial u_i}{\partial x_j}\theta_j\,d\boldsymbol{x}\right| + \left|\int_\Omega w_j\frac{\partial w_i}{\partial x_j}\theta_i\,d\boldsymbol{x}\right|\end{aligned} \tag{5.270}$$

if $2 < p < 3$.

Proof. We begin by observing that on $\Omega \times [0,T)$, $\boldsymbol{w}(t)$ satisfies (5.120) and the associated conditions (5.3b), with $\boldsymbol{v} \to \boldsymbol{w}$, as well as

$$\nabla\cdot\boldsymbol{w} = 0, \text{ in } \Omega\times[0,T), \tag{5.271a}$$

$$\boldsymbol{w}(0) = \boldsymbol{v}_0 - \boldsymbol{u}_0, \text{ in } \Omega. \tag{5.271b}$$

Multiplying (5.120) by w_i, then integrating over Ω, integrating by parts, and applying the periodic boundary conditions, we again obtain (5.122) which, by virtue of the definition of the potential Γ in (5.92), leads to

$$\frac{1}{2}\frac{d}{dt}||\boldsymbol{w}||^2_{\boldsymbol{L}^2(\Omega)} + \int_\Omega w_j\frac{\partial u_i}{\partial x_j}w_i\,d\boldsymbol{x} + \int_\Omega\left(\frac{\partial\Gamma}{\partial e_{ij}}(e_{ij}(\boldsymbol{v})) - \frac{\partial\Gamma}{\partial e_{ij}}(e_{ij}(\boldsymbol{u}))\right)d\boldsymbol{x} \le 0$$

for all $\mu_1 > 0$; from this last inequality which it follows that

$$\frac{1}{2}\frac{d}{dt}\,||\boldsymbol{u}||^2_{\boldsymbol{L}^2(\Omega)} + \int_\Omega w_j \frac{\partial u_i}{\partial x_j} w_i \, d\boldsymbol{x} + \int_\Omega \left(\int_0^1 \frac{\partial^2 \Gamma}{\partial e_{ij} \partial e_{kl}} (\boldsymbol{e}(\boldsymbol{u} + \tau \boldsymbol{w})) d\tau \right) e_{ij}(\boldsymbol{w}) e_{kl}(\boldsymbol{w}) \, d\boldsymbol{x} \le 0. \quad (5.272)$$

However, by virtue of (5.92), with $\alpha = 2 - p$, $p > 2$, for any $\boldsymbol{\zeta} \in R^{n^2}$, $\boldsymbol{\zeta} \neq \boldsymbol{0}$,

$$\begin{aligned}\frac{\partial^2 \Gamma}{\partial e_{ij} \partial e_{kl}} \zeta_{ij} \zeta_{kl} &= (\epsilon + |\boldsymbol{e}|^2)^{\frac{p-2}{2}} \left[|\boldsymbol{\zeta}|^2 + (p-2) \frac{|\boldsymbol{\zeta} \cdot \boldsymbol{e}|^2}{(\epsilon + |\boldsymbol{e}|^2)} \right] \\ &\ge (\epsilon + |\boldsymbol{e}|^2)^{\frac{p-2}{2}} |\boldsymbol{\zeta}|^2 \ge \epsilon^{\frac{p-2}{2}} |\boldsymbol{\zeta}|^2.\end{aligned}$$

Thus, using once more the usual Korn inequality we have

$$\int_\Omega \left(\int_0^1 \frac{\partial^2 \Gamma}{\partial e_{ij} \partial e_{kl}} (\boldsymbol{e}(\boldsymbol{u} + \tau \boldsymbol{w})) d\tau \right) e_{ij}(\boldsymbol{w}) e_{kl}(\boldsymbol{w}) \, d\boldsymbol{x} \ge e^{\frac{p-2}{2}} \int_\Omega e_{ij}(\boldsymbol{w}) e_{ij}(\boldsymbol{w}) \, d\boldsymbol{x} \ge \epsilon^{\frac{p-2}{2}} k_2(\Omega) \, ||\boldsymbol{w}||^2_{\boldsymbol{W}^{1,2}(\Omega)}. \quad (5.273)$$

Employing (5.273) in (5.272) we find that, for any $\mu_1 > 0$,

$$\frac{1}{2}\frac{d}{dt}\,||\boldsymbol{w}||^2_{\boldsymbol{L}^2(\Omega)} + \epsilon^{\frac{p-2}{2}} k_2(\Omega) \, ||\boldsymbol{w}||^2_{\boldsymbol{W}^{1,2}(\Omega)} \le \left| \int_\Omega w_j \frac{\partial u_i}{\partial x_j} w_i \, d\boldsymbol{x} \right|. \quad (5.274)$$

From standard estimates [CFT1] for the convective term on the right-hand side of (5.274) we have

$$\left| \int_\Omega w_j \frac{\partial u_i}{\partial x_j} w_i \, d\boldsymbol{x} \right| \le c_1(\Omega) \, ||\boldsymbol{w}||_{\boldsymbol{W}^{1,2}(\Omega)} \, ||\boldsymbol{u}||_{\boldsymbol{W}^{1,2}(\Omega)} \, ||\boldsymbol{w}||^{1/2}_{\boldsymbol{L}^2(\Omega)} \, ||\boldsymbol{w}||^{3/2}_{\boldsymbol{W}^{1,2}(\Omega)} \quad (5.275)$$

for some $c_1(\Omega) > 0$. Now, as $\boldsymbol{u}_0, \boldsymbol{v}_0, \boldsymbol{w}_0 \in \boldsymbol{\mathcal{A}}_{\mu_1}$ it follows from the definition of $\boldsymbol{\mathcal{A}}_{\mu_1}$ that $\boldsymbol{u}(t)$, $\boldsymbol{v}(t)$, $\boldsymbol{w}(t)$ are all contained in $B^{\rho_{\mu_1}}_{\boldsymbol{W}^{2,2}(\Omega)}$ for all $t \ge 0$. Thus, for any $\mu_1 > 0$,

$$||\boldsymbol{u}(t)||_{\boldsymbol{W}^{1,2}(\Omega)} \le \rho_{\mu_1}, \quad \forall t \ge 0 \quad (5.276)$$

with similar results for $||\boldsymbol{v}(t)||_{\boldsymbol{W}^{1,2}(\Omega)}$ and $||\boldsymbol{w}(t)||_{\boldsymbol{W}^{1,2}(\Omega)}$. From (5.275) we now have, $\forall t \ge 0$,

$$\left| \int_\Omega w_j \frac{\partial u_i}{\partial x_j} w_i \, d\boldsymbol{x} \right| \le c_1(\Omega)\rho_{\mu_1} \, ||\boldsymbol{w}||^{3/2}_{\boldsymbol{W}^{1,2}(\Omega)} \, ||\boldsymbol{w}||^{1/2}_{\boldsymbol{L}^2(\Omega)} \, . \tag{5.277}$$

Employing Young's inequality (see Appendix A) with $a = ||\boldsymbol{w}||^{3/2}_{\boldsymbol{W}^{1,2}(\Omega)}$, $b = ||\boldsymbol{w}||^{1/2}_{\boldsymbol{L}^2(\Omega)}$, and $q = 4/3$, on the right-hand side of (5.277), we see that for some $c_2 = c_2(\mu_1; \Omega) > 0$,

$$\left| \int_\Omega w_j \frac{\partial u_i}{\partial x_j} w_i \, d\boldsymbol{x} \right| \le c_2 \left(\delta^{4/3} \, ||\boldsymbol{w}||^2_{\boldsymbol{W}^{1,2}(\Omega)} + \frac{1}{\delta^4} \, ||\boldsymbol{w}||^2_{\boldsymbol{L}^2(\Omega)} \right) \tag{5.278}$$

in which case the differential inequality (5.274) becomes

$$\frac{1}{2}\frac{d}{dt} \, ||\boldsymbol{w}||^2_{\boldsymbol{L}^2(\Omega)} + \epsilon^{\frac{p-2}{2}} k_2(\Omega) \, ||\boldsymbol{w}||^2_{\boldsymbol{W}^{1,2}(\Omega)} \le c_2 \left(\delta^{4/3} \, ||\boldsymbol{w}||^2_{\boldsymbol{W}^{1,2}(\Omega)} + \frac{1}{\delta^4} \, ||\boldsymbol{w}||^2_{\boldsymbol{L}^2(\Omega)} \right) . \tag{5.279}$$

If we choose δ sufficiently small we obtain from (5.279) an estimate of the form

$$\frac{d}{dt} \, ||\boldsymbol{w}||^2_{\boldsymbol{L}^2(\Omega)} + \epsilon^{\frac{p-2}{2}} k_2(\Omega) \, ||\boldsymbol{w}||^2_{\boldsymbol{W}^{1,2}(\Omega)} \le c_3 \, ||\boldsymbol{w}||^2_{\boldsymbol{L}^2(\Omega)} \tag{5.280}$$

with $c_3 = c_3(\mu_1; \Omega) > 0$. Direct integration of (5.280) now produces the following estimates for $t > 0$,

$$||\boldsymbol{v}(t) - \boldsymbol{u}(t)||^2_{\boldsymbol{L}^2(\Omega)} \le ||\boldsymbol{v}_0 - \boldsymbol{u}_0||^2_{\boldsymbol{L}^2(\Omega)} \exp(c_3 t) \tag{5.281a}$$

and

$$\int_0^t ||\boldsymbol{w}(\tau)||^2_{\boldsymbol{W}^{1,2}(\Omega)} \, d\tau \le \frac{2\epsilon^{\frac{2-p}{2}}}{k_2(\Omega)} \, ||\boldsymbol{v}_0 - \boldsymbol{u}_0||^2_{\boldsymbol{L}^2(\Omega)} \exp(c_3 t). \tag{5.281b}$$

In order to proceed with the proof of the lemma, we now turn to the problem satisfied by $\boldsymbol{\theta}$, as defined by (5.269b). The function $\boldsymbol{\theta}$ satisfies the system of equations

$$\begin{aligned} \frac{\partial \theta_i}{\partial t} + \theta_j \frac{\partial u_i}{\partial x_j} + u_j \frac{\partial \theta}{\partial x_j} + w_j \frac{\partial w_i}{\partial x_j} = -&\frac{\partial p_{\boldsymbol{\theta}}}{\partial x_i} \\ &+ \frac{\partial}{\partial x_j}[\gamma(\boldsymbol{v})e_{ij}(\boldsymbol{v}) - \gamma(\boldsymbol{u})e_{ij}(\boldsymbol{u})] \\ &- \frac{\partial}{\partial x_j}[\gamma(\boldsymbol{u})e_{ij}(\boldsymbol{U}) - \alpha B_{ijkl}(\boldsymbol{u})e_{kl}(\boldsymbol{U})] \\ &- 2\mu_1 \frac{\partial}{\partial x_j}(\Delta e_{ij}(\boldsymbol{\theta})), \text{ in } \Omega \times [0, T), \end{aligned} \tag{5.282}$$

$p_{\boldsymbol{\theta}}$ being the difference of the pressures corresponding to the problems for $\boldsymbol{w}$ and $\boldsymbol{U}$, for $\boldsymbol{\theta}$ satisfying (5.3b), with $\boldsymbol{v} \to \boldsymbol{\theta}$, and

$$\nabla \cdot \boldsymbol{\theta}, \text{ in } \Omega \times [0, T), \tag{5.283a}$$

$$\boldsymbol{\theta}(0) = \mathbf{0}, \text{ in } \Omega. \tag{5.283b}$$

If we multiply (5.282) through by θ_i, integrate over Ω, and then integrate by parts we recover, for the space-periodic problem, the identity (5.129) with $\boldsymbol{\Theta}$ replaced by $\boldsymbol{\theta}$. Using the definitions of Γ, the identity (5.267), the usual Korn inequality, and the fact that $\boldsymbol{U} = \boldsymbol{w} - \boldsymbol{\theta}$, we obtain

$$\begin{aligned}
&\int_\Omega [\gamma(\boldsymbol{v})e_{ij}(\boldsymbol{v}) - \gamma(\boldsymbol{u})e_{ij}(\boldsymbol{u})]e_{ij}(\boldsymbol{\theta})d\boldsymbol{x} \\
&\qquad - \int_\Omega [\gamma(\boldsymbol{u})e_{ij}(\boldsymbol{U}) - \alpha B_{ijkl}(\boldsymbol{u})e_{kl}(\boldsymbol{U})]e_{ij}(\boldsymbol{\theta})d\boldsymbol{x} \\
&\quad = \int_\Omega \left(\int_0^1 \frac{\partial^2\Gamma}{\partial e_{ij}\partial e_{kl}}(\boldsymbol{e}(\boldsymbol{u} + \tau\boldsymbol{w}))d\tau\right) e_{ij}(\boldsymbol{\theta})e_{kl}(\boldsymbol{w})d\boldsymbol{x} \\
&\qquad - \int_\Omega \frac{\partial^2\Gamma}{\partial e_{ij}\partial e_{kl}}(\boldsymbol{e}(\boldsymbol{u}))e_{ij}(\boldsymbol{\theta})e_{kl}(\boldsymbol{U})d\boldsymbol{x} \\
&\quad = \int_\Omega \frac{\partial^2\Gamma}{\partial e_{ij}\partial e_{kl}}(\boldsymbol{e}(\boldsymbol{u}))e_{ij}(\boldsymbol{\theta})e_{kl}(\boldsymbol{\theta})d\boldsymbol{x} \\
&\qquad + \int_\Omega \left(\int_0^1 \left[\frac{\partial^2\Gamma}{\partial e_{ij}\partial e_{kl}}(\boldsymbol{e}(\boldsymbol{u} + \tau\boldsymbol{w})) - \frac{\partial^2\Gamma}{\partial e_{ij}\partial e_{kl}}(\boldsymbol{e}(\boldsymbol{u}))\right] d\tau\right) e_{ij}(\boldsymbol{\theta})e_{kl}(\boldsymbol{w})d\boldsymbol{x} \\
&\quad \geq \epsilon^{p-2}\bar{k}(\Omega)\,||\boldsymbol{\theta}||^2_{\boldsymbol{W}^{1,2}(\Omega)} + \int_\Omega \Gamma_{ijklmn}e_{ij}(\boldsymbol{\theta})e_{kl}(\boldsymbol{w})e_{mn}(\boldsymbol{w})d\boldsymbol{x}
\end{aligned} \tag{5.284}$$

where, as in Sect. 5.3.3,

$$\Gamma_{ijklmn} = \int_0^1 \int_0^1 \frac{\partial^3\Gamma}{\partial e_{ij}\partial e_{kl}\partial e_{mn}}(\boldsymbol{e}(\boldsymbol{u} + \sigma\tau\boldsymbol{w})\tau d\tau d\sigma.$$

Substituting the lower bound in (5.284) into (5.129), after replacing $\boldsymbol{\Theta}$ with $\boldsymbol{\theta}$, we obtain the differential inequality

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\,||\boldsymbol{\theta}||^2_{\boldsymbol{L}^2(\Omega)} + \int_\Omega \theta_i \frac{\partial u_i}{\partial x_j}\theta_j d\boldsymbol{x} + \int_\Omega w_j \frac{\partial w_i}{\partial x_j}\theta_i d\boldsymbol{x} \\
&\qquad + \epsilon^{p-2}\bar{k}(\Omega)\,||\boldsymbol{\theta}||^2_{\boldsymbol{W}^{1,2}(\Omega)} + \int_\Omega \Gamma_{ijklmn}e_{ij}(\boldsymbol{\theta})e_{kl}(\boldsymbol{w})e_{mn}(\boldsymbol{w})d\boldsymbol{x} \\
&\qquad\qquad + 2\mu_1 \int_\Omega \frac{\partial e_{ij}}{\partial x_k}(\boldsymbol{\theta})\frac{\partial e_{ij}}{\partial x_k}(\boldsymbol{\theta})d\boldsymbol{x} \leq 0.
\end{aligned} \tag{5.285}$$

By a direct calculation, based on (5.92), we now easily determine that, for $p > 2$,

$$\frac{\partial^3 \Gamma}{\partial e_{ij} \partial e_{kl} \partial e_{mn}} = (p-2)(\epsilon + |\boldsymbol{e}|^2)^{\frac{p-2}{2}} \times \left\{ \left[\frac{\delta_{im}\delta_{jn} e_{kl} + \delta_{km}\delta_{ln} e_{ij} + \delta_{ij}\delta_{kl} e_{mn}}{(\epsilon + |\boldsymbol{e}|^2)} \right] + \frac{p-4}{(\epsilon + |\boldsymbol{e}|^2)^2} e_{ij} e_{kl} e_{mn} \right\}. \tag{5.286}$$

Therefore, for $p < 3$, $\exists c_p(\Omega) > 0$ such that

$$\left| \frac{\partial^3 \Gamma}{\partial e_{ij} \partial e_{kl} \partial e_{mn}} \right| \le c_p; \quad i, j, k, l, m, n = 1, 2, 3. \tag{5.287}$$

By employing Lemma 5.2 and (5.287) in (5.285) we now arrive at the differential inequality (5.270). □

Remarks. By virtue of (5.276), $\boldsymbol{u} \in B^{\rho_{\mu_1}}_{\boldsymbol{W}^{2,2}(\Omega)}$, $\forall t > 0$; therefore, for some $\bar{c}_1 > 0$, and any $\delta > 0$,

$$\left| \int_\Omega \theta_i \frac{\partial u_i}{\partial x_j} \theta_j \, d\boldsymbol{x} \right| \le \bar{c}_1 \left\{ \delta^{4/3} \, ||\boldsymbol{\theta}||^2_{\boldsymbol{W}^{1,2}(\Omega)} + \frac{1}{\delta^4} \, ||\boldsymbol{\theta}||^2_{\boldsymbol{L}^2(\Omega)} \right\} \tag{5.288a}$$

while

$$\begin{aligned} \left| \int_\Omega \theta_i \frac{\partial w_i}{\partial x_j} w_j \, d\boldsymbol{x} \right| &\le \bar{c}_2 \, ||\boldsymbol{\theta}||_{\boldsymbol{W}^{1,2}(\Omega)} \, ||\boldsymbol{w}||^{3/2}_{\boldsymbol{W}^{1,2}(\Omega)} \, ||\boldsymbol{w}||^{1/2}_{\boldsymbol{L}^2(\Omega)} \\ &\le \bar{c}_3 \, ||\boldsymbol{\theta}||_{\boldsymbol{W}^{1,2}(\Omega)} \, ||\boldsymbol{w}||_{\boldsymbol{W}^{1,2}(\Omega)} \\ &\le \bar{c}_4(\bar{\delta}) \, ||\boldsymbol{\theta}||^2_{\boldsymbol{W}^{1,2}(\Omega)} + \frac{1}{\bar{\delta}} \, ||\boldsymbol{w}||^2_{\boldsymbol{W}^{1,2}(\Omega)} \end{aligned} \tag{5.288b}$$

for some $\bar{c}_4 > 0$, and any $\bar{\delta} > 0$, as $\boldsymbol{w} \in B^{\rho_{\mu_1}}_{\boldsymbol{W}^{1,2}(\Omega)}$, $\forall t > 0$.

Lemma 5.19. *For $\boldsymbol{\theta}(t; \mu_1)$ as defined by (5.269b), $\exists \beta_1, \beta_2 > 0$, depending (at most) on ϵ, μ_1, p, and Ω, such that for fixed $\mu_1 > 0$, $\epsilon^* > 0$, and all $t > 0$,*

$$\sup_{\substack{\boldsymbol{u}_0, \boldsymbol{v}_0 \in \boldsymbol{\mathcal{A}}_{\mu_1} \\ 0 \le ||\boldsymbol{u}_0 - \boldsymbol{v}_0|| \le \epsilon^*}} ||\boldsymbol{\theta}||_{\boldsymbol{L}^2(\Omega)} \le \beta_1 e^{\beta_2 t} \epsilon^*. \tag{5.289}$$

Proof. Once again we return to the identity (5.122) which now, by virtue of the generalized Korn inequality of Lemma 5.2, the definition of the potential Γ, the lower bound (5.273), and an estimate entirely analogous to (5.288a), yields a differential inequality of the form

$$\frac{1}{2}\frac{d}{dt}\,||\boldsymbol{w}||^2_{\boldsymbol{L}^2(\Omega)} + \epsilon^{\frac{p-2}{2}} k_2(\Omega)\,||\boldsymbol{w}||^2_{\boldsymbol{W}^{1,2}(\Omega)} + 2\mu_1\hat{k}(\Omega)\,||\boldsymbol{w}||^2_{\boldsymbol{H}^2(\Omega)}$$
$$\leq c_1'\delta'^{4/3}\,||\boldsymbol{w}||^2_{\boldsymbol{W}^{1,2}(\Omega)} + \frac{c_2'}{\delta'^4}\,||\boldsymbol{w}||^2_{\boldsymbol{L}^2(\Omega)} \quad (5.290)$$

for some $c_1', c_2' > 0$ and any $\delta' > 0$. Choosing δ' sufficiently small, we easily generate for $t > 0$ an estimate similar to (5.281a) as well as the new estimate

$$\mu_1 \int_0^t ||\boldsymbol{w}(\tau)||^2_{\boldsymbol{W}^{2,2}(\Omega)}\,d\tau \leq k^*\,||\boldsymbol{w}(0)||^2_{\boldsymbol{L}^2(\Omega)} \exp(c^* t) \quad (5.291)$$

where $k^* = k^*(\epsilon, p, \mu_1; \Omega) > 0$ with a like dependence for $c^* > 0$.

Returning to the differential inequality (5.270) we estimate, using the Hölder Inequality, the first expression on the right-hand side, i.e., for some $k_*, \tilde{k} > 0$,

$$\int_\Omega |\boldsymbol{e}(\boldsymbol{\theta})||\boldsymbol{e}(\boldsymbol{w})|^2 d\boldsymbol{x} \leq \left(\int_\Omega |\boldsymbol{e}(\boldsymbol{\theta})|^3\right)^{1/3}\left(\int_\Omega |\boldsymbol{e}(\boldsymbol{w})|^3\right)^{2/3}$$
$$\leq k_*\,||\boldsymbol{\theta}||_{\boldsymbol{W}^{1,3}(\Omega)}\,||\boldsymbol{w}||^2_{\boldsymbol{W}^{1,3}(\Omega)} \leq \tilde{k}\,||\boldsymbol{\theta}||_{\boldsymbol{W}^{2,2}(\Omega)}\,||\boldsymbol{w}||^2_{\boldsymbol{W}^{2,2}(\Omega)}, \quad (5.292)$$

the last estimate following from the embedding $\boldsymbol{H}^2(\Omega) \hookrightarrow \boldsymbol{W}^{1,3}(\Omega)$. As $\boldsymbol{w} \in B^{\rho_{\mu_1}}_{\boldsymbol{W}^{2,2}(\Omega)}$, $\forall t > 0$, we may rewrite this last estimate in (5.292) in the form

$$\int_\Omega |\boldsymbol{e}(\boldsymbol{\theta})||\boldsymbol{e}(\boldsymbol{w})|^2 d\boldsymbol{x} \leq \tilde{k}_{\rho_{\mu_1}}\,||\boldsymbol{\theta}||_{\boldsymbol{W}^{2,2}(\Omega)}\,||\boldsymbol{w}||_{\boldsymbol{W}^{2,2}(\Omega)}$$
$$\leq k^\#\left(\delta^\#\,||\boldsymbol{\theta}||^2_{\boldsymbol{W}^{2,2}(\Omega)} + \frac{1}{\delta^\#}\,||\boldsymbol{w}||^2_{\boldsymbol{W}^{2,2}(\Omega)}\right) \quad (5.293)$$

for some $k^\# > 0$ and any $\delta^\# > 0$. Combining (5.270) with the estimates (5.288a,b) and (5.293) we have, $\forall t > 0$, and any fixed $\mu_1 > 0$,

$$\frac{1}{2}\frac{d}{dt}\,||\boldsymbol{\theta}||^2_{\boldsymbol{L}^2(\Omega)} + \epsilon^{p-2} k_2(\Omega)\,||\boldsymbol{\theta}||^2_{\boldsymbol{W}^{2,2}(\Omega)} + 2\mu_1\bar{k}(\Omega)\,||\boldsymbol{\theta}||^2_{\boldsymbol{H}^2(\Omega)}$$
$$\leq \bar{c}_p k^\#\left(\delta^\#\,||\boldsymbol{\theta}||^2_{\boldsymbol{H}^2(\Omega)} + \frac{1}{\delta^\#}\,||\boldsymbol{w}||^2_{\boldsymbol{W}^{2,2}(\Omega)}\right)$$
$$+ \bar{c}_1\left(\delta^{4/3}\,||\boldsymbol{\theta}||^2_{\boldsymbol{W}^{1,2}(\Omega)} + \frac{1}{\delta^4}\,||\boldsymbol{\theta}||^2_{\boldsymbol{L}^2(\Omega)}\right)$$
$$+ \bar{c}_4\left(\bar{\delta}\,||\boldsymbol{\theta}||^2_{\boldsymbol{W}^{1,2}(\Omega)} + \frac{1}{\bar{\delta}}\,||\boldsymbol{w}||^2_{\boldsymbol{W}^{1,2}(\Omega)}\right). \quad (5.294)$$

By choosing δ, $\bar{\delta}$, and $\delta^{\#}$ all sufficiently small in (5.294), we now deduce the existence of $\rho_1, \rho_2 > 0$ which depend (at most) on ϵ, μ_1, p, and Ω, such that $\forall t \geq 0$,

$$\frac{d}{dt} ||\boldsymbol{\theta}||^2_{\boldsymbol{L}^2(\Omega)} \leq \rho_1 ||\boldsymbol{w}||^2_{\boldsymbol{W}^{2,2}(\Omega)} + \rho_2 ||\boldsymbol{\theta}||^2_{\boldsymbol{W}^{2,2}(\Omega)} . \tag{5.295}$$

As $\boldsymbol{\theta}(0) = \boldsymbol{0}$, it follows from (5.295) that, $\forall t \geq 0$,

$$||\boldsymbol{\theta}||^2_{\boldsymbol{L}^2(\Omega)} \leq \rho_1 e^{\rho_2 t} \int_0^t ||\boldsymbol{w}(\tau)||^2_{\boldsymbol{W}^{2,2}(\Omega)} d\tau. \tag{5.296}$$

However, by virtue of the estimate (5.291), and the fact that $||\boldsymbol{w}(0)||_{\boldsymbol{L}^2(\Omega)} \leq \epsilon^*$, it is easy to see that (5.296) implies that

$$||\boldsymbol{\theta}||^2_{\boldsymbol{L}^2(\Omega)} \leq \frac{\rho_1 k^*}{\mu_1} [\exp(c^* + \rho_2)t] \epsilon^{*^2}, \quad \forall t > 0 \tag{5.297}$$

from which (5.289) follows. □

We are now in a position to prove the uniform differentiability of the semigroup $\boldsymbol{S}_{\mu_1}$ on the attractor $\boldsymbol{\mathcal{A}}_{\mu_1}$; more specifically we have the following

Theorem 5.10. *For $\mu_1 > 0$, and $2 < p < 3$, the nonlinear semigroup $\boldsymbol{S}_{\mu_1}$ is uniformly differentiable on the maximal compact attractor $\boldsymbol{\mathcal{A}}_{\mu_1}$ given by* (5.260).

Proof. Having established the estimate (5.289), we now modify the bound (5.288b) to read

$$\begin{aligned}\left| \int_\Omega \theta_i \frac{\partial w_i}{\partial x_j} w_j \, d\boldsymbol{x} \right| &\leq c(\Omega) ||\boldsymbol{w}||_{\boldsymbol{W}^{2,2}(\Omega)} |\boldsymbol{w}|_{\boldsymbol{H}^1(\Omega)} ||\boldsymbol{\theta}||_{\boldsymbol{L}^2(\Omega)} \\ &\leq \beta_1 c(\Omega) e^{\beta_2 t} \epsilon^* ||\boldsymbol{w}||^2_{\boldsymbol{W}^{2,2}(\Omega)} ,\end{aligned} \tag{5.298}$$

the last bound following directly from (5.289). Also, returning to (5.292), we have

$$\begin{aligned}\int_\Omega |\boldsymbol{e}(\boldsymbol{\theta})||\boldsymbol{e}(\boldsymbol{w})|^2 d\boldsymbol{x} &\leq k_*(\Omega) ||\boldsymbol{\theta}||_{\boldsymbol{W}^{1,3}(\Omega)} ||\boldsymbol{w}||^2_{\boldsymbol{W}^{1,3}(\Omega)} \\ &\leq k'(\Omega) ||\boldsymbol{\theta}||_{\boldsymbol{W}^{1,3}(\Omega)} ||\boldsymbol{w}||^2_{\boldsymbol{W}^{2,2}(\Omega)}\end{aligned} \tag{5.299}$$

if we, once again, use the embedding $\boldsymbol{H}^2(\Omega) \hookrightarrow \boldsymbol{W}^{1,3}(\Omega)$. As $\boldsymbol{H}^{3/2,2}(\Omega) \hookrightarrow \boldsymbol{W}^{1,3}(\Omega)$, and we may interpolate $\boldsymbol{H}^{m+s}(\Omega)$, $0 < s < 1$, $m \in N$ (the natural numbers), between $\boldsymbol{H}^{m+1}(\Omega)$ and $\boldsymbol{L}^2(\Omega)$, (see Appendix A), for any $\lambda > 0$,

$$
\begin{aligned}
||\boldsymbol{\theta}||_{\boldsymbol{W}^{1,3}(\Omega)} &\leq \bar{c}(\Omega)\, ||\boldsymbol{\theta}||^{1/4}_{\boldsymbol{L}^2(\Omega)}\, ||\boldsymbol{\theta}||^{3/4}_{\boldsymbol{W}^{2,2}(\Omega)} \\
&\leq \bar{c}(\Omega)[\lambda^{8/3}\, ||\boldsymbol{\theta}||^2_{\boldsymbol{W}^{2,2}(\Omega)} + \lambda^{-8/5}\, ||\boldsymbol{\theta}||^{8/5}_{\boldsymbol{L}^2(\Omega)}]
\end{aligned} \tag{5.300}
$$

where we have used Young's inequality. Employing (5.300) in (5.299) yields the estimate

$$
\int_\Omega |\boldsymbol{e}(\boldsymbol{\theta})||\boldsymbol{e}(\boldsymbol{w})|^2 d\boldsymbol{x} \leq \bar{k}(\Omega)\, ||\boldsymbol{w}||^2_{\boldsymbol{W}^{2,2}(\Omega)} \left\{\lambda^{8/3}\, ||\boldsymbol{\theta}||^2_{\boldsymbol{W}^{2,2}(\Omega)} + \lambda^{-8/3}\, ||\boldsymbol{\theta}||^{2/5}_{\boldsymbol{L}^2(\Omega)}\right\}. \tag{5.301}
$$

Making use of the estimates (5.288a), (5.299), and (5.301) in the differential inequality (5.270) we find that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\, ||\boldsymbol{\theta}||^2_{\boldsymbol{L}^2(\Omega)} + \epsilon^{p-2} k_2(\Omega)\, ||\boldsymbol{\theta}||^2_{\boldsymbol{W}^{1,2}(\Omega)} + 2\mu_1 \hat{k}(\Omega)\, ||\boldsymbol{\theta}||^2_{\boldsymbol{W}^{2,2}(\Omega)} \\
&\qquad \leq \bar{c}_p \bar{k}(\Omega)\, ||\boldsymbol{w}||^2_{\boldsymbol{W}^{2,2}(\Omega)} \left\{\lambda^{8/3}\, ||\boldsymbol{\theta}||^2_{\boldsymbol{W}^{2,2}(\Omega)} + \lambda^{-8/3}\, ||\boldsymbol{\theta}||^{2/5}_{\boldsymbol{L}^2(\Omega)}\right\} \\
&+ \beta_1 c(\Omega) \boldsymbol{e}^{\beta_2 t} \epsilon^*\, ||\boldsymbol{w}||^2_{\boldsymbol{W}^{2,2}(\Omega)} + \bar{c}_1(\Omega) \left\{\delta^{4/3}\, ||\boldsymbol{\theta}||^2_{\boldsymbol{W}^{1,2}(\Omega)} + \frac{1}{\delta^4}\, ||\boldsymbol{\theta}||^2_{\boldsymbol{L}^2(\Omega)}\right\}
\end{aligned} \tag{5.302}
$$

for $t \geq 0$ and arbitrary $\lambda, \delta > 0$. Using, once again, the fact that $\boldsymbol{w}(t) \in B^{\rho_{\mu_1}}_{\boldsymbol{W}^{2,2}(\Omega)}$, $\forall t \geq 0$, and choosing λ, δ sufficiently small in (5.302), we now easily obtain (in view of (5.289)) a differential inequality of the form

$$
\frac{d}{dt}\, ||\boldsymbol{\theta}||^2_{\boldsymbol{L}^2(\Omega)} \leq c(\mu_1; \Omega)[\gamma_1(t)\epsilon^* + \gamma_2(t){\epsilon^*}^{2/5}]\, ||\boldsymbol{w}||^2_{\boldsymbol{W}^{2,2}(\Omega)} + c'(\Omega)\, ||\boldsymbol{\theta}||^2_{\boldsymbol{L}^2(\Omega)} \tag{5.303}
$$

with the $\gamma_i(t) > 0$ and monotonically increasing in t for $t > 0$. As $\boldsymbol{\theta}(0) = \boldsymbol{0}$ it now follows from (5.303) and (5.291) that, for any $t > 0$,

$$
\begin{aligned}
||\boldsymbol{\theta}||^2_{\boldsymbol{L}^2(\Omega)} &\leq c[\gamma_1(t)\epsilon^* + \gamma_2(t){\epsilon^*}^{2/5}]e^{c't} \int_0^t ||\boldsymbol{w}||^2_{\boldsymbol{W}^{2,2}(\Omega)}\, (\Omega) d\tau \\
&\leq \frac{ck^*}{\mu_1}[\gamma_1(t)\epsilon^* + \gamma_2(t){\epsilon^*}^{2/5}]{\epsilon^*}^2 e^{(c'+c^*)t}.
\end{aligned} \tag{5.304}
$$

From (5.304) it is immediate that

$$
||\boldsymbol{\theta}(t)||_{\boldsymbol{L}^2(\Omega)} / \epsilon^* \to 0, \text{ as } \epsilon^* \to 0 \tag{5.305}
$$

which serves to establish the uniform differentiability of $\boldsymbol{S}_{\mu_1}(t)$ on $\boldsymbol{\mathcal{A}}_{\mu_1}$. □

5.4.3 Uniform Boundedness of the Fréchet Differential $\boldsymbol{L}_{\mu_1}(t;\boldsymbol{u}_0)$, $\boldsymbol{u}_0 \in \boldsymbol{\mathcal{A}}_{\mu_1}$

Having established the uniform differentiability of $\boldsymbol{S}_{\mu_1}$, $\mu_1 > 0$, on the maximal compact global attractor $\boldsymbol{\mathcal{A}}_{\mu_1}$, we now seek to prove that the Fréchet differential $\boldsymbol{L}_{\mu_1}(t;\boldsymbol{u}_0)$, $\boldsymbol{u}_0 \in \boldsymbol{\mathcal{A}}_{\mu_1}$, of $\boldsymbol{S}_{\mu_1}(t)$ is uniformly bounded in the strong operator norm of $\mathcal{L}(\boldsymbol{H}_{per}, \boldsymbol{H}_{per})$ for all $t > 0$; with this result and the result of Theorem 5.10 in hand, we will be in a position to apply standard techniques to compute upper bounds for $d_H(\boldsymbol{\mathcal{A}}_{\mu_1})$ and $d_F(\boldsymbol{\mathcal{A}}_{\mu_1})$, $\mu_1 > 0$. We begin by recalling that the linearized equations for $\boldsymbol{U}(t;\mu_1) = \boldsymbol{L}_{\mu_1}(t;\boldsymbol{u}_0)\boldsymbol{U}_0$ are (5.263) with $\boldsymbol{U}(0;\mu_1) = \boldsymbol{U}_0$ for any $\mu_1 > 0$. Also

$$\left|\left|\boldsymbol{L}_{\mu_1}(t;\boldsymbol{u}_0)\right|\right|_{\mathcal{L}(\boldsymbol{H}_{per},\boldsymbol{H}_{per})} = \sup_{\boldsymbol{U}_0 \in \boldsymbol{H}_{per}} \left(\left|\left|\boldsymbol{L}_{\mu_1}(t;\boldsymbol{u}_0)\boldsymbol{U}_0\right|\right|_{\boldsymbol{L}^2(\Omega)} / \,||\boldsymbol{U}_0||_{\boldsymbol{L}^2(\Omega)}\right) \tag{5.306}$$

where the linearization in (5.263) is taken about $\boldsymbol{u}(t) = \boldsymbol{S}_{\mu_1}(t)\boldsymbol{u}_0$ with $\boldsymbol{u}_0 \in \boldsymbol{\mathcal{A}}_{\mu_1}$. Our goal in this subsection is to establish the following result:

Theorem 5.11. *The Fréchet differential of $\boldsymbol{S}_{\mu_1}(t)$, $\boldsymbol{L}_{\mu_1}(t,\boldsymbol{u}_0)$ for $\boldsymbol{u}_0 \in \boldsymbol{\mathcal{A}}_{\mu_1}$, satisfies, for some $l > 0$, and all $t \geq 0$,*

$$\sup_{\boldsymbol{u}_0 \in \boldsymbol{\mathcal{A}}_{\mu_1}} \left|\left|\boldsymbol{L}_{\mu_1}(t;\boldsymbol{u}_0)\right|\right|_{\boldsymbol{\mathcal{L}}(\boldsymbol{H}_{per},\boldsymbol{H}_{per})} \leq l^{[t]+1}. \tag{5.307}$$

Proof. We multiply (5.263) through by U_i, integrate over Ω, sum over $i = 1, 2, 3$ and integrate by parts using the spatial periodicity of $\boldsymbol{U}$ and the incompressibility constraint $\nabla \cdot \boldsymbol{U} = 0$; we are easily led to the differential inequality

$$\frac{1}{2}\frac{d}{dt}\,||\boldsymbol{U}||^2_{\boldsymbol{L}^2(\Omega)} + 2\mu_1 \int_\Omega \frac{\partial e_{ij}}{\partial x_k}(\boldsymbol{U})\frac{\partial e_{ij}}{\partial x_k}(\boldsymbol{U})d\boldsymbol{x} \leq \left|\int_\Omega u_i \frac{\partial U_i}{\partial x_j} U_j \, d\boldsymbol{x}\right| \tag{5.308}$$

where, on the left-hand side of (5.308), we have dropped the nonnegative term

$$\int_\Omega [\gamma(\boldsymbol{u})e_{ij}(\boldsymbol{U}) - \alpha B_{ijkl}(\boldsymbol{u})e_{ij}(\boldsymbol{U})e_{kl}(\boldsymbol{U})]d\boldsymbol{x} = \int_\Omega \frac{\partial^2 \Gamma}{\partial e_{ij}\partial e_{kl}}(\boldsymbol{e}(\boldsymbol{u}))e_{ij}(\boldsymbol{U})e_{kl}(\boldsymbol{U})d\boldsymbol{x} \geq 0.$$

Now,

$$\begin{aligned}
\left|\int_\Omega u_i \frac{\partial U_i}{\partial x_j} U_j \, d\boldsymbol{x}\right| &\leq ||\boldsymbol{u}||_{\boldsymbol{L}^\infty(\Omega)}\,||\boldsymbol{U}||_{\boldsymbol{H}^1(\Omega)}\,||\boldsymbol{U}||_{\boldsymbol{L}^2(\Omega)} \\
&\leq c\,||\boldsymbol{u}||_{\boldsymbol{W}^{2,2}(\Omega)}\,||\boldsymbol{U}||_{\boldsymbol{W}^{2,2}(\Omega)}\,||\boldsymbol{U}||_{\boldsymbol{L}^2(\Omega)} \\
&\leq c\rho_{\mu_1}\,||\boldsymbol{U}||_{\boldsymbol{L}^2(\Omega)}\,||\boldsymbol{U}||_{\boldsymbol{W}^{2,2}(\Omega)}
\end{aligned}$$

as $\boldsymbol{u}(t) \in B^{\rho_{\mu_1}}_{\boldsymbol{W}^{2,2}(\Omega)}$, $t \geq 0$, if $\boldsymbol{u}_0 \in \boldsymbol{\mathcal{A}}_{\mu_1}$ and $\mu_1 > 0$. Thus $\exists \kappa(\mu_1;\Omega) > 0$ such that

$$\frac{1}{2}\frac{d}{dt}\,||\boldsymbol{U}||^2_{\boldsymbol{L}^2(\Omega)} + \kappa(\mu_1;\Omega)\,||\boldsymbol{U}||^2_{\boldsymbol{W}^{2,2}(\Omega)} \leq \frac{\rho'}{2\delta}\,||\boldsymbol{U}||^2_{\boldsymbol{L}^2(\Omega)} + \frac{\rho'\delta}{2}\,||\boldsymbol{U}||^2_{\boldsymbol{W}^{2,2}(\Omega)} \tag{5.309}$$

for all $\delta > 0$, where $\rho' = \rho'(\mu_1;\Omega) > 0$, and we have used the fact that $\boldsymbol{U} \in L^\infty((0,T);\boldsymbol{W}^{2,2}(\Omega))$, $\forall T > 0$, as well as the generalized Korn estimate of Lemma 5.2. For δ sufficiently small, therefore,

$$\frac{d}{dt}\,||\boldsymbol{U}||^2_{\boldsymbol{L}^2(\Omega)} + \kappa(\mu_1;\Omega)\,||\boldsymbol{U}||^2_{\boldsymbol{W}^{2,2}(\Omega)} \leq \left(\frac{\rho'}{2\delta}\right)||\boldsymbol{U}||^2_{\boldsymbol{L}^2(\Omega)} \tag{5.310}$$

from which it follows that $\forall t \geq 0$, $\boldsymbol{u}_0 \in \boldsymbol{\mathcal{A}}_{\mu_1}$, $\mu_1 > 0$, and $\boldsymbol{U}_0 \in \boldsymbol{H}_{per}$

$$\frac{\left|\left|\boldsymbol{L}_{\mu_1}(t;\boldsymbol{u}_0)\right|\right|_{\boldsymbol{L}^2(\Omega)}}{||\boldsymbol{U}_0||_{\boldsymbol{L}^2(\Omega)}} \leq \exp\left(\sqrt{\frac{\rho'}{2\delta}t}\right) \tag{5.311}$$

so that

$$\sup_{0\leq t\leq 1}\ \sup_{\boldsymbol{u}_0\in\boldsymbol{\mathcal{A}}_{\mu_1}} \left|\left|\boldsymbol{L}_{\mu_1}(t;\boldsymbol{u}_0)\right|\right|_{\boldsymbol{\mathcal{L}}(\boldsymbol{H}_{per},\boldsymbol{H}_{per})} \leq e^{\sqrt{\rho'2\delta}} \equiv \ell. \tag{5.312}$$

As $\boldsymbol{S}_{\mu_1}(t) = \boldsymbol{S}_{\mu_1}(t-[t])S(t)^{[t]}$, (5.307) now follows as a direct consequence of (5.312) (see, e.g., Teman [Te4], Sect. V.2) and the uniform boundedness of $\boldsymbol{L}_{\mu_1}(t;\boldsymbol{u}_0)$ on $\boldsymbol{\mathcal{A}}_{\mu_1}$, for each $\mu_1 > 0$, has been established. □

5.4.4 Bounds for $d_H(\mathcal{A}_{\mu_1})$ and $d_F(\mathcal{A}_{\mu_1})$, $\mu_1 > 0$

To derive upper bounds for the Hausdorff and fractal dimensions of the attractor $\boldsymbol{\mathcal{A}}_{\mu_1}$, we first make the obvious modifications concerning the definitions of the Lyapunov numbers and exponents which were introduced in Sect. 5.3.5; the relevant results basically involve replacing the Hilbert space $\boldsymbol{H}$ by $\boldsymbol{H}_{per}$. Thus, for any $\boldsymbol{L} \in \boldsymbol{\mathcal{L}}(\boldsymbol{H}_{per},\boldsymbol{H}_{per})$ and any nonnegative integer k we set

$$\alpha_k(\boldsymbol{L}) = \sup_{\substack{G\subset \boldsymbol{H}_{per}\\ \dim G=k}}\ \inf_{\substack{\zeta\in G\\ ||\zeta||_{\boldsymbol{H}_{per}}=1}} ||\boldsymbol{L}\zeta||_{\boldsymbol{L}^2(\Omega)} \tag{5.313}$$

and

$$\omega_k(\boldsymbol{L}) = \alpha_1(\boldsymbol{L})\cdots\alpha_k(\boldsymbol{L}). \tag{5.314}$$

Then $\{\alpha_k(\boldsymbol{L})\}$ is nonincreasing and, if $\boldsymbol{L}$ is a compact self-adjoint nonnegative linear operator on $\boldsymbol{H}_{per}$, the $\alpha_k(\boldsymbol{L})$ are the eigenvalues of $(\boldsymbol{L}^*\boldsymbol{L})^{1/2}$ with $\alpha_1(\boldsymbol{L}) \geq \alpha_2(\boldsymbol{L}) \geq \cdots \geq 0$ where $\boldsymbol{L}^*$ is the adjoint operator. For $\boldsymbol{L} \in \boldsymbol{\mathcal{L}}(\boldsymbol{H}_{per}, \boldsymbol{H}_{per})$ and $d \in R^+$, $d = n + s$, $n \geq 1$ an integer and $s \in (0, 1)$, we define

$$\omega_d(\boldsymbol{L}) = \omega_n(\boldsymbol{L})^{1-s}\omega_{n+1}(\boldsymbol{L})^s \tag{5.315}$$

so that $d \mapsto \omega_d(\boldsymbol{L})$ is a nonincreasing function from $[1, \infty)$ into R^+. Now, let $\boldsymbol{S}_{\mu_1}(t)$ be the nonlinear semigroup generated by the bipolar problem ($\mu_1 > 0$) and $\boldsymbol{L}_{\mu_1}(t; \boldsymbol{u}_0)$ the associated Fréchet differential with $\boldsymbol{u}_0 \in \boldsymbol{\mathcal{A}}_{\mu_1}$; the numbers $\omega_k(\boldsymbol{L}_{\mu_1}(t; \boldsymbol{u}_0))$ bound the largest distortion of an infinitesmal k-dimensional volume produced by $\boldsymbol{S}_{\mu_1}(t)$ around the point $\boldsymbol{u}_0$. As we have shown in Sect. 5.4.2 that $\boldsymbol{S}_{\mu_1}$, $\mu_1 > 0$, is uniformly differentiable on $\boldsymbol{\mathcal{A}}_{\mu_1}$, the numbers $\omega_k(\boldsymbol{L}_{\mu_1}(t; \boldsymbol{u}_0))$ are well-defined $\forall t \geq 0$, $k \in N$, and $\boldsymbol{u}_0 \in \boldsymbol{\mathcal{A}}_{\mu_1}$. Furthermore, if we set, $\forall t \geq 0$,

$$\bar{\omega}_k^{\mu_1}(t) = \sup_{\boldsymbol{u}_0 \in \boldsymbol{\mathcal{A}}_{\mu_1}} \omega_k(\boldsymbol{L}_{\mu_1}(t; \boldsymbol{u}_0)); \quad k \in N \tag{5.316}$$

then for each $\mu_1 > 0$, the functions $t \mapsto \bar{\omega}_k^{\mu_1}(t)$ are subexponential, i.e.,

$$\bar{\omega}_k^{\mu_1}(t + s) \leq \bar{\omega}_k^{\mu_1}(t)\bar{\omega}_k^{\mu_1}(s); \quad \forall s, t \geq 0. \tag{5.317}$$

As in Sect. 5.3.5, standard results (e.g., [Te4]) now imply that $\lim_{t\to\infty}\{\bar{\omega}_k^{\mu_1}(t)\}^{1/t}$ exists and is equal to

$$\Pi_k^{\mu_1} \equiv \inf_{t>0}\{\bar{\omega}_k^{\mu_1}(t)\}^{1/t}. \tag{5.318}$$

Defining, recursively, the numbers

$$\begin{cases} \Lambda_1^{\mu_1} = \Pi_1^{\mu_1}, \Lambda_1^{\mu_1}\Lambda_2^{\mu_1} = \Pi_2^{\mu_1}, \ldots, \\ \Lambda_1^{\mu_1} \cdots \Lambda_k^{\mu_1} = \Pi_k^{\mu_1}, \end{cases} \tag{5.319}$$

we have

$$\Lambda_1^{\mu_1} = \Pi_1^{\mu_1}, \Lambda_k^{\mu_1} = \frac{\Pi_k^{\mu_1}}{\Pi_{k-1}^{\mu_1}}, \quad k \geq 2. \tag{5.320}$$

The $\Lambda_k^{\mu_1}$ are global (uniform) Lyapunov numbers on $\boldsymbol{\mathcal{A}}_{\mu_1}$, while the numbers

$$\lambda_k^{\mu_1} = \ln \Lambda_k^{\mu_1}, \quad k \geq 1 \tag{5.321}$$

are the global (uniform) Lyapunov exponents; thus, it follows that

$$\inf_{t>0}\{\bar{\omega}_k^{\mu_1}(t)\}^{1/t} = \exp(\lambda_1^{\mu_1} + \cdots + \lambda_k^{\mu_1}). \tag{5.322}$$

Using the linearized bipolar equations (5.263) we now define the linear operator $\boldsymbol{\mathcal{L}}_{\mu_1}(\boldsymbol{u}) : \tilde{\boldsymbol{H}} \to \boldsymbol{H}_{per}$, where

$$\tilde{\boldsymbol{H}} = \{\boldsymbol{\phi} \in \boldsymbol{W}^{2,2}(\Omega) \cap \boldsymbol{H}_{per} \mid \nabla \cdot \boldsymbol{\phi} = 0 \text{ in } \Omega\} \tag{5.323}$$

as follows: for $\alpha = 2 - p$, $2 < p < 3$,

$$(\boldsymbol{\mathcal{L}}_{\mu_1}(\boldsymbol{u})\boldsymbol{\phi})_i = 2\mu_1 \frac{\partial}{\partial x_j}(\Delta e_{ij}(\boldsymbol{\phi})) - \frac{\partial}{\partial x_j}[\gamma(\boldsymbol{u})e_{ij}(\boldsymbol{\phi}) - \alpha B_{ijkl}(\boldsymbol{u})e_{kl}(\boldsymbol{\phi})] + u_j \frac{\partial \phi_i}{\partial x_j} + \phi_j \frac{\partial u_i}{\partial x_j} \tag{5.324}$$

where $\boldsymbol{u} = \boldsymbol{S}_{\mu_1}(t)\boldsymbol{u}_0$, $\mu_1 > 0$, $\mu_0 \in \boldsymbol{\mathcal{A}}_{\mu_1}$. As a consequence of standard results on infinite dimensional dynamical systems (e.g., [Te4])

$$\lambda_1^{\mu_1} + \cdots + \lambda_k^{\mu_1} \le -q_k^{\mu_1} \tag{5.325}$$

with $q_k^{\mu_1}$ given by (5.176), where $\boldsymbol{Q}$ is an orthogonal projection on $\tilde{\boldsymbol{H}}$ of rank k and tr, as in Sect. 5.3.5, denotes the trace operation. We note that $\text{tr}(\boldsymbol{\mathcal{L}}_{\mu_1}(\boldsymbol{u}) \circ \boldsymbol{Q})$ is again computed as in (5.177) with $\{\boldsymbol{\phi}_j\}$ any basis of $\tilde{\boldsymbol{H}}$ such that (i) the $\boldsymbol{\phi}_j$ are orthonormal in $\boldsymbol{L}^2(\Omega)$ and (ii) $\{\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \ldots, \boldsymbol{\phi}_k\}$ is a basis of $\boldsymbol{Q} \circ \boldsymbol{H}_{per}$. As in Sect. 5.3.5, the result expressed by Lemma 5.12 is a consequence of Theorem 3.3 of [Te4]. It then follows that as a consequence of (5.182a,b), we have

$$d_H(\boldsymbol{\mathcal{A}}_{\mu_1}) \le k, \quad d_F(\boldsymbol{\mathcal{A}}_{\mu_1}) \le 2k, \quad \mu_1 > 0 \tag{5.326}$$

if $\lambda_1^{\mu_1} + \cdots + \lambda_k^{\mu_1} < 0$; this latter condition is, in view of (5.325), equivalent to the condition that $q_k^{\mu_1} > 0$ where $q_k^{\mu_1}$ is given by (5.176). Our task, therefore, is to find the smallest positive integer k, for each $\mu_1 > 0$, which satisfies $q_k^{\mu_1} > 0$; to accomplish this goal, we will state and prove the following

Lemma 5.20. *For the semigroup $\boldsymbol{S}_{\mu_1}(t)$ associated with the space-periodic problem* (5.2a,b)*,* (5.3b)*,* (5.4)*, in* $\dim n = 2$*, with* $2 < p < 3$*,* $\exists \hat{K} = \hat{K}(\epsilon, p; \Omega) > 0$*,* $c' = c'(\Omega) > 0$ *such that, for* $\delta > 0$ *chosen sufficiently small, we have for all* $\mu_1 > 0$

$$\lambda_1^{\mu_1} + \cdots + \lambda_k^{\mu_1} \le -\hat{K}(\epsilon, p; \Omega)\zeta_1 k^{5/3} + \frac{c'(\Omega)k}{\delta^4} \lim_{t\to\infty} \sup_{\boldsymbol{u}_0 \in \boldsymbol{\mathcal{A}}_{\mu_1}} \frac{1}{t}\int_0^t ||\boldsymbol{u}||^2_{\boldsymbol{W}^{1,2}(\Omega)}\, d\tau \tag{5.327}$$

where ζ_1 is the first eigenvalue of $-\Delta$ on Ω corresponding to an eigenvector $\boldsymbol{w}_1 \in \boldsymbol{H}^1(\Omega) \cap \boldsymbol{H}_{per}$ such that $\nabla \cdot \boldsymbol{w}_1 = 0$.

Proof. Using the definition of (5.324) of $\boldsymbol{\mathcal{L}}_{\mu_1}$ we compute, for $\boldsymbol{\phi} \in \tilde{\boldsymbol{H}}$, that

$$
\begin{aligned}
(\boldsymbol{\mathcal{L}}_{\mu_1}(\boldsymbol{u})\boldsymbol{\phi}, \boldsymbol{\phi})_{\boldsymbol{L}^2(\Omega)} &= 2\mu_1 \int_\Omega \frac{\partial e_{ij}}{\partial x_k}(\boldsymbol{\phi}) \frac{\partial e_{ij}}{\partial x_k}(\boldsymbol{\phi}) d\boldsymbol{x} \\
&\quad + \int_\Omega \frac{\partial^2 \Gamma}{\partial e_{ij} \partial e_{kl}}(\boldsymbol{u}) e_{ij}(\boldsymbol{\phi}) e_{kl}(\boldsymbol{\phi}) d\boldsymbol{x} + \int_\Omega \phi_j \frac{\partial u_i}{\partial x_j} \phi_i \, d\boldsymbol{x} \\
&\geq 2\mu_1 \int_\Omega \frac{\partial e_{ij}}{\partial x_k}(\boldsymbol{\phi}) \frac{\partial e_{ij}}{\partial x_k}(\boldsymbol{\phi}) d\boldsymbol{x} \\
&\quad + \int_\Omega (\epsilon + |\boldsymbol{e}(\boldsymbol{u})|^2)^{\frac{p-2}{2}} e_{ij}(\boldsymbol{\phi}) e_{ij}(\boldsymbol{\phi}) d\boldsymbol{x} + \int_\Omega \phi_j \frac{\partial u_i}{\partial x_j} \phi_i \, d\boldsymbol{x} \\
&\geq \epsilon^{\frac{p-2}{2}} k_2(\Omega) \, ||\boldsymbol{\phi}||^2_{\boldsymbol{W}^{1,2}(\Omega)} + \int_\Omega \phi_j \frac{\partial u_i}{\partial x_j} \phi_i \, d\boldsymbol{x}
\end{aligned}
\tag{5.328}
$$

for all $\mu_1 > 0$. Also, for $\boldsymbol{\phi} \in \tilde{\boldsymbol{H}}$ such that $||\boldsymbol{\phi}||_{\boldsymbol{L}^2(\Omega)} = 1$, any $\delta > 0$, and some $c'(\Omega) > 0$,

$$
\begin{aligned}
\left| \int_\Omega \phi_j \frac{\partial u_i}{\partial x_j} \phi_i \, d\boldsymbol{x} \right| &\leq c_1(\Omega) \, ||\boldsymbol{\phi}||^{3/2}_{\boldsymbol{W}^{1,2}(\Omega)} \, ||\boldsymbol{u}||_{\boldsymbol{W}^{1,2}(\Omega)} \\
&\leq c'(\Omega) \left(\delta^{4/3} \, ||\boldsymbol{\phi}||^2_{\boldsymbol{W}^{1,2}(\Omega)} + \frac{1}{\delta^4} \, ||\boldsymbol{u}||^2_{\boldsymbol{W}^{1,2}(\Omega)} \right)
\end{aligned}
\tag{5.329}
$$

where we have used (5.275) with $\boldsymbol{w}$ replaced by $\boldsymbol{\phi}$. Therefore, using (5.177) we have, for any $\mu_1 > 0$,

$$
\begin{aligned}
\operatorname{tr}(\boldsymbol{\mathcal{L}}_{\mu_1}(\boldsymbol{u}) \circ \boldsymbol{Q}) &= \sum_{\ell=1}^{k} (\boldsymbol{\mathcal{L}}_{\mu_1}(\boldsymbol{u})\boldsymbol{\phi}_\ell, \boldsymbol{\phi}_\ell)_{\boldsymbol{L}^2(\Omega)} \\
&\geq \epsilon^{\frac{p-2}{2}} k_2(\Omega) \sum_{\ell=1}^{k} ||\boldsymbol{\phi}_\ell||^2_{\boldsymbol{W}^{1,2}(\Omega)} - \frac{c'(\Omega)k}{\delta^4} \, ||\boldsymbol{u}||^2_{\boldsymbol{W}^{1,2}(\Omega)} \\
&\quad - c'(\Omega)\delta^{4/3} \sum_{\ell=1}^{k} ||\boldsymbol{\phi}_\ell||^2_{\boldsymbol{W}^{1,2}(\Omega)}
\end{aligned}
\tag{5.330}
$$

so that for δ chosen positive, and sufficiently small, $\exists K = K(\epsilon, p; \Omega)$ such that

$$
\operatorname{tr}(\boldsymbol{\mathcal{L}}_{\mu_1}(\boldsymbol{u}) \circ \boldsymbol{Q}) \geq K(\epsilon, p; \Omega) \sum_{\ell=1}^{k} |\boldsymbol{\phi}_\ell|^2_{\boldsymbol{H}^1(\Omega)} - \frac{c'(\Omega)k}{\delta^4} \, ||\boldsymbol{u}||^2_{\boldsymbol{W}^{1,2}(\Omega)} \tag{5.331}
$$

for any $\mu_1 > 0$. However, if the ζ_j, $j = 1, \ldots, k$ are the first k eigenvalues of $-\Delta$ on Ω corresponding to eigenvectors $\boldsymbol{w}_j \in \boldsymbol{H}^1(\Omega) \cap \boldsymbol{H}_{per}$ such that $\nabla \cdot \boldsymbol{w}_j = 0$, $j = 1, \ldots, k$, then

$$\sum_{\ell=1}^{k} |\boldsymbol{\phi}_\ell|^2_{\boldsymbol{H}^1(\Omega)} \geq \zeta_1 + \cdots + \zeta_k \tag{5.332a}$$

and, furthermore [Me]

$$\zeta_j \geq \tilde{c}(\Omega)\zeta_j j^{2/3}, \quad \forall j \geq 1 \text{ and some } \tilde{c} > 0. \tag{5.332b}$$

Employing (5.332a,b) in (5.331), we now find that, for any $\mu_1 > 0$,

$$\mathrm{tr}(\boldsymbol{\mathcal{L}}_{\mu_1}(\boldsymbol{u}) \circ \boldsymbol{Q}) \geq \hat{K}(\epsilon, p; \Omega)\zeta_1 k^{5/3} - \frac{c'(\Omega)k}{\delta^4} \, ||\boldsymbol{u}||^2_{\boldsymbol{W}^{1,2}(\Omega)} . \tag{5.333}$$

By virtue of (5.176) and (5.333) we can now state that, for any $\mu_1 > 0$,

$$q_k^{\mu_1} \geq \hat{K}(\epsilon; p; \Omega)\zeta_1 k^{5/3} - \frac{c'(\Omega)k}{\delta^4} \lim_{t\to\infty} \sup_{\boldsymbol{u}_0 \in \boldsymbol{\mathcal{A}}_{\mu_1}} \frac{1}{t} \int_0^t ||\boldsymbol{u}||^2_{\boldsymbol{W}^{1,2}(\Omega)} \, d\tau \tag{5.334}$$

and the Lemma 5.20 then follows directly from (5.325). □

We are now in a position to state and prove the basic results of this subsection, namely, the estimates for the upper bounds of the Hausdorff and fractal dimensions of $\boldsymbol{\mathcal{A}}_{\mu_1}$, $\mu_1 > 0$.

Theorem 5.12. *Let $\boldsymbol{\mathcal{A}}_{\mu_1}$, for $\mu_1 > 0$, be the maximal compact global attractor for the semigroup $\boldsymbol{S}_{\mu_1}(t)$ defined by the space-periodic problem* (5.2a,b)*,* (5.3b)*,* (5.4)*, in* dim $n = 2$*, with* $2 < p < 3$*. Then* $\exists \Psi(\epsilon, p, \zeta_1; \Omega) > 0$ *such that for k_m, the smallest integer for which*

$$k_m - 1 < \Psi |\boldsymbol{f}|^3_\infty < k_m, \tag{5.335a}$$

we have

$$d_H(\boldsymbol{\mathcal{A}}_{\mu_1}) \leq k_m, \quad d_F(\boldsymbol{\mathcal{A}}_{\mu_1}) \leq 2k_m. \tag{5.335b}$$

Proof. From (5.222), with $\boldsymbol{u}$ in place of $\boldsymbol{v}$, we obtain, $\forall \mu_1 \geq 0$ and all $\beta > 0$,

$$\frac{d}{dt} \, ||\boldsymbol{u}||^2_{\boldsymbol{L}^2(\Omega)} + 2\epsilon^{\frac{p-2}{2}} k'(\Omega) \, ||\boldsymbol{u}||^2_{\boldsymbol{W}^{1,2}(\Omega)} \leq \beta |\boldsymbol{f}|^2_\infty + \frac{1}{\beta} \, ||\boldsymbol{u}||^2_{\boldsymbol{L}^2(\Omega)} . \tag{5.336}$$

By choosing β sufficiently large in (5.336) we obtain, for some $k^*(\epsilon, p; \Omega) > 0$, an estimate of the form

$$\frac{d}{dt}\,||\boldsymbol{u}||^2_{\boldsymbol{L}^2(\Omega)} + k^*(\epsilon, p; \Omega)\,||\boldsymbol{u}||^2_{\boldsymbol{W}^{1,2}(\Omega)} \leq \beta|\boldsymbol{f}|^2_\infty, \quad \mu_1 > 0, \tag{5.337}$$

from which it follows that, for any $\mu_1 > 0$,

$$||\boldsymbol{u}||^2_{\boldsymbol{L}^2(\Omega)} + k^*(\epsilon, p; \Omega)\int_0^t ||\boldsymbol{u}||^2_{\boldsymbol{W}^{1,2}(\Omega)}\,d\tau \leq \beta t|\boldsymbol{f}|^2_\infty + ||\boldsymbol{u}_0||^2_{\boldsymbol{L}^2(\Omega)}\,. \tag{5.338}$$

Therefore, for all $\mu_1 > 0$,

$$\frac{1}{t}\int_0^t ||\boldsymbol{u}||^2_{\boldsymbol{W}^{1,2}(\Omega)}\,d\tau \leq \left(\frac{\beta}{k^*}\right)|\boldsymbol{f}|^2_\infty + \frac{1}{k^* t}\,||\boldsymbol{u}_0||^2_{\boldsymbol{L}^2(\Omega)}$$

in which case we see that

$$\lim_{t\to\infty}\sup_{\boldsymbol{u}_0\in\boldsymbol{\mathcal{A}}_{\mu_1}} \frac{1}{t}\int_0^t ||\boldsymbol{u}||^2_{\boldsymbol{W}^{1,2}(\Omega)}\,d\tau \leq \left(\frac{\beta}{k^*}\right)|\boldsymbol{f}|^2_\infty \tag{5.339}$$

with the estimate holding for all $\mu_1 > 0$. Employing (5.339) in (5.327), we now see that for any $\mu_1 > 0$ we have

$$\lambda_1^{\mu_1} + \cdots + \lambda_k^{\mu_1} \leq -\hat{K}(\epsilon, p; \Omega)\zeta_1 k^{5/3} + \frac{c'(\Omega)}{k^*(\epsilon, p; \Omega)}\left(\frac{\beta}{\delta^4}\right)k|\boldsymbol{f}|^2_\infty \tag{5.340}$$

so that $\lambda_1^{\mu_1} + \cdots + \lambda_k^{\mu_1} < 0$, for any $\mu_1 > 0$, provided

$$k > \left[\frac{c'(\Omega)}{c^*(\epsilon, p; \Omega)}\left(\frac{\beta}{\delta^4\zeta_1}\right)\right]^{3/2} |\boldsymbol{f}|^2_\infty \tag{5.341}$$

with $c^* = \hat{K}k^*$. In other words, $\lambda_1^{\mu_1} + \cdots + \lambda_k^{\mu_1} < 0$, for any $\mu_1 > 0$, for k the smallest positive integer such that

$$k - 1 < \left[\frac{c'(\Omega)}{c^*(\epsilon, p; \Omega)}\left(\frac{\beta}{\delta^4\zeta_1}\right)\right]^{3/2} |\boldsymbol{f}|^3_\infty < k. \tag{5.342}$$

The result of the theorem now follows if we take

$$\Psi(\epsilon, p, \zeta_1; \Omega) = \left[\frac{c'(\Omega)}{c^*(\epsilon, p; \Omega)}\left(\frac{\beta}{\delta^4\zeta_1}\right)\right]^{3/2} \tag{5.343}$$

with $\beta > 0$ sufficiently large, and $\delta > 0$ sufficiently small. □

Remarks. The upper bounds in (5.335b) not only hold $\forall\mu_1 > 0$ but are, themselves, independent of μ_1. However, we can not apply the ideas delineated in this subsection to conclude, e.g., that the upper bounds of Theorem 5.12 hold for $\mu_1 = 0$. In other words, it does not follow from the work presented above that

$$d_H(\boldsymbol{\mathcal{A}}_0) \le k_m < 1 + \Psi(\epsilon, p, \zeta_1; \Omega)|\boldsymbol{f}|_\infty^3.$$

The reason for this is that the application of the concepts outlined in this section depend, in an essential way, on the uniform differentiability of the underlying nonlinear semigroup, and this has not been established for $\boldsymbol{S}_0(t)$.

