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# Spectral Properties of Noncommuting Operators

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# Preface

The work described in these notes has had a long gestation. It grew out of my sojourn at Macquarie University, Sydney, 1986-87 and 1989-90, during which time Alan McIntosh was applying Clifford analysis techniques to the study of singular integral operators and irregular boundary value problems. His research group provided a stimulating and convivial environment over the years. I would like to thank my collaborators in this enterprise: Jerry Johnson, Alan McIntosh, Susumu Okada, James Picton-Warlow, Werner Ricker, Frank Sommen and Bernd Straub. The work was supported by two large grants from the Australian Research Council.

Sydney, March 2004 *Brian Jefferies*

# Contents

# 1

# Introduction

The subject of these notes is the spectral theory of systems of operators. Because 'spectral theory' means different things to different workers in functional analysis, it is worthwhile to first set down how the term is used in the present context and the relationship it bears to the spectral theory of a single selfadjoint operator.

The *spectrum* $\sigma(A)$ of a single matrix $A$ is the finite set of all *eigenvalues* of $A$, that is, complex numbers $\lambda$ for which the equation $Av = \lambda v$ has a nonzero vector $v$ as a solution. In order to treat linear operators $A$ acting on some function space, it is preferable to take $\sigma(A)$ to mean the set of all $\lambda \in \mathbb{C}$ for which $\lambda I - A$ is not invertible. The most complete spectral analysis is available for selfadjoint operators $A$ acting in Hilbert space, for then the linear operator $A$ has a spectral decomposition

$$A = \int_{\sigma(A)} \lambda \, dP_A(\lambda) \tag{1.1}$$

with respect to a spectral measure $P_A$ associated with $A$. In the case that $A$ is an hermitian matrix, the integral representation (1.1) becomes a finite sum

$$A = \sum_{\lambda \in \sigma(A)} \lambda \, P_A(\{\lambda\}) \tag{1.2}$$

in which $P_A(\{\lambda\})$ is the orthogonal projection onto the eigenspace of the eigenvalue $\lambda$. The spectral theory of selfadjoint operators lies at the foundation of quantum physics.

The solution of linear operator equations, such as those that arise in quantum mechanics, often requires the formation of functions of operators. For example, in order to solve the linear equation

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} + Au(t) = 0, \quad u(0) = u_0,$$

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we need to form the exponential $e^{-tA}$, $t \geq 0$, of $A$. Because of the importance of linear evolution equations, the theory of exponentiating an operator is well-understood, but in general, the *spectral* properties of $A$ determine the types of functions $f(A)$ of $A$ that can be formed in a reasonable manner.

In the case of a selfadjoint operator $A$, we can take

$$f(A) = \int_{\sigma(A)} f(\lambda)\, dP_A(\lambda) \tag{1.3}$$

for any $P_A$-essentially bounded Borel measurable function $f : \sigma(A) \to \mathbb{C}$. The pleasant spectral properties of a selfadjoint operator $A$ are reflected in the rich class of functions $f(A)$ of $A$ that can be formed.

A basic task of quantum mechanics is to find a quantum representation $f(P, Q)$ of a classical observable $(p, q) \longmapsto f(p, q)$ on phase space. Here $P = \frac{\hbar}{i}\frac{\mathrm{d}}{\mathrm{d}x}$ is the momentum operator and $Q$ is the position operator of 'multiplication by $x$'. They satisfy the commutation relation $QP - PQ = i\hbar I$. For example, if $H(p, q) = \frac{p^2}{2m} + V(q)$ is the classical hamiltonian of the system, then $H(P, Q) = \frac{P^2}{2m} + V(Q)$ is the corresponding quantum observable, provided that the sum of the two unbounded operators is interpreted appropriately. Although it is known that the structure of classical observables is not preserved in the quantum setting for an extensive class of observables $f$, we are left with the problem of forming a function $f(P, Q)$ of a pair $(P, Q)$ of operators which do not commute with each other.

In another context, symmetric hyperbolic systems

$$\frac{\partial u}{\partial t} + \sum_{j=1}^{n} A_j \frac{\partial u}{\partial x_j} = 0 \tag{1.4}$$

of partial differential equations arise in the linearised equations of magnetohydrodynamics [15]. In the case that the matrices $A_1, \dots, A_n$ are hermitian, the fundamental solution is the matrix-valued distribution

$$\frac{1}{(2\pi)^n}\left(e^{it\sum_{j=1}^{n} A_j\xi_j}\right)\hat{}\ .$$

Here the Fourier transform $\hat{}$ is taken in the sense of distributions with respect to the variable $\xi \in \mathbb{R}^n$.

Then the fundamental solution $f \longmapsto f(A_1, \dots, A_n)$ of (1.4) at time $t = 1$ may be viewed as a mapping that forms functions $f(A_1, \dots, A_n)$ of the $n$ matrices $A_1, \dots, A_n$. The snapshot of the support of the fundamental solution at time $t = 1$ determines the propagation cone of solutions of the initial value problem for the symmetric hyperbolic system (1.4). A mapping such as $f \longmapsto f(A_1, \dots, A_n)$ will be termed a *functional calculus* in this work. Although the expression is used somewhat loosely, the idea is common to the areas in functional analysis just mentioned.

In the traditional setting of a single operator $A$, a decent functional calculus $f \longmapsto f(A)$ is a *homomorphism* of Banach algebras: $(fg)(A) = f(A)g(A)$

for two functions $f, g$ belonging to the domain of the functional calculus. In the case of a selfadjoint operator $A$, the domain of the functional calculus defined by formula (1.3) is the Banach algebra $L^\infty(P_A)$ under pointwise multiplication. For two operators $A_1, A_2$ which do not commute, there is a choice in operator ordering. For example, given the function $f(z_1, z_2) = z_1 z_2$, the operator $f(A_1, A_2)$ could be $A_1A_2$, $A_2A_1$, $\frac{1}{2}(A_1A_2 + A_2A_1)$ or some other choice of weighted operator product. Under these circumstances, the homomorphism property fails, but we still use the term 'functional calculus'.

In the noncommutative setting of spectral theory considered in the present work, there is a shift of emphasis from the algebraic formulation of the spectrum to a more analytic formulation of the 'joint spectrum' of operators $(A_1, \dots, A_n)$ as the underlying set on which the 'richest' functional calculus $f \longmapsto f(A_1, \dots, A_n)$ is defined. From this point of view, the 'joint spectrum' of matrices $(A_1, \dots, A_n)$ associated with the symmetric hyperbolic system (1.4) determines the propagation cone of the solution, so it has a natural interpretation. For bounded selfadjoint operators, the 'joint spectrum' of $(A_1, \dots, A_n)$ can be defined algebraically in terms of commutative objects $(\tilde{A}_1, \dots, \tilde{A}_n)$ associated with $(A_1, \dots, A_n)$, see Section 7.1.

The study of functions of noncommuting operators has been extensively developed by V.P. Maslov and co-workers, see [82] for a list of references. The calculus of noncommuting operators has fundamental applications to the asymptotic analysis of differential equations, quantisation and quantum groups. The emphasis in the present work is in a different direction: the properties of the support of functional calculi associated with the operators $(A_1, \dots, A_n)$ is examined and the relationship between the nature of the operators $(A_1, \dots, A_n)$ and possible functional calculi is explored. In the case of a single operator, this is the traditional domain of *spectral theory.* The support of the 'natural' functional calculus is interpreted as the joint spectrum of the operators $(A_1, \dots, A_n)$ and it is in this sense that the work is devoted to the spectral properties of systems of noncommuting operators.

Even for a single bounded selfadjoint operator $A$, there is a choice between the 'richest' functional calculus $f \longmapsto f(A)$ for $f \in L^\infty(P_A)$ and the functional calculus $f(A) = \sum_{j=0}^\infty c_j A^j$ for functions $f$ with a uniformly convergent power series expansion $\sum_{j=0}^\infty c_j z^j$ for all $z \in \mathbb{C}$ belonging to the closed unit disk $D(r)$ of radius $r = \|A\|$ centred at zero. The spectrum $\sigma(A)$ of $A$ is precisely the support of the richest functional calculus rather than the closed disk $D(r)$ – a set much larger than $\sigma(A)$.

When the operators $(A_1, \dots, A_n)$ commute with each other, a general notion of joint spectrum relies on ideas from algebraic topology [104], [111], [25]. However, for the class of operators treated in this work, such considerations are unnecessary (see [76] for a comparison of joint spectra in the commuting case) and we can deal with both the commutative and noncommutative setting simultaneously. Of course, this is at the expense of placing a restriction on the combined spectra of the operators $(A_1, \dots, A_n)$, which should be on (or, in Chapter 6, not be too far from) the real axis. Recent work [10] shows how this restriction can be lifted in the Hilbert space setting. Certain ideas from algebraic geometry do play a part in Chapter 5 in the context of computing the joint spectrum of a system of hermitian matrices.

The subject of these notes is the generalisation of the spectral theory of a single operator to the setting of a finite system of possibly noncommuting bounded (or, in Chapter 6, densely defined) operators. There are other means by which the program can be realised. The noncommuting variable approach is taken in a series of papers by J.L. Taylor [104, 105, 106, 107]. A geometric approach in von Neumann algebras is taken in [2]. One could attempt to compute the Gelfand spectrum of corresponding commuting objects, see [83], [3], [4], [6] and Section 7.1 below. A monograph surveying many results in several variable spectral theory has recently appeared [80].

Another point of view is to see to what extent the Spectral Mapping Theorem for a single operator generalises to a system of operators, especially with weak commutativity assumptions – see [74] and [36, 37] for this approach.

It should be obvious from the description above that the present mathematical work has its roots in physical applications. Indeed, the spectral theory of a single selfadjoint operator was developed by J. von Neumann [113] in order to put quantum mechanics on a firm foundation. The names of the mathematician H. Weyl and the physicist R. Feynman recur in this work. Both were motivated by problems in quantum physics.

In [115], H. Weyl proposed the functional calculus

$$\frac{1}{2\pi}\left(e^{i\xi_1 P + i\xi_2 Q}\right)^{\hat{}} : f \longmapsto f(P, Q)$$

as a quantisation procedure sending the classical observable $f$ on phase space to the quantum observable $f(P, Q)$. Although a real valued function is mapped to a selfadjoint operator, a nonnegative observable need not be mapped to a positive operator, that is, a quantum observable whose expectation values are nonnegative; from this point of view, the procedure is physically unrealistic except for a limited class of classical observables.

An operational calculus for systems of noncommuting operators was proposed by R. Feynman [28] with a view of applications to quantum electrodynamics. The idea is to attach time indices to the operators concerned, treat the resulting operator valued functions as commuting objects in functional calculations and, at the end of the day, 'disentangle' the resulting expressions by restoring time-ordering in which operators with earlier time indices than other operators act *first*. The connection with Weyl's calculus was fleshed out by E. Nelson [83].

A natural approach to forming functions $f(A)$ of a single bounded linear operator $A$ is to apply the Riesz-Dunford formula

$$f(A) = \frac{1}{2\pi i}\int_C (\zeta I - A)^{-1} f(\zeta)\, d\zeta \tag{1.5}$$

to a function $f$ holomorphic in a neighbourhood of the spectrum $\sigma(A)$ of $A$ and a suitable closed contour $C$ about $\sigma(A)$. Although this approach can be generalised to systems $A = (A_1, \dots, A_n)$ of commuting bounded linear operators and holomorphic functions defined in $\mathbb{C}^n$ by defining the joint spectrum in terms of the Koszul complex [104], [111], a different line is taken in these notes.

*Clifford analysis* also possesses an analogue of the Cauchy integral formula in one complex variable for higher dimensions. The Clifford algebra $\mathbb{C}_{(n)}$ is a complex algebra with unit $e_0$, generated by $n$ anti-commuting vectors $e_1, \dots, e_n$. A function $f(x_0, x_1, \dots, x_n)$ of $n+1$ real variables $x_0, x_1, \dots, x_n$, with values in $\mathbb{C}_{(n)}$ and satisfying $Df = 0$ for the operator

$$D = \sum_{j=0}^{n} e_j \frac{\partial}{\partial x_j}$$

is called *left monogenic*. The Cauchy integral formula takes the form

$$f(x) = \int_{\partial\Omega} G_y(x)\boldsymbol{n}(y)f(y)\,d\mu(y), \quad x \in \Omega. \tag{1.6}$$

Here $f$ is left monogenic in a neighbourhhood of $\overline{\Omega}$, where $\Omega$ is a bounded open subset of $\mathbb{R}^{n+1}$ with smooth oriented boundary $\partial\Omega$ and outward unit normal $\boldsymbol{n}(y)$ at $y \in \partial\Omega$. The surface measure of $\partial\Omega$ is denoted by $\mu$. The Cauchy kernel

$$G_y(x) = \frac{1}{\Sigma_n} \frac{\overline{y - x}}{|y - x|^{n+1}}, \quad x, y \in \mathbb{R}^{n+1}, x \neq y, \tag{1.7}$$

with $\Sigma_n = 2\pi^{\frac{n+1}{2}}/\Gamma\left(\frac{n+1}{2}\right)$ the volume of unit $n$-sphere in $\mathbb{R}^{n+1}$, is the analogue of the normalised Cauchy kernel $\frac{1}{2\pi}(\zeta - z)^{-1}$ in complex analysis. The theme of the present notes is to form functions $f(A_1, \dots, A_n)$ of $n$ operators $A_1, \dots, A_n$ via the formula

$$f(A_1, \dots, A_n) = \int_{\partial\Omega} G_y(A_1, \dots, A_n)\boldsymbol{n}(y)f(y)\,d\mu(y), \tag{1.8}$$

which arises by analogy with the Riesz-Dunford formula (1.5). The principal difficulty is making sense of the function $x \longmapsto G_x(A_1, \dots, A_n)$ and determining its singularities, the collection of which may be viewed as the *joint spectrum* of the system $(A_1, \dots, A_n)$ of operators. Along the way to realising this idea, we shall make contact with the Weyl functional calculus for $n$ operators, Feynman's operational calculus and the fundamental solution of the symmetric hyperbolic system (1.4).

It may seem somewhat surprising that Clifford analysis should be a tool in the analysis of the spectral theory of systems of operators. These notes grew out of a desire to bring together the seemingly disparate streams of thought I have been exposed to over the years by my friends and colleagues. On the one hand, A. McIntosh has been an enthusiastic proponent of Clifford techniques in harmonic analysis and the solution of irregular boundary value problems in partial differential equations [73]. A connection with Weyl's calculus appears in joint work with A. Pryde [75], [87], [88], [89]. On the other hand, the joint work of G.W. Johnson and M. Lapidus [59], [60], [61] and G.W. Johnson with myself [48], [49], [50], [51] shows the connection of Feynman's operational calculus with the monogenic functional calculus for systems of operators described in these notes.

Feynman viewed his operational calculus as a procedure to invoke when the *Feynman integral*, as such, cannot be applied. Indeed, there is an allusion to Clifford analysis techniques in [28, Appendix B, p. 126]: *The Pauli matrices (times i) are the basis for the algebra of quaternions so that the solution of such problems* [concerning functional calculi] *might open up the possibility of a true infinitesimal calculus of quantities in the field of hypercomplex numbers.* In the point of view set out here, for the Pauli matrices $\sigma_1,\ \sigma_2,\ \sigma_3$, the key property needed for the construction of a joint functional calculus by the method of these notes is that they are *selfadjoint*, so that $\xi_1\sigma_1 + \xi_2\sigma_2 + \xi_3\sigma_3$ has real spectrum for all $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$. Even if the $n$ operators $A_1, \dots, A_n$ do not have real spectra, it is enough to require that the spectrum of the operator $\sum_{j=1}^n \xi_j A_j$ is contained in a fixed sector in $\mathbb{C}$ for all $\xi \in \mathbb{R}^n$ in order that the functional calculus described here should exist.

It is by utilising the underlying real-variable characteristics of Clifford analysis of monogenic functions defined in $\mathbb{R}^{n+1}$ and the spectral properties of the operators $A_1, \dots, A_n$ that we can bypass homological considerations of [104], [111], [25], leading to a rather straightforward approach to forming functions of systems of noncommuting operators. Even in this restricted setting, there is considerable scope for investigating the properties of joint functional calculi and their relationship with quantisation procedures and the geometric analysis of the support of solutions of the hyperbolic system (1.4) of partial differential equations.

A more detailed description of the contents of the present notes and the connection with the work of these authors follows.

The background to Weyl's functional calculus is given in Section 1 of Chapter 2. A unitary representation of the Heisenberg group is used to form functions $\sigma(\boldsymbol{D}, \boldsymbol{X})$ of position $\boldsymbol{X}$ and momentum operators $\boldsymbol{D}$ in quantum mechanics on $\mathbb{R}^n$. The same idea works for a system $\boldsymbol{A} = (A_1, \dots, A_n)$ of $n$ bounded linear operators on a Banach space provided that the right exponential growth estimates (2.2) are satisfied and this is described carefully in Section 2 of Chapter 2, from work of E. Nelson [83], M. Taylor [108], R.F.V. Anderson [7], [8] and A. Pryde [88]. The *joint spectrum* $\gamma(\boldsymbol{A})$ of $\boldsymbol{A}$ is simply the support of the Weyl functional calculus $\mathcal{W}_{\boldsymbol{A}}$ – an operator valued distribution with compact support.

For $n = 1$, a single bounded linear operator $A$ satisfies the exponential growth estimate (2.2) precisely when it is a *generalised scalar operator* with real spectrum [23]. Such operators may be viewed as generalisations of self-adjoint operators for which spectral measures are replaced by spectral distributions.

Chapter 3 sets down the background in Clifford analysis, such as the Cauchy integral formula (1.6), needed to construct a functional calculus for operators. Most of the material here is from the monograph [19]. Other important formulae include the monogenic representation of distributions (Theorem 3.3) and the plane wave decomposition of the Cauchy kernel (Proposition 3.4). Proposition 3.6 gives an approximation result for real analytic functions with a proof due to F. Sommen.

A natural way to construct the Cauchy kernel for an $n$-tuple $\boldsymbol{A} = (A_1, \ldots, A_n)$ of mutually commuting operators with real spectra is to adapt formula (1.7) in the time-honoured way by replacing the vector $x \in \mathbb{R}^n$ by the $n$-tuple $\boldsymbol{A}$ and writing

$$G_y(\boldsymbol{A}) = \frac{1}{\Sigma_n}\left(\overline{y} + \sum_{j=1}^{n} A_j e_j\right)\left(y_0^2 I + \sum_{j=1}^{n}(y_j I - A_j)^2\right)^{-(n+1)/2}, \tag{1.9}$$

for all $y = (y_0, y_1, \ldots, y_n) \in \mathbb{R}^{n+1}$ with $y_0 \neq 0$. Then the Cauchy kernel $y \longmapsto G_y(\boldsymbol{A})$ will have singularities on the set

$$\gamma(\boldsymbol{A}) = \left\{(0, y_1, \ldots, y_n) \in \mathbb{R}^{n+1} : 0 \in \sigma\left(\sum_{j=1}^{n}(y_j I - A_j)^2\right)\right\}. \tag{1.10}$$

This is the basic idea of the paper [75] of A. McIntosh and A. Pryde. If $n$ is odd, then formula (1.9) is readily interpreted and a functional calculus may be constructed via the Riesz-Dunford formula (1.8). If $n$ is even, it is not clear how the fractional power should be interpreted.

Chapter 4 examines this problem from two viewpoints. If $\boldsymbol{A}$ satisfies the exponential growth estimates (2.2), then $G_y(\boldsymbol{A})$ may be defined as $\mathcal{W}_{\boldsymbol{A}}(G_y)$ for all $y \in \mathbb{R}^{n+1}$ outside the support $\gamma(\boldsymbol{A})$ of the Weyl functional calculus $\mathcal{W}_{\boldsymbol{A}}$. The observation that the operator valued distribution $\mathcal{W}_{\boldsymbol{A}}$ may be passed from outside the Clifford version of the Cauchy integral formula (1.6) into the integrand verifies the Riesz-Dunford formula (1.8). It is proved in Theorem 4.8 that $\gamma(\boldsymbol{A})$ is exactly the set of singularities of the Cauchy kernel $y \longmapsto G_y(\boldsymbol{A})$. Section 4.1 is based on [53]. Unlike formula (1.9), it is not necessary to assume that $\boldsymbol{A}$ consists of commuting operators.

On the other hand, the original motivation for the study of the representation (1.8) was to treat the (commuting) unbounded operators of differentiation on a Lipschitz surface – a system of operators that does not satisfy the exponential estimates (2.2). Soon after the work [75], A. McIntosh realised that the plane wave decomposition of the Cauchy kernel [103] could be used profitably in the present context. Sections 4.2 and 4.3 are based on joint work [54] of the author with A. McIntosh and J. Picton-Warlow and represent the

Cauchy kernel $y \longmapsto G_y(\boldsymbol{A})$ in terms of the plane wave formula. Rather than the the exponential estimates (2.2), what is essential here is the condition (4.10) that real linear combinations of $A_1, \dots, A_n$ should have real spectra. It is not necessary to assume that the bounded linear operators $A_1, \dots, A_n$ commute with each other. Now the joint spectrum $\gamma(\boldsymbol{A})$ is *defined* to be the set of singularities of the Cauchy kernel $y \longmapsto G_y(\boldsymbol{A})$.

A basic property of the notion of a 'spectrum' of an operator or system of operators is that disjoint components should be associated with projections onto subspaces left invariant by the system. That the joint spectrum $\gamma(\boldsymbol{A})$ enjoys this property is proved in Section 4.4 by appealing to formula (1.8). The result is actually a consequence of a general version of the noncommutative Shilov idempotent theorem [4, Theorem 4.1] proved by E. Albrecht, but the Clifford analysis techniques used in Section 4.4 are natural in the present context.

Chapter 5 exploits the complementary viewpoints of the joint spectrum $\gamma(\boldsymbol{A})$ for a system $\boldsymbol{A} = (A_1, \dots, A_n)$ of *matrices* as the set of singularities of the Cauchy kernel $G_{(\cdot)}(\boldsymbol{A})$ and as the support of the Weyl functional calculus $\mathcal{W}_{\boldsymbol{A}}$. For matrices, the spectral reality condition (4.10) is equivalent to the exponential growth estimates (2.2) necessary for the existence of the Weyl functional calculus $\mathcal{W}_{\boldsymbol{A}}$. This is proved in Section 5.2 following [44] although, in another language, the result is known from the techniques of partial differential equations, see [58, p. 153]. An explicit formula for $\mathcal{W}_{\boldsymbol{A}}$ due to E. Nelson [83, Theorem 9] is proved in Section 5.1 for the case that $A_1, \dots, A_n$ are hermitian $N \times N$ matrices. The proof is based on [42].

The 'numerical range' of the system $\boldsymbol{A}$ enters into Nelson's formula. Let $S(\mathbb{C}^N) = \{u \in \mathbb{C}^N : |u| = 1\}$ be the unit sphere in $\mathbb{C}^N$. The numerical range map $W_{\boldsymbol{A}} : S(\mathbb{C}^N) \to \mathbb{R}^n$ is defined by

$$W_{\boldsymbol{A}} : u \longmapsto (\langle A_1 u, u \rangle, \dots, \langle A_n u, u \rangle), \quad u \in S(\mathbb{C}^N),$$

with $\langle \cdot, \cdot \rangle$ representing the inner product of $\mathbb{C}^N$. The range of $W_{\boldsymbol{A}}$ is the 'generalised numerical range' of the system $\boldsymbol{A}$. For the case $n = 2$, the range of the map $W_{\boldsymbol{A}}$ is just the usual numerical range of the $(N \times N)$ matrix $A_1 + iA_2$. Differential properties of the numerical range map and their relationship to spectral properties of the matrix $A_1 + iA_2$ are studied in [38] and [63]. The matrix valued distribution $\mathcal{W}_{\boldsymbol{A}}$ is written out in Theorem 5.1 as a matrix valued differential operator acting on the image $\mu_{\boldsymbol{A}} = \nu \circ W_{\boldsymbol{A}}^{-1}$ of the uniform probability measure $\nu$ on $S(\mathbb{C}^N)$ by the numerical range map $W_{\boldsymbol{A}}$. An alternative representation of the Weyl calculus $\mathcal{W}_{\boldsymbol{A}}$ is based on formulae of Herglotz-Petrovsky-Leray [11] for the fundamental solution of the symmetric hyperbolic system (1.4), but the image measure $\mu_{\boldsymbol{A}}$ is not a feature of this representation.

An explicit calculation of the joint spectrum $\gamma(\boldsymbol{A})$ of a pair $\boldsymbol{A} = (A_1, A_2)$ of hermitian matrices is made in Section 5.3, following the approach of [56]. If the matrices $A_1$ and $A_2$ commute with each other, then $\gamma(\boldsymbol{A})$ can be identified with the finite set of eigenvalues of the normal matrix $A_1 + iA_2$, otherwise $\gamma(\boldsymbol{A})$

exhibits complicated geometric structure. In general, the numerical range of the matrix $A_1 + iA_2$ is the convex hull of certain algebraic plane curves, also associated with singularities of the numerical range map [63]. Then the joint spectrum $\gamma(\boldsymbol{A})$ is exactly the numerical range of the matrix $A_1+iA_2$, possibly omitting some regions bounded by the algebraic plane curves. These omitted regions are called *lacunas*. A brief summary of ideas related to numerical range and algebraic curves is given in Subsection 5.3.1.

Theorem 5.24 gives a geometric characterisation of the joint spectrum $\gamma(\boldsymbol{A})$ and so, the support of the matrix valued distribution $\mathcal{W}_{\boldsymbol{A}}$, for a pair $\boldsymbol{A}$ of hermitian matrices. If the hermitian matrices $A_1, A_2$ commute, then as mentioned above, the joint spectrum $\gamma(\boldsymbol{A})$ is a finite set, otherwise it has nonempty interior.

The proof is achieved by obtaining explicit formulae for the Cauchy kernel $G_y(\boldsymbol{A})$ from its plane wave decomposition by the method of residues. The symmetric hyperbolic system (1.4) with $n = 2$ is of importance in magnetohydrodynamics. A different plane wave decomposition for the fundamental solution of (1.4) is used in [15] and [16] to obtain essentially the same result. With a suitable amount of mathematical translation one direction of Theorem 5.24 can also be deduced from the results of [11] and [12], but the hermitian character of our system does not figure in these works. The proof given here demonstrates that the plane wave decomposition for the Cauchy kernel $G_y(\boldsymbol{A})$ can be a useful tool to determine the support of the Weyl calculus $\mathcal{W}_{\boldsymbol{A}}$ and it is the natural definition of $G_y(\boldsymbol{A})$ in case the exponential growth estimates (2.2) fail.

An explicit calculation of $\gamma(\boldsymbol{A})$ for a pair $\boldsymbol{A}$ of simultaneously triangularisable $(N \times N)$ matrices with real eigenvalues is made in Section 5.4. If $\boldsymbol{D}$ is the pair of diagonal matrices obtained from the ordered eigenvalues of $\boldsymbol{A}$, then Nelson's formula applies to the image measure $\mu_{\boldsymbol{D}}$ which, generically for $N \geq 3$, has a continuous, piecewise polynomial density with respect to two-dimensional Lebesgue measure. The distribution $\mathcal{W}_{\boldsymbol{A}}$ is zero on the polynomial parts of $\mu_{\boldsymbol{D}}$ but may have support on the bounding segments. Unlike the case for hermitian matrices, the joint spectrum $\gamma(\boldsymbol{A})$ may have empty interior without being a finite set.

The situation for an $n$-tuple $\boldsymbol{A} = (A_1, \dots, A_n)$ of matrices satisfying the spectral reality condition (4.10) is briefly considered in Section 5.5 for even integers $n$, following the ideas of Atiyah, Bott and Gårding [11]. The plane wave decomposition of the Cauchy kernel $G_y(\boldsymbol{A})$ yields a representation of the real analytic parts of the distribution $\mathcal{W}_{\boldsymbol{A}}$ away from the wave front surface $W(\boldsymbol{A})$ of $\boldsymbol{A}$ calculated from the polynomial

$$P^{\boldsymbol{A}} : \xi \longmapsto \det(\xi_0 I + \xi_1 A_1 + \cdots + \xi_n A_n), \quad \xi \in \mathbb{R}^{n+1}.$$

The representation of $\mathcal{W}_{\boldsymbol{A}}$ involves rational integrals over certain (Petrovsky) cycles in the $(n-1)$-dimensional complex projective space $\mathbb{CP}^{n-1}$. The elementary argument of Section 5.3 in the case $n = 2$ can now be interpreted in terms of the vanishing of Petrovsky cycles in $\mathbb{CP}$.

Chapter 6 is based on [45],[46],[47] and returns to the original motivation of A. McIntosh for applying the plane wave decomposition of the Cauchy kernel to the definition of $G_y(\boldsymbol{A})$. Up until this point, it has been assumed that $\boldsymbol{A}$ consists of bounded linear operators acting on a Banach space and satisfying either the exponential growth estimates (2.2) or the weaker spectral reality condition (4.10) (which, by virtue of Theorem 5.10, is equivalent to (2.2) for systems of *matrices*). The plane wave formula for $G_y(\boldsymbol{A})$ still works if $\boldsymbol{A}$ consists of unbounded operators whose spectra are contained in a fixed sector in the complex plane and satisfy uniform resolvent estimates outside the sector. The commuting system of directional derivatives on a Lipschitz surface form such an example not satisfying (2.2). Then functions $f(\boldsymbol{A})$ of the system $\boldsymbol{A}$ of unbounded operators can be formed by formula (1.8) for left monogenic functions $f$ with suitable decay at zero and infinity in a sector in $\mathbb{R}^{n+1}$, even if elements of $\boldsymbol{A}$ do not commute with each other.

Subject to the spectral reality condition (4.10), functions $f(\boldsymbol{A})$ of the system $\boldsymbol{A}$ were formed for real analytic functions $f$ defined in a neighbourhood of $\gamma(\boldsymbol{A}) \subset \mathbb{R}^n$ simply by extending $f$ monogenically from an open subset of $\mathbb{R}^n$ into an open subset of $\mathbb{R}^{n+1}$. Once $\gamma(\boldsymbol{A})$ is a closed unbounded set contained in a sector in $\mathbb{R}^{n+1}$, it is not obvious which real analytic functions will actually extend monogenically off $\mathbb{R}^n$ into an open subset of $\mathbb{R}^{n+1}$ containing the joint spectrum $\gamma(\boldsymbol{A})$.

Viewing $\zeta \in \mathbb{C}^n$ as a commuting $n$-tuple of multiplication operators in the Clifford algebra $\mathbb{C}_{(n)}$, it is shown in Section 6.3 that associated with any uniformly bounded left monogenic function $f$ defined in a sector in $\mathbb{R}^{n+1}$, there is a uniformly bounded $\mathbb{C}_{(n)}$-valued holomorphic function $\zeta \longmapsto \tilde{f}(\zeta)$ defined on related sectors in $\mathbb{C}^n$. The association is by analytic continuation of the restriction of $f$ to $\mathbb{R}^n$, onto sectors in $\mathbb{C}^n$, so the mapping $f \longmapsto \tilde{f}$ is surely one-to-one. It is not so obvious that every bounded holomorphic function $b$ is obtained in this way.

In Section 6.4, it is shown that $b = \tilde{f}$ for the left monogenic function $f$ defined by the formula

$$f = \int_0^\infty b._\ell \Phi_t \, \frac{dt}{t}.$$

For each $t > 0$, the function $\Phi_t$ has decay at zero and infinity and $b._\ell\Phi_t$ is the left monogenic extension of the product function $b.\Phi_t$ from $\mathbb{R}^n$ to a sector in $\mathbb{R}^{n+1}$. The proof uses Fourier analysis and the methods are restricted to regions consisting of sectors in $\mathbb{R}^{n+1}$ and $\mathbb{C}^n$. An approach using Clifford wavelets [78] may prove useful for regions with more complicated geometry.

Once the commuting $n$-tuple $\boldsymbol{A} = (A_1, \ldots, A_n)$ satisfies 'square function estimates' in a Hilbert space, we show in Section 6.5 how the Cauchy integral formula (1.8) is used to form operators $b(A_1, \ldots, A_n)$ when $b$ is a uniformly bounded holomorphic function defined on a suitable sector in $\mathbb{C}^n$. The treatment applies to differentiation operators on a Lipschitz surface and leads to a proof of the boundedness of the Cauchy integral operator on a Lipschitz surface. The details of the proof are only sketched here, see for example [72],

[73], but the example illustrates that the Clifford analysis techniques outlined in these notes have diverse applications to operator theory involving several real variables. A Clifford wavelet approach to the boundedness of the Cauchy integral operator on a Lipschitz surface appears in [78].

Chapter 7 returns to ideas related to the Weyl functional calculus considered in Chapter 2. Following work of E. Nelson [83] and E. Albrecht [4], Section 7.1 starts by examining to what extent the joint spectrum $\gamma(\boldsymbol{A})$ of an $n$-tuple of bounded selfadjoint operators can be considered as the Gelfand spectrum of a certain commutative Banach algebra of 'operants'.

It is possible to index whole families of functional calculi for systems $\boldsymbol{A} = (A_1, \dots, A_n)$ of operators by probability measures $\boldsymbol{\mu} = (\mu_1, \dots, \mu_n)$ that keep track of operator ordering. The approach grew out of heuristic ideas advanced by R. Feynman [28] from his consideration of evolving quantum systems, and, in particular, from his work on quantum electrodynamics [27]. He regarded his operational calculus as a kind of generalised path integral (more detail regarding Feynman's heuristic ideas can be found in [48] and in Chapter 14 of [61]). The basic ideas are as follows: time indices are attached to keep track of the order of operations in products. Operators with smaller time indices are to act before operators with larger time indices no matter how they are placed on the page. With time indices attached, functions of the operators are formed just as if the operators were commuting. Finally, the operator expressions must be restored to their natural order or 'disentangled'.

Following joint work of the author with G.W. Johnson [50],[51], Feynman's ideas are implemented in Section 7.2 by assigning a measure on the set of time indices to each of the operators. These measures determine a particular operational calculus and the operational calculus may well change when the set of measures changes. In the case that the measures are all equal, we obtain the equally weighted functional calculus, which, in the case of selfadjoint operators acting on Hilbert space, coincides with the Weyl functional calculus examined in Chapter 2.

Once the exponential growth estimate (2.3) is replaced by the analogous 'disentangled' exponential growth estimate (7.13), many of the arguments concerning the Weyl calculus hold for Feynman's $\boldsymbol{\mu}$-operational calculus $\mathcal{F}_{\boldsymbol{\mu},\boldsymbol{A}}$: the $\boldsymbol{\mu}$-Cauchy kernel $G_{\boldsymbol{\mu}}(y, \boldsymbol{A}) := \mathcal{F}_{\boldsymbol{\mu},\boldsymbol{A}}(G_{\boldsymbol{\mu}}(y, \cdot))$ is defined and its set $\gamma_{\boldsymbol{\mu}}(\boldsymbol{A})$ of singularities – the *$\boldsymbol{\mu}$-joint spectrum of $\boldsymbol{A}$* – coincides with the support of the operator valued distribution $\mathcal{F}_{\boldsymbol{\mu},\boldsymbol{A}}$ (Theorem 7.10). The operator $f_{\boldsymbol{\mu}}(\boldsymbol{A}) := \mathcal{F}_{\boldsymbol{\mu},\boldsymbol{A}}(f)$ has a representation via the Riesz-Dunford integral formula (Proposition 7.9).

In a similar fashion, the spectral reality condition (4.10) is replaced by a condition on the analytic continuation of 'disentangled' resolvents (Definition 7.13), automatically satisfied if, for example, $\boldsymbol{A}$ consists of bounded selfadjoint operators and $\boldsymbol{\mu}$ is any $n$-tuple of continuous Borel probability measures on $[0, 1]$. At this stage, the rest of the procedure for constructing the Cauchy kernel $G_{\boldsymbol{\mu}}(y, \boldsymbol{A})$ from the plane wave decomposition of $G(y, x)$ and a $\boldsymbol{\mu}$-functional calculus for $\boldsymbol{A}$ is familiar and the notes end here.

# 2

# Weyl Calculus

In the case that a system $\boldsymbol{A} = (A_1, \dots, A_n)$ of $n$ bounded linear operators satisfies growth estimates for exponentials of the operators, functions $f(\boldsymbol{A})$ of $\boldsymbol{A}$ can be formed by a type of Fourier inversion. The mapping $f \longmapsto f(\boldsymbol{A})$ is called the *Weyl functional calculus.* The basic idea and properties of the Weyl calculus are outlined in this chapter before considering systems $\boldsymbol{A}$ for which the growth estimates for exponentials may fail.

## 2.1 Background

In this section, the original motivation for the introduction of Weyl's functional calculus is described.

In Hamiltonian mechanics over phase space $\mathbb{R}^{2n}$, states are represented by elements $(p, q)$ of $\mathbb{R}^{2n}$ with $p = (p_1, \dots, p_n)$ the momentum vector and $q = (q_1, \dots, q_n)$ the position vector. For a system of $k$ interacting particles in three dimensional space, we would take $n = 3k$. Observables are represented by functions $f : \mathbb{R}^{2n} \to \mathbb{R}$, so that $f(p, q)$ is the result of the observation of the state $(p, q)$. One distinguished observable is the Hamiltonian function $H : \mathbb{R}^{2n} \to \mathbb{R}$ by which the equations of motion

$$\dot{p} = -\frac{\partial H}{\partial q}, \quad \dot{q} = \frac{\partial H}{\partial p}$$

are represented. A more general discussion would involve symplectic manifolds of dimension $2n$, but for the present purpose it is enough to consider $\mathbb{R}^{2n}$.

The *Poisson bracket* $\{f, g\}$ of two smooth observables $f : \mathbb{R}^{2n} \to \mathbb{R}$ and $g : \mathbb{R}^{2n} \to \mathbb{R}$ is the new observable defined by

$$\{f, g\} = \sum \left( \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q_j} - \frac{\partial f}{\partial q_j} \frac{\partial g}{\partial p_j} \right).$$

If the coordinate functions are denoted by $\boldsymbol{p}_j$, $\boldsymbol{q}_k$ for $j, k = 1, \dots, n$, then their Poisson brackets are given by

B. Jefferies: LNM 1843, pp. 13–25, 2004.

$$\{\boldsymbol{p}_j, \boldsymbol{p}_k\} = \{\boldsymbol{q}_j, \boldsymbol{q}_k\} = 0, \qquad \{\boldsymbol{p}_j, \boldsymbol{q}_k\} = \delta_{jk}.$$

By comparison, in quantum mechanics over $\mathbb{R}^n$, the position operators $Q_j = X_j$ of multiplication by $\boldsymbol{q}_j$ correspond to the classical coordinate observables $\boldsymbol{q}_j$ and the momentum operators $P_k$ corresponding to the coordinate observables $\boldsymbol{p}_k$ are given by $P_k = \hbar D_k = \frac{\hbar}{i}\frac{\partial}{\partial x_k}$. The canonical commutation relations

$$[P_j, P_k] = [Q_j, Q_k] = 0, \ [P_j, Q_k] = \frac{\hbar}{i}\delta_{jk} I$$

hold for the commutator $[A, B] = AB - BA$.

In both classical and quantum mechanics, the position, momentum and constant observables span the Heisenberg Lie algebra $\boldsymbol{h}_n$ over $\mathbb{R}^{2n+1}$. The Heisenberg group $\boldsymbol{H}_n$ corresponding to the Lie algebra $\boldsymbol{h}_n$ is given on $\mathbb{R}^{2n+1}$ by the group law

$$(p, q, t)(p', q', t') = \left(p + p', q + q', t + t' + \frac{1}{2}(pq' - qp')\right).$$

Here we write $\xi a$ for the dot product $\sum \xi_j a_j$ of $\xi \in \mathbb{C}^k$ with the $k$-tuple $a = (a_1, \ldots, a_k)$ of numbers or operators. Set $\boldsymbol{D} = (D_1, \ldots, D_n)$ and $\boldsymbol{X} = (X_1, \ldots, X_n)$.

The map $\rho$ from $\boldsymbol{H}_n$ to the group of unitary operators on $L^2(\mathbb{R}^n)$ formally defined by $\rho(p, q, t) = e^{i(p\boldsymbol{D} + q\boldsymbol{X} + tI)}$ is a unitary representation of the Heisenberg group $\boldsymbol{H}_n$. The operator $e^{i(p\boldsymbol{D} + q\boldsymbol{X} + tI)}$ maps $f \in L^2(\mathbb{R}^n)$ to the function $x \longmapsto e^{it} e^{ipq/2} e^{iqx} f(x + p)$.

If $\hat{f}(\xi) = \int_{\mathbb{R}^{2n}} e^{-ix\xi} f(x)\, dx$ denotes the Fourier transform of a function $f \in L^1(\mathbb{R}^{2n})$, the Fourier inversion formula

$$f(x) = (2\pi)^{-2n} \int_{\mathbb{R}^{2n}} e^{ix\xi} \hat{f}(\xi)\, d\xi$$

retrieves $f$ from its Fourier transform $\hat{f}$ in the case that $\hat{f}$ is also integrable.

Now suppose that $\sigma : \mathbb{R}^{2n} \to \mathbb{C}$ is a function whose Fourier transform $\hat{\sigma}$ belongs to $L^1(\mathbb{R}^{2n})$. Then the bounded linear operator $\sigma(\boldsymbol{D}, \boldsymbol{X})$ is defined by

$$(2\pi)^{-2n} \int_{\mathbb{R}^{2n}} \rho(p, q, 0) \hat{\sigma}(p, q)\, dpdq = (2\pi)^{-2n} \int_{\mathbb{R}^{2n}} e^{i(p\boldsymbol{D} + q\boldsymbol{X})} \hat{\sigma}(p, q)\, dpdq.$$

The *Weyl functional calculus* $\sigma \longmapsto \sigma(\boldsymbol{D}, \boldsymbol{X})$ was proposed by H. Weyl [115, Section IV.14] as a means of associating a quantum observable $\sigma(\boldsymbol{D}, \boldsymbol{X})$ with a classical observable $\sigma$. Weyl's ideas were later developed by H.J. Groenewold [34], J.E. Moyal [79] and J.C.T. Pool [86].

The mapping $\sigma \longmapsto \sigma(\boldsymbol{D}, \boldsymbol{X})$ extends uniquely to a bijection from the Schwartz space $\mathcal{S}'(\mathbb{R}^{2n})$ of tempered distributions to the space of continuous linear maps from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$. Moreover, the application $\sigma \longmapsto \sigma(\boldsymbol{D}, \boldsymbol{X})$ defines a unitary map from $L^2(\mathbb{R}^{2n})$ onto the space of Hilbert-Schmidt operators on $L^2(\mathbb{R}^n)$ and from $L^1(\mathbb{R}^{2n})$ into the space of compact operators on

$L^2(\mathbb{R}^n)$. For $a, b \in \mathbb{C}^n$, the function $\sigma(\xi, x) = (a\xi + bx)^k$ is mapped by the Weyl calculus to the operator $\sigma(\boldsymbol{D}, \boldsymbol{X}) = (a\boldsymbol{D} + b\boldsymbol{X})^k$. In the expression $\sigma(\boldsymbol{D}, \boldsymbol{X})$, the monomial terms in any polynomial $\sigma(\xi, x)$ are replaced by symmetric operator products. *Harmonic analysis in phase space* is a succinct description of this circle of ideas exposed in [29].

On the negative side, the Poisson bracket is mapped by the Weyl calculus to a constant times the commutator only for polynomials $\sigma(\xi, x)$ of degree less than or equal to two. Results of H.J. Groenewold and L. van Hove [29, pp 197–199] show that a quantisation over a space of observables defined on phase space $\mathbb{R}^{2n}$ and reasonably larger than the Heisenberg algebra $\boldsymbol{h}_n$ is not possible. A general discussion of obstructions to quantisation may be found in [33]. Although $\sigma(\boldsymbol{D}, \boldsymbol{X})$ is a selfadjoint operator for reasonable real-valued symbols $\sigma$, it may happen that $\sigma(\boldsymbol{D}, \boldsymbol{X})$ is not a positive operator even if $\sigma$ is a positive function.

In the theory of pseudodifferential operators initiated by J.J. Kohn and L. Nirenberg [70], one associates the symbol $\sigma$ with the operator $\sigma(\boldsymbol{D}, \boldsymbol{X})_{KN}$ given by

$$(2\pi)^{-2n} \int_{\mathbb{R}^{2n}} e^{iq\boldsymbol{X}} e^{ip\boldsymbol{D}} \hat{\sigma}(p, q)\, dpdq,$$

so that if $\sigma$ is a polynomial, differentiation always act *first*. For singular integral operators, the product of symbols corresponds to the composition operators modulo regular integral operators. The symbolic calculus for pseudodifferential operators is studied in [110], [109], [41]. The Weyl calculus has been developed as a theory of pseudodifferential operators by L. Hörmander [40], [41].

In [28], R. Feynman developed another connection of the Weyl calculus with quantum physics, although a considerable amount of mathematical interpretation of Feynman's arguments is required. Rather than the operators $\boldsymbol{X}$ and $\boldsymbol{D}$ that rise in quantum mechanics, Feynman considers an operator calculus for general systems of operators in [28], where the idea is to attach time indices to keep track of the order of operations in products. Operators with smaller time indices are to act before operators with larger time indices. With time indices attached, functions of the operators are formed just as if the operators were commuting. Finally, the operator expressions must be restored to their natural order or 'disentangled'. The final step is often difficult; it consists roughly of manipulating the operator expressions until their order on the page is consistent with the time ordering.

Feynman's ideas were developed in a mathematical setting for bounded selfadjoint operators in [83] and further developed in [108],[7],[8],[9]. The disentangling process considered in [83] results in equally weighted operator products and this gives rise to the Weyl functional calculus considered in more detail in the following section. Other possible weightings for choices of operator products are considered in Chapter 7.

## 2.2 Operators of Paley-Wiener Type $s$

Let $A_1, \dots, A_n$ be bounded selfadjoint operators acting on a Hilbert space $H$. The *Weyl functional calculus* is a means of forming functions $f(A_1, \dots, A_n)$ of the $n$-tuple $\boldsymbol{A} = (A_1, \dots, A_n)$ of operators. The operators $A_1, \dots, A_n$ do not necessarily commute with each other, so there is no fundamentally unique way of forming such functions. However, the Weyl functional calculus is defined in such a way that if, say, $f(x_1, \dots, x_n) = x_1 x_2$, then $f(A_1, \dots, A_n) = \frac{1}{2}(A_1A_2 + A_2A_1)$. More generally, if $f$ is a polynomial, then all possible choices of operator orderings are equally weighted.

In order to form operators $f(A_1, \dots, A_n)$ for functions $f$ more general than polynomials, we adopt the procedure of H. Weyl [115, Section IV.14] described above, but replacing the pair $(\boldsymbol{D}, \boldsymbol{X})$ by the $n$-tuple $\boldsymbol{A}$.

For every $\xi \in \mathbb{R}^n$, $\langle \boldsymbol{A}, \xi \rangle = \langle \xi, \boldsymbol{A} \rangle$ denotes the selfadjoint operator $\sum_{j=1}^n A_j \xi_j$. The operator $e^{i\langle \boldsymbol{A}, \xi \rangle}$ is therefore unitary for each $\xi \in \mathbb{R}^n$. The Fourier transform $\hat{f}$ of a function $f$ integrable over $\mathbb{R}^n$ is defined by $\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\langle x, \xi \rangle} f(x)\, dx$ for all $\xi \in \mathbb{R}^n$. The integral

$$f(\boldsymbol{A}) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle \boldsymbol{A}, \xi \rangle} \hat{f}(\xi)\, d\xi \tag{2.1}$$

is an operator valued Bochner integral for each function $f$ belonging to the space $\mathcal{S}(\mathbb{R}^n)$ of rapidly decreasing functions on $\mathbb{R}^n$. Then the mapping $f \longmapsto f(\boldsymbol{A})$, for all $f \in \mathcal{S}(\mathbb{R}^n)$ is the Weyl functional calculus for the $n$-tuple $\boldsymbol{A}$ of selfadjoint operators.

There exists a unique operator valued distribution $\mathcal{W}_{\boldsymbol{A}} : f \longmapsto f(\boldsymbol{A})$, $f \in C^\infty(\mathbb{R}^n)$ defined over the test function space $C^\infty(\mathbb{R}^n)$ of all infinitely differentiable functions, such that the restriction of $\mathcal{W}_{\boldsymbol{A}}$ to $\mathcal{S}(\mathbb{R}^n)$ is the Weyl calculus for $\boldsymbol{A}$. The support of this distribution is contained in the closed unit ball in $\mathbb{R}^n$ centred at zero and with radius $\left(\sum_{j=1}^n \|A_j\|^2\right)^{1/2}$.

The Weyl calculus for bounded selfadjoint operators is considered in [108], [83]. The key idea of the argument is the application of the Paley-Wiener theorem to growth estimates for exponentials of the operators. In Chapter 4, a functional calculus for systems of operators for which these growth estimates fail is developed. In the remainder of this chapter, the construction of the Weyl calculus is more fully described.

### The Paley-Wiener Theorem

We begin by reviewing some facts about operator valued tempered distributions. Let $\mathcal{S}(\mathbb{R}^n)$ be the space of rapidly decreasing functions on $\mathbb{R}^n$, that is, functions $f : \mathbb{R}^n \to \mathbb{C}$ for which

$$p_{j,k}(f) := \sup\left\{(1+|x|)^j |\partial^\alpha f(x)| : x \in \mathbb{R}^n, |\alpha| \le k\right\}$$

is finite for all nonnegative integers $j, k$. For a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ of nonnegative integers, we use the notation $\partial^\alpha$ to denote the partial differentiation operator $\partial^{|\alpha|}/\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$ with $|\alpha| = \alpha_1 + \cdots + \alpha_n$. The space $\mathcal{S}(\mathbb{R}^n)$ is equipped with the locally convex topology defined by the collection of seminorms $p_{j,k}$ for all $j, k = 0, 1, \dots$. The space $C^\infty(\mathbb{R}^n)$ of smooth functions on $\mathbb{R}^n$ has the topology of uniform convergence of functions and their derivatives on compact subsets of $\mathbb{R}^n$.