The most interesting aspect of the result stated in Theorem 5.12 is the independence of the upper bounds for $d_H(\boldsymbol{\mathcal{A}}_{\mu_1})$, $d_F(\boldsymbol{\mathcal{A}}_{\mu_1})$, $\mu_1 > 0$, with respect to μ_1; this was certainly not the case in our work in Sect. 5.3.5 on the incompressible bipolar viscous fluid, with $1 < p \le 2$, where it was shown, e.g., that the upper bound for $d_H(\boldsymbol{\mathcal{A}}_{\mu_1})$, $\mu_1 > 0$, behaves like μ_1^{-6} for small μ_1. The reason for the difference in the behavior of the upper bounds for $d_H(\boldsymbol{\mathcal{A}}_{\mu_1})$, $d_F(\boldsymbol{\mathcal{A}}_{\mu_1})$, $\mu_1 > 0$, in the two cases $1 < p \le 2$ and $2 < p < 3$ can be traced directly to the estimate (5.328) and, in particular, to the fact that for $\boldsymbol{\phi} \in \tilde{\boldsymbol{H}}$, $\boldsymbol{u} = \boldsymbol{S}_{\mu_1}(t)\boldsymbol{u}_0$, $\boldsymbol{u}_0 \in \boldsymbol{\mathcal{A}}_{\mu_1}$,

$$\int_\Omega (\epsilon + |\boldsymbol{e}(\boldsymbol{u})|^2)^{\frac{p-2}{2}} e_{ij}(\boldsymbol{\phi}) e_{ij}(\boldsymbol{\phi}) d\boldsymbol{x} \ge \epsilon^{\frac{p-2}{2}} k_2(\Omega)\, ||\boldsymbol{\phi}||^2_{\boldsymbol{W}^{1,2}(\Omega)} \tag{5.344a}$$

for some $k_2(\Omega) > 0$, if $p \ge 2$; this is in contrast to the best possible estimate available for the case $1 < p \le 2$, namely, for some $\hat{k}(\Omega) > 0$ it follows from Lemma 5.3 that

$$\int_\Omega (\epsilon + |\boldsymbol{e}(\boldsymbol{u})|^2)^{\frac{p-2}{2}} e_{ij}(\boldsymbol{\phi}) e_{ij}(\boldsymbol{\phi}) d\boldsymbol{x} \ge \hat{k}(\Omega)\, ||\boldsymbol{\phi}||^2_{\boldsymbol{W}^{1,p}(\Omega)}\,. \tag{5.344b}$$

If we have to work with (5.344b) in (5.328), for $1 < p \le 2$, instead of with (5.344a), as we may for $p > 2$, then the expression

$$2\mu_1 \int_\Omega \frac{\partial e_{ij}}{\partial x_k}(\boldsymbol{\phi}) \frac{\partial e_{ij}}{\partial x_k}(\boldsymbol{\phi}) d\boldsymbol{x}$$

would have to be retained in (5.328), thus propelling the constant μ_1 through all the subsequent estimates.

5.5 Lower Semicontinuity of the Attractors for Nonlinear Bipolar Equations

5.5.1 The Convergence Problem

In Sect. 5.4 we showed that for the space-periodic problem (5.2a,b), (5.3b), (5.4), in dim $n = 2$, with $p > 2$, there exist absorbing sets in $\boldsymbol{H}_{per}$ and $\boldsymbol{W}^{1,2}(\Omega)$ for

$\boldsymbol{S}_{\mu_1}(t)$ when $\mu_1 \geq 0$, which are independent of $\mu_1 > 0$, as well as absorbing sets for $\boldsymbol{S}_{\mu_1}(t)$, in $\boldsymbol{W}^{2,2}(\Omega)$, when $\mu_1 > 0$; this is the essential content of Theorem 5.9. The absorbing sets in $\boldsymbol{W}^{1,2}(\Omega)$ turned out to be of the form $B^{\rho'}_{\boldsymbol{W}^{1,2}(\Omega)}$, i.e., balls of sufficiently large radius ρ', independent of μ_1, in $\boldsymbol{W}^{1,2}(\Omega)$ while the absorbing sets in $\boldsymbol{W}^{2,2}(\Omega)$ were of the form $B^{\rho_{\mu_1}}_{\boldsymbol{W}^{2,2}(\Omega)}$, i.e., balls of sufficiently large radius ρ_{μ_1} in $\boldsymbol{W}^{2,2}(\Omega)$, with ρ_{μ_1} satisfying $\rho_{\mu_1} \to \infty$ as $\mu_1 \to 0$. Maximal compact global attractors were then defined in Sect. 5.4 as follows: for

$$\mu_1 > 0 : \boldsymbol{\mathcal{A}}_{\mu_1} = \bigcap_{t>0} \boldsymbol{S}_{\mu_1}(t) B^{\rho_{\mu_1}}_{\boldsymbol{W}^{2,2}(\Omega)},$$

$$\mu_1 = 0 : \boldsymbol{\mathcal{A}}_0 = \bigcap_{t>0} \boldsymbol{S}_0(t) B^{\rho'}_{\boldsymbol{W}^{1,2}(\Omega)},$$

and the existence, for fixed $\mu_1 > 0$, of the absorbing set $B^{\rho_{\mu_1}}_{\boldsymbol{W}^{2,2}(\Omega)}$ then suffices to yield the uniform compactness of the semigroup $\boldsymbol{S}_{\mu_1}(t)$ for large t.

Once the attractors $\boldsymbol{\mathcal{A}}_{\mu_1}$, $\mu_1 > 0$, and $\boldsymbol{\mathcal{A}}_0$ were defined in Sect. 5.4 we were then able to establish, i.e., Theorem 5.10, that for $\mu_1 > 0$, and $2 < p < 3$, the nonlinear semigroup $\boldsymbol{S}_{\mu_1}$ is uniformly differentiable on $\boldsymbol{\mathcal{A}}_{\mu_1}$ and, also, that the Fréchet differential $\boldsymbol{L}_{\mu_1}(t; \boldsymbol{u}_0)$ of $\boldsymbol{S}_{\mu_1}(t)$, for $\boldsymbol{u}_0 \in \boldsymbol{\mathcal{A}}_{\mu_1}$, is uniformly bounded in the strong operator norm of $\boldsymbol{\mathcal{L}}(\boldsymbol{H}_{per}, \boldsymbol{H}_{per})$; this latter result is the essential content of Theorem 5.11.

Our goal in this section is to examine the relationship between the attractors $\boldsymbol{\mathcal{A}}_{\mu_1}$, $\mu_1 > 0$, and $\boldsymbol{\mathcal{A}}_0$ and, in particular, to show that in a well-defined sense the attractors $\boldsymbol{\mathcal{A}}_{\mu_1}$ converge to $\boldsymbol{\mathcal{A}}_0$ as $\mu_1 \to 0$. In this regard, the best that has been done to this point is to exhibit the convergence of $\boldsymbol{\mathcal{A}}_{\mu_1}$ to $\boldsymbol{\mathcal{A}}_0$ as $\mu_1 \to 0$, in the sense of the semidistance measure for sets (as opposed to the actual distance measure). We need the following

Definition 5.8. The semidistance measure $\hat{d}(S_1, S_2)$ in $\boldsymbol{L}^2(\Omega)$ between the sets S_1 and S_2 is given by

$$\hat{d}(S_1, S_2) = \sup_{\boldsymbol{x} \in S_1} \inf_{\boldsymbol{y} \in S_2} ||\boldsymbol{x} - \boldsymbol{y}||_{\boldsymbol{L}^2(\Omega)}. \tag{5.345}$$

Our goal in this section is to prove that $\hat{d}(\boldsymbol{\mathcal{A}}_{\mu_1}, \boldsymbol{\mathcal{A}}_0) \to 0$, as $\mu_1 \to 0$. In order to show that $d(\boldsymbol{\mathcal{A}}_{\mu_1}, \boldsymbol{\mathcal{A}}_0) \to 0$, as $\mu_1 \to 0$, with d the usual distance measure between sets we would have to be able to reverse the roles of $\boldsymbol{\mathcal{A}}_{\mu_1}$, $\boldsymbol{\mathcal{A}}_0$ in terms of semidistance convergence, i.e., to show that $\hat{d}(\boldsymbol{\mathcal{A}}_0, \boldsymbol{\mathcal{A}}_{\mu_1}) \to 0$, as $\mu_1 \to 0$; such a result has proven to be elusive to this point and, in fact, does not appear to be valid. Indeed, if one could prove that $d(\boldsymbol{\mathcal{A}}_{\mu_1}, \boldsymbol{\mathcal{A}}_0) \to 0$, as $\mu_1 \to 0$, then it would be possible to go to the limit, in Theorem 5.12, as $\mu_1 \to 0$, and extract from (5.335a,b) upper bounds for the Hausdorff and fractal dimensions of the attractor $\boldsymbol{\mathcal{A}}_0$.

5.5.2 A Basic Estimate for Solutions of the Space-Periodic Bipolar Problem

In this subsection we will derive an upper bound for the $L^2([0,T];\boldsymbol{H}^3(\Omega))$ norm of the solution of the space-periodic problem for the bipolar fluid flow equations with $\mu_1 > 0$; the result, which is valid in both dimensions $n = 2, 3$, will be a crucial element in establishing the convergence, in the sense of semidistance, of $\boldsymbol{\mathcal{A}}_{\mu_1}$ to $\boldsymbol{\mathcal{A}}_0$, as $\mu_1 \to 0$, and will also be used in our discussion of inertial manifolds in Sect. 6.2. The proof which is presented here follows the proof of the similar result in [Hao]. A different proof of this same estimate will be presented in connection with the discussion of the L^2 squeezing property for nonlinear incompressible bipolar fluids in Sect. 6.3; this result is of central importance to the analysis of bipolar flow problems so we have deemed it worthwhile to offer an alternative proof here which, at the same time, also serves to establish some of the mathematical framework that will be employed in the work on inertial manifolds to be presented in Sect. 6.2.

We are interested in the problem (5.2a,b), with $\rho = 1$, (5.3b), for some $L > 0$, and (5.4) and recall that (5.2a) may be written out in the form

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} = -\nabla\bar{p} - 2\mu_1\nabla\cdot(\Delta\boldsymbol{e}) + \nabla\cdot(2\mu_0(\epsilon + |\boldsymbol{e}(\boldsymbol{v})|^2)^{\frac{p-2}{2}}\boldsymbol{e}) + \boldsymbol{f}. \tag{5.346}$$

where $\bar{p}$ is the pressure. Associated with (5.2a)—or (5.346)—(5.2b), (5.3b), (5.4) we have the following fundamental linear problem: find $(\boldsymbol{u}, \bar{p}) : \Omega \to R^n \times R$ such that

$$\nabla\cdot(\Delta\boldsymbol{e}) + \nabla\bar{p} = \boldsymbol{f}, \text{ in } \Omega \tag{5.347a}$$

$$\nabla\cdot\boldsymbol{u} = 0, \text{ in } \Omega \tag{5.347b}$$

where $\boldsymbol{e} = \boldsymbol{e}(\boldsymbol{u})$ and $\boldsymbol{u}$ satisfies the periodicity conditions (5.3b). Associated with the problem (5.347a,b), (5.3b) is the linear operator $\boldsymbol{A}$ which is defined as follows: Let

$$\boldsymbol{V}_{per}(\Omega) = \{\boldsymbol{u} : \Omega \to R^n \mid \boldsymbol{u} \in \boldsymbol{W}^{2,2}(\Omega),\ \nabla\cdot\boldsymbol{u} = 0,\ \text{and } \boldsymbol{u} \text{ satisfies (5.3b)}\} \tag{5.348}$$

and consider the positive definite $\boldsymbol{V}_{per}(\Omega)$-elliptic symmetric bilinear form $a(\cdot,\cdot)$: $\boldsymbol{V}_{per} \times \boldsymbol{V}_{per} \to R$ given by

$$a(\boldsymbol{u}, \boldsymbol{v}) = \frac{1}{2}(\Delta\boldsymbol{u}, \Delta\boldsymbol{v})_{L^2(\Omega)}. \tag{5.349}$$

Then as a consequence of the Lax-Milgram Lemma (Appendix A) we obtain an isometry $\boldsymbol{A} \in \boldsymbol{\mathcal{L}}(\boldsymbol{V}_{per}, \boldsymbol{V}'_{per})$, with $\boldsymbol{V}'_{per}$ the dual space to $\boldsymbol{V}_{per}$, via

$$\langle \boldsymbol{A}\boldsymbol{u}, \boldsymbol{v}\rangle_{V'_{per}\times V_{per}} = a(\boldsymbol{u}, \boldsymbol{v}) = \langle \boldsymbol{f}, \boldsymbol{v}\rangle_{V'_{per}\times V'_{per}}, \quad \forall \boldsymbol{v} \in \boldsymbol{V}_{per} \tag{5.350}$$

with $\boldsymbol{f} \in \boldsymbol{V}'_{per}$. For the domain of $\boldsymbol{A}$ we have

$$D(\boldsymbol{A}) = \{\boldsymbol{u} \in \boldsymbol{V}_{per} \mid a(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v})_{L^2(\Omega)},\ \boldsymbol{f} \in \boldsymbol{H}_{per} \subset \boldsymbol{V}'_{per},\ \forall \boldsymbol{v} \in \boldsymbol{V}_{per}.\} \tag{5.351}$$

For ease of presentation we will, for the remainder of this subsection, denote the $\boldsymbol{L}^2(\Omega)$ inner product $(\cdot,\cdot)_{L^2(\Omega)}$ simply as $(\cdot,\cdot)$. We also observe that $\boldsymbol{A} \in \mathcal{L}(D(\boldsymbol{A}); \boldsymbol{H}_{per}) \cap \mathcal{L}(\boldsymbol{V}_{per}, \boldsymbol{V}'_{per})$.

As a consequence of standard embedding theorems, $\boldsymbol{A}^{-1}$ is compact as a mapping in $\boldsymbol{V}'_{per}$ (or in $\boldsymbol{H}_{per}$). Therefore, the spectrum of $\boldsymbol{A}$ consists of real eigenvalues λ_j, with the multiplicity of each λ_j finite; these eigenvalues λ_j may be ordered, i.e.,

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots; \quad \lambda_j \to +\infty, \text{ as } j \to +\infty$$

and the only possible accumulation point of $\{\lambda_j\}$ is at infinity. The self-adjoint operator $\boldsymbol{A}$ possesses an orthonormal set of eigenfunctions $\{\boldsymbol{\phi}\}_{j=1}^{\infty}$, which is complete in $\boldsymbol{V}'_{per}$ (or $\boldsymbol{H}_{per}$), and which satisfies

$$\boldsymbol{A}\boldsymbol{\phi}_j = \lambda_j \boldsymbol{\phi}_j, \text{ with } \boldsymbol{\phi}_j \in \boldsymbol{V}_{per} \text{ (or } \boldsymbol{D}(\boldsymbol{A})),\ \forall j. \tag{5.352}$$

Because $\boldsymbol{A}^{-1}$ is compact, we can define the fractional powers of $\boldsymbol{A}$ by using the spectral resolution of $\boldsymbol{A}$: $\forall \alpha \in R$ we define

$$\boldsymbol{A}^{\alpha}\boldsymbol{u} = \sum_{k=1}^{\infty} \lambda_k^{\alpha} (\boldsymbol{u}, \boldsymbol{\phi}_k)_{L^2(\Omega)} \boldsymbol{\phi}_k, \quad \forall \boldsymbol{u} \in D(\boldsymbol{A}^{\alpha}) \tag{5.353}$$

where for $\alpha > 0$,

$$D(\boldsymbol{A}^{\alpha}) = \{\boldsymbol{u} \in \boldsymbol{H}_{per} \mid \sum_{j=1}^{\infty} \lambda_j^{2\alpha} (\boldsymbol{u}, \boldsymbol{\phi})^2_{L^2(\Omega)} < \infty\}, \tag{5.354}$$

while for $\alpha < 0$, $D(\boldsymbol{A}^{\alpha})$ is the completion of $\boldsymbol{H}_{per}$ with respect to the norm

$$||\boldsymbol{u}||_{\alpha} = \left\{ \sum_{j=1}^{\infty} \lambda_j^{2\alpha} \left(\boldsymbol{u}, \boldsymbol{\phi}_j\right)^2_{L^2(\Omega)} \right\}^{1/2} \tag{5.355}$$

which is induced by the scalar product

$$(\boldsymbol{u}, \boldsymbol{v})_{D(A^{\alpha})} = \sum_{j=1}^{\infty} \lambda_j^{2\alpha} \left(\boldsymbol{u}, \boldsymbol{\phi}_j\right)_{L^2(\Omega)} \left(\boldsymbol{v}, \boldsymbol{\phi}_j\right)_{L^2(\Omega)}. \tag{5.356}$$

Using the Fourier transform, it is a straightforward exercise to show that there exist constants $\bar{k}_1, \bar{k}_2 > 0$ such that

$$\bar{k}_1 \, |\boldsymbol{u}|_{\boldsymbol{H}^4(\Omega)} \leq ||\boldsymbol{A}\boldsymbol{u}||_{\boldsymbol{L}^2(\Omega)} \leq \bar{k}_2 \, |\boldsymbol{u}|_{\boldsymbol{H}^4(\Omega)} \,, \ \forall \boldsymbol{u} \in D(\boldsymbol{A}), \tag{5.357a}$$

$$\bar{k}_1 \, |\boldsymbol{u}|_{\boldsymbol{H}^3(\Omega)} \leq \left|\left|\boldsymbol{A}^{3/4}\boldsymbol{u}\right|\right|_{\boldsymbol{L}^2(\Omega)} \leq \bar{k}_2 \, |\boldsymbol{u}|_{\boldsymbol{H}^3(\Omega)} \,, \ \forall \boldsymbol{u} \in D(\boldsymbol{A}), \tag{5.357b}$$

$$\bar{k}_1 \, |\boldsymbol{u}|_{\boldsymbol{H}^2(\Omega)} \leq \left|\left|\boldsymbol{A}^{1/2}\boldsymbol{u}\right|\right|_{\boldsymbol{L}^2(\Omega)} \leq \bar{k}_2 \, |\boldsymbol{u}|_{\boldsymbol{H}^2(\Omega)} \,, \ \forall \boldsymbol{u} \in D(\boldsymbol{A}^{1/2}), \tag{5.357c}$$

and

$$\bar{k}_1 \, |\boldsymbol{u}|_{\boldsymbol{H}^1(\Omega)} \leq \left|\left|\boldsymbol{A}^{1/4}\boldsymbol{u}\right|\right|_{\boldsymbol{L}^2(\Omega)} \leq \bar{k}_2 \, |\boldsymbol{u}|_{\boldsymbol{H}^1(\Omega)} \,, \ \forall \boldsymbol{u} \in D(\boldsymbol{A}^{1/4}). \tag{5.357d}$$

Therefore, we have the equivalences

$$\begin{cases} ||\boldsymbol{A}\boldsymbol{u}||_{\boldsymbol{L}^2(\Omega)} \sim |\boldsymbol{u}|_{\boldsymbol{H}^4(\Omega)} \,, \\ \left|\left|\boldsymbol{A}^{3/4}\boldsymbol{u}\right|\right|_{\boldsymbol{L}^2(\Omega)} \sim |\boldsymbol{u}|_{\boldsymbol{H}^3(\Omega)} \,, \\ \left|\left|\boldsymbol{A}^{1/2}\boldsymbol{u}\right|\right|_{\boldsymbol{L}^2(\Omega)} \sim |\boldsymbol{u}|_{\boldsymbol{H}^2(\Omega)} \,, \\ \left|\left|\boldsymbol{A}^{1/4}\boldsymbol{u}\right|\right|_{\boldsymbol{L}^2(\Omega)} \sim |\boldsymbol{u}|_{\boldsymbol{H}^1(\Omega)} \,. \end{cases} \tag{5.357e}$$

We now set, for $\boldsymbol{e} = \boldsymbol{e}(\boldsymbol{u})$,

$$\left[\boldsymbol{A}_p\boldsymbol{u}\right]_i = \frac{\partial}{\partial x_j}\left[(\epsilon + |\boldsymbol{e}|^2)^{\frac{p-2}{2}} e_{ij}\right], \quad i = 1, \ldots, n, \tag{5.358}$$

$$\boldsymbol{B}(\boldsymbol{u}, \boldsymbol{v}) = \boldsymbol{u} \cdot \nabla \boldsymbol{v} \tag{5.359}$$

and

$$\boldsymbol{R}(\boldsymbol{u}) = -2\mu_0 \boldsymbol{A}_p(\boldsymbol{u}) + \boldsymbol{B}(\boldsymbol{u}, \boldsymbol{u}) - \boldsymbol{f} \tag{5.360}$$

where we take $\boldsymbol{f} \in \boldsymbol{H}_{per}$. Then the bipolar problem (5.356), (5.2b), (5.3b), (5.4) is equivalent to the following abstract initial-value problem posed in $\boldsymbol{V}_{per}(\Omega)$:

$$\frac{d\boldsymbol{u}}{dt} + 2\mu_1 \boldsymbol{A}\boldsymbol{u} + \boldsymbol{R}(\boldsymbol{u}) = \boldsymbol{0}, \tag{5.361a}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0. \tag{5.361b}$$

With the help of the machinery we have just prepared we are now in a position to state and prove the following: set, for $\boldsymbol{v} \in \boldsymbol{V}_{per}(\Omega)$, $||\boldsymbol{v}||_V = ||\boldsymbol{v}||_{\boldsymbol{V}_{per}}$ where $\boldsymbol{V}_{per}$ is given by (5.348); then we have

Lemma 5.21. *Let $\boldsymbol{v}(t)$ be the unique solution of* (5.346)*,* (5.2b)*,* (5.3b)*,* (5.4) *with $\Omega = [0, L]^n$, $n = 2, 3$, and suppose that $||\boldsymbol{v}||_V \leq R$, $t \in [0, T)$. Then $\exists k' > 0$, k' depending only on Ω, μ_0, $\boldsymbol{f}$, R, and T, such that*

$$\mu_1 \int_0^T \left|\left|\boldsymbol{A}^{3/4}\boldsymbol{v}\right|\right|^2_{\boldsymbol{L}^2(\Omega)} dt \leq k' \tag{5.362}$$

with $\boldsymbol{A}$ as defined by (5.350)*.*

Proof. We work with the abstract formulation of the problem (5.346), (5.2b), (5.3b), (5.4), i.e., with (5.361a,b). Thus, let $\boldsymbol{v}(t)$ be the unique solution of (5.361a,b) with $\boldsymbol{v}_0 \in \boldsymbol{V} \equiv \boldsymbol{V}_{per}(\Omega)$ such that $||\boldsymbol{v}||_V \leq R$ for some $R > 0$ and all $t \in [0, T)$. Taking the inner-product, in $\boldsymbol{L}^2(\Omega)$, of (5.361a) with $\boldsymbol{A}^{1/2}\boldsymbol{v}(t)$ we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left|\left|\boldsymbol{A}^{1/4}\boldsymbol{u}\right|\right|^2_{\boldsymbol{L}^2(\Omega)} + 2\mu_1\left|\left|\boldsymbol{A}^{3/4}\boldsymbol{u}\right|\right|^2_{\boldsymbol{L}^2(\Omega)}\\&\quad\leq \left(2\mu_0\left|\left|\boldsymbol{A}_p(\boldsymbol{u})\right|\right|_{\boldsymbol{L}^2(\Omega)} + ||\boldsymbol{B}(\boldsymbol{u},\boldsymbol{u})||_{\boldsymbol{L}^2(\Omega)} + ||\boldsymbol{f}||_{\boldsymbol{L}^2(\Omega)}\right)\cdot\left|\left|\boldsymbol{A}^{1/2}\boldsymbol{u}\right|\right|_{\boldsymbol{L}^2(\Omega)}.\end{aligned} \tag{5.363}$$

Since $||\boldsymbol{u}(t)||_V \leq R$, for some $R > 0$, and all $t \in [0, T)$, as a consequence of the fact that

$$\begin{aligned}&\frac{\partial}{\partial x_j}\left[(\epsilon + |\boldsymbol{e}(\boldsymbol{u})|^2)^{\frac{p-2}{2}} e_{ij}(\boldsymbol{u})\right]\\&\quad= (\epsilon + |\boldsymbol{e}(\boldsymbol{u})|^2)^{\frac{p-2}{2}}\frac{\partial}{\partial x_j}e_{ij} + (p-2)(\epsilon + |\boldsymbol{e}(\boldsymbol{u})|^2)^{\frac{p-4}{2}} e_{ij}e_{kl}\frac{\partial}{\partial x_j}e_{kl},\end{aligned} \tag{5.364}$$

we easily see that there exists a positive constant c_1 such that

$$(2\mu_0\left|\left|\boldsymbol{A}_p(\boldsymbol{u})\right|\right|_{\boldsymbol{L}^2(\Omega)} + ||\boldsymbol{B}(\boldsymbol{u},\boldsymbol{u})||_{\boldsymbol{L}^2(\Omega)} + ||\boldsymbol{f}||_{\boldsymbol{L}^2(\Omega)})\cdot\left|\left|\boldsymbol{A}^{1/2}\boldsymbol{u}\right|\right|_{\boldsymbol{L}^2(\Omega)} \leq c_1. \tag{5.365}$$

By (5.363) and (5.365), we obtain

$$\frac{d}{dt}\left|\left|\boldsymbol{A}^{1/4}\boldsymbol{u}\right|\right|^2_{\boldsymbol{L}^2(\Omega)} + 4\mu_1\left|\left|\boldsymbol{A}^{3/4}\boldsymbol{u}\right|\right|^2_{\boldsymbol{L}^2(\Omega)} \leq 2c_1. \tag{5.366}$$

Integrating (5.366) from 0 to T, we have

$$\left\|A^{1/4}u(T)\right\|^2_{L^2(\Omega)} + 4\mu_1 \int_0^T \left\|A^{3/4}u(t)\right\|^2_{L^2(\Omega)} dt \le \left\|A^{1/4}u(0)\right\|^2_{L^2(\Omega)} + 2c_1 T \tag{5.367}$$

and (5.362) is now a direct consequence of (5.367). □

An immediate consequence of Lemma 5.21 is the following result which will be pivotal not only for the analysis in 5.5.3 but for our discussion of inertial manifolds in Sect. 6.2 as well:

Theorem 5.13. *Under the conditions stated in Lemma 5.21, the unique solution of* (5.346)*,* (5.2b)*,* (5.3b)*,* (5.4) *satisfies*

$$\mu_1 \int_0^T |v(t)|^2_{H^3(\Omega)} dt \le \tilde{k} \tag{5.368}$$

for some $\tilde{k} > 0$, $\tilde{k} = \tilde{k}(\mu_0, \epsilon, f, R, T)$.

Proof. The bound (5.368) is a direct consequence of (5.362) and the lower bound in (5.357b). □

5.5.3 Convergence of $\mathcal{A}_{\mu_1}$ to $\mathcal{A}_0$ as $\mu_1 \to 0$

We now want to establish the convergence of the attractors $\mathcal{A}_{\mu_1}$ to the attractor $\mathcal{A}_0$, for the space-periodic problem in dim $n = 2$, as $\mu_1 \to 0$; convergence will be established in the sense of semidistance, i.e., $\hat{d}(\mathcal{A}_{\mu_1}, \mathcal{A}_0) \to 0$, as $\mu_1 \to 0$, or

$$\sup_{v \in \mathcal{A}_0} \inf_{v_{\mu_1} \in \mathcal{A}_{\mu_1}} \left\|v - v_{\mu_1}\right\|_{L^2(\Omega)} \to 0, \text{ as } \mu_1 \to 0. \tag{5.369}$$

By virtue of [Te4], Theorem 1.2, to prove (5.369) it suffices to show that, as $\mu_1 \to 0$,

$$\delta_{\mu_1}(I) = \sup_{v_0 \in B_{R_0}(\mathbf{0})} \sup_{t \in I} \left\|S_{\mu_1}(t)v_0 - S_0(t)v_0\right\|_{L^2(\Omega)} \to 0 \tag{5.370}$$

for every compact interval $I \subset (0, \infty)$, with $B_{R_0}(\mathbf{0})$ a ball of radius R_0 (independent of μ_1) in $L^2(\Omega)$. To this end we first establish the following:

Lemma 5.22. *Let* $v_{\mu_1}(t)$ *and* $v(t)$ *be, respectively, the unique solutions of* (5.346)*,* (5.2b)*,* (5.3b)*,* (5.4) *for* $\mu_1 > 0$ *and* $\mu_1 = 0$ *(with the same initial data* v_0*). Let*

$$w(t;\mu_1) = v_{\mu_1}(t) - v(t), \quad \mu_1 > 0. \tag{5.371}$$

Then $\exists \tilde{k}(\epsilon, p; \Omega)$, $c(\Omega)$, $\bar{c}(\Omega)$, *all positive constants, such that, for* $p > 2$,

$$\frac{1}{2}\frac{d}{dt}\,||\boldsymbol{w}||^2_{\boldsymbol{L}^2(\Omega)} + \tilde{k}(\epsilon, p; \Omega)\,||\boldsymbol{w}||^2_{\boldsymbol{W}^{1,2}(\Omega)} \le c(\Omega)\,||\boldsymbol{w}||^2_{\boldsymbol{L}^2(\Omega)} + \mu_1\bar{c}(\Omega)\,|\boldsymbol{v}|_{\boldsymbol{H}^3(\Omega)}\,. \tag{5.372}$$

Proof. The system of equations satisfied by $\boldsymbol{w}$ is

$$\frac{\partial w_i}{\partial t} + w_j\frac{\partial v_i}{\partial x_j} + v_j\frac{\partial w_i}{\partial x_j} + w_j\frac{\partial w_i}{\partial x_j} = -\frac{\partial p_{\boldsymbol{w}}}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\gamma(\boldsymbol{v}_{\mu_1})e_{ij}(\boldsymbol{v}_{\mu_1}) - \gamma(\boldsymbol{v})e_{ij}(\boldsymbol{v})\right] - 2\mu_1\frac{\partial}{\partial x_j}(\Delta e_{ij}(\boldsymbol{v}_{\mu_1})) \tag{5.373}$$

where $p_{\boldsymbol{w}}$ is the difference of the pressure fields corresponding to $\boldsymbol{v}_{\mu_1}$ and $\boldsymbol{v}$. Multiplying (5.373) by $w_i(t;\mu_1)$, integrating over Ω, summing on i, $i = 1, 2, 3$, and integrating once by parts we obtain

$$\frac{1}{2}\frac{d}{dt}\,||\boldsymbol{w}(t;\mu_1)||^2_{\boldsymbol{L}^2(\Omega)} + \int_\Omega w_j\frac{\partial v_i}{\partial x_j}w_i\,d\boldsymbol{x} = -\int_\Omega\left(\int_0^1\frac{\partial^2\Gamma}{\partial e_{ij}\partial e_{kl}}(\boldsymbol{v}+\tau\boldsymbol{w})d\tau\right)e_{ij}(\boldsymbol{w})e_{kl}(\boldsymbol{w})d\boldsymbol{x} + 2\mu_1\int_\Omega\Delta e_{ij}(\boldsymbol{u}_{\mu_1})e_{ij}(\boldsymbol{w})d\boldsymbol{x} \tag{5.374}$$

where $\Gamma(e_{ij}e_{ij})$ is the potential defined by (5.92) with $\alpha = 2 - p$. From (5.374) we obtain, directly, the estimate

$$\frac{1}{2}\frac{d}{dt}\,||\boldsymbol{w}(t;\mu_1)||^2_{\boldsymbol{L}^2(\Omega)} + \epsilon^{\frac{p-2}{2}}\int_\Omega e_{ij}(\boldsymbol{w})e_{ij}(\boldsymbol{w})d\boldsymbol{x} \le \left|\int_\Omega w_j\frac{\partial v_i}{\partial x_j}w_i\,d\boldsymbol{x}\right| + 2\mu_1\int_\Omega\Delta e_{ij}(\boldsymbol{v}_{\mu_1})e_{ij}(\boldsymbol{v}_{\mu_1})d\boldsymbol{x} - 2\mu_1\int_\Omega\Delta e_{ij}(\boldsymbol{v}_{\mu_1})e_{ij}(\boldsymbol{v})d\boldsymbol{x}. \tag{5.375}$$

As the expression $\int_\Omega\Delta e_{ij}(\boldsymbol{v}_{\mu_1})e_{ij}(\boldsymbol{v}_{\mu_1})d\boldsymbol{x}$ may be integrated once more, by parts, we find that

$$\frac{1}{2}\frac{d}{dt}\,||\boldsymbol{w}(t;\mu_1)||^2_{\boldsymbol{L}^2(\Omega)} + \epsilon^{\frac{p-2}{2}}k_2(\Omega)\,||\boldsymbol{w}||^2_{\boldsymbol{W}^{1,2}(\Omega)} + 2\mu_1\int_\Omega\frac{\partial e_{ij}}{\partial x_k}(\boldsymbol{v}_{\mu_1})\frac{\partial e_{ij}}{\partial x_k}(\boldsymbol{v}_{\mu_1})d\boldsymbol{x} \le \left|\int_\Omega w_j\frac{\partial v_i}{\partial x_j}w_i\,d\boldsymbol{x}\right| - 2\mu_1\int_\Omega\Delta e_{ij}(\boldsymbol{v}_{\mu_1})e_{ij}(\boldsymbol{v})d\boldsymbol{x} \tag{5.376}$$

so that by dropping the third expression on the left-hand side of this last inequality we have the estimate

$$\frac{1}{2}\frac{d}{dt}\,||\boldsymbol{w}(t;\mu_1)||^2_{\boldsymbol{L}^2(\Omega)} + \epsilon^{\frac{p-2}{2}}k_2(\Omega)\,||\boldsymbol{w}||^2_{\boldsymbol{W}^{1,2}(\Omega)} \leq \left|\int_\Omega w_j\frac{\partial v_i}{\partial x_j}w_i\,d\boldsymbol{x}\right| - 2\mu_1\int_\Omega \Delta e_{ij}(\boldsymbol{v}_{\mu_1})e_{ij}(\boldsymbol{v})d\boldsymbol{x}. \tag{5.377}$$

However,

$$\left|\int_\Omega w_j\frac{\partial v_i}{\partial x_j}w_i\,d\boldsymbol{x}\right| \leq c_1(\Omega)\,||\boldsymbol{v}||_{\boldsymbol{W}^{1,2}(\Omega)}\,||\boldsymbol{w}||^{3/2}_{\boldsymbol{W}^{1,2}(\Omega)}\,||\boldsymbol{w}||^{1/2}_{\boldsymbol{L}^2(\Omega)}. \tag{5.378}$$

As $\boldsymbol{v} \in L^\infty((0,T);\boldsymbol{W}^{1,2}(\Omega))$, $\forall T > 0$, by virtue of the existence of the absorbing set $B^{\rho'}_{\boldsymbol{W}^{1,2}(\Omega)}$, (with ρ' independent of μ_1), for $\mu_1 = 0$ we have, for $t \in I \subset (0,\infty)$,

$$||\boldsymbol{v}||_{\boldsymbol{W}^{1,2}(\Omega)} \leq \rho_0, \quad t \in I \tag{5.379}$$

for some $\rho_0 > 0$. Combining (5.378) with (5.379), and using Young's inequality, we deduce, for some $c_0(\Omega) > 0$, and any $\delta > 0$, the estimate

$$\left|\int_\Omega w_j\frac{\partial v_i}{\partial x_j}w_i\,d\boldsymbol{x}\right| \leq c_0(\Omega)\left[\delta^{3/4}\,||\boldsymbol{w}||^2_{\boldsymbol{W}^{1,2}(\Omega)} + \frac{1}{\delta^4}\,||\boldsymbol{w}||^2_{\boldsymbol{L}^2(\Omega)}\right]$$

whose use in (5.377), for δ chosen sufficiently small, results in a differential inequality of the form

$$\frac{1}{2}\frac{d}{dt}\,||\boldsymbol{w}||^2_{\boldsymbol{L}^2(\Omega)} + \tilde{k}(\epsilon,p;\Omega)\,||\boldsymbol{w}||^2_{\boldsymbol{W}^{1,2}(\Omega)} \leq c(\Omega)\,||\boldsymbol{w}||^2_{\boldsymbol{L}^2(\Omega)} - 2\mu_1\int_\Omega \Delta e_{ij}(\boldsymbol{v}_{\mu_1})e_{ij}(\boldsymbol{v})d\boldsymbol{x}. \tag{5.380}$$

However,

$$\begin{aligned}\left|\int_\Omega \Delta e_{ij}(\boldsymbol{v}_{\mu_1})e_{ij}(\boldsymbol{v})d\boldsymbol{x}\right| &\leq \left(\int_\Omega \Delta e_{ij}(\boldsymbol{v}_{\mu_1})\Delta e_{ij}(\boldsymbol{v}_{\mu_1})d\boldsymbol{x}\right)^{1/2} \\ &\quad\times\left(\int_\Omega e_{ij}(\boldsymbol{v})e_{ij}(\boldsymbol{v})d\boldsymbol{x}\right)^{1/2} \\ &\leq c'(\Omega)\,|\boldsymbol{v}|_{\boldsymbol{H}^3(\Omega)}\,||\boldsymbol{v}||_{\boldsymbol{W}^{1,2}(\Omega)} \leq c'(\Omega)\rho_0\,|\boldsymbol{v}|_{\boldsymbol{H}^3(\Omega)}\end{aligned} \tag{5.381}$$

where we have again used (5.379) for $t \in I \subset (0,\infty)$. Combining (5.380) with (5.381) now yields the estimate (5.372). □

Using the results delineated in Theorem 5.13 and Lemma 5.22 we can now state and prove the main result of this section, namely,

Theorem 5.14. *For* $\mu_1 > 0$ *and* $\mu_1 = 0$*, respectively, let* $\boldsymbol{\mathcal{A}}_{\mu_1}$ *and* $\boldsymbol{\mathcal{A}}_0$ *be the maximal compact global attractors associated with the problem* (5.2a,b)*,* (5.3b)*,* (5.4)*, in* $\dim n = 2$*, with* $p > 2$*; then* $\hat{d}(\boldsymbol{\mathcal{A}}_{\mu_1}, \boldsymbol{\mathcal{A}}_0) \to 0$*, as* $\mu_1 \to 0$*, with* $\hat{d}$ *the semidistance measure given by* (5.345)*.*

Proof. Dropping the term proportional to $||\boldsymbol{w}||^2_{\boldsymbol{W}^{1,2}(\Omega)}$ in (5.372), we may rewrite the resulting differential inequality in the form

$$\frac{d}{dt}\left(e^{-2ct}\,||\boldsymbol{w}||^2_{\boldsymbol{L}^2(\Omega)}\right) \le 2\mu_1\bar{c}(\Omega)e^{-2ct}\,\left|\boldsymbol{v}_{\mu_1}\right|_{\boldsymbol{H}^3(\Omega)} \tag{5.382}$$

which when integrated over $[0, t)$ yields, in view of the fact that $||\boldsymbol{w}(0)||_{\boldsymbol{L}^2(\Omega)} = 0$, the estimate

$$e^{-2ct}\,||\boldsymbol{w}||^2_{\boldsymbol{L}^2(\Omega)} \le 2\mu_1\bar{c}(\Omega)\int_0^t e^{-2c\tau}\,\left|\boldsymbol{v}_{\mu_1}\right|_{\boldsymbol{H}^3(\Omega)}\,d\tau. \tag{5.383}$$

Now, for any $t > 0$, by using (5.368) we may estimate the integral on the right-hand side of (5.383) as follows:

$$\begin{aligned}\int_0^t e^{-2c\tau}\,\left|\left|\boldsymbol{v}_{\mu_1}\right|\right|_{\boldsymbol{H}^3(\Omega)}\,d\tau &\le \left(\int_0^t e^{-4c\tau}\,d\tau\right)^{1/2}\left(\int_0^t \left|\boldsymbol{v}_{\mu_1}\right|^2_{\boldsymbol{H}^3(\Omega)}\,d\tau\right)^{1/2}\\ &\le \frac{1}{2\sqrt{c}}\left(\int_0^t \left|\boldsymbol{v}_{\mu_1}\right|^2_{\boldsymbol{H}^3(\Omega)}\,d\tau\right)^{1/2} \le \frac{1}{2}\left(\frac{\gamma_1(t)}{c\mu_1}\right)^{1/2}\end{aligned} \tag{5.384}$$

and (see (5.368)) we have abbreviated $\tilde{k}(\mu_0, \epsilon, \boldsymbol{f}, R, t) \equiv \gamma_1(t)$. Inserting (5.384) in (5.383) we find that for $t \in I \subset (0, \infty)$

$$||\boldsymbol{w}(t;\mu_1)||^2_{\boldsymbol{L}^2(\Omega)} \le \mu_1\bar{c}(\Omega)e^{2ct}\left(\frac{\gamma(t)}{c\mu_1}\right)^{1/2} \equiv K_\Omega(t)\mu_1^{1/2}. \tag{5.385}$$

As a direct consequence of (5.385) we now find that

$$\begin{aligned}\sup_{\boldsymbol{u}_0\in B_{R_0}(\boldsymbol{0})}\ \sup_{t\in I}||\boldsymbol{w}(t;\mu_1)||_{\boldsymbol{L}^2(\Omega)} &= \sup_{\boldsymbol{u}_0\in B_{R_0}(\boldsymbol{0})}\ \sup_{t\in I} d(\boldsymbol{S}_{\mu_1}(t)\boldsymbol{u}_0, \boldsymbol{S}_0(t)\boldsymbol{u}_0)\\ &\le K_\Omega^{1/2}(t)\mu_1^{1/4}\end{aligned} \tag{5.386}$$

where $d(\cdot,\cdot)$ is the distance function in $\boldsymbol{L}^2(\Omega)$. In view of (5.370), the result stated in the theorem now follows as a direct consequence of (5.386). □

Chapter 6
Inertial Manifolds, Orbit Squeezing, and Attractors for Bipolar Flow in Unbounded Channels

6.1 Introduction

In Chap. 5 we discussed, in considerable detail, the existence of maximal compact global attractors for bipolar and non-Newtonian flows associated with either (5.2a,b), (5.3a), (5.4), $\Omega \subseteq R^n$, $n = 2, 3$, a bounded domain, or (5.2a,b), (5.3b), (5.4) where $\Omega = [0, L]^n$, $n = 2, 3$, $L > 0$. In this chapter we will extend our consideration of the long-time behavior of the solutions of the bipolar, incompressible, viscous flow problem in two basic directions; we will examine (1) the problem of the existence of an inertial manifold, and the associated question of squeezing of orbits, and (2) the problem of establishing the existence of a maximal compact attractor for the flow in an unbounded, parallel-walled, channel whose existence was considered in Sect. 4.4.

The concept of an inertial manifold for dissipative evolutionary equations appears to have been introduced in [FoS]. Formally, an inertial manifold is a finite-dimensional attractor which attracts, exponentially, all orbits of the evolutionary problem in question. Since the initial work in [FoS] the subject of inertial manifolds for dissipative evolutionary equations has attracted considerable interest; among the work of a general character; in this area, we may cite [CFNT1, 2], [Con1, EFNT, FST], [LS1, 2], [Ma1, 2], [Smi], [SM], [Te2], and [SY1]. Work in the literature which is specifically geared towards establishing the existence of an inertial manifold (or, an approximate inertial manifold) for the Navier–Stokes equations in both two and three dimensions includes [CF, CFMT, HR, Kw], and [Ti]. Additionally, the existence of an inertial manifold for the extension of the Navier–Stokes model usually attributed to Ladyzhenskya [La2] was established in [OS2]; the Ladyzhenskya equations have been discussed in Sect. 1.2. The existence of an inertial manifold for the nonlinear system of equations describing the motion of a bipolar incompressible viscous fluid is taken up in Sect. 6.2; we show, following the analysis in [BH3] that, unlike the current situation with regard to the Navier–Stokes equations, an inertial manifold does exist for the case of the space-periodic problem, in both dimensions $n = 2$ and $n = 3$, provided $0 \leq \alpha < 1$ $(1 < p \leq 2)$. In the

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course of establishing the existence of an inertial manifold a squeezing property is proven for the orbits of the semigroup $\boldsymbol{S}_{\mu_1}(t)$; while this particular squeezing property is naturally adopted to establishing the existence of the inertial manifold for incompressible bipolar fluid flow, a more basic L^2 form of the squeezing property is shown to hold in Sect. 6.3. The L^2 squeezing property is proven by using the more delicate analysis of the nonlinear viscosity term which was presented in [BH1].

In Sect. 6.4 we follow the analysis in [BH5] and prove the existence of a global compact attractor for the equations governing nonlinear bipolar fluid flows in unbounded two-dimensional channels; this work depends, in an essential manner, on the corresponding existence theory for such problems which was presented in [BH4] and detailed in Chap. 4. The analysis in Sect. 6.4 is based on the consideration of a sequence of approximating problems, defined on simply connected bounded subdomains of an unbounded two-dimensional channel, which "converge" to the channel in an appropriate sense.

Finally, in Sect. 6.5, we outline some of the related recent work of other authors who have looked at the issue of asymptotic behavior of solutions to problems involving incompressible bipolar and non-Newtonian flow and the existence of global attractors; such work includes, in particular, the analysis in [BaH], [DC1, 2], [DL], [Do1, 2], [Ju, LWW], [LZ1, 2, 3], [LZZ1, 2, 3], [MP1, 2], [NP1, 2, 3], [ZZ1, 2], and [ZZL1, 2, 3].

6.2 Inertial Manifolds for Incompressible Bipolar Viscous Flow

6.2.1 Introduction

In Sect. 5.3 we established the existence of a maximal compact global attractor $\boldsymbol{\mathcal{A}}_{\mu_1}$ of the form

$$\boldsymbol{\mathcal{A}}_{\mu_1} = \bigcap_{t>0} \boldsymbol{S}_{\mu_1}(t) B^{\rho'}_{\boldsymbol{H}^2(\Omega)}; \quad \mu_1 > 0$$

for the solution operator $\boldsymbol{S}_{\mu_1}(t)$ associated with the bipolar initial-boundary value problem (5.2a,b), (5.3a), (5.4), where $B^{\rho'}_{\boldsymbol{H}^2(\Omega)}$ is a ball of radius ρ' in $\boldsymbol{H}^2(\Omega)$ with ρ' dependent on μ_1; this result holds in both dimensions $n = 2$ and $n = 3$ with $1 < p \leq 2$. We also computed, in Sect. 5.3.5, upper bounds for both the Hausdorff and fractal dimension of $\boldsymbol{\mathcal{A}}_{\mu_1}$. Results similar to those proven in Sect. 5.3 can be shown to hold for the related space-periodic problem (5.2a,b), (5.3b), (5.4) with $\mu_1 > 0$; in particular, it is easily established that for the space-periodic problem, in dimensions $n = 2$ or 3, the maximal compact global attractor has the form

$$\boldsymbol{\mathcal{A}}_{\mu_1} = \bigcap_{t>0} \boldsymbol{S}_{\mu_1}(t) B^{\bar{\rho}}_{\boldsymbol{H}^2_{per}(\Omega)}; \quad \mu_1 > 0$$

with $B^{\bar{\rho}}_{\boldsymbol{H}^2_{per}(\Omega)}$ a ball of (μ_1-dependent) radius $\bar{\rho}$ in $\boldsymbol{H}^2_{per}(\Omega)$, $\Omega = [0, L]^n$, $L > 0$, $n = 2$ or 3.

In this section we continue our development of the dynamical systems approach to the asymptotic (large time) behavior of the solutions of the incompressible nonlinear bipolar flow equations; our focus will be on the problem of the existence of an inertial manifold. As already indicated, an inertial manifold for a dissipative nonlinear evolutionary equation is a finite-dimensional attractor which attracts, exponentially, all orbits of the relevant equation. For $\mu_1 > 0$, and $0 \leq \alpha < 1$ (so that $1 < p \leq 2$), let us denote, again, by $\boldsymbol{S}_{\mu_1}(t)$ the solution operator for the bipolar problem (5.2a,b), (5.4), satisfying the spatial periodicity conditions (5.3b) on $\Omega = [0, L]^n$, $L > 0$, $n = 2, 3$. We make the following

Definition 6.1. A set $\mathcal{M}$ is an *inertial manifold* for the bipolar problem on $\Omega \times [0, T)$, $T > 0$, with solution operator $\boldsymbol{S}_{\mu_1}(t)$, if

(i) $\mathcal{M}$ is a finite-dimensional Lipschitz manifold,
(ii) $\mathcal{M}$ is invariant in the sense that $\forall t \geq 0$, $\boldsymbol{S}_{\mu_1}(t)\mathcal{M} \subset \mathcal{M}$, and
(iii) $\mathcal{M}$ attracts exponentially all orbits of $\boldsymbol{S}_{\mu_1}(\cdot)$, i.e.,

$$d(\boldsymbol{S}_{\mu_1}(t)\boldsymbol{v}_0, \mathcal{M}) \to 0 \text{ as } t \to \infty. \tag{6.1}$$

Remarks. The convergence in (6.1) holds $\forall \boldsymbol{v}_0 \in D(\boldsymbol{A}^{3/4})$, $\boldsymbol{A}$ being the linear positive self-adjoint operator defined by (5.350), (5.351) with $a(\cdot,\cdot) : \boldsymbol{V}_{per} \times \boldsymbol{V}_{per} \to R$ the positive definite $\boldsymbol{V}_{per}(\Omega)$-elliptic symmetric bilinear form given by (5.348); in fact, we have $\boldsymbol{A} \in \boldsymbol{\mathcal{L}}(D(\boldsymbol{A}); \boldsymbol{H}_{per}) \cap \boldsymbol{\mathcal{L}}(\boldsymbol{V}_{per}, \boldsymbol{V}'_{per})$. Included in part (iii) of Definition 6.1 is the understanding that the rate of decay in (6.1) is uniformly exponential for $\boldsymbol{v}_0$ in bounded sets of $D(\boldsymbol{A}^{1/4})$ with $\mathcal{M} \subseteq D(\boldsymbol{A}^{1/4})$.

For the analysis in this section we will again express the incompressible bipolar equations, with $\rho = 1$, in the form (5.346), i.e., with the pressure given by $\bar{p}$

$$\begin{aligned}\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\nabla \bar{p} - 2\mu_1 \nabla \cdot (\Delta \boldsymbol{e}) \\ + \nabla \cdot (2\mu_0(\epsilon + |\boldsymbol{e}(\boldsymbol{v})|^2)^{(p-2)/2}\boldsymbol{e}) + \boldsymbol{f}.\end{aligned}$$

We also recall the following fundamental linear problem associated with (5.2a,b), (5.3b), (5.4) which was introduced in Sect. 5.5.2 as (5.347a,b):

$$\nabla \cdot (\Delta \boldsymbol{e}) + \nabla \bar{p} = \boldsymbol{f}, \text{ in } \Omega,$$
$$\nabla \cdot \boldsymbol{u} = 0, \text{ in } \Omega,$$

where $\boldsymbol{e} = \boldsymbol{e}(\boldsymbol{u})$ and $\boldsymbol{u}$ satisfies the spatial periodicity conditions in (5.3b).

All of the properties of the operator $\boldsymbol{A}$ that were made explicit in Sect. 5.5.2 will, of course, apply to our work in this section, e.g., the results of (5.352)–(5.357a–e).

Moreover, it is shown in appendix C that for the space-periodic problem the eigenvalues of $\boldsymbol{A}$ have the form

$$\lambda_k = \frac{8\pi^4}{L^4} |\boldsymbol{k}|^4, \quad \boldsymbol{k} = (k_1, k_2, \ldots, k_n) \tag{6.2}$$

for $n = 2, 3$.

We now define $\boldsymbol{A}_p$, $\boldsymbol{B}(\boldsymbol{u}, \boldsymbol{v})$, and $\boldsymbol{R}(\boldsymbol{u})$ precisely as was done in (5.358), (5.359), and (5.360), respectively, i.e.,

$$[\boldsymbol{A}_p \boldsymbol{u}]_i = \frac{\partial}{\partial x_j}[(\epsilon + |\boldsymbol{e}|^2)^{(p-2)/2} e_{ij}], \quad i = 1, \ldots, n,$$

$$\boldsymbol{B}(\boldsymbol{u}, \boldsymbol{v}) = \boldsymbol{u} \cdot \nabla \boldsymbol{v},$$

and

$$\boldsymbol{R}(\boldsymbol{u}) = 2\mu_0 \boldsymbol{A}_p(\boldsymbol{u}) + \boldsymbol{B}(\boldsymbol{u}, \boldsymbol{u}) - \boldsymbol{f},$$

where $\boldsymbol{f} \in \boldsymbol{H}_{per}$. Then, just as we did in Sect. 5.5.2, the bipolar problem (5.2a,b), (5.3b), (5.4) is now viewed as the abstract initial-value problem (5.361a,b) posed in $\boldsymbol{V}_{per}(\Omega)$, i.e.,

$$\frac{d\boldsymbol{u}}{dt} + 2\mu_1 \boldsymbol{A}\boldsymbol{u} + \boldsymbol{R}(\boldsymbol{u}) = \boldsymbol{0}, \quad t > 0,$$

$$\boldsymbol{u}(\boldsymbol{0}) = \boldsymbol{u}_0 \in \boldsymbol{H}_{per}.$$

6.2.2 An Outline of the Methodology for Proving the Existence of an Inertial Manifold

We sketch, below, the steps that will be followed, in the succeeding subsections, in order to establish the existence of an inertial manifold for the problem (5.2a,b), (5.3b), (5.4) for the bipolar fluid dynamics equations in dimensions $n = 2, 3$, with $\mu_1 > 0$.

A. We first prove, in Sect. 6.2.3, a Lipschitz property for the nonlinear map $\boldsymbol{R}(\cdot)$; in fact, we will show that $\boldsymbol{R}(\boldsymbol{u})$ is Lipschitz on bounded sets of $D(\boldsymbol{A}^{1/4})$ with values in $D(\boldsymbol{A}^{-1/4})$, i.e., for $M > 0$, $\exists C_M > 0$ such that

$$\left|\left|\boldsymbol{A}^{-1/4}\boldsymbol{R}(\boldsymbol{u}) - \boldsymbol{A}^{-1/4}\boldsymbol{R}(\boldsymbol{v})\right|\right|_{\boldsymbol{L}^2(\Omega)} \leq C_M \left|\left|\boldsymbol{A}^{1/4}(\boldsymbol{u} - \boldsymbol{v})\right|\right|_{\boldsymbol{L}^2(\Omega)}, \tag{6.3}$$

$\forall \boldsymbol{u}, \boldsymbol{v} \in D(\boldsymbol{A}^{1/4})$ such that $\left\|\boldsymbol{A}^{1/4}(\boldsymbol{u})\right\|_{\boldsymbol{L}^2(\Omega)} \leq M$ and $\left\|\boldsymbol{A}^{1/4}(\boldsymbol{v})\right\|_{\boldsymbol{L}^2(\Omega)} \leq M$. The result expressed by (6.3) will be stated as a formal theorem and proven in Sect. 6.2.3; it clearly implies that $\boldsymbol{R}$ is bounded on bounded subsets of $D(\boldsymbol{A}^{1/4})$, i.e., that $\exists C'_M > 0$ such that

$$\left\|\boldsymbol{A}^{-1/4}\boldsymbol{R}(\boldsymbol{u})\right\|_{\boldsymbol{L}^2(\Omega)} \leq C'_M, \quad \forall \boldsymbol{u} \in D(\boldsymbol{A}^{1/4}) \text{ with } \left\|\boldsymbol{A}^{1/4}\boldsymbol{u}\right\|_{\boldsymbol{L}^2(\Omega)} \leq M. \tag{6.4}$$

B. Using the Lipschitz property (6.3) we will establish, in Sect. 6.2.4, a squeezing property for orbits of the semigroup generated by (5.361a,b) of the following form. Let $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_N$ be the first N eigenfunctions of the operator $\boldsymbol{A}$. Let $\boldsymbol{P}_N : \boldsymbol{H}_{per} \to \text{span}\{\boldsymbol{w}, \ldots, \boldsymbol{w}_N\}$ be the usual projection operator and set $\boldsymbol{Q}_N = \boldsymbol{I} - \boldsymbol{P}_N$. Let $\gamma > 0$. Then we will prove that if, for $t \in [0, T]$, $T > 0$,

$$\left\|\boldsymbol{A}^{1/4}(\boldsymbol{u}(t))\right\|_{\boldsymbol{L}^2(\Omega)} \leq M \text{ and } \left\|\boldsymbol{A}^{1/4}(\boldsymbol{v}(t))\right\|_{\boldsymbol{L}^2(\Omega)} \leq M, \tag{6.5}$$

$M > 0$ as in the statement of the Lipschitz property (6.3), then $\exists c_i > 0, i = 1, 2$, depending only on $\gamma, M, T, \boldsymbol{f}, \mu_0, \mu_1, \epsilon$, and Ω such that for every N and every $t \in [0, T]$ either

$$\left\|\boldsymbol{Q}_N \boldsymbol{A}^{-1/4}(\boldsymbol{u}(t) - \boldsymbol{v}(t))\right\|_{\boldsymbol{L}^2(\Omega)} \leq \gamma \left\|\boldsymbol{P}_N \boldsymbol{A}^{-1/4}(\boldsymbol{u}(t) - \boldsymbol{v}(t))\right\|_{\boldsymbol{L}^2(\Omega)} \tag{6.6}$$

or else

$$\left\|\boldsymbol{A}^{-1/4}(\boldsymbol{u}(t) - \boldsymbol{v}(t))\right\|_{\boldsymbol{L}^2(\Omega)} \leq c_1 \exp(-c_2 \mu_1 \gamma_{N+1} t) \left\|\boldsymbol{A}^{-1/4}(\boldsymbol{u}(0) - \boldsymbol{v}(0))\right\|_{\boldsymbol{L}^2(\Omega)}. \tag{6.7}$$

We remark, in passing, that an $\boldsymbol{L}^2$ version of the squeezing property expressed by (6.6), (6.7) also holds relative to the orbits of the semigroup generated by (5.361a,b) but is not well-adapted to the proof of the existence of an inertial manifold; the $\boldsymbol{L}^2$ version of the squeezing property for the (space-periodic) bipolar problem has been established in [BH1] and will be discussed in Sect. 6.3.

C. From the analysis in Sect. 5.3 it follows that there exist absorbing balls in $\boldsymbol{H}_{per}$, $D(\boldsymbol{A}^{1/4})$, and $\boldsymbol{V}_{per}$ which attract all the orbits of (5.361a,b). We will want, however, in the subsequent discussion to restrict our attention to the dynamics inside an absorbing ball $B_{r_1} \subseteq D(\boldsymbol{A}^{1/4})$; to this end we shall introduce in Sect. 6.2.5 the smooth cut-off function $\Theta : R^+ \to [0, 1]$ given by

$$\begin{cases} \Theta(\xi) = 1, & 0 \leq \xi \leq 1, \\ \Theta(\xi) = 0, & \xi \geq 2, \\ \left|\Theta'(\xi)\right| \leq 2, & \xi \geq 0, \end{cases} \tag{6.8}$$

and then set $\Theta_{r_1}(r) = \Theta(r/r_1)$. We then modify (5.361a) and consider, in its place, the evolution equation

$$\frac{d\boldsymbol{u}}{dt} + 2\mu_1 \boldsymbol{A}\boldsymbol{u} + \Theta_{r_1}\left(\left|\left|\boldsymbol{A}^{1/4}\boldsymbol{u}\right|\right|_{\boldsymbol{L}^2(\Omega)}\right)\boldsymbol{R}(\boldsymbol{u}) = \boldsymbol{0}. \tag{6.9}$$

It is not difficult to prove the existence and uniqueness of solutions to (6.9), (5.361b) with $\boldsymbol{u}_0 \in \boldsymbol{H}_{per}$; in addition, we will show, in Sect. 6.2.5, that the ball B_{r_2}, $r_2 = 2r_1$, is an absorbing set (for the orbits of (6.9), (5.361b)) in $D(\boldsymbol{A}^{1/4})$; it is also proven, in Sect. 6.2.5, that after a sufficiently large time t^*, the dynamics of the original equation (5.361a) are exactly represented by the dynamics of the modified equation (6.9).

D. In Sect. 6.2.5 we also introduce the space $H_{b,l}$ of the Lipschitz maps ($b > 0$, $l > 0$)

$$\boldsymbol{\phi} : \boldsymbol{P}_N D(\boldsymbol{A}^{1/4}) \to \boldsymbol{Q}_N D(\boldsymbol{A}^{1/4})$$

satisfying

$$\operatorname{supp} \boldsymbol{\phi} \subset \left\{\boldsymbol{p} \in \boldsymbol{P}_N D(\boldsymbol{A}^{1/4}) \mid \left|\left|\boldsymbol{A}^{1/4}\boldsymbol{p}\right|\right|_{\boldsymbol{L}^2(\Omega)} \le r_2\right\}, \tag{6.10a}$$

$$\left|\left|\boldsymbol{A}^{1/4}\boldsymbol{\phi}(\boldsymbol{p})\right|\right|_{\boldsymbol{L}^2(\Omega)} \le b, \quad \forall \boldsymbol{p} \in \boldsymbol{P}_N D(\boldsymbol{A}^{1/4}), \tag{6.10b}$$

and $\forall \boldsymbol{p}_1, \boldsymbol{p}_2 \in \boldsymbol{P}_N D(\boldsymbol{A}^{1/4})$

$$\left|\left|\boldsymbol{A}^{1/4}\boldsymbol{\phi}(\boldsymbol{p}_1) - \boldsymbol{A}^{1/4}\boldsymbol{\phi}(\boldsymbol{p}_2)\right|\right|_{\boldsymbol{L}^2(\Omega)} \le l \left|\left|\boldsymbol{A}^{1/4}(\boldsymbol{p}_1 - \boldsymbol{p}_2)\right|\right|_{\boldsymbol{L}^2(\Omega)}. \tag{6.11}$$

It is not difficult to prove that $H_{b,l}$ is complete with respect to the metric induced by the norm

$$||\boldsymbol{\phi}_1 - \boldsymbol{\phi}_2||_* = \sup_{\boldsymbol{p} \in \boldsymbol{P}_N D(\boldsymbol{A}^{1/4})} \left|\left|\boldsymbol{A}^{1/4}\boldsymbol{\phi}_1(\boldsymbol{p}) - \boldsymbol{A}^{1/4}\boldsymbol{\phi}_2(\boldsymbol{p})\right|\right|_{\boldsymbol{L}^2(\Omega)} \tag{6.12}$$

$\boldsymbol{\phi}_i \in H_{b,l}$, $i = 1, 2$.

E. We next specify a mapping $\boldsymbol{T}$ which associates with each $\boldsymbol{\phi} \in H_{b,l}$ a function $\boldsymbol{T}\boldsymbol{\phi}$ defined on $\boldsymbol{P}_N D(\boldsymbol{A}^{1/4})$; the mapping $\boldsymbol{T}$ arises in the following manner:

(i) We apply the projections $\boldsymbol{P}_N$, $\boldsymbol{Q}_N$ to the modified equation (6.9) to obtain evolution equations for $\boldsymbol{p} = \boldsymbol{P}_N\boldsymbol{u}$ and $\boldsymbol{q} = \boldsymbol{Q}_N\boldsymbol{u}$ of the form

$$\begin{cases} \dfrac{d\boldsymbol{p}}{dt} + 2\mu_1 \boldsymbol{A}\boldsymbol{p} + \boldsymbol{P}_N \boldsymbol{F}(\boldsymbol{u}) = \boldsymbol{0}, & (6.13a) \\ \dfrac{d\boldsymbol{q}}{dt} + 2\mu_1 \boldsymbol{A}\boldsymbol{q} + \boldsymbol{Q}_N \boldsymbol{F}(\boldsymbol{u}) = \boldsymbol{0}, & (6.13b) \end{cases}$$

where $\boldsymbol{F}(\boldsymbol{u}) = \Theta_{r_1}\left(\left\|\boldsymbol{A}^{1/4}\boldsymbol{u}\right\|_{\boldsymbol{L}^2(\Omega)}\right)\boldsymbol{R}(\boldsymbol{u})$.

(ii) Next, we choose $\boldsymbol{\phi} \in H_{b,l}$ and $\boldsymbol{p}_0 \in \boldsymbol{P}_N D(\boldsymbol{A}^{1/4})$ and consider $\boldsymbol{p} = \boldsymbol{p}(t)$ as determined by the initial value problem

$$\begin{cases} \dfrac{d\boldsymbol{p}}{dt} + 2\mu_1 \boldsymbol{A}\boldsymbol{p} + \boldsymbol{P}_N \boldsymbol{F}(\boldsymbol{p} + \boldsymbol{\phi}(\boldsymbol{p})) = \boldsymbol{0}, \\ \boldsymbol{p}(0) = \boldsymbol{p}_0, \end{cases} \tag{6.14}$$

i.e., $\boldsymbol{p}(t) = \boldsymbol{p}(t; \boldsymbol{\phi}, \boldsymbol{p}_0)$.