For a Banach space $X$, the linear space of all continuous linear operators on $X$ is denoted by $\mathcal{L}(X)$. An $\mathcal{L}(X)$*-valued tempered distribution* $T$ is a continuous linear map from $\mathcal{S}(\mathbb{R}^n)$ into the space $\mathcal{L}(X)$ endowed with the uniform operator topology. As usual, the Fourier transform $\hat{T}$ of an $\mathcal{L}(X)$-valued tempered distribution is defined by $\hat{T}(f) = T(\hat{f})$ for all $f \in \mathcal{S}(\mathbb{R}^n)$. The inverse Fourier transform of $T$ is given by $\check{T}(g) = T(\check{g})$ with

$$\check{g}(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x, \xi\rangle} g(\xi)\, d\xi$$

for all $g \in \mathcal{S}(\mathbb{R}^n)$.

An element $\mathcal{L}(C^\infty(\mathbb{R}^n), \mathcal{L}(X))$ is a distribution with compact support [110, Theorem 24.2]. For $T \in \mathcal{L}(C^\infty(\mathbb{R}^n), \mathcal{L}(X))$, the smallest nonnegative integer $k$ such that for every compact subset $K$ of $\mathbb{R}^n$, there exists a number $C_K > 0$ such that

$$\|T(f)\|_{\mathcal{L}(X)} \le C_K \sup\{|\partial^\alpha f(x)| : x \in K, |\alpha| \le k\},$$

for all smooth functions $f$ with support contained in $K$ is called the *order* of $T$. Every element of $\mathcal{L}(C^\infty(\mathbb{R}^n), \mathcal{L}(X))$ has finite order [110, Theorem 24.3, Corollary]. The *support* $\mathrm{supp}(T)$ of $T$ is the complement of the set of all points $x \in \mathbb{R}^n$ for which there exists an open neighborhood $U_x$ such that $T(f) = 0$ for all smooth functions $f$ supported by $U_x$.

The version of the Paley-Wiener Theorem below for scalar valued distributions is proved in [110, Theorem 29.2]. The vector valued version follows readily from the scalar case. Set $B_r = \{x \in \mathbb{R}^n : |x| \le r\}$. The real and imaginary parts of a complex vector $\zeta = (\zeta_1, \dots, \zeta_n) \in \mathbb{C}^n$ are the real vectors $\Re\zeta = (\Re\zeta_1, \dots, \Re\zeta_n) \in \mathbb{R}^n$ and $\Im\zeta = (\Im\zeta_1, \dots, \Im\zeta_n) \in \mathbb{R}^n$, respectively.

**Proposition 2.1.** (Paley-Wiener) *Let $E$ be a Banach space and let $T \in \mathcal{L}(\mathcal{S}(\mathbb{R}^n), E)$ be a tempered distribution. Then there exists $r \ge 0$ such that $T$ has compact support contained in the ball $B_r$ if and only if $T$ is the Fourier transform of an entire function $e : \mathbb{C}^n \to E$ for which there exists $C \ge 0, s \ge 0$ such that $\|e(\zeta)\|_E \le C(1 + |\zeta|)^s e^{r|\Im\zeta|}$, for all $\zeta \in \mathbb{C}^n$.*

The sum $\sum_{j=1}^n \zeta_j A_j$ is also written as $\langle \zeta, \boldsymbol{A}\rangle$ for $\zeta \in \mathbb{C}^n$. The Euclidean norm $\sqrt{\sum_{j=1}^n |\zeta_j|^2}$ of $\zeta \in \mathbb{C}^n$ is denoted by $|\zeta|$.

**Definition 2.2.** Let $A_1, \ldots, A_n$ be bounded linear operators acting on a Banach space $X$. If there exists $C, s \geq 0$ such that

$$\|e^{i\langle \xi, \boldsymbol{A} \rangle}\|_{\mathcal{L}(X)} \leq C(1+|\xi|)^s, \quad \text{for all } \xi \in \mathbb{R}^n, \tag{2.2}$$

then the $n$-tuple $\boldsymbol{A} = (A_1, \ldots, A_n)$ of operators is said to be of *Paley-Wiener type s*.

If there exists $C, r, s \geq 0$ such that

$$\|e^{i\langle \zeta, \boldsymbol{A} \rangle}\|_{\mathcal{L}(X)} \leq C(1+|\zeta|)^s e^{r|\Im \zeta|}, \quad \text{for all } \zeta \in \mathbb{C}^n, \tag{2.3}$$

then the $n$-tuple $\boldsymbol{A} = (A_1, \ldots, A_n)$ of operators is said to be of *Paley-Wiener type* $(s, r)$.

The terminology is adapted from work of A. Pryde [88]. Even if $n = 1$, the index $s$ in (2.2) may be strictly positive. If the bound (2.2) holds, then each operator $A_j$, $j = 1, \ldots, n$ necessarily has real spectrum by [23, Theorem 5.4.5].

*Example 2.3.* (i) [88, Proposition 2.2] Let $A$ be an $N \times N$ matrix with real spectrum. Appealing to the Jordan decomposition theorem, there exists an invertible matrix $T$, a diagonal matrix $D$ with real entries and a nilpotent matrix $N$ commuting with $D$ such that $A = T(D+N)T^{-1}$.

Let $J(A)$ be the size of the largest Jordan block in the Jordan decomposition $D + N$ of $A$ and let $r(A) = \sup |\sigma(A)|$ be the spectral radius of $A$. Then $r(A)$ is the maximum absolute value of the diagonal entries of $D$.

The matrix $A$ is of Paley-Wiener type $(s, r)$ with $s = J(A) - 1$ and $r = r(A)$ because

$$e^{i\zeta A} = Te^{i\zeta(D+N)}T^{-1} = Te^{i\zeta D}e^{i\zeta N}T^{-1} = Te^{i\zeta D}\left(I + \sum_{j=1}^{s} \frac{N^j \zeta^j}{j!}\right)T^{-1},$$

so that

$$\begin{aligned} \|e^{i\zeta A}\| &\leq \|T\|\,\|T^{-1}\|e^{|\Im\zeta|\|D\|}\left(1 + \sum_{j=1}^{s} \frac{|\zeta|^j}{j!}\right) \\ &\leq \|T\|\,\|T^{-1}\|(1+|\zeta|)^s e^{r|\Im\zeta|}, \quad \text{for all} \ \ \zeta \in \mathbb{C}. \end{aligned}$$

(ii) [88, Theorem 4.5] Let $X = \mathbb{C}^N$ and suppose that $A_1, \ldots, A_n$ are simultaneously triangularisable matrices with real spectra. Then $\boldsymbol{A} = (A_1, \ldots, A_n)$ is of Paley-Wiener type $(N-1, r(\boldsymbol{A}))$. Here $r(\boldsymbol{A})$ is the joint spectral radius defined in [88, p. 92]. The conclusion is actually equivalent to the condition that the matrix $\langle \xi, \boldsymbol{A} \rangle$ has real spectrum for each $\xi \in \mathbb{R}^n$, see Theorem 5.10 below. This condition includes both the case of simultaneously triangularisable matrices with real spectra and hermitian matrices.

In the case when $A_1, \ldots, A_n$ are bounded selfadjoint operators acting on a Hilbert space $H$, the operator $\langle \xi, \boldsymbol{A} \rangle$ is selfadjoint for every $\xi \in \mathbb{R}^n$. It follows that $e^{i\langle \xi, \boldsymbol{A} \rangle}$ is a unitary operator, and by the Lie-Kato-Trotter product formula,

$$\|e^{i\langle \xi + i\eta, \boldsymbol{A} \rangle}\| = \|e^{i\langle \xi, \boldsymbol{A} \rangle - \langle \eta, \boldsymbol{A} \rangle}\| \leq e^{\|\langle \eta, \boldsymbol{A} \rangle\|} \leq e^{r|\eta|}$$

for all $\xi, \eta \in \mathbb{R}^n$ [108, Theorem 1]. Hence the bound (2.2) holds with $C = 1$, $s = 0$ and $r = \left(\sum_{j=1}^n \|A_j\|^2\right)^{1/2}$.

An obvious necessary condition that (2.3) holds is that each operator $A_1, \ldots, A_n$ is of Paley-Wiener type $(s, r)$. It is not sufficient to assume that each operator $A_1, \ldots, A_n$ is of Paley-Wiener type $(s, r)$ to conclude the inequality (2.3). For example each of the matrices $A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ is of Paley-Wiener type $(1, 1)$, but $(A_1, A_2)$ is not of Paley-Wiener type $(s, r)$ for any $s, r > 0$, because $e^{i(A_1 - A_2)t} = \cosh(t)I + \sinh(t) \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ for all $t \in \mathbb{R}$.

Once we know that the bound (2.3) is valid, the Paley-Wiener Theorem establishes the existence of a nonempty compact subset $\gamma(\boldsymbol{A})$ of $\mathbb{R}^n$ and a $C^\infty$-functional calculus $f \longmapsto f(\boldsymbol{A})$ defined for all sufficiently smooth functions $f$ given in an open neighborhood of $\gamma(\boldsymbol{A})$ in $\mathbb{R}^n$; see Theorem 2.4 and Definition 2.5. The $C^\infty$-functional calculus agrees with the map $f \longmapsto f(\boldsymbol{A})$ defined for all polynomials $f$ in $n$ variables in which $f(\boldsymbol{A})$ has equally weighted operator products; see equation (2.5). The set $\gamma(\boldsymbol{A})$ serves as the 'joint spectrum' of the $n$-tuple $\boldsymbol{A}$ of operators.

The following result, essentially from [7], shows that if the $n$-tuple $\boldsymbol{A}$ satisfies the bound (2.3), then the Paley-Wiener theorem immediately provides an extension of the symmetric operator calculus from the space of polynomials to a space of smooth functions.

Given nonnegative integers $m_1, \ldots, m_n$, we let $m = m_1 + \cdots + m_n$ and

$$P^{m_1,\ldots,m_n}(z_1, \ldots, z_n) = z_1^{m_1} \cdots z_n^{m_n}. \tag{2.4}$$

**Theorem 2.4.** *Let $A_1, \ldots, A_n$ be bounded linear operators acting on a Banach space $X$. If $r, s \geq 0$ and $\boldsymbol{A} = (A_1, \ldots, A_n)$ is of Paley-Wiener type $(s, r)$, then there exists a unique $\mathcal{L}(X)$-valued distribution $\mathcal{W}_{\boldsymbol{A}} \in \mathcal{L}(C^\infty(\mathbb{R}^n), \mathcal{L}(X))$ such that*

$$\mathcal{W}_{\boldsymbol{A}}(P^{m_1,\ldots,m_n}) = \frac{m_1! \cdots m_n!}{m!} \sum_\pi A_{\pi(1)} \cdots A_{\pi(m)}, \tag{2.5}$$

*where $m_1, \ldots, m_n$ are any nonnegative integers, $m = m_1 + \cdots + m_n$ and the sum is taken over every map $\pi$ of the set $\{1, \ldots, m\}$ into $\{1, \ldots, n\}$ which assumes the value $j$ exactly $m_j$ times, for each $j = 1, \ldots, n$.*

*The distribution $\mathcal{W}_{\boldsymbol{A}}$ is given by*

$$\mathcal{W}_{\boldsymbol{A}}(f) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle \xi, \boldsymbol{A} \rangle} \hat{f}(\xi)\, d\xi, \quad \text{for every } f \in \mathcal{S}(\mathbb{R}^n). \tag{2.6}$$

*The integral converges as a Bochner integral in $\mathcal{L}(X)$. The support $K$ of the distribution $\mathcal{W}_{\boldsymbol{A}}$ is nonempty and contained in the ball $B_r$. The $n$-tuple $\boldsymbol{A}$ is of Paley-Wiener type $(s, \sup|K|)$. The order of $\mathcal{W}_{\boldsymbol{A}}$ is at most the smallest integer strictly greater than $n/2+s$.*

*Proof.* Let $e(\zeta) = (2\pi)^{-n}e^{i\langle\zeta,\boldsymbol{A}\rangle}$ for each $\zeta \in \mathbb{C}^n$. According to the bound (2.2), there exists $C > 0$ and $s \geq 0$ such that $\|e(\xi)\|_{\mathcal{L}(X)} \leq C(1+|\xi|)^s$ for all $\xi \in \mathbb{R}^n$. For every $f \in \mathcal{S}(\mathbb{R}^n)$, the Fourier transform $\hat{f}$ of $f$ again belongs to $\mathcal{S}(\mathbb{R}^n)$ [110], so there exists $C_1 > 0$ such that $\|e(\xi)\|_{\mathcal{L}(X)}|\hat{f}(\xi)| \leq C_1(1+|\xi|)^{-n-1}$ for all $\xi \in \mathbb{R}^n$. Because

$$\int_{\mathbb{R}^n} \|e(\xi)\|_{\mathcal{L}(X)}|\hat{f}(\xi)|\,d\xi \leq C_1 \int_{\mathbb{R}^n} (1+|\xi|)^{-n-1}\,d\xi < \infty,$$

the integral (2.6) exists as a Bochner integral for every $f \in \mathcal{S}(\mathbb{R}^n)$. Moreover, $\mathcal{W}_{\boldsymbol{A}}$ is a tempered distribution, the Fourier transform of $e$. According to the assumption (2.3) and the Paley-Wiener Theorem 2.1, $\mathcal{W}_{\boldsymbol{A}}$ has compact support contained in $B_r$, so it has a unique extension from $\mathcal{S}(\mathbb{R}^n)$ to $C^\infty(\mathbb{R}^n)$. The smallest $r$ possible is $\sup|K|$.

Formula (2.5) follows from the observation [7, Theorem 2.8] that

$$\begin{aligned}\mathcal{W}_{\boldsymbol{A}}(P^{m_1,\dots,m_n}) &= (P^{m_1,\dots,m_n}.[e(\xi)\hat{\ }])(1)\\ &= [P^{m_1,\dots,m_n}(\boldsymbol{D}_\xi)e^{i\langle\xi,\boldsymbol{A}\rangle}]_{\xi=0}.\end{aligned}$$

We show that $K$ is nonempty by showing that the projection of $K$ onto the first coordinate contains the spectrum $\sigma(A_1)$ of $A_1$, which we know to be nonempty. Let $\pi_1 : \mathbb{R}^n \to \mathbb{R}$ be the projection onto the first coordinate.

If $f \in C_c^\infty(\mathbb{R})$ has support disjoint from $\pi_1 K$, then $f \circ \pi_1 \in C^\infty(\mathbb{R}^n)$ has support disjoint from $K$ so that $\mathcal{W}_{\boldsymbol{A}}(f \circ \pi_1) = 0$. Therefore, the distribution $f \longmapsto \mathcal{W}_{\boldsymbol{A}}(f \circ \pi_1)$, $f \in C^\infty(\mathbb{R})$, has support contained in $\pi_1 K$.

Let $g_2, \dots, g_n$ be smooth functions on $\mathbb{R}$ with compact support and equal to one in a neighbourhood of zero. Then the function $f_\epsilon : x \longmapsto f(x_1)g_2(\epsilon x_2)\cdots g_2(\epsilon x_n)$, $x \in \mathbb{R}^n$, converges to $f \circ \pi_1$ in $C^\infty(\mathbb{R}^n)$ as $\epsilon \to 0$. An application of change of variables and dominated convergence ensures that $\mathcal{W}_{\boldsymbol{A}}(f_\epsilon) \to \mathcal{W}_{A_1}(f)$ in $\mathcal{L}(X)$, so the equality $\mathcal{W}_{\boldsymbol{A}}(f \circ \pi_1) = \mathcal{W}_{A_1}(f)$ follows. The support of the distribution $\mathcal{W}_{A_1}$ is $\sigma(A_1)$ [23, Theorem 3.1.6], so the inclusion $\sigma(A_1) \subseteq \pi_1 K$ holds.

It remains to consider the order of the $\mathcal{L}(X)$-valued distribution $\mathcal{W}_{\boldsymbol{A}}$. The argument of [7, Lemma 3.8] works here too. By (2.2), the Cauchy-Schwarz inequality and Plancherel theorem, we have

$$\begin{aligned}\|\mathcal{W}_{\boldsymbol{A}}(f)\| &\leq \int_{\mathbb{R}^n} \|e(\xi)\||\hat{f}(\xi)|d\xi\\ &\leq C\int_{\mathbb{R}^n} (1+|\xi|)^s|\hat{f}(\xi)|d\xi\\ &= C\int_{\mathbb{R}^n} \frac{(1+|\xi|)^s}{(1+|\xi|^k)}|[(1+(-\Delta)^{k/2})f]\hat{\ }(\xi)|d\xi\end{aligned}$$

$$\leq C' \left\| \frac{(1+|\xi|)^s}{(1+|\xi|^k)} \right\|_2 \|(1+(-\Delta)^{k/2})f\|_2.$$

If $k$ is an integer strictly greater than $n/2+s$, then $\left\|(1+|\xi|)^s/(1+|\xi|^k)\right\|_2 < \infty$ and if $f_1, f_2, \ldots$ are smooth functions with support in a fixed relatively compact open set $U \subset \mathbb{R}^n$ and $f_j$ and its derivatives up to order $k$ converge to zero uniformly on $U$ as $j \to \infty$, then $\|(1+(-\Delta)^{k/2})f_j\|_2 \to 0$ as $j \to \infty$. For even $k$, this follows from the estimate

$$\|(1+(-\Delta)^{k/2})f\|_2 \leq \ell(U)^{1/2} \sup_{x \in U, |\alpha| \leq k} |\partial^\alpha f(x)|.$$

For odd $k$, we appeal to the equality $\|(-\Delta)^{1/2}\phi\|_2^2 = \sum_{m=1}^n \|\partial_m \phi\|_2^2$ for all $\phi \in C_c^\infty(\mathbb{R}^n)$. Hence $\mathcal{W}_{\boldsymbol{A}}$ has order at most $k$.

For convenience, we shall sometimes use the notation $e(\xi)\hat{}$ for the Fourier transform of the tempered distribution defined by a suitable function $\xi \longmapsto e(\xi)$, $\xi \in \mathbb{R}^n$. In particular, formula (2.6) is written as

$$\mathcal{W}_{\boldsymbol{A}} = (2\pi)^{-n} \left[ e^{i\langle \xi, \boldsymbol{A} \rangle} \right]\hat{}.$$

A number of properties of the Weyl calculus are listed in [7], at least for hermitian operators acting on a Banach space. Rather than repeat them here, they will appear in Chapter 4 in a more general context when the Paley-Wiener bound (2.3) may fail.

## 2.3 The Joint Spectrum

Even for a system $\boldsymbol{A}$ of commuting bounded linear operators acting on a Banach space, there are many possible approaches to the definition of the joint spectrum of $\boldsymbol{A}$. Fortunately, the definition given below agrees with most [76] in the case that $\boldsymbol{A}$ is a commuting system and the exponential bound (2.3) holds, because each operator $A_j$ necessarily has real spectrum [23, Theorem 5.4.5].

**Definition 2.5.** Let $A_1, \ldots, A_n$ be bounded linear operators acting on a Banach space $X$. Suppose that $r, s \geq 0$ and $\boldsymbol{A} = (A_1, \ldots, A_n)$ is of Paley-Wiener type $(s, r)$. The support of the distribution $\mathcal{W}_{\boldsymbol{A}}$, denoted by $\gamma(\boldsymbol{A})$, is called the *joint spectrum* of the $n$-tuple $\boldsymbol{A} = (A_1, \ldots, A_n)$. The distribution $\mathcal{W}_{\boldsymbol{A}}$ is called the *Weyl calculus for* $\boldsymbol{A}$. The number

$$r(\boldsymbol{A}) = \sup\{|x| : x \in \gamma(\boldsymbol{A})\}$$

is called the *joint spectral radius* of $\boldsymbol{A}$.

According to the Paley-Wiener theorem, $r(\boldsymbol{A}) \leq r$. We shall see in Chapter 4 that the nonempty compact subset $\gamma(\boldsymbol{A})$ of $\mathbb{R}^n$ may be interpreted as the set of singularities of a multidimensional analogue $\omega \longmapsto G_\omega(\boldsymbol{A})$ of the resolvent family of a single operator. In the case that the operators of $\boldsymbol{A}$ commute, the expression $G_\omega(\boldsymbol{A})$ can be written down explicitly. In the noncommuting case, the joint spectrum $\gamma(\boldsymbol{A})$ possesses another layer of complexity and $G_\omega(\boldsymbol{A})$ is necessarily more mysterious.

*Example 2.6.* a) Let $n = 1$. As mentioned in Example 2.3 (i), any matrix with real spectrum is of Paley-Wiener type $(s, r)$. Such a matrix $A$ is diagonalisable if and only if it is of Paley-Wiener type $(0, r)$. To see this, let $N$ be the nilpotent part and $D$ the diagonal part, with real entries, of the Jordan decomposition of $A$. Then there exists an invertible matrix $T$ such that $Te^{i\xi A}T^{-1} = e^{i\xi D}e^{i\xi N}$. Because $N$ is nilpotent, each entry of $e^{i\xi N}$ is a polynomial in $\xi$.

If $A$ is diagonalisable, then $N = 0$ and $A$ is of Paley-Wiener type $(0, r)$, with $r$ the maximum absolute value of the eigenvalues of $\boldsymbol{A}$, all of which are real numbers. If $A$ is of Paley-Wiener type $(0, r)$ for some $r > 0$, then $e^{i\xi D}$ is unitary and $\|e^{i\xi N}\| = \|Te^{i\xi A}T^{-1}\| \leq C$ for all $\xi \in \mathbb{R}$. This is possible only if the nilpotent matrix $N$ is the zero matrix, so that $e^{i\xi N} = 1$ for all $\xi \in \mathbb{R}$.

b) If $A$ is a single operator of Paley-Wiener type $(s, r)$, then $\gamma(A) = \sigma(A)$. The result is proved in [23, Theorem 3.1.6] under quite general circumstances and in [7, Lemma 3.3] for hermitian operators on a Banach space.

The present example when $n = 1$ has an extensive literature. The condition that a single bounded operator $A$ acting on a Banach space $X$ is of Paley-Wiener type $(s, r)$ is equivalent to the condition that $A$ is a *generalised scalar operator* with real spectrum [23, Theorem 5.4.5]. Furthermore, if for each $g \in C^\infty(\mathbb{R})$, we denote the Gelfand representation of $\mathcal{W}_A(g)$ in the Banach algebra $\mathcal{A}$ generated by $\{\mathcal{W}_A(f) : f \in C^\infty(\mathbb{R})\}$ by $\mathcal{W}_A(g)\hat{}$, then the maximal ideal space of $\mathcal{A}$ can be identified with the spectrum $\sigma(A)$ of $A$ in $\mathcal{L}(X)$ and the equalities

i) $\mathcal{W}_A(f)\hat{} = f|\sigma(A)$ and
ii) $\sigma(\mathcal{W}_A(f)) = f(\sigma(A))$

hold for all $f \in \mathbb{C}^\infty(\mathbb{R})$ [23, Theorem 3.2.2].

Let $k$ be a nonnegative integer and $U$ a nonempty open subset of $\mathbb{R}^n$. The space $C^k(U)$ of all functions continuously differentiable in $U$ for derivatives up to order $k$ is given the topology of uniform convergence of functions and their derivatives, up to order $k$, on compact subsets of $U$. If $K$ is a closed subset of $\mathbb{R}^n$, set $C^k(K) = \cup_U C^k(U)$, where the union is over all open sets $U$ containing $K$ and $C^k(K)$ is given the inductive limit topology. Then any distribution $T \in \mathcal{L}(C^\infty(\mathbb{R}^n), \mathcal{L}(X))$ of order $k$ and with support $K$ uniquely defines a distribution $\tilde{T} \in \mathcal{L}(C^k(K), \mathcal{L}(X))$ such that $\tilde{T}(f|U) = T(f)$ for every $f \in C^\infty(\mathbb{R}^n)$ and every open neighborhood $U$ of $K$ [110, Theorem 24.1]. As is customary, the same notation is used for the distribution $T$ as an element of $\mathcal{L}(C^\infty(\mathbb{R}^n), \mathcal{L}(X))$ and of $\mathcal{L}(C^k(K), \mathcal{L}(X))$.

For the Weyl calculus for a system of hermitian operators, the Lie-Trotter product formula together with part of the Paley-Wiener theorem establishes that the joint spectral radius is bounded by $\left(\sum_{j=1}^n \|A_j\|^2\right)^{1/2}$ [7]. The following result showing that the bound (2.2) implies the bound (2.3) is mentioned in [64] although no proof is given. We give a proof below that gives the same bound $\left(\sum_{j=1}^n \|A_j\|^2\right)^{1/2}$ for the joint spectral radius.

**Theorem 2.7.** *If $\boldsymbol{A}$ is an $n$-tuple of bounded linear operators of type $s$, then $\boldsymbol{A}$ is of type $(s,r)$ with $r = \sqrt{\sum_{j=1}^n \|A_j\|^2}$. The support of $\mathcal{W}_{\boldsymbol{A}}$ is contained in the rectangle $[-\|A_1\|, \|A_1\|] \times \cdots \times [-\|A_n\|, \|A_n\|]$ in $\mathbb{R}^n$.*

*Proof.* Let $\pi_1 : \mathbb{R}^n \to \mathbb{R}$ be the projection onto the first coordinate. For any rapidly decreasing functions $g_2, \dots, g_n : \mathbb{R} \to \mathbb{R}$, let $T_{g_2,\dots,g_n} \in \mathcal{L}(\mathcal{S}(\mathbb{R}), \mathcal{L}(X))$ be the distribution

$$T_{g_2,\dots,g_n} : f \longmapsto \mathcal{W}_{\boldsymbol{A}}(f \otimes g_2 \otimes \cdots \otimes g_n), \quad f \in \mathcal{S}(\mathbb{R}).$$

We shall show that $T_{g_2,\dots,g_n}$ has compact support $S(g_2, \dots, g_n)$ contained in $[-\|A_1\|, \|A_1\|]$. If $x \in \cup S(g_2, \dots, g_n)^c$, then for every open neighborhood $U$ of $x$ in $\mathbb{R}$ disjoint from $\cap S(g_2, \dots, g_n)$ and every smooth function $f : \mathbb{R} \to \mathbb{R}$ with support contained in $U$, we have $\mathcal{W}_{\boldsymbol{A}}(f \otimes g_2 \otimes \cdots \otimes g_n) = 0$ for all rapidly decreasing functions $g_2, \dots, g_n : \mathbb{R} \to \mathbb{R}$.

In particular, if $x \in \mathbb{R}$ and $|x| > \|A_1\|$, and $U$ is an open neighbourhood of $x$ disjoint from $[-\|A_1\|, \|A_1\|]$, then $\mathcal{W}_{\boldsymbol{A}}(f \otimes g_2 \otimes \cdots \otimes g_n) = 0$ for every smooth function $f : \mathbb{R} \to \mathbb{R}$ with support contained in $U$ and all rapidly decreasing functions $g_2, \dots, g_n : \mathbb{R} \to \mathbb{R}$.

Because the linear span of all functions $f \otimes g_2 \otimes \cdots \otimes g_n$ with $f, g_j \in C_c^\infty(\mathbb{R})$ is dense in $C_c^\infty(\mathbb{R}^n)$, it follows that the set $\{x\} \times \mathbb{R} \times \cdots \times \mathbb{R}$ is contained in $\mathrm{supp}(\mathcal{W}_{\boldsymbol{A}})^c$. Expressed otherwise,

$$\mathrm{supp}(\mathcal{W}_{\boldsymbol{A}}) \subseteq \pi_1^{-1}\big(\cap S(g_2, \dots, g_n)\big) \subseteq [-\|A_1\|, \|A_1\|] \times \mathbb{R} \times \cdots \times \mathbb{R}.$$

Once we do this for each coordinate, the inclusion $\mathrm{supp}(\mathcal{W}_{\boldsymbol{A}}) \subseteq K$ and the formula for $r$ follow.

Let $B(\xi) = \sum_{j=2}^n A_j \xi_j$ and

$$\Phi^{(\epsilon)}_{\xi_2,\dots,\xi_n}(t) = e^{-\epsilon t^2} e^{iA_1 t + iB(\xi)}$$

and set $T^{(\epsilon)}_{\xi_2,\dots,\xi_n} = \frac{1}{2\pi} \hat{\Phi}^{(\epsilon)}_{\xi_2,\dots,\xi_n}$ for every $\epsilon \geq 0$. Because $\Phi^{(\epsilon)}_{\xi_2,\dots,\xi_n}$ has exponential decay, $T^{(\epsilon)}_{\xi_2,\dots,\xi_n} \in L^1(\mathbb{R}, \mathcal{L}(X))$ for each $\epsilon > 0$ and $T^{(\epsilon)}_{\xi_2,\dots,\xi_n} \to T^{(0)}_{\xi_2,\dots,\xi_n}$ in $\mathcal{L}(C_c^\infty(\mathbb{R}), \mathcal{L}(X))$ as $\epsilon \to 0+$.

Now

$$\int_0^\infty e^{-\lambda t} \|\Phi^{(\epsilon)}_{\xi_2,\dots,\xi_n}(t)\| \, dt \leq e^{\|B(\xi)\|} \int_0^\infty e^{-\lambda t} e^{-\epsilon t^2/2} e^{\|A_1\| t} \, dt$$

for all $\lambda > \|A_1\|$. The function $\lambda \longmapsto \int_0^\infty e^{-\lambda t}\Phi^{(\epsilon)}_{\xi_2,\dots,\xi_n}(t)\,dt$ has an analytic continuation from $\{\mathrm{Re}(\lambda) > \|A_1\|\}$ to $\{|\lambda| > \|A_1\|\}$. To see this, we use the perturbation series expansion

$$e^{iA_1t+iB(\xi)} = I + \sum_{k=1}^{\infty}(it)^k \int_0^1\int_0^{t_k}\cdots\int_0^{t_2} e^{i(1-t_k)B(\xi)}A_1\cdots \\ e^{i(t_2-t_1)B(\xi)}A_1e^{it_1B(\xi)}\,dt_1\dots dt_k.$$

For $\lambda > \|A_1\|$, we have

$$\int_0^\infty e^{-\lambda t}\Phi^{(\epsilon)}_{\xi_2,\dots,\xi_n}(t)\,dt \\ = \frac{I}{\lambda+\epsilon} + \sum_{k=1}^{\infty} i^k \int_0^\infty e^{-\lambda t}t^k e^{-\epsilon t^2}\int_0^1\int_0^{t_k}\cdots\int_0^{t_2} e^{i(1-t_k)B(\xi)}A_1\cdots \\ e^{i(t_2-t_1)B(\xi)}A_1e^{it_1B(\xi)}\,dt_1\dots dt_k. \tag{2.7}$$

The norm of each term of the series on the right is bounded by

$$\left(\frac{\|A_1\|}{\lambda}\right)^k k!\int_0^1\int_0^{t_k}\cdots\int_0^{t_2} e^{(1-t_k)\|B(\xi)\|}\dots e^{t_1\|B(\xi)\|}\,dt_1\dots dt_k \\ = \left(\frac{\|A_1\|}{\lambda}\right)^k e^{\|B(\xi)\|},$$

so the right-hand side of equation (2.7) may be analytically continued to the region $\{|\lambda| > \|A_1\|\}$ for every $\epsilon > 0$.

By the Plancherel formula, we have

$$\int_0^\infty e^{-\lambda t}\Phi^{(\epsilon)}_{\xi_2,\dots,\xi_n}(t)\,dt = \int_{\mathbb{R}}\frac{1}{\lambda+ix}T^{(\epsilon)}_{\xi_2,\dots,\xi_n}(x)\,dx. \tag{2.8}$$

Let $\tilde f(z) = \frac{1}{2\pi i}\int_{\mathbb{R}} f(y)/(z-y)\,dy$ be the Cauchy transform of $f \in L^1(\mathbb{R})$ with $z \in \mathbb{C}$ and $\mathrm{Im}z \neq 0$. It follows from [22, Theorem 5.6] that

$$\langle f,\phi\rangle = \lim_{\epsilon\to 0+}\int_{\mathbb{R}}[\tilde f(x+i\epsilon) - \tilde f(x-i\epsilon)]\phi(x)\,dx \tag{2.9}$$

for all $\phi \in C_c^\infty(\mathbb{R})$. The same notation is used for operator valued functions $f$. The Cauchy transform $\tilde T^{(\epsilon)}_{\xi_2,\dots,\xi_n}$ of $T^{(\epsilon)}_{\xi_2,\dots,\xi_n}$ exists because $T^{(\epsilon)}_{\xi_2,\dots,\xi_n} \in L^1(\mathbb{R},\mathcal{L}(X))$ for each $\epsilon > 0$.

Now according to formula (2.8) and the argument above, the Cauchy transform $\tilde T^{(\epsilon)}_{\xi_2,\dots,\xi_n}$ of $T^{(\epsilon)}_{\xi_2,\dots,\xi_n}$ may be analytically continued into the region $\{|\lambda| > \|A_1\|\}$ for every $\epsilon > 0$. If $\phi$ is a smooth function with support disjoint from $[-\|A_1\|, \|A_1\|]$, then by equation (2.9), we have $\langle T^{(\epsilon)}_{\xi_2,\dots,\xi_n}, \phi\rangle = 0$ for every $\epsilon > 0$. Because $T^{(\epsilon)}_{\xi_2,\dots,\xi_n} \to T^{(0)}_{\xi_2,\dots,\xi_n}$ in $\mathcal{L}(C_c^\infty(\mathbb{R}), \mathcal{L}(X))$ as $\epsilon \to 0+$, it follows that $\langle T^{(0)}_{\xi_2,\dots,\xi_n}, \phi\rangle = 0$ for all $\xi_2, \dots, \xi_n \in \mathbb{R}$. Moreover,

$$\langle T_{g_2,\dots,g_n}, \phi\rangle = (2\pi)^{-(n-1)} \int_{\mathbb{R}^{n-1}} \langle T^{(0)}_{\xi_2,\dots,\xi_n}, \phi\rangle\, \hat{g}_1(\xi_1) \cdots \hat{g}_n(\xi_n)\, d\xi_2 \cdots d\xi_n = 0.$$

It follows that that $T_{g_2,\dots,g_n}$ has compact support $S(g_2, \dots, g_n)$ contained in $[-\|A_1\|, \|A_1\|]$. Doing this for each coordinate, the inclusion $\mathrm{supp}(\mathcal{W}_{\boldsymbol{A}}) \subseteq K$ follows. $\square$

# 3

# Clifford Analysis

In this chapter, we give the necessary background in Clifford analysis that facilitates the representation of functions of systems of operators by a Cauchy formula. More comprehensive accounts of Clifford analysis may be found in the monographs [19], [21].

For a single bounded linear operator $A$ acting on a Banach space $X$, the Riesz-Dunford functional calculus

$$f(A) = \frac{1}{2\pi i}\int_C (\zeta I - A)^{-1} f(\zeta)\, d\zeta$$

represents the function $f(A)$ of the operator $A$ as a contour integral about the spectrum $\sigma(A)$ of $A$. This is what we are looking for in the case that $A$ is replaced by an $n$-tuple $\boldsymbol{A}$ of bounded linear operators. But first we need a higher dimensional analogue of the Cauchy integral formula and a suitable replacement for the Cauchy kernel $\zeta \longmapsto (\zeta I - A)^{-1}$, that is, the resolvent of the operator $A$.

## 3.1 Clifford Algebras

The basic idea of forming a Clifford algebra $\mathcal{A}$ with $n$ generators is to take the smallest real or complex algebra $\mathcal{A}$ with an identity element $e_0$ such that $\mathbb{R} \oplus \mathbb{R}^n$ is embedded in $\mathcal{A}$ via the identification of $(x_0, \boldsymbol{x}) \in \mathbb{R} \oplus \mathbb{R}^n$ with $x_0 e_0 + \boldsymbol{x} \in \mathcal{A}$ and the identity

$$|\boldsymbol{x}|^2 = -|\boldsymbol{x}|^2 e_0 = -(x_1^2 + x_2^2 + \cdots + x_n^2)e_0$$

holds for all $\boldsymbol{x} \in \mathbb{R}^n$. Then we arrive at the following definition.

Let $\mathbb{F}$ be either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. The *Clifford algebra* $\mathbb{F}_{(n)}$ over $\mathbb{F}$ is a $2^n$-dimensional algebra with unit defined as follows. Given the standard basis vectors $e_0, e_1, \ldots, e_n$ of the vector space $\mathbb{F}^{n+1}$, the basis vectors $e_S$ of $\mathbb{F}_{(n)}$ are indexed by all finite subsets

B. Jefferies: LNM 1843, pp. 27–38, 2004.

$S$ of $\{1, 2, \ldots, n\}$. The basis vectors are determined by the following rules for multiplication on $\mathbb{F}_{(n)}$:

$$\begin{aligned} e_0 &= 1, & & \\ e_j^2 &= -1, & &\text{for } \quad 1 \le j \le n \\ e_j e_k = -e_k e_j &= e_{\{j,k\}}, & &\text{for } \quad 1 \le j < k \le n \\ e_{j_1} e_{j_2} \cdots e_{j_s} &= e_S, & &\text{if } \quad 1 \le j_1 < j_2 < \cdots < j_s \le n \\ & & &\qquad\qquad \text{and} \quad S = \{j_1, \ldots, j_s\}. \end{aligned}$$

Here the identifications $e_0 = e_\emptyset$ and $e_j = e_{\{j\}}$ for $1 \le j \le n$ have been made.

Suppose that $m \le n$ are positive integers. The vector space $\mathbb{R}^m$ is identified with a subspace of $\mathbb{F}_{(n)}$ by virtue of the embedding $(x_1, \ldots, x_m) \longmapsto \sum_{j=1}^m x_j e_j$. On writing the coordinates of $x \in \mathbb{R}^{n+1}$ as $x = (x_0, x_1, \ldots, x_n)$, the space $\mathbb{R}^{n+1}$ is identified with a subspace of $\mathbb{F}_{(n)}$ with the embedding $(x_0, x_1, \ldots, x_n) \longmapsto \sum_{j=0}^n x_j e_j$.

The product of two elements $u = \sum_S u_S e_S$ and $v = \sum_S v_S e_S, v_S \in \mathbb{F}$ with coefficients $u_S \in \mathbb{F}$ and $v_S \in \mathbb{F}$ is $uv = \sum_{S,R} u_S v_R e_S e_R$. According to the rules for multiplication, $e_S e_R$ is $\pm 1$ times a basis vector of $\mathbb{F}_{(n)}$. The *scalar part* of $u = \sum_S u_S e_S, u_S \in \mathbb{F}$ is the term $u_\emptyset$, also denoted as $u_0$.

The Clifford algebras $\mathbb{R}_{(0)}, \mathbb{R}_{(1)}$ and $\mathbb{R}_{(2)}$ are the real, complex numbers and the quaternions, respectively. In the case of $\mathbb{R}_{(1)}$, the vector $e_1$ is identified with $i$ and for $\mathbb{R}_{(2)}$, the basis vectors $e_1, e_2, e_1 e_2$ are identified with $i, j, k$ respectively. Because we also consider complex Clifford algebras $\mathbb{C}_{(n)}$, it is less confusing if we avoid these particular identifications.

The conjugate $\overline{e_S}$ of a basis element $e_S$ is defined so that $e_S \overline{e_S} = \overline{e_S} e_S = 1$. Denote the complex conjugate of a number $c \in \mathbb{F}$ by $\overline{c}$. Then the operation of *conjugation* $u \longmapsto \overline{u}$ defined by $\overline{u} = \sum_S \overline{u_S}\, \overline{e_S}$ for every $u = \sum_S u_S e_S, u_S \in \mathbb{F}$ is an involution of the Clifford algebra $\mathbb{F}_{(n)}$. Then $\overline{vu} = \overline{u}\,\overline{v}$ for all elements $u$ and $v$ of $\mathbb{F}_{(n)}$. Because $e_j^2 = -1$, the conjugate $\overline{e_j}$ of $e_j$ is $-e_j$.

An inner product is defined on $\mathbb{F}_{(n)}$ by the formula $(u, v) = [u\overline{v}]_0 = \sum u_S \overline{v_S}$ for every $u = \sum_S u_S e_S$ and $v = \sum_S v_S e_S$ belonging to $\mathbb{F}_{(n)}$. The corresponding norm is written as $|\cdot|$.

Now we are identifying $x = (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$ with the element $\sum_{j=0}^n x_j e_j$ of $\mathbb{R}_{(n)}$, so the conjugate $\overline{x}$ of $x$ in $\mathbb{R}_{(n)}$ is $x_0 e_0 - x_1 e_1 - \cdots - x_n e_n$. A useful feature of Clifford algebras is that a nonzero vector $x \in \mathbb{R}^{n+1}$ has an inverse $x^{-1}$ in the algebra $\mathbb{R}_{(n)}$ (the *Kelvin inverse*) given by

$$x^{-1} = \frac{\overline{x}}{|x|^2} = \frac{x_0 e_0 - x_1 e_1 - \cdots - x_n e_n}{x_0^2 + x_1^2 + \cdots + x_n^2}.$$

We shall tend to write $x = (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$ as $x = x_0 e_0 + \boldsymbol{x}$ with $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

## 3.2 Banach Modules

In the course of forming functions $f(\boldsymbol{A})$ of an $n$-tuple $\boldsymbol{A} = (A_1, \dots, A_n)$ of bounded linear operators acting on a Banach space $X$, we will need to consider expressions like $\tilde{\boldsymbol{A}} = A_1 e_1 + \cdots + A_n e_n$ for the basis vectors $e_j$, $j = 1, \dots, n$ of $\mathbb{R}^n$. Then the same expression $\tilde{\boldsymbol{A}}$ can act on an element $u = \sum_S u_S e_S$ with $u_S \in X$ in two different ways: $\tilde{\boldsymbol{A}}u = \sum_{j,S}(A_j u_S)(e_j e_S)$ and $u\tilde{\boldsymbol{A}} = \sum_{j,S}(A_j u_S)(e_S e_j)$. In the first case, $\tilde{\boldsymbol{A}}(u\lambda) = (\tilde{\boldsymbol{A}}u)\lambda$ for all $\lambda \in \mathbb{F}_{(n)}$, so that $\tilde{A}$ is a right module homomorphism, and in the second, $(\lambda u)\tilde{\boldsymbol{A}} = \lambda(u\tilde{\boldsymbol{A}})$ for all $\lambda \in \mathbb{F}_{(n)}$, so that $\tilde{A}$ is a left module homomorphism. The action of the formal symbol $\tilde{\boldsymbol{A}}$ will depend on the problem at hand. The formal definitions related to this observation follow.

A Banach space $X$ with norm $\|\cdot\|$ over $\mathbb{F}$ with an operation of multiplication by elements of $\mathbb{F}_{(n)}$ turning it into a two-sided module over $\mathbb{F}_{(n)}$ is called a *Banach module* over $\mathbb{F}_{(n)}$, if there exists a $C \geq 1$ such that

$$\|xu\| \leq C|u|\,\|x\| \qquad \text{and} \qquad \|ux\| \leq C|u|\,\|x\|$$

for all $u \in \mathbb{F}_{(n)}$ and $x \in X$. The vector space of all continuous right module homomorphisms from a Banach module $X$ to a Banach module $Y$ is denoted by $\mathcal{L}_{(n)}(X, Y)$. Thus, a bounded linear map $A : X \to Y$ belongs to $\mathcal{L}_{(n)}(X, Y)$ if $(Ax)u = A(xu)$ for all $x \in X$ and $u \in \mathbb{F}_{(n)}$. Both $\mathcal{L}_{(n)}(X, Y)$ and the space $\mathcal{L}(X, Y)$ of continuous linear operators from $X$ to $Y$ are considered as Banach spaces over $\mathbb{F}$ with the uniform operator norm $\|\cdot\|$.

The algebraic tensor product $X_{(n)} = X \otimes \mathbb{F}_{(n)}$ of a Banach space $X$ over $\mathbb{F}$ with $\mathbb{F}_{(n)}$ is a Banach module. Elements of $X_{(n)}$ may be viewed as finite sums $u = \sum_S x_S \otimes e_S$ of tensor products of elements $x_S$ of $X$ with basis vectors $e_S$ of $\mathbb{F}_{(n)}$. Multiplication in $X_{(n)}$ by elements $\lambda$ of the Clifford algebra $\mathbb{F}_{(n)}$ is defined by $u\lambda = \sum_S x_S \otimes (e_S\lambda)$ and $\lambda u = \sum_S x_S \otimes (\lambda e_S)$. The tensor product notation $x_S \otimes e_S$ is written simply as $x_S e_S$. The norm on $X_{(n)}$ is taken to be $\|u\| = \left(\sum_S \|x_S\|_X^2\right)^{1/2}$.

The analogous procedure applies to a locally convex space $E$ to define the module $E_{(n)}$ with its induced locally convex topology. If $E$ and $F$ are two locally convex spaces, then the spaces $\big(\mathcal{L}(E, F)\big)_{(n)}$ and $\mathcal{L}_{(n)}(E_{(n)}, F_{(n)})$ are identified by defining the operation of $T = \sum_S T_S e_S$ on $u = \sum_S u_S e_S$ as $T(u) = \sum_{S,S'} T_S(u_{S'})e_S e_{S'}$, so making $T$ into a right module homomorphism. Because $E_{(n)}$ is a two-sided module, we can also interpret $T$ as a left module homomorphism by writing $T(u) = \sum_{S,S'} T_S(u_{S'})e_{S'} e_S$.

In the case that $E$ and $F$ are equal to a Banach space $X$, the norm of $T$ is given by $\|T\| = \left(\sum_S \|T_S\|_{\mathcal{L}(X)}^2\right)^{1/2}$. In particular, for $n$ bounded linear operators $T_1, \dots, T_n$ acting on $X$, we have

$$\Big\|\sum_{j=1}^n T_j e_j\Big\|_{\mathcal{L}_{(n)}(X_{(n)})} = \Big(\sum_{j=1}^n \|T_j\|_{\mathcal{L}(X)}^2\Big)^{1/2}.$$

Given $x \in E$ and $\xi \in F'$, the element $\langle Tx, \xi\rangle \in \mathbb{F}_{(n)}$ is defined for each $T = \sum_S T_S e_S$ belonging to $\mathcal{L}_{(n)}(E_{(n)}, F_{(n)})$ by $\langle Tx, \xi\rangle = \sum_S \langle T_S x, \xi\rangle e_S$.

## 3.3 Cauchy Formula

What is usually called *Clifford analysis* is the study of functions of finitely many real variables, which take values in a Clifford algebra, and which satisfy higher dimensional analogues of the Cauchy-Riemann equations.

It is worthwhile to spell out the direction this analogy takes. The Cauchy-Riemann equations for a complex valued function $f$ defined in an open subset of the complex plane may be represented as $\overline{\partial} f = 0$ for the operator

$$\overline{\partial} = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}, \quad z = x + iy \in \mathbb{C}.$$

The fundamental solution $E$ of the operator $\overline{\partial}$ is the solution in the sense of Schwartz distributions of the equation $\overline{\partial} E = \delta_0$ for the unit point mass $\delta_0$ at zero. Then

$$E(z) = \frac{1}{2\pi}\frac{1}{z} = \frac{1}{2\pi}\frac{\overline{z}}{|z|^2}, \quad \text{for } z = x + iy \in \mathbb{C} \setminus \{0\}.$$

A function $f$ satisfying $\overline{\partial} f = 0$ in a neighbourhood of a simple closed contour $C$ together with its interior can be represented as

$$f(z) = \frac{1}{i}\int_C E(\zeta - z) f(\zeta)\, d\zeta = \int_C E(\zeta - z)\boldsymbol{n}(\zeta) f(\zeta)\, d|\zeta|$$

at all points $z$ inside $C$. Here $\boldsymbol{n}(\zeta)$ is the outward unit normal at $\zeta \in \mathbb{C}$, $d|\zeta|$ is arclength measure so that $i\boldsymbol{n}(\zeta) d|\zeta| = d\zeta$. The higher dimensional analogue for functions taking values in a Clifford algebra is as follows.

A function $f : U \to \mathbb{F}_{(n)}$ defined in an open subset $U$ of $\mathbb{R}^{n+1}$ has a unique representation $f = \sum_S f_S e_S$ in terms of $\mathbb{F}$-valued functions $f_S$, $S \subseteq \{1, \ldots, n\}$ in the sense that $f(x) = \sum_S f_S(x) e_S$ for all $x \in U$. Then $f$ is continuous, differentiable and so on, in the normed space $\mathbb{F}_{(n)}$, if and only if for all finite subsets $S$ of $\{1, \ldots, n\}$, its scalar component functions $f_S$ have the corresponding property. Let $\partial_j$ be the operator of differentiation of a scalar function in the $j$'th coordinate in $\mathbb{R}^{n+1}$ – the coordinates of $x \in \mathbb{R}^{n+1}$ are written as $x = (x_0, x_1, \ldots, x_n)$. For a continuously differentiable function $f : \mathbb{R}^{n+1} \to \mathbb{F}_{(n)}$ with $f = \sum_S f_S e_S$, the functions $Df$ and $fD$ are defined by

$$Df = \sum_S \left( (\partial_0 f_S) e_S + \sum_{j=1}^n (\partial_j f_S) e_j e_S \right)$$

$$fD = \sum_S \left( (\partial_0 f_S) e_S + \sum_{j=1}^n (\partial_j f_S) e_S e_j \right).$$

Now suppose that $f$ is an $\mathbb{F}_{(n)}$-valued, continuously differentiable function defined in an open subset $U$ of $\mathbb{R}^{n+1}$. Then $f$ is said to be *left monogenic* in $U$ if $Df(x) = 0$ for all $x \in U$ and *right monogenic* in $U$ if $fD(x) = 0$ for all $x \in U$.

For each $x \in \mathbb{R}^{n+1}$, the function $G(\,\cdot\,, x)$ defined by

$$G(\omega, x) = \frac{1}{\Sigma_n} \frac{\overline{\omega - x}}{|\omega - x|^{n+1}} \tag{3.1}$$

for every $\omega \neq x$ is both left and right monogenic as a function of $\omega$. Here the volume $2\pi^{\frac{n+1}{2}}/\Gamma\left(\frac{n+1}{2}\right)$ of the unit $n$-sphere in $\mathbb{R}^{n+1}$ has been denoted by $\Sigma_n$ and we have used the identification of $\mathbb{R}^{n+1}$ with a subspace of $\mathbb{R}_{(n)}$ mentioned earlier.

The function $G(\,\cdot\,, x)$, $x \in \mathbb{R}^{n+1}$ plays the role in Clifford analysis of a *Cauchy kernel*. If we write $G(\omega, x) = E(\omega - x)$ for all $\omega \neq x$ in $\mathbb{R}^n$, then the $\mathbb{R}^{n+1}$-valued function

$$E(x) = \overline{x}/(\Sigma_n |x|^{n+1})$$

defined for all $x \neq 0$ belonging to $\mathbb{R}^{n+1}$ is the fundamental solution of the operator $D$, that is, $DE = \delta_0 e_0$ in the sense of Schwartz distributions. Then a function satisfying $Df = 0$ in an open set can be retrieved from a surface integral involving $E$ as follows.

Suppose that $\Omega \subset \mathbb{R}^{n+1}$ is a bounded open set with smooth boundary $\partial\Omega$ and exterior unit normal $\boldsymbol{n}(\omega)$ defined for all $\omega \in \partial\Omega$. For any left monogenic function $f$ defined in a neighbourhood $U$ of $\overline{\Omega}$, the Cauchy integral formula

$$\int_{\partial\Omega} G(\omega, x)\boldsymbol{n}(\omega) f(\omega)\, d\mu(\omega) = \begin{cases} f(x), \text{ if } & x \in \Omega; \\ \quad 0, \text{ if } & x \in U \setminus \overline{\Omega}. \end{cases} \tag{3.2}$$

is valid. Here $\mu$ is the surface measure of $\partial\Omega$. The result is proved in [19, Corollary 9.6] by appealing to by Stoke's theorem. If $g$ is right monogenic in $U$ then $\int_{\partial\Omega} g(\omega)\boldsymbol{n}(\omega) f(\omega)\, d\mu(\omega) = 0$ [19, Corollary 9.3].

*Example 3.1.* For the case $n = 1$, the Clifford algebra $\mathbb{R}_{(1)}$ is identified with $\mathbb{C}$. A continuously differentiable function $f : U \to \mathbb{R}_{(1)}$ defined in an open subset $U$ of $\mathbb{R}^2$ satisfies $Df = 0$ in $U$ if and only if it satisfies the Cauchy-Riemann equations $\overline{\partial} f = 0$ in $U$. For each $x, \omega \in \mathbb{R}^2$, $x \neq \omega$, we have

$$G(\omega, x) = \frac{1}{2\pi} \frac{1}{\omega - x}.$$

The inverse is taken in $\mathbb{C}$. As indicated above, the tangent at the point $\zeta(t)$ of the portion $\{\zeta(s) : a < s < b\}$ of a positively oriented rectifiable curve $C$ is $i$ times the normal $\boldsymbol{n}(\zeta(t))$ at $\zeta(t)$, so the equality $d\zeta = i.\boldsymbol{n}(\zeta)\, d|\zeta|$ shows that (3.2) is the Cauchy integral formula for a simple closed contour $C$ bounding a region $\Omega$.

## 3.4 Vector Valued Functions

Suppose that $(\Sigma, \mathcal{S}, \mu)$ is a measure space and $E$ is a sequentially complete locally convex space. Let $f : \Sigma \to E$ be a function for which there exist $E$-valued $\mu$-integrable $\mathcal{S}$-simple functions $s_n, n = 1, 2, \dots$ such that $s_n \to f$ $\mu$-a.e., and for every continuous seminorm $p$ on $E$, $\int_\Sigma p(s_n - s_m) d\mu \to 0$ as $n, m \to \infty$. Then the integral $\int_A f \, d\mu$ of $f$ with respect to $\mu$, over a set $A \in \mathcal{S}$, is defined to be the limit $\lim_{n\to\infty} \int_A s_n \, d\mu$. The limit is independent of the approximating sequence $s_n, n = 1, 2, \dots$; such a function $f$ is said to be *Bochner $\mu$-integrable*. It follows immediately that for a continuous linear map $T : E \to F$ between sequentially complete locally convex spaces $E$ and $F$, if $f$ is Bochner $\mu$-integrable, then $T \circ f$ is Bochner $\mu$-integrable and $T\left(\int_A f \, d\mu\right) = \int_A T \circ f d\mu$ for all $A \in \mathcal{S}$. A bounded continuous function with values in a Fréchet space or $LF$-space is Bochner integrable with respect to any finite regular Borel measure.

It is a simple matter to check from the definition of a Bochner integrable function, that for a Banach module $X$ over $\mathbb{F}_{(n)}$, the integral $\int_E f \, d\mu$ of an $X$-valued Bochner $\mu$-integrable function $f$ has the property that

$$u \int_E f \, d\mu = \int_E u f(\sigma) \, d\mu(\sigma),$$
$$\left(\int_E f \, d\mu\right) u = \int_E f(\sigma) u \, d\mu(\sigma).$$

for all $u \in \mathbb{F}_{(n)}$.

Let $X$ be a Banach space. A sequence $\{f_k\}_{k=1}^\infty$ of $X$-valued functions $f_k : \Omega \to X$ is *normally summable* in $X$ if there exists a summable sequence $\{M_k\}_{k=1}^\infty$ of nonnegative real numbers $M_k$ such that $\|f_k(\omega)\| \le M_k$, for all $\omega \in \Omega$ and all $k = 1, 2, \dots$. Thus, a normally summable sequence $\{f_k\}_{k=1}^\infty$ of $X$-valued functions on $\Omega$ is absolutely and uniformly summable on $\Omega$. In the case that $X$ is a Banach module over $\mathbb{F}_{(n)}$, we have

$$u \sum_k f_k = \sum_k u f_k \quad \text{and}$$
$$\left(\sum_k f_k\right) u = \sum_k f_k u$$

for all $u \in \mathbb{F}_{(n)}$.

The definition of monogenicity extends readily to other vector and operator valued functions. In particular, if $g$ is a left monogenic $\mathbb{F}_{(n)}$-valued function, then the tensor product $g \otimes x : \omega \longmapsto g(\omega) \otimes x$ of $g$ with an element $x$ of a Banach space $X$ is left monogenic in $X_{(n)}$. If $\{g_j \otimes x_j\}_{j=1}^\infty$ is normally summable in $X_{(n)}$ and each function $g_j$ is left monogenic, then $\sum_{j=1}^\infty g_j \otimes x_j$ is left monogenic in $X_{(n)}$.

As in the case of vector valued analytic functions [39, Section 3.10], there is a choice of possible topologies in which to take limits. The proof of the following assertion follows the case of vector valued analytic functions [39, Theorem 3.10.1].

**Proposition 3.2.** *A function is monogenic for the weak topology of a locally convex module* $E_{(n)}$ *if and only if it is monogenic for the original topology. Moreover, for a Banach space* $E$*, if* $g : U \to E_{(n)}$ *is right monogenic and* $f : U \to \mathbb{F}_{(n)}$ *is left monogenic and* $\Omega$ *is an open set with smooth oriented boundary* $\partial\Omega$ *such that* $\overline{\Omega} \subset U$*, then the function* $\omega \longmapsto g(\omega)\boldsymbol{n}(\omega)f(\omega)$*,* $\omega \in \partial\Omega$*, is Bochner* $\mu$*-integrable in* $E_{(n)}$ *and*

$$\int_{\partial\Omega} g(\omega)\boldsymbol{n}(\omega)f(\omega)\, d\mu(\omega) = 0.$$

In particular, this is valid in the case that $X$ is a Banach space and $E = \mathcal{L}(X)$ with the uniform operator norm. It follows from the principle of uniform boundedness and the Cauchy integral formula that an $\mathcal{L}(X)$-valued function is norm monogenic when it is monogenic for the weak or strong operator topologies.

## 3.5 Monogenic Expansions

The primary interest in this work is forming functions $f(\boldsymbol{A})$ of a suitable $n$-tuple $\boldsymbol{A}$ of linear operators. In the case that $\boldsymbol{A}$ satisfies an exponential growth estimate (2.3), it is enough to assume $f$ is sufficiently smooth in a neighbourhood of the joint spectrum $\gamma(\boldsymbol{A})$ defined in Definition 2.2. By analogy with the case of a single operator, in other cases it would be reasonable to expect to make sense of $f(\boldsymbol{A})$ just when $f$ is an analytic function of $n$-real variables in a neighbourhood of a compact subset $\gamma(\boldsymbol{A})$ of $\mathbb{R}^n$. In this section we see how to extend an analytic function of $n$-real variables to a monogenic function defined on a subset of $\mathbb{R}^{n+1}$. Once we have our hands on an appropriate Cauchy kernel, this will allow us to form $f(\boldsymbol{A})$ via the analogue of the Cauchy integral formula (3.2).

Suppose that $f$ is an analytic $\mathbb{F}$-valued function defined on an open neighbourhood of zero in $\mathbb{R}^n$ and the Taylor series of $f$ is given by

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{l_1=1}^{n} \cdots \sum_{l_k=1}^{n} a_{l_1 \dots l_k} x_{l_1} \cdots x_{l_k}, \tag{3.3}$$

for all $x \in \mathbb{R}^n$ in a neighbourhood of zero. The coefficients $a_{l_1 \dots l_k} \in \mathbb{F}$ are assumed to be symmetric in $l_1, \dots l_k$. Expansions about other points $p$ in $\mathbb{R}^n$ are treated by translating $x$ to $x - p$.

Then the unique monogenic extension $\tilde{f}$ of $f$ is

$$\tilde{f}(x) = \sum_{k=0}^{\infty} \left( \sum_{(l_1,\dots,l_k)} a_{l_1 \dots l_k} V^{l_1 \dots l_k}(x) \right) \tag{3.4}$$

for all $x$ belonging to some neighbourhood of zero in $\mathbb{R}^{n+1}$. Here, the sum $\sum_{(l_1,\dots,l_k)}\cdots$ is over the set $\{1 \le l_1 \le \cdots \le l_k \le n\}$, and for $(l_1,\dots,l_k) \in \{1,2,\dots,n\}^k$, the function $V^{l_1\dots l_k} : \mathbb{R}^{n+1} \to \mathbb{F}_{(n)}$ is defined as follows. For each $j = 1,\dots,n$, the monogenic extension of the function $\boldsymbol{x}_j : x \longmapsto x_j, x \in \mathbb{R}^n$ is given by $\boldsymbol{z}_j : x \longmapsto x_j e_0 - x_0 e_j, x \in \mathbb{R}^{n+1}$. Then $V^0(x) = e_0, x \in \mathbb{R}^{n+1}$ and

$$V^{l_1\dots l_k} = \frac{1}{k!}\sum_{j_1,\dots,j_k} \boldsymbol{z}_{j_1}\cdots\boldsymbol{z}_{j_k}, \tag{3.5}$$

where the sum is over all distinguishable permutations of all of $(l_1,\dots,l_k)$, and products are in the sense of pointwise multiplication in $\mathbb{F}_{(n)}$.

If $\tilde{f}$ is left monogenic in the open ball $B_R(0)$ of radius $R$ about zero in $\mathbb{R}^{n+1}$, then (3.4) converges normally in $B_R(0)$ [19, p82].

The function $V^{l_1\dots l_k}$ is both left and right monogenic. It is the unique monogenic $\mathbb{F}_{(n)}$-valued extension of the real valued function $\boldsymbol{x}_{l_1}\cdots\boldsymbol{x}_{l_k}$ defined on $\mathbb{R}^n$ to all of $\mathbb{R}^{n+1}$ called the *inner spherical monogenic polynomial* [19]. According to [19, Theorem 11.3.4, Remark 11.2.7 (ii)], the monogenic function $V^{l_1\dots l_k}$ actually takes its values in $\mathbb{R}^{n+1}$ although this is not immediately apparent from formula (3.5).

By locally extending power series like equation (3.3) to expansions like (3.4), any analytic function $f : U \to \mathbb{F}$ defined in an open subset of $\mathbb{R}^n$ is the restriction to $U$ of a function $\tilde{f} : V \to \mathbb{F}^{n+1}$ with is both left and right monogenic in an open subset $V$ of $\mathbb{R}^{n+1}$ such that $U = V \cap \mathbb{R}^n$, see [19, Theorem 14.8, Remark 14.9].

The average over symmetric products in (3.5) is reminiscent of equation (2.5) for the Weyl calculus. This observation by A. McIntosh initiated the present investigation. Indeed, if we define $V^0(\boldsymbol{A}) = Id$ and

$$V^{l_1\dots l_k}(\boldsymbol{A}) = \frac{1}{k!}\sum_{j_1,\dots,j_k} A_{j_1}\cdots A_{j_k}, \tag{3.6}$$

and if the norms $\|A_j\|$ of the operators $A_j$, $j = 1,\dots,n$ are small enough, then the operator

$$f(\boldsymbol{A}) = \sum_{k=0}^{\infty}\left(\sum_{(l_1,\dots,l_k)} a_{l_1\dots l_k} V^{l_1\dots l_k}(\boldsymbol{A})\right) \tag{3.7}$$

is what we would obtain from the Weyl calculus if the exponential growth estimate (2.3) were true. However, the expansion (3.4) is not adequate for the study of spectral properties of $\boldsymbol{A}$, even when $n = 1$, that is, for a single operator, because we are assuming that the power series (3.3) converges in some ball centred at zero in $\mathbb{R}^n$.

## 3.6 Monogenic Representation of Distributions

Equation (2.9) recovers a function or distribution from its Cauchy transform and this is used in the proof of Theorem 2.7 to detect the support of the distribution – an argument that will recur in higher dimensions in Chapter 4. The higher dimensional analogue of equation (2.9) is set out below.

Let $T \in \mathcal{L}_{(n)}(C^\infty(\mathbb{R}^n)_{(n)}, \mathbb{F}_{(n)})$ be an $\mathbb{F}_{(n)}$-valued distribution with compact support. Then $T$ is interpreted as a right module homomorphism from $C^\infty(\mathbb{R}^n)_{(n)}$ into $\mathbb{F}_{(n)}$. Thus, we may represent $T$ as a finite sum $T = \sum_S T_S e_S$ with $T_S \in \mathcal{L}(C^\infty(\mathbb{R}^n), \mathbb{F})$ so that the action of $T$ on $\phi = \sum_S \phi_S e_S$ with $\phi_S \in C^\infty(\mathbb{R}^n)$ is given by $T(\phi) = \sum_{S,S'} T_S(\phi_{S'}) e_S e_{S'}$.

The function $\tilde{T}(\omega) = T(G(\omega, \cdot))$ for all $\omega \in \mathbb{R}^{n+1} \setminus \mathbb{R}^n$ is called the *right monogenic representation* of $T$. The following result is proved in [19, Theorem 27.7] for the left monogenic representation, in which right module homomorphisms are replaced by left module homomorphisms in the obvious way.

**Theorem 3.3.** *Let $T \in \mathcal{L}_{(n)}(C^\infty(\mathbb{R}^n)_{(n)}, \mathbb{F}_{(n)})$. Then $\tilde{T}$ may be extended to a right monogenic function, still denoted by $\tilde{T}$, in $\mathbb{R}^{n+1} \setminus \operatorname{supp}(T)$.*

*Furthermore, $\lim_{|\omega|\to\infty} \tilde{T}(\omega) = 0$ and for any $\phi \in C_c^\infty(\mathbb{R}^n)_{(n)}$, we have*

$$T(\phi) = \lim_{\epsilon\to 0^+} \int_{\mathbb{R}^n} [\tilde{T}(\omega + \epsilon e_0) - \tilde{T}(\omega - \epsilon e_0)]\phi(\omega)\, d\omega.$$

If $T = T_0 e_0$, then $\tilde{T}$ is both left and right monogenic.

## 3.7 Plane Wave Decomposition

The following plane wave decomposition is given in [103, p.111]. Further proofs appear in [101] and [72]. The latter uses a general Fourier transform calculus for monogenic functions reproduced in Section 6.3 below.

The unit sphere in $\mathbb{R}^n$ centred at zero is denoted by $S^{n-1}$ and $ds$ is surface measure on $S^{n-1}$.

**Proposition 3.4.** *Let $\omega = x_0 e_0 + \boldsymbol{x}$ be an element of $\mathbb{R}^{n+1}$ with $\boldsymbol{x} \in \mathbb{R}^n$. If $x_0 > 0$,then*

$$\frac{\overline{\omega}}{\Sigma_n |\omega|^{n+1}} = \frac{(n-1)!}{2} \left(\frac{i}{2\pi}\right)^n \int_{S^{n-1}} (e_0 + is) \left(\langle \boldsymbol{x}, s\rangle - x_0 s\right)^{-n} ds.$$

*If $x_0 < 0$, then*

$$\frac{\overline{\omega}}{\Sigma_n |\omega|^{n+1}} = -\frac{(n-1)!}{2} \left(\frac{-i}{2\pi}\right)^n \int_{S^{n-1}} (e_0 + is) \left(\langle \boldsymbol{x}, s\rangle - x_0 s\right)^{-n} ds.$$

A proof involving Fourier transforms is outlined in Subsection 6.3.2.

The Kelvin inverse $(\langle \boldsymbol{x}, s\rangle - x_0 s)^{-1}$ in the Clifford algebra $\mathbb{R}_{(n)}$ is equal to

$$(\langle \boldsymbol{x}, s\rangle + x_0 s)\left(\langle \boldsymbol{x}, s\rangle^2 + x_0^2\right)^{-1}$$

so that $(\langle \boldsymbol{x}, s\rangle - x_0 s)^{-n} = (\langle \boldsymbol{x}, s\rangle + x_0 s)^n \left(\langle \boldsymbol{x}, s\rangle^2 + x_0^2\right)^{-n}$.

*Remark 3.5.* If $n$ is odd, the integral of $(\langle \boldsymbol{x}, s\rangle - x_0 e_0 s)^{-n}$ over $S^{n-1}$ is zero. As long as $(x_1, \ldots, x_n) \neq 0$, the integral of the other term $s\,(\langle \boldsymbol{x}, s\rangle - x_0 e_0 s)^{-n}$ over $S^{n-1}$ is continuous at $x_0 = 0$.

If $n$ is even, the integral of $s\,(\langle \boldsymbol{x}, s\rangle - x_0 e_0 s)^{-n}$ over $S^{n-1}$ is zero and the integral of $(\langle \boldsymbol{x}, s\rangle - x_0 e_0 s)^{-n}$ suffers a jump as $x_0$ passes through 0.

## 3.8 Approximation

We know from Section 3.5 that an $\mathbb{F}$-valued analytic function $f$ in $n$ real variables has a monogenic extension $\tilde{f}$ to an open subset of $\mathbb{R}^{n+1}$. Moreover, the Stone-Weierstrass Theorem tells us that $f$ may be approximated uniformly on compact subsets of its domain by $\mathbb{F}$-valued polynomials in $n$ real variables. Can we choose these polynomials so that their monogenic extensions approximate $\tilde{f}$ on compact subsets of the domain of $\tilde{f}$ in $\mathbb{R}^{n+1}$? The question is answered in the affirmative in Proposition 3.6 below. The result is used in Chapter 4 to show that $f(\boldsymbol{A}) \in \mathcal{L}(X)$ if $f$ is a real analytic function defined in a neighbourhood of the 'joint spectrum' $\gamma(\boldsymbol{A})$ of $\boldsymbol{A}$.

For any open subset $U$ of $\mathbb{R}^{n+1}$, let $M(U, \mathbb{F}_{(n)})$ be the collection of all $\mathbb{F}_{(n)}$-valued functions which are left monogenic in $U$. It is a right $\mathbb{F}_{(n)}$-module. The space $M(U, \mathbb{F}_{(n)})$ is given the compact-open topology (uniform convergence on every compact subset of $U$). If $K$ is a closed subset of $\mathbb{R}^n$, then $M(K, \mathbb{F}_{(n)})$ is the union of all spaces $M(U, \mathbb{F}_{(n)})$, as $U$ ranges over the open sets in $\mathbb{R}^{n+1}$ containing $K$. The space $M(K, \mathbb{F}_{(n)})$ is equipped with the inductive limit topology.

Equipped with the Cauchy-Kowalewski product [19, p. 113], the space $M(K, \mathbb{F}_{(n)})$ becomes a topological algebra and the closed linear subspace $M(K, \mathbb{F})$ of $M(K, \mathbb{F}_{(n)})$ consisting of left monogenic extensions of $\mathbb{F}$-valued functions on $K$ is a commutative topological algebra. Then the topological algebra $M(K, \mathbb{F})$ is isomorphic, via monogenic extension, to the topological algebra $H_M(K, \mathbb{F})$ of $\mathbb{F}$-valued functions analytic in an open neighbourhood of $K$ in $\mathbb{R}^n$ with pointwise multiplication. The commutativity of $M(K, \mathbb{F})$ arises from pointwise multiplication in $H_M(K, \mathbb{F})$. We write just $H_M(K)$ for $H_M(K, \mathbb{C})$.

The induced topology on $H_M(K)$ is convergence of the left (or right) monogenic extensions on compact subsets of a neighbourhood of $K$ in $\mathbb{R}^{n+1}$, rather than the usual topology of convergence on compact subsets of a neighbourhood of $K$ in $\mathbb{R}^n$. The distinction is emphasised by the subscript 'M'.

The next statement would follow from the Stone-Weierstrass approximation theorem if $H_M(K)$ had the topology of uniform convergence on $K$. The point is that $H_M(K)$ has the topology, inherited from $M(K, \mathbb{F}_{(n)})$, of uniform convergence of monogenic extensions on compact subsets of $\mathbb{R}^{n+1}$. The proof below was suggested by F. Sommen and is more direct than the proof that appears in [54, Proposition 3.2].

**Proposition 3.6.** *Let $K$ be a compact subset of $\mathbb{R}^n$. The linear space of all scalar valued polynomials is dense in $H_M(K)$.*

*Proof.* Let $U$ be an open neighbourhood of $K$ in $\mathbb{R}^n$ and suppose that $f : U \to \mathbb{C}$ is an analytic function of $n$ real variables.

Then there exists a complex valued harmonic function $(x_0, \boldsymbol{x}) \longmapsto h(x_0, \boldsymbol{x})$ defined in a neighbourhood $\tilde{U}$ of $U$ in $\mathbb{R}^{n+1}$ satisfying the two properties

(i) $h(x_0, \boldsymbol{x}) = h(-x_0, \boldsymbol{x})$
(ii) $h(0, \boldsymbol{x}) = f(\boldsymbol{x})$

for all $\boldsymbol{x} \in \mathbb{R}^n$ such that $(\pm x_0, \boldsymbol{x}) \in \tilde{U}$. Moreover it is also sufficient to consider the open sets $\tilde{O}$ such that they are symmetric with respect to the $x_0 \to -x_0$ reflection and that for fixed $x \in \tilde{O}$, the intersection with the vertical line $x + ae_0$, $a \in \mathbb{R}$ with $\tilde{O}$ is convex. In that case there exists a unique harmonic function $h$ satisfying condition (i) and defined in a maximal open domain $\tilde{O}$ of the above type which extends $f$.

By the uniform approximation theorem for harmonic functions due to J.L. Walsh [30, p. 8], for every $\epsilon > 0$ and every symmetric compact subset $\tilde{K}$ of $\tilde{U}$, there exists an entire harmonic function $H(x_0, \boldsymbol{x})$ such $|h(x_0, \boldsymbol{x}) - H(x_0, \boldsymbol{x})| < \epsilon$ for every $(x_0, \boldsymbol{x}) \in \tilde{K}$. The same holds true for $\frac{1}{2}(H(x_0, \boldsymbol{x}) + H(-x_0, \boldsymbol{x}))$, so we can assume $H$ is symmetric in $x_0$. One can also replace the $C_0$-norm on $\tilde{K}$ by any $C_{00}$-norm with derivatives because both topologies are equivalent for harmonic functions (by the mean value theorem).

From Clifford analysis [19, Proposition 14.4, p. 110] we also know that the function $f(\boldsymbol{x})$ extends to a unique monogenic function $f(x_0, \boldsymbol{x})$ for $x_0 e_0 + \boldsymbol{x} \in \tilde{U}$. Now $h(x_0, \boldsymbol{x}) = 1/2(f(x_0, \boldsymbol{x}) + f(-x_0, \boldsymbol{x}))$ is symmetric harmonic and also an extension of $f(\boldsymbol{x})$. Hence it must be the unique symmetric harmonic extension. The function $g(x_0, \boldsymbol{x}) = 1/2(f(x_0, \boldsymbol{x}) - f(-x_0, \boldsymbol{x}))$ on the other hand is antisymmetric monogenic (so it vanishes for $x_0 = 0$) and $f(x_0, \boldsymbol{x}) = h(x_0, \boldsymbol{x}) + g(x_0, \boldsymbol{x})$ is monogenic, so that $(D_{x_0} + D_{\boldsymbol{x}})f = 0$. Hence $D_{\boldsymbol{x}} h(x_0, \boldsymbol{x}) + D_{x_0} g(x_0, \boldsymbol{x}) = 0$, and the representation

$$g(x_0, \boldsymbol{x}) = -\int_0^{x_0} D_{\boldsymbol{x}} h(t, \boldsymbol{x}) dt$$

is valid for the antisymmetric monogenic part $g$ of $f$. In particular, the domain of the above type in which $f$ is monogenic is the same as the domain $\tilde{U}$ in which $h$ is defined.

Hence $g(x_0, \boldsymbol{x})$ is fully expressed in terms of $h$ and its derivatives and so if $H$ is a Walsh approximation for $h$ for the $C^1$-norm say and if $G(x_0, \boldsymbol{x})$ is the function $-\int_0^{x_0} D_{\boldsymbol{x}} H(t, \boldsymbol{x}) dt$ then $F(x_0, \boldsymbol{x}) = H(x_0, \boldsymbol{x}) + G(x_0, \boldsymbol{x})$ is entire monogenic and because in the process of producing $g$ (resp. $G$) there are only first order derivatives of $h$ (resp. $H$) needed, $F(x_0, \boldsymbol{x})$ is a good Walsh approximation for $f(x_0, \boldsymbol{x})$ for the $C_0$-norm on the compact $\tilde{K}$. This proves the claim. □

# 4

# Functional Calculus for Noncommuting Operators

Given a single bounded selfadjoint operator $A$ acting on a Hilbert space, a function $f(A)$ of $A$ may be formed by the Riesz-Dunford functional calculus

$$f(A) = \frac{1}{2\pi i} \int_C (\zeta I - A)^{-1} f(\zeta)\, d\zeta \tag{4.1}$$

as a contour integral about the spectrum $\sigma(A)$ of $A$ if $f$ is analytic in a neighbourhood of $\sigma(A)$, or, if $f \in L^\infty(P_A)$ with respect to the spectral measure $P_A$ of $A$, then $f(A)$ may be represented by the Spectral Theorem for selfadjoint operators as

$$f(A) = \int_{\sigma(A)} f(\lambda)\, P_A(d\lambda).$$

As is well known, both procedures give the same operator $f(A)$ in the case that $f$ is analytic in a neighbourhood of $\sigma(A)$.

As mentioned in the beginning of Chapter 2, we are looking for a higher dimensional analogue of the Riesz-Dunford functional calculus for a system $\boldsymbol{A}$ of $n$ bounded linear operators acting on a Banach space. In the noncommuting case when $\boldsymbol{A}$ is of Paley-Wiener type $s$, the Weyl calculus $\mathcal{W}_{\boldsymbol{A}}$ considered in Chapter 1 plays the role of a spectral measure for a single selfadjoint operator, although $\mathcal{W}_{\boldsymbol{A}}$ is generally an operator valued distribution of order greater than one.

In this chapter, the Weyl calculus $\mathcal{W}_{\boldsymbol{A}}$ (when it exists) is used to define the Cauchy kernel $G_\omega(\boldsymbol{A})$ for a higher dimensional analogue of the Riesz-Dunford functional calculus. As expected, the two calculi agree and the Cauchy kernel $G_\omega(\boldsymbol{A})$ is defined by alternative means when $\boldsymbol{A}$ fails growth estimates for exponentials.

## 4.1 The Weyl Calculus and the Cauchy Kernel

Let $\boldsymbol{A} = (A_1, \dots, A_n)$ be an $n$-tuple of bounded linear operators acting on a Banach space $X$. Suppose that $s > 0$ and that $\boldsymbol{A}$ is of Paley-Wiener

B. Jefferies: LNM 1843, pp. 39–66, 2004.

type $s$, as in Definition 2.2. Then according to Theorem 2.7, $\boldsymbol{A}$ is of Paley-Wiener type $(s, r)$ for some $r > 0$ and by Theorem 2.4, the distribution $\mathcal{W}_{\boldsymbol{A}} = (2\pi)^{-n} \left[e^{i\langle \xi, \boldsymbol{A}\rangle}\right]^{\wedge}$ has compact support.

Suppose that $U$ is an open neighbourhood of the support of $\mathcal{W}_{\boldsymbol{A}}$ in $\mathbb{R}^n$. As is usual in distribution theory, if $\tilde{f}$ is any extension of $f \in C^\infty(U)$ to a smooth function defined on all of $\mathbb{R}^n$, then the operator $f(\boldsymbol{A}) = \mathcal{W}_{\boldsymbol{A}}(f)$ is set equal to $\mathcal{W}_{\boldsymbol{A}}(\tilde{f})$. Then the distribution $\mathcal{W}_{\boldsymbol{A}} : f \longmapsto f(\boldsymbol{A})$ over $C^\infty(U)$ is also called the *Weyl functional calculus* for $\boldsymbol{A}$.

Identifying the set $\mathbb{R}^n$ with the subspace $\{x \in \mathbb{R}^{n+1} : x_0 = 0\}$ of $\mathbb{R}^{n+1}$, the definition of $\mathcal{W}_{\boldsymbol{A}}$ is extended to apply to Clifford algebra valued functions defined in an open neighbourhood $V$ in $\mathbb{R}^{n+1}$ of the support of $\mathcal{W}_{\boldsymbol{A}}$ in $\mathbb{R}^n$.

First, the mapping $\mathcal{W}_{\boldsymbol{A}} : C^\infty(V) \to \mathcal{L}(X)$ is defined by applying $\mathcal{W}_{\boldsymbol{A}}$ to the restriction of functions $f \in C^\infty(V)$ to the open subset $V \cap \mathbb{R}^n$ of $\mathbb{R}^n$.

The algebraic tensor product $\mathcal{W}_{\boldsymbol{A}} \otimes I_{(n)} : C^\infty(V)_{(n)} \to \mathcal{L}(X)_{(n)}$ of $\mathcal{W}_{\boldsymbol{A}}$ with the identity operator $I_{(n)}$ on $\mathbb{F}_{(n)}$ is also denoted just by $\mathcal{W}_{\boldsymbol{A}}$. Here $C^\infty(V)_{(n)}$ is the locally convex module obtained by tensoring the locally convex space $C^\infty(V)$ with $\mathbb{F}_{(n)}$, as mentioned in Section 3.2. So, if $f = \sum_S f_S e_S$ is an element of $C^\infty(V)_{(n)}$, then according to the prescription just given $\mathcal{W}_{\boldsymbol{A}}(f) = \sum_S \mathcal{W}_{\boldsymbol{A}}(f_S|_{V\cap\mathbb{R}^n})e_S$. The map $\mathcal{W}_{\boldsymbol{A}} : C^\infty(V)_{(n)} \to \mathcal{L}(X)_{(n)}$ is a two-sided module homomorphism. The symbols $\mathcal{W}_{\boldsymbol{A}}(f)$ and $f(\boldsymbol{A})$ are used interchangeably.

The support $\operatorname{supp}(\mathcal{W}_{\boldsymbol{A}})$ of the distribution $\mathcal{W}_{\boldsymbol{A}}$, which is independent of the particular meaning attached to it above, is a compact subset of $\mathbb{R}^n$. Let $U$ be an open neighbourhood of $\operatorname{supp}(\mathcal{W}_{\boldsymbol{A}})$ in $\mathbb{R}^n$ and suppose that the function $f : U \to \mathbb{C}$ is (real) analytic, that is, $f$ has a uniformly power series expansion in a neighbourhood of every point belonging to $U$. Let $\tilde{f}$ be a left (and right) monogenic extension of $f$ to an open neighbourhood of $U$ in $\mathbb{R}^{n+1}$. Then according to the definition of $\tilde{f}(\boldsymbol{A})$, the equality

$$\tilde{f}(\boldsymbol{A}) = f(\boldsymbol{A}) \otimes I_{(n)} \equiv f(\boldsymbol{A})e_0 \tag{4.2}$$

is valid. Because $f$ is assumed to be *analytic* in $U$, it certainly belongs to $C^\infty(U)$ and so $f(\boldsymbol{A}) = \mathcal{W}_{\boldsymbol{A}}(f)$ makes sense. Because of the equality (4.2), the element $(\tilde{f})(\boldsymbol{A})$ of $\mathcal{L}(X)_{(n)}$ is written simply as the bounded linear operator $f(\boldsymbol{A})$.

Suppose that $f$ is an analytic $\mathbb{F}$-valued function defined on an open neighbourhood of zero in $\mathbb{R}^n$ and the Taylor series of $f$ is given by equation (3.3). The unique monogenic extension $\tilde{f}$ of $f$ is given by equation (3.4). Then, it follows easily from equation (2.5) that the equality

$$f(\boldsymbol{A}) = \sum_{k=0}^{\infty} \left( \sum_{(l_1,\dots,l_k)} a_{l_1\dots l_k} V^{l_1\dots l_k}(\boldsymbol{A}) \right) \tag{4.3}$$

holds if (3.3) converges in a suitable neighbourhood of $\operatorname{supp}(\mathcal{W}_{\boldsymbol{A}})$. The operators $V^{l_1\dots l_k}(\boldsymbol{A})$ are given by formula (3.6).

In the case in which the monogenic expansion of a function about a point does not converge over all of supp $(\mathcal{W}_{\boldsymbol{A}})$, the *Cauchy integral formula* is useful. Of course, this is the central idea of the Riesz-Dunford functional calculus for a single operator. Moreover, when $\boldsymbol{A}$ is an $n$-tuple of operators acting on a Banach space and the Weyl functional calculus for $\boldsymbol{A}$ is not defined – there is no exponential bound – the Cauchy integral formula can be used to define functions of the $n$-tuple $\boldsymbol{A}$, see Section 4.3 below. Let $\nabla$ denote the vector differential operator $(\partial/\partial x_1, \dots, \partial/\partial x_n)$.

For any $y \in \mathbb{R}^{n+1}$ not belonging to supp $(\mathcal{W}_{\boldsymbol{A}})$, there exists an open neighbourhood $U_y$ of supp $(\mathcal{W}_{\boldsymbol{A}})$ in $\mathbb{R}^{n+1}$, not containing $y$, such that the $\mathbb{R}^{n+1}$-valued function

$$x \longmapsto G_y(x) = \frac{1}{\Sigma_n} \frac{\overline{y-x}}{|y-x|^{n+1}}, \text{ for all } x \in U_y,$$

belongs to $C^\infty(U_y)_{(n)}$. Then $\mathcal{W}_{\boldsymbol{A}}(G_y) = (G_y)(\boldsymbol{A})$ may be viewed as an element of $\mathcal{L}(X)_{(n)}$.

*Example 4.1.* Let $n = 3$ and consider the simplest non-commuting example of the Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

viewed as linear transformations acting on $H = \mathbb{C}^2$. Set $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$. A calculation [7, Theorem 4.1] shows that for all $f \in C^\infty(\mathbb{R}^3)$, the matrix $\mathcal{W}_{\boldsymbol{\sigma}}(f)$ is given by

$$\mathcal{W}_{\boldsymbol{\sigma}}(f) = I \int_{S^2} (f + \boldsymbol{n} \cdot \nabla f)\, d\mu + \int_{S^2} \boldsymbol{\sigma} \cdot \nabla f \, d\mu.$$

Here $d\mu$ is the unit surface measure on the sphere $S^2$ of radius one centred at zero in $\mathbb{R}^3$ and $\boldsymbol{n}(x)$ is the outward unit normal at $x \in S^2$. Thus, supp $(\mathcal{W}_{\boldsymbol{\sigma}}) = S^2$.

For all $\omega \in \mathbb{R}^4$ such that $\omega \notin S^2 \subset \mathbb{R}^3$, $\mathcal{W}_{\boldsymbol{\sigma}}(G_\omega) \in \mathcal{L}(\mathbb{C}^2)_{(3)}$ is given by

$$\mathcal{W}_{\boldsymbol{\sigma}}(G_\omega) = I \int_{S^2} (G_\omega + \boldsymbol{n} \cdot \nabla G_\omega)\, d\mu + \int_{S^2} \boldsymbol{\sigma} \cdot \nabla G_\omega d\mu.$$

Let $v_1, v_2$ be the standard basis vectors of $\mathbb{C}^2$. For each $x_0 \in \mathbb{R}$, the function $(\boldsymbol{x}, t) \longmapsto \mathcal{W}_{tJ}(G_{(\boldsymbol{x}+x_0 e_0)})v_j$ is the solution of the Weyl equation

$$\partial_t u_t + \boldsymbol{\sigma} \cdot \nabla u_t = 0, \qquad t > 0,$$

with initial datum $u_0(x) = -v_j \otimes G_0(\boldsymbol{x}+x_0 e_0) = \Sigma_3^{-1} v_j \otimes (x_0 e_0 - \boldsymbol{x})/|\boldsymbol{x}+x_0 e_0|^4$ for all $\boldsymbol{x} \in \mathbb{R}^3, \boldsymbol{x} + x_0 e_0 \neq 0$. The function $\omega \longmapsto \mathcal{W}_{t\boldsymbol{\sigma}}(G_\omega)v_j$ is left and right monogenic off the sphere of radius $t > 0$.

The following statements are formulated in a context more general than that of the Weyl functional calculus because they will also be used in Chapter 7.

Suppose that $X$ is a Banach space over the field $\mathbb{F}$ and $T : C^\infty(\mathbb{R}^n) \to \mathcal{L}(X)$ is a distribution with compact support $K$. We use the same symbol $T$ to denote the map which sends the element $f = \sum_S f_S e_S$ of $C^\infty(K)_{(n)}$ to the element $\sum_S T(f_S)e_S$ of $\mathcal{L}(X)_{(n)}$, rather than the more descriptive notation $T \otimes I_{(n)}$. In particular, $T(f) \in \mathcal{L}(X)_{(n)}$ is defined for all $f \in M(K, \mathbb{F}_{(n)})$. Note that $T$ is both a left and right module homomorphism from $C^\infty(K)_{(n)}$ to $\mathcal{L}(X)_{(n)}$.

**Proposition 4.2.** *Let $U$ be an open subset of $\mathbb{R}^{n+1}$ containing $K = \mathrm{supp}(T)$. Suppose that $y \longmapsto F_y$ is a continuous map from $U \setminus K$ into $C^\infty(K)_{(n)}$. If for each open set $V$ with $\overline{V} \subset U \setminus K$, there exists a neighbourhood $N_V$ of $K$, such that for each $x \in N_V$, the $\mathbb{F}_{(n)}$-valued function $y \longmapsto F_y(x)$ is left monogenic in $V$, then $y \longmapsto T(F_y)$ is left monogenic in $U \setminus K$.*

*Proof.* By Cauchy's theorem for monogenic functions [19, Theorem 9.6], for all intervals $I$ contained in $U \setminus K$, $\int_{\partial I} \boldsymbol{n}(\omega)F_\omega(x)\, d\mu(\omega) = 0$ for each $x$ belonging to some neighbourhood of $K$. The function $y \longmapsto F_y, y \in U \setminus K$ is continuous, and so Bochner integrable in $C^\infty(K)_{(n)}$ on all boundaries $\partial I$ of intervals $I$ contained in $U \setminus K$. Moreover, the function $\int_{\partial I} n(\omega)F_\omega\, d\mu(\omega)$ belongs to $C^\infty(K)_{(n)}$ and vanishes in a neighbourhood of the support $K$ of $T$.

The distribution $T : C^\infty(K)_{(n)} \to \mathcal{L}(X)_{(n)}$ is a continuous linear map and a left module homomorphism, so as observed in Section 3.4, the equalities

$$\begin{aligned}\int_{\partial I} \boldsymbol{n}(\omega)T\left(F_\omega\right)\, d\mu(\omega) &= \int_{\partial I} T\left(\boldsymbol{n}(\omega)F_\omega\right)\, d\mu(\omega)\\ &= T\left(\int_{\partial I} \boldsymbol{n}(\omega)F_\omega\, d\mu(\omega)\right) = 0\end{aligned}$$

hold. By Morera's theorem for monogenic functions [19, Theorem 10.4], $y \longmapsto T(F_y)$ is left monogenic in $U \setminus K$. □

The same result holds for right monogenic functions.

**Corollary 4.3.** *The $\mathcal{L}(X)_{(n)}$-valued function $y \longmapsto \mathcal{W}_{\boldsymbol{A}}(G_y)$ is left and right monogenic in $\mathbb{R}^{n+1} \setminus \mathrm{supp}\,(\mathcal{W}_{\boldsymbol{A}})$.*

**Theorem 4.4.** *Let $T$ be an $\mathcal{L}(X)$-valued distribution with compact support. Let $\Omega$ be a bounded open neighbourhood of $\mathrm{supp}(T)$ in $\mathbb{R}^{n+1}$ with smooth boundary $\partial\Omega$ and exterior unit normal $\boldsymbol{n}(\omega)$ defined for all $\omega \in \partial\Omega$. Let $\mu$ be the surface measure of $\Omega$.*

*Suppose that $f$ is left monogenic and $g$ is right monogenic in a neighbourhood of the closure $\overline{\Omega} = \Omega \cup \partial\Omega$ of $\Omega$. Then*

$$T(f) = \int_{\partial\Omega} T(G_\omega)\boldsymbol{n}(\omega)f(\omega)\, d\mu(\omega),$$

$$T(g) = \int_{\partial\Omega} g(\omega)\boldsymbol{n}(\omega)T(G_\omega)\,d\mu(\omega).$$

*Proof.* We consider only the case where $f$ is left monogenic. The case where $g$ is right monogenic is similar. The space $C^\infty(\Omega)_{(n)}$ of smooth $\mathbb{F}_{(n)}$-valued functions defined on $\Omega$ is a separable Fréchet space with the topology of uniform convergence of functions, and their derivatives, on compact subsets of $\Omega$. The continuous function $\omega \longmapsto G_\omega \boldsymbol{n}(\omega)f(\omega), \omega \in \partial\Omega$ takes its values in $C^\infty(\Omega)_{(n)}$ and satisfies $\int_{\partial\Omega} p(G_\omega \boldsymbol{n}(\omega))|f(\omega)|\,d\mu(\omega) < \infty$ for each continuous seminorm $p$ on $C^\infty(\Omega)_{(n)}$, that is, it is Bochner integrable in $C^\infty(\Omega)_{(n)}$.

By the Cauchy integral theorem (3.2), the equality

$$f(x) = \int_{\partial\Omega} G_\omega(x)\boldsymbol{n}(\omega)f(\omega)\,d\mu(\omega)$$

holds for all $x$ belonging to the open set $\Omega$. Combining this equation with the observation made in the Section 3.4, and the fact that the distribution $T$ defines a continuous linear map and right *and* left module homomorphism (denoted by the same symbol) from $C^\infty(\Omega)_{(n)}$ into the space $\mathcal{L}(X)_{(n)}$ with the uniform operator norm, it follows that the function $\omega \longmapsto T(G_\omega)\boldsymbol{n}(\omega)f(\omega), \omega \in \partial\Omega$ is Bochner integrable in the space $\mathcal{L}(X)_{(n)}$, with the uniform norm, and the equalities

$$\begin{aligned} T\left(\int_{\partial\Omega} G_\omega \boldsymbol{n}(\omega)f(\omega)\,d\mu(\omega)\right) &= \int_{\partial\Omega} T\left(G_\omega \boldsymbol{n}(\omega)f(\omega)\right)\,d\mu(\omega) \\ &= \int_{\partial\Omega} T(G_\omega)\boldsymbol{n}(\omega)f(\omega)\,d\mu(\omega) \end{aligned}$$

hold. In the second equality, the fact that $T$ is a right module homomorphism has been used. The stated equality $T(f) = \int_{\partial\Omega} T(G_\omega)\boldsymbol{n}(\omega)f(\omega)\,d\mu(\omega)$ therefore holds. □

**Corollary 4.5.** *Let $\Omega$ be a bounded open neighbourhood of* $\operatorname{supp}(\mathcal{W}_{\boldsymbol{A}})$ *in* $\mathbb{R}^{n+1}$ *with smooth boundary $\partial\Omega$ and exterior unit normal $\boldsymbol{n}(\omega)$ defined for all $\omega \in \partial\Omega$. Let $\mu$ be the surface measure of $\Omega$.*

*Suppose that $f$ is left monogenic and $g$ is right monogenic in a neighbourhood of the closure $\overline{\Omega} = \Omega \cup \partial\Omega$ of $\Omega$. Then*

$$f(\boldsymbol{A}) := \mathcal{W}_{\boldsymbol{A}}(f) = \int_{\partial\Omega} \mathcal{W}_{\boldsymbol{A}}(G_\omega)\boldsymbol{n}(\omega)f(\omega)\,d\mu(\omega), \tag{4.4}$$

$$g(\boldsymbol{A}) := \mathcal{W}_{\boldsymbol{A}}(g) = \int_{\partial\Omega} g(\omega)\boldsymbol{n}(\omega)\mathcal{W}_{\boldsymbol{A}}(G_\omega)\,d\mu(\omega). \tag{4.5}$$

**Corollary 4.6.** *Suppose that $f : U \to \mathbb{F}$ is a real analytic function in a neighbourhood $U$ of* $\operatorname{supp}(\mathcal{W}_{\boldsymbol{A}})$ *in $\mathbb{R}^n$. Let $V$ be an open subset of $\mathbb{R}^{n+1}$ such that $V \cap \mathbb{R}^n = U$ and $\tilde{f} : V \to \mathbb{F}_{(n)}$ is a left and right monogenic function such that $\tilde{f}(x) = f(x)$ for all $x \in U$.*

*Let $\Omega$ be a bounded open neighbourhood of* $\mathrm{supp}\,(\mathcal{W}_{\boldsymbol{A}})$ *in* $\mathbb{R}^{n+1}$ *such that* $\overline{\Omega} \subset V$*. Furthermore, suppose that $\Omega$ has a smooth oriented boundary $\partial\Omega$ and exterior unit normal $\boldsymbol{n}(\omega)$ defined for all $\omega \in \partial\Omega$. Let $\mu$ be the surface measure of $\Omega$. Then*

$$f(\boldsymbol{A}) = \int_{\partial\Omega} \mathcal{W}_{\boldsymbol{A}}(G_\omega)\boldsymbol{n}(\omega)\tilde{f}(\omega)\, d\mu(\omega) \tag{4.6}$$

$$= \int_{\partial\Omega} \tilde{f}(\omega)\boldsymbol{n}(\omega)\mathcal{W}_{\boldsymbol{A}}(G_\omega)\, d\mu(\omega). \tag{4.7}$$

We mention here that the extension of these results to $H$-valued functions for a Hilbert space $H$ is straightforward. First, if $f = \sum_{j=1}^n f_j h_j$ for monogenic functions $f_j$ and vectors $h_j \in H$, then $T(f) = \sum_j T(f_j)h_j$ and the above equality holds. In the limit, both sides of the equation converge because $C^\infty(\mathrm{supp}(T)) \otimes H$ is dense in $C^\infty(\mathrm{supp}(T); H)$.

## 4.2 The Joint Spectrum and the Cauchy Kernel

Let $\boldsymbol{A}$ be an $n$-tuple of bounded linear operators acting on a Banach space $X$. Suppose that $\boldsymbol{A}$ is of Paley-Wiener type $s$ for some $s \geq 0$.

Comparison of equations (4.6) and (4.7) with equation (4.1) for the Riesz-Dunford functional calculus shows that the $\mathcal{L}(X)_{(n)}$-valued function $\omega \longmapsto \mathcal{W}_{\boldsymbol{A}}(G_\omega)$ defined for all $\omega \in \mathbb{R}^{n+1}$ not belonging to the joint spectrum $\gamma(\boldsymbol{A})$ of $\boldsymbol{A}$ is the higher-dimensional analogue of the resolvent family $\lambda \longmapsto (\lambda I - A)^{-1}$ of a single bounded linear operator $A$, that is, the Cauchy kernel for the functional calculus given by formula (4.6).

The spectrum $\sigma(T)$ of a single operator $T$ is the set of 'singularities' of the resolvent function $\lambda \longmapsto (\lambda I - T)^{-1}$. It is not immediately obvious that the joint spectrum $\gamma(\boldsymbol{A})$ of $\boldsymbol{A}$ is actually the set of singularities of the Cauchy kernel $\omega \longmapsto G_\omega(\boldsymbol{A})$.

Before looking at this point more closely, we first see that there is another way to define the Cauchy kernel $G_\omega(\boldsymbol{A})$ for all $\omega \in \mathbb{R}^{n+1}$ outside a sufficiently large ball, in such a way that it need not be assumed that $\boldsymbol{A}$ is of Paley-Wiener type $s$ for some $s \geq 0$. A clue is provided by the Neumann series

$$(\lambda I - T)^{-1} = \frac{1}{\lambda}\sum_{k=0}^{\infty} \frac{T^k}{\lambda^k}, \quad \text{for } |\lambda| > \|T\|$$

for the resolvent of $T$.

For each $\omega \in \mathbb{R}^{n+1}$ such that $\omega \neq 0$, let

$$G_\omega(x) = \sum_{k=0}^{\infty} \left( \sum_{(l_1,\dots,l_k)} W_{l_1\dots l_k}(\omega) V^{l_1\dots l_k}(x) \right) \tag{4.8}$$

be the monogenic power series expansion of $G_\omega$ in the region $|x| < |\omega|$ [19, 11.4 pp77-81]. Here $W_{l_1 \dots l_k}(\omega)$ is given for each $\omega \in \mathbb{R}^{n+1}, \omega \neq 0$ by $(-1)^k \partial_{\omega_{l_1}} \cdots \partial_{\omega_{l_k}} G_\omega(0)$ and $V^{l_1 \dots l_k}$ is given by equation (3.5).

It follows from equation (2.5) for the Weyl calculus that

$$(G_\omega)(\boldsymbol{A}) = \sum_{k=0}^{\infty} \left( \sum_{(l_1, \dots, l_k)} W_{l_1 \dots l_k}(\omega) V^{l_1 \dots l_k}(\boldsymbol{A}) \right) \tag{4.9}$$

for all $\omega \in \mathbb{R}^{n+1}$ such that $|\omega| > (1 + \sqrt{2}) \| \sum_{j=1}^{n} A_j e_j \|$. Formula (4.9) is adopted as a *definition* of the Cauchy kernel in [68, Definition 3.11]. The sum converges in $\mathcal{L}(X)_{(n)}$ because of the following result.