(iii) We employ for our operator $\boldsymbol{A}$ the following lemma, a proof of which may be found in [Te4]: for any $\alpha \in R^1$, if $\sigma \in L^\infty(R^1; D(\boldsymbol{A}^{\alpha-1/2}))$ then $\exists!$ function $\boldsymbol{\xi}$, continuous and bounded from R^1 into $D(\boldsymbol{A}^\alpha)$, which satisfies

$$\frac{d\boldsymbol{\xi}}{dt} + \boldsymbol{A}\boldsymbol{\xi} = \boldsymbol{\sigma}. \tag{6.15}$$

Remarks. In establishing the lemma cited above one looks at the initial value problem

$$\begin{cases} \dfrac{d\boldsymbol{\xi}}{dt} + \boldsymbol{A}\boldsymbol{\xi} = \boldsymbol{0}, \\ \boldsymbol{\xi}(0) = \boldsymbol{\xi}_0, \end{cases} \tag{6.16}$$

whose (unique) solution $e^{-t\boldsymbol{A}} : \boldsymbol{\xi}_0 \to \boldsymbol{\xi}(t)$ is continuous as a mapping from $D(\boldsymbol{A}^{\alpha-1/2})$ into $D(\boldsymbol{A}^\alpha)$, $\forall t > 0$; the unique solution of (6.15) then has the form

$$\boldsymbol{\xi}(t) = e^{-(t-t_0)\boldsymbol{A}}\boldsymbol{\xi}(t_0) + \int_{t_0}^{t} e^{-(t-\tau)\boldsymbol{A}}\boldsymbol{\sigma}(\tau)d\boldsymbol{\tau} \tag{6.17}$$

and as $\left\|e^{-t\boldsymbol{A}}\right\|_{\mathcal{L}(D(\boldsymbol{A}^\alpha))} \leq \exp(-\lambda t)$, for some $\lambda > 0$, all $\alpha \in R^1$, and all $t \geq 0$, as $t_0 \to -\infty$, we obtain

$$\boldsymbol{\xi}(t) = \int_{-\infty}^{t} e^{-(t-\tau)\boldsymbol{A}}\boldsymbol{\sigma}(\boldsymbol{\tau})d\boldsymbol{\tau}. \tag{6.18}$$

(iv) We consider, for $\boldsymbol{p} = \boldsymbol{p}(t; \boldsymbol{\phi}, \boldsymbol{p}_0)$ defined by (6.17) an analog of the equation for $\boldsymbol{p}(t)$, namely,

$$\frac{d\boldsymbol{q}}{dt} + 2\mu_1 \boldsymbol{A}\boldsymbol{q} + \boldsymbol{Q}_N \boldsymbol{F}(\boldsymbol{p} + \boldsymbol{\phi}(\boldsymbol{p})) = \mathbf{0}. \tag{6.19}$$

In (6.19), $\boldsymbol{\sigma} \equiv -\boldsymbol{Q}_N \boldsymbol{F}(\boldsymbol{p} + \boldsymbol{\phi}(\boldsymbol{p})) \in L^\infty(R^1; D(\boldsymbol{A}^{-1/4}))$, i.e., (6.19) is of the form (6.15); thus by the lemma referenced in (iii), $\exists!$ solution $\boldsymbol{q} = \boldsymbol{q}(t; \boldsymbol{\phi}, \boldsymbol{p}_0)$ of (6.19) which is continuous and bounded as a mapping from R^1 into $\boldsymbol{Q}_N D(\boldsymbol{A}^{1/4})$. In particular,

$$\boldsymbol{q}(0) = \boldsymbol{q}(0; \boldsymbol{\phi}, \boldsymbol{p}_0) \in \boldsymbol{Q}_N D(\boldsymbol{A}^{1/4}). \tag{6.20}$$

(v) Finally, we consider the function that maps

$$\boldsymbol{p}_0 \in \boldsymbol{P}_N D(\boldsymbol{A}^{1/4}) \to \boldsymbol{q}(0; \boldsymbol{\phi}, \boldsymbol{p}_0) \in \boldsymbol{Q}_N D(\boldsymbol{A}^{1/4}). \tag{6.21}$$

This function, which depends on the choice of $\boldsymbol{\phi} \in H_{b,l}$, will be denoted by $\boldsymbol{T}\boldsymbol{\phi}$; by virtue of (6.19), (6.20), and (6.21), coupled with the representation (6.18), $\boldsymbol{T}\boldsymbol{\phi} : \boldsymbol{p}_0 \to \boldsymbol{q}(0; \boldsymbol{\phi}, \boldsymbol{p}_0)$ has the specific form

$$\boldsymbol{T}\boldsymbol{\phi}(\boldsymbol{p}_0) = -\int_{-\infty}^{0} e^{2\mu_1 \boldsymbol{A}\tau} \boldsymbol{Q}_N \boldsymbol{F}(\boldsymbol{p}(\tau) + \boldsymbol{\phi}(\boldsymbol{p}(\tau))d\tau \equiv \boldsymbol{q}(0; \boldsymbol{\phi}, \boldsymbol{p}_0). \tag{6.22}$$

F. Once the mapping $\boldsymbol{T}$ has been specified, the task at hand is

(i) to prove that for $\lambda_N^{1/2}$ and $\lambda_{N+1}^{1/2} - \lambda_N^{1/2}$ both sufficiently large

$$\boldsymbol{T} : H_{b,l} \xrightarrow{\text{into}} H_{b,l},$$

with $\boldsymbol{T}$ a strict contraction on $H_{b,l}$, and then

(ii) to prove that the manifold $\boldsymbol{\mathcal{M}}$ defined by the graph of the (resulting) fixed point $\boldsymbol{\phi}_0$ of $\boldsymbol{T}$ is an inertial manifold for the bipolar problem.

The manifold, $\boldsymbol{\mathcal{M}}$, as defined above, will be a finite-dimensional Lipschitz manifold by virtue of the definition of $H_{b,l}$.

G. Finally, in Sect. 6.2.6 we will show that

$$d(\boldsymbol{S}_{\mu_1}(t)\boldsymbol{u}_0, \boldsymbol{\mathcal{M}}) \to 0$$

exponentially in t, as $t \to \infty$.

6.2.3 The Lipschitz Property

The purpose of this section is to establish the fact that $\boldsymbol{R}(\boldsymbol{u})$, as defined by (5.360), is Lipschitz on bounded sets of $D(\boldsymbol{A}^{1/4})$ with values in $D(\boldsymbol{A}^{-1/4})$, i.e., given $M > 0$, $\exists C_M > 0$ such that (6.3) holds $\forall \boldsymbol{u}, \boldsymbol{v} \in D(\boldsymbol{A}^{1/4})$ such that $\left|\left|\boldsymbol{A}^{1/4}(\boldsymbol{u})\right|\right|_{\boldsymbol{L}^2(\Omega)} \leq M$ and $\left|\left|\boldsymbol{A}^{1/4}(\boldsymbol{v})\right|\right|_{\boldsymbol{L}^2(\Omega)} \leq M$. Our first result in that direction is

Lemma 6.1. *For $\boldsymbol{A}_p$ defined by (5.358), and $\boldsymbol{u}, \boldsymbol{v}$ in $D(\boldsymbol{A}^{1/4})$, it follows that, for some $C_1 = C_1(\Omega) > 0$,*

$$\left|(\boldsymbol{A}_p(\boldsymbol{u}) - \boldsymbol{A}_p(\boldsymbol{v}), \boldsymbol{w})_{\boldsymbol{L}^2(\Omega)}\right| \leq \left(\frac{2-p}{2} + 1\right) \epsilon^{\frac{p-2}{2}} |\boldsymbol{u} - \boldsymbol{v}|_{\boldsymbol{H}^1(\Omega)} |\boldsymbol{w}|_{\boldsymbol{H}^1(\Omega)}. \quad (6.23)$$

Proof. Let $\boldsymbol{u}, \boldsymbol{v} \in D(\boldsymbol{A}^{1/4})$ and set $\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{v}$. With $\boldsymbol{A}_p$ defined as in (5.358) we have

$$\begin{aligned}\left(\boldsymbol{A}_p(\boldsymbol{u}) - \boldsymbol{A}_p(\boldsymbol{v}), \boldsymbol{w}\right)_{\boldsymbol{L}^2(\Omega)} &= \int_\Omega [\boldsymbol{A}_p(\boldsymbol{u})_i - \boldsymbol{A}_p(\boldsymbol{v})_i] w_i \, d\boldsymbol{x} \\ &= \int_\Omega \left\{ \frac{\partial}{\partial x_j} [(\epsilon + |\boldsymbol{e}(\boldsymbol{u})|^2)^{\frac{p-2}{2}} e_{ij}(\boldsymbol{u})] \right. \\ &\qquad \left. - \frac{\partial}{\partial x_j} [(\epsilon + |\boldsymbol{e}(\boldsymbol{v})|^2)^{\frac{p-2}{2}} e_{ij}(\boldsymbol{v})] \right\} w_i \, d\boldsymbol{x}.\end{aligned} \quad (6.24)$$

Integrating (6.24) by parts, and using the fact that, as a consequence of the space periodicity satisfied by $\boldsymbol{u}$ and $\boldsymbol{v}$,

$$\int_{\partial\Omega} [(\epsilon + |\boldsymbol{e}(\boldsymbol{u})|^2)^{\frac{p-2}{2}} e_{ij}(\boldsymbol{u}) - (\epsilon + |\boldsymbol{e}(\boldsymbol{u})|^2)^{\frac{p-2}{2}} e_{ij}(\boldsymbol{v})] w_i \nu_j \, dS = 0, \quad (6.25)$$

we easily find that

$$\begin{aligned}&\left(\boldsymbol{A}_p(\boldsymbol{u}) - \boldsymbol{A}_p(\boldsymbol{v}), \boldsymbol{w}\right)_{\boldsymbol{L}^2(\Omega)} \\ &\quad = -\int_\Omega [(\epsilon + |\boldsymbol{e}(\boldsymbol{u})|^2)^{\frac{p-2}{2}} e_{ij}(\boldsymbol{u}) - (\epsilon + |\boldsymbol{e}(\boldsymbol{v})|^2)^{\frac{p-2}{2}} e_{ij}(\boldsymbol{v})] \frac{\partial w_i}{\partial x_j} d\boldsymbol{x}.\end{aligned} \quad (6.26)$$

We now set

$$r_p(\boldsymbol{e}) = \frac{1}{p}(\epsilon + |\boldsymbol{e}|^2)^{p/2}, \quad (6.27a)$$

$$\bar{e}_{ij}(t) = e_{ij}(\boldsymbol{u}) + t(e_{ij}(\boldsymbol{v}) - e_{ij}(\boldsymbol{v})), \quad 0 \leq t \leq 1. \quad (6.27b)$$

Then

$$\frac{\partial r_p}{\partial e_{ij}} = (\epsilon + |\boldsymbol{e}|^2)^{\frac{p-2}{2}} e_{ij}, \tag{6.28}$$

so that

$$[(\epsilon + |\boldsymbol{e}(\boldsymbol{u})|^2)^{\frac{p-2}{2}} e_{ij}(\boldsymbol{u}) - (\epsilon + |\boldsymbol{e}(\boldsymbol{v})|^2)^{\frac{p-2}{2}} e_{ij}(\boldsymbol{v})] = \int_0^1 \frac{\partial}{\partial t}\left(\frac{\partial r_p}{\partial e_{ij}}(\bar{e}_{ij}(t))\right) dt. \tag{6.29}$$

Employing (6.29) in (6.26) yields

$$\begin{aligned}\left(\boldsymbol{A}_p(\boldsymbol{u}) - \boldsymbol{A}_p(\boldsymbol{v}), \boldsymbol{w}\right)_{\boldsymbol{L}^2(\Omega)} &= \left|\int_\Omega \left\{\int_0^1 \frac{\partial}{\partial t}\left(\frac{\partial r_p}{\partial e_{ij}}(\bar{e}_{ij}(t))\right) dt\right\} \frac{\partial w_i}{\partial x_j} d\boldsymbol{x}\right| \\ &\le \int_\Omega \int_0^1 \left|\frac{\partial}{\partial t}\left(\frac{\partial r_p}{\partial e_{ij}}(\bar{e}_{ij}(t))\right) \frac{\partial w_i}{\partial x_j}\right| dt\, d\boldsymbol{x}.\end{aligned} \tag{6.30}$$

However, by virtue of (6.27a,b), (6.28),

$$\begin{aligned}\frac{\partial^2 r_p}{\partial t\, \partial e_{ij}}(\bar{e}_{ij}(t)) &= \frac{\partial^2 r_p}{\partial e_{ij}\partial e_{kl}}(\bar{e}_{ij}(t))\frac{\partial \bar{e}_{kl}}{\partial t} \\ &= \frac{\partial^2 r_p}{\partial e_{ij}\partial e_{kl}}(\bar{e}_{ij}(t))(e_{kl}(\boldsymbol{v}) - e_{kl}(\boldsymbol{u})),\end{aligned} \tag{6.31a}$$

while

$$\frac{\partial^2 r_p}{\partial e_{ij}\partial e_{kl}} = \frac{p-2}{2}(\epsilon + |\boldsymbol{e}|^2)^{\frac{p-2}{2}} e_{ij}e_{kl} + (\epsilon + |\boldsymbol{e}|^2)^{\frac{p-2}{2}} \delta_{ij}\delta_{jl}. \tag{6.31b}$$

Therefore, for any $\boldsymbol{\xi} \neq 0$, $\boldsymbol{\eta} \neq 0$,

$$\begin{aligned}\left|\frac{\partial^2 r_p}{\partial e_{ij}\partial e_{kl}}\xi_{ij}\eta_{kl}\right| &= \left|\frac{p-2}{2}(\epsilon + |\boldsymbol{e}|^2)^{\frac{p-4}{2}} e_{ij}e_{kl}\xi_{ij}\eta_{kl} + (\epsilon + |\boldsymbol{e}|^2)^{\frac{p-2}{2}} \xi_{kj}\eta_{kj}\right| \\ &\le \left|\frac{p-2}{2}\right|(\epsilon + |\boldsymbol{e}|^2)^{\frac{p-4}{2}} |e_{ij}\xi_{ij}||e_{kl}\eta_{kl}| + (\epsilon + |\boldsymbol{e}|^2)^{\frac{p-2}{2}} |\xi_{kj}\eta_{kj}| \\ &\le \left|\frac{p-2}{2}\right|(\epsilon + |\boldsymbol{e}|^2)^{\frac{p-4}{2}} |\boldsymbol{e}|^2 |\boldsymbol{\xi}||\boldsymbol{\eta}| + (\epsilon + |\boldsymbol{e}|^2)^{\frac{p-2}{2}} |\boldsymbol{\xi}||\boldsymbol{\eta}| \\ &\le \left(\frac{2-p}{2} + 1\right)\epsilon^{\frac{p-2}{2}} |\boldsymbol{\xi}||\boldsymbol{\eta}|,\end{aligned} \tag{6.32}$$

as $1 < p \le 2$. Combining (6.30)–(6.32) we obtain

$$
\begin{aligned}
&\left(\boldsymbol{A}_p(\boldsymbol{u}) - \boldsymbol{A}_p(\boldsymbol{v}), \boldsymbol{w}\right)_{\boldsymbol{L}^2(\Omega)} \\
&\le \int_\Omega \int_0^1 \left| \frac{\partial^2 r_p}{\partial e_{ij} \partial e_{kl}} (\bar{e}_{ij}(t))(e_{kl}(\boldsymbol{v}) - e_{kl}(\boldsymbol{u})) \cdot \frac{\partial w_i}{\partial x_j} \right| dt\, d\boldsymbol{x} \\
&\le \left(\frac{2-p}{2} + 1 \right) \epsilon^{\frac{p-2}{2}} \int_\Omega \int_0^1 \left(\sum_{k,l} |e_{kl}(\boldsymbol{v}-\boldsymbol{u})|^2 \right)^{1/2} \left(\sum_{i,j} \left| \frac{\partial w_i}{\partial x_j} \right|^2 \right)^{1/2} dt\, d\boldsymbol{x} \\
&\le \left(\frac{2-p}{2} + 1 \right) \epsilon^{\frac{p-2}{2}} \int_\Omega \left(\sum_{k,l} |e_{kl}(\boldsymbol{v}-\boldsymbol{u})|^2 \right)^{1/2} \left(\sum_{i,j} \left| \frac{\partial w_i}{\partial x_j} \right|^2 \right)^{1/2} d\boldsymbol{x}
\end{aligned}
\tag{6.33}
$$

from which (6.23) now follows. □

Our next result concerns the bilinear form $\boldsymbol{B}$, i.e.,

Lemma 6.2. *For $\boldsymbol{B}(\boldsymbol{u}, \boldsymbol{v})$ defined by (5.359), with $\boldsymbol{u}, \boldsymbol{v} \in D(\boldsymbol{A}^{1/4})$, $\exists C_2 = C_2(\Omega) > 0$ such that*

$$
\left| (\boldsymbol{B}(\boldsymbol{u}, \boldsymbol{u}) - \boldsymbol{B}(\boldsymbol{v}, \boldsymbol{v}), \boldsymbol{w})_{\boldsymbol{L}^2(\Omega)} \right| \le C_2(|\boldsymbol{u}|_{\boldsymbol{H}^1(\Omega)} + |\boldsymbol{v}|_{\boldsymbol{H}^1(\Omega)}) |\boldsymbol{u} - \boldsymbol{v}|_{\boldsymbol{H}^1(\Omega)} |\boldsymbol{w}|_{\boldsymbol{H}^1(\Omega)}. \tag{6.34}
$$

Proof.

$$
\begin{aligned}
&\left| (\boldsymbol{B}(\boldsymbol{u}, \boldsymbol{u}) - \boldsymbol{B}(\boldsymbol{v}, \boldsymbol{v}), \boldsymbol{w})_{\boldsymbol{L}^2(\Omega)} \right| \\
&= \left| \int_\Omega \left(u_j \frac{\partial u_i}{\partial x_j} - v_j \frac{\partial v_i}{\partial x_j} - v_j \frac{\partial v_i}{\partial x_j} \right) w_i \, d\boldsymbol{x} \right| \\
&\le \left| \int_\Omega (u_j - v_j) \frac{\partial u_i}{\partial x_j} w_i \, d\boldsymbol{x} \right| + \left| \int_\Omega v_i \frac{\partial}{\partial x_j} (u_i - v_i) w_i \, d\boldsymbol{x} \right| \\
&\le \left(\int_\Omega (u_j - v_j)(u_j - v_j) w_i w_i \, d\boldsymbol{x} \right)^{1/2} \left(\int_\Omega \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} d\boldsymbol{x} \right)^{1/2} \\
&\quad + \left(\int_\Omega v_j v_j w_i w_i \, d\boldsymbol{x} \right)^{1/2} \left(\int_\Omega \sum_{i,j} \left| \frac{\partial (u_i - v_i)}{\partial x_j} \right|^2 d\boldsymbol{x} \right)^{1/2}.
\end{aligned}
\tag{6.35}
$$

Using the Sobolev embedding $\boldsymbol{H}^1(\Omega) \hookrightarrow \boldsymbol{L}^4(\Omega)$, which is valid for $\Omega = [0, L]^n$, $n = 2, 3$, we obtain from this last estimate

$$
\begin{aligned}
&\left|(\boldsymbol{B}(\boldsymbol{u},\boldsymbol{u})-\boldsymbol{B}(\boldsymbol{v},\boldsymbol{v}),\boldsymbol{w})_{\boldsymbol{L}^2(\Omega)}\right| \\
&\quad\le \left(\int_\Omega \sum_j (u_j-v_j)^4 d\boldsymbol{x}\right)^{1/4}\left(\int_\Omega \sum_{i,j}\left|\frac{\partial u_i}{\partial x_j}\right|^2 d\boldsymbol{x}\right)^{1/2}\left(\int_\Omega \sum_i |w_i|^4 d\boldsymbol{x}\right)^{1/4} \\
&\qquad+\left(\int_\Omega \sum_j |v_j|^4 d\boldsymbol{x}\right)^{1/4}\left(\int_\Omega \sum_{i,j}\left|\frac{\partial (u_i-v_i)}{\partial x_j}\right|^2 d\boldsymbol{x}\right)^{1/2}\left(\int_\Omega \sum_i |w_i|^4 d\boldsymbol{x}\right)^{1/4}
\end{aligned}
\tag{6.36}
$$

which serves to establish (6.34). □

We are now in a position to prove the following Lipschitz property for $\boldsymbol{R}$:

Theorem 6.1. *If $\boldsymbol{R}$ is defined on $D(\boldsymbol{A}^{1/4})$ by (5.360), then $\boldsymbol{R}$ is Lipschitz on bounded sets of $D(\boldsymbol{A}^{1/4})$ with values in $D(\boldsymbol{A}^{-1/4})$, i.e., given $M > 0$, $\exists C_M > 0$ such that $\forall \boldsymbol{u},\boldsymbol{v} \in D(\boldsymbol{A}^{1/4})$ for which $\left\|\boldsymbol{A}^{1/4}(\boldsymbol{u})\right\|_{\boldsymbol{L}^2(\Omega)} \le M$ and $\left\|\boldsymbol{A}^{1/4}(\boldsymbol{v})\right\|_{\boldsymbol{L}^2(\Omega)} \le M$,*

$$
\left\|\boldsymbol{A}^{-1/4}\boldsymbol{R}(\boldsymbol{u})-\boldsymbol{A}^{-1/4}\boldsymbol{R}(\boldsymbol{v})\right\|_{\boldsymbol{L}^2(\Omega)} \le C_M\left\|\boldsymbol{A}^{1/4}(\boldsymbol{u}-\boldsymbol{v})\right\|_{\boldsymbol{L}^2(\Omega)}. \tag{6.37}
$$

Proof. By virtue of (5.360) we have

$$
\boldsymbol{R}(\boldsymbol{u})-\boldsymbol{R}(\boldsymbol{v}) = 2\mu_0(\boldsymbol{A}_p(\boldsymbol{u})-\boldsymbol{A}_p(\boldsymbol{v}))+(\boldsymbol{B}(\boldsymbol{u},\boldsymbol{u})-\boldsymbol{B}(\boldsymbol{v},\boldsymbol{v})) \tag{6.38}
$$

so combining (6.38) with (6.23) and (6.35) we find that

$$
\begin{aligned}
&\left|(\boldsymbol{R}(\boldsymbol{u})-\boldsymbol{R}(\boldsymbol{v}),\boldsymbol{w})_{\boldsymbol{L}^2(\Omega)}\right| \\
&\le 2\mu_0\left|(\boldsymbol{A}_p(\boldsymbol{u})-\boldsymbol{A}_p(\boldsymbol{v}),\boldsymbol{w})_{\boldsymbol{L}^2(\Omega)}\right|+\left|(\boldsymbol{B}(\boldsymbol{u},\boldsymbol{u})-\boldsymbol{B}(\boldsymbol{v},\boldsymbol{v}),\boldsymbol{w})_{\boldsymbol{L}^2(\Omega)}\right| \\
&\le \left[2\mu_0 C_1\left(\frac{2-p}{2}+1\right)\epsilon^{\frac{p-2}{2}}+C_2(|\boldsymbol{u}|_{\boldsymbol{H}^1(\Omega)}+|\boldsymbol{v}|_{\boldsymbol{H}^1(\Omega)})\right]|\boldsymbol{u}-\boldsymbol{v}|_{\boldsymbol{H}^1(\Omega)}|\boldsymbol{w}|_{\boldsymbol{H}^1(\Omega)}.
\end{aligned}
\tag{6.39}
$$

Employing (5.357d), i.e., the equivalence between the norms $|\boldsymbol{u}|_{\boldsymbol{H}^1(\Omega)}$ and $\left\|\boldsymbol{A}^{1/4}\boldsymbol{u}\right\|_{\boldsymbol{L}^2(\Omega)}$, for $\boldsymbol{u} \in D(\boldsymbol{A}^{1/4})$, (6.39) yields the following estimate, for some $C_3 = C_3(\Omega) > 0$:

$$\left|(\boldsymbol{R}(\boldsymbol{u})-\boldsymbol{R}(\boldsymbol{v}),\boldsymbol{w})_{L^2(\Omega)}\right|$$
$$\leq C_3\left[2\mu_0 C_1\left(\frac{2-p}{2}+1\right)\epsilon^{\frac{p-2}{2}}+C_2\left(\left\|\boldsymbol{A}^{1/4}\boldsymbol{u}\right\|_{\boldsymbol{L}^2(\Omega)}+\left\|\boldsymbol{A}^{1/4}\boldsymbol{v}\right\|_{\boldsymbol{L}^2(\Omega)}\right)\right]$$
$$\times\left\|\boldsymbol{A}^{1/4}(\boldsymbol{u}-\boldsymbol{v})\right\|_{\boldsymbol{L}^2(\Omega)}\left\|\boldsymbol{A}^{1/4}\boldsymbol{w}\right\|_{\boldsymbol{L}^2(\Omega)}. \tag{6.40}$$

Thus, for $\boldsymbol{u},\boldsymbol{v}\in D(\boldsymbol{A}^{1/4})$ satisfying $\left\|\boldsymbol{A}^{1/4}\boldsymbol{w}\right\|_{\boldsymbol{L}^2(\Omega)}\leq M$, $\left\|\boldsymbol{A}^{1/4}\boldsymbol{v}\right\|_{\boldsymbol{L}^2(\Omega)}\leq M$, for some $M>0$,

$$\left|(\boldsymbol{R}(\boldsymbol{u})-\boldsymbol{R}(\boldsymbol{v}),\boldsymbol{w})_{L^2(\Omega)}\right|\leq C_M\left\|\boldsymbol{A}^{1/4}(\boldsymbol{u}-\boldsymbol{v})\right\|_{\boldsymbol{L}^2(\Omega)}\left\|\boldsymbol{A}^{1/4}\boldsymbol{w}\right\|_{\boldsymbol{L}^2(\Omega)} \tag{6.41}$$

with

$$C_M=C_3\left[2\mu_0 C_1\left(\frac{2-p}{2}+1\right)\epsilon^{\frac{p-2}{2}}+C_2 M\right]. \tag{6.42}$$

The required estimate, i.e., (6.37) is now a direct consequence of (6.41). □

6.2.4 The Squeezing Property for the Orbits of $\boldsymbol{S}_{\mu_1}$

In Sect. 6.2.2 we presented a statement of the squeezing property for the orbits of the semigroup $\boldsymbol{S}_{\mu_1}$ which will be used in the proof of the existence of an inertial manifold for the bipolar problem (5.2a,b), (5.3a), (5.4); we summarize that statement here. If $\boldsymbol{w}_1,\ldots,\boldsymbol{w}_N$ are the first N eigenfunctions of $\boldsymbol{A}$, $\boldsymbol{P}_N:\boldsymbol{H}_{per}\to\text{span}\{\boldsymbol{w}_1,\ldots,\boldsymbol{w}_n\}$ is the projection operator, and $\boldsymbol{Q}_N=\boldsymbol{I}-\boldsymbol{P}_N$, then given $\gamma>0$, and solutions $\boldsymbol{u}(t)$, $\boldsymbol{v}(t)$ of (5.2a,b), (5.3a), (5.4) satisfying the bounds (6.5), for $0\leq t\leq T$, $T>0$, there exists for $i=1,2$, $c_i=c_i(\gamma,M,\boldsymbol{f},\mu_0,\mu_1,\epsilon;\Omega)>0$ such that either (6.6) or (6.7) holds for every N and each $t\in[0,T]$. An $\boldsymbol{L}^2$ version of this squeezing property, which is of independent interest, but is not well-adapted to the proof of the existence of an inertial manifold, will be established in Sect. 6.3. We begin the analysis with the following key

Lemma 6.3. *Let $\boldsymbol{u},\boldsymbol{v}\in D(\boldsymbol{A}^{1/4})$ be the unique solutions of the initial-value problems*

$$\begin{cases}\dfrac{d\boldsymbol{u}}{dt}+2\mu_1\boldsymbol{A}\boldsymbol{u}+\boldsymbol{R}(\boldsymbol{u})=\boldsymbol{0},\\ \boldsymbol{u}(0)=\boldsymbol{u}_0,\end{cases} \tag{6.43a}$$

$$\begin{cases} \dfrac{d\boldsymbol{v}}{dt} + 2\mu_1 \boldsymbol{A}\boldsymbol{v} + \boldsymbol{R}(\boldsymbol{v}) = \boldsymbol{0}, \\ \boldsymbol{v}(0) = \boldsymbol{v}_0. \end{cases} \tag{6.43b}$$

Let $M > 0$ *and suppose that* $\forall t \in [0, T]$, $\left|\left|\boldsymbol{A}^{1/4}\boldsymbol{u}(t)\right|\right|_{\boldsymbol{L}^2(\Omega)} \leq M$, $\left|\left|\boldsymbol{A}^{1/4}\boldsymbol{v}(t)\right|\right|_{\boldsymbol{L}^2(\Omega)} \leq M$. *Then* $\exists c_3 > 0$ *such that* $\boldsymbol{w}(t) = \boldsymbol{u}(t) - \boldsymbol{v}(t)$ *satisfies, for* $0 < t < \tau < T$,

$$\frac{\left|\left|\boldsymbol{A}^{1/4}\boldsymbol{w}(\tau)\right|\right|^2_{\boldsymbol{L}^2(\Omega)}}{\left|\left|\boldsymbol{A}^{-1/4}\boldsymbol{w}(t)\right|\right|^2_{\boldsymbol{L}^2(\Omega)}} \leq \frac{\left|\left|\boldsymbol{A}^{1/4}\boldsymbol{w}(t)\right|\right|^2_{\boldsymbol{L}^2(\Omega)}}{\left|\left|\boldsymbol{A}^{-1/4}\boldsymbol{w}(t)\right|\right|^2_{\boldsymbol{L}^2(\Omega)}} \exp(c_3(\tau - t)). \tag{6.44}$$

Proof. For $t \in [0, T]$ we define the quotient $q(t)$ by

$$q(t) = \frac{\left(\boldsymbol{A}^{1/4}\boldsymbol{w}, \boldsymbol{A}^{1/4}\boldsymbol{w}\right)_{\boldsymbol{L}^2(\Omega)}}{\left(\boldsymbol{A}^{-1/4}\boldsymbol{w}, \boldsymbol{A}^{-1/4}\boldsymbol{w}\right)_{\boldsymbol{L}^2(\Omega)}}. \tag{6.45}$$

Differentiating $q(t)$, we obtain ($' = d/dt$)

$$\begin{aligned} \frac{dq}{dt} = \frac{2}{\left(\boldsymbol{A}^{-1/4}\boldsymbol{w}, \boldsymbol{A}^{-1/4}\boldsymbol{w}\right)^2_{\boldsymbol{L}^2(\Omega)}} \Big[&\left(\boldsymbol{A}^{-1/4}\boldsymbol{w}, \boldsymbol{A}^{-1/4}\boldsymbol{w}\right)_{\boldsymbol{L}^2(\Omega)} \left(\boldsymbol{A}^{1/4}\boldsymbol{w}', \boldsymbol{A}^{1/4}\boldsymbol{w}\right)_{\boldsymbol{L}^2(\Omega)} \\ &- \left(\boldsymbol{A}^{-1/4}\boldsymbol{w}', \boldsymbol{A}^{-1/4}\boldsymbol{w}\right)_{\boldsymbol{L}^2(\Omega)} \left(\boldsymbol{A}^{1/4}\boldsymbol{w}, \boldsymbol{A}^{1/4}\boldsymbol{w}\right)_{\boldsymbol{L}^2(\Omega)} \Big] \end{aligned}$$

or

$$\frac{dq}{dt} = \frac{2}{\left|\left|\boldsymbol{A}^{-1/4}\boldsymbol{w}(t)\right|\right|^2_{\boldsymbol{L}^2(\Omega)}} \left[\left(\boldsymbol{w}', \boldsymbol{A}^{1/4}\boldsymbol{w}\right)_{\boldsymbol{L}^2(\Omega)} - q(t)\left(\boldsymbol{w}, \boldsymbol{A}^{-1/4}\boldsymbol{w}\right)_{\boldsymbol{L}^2(\Omega)} \right]. \tag{6.46}$$

However,

$$\frac{d\boldsymbol{w}}{dt} + 2\mu_1 \boldsymbol{A}(\boldsymbol{w}) + \boldsymbol{R}(\boldsymbol{u}) - \boldsymbol{R}(\boldsymbol{v}) = \boldsymbol{0} \tag{6.47}$$

so

$$\frac{dq}{dt} = \frac{-2}{\left|\left|\boldsymbol{A}^{-1/4}\boldsymbol{w}\right|\right|^2_{\boldsymbol{L}^2(\Omega)}} \left(2\mu_1 \boldsymbol{A}\boldsymbol{w} + \boldsymbol{R}(\boldsymbol{u}), \boldsymbol{A}^{1/4}\boldsymbol{w} - q(t)\boldsymbol{A}^{-1/4}\boldsymbol{w}\right)_{\boldsymbol{L}^2(\Omega)}$$

from which it follows that

$$\frac{dq}{dt} = \frac{-2}{||A^{-1/4}w||_{L^2(\Omega)}{}^2}\left(2\mu_1 A^{3/4}w + A^{-1/4}(R(u) - R(v)), A^{3/4}w - q(t)A^{-1/4}w\right)_{L^2(\Omega)}. \tag{6.48}$$

If we now make note of the fact that

$$\begin{aligned}&\left(qA^{-1/4}w, A^{3/4}w - qA^{-1/4}w\right)_{L^2(\Omega)}\\&\quad = q\left(A^{-1/4}w, A^{3/4}w\right)_{L^2(\Omega)} - q^2\left(A^{-1/4}w, A^{-1/4}w\right)_{L^2(\Omega)}\\&\quad = q\left(A^{1/4}w, A^{1/4}w\right)_{L^2(\Omega)} - q^2\left(A^{-1/4}w, A^{-1/4}w\right)_{L^2(\Omega)} \equiv 0,\end{aligned}$$

then by virtue of the definition of $q(t)$, i.e., (6.45),

$$\begin{aligned}&\left(A^{3/4}w, A^{3/4}w - qA^{-1/4}w\right)_{L^2(\Omega)}\\&\quad = \left(A^{3/4}w - qA^{-1/4}w, A^{3/4}w - qA^{-1/4}w\right)_{L^2(\Omega)}\\&\quad = \left|\left|A^{3/4}w - qA^{-1/4}w\right|\right|_{L^2(\Omega)}{}^2.\end{aligned} \tag{6.49}$$

Employing (6.49) in (6.48), and using the Cauchy-Schwarz inequality, we find that

$$\begin{aligned}&\frac{dq}{dt} + \frac{4\mu_1}{||A^{-1/4}w||_{L^2(\Omega)}{}^2}||A^{3/4}w - qA^{-1/4}w||_{L^2(\Omega)}{}^2\\&= \frac{-2}{||A^{-1/4}w||_{L^2(\Omega)}{}^2}\left(A^{-1/4}(R(u) - R(v)), A^{3/4}w - qA^{-1/4}w\right)_{L^2(\Omega)}\\&\le \frac{2}{||A^{-1/4}w||_{L^2(\Omega)}{}^2}||A^{-1/4}(R(u) - R(v))||_{L^2(\Omega)} \cdot ||A^{3/4}w - qA^{-1/4}w||_{L^2(\Omega)}\\&\le \frac{4\mu_1}{||A^{-1/4}w||_{L^2(\Omega)}{}^2}||A^{3/4}w - qA^{-1/4}||_{L^2(\Omega)}{}^2 + \frac{1}{\mu_1}\frac{||A^{-1/4}(R(u) - R(v))||_{L^2(\Omega)}{}^2}{||A^{-1/4}w||_{L^2(\Omega)}{}^2},\end{aligned} \tag{6.50}$$

where for the last estimate we have used the arithmetic-geometric mean inequality. We now avail ourselves of the Lipschitz property (6.3), which is valid for $u, v \in D(A^{1/4})$ satisfying $\left|\left|A^{1/4}u\right|\right|_{L^2(\Omega)} \le M$, $\left|\left|A^{1/4}v\right|\right|_{L^2(\Omega)} \le M$, for $M > 0$;

employing (6.3) in the last estimate in (6.50), and making use of the definition of $q(t)$, we find that

$$\frac{dq}{dt} \le \frac{1}{\mu_1} \cdot C_M^2 q, \tag{6.51}$$

where C_M is given by (6.42). Integration of (6.51) from τ to t yields (6.44), with $c_3 = \frac{1}{\mu_1} C_M^2$, which serves to establish the lemma. □

With the aid of Lemma 6.3 we are now able to prove the following

Theorem 6.2 (The Squeezing Property). *Let $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n$ be the first N eigenfunctions of the operator $\boldsymbol{A}$ and $\boldsymbol{P}_N : \boldsymbol{H}_{per} \to \text{span}\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}\}$ the projection operator; set $\boldsymbol{Q}_N = \boldsymbol{I} - \boldsymbol{P}_N$ with $\boldsymbol{I}$ the identity map on $\boldsymbol{H}_{per}$. Let $\gamma > 0$ be given. Then for $\boldsymbol{u}, \boldsymbol{v} \in D(\boldsymbol{A}^{1/4})$, solutions of (6.43a,b) satisfying (6.5) for $t \in [0, T]$, with $M > 0$ as in the statement of the Lipschitz property (6.3), $\exists c_i$, $i = 1, 2$, depending only on γ, M, T, $\boldsymbol{f}$, μ_0, μ_1, ϵ, and Ω, such that for every N, and each $t \in [0, T]$, either (6.6) or (6.7) holds. In addition, with C_M defined as in (6.42),*

$$\left|\left|\boldsymbol{A}^{-1/4}(\boldsymbol{u}(t) - \boldsymbol{v}(t))\right|\right|_{\boldsymbol{L}^2(\Omega)} \le \exp\left(\frac{C_M^2}{\mu_1} t\right) \left|\left|\boldsymbol{A}^{-1/4}(\boldsymbol{u}(0) - \boldsymbol{v}(0))\right|\right|_{\boldsymbol{L}^2(\Omega)}. \tag{6.52}$$

Proof. We begin by taking the scalar product in $\boldsymbol{L}^2$ of (6.47) with $\boldsymbol{A}^{-1/4}\boldsymbol{w}$; using the Cauchy-Schwarz inequality and the Lipschitz property (6.3), we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left|\left|\boldsymbol{A}^{-1/4}\boldsymbol{w}\right|\right|_{\boldsymbol{L}^2(\Omega)}^2 &+ 2\mu_1\left|\left|\boldsymbol{A}^{1/4}\boldsymbol{w}\right|\right|_{\boldsymbol{L}^2(\Omega)}^2 \\ &= -\Big(\boldsymbol{A}^{-1/4}(\boldsymbol{R}(\boldsymbol{u}) - \boldsymbol{R}(\boldsymbol{v})), \boldsymbol{A}^{-1/4}\boldsymbol{w}\Big)_{\boldsymbol{L}^2(\Omega)} \\ &\le \left|\left|\boldsymbol{A}^{-1/4}(\boldsymbol{R}(\boldsymbol{u}) - \boldsymbol{R}(\boldsymbol{v}))\right|\right|_{\boldsymbol{L}^2(\Omega)} \cdot \left|\left|\boldsymbol{A}^{-1/4}\boldsymbol{w}\right|\right|_{\boldsymbol{L}^2(\Omega)} \\ &\le C_M \left|\left|\boldsymbol{A}^{1/4}\boldsymbol{w}\right|\right|_{\boldsymbol{L}^2(\Omega)}\left|\left|\boldsymbol{A}^{-1/4}\boldsymbol{w}\right|\right|_{\boldsymbol{L}^2(\Omega)} \end{aligned} \tag{6.53}$$

with C_M given by (6.42). Employing the arithmetic-geometric mean inequality, we are led from (6.53) to the estimate

$$\begin{aligned} \frac{d}{dt}\left|\left|\boldsymbol{A}^{-1/4}\boldsymbol{w}\right|\right|_{\boldsymbol{L}^2(\Omega)}^2 &+ 4\mu_1\left|\left|\boldsymbol{A}^{1/4}\boldsymbol{w}\right|\right|_{\boldsymbol{L}^2(\Omega)}^2 \\ &\le 3\mu_1\left|\left|\boldsymbol{A}^{1/4}\boldsymbol{w}\right|\right|_{\boldsymbol{L}^2(\Omega)}^2 + \frac{4C_M^2}{3\mu_1}\left|\left|\boldsymbol{A}^{-1/4}\boldsymbol{w}\right|\right|_{\boldsymbol{L}^2(\Omega)}^2 \end{aligned} \tag{6.54}$$

from which it follows that

$$\frac{d}{dt}\left\|\boldsymbol{A}^{-1/4}\boldsymbol{w}\right\|_{\boldsymbol{L}^2(\Omega)}^2 + \left\|\boldsymbol{A}^{-1/4}\boldsymbol{w}\right\|_{\boldsymbol{L}^2(\Omega)}^2 \left(\frac{\mu_1\left\|\boldsymbol{A}^{1/4}\boldsymbol{w}\right\|_{\boldsymbol{L}^2(\Omega)}^2}{\left\|\boldsymbol{A}^{-1/4}\boldsymbol{w}\right\|_{\boldsymbol{L}^2(\Omega)}^2} - \frac{4C_M^2}{3\mu_1}\right) \leq 0. \tag{6.55}$$

By virtue of Lemma 6.3, however, for $0 < t < t_0 < T$ we have

$$\frac{\left\|\boldsymbol{A}^{1/4}\boldsymbol{w}(t)\right\|_{\boldsymbol{L}^2(\Omega)}^2}{\left\|\boldsymbol{A}^{-1/4}\boldsymbol{w}(t)\right\|_{\boldsymbol{L}^2(\Omega)}^2} \geq \frac{\left\|\boldsymbol{A}^{1/4}\boldsymbol{w}(t_0)\right\|_{\boldsymbol{L}^2(\Omega)}^2}{\left\|\boldsymbol{A}^{-1/4}\boldsymbol{w}(t_0)\right\|_{\boldsymbol{L}^2(\Omega)}^2} \exp(-c_3(t_0 - t) \geq \eta \exp(-c_3 t_0) \tag{6.56}$$

with

$$\eta = \left\|\boldsymbol{A}^{1/4}\boldsymbol{w}(t_0)\right\|_{\boldsymbol{L}^2(\Omega)}^2 \Big/ \left\|\boldsymbol{A}^{-1/4}\boldsymbol{w}(t_0)\right\|_{\boldsymbol{L}^2(\Omega)}^2. \tag{6.57}$$

Combining (6.55) and (6.56), we find that

$$\frac{d}{dt}\left\|\boldsymbol{A}^{-1/4}\boldsymbol{w}(t)\right\|_{\boldsymbol{L}^2(\Omega)}^2 + \left\|\boldsymbol{A}^{-1/4}\boldsymbol{w}(t)\right\|_{\boldsymbol{L}^2(\Omega)}^2 \left(\mu_1 \eta \exp(-c_3 t_0) - \frac{4C_M^2}{3\mu_1}\right) \leq 0 \tag{6.58}$$

so that upon integrating from zero to t_0 we obtain

$$\left\|\boldsymbol{A}^{-1/4}\boldsymbol{w}(t_0)\right\|_{\boldsymbol{L}^2(\Omega)}^2 \leq \left\|\boldsymbol{A}^{-1/4}\boldsymbol{w}(0)\right\|_{\boldsymbol{L}^2(\Omega)}^2 \exp\left\{-\mu_1 \eta t_0 \exp(-c_3 t_0) + \frac{4C_M^2}{3\mu_1} t_0\right\}. \tag{6.59}$$

We now consider the cases

$$\left\|\boldsymbol{Q}_N \boldsymbol{A}^{-1/4}\boldsymbol{w}(t_0)\right\|_{\boldsymbol{L}^2(\Omega)} > \gamma \left\|\boldsymbol{P}_N \boldsymbol{A}^{-1/4}\boldsymbol{w}(t_0)\right\|_{\boldsymbol{L}^2(\Omega)} \tag{6.60a}$$

and

$$\left\|\boldsymbol{Q}_N \boldsymbol{A}^{-1/4}\boldsymbol{w}(t_0)\right\|_{\boldsymbol{L}^2(\Omega)} \leq \gamma \left\|\boldsymbol{P}_N \boldsymbol{A}^{-1/4}\boldsymbol{w}(t_0)\right\|_{\boldsymbol{L}^2(\Omega)}. \tag{6.60b}$$

In view of the statement of Theorem 6.2, i.e., either (6.6) holds or (6.7) does, it is only necessary to consider what happens if (6.60a) applies; in this case

$$\begin{aligned}\eta &= \frac{\left|\left|\boldsymbol{A}^{1/4}\boldsymbol{w}(t_0)\right|\right|_{\boldsymbol{L}^2(\Omega)}^{\;\;2}}{\left|\left|\boldsymbol{A}^{-1/4}\boldsymbol{w}(t_0)\right|\right|_{\boldsymbol{L}^2(\Omega)}^{\;\;2}} \\ &= \frac{\left|\left|\boldsymbol{P}_N\boldsymbol{A}^{1/4}\boldsymbol{w}(t_0)\right|\right|_{\boldsymbol{L}^2(\Omega)}^{\;\;2} + \left|\left|\boldsymbol{Q}_N\boldsymbol{A}^{1/4}\boldsymbol{w}(t_0)\right|\right|_{\boldsymbol{L}^2(\Omega)}^{\;\;2}}{\left|\left|\boldsymbol{P}_N\boldsymbol{A}^{-1/4}\boldsymbol{w}(t_0)\right|\right|_{\boldsymbol{L}^2(\Omega)}^{\;\;2} + \left|\left|\boldsymbol{Q}_N\boldsymbol{A}^{-1/4}\boldsymbol{w}(t_0)\right|\right|_{\boldsymbol{L}^2(\Omega)}^{\;\;2}} \\ &\geq \frac{\left|\left|\boldsymbol{Q}_N\boldsymbol{A}^{1/4}\boldsymbol{w}(t_0)\right|\right|_{\boldsymbol{L}^2(\Omega)}^{\;\;2}}{\left(1+\frac{1}{\gamma}\right)\left|\left|\boldsymbol{Q}_N\boldsymbol{A}^{-1/4}\boldsymbol{w}(t_0)\right|\right|_{\boldsymbol{L}^2(\Omega)}^{\;\;2}} \\ &\geq \frac{\gamma}{1+\gamma}\cdot\frac{\lambda_{N+1}^{1/2}}{\lambda_{N+1}^{-1/2}} \equiv \frac{\gamma}{1+\gamma}\lambda_{N+1},\end{aligned} \tag{6.61}$$

λ_{N+1} being the $(N+1)$st eigenvalue of $\boldsymbol{A}$. Employing this last lower bound for η in (6.59), we are led to the estimate

$$||\boldsymbol{A}^{-1/4}\boldsymbol{w}(t_0)||_{\boldsymbol{L}^2(\Omega)}^{\;\;2} \leq ||\boldsymbol{A}^{-1/4}\boldsymbol{w}(0)||_{\boldsymbol{L}^2(\Omega)}^{\;\;2}\exp\left\{-\frac{\gamma}{1+\gamma}\mu_1\lambda_{N+1}t_0\exp(-c_3T)+\frac{4C_M^2}{3\mu_1}T\right\} \tag{6.62}$$

as $t_0 < T$. Replacing t_0 by $t < T$ in (6.62), we obtain (6.7) with

$$c_1 = \exp\left(\frac{4C_M^2}{3\mu_1}T\right) \text{ and } c_2 = \frac{\gamma}{\gamma+1}\exp(-c_3T). \tag{6.63}$$

To complete the proof of Theorem 6.2 it remains only to establish (6.52). However, by (6.53),

$$\begin{aligned}\frac{d}{dt}\left|\left|\boldsymbol{A}^{-1/4}\boldsymbol{w}\right|\right|_{\boldsymbol{L}^2(\Omega)}^{\;\;2} + 4\mu_1\left|\left|\boldsymbol{A}^{1/4}\boldsymbol{w}\right|\right|_{\boldsymbol{L}^2(\Omega)}^{\;\;2} &\leq 2C_M\left|\left|\boldsymbol{A}^{1/4}\boldsymbol{w}\right|\right|_{\boldsymbol{L}^2(\Omega)}\left|\left|\boldsymbol{A}^{-1/4}\boldsymbol{w}\right|\right|_{\boldsymbol{L}^2(\Omega)} \\ &\leq 4\mu_1\left|\left|\boldsymbol{A}^{1/4}\boldsymbol{w}\right|\right|_{\boldsymbol{L}^2(\Omega)}^{\;\;2} + \frac{C_M^2}{\mu_1}\left|\left|\boldsymbol{A}^{-1/4}\boldsymbol{w}\right|\right|_{\boldsymbol{L}^2(\Omega)}^{\;\;2},\end{aligned}$$

so that

$$\frac{d}{dt}\left|\left|\boldsymbol{A}^{-1/4}\boldsymbol{w}\right|\right|_{\boldsymbol{L}^2(\Omega)}^{\;\;2} \leq \frac{C_M^2}{\mu_1}\left|\left|\boldsymbol{A}^{-1/4}\boldsymbol{w}\right|\right|_{\boldsymbol{L}^2(\Omega)}^{\;\;2}. \tag{6.64}$$

The estimate (6.52) now follows by integrating (6.64) and using the definition of $\boldsymbol{w}(t)$. □

6.2.5 A Fixed Point Theorem

In this subsection we will examine both the structure of the space of Lipschitz maps $H_{b,l}$ $(b > 0,\ l > 0)$ consisting of those $\boldsymbol{\phi} : \boldsymbol{P}_N D(\boldsymbol{A}^{1/4}) \to \boldsymbol{Q}_N D(\boldsymbol{A}^{1/4})$ that satisfy (6.10a,b), as well as the properties of the mapping $\boldsymbol{T}$ that associates with each $\boldsymbol{\phi} \in H_{b,l}$ the function $\boldsymbol{T\phi}$, defined on $\boldsymbol{P}_N D(\boldsymbol{A}^{1/4})$, which is given by the mapping (6.21). The specific structure of the map $\boldsymbol{T\phi} : \boldsymbol{p}_0 \to \boldsymbol{q}_0(0; \boldsymbol{\phi}, \boldsymbol{p}_0)$ is delineated in (6.22), where $\boldsymbol{F}(\boldsymbol{u}) = \Theta_{r_1}\left(\left\|\boldsymbol{A}^{1/4}\boldsymbol{u}\right\|_{\boldsymbol{L}^2(\Omega)}\right)\boldsymbol{R}(\boldsymbol{u})$.

As was already indicated in Sect. 6.2.2, it is a straightforward matter to prove the existence and uniqueness of solutions for the modified initial-value problem (6.9), (5.361b), with $\boldsymbol{u}_0 \in \boldsymbol{H}_{per}$. Also, the absorbing property of the modified problem may be easily demonstrated by taking the inner-product of (6.9) with $\boldsymbol{A}^{1/4}\boldsymbol{u}$; for $\left\|\boldsymbol{A}^{1/4}\boldsymbol{u}\right\|_{\boldsymbol{L}^2(\Omega)} \geq 2r_1$, $r_1 > 0$, we obtain, with $\Theta_{r_1}(r) = \Theta(r/r_1)$ and $\Theta : R^+ \to [0,1]$ the smooth cut-off function given by (6.8),

$$\frac{1}{2}\frac{d}{dt}\left(\boldsymbol{A}^{1/4}\boldsymbol{u}, \boldsymbol{A}^{1/4}\boldsymbol{u}\right)_{\boldsymbol{L}^2(\Omega)} + 2\mu_1\left(\boldsymbol{A}^{3/4}\boldsymbol{u}, \boldsymbol{A}^{3/4}\boldsymbol{u}\right)_{\boldsymbol{L}^2(\Omega)} = 0 \tag{6.65}$$

because $\Theta_{r_1}\left(\left\|\boldsymbol{A}^{1/4}\boldsymbol{u}\right\|_{\boldsymbol{L}^2(\Omega)}\right) = 0$ for $\left\|\boldsymbol{A}^{1/4}\boldsymbol{u}\right\|_{\boldsymbol{L}^2(\Omega)} \geq 2r_1$. Inasmuch as $\left(\boldsymbol{A}^{3/4}\boldsymbol{u}, \boldsymbol{A}^{3/4}\boldsymbol{u}\right)_{\boldsymbol{L}^2(\Omega)} \geq \lambda_1\left(\boldsymbol{A}^{1/4}\boldsymbol{u}, \boldsymbol{A}^{1/4}\boldsymbol{u}\right)_{\boldsymbol{L}^2(\Omega)}$, with $\lambda_1 > 0$ the smallest eigenvalue of $\boldsymbol{A}$, we have

$$\frac{1}{2}\frac{d}{dt}\left\|\boldsymbol{A}^{1/4}\boldsymbol{u}\right\|_{\boldsymbol{L}^2(\Omega)}^2 + 2\mu_1\lambda_1\left\|\boldsymbol{A}^{1/4}\boldsymbol{u}\right\|_{\boldsymbol{L}^2(\Omega)}^2 \leq 0. \tag{6.66}$$

Thus,

$$\left\|\boldsymbol{A}^{1/4}\boldsymbol{u}(t)\right\|_{\boldsymbol{L}^2(\Omega)}^2 \leq \left\|\boldsymbol{A}^{1/4}\boldsymbol{u}(0)\right\|_{\boldsymbol{L}^2(\Omega)}^2 e^{-4\mu_1\lambda_1 t}, \quad t > 0. \tag{6.67}$$

Therefore, if $\left\|\boldsymbol{A}^{1/4}\boldsymbol{u}_0\right\|_{\boldsymbol{L}^2(\Omega)} > r_2$, where $r_2 \geq 2r_1$, the orbit of $\boldsymbol{u}(t)$ will converge exponentially in $D(\boldsymbol{A}^{1/4})$ to the ball B_{r_2}, while if $\left\|\boldsymbol{A}^{1/4}\boldsymbol{u}_0\right\|_{\boldsymbol{L}^2(\Omega)} \leq r_2$, then $\boldsymbol{u}(t)$ will stay inside the ball B_{r_2} for all $t > 0$. However, $\Theta_{r_1}\left(\left\|\boldsymbol{A}^{1/4}\boldsymbol{u}\right\|_{\boldsymbol{L}^2(\Omega)}\right) = 1$, for $\left\|\boldsymbol{A}^{1/4}\boldsymbol{u}\right\|_{\boldsymbol{L}^2(\Omega)} \leq r_1$; thus, the original equation (5.361a) and the modified equation (6.9) are identical in a neighborhood of the global attractor and the dynamics of (5.368) are exactly represented by those of (6.9) after a sufficiently large time.

We now state and prove a series of five technical lemmas leading to the proof of the main result in this subsection, namely, Theorem 6.3; this result asserts that, under certain conditions on the eigenvalues λ_j of $\boldsymbol{A}$ (including the spectral gap condition of Appendix C), $\boldsymbol{T} : H_{b,l} \overset{\text{into}}{\longrightarrow} H_{b,l}$ and is, in fact, a strict contraction on $H_{b,l}$.

Lemma 6.4. *For $\boldsymbol{\phi} \in H_{b,l}$ we have*

$$\operatorname{supp} \boldsymbol{T\phi} \subseteq \{\boldsymbol{p} \in \boldsymbol{P}_N D(\boldsymbol{A}^{1/4}) \mid \left|\left|\boldsymbol{A}^{1/4}\boldsymbol{p}_2\right|\right|_{\boldsymbol{L}^2(\Omega)} \leq 2r_1\}. \tag{6.68}$$

Proof. the proof follows that of Lemma 3.1, Sect. 3.2 of [Te4], Chap. VIII, almost without change. □

Lemma 6.5. *Let $\boldsymbol{\phi} \in H_{b,l}$ and $\boldsymbol{p}_1, \boldsymbol{p}_2 \in \boldsymbol{P}_N D(\boldsymbol{A}^{1/4})$. If $\boldsymbol{u}_i = \boldsymbol{p}_i + \boldsymbol{\phi}(\boldsymbol{p}_i)$, $i = 1, 2$, then $\exists M_1, M_2 > 0$ such that*

$$\left|\left|\boldsymbol{A}^{-1/4}\boldsymbol{F}(\boldsymbol{u}_1)\right|\right|_{\boldsymbol{L}^2(\Omega)} \leq M_1, \tag{6.69a}$$

$$\left|\left|\boldsymbol{A}^{-1/4}(\boldsymbol{F}(\boldsymbol{u}_1) - \boldsymbol{F}(\boldsymbol{u}_2))\right|\right|_{\boldsymbol{L}^2(\Omega)} \leq M_2(1+l)\left|\left|\boldsymbol{A}^{1/4}(\boldsymbol{p}_1 - \boldsymbol{p}_2)\right|\right|_{\boldsymbol{L}^2(\Omega)}. \tag{6.69b}$$

Proof. The stated results are a direct consequence of the Lipschitz property (6.3) and standard lemmas in [Te4] as follows: we want to establish (6.69a,b), where $\boldsymbol{u}_i = \boldsymbol{p}_i\boldsymbol{\phi}(\boldsymbol{p}_i)$, $i = 1, 2$ with $\boldsymbol{\phi} \in H_{b,l}$ and $\boldsymbol{p}_1, \boldsymbol{p}_2 \in \boldsymbol{P}_N D(\boldsymbol{A}^{1/4})$. However, as a consequence of Lemmas 2.1 and 2.2 of [Te4], Chap. VIII, and the Lipschitz property (6.3) it follows that $\exists M_i > 0$, $i = 1, 2$ such that

$$\left|\left|\boldsymbol{A}^{-1/4}\boldsymbol{F}(\boldsymbol{u}_1)\right|\right|_{\boldsymbol{L}^2(\Omega)} \leq M_1, \tag{6.70a}$$

$$\left|\left|\boldsymbol{A}^{-1/4}(\boldsymbol{F}(\boldsymbol{u}_1) - \boldsymbol{F}(\boldsymbol{u}_2))\right|\right|_{\boldsymbol{L}^2(\Omega)} \leq M_2\left|\left|\boldsymbol{A}^{1/4}(\boldsymbol{u}_1 - \boldsymbol{u}_2)\right|\right|_{\boldsymbol{L}^2(\Omega)}, \tag{6.70b}$$

so that (6.69a) follows. For (6.69b) we use (6.70b), and the definition of $H_{b,l}$, which implies that

$$\begin{aligned}\left|\left|\boldsymbol{A}^{1/4}(\boldsymbol{u}_1 - \boldsymbol{u}_2)\right|\right|_{\boldsymbol{L}^2(\Omega)} &\leq \left|\left|\boldsymbol{A}^{1/4}(\boldsymbol{p}_1 - \boldsymbol{p}_2)\right|\right|_{\boldsymbol{L}^2(\Omega)} + \left|\left|\boldsymbol{A}^{1/4}(\boldsymbol{\phi}(\boldsymbol{p}_1) - \boldsymbol{\phi}(\boldsymbol{p}_2))\right|\right|_{\boldsymbol{L}^2(\Omega)} \\ &\leq (1+l)\left|\left|\boldsymbol{A}^{1/4}(\boldsymbol{p}_1 - \boldsymbol{p}_2)\right|\right|_{\boldsymbol{L}^2(\Omega)}.\end{aligned} \tag{6.71}$$

□

Lemma 6.6. *If $\boldsymbol{p}_0 \in \boldsymbol{P}_N D(\boldsymbol{A}^{1/4})$, then*

$$\boldsymbol{T\phi}(\boldsymbol{p}_0) \in \boldsymbol{Q}_N D(\boldsymbol{A}^{1/4}) \text{ and } \left|\left|\boldsymbol{A}^{1/4}[\boldsymbol{T\phi}(\boldsymbol{p}_0)]\right|\right|_{\boldsymbol{L}^2(\Omega)} \leq b' \tag{6.72}$$

with $b' = e^{-1/2}\mu_1^{-1}\lambda_{N+1}^{-1/2} < b$ (for $\lambda_{N+1}^{1/2}$ sufficiently large).

Proof. Suppose that $\boldsymbol{p}_0 \in \boldsymbol{P}_N D(\boldsymbol{A}^{1/4})$. From the definition of the mapping $\boldsymbol{T}$ it is clear that $\boldsymbol{T}\boldsymbol{\phi}(\boldsymbol{p}_0) \in \boldsymbol{Q}_N D(\boldsymbol{A}^{1/4})$. Also, as a consequence of (6.22) and (6.69a),

$$
\begin{aligned}
\left|\left|\boldsymbol{A}^{1/4}(\boldsymbol{T}\boldsymbol{\phi})(\boldsymbol{p}_0)\right|\right|_{\boldsymbol{L}^2(\Omega)} &\leq \int_{-\infty}^{0} \left|\left|\boldsymbol{A}^{1/4}e^{2\mu_1\boldsymbol{A}\tau}\boldsymbol{Q}_N\boldsymbol{F}(\boldsymbol{p}(\tau)+\boldsymbol{\phi}(\boldsymbol{p}(\tau)))\right|\right|_{\boldsymbol{L}^2(\Omega)} d\tau \\
&\leq (2\mu_1)^{1/2}\int_{-\infty}^{0}\{\left|(2\mu_1\boldsymbol{A}\boldsymbol{Q}_N)^{1/2}e^{2\mu_1\boldsymbol{A}\tau}\right|_{\boldsymbol{\mathcal{L}}(\boldsymbol{Q}_N,\boldsymbol{H}_{per})} \\
&\qquad \times \left|\left|\boldsymbol{A}^{-1/4}\boldsymbol{F}(\boldsymbol{p}(\tau)+\boldsymbol{\phi}(\tau))\right|\right|_{\boldsymbol{L}^2(\Omega)}\} d\tau \\
&\leq (2\mu_1)^{1/2}M_1\int_{-\infty}^{0}\left|(2\mu_1\boldsymbol{A}\boldsymbol{Q}_N)^{1/2}e^{2\mu_1\boldsymbol{A}\tau}\right|_{\boldsymbol{\mathcal{L}}(\boldsymbol{Q}_N,\boldsymbol{H}_{per})} d\tau.
\end{aligned}
\tag{6.73}
$$

However, as a direct consequence of [Te4], Lemma 3.2, Chap. VIII, we have the following result: Let $\delta \in R^1$, and $\tau < 0$, and set $k_2(\delta) = \delta^{\delta}e^{-\delta}$ and

$$
k_3(\delta) = \begin{cases} 1, & \text{if } \delta < 0, \\ e^{-\delta} + \dfrac{k_2(\delta)}{1-\delta}\cdot\delta^{1-\delta}, & \text{if } 0 \leq \delta < 1. \end{cases}
$$

Then $\left|\left|(\boldsymbol{A}\boldsymbol{Q}_N)^{\delta}e^{\tau\boldsymbol{A}\boldsymbol{Q}_N}\right|\right|$ in $\boldsymbol{\mathcal{L}}(\boldsymbol{Q}_N, \boldsymbol{H}_{per})$ is bounded by

$$
\begin{cases} k_2(\delta)|\tau|^{-\delta}, & \text{if } -\delta/\lambda_{N+1} \leq \tau < 0, \\ \lambda_{N+1}^{\delta}e^{\tau\lambda_{N+1}}, & \text{if } \tau < -\delta/(N+1), \end{cases}
\tag{6.74a}
$$

and, moreover, if $\delta < 1$,

$$
\int_{-\infty}^{0}\left|\left|(\boldsymbol{A}\boldsymbol{Q}_N)^{\delta}e^{\tau\boldsymbol{A}\boldsymbol{Q}_N}\right|\right|_{\boldsymbol{\mathcal{L}}(\boldsymbol{Q}_N,\boldsymbol{H}_{per})} d\tau \leq k_3(\delta)\lambda_{N+1}^{\delta-1}.
\tag{6.74b}
$$

Applying (6.74b) with $\delta = \dfrac{1}{2}$ to the last estimate in (6.73), we have, by virtue of the definitions of $k_2(\delta)$, $k_3(\delta)$,

$$
\begin{aligned}
\left|\left|\boldsymbol{A}^{1/4}(\boldsymbol{T}\boldsymbol{\phi}(\boldsymbol{p}_0))\right|\right|_{\boldsymbol{L}^2(\Omega)} &\leq (2\mu_1)^{-1/2}k_3\left(\frac{1}{2}\right)(2\mu_1\lambda_{N+1})^{-1/2}M_1 \\
&\leq e^{-1/2}\mu_1^{-1}M_1\lambda_{N+1}^{-1/2},
\end{aligned}
\tag{6.75}
$$

which completes the proof of (6.72) with $b' = e^{-1/2}\mu_1^{-1}\lambda_{N+1}^{-1/2}$; note that $b' < b$ for λ_{N+1} sufficiently large. □

Lemma 6.7. *Assume that*

$$\sigma_N = 2\mu_1(\lambda_{N+1} - \lambda_N) - M_2(1+l)\lambda_N^{1/2} > 0. \tag{6.76}$$

Then for $\boldsymbol{\phi} \in H_{b,l}$ *and* $\boldsymbol{p}_{01}, \boldsymbol{p}_{02} \in \boldsymbol{P}_N D(\boldsymbol{A}^{1/4})$*, we have*

$$\left|\left|\boldsymbol{A}^{1/4}(\boldsymbol{T\phi}(\boldsymbol{p}_{01}) - \boldsymbol{T\phi}(\boldsymbol{p}_{02}))\right|\right|_{\boldsymbol{L}^2(\Omega)} \leq l' \left|\left|\boldsymbol{A}^{1/4}(\boldsymbol{p}_{01} - \boldsymbol{p}_{02})\right|\right|_{\boldsymbol{L}^2(\Omega)}, \tag{6.77}$$

where

$$l' = M_2(1+l)\lambda_{N+1}^{-1/2}[(2\mu_1)^{-1} + (2\mu_1 - r_N\xi_N)^{-1}]e^{-1/2}\exp\left(\frac{r_N\xi_N}{4\mu_1}\right), \tag{6.78a}$$

$$r_n = \lambda_N/\lambda_{N+1}, \tag{6.78b}$$

$$\xi_N = 2\mu_1 + M_2(1+l)\lambda_N^{-1/2}. \tag{6.78c}$$

Proof. Let $\boldsymbol{\phi}$ be a fixed but arbitrary element in $H_{b,l}$ and $\boldsymbol{p}_1 = \boldsymbol{p}_1(t)$, $\boldsymbol{p}_2 = \boldsymbol{p}_2(t)$ solutions of the initial-value problems

$$\begin{cases} \dfrac{d\boldsymbol{p}_1}{dt} + 2\mu_1\boldsymbol{A}\boldsymbol{p}_1 + \boldsymbol{P}_N\boldsymbol{F}(\boldsymbol{u}_1) = \boldsymbol{0}, \\ \boldsymbol{p}_1(0) = \boldsymbol{p}_{01}, \end{cases} \tag{6.79a}$$

$$\begin{cases} \dfrac{d\boldsymbol{p}_2}{dt} + 2\mu_1\boldsymbol{A}\boldsymbol{p}_2 + \boldsymbol{P}_N\boldsymbol{F}(\boldsymbol{u}_2) = \boldsymbol{0}, \\ \boldsymbol{p}_2(0) = \boldsymbol{p}_{02}, \end{cases} \tag{6.79b}$$

where $\boldsymbol{u}_i = \boldsymbol{p}_i + \boldsymbol{\phi}(\boldsymbol{p}_i)$, $i = 1, 2$. Setting $\hat{\boldsymbol{p}}(t) = \boldsymbol{p}_1(t) - \boldsymbol{p}_2(t)$, we have

$$\begin{cases} \dfrac{d\hat{\boldsymbol{p}}_2}{dt} + 2\mu_1\boldsymbol{A}\hat{\boldsymbol{p}} + \boldsymbol{P}_N(\boldsymbol{F}(\boldsymbol{u}_1) - \boldsymbol{F}(\boldsymbol{u}_2)) = \boldsymbol{0}, \\ \hat{\boldsymbol{p}}(0) = \boldsymbol{p}_{01} - \boldsymbol{p}_{02}. \end{cases} \tag{6.80}$$

Taking the inner product of the equation for $\hat{\boldsymbol{p}}(t)$, above, with $\boldsymbol{A}^{1/2}\hat{\boldsymbol{p}}$, and applying Lemma 6.5 (specifically, (6.69b)), we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}||\boldsymbol{A}^{1/4}\hat{\boldsymbol{p}}||_{\boldsymbol{L}^2(\Omega)}{}^2 + 2\mu_1||\boldsymbol{A}^{3/4}\hat{\boldsymbol{p}}||_{\boldsymbol{L}^2(\Omega)}{}^2 &\geq -||\boldsymbol{A}^{-1/4}(\boldsymbol{F}(\boldsymbol{u}_1) - \boldsymbol{F}(\boldsymbol{u}_2))||_{\boldsymbol{L}^2(\Omega)} \cdot ||\boldsymbol{A}^{3/4}\hat{\boldsymbol{p}}||_{\boldsymbol{L}^2(\Omega)} \\ &\geq -M_2(1+l)||\boldsymbol{A}^{1/4}\hat{\boldsymbol{p}}||_{\boldsymbol{L}^2(\Omega)} \cdot ||\boldsymbol{A}^{3/4}\hat{\boldsymbol{p}}||_{\boldsymbol{L}^2(\Omega)}. \end{aligned} \tag{6.81}$$

However,

$$\left|\left|\boldsymbol{A}^{3/4}\hat{\boldsymbol{p}}\right|\right|_{\boldsymbol{L}^2(\Omega)} = \left|\left|\boldsymbol{A}^{1/2}\boldsymbol{A}^{1/4}\hat{\boldsymbol{p}}\right|\right|_{\boldsymbol{L}^2(\Omega)} \leq \lambda_N^{1/2}\left|\left|\boldsymbol{A}^{1/4}\hat{\boldsymbol{p}}\right|\right|_{\boldsymbol{L}^2(\Omega)}$$

so

$$||\boldsymbol{A}^{1/4}\hat{\boldsymbol{p}}||_{\boldsymbol{L}^2(\Omega)}\frac{d}{dt}||\boldsymbol{A}^{1/4}\hat{\boldsymbol{p}}||_{\boldsymbol{L}^2(\Omega)}+2\mu_1\lambda_N||\boldsymbol{A}^{1/4}\hat{\boldsymbol{p}}_N||_{\boldsymbol{L}^2(\Omega)}{}^2 \geq -M_2(1+l)\lambda_N^{1/2}||\boldsymbol{A}^{1/4}\hat{\boldsymbol{p}}||_{\boldsymbol{L}^2(\Omega)}{}^2 \tag{6.82}$$

or

$$\frac{d}{dt}\left|\left|\boldsymbol{A}^{1/4}\hat{\boldsymbol{p}}\right|\right|_{\boldsymbol{L}^2(\Omega)}+(2\mu_1\lambda_N+M_2(1+l)\lambda_N^{1/2})\left|\left|\boldsymbol{A}^{1/4}\hat{\boldsymbol{p}}\right|\right|_{\boldsymbol{L}^2(\Omega)} \geq 0. \tag{6.83}$$

From (6.83) we easily deduce that, for $\tau \leq 0$,

$$\left|\left|\boldsymbol{A}^{1/4}\hat{\boldsymbol{p}}(\tau)\right|\right|_{\boldsymbol{L}^2(\Omega)} \leq \left|\left|\boldsymbol{A}^{1/4}\hat{\boldsymbol{p}}(0)\right|\right|_{\boldsymbol{L}^2(\Omega)}\exp(-\tau[2\mu_1\lambda_N+M_2(1+l)\lambda_N^{1/2}]). \tag{6.84}$$

Next, using the Lipschitz condition relative to $\boldsymbol{F}(\boldsymbol{u})$, which is given by (6.3), we estimate as follows:

$$\begin{aligned}
&\left|\left|\boldsymbol{A}^{1/4}(\boldsymbol{T}\boldsymbol{\phi}(\boldsymbol{p}_{01})-\boldsymbol{T}\boldsymbol{\phi}(\boldsymbol{p}_{02}))\right|\right|_{\boldsymbol{L}^2(\Omega)}\\
&\quad\leq \int_{-\infty}^{0}\left|\left|\boldsymbol{A}^{1/4}e^{2\mu_1\boldsymbol{A}\tau}\boldsymbol{Q}_N(\boldsymbol{F}(\boldsymbol{u}_1)-\boldsymbol{F}(\boldsymbol{u}_2))\right|\right|_{\boldsymbol{L}^2(\Omega)}d\tau\\
&\quad\leq (2\mu_1)^{-1/2}\int_{-\infty}^{0}||(2\mu_1\boldsymbol{A}\boldsymbol{Q}_N)^{1/2}e^{2\mu_1\boldsymbol{A}\tau}||\cdot\left|\left|\boldsymbol{A}^{-1/4}(\boldsymbol{F}(\boldsymbol{u}_1)-\boldsymbol{F}(\boldsymbol{u}_2))\right|\right|_{\boldsymbol{L}^2(\Omega)}d\tau\\
&\quad\leq M_2(1+l)(2\mu_1)^{-1/2}\int_{-\infty}^{0}||(2\mu_1\boldsymbol{A}\boldsymbol{Q}_N)^{1/2}e^{2\mu_1\boldsymbol{A}\tau}||\cdot\left|\left|\boldsymbol{A}^{1/4}\hat{\boldsymbol{p}}(\tau)\right|\right|_{\boldsymbol{L}^2(\Omega)}d\tau\\
&\quad\leq M_2(1+l)(2\mu_1)^{-1/2}\left|\left|\boldsymbol{A}^{1/4}\hat{\boldsymbol{p}}(0)\right|\right|_{\boldsymbol{L}^2(\Omega)}\int_{-\infty}^{0}||(2\mu_1\boldsymbol{A}\boldsymbol{Q}_N)^{1/2}e^{2\mu_1\boldsymbol{A}\tau}||e^{\tau\lambda_N\xi_N}dt,
\end{aligned} \tag{6.85}$$

where $\xi_N = 2\mu_1 + M_2(1+l)\lambda_N^{-1/2}$; for simplicity we have written $||2\mu_1\boldsymbol{A}\boldsymbol{Q}_N)^{1/2}e^{2\mu_1\boldsymbol{A}\tau}||$ instead of $|(2\mu_1\boldsymbol{A}\boldsymbol{Q}_N)^{1/2}e^{2\mu_1\boldsymbol{A}\tau}|_{\mathcal{L}(\boldsymbol{Q}_N,\boldsymbol{H}_{per})}$, and we have used (6.84), which is valid for $\tau \leq 0$. We now focus our attention on the integral

$$\int_{-\infty}^{0}||(2\mu_1\boldsymbol{A}\boldsymbol{Q}_N)^{1/2}e^{2\mu_1\boldsymbol{A}\tau}||e^{-\tau\lambda_N\xi_N}d\tau$$

in the last estimate of (6.85). By virtue of the bounds for $||(\boldsymbol{A}\boldsymbol{Q}_N)^{\delta}e^{\tau\boldsymbol{A}\boldsymbol{Q}_N}||$ which are given by (6.74a) we have, first of all, that

$$
\begin{aligned}
&\int_{-\infty}^{-1/(4\mu_1\lambda_{N+1})} \left|\left|(2\mu_1 \boldsymbol{A}\boldsymbol{Q}_N)^{1/2} e^{2\mu_1 \boldsymbol{A}\tau}\right|\right| e^{-\tau\lambda_N\xi_N}\,d\tau \\
&\quad \le \int_{-\infty}^{-1/(4\mu_1\lambda_{N+1})} (2\mu_1\lambda_{N+1})^{1/2} e^{2\mu_1\tau\lambda_{N+1}} e^{-\tau\lambda_N\xi_N}\,d\tau \\
&\quad \le \int_{-\infty}^{-1/(4\mu_1\lambda_{N+1})} (2\mu_1\lambda_{N+1})^{1/2} e^{-\tau\sigma_N}\,d\tau \\
&\quad \le (2\mu_1\lambda_{N+1})^{1/2} \frac{1}{\sigma_N} \exp\left[-\frac{\sigma_N}{4\mu_1\lambda_{N+1}}\right],
\end{aligned}
\tag{6.86}
$$

where σ_N is given by (6.76), which is, in fact, equivalent to

$$
\sigma_N = \lambda_{N+1}(2\mu_1 - r_N\xi_N), \quad r_N = \lambda_N/\lambda_{N+1}. \tag{6.87}
$$

Therefore,

$$
\begin{aligned}
&\int_{-\infty}^{-1/(4\mu_1\lambda_{N+1})} \left|\left|2\mu_1 \boldsymbol{A}\boldsymbol{Q}_N)^{1/2} e^{2\mu_1 \boldsymbol{A}\tau}\right|\right| e^{-\tau\lambda_N\xi_N}\,d\tau \\
&\quad \le (2\mu_1)^{1/2} \lambda_{N+1}^{-1/2} e^{-1/2} (2\mu_1 - r_N\xi_N)^{-1} \exp\left(\frac{r_N\xi_N}{4\mu_1}\right).
\end{aligned}
\tag{6.88}
$$

In a like fashion, we have, by again using the bounds for $\left|\left|(\boldsymbol{A}\boldsymbol{Q}_N)^\delta e^{\tau \boldsymbol{A}\boldsymbol{Q}_N}\right|\right|$ implied by (6.74a), the series of estimates

$$
\begin{aligned}
&\int_{-1/(4\mu_1\lambda_{N+1})}^{0} \left|\left|(2\mu_1 \boldsymbol{A}\boldsymbol{Q}_N)^{1/2} e^{2\mu_1 \boldsymbol{A}\tau}\right|\right| e^{-\tau\lambda_N\xi_N}\,d\tau \\
&\quad \le \int_{-1/(4\mu_1\lambda_{N+1})}^{0} k_2\left(\frac{1}{2}\right) |\tau|^{-1/2} e^{-\tau\lambda_N\xi_N}\,d\tau \\
&\quad \le (2e)^{-1/2} \exp\left(\frac{\lambda_N\xi_N}{4\mu_1\lambda_{N+1}}\right) \int_{-1/(4\mu_1\lambda_{N+1})}^{0} |\tau|^{-1/2}\,d\tau \\
&\quad = (2\mu_1)^{-1/2} e^{-1/2} \lambda_{N+1}^{-1/2} \exp\left(\frac{r_N\xi_N}{4\mu_1}\right).
\end{aligned}
\tag{6.89}
$$

Combining (6.88) with the last estimate in (6.89), we are led to the bound

$$
\begin{aligned}
&\int_{-\infty}^{0} \left|\left|(2\mu_1 \boldsymbol{A}\boldsymbol{Q}_N)^{1/2} e^{2\mu_1 \boldsymbol{A}\tau}\right|\right| e^{-\tau\lambda_N\xi_N}\,d\tau \\
&\quad \le [(2\mu_1)^{1/2}(2\mu_1 - r_N\xi_N)^{-1} + (2\mu_1)^{-1/2}] \lambda_{N+1}^{-1/2} e^{-1/2} \exp\left(\frac{r_N\xi_N}{4\mu_1}\right)
\end{aligned}
\tag{6.90}
$$

which, in conjunction with the last estimate in (6.85), i.e.,

$$\left|\left|A^{1/4}(T\phi(p_{01})-T\phi(p_{02})\right|\right|_{L^2(\Omega)}$$
$$\le M_2(1+l)(2\mu_1)^{-1/2}\left|\left|A^{1/4}\hat{p}(0)\right|\right|_{L^2(\Omega)}\int_{-\infty}^{0}||(2\mu_1 A Q_N)^{1/2}e^{2\mu_1 A\tau}||e^{-\tau\lambda_N\xi_N}\,d\tau$$

serves to establish (6.77) with l' given in (6.78a,b,c). □

Remarks. As a direct consequence of Lemma 6.7 it follows that $T\phi$, as defined by (6.22), belongs to the space $H_{b,l'}$.

Our last result in the current sequence is

Lemma 6.8. *Suppose that* $\sigma_N > 0$, *with* σ_N *defined by* (6.76)*. Then for* $\phi_1, \phi_2 \in H_{b,l}$ *and* $p_0 \in P_N(D(A^{1/4}))$ *we have*

$$\left|\left|A^{1/4}(T\phi_1(p_0)-T\phi_2(p_0))\right|\right|_{L^2(\Omega)} \le L||\phi_1-\phi_2||_* \tag{6.91}$$

with $L = \dfrac{M_2}{2\mu_1}(2e^{-1/2}\lambda_{N+1}^{-1/2}-\lambda_N^{-1/2}l')$ *and* $||\phi_1-\phi_2||_*$ *given by* (6.12)*.*

Proof. We begin by setting

$$p_i = p(t,\phi_i,p_0), \quad u_i = p_i + \phi_i(p_i) \tag{6.92}$$

for $i = 1,2$ and $\hat{p} = p_1 - p_2$. For $\hat{p}(t)$ the initial-value problem (6.80) is again applicable and, thus, so is the first estimate in (6.81). However, if we once again make use of Lemma 6.5, i.e., of (6.70b), as well as (6.71), we have

$$\begin{aligned}
\left|\left|A^{1/4}(F(u_1)-F(u_2))\right|\right|_{L^2(\Omega)} &\le M_2\left|\left|A^{1/4}u_1 - A^{1/4}u_2\right|\right|_{L^2(\Omega)}\\
&\le M_2\left(\left|\left|A^{1/4}(p_1-p_2)\right|\right|_{L^2(\Omega)} + \left|\left|A^{1/4}\phi_1(p_1)-A^{1/4}\phi_2(p_2)\right|\right|_{L^2(\Omega)}\right)\\
&\le M_2\left(\left|\left|A^{1/4}(p_1-p_2)\right|\right|_{L^2(\Omega)} + \left|\left|A^{1/4}\phi_1(p_1)-A^{1/4}\phi_1(p_1)\right|\right|_{L^2(\Omega)}\right.\\
&\qquad\left. + \left|\left|A^{1/4}\phi_1(p_2)-A^{1/4}\phi_2(p_2)\right|\right|_{L^2(\Omega)}\right)\\
&\le M_2\left[(1+l)\left|\left|A^{1/4}(p_1-p_2)\right|\right|_{L^2(\Omega)} + ||\phi_1-\phi_2||_*\right]\\
&= M_2\left[(1+l)\left|\left|A^{1/4}\hat{p}\right|\right|_{L^2(\Omega)} + ||\phi_1-\phi_2||_*\right]
\end{aligned} \tag{6.93}$$

where $||\phi_1-\phi_2||_*$ is given by (6.12).

Using, once more, the elementary estimate

$$\left|\left|\boldsymbol{A}^{3/4}\hat{\boldsymbol{p}}\right|\right|_{\boldsymbol{L}^2(\Omega)} = \left|\left|\boldsymbol{A}^{1/2}\boldsymbol{A}^{1/4}\hat{\boldsymbol{p}}\right|\right|_{\boldsymbol{L}^2(\Omega)} \leq \lambda_N^{1/2}\left|\left|\boldsymbol{A}^{1/4}\hat{\boldsymbol{p}}\right|\right|_{\boldsymbol{L}^2(\Omega)}$$

we can now combine the first estimate in (6.81) and the last estimate in (6.93) to produce the differential inequality

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left|\left|\boldsymbol{A}^{1/4}\hat{\boldsymbol{p}}\right|\right|_{\boldsymbol{L}^2(\Omega)}^2 + 2\mu_1\lambda_N\left|\left|\boldsymbol{A}^{1/4}\hat{\boldsymbol{p}}\right|\right|_{\boldsymbol{L}^2(\Omega)}^2 \\ \geq -M_2(1+l)\lambda_N^{1/2}\left|\left|\boldsymbol{A}^{1/4}\hat{\boldsymbol{p}}\right|\right|_{\boldsymbol{L}^2(\Omega)}^2 - M_2\lambda_N^{1/2}||\boldsymbol{\phi}_1-\boldsymbol{\phi}_2||_*\left|\left|\boldsymbol{A}^{1/4}\hat{\boldsymbol{p}}\right|\right|_{\boldsymbol{L}^2(\Omega)}.\end{aligned} \tag{6.94}$$

From (6.94) we easily obtain

$$\frac{d}{dt}\left|\left|\boldsymbol{A}^{1/4}\hat{\boldsymbol{p}}\right|\right|_{\boldsymbol{L}^2(\Omega)} + (2\mu_1\lambda_N + M_2(1+l)\lambda_N^{1/2})\left|\left|\boldsymbol{A}^{1/4}\hat{\boldsymbol{p}}\right|\right|_{\boldsymbol{L}^2(\Omega)} \geq -M_2\lambda_N^{1/2}||\boldsymbol{\phi}_1-\boldsymbol{\phi}_2||_*. \tag{6.95}$$

However, $\hat{\boldsymbol{p}}(0) = 0$; so integration of (6.95) from zero to $\tau < 0$ yields the estimate

$$\left|\left|\boldsymbol{A}^{1/4}\hat{\boldsymbol{p}}(\tau)\right|\right|_{\boldsymbol{L}^2(\Omega)} \leq M_2\lambda_N^{1/2}(\xi_N\lambda_N)^{-1}(\exp(-\xi_N\lambda_N\tau)-1)||\boldsymbol{\phi}_1-\boldsymbol{\phi}_2||_* \tag{6.96}$$

where $\xi_N = 2\mu_1 + M_2(1+l)\lambda_N^{-1/2}$. From (6.93) and (6.95) we now deduce the following sequence of estimates (using, once more, the explicit representation of $\boldsymbol{T}$ given by (6.22)):

$$\begin{aligned}&\left|\left|\boldsymbol{A}^{1/4}(\boldsymbol{T}\boldsymbol{\phi}_1(\boldsymbol{p}_0) - \boldsymbol{T}\boldsymbol{\phi}(\boldsymbol{p}_0))\right|\right|_{\boldsymbol{L}^2(\Omega)} \\
&\leq \int_{-\infty}^0 \left|\left|\boldsymbol{A}^{1/4}e^{2\mu_1\boldsymbol{A}\tau}\boldsymbol{Q}_N(\boldsymbol{F}(\boldsymbol{u}_1)-\boldsymbol{F}(\boldsymbol{u}_2))\right|\right|_{\boldsymbol{L}^2(\Omega)} d\tau \\
&\leq (2\mu_1)^{-1/2}\int_{-\infty}^0 ||2\mu_1\boldsymbol{A}\boldsymbol{Q}_N)^{1/2}e^{2\mu_1\boldsymbol{A}\tau}||\cdot\left|\left|\boldsymbol{A}^{-1/4}(\boldsymbol{F}(\boldsymbol{u}_1)-\boldsymbol{F}(\boldsymbol{u}_2))\right|\right|_{\boldsymbol{L}^2(\Omega)} d\tau \\
&\leq (2\mu_1)^{-1/2}M_2\int_{-\infty}^0 \Big\{||(2\mu_1\boldsymbol{A}\boldsymbol{Q}_N)^{1/2}e^{2\mu_1\boldsymbol{A}\tau}|| \\
&\qquad\qquad \times\left[(1+l)\left|\left|\boldsymbol{A}^{1/4}\hat{\boldsymbol{p}}(t)\right|\right|_{\boldsymbol{L}^2(\Omega)} + ||\boldsymbol{\phi}_1-\boldsymbol{\phi}_2||_*\right]\Big\}\, d\tau \\
&\leq (2\mu_1)^{-1/2}M_2||\boldsymbol{\phi}_1-\boldsymbol{\phi}_2||_* \\
&\quad \times\int_{-\infty}^0 ||(2\mu_1\boldsymbol{A}\boldsymbol{Q}_N)^{1/2}e^{2\mu_1\boldsymbol{A}\tau}||\left[1+(1+l)\frac{M_2}{2\mu_1}\lambda_N^{-1/2}e^{-\tau\lambda_N\xi_N}\right]d\tau\end{aligned}$$

$$\leq (2\mu_1)^{-1/2} M_2 ||\boldsymbol{\phi}_1 - \boldsymbol{\phi}_2||_* \\ \times \left[\int_{-\infty}^{0} \left|\left|(2\mu_1 \boldsymbol{A}\boldsymbol{Q}_N)^{1/2} e^{2\mu_1 \boldsymbol{A}\tau}\right|\right| d\tau \right. \\ \left. + \int_{-\infty}^{0} \left|\left|(2\mu_1 \boldsymbol{A}\boldsymbol{Q}_N)^{1/2} e^{2\mu_1 \boldsymbol{A}\tau}\right|\right| (1+l) \frac{M_2}{2\mu_1} \lambda_N^{-1/2} e^{-\tau\lambda_N \xi_N} d\tau \right].$$

Applying (6.90) and the bounds for $\left|\left|(\boldsymbol{A}\boldsymbol{Q}_N)^{\delta} e^{\tau \boldsymbol{A}\boldsymbol{Q}_N}\right|\right|$ expressed by (6.74a), to the last estimate above, we deduce that

$$\left|\left|\boldsymbol{A}^{1/4}(\boldsymbol{T}\boldsymbol{\phi}_1(\boldsymbol{p}_0) - \boldsymbol{T}\boldsymbol{\phi}_2(\boldsymbol{p}_0))\right|\right|_{\boldsymbol{L}^2(\Omega)} \\ \leq (2\mu_1)^{-1/2} M_2 ||\boldsymbol{\phi}_1 - \boldsymbol{\phi}_2||_* \\ \times \left[2e^{-1/2}(2\mu_1)^{-1/2} \lambda_{N+1}^{-1/2} + \left((2\mu_1)^{1/2}(2\mu_1 - \lambda_N \xi_N)^{-1} \right. \right. \\ \left. \left. + \ (2\mu_1^{-1/2})\right) e^{-1/2} \lambda_{N+1}^{-1/2} e^{r_N \xi_N/(4\mu_1)} \right] \tag{6.97}$$

or

$$\left|\left|\boldsymbol{A}^{1/4}(\boldsymbol{T}\boldsymbol{\phi}_1(\boldsymbol{p}_0) - \boldsymbol{T}\boldsymbol{\phi}_2(\boldsymbol{p}_0))\right|\right|_{\boldsymbol{L}^2(\Omega)} \\ \leq \frac{M_2}{2\mu_1}(2e^{-1/2}\lambda_{N+1}^{-1/2} + \lambda_N l')||\boldsymbol{\phi}_1 - \boldsymbol{\phi}_2||_* \\ = L||\boldsymbol{\phi}_1 - \boldsymbol{\phi}_2||_* \tag{6.98}$$

where

$$L = \frac{M_2}{2\mu_1}(2e^{-1/2}\lambda_{N+1}^{-1/2} + \lambda_N^{-1/2} l')$$

and l' is given by (6.78a). □

With Lemmas 6.4–6.8 in hand, we are now ready to state and prove the main result of this subsection, namely, that the map $\boldsymbol{T}$, as defined by (6.22), satisfies $\boldsymbol{T} : H_{b,l} \xrightarrow{\text{into}} H_{b,l}$ and is, in fact, a strict contraction on $H_{b,l}$.

Theorem 6.3. *Let $H_{b,l}$ be the space of Lipschitz maps $\boldsymbol{\phi} : \boldsymbol{P}_N D(\boldsymbol{A}^{1/4}) \to \boldsymbol{Q}_N D(\boldsymbol{A}^{1/4})$ that satisfy (6.10a,b) and (6.11), where $b > 0$, $l > 0$. Define the mapping $\boldsymbol{T}$ by (6.22) with $\boldsymbol{\phi} \in H_{b,l}$, $\boldsymbol{p}_0 \in \boldsymbol{P}_N D(\boldsymbol{A}^{1/4})$ and where $\boldsymbol{q}(0; \boldsymbol{\phi}, \boldsymbol{p}_0) \in \boldsymbol{Q}_N D(\boldsymbol{A}^{1/4})$ is the value at $t = 0$ of the unique continuous solution of (6.19) and $\boldsymbol{p}(t)$ is the unique solution of the initial-value problem (6.14). Then ∃ constants $k_1 > 0$, $k_2 > 0$ such that if*

(i) $\lambda_{N+1}^{1/2} - \lambda_N^{1/2} \geq k_1/2\mu_1$ *(Spectral Gap Condition),*
(ii) $\lambda_N^{1/2} \geq k_2/2\mu_1$,

then $\boldsymbol{T} : H_{b,l} \overset{into}{\longrightarrow} H_{b,l}$ *and is a strict contraction on* $H_{b,l}$.

Proof. Choose l so that $0 < l < 1$. We will show that if conditions (i) and (ii) of the theorem are satisfied with

$$k_1 = 2M_2(1+l)l^{-1}, \quad k_2 = 2M_2(2e^{-1/2} + l) \tag{6.99}$$

($M_2 > 0$ the constant in (6.70b)) then $\sigma_N > 0$ (σ_N given by (6.76)), l' as defined by (6.78a,b,c) satisfies $l' < l$, and $L \leq \frac{1}{2}$ ($L > 0$ the constant appearing in (6.91)). It will then follow directly from (6.91) that $\boldsymbol{T}$ is a strict contraction on $H_{b,l}$. We note that the positivity of σ_N will enable us to deduce that the estimate (6.98) is valid while $l' < l$, coupled with (6.77), yields the fact that $\boldsymbol{T}\boldsymbol{\phi} \in H_{b,l}$ if $\boldsymbol{\phi} \in H_{b,l}$. We begin with the sign of σ_N and note the inequality

$$\sigma_N = 2\mu_1(\lambda_{N+1} - \lambda_N) - M_2(1+l)\lambda_N^{1/2} > 0$$

is equivalent to the statement that

$$2\mu_1 - r_N\xi_N > 0 \tag{6.100}$$

where r_N and ξ_N are given, respectively, by (6.78a,b,c). If (6.100) holds, however, then

$$\begin{aligned} l' &= M_2(1+l)\lambda_{N+1}^{-1/2}\left[(2\mu_1^{-1} + (2\mu_1 - r_N\xi_N)^{-1}\right]e^{-1/2}e^{r_N\xi_N/(4\mu_1)} \\ &\leq M_2(1+l)\lambda_{N+1}^{-1/2}\left[(2\mu_1)^{-1} + (2\mu_1 - r_N\xi_N)^{-1}\right], \end{aligned} \tag{6.101}$$

in which case, to deduce that $l' < l$, it suffices to show that

$$(2\mu_1)^{-1}M_2(1+l)\lambda_{N+1}^{-1/2} < l/2 \tag{6.102a}$$

and

$$M_2(1+l)\lambda_{N+1}^{-1/2} \leq \frac{1}{2}l(2\mu_1 - r_N\xi_N). \tag{6.102b}$$

Now, (6.102a) can be written in the form

$$(2\mu_1)^{-1}k_1 \leq \lambda_{N+1}^{1/2}, \quad k_1 = 2M_2(1+l)l^{-1}, \tag{6.103}$$

and if (6.103) holds then (6.102b) can be written as

$$k_1\lambda_{N+1}^{-1/2} \leq 2\mu_1 - r_N\xi_N$$

or, equivalently, as

$$k_1\lambda_{N+1}^{-1/2} - 2\mu_1 + 2\mu_1 r_N + M_2(1+l)\lambda_{N+1}^{-1/2} r_N^{1/2} \le 0. \tag{6.104}$$

Assuming that condition (i) of the theorem holds or, equivalently, that

$$2\mu_1 r_N^{1/2} + k_1\lambda_{N+1}^{-1/2} \le 2\mu_1, \tag{6.105}$$

we find that (6.104) is valid, i.e.,

$$\begin{aligned} &k_1\lambda_{N+1}^{-1/2} - 2\mu_1 + 2\mu_1 r_N + M_2(1+l)\lambda_{N+1}^{-1/2} r_N^{1/2} \\ &\quad\le k_1\lambda_{N+1}^{-1/2} - 2\mu_1 + 2\mu_1 r_N + k_1\lambda_{N+1}^{-1/2} r_N^{1/2} \\ &\quad\le k_1\lambda_{N+1}^{-1/2} - 2\mu_1 + 2\mu_1 r_N^{1/2} \le 0. \end{aligned} \tag{6.106}$$

However, it is easily seen that both (6.100) and (6.103) are direct consequences of the spectral gap condition as expressed by hypothesis (i) of the Theorem 6.3; thus, if this condition holds then both $\sigma_N > 0$ and $l' < l$. Denoting the spectral gap condition expressed by condition (i) of the theorem as SGC, the precise consequence of the steps delineated above may be ordered as follows:

(a) SGC $\Rightarrow$ (6.100) $\Rightarrow$ $l' < l$ if (6.102a,b) hold,
(b) SGC $\Rightarrow$ (6.103) $\Leftrightarrow$ (6.102a),
(c) SGC $\Leftrightarrow$ (6.105) $\Rightarrow$ (6.104) $\Leftrightarrow$ (6.102b).

Finally, in order to show that $L < 1$, so that $\boldsymbol{T}$ is, by virtue of the estimate (6.98), a contraction map on the complete metric space $H_{b,l}$, it suffices to demonstrate that

$$L = \frac{M_2}{2\mu_1}(2e^{-1/2}\lambda_{N+1}^{-1/2} + \lambda_N^{-1/2} l') < \frac{1}{2}. \tag{6.107}$$

Since $l' < l$, however, and $\lambda_{N+1}^{1/2} \ge \lambda_N^{1/2}$,

$$L < \frac{M_2}{2\mu_1}(2e^{-1/2} + l)\lambda_N^{-1/2} < \frac{1}{2} \tag{6.108}$$

by virtue of the hypothesis (ii) of the theorem and the explicit form of k_2 as given in (6.99). □

Remarks. The validity of the spectral gap condition (SGC) with respect to the operator $\boldsymbol{A}$, as defined by (5.349)–(5.351), where $\Omega = [0, L]^n$, $L > 0$, $n = 2, 3$, and $\boldsymbol{V}_{per}(\Omega)$ is given by (5.348), is considered in Appendix C. It is, in fact, proven in C that condition (i) of Theorem 6.3 (the SGC) is satisfied for the bipolar problem in dim $n = 2$, for arbitrary $\mu_1 > 0$, if N is sufficiently large; in dim $n = 3$, however, SGC is satisfied only for μ_1 sufficiently large.

As a direct consequence of Theorem 6.3, namely, that the mapping $\boldsymbol{T}$ is a contraction map of the complete metric space $H_{b,l}$ into itself, it follows that $\boldsymbol{T}$ has a fixed point $\boldsymbol{\phi}^* \in H_{b,l}$. From the analysis in Sect. 6.2.5 it follows that $\boldsymbol{\mathcal{M}} = \text{graph}\,\boldsymbol{\phi}^*$ is a finite-dimensional Lipschitz manifold and it is a simple exercise to show directly that $\boldsymbol{\mathcal{M}}$ is invariant under the action of the solution operator $\boldsymbol{S}_{\mu_1}$, i.e., $\boldsymbol{S}_{\mu_1}(t)\boldsymbol{\mathcal{M}} \subset \boldsymbol{\mathcal{M}}$. In the following subsection, therefore, we need only prove that $\boldsymbol{\mathcal{M}}$ attracts, exponentially, all orbits of the modified initial-value problem (6.9), (5.361b).

6.2.6 Existence of the Inertial Manifold $\boldsymbol{\mathcal{M}}$

To complete the proof of the existence of an inertial manifold $\boldsymbol{\mathcal{M}}$, for the bipolar viscous fluid flow problem governed by (5.2a,b), (5.3a), (5.4), we set $\boldsymbol{\mathcal{M}} = \text{graph}\,\boldsymbol{\phi}^*$, $\boldsymbol{\phi}^*$ the unique fixed point of $\boldsymbol{T}$ in $H_{b,l}$, and show that $\exists t_0 > 0$ such that, for $t \geq t_0$ and $\boldsymbol{u}_0 \in D(\boldsymbol{A}^{1/4})$, all orbits of the intial-value problem (6.9), (5.361b) are attracted exponentially by $\boldsymbol{\mathcal{M}}$. We have, in fact, the following specific result:

Theorem 6.4. *Let* $\boldsymbol{\mathcal{M}} = \text{graph}\,\boldsymbol{\phi}^*$, *where* $\boldsymbol{\phi}^* \in H_{b,l}$ *is the unique fixed point of* $\boldsymbol{T}$ *in* $H_{b,l}$ *whose existence is guaranteed by Theorem 6.3 and the completeness of the space* $H_{b,l}$ *with respect to the norm specified in* (6.12)*. Then* $\exists t_0 > 0$ *such that for* $\boldsymbol{u}_0 \in D(\boldsymbol{A}^{1/4})$ *and* $t \geq t_0$

$$d(\boldsymbol{S}_{\mu_1}(t)\boldsymbol{u}_0, \boldsymbol{\mathcal{M}}) \leq \exp\left(\frac{-t}{2t_0}\ln 2\right) d(\boldsymbol{u}_0, \boldsymbol{\mathcal{M}}). \tag{6.109}$$

Proof. We begin by noting that it is a straightforward matter to establish a squeezing property for orbits of the modified problem (6.9), (5.361b) which is entirely analogous to the one proven in Theorem 6.3 for the initial-value problem (5.361a,b); more specifically, for solutions $\boldsymbol{u}(t)$, $\boldsymbol{v}(t)$ of (6.9), (5.361b) satisfying (6.5), for some $M > 0$, if we are given $\gamma > 0$, then for any $t \in [0, T]$ and every N, $\exists \bar{c}_i$, $i = 1, 2$ such that either (6.6) holds or (6.7) does (with $\bar{c}_i$ replacing c_i, $i = 1, 2$). For the orbits of (6.9), (5.361b) satisfying (6.5), for some $M > 0$, (6.52) will also hold, with C_M replaced by some $\bar{C}_M > 0$ for $t \in [0, T]$; thus, setting

$$t_0 = \min\left(\frac{\mu_1 \ln 2}{\bar{C}_M^2}, \frac{T}{2}\right) \tag{6.110}$$

we obtain, from this modified version of (6.52), the estimate

$$\left|\left|\boldsymbol{A}^{1/4}(\boldsymbol{u}(t) - \boldsymbol{v}(t))\right|\right|_{\boldsymbol{L}^2(\Omega)} \leq 2\left|\left|\boldsymbol{A}^{-1/4}(\boldsymbol{u}(0) - \boldsymbol{v}(0))\right|\right|_{\boldsymbol{L}^2(\Omega)}, \quad t < 2t_0. \tag{6.111}$$

If we set $\gamma = \frac{1}{8}$ and choose $N > N_0$, where N_0 satisfies

$$\lambda_{N_0+1} \geq (\bar{c}_1 \mu_1 t_0)^{-1} \ln(2\bar{c}_2), \tag{6.112}$$

then from the modified forms of (6.6), (6.7) we will have either

$$\left|\left|\boldsymbol{Q}_N \boldsymbol{A}^{-1/4}(\boldsymbol{u}(t) - \boldsymbol{v}(t))\right|\right|_{\boldsymbol{L}^2(\Omega)} \leq \frac{1}{8}\left|\left|\boldsymbol{P}_N \boldsymbol{A}^{-1/4}(\boldsymbol{u}(t) - \boldsymbol{v}(t))\right|\right|_{\boldsymbol{L}^2(\Omega)} \tag{6.113}$$

or

$$\left|\left|\boldsymbol{A}^{-1/4}(\boldsymbol{u}(t) - \boldsymbol{v}(t))\right|\right|_{\boldsymbol{L}^2(\Omega)} \leq \frac{1}{2}\left|\left|\boldsymbol{A}^{-1/4}(\boldsymbol{u}(0) - \boldsymbol{v}(0))\right|\right|_{\boldsymbol{L}^2(\Omega)} \tag{6.114}$$

where $\boldsymbol{u}_0, \boldsymbol{v}_0 \in D(\boldsymbol{A}^{1/4})$, $\left|\left|\boldsymbol{A}^{1/4}\boldsymbol{u}(0)\right|\right|_{\boldsymbol{L}^2(\Omega)} \leq M$, $\left|\left|\boldsymbol{A}^{1/4}\boldsymbol{v}(0)\right|\right|_{\boldsymbol{L}^2(\Omega)} \leq M$, and $t_0 \leq t \leq 2t_0$.

We now denote the distance between any point $\boldsymbol{w}$ in the absorbing ball B_{r_2} in $D(\boldsymbol{A}^{1/4})$ and the manifold $\boldsymbol{\mathcal{M}}$ by

$$d(\boldsymbol{w}, \boldsymbol{\mathcal{M}}) = \inf_{\boldsymbol{v} \in \boldsymbol{\mathcal{M}}} \left\{ \left|\left|\boldsymbol{A}^{-1/4}(\boldsymbol{w} - \boldsymbol{v}\right|\right|_{\boldsymbol{L}^2(\Omega)} \right\}. \tag{6.115}$$

To show that $\boldsymbol{\mathcal{M}}$ attracts all orbits of the modified intial-value problem (6.9), (5.361b) exponentially, it suffices to prove that $\boldsymbol{\mathcal{M}}$ attracts, exponentially, all orbits contained in the absorbing ball B_{r_2}, i.e., all orbits $\boldsymbol{u}(t)$ such that $\left|\left|\boldsymbol{A}^{1/4}\boldsymbol{u}(t)\right|\right|_{\boldsymbol{L}^2(\Omega)} \leq r_2$, $t \in [0, \infty)$. Therefore, let $\boldsymbol{v}(0) = \boldsymbol{v}_0 \in \boldsymbol{\mathcal{M}}$, $\boldsymbol{v}_0 = \boldsymbol{P}_N \boldsymbol{v}_0 + \boldsymbol{\phi}(\boldsymbol{P}_N \boldsymbol{v}_0)$, be such that

$$\text{dist}(\boldsymbol{u}(0), \boldsymbol{\mathcal{M}}) = \left|\left|\boldsymbol{A}^{-1/4}\boldsymbol{u}(0) - \boldsymbol{v}(0))\right|\right|_{\boldsymbol{L}^2(\Omega)}. \tag{6.116}$$

Obviously,

$$\left|\left|\boldsymbol{P}_N \boldsymbol{A}^{1/4}\boldsymbol{v}(0)\right|\right|_{\boldsymbol{L}^2(\Omega)} \leq r_2, \tag{6.117}$$

so that, with $b > 0$,

$$\left|\left|\boldsymbol{A}^{1/4}\boldsymbol{v}(0)\right|\right|_{\boldsymbol{L}^2(\Omega)} \leq r_2 + b, \tag{6.118}$$

in which case, for $t \geq 0$,

$$\left|\left|\boldsymbol{A}^{1/4}\boldsymbol{v}(t)\right|\right|_{\boldsymbol{L}^2(\Omega)} = \left|\left|\boldsymbol{A}^{1/4}\boldsymbol{S}_{\mu_1}(t)\boldsymbol{v}(0)\right|\right|_{\boldsymbol{L}^2(\Omega)} \leq r_2 + b, \quad \forall t \geq 0. \tag{6.119}$$

Choosing M (in the statement of the squeezing property for the orbits of (6.9), (5.361b)) to be $M = r_2 + b$, we apply the estimates recorded in (6.113), (6.114) to $\boldsymbol{S}_{\mu_1}(t_1)\boldsymbol{u}_0$ and $\boldsymbol{S}_{\mu_1}(t_1)\boldsymbol{v}_0$, with $t_0 \le t_1 \le 2t_0$: if (6.114) applies, then

$$\begin{aligned} d(\boldsymbol{S}_{\mu_1}(t_1)\boldsymbol{u}_0, \boldsymbol{\mathcal{M}}) &\le \left|\left|\boldsymbol{A}^{-1/4}(\boldsymbol{S}_{\mu_1}(t_1)\boldsymbol{u}_0 - \boldsymbol{S}_{\mu_1}(t_1)\boldsymbol{v}_0)\right|\right|_{\boldsymbol{L}^2(\Omega)} \\ &\le \frac{1}{2}\left|\left|\boldsymbol{A}^{-1/4}(\boldsymbol{u}_0 - \boldsymbol{v}_0)\right|\right|_{\boldsymbol{L}^2(\Omega)} = \frac{1}{2} d(\boldsymbol{u}_0, \boldsymbol{\mathcal{M}}). \end{aligned} \tag{6.120}$$

On the other hand, if (6.113) holds, then we have the following sequence of estimates:

$$\begin{aligned} & d(\boldsymbol{S}_{\mu_1}(t_1)\boldsymbol{u}_0, \boldsymbol{\mathcal{M}}) \\ &\quad \le \left|\left|\boldsymbol{A}^{-1/4}(\boldsymbol{S}_{\mu_1}(t_1)\boldsymbol{u}_0 - (\boldsymbol{P}_N\boldsymbol{S}_{\mu_1}(t_1)\boldsymbol{v}_0 + \boldsymbol{\phi}(\boldsymbol{P}_N\boldsymbol{S}_{\mu_1}(t_1)\boldsymbol{v}_0)))\right|\right|_{\boldsymbol{L}^2(\Omega)} \\ &\quad \le \left|\left|\boldsymbol{A}^{-1/4}(\boldsymbol{Q}_N\boldsymbol{S}_{\mu_1}(t_1)\boldsymbol{u}_0 - \boldsymbol{\phi}(\boldsymbol{P}_N\boldsymbol{S}_{\mu_1}(t_1)\boldsymbol{v}_0))\right|\right|_{\boldsymbol{L}^2(\Omega)} \\ &\quad \le \left|\left|\boldsymbol{A}^{-1/4}(\boldsymbol{Q}_N\boldsymbol{S}_{\mu_1}(t_1)\boldsymbol{u}_0 - \boldsymbol{Q}_N\boldsymbol{S}_{\mu_1}(t_1)\boldsymbol{v}_0)\right|\right|_{\boldsymbol{L}^2(\Omega)} \\ &\quad \le \left|\left|\boldsymbol{A}^{-1/4}(\boldsymbol{\phi}(\boldsymbol{P}_N\boldsymbol{S}_{\mu_1}(t_1)\boldsymbol{u}_0 - \boldsymbol{\phi}(\boldsymbol{P}_N\boldsymbol{S}_{\mu_1}(t_1)\boldsymbol{v}_0)))\right|\right|_{\boldsymbol{L}^2(\Omega)} \\ &\quad \le \left(l + \frac{1}{8}\right)\left|\left|\boldsymbol{A}^{-1/4}(\boldsymbol{P}_N\boldsymbol{S}_{\mu_1}(t_1)\boldsymbol{u}_0 - \boldsymbol{P}_N\boldsymbol{S}_{\mu_1}(t_1)\boldsymbol{v}_0)\right|\right|_{\boldsymbol{L}^2(\Omega)}. \end{aligned} \tag{6.121}$$

Taking $l = \dfrac{1}{8}$ in the last estimate, we find that

$$\begin{aligned} d(\boldsymbol{S}_{\mu_1}(t_1)\boldsymbol{u}_0, \boldsymbol{\mathcal{M}}) &\le \frac{1}{4}\left|\left|\boldsymbol{A}^{-1/4}(\boldsymbol{S}_{\mu_1}(t_1)\boldsymbol{u}_0 - \boldsymbol{S}_{\mu_1}(t_1)\boldsymbol{v}_0)\right|\right|_{\boldsymbol{L}^2(\Omega)} \\ &\le \frac{1}{4}\cdot 2\left|\left|\boldsymbol{A}^{-1/4}(\boldsymbol{u}_0 - \boldsymbol{v}_0)\right|\right|_{\boldsymbol{L}^2(\Omega)} \le \frac{1}{2} d(\boldsymbol{u}_0, \boldsymbol{\mathcal{M}}) \end{aligned} \tag{6.122}$$

for $t_0 \le t_1 \le 2t_0$. Iterating upon the procedure delineated above we have, therefore, for $t_0 \le t_1 \le 2t_0$,

$$d(\boldsymbol{S}_{\mu_1}(nt_1)\boldsymbol{u}_0, \boldsymbol{\mathcal{M}}) \le \left(\frac{1}{2}\right)^n d(\boldsymbol{u}_0, \boldsymbol{\mathcal{M}}) \to 0 \tag{6.123}$$

as $n \to \infty$. For arbitrary $t \ge t_0$ we may write $t = nt_1$ for some $t_1, t_0 \le t_1 \le 2t_0$, in which case

$$
\begin{aligned}
d(S_{\mu_1}(t)u_0, \mathcal{M}) &\leq \left(\frac{1}{2}\right)^n d(u_0, \mathcal{M}) \\
&\leq \exp\left(-\frac{t}{t_1}\ln 2\right) d(u_0, \mathcal{M}) \\
&\leq \exp\left(-\frac{t}{2t_0}\ln 2\right) d(u_0, \mathcal{M}),
\end{aligned}
\tag{6.124}
$$

thus establishing the required exponential convergence of orbits of the modified initial-value problem (6.9), (5.361b) and, hence, of the original problem (5.361a,b), to the manifold $\mathcal{M}$ that is generated as the graph of the unique fixed point of T. The proof of the existence of an inertial manifold for the space-periodic version of the nonlinear, incompressible, bipolar viscous model is now complete. □

6.3 The L^2 Squeezing Property for Bipolar Viscous Fluids

6.3.1 Introduction

In Sect. 6.2 we established, in conjunction with the proof of the existence of an inertial manifold for the incompressible bipolar viscous fluid satisfying periodic boundary conditions, i.e., for (5.2a,b), (5.3b), (5.4), a squeezing property for orbits; the abstract formulation of the problem assumes the form (5.361a,b), where A is the linear, positive self-adjoint operator defined by (5.350), (5.351), with $a(u, v)$ given by (5.349), while $R(u)$ is defined by (5.360), with $A_p u$ and $B(u, v)$ given, respectively, by (5.358), (5.359). The precise statement of this squeezing property for the orbits of the problem (5.361a,b) is, essentially, the content of Theorem 6.2; this result holds in both dimensions $n = 2, 3$, when $1 < p \leq 2$, and says that provided two solutions $u(t)$, $v(t)$ of (5.361a,b), satisfying initial conditions $u(0) = u_0$, $v(0) = v_0$, obey (5.39) for some $M > 0$, either (6.6) or (6.7) must hold.

Although it is the squeezing property represented by (6.6), (6.7) which is naturally adopted to establishing the existence of the inertial manifold $\mathcal{M}$ in 6.2, it is of interest to inquire as to whether or not a more basic, i.e., L^2 form of the squeezing property holds with respect to the orbits of the incompressible bipolar problem (5.2a,b), (5.3b), (5.4); this question will be answered in the affirmative in this section. In particular, if for $u \in D(A)$ we set

$$
||u||_V = (Au, u)_{L^2(\Omega)}{}^{1/2} \tag{6.125}
$$

then it will be shown that for solutions $\boldsymbol{u}(t)$, $\boldsymbol{v}(t)$ of (5.361a,b) such that $||\boldsymbol{u}||_V \le M$, $||\boldsymbol{v}||_V \le M$, for some $M > 0$, $\exists \bar{c}_i > 0$, $i = 1, 2$, depending only on M, T, $\boldsymbol{f}$, μ_0, μ_1, ϵ, and Ω such that for every N, and all $t \in [0, T]$, either

$$||\boldsymbol{Q}_N(\boldsymbol{u}(t) - \boldsymbol{v}(t))||_{\boldsymbol{L}^2(\Omega)} \le ||\boldsymbol{P}_N(\boldsymbol{u}(t) - \boldsymbol{v}(t))||_{\boldsymbol{L}^2(\Omega)} \tag{6.126}$$

or

$$||\boldsymbol{u}(t) - \boldsymbol{v}(t)||_{\boldsymbol{L}^2(\Omega)}{}^2 \le \bar{c}_1 e^{-\bar{c}_2 \mu_1 \lambda_{N+1} t} ||\boldsymbol{u}(0) - \boldsymbol{v}(0)||_{\boldsymbol{L}^2(\Omega)}{}^2 \tag{6.127}$$

where $\boldsymbol{P}_N$ is the projection operator defined in Sect. 6.2.2 and $\boldsymbol{Q}_N = \boldsymbol{I} - \boldsymbol{P}_N$. The proof of this $\boldsymbol{L}^2$ squeezing property hinges on a more careful study of the nonlinear viscosity term, embodied in the nonlinear operator $\boldsymbol{A}_p$ defined by (5.358), than that which was required in Sect. 6.2 and will proceed via a series of lemmas.

6.3.2 An Estimate for Nonlinear Viscosity

Our goal in this subsection is to derive (what will turn out to be) a suitable estimate for the $\boldsymbol{L}^2(\Omega)$ norm of the difference $\boldsymbol{A}_p(\boldsymbol{u}) - \boldsymbol{A}_p(\boldsymbol{v})$, where $\boldsymbol{A}_p$ is the nonlinear operator given by (5.358), $\boldsymbol{u}$, $\boldsymbol{v}$ satisfy $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega)$, $\boldsymbol{v} \in \boldsymbol{H}^3(\Omega)$, respectively, and $1 < p \le 2$.

Lemma 6.9. *For* $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega)$, $\boldsymbol{v} \in \boldsymbol{H}^3(\Omega)$, *and* $1 < p \le 2$,

$$\begin{aligned}
&\left| \left(\epsilon + \sum_{k,l} e_{kl}^2(\boldsymbol{u}) \right)^{\frac{p-2}{2}} \frac{\partial}{\partial x_j} e_{ij}(\boldsymbol{u}) - \left(\epsilon + \sum_{k,l} e_{kl}^2(\boldsymbol{v}) \right)^{\frac{p-2}{2}} \frac{\partial}{\partial x_j} e_{ij}(\boldsymbol{v}) \right| \\
&\quad \le \epsilon^{\frac{p-2}{2}} \left| \frac{\partial}{\partial x_j} e_{ij}(\boldsymbol{u} - \boldsymbol{v}) \right| \\
&\qquad + \left| \frac{p-2}{2} \right| \epsilon^{\frac{p-4}{2}} \left(\sum_{k,l} |e_{kl}(\boldsymbol{u} + \boldsymbol{v})||e_{kl}(\boldsymbol{u} - \boldsymbol{v})| \right) \left| \frac{\partial}{\partial x_j} e_{ij}(\boldsymbol{v}) \right|.
\end{aligned} \tag{6.128}$$

Proof. As $1 < p \le 2$, for fixed i, j (no sum on j)

$$\begin{aligned}
&\left| \left(\epsilon + \sum_{k,l} e_{kl}^2(\boldsymbol{u}) \right)^{\frac{p-2}{2}} \frac{\partial}{\partial x_j} e_{ij}(\boldsymbol{u}) - \left(\epsilon + \sum_{k,l} e_{kl}^2(\boldsymbol{v}) \right)^{\frac{p-2}{2}} \frac{\partial}{\partial x_j} e_{ij}(\boldsymbol{v}) \right| \\
&\quad = \left| \left(\epsilon + \sum_{k,l} e_{kl}^2(\boldsymbol{u}) \right)^{\frac{p-2}{2}} \left(\frac{\partial}{\partial x_j} e_{ij}(\boldsymbol{u}) - \frac{\partial}{\partial x_j} e_{ij}(\boldsymbol{v}) \right) \right.
\end{aligned}$$

$$+\left[\left(\epsilon+\sum_{k,l}e_{kl}^2(\boldsymbol{u})\right)^{\frac{p-2}{2}}-\left(\epsilon+\sum_{k,l}e_{kl}^2(\boldsymbol{v})\right)^{\frac{p-2}{2}}\right]\frac{\partial}{\partial x_j}e_{ij}(\boldsymbol{v})\Bigg| \tag{6.129}$$

$$\leq \epsilon^{\frac{p-2}{2}}\left|\frac{\partial}{\partial x_j}e_{ij}(\boldsymbol{u}-\boldsymbol{v})\right|$$

$$+\left|\left[\left(\epsilon+\sum_{k,l}e_{kl}^2(\boldsymbol{u})\right)^{\frac{p-2}{2}}-\left(\epsilon+\sum_{k,l}e_{kl}^2(\boldsymbol{v})\right)^{\frac{p-2}{2}}\right]\frac{\partial}{\partial x_j}e_{ij}(\boldsymbol{v})\right|.$$

By the mean-value theorem in the form

$$(\epsilon+a)^{\frac{p-2}{2}}-(\epsilon+b)^{\frac{p-2}{2}}=\left(\frac{p-2}{2}\right)(\epsilon+s)^{\frac{p-2}{4}}(a-b) \tag{6.130}$$

for some s, $a<s<b$, we infer the existence of s^*,

$$\sum_{k,l}e_{kl}^2(\boldsymbol{u})<s^*<\sum_{k,l}e_{kl}^2(\boldsymbol{v}) \tag{6.131}$$

such that, in the second term on the right-hand side of the last estimate in (6.129),

$$\begin{aligned}
&\left|\left(\epsilon+\sum_{k,l}e_{kl}^2(\boldsymbol{u})\right)^{\frac{p-2}{2}}-\left(\epsilon+\sum_{k,l}e_{kl}^2(\boldsymbol{v})\right)^{\frac{p-2}{2}}\right|\\
&\quad\leq\left|\frac{p-2}{2}\right|(\epsilon+s^*)^{\frac{p-4}{2}}\left|\sum_{k,l}e_{kl}^2(\boldsymbol{u})-\sum_{k,l}e_{kl}^2(\boldsymbol{v})\right|\\
&\quad\leq\left|\frac{p-2}{2}\right|\epsilon^{\frac{p-4}{2}}\sum_{k,l}|e_{kl}(\boldsymbol{u}+\boldsymbol{v})||e_{kl}(\boldsymbol{u}-\boldsymbol{v})|.
\end{aligned} \tag{6.132}$$

In (6.132) we have used the fact that

$$\begin{aligned}
\sum_{k,l}e_{kl}^2(\boldsymbol{u})-\sum_{k,l}e_{kl}^2(\boldsymbol{v})&=\sum_{k,l}e_{kl}^2(\boldsymbol{u})-e_{kl}^2(\boldsymbol{v})\\
&=\sum_{k,l}(e_{kl}(\boldsymbol{u})+e_{kl}(\boldsymbol{v}))(e_{kl}(\boldsymbol{u})-e_{kl}(\boldsymbol{v}))\\
&=\sum_{k,l}e_{kl}(\boldsymbol{u}+\boldsymbol{v})e_{kl}(\boldsymbol{u}-\boldsymbol{v}).
\end{aligned}$$

Combining (6.129) with (6.132) then yields (6.128). □

The second basic estimate that we need in this subsection, before we can state and prove the required bound for $\left|\left|\boldsymbol{A}_p(\boldsymbol{u}) - \boldsymbol{A}_p(\boldsymbol{v})\right|\right|_{\boldsymbol{L}^2(\Omega)}$, is embodied in the following

Lemma 6.10. *For* $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega)$, $\boldsymbol{v} \in \boldsymbol{H}^3(\Omega)$, *and* $1 < p \le 2$, *it follows that for fixed* i, j *(no sum on* j*) we have the estimate*

$$\begin{aligned}
&\Bigg|\left(\epsilon + \sum_{k,l} e_{kl}^2(\boldsymbol{u})\right)^{\frac{p-4}{2}} e_{ij}(\boldsymbol{u}) \left(\sum_{k,l} e_{kl}(\boldsymbol{u}) \frac{\partial}{\partial x_j} e_{kl}(\boldsymbol{u})\right) \\
&\quad - \left(\epsilon + \sum_{k,l} e_{kl}^2(\boldsymbol{v})\right)^{\frac{p-4}{2}} e_{ij}(\boldsymbol{v}) \left(\sum_{k,l} e_{kl}(\boldsymbol{v}) \frac{\partial}{\partial x_j} e_{kl}(\boldsymbol{v})\right)\Bigg| \\
&\le \epsilon^{\frac{p-2}{2}} \sum_{k,l} \left|\frac{\partial}{\partial x_j} e_{kl}(\boldsymbol{u}-\boldsymbol{v})\right| \\
&\quad + \left|\frac{p-4}{2}\right| \epsilon^{\frac{p-4}{2}} \sum_{k,l} \sum_{m,n} |e_{mn}(\boldsymbol{u}+\boldsymbol{v})||e_{mn}(\boldsymbol{u}-\boldsymbol{v})| \left|\frac{\partial}{\partial x_j} e_{kl}(\boldsymbol{v})\right| \\
&\quad + \epsilon^{\frac{p-4}{2}} \sum_{k,l} \Big(\left|e_{ij}(\boldsymbol{u})\right| \cdot |e_{kl}(\boldsymbol{u}-\boldsymbol{v})| + \left|e_{ij}(\boldsymbol{u}-\boldsymbol{v})\right| \cdot |e_{kl}(\boldsymbol{v})|\Big) \left|\frac{\partial}{\partial x_j} e_{kl}(\boldsymbol{v})\right|.
\end{aligned} \tag{6.133}$$

Proof. We begin with the set of elementary estimates

$$\begin{aligned}
&\Bigg|\left(\epsilon + \sum_{k,l} e_{kl}^2(\boldsymbol{u})\right)^{\frac{p-4}{2}} e_{ij}(\boldsymbol{u}) \left(\sum_{k,l} e_{kl}(\boldsymbol{u}) \frac{\partial}{\partial x_j} e_{kl}(\boldsymbol{u})\right) \\
&\quad - \left(\epsilon + \sum_{k,l} e_{kl}^2(\boldsymbol{v})\right)^{\frac{p-4}{2}} e_{ij}(\boldsymbol{v}) \left(\sum_{k,l} e_{kl}(\boldsymbol{v}) \frac{\partial}{\partial x_j} e_{kl}(\boldsymbol{v})\right)\Bigg| \\
&= \Bigg|\sum_{k,l} \Bigg[\left(\epsilon + |\boldsymbol{e}(\boldsymbol{u})|^2\right)^{\frac{p-4}{2}} e_{ij}(\boldsymbol{u}) e_{kl}(\boldsymbol{u}) \frac{\partial}{\partial x_j} e_{kl}(\boldsymbol{u}) \\
&\quad - \left(\epsilon + |\boldsymbol{e}(\boldsymbol{u})|^2\right)^{\frac{p-4}{2}} e_{ij}(\boldsymbol{v}) e_{kl}(\boldsymbol{v}) \frac{\partial}{\partial x_j} e_{kl}(\boldsymbol{v})\Bigg]\Bigg| \\
&\le \sum_{k,l} \left|\left(\epsilon + |\boldsymbol{e}(\boldsymbol{u})|^2\right)^{\frac{p-4}{2}} e_{ij}(\boldsymbol{u}) e_{kl}(\boldsymbol{u}) \frac{\partial}{\partial x_j} e_{kl}(\boldsymbol{u}-\boldsymbol{v})\right|
\end{aligned} \tag{6.134}$$

$$+\sum_{k,l}\left|\left(\epsilon+|e(u)|^2\right)^{\frac{p-4}{2}} e_{ij}(u)e_{kl}(u)-\left(\epsilon+|e(v)|^2\right)^{\frac{p-4}{2}} e_{ij}(v)e_{kl}(v)\right|\left|\frac{\partial}{\partial x_j}e_{kl}(v)\right|$$

where in going from the first to the second expression on the right-hand side of (6.134) we have added and subtracted the term

$$\sum_{k,l}\left(\epsilon+|e(u)|^2\right)^{\frac{p-4}{2}} e_{ij}(u)e_{kl}(u)\frac{\partial}{\partial x_j}e_{kl}(v).$$

We now work on the expression

$$\left|\left(\epsilon+|e(u)|^2\right)^{\frac{p-4}{2}} e_{ij}(u)e_{kl}(u)-\left(\epsilon+|e(v)|^2\right)^{\frac{p-4}{2}} e_{ij}(v)e_{kl}(v)\right|$$

which appears in the second term on the right-hand side of the last estimate in (6.134). We begin by noting that, for fixed i, j, k, l,

$$\begin{aligned}\left|e_{ij}(u)e_{kl}(u)\right| &\le \frac{1}{2}e_{ij}^2(u)+\frac{1}{2}e_{kl}^2(u)\\ &\le \frac{1}{2}|e(u)|^2+\frac{1}{2}|e(u)|^2=|e(u)|^2.\end{aligned} \tag{6.135}$$

Now, suppose that $|e(u)|^2 \le |e(v)|^2$; then

$$\begin{aligned}&\left|\left(\epsilon+|e(u)|^2\right)^{\frac{p-4}{2}} e_{ij}(u)e_{kl}(u)-\left(\epsilon+|e(v)|^2\right)^{\frac{p-4}{2}} e_{ij}(v)e_{kl}(v)\right|\\ &\quad\le\left|\left[\left(\epsilon+|e(u)|^2\right)^{\frac{p-4}{2}}-\left(\epsilon+|e(v)|^2\right)^{\frac{p-4}{2}}\right]e_{ij}(u)e_{kl}(u)\right|\\ &\qquad+\left(\epsilon+|e(u)|^2\right)^{\frac{p-4}{2}}\left|e_{ij}(u)e_{kl}(u)-e_{ij}(v)e_{kl}(v)\right|\end{aligned} \tag{6.136}$$

where we have added and subtracted the expression

$$\left(\epsilon+|e(v)|^2\right)^{\frac{p-4}{2}} e_{ij}(u)e_{kl}(u).$$

By the mean-value theorem, $\exists\xi^*$, with

$$|e(u)|^2 \le \xi^* \le |e(v)|^2 \;\; (\text{as } |e(u)|^2 \le |e(v)|^2)$$

such that

$$
\begin{aligned}
&\left|\left(\epsilon+|\boldsymbol{e}(\boldsymbol{u})|^2\right)^{\frac{p-4}{2}}-\left(\epsilon+|\boldsymbol{e}(\boldsymbol{v})|^2\right)^{\frac{p-4}{2}}\right| \\
&\quad=\left|\frac{p-4}{2}\right|(\epsilon+\xi^*)^{\frac{p-6}{2}}\left||\boldsymbol{e}(\boldsymbol{u})|^2-|\boldsymbol{e}(\boldsymbol{v})|^2\right| \\
&\quad\le\left|\frac{p-4}{2}\right|\left(\epsilon+|\boldsymbol{e}(\boldsymbol{u})|^2\right)^{\frac{p-6}{2}}\left||\boldsymbol{e}(\boldsymbol{u})|^2-|\boldsymbol{e}(\boldsymbol{v})|^2\right|
\end{aligned}
\tag{6.137}
$$

by virtue of the fact that $\xi^* \ge |\boldsymbol{e}(\boldsymbol{u})|^2$ and $1 < p \le 2$. Combining (6.135)–(6.137) we find that, for fixed i, j, k, l,

$$
\begin{aligned}
&\left|\left(\epsilon+|\boldsymbol{e}(\boldsymbol{u})|^2\right)^{\frac{p-4}{2}}e_{ij}(\boldsymbol{u})e_{kl}(\boldsymbol{u})-\left(\epsilon+|\boldsymbol{e}(\boldsymbol{v})|^2\right)^{\frac{p-4}{2}}e_{ij}(\boldsymbol{v})e_{kl}(\boldsymbol{v})\right| \\
&\le\left|\frac{p-4}{2}\right|\left(\epsilon+|\boldsymbol{e}(\boldsymbol{u})|^2\right)^{\frac{p-6}{2}}\left||\boldsymbol{e}(\boldsymbol{u})|^2-|\boldsymbol{e}(\boldsymbol{v})|^2\right||\boldsymbol{e}(\boldsymbol{u})|^2 \\
&\quad+\left(\epsilon+|\boldsymbol{e}(\boldsymbol{v})|^2\right)^{\frac{p-4}{2}}\left|e_{ij}(\boldsymbol{u})e_{kl}(\boldsymbol{u})-e_{ij}(\boldsymbol{v})e_{kl}(\boldsymbol{v})\right| \\
&=\left|\frac{p-4}{2}\right|\frac{|\boldsymbol{e}(\boldsymbol{u})|^2}{(\epsilon+|\boldsymbol{e}(\boldsymbol{u})|)^2}\left(\epsilon+|\boldsymbol{e}(\boldsymbol{u})|^2\right)^{\frac{p-4}{2}}\left||\boldsymbol{e}(\boldsymbol{u})|^2-|\boldsymbol{e}(\boldsymbol{u})|^2\right| \\
&\quad+\left(\epsilon+|\boldsymbol{e}(\boldsymbol{v})|^2\right)^{\frac{p-4}{2}}\left|e_{ij}(\boldsymbol{u})e_{kl}(\boldsymbol{u})-e_{ij}(\boldsymbol{v})e_{kl}(\boldsymbol{v})\right| \\
&\le\left|\frac{p-4}{2}\right|\epsilon^{\frac{p-4}{2}}\left|\sum_{m,n}e_{mn}^2(\boldsymbol{u})-\sum_{m,n}e_{mn}^2(\boldsymbol{v})\right| \\
&\quad+\epsilon^{\frac{p-4}{2}}\left(\left|e_{ij}(\boldsymbol{u})\right||e_{kl}(\boldsymbol{u}-\boldsymbol{v})|+\left|e_{ij}(\boldsymbol{u}-\boldsymbol{v})\right||e_{kl}(\boldsymbol{v})|\right).
\end{aligned}
\tag{6.138}
$$