**Lemma 4.7.** *Let* $\boldsymbol{A}$ *be an* $n$*-tuple of bounded linear operators acting on a Banach space* $X$*. Let* $R > (1 + \sqrt{2}) \| \sum_{j=1}^{n} A_j e_j \|$*. Then the sum*

$$\sum_{k=0}^{\infty} \sum_{(l_1, \dots, l_k)} |W_{l_1 \dots l_k}(\omega)| \, \|V^{l_1 \dots l_k}(\boldsymbol{A})\|$$

*converges uniformly for all* $\omega \in \mathbb{R}^{n+1}$ *such that* $|\omega| \geq R$*.*

*Proof.* The norm $\|V^{l_1 \dots l_k}(\boldsymbol{A})\|$ of $V^{l_1 \dots l_k}(\boldsymbol{A})$ is bounded by

$$\frac{1}{k!} \sum_{j_1, \dots, j_k} \|A_{j_1}\| \cdots \|A_{j_k}\|,$$

where the sum is over all distinguishable permutations of $(l_1, \dots, l_k)$. Suppose that for each $j = 1, \dots, n$, the index $j$ appears exactly $k_j$ times in the $k$-tuple $(l_1, \dots, l_k)$. Then $k = k_1 + \cdots + k_n$ and there are $\frac{k!}{k_1! \cdots k_n!}$ distinguishable permutations of $(l_1, \dots, l_k)$. Thus, $\|V^{l_1 \dots l_k}(\boldsymbol{A})\| \leq \frac{1}{k_1! \cdots k_n!} \|A_1\|^{k_1} \cdots \|A_n\|^{k_n}$. It suffices to show that for each $R > (1 + \sqrt{2}) \| \sum_{j=1}^{n} A_j e_j \|$, the sum

$$\sum_{k_1, \dots, k_n = 0}^{\infty} \frac{1}{k_1! \cdots k_n!} |\partial_{\omega_1}^{k_1} \cdots \partial_{\omega_n}^{k_n} G_\omega(0)| \, \|A_1\|^{k_1} \cdots \|A_n\|^{k_n}$$

converges uniformly for all $|\omega| \geq R, \omega \in \mathbb{R}^{n+1}$. However, this follows from the normal convergence of the multiple power series

$$\frac{1}{|y - x|^{n-1}} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \langle x, \nabla_y \rangle^k \frac{1}{|y|^{n-1}}$$

for $|x| < (\sqrt{2} - 1)|y|$ [19, p. 83] and the equality $G_\omega(x) = \frac{1}{\Sigma_n} \overline{D_x} \frac{1}{|\omega - x|^{n-1}}$, valid for all $\omega \neq x$. $\square$

We know from Corollary 5.3 that the function defined by formula (4.9) for all $|\omega| > (1+\sqrt{2})\|\sum_{j=1}^{n} A_j e_j\|$ is actually the restriction of an $\mathcal{L}(X)_{(n)}$-valued function monogenic in $\mathbb{R}^{n+1} \setminus \operatorname{supp}(\mathcal{W}_{\boldsymbol{A}})$. The question remains as to whether there is a larger open set on which this function has a monogenic extension.

Let us say that the *monogenic spectrum* $\tilde{\gamma}(\boldsymbol{A})$ of the $n$-tuple $\boldsymbol{A}$ of bounded liner operators is the complement of the largest open set $U \subset \mathbb{R}^{n+1}$ in which the function $\omega \longmapsto (G_\omega)(\boldsymbol{A})$ defined by the series (4.9) is the restriction of a monogenic function with domain $U$. Note that this definition does not require $\boldsymbol{A}$ to be of Paley-Wiener type $s$ for some $s \geq 0$.

The proof of the following theorem uses a higher-dimensional version of the argument used in the proof of Theorem 2.7 (cf. equation (2.9)).

**Theorem 4.8.** *Let $s \geq 0$ and let $\boldsymbol{A}$ be an $n$-tuple of bounded linear operators of Paley-Wiener type $s$. Then $\tilde{\gamma}(\boldsymbol{A}) = \operatorname{supp}(\mathcal{W}_{\boldsymbol{A}}) = \gamma(\boldsymbol{A})$.*

*Proof.* By Definition 2.5, the joint spectrum $\gamma(\boldsymbol{A})$ is the support of the operator valued distribution $\mathcal{W}_{\boldsymbol{A}}$.

We have established in Corollary 4.3 that $\tilde{\gamma}(\boldsymbol{A}) \subseteq \operatorname{supp}(\mathcal{W}_{\boldsymbol{A}})$. Let $x \in \tilde{\gamma}(\boldsymbol{A})^c$, let $U \subset \tilde{\gamma}(\boldsymbol{A})^c$ be an open neighbourhood of $x$ in $\mathbb{R}^n$ and suppose that $\phi$ is a smooth function with compact support in $U$.

Let $g \in X$, $h \in X'$. A comparison with [19, Definition 27.6] shows that the $\mathbb{F}_{(n)}$-valued monogenic function $\omega \longmapsto \langle (G_\omega)(\boldsymbol{A})g, h\rangle$, $\omega \in \mathbb{R}^{n+1} \setminus \operatorname{supp}(\mathcal{W}_{\boldsymbol{A}})$, is actually the monogenic representation of the distribution $\langle \mathcal{W}_{\boldsymbol{A}} g, h\rangle : f \longmapsto \langle \mathcal{W}_{\boldsymbol{A}}(f)g, h\rangle$, for all smooth $f$ defined in an open neighbourhood of $\operatorname{supp}(\mathcal{W}_{\boldsymbol{A}})$. Then $\langle \mathcal{W}_{\boldsymbol{A}} g, h\rangle(G_\omega) = \langle (G_\omega)(\boldsymbol{A})g, h\rangle$ and by [19, Theorem 27.7],

$$\langle \mathcal{W}_{\boldsymbol{A}} g, h\rangle(\phi) = \lim_{y_0 \to 0^+} \int_U [\langle (G_{\boldsymbol{y}+y_0 e_0})(\boldsymbol{A})g, h\rangle - \langle (G_{\boldsymbol{y}-y_0 e_0})(\boldsymbol{A})g, h\rangle]\phi(\boldsymbol{y})\, d\boldsymbol{y}.$$

Because $\omega \longmapsto (G_\omega)(\boldsymbol{A})$ is monogenic (hence continuous) for all $\omega$ in $U$, the limit is zero, that is, $\langle \mathcal{W}_{\boldsymbol{A}} g, h\rangle(\phi) = 0$ for all $g \in X$, $h \in X'$ and all smooth $\phi$ supported by $U$. Hence $x \in \operatorname{supp}(\mathcal{W}_{\boldsymbol{A}})^c$, as was to be proved. □

Henceforth, the tilde is omitted and $\gamma(\boldsymbol{A})$ denotes both the joint spectrum and the monogenic spectrum of $\boldsymbol{A}$. The term *joint spectrum* is used for both concepts.

*Remark 4.9.* The significance of the Cauchy kernel $\omega \longmapsto (G_\omega)(\boldsymbol{A})$ is that it is the monogenic representation or 'Cauchy transform' of the distribution $\mathcal{W}_{\boldsymbol{A}}$ off $\operatorname{supp}(\mathcal{W}_{\boldsymbol{A}})$ – the distribution $\mathcal{W}_{\boldsymbol{A}}$ represents the 'boundary values' on $\mathbb{R}^n$ of the monogenic function $\omega \longmapsto (G_\omega)(\boldsymbol{A})$. In Section 5.3, the support of the distribution $\mathcal{W}_A$ for a pair $\boldsymbol{A}$ of hermitian matrices is determined by examining discontinuities of the Cauchy kernel $(G_\omega)(\boldsymbol{A})$.

*Example 4.10.* Let $A = (\sigma_3, \sigma_1)$. It follows by applying [7, Theorem 2.9 (a)] to Example 4.1 that the support of $\mathcal{W}_{\boldsymbol{A}}$ is the closed unit disk $\mathbf{D} \subset \mathbb{R}^2$ centred at zero, so $\gamma(\boldsymbol{A}) = \mathbf{D}$. An explicit calculation is given in [32, Example 2]. The *Clifford spectrum* $\sigma(\boldsymbol{A})$ of [68, Definition 3.1] is $\sigma(\boldsymbol{A}) = \{(0,0)\}$.

In order to define the Cauchy kernel $G_\omega(\boldsymbol{A}) := \mathcal{W}_{\boldsymbol{A}}(G_\omega)$ so that equations (4.6) and (4.7) hold, the assumption that $\boldsymbol{A}$ is of Paley-Wiener type $s$ is used to construct the distribution $\mathcal{W}_{\boldsymbol{A}}$. It is a simple matter to write down an example of a commuting pair $\boldsymbol{A} = (A_1, A_2)$ of bounded linear operators acting on $l^2(\mathbb{N})$ for which the bound

$$\|e^{i(\xi_1 A_1 + \xi_2 A_2)}\| \leq C(1 + |\xi|)^s, \qquad \text{for all } \xi \in \mathbb{R}^2$$

fails, but $\sigma(\xi_1 A_1 + \xi_2 A_2) \subset \mathbb{R}$ for all $\xi \in \mathbb{R}^2$.

*Example 4.11.* For each $n = 1, 2, \ldots$, let $U_n$ be the $n \times n$ matrix such that $(U_n)_{j,j+1} = 1$ for all $j = 1, \ldots, n-1$, and $(U_n)_{k,j} = 0$ otherwise. Let $I_n$ be the $n \times n$ identity matrix. Let $A_1 : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be the direct sum of $(-1)^n I_n$ for $n = 2, 3, \ldots$ and let $A_2 : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be the direct sum of $U_n$ for $n = 2, 3, \ldots$. There exists no $C > 0$ and no $s > 0$ for which the commuting pair $\boldsymbol{A} = (A_1, A_2)$ of operators on $l^2(\mathbb{N})$ satisfies the bound (2.2). Nevertheless, the spectrum $\sigma(\xi_1 A_1 + \xi_2 A_2)$ of the operator $\xi_1 A_1 + \xi_2 A_2$ is real for all $\xi \in \mathbb{R}^2$ because it is real on each common invariant subspace.

Let $x \longmapsto G_\omega(x), x = (x_0, x_1, x_2) \in \mathbb{R}^3$ be the Cauchy kernel on $\mathbb{R}^3$ for $\omega \neq x$. The natural definition of $G_\omega(\boldsymbol{A})$ suggested by the matrix functional calculus is obtained by taking the direct sum of

$$\sum_{k=0}^{n} \frac{1}{k!} \partial_2^k G_\omega\big((0, (-1)^{n+1}, 0)\big)(U_{n+1})^k$$

for $n = 1, 2, \ldots$ for each $\omega \in \mathbb{R}^3 \setminus (\{0\} \times \{-1, 1\} \times \{0\})$.

The example above suggests adopting the power series expansion (4.9) as the definition of $G_\omega(\boldsymbol{A})$ for general $n$-tuples $\boldsymbol{A}$.

Although (4.9) makes sense for any $n$-tuple of bounded operators, the problem remains of enlarging the domain of definition of the monogenic function defined by (4.9) to be as large as possible in a unique way, such as in the case when the natural domain is connected. The following assertion allows us to define the monogenic functional calculus.

**Theorem 4.12.** *Let $\boldsymbol{A} = (A_1, \ldots, A_n)$ be an $n$-tuple of bounded operators acting on a Banach space $X$. Suppose that*

$$\sigma(\langle \boldsymbol{A}, \xi \rangle) \subseteq \mathbb{R}, \quad \text{for all } \xi \in \mathbb{R}^n. \tag{4.10}$$

*Then the $\mathcal{L}_{(n)}(X_{(n)})$-valued function $\omega \longmapsto G_\omega(\boldsymbol{A})$ defined by formula (4.9) is the restriction to the region $|\omega| > (1 + \sqrt{2})\|\sum_{j=1}^n A_j e_j\|$ of a function which is two-sided monogenic on the set $\mathbb{R}^{n+1} \setminus \mathbb{R}^n$.*

To prove the theorem, we appeal to a series of key lemmas, in which we suppose that the $n$-tuple $\boldsymbol{A}$ satisfies the conditions of the theorem. Let $ds$ be the surface measure of the unit $(n-1)$-sphere $S^{n-1}$ in $\mathbb{R}^n$.

We view the $n$-tuple $\boldsymbol{A} = (A_1, \ldots, A_n)$ of bounded linear operators acting on a Banach space $X$ as an element $\boldsymbol{A} = \sum_{j=1}^n A_j e_j$ of the Banach module $\mathcal{L}_{(n)}(X_{(n)})$. In the following statement, $\mathbb{R}^n$ is identified, as usual, with the set of all $x \in \mathbb{R}^{n+1}$ for which $x = (0, x_1, \ldots, x_n)$, and in turn, $\mathbb{R}^{n+1}$ is identified with a subspace of the Clifford algebra $\mathbb{R}_{(n)}$.

**Lemma 4.13.** *Let* $\boldsymbol{y} = \sum_{j=1}^n y_j e_j$ *and* $y_0 \neq 0$. *Then for each* $s \in S^{n-1}$, $\langle \boldsymbol{y}I - A, s\rangle - y_0 sI$ *is invertible in* $\mathcal{L}_{(n)}(X_{(n)})$.

*Proof.* The inverse of $\langle \boldsymbol{y}I - A, s\rangle - y_0 sI$ is given by

$$(\langle \boldsymbol{y}I - A, s\rangle - y_0 sI)^{-1} = (\langle \boldsymbol{y}I - A, s\rangle + y_0 sI)(\langle \boldsymbol{y}I - A, s\rangle^2 + y_0^2 I)^{-1}.$$

We see that this makes sense as follows.

Let $s \in S^{n-1}$, $y_0 \in \mathbb{R}, y_0 \neq 0$ and $\boldsymbol{y} \in \mathbb{R}^n$. Let $f : \mathbb{R} \to (0, \infty)$ be defined by $f(x) = (\langle \boldsymbol{y}, s\rangle - x)^2 + y_0^2$ for all $x \in \mathbb{R}$. Then applying the Spectral Mapping Theorem to the bounded operator $\langle A, s\rangle$,

$$\sigma(\langle \boldsymbol{y}I - A, s\rangle^2 + y_0^2 I) = f[\sigma(\langle A, s\rangle)] \subset f(\mathbb{R}) \subset (0, \infty).$$

Hence, the operator $\langle \boldsymbol{y}I - A, s\rangle^2 + y_0^2 I$ is invertible for $y_0 \neq 0$. Moreover, it commutes with $\langle \boldsymbol{y}I - A, s\rangle \pm y_0 sI$, since all three operators involve only multiples of the identity $I$ and the single operator $\langle \boldsymbol{y}I - A, s\rangle$. By direct calculation,

$$(\langle \boldsymbol{y}I - A, s\rangle + y_0 sI)(\langle \boldsymbol{y}I - A, s\rangle - y_0 sI) = (\langle \boldsymbol{y}I - A, s\rangle^2 + y_0^2 I),$$

because under Clifford multiplication, $s^2 = -1$ for all $s \in S^{n-1}$. □

Thus, for each $s \in S^{n-1}$, the inverse $\big(\langle \boldsymbol{y}I - A, s\rangle - y_0 s\big)^{-1}$ exists as an element of $\mathcal{L}_{(n)}(X_{(n)})$ and so

$$\big(\langle \boldsymbol{y}I - A, s\rangle - y_0 s\big)^{-n} = \Big(\big(\langle \boldsymbol{y}I - A, s\rangle - y_0 s\big)^{-1}\Big)^n$$

is an element of $\mathcal{L}_{(n)}(X_{(n)})$ too.

The following lemma completes the proof of Theorem 4.12.

**Lemma 4.14.** *For each real number* $y_0 \neq 0$, *and each* $\boldsymbol{y} \in \mathbb{R}^n$, *the* $\mathcal{L}_{(n)}(X_{(n)})$-*valued function* $s \longmapsto (e_0 + is)\big(\langle \boldsymbol{y}I - \boldsymbol{A}, s\rangle - y_0 s\big)^{-n}$ *defined for* $s \in S^{n-1}$ *is Bochner* $\mu$-*integrable on* $S^{n-1}$, *and the function*

$$\boldsymbol{y} + y_0 e_0 \longmapsto \int_{S^{n-1}} (e_0 + is)\big(\langle \boldsymbol{y}I - \boldsymbol{A}, s\rangle - y_0 s\big)^{-n}\, ds$$

*is left and right monogenic on* $\mathbb{R}^{n+1} \setminus \mathbb{R}^n$.

*Furthermore, if* $y_0 > 0$ *and* $|\boldsymbol{y}| > (1 + \sqrt{2})\|A\|$, *then*

$$G_{\boldsymbol{y}+y_0 e_0}(\boldsymbol{A}) = \frac{(n-1)!}{2}\left(\frac{i}{2\pi}\right)^n \int_{S^{n-1}} (e_0 + is)\left(\langle \boldsymbol{y}I - \boldsymbol{A}, s\rangle - y_0 s\right)^{-n} ds.$$

*If* $y_0 < 0$ *and* $|y| > (1+\sqrt{2})\|A\|$*, then*

$$G_{\boldsymbol{y}+y_0e_0}(\boldsymbol{A}) = -\frac{(n-1)!}{2}\left(\frac{-i}{2\pi}\right)^n \int_{S^{n-1}} (e_0+is)\left(\langle \boldsymbol{y}I - \boldsymbol{A}, s\rangle - y_0 s\right)^{-n} ds.$$

*Here the left-hand sides of the equations are defined by (4.9).*

*Proof.* The function $s \longmapsto (e_0+is)\big(\langle \boldsymbol{y}I - \boldsymbol{A}, s\rangle - y_0 s\big)^{-n}$ is continuous on $S^{n-1}$, and so Bochner $\mu$-integrable. The monogenicity of the function follows by differentiation under the integral sign.

We shall establish the equality

$$\begin{aligned} G_{\boldsymbol{y}+y_0e_0}(tA) = {} & \frac{(n-1)!}{2}\left(\frac{i}{2\pi}\right)^n \\ & \times \int_{S^{n-1}} (e_0+is)\left(\langle \boldsymbol{y}I - t\boldsymbol{A}, s\rangle - y_0 s\right)^{-n} ds \end{aligned} \tag{4.11}$$

for all $0 \le t \le 1$, $y_0 > 0$ and $|\boldsymbol{y}| > (1+\sqrt{2})\|A\|$. The case $y_0 < 0$ is similar.

For $t = 0$, the left hand side of equation (4.11) is equal to

$$G_{\boldsymbol{y}+y_0e_0}(0) = \frac{1}{\Sigma_n}\frac{y_0e_0 - \boldsymbol{y}}{|\boldsymbol{y}+y_0e_0|^{n+1}}I.$$

An appeal to Proposition 3.4 shows that the right hand side equals $G_{\boldsymbol{y}+y_0e_0}(0)$ at $t = 0$. By differentiation under the integral sign, for $y_0 > 0$, the right hand side of (4.11) is a solution of the initial value problem

$$\partial_t u(\boldsymbol{y},t) + \langle \boldsymbol{A}, \nabla_{\boldsymbol{y}}\rangle u(\boldsymbol{y},t) = 0, \quad u(\boldsymbol{y},0) = G_{\boldsymbol{y}+y_0e_0}(0)I, \tag{4.12}$$

in the Banach module $\mathcal{L}_{(n)}(X_{(n)})$. Then

$$u(\boldsymbol{y},t) = G_{\boldsymbol{y}+y_0e_0}(0)I - \int_0^t \langle \boldsymbol{A}, \nabla_{\boldsymbol{y}}\rangle u(\boldsymbol{y},s)\,ds. \tag{4.13}$$

In the case that $|y_0| > |\boldsymbol{y}| + \|A\|$, a power series expansion shows that the right hand side of (4.11) is analytic in $t$ for all $|t| \le 1$.

Let $y \in \mathbb{R}^n$ satisfy $|\boldsymbol{y}| > (1+\sqrt{2})\|A\|$ and set $\omega = y_0e_0 + \boldsymbol{y}$. In the notation used in formula (4.9), the series

$$\sum_{k=0}^{\infty} t^k \left(\sum_{(l_1,\dots,l_k)} W_{l_1\dots l_k}(\omega) V^{l_1\dots l_k}(\boldsymbol{A})\right) \tag{4.14}$$

represents $e^{-\langle \boldsymbol{A},\nabla_{\boldsymbol{y}}\rangle t} G_{\boldsymbol{y}+y_0e_0}(0)$ and iterating equation (4.13), we find that

$$u(\boldsymbol{y},t) = e^{-\langle \boldsymbol{A},\nabla_{\boldsymbol{y}}\rangle t} G_{\boldsymbol{y}+y_0e_0}(0),$$

that is, the solution of equation (4.12) with the initial condition $u(y,0) = G_{\boldsymbol{y}+y_0e_0}(0)I$ has the series representation (4.14).

In the region $\Gamma \subset \mathbb{R}^{n+1}$ where $|\boldsymbol{y}| > (1+\sqrt{2})\|A\|$ and $|y_0| > |\boldsymbol{y}| + \|A\|$, the right-hand side of equation (4.11) and the expression (4.14) are analytic in $t$ for $0 \le |t| \le 1$, so equality follows for all $0 \le |t| \le 1$ in the region $\Gamma$ by the uniqueness of the Taylor series expansion. Both sides of equation (4.11) are monogenic in their domains, so by unique continuation, the equality (4.11) must be true for all $0 \le |t| \le 1$ and all $y_0 > 0$ and $|\boldsymbol{y}| > (1+\sqrt{2})\|A\|$. □

The maximal monogenic extension of the function $\omega \longmapsto G_\omega(\boldsymbol{A})$ is denoted by the same symbol, that is, let $\Omega$ be the union of all open sets containing the open set $\Gamma = \{|\omega| > (1+\sqrt{2})\|A\|\}$ on which is defined a two-sided monogenic function equal to $\omega \longmapsto G_\omega(\boldsymbol{A})$ on $\Gamma$. Then a two-sided monogenic function equal to $\omega \longmapsto G_\omega(\boldsymbol{A})$ on $\Gamma$ is defined on all of $\Omega$. It is unique because $\Omega$ is connected and contains $\Gamma$ – a compact subset of $\mathbb{R}^n$ cannot disconnect $\mathbb{R}^{n+1}$.

**Definition 4.15.** Let $\boldsymbol{A}$ be an $n$-tuple of bounded operators acting on a nonzero Banach space $X$. Suppose that condition (4.10) holds.

The complement $\gamma(\boldsymbol{A})$ of the domain $\Omega$ of $\omega \longmapsto G_\omega(\boldsymbol{A})$ is called the *joint spectrum* of $\boldsymbol{A}$.

By Theorem 4.8, this definition of $\gamma(\boldsymbol{A})$ is consistent with Definition 2.5 in the case that $\boldsymbol{A}$ is of Paley-Wiener type $s$. Moreover, the maximal monogenic extension of the function $\omega \longmapsto G_\omega(\boldsymbol{A})$ defined by formula (4.9) is equal to $\mathcal{W}_{\boldsymbol{A}}(G_\omega)$ for all $\omega \in \mathbb{R}^{n+1} \setminus \gamma(\boldsymbol{A})$.

According to Lemma 4.14, the joint spectrum $\gamma(\boldsymbol{A})$ is contained in the closed ball of radius $(1+\sqrt{2})\big(\sum_{j=1}^n \|A_j\|^2\big)^{1/2}$ about zero in $\mathbb{R}^n$, so it is compact by the Heine-Borel theorem. The following result was mentioned in [68, Lemma 3.13], but with a different definition of the spectrum.

**Theorem 4.16.** *Let $\boldsymbol{A}$ be an $n$-tuple of bounded operators acting on a nonzero Banach space $X$ such that condition* (4.10) *holds. Then $\gamma(\boldsymbol{A})$ is a nonempty compact subset of $\mathbb{R}^n$.*

*Proof.* It only remains to show that $\gamma(\boldsymbol{A})$ is nonempty. The norms of the coefficients $W_{l_1 \dots l_k}(\omega)$ of the expansion (4.9) decrease monotonically with $|\omega|$, so the function $\omega \longmapsto G_\omega(\boldsymbol{A})$ is bounded and monogenic outside a ball. If $\gamma(\boldsymbol{A}) = \emptyset$, then for each $x \in X$ and $\xi \in X'$, the function $\omega \longmapsto \langle G_\omega(\boldsymbol{A})x, \xi\rangle$ is two-sided monogenic inside any ball, and so it is bounded and two-sided monogenic everywhere. By Liouville's Theorem [19, 12.3.11], it is a constant function. However, by the Hahn-Banach theorem we can obtain $x \in X$ and $\xi \in X'$ and $\omega_1, \omega_2 \in \mathbb{R}^{n+1}$ such that $\langle G_{\omega_1}(\boldsymbol{A})x, \xi\rangle \neq \langle G_{\omega_2}(\boldsymbol{A})x, \xi\rangle$, a contradiction. □

The following result shows that singularities in the Cauchy kernel $\omega \longmapsto G_\omega(\boldsymbol{A})$ may be detected just from discontinuities.

**Proposition 4.17.** *Let $\boldsymbol{A}$ be an $n$-tuple of bounded operators acting on a Banach space $X$ such that $\sigma\big(\langle \boldsymbol{A}, \xi\rangle\big) \subset \mathbb{R}$ for all $\xi \in \mathbb{R}^n$. Then $\gamma(\boldsymbol{A}) \subset \mathbb{R}^n$*

*is the complement in* $\mathbb{R}^{n+1}$ *of the set of all points* $\omega \in \mathbb{R}^{n+1}$ *at which the function*

$$(\boldsymbol{y} + y_0 e_0) \longmapsto \operatorname{sgn}(y_0)^{n-1} \int_{S^{n-1}} (e_0 + is) \left( \langle \boldsymbol{y}I - \boldsymbol{A}, s\rangle - y_0 s \right)^{-n} ds$$

*is continuous in a neighbourhood of* $\omega$*.*

*Proof.* Suppose that the function is continuous in a neighbourhood $U \subset \mathbb{R}^{n+1}$ of $\omega \in \mathbb{R}^{n+1}$. By Lemma 2.5 and Painlevé's Theorem [19, Theorem 10.6, p. 64], the function

$$\boldsymbol{y} + y_0 e_0 \longmapsto \operatorname{sgn}(y_0)^{n-1} \int_{S^{n-1}} (e_0 + is) \left( \langle \boldsymbol{y}I - \boldsymbol{A}, s\rangle - y_0 s \right)^{-n} x, \xi \rangle ds$$

is two-sided monogenic for each $x \in X$ and $\xi \in X'$. The statement now follows from the equality

$$\begin{aligned} &\int_{S^{n-1}} (e_0 + is) \left( \langle \boldsymbol{y}I - \boldsymbol{A}, s\rangle - y_0 s \right)^{-n} x, \xi \rangle ds \\ &\qquad = \left\langle \left( \int_{S^{n-1}} (e_0 + is) \left( \langle \boldsymbol{y}I - \boldsymbol{A}, s\rangle - y_0 s \right)^{-n} ds \right) x, \xi \right\rangle \end{aligned} \tag{4.15}$$

and the observation that an $\mathcal{L}_{(n)}(X_{(n)})$-valued function is left or right monogenic for the norm topology if and only if it is left or right monogenic for the weak operator topology. □

As a consequence of Proposition 4.17, the set $\gamma(\boldsymbol{A})$ remains the same, if, in the definition of $\gamma(\boldsymbol{A})$, the term "two-sided monogenic" is replaced by either 'left monogenic' or 'right monogenic'.

We have established the following plane wave representation for the Cauchy kernel $G_\omega(\boldsymbol{A})$, $\omega \in \mathbb{R}^{n+1} \setminus \gamma(\boldsymbol{A})$, of an $n$-tuple $\boldsymbol{A}$ of bounded linear operators on $X$ with the property that $\sigma(\langle \boldsymbol{A}, \xi\rangle) \subset \mathbb{R}$ for all $\xi \in \mathbb{R}^n$.

In the case $\omega \in \mathbb{R}^{n+1}$ and $\omega = \boldsymbol{y} + y_0 e_0$ with $\boldsymbol{y} \in \mathbb{R}^n$ and $y_0$ a nonzero real number, we have

$$\begin{aligned} G_\omega(\boldsymbol{A}) = \frac{(n-1)!}{2} \left( \frac{i}{2\pi} \right)^n &\operatorname{sgn}(y_0)^{n-1} \\ &\times \int_{S^{n-1}} (e_0 + is) \left( \langle \boldsymbol{y}I - \boldsymbol{A}, s\rangle - y_0 s \right)^{-n} ds. \end{aligned} \tag{4.16}$$

If $\omega \in \mathbb{R}^n \setminus \gamma(\boldsymbol{A})$, then $G_\omega(\boldsymbol{A})$ is equal to the limits

$$\begin{aligned} &\frac{(n-1)!}{2} \left( \frac{i}{2\pi} \right)^n \lim_{y_0 \to 0+} \int_{S^{n-1}} (e_0 + is) \left( \langle \omega I - \boldsymbol{A}, s\rangle - y_0 s \right)^{-n} ds. \\ &= -\frac{(n-1)!}{2} \left( \frac{-i}{2\pi} \right)^n \lim_{y_0 \to 0-} \int_{S^{n-1}} (e_0 + is) \left( \langle \omega I - \boldsymbol{A}, s\rangle - y_0 s \right)^{-n} ds. \end{aligned}$$

## 4.3 The Monogenic Functional Calculus

Let $\boldsymbol{A}$ be an $n$-tuple of bounded operators acting on a Banach space $X$ such that $\sigma\big(\langle \boldsymbol{A}, \xi\rangle\big) \subset \mathbb{R}$ for all $\xi \in \mathbb{R}^n$. Let $\Omega$ be a bounded open neighbourhood of $\gamma(\boldsymbol{A})$ in $\mathbb{R}^{n+1}$ with smooth boundary $\partial\Omega$ and exterior unit normal $\boldsymbol{n}(\omega)$ defined for all $\omega \in \partial\Omega$. Let $\mu$ be the surface measure of $\partial\Omega$. Suppose that $f$ is left monogenic in a neighbourhood of the closure $\overline{\Omega} = \Omega \cup \partial\Omega$ of $\Omega$. Then we *define*

$$f(\boldsymbol{A}) = \int_{\partial\Omega} G_\omega(\boldsymbol{A})\boldsymbol{n}(\omega) f(\omega)\, d\mu(\omega) \tag{4.17}$$

In Section 2.8, the space of all left monogenic functions defined in a neighbourhood of $\gamma(\boldsymbol{A})$ in $\mathbb{R}^{n+1}$ was denoted by $M(\gamma(\boldsymbol{A}), \mathbb{F}_{(n)})$. It is a right module because $D(f\lambda) = (Df)\lambda = 0$ for all $f \in M(\gamma(\boldsymbol{A}), \mathbb{F}_{(n)})$ and $\lambda \in \mathbb{F}_{(n)}$.

**Definition 4.18.** The linear mapping and right-module homomorphism $f \longmapsto f(\boldsymbol{A})$ defined for all $f \in M(\gamma(\boldsymbol{A}), \mathbb{F}_{(n)})$ by equation (4.17) is called *the monogenic functional calculus.*

Our interest is mainly in the application of the monogenic functional calculus to the closed linear subspace $H_M(\gamma(\boldsymbol{A}))$ of $M(\gamma(\boldsymbol{A}), \mathbb{F}_{(n)})$ so that operators $f(\boldsymbol{A}) \in \mathcal{L}(X)$ are formed for any analytic functions $f$ of $n$ real variables defined in a neighbourhood of the compact subset $\gamma(\boldsymbol{A})$ of $\mathbb{R}^n$. That the monogenic functional calculus is actually well-defined is considered below.

Because $\omega \longmapsto G_\omega(\boldsymbol{A})$ is right monogenic, the element $f(\boldsymbol{A})$ of $\mathcal{L}_{(n)}(X_{(n)})$ is defined independently of the set $\Omega$ with the properties mentioned above. This may be seen by taking $x \in X$ and $\xi \in X'$. Then by the properties of Bochner integrals mentioned in Section 3.4, the equality

$$\langle f(\boldsymbol{A})x, \xi\rangle = \int_{\partial\Omega} \langle G_\omega(\boldsymbol{A})x, \xi\rangle \boldsymbol{n}(\omega) f(\omega)\, d\mu(\omega)$$

holds and the $\mathbb{F}_{(n)}$-valued function $\omega \longmapsto \langle G_\omega(\boldsymbol{A})x, \xi\rangle$ is two-sided monogenic off $\gamma(\boldsymbol{A})$. The analogue for monogenic functions of Cauchy's Theorem [19, Corollary 9.3] mentioned in Section 2.4, ensures that the open set $\Omega$ can be changed as long as the boundary of the set $\Omega$ does not cross $\gamma(\boldsymbol{A})$. Because this is true for all $x \in X$ and $\xi \in X'$, the Hahn-Banach theorem ensures that the value of the integral (4.17) does not change when $\Omega$ is so modified.

Moreover, a similar argument shows that if $f : V \to \mathbb{C}$ is a function analytic in a neighbourhood $V$ of $\gamma(\boldsymbol{A})$ in $\mathbb{R}^n$ and $\tilde{f}_1 : U_1 \to \mathbb{C}_{(n)}$ and $\tilde{f}_2 : U_2 \to \mathbb{C}_{(n)}$ are left monogenic functions defined in neigbourhoods $U_1, U_2$ of $\gamma(\boldsymbol{A})$ in $\mathbb{R}^{n+1}$ such that $\tilde{f}_1(x) = f(x)$ for all $x \in U_1 \cap V$ and $\tilde{f}_2(x) = f(x)$ for all $x \in U_2 \cap V$, then $\tilde{f}_1(\boldsymbol{A}) = \tilde{f}_2(\boldsymbol{A})$. It therefore makes sense to define $f(\boldsymbol{A}) = \tilde{f}_1(\boldsymbol{A})$. In Theorem 4.22 (iv), we show that $f(\boldsymbol{A})$ actually belongs to the closed linear subspace $\mathcal{L}(X)$ of the Banach module $\mathcal{L}_{(n)}(X_{(n)})$.

The operation $f \longmapsto f(\boldsymbol{A})$ defined on $H_M(\gamma(\boldsymbol{A}))$ extends to analytic functions with values in a finite dimensional vector space $V$ over $\mathbb{C}$ by application to the component functions of $f$. In particular, if $f : U \to \mathbb{C}_{(n)}$ is an analytic function defined on a neighbourhood $U$ of $\tilde{\gamma}(\boldsymbol{A})$ in $\mathbb{R}^n$ and $f = \sum_S f_S e_S$ for the scalar component functions $f_S$ defined for $S \subset \{1, \dots, n\}$, then $f(\boldsymbol{A}) = \sum_S f_S(\boldsymbol{A}) e_S$. If the term 'analytic' is replaced by '$C^\infty$', then this property is shared with the Weyl functional calculus, see [53].

The following proposition follows directly from equation (4.4). In particular, the earlier notation is consistent with the present notation.

**Proposition 4.19.** *Let $\boldsymbol{A}$ be an $n$-tuple of bounded linear operators acting on a Banach space $X$. Suppose $s \geq 0$ and that $\boldsymbol{A}$ is of Paley-Wiener type $s$.*

*Then for each $f \in M(\gamma(\boldsymbol{A}), \mathbb{F}_{(n)})$, the element $f(\boldsymbol{A})$ of $\mathcal{L}_{(n)}(X_{(n)})$ defined by formula* (4.13) *is equal to $\mathcal{W}_{\boldsymbol{A}}(f)$.*

In this special case, it follows immediately that $f(\boldsymbol{A}) \in \mathcal{L}(X)$ for $f \in H_M(\gamma(\boldsymbol{A}))$.

The following statement follows from formula (4.13) and the estimate

$$\|f(\boldsymbol{A})\| \leq 2^{n/2} \mu(\partial \Omega) \sup_{\omega \in \partial \Omega} \|G_\omega(\boldsymbol{A})\| \sup_{\omega \in \partial \Omega} |f(\omega)|.$$

**Proposition 4.20.** *Let $\boldsymbol{A}$ be an $n$-tuple of bounded operators acting on a Banach space $X$. Suppose that $\sigma\big(\langle \boldsymbol{A}, \xi \rangle\big) \subset \mathbb{R}$ for all $\xi \in \mathbb{R}^n$. Then the mapping $f \longmapsto f(\boldsymbol{A})$ is continuous from $M(\gamma(\boldsymbol{A}), \mathbb{F}_{(n)})$ to $\mathcal{L}_{(n)}(X_{(n)})$.*

**Proposition 4.21.** *Let $\boldsymbol{A}$ be an $n$-tuple of bounded operators acting on a Banach space $X$ such that $\sigma\big(\langle \boldsymbol{A}, \xi \rangle\big) \subset \mathbb{R}$ for all $\xi \in \mathbb{R}^n$. Suppose that $f : U \to \mathbb{C}_{(n)}$ is left monogenic in an open neighbourhood $U$ in $\mathbb{R}^{n+1}$ of the closed unit ball of radius $(1 + \sqrt{2})\big(\sum_{j=1}^n \|A_j\|^2\big)^{1/2}$ about zero.*

*If the Taylor series of $f$ restricted to $U \cap \mathbb{R}^n$ is given by*

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{l_1=1}^{n} \cdots \sum_{l_k=1}^{n} a_{l_1 \dots l_k} x_{l_1} \cdots x_{l_k}, \tag{4.18}$$

*with $a_{l_1 \dots l_k} \in \mathbb{C}_{(n)}$, then*

$$f(\boldsymbol{A}) = \sum_{k=0}^{\infty} \left( \sum_{(l_1, \dots, l_k)} V^{l_1 \dots l_k}(\boldsymbol{A}) \right) a_{l_1 \dots l_k}. \tag{4.19}$$

*Proof.* Let $\Omega$ be an open set in $\mathbb{R}^{n+1}$ with smooth boundary $\partial \Omega$ such that $\Omega \subset B_r(0) \subset U$ and $\Omega$ contains the closed unit ball of radius $(1 + \sqrt{2})\big(\sum_{j=1}^n \|A_j\|^2\big)^{1/2}$ in $\mathbb{R}^{n+1}$. The series

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{l_1=1}^{n} \cdots \sum_{l_k=1}^{n} V^{l_1 \dots l_k}(x) a_{l_1 \dots l_k},$$

representing the left monogenic extension of (4.18), converges normally in $\Omega$ [19, 11.5.2], so

$$f(\boldsymbol{A}) = \sum_{k=0}^{\infty} \sum_{(l_1,\dots,l_k)} \left( \int_{\partial\Omega} G_\omega(\boldsymbol{A})\boldsymbol{n}(\omega) V^{l_1\dots l_k}(\omega)\, d\mu(\omega) \right) a_{l_1\dots l_k}.$$

It follows from the expansion (4.9) and formula (12.2) of [19, p. 86] that

$$\int_{\partial\Omega} G_\omega(\boldsymbol{A})\boldsymbol{n}(\omega) V^{l_1\dots l_k}(\omega)\, d\mu(\omega) = V^{l_1\dots l_k}(\boldsymbol{A})$$

for all $l_1,\dots,l_k = 1,\dots,n$ and $k = 1,2,\dots$. The equality (4.19) follows. □

**Theorem 4.22.** *Let $\boldsymbol{A}$ be an $n$-tuple of bounded operators acting on a Banach space $X$ such that $\sigma(\langle \boldsymbol{A}, \xi\rangle) \subset \mathbb{R}$ for all $\xi \in \mathbb{R}^n$.*

*(i) Suppose that $k_1,\dots,k_n = 0,1,2,\dots,k = k_1 + \cdots + k_n$ and $f(x) = x_1^{k_1}\cdots x_n^{k_n}$ for all $x = (x_1,\dots,x_n) \in \mathbb{R}^n$. Then*

$$f(\boldsymbol{A}) = \frac{k_1!\cdots k_n!}{k!} \sum_\pi A_{\pi(1)}\cdots A_{\pi(k)},$$

*where the sum is taken over every map $\pi$ of the set $\{1,\dots,k\}$ into $\{1,\dots,n\}$ which assumes the value $j$ exactly $k_j$ times, for each $j = 1,\dots,n$.*

*(ii) Let $p : \mathbb{C} \to \mathbb{C}$ be a polynomial and $\zeta \in \mathbb{C}^n$. Set $f(z) = p(\langle z, \zeta\rangle)$, for all $z \in \mathbb{C}^n$. Then $f(\boldsymbol{A}) = p(\langle \boldsymbol{A}, \zeta\rangle)$.*

*(iii) Let $\Omega$ be an open set in $\mathbb{R}^{n+1}$ containing $\gamma(\boldsymbol{A})$ with a smooth boundary $\partial\Omega$. Then for all $\omega \notin \overline{\Omega}$,*

$$G_\omega(\boldsymbol{A}) = \int_{\partial\Omega} G_\zeta(\boldsymbol{A})\boldsymbol{n}(\zeta) G_\omega(\zeta)\, d\mu(\zeta),$$

*where the left-hand side is defined by equation (4.16).*

*(iv) Suppose that $U$ is an open neighbourhood of $\gamma(\boldsymbol{A})$ in $\mathbb{R}^n$ and $f : U \to \mathbb{C}$ is an analytic function. Then $f(\boldsymbol{A}) \in \mathcal{L}(X)$.*

*Proof.* (i) Let $\Gamma$ be the set of all $k$-tuples $(l_1,\dots,l_k)$ in $\{1,\dots,n\}^k$ for which $j$ appears exactly $k_j$ times, for each $j = 1,\dots,n$. Let $a_\gamma = k_1!\cdots k_n!$ for all $\gamma \in \Gamma$ and $a_\gamma = 0$ for all $\gamma \in \{1,\dots,n\}^k \setminus \Gamma$. Then $x_1^{k_1}\cdots x_n^{k_n} = \frac{1}{k!}\sum_{l_1=1}^n \cdots \sum_{l_k=1}^n a_{(l_1,\dots,l_k)} x_{l_1}\cdots x_{l_k}$, so by the proposition above,

$$f(\boldsymbol{A}) = \sum_{(l_1,\dots,l_k)} a_{(l_1\dots l_k)} V^{l_1\dots l_k}(\boldsymbol{A}) = \frac{k_1!\cdots k_n!}{k!} \sum_\pi A_{\pi(1)}\cdots A_{\pi(k)}.$$

Statement (ii) follows from (i) because only symmetric products of the $\langle A_j\rangle$ appear in both $f(\boldsymbol{A})$ and $p(\langle \boldsymbol{A}, \zeta\rangle)$.

(iii) On appealing to equations (4.8) and (4.9), the equality follows directly from Proposition 4.21 for all $\omega \notin \overline{\Omega}$ such that $|\omega| > (1+\sqrt{2})\|A\|$. Both sides of the equation are right monogenic in $\Omega$ in the complement of the set $\overline{\Omega}$, so equality follows there by unique continuation.

(iv) According to (i), $p(\boldsymbol{A}) \in \mathcal{L}(X)$ for any scalar valued polynomial $p$ on $\mathbb{R}^n$. By Proposition 3.6, there exists an open neighbourhood $V$ of $U$ in $\mathbb{R}^{n+1}$ such that the left monogenic extension $\tilde{f}$ of $f$ can be approximated on compact subsets of $V$ by monogenic extensions of scalar polynomials on $\mathbb{R}^n$. An appeal to Proposition 4.20 shows that $f(\boldsymbol{A})$ belongs to the closed linear subspace $\mathcal{L}(X)$ of $\mathcal{L}_{(n)}(X_{(n)})$. □

As follows from [7], the Weyl functional calculus $\mathcal{W}_{\boldsymbol{A}}$ for an $n$-tuple $\boldsymbol{A}$ of bounded operators acting on a Banach space $X$ is determined by the following two properties:

a) $\mathcal{W}_{\boldsymbol{A}} : C^\infty(\mathbb{R}^n) \to \mathcal{L}(X)$ is a continuous linear map for the operator norm;
b) $\mathcal{W}_{\boldsymbol{A}}(p(\langle \cdot, \xi\rangle)) = p(\langle \boldsymbol{A}, \xi\rangle)$ for every polynomial $p : \mathbb{R} \to \mathbb{R}$ and $\xi \in \mathbb{R}^n$.

The Paley-Wiener Theorem ensures that the inverse Fourier transform $(\mathcal{W}_{\boldsymbol{A}})\check{}$ of $\mathcal{W}_{\boldsymbol{A}}$ extends to an entire analytic function on $\mathbb{C}^n$ satisfying an exponential bound and b) guarantees that that $(\mathcal{W}_{\boldsymbol{A}})\check{}(\xi) = (2\pi)^{-n} e^{i\langle \boldsymbol{A}, \xi\rangle}$ for all $\xi \in \mathbb{R}^n$. Hence $\mathcal{W}_{\boldsymbol{A}} = (2\pi)^{-n}(e^{i\langle \boldsymbol{A},\xi\rangle})\hat{}$. In particular, $\sigma(\langle \boldsymbol{A}, \xi\rangle) \subset \mathbb{R}$ for all $\xi \in \mathbb{R}^n$ (see, for example, [75, Corollary 7.5]).

The analogous statement for the monogenic functional calculus follows.

**Theorem 4.23.** *Let $\boldsymbol{A}$ be an $n$-tuple of bounded linear operators acting on a Banach space $X$. Suppose that there exists a compact subset $K$ of $\mathbb{R}^n$ and a map $T$ such that*

*a) $T : H_M(K) \to \mathcal{L}(X)$ is a continuous linear map;*
*b) $T(p(\langle \cdot, \xi\rangle)) = p(\langle \boldsymbol{A}, \xi\rangle)$ for every polynomial $p : \mathbb{R} \to \mathbb{R}$ and $\xi \in \mathbb{R}^n$.*

*Then $\sigma(\langle \boldsymbol{A}, \xi\rangle)$ is real for each $\xi \in \mathbb{R}^n$, $\gamma(\boldsymbol{A}) \subseteq K$ and $T(f) = f(\boldsymbol{A})$ for every $f \in H_M(K)$.*

*Proof.* Denote the tensor product $T \otimes I_{(n)}$ of $T$ with the identity $I_{(n)}$ on $\mathbb{F}_{(n)}$ by $T$ again and define $T : M(K, \mathbb{F}_{(n)}) \to \mathcal{L}_{(n)}(X_{(n)})$ by $T(f) = T(f|_U)$, $f \in M(K, \mathbb{F}_{(n)})$, for an open neighbourhood $U$ of $K$ in $\mathbb{R}^n$ in which $f$ is defined.

Let $\xi \in \mathbb{R}^n$ and $\langle K, \xi\rangle := \{\langle x, \xi\rangle : x \in K\} \subset \mathbb{R}$. For all $\lambda \in \mathbb{C} \setminus \langle K, \xi\rangle$, the function $x \longmapsto (\lambda - \langle x, \xi\rangle)^{-1}$ belongs to $H_M(K)$ and the function $\lambda \longmapsto (\lambda - \langle \cdot, \xi\rangle)^{-1}$ is an $H_M(K)$-valued analytic function on $\mathbb{C} \setminus \langle K, \xi\rangle$, so

$$\int_\Gamma (\lambda - \langle \cdot, \xi\rangle)^{-1}\, d\lambda = 0$$

in $H_M(K)$ for all closed contours $\Gamma$ contained in $\mathbb{C} \setminus \langle K, \xi\rangle$. The integral converges as a Bochner integral, so that

$$\int_\Gamma T\big((\lambda - \langle\,\cdot\,,\xi\rangle)^{-1}\big)\,d\lambda = T\int_\Gamma (\lambda - \langle\,\cdot\,,\xi\rangle)^{-1}\,d\lambda = 0.$$

By Morera's Theorem, $\lambda \longmapsto T\big((\lambda - \langle\,\cdot\,,\xi\rangle)^{-1}\big)$ is an $\mathcal{L}(X)$-valued analytic function defined in $\mathbb{C}\setminus\langle K,\xi\rangle$. By b) and the continuity of $T$, the equality

$$(\lambda - \langle \boldsymbol{A},\xi\rangle)^{-1} = T\big((\lambda - \langle\,\cdot\,,\xi\rangle)^{-1}\big)$$

holds for all $\lambda\in\mathbb{C}$ such that $|\lambda| > \sup|\langle K,\xi\rangle|$. It follows that the resolvent set of the operator $\langle \boldsymbol{A},\xi\rangle$ contains the set $\mathbb{C}\setminus\langle K,\xi\rangle$, that is, $\sigma(\langle \boldsymbol{A},\xi\rangle)\subseteq\langle K,\xi\rangle\subset\mathbb{R}$.

As in the proof of [7, Theorem 2.4], property b) and the continuity of $T$ on $H_M(K)$ guarantee that $T(f)$ is equal to (4.19) for all complex valued analytic functions $f$ with a power series given by (4.18) in an open neighbourhood of $K$ with $a_{l_1\dots l_k}\in\mathbb{C}$.

Let $R > (1+\sqrt{2})\|A\|$ be so large that $K$ is contained in the open ball $B_R(0)$ of radius $R$ in $\mathbb{R}^{n+1}$. According to equations (4.8) and (4.9), it follows that $G_\omega(\boldsymbol{A}) = T(G_\omega)$ for all $\omega\in\mathbb{R}^{n+1}$ with $|\omega|\ge R$.

Now the function $\omega\longmapsto G_\omega$ is monogenic from $\mathbb{R}^{n+1}\setminus K$ into $M(K,\mathbb{F}_{(n)})$, because for each $\alpha\in\mathbb{R}^{n+1}\setminus K$ there exist disjoint open sets $U$ and $V$ in $\mathbb{R}^{n+1}$ such that $\alpha\in U$, $K\subset V$ and $\nabla_\omega G_\omega(x)$ is uniformly bounded and uniformly continuous for all $\omega\in U$ and $x\in V$. Consequently, $\omega\longmapsto T(G_\omega)$ is monogenic from $\mathbb{R}^{n+1}\setminus K$ into $\mathcal{L}_{(n)}(X_{(n)})$ and the function defined by equation ((4.9)) has a monogenic extension off $K$, that is, $\gamma(\boldsymbol{A})\subseteq K$ and $G_\omega(\boldsymbol{A}) = T(G_\omega)$ for all $\omega\in\mathbb{R}^{n+1}\setminus K$.

Let $f\in H_M(K)$ and suppose that $\tilde f$ is a left monogenic extension of $f$ to an open neighbourhood of $K$ in $\mathbb{R}^{n+1}$. We may suppose further that $\tilde f$ is defined in a neighbourhood of the closure $\overline{\Omega}$ of a bounded open set $\Omega\supset K$ in $\mathbb{R}^{n+1}$, for which the Cauchy integral formula (3.2) holds for $\tilde f$. Then by formula (3.2), we have

$$\begin{aligned}
T(f) &= T\left(\int_{\partial\Omega} G_\omega(\,\cdot\,)\boldsymbol{n}(\omega)\tilde f(\omega)\,d\mu(\omega)\right)\\
&= \int_{\partial\Omega} T\left(G_\omega(\,\cdot\,)\boldsymbol{n}(\omega)\tilde f(\omega)\right)\,d\mu(\omega)\\
&= \int_{\partial\Omega} T(G_\omega)\boldsymbol{n}(\omega)\tilde f(\omega)\,d\mu(\omega)\\
&= \int_{\partial\Omega} G_\omega(\boldsymbol{A})\boldsymbol{n}(\omega)\tilde f(\omega)\,d\mu(\omega)\\
&= f(\boldsymbol{A}).
\end{aligned}\qquad\square$$

The monogenic functional calculus, when it exists, is therefore the *richest* analytic functional calculus satisfying b) that can be defined over a compact subset of $\mathbb{R}^n$.

Suppose that $L:\mathbb{R}^n\to\mathbb{R}^m$ is an affine transformation given by $(Lx)_k = \sum_{j=1}^n c_{kj}x_j + d_k$ for all $x\in\mathbb{R}^n$ and $k=1,\dots,m$. The $m$-tuple $L\boldsymbol{A}$ is given by $(L\boldsymbol{A})_k = \sum_{j=1}^n c_{kj}A_j + d_k I$ and $Lf = f \circ L$ for a function defined on a subset of $\mathbb{R}^m$. Let $\pi_j : \mathbb{R}^n \to \mathbb{R}$ be the $j$'th projection $\pi_j(x) = x_j$ for all $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

The following properties of the Weyl functional calculus [7, Theorem 2.9], suitably interpreted, are also enjoyed by the monogenic functional calculus.

**Theorem 4.24.** *Let $\boldsymbol{A}$ be an $n$-tuple of bounded operators acting on a Banach space $X$ such that $\sigma(\langle \boldsymbol{A}, \xi\rangle) \subset \mathbb{R}$ for all $\xi \in \mathbb{R}^n$.*

*(a)* **Affine covariance:** *if $L : \mathbb{R}^n \to \mathbb{R}^m$ is an affine map, then $\gamma(L\boldsymbol{A}) \subseteq L\gamma(\boldsymbol{A})$ and for any function $f$ analytic in a neighbourhood in $\mathbb{R}^m$ of $L\gamma(\boldsymbol{A})$, the equality $f(L\boldsymbol{A}) = (f \circ L)(\boldsymbol{A})$ holds.*

*(b)* **Consistency with the one-dimensional calculus:** *if $g : \mathbb{R} \to \mathbb{C}$ is analytic in a neighbourhood of the projection $\pi_1\gamma(\boldsymbol{A})$ of $\gamma(\boldsymbol{A})$ onto the first ordinate, and $f = g \circ \pi_1$, then $f(\boldsymbol{A}) = g(A_1)$. We also have consistency with the $k$-dimensional calculus, $1 < k < n$.*

*(c)* **Continuity:** *The mapping $(T, f) \longmapsto f(T)$ is continuous for $T = \sum_{j=1}^n T_j e_j$ from $\mathcal{L}_{(n)}(X_{(n)}) \times M(\mathbb{R}^{n+1}, \mathbb{C}_{(n+1)})$ to $\mathcal{L}_{(n)}(X_{(n)})$ and from $\mathcal{L}(X) \times H_M(\mathbb{R}^n)$ to $\mathcal{L}(X)$.*

*(d)* **Covariance of the Range:** *If $T$ is an invertible continuous linear map on $X$ and $TAT^{-1}$ denotes the $n$-tuple with entries $TA_jT^{-1}$ for $j = 1, \ldots n$, then $\gamma(TAT^{-1}) = \gamma(\boldsymbol{A})$ and $f(TAT^{-1}) = Tf(\boldsymbol{A})T^{-1}$ for all functions $f$ analytic in a neighbourhood of $\gamma(\boldsymbol{A})$ in $\mathbb{R}^n$.*

*Proof.* (a) The mapping $f \longmapsto f \circ L(\boldsymbol{A})$ defined for all $f \in H_M(L\gamma(\boldsymbol{A}))$ satisfies the conditions of Theorem 4.23 for the $m$-tuple $L\boldsymbol{A}$, so $\gamma(L\boldsymbol{A}) \subseteq L\gamma(\boldsymbol{A})$ and $f \circ L(\boldsymbol{A}) = f(L\boldsymbol{A})$ for all $f \in H_M(L\gamma(\boldsymbol{A}))$.

(b) Set $L = \pi_1$ and apply (a).

(c) Let $\boldsymbol{A} = \sum_{j=1}^n A_j e_j$ and choose $R > (\sqrt{2} + 1)\|A\|$. Let $U_R$ be the intersection of the open unit ball of radius $R$ in $\mathcal{L}_{(n)}(X_{(n)})$ with the subspace $\{\sum_{j=1}^n S_j e_j : S_j \in \mathcal{L}(X)\}$. According to equation (4.9), the mapping $(\omega, T) \longmapsto G_\omega(T)$ is continuous from $\mathbb{R}^{n+1} \times U_R$ into $\mathcal{L}_{(n)}(X_{(n)})$ for all $|\omega| > R$.

Let $B_r(0)$ be the open ball of radius $r > R$ in $\mathbb{R}^{n+1}$. Then from (4.18) we have

$$
\begin{aligned}
\|f_1(T_1) - f_2(T_2)\| &\le \int_{\partial B_r(0)} \|G_\omega(T_1)\boldsymbol{n}(\omega)f_1(\omega) - G_\omega(T_2)\boldsymbol{n}(\omega)f_2(\omega)\|\, d\mu(\omega) \\
&\le 2^{n/2}\mu(\partial B_r(0))\Big( \sup_{\omega\in\partial B_r(0)} \|G_\omega(T_1) - G_\omega(T_2)\| \\
&\quad \times \max\{\sup_{\omega\in\partial B_r(0)} |f_1(\omega)|, \sup_{\omega\in\partial B_r(0)} |f_2(\omega)|\} \\
&\quad + \sup_{\omega\in\partial B_r(0)} \|f_1(\omega) - f_2(\omega)\| \\
&\quad \times \max\{\sup_{\omega\in\partial B_r(0)} |G_\omega(T_1)|, \sup_{\omega\in\partial B_r(0)} |G_\omega(T_2)|\}\Big)
\end{aligned}
$$

for all $T_1, T_2 \in U_R$. The spaces $M(\mathbb{R}^n, \mathbb{C}_{(n)})$ and $M(\mathbb{R}^{n+1}, \mathbb{C}_{(n)})$ are isomorphic [19, Corollary 14.6]. Combined with Theorem 4.22 (iii), this completes the proof of (c).

(d) follows from the identity $G_\omega(TAT^{-1}) = TG_\omega(\boldsymbol{A})T^{-1}$ valid from (4.18) for $|\omega|$ large enough. Then $\gamma(TAT^{-1}) \subseteq \gamma(\boldsymbol{A})$. The reverse inclusion comes from writing $G_\omega(\boldsymbol{A}) = T^{-1}G_\omega(TAT^{-1})T$ for $|\omega|$ large enough. □

The inclusion in (a) may be proper, as may be seen from the equality $\gamma(\pi_1 A) = \sigma(A_1)$. The next assertion shows that property ii) of Theorem 4.22 can be extended from polynomials to analytic functions.

**Proposition 4.25.** *Let $\boldsymbol{A}$ be an $n$-tuple of bounded operators acting on a Banach space $X$ such that $\sigma\big(\langle \boldsymbol{A}, \xi\rangle\big) \subset \mathbb{R}$ for all $\xi \in \mathbb{R}^n$. Let $\zeta \in \mathbb{C}^n$ and set $\langle \gamma(\boldsymbol{A}), \zeta\rangle := \{\langle x, \zeta\rangle : x \in \gamma(\boldsymbol{A})\}$. Then $\sigma(\langle \boldsymbol{A}, \zeta\rangle) \subseteq \langle \gamma(\boldsymbol{A}), \zeta\rangle$.*

*Suppose that $U \subset \mathbb{C}$ is a bounded open set with connected complement containing the set $\langle \gamma(\boldsymbol{A}), \zeta\rangle$. Suppose that $g : U \to \mathbb{C}$ is analytic. Set $f(z) = g(\langle z, \zeta\rangle)$, for all $z \in \mathbb{C}^n$ such that $\langle z, \zeta\rangle \in U$. Then $f(\boldsymbol{A}) = g(\langle \boldsymbol{A}, \zeta\rangle)$.*

*Proof.* The proof of the inclusion $\sigma(\langle \boldsymbol{A}, \zeta\rangle) \subseteq \langle \gamma(\boldsymbol{A}), \zeta\rangle$ follows the argument of Theorem 4.23. By Runge's Theorem for functions of a single complex variable, $g$ can be approximated uniformly on compact subsets of $U$ by polynomials $\langle p_n\rangle_n$ on $\mathbb{C}$. Hence $f$ can be approximated by $\{p_n \circ \zeta\}_n$ uniformly on sets $\langle \cdot, \zeta\rangle^{-1}K$ for $K \subset U$ compact.

Now take $K$ to be a compact subset of $U$ whose interior $K^\circ$ contains $\langle \gamma(\boldsymbol{A}), \zeta\rangle$. Let $V$ be an open subset of $\mathbb{R}^{n+1}$ such that $\gamma(\boldsymbol{A}) \subset V$ and $\overline{V}$ is contained in $\langle \cdot, \zeta\rangle^{-1}K^\circ$. Then $f$ can be approximated uniformly on $\overline{V}$ by functions $\{p_n \circ \zeta\}_n$ with $\{p_n\}_n$ a sequence of polynomials on $\mathbb{C}$. The equality $f(\boldsymbol{A}) = g(\langle \boldsymbol{A}, \zeta\rangle)$ is a consequence of Theorem 4.22 (ii) and Proposition 4.20. □

In the case that $\boldsymbol{A}$ is a commuting $n$-tuple of bounded operators acting on a Banach space $X$, it is shown in [75, Corollary 3.4] that for $\lambda \in \mathbb{R}^n$, the operator $\sum_{j=1}^n (\lambda_j I - A_j)^2$ is invertible in $\mathcal{L}(X)$ if and only if $\sum_{j=1}^n (\lambda_j I - A_j)e_j$ is an invertible element of $\mathcal{L}_{(n)}(X_{(n)})$.

The following result was announced in [68, Lemma 3.2, Corollary 3.17] for commuting selfadjoint operators.

**Theorem 4.26.** *Let $\boldsymbol{A}$ be a commuting $n$-tuple of bounded operators acting on a Banach space $X$ such that $\sigma(A_j) \subset \mathbb{R}$ for all $j = 1, \ldots, n$.*

*Then $\gamma(\boldsymbol{A})$ is the complement in $\mathbb{R}^n$ of the set of all $\lambda \in \mathbb{R}^n$ for which the operator $\sum_{j=1}^n (\lambda_j I - A_j)^2$ is invertible in $\mathcal{L}(X)$.*

*Moreover, $\gamma(\boldsymbol{A})$ is the Taylor spectrum of $\boldsymbol{A}$. If the complex valued function $f$ is analytic in a neighbourhood of $\gamma(\boldsymbol{A})$ in $\mathbb{R}^n$, then the operator $f(\boldsymbol{A}) \in \mathcal{L}(X)$ coincides with the operator obtained from Taylor's functional calculus [104].*

*Proof.* Let $\rho_{(n)}(\boldsymbol{A})$ be the set of all $\lambda \in \mathbb{R}^{n+1}$ such that either $\lambda_0 \neq 0$ or if $\lambda_0 = 0$, then the operator $\sum_{j=1}^n (\lambda_j I - A_j)^2$ is invertible in $\mathcal{L}(X)$. Set $\sigma_{(n)}(\boldsymbol{A}) = \mathbb{R}^n \setminus \rho_{(n)}(\boldsymbol{A})$.

Each of the operators $A_j$ has real spectrum, so $\sigma(\langle \boldsymbol{A}, \xi \rangle) \subset \mathbb{R}$ [75, Proposition 10.1]. Suppose first that $n$ is odd. In this case, the Cauchy kernel $G_\omega(\boldsymbol{A})$ for $\boldsymbol{A}$ can be written down directly. The element

$$\frac{1}{\Sigma_n} |\omega I - \boldsymbol{A}|^{-n-1} (\overline{\omega I - \boldsymbol{A}}) \tag{4.20}$$

of $\mathcal{L}_{(n)}(X_{(n)})$ has the power series expansion (4.9) for $|\omega|$ large enough. Here

$$|\omega I - A|^{-m} = \left( \left( \omega_0^2 I + \sum_{j=1}^n (\omega_j I - A_j)^2 \right)^{-1} \right)^{m/2}$$

for an even integer $m$ and $\overline{\omega I - A} = \omega_0 I - \sum_{j=1}^n (\omega_j I - A_j) e_j$.

The operator $\omega_0^2 I + \sum_{j=1}^n (\omega_j I - A_j)^2$ is invertible for each $\omega \in \rho_{(n)}(\boldsymbol{A})$ because $A_j$ has real spectrum for each $j = 1, \ldots, n$ [75, Proposition 10.1]. As stated in [75, Example 5.4], it is easily verified that the function $\omega \longmapsto 1/\Sigma_n |\omega I - \boldsymbol{A}|^{-n-1} \overline{\omega I - \boldsymbol{A}}$, $\omega \in \rho_{(n)}(\boldsymbol{A})$, is monogenic in $\mathcal{L}_{(n)}(X_{(n)})$. Hence $\gamma(\boldsymbol{A}) \subseteq \sigma_{(n)}(\boldsymbol{A})$ and $G_\omega(\boldsymbol{A})$ is given by the expression (4.20) for all $\omega \in \rho_{(n)}(\boldsymbol{A})$.

Now suppose that $x \in \mathbb{R}^n \setminus \gamma(\boldsymbol{A})$. Then $\omega \longmapsto G_\omega(\boldsymbol{A})$ is norm-continuous in a neighbourhood $U$ of $x$ in $\mathbb{R}^{n+1}$ and it is given by (4.20) for $\omega_0 \neq 0$. The function

$$\omega \longmapsto \Sigma_n |\omega I - \boldsymbol{A}|^{n-1} G_\omega(\boldsymbol{A})$$

is also continuous in $U$. For $\omega_0 \neq 0$, we have

$$\Sigma_n |\omega I - \boldsymbol{A}|^{n-1} G_\omega(\boldsymbol{A}) = |\omega I - \boldsymbol{A}|^{-2} \overline{\omega I - \boldsymbol{A}}$$

and the equality $(\omega I - \boldsymbol{A})^{-1} = |\omega I - \boldsymbol{A}|^{-2} \overline{\omega I - \boldsymbol{A}}$ holds in $\mathcal{L}_{(n)}(X_{(n)})$, so the $\mathcal{L}_{(n)}(X_{(n)})$-valued function $\omega \longmapsto (\omega I - \boldsymbol{A})^{-1}$ has a continuous extension $J$ from $U \setminus \mathbb{R}^n$ to $U$. Continuity ensures that the equalities $J(\omega)(\omega I - \boldsymbol{A}) = (\omega I - \boldsymbol{A}) J(\omega) = I e_0$ hold for all $\omega \in U$, so $xI - \boldsymbol{A}$ is invertible in $\mathcal{L}_{(n)}(X_{(n)})$, that is, $x \in \rho_{(n)}(\boldsymbol{A})$. This completes the proof that $\gamma(\boldsymbol{A}) = \sigma_{(n)}(\boldsymbol{A})$ for the case in which $n$ is odd.

For $n$ even, we have to define $(\omega_0^2 I + \sum_{j=1}^n (\omega_j I - A_j)^2)^{-(n+1)/2}$ in some fashion. A convenient way is to use the plane wave decomposition formula (4.16) to define $G_\omega(\boldsymbol{A})$. To identify the set $\gamma(\boldsymbol{A})$, we use Taylor's functional calculus [104].

That $\sigma_{(n)}(\boldsymbol{A})$ is the Taylor spectrum of $\boldsymbol{A}$ is proved in [76, Theorem 1]. A continuous linear map $T : H_M(\sigma_{(n)}(\boldsymbol{A})) \to \mathcal{L}(X)$ such that $T(p) = p(\boldsymbol{A})$ for all polynomials $p : \mathbb{R}^n \to \mathbb{C}$ is constructed in [104].

The function $\omega \longmapsto |\omega - \cdot|^{-n-1}$ is analytic from $\rho_{(n)}(\boldsymbol{A})$ into $H_M(\sigma_{(n)}(\boldsymbol{A}))$, so on application of the mapping $T$, it follows that $\omega \longmapsto T\big(|\omega - \cdot|^{-n-1}\big)$ is analytic from $\rho_{(n)}(\boldsymbol{A})$ into $\mathcal{L}(X)$. The analytic functional calculus ensures that the function

$$\omega \longmapsto 1/\Sigma_n T\big(|\omega - \cdot|^{-n-1}\big)\overline{\omega I - A}$$

has the power series expansion (4.9) for $|\omega|$ large enough and is monogenic in $\rho_{(n)}(\boldsymbol{A})$. Hence $\gamma(\boldsymbol{A}) \subseteq \sigma_{(n)}(\boldsymbol{A})$ and $G_\omega(\boldsymbol{A})$ is given by formula (4.20) for all $\omega \in \rho_{(n)}(\boldsymbol{A})$. The proof that $\sigma_{(n)}(\boldsymbol{A}) \subseteq \gamma(\boldsymbol{A})$ follows the case for $n$ odd.

Equality of the monogenic functional calculus and Taylor's functional calculus $T$ [104] is a consequence of Theorem 4.23. □

## 4.4 Spectral Decomposition

The Riesz-Dunford functional calculus for a single bounded operator $A$ acting on a Banach space $X$ is used to construct the projections associated with the components of the spectrum $\sigma(A)$ of $A$. Given a simple closed contour $C$ about $\sigma(A)$ and a function $f$ analytic in a neighbourhood of the closure of the interior of $C$, the bounded linear operator $f(A)$ is defined by means of formula (4.1). The mapping $f \longmapsto f(A)$ is a homomorphism from the space of germs of functions analytic in a neighbourhood of $\sigma(\boldsymbol{A})$ into the space of bounded linear operators on $X$.

For each connected component $\sigma_j$ of $\sigma(A)$ and simple closed contour $\Gamma_j$ about $\sigma_j$ not surrounding any other component of $\sigma(A)$, the operator

$$P_j = \int_{\Gamma_j} (\zeta I - A)^{-1}\, d\zeta \tag{4.21}$$

is a projection. Whenever $\sigma_j$ is a single point $\{\lambda\}$, the operator $P_j$ is the projection onto the eigenspace of the operator $A$ corresponding to the eigenvalue $\lambda$.

In the case that $\boldsymbol{A} = (A_1, \dots, A_n)$ is a system of bounded linear operators acting on a Banach space $X$ and $\sigma(\langle \boldsymbol{A}, \xi\rangle) \subset \mathbb{R}$ for all $\xi \in \mathbb{R}^n$, we can use the monogenic functional calculus to define the projections associated with components of the joint spectrum. An important feature is that the operators $A_1, \dots, A_n$ do not necessarily commute with each other.

In passing from a single operator $A$ to a finite system $\boldsymbol{A}$ of operators, formula (4.21) is modified by replacing the resolvent family $(\lambda I - A)^{-1}$, $\lambda \in \mathbb{C} \setminus \sigma(A)$, with the Cauchy kernel $G_\omega(\boldsymbol{A})$, $\omega \in \mathbb{R}^{n+1} \setminus \gamma(\boldsymbol{A})$, taking values in $\mathcal{L}(X) \otimes \mathbb{F}_{(n)}$, the elements of which are identified with right-module homomorphisms of the Clifford module $X \otimes \mathbb{F}_{(n)}$. The Cauchy kernel $G_\omega(\boldsymbol{A})$ for a general noncommutative system $\boldsymbol{A}$ is no longer defined by a simple algebraic formula. It is represented by means of a plane wave decomposition (4.16).

In the present section, the *projections* $P_j$ associated with the components $\gamma_j$ of the set $\gamma(\boldsymbol{A})$ are constructed by means of the formula

$$P_j = \int_{\partial\Omega_j} G_\omega(\boldsymbol{A})\boldsymbol{n}(\omega)\, d\mu(\omega), \tag{4.22}$$

analogously to the formula (4.21) obtained from the Riesz-Dunford functional calculus. The integral on the right-hand side of (4.22) is apparently an element of the Clifford module $\mathcal{L}(X) \otimes \mathbb{F}_{(n)}$, but from Theorem 4.22 (iv), all components other than the scalar component $P_j e_0$ are zero.

If $\boldsymbol{A}$ has a Weyl functional calculus $\mathcal{W}_{\boldsymbol{A}}$, it is proved in Theorem 4.8 that $\gamma(\boldsymbol{A}) = \operatorname{supp} \mathcal{W}_{\boldsymbol{A}}$ and in this case, the projections $P_j$ may be constructed by techniques of distribution theory [8].

Furthermore, a very general result of E. Albrecht [4, Theorem 4.1] asserts that once we have an analytic functional calculus in $n$ variables satisfying certain symmetry conditions – such as may be obtained from formula (4.17) by extending analytic functions in $n$ real variables to $\mathbb{C}_{(n)}$-valued monogenic functions in $(n+1)$ real variables – the operators defined by (4.22) are the required spectral projections. The purpose of the present section is to obtain a direct proof of this fact using the monogenic functional calculus.

For the Riesz-Dunford functional calculus, the operator (4.21) is proved to be a projection by appealing to the resolvent relation. For a system of operators, this argument is unavailable.

In the case that $\boldsymbol{A}$ is of Paley-Wiener type $s$ and the Weyl functional calculus actually exists, the spectral decomposition for the support of the Weyl calculus follows from the formula $\mathcal{W}_{s\boldsymbol{A}} * \mathcal{W}_{t\boldsymbol{A}} = \mathcal{W}_{(s+t)\boldsymbol{A}}$, $s, t > 0$, interpreted in the sense of the convolution of operator valued distributions [8]. In Lemma 4.29, we find a substitute for this formula in the case that only the monogenic functional calculus exists.

**Theorem 4.27.** *Let $\boldsymbol{A}$ be an $n$-tuple of bounded operators acting on a Banach space $X$, with the property that $\sigma(\langle \boldsymbol{A}, \xi\rangle) \subset \mathbb{R}$ for all $\xi \in \mathbb{R}^n$.*

*Suppose that $k > 1$, $\gamma_1, \dots, \gamma_k$ are non-empty, disjoint closed subsets of $\gamma(\boldsymbol{A})$, and $\cup_{j=1}^k \gamma_j = \gamma(\boldsymbol{A})$. For each $j = 1, \dots, k$, let $\Omega_j$ be a bounded open neighbourhood of $\gamma_j$ in $\mathbb{R}^{n+1}$ with smooth boundary $\partial\Omega_j$ and exterior unit normal $\boldsymbol{n}(\omega)$ defined for all $\omega \in \partial\Omega_j$. Let $\mu_j$ be the surface measure of $\Omega_j$. Further, suppose that the sets $\overline{\Omega_j}, j = 1, \dots, k$ are pairwise disjoint. For each $j = 1, \dots, k$, set*

$$P_j = \int_{\partial\Omega_j} G_\omega(\boldsymbol{A})\boldsymbol{n}(\omega)\, d\mu_j(\omega).$$

*Then $P_j \in \mathcal{L}(X)$ is a bounded operator, the equality $P_j f(\boldsymbol{A}) = f(\boldsymbol{A}) P_j$ holds for any analytic function $f$ in $n$ real variables defined in a neighbourhood of $\gamma(\boldsymbol{A})$ in $\mathbb{R}^n$, $P_j^2 = P_j$, and for each $l = 1, \dots, k$ such that $l \neq j$, $P_j P_l = 0$. Moreover, $\sum_{j=1}^k P_j = I$, and for each $j = 1, \dots, k$ the operator $P_j$ is neither the identity nor the zero operator.*

Before proving the theorem, we note some elementary facts. Let $f : U \to \mathbb{F}_{(n)}$ be left monogenic in a neighbourhood $U$ of $\overline{\Omega} = \cup_j \overline{\Omega_j}$ in $\mathbb{R}^{n+1}$. Suppose that the function $f_{\Omega_j}$ is equal to $f$ in a neighbourhood of $\overline{\Omega_j}$, and is identically zero in a neighbourhood of $\overline{\Omega_l}$ for all $l \neq j$, $l = 1, \dots, k$. Then $f_{\Omega_j}$ is also left monogenic in a neighbourhood of $\gamma(\boldsymbol{A})$.

Suppose now that $f$ is monogenic in a neighbourhood of $\gamma(\boldsymbol{A})$, the function $f|\mathbb{R}^n$ is a complex valued analytic function in $n$ real variables and that $f_{\Omega_j}$, defined as above, is left monogenic in a neighbourhood $U$ of $\overline{\Omega_j}$ and zero elsewhere. For such a function $f$, the element $f(\boldsymbol{A})$ of $\mathcal{L}(X)_{(n)}$ defined by formula (4.22) is actually an element of $\mathcal{L}(X)$ by Theorem 4.22 (iv).

Let $\delta$ be a positive number and $V$ a neighbourhood of $\overline{\Omega}$ such that $\delta V + V \subset U$ and $(1+\delta)\gamma(\boldsymbol{A}) \subset \Omega$. Then for all $0 < |s| < \delta$ the function $x \longmapsto f_{\Omega_j}(sx+\omega)$ is monogenic in $V$ for each $\omega \in V$. The formula

$$G_\omega(s\boldsymbol{A}) = s^{-n} G_{\omega/s}(\boldsymbol{A})$$

for $\omega \in \mathbb{R}^{n+1}$ with large norm follows from formula (4.9). It follows that $\gamma(s\boldsymbol{A}) = s\gamma(\boldsymbol{A})$ and the equality (4.17) holds for all $\omega$ not in $s\gamma(\boldsymbol{A})$. Hence, for each $\omega \in V$, the function $x \longmapsto f_{\Omega_j}(x+\omega)$ is monogenic in the neighbourhood $sV$ of the joint monogenic spectrum $\gamma(s\boldsymbol{A}) = s\gamma(\boldsymbol{A})$ of the $n$-tuple $s\boldsymbol{A} = (s\boldsymbol{A}_1, \dots, s\boldsymbol{A}_n)$. Then $f_{\Omega_j}(s\boldsymbol{A} + \omega)$ is defined from the monogenic functional calculus for $s\boldsymbol{A}$ by the formula

$$\begin{aligned} f_{\Omega_j}(s\boldsymbol{A} + \omega) &= \int_{\partial(s\Omega)} G_\zeta(s\boldsymbol{A})\boldsymbol{n}(\zeta) f_{\Omega_j}(\zeta + \omega)\, d\mu^{(s)}(\zeta) \\ &= \int_{\partial(s\Omega_j)} G_\zeta(s\boldsymbol{A})\boldsymbol{n}(\zeta) f_{\Omega_j}(\zeta + \omega)\, d\mu_j^{(s)}(\zeta) \\ &= \int_{\partial\Omega_j} G_\zeta(\boldsymbol{A})\boldsymbol{n}(\zeta) f_{\Omega_j}(s\zeta + \omega)\, d\mu_j(\zeta) \end{aligned} \tag{4.23}$$

for the surface measure $\mu^{(s)}$ of $\partial(s\Omega)$, and the surface measure $\mu_j^{(s)}$ of $\partial(s\Omega_j)$.

For each $\omega \in \mathbb{R}^{n+1}$ with $\omega = (\omega_0, \omega_1, \dots, \omega_n)$, set $\tilde{\omega} = \sum_{j=1}^n \omega_j e_j$. Then $\omega = \tilde{\omega} + \omega_0 e_0$.

**Lemma 4.28.** *For each* $\xi \in \mathbb{R}^n$ *and* $s \in \mathbb{R}$*, the equality*

$$\begin{aligned} &\int_{\partial\Omega} G_\omega(\boldsymbol{A})\boldsymbol{n}(\omega) e^{i(1+s)(\langle\tilde{\omega},\xi\rangle - \omega_0\xi)}\, d\mu(\omega) \\ &\qquad = \int_{\partial\Omega} G_\omega(\boldsymbol{A})\boldsymbol{n}(\omega) e^{i(\langle s\boldsymbol{A} + \tilde{\omega}I,\xi\rangle - \omega_0\xi I)}\, d\mu(\omega) \end{aligned}$$

*is valid.*

*Proof.* Let $\tilde{\xi} : \mathbb{R}^{n+1} \to \mathbb{R}_{(n)}$ be the function defined by $\tilde{\xi}(\omega) = \langle\omega, \xi\rangle e_0 - \omega_0\xi$ for each $\xi \in \mathbb{R}^n$ and $\omega \in \mathbb{R}^{n+1}$. An application of the Dirac operator $D$ to the left and right of the function $\tilde{\xi}$ verifies that it is monogenic. Furthermore, $e^{i\tilde{\xi}} : \mathbb{R}^{n+1} \to \mathbb{R}_{(n)}$ is the unique monogenic extension from $\mathbb{R}^n$ to $\mathbb{R}^{n+1}$ of the complex valued function $x \longmapsto e^{i\langle x,\xi\rangle}$, $x \in \mathbb{R}^n$.

As elements of the Banach module $\mathcal{L}(X)_{(n)}$, the operator $\langle s\boldsymbol{A}, \xi\rangle e_0$ commutes with $\tilde{\xi}(\omega)I$ for each $\omega \in \mathbb{R}^{n+1}$, so $e^{i(\langle s\boldsymbol{A}+\tilde{\omega}I,\xi\rangle - \omega_0 \xi I)} = e^{i\tilde{\xi}(\omega)}e^{i\langle s\boldsymbol{A},\xi\rangle}$ and we have the equality

$$\int_{\partial\Omega} G_\omega(\boldsymbol{A})\boldsymbol{n}(\omega)e^{i(\langle s\boldsymbol{A}+\tilde{\omega}I,\xi\rangle-\omega_0\xi I)}\,d\mu(\omega)$$
$$= \left(\int_{\partial\Omega} G_\omega(\boldsymbol{A})\boldsymbol{n}(\omega)e^{i\tilde{\xi}(\omega)}\,d\mu(\omega)\right)e^{i\langle s\boldsymbol{A},\xi\rangle}.$$

We can take the operator $e^{i\langle s\boldsymbol{A},\xi\rangle}$ outside the integral, because the operation of left multiplication by elements of $\mathbb{F}_{(n)}$ is continuous in the Banach module $\mathcal{L}(X)_{(n)}$.

Now apply Proposition 4.25 to obtain

$$\int_{\partial\Omega} G_\omega(\boldsymbol{A})\boldsymbol{n}(\omega)e^{i(\langle s\boldsymbol{A}+\tilde{\omega}I,\xi\rangle-\omega_0\xi I)}\,d\mu(\omega)$$
$$= e^{i\langle \boldsymbol{A},\xi\rangle}e^{i\langle s\boldsymbol{A},\xi\rangle}$$
$$= e^{i\langle \xi, A\rangle(1+s)}$$
$$= \int_{\partial\Omega} G_\omega(\boldsymbol{A})\boldsymbol{n}(\omega)e^{i(1+s)(\langle\tilde{\omega},\xi\rangle-\omega_0\xi)}\,d\mu(\omega). \quad \square$$

The monogenic extension $\tilde{\phi}$ of an analytic function $\phi$ in $n$ real variables $x_1, \dots, x_n$ is obtained by replacing the products $(x_{j_1}-a_{j_1})\cdots(x_{j_k}-a_{j_k})$ in the power series expansions for $\phi$ about a point $a \in \mathbb{R}^n$ by symmetric products in the monogenic functions $(z_{j_1}-a_{j_1}), \dots, (z_{j_k}-a_{j_k})$. Here $z_j(\omega) = \omega_j e_0 - \omega_0 e_j$, $\omega \in \mathbb{R}^{n+1}$, is the monogenic extension to $\mathbb{R}^{n+1}$ of the projection $\pi_j : \mathbb{R}^n \to \mathbb{R}$ onto the $j$'th coordinate, see [19, p113].

Let $\mathcal{E}$ denote the collection of all functions $\xi \longmapsto q(\xi)e^{-|\xi|^2/2}$, $\xi \in \mathbb{R}^n$, with $q(\xi)$ a complex valued polynomial in $\xi \in \mathbb{R}^n$. If $\phi$ is the Fourier transform of a function $\psi \in \mathcal{E}$ given by

$$\phi(z) = \int_{\mathbb{R}^n} e^{-i\langle z,\xi\rangle}\psi(\xi)\,d\xi$$

for all $z \in \mathbb{C}^n$, then $\phi|\mathbb{R}^n$ again belongs to $\mathcal{E}$. The monogenic extension $\tilde{\phi} : \mathbb{R}^{n+1} \to \mathbb{C}_{(n)}$ of $\phi|\mathbb{R}^n$ is defined on all of $\mathbb{R}^{n+1}$ and is obtained from the formula

$$\tilde{\phi}(\omega) = \int_{\mathbb{R}^n} e^{-i\langle\tilde{\omega},\xi\rangle+i\omega_0\xi}\,\psi(\xi)\,d\xi, \quad \omega \in \mathbb{R}^{n+1}.$$

**Lemma 4.29.** *If $\phi \in \mathcal{E}$, then*

$$\phi((1+s)\boldsymbol{A}) = \int_{\partial\Omega} G_\omega(\boldsymbol{A})\boldsymbol{n}(\omega)\tilde{\phi}(s\boldsymbol{A}+\omega)\,d\mu(\omega)$$

*for every $0 < s < \delta$.*

*Proof.* Suppose that $\psi \in \mathcal{E}$ and $\phi(x) = \hat{\psi}(x)$ for all $x \in \mathbb{R}^n$. For each $t \in \mathbb{R}$, we have

$$\begin{aligned}\phi(tA) &= \int_{\partial\Omega} G_\omega(\boldsymbol{A})\boldsymbol{n}(\omega)\tilde{\phi}(t\omega)\,d\mu(\omega)\\ &= \int_{\partial\Omega} G_\omega(\boldsymbol{A})\boldsymbol{n}(\omega)\left[\int_{\mathbb{R}^n} e^{-it(\langle\tilde{\omega},\xi\rangle-\omega_0\xi)}\,\psi(\xi)\,d\xi\right]d\mu(\omega)\\ &= \int_{\mathbb{R}^n}\left[\int_{\partial\Omega} G_\omega(\boldsymbol{A})\boldsymbol{n}(\omega)e^{-it(\langle\tilde{\omega},\xi\rangle-\omega_0\xi)}\,d\mu(\omega)\right]\psi(\xi)\,d\xi \qquad (4.24)\end{aligned}$$

The interchange of the order of integration follows from the Hahn-Banach Theorem and the scalar Fubini's theorem.

The element $\tilde{\phi}(s\boldsymbol{A}+\omega)$ of $\mathcal{L}(X)_{(n)}$ is defined by the formula

$$\begin{aligned}\tilde{\phi}(s\boldsymbol{A}+\omega) &= \int_{\partial\Omega} G_\zeta(\boldsymbol{A})\boldsymbol{n}(\zeta)\tilde{\phi}(s\zeta+\omega)\,d\mu(\zeta)\\ &= \int_{\partial\Omega} G_\zeta(\boldsymbol{A})\boldsymbol{n}(\zeta)\left[\int_{\mathbb{R}^n} e^{-i\langle s\tilde{\zeta}+\tilde{\omega},\xi\rangle+i(s\zeta_0+\omega_0)\xi}\,\psi(\xi)\,d\xi\right]d\mu(\zeta)\\ &= \int_{\mathbb{R}^n}\left[\int_{\partial\Omega} G_\zeta(\boldsymbol{A})\boldsymbol{n}(\zeta)e^{-i\langle s\tilde{\zeta}+\tilde{\omega},\xi\rangle+i(s\zeta_0+\omega_0)\xi}\,d\mu(\zeta)\right]\psi(\xi)\,d\xi\\ &= \int_{\mathbb{R}^n} e^{-i(\langle s\boldsymbol{A}+\tilde{\omega}I,\xi\rangle+i\omega_0\xi I)}\,\psi(\xi)\,d\xi.\end{aligned}$$

Then Lemmas 4.28 and 4.29 imply that

$$\begin{aligned}\phi((1+s)\boldsymbol{A}) &= \int_{\mathbb{R}^n}\left[\int_{\partial\Omega} G_\omega(\boldsymbol{A})\boldsymbol{n}(\omega)e^{-i(1+s)(\langle\tilde{\omega}I,\xi\rangle-\omega_0\xi)}\,d\mu(\omega)\right]\psi(\xi)\,d\xi\\ &= \int_{\mathbb{R}^n}\left[\int_{\partial\Omega} G_\omega(\boldsymbol{A})\boldsymbol{n}(\omega)e^{-i(\langle s\boldsymbol{A}+\tilde{\omega}I,\xi\rangle-\omega_0\xi I)}\,d\mu(\omega)\right]\psi(\xi)\,d\xi\\ &= \int_{\partial\Omega} G_\omega(\boldsymbol{A})\boldsymbol{n}(\omega)\left[\int_{\mathbb{R}^n} e^{-i(\langle s\boldsymbol{A}+\tilde{\omega}I,\xi\rangle-\omega_0\xi I)}\,\psi(\xi)\,d\xi\right]d\mu(\omega)\\ &= \int_{\partial\Omega} G_\omega(\boldsymbol{A})\boldsymbol{n}(\omega)\tilde{\phi}(s\boldsymbol{A}+\omega)\,d\mu(\omega). \qquad \square\end{aligned}$$

**Lemma 4.30.** *There exists* $\delta' > 0$ *such that*

$$f_{\Omega_j}((1+s)\boldsymbol{A}) = \int_{\partial\Omega_j} G_\omega(\boldsymbol{A})\boldsymbol{n}(\omega)f_{\Omega_j}(s\boldsymbol{A}+\omega)\,d\mu_j(\omega)$$

*for all* $0 < s < \delta'$.

*Proof.* We have

$$\begin{aligned}f_{\Omega_j}((1+s)\boldsymbol{A}) &= \int_{\partial\Omega_j} G_\omega(\boldsymbol{A})\boldsymbol{n}(\omega)f_{\Omega_j}((1+s)\omega)\,d\mu_j(\omega)\\ &= \int_{\partial\Omega_j} G_\omega(\boldsymbol{A})\boldsymbol{n}(\omega)f((1+s)\omega)\,d\mu_j(\omega)\end{aligned}$$

$$f_{\Omega_j}(s\boldsymbol{A}+\omega) = \int_{\partial\Omega_j} G_\zeta(\boldsymbol{A})\boldsymbol{n}(\zeta)f_{\Omega_j}(s\zeta+\omega)\,d\mu_j(\zeta). \tag{4.25}$$

By Proposition 3.6, we can approximate $f_{\Omega_j}$ uniformly on $\overline{\Omega}$ by monogenic polynomials, and hence, by monogenic extensions of elements of $\mathcal{E}$, so Lemma 4.29 is valid for $\phi$ replaced by $f_{\Omega_j}$, proving that

$$f_{\Omega_j}((1+s)\boldsymbol{A}) = \int_{\partial\Omega} G_\omega(\boldsymbol{A})\boldsymbol{n}(\omega)f_{\Omega_j}(s\boldsymbol{A}+\omega)\,d\mu_j(\omega)$$

for all $0 < s < \delta$.

Suppose that $f_{\Omega_j}$ vanishes outside a neighbourhood $U$ of $\Omega_j$ disjoint from $\overline{\Omega}_k$, $k \neq j$. Now choose $0 < \delta' \leq \delta$ such that $\delta' \sup_{\zeta\in\partial\Omega_j}|\zeta| < \mathrm{dist}(\overline{U}, \overline{\Omega}\setminus\Omega_j)$. Then for all elements $\omega$ belonging to $\partial\Omega_k$, with $k \neq j$, we have $f_{\Omega_j}(s\boldsymbol{A}+\omega) = 0$ for all $0 < s < \delta'$, by equation 4.25. The integral over $\partial\Omega$ is actually an integral over $\partial\Omega_j$. □

*Proof (of Theorem 4.27).* Now

$$\begin{aligned}\lim_{s\to 0^+} f_{\Omega_j}(s\boldsymbol{A}+\omega) &= \lim_{s\to 0^+}\int_{\partial\Omega_j} G_\zeta(\boldsymbol{A})\boldsymbol{n}(\zeta)f_{\Omega_j}(s\zeta+\omega)\,d\mu_j(\zeta)\\ &= P_j f(\omega) = f(\omega)P_j \end{aligned} \tag{4.26}$$

uniformly for $\omega \in V$. Here we have used the fact, mentioned above, that $P_j \in \mathcal{L}(X)$.

An appeal to Lemma 4.30 shows that

$$\lim_{s\to 0^+} f_{\Omega_j}((1+s)\boldsymbol{A}) = f_{\Omega_j}(\boldsymbol{A})P_j.$$

Similarly, on replacing 'left monogenic' by 'right monogenic', we obtain

$$f_{\Omega_j}((1+s)\boldsymbol{A}) = \textstyle\int_{\partial\Omega_j} f_{\Omega_j}(s\boldsymbol{A}+\omega)\boldsymbol{n}(\omega)G_\omega(\boldsymbol{A})\,d\mu_j(\omega), \tag{4.27}$$

so that $\lim_{s\to 0^+} f_{\Omega_j}((1+s)\boldsymbol{A}) = P_j f_{\Omega_j}(\boldsymbol{A})$.

However, $f_{\Omega_j}((1+s)\boldsymbol{A})$ is also given by the formula

$$f_{\Omega_j}((1+s)\boldsymbol{A}) = \int_{\partial\Omega_j} G_\omega((1+s)\boldsymbol{A})\boldsymbol{n}(\omega)f(\omega)\,d\mu_j(\omega),$$

for all $0 < s < \delta$, because $(1+\delta)\gamma(\boldsymbol{A}) \subset \Omega$. By continuity, we have

$$\lim_{s\to 0^+} f_{\Omega_j}((1+s)\boldsymbol{A}) = f_{\Omega_j}(\boldsymbol{A})$$

and so,

$$f_{\Omega_j}(\boldsymbol{A}) = P_j f_{\Omega_j}(\boldsymbol{A}) = f_{\Omega_j}(\boldsymbol{A})P_j.$$

On taking $f$ to be identically equal to one in a neighbourhood of $\overline{\Omega_j}$ not intersecting $\overline{\Omega_l}$, for all $l = 1, \ldots, k$ such that $l \neq j$, we have $P_j = P_j^2$. For each $j = 1, \ldots, k$, the operator $P_j$ is a projection for which $(I - P_j)f_{\Omega_j}(\boldsymbol{A}) = f_{\Omega_j}(\boldsymbol{A})(I - P_j) = 0$.

The identity $I = \sum_{j=1}^{k} 1_{\Omega_j}(\boldsymbol{A}) = \sum_{j=1}^{k} P_j$ follows from equation (4.19), so for each $j = 1, \ldots, k$, we have $P_l f_{\Omega_j}(\boldsymbol{A}) = f_{\Omega_j}(\boldsymbol{A})P_l = 0$ for all $l = 1, \ldots, k$ such that $l \neq j$. Hence, from the equality $f(\boldsymbol{A}) = \sum_{j=1}^{k} f_{\Omega_j}(\boldsymbol{A})$, we have $P_j f(\boldsymbol{A}) = f(\boldsymbol{A})P_j = f_{\Omega_j}(\boldsymbol{A})$.

Finally, we need to show that $P_j \neq 0$ for each $j = 1, \ldots, k$. From Theorem 4.22 (iii), the equality

$$G_z(\boldsymbol{A}) = \sum_{j=1}^{k} \int_{\partial \Omega_j} G_\omega(\boldsymbol{A})\boldsymbol{n}(\omega)G_z(\omega)\, d\mu(\omega) = \sum_{j=1}^{k} (G_z)_{\Omega_j}(\boldsymbol{A})$$

holds for all $z \in \mathbb{R}^{n+1} \setminus \gamma(\boldsymbol{A})$. But the argument above shows that the equality

$$(G_z)_{\Omega_j}(\boldsymbol{A}) = G_z(\boldsymbol{A})P_j$$

holds. If $P_j$ were zero, then $z \longmapsto G_z(\boldsymbol{A})$ would be the restriction of a monogenic function with the set $\gamma_j$ in its domain, contradicting the definition of $\gamma(\boldsymbol{A})$ as the set of singularities of the function $G_{(\cdot)}(\boldsymbol{A})$. $\square$

# 5

# The Joint Spectrum of Matrices

A number of more or less explicit computations of the joint spectrum of a system of matrices can be made. A more detailed understanding of the joint spectrum of a system of matrices requires ideas from algebraic geometry. For the simplest case of a pair $\boldsymbol{A} = (A_1, A_2)$ of hermitian matrices, the numerical range of the matrix $A_1 + iA_2$ is the convex hull of certain plain algebraic curves which feature in the joint spectrum $\gamma(\boldsymbol{A})$ of $\boldsymbol{A}$. If $A_1$ and $A_2$ commute, then $\gamma(\boldsymbol{A})$ may be identified with the finite set of complex eigenvalues of the normal matrix $A_1 + iA_2$, otherwise $\gamma(\boldsymbol{A})$ consists of the numerical range of $A_1 + iA_2$ minus certain plain regions or *lacunas*.

## 5.1 Nelson's Formula for Hermitian Matrices

In the finite dimensional setting, E. Nelson [83, Theorem 9] gave an explicit formula for the Weyl calculus. As pointed out in [7, p. 241], this amounts to calculating the fundamental solution for a hyperbolic system (1.4) of partial differential equations. The purpose of this section is to set out a proof of Nelson's formula. The essential ingredients of the proof follow.

As is well known from matrix theory, a function of an $(N \times N)$ matrix $M$ can be expressed as a polynomial in $M$ of degree less than $N$. The key to calculating the Weyl calculus in the finite dimensional setting is to find a suitable expression

$$e^{iM} = \gamma_0(M) + \gamma_1(M)M + \cdots + \gamma_{N-1}(M)M^{N-1} \tag{5.1}$$

for all $N \times N$ hermitian matrices $M$. This was achieved in the proof of [83, Theorem 9] by an ingenious application of recursion relations and induction. The proof below is more pedestrian but perhaps easier to understand. The Cayley-Hamilton theorem and binomial expansions are invoked to find an expression

$$(\zeta I - M)^{-1} = \alpha_0(M,\zeta) + \alpha_1(M,\zeta)M + \cdots + \alpha_{N-1}(M,\zeta)M^{N-1}$$

B. Jefferies: LNM 1843, pp. 67–121, 2004.

for all complex numbers $\zeta$ lying outside the spectrum $\sigma(M)$ of $M$. By applying the formula $e^{iM} = \frac{1}{2\pi i}\int_C e^{i\zeta}(\zeta I - M)^{-1}d\zeta$, obtained from the Cauchy-Riesz calculus for a simple closed curve $C$ about $\sigma(M)$, the representation (5.1) follows.

According to formula (2.1), the Weyl calculus $\mathcal{W}_{\boldsymbol{A}}$ for an $n$-tuple $\boldsymbol{A} = (A_1, \dots, A_n)$ of $(N\times N)$ hermitian matrices is $(2\pi)^{-n}$ times the Fourier transform of the matrix valued function $\xi \longmapsto e^{i\langle \boldsymbol{A},\xi\rangle}$, $\xi \in \mathbb{R}^n$, in the sense of distributions. To calculate $\mathcal{W}_{\boldsymbol{A}}$, it is then necessary to obtain the Fourier transform of the function (5.1) in the case that $M = \langle \boldsymbol{A}, \xi\rangle$. The representation

$$\frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\det(\zeta I - M)}\,d\zeta = \frac{1}{(N-1)!}\int_{S(\mathbb{C}^N)} f^{(N-1)}(\langle Mu, u\rangle)\,d\nu(u)$$

facilitates this calculation. Here $f$ is a function analytic in a neighbourhood of the convex hull $\mathrm{co}(\sigma(M))$ of $\sigma(M)$, $C$ is a simple closed curve about $\sigma(M)$ and $\nu$ is the unitarily invariant probability measure on the unit sphere $S(\mathbb{C}^N)$ in $\mathbb{C}^N$. Only the case $f(\zeta) = e^{i\zeta}$ is needed in the calculation of $\mathcal{W}_{\boldsymbol{A}}$, but the general formula is proved in Proposition 5.4 below.

In order to state Nelson's formula for the Weyl calculus, we need to fix some terminology. Let $\boldsymbol{A} = (A_1, \dots, A_n)$ be an $n$-tuple of $N \times N$ hermitian matrices. Let $\nu$ be the unitarily invariant probability measure on the unit sphere $S(\mathbb{C}^N) = \{u \in \mathbb{C}^N : |u| = 1\}$ in $\mathbb{C}^N$. Let $W_{\boldsymbol{A}} : u \longmapsto (\langle A_1 u, u\rangle, \dots, \langle A_n u, u\rangle) \in \mathbb{R}^n$, for all $u \in S(\mathbb{C}^N)$. Then

$$\int_{S(\mathbb{C}^N)} e^{i\langle\langle \boldsymbol{A},\xi\rangle u,u\rangle}\,d\nu(u) = \int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle}\,d\nu\circ W_{\boldsymbol{A}}^{-1}(x).$$

Here $\langle x, \xi\rangle = \sum_{j=1}^n x_j\xi_j$ is the inner product of $\mathbb{R}^n$. Let $\mu_{\boldsymbol{A}} = \nu\circ W_{\boldsymbol{A}}^{-1}$ and set $\check{\mu}_{\boldsymbol{A}}(\xi) = (2\pi)^{-n}\int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle}\,d\mu_{\boldsymbol{A}}(x)$ for all $\xi \in \mathbb{R}^n$. The measure $\mu_{\boldsymbol{A}}$ is the image of the uniform probability measure on the unit sphere $S(\mathbb{C}^N)$ in $\mathbb{C}^N$ by the *numerical range map* $W_{\boldsymbol{A}}$.

The space of smooth functions on $\mathbb{R}^n$ is denoted by $C^\infty(\mathbb{R}^n)$ and the smooth functions on $\mathbb{R}^n$ with compact support, by $C_c^\infty(\mathbb{R}^n)$ . If $T$ is a matrix valued distribution on $\mathcal{S}(\mathbb{R}^n)$, then the Fourier transform $\hat{T}$ of $T$ is defined by $\hat{T}(f) = T(\hat{f})$ for all $f \in \mathcal{S}(\mathbb{R}^n)$. Similarly, the inverse Fourier transform $\check{T}$ of $T$ is defined by $\check{T}(f) = T(\check{f})$ for all $f \in \mathcal{S}(\mathbb{R}^n)$. For a distribution on $C^\infty(\mathbb{R}^n)$, $\hat{T}$ is used to denote the Fourier transform of the restriction of $T$ to $\mathcal{S}(\mathbb{R}^n)$. As is customary, a function $\phi$ is confounded with the distribution $T_\phi : f \longmapsto \int_{\mathbb{R}^n} f(x)\phi(x)\,dx$ it defines. Thus, $\check{\mu}_{\boldsymbol{A}}$ is the inverse Fourier transform of $\mu_{\boldsymbol{A}}$ in the sense of distributions.