In a similar manner, it is easily demonstrated that the last estimate in (6.138) also holds in the case where $|\boldsymbol{e}(\boldsymbol{u})|^2 \ge |\boldsymbol{e}(\boldsymbol{v})|^2$. In view of (6.134) and (6.138) we now have, for fixed i, j,

$$
\begin{aligned}
&\left|\left(\epsilon+\sum_{k,l}e_{kl}^2(\boldsymbol{u})\right)^{\frac{p-4}{2}}e_{ij}(\boldsymbol{u})\left(\sum_{k,l}e_{kl}(\boldsymbol{u})\frac{\partial}{\partial x_j}e_{kl}(\boldsymbol{u})\right)\right. \\
&\qquad\left.-\left(\epsilon+\sum_{k,l}e_{kl}^2(\boldsymbol{v})\right)^{\frac{p-4}{2}}e_{ij}(\boldsymbol{v})\left(\sum_{k,l}e_{kl}(\boldsymbol{v})\frac{\partial}{\partial x_j}e_{kl}(\boldsymbol{v})\right)\right|
\end{aligned}
$$

$$\begin{aligned}
&\leq \sum_{k,l}\left(\epsilon+|e(u)|^2\right)^{\frac{p-4}{2}}|e(u)|^2\left|\frac{\partial}{\partial x_j}e_{kl}(u-v)\right| \\
&+\left|\frac{p-4}{2}\right|\epsilon^{\frac{p-4}{2}}\sum_{k,l}\sum_{m,n}|e_{mn}(u+v)||e_{mn}(u-v)|\left|\frac{\partial}{\partial x_j}e_{kl}(v)\right| \\
&+\epsilon^{\frac{p-4}{2}}\sum_{k,l}\left(\left|e_{ij}(u)\right|\cdot|e_{kl}(u-v)|+\left|e_{ij}(u-v)\right|\cdot|e_{kl}(v)|\right)\left|\frac{\partial}{\partial x_j}e_{kl}(v)\right|
\end{aligned} \tag{6.139}$$

from which the required conclusion, i.e., (6.133) follows immediately. □

Finally, we may state the estimate for the norm of $A_p(u) - A_p(v)$ which has been hinted at, above; we formulate this result as

Theorem 6.5. *Let* $u \in H^2(\Omega)$, $v \in H^3(\Omega)$, *and assume that* p *satisfies* $1 < p \leq 2$. *Then for* A_p *defined by* (5.358), $\exists k_i > 0$, $i = 1, 2$, *independent of* u, v, *such that*

$$\begin{aligned}
\left|\left|A_p(u)-A_p(v)\right|\right|_{L^2(\Omega)} &\leq k_1||u-v||_{H^2(\Omega)} \\
&+k_2\left(||u||_{H^2(\Omega)}+||v||_{H^2(\Omega)}\right)|v|_{H^3(\Omega)}||u-v||_{H^2(\Omega)}
\end{aligned} \tag{6.140}$$

Proof. We begin by fixing i, j and noting that (with $1 \leq p < 2$) we have, directly,

$$\begin{aligned}
\frac{\partial}{\partial x_j}\left[(\epsilon+|e(u)|^2)^{\frac{p-2}{2}}e_{ij}(u)\right] &= \frac{\partial}{\partial x_j}\left[\left(\epsilon+\sum_{k,l}e_{kl}^2(u)\right)^{\frac{p-2}{2}}e_{ij}(u)\right] \\
&= \left(\epsilon+\sum_{k,l}e_{kl}^2(u)\right)^{\frac{p-2}{2}}\frac{\partial}{\partial x_j}e_{ij}(u) \\
&+(p-2)\left(\epsilon+\sum_{k,l}e_{kl}^2(u)\right)^{\frac{p-4}{2}}e_{ij}(u)\left(\sum_{k,l}e_{kl}(u)\frac{\partial}{\partial x_j}e_{kl}(u)\right)
\end{aligned} \tag{6.141}$$

(no sum on j). In a similar fashion,

$$\frac{\partial}{\partial x_j}\left[(\epsilon+|e(v)|^2)^{\frac{p-2}{2}}e_{ij}(v)\right] = \left(\epsilon+\sum_{k,l}e_{kl}^2(v)\right)^{\frac{p-2}{2}}\frac{\partial}{\partial x_j}e_{ij}(v)$$

$$+ (p-2)\left(\epsilon + \sum_{k,l} e_{kl}^2(\boldsymbol{v})\right)^{\frac{p-4}{2}} e_{ij}(\boldsymbol{v}) \left(\sum_{k,l} e_{kl}(\boldsymbol{v}) \frac{\partial}{\partial x_j} e_{kl}(\boldsymbol{v})\right) \tag{6.142}$$

(no sum on j). Subtracting (6.142) from (6.141) we have (with no sum on j):

$$\begin{aligned}
&\frac{\partial}{\partial x_j}\left[\left(\epsilon + |\boldsymbol{e}(\boldsymbol{u})|^2\right)^{\frac{p-2}{2}} e_{ij}(\boldsymbol{u})\right] - \frac{\partial}{\partial x_j}\left[\left(\epsilon + |\boldsymbol{e}(\boldsymbol{v})|^2\right)^{\frac{p-2}{2}} e_{ij}(\boldsymbol{v})\right] \\
&= \left[\left(\epsilon + \sum_{k,l} e_{kl}^2(\boldsymbol{u})\right)^{\frac{p-2}{2}} \frac{\partial}{\partial x_j} e_{ij}(\boldsymbol{u}) - \left(\epsilon + \sum_{k,l} e_{kl}^2(\boldsymbol{v})\right)^{\frac{p-2}{2}} \frac{\partial}{\partial x_j} e_{ij}(\boldsymbol{v})\right] \\
&\quad + (p-2)\left[\left(\epsilon + \sum_{k,l} e_{kl}^2(\boldsymbol{u})\right)^{\frac{p-4}{2}} e_{ij}(\boldsymbol{u}) \left(\sum_{k,l} e_{kl}(\boldsymbol{u}) \frac{\partial}{\partial x_j} e_{kl}(\boldsymbol{u})\right)\right. \\
&\quad \left. - \left(\epsilon + \sum_{k,l} e_{kl}^2(\boldsymbol{v})\right)^{\frac{p-4}{2}} e_{ij}(\boldsymbol{v}) \left(\sum_{k,l} e_{kl}(\boldsymbol{v}) \frac{\partial}{\partial x_j} e_{kl}(\boldsymbol{v})\right)\right].
\end{aligned} \tag{6.143}$$

Therefore, if we hold i, j fixed, and combine (6.128) and (6.133) with (6.143), we obtain

$$\begin{aligned}
&\left|\frac{\partial}{\partial x_j}\left[\left(\epsilon + |\boldsymbol{e}(\boldsymbol{u})|^2\right)^{\frac{p-2}{2}} e_{ij}(\boldsymbol{u})\right] - \frac{\partial}{\partial x_j}\left[\left(\epsilon + |\boldsymbol{e}(\boldsymbol{v})|^2\right)^{\frac{p-2}{2}} e_{ij}(\boldsymbol{v})\right]\right| \\
&\le \epsilon^{\frac{p-2}{2}} \left|\frac{\partial}{\partial x_j} e_{ij}(\boldsymbol{u}-\boldsymbol{v})\right| \\
&\quad + \left|\frac{p-2}{2}\right| \epsilon^{\frac{p-4}{2}} \sum_{k,l} |e_{kl}(\boldsymbol{u}+\boldsymbol{v})||e_{kl}(\boldsymbol{u}-\boldsymbol{v})| \left|\frac{\partial}{\partial x_j} e_{ij}(\boldsymbol{v})\right| \\
&\quad + |p-2| \epsilon^{\frac{p-2}{2}} \sum_{k,l} \left|\frac{\partial}{\partial x_j} e_{kl}(\boldsymbol{u}-\boldsymbol{v})\right| \\
&\quad + |p-2| \left|\frac{p-4}{2}\right| \epsilon^{\frac{p-4}{2}} \sum_{k,l}\sum_{m,n} |e_{mn}(\boldsymbol{u}+\boldsymbol{v})||e_{mn}(\boldsymbol{u}-\boldsymbol{v})| \left|\frac{\partial}{\partial x_j} e_{kl}(\boldsymbol{v})\right| \\
&\quad + |p-2| \epsilon^{\frac{p-4}{2}} \sum_{k,l} \left(|e_{ij}(\boldsymbol{u})||e_{kl}(\boldsymbol{u}-\boldsymbol{v})| + |e_{ij}(\boldsymbol{u}-\boldsymbol{v})||e_{kl}(\boldsymbol{v})|\right) \left|\frac{\partial}{\partial x_j} e_{kl}(\boldsymbol{v})\right|.
\end{aligned} \tag{6.144}$$

If we now apply Lemma C.3 to the estimate (6.144) we may infer the existence of a constant $c_1(\Omega) > 0$ such that the following estimate holds, again for fixed i, j:

$$\left[\int_\Omega \left|\frac{\partial}{\partial x_j}\left[\left(\epsilon + |\boldsymbol{e}(\boldsymbol{u})|^2\right)^{\frac{p-2}{2}} e_{ij}(\boldsymbol{u})\right] - \frac{\partial}{\partial x_j}\left[\left(\epsilon + |\boldsymbol{e}(\boldsymbol{v})|^2\right)^{\frac{p-2}{2}} e_{ij}(\boldsymbol{v})\right]\right|^2 d\boldsymbol{x}\right]^{1/2}$$

$$\le \epsilon^{\frac{p-2}{2}}\left[\int_\Omega \left|\frac{\partial}{\partial x_j} e_{ij}(\boldsymbol{u}-\boldsymbol{v})\right|^2 d\boldsymbol{x}\right]^{1/2}$$

$$+ \left|\frac{p-2}{2}\right| \epsilon^{\frac{p-4}{2}} \sum_{k,l}\left[\int_\Omega |e_{kl}(\boldsymbol{u}+\boldsymbol{v})|^2 |e_{kl}(\boldsymbol{u}-\boldsymbol{v})|^2 \left|\frac{\partial}{\partial x_j} e_{ij}(\boldsymbol{v})\right|^2 d\boldsymbol{x}\right]^{1/2}$$

$$+ |p-2|\epsilon^{\frac{p-2}{2}} \sum_{k,l}\left[\int_\Omega \left|\frac{\partial}{\partial x_j} e_{kl}(\boldsymbol{u}-\boldsymbol{v})\right|^2 d\boldsymbol{x}\right]^{1/2}$$

$$+ |p-2|\left|\frac{p-4}{2}\right| \epsilon^{\frac{p-4}{2}} \sum_{k,l}\sum_{m,n}\left[\int_\Omega |e_{mn}(\boldsymbol{u}+\boldsymbol{v})|^2 |e_{mn}(\boldsymbol{u}-\boldsymbol{v})|^2 \left|\frac{\partial}{\partial x_j} e_{kl}(\boldsymbol{v})\right|^2 d\boldsymbol{x}\right]^{1/2}$$

$$+ |p-2|\epsilon^{\frac{p-4}{2}} \sum_{k,l}\left[\int_\Omega |e_{ij}(\boldsymbol{u})|^2 |e_{kl}(\boldsymbol{u}-\boldsymbol{v})|^2 \left|\frac{\partial}{\partial x_j} e_{kl}(\boldsymbol{v})\right|^2 d\boldsymbol{x}\right]^{1/2}$$

$$+ |p-2|\epsilon^{\frac{p-4}{2}} \sum_{k,l}\left[\int_\Omega |e_{ij}(\boldsymbol{u}-\boldsymbol{v})|^2 |e_{kl}(\boldsymbol{v})|^2 d\boldsymbol{x}\right]^{1/2}.$$

Therefore,

$$\left[\int_\Omega \left|\frac{\partial}{\partial x_j}\left[\left(\epsilon + |\boldsymbol{e}(\boldsymbol{u})|^2\right)^{\frac{p-2}{2}} e_{ij}(\boldsymbol{u})\right] - \frac{\partial}{\partial x_j}\left[\left(\epsilon + |\boldsymbol{e}(\boldsymbol{v})|^2\right)^{\frac{p-2}{2}} e_{ij}(\boldsymbol{v})\right]\right|^2 d\boldsymbol{x}\right]^{1/2}$$

$$\le c_1 \epsilon^{\frac{p-2}{2}} |e_{ij}(\boldsymbol{u}-\boldsymbol{v})|_{H^1(\Omega)}$$

$$+ c_1\left|\frac{p-2}{2}\right| \epsilon^{\frac{p-4}{2}} \sum_{k,l} |e_{kl}(\boldsymbol{u}+\boldsymbol{v})|_{H^1(\Omega)} |e_{kl}(\boldsymbol{u}-\boldsymbol{v})|_{H^1(\Omega)} \left|\frac{\partial}{\partial x_j} e_{ij}(\boldsymbol{v})\right|_{H^1(\Omega)}$$

$$+ c_1|p-2|\epsilon^{\frac{p-2}{2}} \sum_{k,l} |e_{kl}(\boldsymbol{u}-\boldsymbol{v})|_{H^1(\Omega)}$$

$$+ c_1|p-2|\left|\frac{p-4}{2}\right| \epsilon^{\frac{p-4}{2}} \sum_{k,l}\sum_{m,n} |e_{mn}(\boldsymbol{u}+\boldsymbol{v})|_{H^1(\Omega)} |e_{mn}(\boldsymbol{u}-\boldsymbol{v})|_{H^1(\Omega)} \left|\frac{\partial}{\partial x_j} e_{kl}(\boldsymbol{v})\right|_{H^1(\Omega)}$$

$$+ c_1|p-2|\epsilon^{\frac{p-4}{2}} \sum_{k,l} |e_{ij}(\boldsymbol{u})|_{H^1(\Omega)} |e_{kl}(\boldsymbol{u}-\boldsymbol{v})|_{H^1(\Omega)} \left|\frac{\partial}{\partial x_j} e_{kl}(\boldsymbol{v})\right|_{H^1(\Omega)}$$

$$+ c_1|p-2|\epsilon^{\frac{p-4}{2}} \sum_{k,l} \left|e_{ij}(\boldsymbol{u}-\boldsymbol{v})\right|_{H^1(\Omega)} |e_{kl}(\boldsymbol{v})|_{H^1(\Omega)} \left|\frac{\partial}{\partial x_j} e_{kl}(\boldsymbol{v})\right|_{H^1(\Omega)}.$$

Thus, for some constant $c_2(\Omega) > 0$,

$$\begin{aligned}
&\left[\int_\Omega \left|\frac{\partial}{\partial x_j}\left[\left(\epsilon+|\boldsymbol{e}(\boldsymbol{u})|^2\right)^{\frac{p-2}{2}} e_{ij}(\boldsymbol{u})\right] - \frac{\partial}{\partial x_j}\left[\left(\epsilon+|\boldsymbol{e}(\boldsymbol{v})|^2\right)^{\frac{p-2}{2}} e_{ij}(\boldsymbol{v})\right]\right|^2 d\boldsymbol{x}\right]^{1/2} \\
&\le c_2\epsilon^{\frac{p-2}{2}} ||\boldsymbol{u}-\boldsymbol{v}||_{\boldsymbol{H}^2(\Omega)} \\
&\quad + c_2\left|\frac{p-2}{2}\right|\epsilon^{\frac{p-4}{2}} ||\boldsymbol{u}+\boldsymbol{v}||_{\boldsymbol{H}^2(\Omega)} ||\boldsymbol{u}-\boldsymbol{v}||_{\boldsymbol{H}^2(\Omega)} |\boldsymbol{v}|_{\boldsymbol{H}^3(\Omega)} \\
&\quad + c_2|p-2|\epsilon^{\frac{p-2}{2}} ||\boldsymbol{u}-\boldsymbol{v}||_{\boldsymbol{H}^2(\Omega)} \\
&\quad + c_2|p-2|\left|\frac{p-4}{2}\right|\epsilon^{\frac{p-4}{2}} ||\boldsymbol{u}+\boldsymbol{v}||_{\boldsymbol{H}^2(\Omega)} ||\boldsymbol{u}+\boldsymbol{v}||_{\boldsymbol{H}^2(\Omega)} |\boldsymbol{v}|_{\boldsymbol{H}^3(\Omega)} \\
&\quad + c_2|p-2|\epsilon^{\frac{p-4}{2}} ||\boldsymbol{u}||_{\boldsymbol{H}^2(\Omega)} ||\boldsymbol{u}-\boldsymbol{v}||_{\boldsymbol{H}^2(\Omega)} |\boldsymbol{v}|_{\boldsymbol{H}^3(\Omega)} \\
&\quad + c_2|p-2|\epsilon^{\frac{p-4}{2}} ||\boldsymbol{v}||_{\boldsymbol{H}^2(\Omega)} ||\boldsymbol{u}-\boldsymbol{v}||_{\boldsymbol{H}^2(\Omega)} |\boldsymbol{v}|_{\boldsymbol{H}^3(\Omega)} \\
&\le c_2(1+|p-2|)\epsilon^{\frac{p-2}{2}} ||\boldsymbol{u}-\boldsymbol{v}||_{\boldsymbol{H}^2(\Omega)} \\
&\quad + c_2|p-2|\epsilon^{\frac{p-4}{2}} \left(\frac{3}{2}+\left|\frac{p-4}{2}\right|\right) \left(||\boldsymbol{u}||_{\boldsymbol{H}^2(\Omega)} + ||\boldsymbol{v}||_{\boldsymbol{H}^2(\Omega)}\right) \\
&\qquad \times \left(||\boldsymbol{u}-\boldsymbol{v}||_{\boldsymbol{H}^2(\Omega)} |\boldsymbol{v}|_{\boldsymbol{H}^3(\Omega)}\right).
\end{aligned} \tag{6.145}$$

From (6.145), therefore, we infer the existence of constants $c_3(\Omega) > 0$, $c_4(\Omega) > 0$, such that

$$\begin{aligned}
&\left[\int_\Omega \left|\frac{\partial}{\partial x_j}\left[\left(\epsilon+|\boldsymbol{e}(\boldsymbol{u})|^2\right)^{\frac{p-2}{2}} e_{ij}(\boldsymbol{u})\right] - \frac{\partial}{\partial x_j}\left[\left(\epsilon+|\boldsymbol{e}(\boldsymbol{v})|^2\right)^{\frac{p-2}{2}} e_{ij}(\boldsymbol{v})\right]\right|^2 d\boldsymbol{x}\right]^{1/2} \\
&\le c_3||\boldsymbol{u}-\boldsymbol{v}||_{\boldsymbol{H}^2(\Omega)} + c_4\left(||\boldsymbol{u}||_{\boldsymbol{H}^2(\Omega)} + ||\boldsymbol{v}||_{\boldsymbol{H}^2(\Omega)}\right)\left(||\boldsymbol{u}-\boldsymbol{v}||_{\boldsymbol{H}^2(\Omega)} |\boldsymbol{v}|_{\boldsymbol{H}^3(\Omega)}\right)
\end{aligned} \tag{6.146}$$

where

$$\begin{cases} c_3 = c_2(1+|p-2|)\epsilon^{\frac{p-2}{2}}, \\ c_4 = c_2|p-2|\left(\frac{3}{2}+\left|\frac{p-4}{2}\right|\right)\epsilon^{\frac{p-4}{2}}. \end{cases} \tag{6.147}$$

Now, by virtue of the definition of $\boldsymbol{A}_p$, i.e., (5.358), we obtain from (6.146), by summing over i and j,

$$
\begin{aligned}
&||\boldsymbol{A}_p(\boldsymbol{u}) - \boldsymbol{A}_p(\boldsymbol{v})||_{L^2(\Omega)} \\
&= \left[\int_\Omega \sum_i \left(\sum_j \left[\frac{\partial}{\partial x_j}\left(\left(\epsilon + |\boldsymbol{e}(\boldsymbol{u})|^2\right)^{\frac{p-2}{2}} e_{ij}(\boldsymbol{u})\right) - \frac{\partial}{\partial x_j}\left(\left(\epsilon + |\boldsymbol{e}(\boldsymbol{v})|^2\right)^{\frac{p-2}{2}} e_{ij}(\boldsymbol{v})\right)\right]^2\right) d\boldsymbol{x}\right]^{1/2} \\
&\le \sum_i \left[\int_\Omega \left(\sum_j \left[\frac{\partial}{\partial x_j}\left(\left(\epsilon + |\boldsymbol{e}(\boldsymbol{u})|^2\right)^{\frac{p-2}{2}} e_{ij}(\boldsymbol{u})\right) - \frac{\partial}{\partial x_j}\left(\left(\epsilon + |\boldsymbol{e}(\boldsymbol{v})|^2\right)^{\frac{p-2}{2}} e_{ij}(\boldsymbol{v})\right)\right]^2\right) d\boldsymbol{x}\right]^{1/2} \\
&\le \sum_{i,j} \left[\int_\Omega \left|\frac{\partial}{\partial x_j}\left[\left(\epsilon + |\boldsymbol{e}(\boldsymbol{u})|^2\right)^{\frac{p-2}{2}} e_{ij}(\boldsymbol{u})\right] - \frac{\partial}{\partial x_j}\left[\left(\epsilon + |\boldsymbol{e}(\boldsymbol{v})|^2\right)^{\frac{p-2}{2}} e_{ij}(\boldsymbol{v})\right]\right|^2 d\boldsymbol{x}\right]^{1/2} \\
&\le \sum_{i,j} \left[c_3||\boldsymbol{u}-\boldsymbol{v}||_{\boldsymbol{H}^2(\Omega)} + c_4\left(||\boldsymbol{u}||_{\boldsymbol{H}^2(\Omega)} + ||\boldsymbol{v}||_{\boldsymbol{H}^2(\Omega)}\right)||\boldsymbol{u}-\boldsymbol{v}||_{\boldsymbol{H}^2(\Omega)}|\boldsymbol{v}|_{\boldsymbol{H}^3(\Omega)}\right] \\
&\le c_3 n^2||\boldsymbol{u}-\boldsymbol{v}||_{\boldsymbol{H}^2(\Omega)} + c_4 n^2\left(||\boldsymbol{u}||_{\boldsymbol{H}^2(\Omega)} + ||\boldsymbol{v}||_{\boldsymbol{H}^2(\Omega)}\right)||\boldsymbol{u}-\boldsymbol{v}||_{\boldsymbol{H}^2(\Omega)}|\boldsymbol{v}|_{\boldsymbol{H}^3(\Omega)} \\
&\le k_1||\boldsymbol{u}-\boldsymbol{v}||_{\boldsymbol{H}^2(\Omega)} + k_2\left(||\boldsymbol{u}||_{\boldsymbol{H}^2(\Omega)} + ||\boldsymbol{v}||_{\boldsymbol{H}^2(\Omega)}\right)||\boldsymbol{u}-\boldsymbol{v}||_{\boldsymbol{H}^2(\Omega)}|\boldsymbol{v}|_{\boldsymbol{H}^3(\Omega)} \qquad (6.148)
\end{aligned}
$$

thus establishing the estimate (6.140) with $k_1 = n^2 c_3$ and $k_2 = n^2 c_4$. □

6.3.3 An Estimate for the Quotient $||\boldsymbol{u} - \boldsymbol{v}||_V / ||\boldsymbol{u} - \boldsymbol{v}||_{L^2(\Omega)}$

Let $\boldsymbol{u}, \boldsymbol{v}$ be, respectively, solutions of the initial-value problem (5.361a,b) corresponding to the initial data $\boldsymbol{u}(0) = \boldsymbol{u}_0$ and $\boldsymbol{v}(0) = \boldsymbol{v}_0$. Using the definitions (5.358)–(5.360) we rewrite these two problems in the form

$$\frac{d\boldsymbol{u}}{dt} + 2\mu_1 \boldsymbol{A}\boldsymbol{u} - 2\mu_0 \boldsymbol{A}_p(\boldsymbol{u}) + \boldsymbol{B}(\boldsymbol{u}, \boldsymbol{u}) = \boldsymbol{f}, \qquad (6.149a)$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \qquad (6.149b)$$

$$\frac{d\boldsymbol{v}}{dt} + 2\mu_1 \boldsymbol{A}\boldsymbol{v} - 2\mu_0 \boldsymbol{A}_p(\boldsymbol{v}) + \boldsymbol{B}(\boldsymbol{v}, \boldsymbol{v}) = \boldsymbol{f}, \qquad (6.150a)$$

$$\boldsymbol{v}(0) = \boldsymbol{v}_0, \qquad (6.150b)$$

where $\boldsymbol{u}_0, \boldsymbol{v}_0 \in \boldsymbol{V} = \boldsymbol{V}_{per}(\Omega)$. For $\boldsymbol{u} \in \boldsymbol{V}$, $||\boldsymbol{u}||_V$ is given by (6.125). Also, in (6.149a,b), (6.150a,b), $\boldsymbol{f} \in \boldsymbol{L}^2(Q_T)$ with $Q_T = \Omega \times [0, T)$, $T > 0$. We set, for $0 < t < T$, $\boldsymbol{w}(t) = \boldsymbol{u}(t) - \boldsymbol{v}(t)$. In this section our task will be to derive an estimate for the evolution of the quotient of norms

$$q(t) = ||\boldsymbol{w}(t)||_V{}^2 / ||\boldsymbol{w}(t)||_{\boldsymbol{L}^2(\Omega)}{}^2. \tag{6.151}$$

The resulting estimate will occupy a central position in the proof of the $\boldsymbol{L}^2$ squeezing property; the precise result is the following:

Theorem 6.6. *Let* $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}_{per}(\Omega)$ *be the unique solutions of* (6.149a,b)*,* (6.150a,b) *on* $[0, T)$*,* $T > 0$*, and suppose* $\exists M > 0$ *such that*

$$||\boldsymbol{u}||_V \le M, \quad ||\boldsymbol{v}||_V \le M, \quad t \in [0, T). \tag{6.152}$$

Then $\exists \hat{c}_2 > 0$*,* $i = 1, 2$*, such that* $\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{v}$ *satisfies*

$$q(\tau) \le q(t) \exp\left(\hat{c}_1(\tau - t) + \hat{c}_2 \int_t^\tau |\boldsymbol{v}(\tau)|_{\boldsymbol{H}^3(\Omega)}{}^2 d\tau\right) \tag{6.153}$$

for $0 < t < \tau < T$*, where* $q(t)$ *is given by* (6.151)*.*

Proof. A direct calculation, which follows, e.g., the pattern of the analogous calculation in [Te4] yields the relation

$$\frac{dq}{dt} = \frac{2}{||\boldsymbol{w}||_{\boldsymbol{L}^2(\Omega)}{}^2} \left[\left(\boldsymbol{w}', \boldsymbol{A}\boldsymbol{w}\right)_{\boldsymbol{L}^2(\Omega)} - \left(\boldsymbol{w}', q\boldsymbol{w}\right)_{\boldsymbol{L}^2(\Omega)}\right] \tag{6.154}$$

where $' = \dfrac{d}{dt}$. Subtracting (6.150a) from (6.149a) we obtain

$$\begin{aligned} &\frac{d\boldsymbol{w}}{dt} + 2\mu_1 \boldsymbol{A}\boldsymbol{w} - 2\mu_0(\boldsymbol{A}_p(\boldsymbol{u}) - \boldsymbol{A}_p(\boldsymbol{v})) \\ &\qquad + \boldsymbol{B}(\boldsymbol{u}, \boldsymbol{w}) + \boldsymbol{B}(\boldsymbol{w}, \boldsymbol{v}) = \boldsymbol{0}. \end{aligned} \tag{6.155}$$

Combining (6.155) with (6.154) we find that

$$\begin{aligned} \frac{dq}{dt} = &-\frac{2}{||\boldsymbol{w}||_{L^2(\Omega)}{}^2}\left(2\mu_1 \boldsymbol{A}\boldsymbol{w} - 2\mu_0(\boldsymbol{A}_p(\boldsymbol{u}) - \boldsymbol{A}_p(\boldsymbol{v})), \boldsymbol{A}\boldsymbol{w} - q\boldsymbol{w}\right)_{\boldsymbol{L}^2(\Omega)} \\ &- \frac{2}{||\boldsymbol{w}||_{L^2(\Omega)}{}^2}(\boldsymbol{B}(\boldsymbol{u}, \boldsymbol{w}) + \boldsymbol{B}(\boldsymbol{w}, \boldsymbol{v}), \boldsymbol{A}\boldsymbol{w} - q\boldsymbol{w})_{\boldsymbol{L}^2(\Omega)}. \end{aligned} \tag{6.156}$$

However, by virtue of the definition of $q(t)$, i.e., (6.151),

$$\begin{aligned} (q\boldsymbol{w}, \boldsymbol{A}\boldsymbol{w} - q\boldsymbol{w})_{\boldsymbol{L}^2(\Omega)} &= (q\boldsymbol{w}, \boldsymbol{A}\boldsymbol{w})_{\boldsymbol{L}^2(\Omega)} - (q\boldsymbol{w}, q\boldsymbol{w})_{\boldsymbol{L}^2(\Omega)} \\ &= q(\boldsymbol{w}, \boldsymbol{A}\boldsymbol{w})_{\boldsymbol{L}^2(\Omega)} - q^2(\boldsymbol{w}, \boldsymbol{w})_{\boldsymbol{L}^2(\Omega)} \\ &\equiv 0 \end{aligned}$$

so

$$(Aw, Aw - qw)_{L^2(\Omega)} = ||Aw - qw||_{L^2(\Omega)}{}^2. \tag{6.157}$$

Using (6.157) in (6.156), and applying the Cauchy-Schwarz inequality, we obtain the estimate

$$\begin{aligned}
&\frac{dq}{dt} + \frac{4\mu_1}{||w||_{L^2(\Omega)}{}^2}||Aw - qw||_{L^2(\Omega)}{}^2 \\
&= -\frac{2}{||w||_{L^2(\Omega)}{}^2}\left(-2\mu_0(A_p(u) - A_p(v)) + B(u,w) + B(w,v), Aw - qw\right)_{L^2(\Omega)} \\
&\leq \frac{2||Aw - qw||_{L^2(\Omega)}}{||w||_{L^2(\Omega)}{}^2}\left(2\mu_0\left|\left|A_p(u) - A_p(v)\right|\right|_{L^2(\Omega)} + ||B(u,w)||_{L^2(\Omega)} + ||B(w,v)||_{L^2(\Omega)}\right).
\end{aligned} \tag{6.158}$$

By virtue of the definition (5.359) of $B(u, v)$, Sobolev embedding, and the equivalence relation

$$\left|\left|A^{1/2}u\right|\right|_{L^2(\Omega)} \sim ||u||_{H^2(\Omega)}$$

we have for some constants $\bar{C}(\Omega) > 0$, $C'(\Omega) > 0$,

$$\begin{aligned}
||B(u,w)||_{L^2(\Omega)} &\leq ||u||_{L^2(\Omega)}|w|_{H^1(\Omega)} \\
&\leq \bar{C}(\Omega)||u||_{L^2(\Omega)}|w|_{H^1(\Omega)} \\
&\leq C'(\Omega)||u||_V||w||_V \\
&\leq C'(\Omega)M||w||_V
\end{aligned} \tag{6.159}$$

where we have also used the hypothesis (6.152). In a similar fashion

$$||B(w,v)||_{L^2(\Omega)} \leq C'(\Omega)M||w||_V \tag{6.160}$$

Also, if we use (6.140), then for $t \in [0, T)$ we have, again by virtue of (6.152),

$$\begin{aligned}
\left|\left|A_p(u) - A_p(v)\right|\right|_{L^2(\Omega)} &\leq k_1||w||_{H^2(\Omega)} \\
&\quad + k_2\left(||u||_{H^2(\Omega)} + ||v||_{H^2(\Omega)}\right)||w||_{H^2(\Omega)}|v|_{H^3(\Omega)} \\
&\leq k_3||w||_V + k_4M||w||_V|v|_{H^3(\Omega)}
\end{aligned} \tag{6.161}$$

for some $k_i(\Omega) > 0, i = 3, 4$. Combining (6.158) with (6.159)–(6.161) we find that $q(t)$ satisfies the following differential inequality:

$$\frac{dq}{dt} + \frac{4\mu_1}{||\boldsymbol{w}||_{\boldsymbol{L}^2(\Omega)}{}^2}||\boldsymbol{A}\boldsymbol{w} - q\boldsymbol{w}||_{\boldsymbol{L}^2(\Omega)}{}^2 \leq \frac{2||\boldsymbol{A}\boldsymbol{w} - q\boldsymbol{w}||_{\boldsymbol{L}^2(\Omega)}}{||\boldsymbol{w}||_{\boldsymbol{L}^2(\Omega)}} \left(2\mu_0 k_3 + 2\mu_0 k_4 M|\boldsymbol{v}|_{\boldsymbol{H}^3(\Omega)} + 2C'(\Omega)M\right) q^{1/2}(t). \tag{6.162}$$

Applying the elementary inequality $ab \leq \frac{\epsilon a^2}{2} + \frac{b^2}{2\epsilon}$ to (6.162), with $\epsilon = 2\mu_1$, yields

$$\frac{dq}{dt} + \frac{4\mu_1}{||\boldsymbol{w}||_{\boldsymbol{L}^2(\Omega)}{}^2}||\boldsymbol{A}\boldsymbol{w} - q\boldsymbol{w}||_{\boldsymbol{L}^2(\Omega)}{}^2 \leq \frac{4\mu_1||\boldsymbol{A}\boldsymbol{w} - q\boldsymbol{w}||_{\boldsymbol{L}^2(\Omega)}{}^2}{||\boldsymbol{w}||_{\boldsymbol{L}^2(\Omega)}{}^2} + \frac{1}{\mu_1}\left(\mu_0 k_3 + \mu_0 k_4 M|\boldsymbol{v}|_{\boldsymbol{H}^3(\Omega)} + C'(\Omega)M\right)^2 q(t). \tag{6.163}$$

Therefore, with

$$\lambda(t) = \frac{1}{\mu_1}\left(\mu_0 k_3 + \mu_0 k_4 M|\boldsymbol{v}|_{\boldsymbol{H}^3(\Omega)} + C'(\Omega)M\right)^2 \tag{6.164}$$

we have, for $0 \leq t < T$,

$$\frac{dq}{dt} \leq \lambda q. \tag{6.165}$$

Integration of (6.165) from t to τ, $0 < t < \tau < T$, produces

$$q(\tau) \leq q(t) \exp\left[\int_t^\tau \lambda(s)ds\right]$$

or

$$\frac{||\boldsymbol{w}(\tau)||_V{}^2}{||\boldsymbol{w}(\tau)||_{\boldsymbol{L}^2(\Omega)}{}^2} \leq \frac{||\boldsymbol{w}(t)||_V{}^2}{||\boldsymbol{w}(t)||_{\boldsymbol{L}^2(\Omega)}{}^2} \exp\left[\int_t^\tau \lambda(s)ds\right]. \tag{6.166}$$

However, by (6.164),

$$\int_t^\tau \lambda(s)ds \leq \frac{2}{\mu_1}\left(\mu_0 k_3 + C'(\Omega)M\right)^2(\tau - t) + \left(\frac{2}{\mu_1}\right)\mu_0^2 k_4^2 M^2 \int_t^\tau |\boldsymbol{v}(s)|_{\boldsymbol{H}^3(\Omega)}{}^2 ds. \tag{6.167}$$

Combining (6.166) with (6.167) we see that (6.153) holds, for $0 < t < \tau < T$, with

$$\begin{cases} \hat{c}_1 = \dfrac{2}{\mu_1}(\mu_0 k_3 + C'(\Omega)M)^2, \\ \hat{c}_2 = \dfrac{2}{\mu_1}\mu_0^2 k_4^2 M^2, \end{cases} \tag{6.168}$$

thus completing the proof of Theorem 6.6. □

In order to derive the stated L^2 squeezing property for the solutions of (5.361a,b) we will need to deal with the term $\int_t^\tau |v(\tau)|_{H^3(\Omega)}{}^2 d\tau$ in the estimate (6.153); in essence, we are in a position to do this now as a direct consequence of Lemma 5.21 in Sect. 5.5.2. However, because this pivotal result is of independent interest, we will pause, in the following subsection, to give an alternative proof of the bound reflected in (5.362); this proof is based directly on the original formulation of our problem, i.e., (5.2a,b), (5.3b), (5.4) as opposed to its abstract reformulation as the initial value problem (5.361a,b).

6.3.4 The $L^2([0, T]; H^3(\Omega))$ Norm of *u*: An Alternate Derivation of the Estimate

We reconsider Lemma 5.21, positing it now in the following form:

Lemma 6.11. *Let $u(t)$ be the unique solution of* (5.2a,b), (5.3b), (5.4) *with $\Omega = [0, L]^n$, $L > 0$, $n = 2, 3$ and suppose that $||u||_V \le R$ for some $R > 0$ and $t \in [0, T)$. Then $\exists \hat{k} > 0$, depending at most on Ω, μ_0, μ_1, f, R, and T, such that*

$$\int_0^T \int_\Omega \frac{\partial^2 e_{ij}}{\partial x_k \partial x_l} \frac{\partial^2 e_{ij}}{\partial x_k \partial x_l}\, d\boldsymbol{x}\, dt \le \hat{k} \tag{6.169}$$

where $e = e(u)$ and we sum on all repeated indices.

Proof. We begin by rewriting (5.2a) in coordinate form for the velocity field u (instead of v) and replace the pressure field p by $\tilde{p}$ (so as to avoid any possible confusion with the index p, $1 < p \le 2$, in the nonlinear viscosity term). Fixing the index l, we set $w_i = \dfrac{\partial^2 u_i}{\partial x_l^2}$, multiply the bipolar evolution equation by w_i, sum over i, and then integrate the resulting equation over Ω so as to obtain (for density $\rho = 1$) the following identity:

$$
\begin{aligned}
&\int_{\Omega} \frac{\partial u_i}{\partial t} \frac{\partial^2 u_i}{\partial x_l^2} d\boldsymbol{x} + 2\mu_1 \int_{\Omega} \frac{\partial}{\partial x_j} (\Delta e_{ij}) \frac{\partial^2 u_i}{\partial x_l^2} d\boldsymbol{x} \\
&\quad - 2\mu_0 \int_{\Omega} \frac{\partial}{\partial x_j} \left[(\epsilon + |\boldsymbol{e}(\boldsymbol{u})|^2)^{\frac{p-2}{2}} \right] \frac{\partial^2 u_i}{\partial x_l^2} d\boldsymbol{x} \qquad (6.170) \\
&\qquad + \int_{\Omega} u_j \frac{\partial u_i}{\partial x_j} \frac{\partial^2 u_i}{\partial x_l^2} d\boldsymbol{x} + \int_{\Omega} \frac{\partial \tilde{p}}{\partial x_l} \frac{\partial^2 u_i}{\partial x_l^2} d\boldsymbol{x} = \int_{\Omega} f_i \frac{\partial^2 u_i}{\partial x_l^2} d\boldsymbol{x}.
\end{aligned}
$$

In (6.170) we hold l fixed and sum over the repeated indices i, j. We now want to integrate, by parts, in the first, second, and fifth terms on the left-hand side of (6.170); we note that

$$
\int_{\Omega} \frac{\partial u_i}{\partial t} \frac{\partial^2 u_i}{\partial x_l^2} d\boldsymbol{x} = -\frac{1}{2} \frac{\partial}{\partial t} \left(\frac{\partial u_i}{\partial x_l}, \frac{\partial u_i}{\partial x_l} \right)_{\boldsymbol{L}^2(\Omega)} \qquad (6.171a)
$$

with a sum on i but no sum on l,

$$
\int_{\Omega} \frac{\partial \tilde{p}}{\partial x_i} \frac{\partial^2 u_i}{\partial x_l^2} d\boldsymbol{x} = 0 \qquad (6.171b)
$$

by virtue of the spatial-periodicity condition (5.3b) and let the fact that $\nabla \cdot \boldsymbol{u} = 0$, and

$$
\begin{aligned}
\int_{\Omega} \frac{\partial}{\partial x_j} (\Delta e_{ij}) \frac{\partial^2 u_i}{\partial x_l^2} d\boldsymbol{x} &= \int_{\Omega} \frac{\partial}{\partial x_j} \left(\sum_k \frac{\partial^2}{\partial x_k^2} e_{ij} \right) \frac{\partial^2 u_i}{\partial x_l^2} d\boldsymbol{x} \\
&= -\int_{\Omega} \sum_k \left(\frac{\partial^2}{\partial x_k^2} e_{ij} \right) \frac{\partial}{\partial x_j} \left(\frac{\partial^2 u_i}{\partial x_l^2} \right) d\boldsymbol{x} \\
&= \int_{\Omega} \frac{\partial}{\partial x_l} \left(\sum_k \frac{\partial^2}{\partial x_k^2} e_{ij} \right) \frac{\partial}{\partial x_j} \left(\frac{\partial u_i}{\partial x_l} \right) d\boldsymbol{x} \\
&= \int_{\Omega} \frac{\partial}{\partial x_l} \left(\sum_k \frac{\partial^2}{\partial x_k^2} e_{ij} \right) \frac{\partial}{\partial x_l} \left(\frac{\partial u_i}{\partial x_j} \right) d\boldsymbol{x} \qquad (6.171c) \\
&= \int_{\Omega} \frac{\partial}{\partial x_l} \left(\sum_k \frac{\partial^2}{\partial x_k^2} e_{ij} \right) \frac{\partial}{\partial x_l} e_{ij} d\boldsymbol{x} \\
&= -\int_{\Omega} \frac{\partial}{\partial x_k} \left(\frac{\partial e_{ij}}{\partial x_l} \right) \frac{\partial}{\partial x_k} \left(\frac{\partial e_{ij}}{\partial x_l} \right) d\boldsymbol{x}
\end{aligned}
$$

where in the last line on the right-hand side of (6.171c) we sum on the repeated indices i, j, k but not on l. Thus, if we carry out the indicated integrations by parts in (6.170), and employ (6.171a,b,c), we obtain

$$-\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial u_i}{\partial x_l}, \frac{\partial u_i}{\partial x_l}\right)_{L^2(\Omega)} - 2\mu_1 \int_\Omega \frac{\partial}{\partial x_k}\left(\frac{\partial e_{ij}}{\partial x_l}\right)\frac{\partial}{\partial x_k}\left(\frac{\partial e_{ij}}{\partial x_l}\right) d\boldsymbol{x}$$
$$-2\mu_0 \int_\Omega \frac{\partial}{\partial x_j}\left[\left(\epsilon + |\boldsymbol{e}(\boldsymbol{u})|^2\right)^{\frac{p-2}{2}} e_{ij}\right]\frac{\partial^2 u_i}{\partial x_l^2} d\boldsymbol{x} + \int_\Omega u_j \frac{\partial u_i}{\partial x_j}\frac{\partial^2 u_i}{\partial x_l^2} d\boldsymbol{x} = \int_\Omega f_i \frac{\partial^2 u_i}{\partial x_l^2} d\boldsymbol{x}. \tag{6.172}$$

From (6.172) we now have the series of estimates

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial u_i}{\partial x_l}, \frac{\partial u_i}{\partial x_l}\right)_{L^2(\Omega)} + 2\mu_1 \int_\Omega \frac{\partial}{\partial x_k}\left(\frac{\partial e_{ij}}{\partial x_l}\right)\frac{\partial}{\partial x_k}\left(\frac{\partial e_{ij}}{\partial x_l}\right) d\boldsymbol{x} \\
&\quad\le -2\mu_0 \int_\Omega \frac{\partial}{\partial x_j}\left[(\epsilon + |\boldsymbol{e}(\boldsymbol{u})|^2)^{\frac{p-2}{2}} e_{ij}\right]\frac{\partial^2 u_i}{\partial x_l^2} d\boldsymbol{x} \\
&\qquad + \int_\Omega u_j \frac{\partial u_i}{\partial x_j}\frac{\partial^2 u_i}{\partial x_l^2} d\boldsymbol{x} - \int_\Omega f_i \frac{\partial^2 u_i}{\partial x_l^2} d\boldsymbol{x} \\
&\quad\le 2\mu_0 \int_\Omega \left|\frac{\partial}{\partial x_j}\left[(\epsilon + |\boldsymbol{e}(\boldsymbol{u})|^2)^{\frac{p-2}{2}} e_{ij}\right]\right|\left|\frac{\partial^2 u_i}{\partial x_l^2}\right| d\boldsymbol{x} \\
&\qquad + \int_\Omega |u_j|\left|\frac{\partial u_i}{\partial x_j}\right|\left|\frac{\partial^2 u_i}{\partial x_l^2}\right| d\boldsymbol{x} + \int_\Omega |f_i|\left|\frac{\partial^2 u_i}{\partial x_l^2}\right|^2 d\boldsymbol{x} \\
&\quad\le 2\mu_0 \left[\int_\Omega \sum_i \left|\frac{\partial}{\partial x_j}\left[(\epsilon + |\boldsymbol{e}(\boldsymbol{u})|^2)^{\frac{p-2}{2}} e_{ij}\right]\right|^2 d\boldsymbol{x}\right]^{1/2}\left[\int_\Omega \sum_i \left|\frac{\partial^2 u_i}{\partial x_l^2}\right|^2 d\boldsymbol{x}\right]^{1/2} \\
&\qquad + ||\boldsymbol{u}||_{\boldsymbol{L}^\infty(\Omega)}\left[\int_\Omega \sum_i |\nabla u_i|^2 d\boldsymbol{x}\right]^{1/2}\left[\int_\Omega \sum_i \left|\frac{\partial^2 u_i}{\partial x_l^2}\right|^2 d\boldsymbol{x}\right]^{1/2} \\
&\qquad + \left[\int_\Omega \sum_i |f_i|^2 d\boldsymbol{x}\right]^{1/2}\left[\int_\Omega \sum_i \left|\frac{\partial^2 u_i}{\partial x_l^2}\right|^2 d\boldsymbol{x}\right]^{1/2}.
\end{aligned} \tag{6.173}$$

If we set

$$g(|\boldsymbol{e}|) = (\epsilon + |\boldsymbol{e}|^2)^{\frac{p-2}{2}} \tag{6.174}$$

so that $\mu(|\boldsymbol{e}|) = \mu_0 g(|\boldsymbol{e}|)$, then

$$\int_\Omega \sum_i \left| \frac{\partial}{\partial x_j} \left[(\epsilon + |\boldsymbol{e}(\boldsymbol{u})|^2)^{\frac{p-2}{2}} e_{ij} \right] \right|^2 d\boldsymbol{x} = \int_\Omega \frac{\partial}{\partial x_j} [g(|\boldsymbol{e}|) e_{ij}] \frac{\partial}{\partial x_j} [g(|\boldsymbol{e}|) e_{ij}] d\boldsymbol{x}$$
$$= \int_\Omega \left[g \frac{\partial e_{ij}}{\partial x_j} + \frac{\partial g}{\partial x_j} e_{ij} \right] \left[g \frac{\partial e_{ij}}{\partial x_j} + \frac{\partial g}{\partial x_j} e_{ij} \right] d\boldsymbol{x}$$
$$\leq 2 \int_\Omega g^2(|\boldsymbol{e}|) \frac{\partial e_{ij}}{\partial x_j} \frac{\partial e_{ij}}{\partial x_j} d\boldsymbol{x} + 2 \int_\Omega \frac{\partial g}{\partial x_j} \frac{\partial g}{\partial x_k} e_{ij} e_{ik} d\boldsymbol{x}$$
$$\leq 2\epsilon^{p-2} \int_\Omega \frac{\partial e_{ij}}{\partial x_j} \frac{\partial e_{ij}}{\partial x_j} d\boldsymbol{x} + 2 \int_\Omega \frac{\partial g}{\partial x_j} \frac{\partial g}{\partial x_k} e_{ij} e_{ik} d\boldsymbol{x} \tag{6.175}$$

where we sum on all repeated indices. Therefore,

$$\int_\Omega \sum_i \left| \frac{\partial}{\partial x_j} \left[(\epsilon + |\boldsymbol{e}(\boldsymbol{u})|^2)^{\frac{p-2}{2}} e_{ij} \right] \right|^2 d\boldsymbol{x}$$
$$\leq 2\epsilon^{\frac{p-2}{2}} \int_\Omega \frac{\partial e_{ij}}{\partial x_j} \frac{\partial e_{ij}}{\partial x_j} d\boldsymbol{x} + 2 \int_\Omega \sum_i \left| \frac{\partial g}{\partial x_j} e_{ij} \right|^2 d\boldsymbol{x}. \tag{6.176}$$

However, for fixed i,

$$\sum_j \left| \frac{\partial g}{\partial x_j} e_{ij} \right|^2 \leq 2 \sum_j \frac{\partial g}{\partial x_j} \frac{\partial g}{\partial x_j} + 2 \sum_j e_{ij} e_{ij} \tag{6.177a}$$

and

$$\frac{\partial g}{\partial x_j} = \frac{p-2}{2} (\epsilon + |\boldsymbol{e}|^2)^{\frac{p-4}{2}} e_{ms} \frac{\partial}{\partial x_j} e_{ms} \quad (\text{sum on } m, s) \tag{6.177b}$$

so

$$\begin{aligned}
\sum_j \frac{\partial g}{\partial x_j} \frac{\partial g}{\partial x_j} &= \left(\frac{p-2}{2} \right)^2 (\epsilon + |\boldsymbol{e}|^2)^{p-4} \sum_j \left(e_{ms} \frac{\partial}{\partial x_j} e_{ms} \right)^2 \\
&\leq \left(\frac{p-2}{2} \right)^2 (\epsilon + |\boldsymbol{e}|^2)^{p-4} |\boldsymbol{e}|^2 \left(\frac{\partial}{\partial x_j} e_{ms} \frac{\partial}{\partial x_j} e_{ms} \right) \\
&\leq \left(\frac{p-2}{2} \right) (\epsilon + |\boldsymbol{e}|^2)^{p-2} \left(\frac{\partial}{\partial x_j} e_{ms} \frac{\partial}{\partial x_j} e_{ms} \right) \\
&\leq \left(\frac{p-2}{2} \right)^2 \epsilon^{p-2} \left(\frac{\partial}{\partial x_j} e_{ms} \frac{\partial}{\partial x_j} e_{ms} \right).
\end{aligned} \tag{6.178}$$

Combining (6.178) with (6.177a) yields, for fixed i,

$$\sum_j \left| \frac{\partial g}{\partial x_j} e_{ij} \right|^2 \leq 2 \sum_j e_{ij} e_{ij} + \left(\frac{p-2}{2} \right)^2 \epsilon^{p-2} \sum_j \frac{\partial e_{ms}}{\partial x_j} \frac{\partial e_{ms}}{\partial x_j} \tag{6.179}$$

where, in the last term of (6.179), we also sum on m, n. Inserting (6.179) into (6.176) now produces the estimate

$$\begin{aligned} &\int_\Omega \sum_j \left| \frac{\partial}{\partial x_j} \left[(\epsilon + |\boldsymbol{e}(\boldsymbol{u})|^2)^{\frac{p-2}{2}} e_{ij} \right] \right|^2 d\boldsymbol{x} \\ &\quad \leq 2\epsilon^{\frac{p-2}{2}} \int_\Omega \frac{\partial e_{ij}}{\partial x_j} \frac{\partial e_{ij}}{\partial x_j} d\boldsymbol{x} + 4 \int_\Omega e_{ij} e_{ij} d\boldsymbol{x} \\ &\quad\quad + (p-2)^2 \cdot n \cdot \epsilon^{\frac{p-2}{2}} \int_\Omega \frac{\partial e_{ms}}{\partial x_j} \cdot \frac{\partial e_{ms}}{\partial x_j} d\boldsymbol{x} \end{aligned} \tag{6.180}$$

where n, the dimension of the physical space, is either 2 or 3 and we sum, in (6.180), over all repeated indices i, j, m, s. Therefore, for some constant $\tilde{C}$ depending on ϵ, p, n and Ω, we have

$$\left[\int_\Omega \sum_i \left| \frac{\partial}{\partial x_j} \left[(\epsilon + |\boldsymbol{e}(\boldsymbol{u})|^2)^{\frac{p-2}{2}} e_{ij} \right] \right|^2 d\boldsymbol{x} \right]^{1/2} \leq \tilde{C} ||\boldsymbol{u}||_V. \tag{6.181}$$

Employing (6.181) in the last estimate of (6.173) we now find that for fixed l, but with summation on all other repeated indices,

$$\begin{aligned} &\frac{\partial}{\partial t} \left(\frac{\partial u_i}{\partial x_l}, \frac{\partial u_i}{\partial x_l} \right)_{L^2(\Omega)} + 2\mu_1 \int_\Omega \frac{\partial}{\partial x_k} \left(\frac{\partial e_{ij}}{\partial x_l} \right) \frac{\partial}{\partial x_k} \left(\frac{\partial e_{ij}}{\partial x_l} \right) d\boldsymbol{x} \\ &\quad \leq 2\mu_0 \tilde{C} ||\boldsymbol{u}||_V \left[\int_\Omega \sum_i \left| \frac{\partial^2 u_i}{\partial x_l^2} \right|^2 d\boldsymbol{x} \right]^{1/2} + \tilde{C} ||\boldsymbol{u}||_V{}^2 \left[\int_\Omega \sum_i \left| \frac{\partial^2 u_i}{\partial x_l^2} \right|^2 d\boldsymbol{x} \right]^{1/2} \\ &\quad\quad + ||\boldsymbol{f}||_{\boldsymbol{L}^2(\Omega)} \left[\int_\Omega \sum_i \left| \frac{\partial^2 u_i}{\partial x_l^2} \right|^2 d\boldsymbol{x} \right]^{1/2} \end{aligned} \tag{6.182}$$

for some $\tilde{C}_1(\Omega) > 0$. However, $||\boldsymbol{u}||_V \leq R$, for $t \in [0, T)$, so

$$\begin{aligned} &\frac{\partial}{\partial t} \left(\frac{\partial u_i}{\partial x_l}, \frac{\partial u_i}{\partial x_l} \right)_{L^2(\Omega)} + 2\mu_1 \int_\Omega \frac{\partial}{\partial x_k} \left(\frac{\partial e_{ij}}{\partial x_l} \right) \frac{\partial}{\partial x_k} \left(\frac{\partial e_{ij}}{\partial x_l} \right) d\boldsymbol{x} \\ &\quad \leq \left(2\mu_0 \tilde{C} R + \tilde{C}_1 R^2 + \rho ||\boldsymbol{f}||_{\boldsymbol{L}^2(\Omega)} \right) \left[\int_\Omega \sum_i \left| \frac{\partial^2 u_i}{\partial x_l^2} \right|^2 d\boldsymbol{x} \right]^{1/2}. \end{aligned} \tag{6.183}$$

Summing over l now in (6.183) produces the estimate:

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{\partial u_i}{\partial x_l}, \frac{\partial u_i}{\partial x_l}\right)_{L^2(\Omega)} + 2\mu_1 \int_\Omega \frac{\partial}{\partial x_k}\left(\frac{\partial e_{ij}}{\partial x_l}\right)\frac{\partial}{\partial x_k}\left(\frac{\partial e_{ij}}{\partial x_l}\right) d\boldsymbol{x} \\
&\qquad \leq \tilde{C}_2\left(2\mu_0 \tilde{C} R + \tilde{C}_1 R^2 + ||\boldsymbol{f}||_{\boldsymbol{L}^2(\Omega)}\right) ||\boldsymbol{f}||_V \\
&\qquad \leq \tilde{C}_2\left(2\mu_0 \tilde{C} R + \tilde{C}_1 R^2 + ||\boldsymbol{f}||_{\boldsymbol{L}^2(\Omega)}\right) R \\
&\qquad \leq \tilde{C}_3\left(2\mu_0 R^2 + R^3 + ||\boldsymbol{f}||_{\boldsymbol{L}^2(\Omega)} \cdot R\right)
\end{aligned} \tag{6.184}$$

where $\tilde{C}_2(\Omega), \tilde{C}_3(\Omega) > 0$. In (6.184) we now sum on all repeated indices i, j, k, l. Integration of (6.184) over $[0, T)$ then produces the estimate

$$\begin{aligned}
&\frac{1}{2}\left(\frac{\partial u_i(T)}{\partial x_l}, \frac{\partial u_i(T)}{\partial x_l}\right)_{L^2(\Omega)} + 2\mu_1 \int_0^T \int_\Omega \frac{\partial}{\partial x_k}\left(\frac{\partial e_{ij}}{\partial x_l}\right)\frac{\partial}{\partial x_k}\left(\frac{\partial e_{ij}}{\partial x_l}\right) d\boldsymbol{x}\, dt \\
&\quad \leq \frac{1}{2}\left(\frac{\partial u_i(0)}{\partial x_l}, \frac{\partial u_i(0)}{\partial x_l}\right)_{L^2(\Omega)} + \tilde{C}_3(2\mu_0 R^3 + R^3)T + \tilde{C}_3 R \int_0^T ||\boldsymbol{f}||_{\boldsymbol{L}^2(\Omega)}(t)\, dt \\
&\quad \leq \tilde{C}_4 R^3 + \tilde{C}_3(2\mu_0 R^3 + R^3)T + \tilde{C}_3 R \int_0^T ||\boldsymbol{f}||_{\boldsymbol{L}^2(\Omega)}(t)\, dt
\end{aligned} \tag{6.185}$$

for some $\tilde{C}_4(\Omega) > 0$. However, as a direct consequence of (6.185), it follows that

$$\begin{aligned}
&\int_0^T \int_\Omega \frac{\partial}{\partial x_k}\left(\frac{\partial e_{ij}}{\partial x_l}\right)\frac{\partial}{\partial x_k}\left(\frac{\partial e_{ij}}{\partial x_l}\right) d\boldsymbol{x}\, dt \\
&\qquad \leq \frac{1}{2\mu_1}\left[\tilde{C}_4 R^3 + \tilde{C}_3 T(2\mu_0 R^2 + R^3) + \tilde{C}_3 R \int_0^T ||\boldsymbol{f}||_{\boldsymbol{L}^2(\Omega)}(t)\, dt\right]
\end{aligned} \tag{6.186}$$

which serves to establish (6.169) with $\hat{k}(\mu_0, \mu_1, \Omega, \boldsymbol{f}, R, T)$ as given by the right-hand side of (6.186). □

In view of the bound expressed by (6.186), and the equivalence

$$|\boldsymbol{u}|_{\boldsymbol{H}^3(\Omega)}{}^2 \sim \int_\Omega \frac{\partial}{\partial x_k}\left(\frac{\partial e_{ij}(\boldsymbol{u})}{\partial x_l}\right)\frac{\partial}{\partial x_k}\left(\frac{\partial e_{ij}(\boldsymbol{u})}{\partial x_l}\right) d\boldsymbol{x}$$

we may state the following

Theorem 6.7. *Under the same hypotheses as those which apply in the statement of Lemma 6.11,* $\exists k^* > 0$*,* k^* *depending only on* Ω*,* μ_0*,* μ_1*,* $\boldsymbol{f}$*,* R*, and* T*, such that*

$$\int_0^T |\boldsymbol{u}|_{\boldsymbol{H}^3(\Omega)}{}^2(t)\, dt \leq k^*. \tag{6.187}$$

Remarks. From the manner in which $\hat{k}$ was defined at the conclusion of the proof of Lemma 6.11, it is clear that the result expressed by Theorem 6.7 is entirely equivalent to that expressed by Theorem 5.13.

6.3.5 The L^2 Squeezing Property for the Orbits of S_{μ_1}

We are now in a position to prove the validity of the $\boldsymbol{L}^2$ squeezing property for solutions of the incomrpessible bipolar equations, under the condition of spatial periodicity of the velocity field, when $1 < p \leq 2$; the crucial result in this direction is the following lemma:

Lemma 6.12. *With* $1 < p \leq 2$*, let* $\boldsymbol{u}(t)$*,* $\boldsymbol{v}(t)$ *be two solutions of* (5.2a,b)*,* (5.3b)*,* (5.4)*, corresponding to initial data* $\boldsymbol{u}_0, \boldsymbol{v}_0$*, such that for some* $M > 0$*,* $||\boldsymbol{u}(t)||_{\boldsymbol{V}} \leq M$*,* $||\boldsymbol{v}(t)||_{\boldsymbol{V}} \leq M$*, for* $0 \leq t < T$*. Then for any* $t_0 < T$*,* $\exists k_1' > 0$*,* $k_2' > 0$*, and* $\nu_0 > 0$ *such that*

$$||\boldsymbol{w}(t_0)||_{\boldsymbol{L}^2(\Omega)}{}^2 \leq ||\boldsymbol{w}_0||_{\boldsymbol{L}^2(\Omega)}{}^2 \exp(-\mu_1 \nu_0 t_0 k_1' + k_2'). \tag{6.188}$$

where $\boldsymbol{w}(t) = \boldsymbol{u}(t) - \boldsymbol{v}(t)$*.*

Proof. We work with the abstract formulation of (5.2a,b), (5.3b), (5.4), i.e., $\boldsymbol{u}(t)$, $\boldsymbol{v}(t)$, respectively, are assumed to satisfy (6.149a,b) and (6.150a,b), with $\boldsymbol{u}_0, \boldsymbol{v}_0 \in \boldsymbol{V}$ and $||\boldsymbol{u}||_{\boldsymbol{V}} \leq M$, $||\boldsymbol{v}||_{\boldsymbol{V}} \leq M$ for some $M > 0$ and all $t \in [0, T)$. The difference $\boldsymbol{w}(t) = \boldsymbol{u}(t) - \boldsymbol{v}(t)$ then satisfies (6.155) and $\boldsymbol{w}(0) = \boldsymbol{u}_0 - \boldsymbol{v}_0 \equiv \boldsymbol{w}_0$. If we take the inner-product, in $\boldsymbol{L}^2(\Omega)$, of (6.155) with $\boldsymbol{w}(t)$ we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}||\boldsymbol{w}||_{\boldsymbol{L}^2(\Omega)}{}^2 + 2\mu_1 ||\boldsymbol{w}||_{\boldsymbol{V}}{}^2 \\ &\quad \leq \left(2\mu_0 \left|\left|\boldsymbol{A}_p(\boldsymbol{u}) - \boldsymbol{A}_p(\boldsymbol{v})\right|\right|_{\boldsymbol{L}^2(\Omega)} + ||\boldsymbol{B}(\boldsymbol{u},\boldsymbol{w})||_{\boldsymbol{L}^2(\Omega)} + ||\boldsymbol{B}(\boldsymbol{w},\boldsymbol{v})||_{\boldsymbol{L}^2(\Omega)}\right) ||\boldsymbol{w}||_{\boldsymbol{L}^2(\Omega)}. \end{aligned} \tag{6.189}$$

Employing the estimates (6.159)–(6.161) in (6.189) we then obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}||\boldsymbol{w}||_{\boldsymbol{L}^2(\Omega)}{}^2 + 2\mu_1 ||\boldsymbol{w}||_{\boldsymbol{V}}{}^2 \\ &\quad \leq (2\mu_0 k_3 + 2\mu_0 k_4 M |\boldsymbol{v}|_{\boldsymbol{H}^3(\Omega)} + 2C'(\Omega) M) ||\boldsymbol{w}||_{\boldsymbol{V}} ||\boldsymbol{w}||_{\boldsymbol{L}^2(\Omega)} \end{aligned} \tag{6.190}$$

so that

$$\begin{aligned} &\frac{d}{dt}||\boldsymbol{w}||_{\boldsymbol{L}^2(\Omega)}{}^2 + 4\mu_1 ||\boldsymbol{w}||_{\boldsymbol{V}}{}^2 \\ &\qquad \leq \left(2\mu_0 k_3 + 2\mu_0 k_4 M |\boldsymbol{v}|_{\boldsymbol{H}^3(\Omega)} + 2C'(\Omega) M\right) (2||\boldsymbol{w}||_{\boldsymbol{V}}) ||\boldsymbol{w}||_{\boldsymbol{L}^2(\Omega)} \end{aligned}$$

or

$$\begin{aligned}&\frac{d}{dt}||\boldsymbol{w}||_{\boldsymbol{L}^2(\Omega)}{}^2 + 4\mu_1||\boldsymbol{w}||_V{}^2\\&\leq 3\mu_1||\boldsymbol{w}||_V{}^2 + \frac{16}{3\mu_1}\left(\mu_0^2k_3^2 + \mu_0^2k_4^2M^2|\boldsymbol{v}|_{\boldsymbol{H}^3(\Omega)}{}^2 + C'^2(\Omega)M^2\right)||\boldsymbol{w}||_{\boldsymbol{L}^2(\Omega)}{}^2.\end{aligned} \tag{6.191}$$

If we now set

$$\eta(t) = \frac{16}{3\mu_1}\left(\mu_0^2k_3^2 + \mu_0^2k_4^2M^2|\boldsymbol{v}(t)|_{\boldsymbol{H}^3(\Omega)}{}^2 + C'^2(\Omega)M^2\right) \tag{6.192}$$

we find that

$$\frac{d}{dt}||\boldsymbol{w}||_{\boldsymbol{L}^2(\Omega)}{}^2 + \left(\frac{\mu_1||\boldsymbol{w}||_V{}^2}{||\boldsymbol{w}||_{\boldsymbol{L}^2(\Omega)}{}^2} - \eta\right)||\boldsymbol{w}||_{\boldsymbol{L}^2(\Omega)}{}^2 \leq 0. \tag{6.193}$$

However, by Theorem 6.6 and Theorem 6.7 (with $R \equiv M$) we have, for $0 < t < t_0 < T$,

$$\begin{aligned}\frac{||\boldsymbol{w}(t)||_V{}^2}{||\boldsymbol{w}(t)||_{\boldsymbol{L}^2(\Omega)}{}^2} &\geq \frac{||\boldsymbol{w}(t_0)||_V{}^2}{||\boldsymbol{w}(t_0)||_{\boldsymbol{L}^2(\Omega)}{}^2}\exp\left(-\hat{c}_1(t_0 - t) - \hat{c}_2\int_t^{t_0}|\boldsymbol{v}|_{\boldsymbol{H}^3(\Omega)}{}^2 d\tau\right)\\&\geq \frac{||\boldsymbol{w}(t_0)||_V{}^2}{||\boldsymbol{w}(t_0)||_{\boldsymbol{L}^2(\Omega)}{}^2}\exp(-\hat{c}_1t_0 - \hat{c}_2k^*)\\&\geq \frac{||\boldsymbol{w}(t_0)||_V{}^2}{||\boldsymbol{w}(t_0)||_{\boldsymbol{L}^2(\Omega)}{}^2}\exp(-\hat{c}_1T - \hat{c}_2k^*)\end{aligned}$$

or

$$\frac{||\boldsymbol{w}(t)||_V{}^2}{||\boldsymbol{w}(t)||_{\boldsymbol{L}^2(\Omega)}{}^2} \geq \nu_0\exp(-\hat{c}_1T - \hat{c}_2k^*), \quad 0 < t < t_0 < T \tag{6.194}$$

with $\nu_0 = ||\boldsymbol{w}(t_0)||_V{}^2/||\boldsymbol{w}(t_0)||_{\boldsymbol{L}^2(\Omega)}{}^2$. Employing (6.194) in (6.193) we obtain

$$\frac{d}{dt}||\boldsymbol{w}||_{\boldsymbol{L}^2(\Omega)}{}^2 + \left(\mu_1\nu_0\exp(-\hat{c}_1T - \hat{c}_2k^*) - \eta\right)||\boldsymbol{w}||_{\boldsymbol{L}^2(\Omega)}{}^2 \leq 0. \tag{6.195}$$

We now integrate (6.195) from 0 to t_0 and find that

$$
\begin{aligned}
||w(t_0)||_{L^2(\Omega)}{}^2 &\le ||w_0||_{L^2(\Omega)}{}^2 \exp\left[-\int_0^{t_0} \mu_1\nu_0 \exp(-\hat{c}_1 T - \hat{c}_2 k^*)d\tau + \int_0^{t_0} \eta(\tau)d\tau\right] \\
&\le ||w_0||_{L^2(\Omega)}{}^2 \exp\left[-\mu_1\nu_0 t_0 \exp(-\hat{c}_1 T - \hat{c}_2 k^*) + \int_0^{T} \eta(\tau)d\tau\right] \\
&= ||w_0||_{L^2(\Omega)}{}^2 \exp\left[-\mu_1\nu_0 t_0 k_1' + \int_0^{T} \eta(\tau)d\tau\right]
\end{aligned}
\tag{6.196}
$$

with $k_1' = \exp(-\hat{c}_1 T - \hat{c}_2 k^*)$. Also, by virtue of (6.192) and (6.187)

$$
\begin{aligned}
\int_0^T \eta(\tau)d\tau &= \frac{16}{3\mu_1}(\mu_0^2 k_3^2 + C'^2(\Omega)M^2)T \\
&\quad + \frac{16\mu_0^2 k_4^2 M^2}{3\mu_1}\int_0^T |v(\tau)|_{H^3(\Omega)}{}^2 d\tau \\
&\le \frac{16}{3\mu_1}(\mu_0^2 k_3^2 + C'^2(\Omega)M^2)T + \frac{16\mu_0^2 k_4^2 M^2}{3\mu_1}k^* \\
&\equiv k_2'
\end{aligned}
\tag{6.197}
$$

and the lemma, i.e., (6.188) now follows by combining (6.196) and (6.197). □

The L^2 squeezing property for the orbits of S_{μ_1} follows (almost) immediately from Lemma 6.12; we state it as follows:

Theorem 6.8. *Let $u(t)$, $v(t)$ be solutions of* (6.149a,b), (6.150a,b) *(alternatively,* (5.2a,b)*,* (5.3b) *and* (5.4)*, corresponding to initial data $u(0) = u_0$ and $v(0) = v_0$, respectively) such that $||u(t)||_V \le M$, $||v(t)||_V \le M$, for some $M > 0$ and all $t \in [0, T)$. Assume that $1 < p \le 2$ and that u_0, v_0 are in V. Let $\phi_1, \ldots, \phi_N$ be the first N eigenfunctions of A, $P_N : H_{per} \to \mathrm{span}\{\phi_1, \ldots, \phi_N\}$ the projection operator, and $Q_N = I - P_N$. Then $\exists \bar{c}_i > 0$, $i = 1, 2$, depending only on M, T, f, μ_0, μ_1, ϵ, and Ω, such that for every N, and all $t \in [0, T)$, either* (6.126) *holds or* (6.127) *does.*

Proof. Suppose that $||Q_N w(t_0)||_{L^2(\Omega)} > ||P_N w(t_0)||_{L^2(\Omega)}$, which corresponds to the assumption that (6.126) does not hold; then

$$
\begin{aligned}
\nu_0 \equiv \frac{||w(t_0)||_V{}^2}{||w(t_0)||_{L^2(\Omega)}{}^2} &= \frac{||P_N w(t_0)||_V{}^2 + ||Q_N w(t_0)||_V{}^2}{||P_N w(t_0)||_{L^2(\Omega)}{}^2 + ||Q_N w(t_0)||_{L^2(\Omega)}{}^2} \\
&\ge \frac{||Q_N w(t_0)||_V{}^2}{2||Q_N w(t_0)||_{L^2(\Omega)}{}^2} \ge \frac{1}{2}\lambda_{N+1}
\end{aligned}
\tag{6.198}
$$

as

$$\lambda_{N+1} = \min_{w \in M_N^{\perp}} \left(\frac{||w||_V{}^2}{||w||_{L^2(\Omega)}{}^2} \right) \tag{6.199}$$

with $M_N = \text{span}\{\phi_1, \ldots, \phi_N\}$. Using (6.198) in the estimate (6.188) we have

$$||w(t_0)||_{L^2(\Omega)}{}^2 \le ||w_0||_{L^2(\Omega)}{}^2 \exp\left(-\frac{k_1'}{2}\mu_1 \lambda_{N+1} t_0 + k_2'\right) \tag{6.200}$$

thus serving to demonstrate that (6.127) holds at an arbitrary $t = t_0 < T$ with $\bar{c}_1 = \exp(k_2')$ and $\bar{c}_2 = \frac{1}{2}k_1'$; the proof of the L^2 squeezing property for the bipolar (incompressible) fluid model, in the spatially periodic case, is now complete. □

6.4 Existence of Maximal Compact Attractors for Incompressible Viscous Bipolar Fluids in Unbounded Channels

6.4.1 Introduction

In Chap. 5 we looked at the problem of the existence of maximal compact global attractors for the incompressible, nonlinear, bipolar fluid within the context of two distinct scenarios: (1) in bounded domains, with a smooth boundary, where the boundary conditions (5.3a) apply if $\mu_1 > 0$ (and only the non-slip condition $v = \mathbf{0}$ holds for $\mu_1 = 0$) and (2) in the domain $\Omega = [0, L]^n$, $n = 2, 3$, $L > 0$ with periodic boundary conditions. In this section we turn to the problem of existence of a maximal compact global attractor for the bipolar ($\mu_1 > 0$) fluid in an unbounded channel; an existence theorem for this problem was established in Sect. 4.4. In order to keep the analysis in this section as self-contained as possible, in this section we review, briefly, the existence and uniqueness results established in Sect. 4.4 for the system (5.2a,b), (5.4) in an unbounded two-dimensional channel $\Omega \subseteq R^2$ of the form $\Omega = R \times (-a, a)$, $a > 0$; with ν the exterior unit normal to $\partial\Omega$; the boundary conditions are once again assumed to be of the form (5.3a). As in Sect. 4.4, we introduce the spaces

$$\bar{V} \equiv \text{ the closure of } J(\Omega) \text{ in } H^2(\Omega) \tag{6.201a}$$

and

$$\bar{H} \equiv \text{ the closure of } J(\Omega) \text{ in } L^2(\Omega) \tag{6.201b}$$

where

$$\boldsymbol{J}(\Omega) = \{\boldsymbol{\varphi} \in \boldsymbol{C}_0^\infty(\bar{\Omega}) \mid \boldsymbol{\varphi} = \boldsymbol{0} \text{ on } \partial\Omega \ \& \ \operatorname{div}\boldsymbol{\varphi} = 0 \text{ in } \Omega\}. \tag{6.201c}$$

Also, as in Sect. 4.4 we let $\bar{\boldsymbol{V}}'$ and $\bar{\boldsymbol{H}}'$ be the dual spaces, respectively of $\bar{\boldsymbol{V}}$ and $\bar{\boldsymbol{H}}$. The inner product and the norm in $\bar{\boldsymbol{H}}$ are those inherited from $\boldsymbol{L}^2$, i.e., $(\cdot,\ )_{\boldsymbol{L}^2(\Omega)}$ and $||\cdot||_{\boldsymbol{L}^2(\Omega)}$. By duality, if $\bar{\boldsymbol{H}}'$ is the dual of $\bar{\boldsymbol{H}}$, then the adjoint $\boldsymbol{i}^*$ of the identity is injective, $\boldsymbol{i}^*(\bar{\boldsymbol{H}}')$ is dense in $\bar{\boldsymbol{V}}'$, and we can identify $\bar{\boldsymbol{H}}'$ with a dense subspace of $\bar{\boldsymbol{V}}'$. By identifying $\bar{\boldsymbol{H}}$ with its dual $\bar{\boldsymbol{H}}'$ we obtain

$$\bar{\boldsymbol{V}} \subset \bar{\boldsymbol{H}} \equiv \bar{\boldsymbol{H}}' \subset \bar{\boldsymbol{V}}' \tag{6.202}$$

where each space is dense in the following, the injection being continuous.

We also introduce the linear operator $\bar{\boldsymbol{A}}$ by considering the positive definite $\bar{\boldsymbol{V}}$-elliptic symmetric bilinear form $\bar{a}(\cdot,\cdot) : \bar{\boldsymbol{V}} \times \bar{\boldsymbol{V}} \to R$ given by

$$\bar{a}(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \frac{\partial e_{ij}}{\partial x_j} \frac{\partial e_{ij}}{\partial x_j} d\boldsymbol{x}. \tag{6.203}$$

As a consequence of the Lax-Milgram Lemma we obtain an isometry $\bar{\boldsymbol{A}} \in \boldsymbol{\mathcal{L}}(\bar{\boldsymbol{V}}; \bar{\boldsymbol{V}}')$, via

$$\left\langle \bar{\boldsymbol{A}}\boldsymbol{u}, \boldsymbol{v} \right\rangle_{\bar{V}' \times \bar{V}} = \bar{a}(\boldsymbol{u}, \boldsymbol{v}) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{\bar{V}' \times \bar{V}'}, \quad \forall \boldsymbol{v} \in \bar{\boldsymbol{V}} \tag{6.204}$$

with $\boldsymbol{f} \in \bar{\boldsymbol{V}}'$, where the domain of $\bar{\boldsymbol{A}}$ is

$$D(\bar{\boldsymbol{A}}) = \{\boldsymbol{u} \in \boldsymbol{V} \mid \bar{a}(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v})_{L^2(\Omega)}, \boldsymbol{f} \in \bar{\boldsymbol{H}} \subset \bar{\boldsymbol{V}}', \quad \forall \boldsymbol{v} \in \bar{\boldsymbol{V}}\}. \tag{6.205}$$

Thus $\bar{\boldsymbol{A}} \in \boldsymbol{\mathcal{L}}(D(\boldsymbol{A}); \bar{\boldsymbol{H}}) \cap \boldsymbol{\mathcal{L}}(\bar{\boldsymbol{V}}, \bar{\boldsymbol{V}}')$.

To reformulate our problem in a Hilbert space setting, we define on $\boldsymbol{H}_0^1(\Omega)$ and, thus, on $\bar{\boldsymbol{V}}$, the trilinear continuous form $b(\cdot,\cdot,\cdot)$ by setting

$$b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j \, d\boldsymbol{x}, \text{ for } \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega). \tag{6.206}$$

and recall that, $\forall \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega)$,

$$b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{v}) = 0 \text{ and } b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = -b(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v}). \tag{6.207}$$

For $\boldsymbol{u}, \boldsymbol{v} \in \bar{\boldsymbol{V}}$, we denote by $\bar{\boldsymbol{B}}(\boldsymbol{u}, \boldsymbol{v})$ the element of $\bar{\boldsymbol{V}}'$ defined by

$$\left\langle \bar{\boldsymbol{B}}(\boldsymbol{u}, \boldsymbol{v}), \boldsymbol{w} \right\rangle = b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}), \quad \forall \boldsymbol{w} \in \bar{\boldsymbol{V}} \tag{6.208a}$$

and set

$$\bar{\boldsymbol{B}}(\boldsymbol{u}) = \bar{\boldsymbol{B}}(\boldsymbol{u}, \boldsymbol{u}) \in \bar{\boldsymbol{V}}', \quad \forall \boldsymbol{u} \in \bar{\boldsymbol{V}}. \tag{6.208b}$$

For $\boldsymbol{u} \in \bar{\boldsymbol{V}}$, we let $\bar{\boldsymbol{N}}(\boldsymbol{u})$ be the element of $\bar{\boldsymbol{V}}'$ defined by

$$\langle \bar{\boldsymbol{N}}(\boldsymbol{u}), \boldsymbol{v} \rangle = 2 \int_\Omega \mu(\boldsymbol{u}) e_{ij}(\boldsymbol{u}) e_{ij}(\boldsymbol{v}) d\boldsymbol{x}, \quad \forall \boldsymbol{v} \in \bar{\boldsymbol{V}}. \tag{6.209}$$

We recall the weak formulation of the problem considered in Sect. 4.4, namely, given $\boldsymbol{f}$ and $\boldsymbol{v}_0$ satisfying $\boldsymbol{f} \in L^2([0,T); \bar{\boldsymbol{H}})$ and $\boldsymbol{v}_0 \in \bar{\boldsymbol{H}}$, find $\boldsymbol{v}$ satisfying

$$\boldsymbol{v} \in L^2([0,T); \bar{\boldsymbol{V}}) \cap L^\infty([0,T); \bar{\boldsymbol{H}}), \quad \boldsymbol{v}' \in L^2([0,T); \bar{\boldsymbol{V}}'), \tag{6.210}$$

$$\boldsymbol{v}' + 2\mu_1 \bar{\boldsymbol{A}} \boldsymbol{v} + \bar{\boldsymbol{N}}(\boldsymbol{v}) + \bar{\boldsymbol{B}}(\boldsymbol{v}) = \boldsymbol{f} \tag{6.211}$$

$$\boldsymbol{v}(0) = \boldsymbol{v}_0. \tag{6.212}$$

Also, it was shown in Sect. 4.4 that any $\boldsymbol{v}$ satisfying (6.210)–(6.212), for which $\boldsymbol{V}' \in L^2([0,T); \bar{\boldsymbol{V}}')$, also satisfies

$$\left(\frac{\partial \boldsymbol{v}}{\partial t}, \boldsymbol{\phi}\right)_{L^2(\Omega)} + 2\mu_1 \left(\frac{\partial e_{ij}(\boldsymbol{v})}{\partial x_k}, \frac{\partial e_{ij}(\boldsymbol{\phi})}{\partial x_k}\right)_{L^2(\Omega)} + \langle \bar{\boldsymbol{N}}(\boldsymbol{v}), \boldsymbol{\phi} \rangle + b(\boldsymbol{v}, \boldsymbol{v}, \boldsymbol{\phi}) = (\boldsymbol{f}, \boldsymbol{\phi}). \tag{6.213}$$

$\forall \boldsymbol{\varphi} \in \bar{\boldsymbol{V}}$, with $\boldsymbol{v} \in L^2([0,T); \bar{\boldsymbol{V}}) \cap L^\infty([0,T); \bar{\boldsymbol{H}})$ and $\boldsymbol{v}(0) = \boldsymbol{v}_0$. For the abstract version of our problem, i.e., (6.210)–(6.212) the following result on existence and uniqueness was established in Sect. 4.4 (see Theorems 4.18 and 4.19): For $\boldsymbol{f} \in L^2([0,T); \bar{\boldsymbol{H}})$ and $\boldsymbol{v}_0 \in \bar{\boldsymbol{H}}$ there exists a unique solution $\boldsymbol{v}$ of (6.211), (6.212) for which (6.210) is satisfied.

As a consequence of the analysis in Sect. 4.4.4 it follows that the unique solution of (6.211), (6.212), satisfying (6.210), also satisfies $\boldsymbol{v} \in \boldsymbol{C}([0,T]; \bar{\boldsymbol{H}})$. As indicated in Sect. 4.4, because Ω is unbounded, the embeddings $\boldsymbol{H}^2(\Omega) \hookrightarrow \boldsymbol{L}^2(\Omega)$ and $\boldsymbol{H}^1(\Omega) \hookrightarrow \boldsymbol{L}^2(\Omega)$ are not compact and, thus, the existence of solutions of (6.211), (6.212) can not be established directly by using the Galerkin method. Instead, by following the approach in [BB1, 2] and letting $\{\Omega_N\}$, $N = 1, 2, \ldots$ be an expanding sequence of simply connected bounded subdomains of Ω such that $\Omega_N \to \Omega$, as $N \to \infty$, with $\partial\Omega_N$ of class C^∞, it was shown that there exists a convergent subsequence of approximate solutions whose limit is a solution of (6.211), (6.212).

In this section we will establish the existence of a maximal compact attractor for (6.211), (6.212); in order to accomplish this task we need to introduce the sequence of approximate problems employed in the existence and uniqueness proof in Sect. 4.4. Thus, with $\boldsymbol{\mathcal{D}}(\Omega_N) = \boldsymbol{C}_0^\infty(\Omega_N)$, we set

$$\begin{cases} \Gamma_N^+ = \{(x_1, a) \mid (x_1, a) \in \bar{\Omega}_N\}, \\ \Gamma_N^- = \{(x_1, -a) \mid (x_1, -a) \in \bar{\Omega}_N\}, \end{cases} \tag{6.214}$$

$$\boldsymbol{J}(\Omega_N) = \{\boldsymbol{\varphi} \in \boldsymbol{J}(\Omega) \cap (\boldsymbol{\mathcal{D}}(\Omega_N) \cup \Gamma_N^+ \cup \Gamma_N^-)\}, \tag{6.215}$$

$$\begin{cases} \bar{\boldsymbol{V}}_N = \text{ the closure of } \boldsymbol{J}(\Omega_N) \text{ in } \boldsymbol{H}^2(\Omega_N), \\ \bar{\boldsymbol{H}}_N = \text{ the closure of } \boldsymbol{J}(\Omega_N) \text{ in } \boldsymbol{L}^2(\Omega_N). \end{cases} \tag{6.216}$$

and denote by $\bar{\boldsymbol{V}}'_N$ and $\bar{\boldsymbol{H}}'_N$, respectively, the dual of $\bar{\boldsymbol{V}}_N$ and $\bar{\boldsymbol{H}}_N$. We also define $\bar{\boldsymbol{A}}_N$ by

$$\langle \bar{\boldsymbol{A}}_N \boldsymbol{v}, \boldsymbol{\varphi} \rangle = \left(\frac{\partial e_{ij}(\boldsymbol{v})}{\partial x_k}, \frac{\partial e_{ij}(\boldsymbol{\varphi})}{\partial x_k} \right)_{L^2(\Omega)}, \quad \forall \boldsymbol{v}, \boldsymbol{\varphi} \in \bar{\boldsymbol{V}}_N. \tag{6.217}$$

We recall from Sect. 4.4 that

(i) $\boldsymbol{J}(\Omega_1) \subset \boldsymbol{J}(\Omega_2) \subset \cdots \subset \boldsymbol{J}(\Omega)$.
(ii) $\forall \boldsymbol{v} \in \bar{\boldsymbol{V}}_N$ ($\boldsymbol{v} \in \bar{\boldsymbol{H}}_N$), if we extend $\boldsymbol{v}$ by setting $\boldsymbol{v} = \boldsymbol{0}$ outside Ω_N, then $\boldsymbol{v} \in \bar{\boldsymbol{V}}_{N+j} \subset \bar{\boldsymbol{V}}$, $j = 0, 1, \ldots$ ($\boldsymbol{v} \in \bar{\boldsymbol{H}}_{N+j} \subset \bar{\boldsymbol{H}}$, $j = 1, 2, \ldots$), i.e., $\bar{\boldsymbol{V}}_1 \subset \bar{\boldsymbol{V}}_2 \subset \cdots \subset \bar{\boldsymbol{V}}$ (and $\bar{\boldsymbol{H}}_1 \subset \bar{\boldsymbol{H}}_2 \subset \cdots \subset \bar{\boldsymbol{H}}$).

and that the Nth member of the sequence of approximating problems which corresponds to (6.210), (6.211), (6.212) consists of the following: for $\boldsymbol{f}^N \in L^2([0, T); \bar{\boldsymbol{H}}_N)$, $\boldsymbol{v}_0^N \in \bar{\boldsymbol{H}}_N$ find $\boldsymbol{v}^N$ such that

$$\boldsymbol{v}^N \in L^2([0, T); \bar{\boldsymbol{V}}_N) \cap L^\infty([0, T); \bar{\boldsymbol{H}}_N), \quad (\boldsymbol{v}^N)' \in L^2([0, T); \bar{\boldsymbol{V}}'_N) \tag{6.218a}$$

$$(\boldsymbol{v}^N)' + 2\mu_1 \bar{\boldsymbol{A}} \boldsymbol{v}^N + \bar{\boldsymbol{N}}(\boldsymbol{v}^N) + \bar{\boldsymbol{B}}(\boldsymbol{v}^N) = \boldsymbol{f}^N \tag{6.218b}$$

$$\boldsymbol{v}^N(0) = \boldsymbol{v}_0^N. \tag{6.218c}$$

The existence of a solution $\boldsymbol{v}^N$ of (6.218a,b,c) was established in Sect. 4.4; moreover, it was shown there that for any $t \in (0, T)$, $\boldsymbol{v}^N$ satisfies the energy identity

$$||\boldsymbol{v}^N||_{L^2(\Omega_N)}{}^2 + 4\mu_1 \int_0^t \int_{\Omega_N} \frac{\partial e_{ij}(\boldsymbol{v}^N)}{\partial x_k} \cdot \frac{\partial e_{ij}(\boldsymbol{v}^N)}{\partial x_k} d\boldsymbol{x}\, d\tau + 2 \int_0^t \langle \mu(\boldsymbol{v}^N), \boldsymbol{v}^N \rangle d\tau$$
$$= ||\boldsymbol{v}_0^N||_{L^2(\Omega_N)}{}^2 + 2 \int_0^t \left(\boldsymbol{f}^N(\tau), \boldsymbol{v}^N \right)_{L^2(\Omega_N)} d\tau. \tag{6.219}$$

By virtue of Theorems 4.18 and 4.19, and the remark which follows the restatement of those theorems, above, it follows that there exists a unique solution $\boldsymbol{v} \in L^2([0, T); \bar{\boldsymbol{V}}) \cap C([0, T); \bar{\boldsymbol{H}})$ when $\boldsymbol{v}_0 \in \bar{\boldsymbol{H}}$ and $\boldsymbol{f} \in L^2([0, T); \bar{\boldsymbol{H}})$. We may then conclude that the associated solution operator $\boldsymbol{S}_{\mu_1}(t)$ is a continuous nonlinear map from $\bar{\boldsymbol{H}}$ into itself for each $t \in [0, T)$ and it is natural to ask whether there exists a global attractor in this case. It is easy to prove the existence of an absorbing set in $\bar{\boldsymbol{V}}$, i.e., a bounded set reached by every trajectory in a finite time; unfortunately, the injection of $\bar{\boldsymbol{V}}$ into $\bar{\boldsymbol{H}}$ is not compact, so one can not apply general results (e.g. [Te4]) in this situation. However, following some ideas in [Ab1, 2] we are able to prove time-dependent weighted estimates for the solution $\boldsymbol{v}$ so as to establish the existence of the global attractor when the function $\boldsymbol{f}$ satisfies some suitable assumption relative to decay at infinity. In this section we start by proving the existence of absorbing sets in $\bar{\boldsymbol{H}}$ and $\boldsymbol{H}^2$ in Sect. 6.4.2. Then in Sect. 6.4.3 we introduce a weighted function space and prove the time-dependent weighted estimates for the solution $\boldsymbol{v}$ which serve to establish the existence of the global attractor.

6.4.2 The Absorbing Sets in $\bar{H}$ and H^2

Let $\boldsymbol{v}_0 \in \bar{\boldsymbol{H}}$ and $\boldsymbol{f} \in L^\infty([0,\infty); \bar{\boldsymbol{H}})$. Also, let $\boldsymbol{v}_0^N, \boldsymbol{f}^N$ be, respectively, the projections of $\boldsymbol{v}_0$ and $\boldsymbol{f}$ on $\bar{\boldsymbol{H}}$ and let $\boldsymbol{v}^N$ be the unique solution of (6.218b,c) which satisfies (6.218a). Our first task is to establish the existence of an absorbing set in $\bar{\boldsymbol{H}}$. We begin with

Lemma 6.13. *If $\boldsymbol{v}$ satisfying (6.210) is the unique solution of the initial-value problem (6.211), (6.212), with $\boldsymbol{v}_0 \in \bar{\boldsymbol{H}}$ and $\boldsymbol{f} \in L^\infty([0,T); \bar{\boldsymbol{H}})$, respectively, then $\exists \beta = \beta(\mu_1; \Omega) > 0$ and $t_0 = t_0(\mu_1, \boldsymbol{v}_0) > 0$ such that $\forall t \geq t_0$,*

$$||\boldsymbol{v}||_{\boldsymbol{L}^2(\Omega)}{}^2 \leq 2\beta^2 ||\boldsymbol{f}||_\infty{}^2 \equiv \rho \tag{6.220}$$

where

$$||\boldsymbol{f}||_\infty = \sup_{[0,\infty)} ||\boldsymbol{f}||_{\boldsymbol{L}^2(\Omega)} \equiv ||\boldsymbol{f}||_{L^\infty([0,\infty);\bar{\boldsymbol{H}})}. \tag{6.221}$$

Proof. We begin by taking the inner-product of (6.218b) with $\boldsymbol{v}^N$ so as to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||\boldsymbol{v}^N||_{\boldsymbol{L}^2(\Omega_N)}{}^2 + 2\mu_1\left(\frac{\partial e_{ij}(\boldsymbol{v}^N)}{\partial x_k}, \frac{\partial e_{ij}(\boldsymbol{v}^N)}{\partial x_k}\right)_{L^2(\Omega_N)} &+ \left\langle \bar{\boldsymbol{N}}(\boldsymbol{v}^N), \boldsymbol{v}^N\right\rangle \\ &= \left(\boldsymbol{f}^N, \boldsymbol{v}^N\right)_{\boldsymbol{L}^2(\Omega_N)}\end{aligned} \tag{6.222}$$

where we have used the fact that $b(\boldsymbol{v}^N, \boldsymbol{v}^N, \boldsymbol{v}^N) = 0$. Dropping the positive term $\left\langle \bar{\boldsymbol{N}}(\boldsymbol{v}^N), \boldsymbol{v}^N\right\rangle$ in (6.222) we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||\boldsymbol{v}^N||_{\boldsymbol{L}^2(\Omega_N)}{}^2 + 2\mu_1\left(\frac{\partial e_{ij}(\boldsymbol{v}^N)}{\partial x_k}, \frac{\partial e_{ij}(\boldsymbol{v}^N)}{\partial x_k}\right)_{L^2(\Omega_N)} &\leq \left(\boldsymbol{f}^N, \boldsymbol{v}^N\right)_{\boldsymbol{L}^2(\Omega_N)} \\ &\leq ||\boldsymbol{f}^N||_\infty ||\boldsymbol{v}^N||_{\boldsymbol{L}^2(\Omega_N)} \\ &\leq ||\boldsymbol{f}||_\infty ||\boldsymbol{v}^N||_{\boldsymbol{L}^2(\Omega_N)}.\end{aligned} \tag{6.223}$$

Applying Lemma B.2 with $\Omega = \Omega_N$, and letting $N \to \infty$ in the terms representing the upper and lower bounds, after extending $\boldsymbol{v}^N$ to all of Ω by taking $\boldsymbol{v}^N \equiv \boldsymbol{0}$ in Ω/Ω_N, it follows that $\exists k = k(\Omega) > 0$ such that

$$\frac{1}{2}\frac{d}{dt}||\boldsymbol{v}^N||_{\boldsymbol{L}^2(\Omega)}{}^2 + 2\mu_1 k(\Omega)||\boldsymbol{v}^N||_{\boldsymbol{H}^2(\Omega)}{}^2 \leq ||\boldsymbol{f}||_\infty ||\boldsymbol{v}^N||_{\boldsymbol{L}^2(\Omega)} \tag{6.224}$$

so that for any $\beta > 0$,

$$\frac{1}{2}\frac{d}{dt}||\boldsymbol{v}^N||_{\boldsymbol{L}^2(\Omega)}{}^2 + 2\mu_1 k(\Omega)||\boldsymbol{v}^N||_{\boldsymbol{H}^2(\Omega)}{}^2 \leq \frac{\beta}{2}||\boldsymbol{f}||_\infty + \frac{1}{2\beta}||\boldsymbol{v}^N||_{\boldsymbol{L}^2(\Omega)}. \tag{6.225}$$

If we set $\beta = \dfrac{1}{2\mu_1 k(\Omega)}$ in (6.225), then we find that

$$\frac{1}{2}\frac{d}{dt}||\boldsymbol{v}^N||_{\boldsymbol{L}^2(\Omega)}{}^2 + 2\mu_1 k(\Omega)||\boldsymbol{v}^N||_{\boldsymbol{H}^2(\Omega)}{}^2 \leq \beta||\boldsymbol{f}||_\infty. \tag{6.226}$$

We now multiply (6.226) by $e^{2\mu_1 k(\Omega)t}$, and integrate from 0 to t, so as to obtain the estimates

$$\begin{aligned}
||\boldsymbol{v}^N||_{\boldsymbol{L}^2(\Omega)}{}^2 &\leq e^{-2\mu_1 k(\Omega)t}\left[||\boldsymbol{v}^N(0)||_{\boldsymbol{L}^2(\Omega)}{}^2 + \beta\int_0^t e^{2\mu_1 k(\Omega)s}||\boldsymbol{f}||_\infty{}^2 ds\right] \\
&\leq e^{-2\mu_1 k(\Omega)t}\left\{||\boldsymbol{v}(0)||_{\boldsymbol{L}^2(\Omega)}{}^2 + \frac{\beta||\boldsymbol{f}||_\infty{}^2}{2\mu_1 k(\Omega)t}\left[1 - e^{-2\mu_1 k(\Omega)t}\right]\right\} \\
&\leq e^{-2\mu_1 k(\Omega)t}\left[||\boldsymbol{v}(0)||_{\boldsymbol{L}^2(\Omega)}{}^2 + \beta^2||\boldsymbol{f}||_\infty{}^2\right].
\end{aligned} \tag{6.227}$$

It follows that there exists $t_0 = t_0(\mu_1; ||\boldsymbol{v}_0||_{\boldsymbol{L}^2(\Omega)}) > 0$ such that $\forall t \geq t_0$,

$$||\boldsymbol{v}^N||_{\boldsymbol{L}^2(\Omega)}{}^2 \leq 2\beta^2||\boldsymbol{f}||_\infty{}^2. \tag{6.228}$$

However, β is independent of N, so the required result, i.e., (6.220) follows from (6.228) by letting $N \to \infty$. □

Theorem 6.9. *There exists an absorbing set in* $\bar{\boldsymbol{H}}$ *for the nonlinear semigroup* $\boldsymbol{S}_{\mu_1}(t)$ *generated by the solution of the initial-value problem* (6.211)*,* (6.212) *if* $\boldsymbol{v}_0 \in \bar{\boldsymbol{H}}$ *and* $\boldsymbol{f} \in L^\infty([0,T); \bar{\boldsymbol{H}})$.

Proof. The estimate (6.220) suffices to establish the existence of an absorbing set in $\bar{\boldsymbol{H}}$ because, if $\boldsymbol{v}_0$ is in a bounded set $B_0 \subset \bar{\boldsymbol{H}}$, with $||\boldsymbol{v}_0||_{\boldsymbol{L}^2(\Omega)} \leq R_0$, for some $R_0 > 0$, $\boldsymbol{S}_{\mu_1}(t)\boldsymbol{v}_0 \in B^\rho_{\bar{\boldsymbol{H}}}$, for $t \geq t_0(\mu_1; R_0)$, where $\rho = 2\beta^2||\boldsymbol{f}||_\infty{}^2$. □

Having established the existence of an absorbing set in $\bar{\boldsymbol{H}}$ for the solutions of (6.211), (6.212), we now want to prove the existence of an absorbing set in $\boldsymbol{H}^2(\Omega)$. To this end we first prove

Lemma 6.14. *Let $\boldsymbol{v}^N$, satisfying (6.218a), be the unique solution of (6.218b,c) with $\boldsymbol{v}_0^N$, $\boldsymbol{f}^N$, respectively, the projections of $\boldsymbol{v}_0$ and $\boldsymbol{f}$ on $\bar{\boldsymbol{H}}_N$. Set*

$$y_N(t) = \mu_1 \int_{\Omega_N} \frac{\partial e_{ij}(\boldsymbol{v}^N)}{\partial x_k} \frac{\partial e_{ij}(\boldsymbol{v}^N)}{\partial x_k} d\boldsymbol{x} + \int_{\Omega_N} \Gamma(e_{ij}e_{ij}) d\boldsymbol{x}. \tag{6.229}$$

Then $\exists \tilde{c} = \tilde{c}(\Omega) > 0$, and independent of N, such that y_N satisfies the differential inequality

$$\frac{dy_N}{dt} \le a(t) y_N(t) + b(t), \tag{6.230}$$

with

$$a(t) = \frac{\tilde{c}(\Omega)}{\mu_1} ||\boldsymbol{v}^N||_{\boldsymbol{H}^2(\Omega)}{}^2 \text{ and } b(t) = \frac{1}{2} ||\boldsymbol{f}||_\infty{}^2. \tag{6.231}$$

Proof. We take the inner-product in $\boldsymbol{L}^2(\Omega_N)$ of (6.218b) with $\dfrac{d\boldsymbol{v}^N}{dt}$, and again extend $\boldsymbol{v}^N$ to all of Ω by taking $\boldsymbol{v}^N = \boldsymbol{0}$ in Ω/Ω_N, so as to obtain the energy identity

$$\left|\left|\frac{d\boldsymbol{v}^N}{dt}\right|\right|_{\boldsymbol{L}^2(\Omega)}^2 + 2\mu_1 \left(\frac{\partial e_{ij}(\boldsymbol{v}^N)}{\partial x_k}, \frac{\partial e_{ij}\left(\frac{d\boldsymbol{v}^N}{dt}\right)}{\partial x_k} \right)_{L^2(\Omega)} + \left\langle N(\boldsymbol{v}^N), \frac{d\boldsymbol{v}^N}{dt} \right\rangle + b\left(\boldsymbol{v}^N, \boldsymbol{v}^N, \frac{d\boldsymbol{v}^N}{dt}\right)$$
$$= \left(\boldsymbol{f}^N, \frac{d\boldsymbol{v}^N}{dt}\right)_{\boldsymbol{L}^2(\Omega)}. \tag{6.232}$$

If we introduce the usual potential ($e_{ij} = e_{ij}(\boldsymbol{v}^N)$),

$$\Gamma(e_{ij}e_{ij}) = \int_0^{e_{ij}e_{ij}} \mu_0(\epsilon + s)^{-\frac{\alpha}{2}}\, ds$$

so that

$$\frac{d\Gamma}{dt} = 2\mu(\boldsymbol{v}^N) e_{ij} \frac{\partial e_{ij}}{\partial t}, \tag{6.233}$$

then we have

$$\begin{aligned}\left\langle N(\boldsymbol{v}^N), \frac{d\boldsymbol{v}^N}{dt}\right\rangle &= 2\int_\Omega \mu(\boldsymbol{v}^N) e_{ij}\frac{\partial e_{ij}}{\partial t} d\boldsymbol{x} \\ &= \frac{d}{dt}\left\{\int_\Omega \Gamma(e_{ij}e_{ij}) d\boldsymbol{x}\right\}.\end{aligned} \tag{6.234}$$

From (6.232), (6.233) we obtain

$$\begin{aligned}&\left\|\frac{d\boldsymbol{v}^N}{dt}\right\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{d}{dt}\left\{\mu_1\int_\Omega \frac{\partial e_{ij}(\boldsymbol{v}^N)}{\partial x_k}\frac{\partial e_{ij}(\boldsymbol{v}^N)}{\partial x_k} d\boldsymbol{x} + \int_\Omega \Gamma(e_{ij}e_{ij}) d\boldsymbol{x}\right\} \\ &\quad = -b\left(\boldsymbol{v}^N, \boldsymbol{v}^N, \frac{d\boldsymbol{v}^N}{dt}\right) + \left(\boldsymbol{f}^N, \frac{d\boldsymbol{v}^N}{dt}\right)_{\boldsymbol{L}^2(\Omega)} \\ &\quad \le \left|\int_\Omega v_j^N \frac{\partial v_i^N}{\partial x_j}\frac{dv_i^N}{dt} d\boldsymbol{x}\right| + ||\boldsymbol{f}||_\infty \left\|\frac{d\boldsymbol{v}^N}{dt}\right\|_{\boldsymbol{L}^2(\Omega)} \\ &\quad \le \left\|v_j^N\right\|_{L^4(\Omega)}\left\|\frac{\partial v_i^N}{\partial x_j}\right\|_{L^4(\Omega)}\left\|\frac{dv_i^N}{dt}\right\|_{L^2(\Omega)} + ||\boldsymbol{f}||_\infty\left\|\frac{d\boldsymbol{v}^N}{dt}\right\|_{\boldsymbol{L}^2(\Omega)}.\end{aligned} \tag{6.235}$$

By Lemmas 4.8 and 4.9, there exist $\hat{c}(\Omega)$ and $\tilde{c}(\Omega)$, both positive and independent of N, such that

$$\begin{aligned}&\left\|v_j^N\right\|_{L^4(\Omega)}\left\|\frac{\partial v_i^N}{\partial x_j}\right\|_{L^4(\Omega)}\left\|\frac{dv_i^N}{dt}\right\|_{L^2(\Omega)} \\ &\quad \le \hat{c}(\Omega)||\boldsymbol{v}^N||_{\boldsymbol{L}^2(\Omega)}{}^{\frac{1}{2}}||\boldsymbol{v}^N||_{\boldsymbol{H}^1(\Omega)}||\boldsymbol{v}^N||_{\boldsymbol{H}^2(\Omega)}{}^{\frac{1}{2}}\left\|\frac{dv_i^N}{dt}\right\|_{L^2(\Omega)} \\ &\quad \le \bar{c}(\Omega)||\boldsymbol{v}^N||_{\boldsymbol{H}^2(\Omega)}{}^2\left\|\frac{dv_i^N}{dt}\right\|_{L^2(\Omega)}.\end{aligned} \tag{6.236}$$

Combining (6.235) and (6.236), and applying Young's inequality in conjunction with Lemma B.2 we obtain the estimates

$$\begin{aligned}&\frac{1}{2}\left\|\frac{d\boldsymbol{v}^N}{dt}\right\|_{\boldsymbol{L}^2(\Omega)}{}^2 + \frac{d}{dt}\left\{\mu_1\int_{\Omega_N} \frac{\partial e_{ij}(\boldsymbol{v}^N)}{\partial x_k}\frac{\partial e_{ij}(\boldsymbol{v}^N)}{\partial x_k} d\boldsymbol{x} + \int_{\Omega_N} \Gamma(e_{ij}e_{ij}) d\boldsymbol{x}\right\} \\ &\quad \le c^2(\Omega)||\boldsymbol{v}^N||_{\boldsymbol{H}^2(\Omega)}{}^4 + ||\boldsymbol{f}||_\infty \\ &\quad \le \frac{\tilde{c}(\Omega)}{\mu_1}||\boldsymbol{v}^N||_{\boldsymbol{H}^2(\Omega)}{}^2\left\{\mu_1\int_{\Omega_N} \frac{\partial e_{ij}(\boldsymbol{v}^N)}{\partial x_k}\frac{\partial e_{ij}(\boldsymbol{v}^N)}{\partial x_k} d\boldsymbol{x}\right\} + \frac{1}{2}||\boldsymbol{f}||_\infty{}^2\end{aligned} \tag{6.237}$$

from which (6.230), (6.231) follows. □

Having established the differential inequality (6.230) we now are in a position to apply the Uniform Gronwall Lemma (Appendix A) once we prove the following:

Lemma 6.15. *For the coefficients $a(t)$, $b(t)$ defined by (6.231), there exists r and $k_i(r)$, $i = 1,2,3$, all positive constants, independent of N, such that for all $t \geq t_0(\mu_1; ||\boldsymbol{v}_0||_{\boldsymbol{L}^2(\Omega)})$*

$$\int_t^{t+r} a(s)\,ds \leq k_1(r), \quad \int_t^{t+r} b(s)\,ds \leq k_2(r), \quad \int_t^{t+r} y_N(s)\,ds \leq k_3(r). \tag{6.238}$$

Proof. We start by choosing $r > 0$ arbitrary and then integrate (6.224) from t to $t + r$ so as to obtain

$$\begin{aligned}\frac{1}{2}||\boldsymbol{v}^N(t+r)||_{\boldsymbol{L}^2(\Omega)}{}^2 &- \frac{1}{2}||\boldsymbol{v}^N(t)||_{\boldsymbol{L}^2(\Omega)}{}^2\\ &+ \int_t^{t+r} 2\mu_1 k(\Omega)||\boldsymbol{v}^N||_{\boldsymbol{H}^2(\Omega)}{}^2 ds \leq \int_t^{t+r} ||\boldsymbol{f}||_\infty ||\boldsymbol{v}^N||_{\boldsymbol{L}^2(\Omega)}.\end{aligned} \tag{6.239}$$

An immediate consequence of (6.239) is the estimate

$$\int_t^{t+r} 2\mu_1 k(\Omega)||\boldsymbol{v}^N||_{\boldsymbol{H}^2(\Omega)}{}^2 ds \leq \int_t^{t+r} ||\boldsymbol{f}||_\infty ||\boldsymbol{v}^N||_{\boldsymbol{L}^2(\Omega)} ds + \frac{1}{2}||\boldsymbol{v}^N(t)||_{\boldsymbol{L}^2(\Omega)}{}^2. \tag{6.240}$$

However, by virtue of the definition of $a(t)$, i.e., (6.231),

$$\begin{aligned}\int_t^{t+r} a(s)\,ds &= \frac{\tilde{c}(\Omega)}{\mu_1}\int_t^{t+r} ||\boldsymbol{v}^N||_{\boldsymbol{H}^2(\Omega)}{}^2 ds\\ &\equiv \frac{\tilde{c}(\Omega)}{2\mu_1^2 k(\Omega)}\int_t^{t+r} 2\mu_1 k(\Omega)||\boldsymbol{v}^N||_{\boldsymbol{H}^2(\Omega)}{}^2 ds\end{aligned}$$

so an application of (6.240) yields

$$\begin{aligned}\int_t^{t+r} a(s)\,ds &\leq \frac{\tilde{c}(\Omega)}{2\mu_1^2 k(\Omega)}\left[\int_t^{t+r} ||\boldsymbol{f}||_\infty ||\boldsymbol{v}^N||_{\boldsymbol{L}^2(\Omega)} ds + \frac{1}{2}||\boldsymbol{v}^N(t)||_{\boldsymbol{L}^2(\Omega)}{}^2\right]\\ &\leq \frac{\tilde{c}(\Omega)}{2\mu_1^2 k(\Omega)}\left[||\boldsymbol{f}||_\infty \rho r + \frac{1}{2}\rho^2\right] \equiv k_1(r)\end{aligned} \tag{6.241}$$

where we have made use of (6.228) and, as in (6.220), set $\rho = 2\beta^2||\boldsymbol{f}||_\infty$. Also from the definition of $b(t)$, we have

$$\int_t^{t+r} b(s)\,ds = \int_t^{t+r} ||\boldsymbol{f}||_\infty ds = r||\boldsymbol{f}||_\infty \equiv k_2(r). \tag{6.242}$$

It remains, therefore, to show that for $t \geq t_0(\mu_1; ||\boldsymbol{v}_0||_{\boldsymbol{L}^2(\Omega)})$,

$$\begin{aligned}\int_t^{t+r} y(s)\,ds\mu_1 \int_t^{t+r}\int_\Omega \frac{\partial e_{ij}(\boldsymbol{v}^N)}{\partial x_k}\frac{\partial e_{ij}(\boldsymbol{v}^N)}{\partial x_k}d\boldsymbol{x}\,d\tau \\ + \int_t^{t+r}\int_\Omega \Gamma(e_{ij}(\boldsymbol{v}^N)e_{ij}(\boldsymbol{v}^N))d\boldsymbol{x}\,d\tau \leq k_3(r)\end{aligned} \tag{6.243}$$

for some $k_3(r) \geq 0$; to this end, we integrate the last estimate in (6.223) from t to $t + r$, apply (6.228), and thereby obtain

$$\begin{aligned}&\frac{1}{2}\left|\left|\boldsymbol{v}^N(t+r)\right|\right|_{\boldsymbol{L}^2(\Omega)}^2 + 2\mu_1\int_t^{t+r}\int_\Omega \frac{\partial e_{ij}(\boldsymbol{v}^N)}{\partial x_k}\frac{\partial e_{ij}(\boldsymbol{v}^N)}{\partial x_k}d\boldsymbol{x}\,d\tau \\ &\quad\leq ||\boldsymbol{f}||_\infty \int_t^{t+r}\left|\left|\boldsymbol{v}^N(s)\right|\right|^2 ds + \frac{1}{2}\left|\left|\boldsymbol{v}^N(t)\right|\right|_{\boldsymbol{L}^2(\Omega)}^2 \\ &\quad\leq ||\boldsymbol{f}||_\infty\rho r + \frac{1}{2}\rho^2, \quad \forall t \geq t_0(\mu_1; ||\boldsymbol{v}_0||_{\boldsymbol{L}^2(\Omega)}).\end{aligned} \tag{6.244}$$

By dropping the nonnegative term $\frac{1}{2}\left|\left|\boldsymbol{v}^N(t+r)\right|\right|_{\boldsymbol{L}^2(\Omega)}^2$ in (6.244), we obtain the bound

$$\mu_1\int_t^{t+r}\int_\Omega \frac{\partial e_{ij}(\boldsymbol{v}^N)}{\partial x_k}\frac{\partial e_{ij}(\boldsymbol{v}^N)}{\partial x_k}d\boldsymbol{x} \leq \frac{1}{2}||\boldsymbol{f}||_\infty\rho r + \frac{1}{4}\rho^2. \tag{6.245}$$

As we are assuming that $1 < p \leq 2$, so that $\alpha = 2 - p$ satisfies $0 \leq \alpha < 1$, $(\epsilon + s)^{-\alpha/2} \leq \epsilon^{-\alpha/2}$, $\forall s \geq 0$, and

$$\Gamma(e_{ij}e_{ij}) = \int_0^{e_{ij}e_{ij}} \mu_0(\epsilon + s)^{-\alpha/2}ds \leq \mu_0\epsilon^{-\alpha/2}e_{ij}e_{ij}.$$

Therefore, for $t \geq t_0(\mu_1; ||\boldsymbol{v}_0||_{\boldsymbol{L}^2(\Omega)})$, $\exists c'(\Omega) > 0$ such that

$$\begin{aligned}\int_t^{t+r}\int_\Omega \Gamma(e_{ij}e_{ij})d\boldsymbol{x}\,ds &\le \frac{\mu_0}{\epsilon^{\frac{\alpha}{2}}}\int_t^{t+r}\left(\int_\Omega e_{ij}e_{ij}\,d\boldsymbol{x}\right)ds\\ &\le \frac{\mu_0 c'(\Omega)}{\epsilon^{\frac{\alpha}{2}}}\int_t^{t+r}\left|\boldsymbol{v}^N\right|_{\boldsymbol{H}^1(\Omega)}{}^2 ds\\ &\le \frac{\mu_0 c'(\Omega)}{\epsilon^{\frac{\alpha}{2}}}\int_t^{t+r}\left|\left|\boldsymbol{v}^N\right|\right|_{\boldsymbol{H}^2(\Omega)}{}^2 ds \qquad (6.246)\\ &\le \frac{\mu_1\mu_0 c'(\Omega)}{\tilde{c}(\Omega)\epsilon^{\frac{\alpha}{2}}}\int_t^{t+r} a(s)\,ds\\ &\le \frac{\mu_1\mu_0 c'(\Omega)}{\tilde{c}(\Omega)\epsilon^{\frac{\alpha}{2}}}k_1(r)\end{aligned}$$

where we have used (6.241). Combining (6.245) and (6.244) we now see that (6.243) holds for $t \ge t_0(\mu_1; ||\boldsymbol{v}_0||_{\boldsymbol{L}^2(\Omega)})$ with

$$k_3(r) = \frac{1}{2}||\boldsymbol{f}||_\infty \rho r + \frac{1}{4}\rho^2 + \frac{\mu_1\mu_0 c'(\Omega)}{\tilde{c}(\Omega)\epsilon^{\frac{\alpha}{2}}}k_1(r). \qquad (6.247)$$

□

With the bounds displayed in (6.238) in hand, we may apply the Uniform Gronwall Lemma to (6.230) so as to conclude that, for $t \ge t_0(\mu_1; ||\boldsymbol{v}_0||_{\boldsymbol{L}^2(\Omega)})$,

$$y_N(t+r) \le \left(\frac{k_3(r)}{r} + k_2(r)\right)\exp(k_1(r))$$

in which case, $\forall t \ge t_0(\mu_1; ||\boldsymbol{v}||_{\boldsymbol{L}^2(\Omega)}) + r$,

$$y_N(t) \le \left(\frac{k_3(r)}{r} + k_2(r)\right)\exp(k_1(r)). \qquad (6.248)$$

By the definition of $y_N(t)$, Lemma B.2, and (6.248), there exists a constant $k(r) > 0$ such that

$$\left|\left|\boldsymbol{v}^N(t)\right|\right|_{\boldsymbol{H}^2(\Omega)} \le k(r), \quad \forall t \ge t_0(\mu_1; ||\boldsymbol{v}_0||_{\boldsymbol{L}^2(\Omega)}) + r. \qquad (6.249)$$

But the $k_i(r)$, $i = 1, 2, 3$ are independent of N and so, therefore, is $k(r)$. Thus, we may let $N \to \infty$ in (6.249) so as to obtain the following result:

Theorem 6.10. *There exists an absorbing set in $\boldsymbol{H}^2(\Omega)$ for the nonlinear semigroup $\boldsymbol{S}_{\mu_1}(t)$ generated by the solution of the initial-value problem* (6.211)*,* (6.212)*, if $\boldsymbol{v}_0 \in \bar{\boldsymbol{H}}$ and $\boldsymbol{f} \in L^\infty([0,T); \bar{\boldsymbol{H}})$.*

Proof. The estimate (6.249) shows that for $\boldsymbol{v}_0$ in a bounded set $B_0' \subset \bar{\boldsymbol{H}}$, with $||\boldsymbol{v}_0||_{\boldsymbol{L}^2(\Omega)} \leq R_0'$, for some $R_0' > 0$, $\boldsymbol{S}_{\mu_1}(t)\boldsymbol{v}_0 \in B^{\rho'}_{\boldsymbol{H}^2(\Omega)}$ for $t \geq t_0(\mu_1; R_0')$ where $\rho' = k(r)$. □

6.4.3 Existence of a Global Attractor for Flow in an Unbounded Channel

Although we have established, in Sect. 6.4.2, the existence of absorbing sets in $B^{\rho}_{\bar{\boldsymbol{H}}}$ and $B^{\rho'}_{\boldsymbol{H}^2(\Omega)}$, because Ω is not bounded the embedding $\boldsymbol{H}^2(\Omega) \hookrightarrow \bar{\boldsymbol{H}}$ is not compact; thus, at this point, we can not deduce the existence of an attractor. In order to prove the existence of a maximal compact global attractor, for the orbits of the semigroup $\boldsymbol{S}_{\mu_1}(t)$ generated by the solutions of (6.211), (6.212), we will appeal to a result which has been proven in [Ab2], namely,

Lemma 6.16. *Let $h(x_1, t)$ be a smooth (weight) function satisfying*

(i) $h(x_1, t) \geq 0$, $\forall t \geq 0$, $-\infty < x_1 < +\infty$, *and* $h(\boldsymbol{x}, 0) = 0$, $-\infty < x_1 < +\infty$,
(ii) each derivative of order ≥ 1 *of* h *is a bounded function,*
(iii) $h(x_1, t) \to +\infty$ *as* $|x_1| \to \infty$.

Then there exists a positive constant C, such that if the velocity field $\boldsymbol{v}$ in (6.211), (6.212) satisfies

$$\sup_{t \geq \bar{t}_0} \int_{\Omega} ||\boldsymbol{v}(x_1, x_2, t)||_{\boldsymbol{L}^2(\Omega)}{}^2 h(x_1, t) dx_1\, dx_2 \leq C \tag{6.250}$$

for some $\bar{t}_0 > 0$ and all $\boldsymbol{v}_0 \in B^{\rho'}_{\boldsymbol{H}^2(\Omega)}$, the dynamical system defined by (6.211), (6.212) possesses a global attractor $\boldsymbol{\mathcal{A}}_{\mu_1}$, i.e., a compact invariant set in $\bar{\boldsymbol{H}}$ which attracts every bounded set of $\bar{\boldsymbol{H}}$, and is maximal with respect to these properties.

By Lemma 6.16 it follows that to prove the existence of a maximal compact global attractor $\boldsymbol{\mathcal{A}}_{\mu_1}$, for the problem at hand, it suffices to establish (6.250). We will prove (6.250) with a specific choice of $h(x_1, t)$ under some assumptions on the function $\boldsymbol{f} = \boldsymbol{f}(t)$. We begin by setting

$$\varphi(x_1) = \ln(2 + x_1^2) \tag{6.251a}$$

and

$$h(x_1, t) = \varphi(x_1)\left[1 - \exp\left(\frac{-(t - \bar{t}_0)}{\varphi(x_1)}\right)\right] \tag{6.251b}$$

where $\bar{t}_0$, independent of N, will be determined, below; it is a straightforward exercise to verify that (i) and (iii) of Lemma 6.16 hold for this choice of h. To prove (ii), we take derivatives of φ and h so as to get

$$\varphi'(x_1) = \frac{2x_1}{2 + x_1^2}, \quad \varphi'' = \frac{4 - 2x_1^2}{(2 + x_1^2)^2} \tag{6.252}$$

and

$$\begin{aligned} \frac{\partial h}{\partial t} &= \varphi(x_1)\left[-\exp\left(\frac{-(t-\bar{t}_0)}{\varphi(x_1)}\right)\right]\left(-\frac{1}{\varphi(x_1)}\right) \\ &= \exp\left(\frac{-(t-\bar{t}_0)}{\varphi(x_1)}\right) \le 1, \end{aligned} \tag{6.253}$$

$\forall t \ge \bar{t}_0, -\infty \le x_1 \le +\infty$. Also

$$\frac{\partial h}{\partial x_1} = \varphi'(x_1)\left[1 - \exp\left(\frac{-(t-\bar{t}_0)}{\varphi(x_1)}\right) + \frac{(t-\bar{t}_0)}{\varphi^2(x_1)}\exp\left(\frac{-(t-\bar{t}_0)}{\varphi(x_1)}\right)\right] \tag{6.254}$$

and

$$\begin{aligned} \frac{\partial^2 h}{\partial x_1^2} &= \varphi''(x_1)\left[1 - \exp\left(\frac{-(t-\bar{t}_0)}{\varphi(x_1)}\right) + \frac{(t-\bar{t}_0)}{\varphi^2(x_1)}\exp\left(\frac{-(t-\bar{t}_0)}{\varphi(x_1)}\right)\right] \\ &\quad - \frac{(t-\bar{t}_0)(\varphi'(x_1))^2}{\varphi^3(x_1)}\exp\left(\frac{-(t-\bar{t}_0)}{\varphi(x_1)}\right). \end{aligned} \tag{6.255}$$

From (6.253)–(6.255), we see that there is a constant $c_1 > 0$ such that

$$\left|\frac{\partial h}{\partial t}\right|, \quad \left|\frac{\partial h}{\partial x_1}\right|, \quad \left|\frac{\partial^2 h}{\partial x_1^{2\,2}}\right| \le c_1, \quad \forall t \ge \bar{t}_0, \quad -\infty < x_1 < +\infty. \tag{6.256}$$

In order to prove (6.250), we will need the following Poincaré type inequality:

Lemma 6.17. *Let $g \in C_0^2(\Omega)$ and $\bar{h} \in C(R^1)$ with $\bar{h}(x_1) \ge 0$, $-\infty < x_1 < \infty$. Then $\exists c_2 > 0$ such that*

$$\int_\Omega |g(x_1, x_2)|^2 \bar{h}(x_1) dx_1\, dx_2 \le c_2 \int_\Omega \left|\frac{\partial^2 g}{\partial x_2^2}(x_1, x_2)\right|^2 \bar{h}(x_1) dx_1\, dx_2. \tag{6.257}$$

Proof. We write $g(x_1, x_2) = \int_{-a}^{x_2} \frac{\partial g}{\partial \eta}(x_1, \eta) d\eta$. Then

$$\begin{aligned} |g(x_1, x_2)|^2 &= \left| \int_{-a}^{x_2} \frac{\partial g}{\partial \eta}(x_1, \eta) d\eta \right|^2 \\ &\le 2a \left(\int_{-a}^{a} \left| \frac{\partial g}{\partial \eta}(x_1, \eta) \right|^2 d\eta \right) \end{aligned} \tag{6.258}$$

and

$$\begin{aligned} \int_{-a}^{a} \left| \frac{\partial g}{\partial \eta}(x_1, \eta) \right|^2 d\eta &= - \int_{-a}^{a} g(x_1, \eta) \frac{\partial^2 g}{\partial \eta^2}(x_1, \eta) d\eta \\ &\le \left(\int_{-a}^{a} |g(x_1, \eta)|^2 d\eta \right)^{\frac{1}{2}} \left(\int_{-a}^{a} \left| \frac{\partial^2 g}{\partial \eta^2}(x_1, \eta) \right|^2 d\eta \right)^{\frac{1}{2}} \\ &\le \frac{\epsilon}{2} \int_{-a}^{a} |g(x_1, \eta)|^2 d\eta + \frac{1}{2\epsilon} \int_{-a}^{a} \left| \frac{\partial^2 g}{\partial \eta^2}(x_1, \eta) \right|^2 d\eta, \quad \forall \epsilon > 0. \end{aligned} \tag{6.259}$$

From (6.258) and (6.259), we obtain, for any $\epsilon > 0$,

$$\begin{aligned} \int_\Omega |g(x_1, x_2)|^2 \bar{h}(x_1) dx_1\, dx_2 &\le 2a \int_\Omega \left[\int_{-a}^{a} \left| \frac{\partial g}{\partial \eta}(x_1, \eta) \right|^2 d\eta \right] \bar{h}(x_1) dx_1\, dx_2 \\ &\le a\epsilon \int_\Omega \left[\int_{-a}^{a} |g(x_1, \eta)|^2 d\eta \right] \bar{h}(x_1) dx_1\, dx_2 \\ &\quad + \frac{a}{\epsilon} \int_\Omega \left[\int_{-a}^{a} \left| \frac{\partial^2 g}{\partial \eta^2}(x_1, \eta) \right|^2 d\eta \right] \bar{h}(x_1) dx_1\, dx_2 \\ &= 2a^2 \epsilon \int_\Omega |g(x_1, x_2)|^2 \bar{h}(x_1) dx_1\, dx_2 \\ &\quad + \frac{2a^2}{\epsilon} \int_\Omega \left| \frac{\partial^2 g}{\partial x_2^2}(x_1, x_2) \right|^2 \bar{h}(x_1) dx_1\, dx_2. \end{aligned} \tag{6.260}$$

If we now set $\epsilon = \frac{1}{4a^2}$ in (6.260), we obtain the estimate

$$\int_\Omega |g(x_1, x_2)|^2 \bar{h}(x_1) dx_1\, dx_2 \le c_2 \int_\Omega \left| \frac{\partial^2 g}{\partial x_2^2}(x_1, x_2) \right|^2 \bar{h}(x_1) dx_1\, dx_2, \tag{6.261}$$

with $c_2 = 16a^4$. □

Lemma 6.17 can now be used to produce the following result:

Lemma 6.18. *There exists a positive constant c_3 such that*

$$\int_\Omega |\boldsymbol{v}(x_1, x_2)|^2 \bar{h}(x_1) dx_1 dx_2 \le c_3 \int_\Omega \frac{\partial e_{ij}(\boldsymbol{v})}{\partial x_k} \frac{\partial e_{ij}(\boldsymbol{v})}{\partial x_k} \bar{h}(x_1) dx_1 dx_2, \tag{6.262}$$

$\forall \boldsymbol{v} \in \bar{\boldsymbol{V}}$ *with* $\bar{h}(x_1) \ge 0$.

Proof. As $\dfrac{\partial e_{11}}{\partial x_1} = \dfrac{\partial^2 v_1}{\partial x_1^2}$, $\dfrac{\partial e_{22}}{\partial x_1} = \dfrac{\partial^2 v_2}{\partial x_1^2}$, and $\dfrac{\partial v_1}{\partial x_1} + \dfrac{\partial v_2}{\partial x_2} = 0$, we also have

$$\frac{\partial e_{12}}{\partial x_2} = \frac{1}{2}\left(\frac{\partial^2 v_1}{\partial x_2^2} - \frac{\partial^2 v_1}{\partial x_1^2}\right).$$

Therefore,

$$\begin{aligned}
\frac{\partial e_{ij}(\boldsymbol{v})}{\partial x_k} \frac{\partial e_{ij}(\boldsymbol{v})}{\partial x_k} &\ge \left|\frac{\partial e_{11}}{\partial x_1}\right|^2 + \left|\frac{\partial e_{12}}{\partial x_2}\right|^2 + \left|\frac{\partial e_{22}}{\partial x_1}\right|^2 \\
&\ge \left|\frac{\partial^2 v_1}{\partial x_1^2}\right|^2 + \frac{1}{4}\left|\frac{\partial^2 v_1}{\partial x_2^2} - \frac{\partial^2 v_2}{\partial x_1^2}\right|^2 + \left|\frac{\partial^2 v_2}{\partial x_1^2}\right|^2 \\
&\ge \frac{3}{16}\left|\frac{\partial^2 v_1}{\partial x_1^2}\right|^2 + \left|\frac{\partial^2 v_2}{\partial x_1^2}\right|^2 \ge \frac{3}{16}\left[\left|\frac{\partial^2 v_1}{\partial x_1^2}\right|^2 + \left|\frac{\partial^2 v_2}{\partial x_1^2}\right|^2\right].
\end{aligned} \tag{6.263}$$

The estimate (6.262) is now a direct consequence of (6.263) and Lemma 6.17. □

We are now in a position to establish (6.250). We will start by studying the approximate solutions $\boldsymbol{v}^N$.