For any $N\times N$ matrix $M$, and $k = 1, \dots, N$, let $\phi_k(M)$ be the sum of the principal minors in $M$ of order $k$ and set $\phi_0(M) = 1$. The same expression is adopted if $M$ is a matrix differential operator whose entries are complex linear combinations of the partial differential operators $\partial/\partial x_1, \dots, \partial/\partial x_n$ acting on the space of distributions $C^\infty(\mathbb{R}^n)'$. Of particular interest is the matrix differential operator $M = \langle \boldsymbol{A}, \nabla\rangle$ defined by $\langle \boldsymbol{A}, \nabla\rangle = \sum_{j=1}^n A_j\partial/\partial x_j$. Because

$(\xi_j f(\xi))\hat{}(x) = i\partial/\partial x_j \hat{f}(x)$ for all $f \in \mathcal{S}(\mathbb{R}^n)$, it follows that

$$\big(\phi_k(\langle \boldsymbol{A}, \xi\rangle)\check{T}(\xi)\big)\hat{} = \phi_k(i\langle \boldsymbol{A}, \nabla\rangle)T$$

for every distribution $T \in C^\infty(\mathbb{R}^n)'$ whose inverse Fourier transform $\check{T}$ is a locally integrable function $\xi \longmapsto \check{T}(\xi)$ with at most polynomial growth at infinity. We write $id$ for the identity map on $\mathbb{R}^n$. Moreover

$$\big(\xi \cdot \nabla_\xi \check{T}(\xi)\big)\hat{} = -\nabla \cdot id\, T,$$

where $\nabla \cdot id$ is the operator on $C_c^\infty(\mathbb{R}^n)'$ defined on the dense subspace $C_c^\infty(\mathbb{R}^n)$ by $(\nabla \cdot id\, f)(x) = \sum_{j=1}^n \partial_j(x_j f(x))$ for all $f \in C_c^\infty(\mathbb{R}^n)$.

The remainder of the present section is devoted to a proof of the following result of E. Nelson [83, Theorem 9].

**Theorem 5.1.** *Let $\boldsymbol{A} = (A_1, \dots, A_n)$ be an $n$-tuple of $N \times N$ hermitian matrices. The Weyl calculus for the $n$-tuple $\boldsymbol{A}$ is given by*

$$\mathcal{W}_{\boldsymbol{A}} = \sum_{k=0}^{N-1} \sum_{j=0}^{N-k-1} \sum_{m=0}^{j} (-1)^{k+m} \binom{j}{m} \frac{1}{(N-1-j+m)!} \times \\ \langle \boldsymbol{A}, \nabla\rangle^k \phi_{N-j-k-1}(\langle \boldsymbol{A}, \nabla\rangle)(\nabla \cdot id)^m \mu_{\boldsymbol{A}}.$$

*Remark 5.2.* (i) The statement of [83, Theorem 9] is essentially concerned with the situation in which $\boldsymbol{A}$ is a basis of the real vector space of all $N \times N$ hermitian matrices, so that $n = N^2$. However, general properties of the Weyl calculus enable a derivation of the statement above from [83, Theorem 9] for other values of $n$.

(ii) The operator $\nabla$ should be replaced by $-\nabla$ in equation (16) of [83], because $-\nabla$ is the operator corresponding to multiplication of the *inverse* Fourier transform by $i\lambda$ in equation (24) of [83]. The following example verifies this observation.

*Example 5.3.* Let $n = 3$, $N = 2$ and consider the the Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$, as in Example 4.1. By the theorem above,

$$\mathcal{W}_{\boldsymbol{\sigma}} = \mu_{\boldsymbol{\sigma}} - (\nabla \cdot id)\mu_{\boldsymbol{\sigma}} - \langle \sigma, \nabla\rangle \mu_{\boldsymbol{\sigma}} + \phi_1(\langle \boldsymbol{\sigma}, \nabla\rangle)\mu_{\boldsymbol{\sigma}}.$$

Because $\phi_1(\langle \boldsymbol{\sigma}, \nabla\rangle) = \mathrm{trace}(\langle \boldsymbol{\sigma}, \nabla\rangle) = 0$, the matrix $\mathcal{W}_{\boldsymbol{\sigma}}(f)$ is given for every $f \in C^\infty(\mathbb{R}^3)$ by the equation

$$\mathcal{W}_{\boldsymbol{\sigma}}(f) = I \int_{S^2} (f + \boldsymbol{n} \cdot \nabla f)\, d\mu + \int_{S^2} \langle \boldsymbol{\sigma}, \nabla\rangle f\, d\mu$$

of Example 4.1.

Let $M$ be an $N \times N$ matrix. The characteristic polynomial $p_M$ of $M$ is defined by $p_M(z) = \det(M - zI)$ for all $z \in \mathbb{C}$. For each $N \times N$ matrix $M$, let the complex numbers $a_0(M), \dots, a_n(M)$ be the coefficients of the characteristic polynomial $p_M$ of $M$.

Given $\zeta \in \mathbb{C}$, the characteristic polynomial $p_{M-\zeta I}$ of $M - \zeta I$ is given by $p_{M-\zeta I}(z) = \det(M - \zeta I - zI) = p_M(z + \zeta)$ for all $z \in \mathbb{C}$, so that

$$p_{M-\zeta I}(z) = \sum_{h=0}^{N} a_h(M)(z+\zeta)^h = \sum_{h=0}^{N} a_h(M) \sum_{l=0}^{h} \binom{h}{l} \zeta^{h-l} z^l.$$

By the Cayley-Hamilton theorem, $p_{M-\zeta I}(M - \zeta I) = 0$, so if $\zeta$ is not an eigenvalue of the matrix $M$, then on multiplying the resulting equation by $(M - \zeta I)^{-1}$ and rearranging the sums, we obtain

$$\begin{aligned}(M - \zeta I)^{-1} &= -p_M(\zeta)^{-1} \sum_{h=1}^{N} a_h(M) \sum_{l=1}^{h} \binom{h}{l} \zeta^{h-l} (M - \zeta I)^{l-1} \\ &= p_M(\zeta)^{-1} \sum_{k=0}^{N-1} M^k \times \\ &\qquad \left( \sum_{l=k+1}^{N} (-1)^{l-k} \sum_{h=l}^{N} a_h(M) \binom{h}{l} \binom{l-1}{k} \zeta^{h-k-1} \right).\end{aligned}$$

The equality $p_M(\zeta)I = (\zeta I - M) \sum_{k=0}^{N-1} \left( \sum_{j=0}^{N-k-1} a_{j+k+1}(M)\zeta^j \right) M^k$ is easily verified from the Cayley-Hamilton theorem.

Now suppose that $C$ is a simple closed curved in $\mathbb{C}$ surrounding the set $\sigma(M)$ of eigenvalues of the $N \times N$ matrix $M$. Then by the Riesz functional calculus,

$$\begin{aligned}e^{iM} &= \frac{1}{2\pi i} \int_C e^{i\zeta} (\zeta I - M)^{-1} \, d\zeta \\ &= \frac{1}{2\pi i} \sum_{k=0}^{N-1} \left( \sum_{j=0}^{N-k-1} a_{j+k+1}(M) \int_C \frac{e^{i\zeta} \zeta^j}{p_M(\zeta)} \, d\zeta \right) M^k. \end{aligned} \tag{5.2}$$

For all $t \in \mathbb{R}$ so close to one that $C$ surrounds $t\sigma(M)$ too, we have

$$\begin{aligned}\int_C \frac{e^{it\zeta}(i\zeta)^j}{p_M(\zeta)} \, d\zeta &= \frac{\partial^j}{\partial t^j} \int_C \frac{e^{it\zeta}}{p_M(\zeta)} \, d\zeta \\ &= \frac{\partial^j}{\partial t^j} t^{N-1} \int_C \frac{e^{i\zeta}}{p_{tM}(\zeta)} \, d\zeta.\end{aligned}$$

An appeal to Leibniz's formula for the differentiation of products yields

$$e^{iM} = \frac{1}{2\pi i} \sum_{k=0}^{N-1} M^k \sum_{j=0}^{N-k-1} i^{-j} a_{j+k+1}(M) \times$$

$$\sum_{m=0}^{j} \binom{j}{m} \frac{(N-1)!}{(N-1-j+m)!} \left[\frac{\partial^m}{\partial t^m} \int_C \frac{e^{iz}}{p_{tM}(z)}\,dz\right]_{t=1}. \quad (5.3)$$

To calculate the Fourier transform of (5.3) with $M = \langle \boldsymbol{A}, \xi\rangle$, the following observation is useful.

**Proposition 5.4.** *Let $\nu$ be the unitarily invariant probability measure on the unit sphere $S(\mathbb{C}^N)$ in $\mathbb{C}^N$. Let $M$ be a normal $N \times N$ matrix and let $U$ be a simply connected open subset of $\mathbb{C}$ containing the convex hull $co(\sigma(M))$ of the spectrum $\sigma(M)$ of $M$.*

*Let $C$ be a simple closed curve around $\sigma(M)$ contained in $U$ and suppose that $f : U \to \mathbb{C}$ is analytic. Then*

$$\frac{1}{2\pi i}\int_C f(\zeta) p_M(\zeta)\,d\zeta = (-1)^N (N-1)! \int_{S(\mathbb{C}^N)} f^{(N-1)}\big(\langle Mu, u\rangle\big)\,d\nu(u) \quad (5.4)$$

*Proof.* Suppose first that $x \in \mathbb{C}, x \neq 0$ and $f(z) = e^{-xz}$ for all $z \in \mathbb{C}$. If $\lambda_1, \dots, \lambda_N$ are distinct complex numbers and $M$ is the diagonal matrix with entries $\lambda_1, \dots, \lambda_N$, then $f\big(\langle Mu, u\rangle\big) = \exp(-x\sum_{j=1}^N \lambda_j |u_j|^2)$ for $u = (u_1, \dots, u_N) \in \mathbb{C}^N$ and

$$\begin{aligned}
\frac{1}{2\pi i}\int_C f(\zeta) p_M(\zeta)\,d\zeta &= \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\lambda_1 - \zeta)\cdots(\lambda_N - \zeta)}\,d\zeta \\
&= \frac{(-1)^N}{2\pi i}\sum_{k=1}^N \frac{1}{\prod_{j\neq k}(\lambda_k - \lambda_j)} \int_C \frac{f(\zeta)}{\zeta - \lambda_k}\,d\zeta \\
&= (-1)^N \sum_{k=1}^N \frac{e^{-x\lambda_k}}{\prod_{j\neq k}(\lambda_k - \lambda_j)}. \qquad (5.5)
\end{aligned}$$

Let $\Delta_{N-1} = \{w \in \mathbb{R}^N : w_j \geq 0, \sum_{j=1}^N w_j = 1\}$ be the unit simplex in $\mathbb{R}^N$ and denote the normalised Lebesgue measure on $\Delta_{N-1}$ by $\sigma$. Then, as argued in [83, p. 186], $\sigma$ is the image of the measure $\nu$ under the map $u \longmapsto (|u_1|^2, \dots, |u_n|^2), u \in \mathbb{C}^N$. The measure $\frac{1}{(N-1)!}\sigma$ is the image of the Lebesgue measure on the set $\{0 \leq t_1 \leq \cdots \leq t_{N-1} \leq 1\}$ by the affine bijection $w_j = t_j - t_{j-1}$, $j = 1, \dots, N$ with $t_0 = 0$ and $t_N = 1$, so by the change of variables formula,

$$\begin{aligned}
&\frac{1}{(N-1)!}\int_{S(\mathbb{C}^N)} \exp(-x\sum_{j=1}^N \lambda_j |u_j|^2)\,d\nu(u) \\
&\quad = \frac{1}{(N-1)!}\int_{\Delta_{N-1}} \exp(-x\sum_{j=1}^N \lambda_j w_j)\,d\sigma(w) \\
&\quad = e^{-x\lambda_N}\int_0^1 \int_0^{t_{N-1}} \cdots \int_0^{t_2} e^{x\sum_{j=1}^{N-1}(\lambda_{j+1} - \lambda_j)t_j}\,dt_1 \cdots dt_{N-1}
\end{aligned}$$

$$= \frac{(-1)^{N-1}}{x^{N-1}} \sum_{k=1}^{N} \frac{e^{-x\lambda_k}}{\prod_{j\neq k}(\lambda_k - \lambda_j)}.$$

Combined with the preceding calculation, we obtain

$$\frac{1}{2\pi i}\int_C \frac{f(\zeta)}{p_M(\zeta)}\, d\zeta = -\frac{x^{N-1}}{(N-1)!}\int_{S(\mathbb{C}^N)} \exp(-x\sum_{j=1}^{N}\lambda_j|u_j|^2)\, d\nu(u) \tag{5.6}$$

The continuity of both sides in $\lambda = (\lambda_1, \ldots, \lambda_N)$ ensures the equality (5.6) for all $\lambda \in \mathbb{C}^N$ with $M = diag(\lambda_1, \ldots, \lambda_N)$.

Now suppose that $q$ is a polynomial, $\phi : \mathbb{R} \to \mathbb{R}$ is a function of the form $\phi(x) = q(x)e^{-x^2/2}, x \in \mathbb{R}$, and $f(z) = \int_{\mathbb{R}} e^{-ixz}\phi(x)\, dx$ for all $z \in \mathbb{C}$. Then the equality (5.4) is valid for $f$ by Fubini's theorem and equation (5.6), which is also valid with $x \in \mathbb{R}$ replaced by $ix$.

The set $U$ is simply connected, so by Runge's theorem [100, Theorem13.11], an analytic function on $U$ can be approximated uniformly on any compact subset of $U$ by polynomials, and so also by functions of the form $z \longmapsto p(z)e^{-z^2/2}, z \in \mathbb{C}$ with $p$ a polynomial. Any such function is (the analytic continuation of) the Fourier transform of a function of the form $\phi$ above.

Thus, the set of all functions $f$ for which (5.4) is true is dense in the space $H(U)$ of functions analytic in the set $U$, with respect to the topology of uniform convergence on compact subsets of $U$. The right-hand side of (5.4) is continuous for the topology of $H(U)$ because $U$ contains the convex hull of $\sigma(M)$. It follows that equation (5.4) is true for all $f \in H(U)$ and all diagonal matrices $M$. Both sides of (5.4) are unchanged if $M$ is replaced by $UMU^*$ for a unitary transformation $U$ of $\mathbb{C}^N$, so (5.4) is valid for all normal matrices $M$. □

*Proof (of Theorem 5.1).* If we apply the proposition to the function $f(z) = e^{iz}, z \in \mathbb{C}$ and the hermitian matrix $M = \langle \boldsymbol{A}, \xi\rangle$ and note, as mentioned earlier, that $\check{\mu}_{\boldsymbol{A}}(\xi) = (2\pi)^{-n}\int_{S(\mathbb{C}^N)} e^{i\langle\langle \boldsymbol{A},\xi\rangle u,u\rangle}\, d\nu(u)$, then it follows that for every $\xi \in \mathbb{R}^n$, the equality

$$\frac{1}{2\pi i}\int_{C(\xi)} \frac{e^{iz}}{p_{\langle \boldsymbol{A},\xi\rangle}(z)}\, dz = -\frac{(-i)^{N-1}(2\pi)^n}{(N-1)!}\check{\mu}_{\boldsymbol{A}}(\xi). \tag{5.7}$$

holds for any simple closed contour $C(\xi)$ around $\sigma(\langle \boldsymbol{A}, \xi\rangle)$.

The equality $\left[\frac{d}{dt}g(tx)\right]_{t=1} = x \cdot \nabla_x g(x)$ is valid for all differentiable functions $g$ on $\mathbb{R}^N$, so for $M = \langle \boldsymbol{A}, \xi\rangle$, equation (5.3) becomes

$$e^{i\langle \boldsymbol{A},\xi\rangle} = (2\pi)^n(-1)^N \textstyle\sum_{k=0}^{N-1}\langle \boldsymbol{A}, \xi\rangle^k \sum_{j=0}^{N-k-1} i^{N-j-1}a_{j+k+1}(\langle \boldsymbol{A}, \xi\rangle)\times \\ \textstyle\sum_{m=0}^{j}\begin{pmatrix} j \\ m\end{pmatrix} \frac{(\xi\cdot\nabla_\xi)^m\check{\mu}_{\boldsymbol{A}}(\xi)}{(N-1-j+m)!}. \tag{5.8}$$

The coefficients of the characteristic polynomial of an $N \times N$ matrix $M$ are calculated from the sums of the principal minors by virtue of the equality $a_s(M) = (-1)^s\phi_{N-s}(M)$, for all $s = 0, \ldots, n-1$. If we substitute for

$a_s(M), s = 1, \dots, n-1$ in (5.8), and note that $\mathcal{W}_{\boldsymbol{A}}$ is $(2\pi)^{-n}$ times the Fourier transform of the distribution defined by the right-hand side of (5.8), then it follows from the discussion at the beginning of this section that the Weyl calculus for the $n$-tuple $\boldsymbol{A}$ is given by

$$\begin{aligned}\mathcal{W}_{\boldsymbol{A}} &= (-1)^N \sum_{k=0}^{N-1} \sum_{j=0}^{N-k-1} \sum_{m=0}^{j} (-1)^{j+k+1} i^{N-j-1} \binom{j}{m} \frac{1}{(N-1-j+m)!} \times \\ &\qquad (i\langle \boldsymbol{A}, \nabla\rangle)^k \phi_{N-j-k-1}(i\langle \boldsymbol{A}, \nabla\rangle)(-\nabla \cdot id)^m \mu_{\boldsymbol{A}} \\ &= \sum_{k=0}^{N-1} \sum_{j=0}^{N-k-1} \sum_{m=0}^{j} (-1)^{k+m} \binom{j}{m} \frac{1}{(N-1-j+m)!} \times \\ &\qquad \langle \boldsymbol{A}, \nabla\rangle^k \phi_{N-j-k-1}(\langle \boldsymbol{A}, \nabla\rangle)(\nabla \cdot id)^m \mu_{\boldsymbol{A}}. \qquad \Box\end{aligned}$$

*Remark 5.5.* If $\boldsymbol{A}$ is a $n$-tuple of matrices for which there exists an $n$-tuple $\boldsymbol{A}'$ of *hermitian* matrices such that $\langle \boldsymbol{A}, \xi\rangle$ and $\langle \boldsymbol{A}', \xi\rangle$ have the same characteristic polynomial for every $\xi \in \mathbb{R}^n$, then the same formula obviously applies with $\mu_{\boldsymbol{A}}$ replaced by $\mu_{\boldsymbol{A}'}$. For example, $\boldsymbol{A}$ consists of mutually commuting or simultaneously triangularisable matrices with real eigenvalues, see Section 5.4 below.

## 5.2 Exponential Bounds for Matrices

A system $\boldsymbol{A} = (A_1, \dots, A_n)$ of $(N \times N)$ matrices of type $s$ in the sense of Definition 2.2 has the property that

$$\sigma(\langle \boldsymbol{A}, \xi\rangle) \subseteq \mathbb{R}, \quad \text{for all } \xi \in \mathbb{R}^n. \tag{5.9}$$

Indeed, this follows even for a system $\boldsymbol{A}$ of bounded linear operators acting on a Banach space [23, Theorem 4.5, p.160]. The purpose of this section is to prove that condition (5.9) implies the bound (2.3) for a system of *matrices*. A commuting pair $\boldsymbol{A}$ of bounded linear operators acting on $\ell^2(\mathbb{N})$ such that condition (5.9) holds but the bound (2.2) fails is given in Example 4.11.

Although $\langle \boldsymbol{A}, \zeta\rangle$ may be put into its Jordan normal form for each $\zeta \in \mathbb{C}^n$ so that the eigenvalues are real if $\zeta \in \mathbb{R}^n$, it is not obvious how the imaginary parts of the eigenvalues will grow as $|\Im\zeta| \to \infty$. A further difficulty is that the similarity transformations $U(\zeta)$ which put $\langle T, \zeta\rangle$ into its Jordan normal form $J(\zeta)$ are only holomorphic in an open set in which $J(\zeta)$ has constant structure [13, Theorem 3, p. 387]. This difficulty is bypassed by appealing to a formula of E. Nelson in the preceding section expressing the exponential $e^{i\langle \boldsymbol{A}, \zeta\rangle}$ in terms of powers of $\langle \boldsymbol{A}, \zeta\rangle$ up to order $N-1$.

In Theorem 5.10 below it is shown that a system of matrices satisfying (5.9) is of *Paley-Wiener type* $(s, r)$, that is, the bound (2.3) holds. A system of simultaneously triangularisable matrices with real spectrum satisfies (5.9), so Theorem 5.10 is a generalisation of [88, Theorem 4.5].

In order to prove Theorem 5.10, we use the fact, proved in Theorem 5.7, that the imaginary part of the spectrum of $At+iB$ is uniformly bounded for all $t \in \mathbb{R}$ provided that the bounded linear operators $A, B$ have the property that every real linear combination of them has real spectrum. This observation may be of independent interest. The proof appeals to properties of the monogenic functional calculus described in Chapter 4.

Another proof that the bound (2.3) follows from condition (5.9) may be deduced from the theory of the hyperbolic system

$$\frac{\partial u}{\partial t} + \sum_{j=1}^{n} A_j \frac{\partial u}{\partial x_j} = 0, \quad u(x,0) = \delta_0(x)v, \ v \in \mathbb{R}^n, \tag{5.10}$$

of partial differential equations. Condition (5.9) is an expression of Gårding's hyperbolicity condition [58, pp. 149–151], which is equivalent to the statement that the system (5.10) has a suitable fundamental solution. From Holmgren's uniqueness theorem [58, p. 83], we know that the domain of dependence of the distributional solution $u$ of (5.10) is a cone with finite diameter in $\mathbb{R}^{n+1}$, see [58, p. 153], from which the bound (2.3) follows. Estimates for the parameters $C$ and $r$ in (2.3) are obtained in the course of the proof of Theorem 5.10 below.

### 5.2.1 Perturbation

Suppose that $A, B$ are $(N \times N)$ matrices and the spectrum $\sigma(A)$ of $\boldsymbol{A}$ is real. If $A$ and $B$ commute, then for each $t \in \mathbb{R}$, the spectrum $\sigma(At+iB)$ of the matrix $At+iB$ is contained in the set of complex numbers $\lambda t + i\mu$ with $\lambda \in \sigma(A)$ and $\mu \in \sigma(B)$ . It follows that $\sup_{t\in\mathbb{R}} |\Im(\sigma(At+iB))| < \infty$. However, if $A$ and $B$ do not commute, this bound can fail.

By finite dimensional perturbation theory [65, Theorem II.5.1], the unordered set of eigenvalues of $At+iB$ is a continuous function of $t$, so for any $T > 0$, the set

$$\bigcup_{|t|\le T} \Im(\sigma(At+iB))$$

is a compact subset of $\mathbb{R}$. On the other hand, because $\sigma(A)$ is real, $\Im(\sigma(A+iB/t)) \to \{0\}$ as $|t| \to \infty$, so that $\Im(\sigma(At+iB)) = \{o(|t|)\}$ as $|t| \to \infty$. The following example shows that if $N \ge 2$, the set $\Im(\sigma(At+iB))$ may not be bounded as $|t| \to \infty$.

*Example 5.6.* Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Then

$$\sigma(At+iB) = \{t + i \pm e^{i\pi/4}\sqrt{t}\}$$

for all $t \ge 0$. On the other hand, if $B' = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$, then $\sigma(At+iB') = \{t \pm i\}$.

The matrices $A$ and $B'$ above are both upper triangular and each real linear combination of them has real spectrum. By contrast, the matrices $A$ and $B$ each have real spectrum but $\sigma(A-B) = \{\pm i\}$. Moreover,

$$e^{i(A-B)t} = \cosh(t)I + \sinh(t)\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix},$$

so the bound (2.3) certainly fails.

We only need the following result for matrices in the proof of Theorem 5.10, but the proof below is valid for bounded linear operators acting on a Banach space.

**Theorem 5.7.** *Let $A, B$ be bounded linear operators acting on the Banach space $X$ with the property that $\sigma(A\xi_1 + B\xi_2) \subset \mathbb{R}$ for all $\xi \in \mathbb{R}^2$. Then there exist $q, r > 0$ such that*

$$\sigma(At + iB) \subseteq [-q, q]t + i[-r, r], \quad \textit{for all } t \in \mathbb{R}. \tag{5.11}$$

*Moreover, $q, r$ are bounded by $(1+\sqrt{2})(\|A\|^2 + \|B\|^2)^{1/2}$.*

Once we have the notion of the monogenic functional calculus considered in Chapter 4, the proof of the following lemma is straightforward.

**Lemma 5.8.** *Let $A, B$ be bounded linear operators acting on the Banach space $X$ with the property that $\sigma(A\xi_1 + B\xi_2) \subset \mathbb{R}$ for all $\xi \in \mathbb{R}^2$. Let $\gamma(A, B)$ be the monogenic spectrum of the pair $(A, B)$.*

*For every $\lambda \in \mathbb{C}$ and $t \in \mathbb{R}$, set*

$$f_{\lambda,t}(x) = (\lambda - x_1 t - i x_2)^{-1}$$

*for all $x = (x_1, x_2) \in \mathbb{R}^2$ such that $x_1 t + i x_2 \neq \lambda$.*

*Then for every $t \in \mathbb{R}$, the complement $R_t(A, B)$ of the set $\{x_1 t + i x_2 : (x_1, x_2) \in \gamma(A, B)\}$ is contained in the resolvent set of the operator $At + iB$ and the equality*

$$f_{\lambda,t}(A, B) = (\lambda I - At - iB)^{-1}, \quad \lambda \in R_t(A, B), \tag{5.12}$$

*is valid.*

*Proof.* For each $j = 1, 2$, the unique left and right monogenic extension of the coordinate function $x \longmapsto x_j$, $x \in \mathbb{R}^2$, is $\omega \longmapsto \omega_j e_0 - \omega_0 e_j$, $\omega = (\omega_0, \omega_1, \omega_2) \in \mathbb{R}^3$.

Let $\tilde{f}_{\lambda,t}$ be the $\mathbb{C}_{(2)}$-valued function given by

$$\begin{aligned} \tilde{f}_{\lambda,t}(\omega) &= (\lambda e_0 - (\omega_1 e_0 - \omega_0 e_1)t - i(\omega_2 e_0 - \omega_0 e_2))^{-1} \\ &= \frac{(\overline{\lambda} - \omega_1 t + i\omega_2)e_0 - \omega_0 t e_1 + i\omega_0 e_2}{|\lambda - \omega_1 t - i\omega_2|^2 + \omega_0^2(t^2+1)} \end{aligned}$$

for all $\omega \in \mathbb{R}^3$ for which the denominator is nonzero.

The restriction of $\tilde{f}_{\lambda,t}$ to $\mathbb{R}^2$ is equal to $f_{\lambda,t}$, that is, on putting $\omega_0 = 0$. The function $\tilde{f}_{\lambda,t}$ takes its values in the linear subspace spanned by $e_0, e_1, e_2$ in $\mathbb{C}_{(2)}$ and is left and right monogenic. Then for every complex number $\lambda \in R_t(A,B)$, the operator $\tilde{f}_{\lambda,t}(A,B)$ is defined by formula (4.17) for a suitable choice of the open set $\Omega$.

The Neumann series expansion

$$\sum_{k=0}^{\infty} \frac{1}{\lambda^{k+1}}(At+iB)^k$$

of $(\lambda I - At - iB)^{-1}$ converges for $|\lambda|$ large enough. Moreover, the sums

$$f_{\lambda,t}(x) = \sum_{k=0}^{\infty} \frac{1}{\lambda^{k+1}}(x_1 t + i x_2)^k$$
$$\tilde{f}_{\lambda,t}(\omega) = \sum_{k=0}^{\infty} \frac{1}{\lambda^{k+1}}((\omega_1 e_0 - \omega_0 e_1)t + i(\omega_2 e_0 - \omega_0 e_2))^k$$

converge uniformly as $x \in \mathbb{R}^2$ and $\omega \in \mathbb{R}^3$ range over compact sets and $\lambda$ is outside a sufficiently large ball. Set

$$p_{k,t}(x) = (x_1 t + i x_2)^k, \quad x \in \mathbb{R}^2,$$
$$\tilde{p}_{k,t}(\omega) = ((\omega_1 e_0 - \omega_0 e_1)t + i(\omega_2 e_0 - \omega_0 e_2))^k, \quad \omega \in \mathbb{R}^3,$$

for each $k = 0, 1, 2 \ldots$ .

Now $f_{\lambda,t}(A,B) = \tilde{f}_{\lambda,t}(A,B)$, by definition, for all $\lambda \in R_t(A,B)$. According to Theorem 4.22 (ii), the equality

$$p_{k,t}(A,B) = \tilde{p}_{k,t}(A,B) = (At+iB)^k$$

holds for all $k = 0, 1, 2 \ldots$, so by the continuity of the monogenic functional calculus $f \longmapsto f(A,B)$ (Proposition 4.20), we have

$$f_{\lambda,t}(A,B) = \sum_{k=0}^{\infty} \frac{1}{\lambda^{k+1}}(At+iB)^k = (\lambda I - At - iB)^{-1},$$

that is,

$$(\lambda I - At - iB) f_{\lambda,t}(A,B) = f_{\lambda,t}(A,B)(\lambda I - At - iB) = I, \tag{5.13}$$

for all $\lambda \in \mathbb{C}$ with $|\lambda|$ sufficiently large.

Now $f_{\lambda,t}(A,B)$ is defined by formula (4.17) for all $\lambda \in R_t(A,B)$ and by differentiation under the integral (4.17), we see that $\lambda \longmapsto f_{\lambda,t}(A,B)$, $\lambda \in R_t(A,B)$, is a complex-analytic $\mathcal{L}(X)$-valued function. It follows that equation (5.13) holds for all $\lambda \in R_t(A,B)$, so that the resolvent set of the operator $At + iB$ contains the $R_t(A,B)$ and the equality (5.12) holds. □

*Proof (Proof of Theorem 5.7).* Put

$$q = \sup\{|x_1| : (x_1, x_2) \in \gamma(A, B)\},$$
$$r = \sup\{|x_2| : (x_1, x_2) \in \gamma(A, B)\}.$$

Then for every $t \in \mathbb{R}$, the complement of the rectangle $[-q, q]t + i[-r, r]$ is contained in the set $R_t(A, B)$ defined in Lemma 5.8. The inclusion (5.11) follows from Lemma 5.8.

The bound for $q, r$ follows from the expansion (4.9) for $G_\omega(A, B)$, which converges for $|\omega| > (1 + \sqrt{2})(\|A\|^2 + \|B\|^2)^{1/2}$ by Lemma 4.7. □

*Remark 5.9.* For bounded selfadjoint operators $A, B$ acting on a Hilbert space, the spectrum $\sigma(At + iB)$ is contained in the numerical range of $At + iB$, so the result follows immediately.

### 5.2.2 The Exponential Bound

This section is devoted to proving the following result in which an algebraic condition implies a matrix-norm bound on exponentials. Let $N \geq 2$ be an integer.

**Theorem 5.10.** *Let $\boldsymbol{A} = (A_1, \ldots, A_n)$ be an $n$-tuple of $N \times N$ matrices satisfying the spectral condition (5.9).*

*Then there exist numbers $C > 0$ and $r \geq 0$ such that*

$$\|e^{i\langle \boldsymbol{A}, \zeta\rangle}\| \leq C(1 + |\zeta|)^{N-1} e^{r|\Im\zeta|} \quad \textit{for all } \zeta \in \mathbb{C}^n.$$

*Proof.* We first observe that for all $\zeta \in \mathbb{C}^n$ satisfying $|\Re\zeta| \leq 1$, the bound

$$\|e^{i\langle \boldsymbol{A}, \zeta\rangle}\| \leq e^R e^{R|\Im\zeta|}$$

for $R = \left(\sum_{j=1}^n \|A_j\|^2\right)^{1/2}$, so we need only consider the case $|\Re\zeta| \geq 1$.

On setting $M = \langle \boldsymbol{A}, \zeta\rangle$ with $\zeta \in \mathbb{C}^n$ in equation (5.3), we have

$$e^{i\langle \boldsymbol{A}, \zeta\rangle} = \frac{1}{2\pi i} \sum_{k=0}^{N-1} \left( \sum_{j=0}^{N-k-1} a_{j+k+1}(\langle \boldsymbol{A}, \zeta\rangle) \int_C \frac{e^{iz} z^j}{p_{\langle \boldsymbol{A}, \zeta\rangle}(z)} \, dz \right) \langle \boldsymbol{A}, \zeta\rangle^k. \quad (5.14)$$

Here $C$ is any simple closed contour with the finite set $\sigma(\langle \boldsymbol{A}, \zeta\rangle)$ in its interior.

Let $S(\mathbb{C}^n) = \{\zeta \in \mathbb{C}^n : |\zeta| = 1\}$ and $u, v \geq 0$. It follows from [65, Theorem 4.14] and the compactness of $S(\mathbb{C}^n)$ in $\mathbb{C}^n$ that

$$\bigcup \left\{\sigma\big(\langle \boldsymbol{A}, \Re\zeta\rangle u + i\langle \boldsymbol{A}, \Im\zeta\rangle v\big) : \zeta \in S(\mathbb{C}^n)\right\} \quad (5.15)$$

is a compact subset of $\mathbb{C}$.

According to Theorem 5.7, there exists $r > 0$ such that for every $\xi, \eta \in \mathbb{R}^n$ with $\xi + i\eta \in S(\mathbb{C}^n)$ and $t > 0$, the spectrum $\sigma\left(\langle \boldsymbol{A}, \xi\rangle t + i\langle \boldsymbol{A}, \eta\rangle\right)$ of the matrix $\langle \boldsymbol{A}, \xi\rangle t + i\langle \boldsymbol{A}, \eta\rangle$ is contained in $t[-r, r] + i[-r, r]$. The number

$$r = \sup\{|x| : x \in \gamma(\langle \boldsymbol{A}, \xi\rangle, \langle \boldsymbol{A}, \eta\rangle),\ \xi + i\eta \in S(\mathbb{C}^n)\ \} \tag{5.16}$$

is bounded by $(1+\sqrt{2})\big(\sum_{j=1}^n \|A_j\|^2\big)^{1/2}$.

If $u, v > 0$, then setting $t = u/v$, we find that the spectrum of $\langle \boldsymbol{A}, \xi\rangle u + i\langle \boldsymbol{A}, \eta\rangle v$ is contained in $u[-r, r] + iv[-r, r]$. Hence, for every $u, v > 0$, the set (5.15) is contained in the rectangle $u[-r, r] + iv[-r, r]$.

Fix $\xi, \eta \in \mathbb{R}^n$ with $\xi + i\eta \in S(\mathbb{C}^n)$ and set $A = \langle \boldsymbol{A}, \xi\rangle$ and $B = \langle \boldsymbol{A}, \eta\rangle$. Let $r' > r$ and let $C[u, v]$ be the positively oriented closed contour bounding the rectangle

$$u[-r', r'] + i(1+v)[-r', r'].$$

Then for every $u, v \geq 0$, the contour $C[u, v]$ contains the spectrum $\sigma(Au+iBv)$ in its interior and it is bounded away from the $x$-axis as $v \to 0+$.

Furthermore, the bound

$$\begin{aligned}
&\left|\int_{C[u,v]} \frac{e^{iz} z^j}{p_{Au+iBv}(z)}\, dz\right| \\
&\quad \leq e^{r'} e^{vr'}((r'u)^2 + (r'(1+v))^2))^{j/2} \int_{C[u,v]} \frac{|dz|}{|p_{Au+iBv}(z)|} \\
&\quad \leq e^{r'} e^{vr'} (r')^j \left(1 + \sqrt{u^2+v^2}\right)^j \int_{C[u,v]} \frac{|dz|}{|p_{Au+iBv}(z)|}. 
\end{aligned} \tag{5.17}$$

holds. Arclength measure is denoted by $|dz|$. All the roots of the polynomial $p_{Au+iBv}$ are contained in the rectangle $u[-r, r] + iv[-r, r]$.

For all $t > 0$ and $0 \leq w \leq 1$, the function $x \longmapsto p_{At+iBw}(-x + ir')$ is a polynomial of degree $N \geq 2$ whose modulus is bounded below by $(r' - wr)^N$, so for all $u \geq 0$ and $v \geq 0$, the upper part of the integral (5.17) about $C[u, v]$ is bounded by

$$\begin{aligned}
\int_{-r'u}^{r'u} \frac{dx}{|p_{Au+iBv}(-x + i(1+v)r')|} &\leq \int_{-\infty}^{\infty} \frac{dx}{|p_{Au+iBv}(-x + i(1+v)r')|} \\
&= \frac{1}{(1+v)^{N-1}} \int_{-\infty}^{\infty} \frac{dx}{|p_{Au'+iBv'}(-x + ir')|}, \\
&\qquad \text{where } u' = u/(1+v),\ v' = v/(1+v), \\
&\leq \sup_{t>0, 0\leq w\leq 1} \int_{-\infty}^{\infty} \frac{dx}{|p_{At+iBw}(-x + ir')|}.
\end{aligned}$$

To see that the supremum is finite, for each $t > 0$ and $0 \leq w \leq 1$, let $\Gamma(t, w)$ be the unordered $N$-tuple of eigenvalues of the $(N \times N)$ matrix $At + iBw$, counting the possible multiplicity of eigenvalues. Then $|\Re\lambda| \leq rt$ and $|\Im\lambda| \leq r$ for each element $\lambda \in \Gamma(t, w)$.

Because $|p_{At+iBw}(z)| = \prod_{\lambda\in\Gamma(t,w)} |z - \lambda|$, Hölder's inequality implies that

$$\int_{-\infty}^{\infty} \frac{dx}{|p_{At+iBw}(-x + ir')|} \leq \prod_{\lambda\in\Gamma(t,w)} \left(\int_{-\infty}^{\infty} \frac{dx}{|-x + ir' - \lambda|^N}\right)^{1/N}$$

$$= \prod_{\lambda\in\Gamma(t,w)} \left(\int_{-\infty}^{\infty} \frac{dx}{(x^2+(r'-\Im\lambda)^2)^{N/2}}\right)^{1/N}$$
$$\leq \int_{-\infty}^{\infty} \frac{dx}{(x^2+(r'-r)^2)^{N/2}} = \frac{\sqrt{\pi}\Gamma(\frac{N-1}{2})}{(r'-r)^{N-1}\Gamma(\frac{N}{2})}.$$

For all $t>0$, the function $y\longmapsto p_{A+iBt}(r'+iy)$, $y\in\mathbb{R}$, is a polynomial of degree $N$ whose modulus is bounded below by $(r'-r)^N$, so an argument similar to that above shows that for all $u\geq 1$, the right hand part of the integral (5.17) is bounded by

$$\int_{-r'(1+v)}^{r'(1+v)} \frac{dy}{|p_{Au+iBv}(ur'+iy)|} \leq \int_{-\infty}^{\infty} \frac{dy}{|p_{Au+iBv}(ur'+iy)|}$$
$$\leq \sup_{t>0} \int_{-\infty}^{\infty} \frac{dy}{|p_{A+iBt}(r'+iy)|}.$$

Let $r$ be given by the supremum (5.16). For each $r'>r$, let $N_{r'}(\xi+i\eta)$ be the maximum of the numbers

$$\sup_{t>0,0\leq w\leq 1} \int_{-\infty}^{\infty} \frac{dx}{|p_{At+iBw}(\pm(-x+ir'))|}, \quad \sup_{t>0} \int_{-\infty}^{\infty} \frac{dy}{|p_{A+iBt}(\pm(r'+iy))|}.$$

For $\zeta\in\mathbb{C}^N$ with $|\Re\zeta|\geq 1$, let $\zeta'=\zeta/|\zeta|$. Then $\|e^{i\langle \boldsymbol{A},\zeta\rangle}\|$ is bounded by

$$\frac{1}{2\pi}\sum_{k=0}^{N-1} |\langle \boldsymbol{A},\zeta'\rangle|^k| \sum_{j=0}^{N-k-1} |a_{j+k+1}(\langle \boldsymbol{A},\zeta'\rangle)| \left|\int_C \frac{e^{iz}z^j}{p_{\langle \boldsymbol{A},\zeta\rangle}(z)}\,dz\right| |\zeta|^{N-j-1}$$
$$\leq \frac{N_{r'}(\zeta')e^{r'}e^{r'|\Im\zeta|}}{2\pi} \sum_{k=0}^{N-1} |\langle \boldsymbol{A},\zeta'\rangle|^k$$
$$\times \sum_{j=0}^{N-k-1} |a_{j+k+1}(\langle \boldsymbol{A},\zeta'\rangle)|(r')^j(1+|\zeta|)^j||\zeta|^{N-j-1}$$
$$\leq C_{r'}(1+|\zeta|)^{N-1}e^{r'|\Im\zeta|}.$$

The constant $C_{r'}$ is the supremum of

$$\frac{N_{r'}(\zeta')e^{r'}}{2\pi}\left(\sum_{k=0}^{N-1} |\langle \boldsymbol{A},\zeta'\rangle|^k \sum_{j=0}^{N-k-1} |a_{j+k+1}(\langle \boldsymbol{A},\zeta'\rangle)|(r')^j\right)$$

for $\zeta'\in S(\mathbb{C}^n)$. $\square$

The best estimate for general $r$ is given in Theorem 2.7.

**Corollary 5.11.** *Let* $\boldsymbol{A}=(A_1,\dots,A_n)$ *be an* $n$*-tuple of* $N\times N$ *matrices satisfying the spectral condition (5.9).*

*Then* $\boldsymbol{A}$ *is of type* $(s,r)$ *with* $0 \le s \le N-1$ *and* $0 \le r \le \sqrt{\sum_{j=1}^n \|A_j\|^2}$. *The support of* $\mathcal{W}_{\boldsymbol{A}}$ *is contained in the rectangle*

$$[-\|A_1\|, \|A_1\|] \times \cdots \times [-\|A_n\|, \|A_n\|]$$

*in* $\mathbb{R}^n$.

*Let* $T_{\boldsymbol{A}} \in \mathcal{S}'(\mathbb{R}^n)$ *be the Fourier transform of the uniformly bounded function*

$$\xi \longmapsto \frac{i^N (N-1)!}{(2\pi)^{n+1}} \int_{C(\xi)} \frac{e^{iz}}{p_{\langle \boldsymbol{A},\xi\rangle}(z)}\, dz, \quad \xi \in \mathbb{R}^n. \tag{5.18}$$

*For any* $\xi \in \mathbb{R}^n$, *the simple closed contour* $C(\xi)$ *contains* $\sigma(\langle \boldsymbol{A}, \xi\rangle)$ *in its interior. Then the Weyl calculus for the* $n$*-tuple* $\boldsymbol{A}$ *is given by*

$$\mathcal{W}_{\boldsymbol{A}} = \sum_{k=0}^{N-1} \sum_{j=0}^{N-k-1} \sum_{m=0}^{j} (-1)^{k+m} \binom{j}{m} \frac{1}{(N-1-j+m)!} \times \\ \langle \boldsymbol{A}, \nabla\rangle^k \phi_{N-j-k-1}(\langle \boldsymbol{A}, \nabla\rangle)(\nabla \cdot id)^m T_{\boldsymbol{A}}. \tag{5.19}$$

*Proof.* According to Theorem 5.10, the $n$-tuple $\boldsymbol{A}$ is of type $(N-1, r')$ for $r' > (1+\sqrt{2})\big(\sum_{j=1}^n \|A_j\|^2\big)^{1/2}$, so Theorem 2.7 gives the stated bounds. Formula (5.19) for $\mathcal{W}_{\boldsymbol{A}}$ follows from equation (5.3) by taking the Fourier transform in the sense of distributions. □

If $\boldsymbol{A}$ is an $n$-tuple of hermitian matrices, then according to equation (5.7) we have $T_{\boldsymbol{A}} = \mu_{\boldsymbol{A}}$. Another example for which the distribution $T_{\boldsymbol{A}}$ can be calculated explicitly is for an $n$-tuple $\boldsymbol{A}$ of simultaneously triangularisable matrices with real eigenvalues, for then $T_{\boldsymbol{A}} = T_{\tilde{\boldsymbol{A}}} = \mu_{\tilde{\boldsymbol{A}}}$ for an $n$-tuple $\tilde{\boldsymbol{A}}$ of *diagonal* matrices with real entries. We shall look at this example more closely in Section 5.4 below.

## 5.3 The Joint Spectrum of Pairs of Hermitian Matrices

For a pair $\boldsymbol{A} = (A_1, A_2)$ of hermitian matrices, it is possible to determine the joint spectrum $\gamma(\boldsymbol{A})$ by computing the plane wave decomposition (4.16) for the Cauchy kernel $\omega \longmapsto G_\omega(\boldsymbol{A})$ and determining its set $\gamma(\boldsymbol{A})$ of singularities. The plane wave formula (4.16) can be converted to a contour integral in the complex plane and evaluated by residues. The calculation is facilitated by a perturbation result of Rellich [65, Theorem II.6.1] applicable to hermitian matrices. On the other hand, Example 5.31 in Section 5.4 shows that the main result of this section, Theorem 5.24, may fail for two $(2 \times 2)$ upper triangular matrices with real eigenvalues.

### 5.3.1 The Numerical Range of Matrices

Let $\boldsymbol{A} = (A_1, A_2)$ be a pair of $(N \times N)$ hermitian matrices. Set $A = A_1 + iA_2$. An application of the Paley-Wiener Theorem yields that the convex hull of the support $\operatorname{supp}(\mathcal{W}_{\boldsymbol{A}})$ of the associated Weyl distribution $\mathcal{W}_{\boldsymbol{A}}$ coincides with the *numerical range* $K(A) = \{\langle Ax, x\rangle \mid x \in \mathbb{C}^N,\ |x| = 1\}$ of the matrix $A$. For more precise information on the location of $\operatorname{supp}(\mathcal{W}_{\boldsymbol{A}})$ within the numerical range of $A$, we need to have a closer look at the fine structure of $K(A)$.

Of particular interest are certain plane algebraic curves associated with $A$ that were first investigated by R. Kippenhahn in 1952. We briefly recall the concepts involved.

Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. For $0 \le k \le 3$, the *Grassmannian* $G_{3,k}\mathbb{F}$, defined as the set of all $k$-dimensional $\mathbb{F}$-subspaces of $\mathbb{F}^3$, is a compact analytic $\mathbb{F}$-manifold of dimension $k(3-k)$. It has a natural topology, induced by the differential structure of the manifold, which is determined, for example, by the metric $h$ on $G_{3,k}\mathbb{F}$ with

$$h(U,V) \;=\; \sup_{v\in V,\ |v|=1}\ \inf_{u\in U,\ |u|=1} \|u-v\| \qquad \text{for all } U,V \in G_{3,k}\mathbb{F}\,.$$

The *projective plane* $\mathrm{PG}(\mathbb{F}^3)$ over $\mathbb{F}$ is given by

$$\mathrm{PG}(\mathbb{F}^3) \;=\; \bigcup_{0\le k\le 3} G_{3,k}\mathbb{F}\,.$$

The one and two dimensional subspaces of $\mathbb{F}^3$ are usually called the *points* and *lines* in $\mathrm{PG}(\mathbb{F}^3)$, respectively.

By common abuse of notation we introduce *homogeneous coordinates* for the points in $\mathrm{PG}(\mathbb{F}^3)$ as $(u_1 : u_2 : u_3) = \mathbb{F}(u_1, u_2, u_3)$. The coordinates of a vector in $\mathbb{F}^3$ are expressed with respect to the standard basis for $\mathbb{F}^3$.

A *polarity* of $\mathrm{PG}(\mathbb{F}^3)$ is a bijection on $\mathrm{PG}(\mathbb{F}^3)$ which reverses the inclusion of subspaces and the square of which equals the identity mapping. The *standard polarity* $\pi$ is characterised by

$$u^\pi \;=\; \Big\{v \in \mathbb{F}^3 \;\Big|\; \sum_{j=1}^{3} u_j v_j = 0\Big\} \qquad \text{for all } u \in G_{3,1}\mathbb{F}\,,$$

which gives $u^\pi \in G_{3,2}\mathbb{F}$. Using the polarity $\pi$, we can also introduce homogeneous coordinates for the lines in $\mathrm{PG}(\mathbb{F}^3)$ by setting $[v_1 : v_2 : v_3] = (v_1 : v_2 : v_3)^\pi$.

A nonempty subset $C$ of $G_{3,1}\mathbb{F}$ is called a *plane* $\mathbb{F}$*-algebraic curve* if it is the zero locus of a homogeneous 3-variate polynomial over $\mathbb{F}$. The defining polynomial of $C$ is not uniquely determined: if $f$ defines the curve, then so does, for example, $f^k$ for any $k \ge 1$. However, every curve $C$ has a defining polynomial of minimal degree which is unique up to a constant factor. A curve is said to be *irreducible* if it has an irreducible defining polynomial. Since a polynomial ring over a field is a unique factorisation domain, each algebraic curve $C$ is the union of finitely many irreducible curves. If $C_1, \dots, C_k$ are the irreducible components of $C$ with irreducible defining polynomials $f_1, \dots, f_k$, then $f = f_1 \cdots f_k$ is a defining polynomial of $C$ of minimal degree. We call $f$ a *minimal polynomial of* $C$. Note that an irreducible real algebraic curve is not necessarily connected.

Let $f$ be a minimal polynomial of the algebraic curve

$$C = \{u \in G_{3,1}\mathbb{F} \mid f(u) = 0\}.$$

A point $u \in C$ is called *singular* or a *singularity* of $C$ if $(\partial f/\partial u_j)(u) = 0$ for $j = 1, 2, 3$. Observe that $C$ has at most finitely many singular points. These are the singular points of the irreducible components of $C$ together with the points of intersection of any two of these components. A nonsingular point $u \in C$ is called a *simple* point of $C$. The curve $C$ is the topological closure of its simple points. Also, to every simple point $u \in C$, there exists a neighbourhood of $u$ in which $C$ admits a smooth parametrization.

Let $C$ be an irreducible plane algebraic curve with minimal polynomial $f$. At each simple point $u \in C$, we have a unique tangent line to $C$ which is given by

$$\mathcal{T}_u C \;=\; \left[\frac{\partial f}{\partial u_1}(u) : \frac{\partial f}{\partial u_2}(u) : \frac{\partial f}{\partial u_3}(u)\right].$$

If $C$ is not a projective line or a point, then it is well-known that the set $\{(\mathcal{T}_u C)^\pi \mid u \in C \text{ simple}\}$ is contained in a unique irreducible algebraic curve $C^*$, the so-called *dual curve* of $C$. In fact, since an algebraic curve has at most finitely many singularities, the dual curve is the topological closure of the set $\{(\mathcal{T}_u C)^\pi \mid u \in C \text{ simple}\}$. We have $C^{**} = C$. If $C$ is a projective line, then $\{(\mathcal{T}_u C)^\pi \mid u \in C\}$ consists of a single point $u$ in $\mathrm{PG}(\mathbb{F}^3)$. In this case, we set $C^* = \{u\}$ and define $C^{**}$ to be the image under $\pi$ of the set of all lines in $\mathrm{PG}(\mathbb{F}^3)$ which pass through $u$. This again yields $C^{**} = C$. The dual curve of a general plane algebraic curve $C$ is the union of the dual curves of its irreducible components. In particular, $C$ and $C^*$ have the same number of irreducible components.

In general, it is difficult to derive an explicit equation for the dual curve $C^*$ from the given equation of a curve $C$. However, from the above we obtain the following criterion for a point in $\mathrm{PG}(\mathbb{F}^3)$ to belong to $C^*$.

**Lemma 5.12.** *Let* $(x_1 : x_2 : 1) \in G_{3,1}\mathbb{F}$. *If there exists a smooth local parametrization* $\zeta \longmapsto (c(\zeta) : s(\zeta) : \mu(\zeta))$ *of* $C$, *for* $\zeta$ *in an open set* $U \subseteq \mathbb{F}$, *and a point* $z \in U$ *such that* $x_1 c(z) + x_2 s(z) + \mu(z) = 0$ *and* $x_1 c'(z) + x_2 s'(z) + \mu'(z) = 0$, *then the point* $(x_1 : x_2 : 1)$ *belongs to* $C^*$.

*Proof.* The two points $(c(z) : s(z) : \mu(z))$ and $(c'(z) : s'(z) : \mu'(z))$ span the tangent line $\mathcal{T}_{(c(z):s(z):\mu(z))}C$ to $C$ at $(c(z) : s(z) : \mu(z))$. The equations $x_1 c(z) + x_2 s(z) + \mu(z) = 0$ and $x_1 c'(z) + x_2 s'(z) + \mu'(z) = 0$ imply that $(x_1 : x_2 : 1) = (\mathcal{T}_{(c(z):s(z):\mu(z))}C)^\pi$. Hence $(x_1 : x_2 : 1)$ belongs to the dual curve $C^*$ of $C$. □

The details and further information on complex algebraic curves can be found, for example, in [102]. The literature for the real case is somewhat less easy to access. As a general reference to the theory of real algebraic geometry, see [18].

Let $A = A_1 + iA_2 \in \mathcal{L}(\mathbb{C}^N)$. Following R. Kippenhahn [67], we define the complex algebraic curve $C_{\mathbb{C}}(A)$ in the complex projective plane $\mathrm{PG}(\mathbb{C}^3)$ by setting its dual curve to be

$$D(A) \;=\; \big\{(c:d:\mu) \in G_{3,1}\mathbb{C} \;\big|\; \det(cA_1 + dA_2 + \mu I) = 0\big\}.$$

In [67], Kippenhahn showed that the real part $C_{\mathbb{R}}(A)$ of the curve $C_{\mathbb{C}}(A) = D(A)^*$ is contained in the affine subplane $F = \big\{(\alpha_1 : \alpha_2 : 1) \;\big|\; (\alpha_1, \alpha_2) \in \mathbb{R}^2\big\}$ of $\mathrm{PG}(\mathbb{R}^3)$ and, identifying $F$ with $\mathbb{R}^2$ in the canonical way, that the convex hull $\mathrm{co}(C_{\mathbb{R}}(A))$ of $C_{\mathbb{R}}(A)$ is precisely the numerical range of $\boldsymbol{A}$.

The curve $C_{\mathbb{R}}(A)$ considered as a real algebraic curve in $\mathrm{PG}(\mathbb{R}^3)$ is the dual curve of the real part of $D(A)$ given by

$$D_{\mathbb{R}}(A) \;=\; \big\{(c:d:\mu) \in G_{3,1}\mathbb{R} \;\big|\; \det(cA_1 + dA_2 + \mu I) = 0\big\}.$$

Every point $u \in D_{\mathbb{R}}(A)$ has a representation $(\cos\theta : \sin\theta : \mu)$ for some $\theta \in [0, \pi)$ and $\mu \in \mathbb{R}$. As $u$ is a zero of $\det(cA_1 + dA_2 + \mu I)$, it follows that $-\mu$ is an eigenvalue of the operator $\mathcal{A}(\theta) = \cos\theta\, A_1 + \sin\theta\, A_2$.

Note that the points in $D_{\mathbb{R}}(A)$ are in one-to-one correspondence with the lines $L_{y,t}$ in $\mathbb{R}^2$ defined in (5.27) below, satisfying $\langle x, t\rangle \in \sigma(\langle \boldsymbol{A}, t\rangle)$ for all $x \in L_{y,t}$. For $u = (\cos\theta : \sin\theta : \mu) \in D_{\mathbb{R}}(A)$, take $t = (\cos\theta, \sin\theta) \in \mathbb{T}$ and $\boldsymbol{y} \in \mathbb{R}^2$ such that $\langle y, t\rangle = -\mu$. Then $u^\pi$ is the two dimensional subspace

$$\bigcup \big\{(x_1 : x_2 : 1) \;\big|\; (x_1, x_2) \in L_{y,t}\big\}$$

of $\mathbb{R}^3$, that is, $L_{y,t} \times \{1\}$ is the line in which the plane $u^\pi$ normal to $u$ in $\mathbb{R}^3$ cuts the plane $\{x_3 = 1\}$.

With the following result due to F. Rellich [92, Satz 1]), [65, Theorem 6.1, p.120], we obtain local parametrizations of the curve $D_{\mathbb{R}}(A)$.

Let $S(\mathbb{C}^N) = \big\{x \in \mathbb{C}^N \;\big|\; |x| = 1\big\}$ be the unit sphere in $\mathbb{C}^N$.

**Lemma 5.13.** *Let the map $\mathcal{A} : \mathbb{R} \longrightarrow \mathcal{L}(\mathbb{C}^N)$ be given by $\mathcal{A}(\theta) = \cos\theta\, A_1 + \sin\theta\, A_2$ for $\theta \in \mathbb{R}$. Let $\theta_0 \in \mathbb{R}$ and $\mu_0$ be an eigenvalue with multiplicity $r$ of the operator $\mathcal{A}(\theta_0)$. Then there exists a neighborhood $U$ of $\theta_0$ and regular analytic functions $\mu_j : U \longrightarrow \mathbb{R}$ and $x_j : U \longrightarrow S(\mathbb{C}^N)$ with $1 \le j \le r$, such that $\mu_j(\theta_0) = \mu_0$, $\mathcal{A}(\theta)x_j(\theta) = \mu_j(\theta)x_j(\theta)$ and $\langle x_j(\theta), x_k(\theta)\rangle = \delta_{jk}$ for every $\theta \in U$ and $1 \le j, k \le r$.*

Given a point $u_0 = (\cos\theta_0 : \sin\theta_0 : -\mu_0) \in D_{\mathbb{R}}(A)$, any of the maps $\theta \longmapsto (\cos\theta : \sin\theta : -\mu(\theta))$ with $\theta$ in the neighbourhood $U$ of $\theta_0$ as given by Lemma 5.13, is then a smooth local parametrization of a component of $D_{\mathbb{R}}(A)$ in a neighbourhood of $u_0$. With Lemma 5.12, this yields immediately a complete characterisation of the curve $C_{\mathbb{R}}(A)$.

**Lemma 5.14.** *A point* $(x_1 : x_2 : 1) \in G_{3,1}\mathbb{F}$ *belongs to the curve* $C_{\mathbb{R}}(A)$ *if and only if there exists a point* $u_0 = (\cos\theta_0 : \sin\theta_0 : -\mu_0) \in D_{\mathbb{R}}(A)$*, and a local parametrization* $\theta \longmapsto (\cos\theta : \sin\theta : -\mu(\theta))$ *of a component of* $D_{\mathbb{R}}(A)$ *in a neighbourhood* $U$ *of* $u_0$ *such that* $x_1\cos\theta_0 + x_2\sin\theta_0 - \mu(\theta_0) = 0$ *and* $-x_1\sin\theta_0 + x_2\cos\theta_0 - \mu'(\theta_0) = 0$*. Then*

$$(x_1, x_2) \;=\; \mu(\theta_0)(\cos\theta_0, \sin\theta_0) + \mu'(\theta_0)(-\sin\theta_0, \cos\theta_0). \tag{5.20}$$

The line $L_{y,t}$ associated with $u_0^\pi$, as described above, is therefore tangential to the image of $C_{\mathbb{R}}(A)$ in $\mathbb{R}^2$ at $(x_1, x_2)$ except in the case that $\mu(\theta) = a_1\cos\theta + a_2\sin\theta$ in a neighbourhood $U$ of $\theta_0$. Then the set $\{(\cos\theta : \sin\theta : \mu(\theta))^\pi \mid \theta \in U\}$ corresponds to a family of lines passing through the point $(x_1, x_2) = (a_1, a_2)$.

**Lemma 5.15.** *With the exception of a finite set of points in* $C_{\mathbb{R}}(A)$*, if* $(x_1 : x_2 : 1) \in G_{3,1}\mathbb{F}$ *belongs to* $C_{\mathbb{R}}(A)$*, and* $u_0 = (\cos\theta_0 : \sin\theta_0 : \mu_0)$ *is one of the corresponding points in* $D_{\mathbb{R}}(A)$ *and* $\theta \longmapsto (\cos\theta : \sin\theta : -\mu(\theta))$*,* $\theta \in U$*, is one of the corresponding local parametrizations of a component of* $D_{\mathbb{R}}(A)$ *in a neighbourhood of* $u_0$ *as given by Lemma 5.14, then the equation*

$$(x_1 - t\sin\theta_0)\cos\theta + (x_2 + t\cos\theta_0)\sin\theta - \mu(\theta) \;=\; 0 \tag{5.21}$$

*has two real solutions* $\theta \in U$ *for either small positive* $t$ *or small negative* $t$ *and none in* $U$ *for* $t$ *of the opposite sign.*

*Proof.* By Lemma 5.14, the image of the curve $C_{\mathbb{R}}(A)$ in $\mathbb{R}^2$ has the local parametrization $(x_1(\theta), x_2(\theta)) = \mu(\theta)(\cos\theta, \sin\theta) + \mu'(\theta)(-\sin\theta, \cos\theta)$ with $\theta \in U$. Hence, its signed curvature at $(x_1, x_2)$ is given by $|\mu(\phi_0) + \mu''(\phi_0)|^{-1}$ (see, for example, [16, formula (3.9)]). So if $\mu(\phi_0) + \mu''(\phi_0) \neq 0$, then the image of $C_{\mathbb{R}}(A)$ in $\mathbb{R}^2$ is a smooth curve with nonzero curvature in a neighbourhood of $(x_1, x_2)$. Hence, there are two tangents with points of tangency in $U$ on one side of the curve and none on the other for $|t| > 0$ small enough. The solutions $\theta \in U$ of equation (5.21) correspond to the directions of the normals to the tangents.

The points of $C_{\mathbb{R}}(A)$ that we have to exclude correspond to the ones at which the image of $C_{\mathbb{R}}(A)$ in $\mathbb{R}^2$ has infinite curvature. Unless $\mu + \mu''$ vanishes identically, there exist at most finitely many solutions $\theta$ of $\mu(\theta) + \mu''(\theta) = 0$ in any compact interval. If $(x_1 : x_2 : 1) \in C_{\mathbb{R}}(A)$ is a point for which the analytic function $\mu + \mu''$ vanishes in a neighbourhood $U$ of $\theta_0$ in $\mathbb{C}$, there exists $(a_1, a_2) \in \mathbb{R}^2$ such that $\mu(\theta) = a_1\cos\theta + a_2\sin\theta$ for all $\theta \in U$. However, inspection of equation (5.20) shows that then $x = (x_1, x_2) = (a_1, a_2)$ is a point of $C_{\mathbb{R}}(A)$ through which the family of lines $L_{x,t}$, $t \in \mathbb{T}$, passes. In particular, $x$ belongs to the finite set $\sigma(A_1) \times \sigma(A_2)$. □

**Local Coordinates.** As mentioned earlier, our aim is to evaluate the plane wave formula (4.16) by the method of residues, so we need to analytically continue the functions $\Re$ and $\Im$ off the unit circle $\mathbb{T}$. To this end, define the holomorphic function $s : \mathbb{C} \setminus \{0\} \to \mathbb{C}^2$ by

$$s(z) \;=\; \left(\frac{1}{2}(z + 1/z), \frac{1}{2i}(z - 1/z)\right), \quad \text{for } z \in \mathbb{C} \setminus \{0\}. \tag{5.22}$$

then $s(z) = (\Re z, \Im z)$ for every $z \in \mathbb{T}$. From now on, we drop the subscript $\mathbb{R}$ from the Kippenhahn curves $C_{\mathbb{R}}(A)$ and denote them by $C(A)$. Furthermore, we identify $(y_1 : y_2 : 1) \in C_{\mathbb{R}}(A)$ with $\boldsymbol{y} = (y_1, y_2) \in \mathbb{R}^2$, so that $C(A)$ is a subset of $\mathbb{R}^2$.

Let $\boldsymbol{y} \in \mathbb{R}^2$ and suppose that $\zeta \in \mathbb{C} \setminus \{0\}$ is a point at which

$$\det(\langle \boldsymbol{y}I - \boldsymbol{A}, s(\zeta)\rangle) \;=\; 0. \tag{5.23}$$

Suppose that $\zeta \in \mathbb{T}$. Then $s(\zeta) \in \mathbb{T}$ and the matrix $\langle \boldsymbol{A}, s(\zeta)\rangle$ is hermitian. By a result of Rellich [65, Theorem II.6.1], there exists a neighbourhood $V_\zeta$ of $\zeta$ in $\mathbb{C} \setminus \{0\}$, a positive integer $m \le N$, analytic $\mathcal{L}(\mathbb{C}^n)$-valued projections $P_1(z), \ldots, P_m(z)$ with $\sum_{j=1}^m P_j(z) = I$ and analytic functions $\mu_1(z), \ldots, \mu_m(z)$ defined for $z \in V_\zeta$ such that for each $j = 1, \ldots, m$, the equation

$$\det(\mu_j(z)I - \langle \boldsymbol{A}, s(z)\rangle) \;=\; 0, \qquad z \in V_\zeta,$$

holds, $\langle y, s(\zeta)\rangle - \mu_1(\zeta) = 0$ and

$$\langle \boldsymbol{A}, s(z)\rangle \;=\; \sum_{j=1}^m \mu_j(z) P_j(z), \qquad z \in V_\zeta.$$

Here $P_j(z)$ is the projection onto an eigenspace for the eigenvalue $\mu_j(z)$ of $\langle \boldsymbol{A}, s(z)\rangle$.

Set $\lambda_{j,\boldsymbol{y}}(z) = \langle y, s(z)\rangle - \mu_j(z)$ for $j = 1, \ldots, m$ and $z \in V_\zeta$. Then

$$\langle \boldsymbol{y}I - \boldsymbol{A}, s(z)\rangle \;=\; \sum_{j=1}^m \lambda_{j,\boldsymbol{y}}(z) P_j(z), \qquad z \in V_\zeta. \tag{5.24}$$

It turns out that the functions $\mu_j$ and projections $P_j$, $j = 1, \ldots, m$, can be analytically continued along any arc that avoids a certain finite exceptional set of points [13, Theorem 3.3.12] . Therefore, formula (5.24) may also be valid in a neighbourhood $V_\zeta$ of points $\zeta \in \mathbb{C} \setminus \{0\}$ not on the unit circle $\mathbb{T}$.

It can happen that two of the eigenvalues $\lambda_{j,\boldsymbol{y}}(z)$ and $\lambda_{\ell,\boldsymbol{y}}(z)$ of $\langle \boldsymbol{y}I - \boldsymbol{A}, s(z)\rangle$ are equal at a particular complex number $z$. In particular, there may exist an integer $1 < k \le m$ such that $\lambda_{j,\boldsymbol{y}}(\zeta) = 0$ for all $j = 1, \ldots, k$. According to the interpretation preceding Definition 5.19 and the definition of $C(A)$, if $\zeta \in \mathbb{T}$, then there exist $k$ coincident tangent lines from $\boldsymbol{y}$ to $C(A)$ with normal $\zeta$.

**Lemma 5.16.** *Let $x \in \mathbb{R}^2$, let $\zeta \in \mathbb{C} \setminus \{0\}$ be a complex number and $V_\zeta$ an open neighbourhood of $\zeta$ in $\mathbb{C}$ for which (5.24) is an analytic parametrization* *in* $V_\zeta$ *with* $\lambda_{1,\boldsymbol{x}}(\zeta) = 0$ *and* $\lambda'_{1,\boldsymbol{x}}(\zeta) \neq 0$. *Then there exists a unique* $C^\infty$-*function* $\phi : U_x \to \mathbb{C}$ *defined in a neighbourhood* $U_x$ *of* $(0, \boldsymbol{x})$ *in* $\mathbb{R}^3$ *such that* $\phi(0, \boldsymbol{x}) = \zeta$ *and* $\lambda_{1,\boldsymbol{y}}(\phi(\xi, \boldsymbol{y})) = i\xi$ *for all* $(\xi, \boldsymbol{y}) \in U_x$.

*Moreover, for* $\boldsymbol{y}$ *fixed, the function* $\xi \longmapsto \phi(\xi, \boldsymbol{y})$, $(\xi, \boldsymbol{y}) \in U_x$ *is one-to-one and* $\lambda'_{1,\boldsymbol{y}}(\phi(\xi, \boldsymbol{y})) \neq 0$ *for all* $(\xi, \boldsymbol{y}) \in U_x$. *If* $\zeta \in \mathbb{T}$, *then* $\phi(0, \boldsymbol{y}) \in \mathbb{T}$ *for all* $(0, \boldsymbol{y}) \in U_x$.

*Proof.* Let $U \subset \mathbb{R}^4$ be the set $U = V_\zeta \times \mathbb{R}^2$ and let $\Phi : U \to \mathbb{R}^4$ be defined by

$$\Phi(z, \boldsymbol{y}) \;=\; (\lambda_{1,\boldsymbol{y}}(z), \boldsymbol{y}) = \;(\langle y, s(z)\rangle - \mu_1(z), \boldsymbol{y})$$

for all $(z, \boldsymbol{y}) \in U$. Here we identify $\mathbb{C}$ with $\mathbb{R}^2$ on the right hand side of the equation. The derivative $\Phi'(\zeta, \boldsymbol{x})$ of the function $\Phi$ on the open subset $U$ of $\mathbb{R}^4$, as a function of four real variables, is nonsingular at $(\zeta, \boldsymbol{x}) \in U$ because

$$\det\left(\Phi'(\zeta, \boldsymbol{x})\right) \;=\; \left|\lambda'_{1,\boldsymbol{x}}(\zeta)\right|^2 \neq 0.$$

By the inverse function theorem, there exists an open neighbourhood $W$ of $(0, \boldsymbol{x})$ in $\mathbb{R}^4$, an open neighbourhood $U'$ of $(\zeta, \boldsymbol{x})$ in $\mathbb{R}^4$ and a diffeomorphism $f : W \to U'$ such that $\Phi \circ f(\alpha, \boldsymbol{y}) = (\alpha, \boldsymbol{y})$ for all $(\alpha, \boldsymbol{y}) \in W$. In particular, $\Phi'$ is nonsingular on $U'$.

Then $\phi(\xi, \boldsymbol{y}) \in \mathbb{C}$ is defined on the set $U_x$ of all $(\xi, y_1, y_2) \in \mathbb{R}^3$ such that $(0, \xi, y_1, y_2) \in W$, by the formula

$$f(0, \xi, y_1, y_2)) \;=\; (\phi(\xi, \boldsymbol{y}), \boldsymbol{y}),$$

so that $\lambda_{1,\boldsymbol{y}}(\phi(\xi, \boldsymbol{y})) = i\xi$. Because $f$ is a diffeomorphism, the function $(\xi, \boldsymbol{y}) \longmapsto \phi(\xi, \boldsymbol{y})$ is $C^\infty$ on $U_x$. Furthermore, $(\phi(\xi, \boldsymbol{y}), \boldsymbol{y}) \in U'$, so

$$\left|\lambda'_{1,\boldsymbol{y}}(\phi(\xi, \boldsymbol{y}))\right|^2 \;=\; \det\left(\Phi'(\phi(\xi, \boldsymbol{y}), \boldsymbol{y})\right) \;\neq\; 0$$

for all $(\xi, \boldsymbol{y}) \in U_x$.

Now suppose that $\zeta \in \mathbb{T}$. There exists an open neighbourhood $N_\zeta$ of $\zeta$ in $\mathbb{T}$ on which $\lambda_{1,\boldsymbol{x}}$ is defined. Let $\Psi : N_\zeta \times \mathbb{R}^2 \to \mathbb{R}^3$ be defined by

$$\Psi(s, \boldsymbol{y}) \;=\; (\lambda_{1,\boldsymbol{y}}(t), \boldsymbol{y}) \;=\; (\langle y, t\rangle - \mu_1(t), \boldsymbol{y})$$

for all $(t, \boldsymbol{y}) \in N_\zeta \times \mathbb{R}^2$. Then $\Psi(\zeta, \boldsymbol{x}) = (0, \boldsymbol{x})$ and the derivative

$$\Psi'(\zeta, \boldsymbol{x}) : T_{(\zeta, \boldsymbol{x})}(\mathbb{T} \times \mathbb{R}^2) \;\longrightarrow\; \mathbb{R}^3$$

of $\Psi$ at $(\zeta, \boldsymbol{x}) \in \mathbb{T} \times \mathbb{R}^2$ is nonsingular. Here $T_{(\zeta, \boldsymbol{x})}(\mathbb{T} \times \mathbb{R}^2)$ is the tangent space of $\mathbb{T} \times \mathbb{R}^2$ at $(\zeta, \boldsymbol{x})$. As above, there exists a diffeomorphism $g$ from an open neighbourhood of $(0, \boldsymbol{x})$ in $\mathbb{R}^3$ onto an open neighbourhood of $(t, \boldsymbol{x})$ is $\mathbb{T} \times \mathbb{R}^2$ such that $\Psi \circ g = Id$. Because $\Psi = \Phi \,|\, N_\zeta \times \mathbb{R}^2$, we must have $g(\alpha, \boldsymbol{y}) = f(\alpha, 0, \boldsymbol{y})$. Hence

$$(\phi(0, \boldsymbol{y}), \boldsymbol{y}) \;=\; f(0, 0, \boldsymbol{y}) \;\in\; \mathbb{T} \times \mathbb{R}^2,$$

proving that $\phi(0, \boldsymbol{y}) \in \mathbb{T}$. □

If $\tilde{\Phi} : V_\zeta \times \mathbb{C}^2 \to \mathbb{C}^3$ is defined by $\tilde{\Phi}(z, \eta) = \left(\sum_{j=1}^2 \eta_j s_j(z) - \mu_1(z), \eta\right)$ for all $z \in V_\zeta$ and $\eta \in \mathbb{C}^2$, then a similar argument to that above, but replacing $\Phi$ by $\tilde{\Phi}$ and appealing to the inverse function theorem for analytic functions of several variables, shows that $\phi$ is actually the restriction to $U_x$ of a function analytic in an open subset of $\mathbb{C}^3$.

According to a rephrasing of Lemma 5.14 in terms of our local coordinates, the Kippenhahn curves $C(A)$ for a matrix $\boldsymbol{A}$ are characterised locally by the following proposition.

**Proposition 5.17.** *The Kippenhahn curves $C(A)$ consist of all points $\boldsymbol{y} \in \mathbb{R}^2$ for which there exists a point $\zeta$ belonging to the unit circle $\mathbb{T}$ and a neighbourhood $V_\zeta$ of $\zeta$ in $\mathbb{C}$ such that there exists an analytic parametrization (5.24) on $V_\zeta$ for which*

$$\lambda_{1,\boldsymbol{y}}(\zeta) = \lambda'_{1,\boldsymbol{y}}(\zeta) \;=\; 0. \tag{5.25}$$

Of course, in any such parametrization (5.24), we are at liberty to choose the indices $j = 1, \ldots, m$ for the analytic functions $\lambda_{j,\boldsymbol{y}} : V_\zeta \to \mathbb{C}$. In particular, for any $\boldsymbol{y} \in C(A)$, we can choose a neighbourhood $V_\zeta$ of $\zeta$ in $\mathbb{C}$ and indices for which (5.25) holds for $j = 1$.

**Corollary 5.18.** *Let $x \in \mathbb{R}^2 \setminus C(A)$, let $\zeta \in \mathbb{C} \setminus \{0\}$ be a complex number and $V_\zeta$ a neighbourhood of $\zeta$ in $\mathbb{C}$ for which (5.24) is a parametrization with $\lambda_{1,\boldsymbol{x}}(\zeta) = 0$. Then there exists a unique $C^\infty$-function $\phi : U_x \to \mathbb{C}$ defined in a neighbourhood $U_x$ of $(0, \boldsymbol{x})$ in $\mathbb{R}^3$ such that such that $\phi(0, \boldsymbol{x}) = \zeta$ and $\lambda_{1,\boldsymbol{y}}(\phi(\xi, \boldsymbol{y})) = i\xi$ for all $(\xi, \boldsymbol{y}) \in U_x$.*

*Moreover, for $\boldsymbol{y}$ fixed, the function $\xi \longmapsto \phi(\xi, \boldsymbol{y})$, $(\xi, \boldsymbol{y}) \in U_x$ is one-to-one and $\lambda'_{1,\boldsymbol{y}}(\phi(\xi, \boldsymbol{y})) \neq 0$ for all $(\xi, \boldsymbol{y}) \in U_x$. If $\zeta \in \mathbb{T}$, then $z(0, \boldsymbol{y}) \in \mathbb{T}$ for all $(0, \boldsymbol{y}) \in U_x$.*

*Proof.* By Proposition 5.17, $\lambda'_{1,\boldsymbol{x}}(\zeta) \neq 0$, so Lemma 5.16 is applicable. □

We informally state alternative characterisations of the Kippenhahn curves $C(A)$:

- The real part of the curve $D(A)^*$ dual to

$$D(A) \;=\; \big\{(c : d : \mu) \in G_{3,1}\mathbb{C} \;\big|\; \det(cA_1 + dA_2 + \mu I) = 0\big\},$$

  identifying $(\alpha_1, \alpha_2) \in \mathbb{R}^2$ with $(\alpha_1 : \alpha_2 : 1) \in G_{3,1}\mathbb{C}$.
- The real algebraic curve dual to

$$D_{\mathbb{R}}(A) \;=\; \big\{(c : d : \mu) \in G_{3,1}\mathbb{R} \;\big|\; \det(cA_1 + dA_2 + \mu I) = 0\big\},$$

  identifying $(\alpha_1, \alpha_2) \in \mathbb{R}^2$ with $(\alpha_1 : \alpha_2 : 1) \in G_{3,1}\mathbb{R}$.
- All points $\boldsymbol{y} \in \mathbb{R}^2$ for which there exists $\zeta \in \mathbb{T}$ and a neighbourhood $V_\zeta$ of $\zeta$ in $\mathbb{C}$ such that there exists an analytic parametrization (5.24) on $V_\zeta$ for which $\lambda_{1,\boldsymbol{y}}(\zeta) = \lambda'_{1,\boldsymbol{y}}(\zeta) = 0$ [Proposition 5.17].

- The envelope of all lines $L_{y,s}$ given by (5.27) for each $\boldsymbol{y} \in \mathbb{R}^2$ and $s \in \mathbb{T}$ such that
$$\langle y, s\rangle \in \sigma(\langle \boldsymbol{A}, s\rangle).$$
- The singular values of the numerical range map $W_{\boldsymbol{A}}$ associated with the matrix $A_1 + iA_2$ (see [38] and [63]), with the possible exception of "double tangents" [63, Theorem 3.5].

### 5.3.2 Examples

The Weyl functional calculus $\mathcal{W}_{\boldsymbol{A}}$ for a pair $\boldsymbol{A} = (A_1, A_2)$ of $(2 \times 2)$ hermitian matrices can be calculated explicitly. The support $\gamma(\boldsymbol{A})$ of $\mathcal{W}_{\boldsymbol{A}}$ is either the numerical range $K(A_1 + iA_2)$ of the matrix $A_1 + iA_2$, an elliptical plane region with nonempty interior in the case that $A_1, A_2$ do not commute with each other, or $\gamma(\boldsymbol{A})$ consists of a single point $\sigma \in \mathbb{R}^2$ if $\boldsymbol{A} = \sigma I$, or otherwise, two distinct joint eigenvalues $\sigma_1, \sigma_2 \in \mathbb{R}^2$. Calculations of this nature follow from [7] and are given explicitly in [55].

The case of a pair $\boldsymbol{A}$ of noncommuting $(3 \times 3)$ hermitian matrices reveals greater geometric structure. If $\boldsymbol{A}$ has a joint eigenvalue $\sigma \in \mathbb{R}^2$, then $\gamma(\boldsymbol{A})$ consists of $\sigma$ together with the support of the Weyl functional calculus associated with the pair of reduced $(2 \times 2)$ matrices, possibly consisting of the point $\sigma$ together with a disjoint elliptical region. Diagrams are exhibited in [56].

### 5.3.3 The Joint Spectrum of Two Hermitian Matrices

Let $\boldsymbol{A} = (A_1, A_2)$ be a pair of hermitian matrices. Where convenient, we shall represent the $(N \times N)$ matrix associated with $\boldsymbol{A}$ as $A = A_1 + iA_2$ in order to avoid hats and tildes. In the same spirit, $\mathbb{C}$ is identified with $\mathbb{R}^2$ and $\mathbb{R}^2$ is identified with the subspace $\{0\} \times \mathbb{R}^2$ of $\mathbb{R}^3$. We adopt the convention that a point $\omega \in \mathbb{R}^3$ is written as $\omega = \boldsymbol{y} + y_0 e_0$ for $\boldsymbol{y} \in \mathbb{R}^2$ and $y_0 \in \mathbb{R}$. For a $n$-tuple $\boldsymbol{B} = (B_1, \ldots, B_n)$ of $(N \times N)$ matrices and $\xi \in \mathbb{C}^n$, the notation $\langle \boldsymbol{B}, \xi\rangle$ is used to denote the matrix $\sum_{j=1}^n B_j \xi_j$.

We are concerned with the compact set $\gamma(\boldsymbol{A}) \subset \mathbb{R}^2$ of points at which the Cauchy kernel $\omega \longmapsto G_\omega(\boldsymbol{A})$ has a discontinuity as $\omega \in \mathbb{R}^3$ approaches the subspace $\{0\} \times \mathbb{R}^2$ of $\mathbb{R}^3$ from above $(\omega_0 \to 0^+)$ and below $(\omega_0 \to 0^-)$.

To this end, we examine the integral (4.16) more closely. The unit circle $S^1$ in $\mathbb{R}^2$ is written as $\mathbb{T}$. Let $\boldsymbol{y} = (y_1, y_2) \in \mathbb{R}^2$. We interpret $\boldsymbol{B}(\boldsymbol{y}) = \boldsymbol{y}I - \boldsymbol{A}$ as the pair $(B_1(\boldsymbol{y}), B_2(\boldsymbol{y}))$ of matrices with $B_j(\boldsymbol{y}) = y_j I - A_j$ for $j = 1, 2$. Then, appealing to the identity $t^2 = -1$ for $t \in \mathbb{T} \subset \mathbb{R}_{(2)}$ with respect to multiplication in the Clifford algebra, for $y_0 \neq 0$ the integrand of (4.16) can be written down explicitly as

$$\begin{aligned}(\langle \boldsymbol{B}(\boldsymbol{y}), t\rangle - y_0 tI)^{-2} &= (\langle \boldsymbol{B}(\boldsymbol{y}), t\rangle + y_0 tI)^2 (\langle \boldsymbol{B}(\boldsymbol{y}), t\rangle^2 + y_0^2 I)^{-2} \\ &= (\langle \boldsymbol{B}(\boldsymbol{y}), t\rangle^2 - y_0^2 I)(\langle \boldsymbol{B}(\boldsymbol{y}), t\rangle^2 + y_0^2 I)^{-2} \\ &\quad + 2y_0 t \langle \boldsymbol{B}(\boldsymbol{y}), t\rangle (\langle \boldsymbol{B}(\boldsymbol{y}), t\rangle^2 + y_0^2 I)^{-2}.\end{aligned} \tag{5.26}$$

The points $t \in \mathbb{T}$, where $\langle \boldsymbol{B}(\boldsymbol{y}), t\rangle$ is not invertible, will dominate the integral (4.16) as $y_0 \to 0^+$ and $y_0 \to 0^-$, respectively. This suggests to investigate the zeroes of

$$\det\langle \boldsymbol{B}(\boldsymbol{y}), t\rangle = \det(B_1(\boldsymbol{y})t_1 + B_2(\boldsymbol{y})t_2).$$

Now suppose that $t = (t_1, t_2) = (\cos\theta, \sin\theta)$ for $-\pi < \theta < \pi$ and let $z = e^{i\theta}$. Then $t_1 = (z + z^{-1})/2$ and $t_2 = (z - z^{-1})/2i$, so that

$$\begin{aligned} \det\langle \boldsymbol{B}(\boldsymbol{y}), t\rangle &= (2z)^{-n}\det(B_1(\boldsymbol{y})(z^2+1) - iB_2(\boldsymbol{y})(z^2-1)) \\ &= (2z)^{-n}\det((B_1(\boldsymbol{y}) - iB_2(\boldsymbol{y}))z^2 + (B_1(\boldsymbol{y}) + iB_2(\boldsymbol{y}))) \\ &= (2z)^{-n}\det(B_1(\boldsymbol{y}) - iB_2(\boldsymbol{y})) \\ &\qquad \times \det(z^2 I + (B_1(\boldsymbol{y}) - iB_2(\boldsymbol{y}))^{-1}(B_1(\boldsymbol{y}) + iB_2(\boldsymbol{y}))) \end{aligned}$$

if $B_1(\boldsymbol{y}) - iB_2(\boldsymbol{y})$ is invertible.

Fix $\boldsymbol{y} \in \mathbb{R}^2$ and let $T = B_1(\boldsymbol{y}) + iB_2(\boldsymbol{y})$. Then in the case that $T$ and hence, $T^*$, is invertible, the set of points $t \in \mathbb{T}$ where $\det\langle \boldsymbol{B}(\boldsymbol{y}), t\rangle = 0$ is in two-to-one correspondence with $\left[\sigma\left(-(T^*)^{-1}T\right)\right] \cap \mathbb{T}$: if $\zeta$ is an element of the set $\left[\sigma\left(-(T^*)^{-1}T\right)\right] \cap \mathbb{T}$, then the corresponding $t \in \mathbb{T}$ is $\pm\zeta^{1/2}$.

For $t \in \mathbb{T}$, the equation $\det\langle \boldsymbol{B}(\boldsymbol{y}), t\rangle = \det\langle \boldsymbol{y}I - \boldsymbol{A}, t\rangle = 0$ has a geometric interpretation. Let $t^\perp \in \mathbb{T}$ be orthogonal to $t$ in $\mathbb{R}^2$. Then the line

$$L_{\boldsymbol{y},t} = \left\{\boldsymbol{y} + \lambda t^\perp \;\middle|\; \lambda \in \mathbb{R}\right\} \tag{5.27}$$

passes through $\boldsymbol{y} \in \mathbb{R}^2$ and has the property that $\langle x, t\rangle \in \sigma(\langle \boldsymbol{A}, t\rangle)$ for all $x \in L_{\boldsymbol{y},t}$.

As we will see later, the number of such lines that exist for a point $\boldsymbol{y}$ and for all points in a neighbourhood of $\boldsymbol{y}$, is decisive for whether the point $\boldsymbol{y}$ belongs to $\mathrm{supp}(\mathcal{W}_{\boldsymbol{A}})$. We introduce the following definition to isolate those points $\boldsymbol{y} \in \mathbb{R}^2$ for which this is the maximum number possible. The *resolvent set* $\rho(A)$ of $A$ is the complement in $\mathbb{C}$ of the set $\sigma(A)$ of eigenvalues of $A$.

**Definition 5.19.** Let $A$ be a $(N \times N)$ matrix and let $\mathbf{R}(A)$ be the set of all points $\lambda \in \rho(A)$ such that in some neighbourhood $U \subset \rho(A)$ of $\lambda$ in $\mathbb{C}$,

$$\sigma\left(\left((xI - A)^*\right)^{-1}(xI - A)\right) \subset \mathbb{T} \tag{5.28}$$

for each $x \in U$.

The set $\mathbf{R}(A)$ is necessarily an open set. If the matrices $A_1$ and $A_2$ commute, that is, if $A = A_1 + iA_2$ is a normal matrix, then the set $\mathbf{R}(A)$ is readily described. In this case $((xI - A)^*)^{-1}(xI - A)$ is a unitary matrix for *all* $x \in \rho(A)$, so that $\mathbf{R}(A) = \rho(A)$.

Condition (5.28) may be restated by saying that $\mathbf{R}(A)$ is the set of all $\lambda \in \mathbb{C}$ such that in some neighbourhood $U$ of $\lambda$ in $\mathbb{C}$, every solution $u \in \mathbb{C}$ of the equation

$$\det\left((xI - A)^* u + (xI - A)\right) = 0 \tag{5.29}$$

with $x \in U$ satisfies $|u| = 1$.

If $x \in \sigma(A)$, then $u = 0$ is a solution of (5.29) – such points are excluded. Note that, in the notation above, this covers the case where $B_1(\boldsymbol{y}) - iB_2(\boldsymbol{y})$ is not invertible. The determinant $\det\left((\boldsymbol{y}I - A)^* u + (\boldsymbol{y}I - A)\right)$ equals

$$\overline{\det(\boldsymbol{y}I - A)}\ \det\left(uI + ((\boldsymbol{y}I - A)^*)^{-1}(\boldsymbol{y}I - A)\right).$$

for $\boldsymbol{y} \in \rho(A)$. Hence $\det\left((\boldsymbol{y}I - A)^* u + (\boldsymbol{y}I - A)\right)$ is a polynomial of degree $N$ in $u$ and there are $N$ solutions $u \in \mathbb{C}$ of (5.29) counting multiplicity. To each $u \in \mathbb{T}$, there corresponds a line $L_{\boldsymbol{y},u^{1/2}}$ in $\mathbb{R}^2$. If *all* the solutions $u \in \mathbb{C}$ satisfy $|u| = 1$, that is, if $\boldsymbol{y} \in \mathbf{R}(A)$, then this says that the number of lines $L_{\boldsymbol{y},t}$, $t \in \mathbb{T}$, passing through $\boldsymbol{y}$, is the maximum possible. In particular, counting multiplicity, the maximum number of lines $L_{\boldsymbol{y},t}$, $t \in \mathbb{T}$, that can possibly pass through $\boldsymbol{y}$ is $N$. As we shall see in Section 5.5 below, the situation is best described in real projective space $\mathbb{RP}$.

According to the discussion preceding Definition 5.19, we have the following alternative formulation of the set $\mathbf{R}(A)$.

**Proposition 5.20.** *Let $\boldsymbol{A} = (A_1, A_2)$ be a pair of $(N \times N)$ hermitian matrices and $A = A_1 + iA_2$.*

*Then $\mathbf{R}(A)$ is the set of all $\lambda \in \mathbb{R}^2$ for which there exists a neighbourhood $U$ of $\lambda$ in $\mathbb{R}^2$, with the property that for each $x \in U$, every solution $z \in \mathbb{C} \setminus \{0\}$ of the equation*

$$\det(\langle xI - \boldsymbol{A}, s(z)\rangle) = 0$$

*satisfies $|z| = 1$.*

The following result describes the relation between the set $\mathbf{R}(A)$ and the Kippenhahn curves $C(A)$.

**Proposition 5.21.** $\partial\mathbf{R}(A) \subseteq C(A) \subseteq \mathbf{R}(A)^c$.

*Proof.* Let $x \in \overline{\mathbf{R}(A)}$. All solutions $\zeta$ of $\det(\langle x, \zeta\rangle I - \langle \boldsymbol{A}, \zeta\rangle) = 0$ satisfy $|\zeta| = 1$ because the set-valued function

$$y \longmapsto \sigma(((yI - A)^*)^{-1}(yI - A)), \quad y \in \rho(A),$$

is continuous in the metric of unordered $N$-tuples [65, Theorem II.5.1] and by definition, $\sigma(((yI - A)^*)^{-1}(yI - A)) \subset \mathbb{T}$ for all $\boldsymbol{y} \in \mathbf{R}(A)$.

For any such $\zeta \in \mathbb{T}$, there exists an analytic parametrization (5.24) such that $\lambda_{1,\boldsymbol{x}}(\zeta) = 0$. Suppose $\lambda'_{1,\boldsymbol{x}}(\zeta) \neq 0$. Then by Lemma 5.16, for all $\boldsymbol{y}$ in an open neighbourhood of $x$, we can find $\phi(0, \boldsymbol{y}) \in \mathbb{T}$ such that $\lambda_{1,\boldsymbol{y}}(\phi(0, \boldsymbol{y})) = 0$ and $\lambda'_{1,\boldsymbol{y}}(\phi(0, \boldsymbol{y})) \neq 0$.

It follows that if $\lambda'_{1,\boldsymbol{x}}(\zeta) \neq 0$ holds for the parametrizations of all solutions $\zeta$, then there is a neighbourhood $U$ of $x$ such that for every $\boldsymbol{y} \in U$, all nonzero complex solutions $z$ of $\det(\langle y, s(z)\rangle I - \langle \boldsymbol{A}, s(z)\rangle) = 0$ satisfy $|z| = 1$. This means that $x \in \mathbf{R}(A)$.

Therefore, for every element $x$ of $\partial\mathbf{R}(A) = \overline{\mathbf{R}(A)} \setminus \mathbf{R}(A)$, there must exist a solution $\zeta$ and an analytic parametrization (5.24) such that $\lambda_{1,\boldsymbol{x}}(\zeta) = 0$ and $\lambda'_{1,\boldsymbol{x}}(\zeta) = 0$. Proposition 5.17 yields that $x \in C(A)$.

To establish the inclusion $C(A) \subseteq \mathbf{R}(A)^c$, suppose that $x \in C(A)$. By Lemma 5.15, except possibly for a finite subset $J$ of $C(A)$, there exists a neighbourhood $U$ of $x$ in $\mathbb{R}^2$ in which not every solution $z \in \mathbb{C}$ of

$$\det(\langle \boldsymbol{y}I - \boldsymbol{A}, s(z)\rangle) \;=\; 0$$

for $\boldsymbol{y} \in U$, belongs to $\mathbb{T}$. More precisely, for $\boldsymbol{y}$ on one side of $C(A)$, there exist at least two solutions belonging to $\mathbb{T}$ – two unit normal vectors to the local tangents to $C(A)$ passing through $\boldsymbol{y}$ – and for $\boldsymbol{y}$ on the other side of $C(A)$, two solutions that do not belong to $\mathbb{T}$. Moreover, if $x \in J$, then either $x$ is isolated, or any neighbourhood of $x$ contains a point $\boldsymbol{y} \in C(A) \setminus J$ to which the conclusion above applies. In either case, $x \in \mathbf{R}(A)^c$. $\square$

By considering the direct sum of suitable matrices, the inclusions of Proposition 5.21 can be made to be *proper* inclusions.

The following simple condition guarantees that a point $\boldsymbol{y} \in \mathbb{R}^2$ belongs to $\mathbf{R}(A)$. Let $K(A)$ denote the *numerical range*

$$\left\{\langle Ax, x\rangle \;\middle|\; x \in \mathbb{C}^N,\ |x| = 1 \right\}$$

of the $(N \times N)$ matrix $A = A_1 + iA_2$.

**Proposition 5.22.** *Let $A$ be an $(N \times N)$ matrix. Then $\mathbb{C} \setminus K(A) \subseteq \mathbf{R}(A)$. Consequently, $\mathbb{C} \setminus \mathbf{R}(A)$ is a nonempty compact subset of the numerical range $K(A)$ of the matrix $A$.*

*Proof.* Firstly, $\sigma(A) \subset K(A)$, so if $\lambda \in \mathbb{C}$ lies outside $K(A)$ then $\lambda \in \rho(A)$. Moreover, for every $u \in \mathbb{C}$, the inclusion

$$\sigma((\lambda I - A)^* u + (\lambda I - A)) \;\subseteq\; K((\lambda I - A)^* u + (\lambda I - A))$$

holds. Hence, for any complex number $u$ for which $0 \in \sigma((\lambda I - A)^* u + (\lambda I - A))$, there exists $x \in \mathbb{C}^N$ with $|x| = 1$ such that

$$\langle(\lambda I - A)^* u x, x\rangle + \langle(\lambda I - A)x, x\rangle \;=\; 0.$$

Here $\langle\,\cdot\,,\,\cdot\,\rangle$ is the inner product of $\mathbb{C}^N$. Because $\lambda - \langle \boldsymbol{A}v, v\rangle \neq 0$ for all $v \in \mathbb{C}^N$ with $|v| = 1$, the complex number

$$u \;=\; -\frac{\lambda - \langle \boldsymbol{A}x, x\rangle}{\overline{\lambda} - \overline{\langle \boldsymbol{A}x, x\rangle}}$$

has modulus one. Consequently, $\lambda \in \mathbf{R}(A)$. $\square$

*Remark 5.23.* The same proof works with the analogous definition of $\mathbf{R}(A)$ if $\boldsymbol{A}$ is a bounded linear operator on a Hilbert space. If $\boldsymbol{A}$ is normal, then $\mathbf{R}(A) = \rho(A)$ and $\mathcal{W}_{\boldsymbol{A}}$ is the spectral measure of $\boldsymbol{A}$ supported on $\sigma(A)$.

Our aim is to prove the following result strengthening Proposition 5.22 above and providing a geometric characterisation of the support $\mathrm{supp}(\mathcal{W}_{\boldsymbol{A}})$ of the Weyl functional calculus and for the joint spectrum $\gamma(\boldsymbol{A})$ of a *pair* of hermitian matrices.

**Theorem 5.24.** *Let* $\boldsymbol{A} = (A_1, A_2)$ *be a pair of hermitian matrices and* $A = A_1 + iA_2$. *Then the equalities* $\mathrm{supp}(\mathcal{W}_{\boldsymbol{A}}) = \gamma(\boldsymbol{A}) = \mathbb{R}^2 \setminus \mathbf{R}(A)$ *hold.*

The equality $\mathrm{supp}(\mathcal{W}_{\boldsymbol{A}}) = \gamma(\boldsymbol{A})$ is proved in Theorem 4.8 for any $n$-tuple $\boldsymbol{A}$ of bounded selfadjoint operators, so we are now concerned with the second equality for hermitian matrices $A_1, A_2$ – the *geometric* characterisation of $\gamma(\boldsymbol{A})$.

The spectrum $\sigma(A)$ of the matrix $A = A_1 + iA_2$ is clearly contained in the numerical range $K(A) = \mathrm{co}\,(\mathrm{supp}(\mathcal{W}_{\boldsymbol{A}}))$ of $\boldsymbol{A}$. The following immediate consequence of Theorem 5.24 and the fact that $\mathbf{R}(A) \subseteq \rho(A)$ strengthens this observation.

**Corollary 5.25.** *Let* $\boldsymbol{A} = (A_1, A_2)$ *be a pair of hermitian matrices and* $A = A_1 + iA_2$. *Then* $\sigma(A) \subseteq \mathrm{supp}(\mathcal{W}_{\boldsymbol{A}})$.

A bounded linear operator on a Hilbert space is called *normal* if it commutes with its adjoint. The following consequence of Theorem 5.24 characterises the situation in which the inclusion in Corollary 5.25 is an equality.

**Corollary 5.26.** *Let* $\boldsymbol{A} = (A_1, A_2)$ *be a pair of hermitian matrices. Set* $A = A_1 + iA_2$. *The following conditions are equivalent.*

i) $A$ *is a normal matrix.*
ii) $\mathrm{supp}(\mathcal{W}_{\boldsymbol{A}})$ *has empty interior.*
iii) $\sigma\left(\left((xI - A)^*\right)^{-1}(xI - A)\right) \subset \mathbb{T}$ *for all* $x \in \rho(A)$.
iv) $\sigma(A) = \mathrm{supp}(\mathcal{W}_{\boldsymbol{A}})$.

*Proof.* If $A$ is a normal matrix, then the distribution $\mathcal{W}_{\boldsymbol{A}}$ is associated with the spectral measure of $A$ supported by the finite set of joint eigenvalues of $A$, so the implication i) $\Longrightarrow$ ii) is immediate. The definition of the set $\mathbf{R}(A)$ and Theorem 5.24 shows that iv) follows from iii). The implication iv) $\Longrightarrow$ i) is proved in [32], [95], so it remains to establish ii) $\Longrightarrow$ iii).

Suppose that the negation of iii) holds and that $\lambda \in \rho(A)$ has an eigenvalue of the matrix $\left((\lambda I - A)^*\right)^{-1}(\lambda I - A)$ lying outside the unit circle $\mathbb{T}$. Then the same holds in a neighbourhood of $\lambda$ because the unordered $n$-tuple of eigenvalues of the matrix valued function $\lambda \longmapsto \left((\lambda I - A)^*\right)^{-1}(\lambda I - A)$ depends *continuously* on the parameter $\lambda$ [65, Theorem II.5.1]. Hence, $\mathbb{C} \setminus \mathbf{R}(A)$ has nonempty interior. According to Theorem 5.24, $\mathrm{supp}(\mathcal{W}_{\boldsymbol{A}})$ has nonempty interior. □

The remainder of this section is devoted to a proof of Theorem 5.24. We first show that $\mathbf{R}(A) \subseteq \gamma(\boldsymbol{A})^c$. Let $\boldsymbol{x} \in \mathbf{R}(A)$. We must find an open neighbourhood $U$ of $(0, \boldsymbol{x})$ in $\mathbb{R}^3$ such that the function

$$(\epsilon, \boldsymbol{y}) \longmapsto G_{\boldsymbol{y}+\epsilon e_0}(\boldsymbol{A}), \qquad (\epsilon, \boldsymbol{y}) \in U \setminus (\{0\} \times \mathbb{R}^2)$$

is the restriction to $U \setminus (\{0\} \times \mathbb{R}^2)$ of a continuous function defined in $U$. Then by Painlevé's Theorem [19, Theorem 10.6, p. 64], $G_\omega(\boldsymbol{A})$ is monogenic in a neighbourhood of $(0, \boldsymbol{x})$, because $G_\omega(\boldsymbol{A})$ is monogenic above and below $\{0\} \times \mathbb{R}^2$. Hence $\boldsymbol{x} \in \gamma(\boldsymbol{A})^c$.