Lemma 6.19. *Let $\boldsymbol{v}^N$ satisfy* (6.218a,b,c) *where $\boldsymbol{f}^N \in \boldsymbol{L}^2([0, T); \bar{\boldsymbol{H}}_N)$ and $\boldsymbol{v}_0^N \in \bar{\boldsymbol{H}}_N$. Then $\exists C > 0$, and $\bar{t}_0 > 0$, both of which are independent of N, such that, for $t \ge \bar{t}_0$,*

$$\int_{\Omega_N} \left|\left|\boldsymbol{v}^N(x_1, x_2, t)\right|\right|_{\boldsymbol{L}^2(\Omega_N)}^2 h(x_1, t) dx_1 dx_2 \le C \tag{6.264}$$

where $h(x_1, t)$ is given by (6.251b)*, with $\bar{t}_0$ as indicated below.*

Proof. The divergence free condition with respect to $\boldsymbol{v}^N$ indicates the existence of a stream function $\varphi(x_1, x_2, t)$ such that

$$\boldsymbol{v}^N = \left(\frac{\partial \varphi^N}{\partial x_2}, -\frac{\partial \varphi^N}{\partial x_1}\right) \tag{6.265}$$

and a unique φ^N can be obtained by solving the boundary-value problem

$$\begin{cases} \Delta\varphi^N = -\dfrac{\partial v_2^N}{\partial x_1} + \dfrac{\partial v_1^N}{\partial x_2}, & \text{in } \Omega_N, \\ \varphi^N = 0, & \text{on } \partial\Omega_N. \end{cases} \tag{6.266}$$

By (6.249), $\forall \boldsymbol{v}_0^N \in B^{\rho'}_{\boldsymbol{H}^2(\Omega_N)}$, there exists $\bar{t}_0 > 0$, independent of N, such that

$$||\boldsymbol{v}^N||_{\boldsymbol{H}^2(\Omega_N)} \le \rho', \quad \forall t \ge \bar{t}_0, \text{ with } N = 1, 2, \ldots. \tag{6.267}$$

In fact, by virtue of (6.249), $\bar{t}_0 = t_0(\mu_1; ||\boldsymbol{v}_0||_{\boldsymbol{L}^2(\Omega)}) + r$, with $||\boldsymbol{v}_0||_{\boldsymbol{L}^2(\Omega)} = \lim_{N\to\infty} ||\boldsymbol{v}_0^N||_{\boldsymbol{L}^2(\Omega)}$, if we again choose $\boldsymbol{v}_0^N$ to be the projection of $\boldsymbol{v}_0$ on $\bar{\boldsymbol{H}}_N$. Therefore, as a consequence of (6.265) and (6.267), $\exists c_4 > 0$, independent of N, such that

$$|\varphi^N|_{H^3(\Omega_N)} \le c_4, \quad \forall t \ge \bar{t}_0. \tag{6.268}$$

Now, let

$$\boldsymbol{w} = (w_1, w_2) = \left(\frac{\partial(\varphi h)}{\partial x_2}, -\frac{\partial(\varphi h)}{\partial x_1}\right). \tag{6.269}$$

Noting that $\boldsymbol{w} \in \bar{\boldsymbol{V}}_N$ and taking the inner product of (6.218b) with $\boldsymbol{w}$, we obtain

$$\left(\frac{\partial}{\partial t} v_i^N, w_i\right)_{L^2(\Omega)} + 2\mu_1\left(\frac{\partial e_{ij}(\boldsymbol{v}^N)}{\partial x_k}, \frac{\partial e_{ij}(\boldsymbol{v}^N)}{\partial x_k}\right)_{L^2(\Omega)} + \langle \bar{\boldsymbol{N}}(\boldsymbol{v}^N), \boldsymbol{w}\rangle + B(\boldsymbol{v}^N, \boldsymbol{v}^N, \boldsymbol{w}) = (\boldsymbol{f}^N, \boldsymbol{w})_{\boldsymbol{L}^2(\Omega)}. \tag{6.270}$$

By (6.265) and the fact that $\dfrac{\partial h}{\partial x_2} = 0$, we have

$$w_1 = v_1^N h \tag{6.271a}$$

and

$$w_2 = v_2^N h - \varphi\frac{\partial h}{\partial x_1}. \tag{6.271b}$$

A direct calculation, employing (6.256) in conjunction with (6.271a,b) yields the following estimate for the first term on the left-hand side of (6.270) for some $k_i > 0$, $i = 1, 2$:

$$
\begin{aligned}
\left(\frac{\partial v_i^N}{\partial t}, w_i\right)_{L^2(\Omega)} &= \frac{\partial v_1^N}{\partial t} w_1 + \frac{\partial v_2^N}{\partial t} w_2 \\
&= \left(\frac{\partial v_i^N}{\partial t}, v_i^N h\right)_{L^2(\Omega)} - \frac{\partial v_2^N}{\partial t}\frac{\partial h}{\partial x_1}\varphi \\
&= \frac{1}{2}\int_{\Omega_N} \left|\left|\boldsymbol{v}^N\right|\right|_{\boldsymbol{L}^2(\Omega)}{}^2 h\, dx_1 dx_2 \\
&\quad - \frac{1}{2}\int_{\Omega_N} \left|\left|\boldsymbol{v}^N\right|\right|_{\boldsymbol{L}^2(\Omega)}{}^2 \frac{\partial h}{\partial t} dx_1 dx_2 - \frac{\partial v_2^N}{\partial t}\frac{\partial h}{\partial x_1}\varphi \\
&\geq \frac{1}{2}\int_{\Omega_N} \left|\left|\boldsymbol{v}^N\right|\right|_{\boldsymbol{L}^2(\Omega)}{}^2 h\, dx_1 dx_2 - k_1 \left|\left|\frac{d\boldsymbol{v}^N}{dt}\right|\right|_{\boldsymbol{L}^2(\Omega_N)}^2 - k_2 .
\end{aligned} \tag{6.272}
$$

However,

$$
\frac{\partial e_{ij}(\boldsymbol{w})}{\partial x_k} = h\frac{\partial e_{ij}(\boldsymbol{v}^N)}{\partial x_k} + R_{ijk}, \tag{6.273}
$$

R_{ijk} being a sum of terms which involve products of the derivatives of h and φ; therefore, there exists a positive constant k_3 such that

$$
\left|\left|R_{ijk}(\boldsymbol{w})\right|\right|_{L^2(\Omega_N)} \leq k_3. \tag{6.274}
$$

We also have

$$
\begin{aligned}
&2\mu_1\left(\frac{\partial e_{ij}(\boldsymbol{v}^N)}{\partial x_k}, \frac{\partial e_{ij}(\boldsymbol{w})}{\partial x_k}\right)_{L^2(\Omega_N)} \\
&\quad = 2\mu_1\left(\frac{\partial e_{ij}(\boldsymbol{v}^N)}{\partial x_k}, h\frac{\partial e_{ij}(\boldsymbol{v}^N)}{\partial x_k}\right)_{L^2(\Omega_N)} + 2\mu_1\left(\frac{\partial e_{ij}(\boldsymbol{v}^N)}{\partial x_k}, R_{ijk}(\boldsymbol{w})\right)_{L^2(\Omega_N)} \\
&\quad \geq 2\mu_1\left(\frac{\partial e_{ij}(\boldsymbol{v}^N)}{\partial x_k}, h\frac{\partial e_{ij}(\boldsymbol{v}^N)}{\partial x_k}\right)_{L^2(\Omega_N)} - 2\mu_1\left|\left|\frac{\partial e_{ij}(\boldsymbol{v}^N)}{\partial x_k}\right|\right|_{L^2(\Omega_N)} \left|\left|R_{ijk}(\boldsymbol{w})\right|\right|_{L^2(\Omega_N)}
\end{aligned}
$$

so that

$$
2\mu_1\left(\frac{\partial e_{ij}}{\partial x_k}(\boldsymbol{v}^N), \frac{\partial e_{ij}(\boldsymbol{w})}{\partial x_k}\right)_{L^2(\Omega_N)} \geq 2\mu_1\left(\frac{\partial e_{ij}(\boldsymbol{v}^N)}{\partial x_k}, h\frac{\partial e_{ij}(\boldsymbol{v}^N)}{\partial x_k}\right)_{L^2(\Omega_N)} - k_4 \tag{6.275}
$$

for some positive constant k_4. In a similar fashion, for some $k_5 > 0$,

$$\begin{aligned}\langle \bar{\boldsymbol{N}}(\boldsymbol{v}^N), \boldsymbol{w}\rangle &= 2\int_{\Omega_N} \mu(\boldsymbol{e}) e_{ij}(\boldsymbol{v}^N) e_{ij}(\boldsymbol{w}) dx_1 dx_2 \\ &\geq 2\int_{\Omega_N} \mu(\boldsymbol{e}) e_{ij}(\boldsymbol{v}^N) e_{ij}(\boldsymbol{v}^N) h\, dx_1 dx_2 - k_5.\end{aligned} \tag{6.276}$$

Employing the fact that $\dfrac{\partial \boldsymbol{v}_j^N}{\partial x_j} = 0$, we compute that

$$\begin{aligned} b(\boldsymbol{v}^N, \boldsymbol{v}^N, \boldsymbol{w}) &= \int_{\Omega_N} v_j^N \frac{\partial v_i^N}{\partial x_j} v_i^N h\, dx_1 dx_2 - \int_{\Omega_N} v_j^N \frac{\partial v_2^N}{\partial x_j} \varphi \frac{\partial h}{\partial x_1} dx_1 dx_2 \\ &= \frac{1}{2}\int_{\Omega_N} v_j^N h \frac{\partial (v_i^N v_i^N)}{\partial x_j} dx_1 dx_2 - \int_{\Omega_N} v_j^N \frac{\partial v_2^N}{\partial x_j} \varphi \frac{\partial h}{\partial x_1} dx_1 dx_2 \\ &= -\frac{1}{2}\int_{\Omega_N} v_i^N v_i^N \frac{\partial v_j^N}{\partial x_j} h\, dx_1 dx_2 - \frac{1}{2}\int_{\Omega_N} v_i^N v_i^N v_j^N \frac{\partial h}{\partial x_j} dx_1 dx_2 \\ &\quad - \int_{\Omega_N} v_j^N \frac{\partial v_2^N}{\partial x_j} \varphi \frac{\partial h}{\partial x_j} dx_1 dx_2 \end{aligned}$$

or

$$b(\boldsymbol{v}^N, \boldsymbol{v}^N, \boldsymbol{w}) = -\frac{1}{2}\int_{\Omega_N} v_i^N v_i^N v_j^N \frac{\partial h}{\partial x_j} dx_1 dx_2 - \int_{\Omega_N} v_j^N \frac{\partial v_2^N}{\partial x_j} \varphi \frac{\partial h}{\partial x_1} dx_1 dx_2 \tag{6.277}$$

so that, for some $k_6 > 0$,

$$\begin{aligned} \left|b(\boldsymbol{v}^N, \boldsymbol{v}^N, \boldsymbol{w})\right| &\leq \frac{1}{2}\left|\int_{\Omega_N} v_i^N v_i^N v_j^N \frac{\partial h}{\partial x_j} dx_1 dx_2\right| + \left|\int_{\Omega_N} v_j^N \frac{\partial v_2^N}{\partial x_j} \varphi \frac{\partial h}{\partial x_1} dx_1 dx_2\right|. \\ &\leq k_6 \end{aligned} \tag{6.278}$$

Finally, we have, for any $\eta > 0$, and some $k_7 > 0$,

$$\begin{aligned} \left|(\boldsymbol{f}^N, \boldsymbol{w})_{L^2(\Omega_N)}\right| &\leq \left|\int_{\Omega_N} f_i^N v_i^N h\, dx_1 dx_2\right| + \left|\int_{\Omega_N} f_2^N \varphi \frac{\partial h}{\partial x_1} dx_1 dx_2\right| \\ &\leq \left[\left(\int_{\Omega_N} \left|\left|\boldsymbol{f}^N(x_1, x_2)\right|\right|_{\boldsymbol{L}^2(\Omega_N)}{}^2 h(x_1, t) dx_1 dx_2\right)^{\frac{1}{2}}\right] \\ &\quad \times \left[\left(\int_{\Omega_N} \left|\left|\boldsymbol{v}^N(x_1, x_2, t)\right|\right|_{\boldsymbol{L}^2(\Omega_N)}{}^2 h(x_1, t) dx_1 dx_2\right)^{\frac{1}{2}}\right] + k_7 \\ &\leq \frac{\eta}{2}\int_{\Omega} \left|\left|\boldsymbol{f}^N(x_1, x_2)\right|\right|_{\boldsymbol{L}^2(\Omega_N)}{}^2 h(x_1, t) dx_1 dx_2 \\ &\quad + \frac{1}{2\eta}\int_{\Omega_N} \left|\left|\boldsymbol{v}^N(x_1, x_2, t)\right|\right|_{\boldsymbol{L}^2(\Omega_N)}{}^2 h(x_1, t) dx_1 dx_2 + k_7. \end{aligned} \tag{6.279}$$

Returning to (6.270) and combining the above estimates, we obtain, $\forall \eta > 0$ and some $k_8 > 0$,

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_N} \left|\left|\boldsymbol{v}^N\right|\right|_{\boldsymbol{L}^2(\Omega_N)}^2 h\, dx_1 dx_2 + 2\mu_1 \int_{\Omega_N} \frac{\partial e_{ij}(\boldsymbol{v}^N)}{\partial x_k}\frac{\partial e_{ij}(\boldsymbol{v}^N)}{\partial x_k} h\, dx_1 dx_2$$
$$+ 2\int_{\Omega_N} \mu(\boldsymbol{e})e_{ij}(\boldsymbol{v}^N)e_{ij}(\boldsymbol{v}^N)h\, dx_1 dx_2$$
$$\leq k_1 \left|\left|\frac{d\boldsymbol{v}^N}{dt}\right|\right|_{\boldsymbol{L}^2(\Omega_N)}^2 + k_8 + \frac{\eta}{2}\int_{\Omega_N} \left|\left|\boldsymbol{f}^N(x_1,x_2)\right|\right|_{\boldsymbol{L}^2(\Omega_N)}^2 h(x_1,t)\, dx_1 dx_2$$
$$+ \frac{1}{2\eta}\int_{\Omega_N} \left|\left|\boldsymbol{v}^N(x_1,x_2,t)\right|\right|_{\boldsymbol{L}^2(\Omega_N)}^2 h(x_1,t)\, dx_1 dx_2. \tag{6.280}$$

If we now employ Lemma 6.18, taking $\bar{h}(x_1) = h(x_1,t)$ at a fixed but arbitrary $t \geq \bar{t}_0$, set $\eta = 2\mu_1 c_3$, and then drop the term

$$2\int_{\Omega_N} \mu(\boldsymbol{e})e_{ij}(\boldsymbol{v}^N)e_{ij}(\boldsymbol{v}^N)h\, dx_1 dx_2$$

in (6.280), we obtain the inequality

$$\frac{d}{dt}\int_{\Omega_N} \left|\left|\boldsymbol{v}^N\right|\right|_{\boldsymbol{L}^2(\Omega_N)}^2 h\, dx_1 dx_2 + \frac{2\mu_1}{c_3}\int_{\Omega_N} \left|\left|\boldsymbol{v}^N\right|\right|_{\boldsymbol{L}^2(\Omega_N)}^2 h\, dx_1 dx_2$$
$$\leq 2k_1 \left|\left|\frac{d\boldsymbol{v}^N}{dt}\right|\right|_{\boldsymbol{L}^2(\Omega_N)}^2 + k_9 \tag{6.281}$$

for some $k_9 > 0$. Therefore, $\forall t \geq \bar{t}_0$,

$$\frac{d}{dt}\int_{\Omega_N} \left|\left|\boldsymbol{v}^N\right|\right|_{\boldsymbol{L}^2(\Omega_N)}^2 h\, dx_1 dx_2 + \beta_1 \int_{\Omega_N} \left|\left|\boldsymbol{v}^N\right|\right|_{\boldsymbol{L}^2(\Omega_N)}^2 h\, dx_1 dx_2$$
$$\leq \beta_2 \left|\left|\frac{d\boldsymbol{v}^N}{dt}\right|\right|_{\boldsymbol{L}^2(\Omega_N)}^2 + \beta_3 \tag{6.282}$$

with $\beta_1 = 2\mu_1/c_3$, $\beta_2 = 2k_1$, and $\beta_3 = k_9$, all independent of N. Next we multiply (6.282) by $e^{\beta_1 t}$ and integrate from t_0 to s so as to get, $\forall s \geq \bar{t}_0$,

$$e^{\beta_1 s}\int_{\Omega_N} \left|\left|\boldsymbol{v}^N\right|\right|_{\boldsymbol{L}^2(\Omega_N)}^2 h(x_1,s)dx_1 dx_2 - e^{\beta_1 t_0}\int_{\Omega_N} \left|\left|\boldsymbol{v}^N\right|\right|_{\boldsymbol{L}^2(\Omega_N)}^2 h(x_1,t_2)dx_1 dx_2$$
$$\leq \beta_2 \int_{t_0}^{s} e^{\beta_1 t}\left|\left|\frac{d\boldsymbol{v}^N}{dt}\right|\right|_{\boldsymbol{L}^2(\Omega_N)}^2 dt + \beta_3 \int_{t_0}^{s} e^{\beta_1 t}dt. \tag{6.283}$$

Since $h(x_1, t_0) = 0$, if we multiply both sides of (6.283) by $e^{-\beta_1 s}$, we find that, $\forall s \geq \bar{t}_0$,

$$\int_{\Omega_N} \|\boldsymbol{v}^N\|_{\boldsymbol{L}^2(\Omega_N)}{}^2 h(x_1, s)dx_1dx_2$$
$$\leq \beta_2 e^{-\beta_1 s}\int_{t_0}^{s}\left\|\frac{d\boldsymbol{v}^N}{dt}\right\|_{\boldsymbol{L}^2(\Omega_N)}^2 dt + \frac{\beta_3}{\beta_1}\left(e^{\beta_1 s} - e^{\beta_1 \bar{t}_0}\right)e^{-\beta_1 s}. \tag{6.284}$$

We now note that there exists a constant $\beta_4 > 0$ such that

$$\frac{\beta_3}{\beta_1}\left(e^{\beta_1 s} - e^{\beta_1 \bar{t}_0}\right)e^{-\beta_1 s} \leq \beta_4, \quad \forall s \geq \bar{t}_0. \tag{6.285}$$

To prove (6.250), it will suffice to show that there is a constant $\beta_5 > 0$ such that

$$e^{-\beta_1 s}\int_{t_0}^{s} e^{\beta_1 t}\left\|\frac{d\boldsymbol{v}^N}{dt}\right\|_{\boldsymbol{L}^2(\Omega_N)}^2 dt \leq \beta_5, \quad \forall s \geq \bar{t}_0. \tag{6.286}$$

We begin by noting that, as a consequence of our previous estimates, $\forall t \geq 0$, and some $\beta_6 > 0$,

$$\left\|\frac{d\boldsymbol{v}^N}{dt}\right\|_{\boldsymbol{L}^2(\Omega_N)}^2 + \frac{d}{dt}\left\{2\mu_1\int_{\Omega_N}\frac{\partial e_{ij}(\boldsymbol{v}^N)}{\partial x_k}\frac{\partial e_{ij}(\boldsymbol{v}^N)}{\partial x_k}dx_1dx_2\right.$$
$$\left. + \int_{\Omega_N}\Gamma(e_{ij}e_{ij})dx_1dx_2\right\} \leq \beta_6. \tag{6.287}$$

Therefore,

$$\left\|\frac{d\boldsymbol{v}^N}{dt}\right\|_{\boldsymbol{L}^2(\Omega_N)}^2 + \frac{d}{dt}z(t) \leq \beta_6, \quad \forall t \geq \bar{t}_0 \tag{6.288}$$

with

$$z(t) = 2\mu_1\int_{\Omega_N}\frac{\partial e_{ij}(\boldsymbol{v}^N)}{\partial x_k}\frac{\partial e_{ij}(\boldsymbol{v}^N)}{\partial x_k}dx_1dx_2 + 2\int_{\Omega_N}\Gamma(e_{ij}e_{ij})dx_1dx_2. \tag{6.289}$$

Multiplying (6.288) by $e^{\beta_1 t}$, and adding the term $\beta_1 e^{\beta_1 t}z(t)$ to both sides of the resulting inequality, we find that

$$\begin{aligned} e^{\beta_1 t}\left\|\frac{d\boldsymbol{v}^N}{dt}\right\|_{\boldsymbol{L}^2(\Omega_N)}^2 + \frac{d}{dt}\left[e^{\beta_1 t}z(t)\right] &\leq \beta_1 e^{\beta_1 t}z(t) + \beta_6 e^{\beta_1 t} \\ &\leq \beta_7 e^{\beta_1 t}. \end{aligned} \tag{6.290}$$

Integrating (6.290) from $\bar{t}_0$ to s, and multiplying the resulting inequality by $e^{-\beta_1 s}$, we then obtain, $\forall s \geq \bar{t}_0$, the estimate

$$
\begin{aligned}
e^{-\beta_1 s} \int_{t_0}^{s} e^{\beta_1 t} \left\| \frac{d\boldsymbol{v}^N}{dt} \right\|_{\boldsymbol{L}^2(\Omega_N)}^2 dt + z(s) &\leq e^{-\beta_1(s-t_0)} z(t_0) + \beta_7 e^{-\beta_1 s} \int_{t_0}^{s} e^{\beta_1 t} dt \\
&\leq z(t_0) + \frac{\beta_7}{\beta_1} - \frac{\beta_7}{\beta_1} e^{-\beta_1(s-t_0)} \\
&\leq z(t_0) + \frac{\beta_7}{\beta_1} \equiv \beta_5 \qquad (6.291)
\end{aligned}
$$

which serves to establish (6.286). If we now employ the upper bound (6.286) in (6.284) we arrive at (6.264) with $C = \beta_5 + \dfrac{\beta_3}{\beta_1}$. □

With Lemma 6.19 established it is now a simple matter to prove the following result:

Theorem 6.11. *There exists a maximal compact global attractor $\boldsymbol{\mathcal{A}}_{\mu_1}$ for the orbits of the semigroup $\boldsymbol{S}_{\mu_1}(t)$ generated by the solutions of* (6.211)*,* (6.212)*.*

Proof. As $C > 0$ in (6.264) is independent of N, if we let $N \to \infty$ in Lemma 6.19 we obtain the bound (6.250) with $\bar{t}_0 = t_0(\mu_1, ||\boldsymbol{v}_0||_{\boldsymbol{L}^2(\Omega)}) + r$, where $\boldsymbol{v}_0 \in B^{\rho'}_{\boldsymbol{H}^2(\Omega)}$; the theorem now follows from Lemma 6.16. □

6.5 Some Related Work on Attractors and Inertial Manifolds for Incompressible Bipolar and Non-Newtonian Flow

In the concluding section of this chapter, we want to survey some of the related work that has been done, to date, on analyzing the large time behavior of solutions for problems governed either by the bipolar model (5.1a,b) or the non-Newtonian specialization thereof which corresponds to taking $\mu_1 = 0$; before doing so, however, it is appropriate to indicate the correlation between the various subsections of Chap. 5 and published work by the authors and their colleagues. Much of the work in Sect. 5.2, dealing with the linearized stability of the incompressible bipolar equations, may be found in [Bl4]. The essential content of Sect. 5.3, including the estimates for the Hausdorff and fractal dimensions of the global attractor $\boldsymbol{\mathcal{A}}_{\mu_1}$, $\mu_1 > 0$, and $0 \leq \alpha < 1$, was established in [BBN5]. The papers [Bl3, Bl5] cover much of the same material as that presented in Sect. 5.4 for the bipolar and non-Newtonian problems when $-1 < \alpha < 0$. The lower semicontinuity of the attractors for the bipolar problem, which was proven in Sect. 5.5, has been presented in [Bl2] and was also discussed in [Bl5]. Almost all the material presented in Sect. 6.2 dealing with the existence of an inertial manifold for incompressible, viscous, bipolar fluid flow is based on the Ph.D. thesis [Hao] and may also be found in [BH3]. The work in Sect. 6.3, concerning the $\boldsymbol{L}^2$ squeezing property, is based on the paper [BH1]. Finally, much of what was presented in Sect. 6.4, concerning the existence of a maximal compact global attractor for bipolar fluid flow in an unbounded parallel-wall channel, appears in [BH5].

The bulk of the work, to date, on the existence of attractors for either the bipolar, incompressible, fluid model governed by (5.2a,b), (5.3a) or (5.3b), and (5.4)—or the special non-Newtonian case obtained by taking $\mu_1 = 0$ and, in the case of the boundary-value problem, retaining only the condition $\boldsymbol{v} = 0$ in (5.3a)—has been carried out by researchers in either eastern Europe or China. We now offer a brief synposis of this work as well as some related work on similar modifications of the Navier–Stokes equations.

In [BaH] the existence and regularity of Young measured-valued solutions for the non-Newtonian equations (5.2a,b), with $\mu_1 = 0$ and $\mu(|\boldsymbol{e}|)$ given by (5.1d), was studied and some results on the asymptotic behavior of solutions were obtained. In a bounded domain of R^n, $n = 2, 3$, Málek and Nečas [MN] obtained the existence of a global finite-dimensional attractor for the non-Newtonian model using a method of short δ-trajectories with initial values in the attractor. In [NP3] the authors study the asymptotic behavior of solutions for the non-Newtonian model in the whole space with zero external force; with initial data in $L^1 \cap L^2$ it is shown that solutions exhibit L^2 decay in time like $t^{-1/4}$. Málek and Prazák [MP1] prove, for the non-Newtonian model, with space-periodic boundary conditions, the existence of a global attractor with finite fractal dimension for the case $n = 2$ when $p \geq 2$ and for the case $n = 3$ when $p \geq 11/5$. The same authors [MP2] apply the so-called method of l-trajectories to study the large time behavior of solutions to a class of abstract nonlinear dissipative evolution equations of the first order; their results are then extended so as to establish the existence of a finite-dimensional (exponential) attractor for a class of equations in nonlinear fluid mechanics which includes the non-Newtonian model (5.2a,b) with $\mu_1 = 0$. For the Ladyzhenskaya [La2] modification of the Navier–Stokes equations, the existence of a global attractor in the three-dimensional case was proven for a bounded domain; these results were then extended to the case of unbounded channel-like domains.

In [LWW] the authors study the large time behavior of solutions of the non-Newtonian model in $\mathbb{R}^3$ and obtain estimates in $L^2(\mathbb{R}^3)$ for the decay of solutions as $t \to \infty$. The large time behavior of weak solutions for the incompressible, non-Newtonian problem in R^2 is analyzed in [DL]; it is proven that weak solutions decay in the L^2 norm like $(1 + t)^{-1/2}$ and that the decay rate is sharp in the sense that it coincides with the decay rate of a solution to the heat equation. In [LZ3] the authors consider the regularity of the global attractor for the non-Newtonian system in two-dimensional unbounded domains; it is shown that the $\boldsymbol{L}^2$ compact attractor and the $\boldsymbol{H}^2$ compact attractor of the system are the same. In a related work, Liu and Zhao [LZ2] study the long time behavior of the non-Newtonian system in two-dimensional unbounded domains and prove the existence of an $\boldsymbol{H}^2$ compact attractor for the system by showing that the corresponding semigroup is asymptotically compact. In [Do2] time decay rates are established for weak solutions to the incompressible non-Newtonian problem in $\mathbb{R}^n$; using spectral decomposition methods, applied to the Stokes operator, optimal decay estimates of weak solutions in the $\boldsymbol{L}^2$ norm are derived under different conditions on the initial velocity; also error estimates for the difference between the non-Newtonian flow and the corresponding Navier–Stokes flow are obtained. In a related work [Do2] the same author uses a Fourier splitting method to show that weak solutions for the non-Newtonian problem in $\mathbb{R}^n$ decay at the rate $(1+t)^{-n/2}$, in the $\boldsymbol{L}^2$ norm, as $t \to \infty$. In [DC2], the asymptotic stability of weak solutions for the non-Newtonian system in $\mathbb{R}^2$ is studied; even if $||\boldsymbol{u}_0 - \boldsymbol{v}_0||_{\boldsymbol{L}^2(\mathbb{R}^2)}$ is not small, as long as a perturbed flow $\boldsymbol{v}(t)$, corresponding to the initial data $\boldsymbol{v}_0$, satisfies an appropriate energy inequality, it is proven that $||\boldsymbol{u}(t) - \boldsymbol{v}(t)||_{\boldsymbol{L}^2(\mathbb{R}^2)} \to 0$ as $t \to \infty$. The time decay rates of non-Newtonian flows, which conform to the model (5.1a,b) with $\mu_1 = 0$, are studied in $\mathbb{R}^n_+$ in [DC1] for $n \geq 3$; using the spectral decomposition of the Stokes operator, and $\boldsymbol{L}^p - \boldsymbol{L}^q$ estimates, it is shown that weak solutions decay in the $\boldsymbol{L}^2$ norm like $t^{-n/2(\frac{1}{r}-\frac{1}{2})}$, when the initial velocity $\boldsymbol{u}_0 \in L^2 \cap L^r$, for $1 \leq r < 2$ and better decay rates are obtained when $\boldsymbol{u}_0$ satisfies an additional moment condition of the form $\int_{R^n_+} |x_n \boldsymbol{u}_0(\boldsymbol{x})|^r d\boldsymbol{x} < \infty$, $1 < r \leq 2$.

In somewhat more recent work, Zhao, Zhou, and Liao [ZZL1] discuss the long time behavior of solutions for two-dimensional flow of an incompressible non-Newtonian fluid in a bounded domain subjected to a (locally) uniformly integrable external force; they obtain results on the existence and structure of uniform attractors in the general case without restrictions on the size of the external force and then specialize to cover the case where the $\boldsymbol{L}^2$ norm of the external force is small. In [NP2] the authors study the asymptotic time behavior of incompressible non-Newtonian fluids in the whole space assuming initial data in $\boldsymbol{L}^1$; the analysis is focused on the behavior of weak solutions for the problem governed by (5.1a,b), with $\mu_1 = 0$, when $n = 3$ and $p \geq \frac{11}{5}$. In [LZZ1] trajectory attractors and global attractors are constructed for an autonomous two-dimensional non-Newtonian fluid while the same authors, in [LZZ3], discuss results for incompressible non-Newtonian flow subject to external forces which are rapidly oscillating in time but have a smooth average. Furthermore, LLi et al. [LZZ2] have recently studied the long time behavior of solutions for two-dimensional, nonautonomous, incompressible non-Newtonian flows in bounded domains when the external force is translation compact; when the $\boldsymbol{L}^2$ norm of the forcing function is appropriately small it is shown that there exists a unique, bounded, asymptotically stable solution to the initial-boundary value problem. In [ZZ1] the authors study the so-called pullback asymptotic behavior of solutions for a non-autonomous, incompressible, non-Newtonian fluid in two-dimensional bounded domains after first proving the existence of pullback attractors; they establish regularity for the pullback attractors which, in turn, implies the (pullback) asymptotic smoothing effect of the fluid in the sense that solutions become eventually more regular than the initial data. Similar results are established in [ZZL3], this time with respect to the existence and regularity of pullback attractors for a non-Newtonian fluid with delays. Finally, in [ZZL3], upper semicontinuity of the global attractor for an incompressible, non-Newtonian fluid, in the two-dimensional domain $\Omega = R^1 \times (-L, L)$, is proven by considering an expanding sequence $\{\Omega_m\}_{m=1}^{\infty}$ of simply connected, bounded, smooth subdomains of Ω such that $\Omega_M \to \Omega$ as $m \to \infty$; in particular, it is shown that if $\mathcal{A}$ and $\mathcal{A}_m$ are the global attractors of the fluid corresponding to Ω and Ω_m, respectively, and $O(\mathcal{A})$ is a neighborhood of $\mathcal{A}$, then the global attractor $\mathcal{A}_m \subset O(\mathcal{A})$ for m sufficiently large.

Appendix A
Notation, Definitions, and Results from Analysis

A.1 Notation and Definitions

For $x_i \in R^1$, $i = 1, \dots, n$, we denote points in R^n, $n > 1$, by $\boldsymbol{x} = (x_1, \dots, x_n)$. When coordinates are, e.g., polar, cylindrical, etc., instead of Cartesian, this is noted explicitly in the text. Similarly, vectors will be denoted by $\boldsymbol{v} = (v_1, \dots, v_n)$ and operators (mappings), whether linear or nonlinear, are also denoted by boldface letters, e.g., $\boldsymbol{\mathcal{L}}$, $\boldsymbol{S}$, $\boldsymbol{A}$, etc. All inner products are explicitly defined the first time they appear and are expressed in what is now a standard formulation, e.g., for $\boldsymbol{u}, \boldsymbol{v} \in R^n$, $\boldsymbol{u} \cdot \boldsymbol{v} = \sum_{i=1}^{n} u_i v_i$.

For functions $u(\boldsymbol{x})$, $v(\boldsymbol{x})$ defined on a domain $\Omega \subseteq R^n$, the $L^2(\Omega)$ inner-product is given by

$$(u, v)_{L^2(\Omega)} = \int_\Omega u(\boldsymbol{x}) v(\boldsymbol{x})\, d\boldsymbol{x}$$

but for vector-valued functions $\boldsymbol{u}(\boldsymbol{x})$, $\boldsymbol{v}(\boldsymbol{x})$ defined on Ω we employ a bold-face $\boldsymbol{L}^2$, i.e.,

$$(\boldsymbol{u}, \boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} = \int_\Omega \boldsymbol{u}(\boldsymbol{x}) \boldsymbol{v}(\boldsymbol{x})\, d\boldsymbol{x}.$$

For the L^2 inner-product we assume, of course, that $\int_\Omega u^2(\boldsymbol{x})\, d\boldsymbol{x} < \infty$, $\int_\Omega v^2(\boldsymbol{x})\, d\boldsymbol{x} < \infty$ with the corresponding assumption in the vector-valued case being that $\int_\Omega ||\boldsymbol{u}(\boldsymbol{x})||^2\, d\boldsymbol{x} < \infty$, $\int_\Omega ||\boldsymbol{v}(\boldsymbol{x})||^2\, d\boldsymbol{x} < \infty$, where $||\boldsymbol{u}||^2 = \boldsymbol{u} \cdot \boldsymbol{u}$.

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For a tensor-valued function, such as the rate of deformation tensor $\boldsymbol{e}$, we write $|\boldsymbol{e}|^2 = \sum_{i,j} e_{ij}e_{ij}$, where $i, j = 1, \ldots, n$.

Standard notation is used throughout the book for the space $C^k(\Omega)$ of continuously differentiable functions, namely,

$$\begin{aligned} C(\Omega) &= \{u : \Omega \to R^1 \mid u \text{ is continuous}\}, \\ C(\bar{\Omega}) &= \{u \in C(\Omega) \mid u \text{ is continuous up to } \partial\Omega\}, \\ C^k(\Omega) &= \{u \in C(\Omega) \mid u \text{ is } k\text{-times continuously differentiable}\} \end{aligned}$$

and

$$\begin{aligned} C^k(\bar{\Omega}) = \{u \in C^k(\Omega) \mid \boldsymbol{D}^{\boldsymbol{\alpha}} u \text{ is uniformly continuous on bounded} \\ \text{subsets of } \Omega,\ \forall \boldsymbol{\alpha} \text{ such that } |\boldsymbol{\alpha}| \le k\} \end{aligned}$$

where the multi-index $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$, $|\boldsymbol{\alpha}| = \alpha_1 + \cdots + \alpha_n$, the α_i, $i = 1, \ldots, n$ are non-negative integers, and

$$\boldsymbol{D}^{\boldsymbol{\alpha}} u = \frac{\partial^{|\boldsymbol{\alpha}|} u(\boldsymbol{x})}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$$

with $\boldsymbol{D} = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)$. These definitions easily generalize to vector-valued functions $\boldsymbol{u}(\boldsymbol{x}) = (u_1(\boldsymbol{x}), \ldots, u_n(\boldsymbol{x}))$ defined on $\Omega \subset R^n$, e.g., $\boldsymbol{u} \in \boldsymbol{C}^k(\Omega)$ iff $u_i \in C^k(\Omega)$, $i = 1, \ldots, n$. Also, $C^\infty(\Omega) = \bigcap_{k=0}^{\infty} C^k(\Omega)$, with the obvious extensions to the cases $C^\infty(\bar{\Omega})$, $\boldsymbol{C}^\infty(\Omega)$ and $\boldsymbol{C}^\infty(\bar{\Omega})$.

The L^p spaces are also defined in the usual manner, i.e., for $1 \le p < \infty$,

$$L^p(\Omega) = \{u : U \to R^1 \mid u \text{ is Lebesque measurable and } ||u||_{L^p(\Omega)} < \infty\}$$

where

$$||u||_{L^p(\Omega)} = \left(\int_\Omega |u(\boldsymbol{x})|^p \, d\boldsymbol{x}\right)^{1/p}.$$

The generalization to the vector-valued case is denoted as $\boldsymbol{L}^p(\Omega)$ where

$$||\boldsymbol{u}||_{\boldsymbol{L}^p(\Omega)} = \left(\int_\Omega ||\boldsymbol{u}(\boldsymbol{x})||^p \, d\boldsymbol{x}\right)^{1/p}$$

for $1 \leq p < \infty$; these definitions include the $L^2(\Omega)$ and $\boldsymbol{L}^2(\Omega)$ cases covered, separately, above. By $L^p_{loc}(\Omega)$ we denote the set of all $u : \Omega \to R^1$ such that $u \in L^p(\tilde{\Omega})$ for each compact subset $\tilde{\Omega}$ of Ω with an analogous definition for $\boldsymbol{L}^p_{loc}(\Omega)$. For $p = \infty$ we have

$$L^\infty = \{u : \Omega \to R^1 \mid u \text{ is Lebesque measurable and } ||u||_{L^\infty(\Omega)} < \infty\}$$

with

$$||u||_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{\Omega} |u|.$$

In the vector-valued case, denoted by $\boldsymbol{L}^\infty(\Omega)$, we have for $\boldsymbol{u} : \Omega \to R^n$,

$$||\boldsymbol{u}||_{\boldsymbol{L}^\infty(\Omega)} = \operatorname*{ess\,sup}_{\Omega} ||\boldsymbol{u}||.$$

For vector-valued functions $\boldsymbol{u} : \Omega \to R^n$ we have, for each multi-index $\boldsymbol{\alpha}$,

$$\boldsymbol{D}^{\boldsymbol{\alpha}} \boldsymbol{u} = (\boldsymbol{D}^{\boldsymbol{\alpha}} u_1, \ldots, \boldsymbol{D}^{\boldsymbol{\alpha}} u_n)$$

so that

$$\boldsymbol{D}^k \boldsymbol{u} = \{\boldsymbol{D}^{\boldsymbol{\alpha}} \boldsymbol{u} \mid |\boldsymbol{\alpha}| = k\}.$$

Also, for $k = 1$, the gradient of the vector-valued function $\boldsymbol{u} : \Omega \to R^n$ in Cartesian coordinates x_i is the tensor-valued function with components

$$(\boldsymbol{D}\boldsymbol{u})_{ij} \equiv (\nabla \boldsymbol{u})_{ij} = \frac{\partial u_i}{\partial x_j}, \quad i, j = 1, \ldots, n$$

and

$$\operatorname{div} \boldsymbol{u} = \nabla \cdot \boldsymbol{u} = \operatorname{tr}(\nabla \boldsymbol{u}) = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i}.$$

When non-Cartesian coordinates are used in the book, the components of expressions such as $\nabla \boldsymbol{u}$, $\nabla \cdot \boldsymbol{u}$, etc., are always made explicit.

To define the standard Sobolev spaces we first recall the definition of weak derivative; for $u, v : \Omega \to R^1$, with $u, v \in L^1_{loc}(\Omega)$, and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ a multi-index, v is the $\boldsymbol{\alpha}$th weak partial derivative of u, i.e., $v = \boldsymbol{D}^{\boldsymbol{\alpha}} u$ if

$$\int_\Omega u \boldsymbol{D}^{\boldsymbol{\alpha}} \phi \, d\boldsymbol{x} = (-1)^{|\boldsymbol{\alpha}|} \int_\Omega v \phi \, d\boldsymbol{x}$$

for all $\phi : \Omega \to R^1$ such that $\phi \in C_0^\infty(\Omega) \equiv \mathcal{D}(\Omega)$, where $C_0^\infty(\Omega)$ is the space of all infinitely differentiable functions on Ω with compact support. For $\boldsymbol{u}, \boldsymbol{v} : \Omega \to R^n$, with $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{L}^1_{loc}(\Omega)$, we have an analogous definition for $\boldsymbol{v} = \boldsymbol{D}^{\boldsymbol{\alpha}}\boldsymbol{u}$. If $1 \le p < \infty$, and m is a non-negative integer, then the Sobolev space $W^{m,p}(\Omega)$ is defined to be the set of all $u : \Omega \to R^1$ such that $u \in L^1_{loc}(\Omega)$ and, for each multi-index $\boldsymbol{\alpha}$ with $|\boldsymbol{\alpha}| < m$, the weak derivative $\boldsymbol{D}^{\boldsymbol{\alpha}}u$ exists and is in $L^p(\Omega)$. For $u : \Omega \to R^1$ in $W^{m,p}(\Omega)$ we define, for $1 \le p < \infty$,

$$||u||_{W^{m,p}(\Omega)} = \left(\sum_{|\boldsymbol{\alpha}| \le m} \int_\Omega |\boldsymbol{D}^{\boldsymbol{\alpha}} u|^p \, d\boldsymbol{x} \right)^{1/p}$$

while for $p = \infty$,

$$||u||_{W^{m,p}(\Omega)} = \sum_{|\boldsymbol{\alpha}| \le m} \operatorname*{ess\,sup}_{\Omega} |\boldsymbol{D}^{\boldsymbol{\alpha}} u|.$$

We will sometimes use the notation $|| \quad ||_{m,p}$ for $|| \quad ||_{W^{m,p}(\Omega)}$ if there is no chance of confusion. For $\boldsymbol{u} : \Omega \to R^n$, with $n > 1$, we will use the notation $||\boldsymbol{u}||_{\boldsymbol{W}^{m,p}(\Omega)}$ with the obvious modifications to the definitions given above. When $p = 2$ it is a common practice to write, for $u : \Omega \to R^1$, that $W^{m,2}(\Omega) = H^m(\Omega)$, with $H^m(\Omega)$ a Hilbert space with inner-product

$$(u, v)_{H^m(\Omega)} = \sum_{|\boldsymbol{\alpha}| \le m} \int_\Omega \boldsymbol{D}^{\boldsymbol{\alpha}} u(\boldsymbol{x}) D^{\boldsymbol{\alpha}} v(\boldsymbol{x}) \, d\boldsymbol{x}$$

for $u, v : \Omega \to R^1$. Again, the obvious modifications to $\boldsymbol{H}^m(\Omega)$ and $(\boldsymbol{u}, \boldsymbol{v})_{\boldsymbol{H}^m(\Omega)}$ apply when $\boldsymbol{u}, \boldsymbol{v} : \Omega \to R^n$ and $n > 1$. By the Sobolev space $W_0^{m,p}(\Omega)$ we indicate the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$, so that $u : \Omega \to R^1$ is in $W_0^{m,p}(\Omega)$ iff there exists $\{u_k\} \subset C_0^\infty(\Omega)$ such that $u_k \to u$ in $W^{m,p}(\Omega)$, as $k \to \infty$. A careful discussion of the trace theorem (see, e.g., [Ev] and Sect. A.2.7) shows that, loosely speaking, $W_0^{m,p}(\Omega)$ consists of all those $u \in W^{m,p}(\Omega)$ such that $\boldsymbol{D}^{\boldsymbol{\alpha}}u = 0$ on $\partial\Omega$, for all $\boldsymbol{\alpha}$ such that $|\boldsymbol{\alpha}| \le m - 1$. With $p = 2$ we usually write $W_0^{m,2}(\Omega) = H_0^m(\Omega)$. For $\boldsymbol{u} : \Omega \to R^n$, $n > 1$, we use the obvious modifications of the above definitions and write $\boldsymbol{W}_0^{m,p}(\Omega)$ and $\boldsymbol{H}_0^m(\Omega)$, respectively. By $|u|_{H^m(\Omega)}$, for $u : \Omega \to R^1$, we denote the semi-norm given by

$$|u|^2_{H^m(\Omega)} = \left(\sum_{|\boldsymbol{\alpha}| = m} \int_\Omega |\boldsymbol{D}^{\boldsymbol{\alpha}} u(x)|^2 \, d\boldsymbol{x} \right)^{1/2}$$

with an obvious modification in order to obtain $|\boldsymbol{u}|_{\boldsymbol{H}^m(\Omega)}$ for $\boldsymbol{u} : \Omega \to R^n$ with $n > 1$.

For $k \in N$, the Sobolev spaces with negative index, $W^{-k,p}(\Omega)$, are defined to be the dual spaces to $W_0^{k,p}(\Omega)$, where q is conjugate to p, i.e., $\frac{1}{p} + \frac{1}{q} = 1$. The elements of $W^{-k,p}(\Omega)$ are distributions. Another definition of the Sobolev spaces with negative index is

$$W^{-k,p}(\Omega) = \{u \in \mathcal{D}'(\Omega) \mid u = \sum_{|\boldsymbol{\alpha}| \le k} \boldsymbol{D}^{\boldsymbol{\alpha}} u_{\boldsymbol{\alpha}}, \text{ for some } u_{\boldsymbol{\alpha}} \in L_p(\Omega)\}$$

with the $\boldsymbol{D}^{\boldsymbol{\alpha}} u_{\boldsymbol{\alpha}}$ computed in the sense of distributions in $\mathcal{D}'(\Omega)$. The two definitions of $W^{-k,p}(\Omega)$, for $k \in N$, are equivalent. We note that $u \in W^{-k,p}(\Omega)$, $k \in N$, defines a linear operator on $W_0^{k,q}(\Omega)$, if $\frac{1}{p} + \frac{1}{q} = 1$, i.e., for $v \in W_0^{k,q}(\Omega)$

$$\begin{aligned} \langle u, v \rangle &= \sum_{|\boldsymbol{\alpha}| \le k} \langle \boldsymbol{D}^{\boldsymbol{\alpha}} u_{\boldsymbol{\alpha}}, v \rangle \\ &= \sum_{|\boldsymbol{\alpha}| \le k} (-1)^{|\boldsymbol{\alpha}|} \langle u_{\boldsymbol{\alpha}}, \boldsymbol{D}^{\boldsymbol{\alpha}} v \rangle = \sum_{|\boldsymbol{\alpha}| \le k} (-1)^{|\boldsymbol{\alpha}|} \int_{\Omega} u_{\boldsymbol{\alpha}} \boldsymbol{D}^{\boldsymbol{\alpha}} v \, d\boldsymbol{x}. \end{aligned}$$

Also, $W^{-k,p}(\Omega)$ is a Banach space when equipped with the followng norm: for $u \in W^{-k,p}(\Omega)$,

$$||u||_{W^{-k,p}(\Omega)} = \sup_{\substack{v \in W^{k,q}(\Omega) \\ ||v||_{W^{k,q}(\Omega)} \neq 0}} \frac{|\langle u, v \rangle|}{||v||_{W^{k,q}(\Omega)}}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Remarks. We leave for Sect. A.3 the definition of the fractional Sobolev spaces, $W^{s,p}(\Omega)$, where $s \in R^1$ is not necessarily an integer. The spaces $W^{s,p}(\Omega)$ may be defined either by using the Fourier transform (in the sense of distributions) or by introducing the Slobodeckij seminorm defined for functions $f \in L^p(\Omega)$, $1 \le p < \infty$; it is the latter approach which is followed in Sect. A.3.

A.2 Basic Analysis Results

This subsection reviews the statement of a number of standard results, the proofs of which can be found in many texts covering analysis, partial differential equations, or functional analysis, e.g., [Ev, Yos, Te1], or [N2].

A.2.1 The Hölder Inequality

Let $p > 1$ and let, as in Sect. A.1, the space $L^p(\Omega)$ be the Banach space consisting of all measurable functions on Ω (a bounded domain in R^n) whose p-powers are integrable with $||u||_{L^p(\Omega)} \equiv ||u||_p = \left(\int_\Omega |u|^p\right)^{1/p}$, $p > 1$. Suppose that q is conjugate to p, i.e., $\frac{1}{p} + \frac{1}{q} = 1$; then, for $u \in L^p(\Omega)$, and $v \in L^q(\Omega)$, the Hölder Inequality is

$$\int_\Omega uv\, d\boldsymbol{x} \le ||u||_{L^p(\Omega)}||v||_{L^q(\Omega)}.$$

For $p = 1$, $q = \infty$ and $||v||_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_\Omega |v|$. Hölder's inequality is a direct consequence of Young's inequality which we state next. As a consequence of the Hölder Inequality we have the following interpolation inequality for the L^p spaces: For $u \in L^r(\Omega)$, with $p \le q \le r$ and $\frac{1}{q} = \frac{\lambda}{p} + \frac{1-\lambda}{r}$,

$$||u||_{L^q(\Omega)} \le ||u||^{\lambda}_{L^p(\Omega)}||u||^{1-\lambda}_{L^r(\Omega)}.$$

A.2.2 Cauchy's and Young's Inequalities

Cauchy's inequality with $\epsilon > 0$, $a, b > 0$ is the statement that

$$ab \le \epsilon a^2 + \frac{1}{4\epsilon} b^2.$$

The standard case consists of taking $\epsilon = \frac{1}{2}$. For $a, b > 0$ and $1 < p, q < \infty$, with $\frac{1}{p} + \frac{1}{q} = 1$, Young's inequality says that

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}.$$

If $a, b > 0$, $\epsilon > 0$, and $C(\epsilon) = (\epsilon p)^{-q/p} q^{-1}$, then we have Young's inequality with ϵ,

$$ab \le \epsilon a^p + C(\epsilon) b^q.$$

Young's inequality is a consequence of the fact that $f(x) = e^x$ is a convex function (see, e.g., [Ev]) while Young's inequality with ϵ follows from Young's inequality if we write $ab = \left((\epsilon p)^{1/p} a\right)\left(\frac{b}{(\epsilon p)^{1/p}}\right)$.

A.2.3 The Lax-Milgram Lemma

The Lax-Milgram Lemma is a generalization of the Riesz Representation Theorem [Yos] to bilinear forms which do not need to be symmetric; its full statement is as follows: Let H be a Hilbert space with norm $||\cdot||_H$ and $B : H \times H \to R$ a bilinear map. If there exist $c_1 > 0$ and $c_2 > 0$ such that

$$|B(u, v)| \leq c_1 ||u||_H ||v||_H, \quad \forall u, v \in H$$

and

$$B(u, u) \geq c_2 ||u||_H^2, \quad \forall u \in H$$

then, for each $f \in H^*$, $\exists v \in H$ (unique) such that $B(u, v) = f(u)$, $\forall u \in H$.

A.2.4 The Poincaré Inequality

There are various versions of the Poincaré inequality; we state three of the most common versions here:

(i) Let $\Omega \subseteq R^n$ be an open bounded subset and $u \in W_0^{1,p}(\Omega)$, $1 \leq p < n$. Then for each $q \in [1, p^*]$, $p^* = np/n - p$,

$$||u||_{L^q(\Omega)} \leq C ||Du||_{L^p(\Omega)}$$

with $C = C(p, q, n; \Omega) > 0$. If $1 \leq p \leq \infty$, then

$$||u||_{L^p(\Omega)} \leq C ||Du||_{L^p(\Omega)}.$$

(ii) Let $\Omega \subseteq R^n$ be an open bounded domain with C^1 boundary $\partial\Omega$. Then for any $u \in W^{1,p}(\Omega)$, with $1 \leq p < \infty$, and some $c > 0$,

$$\int_\Omega \left| u(\boldsymbol{x}) - \frac{1}{|\Omega|} \int_\Omega u(\boldsymbol{y})\, d\boldsymbol{y} \right|^p d\boldsymbol{x} \leq c \int_\Omega \sum_{i=1}^n \left| \frac{\partial u}{\partial x_i} \right|^p d\boldsymbol{x}.$$

(iii) Let $a > 0$ and

$$\Omega \subseteq \{(x_1, x_2, \ldots, x_n) \mid |x_1| \leq a < \infty,\ -\infty < x_i < \infty,\ i = 2, \ldots, n\}$$

then $\exists c > 0$, $c = c(k, n)$, such that for every $u \in H_0^k(\Omega)$,

$$||u||^2_{W^{k,2}(\Omega)} \leq c \sum_{|\boldsymbol{\alpha}|=k} ||D^{\boldsymbol{\alpha}} u||_{L^2(\Omega)}{}^2.$$

A.2.5 Friedrich's Inequality

Suppose that Ω is a bounded subset of R^n with diameter d and $1 \leq p < \infty$. Then $\forall u \in W_0^{k,p}(\Omega)$,

$$||u||_{L^p(\Omega)} \leq d^k \left(\sum_{|\boldsymbol{\alpha}|=k} ||D^{\boldsymbol{\alpha}} u||^p_{L^p(\Omega)} \right)^{1/p}$$

where $\boldsymbol{\alpha} = (\alpha, \ldots, \alpha_n)$.

A.2.6 Fréchet Derivative

Let $\boldsymbol{X}, \boldsymbol{Y}$ be real Banach spaces and $\boldsymbol{F}$ a nonlinear operator mapping $D(\boldsymbol{F}) \subset \boldsymbol{X}$ to $R(\boldsymbol{F}) \subset \boldsymbol{Y}$ with $D(\boldsymbol{F})$ open in $\boldsymbol{X}$. Then $\boldsymbol{F}$ is Fréchet differentiable at $\boldsymbol{x}_0 \in D(\boldsymbol{F})$ if there exists a bounded linear operator $\boldsymbol{F}'(x_0)$ such that

$$\boldsymbol{F}(\boldsymbol{x}_0 + \boldsymbol{h}) - \boldsymbol{F}(\boldsymbol{x}_0) = \boldsymbol{F}'(\boldsymbol{x}_0)\boldsymbol{h} + \boldsymbol{w}(\boldsymbol{x}_0, \boldsymbol{h})$$

for all $\boldsymbol{h}$ with $||\boldsymbol{h}|| < \epsilon$, for some $\epsilon > 0$, where $||\boldsymbol{w}(\boldsymbol{x}_0, h)||/||\boldsymbol{h}|| \to 0$ as $||\boldsymbol{h}|| \to 0$. We call $\boldsymbol{F}'(\boldsymbol{x}_0)$ the Fréchet derivative of $\boldsymbol{F}(\boldsymbol{x})$ at $\boldsymbol{x}_0$ while $\boldsymbol{d}\boldsymbol{F}(\boldsymbol{x}_0, h) = \boldsymbol{F}'(\boldsymbol{x}_0)h$ is the corresponding Fréchet differential.

A.2.7 $\boldsymbol{\omega}$-Limit Sets

Let $S(t)$ be a semigroup defined on some subset $\boldsymbol{X} \subseteq \boldsymbol{H}$, $\boldsymbol{H}$ a real-Hilbert space. The ω-limit set of $\boldsymbol{X}$ consists of all the limit points of orbits through points in $\boldsymbol{X}$, i.e.,

$$\omega(X) = \{\boldsymbol{y} \in \boldsymbol{H} \mid \exists\{t_n\} \subset R^1, \text{ with } t_n \to \infty \text{ as } n \to \infty, \\ \text{and } \{\boldsymbol{x}_n\} \subset \boldsymbol{H} \text{ such that } \boldsymbol{S}(t_n)\boldsymbol{x}_n \to \boldsymbol{y}, \\ \text{as } n \to \infty\}.$$

A.2.8 The Trace Theorem

Let $\Omega \subseteq R^n$ be an open bounded domain with Lipschitz boundary $\partial\Omega$. Then, the trace theorem says that there exists a bounded linear operator $T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$, $1 \leq p < \infty$, such that

$$Tu = u|_{\partial\Omega}, \quad \forall u \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$$

and

$$||Tu||_{L^p(\partial\Omega)} \leq C(p, \Omega)||u||_{W^{1,p}(\Omega)}, \quad \forall u \in W^{1,p}(\Omega).$$

Tu is called the trace of u. The functions in $W^{1,p}(\Omega)$ with zero trace constitute $W_0^{1,p}(\Omega)$, i.e., $W_0^{1,p}(\Omega) = \{u \in W^{1,p}(\Omega) \mid Tu = 0\}$ and, in fact, $W_0^{1,p}(\Omega) = \{u \in W^{1,p}(\Omega) \mid \exists\{u_n\} \subset C_0^\infty(\Omega)$ such that $u_m \to u$ in $W^{1,p}(\Omega)\}$; thus, for bounded Ω with Lipschitz $\partial\Omega$, trace-zero functions in $W^{1,p}(\Omega)$ can be approximated by smooth functions with compact support.

A.2.9 Aubin's Lemma

Aubin's Lemma (sometimes termed the Aubin-Lions Lemma) is a result in the theory of Banach space-valued functions; it is a compactness criterion which is useful in studying nonlinear evolutionary differential equations. The precise statement of the lemma is as follows (the precise definitions of continuous and compactly embedded are given in Sect. A.3): Let $\boldsymbol{X}_0 \subseteq \boldsymbol{X} \subseteq \boldsymbol{X}_1$ be three Banach spaces with $\boldsymbol{X}_0$ compactly embedded in $\boldsymbol{X}$ and $\boldsymbol{X}$ continuously embedded in $\boldsymbol{X}_1$; suppose, also, that $\boldsymbol{X}_0$ and $\boldsymbol{X}_1$ are reflexive. For $1 < p, q < \infty$ let

$$\boldsymbol{W} = \{\boldsymbol{u} \in L^p([0, T]; \boldsymbol{X}_0 \mid \dot{\boldsymbol{u}} \in L^q([0, T]; \boldsymbol{X}_1)\}.$$

Then the embedding of $\boldsymbol{W}$ into $L^p([0, T]; \boldsymbol{X})$ is also compact (see, also, Lemma A.9).

A.2.10 Estimates for $b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$

The trilinear form

$$b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \int_\Omega u_i \frac{\partial v_i}{\partial x_j} w_j \, d\boldsymbol{x}$$

arises in studies of the Navier–Stokes equations as well as in studies of all related fluid dynamics models (e.g., the bipolar fluid) whenever the convective nonlinearity is present in the acceleration term. A wealth of estimates exist for $|b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})|$ which depend on the assumptions relative to $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$; good references, in this regard, are [CFT1] and [Te1]; we have, for example the following result from Sect. 1.2 of [Te1]: The trilinear form b is defined and (trilinear) continuous on $\boldsymbol{H}_0^1(\Omega) \times \boldsymbol{H}_0^1(\Omega) \times (\boldsymbol{H}_0^1(\Omega) \cap \boldsymbol{L}^n(\Omega))$, Ω a bounded or unbounded domain in R^n, and for some $c(n) > 0$,

$$|b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| \le c(n) ||\boldsymbol{u}||_{\boldsymbol{H}_0^1(\Omega)} ||\boldsymbol{v}||_{\boldsymbol{H}_0^1(\Omega)} ||\boldsymbol{w}||_{\boldsymbol{H}_0^1(\Omega) \cap \boldsymbol{L}^n(\Omega)}.$$

Other results which have been used in this book are (see [CFT1]), e.g.,

(i) $\left| \int_\Omega u_i \frac{\partial v_i}{\partial x_j} v_j \, d\boldsymbol{x} \right| \le ||\boldsymbol{u}||_{\boldsymbol{L}^\infty(\Omega)} |\boldsymbol{w}|_{\boldsymbol{H}^1(\Omega)} ||\boldsymbol{w}||_{\boldsymbol{L}^2(\Omega)}$,

(ii) $\left| \int_\Omega u_i \frac{\partial v_i}{\partial x_j} u_j \, d\boldsymbol{x} \right| \le c_1(\Omega) ||\boldsymbol{v}||_{\boldsymbol{W}^{1,2}(\Omega)} ||\boldsymbol{u}||^2_{\boldsymbol{W}^{1,2}(\Omega)} ||\boldsymbol{u}||_{\boldsymbol{L}^2(\Omega)}{}^{1/2}$, for some $c_1(\Omega) > 0$

where the $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ lie in the indicated spaces. Other results are generated by using interpolation and embedding theorems for Sobolev spaces.

A.2.11 The Uniform Gronwall Lemma

The Gronwall inequality can be written in integral as well as differential form as follows:

(i) Let $K \ge 0$ and let f and g be continuous nonnegative functions for $a \le t \le b$ which satisfy

$$f(t) \le K + \int_a^b f(s)\, g(s)\, ds, \quad a \le t \le b.$$

Then

$$f(t) \le K e^{\int_a^t g(s)\, ds}, \quad a \le t \le b.$$

An alternative form of the integral version of Gronwall is: suppose $g(t)$ is a continuous nonnegative function for $t > 0$ and

$$g(t) \le K + C \int_0^t g(s)\,ds, \quad 0 \le t \le b$$

with $C, K > 0$. Then, for all $t \in [0, b]$

$$g(t) \le K e^{Ct}, \quad 0 \le t \le b.$$

This second version follows from the first by interchanging f and g in the first version, assuming $|f(t)| \le C$, and setting $a = 0$.

(ii) Let $f(t)$ be a nonnegative, absolutely continuous function on $[0, b]$ which satisfies, a.e. on $[0, b]$, the differential inequality

$$f'(t) \le \phi(t) f(t) + \psi(t), \quad 0 < t < b$$

with $\phi(t)$, $\psi(t)$ nonnegative, summable functions on $[0, b]$. Then,

$$f(t) \le \left[f(0) + \int_0^t \psi(\tau)\,d\tau \right] e^{\int_0^t \phi(\tau)\,d\tau}, \quad 0 < t < b.$$

A.3 Some Sobolev Space Embeddings, Estimates, and Interpolation Results

In this subsection of Appendix A we present some of the more standard results concerning continuous and compact embeddings of Sobolev spaces, the estimates implied by these embeddings, and a basic interpolation result for Sobolev spaces; various forms of the results delineated here have been used throughout this book, especially in Chaps. 4–6. Two standard, comprehensive treatises covering the material reviewed in this subsection are [Ad] and [Tr]. In all the results stated below, unless specifically indicated, $\Omega \subset R^n$ will be a bounded, open subset with a smooth (C^1) boundary $\partial\Omega$. Many of the results stated hold under somewhat weaker conditions, e.g., that $\partial\Omega$ satisfy the uniform cone property (see, e.g., [McO], Sect. 6.5); a bounded domain Ω with Lipschitz continuous boundary satisfies this condition and such a weakning of the assumptions, relative to many of the results stated, below, is needed, e.g., in Sect. 3.5 where we have considered perturbations of the boundary of Ω by Lipschitz curves.

We begin by recalling that both the Sobolev spaces $W^{k,p}(\Omega)$ and the Hölder spaces $C^{k,\mu}(\bar{\Omega})$ are Banach spaces when equipped with their respective norms (the norm on $C^{k,\mu}(\bar{\Omega})$ will be reviewed below). We begin with the following definitions.

Definition A.1. Let $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ be two Banach spaces. Then $\boldsymbol{B}_1$ is continuously embedded into $\boldsymbol{B}_2$ (and we write $\boldsymbol{B}_1 \hookrightarrow \boldsymbol{B}_2$) if $\forall \boldsymbol{u} \in \boldsymbol{B}_1$ we have $\boldsymbol{u} \in \boldsymbol{B}_2$ and, for some $c > 0$, $||\boldsymbol{u}||_{\boldsymbol{B}_2} \le c||\boldsymbol{u}||_{\boldsymbol{B}_1}$ where c does not depend on $\boldsymbol{u} \in \boldsymbol{B}_1$. The embedding operator $\boldsymbol{J} : \boldsymbol{B}_1 \to \boldsymbol{B}_2$ takes $\boldsymbol{u} \in \boldsymbol{B}_1$ into the same element $\boldsymbol{u}$ considered as an element of $\boldsymbol{B}_2$.

Definition A.2. If $\boldsymbol{B}_1 \hookrightarrow \boldsymbol{B}_2$, and the embedding operator $\boldsymbol{J} : \boldsymbol{B}_1 \to \boldsymbol{B}_2$ is a compact operator, we say that $\boldsymbol{B}_1$ is compactly embedded into $\boldsymbol{B}_2$; often this is denoted by $\boldsymbol{B}_1 \hookrightarrow\hookrightarrow \boldsymbol{B}_2$.

Our results in this subsection will be stated in the form of a series of lemmas; we first note the obvious continuous embeddings for $1 \le p < \infty$,

$$W_0^{m,p}(\Omega) \hookrightarrow W^{m,p}(\Omega) \hookrightarrow L^p(\Omega).$$

Lemma A.1 (Sobolev Embedding Theorem). *Suppose that* $k > l$, $1 \le p < q \le \infty$, $(k-l)p < n$ *and* $\frac{1}{q} = \frac{1}{p} - \frac{(k-l)}{n}$; *then*

$$W^{k,p}(\Omega) \hookrightarrow W^{l,q}(\Omega).$$

As a special case (take $l = 0$*) we have the following result: if* $k < \frac{n}{p}$ *and* $\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$, *then*

$$W^{k,p}(\Omega) \hookrightarrow L^q(\Omega).$$

In particular, $\exists C = C(k, p, n; \Omega) > 0$ *such that*

$$||u||_{L^q(\Omega)} \le C||u||_{W^{k,p}(\Omega)}.$$

Remarks. A variation on this last lemma is the following result: if $p \le q \le np/n - mp$, then

$$W^{j+m,p}(\Omega) \hookrightarrow W^{j,p}(\Omega).$$

Remarks. An important special case of Lemma A.1, with $l = 0$, arises by taking $k = 1$; then, for $1 \le p < n$, and $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$,

$$W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$$

so that for some $C = C(p, n; \Omega) > 0$,

$$||u||_{L^q(\Omega)} \le C||u||_{W^{1,p}(\Omega)}.$$

This latter result is often established as a consequence of the following estimate (the Gagliardo-Nirenberg-Sobolev Inequality): If $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$, $1 \le p < n$, then $\forall u \in C_0^1(R^n)$,

$$||u||_{L^q(R^n)} \leq C||Du||_{L^p(R^n)}$$

where $C = C(p, n) > 0$. It may also be shown (see [Ev], Sect. 5.6) that for $u \in W_0^{1,p}(\Omega)$ with $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$, and $1 \leq p < n$, that

$$||u||_{L^{q'}(\Omega)} \leq C'||Du||_{L^p(\Omega)}$$

for each $q' \in [1, q]$, where $C' = C'(p, q, n; \Omega) > 0$. Finally, if we take $q' = p$ in this last estimate we find that for $u \in W_0^{1,p}(\Omega)$ and $1 \leq p \leq \infty$,

$$||u||_{L^p(\Omega)} \leq \bar{C}||Du||_{L^p(\Omega)}$$

with $\bar{C} = \bar{C}(p, n; \Omega) > 0$ (which is, of course, one form of the Poincaré Inequality).

Lemma A.2 (Morrey's Inequality). *If $\mu = 1 - n/p$ with $p > n$, then*

$$W^{1,p}(\Omega) \hookrightarrow C^{0,\mu}$$

where $C^{k,\mu}(\bar{\Omega}) \subset C^k(\Omega)$ consists of all those functions in $C^k(\Omega)$ whose kth partial derivatives are μ-Hölder continuous. The norm on $C^{k,\mu}(\bar{\Omega})$ is given by

$$||u||_{C^{k,\mu}(\bar{\Omega})} = ||u||_{C^k(\Omega)} + \sum_{|\boldsymbol{\alpha}|=k} \sup_{x \neq y} \frac{||\boldsymbol{D}^{\boldsymbol{\alpha}} u(\boldsymbol{x}) - \boldsymbol{D}^{\boldsymbol{\alpha}} u(\boldsymbol{y})||}{||\boldsymbol{x} - \boldsymbol{y}||^{\mu}}.$$

Thus, for some $C = C(n, p; \Omega) > 0$,

$$||u||_{C^{0,\mu}(\bar{\Omega})} \leq C||u||_{W^{1,p}(\Omega)}.$$

Remarks. As $C^{0,\mu}(\bar{\Omega}) \hookrightarrow C^0(\Omega)$ we observe the chain of continuous embeddings

$$W_0^{1,p}(\Omega) \hookrightarrow W^{1,p}(\Omega) \hookrightarrow C^{0,\mu}(\bar{\Omega}) \hookrightarrow C^0(\bar{\Omega}).$$

Remarks. Lemma A.2 is extended by either of the following two results:

(i) If $0 < \lambda < m - \frac{n}{p}$, and $p > n$, then

$$W^{j+m,p}(\Omega) \hookrightarrow C^{j,\lambda}(\bar{\Omega}).$$

(ii) If $\alpha \in (0, 1)$ and $k - r - \alpha = \frac{n}{p}$, $p > n$

$$W^{k,p}(\Omega) \hookrightarrow C^{r,\alpha}(\bar{\Omega}).$$

We now delineate a series of results concerning compact embeddings, the first of which is

Lemma A.3 (Kondrachov Embedding Theorem). *Suppose that* $\Omega \subseteq R^n$ *is a compact subset with* C^1 *boundary* $\partial\Omega$*. Then, if* $k > 0$*, and* $k - \frac{n}{p} > l - \frac{n}{q}$,

$$W^{k,p}(\Omega) \hookrightarrow\hookrightarrow W^{l,q}(\Omega).$$

Remarks. If we weaken the conditions on Ω in Lemma A.3 slightly it is possible to establish the following two results: Let $\Omega \subseteq R^n$ be a bounded open subset with $\partial\Omega$ of class C^1; then,

(i) If $0 < n - mp$, and $j + m - \frac{n}{p} \geq j - \frac{n}{q}$,

$$W^{j+m,p}(\Omega) \hookrightarrow\hookrightarrow W^{j,q}(\Omega).$$

(ii) If $mp > n$, then

$$W^{j+m,p}(\Omega) \hookrightarrow\hookrightarrow C^j(\bar{\Omega}).$$

A more specialized version of Lemma A.3 is the following result:

Lemma A.4 (Rellich-Kondrachov Compactness Theorem). *Assume that* Ω *is a bounded open subset of* R^n *with* $\partial\Omega$ *of class* C^1 *and* $1 \leq p < n$*. Then for each* q' *such that* $1 \leq q' < q$, $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$,

$$W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^{q'}(\Omega).$$

In particular, if we take $q' = p$ *we have* $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega)$*. Finally, as* $p \to n$, $q \to \infty$ *so, in fact,* $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega)$ *for all* p *such that* $1 \leq p \leq \infty$*.*

Fractional Sobolev spaces have figured in the analysis in Chaps. 4 and 5; they can be defined by making use of the Fourier transform or, equivalently, by introducing the Slobodeckij seminorm, which is roughly analogous to the Hölder seminorm. Fractional order Sobolev spaces have also been referenced in the literature as Aronszajn spaces, Gagliardo spaces, and Slobodeckij spaces (see, e.g., [Ar, Gag], or [Slo]). For $\Omega \subseteq R^n$ an open bounded subset, with C^1 boundary $\partial\Omega$, $f \in L^p(\Omega)$, $1 \leq p < \infty$, and $0 \leq \theta < 1$, the Slobodeckij seminorm of f is defined by

$$[f]_{\theta,p,\Omega} = \int_\Omega \int_\Omega \frac{|f(\boldsymbol{x}) - f(\boldsymbol{y})|^p}{||\boldsymbol{x} - \boldsymbol{y}||^{\theta p + n}} \, d\boldsymbol{x} \, d\boldsymbol{y}.$$

Now, let $s > 0$ (not an integer) and set $\theta = s - [s]$, so that $\theta \in (0, 1)$; then the Sobolev-Slobodeckij space $W^{s,p}(\Omega)$ is defined as

$$W^{s,p} = \{f \in W^{[s],p}(\Omega) \mid \sup_{|\boldsymbol{\alpha}|=[s]} [\boldsymbol{D}^{\boldsymbol{\alpha}} f]_{\theta,p,\Omega} < \infty\}.$$

$W^{s,p}(\Omega)$ is a Banach space for the norm

$$||f||_{W^{s,p}(\Omega)} = ||f||_{W^{[s],p}} + \sup_{|\boldsymbol{\alpha}|=[s]} [\boldsymbol{D}^{\boldsymbol{\alpha}} f]_{\theta,p,\Omega}.$$

The Sobolev-Slobodeckij spaces provide a continuous scale between the Sobolev spaces, i.e., one has the embeddings

$$W^{k+1,p}(\Omega) \hookrightarrow W^{s',p}(\Omega) \hookrightarrow W^{s,p}(\Omega) \hookrightarrow W^{k,p}(\Omega)$$

for $k \leq s \leq s' \leq k+1$.

Remarks. The spaces $W^{s,p}(\Omega)$ coincide with the real interpolation spaces of Sobolev spaces, i.e., in the sense of equivalent norms

$$W^{s,p}(\Omega) = \left(W^{k,p}(\Omega), W^{k+1,p}(\Omega)\right)_{\theta,p}$$

if $k \in N$, $s \in (k, k+1)$, $\theta = s - [s]$ (see [Tr] or [Ta2] for the notation and definitions).

We close this subsection by stating the following fundamental interpolation result which has been used repeatedly in this volume, especially in Chaps. 4 and 5:

Lemma A.5. *If* $0 \leq \theta \leq 1$, $\dfrac{1}{r} = \dfrac{\theta}{r_1} + \dfrac{(1+\theta)}{r_2}$, *and* $s = \theta s_1 + (1-\theta)s_2$, *then for some* $C > 0$,

$$||u||_{W^{s,r}(\Omega)} \leq C ||u||^{\theta}_{W^{s_1,r_1}(\Omega)} ||u||^{1-\theta}_{W^{s_2,r_2}(\Omega)}.$$

Remarks. With $H^m(\Omega) \equiv W^{m,2}(\Omega)$ it follows, in particular, from Lemma A.5 that for $m \in N$ and $0 < s < 1$, we may interpolate $H^{m+s}(\Omega)$ between $H^{m+1}(\Omega)$ and $L^2(\Omega)$.

A.4 Some Useful Lemmas in Functional Analysis

Lemma A.6. *Suppose that* $1 < p < 2$ *and* $\boldsymbol{\phi} \in \boldsymbol{W}^{1,p}(\Omega) \cap \boldsymbol{W}^{1,2}_0(\Omega) \cap \boldsymbol{W}^{2,2}(\Omega)$, *with* $\Omega \subset R^3$ *a bounded domain with smooth boundary. Then* $\exists \delta' > 0$ *such that for any* $\zeta > 0$, *and some* $d_{\delta'}(\Omega) > 0$,

$$||\boldsymbol{\phi}||^2_{\boldsymbol{W}^{1,p}(\Omega)} \geq \frac{\zeta^{\frac{1}{\delta'}}}{\delta' d_{\delta'}} |\boldsymbol{\phi}|_{\boldsymbol{H}^1(\Omega)}{}^2 - \zeta^{\frac{1}{\delta'(1-\delta')}} \left(\frac{1-\delta'}{\delta'}\right) ||\boldsymbol{\phi}||_{\boldsymbol{H}^2(\Omega)}{}^2. \tag{A.1}$$

Proof. For $1 < p < 2$ interpolation estimates (see [Tr]) yield, for some $\bar{c} = \bar{c}(\delta') > 0$,

$$||\boldsymbol{\phi}||_{\boldsymbol{W}^{1,2}(\Omega)} \leq \bar{c}(\delta')||\boldsymbol{\phi}||^{\delta'}_{\boldsymbol{W}^{1,p}(\Omega)}||\boldsymbol{\phi}||^{1-\delta'}_{\boldsymbol{W}^{1,6}(\Omega)} \tag{A.2}$$

with

$$\delta' = \frac{2p}{6-p} = 2\left(\frac{2-\alpha}{4+\alpha}\right); \quad p = 2-\alpha. \tag{A.3}$$

From (A.3), if $1 < p < 2$, then $2/5 < \delta' < 1$. By virtue of the embedding $\boldsymbol{W}^{1,2}(\Omega) \hookrightarrow \boldsymbol{L}^6(\Omega)$, in $\dim n = 3$, $\boldsymbol{W}^{2,2}(\Omega) \hookrightarrow \boldsymbol{W}^{1,6}(\Omega)$; using this and this equivalence of the $\boldsymbol{W}^{1,2}(\Omega)$ and $\boldsymbol{H}^1(\Omega)$ norms, for $\boldsymbol{\phi} \in \boldsymbol{W}^{1,2}_0(\Omega)$, we obtain from (A.2) the estimate

$$||\boldsymbol{\phi}||_{\boldsymbol{H}^1(\Omega)} \leq \hat{c}_{\delta'}(\Omega)||\boldsymbol{\phi}||^{\delta'}_{\boldsymbol{W}^{1,p}(\Omega)}||\boldsymbol{\phi}||_{\boldsymbol{H}^2(\Omega)}{}^{1-\delta'} \tag{A.4}$$

for some $\hat{c}_{\delta'} > 0$. We now apply Young's inequality to (A.4); setting $d_{\delta'}(\Omega) = \hat{c}^2_{\delta'}(\Omega)$, we find that for any $\zeta > 0$ and $q > 1$,

$$\begin{aligned} \frac{1}{d_{\delta'}(\Omega)}|\boldsymbol{\phi}|_{\boldsymbol{H}^1(\Omega)}{}^2 &\leq ||\boldsymbol{\phi}||^{2\delta'}_{\boldsymbol{W}^{1,p}(\Omega)} \cdot ||\boldsymbol{\phi}||_{\boldsymbol{H}^2(\Omega)}{}^{2(1-\delta')} \\ &\leq \frac{\zeta^q}{q}||\boldsymbol{\phi}||_{\boldsymbol{H}^2(\Omega)}{}^{2(1-\delta')} + \left[\frac{q-1}{q\zeta^{q/q-1}}\right]||\boldsymbol{\phi}||^{2\delta' q/q-1}_{\boldsymbol{W}^{1,p}(\Omega)}. \end{aligned} \tag{A.5}$$

The stated result now follows by choosing $q = 1/1-\delta'$ in (A.5), (as $\delta' > 2/5$ and $q > 5/3$), so that $\delta' q/(q-1) = 1$. □

Lemma A.7. *Let $\Omega \subset R^n$, $n = 2, 3$, be a Lipschitz domain and suppose that $u, v, w \in H^1(\Omega)$. Then there exists a constant $C > 0$, independent of u, v, w, such that*

$$\left[\int_\Omega u^2 v^2 w^2 \, d\boldsymbol{x}\right]^{1/2} \leq C|u|_{H^1(\Omega)}|v|_{H^1(\Omega)}|w|_{H^1(\Omega)}. \tag{A.6}$$

Proof. Employing standard results from Sobolev embedding theory (see [Ev] or Sect. A.3) we infer the existence of $c_1(\Omega) > 0$ such that

$$\begin{cases} ||u||_{L^6(\Omega)} \leq c_1(\Omega)||u||_{H^1(\Omega)}, \\ ||v||_{L^6(\Omega)} \leq c_1(\Omega)||v||_{H^1(\Omega)}, \\ ||w||_{L^6(\Omega)} \leq c_1(\Omega)||w||_{H^1(\Omega)}. \end{cases} \tag{A.7}$$

A first application of the Hölder Inequality yields

$$\left[\int_\Omega u^2 v^2 w^2\, d\boldsymbol{x}\right]^{1/2} \le \left[\int_\Omega |uv|^3\, d\boldsymbol{x}\right]^{1/3} \left[\int_\Omega |w|^6\, d\boldsymbol{x}\right]^{1/6} \tag{A.8}$$

and a second application then produces

$$\begin{aligned}\left[\int_\Omega u^2 v^2 w^2\, d\boldsymbol{x}\right]^{1/2} &\le \left(\int_\Omega |u|^2\, d\boldsymbol{x}\right)^{1/6} \left(\int_\Omega |v|^2\, d\boldsymbol{x}\right)^{1/6} \left(\int_\Omega |w|^2\, d\boldsymbol{x}\right)^{1/6} \\ &\le c_1^3(\Omega)|u|_{H^1(\Omega)}|v|_{H^1(\Omega)}|w|_{H^1(\Omega)}\end{aligned} \tag{A.9}$$

by virtue of (A.7); this establishes (A.6) with $C = c_1^3(\Omega)$. □

Lemma A.8. *Let $\boldsymbol{X}$ be a given Banach space with dual $\boldsymbol{X}'$ and let $\boldsymbol{u}$ and $\boldsymbol{g}$ be two functions belong to $L^1((a,b),\boldsymbol{X})$. The following three conditions are equivalent:*

(i) $\boldsymbol{u}$ is a.e. equal to a primitive function of $\boldsymbol{g}$, i.e.,

$$\boldsymbol{u}(t) = \boldsymbol{\xi} + \int_a^t \boldsymbol{g}(s)\, ds, \quad \boldsymbol{\xi} \in \boldsymbol{X}, \textit{for almost every } t \in [a,b]. \tag{A.10}$$

(ii) For each test function $\boldsymbol{\phi} \in \mathcal{D}(a,b)$,

$$\int_a^b \boldsymbol{u}(t)\boldsymbol{\phi}'(t)\, dt = -\int_a^b \boldsymbol{g}(t)\boldsymbol{\phi}(t)\, dt, \quad \left(\boldsymbol{\phi}' = \frac{d\boldsymbol{\phi}}{dt}\right). \tag{A.11}$$

(iii) For each $\boldsymbol{\eta} \in \boldsymbol{X}'$,

$$\frac{d}{dt}\langle \boldsymbol{u}, \boldsymbol{\eta}\rangle = \langle \boldsymbol{g}, \boldsymbol{\eta}\rangle \tag{A.12}$$

in the scalar distribution sense, on (a,b).

If (A.10)–(A.12) are satisfied $\boldsymbol{u}$ is, in particular, a.e. equal to a continuous function from $[a,b]$ into $\boldsymbol{X}$.

Proof. (See [Te4].) □

Lemma A.9. *Let $\boldsymbol{X}_0$, $\boldsymbol{X}$, $\boldsymbol{X}_1$ be three Banach spaces such that*

$$\boldsymbol{X}_0 \subset \boldsymbol{X} \subset \boldsymbol{X}_1$$

where the injections are continuous and

$$\boldsymbol{X}_i \textit{ is reflexive, } i = 0, 1,$$

$$\textit{the injection } \boldsymbol{X}_0 \to \boldsymbol{X} \textit{ is compact.}$$

Let $T > 0$ *be a fixed finite number, and let* $\alpha_i > 1$, $i = 0, 1$. *If*

$$\boldsymbol{\mathcal{Y}} = \{\boldsymbol{v} \in \boldsymbol{L}^{\alpha_0}((0,T);\boldsymbol{X}_0) \mid \boldsymbol{v}' = \frac{d\boldsymbol{v}}{dt} \in \boldsymbol{L}^{\alpha_1}((0,T);\boldsymbol{X}_1)\} \tag{A.13}$$

then $\boldsymbol{\mathcal{Y}}$ *is a Banach space with the norm*

$$||\boldsymbol{v}||_{\boldsymbol{\mathcal{Y}}} = ||\boldsymbol{v}||_{\boldsymbol{L}^{\alpha_1}((0,T);\boldsymbol{X}_0)} + ||\boldsymbol{v}||_{\boldsymbol{L}^{\alpha_1}((0,T);\boldsymbol{X}_1)}. \tag{A.14}$$

Furthermore

$$\boldsymbol{\mathcal{Y}} \subset L^{\alpha_0}((0,T);\boldsymbol{X}) \text{ is compact.}$$

Remarks. Lemma A.9 is the explicit version of Aubin's Lemma, A.2.9, which has been used in this volume.

Appendix B
Estimates Involving the Rate of Deformation Tensor

In this appendix we will state and prove several of the lemmas involving the rate of deformation tensor which have been used in both Chaps. 4 and 5 and which are cross-listed with the identical lemmas stated in this appendix. The first lemma is an $\boldsymbol{L}^p$ version of the Korn inequality; for the cases of interest, $1 < p < \infty$, proofs may be found in [Go1, 2], [Fu, N1], and [Te5]. Here we follow the general scheme of the proof presented in the monograph [MNRR] and we state the result only for a domain Ω which is open and bounded in $\mathbb{R}^n$, $n = 2, 3$, with smooth boundary $\partial\Omega$ (i.e., $\partial\Omega \in C^{0,1}$); the result also holds for $\boldsymbol{v} \in \boldsymbol{W}^{1,p}_{per}(\Omega)$, with $\Omega = [0, L]^n$, $n = 2, 3$, and $\boldsymbol{v}$ satisfying the conditions in (5.3b).

Lemma B.1. *Let* $\boldsymbol{v} \in \boldsymbol{W}^{1,p}_0(\Omega)$, $1 < p < \infty$, *where* Ω *is a bounded domain in* $\mathbb{R}^n$, $n = 2, 3$, *with smooth boundary* $\partial\Omega \in C^{0,1}$. *Then* $\exists c_1 = c_1(p; \Omega)$, $c_1 > 0$, *such that*

$$\int_\Omega \left[e_{ij}(\boldsymbol{v})e_{ij}(\boldsymbol{v})\right]^{p/2} d\boldsymbol{x} \geq c_1 ||\boldsymbol{v}||_{\boldsymbol{W}^{1,p}(\Omega)}{}^p. \tag{B.1}$$

Proof. We note that (B.1) may be written in the equivalent form

$$||\boldsymbol{e}(\boldsymbol{v})||_p \geq c_1 ||\boldsymbol{v}||_{1,p} \tag{B.2}$$

where we have used the abbreviated notation $||\cdot||_p = ||\cdot||_{\boldsymbol{L}^p(\Omega)}$ and $||\cdot||_{1,p} = ||\cdot||_{\boldsymbol{W}^{1,p}(\Omega)}$. The proof of (B.2) depends upon the following general result which has been established, e.g., in [N1]: Let $\boldsymbol{\mathcal{D}}(\Omega) = \boldsymbol{C}^\infty_0(\Omega)$ be the usual space of test functions and $\boldsymbol{\mathcal{D}}'(\Omega)$ the space of distributions. Let $\Omega \subset R^n$ be an open bounded subset with $\partial\Omega \in C^{0,1}$ and $T \in \boldsymbol{\mathcal{D}}'(\Omega)$. If $T, \dfrac{\partial T}{\partial x_i} \in (\boldsymbol{W}^{1,q}_0(\Omega))'$, for some $q \in (1, \infty)$ and all $i = 1, 2, \ldots, n$, then $\exists \boldsymbol{u} \in \boldsymbol{L}^{q'}(\Omega)$, $q' = q/q - 1$, such that

$$\langle T, \boldsymbol{\varphi} \rangle = \int_\Omega \boldsymbol{u} \cdot \boldsymbol{\varphi} d\boldsymbol{x}, \quad \forall \boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\Omega). \tag{B.3}$$

H. Bellout and F. Bloom, *Incompressible Bipolar and Non-Newtonian Viscous Fluid Flow*, Advances in Mathematical Fluid Mechanics, DOI 10.1007/978-3-319-00891-2,

Moreover, $\exists C > 0$ such that

$$||\boldsymbol{u}||^{q'}_{\boldsymbol{L}^{q'}(\Omega)} \leq C \left(||T||^{q'}_{-1,q'} + \sum_{i=1}^{n} \left|\left| \frac{\partial T}{\partial x_i} \right|\right|^{q'}_{-1,q'} \right). \tag{B.4}$$

Using the above stated result from [N1], we now proceed as follows: First, we define the space

$$\boldsymbol{E}(\Omega) \equiv \left\{ \boldsymbol{u} \in \boldsymbol{L}^p(\Omega) \mid \boldsymbol{e}(\boldsymbol{u}) \in (\boldsymbol{L}^p(\Omega))^2 \right\} \tag{B.5}$$

where $\boldsymbol{e} \in (\boldsymbol{L}^p(\Omega))^2$ means $e_{ij} \in L^p(\Omega)$, $i, j = 1, \ldots, n$. We also define, on $\boldsymbol{E}(\Omega)$, the norm

$$||\boldsymbol{u}||_{\boldsymbol{E}(\Omega)} \equiv ||\boldsymbol{u}||_p + ||\boldsymbol{e}(\boldsymbol{u})||_p. \tag{B.6}$$

Then, $\boldsymbol{E}(\Omega)$ is a Banach space. Let

$$\boldsymbol{I} : \boldsymbol{W}^{1,p}(\Omega) \to \boldsymbol{E}(\Omega) \tag{B.7}$$

be the identity mapping; this is clearly a continuous map and we want to show that $\boldsymbol{I}$ is, in fact, surjective. If we take $\boldsymbol{v} \in \boldsymbol{E}(\Omega)$ then, in the sense of distributions, for all $i, j, k = 1, 2, \ldots, n$,

$$\frac{\partial^2 v_i}{\partial x_j \, \partial x_k} = \frac{\partial e_{ik}(\boldsymbol{v})}{\partial x_j} + \frac{\partial e_{ij}(\boldsymbol{v})}{\partial x_k} - \frac{\partial e_{jk}(\boldsymbol{v})}{\partial x_i}. \tag{B.8}$$

As $\boldsymbol{e}(\boldsymbol{v}) \in (\boldsymbol{L}^p(\Omega))^2$, (B.8) implies that

$$\frac{\partial^2 v_i}{\partial x_j \, \partial x_k} \in (\boldsymbol{W}_0^{1,p'}(\Omega))' \tag{B.9}$$

with $p' = p/p - 1$. Furthermore, as $\boldsymbol{v} \in \boldsymbol{L}^p(\Omega)$, it follows that

$$\frac{\partial v_i}{\partial x_j} \in (\boldsymbol{W}_0^{1,p'}(\Omega))'. \tag{B.10}$$

By virtue of (B.3), (B.4), $\dfrac{\partial v_i}{\partial x_j} \in \boldsymbol{L}^p(\Omega)$ for $i, j = 1, 2, \ldots, n$ and, therefore, $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)$ and $\boldsymbol{I}$ is surjective. Thus, $\boldsymbol{W}^{1,p}(\Omega)$ coincides with $\boldsymbol{E}(\Omega)$ and the Open Mapping Theorem then implies that $\exists c(p; \Omega) > 0$ such that

$$||\boldsymbol{v}||_{1,p} \leq c(p; \Omega)(||\boldsymbol{v}||_p + ||\boldsymbol{e}(\boldsymbol{v})||_p). \tag{B.11}$$

To prove the validity of (B.2) it is necessary only to show that $\forall \boldsymbol{v} \in \boldsymbol{W}_0^{1,p}(\Omega)$, $\exists \tilde{c}(p; \Omega) > 0$ such that

$$||\boldsymbol{v}||_p \le \tilde{c}(p;\Omega)||\boldsymbol{e}(\boldsymbol{v})||_p \tag{B.12}$$

in which case (B.2) will follow with

$$c_1(p;\Omega) = 1/c(p;\Omega)(\tilde{c}(p;\Omega)+1). \tag{B.13}$$

We argue by contradiction, i.e., we assume that $\exists\{\boldsymbol{v}^n\}_{n=1}^{\infty} \subset \boldsymbol{W}_0^{1,p}(\Omega)$ such that $||\boldsymbol{v}^n||_p = 1$ and $n||\boldsymbol{e}(\boldsymbol{v}^n)||_p < 1$. Then, $\boldsymbol{e}(\boldsymbol{v}^n) \to \boldsymbol{0}$ in $(\boldsymbol{L}^p(\Omega))^2$ as $n \to \infty$. Using (B.11), we infer the existence of a subsequence of $\{\boldsymbol{v}^n\}$, still labeled as $\{\boldsymbol{v}^n\}$, such that $\boldsymbol{v}^n \rightharpoonup \boldsymbol{v}$ in $\boldsymbol{W}^{1,p}(\Omega)$ and $\boldsymbol{v}^n \to \boldsymbol{v}$ in $\boldsymbol{L}^p(\Omega)$. Therefore, $||\boldsymbol{v}||_p = 1$, $\boldsymbol{v}|_{\partial\Omega} = \boldsymbol{0}$, and $\boldsymbol{e}(\boldsymbol{v}) = \boldsymbol{0}$. However, it is known [NH] that a vector field $\boldsymbol{v}$ satisfying $\boldsymbol{e}(\boldsymbol{v}) = \boldsymbol{0}$ may be written in the form $\boldsymbol{v} = \boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{x}$ for some vectors $\boldsymbol{a}$, $\boldsymbol{b}$. However, $\boldsymbol{v}$ satisfies the homogeneous boundary condition, in which case $\boldsymbol{v} \equiv \boldsymbol{0}$; this contradicts $||\boldsymbol{v}||_p = 1$ and serves to establish the lemma. □

The next result is another (more elementary) inequality of the Korn variety; although a standard proof may be known, because we could not find a specific reference in which it is established, we will provide the simple proof here.

Lemma B.2. *If* $\boldsymbol{v} \in \boldsymbol{W}^{2,2}(\Omega) \cap \boldsymbol{W}_0^{1,2}(\Omega)$, $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, *a bounded domain with smooth boundary, then* $\exists c_2 = c_2(\Omega) > 0$ *such that*

$$\int_\Omega \frac{\partial e_{ij}(\boldsymbol{v})}{\partial x_k}\frac{\partial e_{ij}(\boldsymbol{v})}{\partial x_k} d\boldsymbol{x} \ge c_2||\boldsymbol{v}||^2_{\boldsymbol{W}^{2,2}(\Omega)}. \tag{B.14}$$

Proof. Following the procedure used to establish (6.263) in the two-dimensional case, a direct calculation yields, for some $\hat{c}_2 > 0$,

$$\frac{\partial e_{ij}(\boldsymbol{v})}{\partial x_k}\frac{\partial e_{ij}(\boldsymbol{v})}{\partial x_k} \ge \hat{c}_2 \sum_i \sum_k \left(\frac{\partial^2 v_i}{\partial x_k^2}\right) \tag{B.15}$$

where we sum, on the left-hand side of (B.15) over i, j, k and where the value of $\hat{c}_2$ depends on whether $n = 2$ or 3. However, for each i,

$$\sum_k \left(\frac{\partial^2 v_i}{\partial x_k^2}\right)^2 \ge \frac{1}{2}(\Delta v_i)^2 \tag{B.16}$$

so that, for some $\bar{c}_2 > 0$,

$$\int_\Omega \frac{\partial e_{ij}(\boldsymbol{v})}{\partial x_k}\frac{\partial e_{ij}(\boldsymbol{v})}{\partial x_k} d\boldsymbol{x} \ge \bar{c}_2 \int_\Omega \sum_i (\Delta v_i)^2 d\boldsymbol{x} \tag{B.17}$$

and (B.14) now follows from standard regularity results for elliptic equations, e.g., [Ev]. □

The result stated in the next lemma is absolutely critical for the analysis of the initial-boundary value problem (5.2a,b), (5.3a), (5.4) because in doing integrations by parts, involving the higher-order derivatives in (5.1a), when $\mu_1 \neq 0$, we need to make use of the fact that the higher-order boundary conditions in (5.3a) imply the vanishing of $\tau_{ijk} e_{ij} \nu_k$ on $\partial\Omega \times [0, T)$; the specific statement of this important result is the following lemma which is also valid for a bounded domain in $\mathbb{R}^2$:

Lemma B.3. *Let $S \subseteq \mathbb{R}^3$ be a smooth surface and $\boldsymbol{v}(\cdot)$ a divergence free C^2 vector field defined on a neighborhood of S, with $\boldsymbol{v} = \boldsymbol{0}$ on S. If $\tau_{ijk}(\boldsymbol{v})\nu_j \nu_k - \tau_{jkl}\nu_j \nu_k \nu_l \nu_i|_S = 0$, for $i = 1, 2, 3$, where $\boldsymbol{\nu}$ is the exterior unit normal on S, then $\tau_{ijk}(\boldsymbol{v}) e_{ij}(\boldsymbol{v})\nu_k|_S = 0$.*

Proof. As τ_{ijk} and e_{ij} are both symmetric in i and j:

$$\tau_{ijk} e_{ij} \nu_k = \tau_{ijk} \frac{\partial v_i}{\partial x_j} \nu_k. \tag{B.18}$$

Let $\boldsymbol{p} \in S$ and let $(\boldsymbol{t}, \boldsymbol{\tau})$ be a pair of orthonormal vectors at $\boldsymbol{p}$ which lie in the tangent plane to S at $\boldsymbol{p}$; thus, $(\boldsymbol{t}, \boldsymbol{\tau}, \boldsymbol{\nu})$ form an orthonormal triplet at $\boldsymbol{p}$. Define the vectors $\boldsymbol{\lambda}^{(i)}$, $i = 1, 2, 3$, by

$$\lambda_j^{(i)} \equiv \frac{\partial v_i}{\partial x_j} = \alpha^{(i)} \nu_j + \beta^{(i)} t_j + \gamma^{(i)} \tau_j. \tag{B.19}$$

However, by virtue of (B.18), (B.19) it follows that

$$\begin{aligned} \tau_{ijk}(\boldsymbol{v}) e_{ij} \nu_k|_{\boldsymbol{p}} &= \tau_{ijk}(\boldsymbol{v}) \lambda_j^{(i)} \nu_k|_{\boldsymbol{p}} \\ &= \alpha^{(i)} \tau_{ijk}(\boldsymbol{v}) \nu_j \nu_k|_{\boldsymbol{p}} \\ &\quad + \beta^{(i)} \tau_{ijk}(\boldsymbol{v}) t_j \nu_k|_{\boldsymbol{p}} + \gamma^{(i)} \tau_{ijk}(\boldsymbol{v}) \tau_j \nu_k|_{\boldsymbol{p}}. \end{aligned} \tag{B.20}$$

We now choose curves $\boldsymbol{\eta}_1(\xi) \subset S$, $\boldsymbol{\eta}_2(\xi) \subset S$, $|\xi| \leq \bar{\xi}$, such that $\boldsymbol{\eta}_1(0) = \boldsymbol{\eta}_2(0) = \boldsymbol{p}$ and $\boldsymbol{\eta}_1(0) = \boldsymbol{t}$, $\boldsymbol{\eta}_2(0) = \boldsymbol{\tau}$. As $\boldsymbol{v}|_S = \boldsymbol{0}$, $v_i(\boldsymbol{\eta}_1(\xi)) = v_i(\boldsymbol{\eta}_2(\xi)) = 0$, $|\xi| \leq \xi$, for $i = 1, 2, 3$; thus

$$\begin{cases} \dfrac{dv_i}{d\xi}|_{\eta_i} \equiv \dfrac{\partial v_i}{\partial x_j}(\boldsymbol{\eta}_1(\xi)) \dfrac{dx_j}{d\xi} = 0, \\ \dfrac{dv_i}{d\xi}|_{\eta_2} \equiv \dfrac{\partial v_i}{\partial x_j}(\boldsymbol{\eta}_2(\xi)) \dfrac{dx_j}{d\xi} = 0, \end{cases} \qquad (i = 1, 2, 3). \tag{B.21}$$

Setting $\xi = 0$ in (B.21) we have

$$\left.\frac{\partial v_i}{\partial x_j}\right|_{\boldsymbol{p}} \cdot t_j = \left.\frac{\partial v_i}{\partial x_j}\right|_{\boldsymbol{p}} \cdot \tau_j = 0, \quad i = 1, 2, 3. \tag{B.22}$$

However, by virtue of (B.19), $\beta^{(i)} = \frac{\partial v_i}{\partial x_j} t_j$ and $\gamma^{(i)} = \frac{\partial v_i}{\partial x_j} \tau_j$, for $i = 1, 2, 3$. Thus, at $\boldsymbol{p}$, $\gamma^{(i)} = \beta^{(i)} = 0$, for $i = 1, 2, 3$, so that by (B.20), and the higher-order boundary conditions in (5.3a)

$$\tau_{ijk}(\boldsymbol{v}) e_{ij} \nu_k|_{\boldsymbol{p}} = \alpha^{(i)} \nu_i \tau_{ljk}(\boldsymbol{v}) \nu_l \nu_j \nu_k|_{\boldsymbol{p}}. \tag{B.23}$$

But, as $\beta^{(i)} = \gamma^{(i)} = 0$, $i = 1, 2, 3$, (B.19) reduces to

$$\lambda_j^{(i)} = \frac{\partial v_i}{\partial x_j} = \alpha^{(i)} \nu_j. \tag{B.24}$$

Therefore, setting $i = j$ in (B.24), summing on i, and using the fact that $\boldsymbol{v}$ is a solenoidal vector field, we obtain

$$\frac{\partial v_i}{\partial x_i} \equiv \nabla \cdot \boldsymbol{v} = \alpha^{(i)} \nu_i = 0 \tag{B.25}$$

in which case (B.23) yields the required result, i.e., $\tau_{ijk}(\boldsymbol{v}) e_{ij} \nu_k|_{\boldsymbol{p}} = 0$. □

Remarks. Lemma B.3 has been used, repeatedly, in the text to effectuate the following integration by parts computation: for $\boldsymbol{v}$ sufficiently smooth and satisfying (5.3a)

$$\begin{aligned} \int_\Omega \frac{\partial}{\partial x_j}(\Delta e_{ij}) v_i \, d\boldsymbol{x} &= \int_\Omega \left[\frac{\partial}{\partial x_j}(\Delta e_{ij} v_i) - \Delta e_{ij} \frac{\partial v_i}{\partial x_j} \right] d\boldsymbol{x} \\ &= \oint_{\partial\Omega} \Delta e_{ij} v_i \nu_j \, dS - \int_\Omega \frac{\partial}{\partial x_k} \left(\frac{\partial e_{ij}}{\partial x_k} \right) \frac{\partial v_i}{\partial x_j} d\boldsymbol{x} \\ &= -\int_\Omega \left[\frac{\partial}{\partial x_k} \left(\frac{\partial e_{ij}}{\partial x_k} \frac{\partial v_i}{\partial x_j} \right) - \frac{\partial e_{ij}}{\partial x_k} \frac{\partial v_i}{\partial x_j \partial x_k} \right] d\boldsymbol{x} \text{ (as } \boldsymbol{v}|_{\partial\Omega} = \boldsymbol{0}) \\ &= -\oint_{\partial\Omega} \frac{\partial e_{ij}}{\partial x_k} \frac{\partial v_i}{\partial x_j} \nu_k \, dS + \int_\Omega \frac{\partial e_{ij}}{\partial x_k} \frac{\partial e_{ij}}{\partial x_k} d\boldsymbol{x} \end{aligned} \tag{B.26}$$

so

$$\begin{aligned} 2\mu_1 \int_\Omega \frac{\partial}{\partial x_j}(\Delta e_{ij}) v_i \, d\boldsymbol{x} &= -\oint_{\partial\Omega} \tau_{ijk} e_{ij} \nu_k \, dS + 2\mu_1 \int_\Omega \frac{\partial e_{ij}}{\partial x_k} \frac{\partial e_{ij}}{\partial x_k} d\boldsymbol{x} \\ &= 2\mu_1 \int_\Omega \frac{\partial e_{ij}}{\partial x_k} \frac{\partial e_{ij}}{\partial x_k} d\boldsymbol{x} \quad (\text{as } \tau_{ijk}(\boldsymbol{v}) e_{ij} \nu_k|_{\partial\Omega} = 0). \end{aligned} \tag{B.27}$$

Remarks. In (B.27), $\tau_{ijk} e_{ij} \nu_k|_{\partial\Omega} = 0$ is a consequence of the boundary conditions (5.3a) which, by (1.127), apply whenever $M_k \tau_k = 0$ on $\partial\Omega$; in particular, $\tau_{ijk} e_{ij} \nu_k|_{\partial\Omega} = 0$ if $M_i = 0$, on $\partial\Omega$, $i = 1, 2, 3$.

Suppose, however, that $M_k\tau_k \neq 0$ on $\partial\Omega$. Then by virtue of (B.20) with $\beta^{(i)} = \nu^{(i)} = 0$, at any $\boldsymbol{p} \in \partial\Omega$

$$\tau_{ijk}(\boldsymbol{v})e_{ij}\nu_k|_{\boldsymbol{p}} = \alpha^{(i)}\tau_{ijk}\nu_j\nu_k|_{\boldsymbol{p}}. \tag{B.28}$$

However, if $M_k\tau_k \neq 0$, then by (1.127),

$$\tau_{ijk}(\boldsymbol{v})\nu_j\nu_k - (\tau_{ljk}(\boldsymbol{v})\nu_l\nu_j\nu_k)\nu_i = (M_k\tau_k)\tau_i$$

at each $\boldsymbol{p} \in \partial\Omega$, and (B.28) becomes

$$\tau_{ijk}(\boldsymbol{v})e_{ij}\nu_k|_p = \alpha^{(i)}\left[(\tau_{ljk}\nu_l\nu_j\nu_k)\nu_i - (M_k\tau_k)\tau_i\right]_{\boldsymbol{p}}. \tag{B.29}$$

From (B.24) we obtain

$$\alpha^{(i)} = \frac{\partial v_i}{\partial x_j}\nu_j = \frac{\partial v_i}{\partial \boldsymbol{\nu}}, \text{ on } \partial\Omega. \tag{B.30}$$

Also

$$\alpha^{(i)}\nu_i = \nabla \cdot \boldsymbol{v} = 0. \tag{B.31}$$

Combining (B.29), (B.30), and (B.31) we obtain, at each $\boldsymbol{p} \in \partial\Omega$,

$$\begin{aligned} \tau_{ijk}(\boldsymbol{v})e_{ij}\nu_k &= (M_k\tau_k)\alpha^{(i)}\tau_i \\ &= (M_k\tau_k)\tau_i\frac{\partial v_i}{\partial \boldsymbol{\nu}} \end{aligned} \tag{B.32}$$

in which case the last result in (B.27) becomes

$$2\mu_1\int_\Omega \frac{\partial}{\partial x_j}(\Delta e_{ij})v_i\,d\boldsymbol{x} = 2\mu_1\int_\Omega \frac{\partial e_{ij}}{\partial x_k}\frac{\partial e_{ij}}{\partial x_k}d\boldsymbol{x} - \oint_{\partial\Omega}(M_k\tau_k)\tau_i\frac{\partial v_i}{\partial \boldsymbol{\nu}}dS. \tag{B.33}$$

In this calculation, e_{ij} is based on the velocity field $\boldsymbol{v}$. In Sect. 4.2 we have multiplied the bipolar equations (4.1) through by the ith component ψ_i of a test function $\boldsymbol{\psi} \in W^{1,2}((0,\infty);\boldsymbol{H})$ and summed on i, where $\boldsymbol{H}$ is given by (4.12); in this case, it is easily seen that (B.33) is to be replaced by

$$2\mu_1\int_\Omega \frac{\partial}{\partial x_j}(\Delta e_{ij}(\boldsymbol{v}))\psi_i\,d\boldsymbol{x} = 2\mu_1\int_\Omega \frac{\partial e_{ij}}{\partial x_k}(\boldsymbol{v})\frac{\partial e_{ij}}{\partial x_k}(\boldsymbol{\psi})d\boldsymbol{x} - \oint_{\partial\Omega}(M_k\tau_k)\tau_i\frac{\partial \psi_i}{\partial \boldsymbol{\nu}}dS. \tag{B.34}$$

Our next result provides a simple lower bound for the integral $\int_\Omega \boldsymbol{w} \cdot \boldsymbol{e}(\boldsymbol{U}) \cdot \boldsymbol{w}\, d\boldsymbol{x}$ whenever $\Omega \subseteq \mathbb{R}^3$ is a bounded domain, with smooth boundary $\partial\Omega$, $\boldsymbol{w} \in \boldsymbol{L}^2(\Omega)$, and $\boldsymbol{U} \in \boldsymbol{W}^{1,2}(\Omega)$; the specific result is

Lemma B.4. *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$. Then for any $\boldsymbol{w} \in \boldsymbol{L}^2(\Omega)$, and $\boldsymbol{U} \in \boldsymbol{W}^{1,2}(\Omega)$, $\exists \Lambda(\boldsymbol{U}) > 0$ such that*

$$||\boldsymbol{w}||_{\boldsymbol{L}^2(\Omega)}{}^2 \geq -\left(\frac{1}{\Lambda}\right) \int_\Omega \boldsymbol{w} \cdot \boldsymbol{e}(\boldsymbol{U}) \cdot \boldsymbol{w}\, d\boldsymbol{x}. \tag{B.35}$$

Proof. Let $\boldsymbol{x} \in \Omega$ and refer the (symmetric) rate of deformation tensor $e_{ij}(\boldsymbol{U})$ to its principal axes at $\boldsymbol{x}$. Then, at $\boldsymbol{x}$,

$$\boldsymbol{w} \cdot \boldsymbol{e}(\boldsymbol{U}) \cdot \boldsymbol{w} = e_{ij}(\boldsymbol{U}) w_i w_j \geq ||\boldsymbol{w}||^2 \min_{1 \leq j \leq 3} [e_{jj}(\boldsymbol{U})] \tag{B.36}$$

where $||\boldsymbol{w}|| = (w_i w_i)^{\frac{1}{2}}$ is the Euclidean norm on $\mathbb{R}^3$ and the $e_{ii}(\boldsymbol{U})$, $i = 1, 2, 3$, are the (real) eigenvalues of $\boldsymbol{e}(\boldsymbol{U})$ at $\boldsymbol{x}$. As

$$\operatorname{div} \boldsymbol{U} = \operatorname{tr} e(\boldsymbol{U}) \equiv \sum_{j=1}^{3} e_{jj}(\boldsymbol{U}) = 0 \tag{B.37}$$

at least one of the $e_{ii}(\boldsymbol{U})$, $i = 1, 2, 3$ must be negative at $\boldsymbol{x}$. We denote the largest (in magnitude) negative eigenvalue of $\boldsymbol{e}(\boldsymbol{U})$ at $\boldsymbol{x} \in \Omega$ by $\Lambda(\boldsymbol{x}, \boldsymbol{U}) = -|\Lambda(\boldsymbol{x}, \boldsymbol{U})|$. However, $\Lambda(\cdot, \boldsymbol{U})$ is continuous on $\bar{\Omega}$, which is compact on $\mathbb{R}^3$; thus, if we set

$$\Lambda(\boldsymbol{U}) = \max_{\boldsymbol{x} \in \bar{\Omega}} |\Lambda(\boldsymbol{x}, \boldsymbol{U})| \tag{B.38}$$

then, as a direct consequence of (B.36),

$$\int_\Omega \boldsymbol{w} \cdot \boldsymbol{e}(\boldsymbol{U}) \cdot \boldsymbol{w}\, d\boldsymbol{x} \geq -\Lambda(\boldsymbol{U}) \tag{B.39}$$

which is equivalent to (B.35). □

Remarks. Lemma B.4 has an analogous statement in space dim $n = 2$. Also, as pointed out in Sect. 5.2.2, this result retains its validity, with obvious modifications, if $w_i, U_i \in \mathbb{C}$, $\boldsymbol{L}^2(\Omega)$ is replaced by its complex counterpart $\boldsymbol{L}_c^2(\Omega)$, etc., and

$$||\boldsymbol{w}||^2_{\boldsymbol{L}_c^2(\Omega)} = \int_\Omega w_i w_i^* \, d\boldsymbol{x}, \tag{B.40a}$$

$$\boldsymbol{w} \cdot \boldsymbol{e}(\boldsymbol{U}) \cdot \boldsymbol{w} = e_{ij}(\boldsymbol{U}) w_i w_j^* \tag{B.40b}$$

where $e_{ij} = \dfrac{1}{2}\left(\dfrac{\partial U_i}{\partial x_j} + \dfrac{\partial U_j^*}{\partial x_i}\right) = e_{ji}^*$, the asterisk denoting complex conjugation.

Our final result in this appendix is the following:

Lemma B.5. *Suppose that* $\boldsymbol{u}(t)$, $\boldsymbol{v}(t)$ *are the unique solutions of* (5.2a,b), (5.3a), (5.4) *which correspond, respectively, to initial data* $\boldsymbol{u}(0) = \boldsymbol{u}_0$ *and* $\boldsymbol{v}(0) = \boldsymbol{v}_0$. *Then, for* $1 < p \leq 2$,

$$\int_\Omega \left[\gamma(\boldsymbol{v})e_{ij}(\boldsymbol{v}) - \gamma(\boldsymbol{u})e_{ij}(\boldsymbol{u})\right]\left[e_{ij}(\boldsymbol{v}) - e_{ij}(\boldsymbol{u})\right] d\boldsymbol{x} \geq 0 \tag{B.41}$$

where $\gamma(\boldsymbol{v}) = \mu(|\boldsymbol{e}(\boldsymbol{v})|)$.

Proof. With $\alpha = 2 - p$ we have introduced the potential (see, e.g., (5.92))

$$\Gamma(e_{ij}e_{ij}) = \int_0^{e_{ij}e_{ij}} \mu_0(\epsilon + s)^{-\alpha/2} ds$$

so that

$$\frac{\partial \Gamma}{\partial e_{ij}} = \gamma(\boldsymbol{e})e_{ij}. \tag{B.42}$$

Therefore,

$$\begin{aligned}
&\left[\gamma(\boldsymbol{v})e_{ij}(\boldsymbol{v}) - \gamma(\boldsymbol{u})e_{ij}(\boldsymbol{u})\right]\left[e_{ij}(\boldsymbol{v}) - e_{ij}(\boldsymbol{u})\right] \\
&\quad = \int_0^1 \frac{d}{d\lambda}\left(\frac{\partial \Gamma(e_{ij}(\boldsymbol{u}) + \lambda(e_{ij}(\boldsymbol{v}) - e_{ij}(\boldsymbol{u}))}{\partial e_{ij}}\right) d\lambda \times [e_{ij}(\boldsymbol{v}) - e_{ij}(\boldsymbol{u})] \\
&\quad = \int_0^1 \left\{ \frac{\partial^2 \Gamma}{\partial e_{ij}\partial e_{kl}}(e_{ij}(\boldsymbol{u}) + \lambda(e_{ij}(\boldsymbol{v}) - e_{ij}(\boldsymbol{u}))[e_{ij}(\boldsymbol{v}) - e_{ij}(\boldsymbol{u})] \right. \\
&\qquad\qquad \left. \times [e_{kl}(\boldsymbol{v}) - e_{kl}(\boldsymbol{u})] \right\} d\lambda.
\end{aligned} \tag{B.43}$$

However, it is easily seen that the potential Γ (e.g., see [MNRR], Chap. 5, estimate (1.8)) satisfies

$$\frac{\partial^2 \Gamma(\boldsymbol{e})}{\partial e_{ij}\partial e_{kl}}\eta_{ij}\eta_{kl} \geq c_1(1 + |\boldsymbol{e}|)^{p-2}|\boldsymbol{\eta}|^2 \tag{B.44}$$

for $p > 1$ and some $c_1 > 0$. Combining (B.43) and (B.44) one obtains

$$\begin{aligned}&\left[\gamma(\boldsymbol{v})e_{ij}(\boldsymbol{v}) - \gamma(\boldsymbol{u})e_{ij}(\boldsymbol{u})\right]\left[e_{ij}(\boldsymbol{v}) - e_{ij}(\boldsymbol{u})\right] \\ &\qquad \geq c_1|\boldsymbol{e}(\boldsymbol{v}) - \boldsymbol{e}(\boldsymbol{u})|^2 \int_0^1 (1 + |\boldsymbol{e}(\boldsymbol{u}) + \lambda(\boldsymbol{e}(\boldsymbol{v}) - \boldsymbol{e}(\boldsymbol{u}))|)^{p-2}\, d\lambda \end{aligned} \tag{B.45}$$

with $|\boldsymbol{e}| = \sqrt{e_{ij}e_{ij}}$, and (B.41) follows as an immediate consequence. □

Lemma B.6 (Korn inequality for the Exterior Problem in the Plane). *For some* $c > 0$,

$$\iint \sum_{i,j} \left|e_{ij}(\boldsymbol{u})\right|^{2-\alpha} d\boldsymbol{x} \geq c||\boldsymbol{D}\boldsymbol{u}||^{2-\alpha}_{\boldsymbol{L}^{2-\alpha}}.$$

Proof. For a proof see the volume [HN]. □

Lemma B.7.

$$\int_\Omega e_{ij}(\boldsymbol{v})e_{ij}(\boldsymbol{v})\, d\boldsymbol{x} \geq \frac{1}{2}||\boldsymbol{v}||_{\boldsymbol{H}^1(\Omega)}{}^2, \quad \forall \boldsymbol{v} \in \boldsymbol{H}^1_0(\Omega).$$

Proof. The proof is an elementary exercise based on the definition of $e_{ij}(\boldsymbol{v})$ and the $\boldsymbol{H}^1(\Omega)$ norm, as well as the "vanishing" of elements of $\boldsymbol{H}^1_0(\Omega)$ on $\partial\Omega$. □

Appendix C
The Spectral Gap Condition

We examine, here, the validity of the spectral gap condition (SGC) with respect to the operator $\boldsymbol{A}$ defined by (5.349)–(5.351), where $\Omega = [0, L]^n$, $L > 0$, $n = 2, 3$ and $\boldsymbol{V}_{per}(\Omega)$ is given by (5.348). We consider, in $\boldsymbol{V}_{per}(\Omega)$, the eigenvalue problem

$$\begin{aligned} \Delta\Delta \boldsymbol{u} + \nabla p &= \lambda \boldsymbol{u}, \\ \nabla \cdot \boldsymbol{u} &= 0. \end{aligned} \tag{C.1}$$

Definition C.1. The number λ is an eigenvalue of $\boldsymbol{A}$ if $\exists \boldsymbol{u} \in \boldsymbol{V}_{per}(\Omega)$, $\boldsymbol{u} \neq 0$, such that

$$\int_\Omega \Delta \boldsymbol{u} \cdot \Delta \boldsymbol{v} d\boldsymbol{x} = \lambda \int_\Omega \boldsymbol{u} \cdot \boldsymbol{v} d\boldsymbol{x}, \ \forall \boldsymbol{v} \in \boldsymbol{V}_{per}(\Omega). \tag{C.2}$$

As is standard, $\boldsymbol{u}$ is then called an eigenfunction of $\boldsymbol{A}$ corresponding to the eigenvalue λ.

We begin with the case $n = 3$.

Lemma C.1. *If $n = 3$, then the numbers*

$$\left\{ \frac{16\pi^4}{L^4}(n_1^2 + n_2^2 + n_3^2) \neq 0 \mid n_i, \ i = 1, 2, 3 \text{ nonnegative integers} \right\}$$

are eigenvalues of $\boldsymbol{A}$.

Proof. Let n_1, n_2, n_3 be three nonnegative integers with $n_1^2 + n_2^2 + n_3^2 \neq 0$. We first show that $\lambda = (16\pi^4/L^4)(n_1^2 + n_2^2 + n_3^2)$ is an eigenvalue of $\boldsymbol{A}$ with corresponding eigenfunction $\boldsymbol{u}$ of the form

$$u_i = \sum_{j=1}^{8} C_{ij} f_j(x_1, x_2, x_3), \quad i = 1, 2, 3, \tag{C.3}$$

H. Bellout and F. Bloom, *Incompressible Bipolar and Non-Newtonian Viscous Fluid Flow*, Advances in Mathematical Fluid Mechanics, DOI 10.1007/978-3-319-00891-2,

where

$$\begin{cases} f_1(x_1, x_2, x_3 = \cos\frac{2\pi n_1}{L}x_1 \cos\frac{2\pi n_2}{L}x_2 \cos\frac{2\pi n_3}{L}x_3, \\ f_2(x_1, x_2, x_3 = \cos\frac{2\pi n_1}{L}x_1 \cos\frac{2\pi n_2}{L}x_2 \sin\frac{2\pi n_3}{L}x_3, \\ f_3(x_1, x_2, x_3 = \cos\frac{2\pi n_1}{L}x_1 \sin\frac{2\pi n_2}{L}x_2 \cos\frac{2\pi n_3}{L}x_3, \\ f_4(x_1, x_2, x_3 = \cos\frac{2\pi n_1}{L}x_1 \sin\frac{2\pi n_2}{L}x_2 \sin\frac{2\pi n_3}{L}x_3, \\ f_5(x_1, x_2, x_3 = \sin\frac{2\pi n_1}{L}x_1 \cos\frac{2\pi n_2}{L}x_2 \cos\frac{2\pi n_3}{L}x_3, \\ f_6(x_1, x_2, x_3 = \sin\frac{2\pi n_1}{L}x_1 \cos\frac{2\pi n_2}{L}x_2 \sin\frac{2\pi n_3}{L}x_3, \\ f_7(x_1, x_2, x_3 = \sin\frac{2\pi n_1}{L}x_1 \sin\frac{2\pi n_2}{L}x_2 \cos\frac{2\pi n_3}{L}x_3, \\ f_8(x_1, x_2, x_3 = \sin\frac{2\pi n_1}{L}x_1 \sin\frac{2\pi n_2}{L}x_2 \sin\frac{2\pi n_3}{L}x_3. \end{cases} \tag{C.4}$$

By direct calculation, (C.3) and (C.4) lead to

$$\begin{aligned} \frac{\partial u_i}{\partial x_1} = \frac{2\pi n_1}{L}\{ & -C_{i1}f_5 - C_{i2}f_6 - C_{i3}f_7 - C_{i4}f_8 \\ & + C_{i5}f_1 + C_{i6}f_2 + C_{i7}f_3 + C_{i8}f_4\}, \end{aligned} \tag{C.5}$$

$$\begin{aligned} \frac{\partial u_i}{\partial x_2} = \frac{2\pi n_2}{L}\{ & -C_{i1}f_3 - C_{i2}f_4 + C_{i3}f_1 + C_{i4}f_2 \\ & - C_{i5}f_7 - C_{i6}f_8 + C_{i7}f_5 + C_{i8}f_6\}, \end{aligned} \tag{C.6}$$

$$\begin{aligned} \frac{\partial u_i}{\partial x_3} = \frac{2\pi n_3}{L}\{ & -C_{i1}f_2 + C_{i2}f_1 - C_{i3}f_4 - C_{i4}f_3 \\ & C_{i5}f_6 + C_{i6}f_5 - C_{i7}f_8 + C_{i8}f_7\}, \end{aligned} \tag{C.7}$$

so that for $i, j = 1, 2, 3$ it is easily verified that

$$\frac{\partial^2 u_i}{\partial x_j^2} = -\left(\frac{2\pi n_j}{L}\right)^2 u_i. \tag{C.8}$$

Therefore,

$$\Delta\Delta u_i = \left(\frac{2\pi}{L}\right)^4 (n_1^2 + n_2^2 + n_3^2)^2 u_i, \quad i = 1, 2, 3. \tag{C.9}$$

Now, employing (C.5)–(C.7), we have

$$
\begin{aligned}
\nabla \cdot \boldsymbol{u} \equiv \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i} = \frac{2\pi}{L}&(n_1 C_{15} + n_2 C_{23} + n_3 C_{32}) f_1 \\
&+ \frac{2\pi}{L}(n_1 C_{16} + n_2 C_{24} - n_3 C_{31}) f_2 \\
&+ \frac{2\pi}{L}(n_1 C_{17} - n_2 C_{21} + n_3 C_{34}) f_3 \\
&+ \frac{2\pi}{L}(n_1 C_{18} - n_2 C_{22} - n_3 C_{33}) f_4 \\
&+ \frac{2\pi}{L}(-n_1 C_{11} + n_2 C_{27} + n_3 C_{36}) f_5 \\
&+ \frac{2\pi}{L}(-n_1 C_{12} + n_2 C_{28} - n_3 C_{35}) f_6 \\
&+ \frac{2\pi}{L}(-n_1 C_{13} - n_2 C_{25} + n_3 C_{38}) f_7 \\
&+ \frac{2\pi}{L}(-n_1 C_{14} - n_2 C_{26} + n_3 C_{37}) f_8.
\end{aligned} \tag{C.10}
$$

For $n_1^2 + n_2^2 + n_3^2 \neq 0$ the condition $\nabla \cdot \boldsymbol{u} = 0$ yields the algebraic system

$$
\begin{cases}
n_1 C_{15} + n_2 C_{23} + n_3 C_{32} = 0, \\
n_1 C_{16} + n_2 C_{24} - n_3 C_{31} = 0, \\
n_1 C_{17} - n_2 C_{21} + n_3 C_{34} = 0, \\
n_1 C_{18} - n_2 C_{22} - n_3 C_{33} = 0, \\
-n_1 C_{11} + n_2 C_{27} + n_3 C_{36} = 0, \\
-n_1 C_{12} + n_2 C_{28} - n_3 C_{35} = 0, \\
-n_1 C_{13} - n_2 C_{25} + n_3 C_{38} = 0, \\
-n_1 C_{14} - n_2 C_{26} + n_3 C_{37} = 0.
\end{cases} \tag{C.11}
$$

Since (C.11) is a homogeneous system of eight equations in twenty-four unknowns, it follows that $(16\pi^4/L^4)(n_1^2 + n_2^2 + n_3^2)$ is an eigenvalue of $\boldsymbol{A}$ with sixteen corresponding independent eigenfunctions of the form given by (C.3)–(C.4). Next, if $n_1 = 0$ but $n_2^2 + n_3^2 \neq 0$, then by virtue of (C.10) we will have $f_5 = f_6 = f_7 = f_8 = 0$. By virtue of (C.9), $\nabla \cdot \boldsymbol{u} = 0$ now implies that

$$\begin{cases} n_2 C_{23} + n_3 C_{32} = 0, \\ n_2 C_{24} - n_3 C_{31} = 0, \\ -n_2 C_{21} + n_3 C_{34} = 0, \\ -n_2 C_{22} - n_3 C_{33} = 0. \end{cases} \tag{C.12}$$

Therefore, as a consequence of (C.9) and (C.12) it follows that, with $n_1 = 0$ and $n_2^2 + n_3^2 \neq 0$, $(16\pi^4/L^4)(n_1^2 + n_2^2 + n_3^2)$ is an eigenvalue of $\boldsymbol{A}$ with eight corresponding independent eigenfunctions of the form (C.3). In an entirely analogous manner we can show that if $n_2 = 0, n_1^2 + n_3^2 \neq 0$, or $n_3 = 0, n_1^2, n_2^2 \neq 0$, then $(16\pi^4/L^4)(n_1^2 + n_2^2 + n_3^2)$ is still an eigenvalue of $\boldsymbol{A}$ with eight corresponding independent eigenfunctions of the form (C.3). Finally, if $n_1 = n_2 = 0, n_3 \neq 0$, then $f_3 = f_1 = f_5 = f_7 = f_8 = 0$, while $\nabla \cdot \boldsymbol{u} = 0$ implies that

$$n_3 C_{32} = 0. \ -n_3 C_{31} = 0 \Leftrightarrow C_{32} = C_{31} = 0. \tag{C.13}$$

Thus, for $n_1 = n_2 = 0$ and $n_3 \neq 0$, $(16\pi^4/L^4)(n_1^2 + n_2^2 + n_3^2)$ is an eigenvalue of $\boldsymbol{A}$ with four corresponding, independent, eigenfunctions of the form (C.3); the analogous result holds for the remaining two cases, i.e., $n_1 = n_3 = 0, n_2 \neq 0$, or $n_2 = n_3 = 0, n_1 \neq 0$, and the proof of the lemma is complete. □

Lemma C.2. *The set of numbers*

$$\left\{ \frac{16\pi^4}{L^4}(n_1^2 + n_2^2 + n_3^2) \neq 0 \mid n_i,\ i = 1, 2, 3 \textit{ nonnegative integers} \right\}$$

exhaust all of the eigenvalues of $\boldsymbol{A}$.

Proof. Suppose that λ is an eigenvalue of $\boldsymbol{A}$ and $\boldsymbol{u} \neq 0$ is a corresponding eigenfunction (in $\boldsymbol{V}_{per}(\Omega)$). Then $\boldsymbol{u}$ possesses the Fourier expansion

$$\boldsymbol{u} = \sum_{\substack{n_1, n_2, n_3 \\ \text{nonnegative integers}}} \boldsymbol{u}_{(n_1, n_2, n_3)} \tag{C.14}$$

with each $\boldsymbol{u}_{(n_1,n_2,n_3)}$ of the form (C.3), (C.4). If $\boldsymbol{u} \in \boldsymbol{V}_{per}(\Omega)$, then

$$\int_\Omega \boldsymbol{u} d\boldsymbol{x} = \boldsymbol{0} \tag{C.15}$$

and $\forall \boldsymbol{n} \neq \boldsymbol{0}$, $\boldsymbol{n} = (n_1, n_2, n_3)$, we also have

$$\int_\Omega \boldsymbol{u}_{(n_1,n_2,n_3)} d\boldsymbol{x} = \boldsymbol{0}, \quad \boldsymbol{n} \neq \boldsymbol{0}. \tag{C.16}$$

Therefore, by virtue of (C.14) and (C.15),

$$\int_{\Omega} \boldsymbol{u}_{(0,0,0)} d\boldsymbol{x} = \int_{\Omega} \boldsymbol{u} d\boldsymbol{x} - \int_{\Omega} \sum_{\boldsymbol{n} \neq \boldsymbol{0}} \boldsymbol{u}_{(n_1,n_2,n_3)} d\boldsymbol{x}, \tag{C.17}$$

so that

$$\int_{\Omega} \boldsymbol{u}_{(0,0,0)} d\boldsymbol{x} = \boldsymbol{0}. \tag{C.18}$$

Since $\boldsymbol{u}_{(0,0,0)}$ is a constant vector, it follows from (C.18) that $\boldsymbol{u}_{(0,0,0)} = \boldsymbol{0}$. Also, since we require that $\nabla \cdot \boldsymbol{u} = 0$, $\forall \boldsymbol{n} = (n_1, n_2, n_3)$, with the n_j nonnegative integers, $\nabla \cdot \boldsymbol{u}_{(n_1,n_2,n_3)} = 0$; that this last statement is true follows from the fact that

$$\begin{aligned} \nabla \cdot \boldsymbol{u} = \boldsymbol{0} &\Rightarrow \int_{\Omega} (\nabla \cdot \boldsymbol{u})^2 d\boldsymbol{x} = 0 \\ &\Rightarrow \sum_{\boldsymbol{n}=(n_1,n_2,n_3)} \int_{\Omega} (\nabla \cdot \boldsymbol{u}_{(n_1,n_2,n_3)})^2 d\boldsymbol{x} = 0 \end{aligned} \tag{C.19}$$

as well as the fact that, by virtue of (C.3) and (C.4), for $\boldsymbol{n} \neq \boldsymbol{n}'$,

$$\int_{\Omega} (\nabla \cdot \boldsymbol{n}_{(n_1,n_2,n_3)}) \cdot (\nabla \cdot \boldsymbol{u}_{(n_1',n_2',n_3')}) d\boldsymbol{x} = 0. \tag{C.20}$$

Since $\nabla \cdot \boldsymbol{u}_{(n_1,n_2,n_3)} = 0$, a.e. in Ω, we have $\boldsymbol{u}_{(n_1,n_2,n_3)} \in \boldsymbol{V}_{per}(\Omega)$, $\forall \boldsymbol{n} \neq \boldsymbol{0}$. However, $\boldsymbol{u} \neq \boldsymbol{0}$ implies that there exists $(n_1', n_2', n_3') \neq \boldsymbol{0}$ such that $\boldsymbol{u}_{(n_1,n_2,n_3)} \neq \boldsymbol{0}$. We now claim that λ must be of the form

$$\lambda = \frac{16\pi^4}{L^4}(n_1'^2 + n_2'^2 + n_3'^2).$$

To see this we note that, by virtue of the definition given above,

$$\int_{\Omega} \Delta \boldsymbol{u} \cdot \Delta \boldsymbol{v} d\boldsymbol{x} = \lambda \int_{\Omega} \boldsymbol{u} \cdot \boldsymbol{v} d\boldsymbol{x}, \quad \forall \boldsymbol{v} \in \boldsymbol{V}_{per}(\Omega);$$

so, taking $\boldsymbol{v} = \boldsymbol{u}_{(n_1',n_2',n_3')}$, we obtain

$$\int_{\Omega} \Delta \boldsymbol{u} \cdot \Delta \boldsymbol{u}_{(n_1',n_2',n_3')} d\boldsymbol{x} = \lambda \int_{\Omega} \boldsymbol{u} \cdot \boldsymbol{u}_{(n_1',n_2',n_3')} d\boldsymbol{x}. \tag{C.21}$$

Integrating the first term on the left-hand side of (C.21) by parts, and using the fact that $\boldsymbol{u}_{(n_1',n_2',n_3')}$ satisfies (C.9), we have

$$\begin{aligned}\int_\Omega \Delta\boldsymbol{u}\cdot\Delta\boldsymbol{u}_{(n'_1,n'_2,n'_3)}d\boldsymbol{x} &= \int_\Omega \boldsymbol{u}\cdot\Delta\Delta\boldsymbol{u}_{(n'_1,n'_2,n'_3)}d\boldsymbol{x}\\ &= \frac{16\pi^4}{L^4}(n_1'^2+n_2'^2+n_3'^2)\int_\Omega \boldsymbol{u}\cdot\boldsymbol{u}_{(n'_1,n'_2,n'_3)}d\boldsymbol{x};\end{aligned} \tag{C.22}$$

so, by (C.21), (C.22)

$$\left[\frac{16\pi^4}{L^4}(n_1'^2+n_2'^2+n_3'^2)-\lambda\right]\int_\Omega \boldsymbol{u}\cdot\boldsymbol{u}_{(n_1'^2,n_2'^2,n_3'^2)}d\boldsymbol{x}=0. \tag{C.23}$$

However, using (C.14) in conjunction with (C.3), (C.4), we get

$$\int_\Omega \boldsymbol{u}\cdot\boldsymbol{u}_{(n'_1,n'_2,n'_3)}d\boldsymbol{x} = \int_\Omega \left|\boldsymbol{u}_{(n'_1,n'_2,n'_3)}\right|^2 d\boldsymbol{x}\neq 0, \tag{C.24}$$

and the required result follows directly from (C.24). □

Remarks. The case $N = 2$ may be handled in a manner similar to the case $n = 3$. We consider functions of the form

$$u_i = \sum_{i=1}^{4} C_{ij} f_j(x_1,x_2), \quad i = 1,2, \tag{C.25}$$

with

$$\begin{cases} f_1(x_1,x_2) = \cos\dfrac{2\pi n_1}{L}x_1\cos\dfrac{2\pi n_2}{L}x_2,\\ f_2(x_1,x_2) = \cos\dfrac{2\pi n_1}{L}x_1\sin\dfrac{2\pi n_2}{L}x_2,\\ f_3(x_1,x_2) = \sin\dfrac{2\pi n_1}{L}x_1\cos\dfrac{2\pi n_2}{L}x_2,\\ f_4(x_1,x_2) = \sin\dfrac{2\pi n_1}{L}x_1\sin\dfrac{2\pi n_2}{L}x_2. \end{cases} \tag{C.26}$$

The result for $n = 2$, which corresponds to Lemmas C.1 and C.2 for $n = 3$, is

Lemma C.3. *For $n = 2$, the set of numbers*

$$\left\{\frac{16\pi^4}{L^4}(n_1^2+n_2^2)\neq 0 \mid n_1,n_2 \text{ nonnegative integers}\right\}$$

contains all the eigenvalues of $\boldsymbol{A}$.

Remarks. Once we have shown that for nonnegative integers n_i $(i = 1,2,3)$ the eigenvalues of $\boldsymbol{A}$ consist of the numbers

$$\lambda = \frac{16\pi^4}{L^4}(n_1^2 + n_2^2 + n_3^2) \neq 0 \quad (n = 3),$$
$$\lambda = \frac{16\pi^4}{L^4}(n_1^2 + n_2^2) \neq 0 \quad (n = 2),$$

the validity of the spectral gap condition is a consequence of standard known results on the difference of consecutive numbers which can be expressed as the sum of squares of nonnegative integers, e.g. [Ric]; in fact, as a consequence of such results it follows that, for the bipolar problem, condition (i) of Theorem 6.3 is satisfied, in dim $n = 2$, for arbitrary $\mu_1 > 0$, if N is sufficiently large, but in dim $n = 3$ only when μ_1 is sufficiently large (see, also, [Me]).

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