We start by examining the plane wave decomposition (4.16). Let $\boldsymbol{y} \in \mathbb{R}^2$ and set $\boldsymbol{B}(\boldsymbol{y}) = \boldsymbol{y}I - \boldsymbol{A}$. First, we convert the integral (4.16) to a contour integral

$$\int_{\mathbb{T}} (\langle \boldsymbol{B}(\boldsymbol{y}), s\rangle - \epsilon s I)^{-2} d\mu(s) \; = \; -i \int_{\mathbb{T}} (\langle \boldsymbol{B}(\boldsymbol{y}), s(z)\rangle - \epsilon s(z) I)^{-2} z^{-1} \, dz \quad (5.30)$$

for the function $s : \mathbb{C}\setminus\{0\} \to \mathbb{C}^2$ defined by formula (5.22). The integral (5.30) may be evaluated using Cauchy's Residue Theorem by finding the residues of the function

$$z \longmapsto (\langle \boldsymbol{B}(\boldsymbol{y}), zs(z)\rangle - \epsilon z s(z) I)^{-2} z \qquad (5.31)$$

in the open unit disk $D = \{z \in \mathbb{C} \mid |z| < 1\}$. The formula (5.26) holds for any $s \in \mathbb{T}$ and $\epsilon = y_0 \neq 0$. We split the integral (5.30) accordingly into its scalar part belonging to the linear subspace $\{Te_0 \mid T \in \mathcal{L}(\mathbb{C}^N)\}$ of $\mathcal{L}(\mathbb{C}^N)_{(2)}$ and its vector part belonging to the linear subspace $\{T_1 e_1 + T_2 e_2 \mid T_1, T_2 \in \mathcal{L}(\mathbb{C}^N)\}$ of $\mathcal{L}(\mathbb{C}^N)_{(2)}$. There is no component belonging to the linear subspace $\{T e_1 e_2 \mid T \in \mathcal{L}(\mathbb{C}^N)\}$ of $\mathcal{L}(\mathbb{C}^N)_{(2)}$.

We make a few observations. If the limit of the scalar part

$$-i \int_{\mathbb{T}} (\langle \boldsymbol{B}(\boldsymbol{y}), s\rangle^2 - \epsilon^2 I)(\langle \boldsymbol{B}(\boldsymbol{y}), s\rangle^2 + \epsilon^2 I)^{-2} z^{-1} \, dz \qquad (5.32)$$

of the integral (5.30) exists in $\mathcal{L}(\mathbb{C}^N)$ and is nonzero as $\epsilon \to 0$, then by formula (5.4), the Cauchy kernel $G_{\boldsymbol{y}+\epsilon e_0}(\boldsymbol{A})$ has a jump discontinuity at $\boldsymbol{y} \in \mathbb{R}^2$ as $\epsilon \to 0$. In this case $\boldsymbol{y} \in \gamma(\boldsymbol{A})$. The formula

$$\mathcal{W}_{\boldsymbol{A}}(\phi) \; = \; \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^2} [G_{\boldsymbol{y}+\varepsilon e_0}(\boldsymbol{A}) - G_{\boldsymbol{y}-\varepsilon e_0}(\boldsymbol{A})]\phi(\boldsymbol{y}) \,.., \qquad \phi \in C_c^\infty(\mathbb{R}^2),$$

mentioned in the proof of Theorem 4.8 shows that the jump

$$y \longmapsto \lim_{\varepsilon \to 0^+} [G_{\boldsymbol{y}+\varepsilon e_0}(\boldsymbol{A}) - G_{\boldsymbol{y}-\varepsilon e_0}(\boldsymbol{A})],$$

where it exists, is the Schwartz kernel of the matrix valued distribution $\mathcal{W}_{\boldsymbol{A}}$. The vector part

$$\frac{i|\epsilon|}{4\pi^2}\int_{\mathbb{T}} s\langle \boldsymbol{B}(\boldsymbol{y}), s\rangle(\langle \boldsymbol{B}(\boldsymbol{y}), s\rangle^2 + \epsilon^2 I)^{-2} z^{-1}\, dz \tag{5.33}$$

of the integral (5.30) depends only on $|\epsilon|$ for $\epsilon \neq 0$, so the vector part of

$$G_{\boldsymbol{y}+\varepsilon e_0}(\boldsymbol{A}) - G_{\boldsymbol{y}-\varepsilon e_0}(\boldsymbol{A})$$

is zero for all $\varepsilon > 0$, in accordance with the fact that the distribution $\mathcal{W}_{\boldsymbol{A}}$ takes its values in the subspace $\mathcal{L}(\mathbb{C}^N)$ of $\mathcal{L}(\mathbb{C}^N)_{(2)}$.

The strategy used to prove that $\boldsymbol{x} \in \gamma(\boldsymbol{A})^c$ is to show that the matrix-valued integral (5.32) converges to zero as $\epsilon \to 0+$, whereas the integral (5.33) converges in $\mathcal{L}(\mathbb{C}^N)_{(2)}$ uniformly for all $\boldsymbol{y} \in \mathbb{R}^2$ in a neighbourhood of $\boldsymbol{x}$.

We first examine the residues of the matrix-valued integrand

$$z \longmapsto (\langle \boldsymbol{B}(\boldsymbol{y}), s(z)\rangle^2 - \epsilon^2 I)(\langle \boldsymbol{B}(\boldsymbol{y}), s(z)\rangle^2 + \epsilon^2 I)^{-2} z^{-1} \tag{5.34}$$

of (5.32). Note that if $\boldsymbol{y}$ belongs to an open neighbourhood of $\boldsymbol{x}$ in $\mathbf{R}(A) \subset \rho(A)$, then the point $z = 0$ is a removable singularity, for $B_1(\boldsymbol{y}) + iB_2(\boldsymbol{y}) = (y_1 + iy_2)I - (A_1 + iA_2)$ is invertible and we may write $(\langle \boldsymbol{B}(\boldsymbol{y}), s(z)\rangle^2 - \epsilon^2 I)(\langle \boldsymbol{B}(\boldsymbol{y}), s(z)\rangle^2 + \epsilon^2 I)^{-2} z^{-1}$ as

$$(\langle \boldsymbol{B}(\boldsymbol{y}), zs(z)\rangle^2 - z\epsilon^2 I)(\langle \boldsymbol{B}(\boldsymbol{y}), zs(z)\rangle^2 + z\epsilon^2 I)^{-2} z,$$

where $(\langle \boldsymbol{B}(\boldsymbol{y}), zs(z)\rangle^2 - z\epsilon^2 I)(\langle \boldsymbol{B}(\boldsymbol{y}), zs(z)\rangle^2 + z\epsilon^2 I)^{-2} \to 4(B_1(\boldsymbol{y}) + iB_2(\boldsymbol{y}))^{-2}$ as $z \to 0$.

**Lemma 5.27.** *Let $\epsilon > 0$. If $z \neq 0$ is a solution of* $\det(\langle \boldsymbol{B}(\boldsymbol{y}), s(z)\rangle + i\epsilon I) = 0$, *then $\overline{z}^{-1}$ satisfies* $\det\left(\langle \boldsymbol{B}(\boldsymbol{y}), s\left(\overline{z}^{-1}\right)\rangle - i\epsilon I\right) = 0$. *In particular, if $\phi$ is the function defined in Corollary 5.18, then $\phi(-\epsilon, \boldsymbol{y}) = \overline{\phi(\epsilon, \boldsymbol{y})}^{-1}$.*

*Proof.* The identity $\langle \boldsymbol{B}(\boldsymbol{y}), s\left(\overline{z}^{-1}\right)\rangle = \langle \boldsymbol{B}(\boldsymbol{y}), s(z)\rangle^*$ holds because $A_1$ and $A_2$ are hermitian matrices, so

$$\begin{aligned}\det\left(\langle \boldsymbol{B}(\boldsymbol{y}), s\left(\overline{z}^{-1}\right)\rangle - i\epsilon I\right) &= \det\left(\langle \boldsymbol{B}(\boldsymbol{y}), s(z)\rangle^* - i\epsilon I\right)\\ &= \overline{\det\left(\langle \boldsymbol{B}(\boldsymbol{y}), s(z)\rangle + i\epsilon I\right)}.\end{aligned}$$

Let $\lambda_{1,\boldsymbol{x}} : V_\zeta \to \mathbb{C}$ and $\phi : U_x \to \mathbb{C}$ be the functions defined in Corollary 5.18. Then $z \longmapsto \overline{\lambda_{1,\boldsymbol{x}}(\overline{z}^{-1})}$, $z \in V_\zeta$ is analytic and equal to $\lambda_{1,\boldsymbol{x}}$ on $V_\zeta \cap \mathbb{T}$ where $\lambda_{1,\boldsymbol{x}}$ has real values. By analytic continuation, it follows that $\overline{\lambda_{1,\boldsymbol{x}}(\overline{z}^{-1})} = \lambda_{1,\boldsymbol{x}}(z)$ for all $z \in V_\zeta$. According to the definition of $\phi$ we have $\lambda_{1,\boldsymbol{x}}\left(\overline{\phi(\epsilon, \boldsymbol{y})}^{-1}\right) = \overline{\lambda_{1,\boldsymbol{x}}(\phi(\epsilon, \boldsymbol{y}))} = -i\epsilon$ and $\lambda_{1,\boldsymbol{x}}(\phi(-\epsilon, \boldsymbol{y})) = -i\epsilon$. The uniqueness of $\phi$ ensures that $\phi(-\epsilon, \boldsymbol{y}) = \overline{\phi(\epsilon, \boldsymbol{y})}^{-1}$ for all $(\epsilon, \boldsymbol{y}) \in U_x$ □

Hence, solutions $z \in \mathbb{C} \setminus \{0\}$ of

$$\begin{aligned}\det\left(\langle \boldsymbol{B}(\boldsymbol{y}), s(z)\rangle^2 + \epsilon^2 I\right) &= \det\left(\langle \boldsymbol{B}(\boldsymbol{y}), s(z)\rangle + i\epsilon I\right)\det\left(\langle \boldsymbol{B}(\boldsymbol{y}), s(z)\rangle - i\epsilon I\right)\\ &= 0\end{aligned}$$

either satisfy $z \in \mathbb{T}$ (if $\epsilon = 0$) or come in pairs $z = \xi$ and $z = \overline{\xi}^{-1}$, one inside the open unit disk $D$ and the other outside the closed unit disk $\overline{D}$.

The following representation was obtained in [16, Equation (4.4a)] using a plane wave decomposition different to the one used here.

**Lemma 5.28.** *Suppose that $\boldsymbol{x} \in \mathbb{R}^2 \setminus \sigma(A)$ does not belong to the Kippenhahn curves $C(A)$. Then there exists an open neighbourhood $U$ of $\boldsymbol{x}$ in $\mathbb{R}^2$ disjoint from $C(A)$ and two contours, $\Gamma_1(\boldsymbol{x})$ surrounding $\overline{D}$ and $\Gamma_2(\boldsymbol{x})$ contained in $D$, both anticlockwise oriented, such that $\langle \boldsymbol{B}(\boldsymbol{y}), s(z)\rangle$ is invertible in $\mathcal{L}(\mathbb{C}^N)$ for all $z \in \Gamma_1(\boldsymbol{x}) \cup \Gamma_2(\boldsymbol{x})$ and $\boldsymbol{y} \in U$, and the limit*

$$\begin{aligned}
&\lim_{\epsilon\to 0+} \int_{\mathbb{T}} (\langle \boldsymbol{B}(\boldsymbol{y}), s(z)\rangle^2 - \epsilon^2 I)(\langle \boldsymbol{B}(\boldsymbol{y}), s(z)\rangle^2 + \epsilon^2 I)^{-2} z^{-1}\, dz \\
&\quad = \frac{1}{2} \lim_{\epsilon\to 0} \int_{\Gamma_1(\boldsymbol{x})+\Gamma_2(\boldsymbol{x})} (\langle \boldsymbol{B}(\boldsymbol{y}), s(z)\rangle^2 - \epsilon^2 I) \times \\
&\qquad\qquad (\langle \boldsymbol{B}(\boldsymbol{y}), s(z)\rangle^2 + \epsilon^2 I)^{-2} z^{-1}\, dz \qquad (5.35)\\
&\quad = \frac{1}{2} \int_{\Gamma_1(\boldsymbol{x})+\Gamma_2(\boldsymbol{x})} \langle \boldsymbol{y}I - \boldsymbol{A}, s(z)\rangle^{-2} z^{-1}\, dz, \qquad (5.36)
\end{aligned}$$

*exists and the convergence is uniform for all $\boldsymbol{y} \in U$.*

*Proof.* Suppose that $\zeta \in \overline{D}$ satisfies

$$\det\left(\langle \boldsymbol{x}I - \boldsymbol{A}, s(\zeta)\rangle\right) \;=\; 0. \qquad (5.37)$$

If $\zeta \in \mathbb{T}$, then we know that an analytic parametrization (5.24) exists in an open neighbourhood $V_\zeta$ of $\zeta$ in $\mathbb{C}$ for which $\lambda_{1,\boldsymbol{x}}(\zeta) = 0$. By assumption, $\boldsymbol{x} \in \mathbb{R}^2 \setminus C(A)$, so Corollary 5.18 implies that there exists a smooth function $\boldsymbol{y} \longmapsto \phi(0, \boldsymbol{y})$ defined in a neighbourhood $U$ of $\boldsymbol{x}$ in $\mathbb{R}^2$ disjoint from $C(A)$, such that $\phi(0, \boldsymbol{x}) = \zeta$, $\phi(0, \boldsymbol{y}) \in \mathbb{T}$ and $\lambda_{1,\boldsymbol{y}}(\phi(0, \boldsymbol{y})) = 0$ for all $\boldsymbol{y} \in U$.

Furthermore, the solution $\phi(\xi, \boldsymbol{y})$ of $\lambda_{1,\boldsymbol{y}}(\phi(\xi, \boldsymbol{y})) = i\xi$ is a smooth function for $(\xi, \boldsymbol{y})$ in a neighbourhood of $(0, \boldsymbol{x})$ in $\mathbb{R}^3$. Hence, given any contours $\Gamma_1(\boldsymbol{x})$ and $\Gamma_2(\boldsymbol{x})$ satisfying the conditions above, there exists an open neighbourhood $V$ of $(0, \boldsymbol{x}) \in \mathbb{R}^3$ such that $\phi(\pm\epsilon, \boldsymbol{y})$ lies in the region between the contours $\Gamma_1(\boldsymbol{x})$ and $\Gamma_2(\boldsymbol{x})$ for all $(\pm\epsilon, \boldsymbol{y}) \in V$. According to Corollary 5.18, the complex numbers $\phi(\pm\epsilon, \boldsymbol{y})$ are distinct and both converge to $\phi(0, \boldsymbol{y})$ as $\epsilon \to 0+$.

On the other hand,

$$X_{\boldsymbol{x}} \;=\; \left\{ z \in D \;\middle|\; \det\left(\langle \boldsymbol{x}I - \boldsymbol{A}, s(z)\rangle\right) = 0 \right\}$$

is a finite subset $\langle \zeta_j \rangle_{j=1}^k$ of the open unit disk $D$. We claim that there exists an open neighbourhood $W$ of $(0, \boldsymbol{x})$ and disjoint closed disks $D_j \subset D$ centred at $\zeta_j \in X_{\boldsymbol{x}}$ such that for every $(\xi, \boldsymbol{y}) \in W$, all solutions $z$ of the equation

$$\det\left(\langle \boldsymbol{y}I - \boldsymbol{A}, s(z)\rangle + i\xi I\right) \;=\; 0 \qquad (5.38)$$

lie in the union $\cup_{j=1}^k D_j$ of the disjoint closed disks.

This would again follow from Corollary 5.18 if we knew that an analytic parametrization (5.24) exists in an open neighbourhood $V_\zeta$ of $\zeta \in X_{\boldsymbol{x}}$. We have already noted that, except for a finite set of points, such an analytic parametrization is possible [13, Theorem 3.3.12]. More simply, setting $\boldsymbol{B}(\boldsymbol{y}) = \boldsymbol{y}I - \boldsymbol{A}$, equation (5.38) can be written as

$$\begin{aligned}
&\det\left(\langle \boldsymbol{y}I - \boldsymbol{A}, s(z)\rangle + i\xi I\right)\\
&\quad = (2z)^{-n}\det\left(\langle \boldsymbol{B}(\boldsymbol{y}), 2zs(z)\rangle + i2z\xi I\right)\\
&\quad = (2z)^{-n}\det(B_1(\boldsymbol{y}) - iB_2(\boldsymbol{y}))\det\big(z^2I - (B_1(\boldsymbol{y}) - iB_2(\boldsymbol{y}))^{-1}\times\\
&\qquad (B_1(\boldsymbol{y}) + iB_2(\boldsymbol{y})) + i2z\xi(B_1(\boldsymbol{y}) - iB_2(\boldsymbol{y}))^{-1}\big)\\
&\quad = (2z)^{-n}\det(B_1(\boldsymbol{y}) - iB_2(\boldsymbol{y}))\det\big[(zI + i\xi(B_1(\boldsymbol{y}) - iB_2(\boldsymbol{y}))^{-1})^2\\
&\qquad\qquad -(B_1(\boldsymbol{y}) - iB_2(\boldsymbol{y}))^{-1}(B_1(\boldsymbol{y}) + iB_2(\boldsymbol{y})) + \xi^2(B_1(\boldsymbol{y}) - iB_2(\boldsymbol{y}))^{-2}\big]\\
&\quad = 0,
\end{aligned}$$

provided that $\boldsymbol{y} \in \rho(A)$. By assumption $\boldsymbol{x} \in \rho(A)$, so the equation is valid for all $\boldsymbol{y}$ in a neighbourhood of $\boldsymbol{x}$ and the solutions $z$ of (5.38) can be expressed in terms of the eigenvalues of an $(N \times N)$ matrix depending continuously on $(\xi, \boldsymbol{y})$. The unordered $N$-tuple of eigenvalues of this matrix valued function depends *continuously* on the parameters $(\xi, \boldsymbol{y})$ [65, Theorem II.5.1] facilitating the construction of the required disjoint closed disks $D_j$, $j = 1, \ldots, k$.

According to Lemma 5.27, poles of the function (5.34) come in pairs $(z, \overline{z}^{-1})$ lying either inside the open unit disk $D$ or outside the closed unit disk $\overline{D}$ for all $\boldsymbol{y}$ in a neighbourhood of $\boldsymbol{x}$. Now choose the inner contour $\Gamma_2(\boldsymbol{x})$ to surround every closed disk $D_j$, and choose $\Gamma_1(\boldsymbol{x})$ to lie between $\mathbb{T}$ and points $\overline{z}^{-1}$, $z \in \cup_{j=1}^k D_j$, outside $\overline{D}$. Next choose the intersection of all open neighbourhoods $V$ of $(0, \boldsymbol{x})$ corresponding to the finitely many solutions $\zeta \in \mathbb{T}$ of equation (5.37) and take the intersection $V'$ of this open set with the open set $W$ corresponding to the finitely many solutions $\zeta_j \in D$, $j = 1, \ldots, k$ of equation (5.38).

Then for every $(\epsilon, \boldsymbol{y}) \in V'$, the contour integral

$$\int_{\Gamma_1(\boldsymbol{x})+\Gamma_2(\boldsymbol{x})} (\langle \boldsymbol{B}(\boldsymbol{y}), s(z)\rangle^2 - \epsilon^2 I)(\langle \boldsymbol{B}(\boldsymbol{y}), s(z)\rangle^2 + \epsilon^2 I)^{-2} z^{-1}\, dz$$

is $2\pi i$ times the sum of the residues of the integrand at the distinct poles $\phi(\pm\epsilon, \boldsymbol{y})$ and $4\pi i$ times the sum of the residues at poles near solutions $\zeta \in D$ of equation (5.37), because both contours $\Gamma_1(\boldsymbol{x})$ and $\Gamma_2(\boldsymbol{x})$ surround these. The possibility of a pole at zero in the case that $\boldsymbol{y} \in \sigma(A)$ is excluded.

The function

$$(\epsilon, \boldsymbol{y}) \longmapsto \int_{\Gamma_1(\boldsymbol{x})+\Gamma_2(\boldsymbol{x})} (\langle \boldsymbol{B}(\boldsymbol{y}), s(z)\rangle^2 - \epsilon^2 I)(\langle \boldsymbol{B}(\boldsymbol{y}), s(z)\rangle^2 + \epsilon^2 I)^{-2} z^{-1}\, dz$$

is continuous on $V'$, so the equality (5.36) is immediate. To prove the equality (5.35), we need to look separately at those poles $\zeta$ of (5.34) satisfying (5.37) lying inside the open disk $D$ and those lying on $\mathbb{T}$.

The sum of the residues of the function (5.34) belonging to $\cup_{j=1}^k D_j \subset D$ is equal to $\frac{1}{2\pi i}\int_{\Gamma_2(\boldsymbol{x})}(\langle \boldsymbol{B}(\boldsymbol{y}), s(z)\rangle^2 - \epsilon^2 I)(\langle \boldsymbol{B}(\boldsymbol{y}), s(z)\rangle^2 + \epsilon^2 I)^{-2} z^{-1}\, dz$ for all $(\epsilon, \boldsymbol{y}) \in V'$, so this is uniformly continuous in $(\epsilon, \boldsymbol{y}) \in V'$.

Now we need to show that the sum of the residues of the function (5.34) over all the poles $\phi(\pm\epsilon, \boldsymbol{y})$ converges uniformly in $\boldsymbol{y}$ as $\epsilon \to 0+$ to twice the sum of the residues of $z \longmapsto \langle \boldsymbol{B}(\boldsymbol{y}), s(z)\rangle^{-2} z^{-1}$ over all the poles $\phi(0, \boldsymbol{y})$. According to Lemma 5.27, one of the poles $\phi(\pm\epsilon, \boldsymbol{y})$ lies in $D$ and the other is outside $D$, so then equality (5.35) will be established.

The set $Z_{\boldsymbol{y}}$ of all solutions $\zeta \in \mathbb{T}$ of equation (5.37), is finite for each $\boldsymbol{y}$ in a neighbourhood of $\boldsymbol{x}$, so it suffices to prove that each residue of (5.34) at $\phi(\pm\epsilon, \boldsymbol{y})$ converges uniformly to the residue of $z \longmapsto \langle \boldsymbol{B}(\boldsymbol{y}), s(z)\rangle^{-2} z^{-1}$ at $\phi(0, \boldsymbol{y})$.

For every solution $\zeta \in Z_{\boldsymbol{y}}$, there exists a neighbourhood $V_\zeta$ in $\mathbb{C}$ such that $V_\zeta \cap (Z_{\boldsymbol{y}} \setminus \{\zeta\}) = \emptyset$ and the parametrization (5.24) holds. Then, writing $\lambda_j$ for the eigenvalues $\lambda_{j,\boldsymbol{y}}(z)$ of $\langle \boldsymbol{y}I - \boldsymbol{A}, s(z)\rangle$ in (5.24), the equality

$$\begin{aligned}(\langle \boldsymbol{B}(\boldsymbol{y}), s(z)\rangle^2 - \epsilon^2 I)&(\langle \boldsymbol{B}(\boldsymbol{y}), s(z)\rangle^2 + \epsilon^2 I)^{-2} z^{-1} \\ &= \sum_{j=1}^m \frac{\lambda_j(z)^2 - \epsilon^2}{(\lambda_j(z)^2 + \epsilon^2)^2 z} P_j(z) \qquad (5.39)\end{aligned}$$

holds for all $z \in V_\zeta$.

By assumption, the eigenvalue functions $\lambda_j$ have at most one zero, $z = \zeta$, in $V_\zeta$. We may suppose that for some integer $k$, $1 \le k \le m$, we have $\lambda_1(\zeta) = \ldots = \lambda_k(\zeta) = 0$ and $\lambda_j(\zeta) \neq 0$ for $j > k$. The terms in the sum (5.39) corresponding to the latter are analytic in the open set $V_\zeta$.

By Corollary 5.18, there exists a neighbourhood $U_{\boldsymbol{x}}$ of $(0, \boldsymbol{x})$ in $\mathbb{R}^3$ such that for all $j$ with $1 \le j \le k$, $\phi_j(\epsilon, \boldsymbol{y}) = \lambda_j^{-1}(i\epsilon)$ defines a $C^\infty$-function on $U_{\boldsymbol{x}}$ satisfying $\lambda_j'(\phi_j(\epsilon, \boldsymbol{y})) \neq 0$ for all $(\epsilon, \boldsymbol{y}) \in U_{\boldsymbol{x}}$. In particular, the set of all $\boldsymbol{y} \in \mathbb{R}^2$ such that $(\xi, \boldsymbol{y}) \in U_{\boldsymbol{x}}$ for some $\xi \in \mathbb{R}$, is disjoint from $C(A)$. Then for $\epsilon > 0$, we have

$$\begin{aligned}\mathrm{Res}&\left(\frac{\lambda_j(z)^2 - \epsilon^2}{(\lambda_j(z)^2 + \epsilon^2)^2 z} P_j(z); \phi_j(\epsilon, \boldsymbol{y})\right) \\ &= \frac{1}{\lambda_j'(\phi_j(\epsilon, \boldsymbol{y}))^2}\left[\frac{d}{dz}\left(\frac{\lambda_j(z)^2 - \epsilon^2}{(\lambda_j(z) + i\epsilon)^2 z} P_j(z)\right)\right]_{\phi_j(\epsilon, \boldsymbol{y})} \\ &\quad - \frac{\lambda_j''(\phi_j(\epsilon, \boldsymbol{y}))\, P_j(\phi_j(\epsilon, \boldsymbol{y}))}{\lambda_j'(\phi_j(\epsilon, \boldsymbol{y}))^3 \phi_j(\epsilon, \boldsymbol{y})}.\end{aligned}$$

Here we have written

$$(\lambda_j(z)^2 + \epsilon^2)^2 = (\lambda_j(z) + i\epsilon)^2 (\lambda_j(z) - i\epsilon)^2$$

and noted that

$$(\lambda_j(z) - i\epsilon)^2 = \left(\frac{\lambda_j(z) - \lambda_j(\phi_j(\epsilon, \boldsymbol{y}))}{z - \phi_j(\epsilon, \boldsymbol{y})}\right)^2 (z - \phi_j(\epsilon, \boldsymbol{y}))^2$$

gives rise to a pole of order two at $\phi_j(\epsilon, \boldsymbol{y})$. Now

$$\frac{d}{dz}\frac{\lambda_j^2-\epsilon^2}{(\lambda_j+i\epsilon)^2} = 2\lambda_j'\,\frac{\lambda_j(\lambda_j+i\epsilon)-(\lambda_j^2-\epsilon^2)}{(\lambda_j+i\epsilon)^3} = 2i\epsilon\lambda_j'\,\frac{\lambda_j-i\epsilon}{(\lambda_j+i\epsilon)^3}$$

is zero at $\phi_j(\epsilon, \boldsymbol{y})$. According to Corollary 5.18, the function $(\epsilon, \boldsymbol{y}) \longmapsto \lambda_j'(\phi_j(\epsilon, \boldsymbol{y}))$ is $C^\infty$ and nonzero in a neighbourhood of $(0, \boldsymbol{x})$. It follows that the matrix

$$\operatorname{Res}\left(\frac{\lambda_j(z)^2-\epsilon^2}{(\lambda_j(z)^2+\epsilon^2)^2 z}\,P_j(z);\phi_j(\epsilon,\boldsymbol{y})\right) \tag{5.40}$$

$$= \frac{1}{2\lambda_j'(\phi_j(\epsilon,\boldsymbol{y}))^2}\left[\frac{d}{dz}\,\frac{P_j(z)}{z}\right]_{\phi_j(\epsilon,\boldsymbol{y})} - \frac{\lambda_j''(\phi_j(\epsilon,\boldsymbol{y}))\,P_j(\phi_j(\epsilon,\boldsymbol{y}))}{2\lambda_j'(\phi_j(\epsilon,\boldsymbol{y}))^3\phi_j(\epsilon,\boldsymbol{y})} \tag{5.41}$$

converges uniformly for all $\boldsymbol{y}$ in a neighbourhood of $\boldsymbol{x}$ as $\epsilon \to 0^+$.

The residue at each of the poles $\phi_j(\pm\epsilon, \boldsymbol{y})$ contributes to the integral over $\Gamma_1(\boldsymbol{x}) + \Gamma_2(\boldsymbol{x})$, so in the limit, we obtain twice the sum of the residues of the matrix-valued function $z \longmapsto \langle \boldsymbol{B}(\boldsymbol{y}), s(z)\rangle^{-2} z^{-1}$ at poles $\zeta \in \mathbb{T}$ and inside $\mathbb{T}$. We have proved the required formula. □

The next lemma establishes that the scalar component of $\omega \longmapsto G_\omega(\boldsymbol{A})$ is continuous in a neighbourhood of $\boldsymbol{x} \in \mathbf{R}(A)$ in $\mathbb{R}^3$.

**Lemma 5.29.** *For every $\boldsymbol{x} \in \mathbf{R}(A)$ the matrix valued function*

$$y \longmapsto G_{\boldsymbol{y}+\epsilon e_0}(\boldsymbol{A}) - G_{\boldsymbol{y}-\epsilon e_0}(\boldsymbol{A}), \quad y \in \mathbb{R}^2,$$

*converges to zero as $\epsilon \to 0+$, uniformly for all $\boldsymbol{y}$ in a neighbourhood of $\boldsymbol{x}$.*

*Proof.* By Corollary 5.18, $\boldsymbol{x} \in C(A)^c$ and $\mathbf{R}(A) \subseteq \rho(A)$, so the representation (5.36) is valid. But there are no poles interior to $\mathbb{T}$ or exterior to $\mathbb{T}$. Hence, the integral over $\Gamma_2(\boldsymbol{x})$ is zero and we can deform $\Gamma_1(\boldsymbol{x})$ to $\infty$. It follows that integral (5.36) is zero. □

The following argument treats the residues of the integrand

$$z \longmapsto s(z)\langle \boldsymbol{B}(\boldsymbol{y}), s(z)\rangle(\langle \boldsymbol{B}(\boldsymbol{y}), s(z)\rangle^2 + \epsilon^2 I)^{-2} z^{-1} \tag{5.42}$$

of the contour integral (5.33), the vector part of the Cauchy kernel.

Let $\boldsymbol{x} \in \mathbb{R}^2 \setminus (C(A) \cup \sigma(A))$. As in the proof of Lemma 5.28, there exists an open set $V_\zeta \subset D$ about each solution $\zeta \in D$ of $\det(\langle \boldsymbol{x}I - \boldsymbol{A}, s(z)\rangle) = 0$ and a neighbourhood $W$ of $(0, \boldsymbol{x})$ in $\mathbb{R}^3$, such that for every $(\xi, \boldsymbol{y}) \in W$, all solutions $z$ of the equation $\det(\langle \boldsymbol{y}I - \boldsymbol{A}, s(z)\rangle - i\xi I) = 0$ belong to $\cup_\zeta V_\zeta$. Moreover, the closures of the open sets $V_\zeta$ are pairwise disjoint.

The sum $R_\zeta(\epsilon, \boldsymbol{y})$ of the residues of the function (5.42) at poles in $V_\zeta$ is a continuous function of $(\epsilon, \boldsymbol{y})$, because it can be represented as a contour integral of the continuous function (5.42) over a contour inside the open unit disk $D$ surrounding $V_\zeta$. Then

$$\lim_{\epsilon\to 0+} \epsilon\, R_\zeta(\epsilon, \boldsymbol{y}) \;=\; 0$$

uniformly for $\boldsymbol{y}$ in a neighbourhood of $\boldsymbol{x}$.

Now let $\zeta \in \mathbb{T}$ be a solution of $\det(\langle \boldsymbol{x}I - \boldsymbol{A}, s(z)\rangle) = 0$. Suppose that $j$, $1 \le j \le m$, is an index for which $\lambda_j(\zeta) = 0$ and $\phi(\epsilon, \boldsymbol{y}) = \lambda_j^{-1}(i\epsilon)$ lies in $D$ for all $0 < \epsilon < \delta$, otherwise, replace $i\epsilon$ by $-i\epsilon$. Such a solution exists by Corollary 5.18 and the assumption that $\boldsymbol{x} \in \mathbb{R}^2 \setminus C(A)$. Furthermore, $(\epsilon, \boldsymbol{y}) \longmapsto \phi(\epsilon, \boldsymbol{y})$ is $C^\infty$ in a neighbourhood of $(0, \boldsymbol{x})$ and $|\lambda_j'(\phi(\epsilon, \boldsymbol{y}))|$ is bounded below.

**Lemma 5.30.** *Let $\boldsymbol{x} \in \mathbb{R}^2 \setminus C(A)$ and suppose that $\phi(\epsilon, \boldsymbol{y})$ is a pole of (5.42) belonging to the open unit disk $D$, as defined above. Then*

$$\epsilon \operatorname{Res}\left(\frac{s(z)\langle \boldsymbol{B}(\boldsymbol{y}), s(z)\rangle(\langle \boldsymbol{B}(\boldsymbol{y}), s(z)\rangle^2 + \epsilon^2 I)^{-2}}{z}; \phi(\epsilon, \boldsymbol{y})\right)$$

*converges as $\epsilon \to 0+$, uniformly for all $\boldsymbol{y}$ in a neighbourhood of $\boldsymbol{x}$.*

*Proof.* As in the proof of Lemma 5.28, it suffices to prove that

$$\epsilon \operatorname{Res}\left(\frac{s(z)\lambda_j(z)}{(\lambda_j(z)^2 + \epsilon^2)^2 z} P_j(z); \phi(\epsilon, \boldsymbol{y})\right)$$

converges as $\epsilon \to 0+$ uniformly for $\boldsymbol{y}$ in a neighbourhood of $\boldsymbol{x}$.

By assumption, $\phi(\epsilon, \boldsymbol{y}) = \lambda_j^{-1}(i\epsilon)$ lies in the open unit disk $D$ for all $0 < \epsilon < \delta$. Then

$$\begin{aligned}
&\epsilon \operatorname{Res}\left(\frac{s(z)\lambda_j(z)}{(\lambda_j(z)^2 + \epsilon^2)^2 z} P_j(z); \phi(\epsilon, \boldsymbol{y})\right) \\
&\quad = \frac{\epsilon}{\lambda_j'(\phi(\epsilon, \boldsymbol{y}))^2}\left[\frac{d}{dz}\left(\frac{s(z)\lambda_j(z)}{(\lambda_j(z) + i\epsilon)^2 z}\; P_j(z)\right)\right]_{\phi(\epsilon, \boldsymbol{y})} + \\
&\qquad \frac{i\lambda_j''(\phi(\epsilon, \boldsymbol{y}))s(\phi(\epsilon, \boldsymbol{y}))P_j(\phi(\epsilon, \boldsymbol{y}))}{4\lambda_j'(\phi(\epsilon, \boldsymbol{y}))^3 \phi(\epsilon, \boldsymbol{y})}.
\end{aligned}$$

Note that

$$\frac{d}{dz}\frac{\lambda_j}{(\lambda_j + i\epsilon)^2} \;=\; \lambda_j' \,\frac{(\lambda_j + i\epsilon) - 2\lambda_j}{(\lambda_j + i\epsilon)^3} \;=\; -\lambda_j' \,\frac{\lambda_j - i\epsilon}{(\lambda_j + i\epsilon)^3}$$

is zero at $\phi(\epsilon, \boldsymbol{y})$. On the other hand,

$$\frac{\epsilon\lambda_j(\phi(\epsilon, \boldsymbol{y}))}{(\lambda_j(\phi(\epsilon, \boldsymbol{y})) + i\epsilon)^2}\left[\frac{d}{dz}\left(\frac{s(z)}{z}\; P_j(z)\right)\right]_{\phi(\epsilon, \boldsymbol{y})}$$

is equal to

$$-\frac{i}{4}\left[\frac{d}{dz}\left(\frac{s(z)}{z}\; P_j(z)\right)\right]_{\phi(\epsilon, \boldsymbol{y})}$$

and the other terms in the residue formula converge uniformly for $\boldsymbol{y}$ in a neighbourhood of $\boldsymbol{x}$ as $\epsilon \to 0+$. □

Consequently, for every $\boldsymbol{x} \in \mathbf{R}(A)$, the matrix-valued integral (5.32) converges to zero as $\epsilon \to 0$, whereas the integral (5.33) converges in $\mathcal{L}(\mathbb{C}^N)$ uniformly in a neighbourhood of $\boldsymbol{x}$. The Cauchy kernel $\omega \longmapsto G_\omega(\boldsymbol{A})$ is therefore continuous in a neighbourhood of $(0, \boldsymbol{x})$ in $\mathbb{R}^3$, proving that $\boldsymbol{x} \in \gamma(\boldsymbol{A})^c$.

To complete the proof of Theorem 5.24, it still remains to prove that $\boldsymbol{x} \in \gamma(\boldsymbol{A})$ for all $\boldsymbol{x} \in \mathbb{R}^2 \setminus \mathbf{R}(A)$. We essentially follow the somewhat abbreviated proof of [16, Theorem 4.3] after noting that Condition II of [16, Theorem 4.3] is superfluous by appealing to our Lemma 5.12. As mentioned in [16, p. 316], the proof is based on a closely related argument of Petrovsky [85, p. 348].

Let $\Delta(\boldsymbol{A})$ be the set of all $\boldsymbol{x} \in \mathbb{R}^2 \setminus C(A)$ such that $\lim_{\varepsilon \to 0+}[G_{\boldsymbol{y}+\varepsilon e_0}(\boldsymbol{A}) - G_{\boldsymbol{y}-\varepsilon e_0}(\boldsymbol{A})]$ converges uniformly to zero for all $\boldsymbol{y}$ in an open neighbourhood of $\boldsymbol{x}$ disjoint from $C(A)$. Then $\Delta(\boldsymbol{A})$ is an open subset of $\mathbb{R}^2$ containing $\gamma(\boldsymbol{A})^c$, because for every $\boldsymbol{x} \in \gamma(\boldsymbol{A})^c$, the Cauchy kernel $\omega \longmapsto G_\omega(\boldsymbol{A})$ is continuous for every $\omega$ in a neighbourhood of $(0, \boldsymbol{x})$ in $\mathbb{R}^3$.

Suppose that

$$\left(\mathbb{R}^2 \setminus (\overline{\mathbf{R}(A)} \cup C(A))\right) \cap \Delta(\boldsymbol{A}) \neq \emptyset. \tag{5.43}$$

We shall obtain a contradiction from the assumption (5.43), so showing that

$$\mathbb{R}^2 \setminus \left(\overline{\mathbf{R}(A)} \cup C(A)\right) \subseteq \Delta(\boldsymbol{A})^c \subset \gamma(\boldsymbol{A}).$$

Because

$$\left(\left(\mathbb{R}^2 \setminus (\overline{\mathbf{R}(A)} \cup C(A))\right) \cap \Delta(\boldsymbol{A})\right) \setminus \sigma(A) \tag{5.44}$$

is a nonempty open set, there exists a nonempty open subset $U$ of the set (5.44) such that $\lim_{\varepsilon \to 0+}[G_{\boldsymbol{y}+\varepsilon e_0}(\boldsymbol{A}) - G_{\boldsymbol{y}-\varepsilon e_0}(\boldsymbol{A})]$ converges uniformly to zero for all $\boldsymbol{y} \in U$.

Now $U$ is disjoint from $\mathbf{R}(A)$ and $\sigma(A)$. If for every $\boldsymbol{x} \in U$, every pole of the function

$$z \longmapsto \langle \boldsymbol{x}I - \boldsymbol{A}, s(z)\rangle^{-1} \tag{5.45}$$

lies on $\mathbb{T}$ , then $U \subset \mathbf{R}(A)$. By Lemma 5.27, poles $z \notin \mathbb{T}$ of (5.45) come in pairs $z \in D$ and $\overline{z}^{-1} \in \overline{D}^c$, so there must exist $\boldsymbol{x} \in U$ such that (5.45) has a pole inside $D$. Moreover, by the argument of Lemma 5.28 , the set

$$\left\{\boldsymbol{y} \in \mathbb{R}^2 \;\middle|\; \sigma\left(\left((\boldsymbol{y}I - A)^*\right)^{-1}(\boldsymbol{y}I - A)\right) \bigcap D \neq \emptyset\right\}$$

is an open subset of $\mathbb{R}^2$, so for every $\boldsymbol{y}$ belonging to some neighbourhood of $\boldsymbol{x}$, the function $z \longmapsto \langle \boldsymbol{y}I - \boldsymbol{A}, s(z)\rangle^{-1}$ has poles inside $D$. By shrinking $U$ if necessary, we suppose that $U$ has this property.

Then the calculation of the residues in Lemmas 5.28 and 5.30 is still valid because $U$ is disjoint from both $\sigma(A)$ and $C(A)$. By Lemma 5.28, the limit

$$\begin{aligned}\lim_{\varepsilon\to 0}\textstyle\int_{\mathbb{T}}(\langle \boldsymbol{y}I-\boldsymbol{A},s\rangle^2-\varepsilon^2 I)(\langle \boldsymbol{y}I-\boldsymbol{A},s\rangle^2+\varepsilon^2 I)^{-2}\,d\mu(s)\\ = -\tfrac{i}{2}\textstyle\int_{\Gamma_1(\boldsymbol{x})+\Gamma_2(\boldsymbol{x})}\langle \boldsymbol{y}I-\boldsymbol{A},s(z)\rangle^{-2}z^{-1}\,dz\end{aligned}$$

is a matrix-valued real analytic function for all $\boldsymbol{y}$ in a neighbourhood $U_{\boldsymbol{x}}$ of $\boldsymbol{x}$ contained in $U$ – a constant times the function

$$\boldsymbol{y} \longmapsto \lim_{\varepsilon\to 0^+}[G_{\boldsymbol{y}+\varepsilon e_0}(\boldsymbol{A})-G_{\boldsymbol{y}-\varepsilon e_0}(\boldsymbol{A})], \qquad \boldsymbol{y}\in U_{\boldsymbol{x}}.$$

By assumption, $U_{\boldsymbol{x}}\subset \Delta(\boldsymbol{A})$, so for all $\boldsymbol{y}\in U_{\boldsymbol{x}}$, we have

$$\int_{\Gamma_1(\boldsymbol{x})+\Gamma_2(\boldsymbol{x})}\langle \boldsymbol{y}I-\boldsymbol{A},s(z)\rangle^{-2}z^{-1}\,dz \;=\; 0. \tag{5.46}$$

The point $z=0$ is a removable singularity of the integrand in equation (5.46) because $\boldsymbol{y}\in U_{\boldsymbol{x}}\subset\rho(A)$.

Up until this point, we have worked locally with solutions $\phi(\boldsymbol{y})$ of the equation

$$\det(\langle \boldsymbol{y}I-\boldsymbol{A},s(z)\rangle) \;=\; 0$$

for $\phi(\boldsymbol{y})$ belonging to a neighbourhood of $\mathbb{T}$.

Now let us consider all solutions $\phi(\boldsymbol{y})\in\mathbb{C}$ of the simultaneous equations

$$\det(\mu I-\langle \boldsymbol{A},s(z)\rangle) \;=\; 0 \tag{5.47}$$

$$\mu-\langle \boldsymbol{y},s(z)\rangle \;=\; 0, \tag{5.48}$$

for $\boldsymbol{y}\in\mathbb{R}^2$.

For $z\neq 0$, equation (5.47) is equivalent to $\det(z\mu I-\langle \boldsymbol{A},zs(z)\rangle)=0$ and the function $(\mu,z)\longmapsto\det(z\mu-\langle \boldsymbol{A},zs(z)\rangle)$ is a polynomial in two variables. Equation (5.47) therefore determines an algebraic function $z\mu(z)$ of $z$ [1, Chapter 8, Definition 2]. Except for a finite set $\Xi$ of points in $\mathbb{C}$, each function element $(\mu,\Omega)$ of $\mu$ can be continued along any arc not passing through one of the exceptional points belonging to $\Xi$ [1, p294]. It follows from Rellich's Theorem and equation (5.24) that $\Xi$ is disjoint from $\mathbb{T}$.

Suppose that $(\mu_j,\Omega_j)$ is a function element of $\mu$ such that $\Omega_j$ is disjoint from $\Xi\cup\{0\}$. Then $\zeta\longmapsto(s_1(\zeta):s_2(\zeta):-\mu_j(\zeta))$, $\zeta\in\Omega_j$, is a smooth local parametrization of the algebraic curve $C(A)^*$ of Subsection 5.3.1. If $\boldsymbol{y}\in\mathbb{R}^2$ and $z\in\mathbb{C}$ satisfy equation (5.48) for $\mu=\mu_j(z)$, and $\mu_j'(z)-\langle y,s\rangle'(z)=0$, then by Lemma 5.12, $\boldsymbol{y}\in C(A)$. Consequently, if $\boldsymbol{y}\notin C(A)$, then any solution $z_0$ of equations (5.47), (5.48) with $\mu=\mu_j(z)$ has the property that

$$\mu_j'(z_0)-\langle y,s\rangle'(z_0) \;\neq\; 0.$$

Suppose that $\boldsymbol{y}\notin C(A)$. By the remark after Lemma 5.16, there exists an open neighbourhood $V_{\boldsymbol{y}}$ of $\boldsymbol{y}$ in $\mathbb{R}^2$ and an analytic function $w\longmapsto\phi_j(w)$, $w\in V_{\boldsymbol{y}}$, of two real variables such that $\langle w,s(\phi_j(w))\rangle=\mu_j(\phi_j(w))$ for all $w\in V_{\boldsymbol{y}}$. Hence,

$$\det(\langle w, s(\phi_j(w))\rangle I - \langle \boldsymbol{A}, s(\phi_j(w))\rangle) \;=\; 0$$

and for every $w \in V_{\boldsymbol{y}}$ the complex number $\phi_j(w)$ is a pole of the function

$$z \;\longmapsto\; \langle wI - \boldsymbol{A}, s(z)\rangle^{-1}.$$

Now according to (5.43), we are assuming that poles of the function (5.45) exist inside $D$. So there exist a nonzero integer $k$ and $2k$ functions $\boldsymbol{y} \longmapsto \pm\phi_j(\boldsymbol{y})$, $j = 1, \ldots, k$, defined for $\boldsymbol{y} \in U_{\boldsymbol{x}}$, that are analytic in two real variables and poles of (5.45) belonging to $D$. We can also assume that they have the property that $\pm\phi_j(\boldsymbol{y}) \notin \Xi \cup \{0\}$ for all $\boldsymbol{y} \in U_{\boldsymbol{x}}$ and and that they are constructed, as above, from the algebraic function $z\mu(z)$.

This is valid, because to any nonzero exceptional point $z \in \Xi$, there corresponds a unique solution $\boldsymbol{y} \in \mathbb{R}^2$ of (5.48) satisfying the equations

$$y_1 s_1(z) + y_2 s_2(z) \;=\; \mu \tag{5.49}$$
$$y_1 s_1(\overline{z}) - y_2 s_2(\overline{z}) \;=\; \overline{\mu}. \tag{5.50}$$

Here we use the observation that $s_1(z)s_2(\overline{z}) + s_1(\overline{z})s_2(z) = 0$ if and only if $|z| = 1$ and $\Xi$ is disjoint from $\mathbb{T}$. The point $z = 0$ is associated with points $\zeta \in \sigma(A)$ with $y_1 + iy_2 = \zeta$ and $\lim_{z\to 0} z\mu_j(z) = \zeta/2$, for some function element $(\mu_j, \Omega_j)$ of $\mu$ with $0 \in \overline{\Omega}_j$.

With these preliminary observations out of the way, we will obtain a contradiction from the assumption that equation (5.46) holds in a neighbourhood $U_{\boldsymbol{x}}$ of $\boldsymbol{x}$.

Let $x_1 \in \mathbf{R}(A)$ and suppose that $t \longmapsto \gamma(t)$, $0 \le t \le 1$, is a smooth curve in $\mathbb{R}^2$ such that $\gamma(0) = x$ and $\gamma(1) = x_1$. Suppose further that where $\gamma$ crosses a curve belonging to $C(A)$, it does so nontangentially and avoids all intersections, cusps and isolated points. This is possible because there are only finitely many such points. Furthermore, we suppose that $\gamma$ also avoids the image in $\mathbb{R}^2$ of the exceptional points $\Xi$ and the spectrum $\sigma(A)$ of $\boldsymbol{A}$. Then in a neighbourhood of any point in $\gamma([0,1])$, the functions $\{\phi_j\}_{j=1}^k$ defined by the algebraic function $z\mu(z)$ from equations (5.47) and (5.48) in the manner described above, do not take values in $\Xi\cup\{0\}$. Moreover, we have $\phi_j(\gamma(1)) \in \mathbb{T}$ and $\phi_j(\gamma(0)) \in U_{\boldsymbol{x}} \subset D$ for $j = 1, \ldots, k$. Let

$$t_0 \;=\; \sup\{t > 0 : \phi_j(\gamma(s)) \in D \text{ for every } 0 \le s \le t \text{ and } j = 1, \ldots, k\ \}.$$

Then $0 < t_0 \le 1$ and, by continuity, for some $m = 1, \ldots, k$, we must have $\phi_m(\gamma(t_0)) \in \mathbb{T}$. If $\mu_j'(\phi_m(\gamma(t_0))) - \langle \gamma(t_0), s\rangle'(\phi_m(\gamma(t_0))) \neq 0$, then by Rellich's Theorem and Lemma 5.16, there exists $\delta > 0$ such that $\phi_m(\gamma(t)) \in \mathbb{T}$ for all $t \in (t_0 - \delta, t_0 + \delta)$, contradicting the definition of $t_0$. Hence $\gamma(t_0) \in C(A)$ by Proposition 5.17.

According to our assumption, equation (5.46), the sum $Res(\boldsymbol{y})$ of the residues of the function $z \longmapsto \langle \boldsymbol{y}I - \boldsymbol{A}, s(z)\rangle^{-2} z^{-1}$ at $\pm\phi_j(\boldsymbol{y})$ and $\pm\overline{\phi_j}(\boldsymbol{y})^{-1}$, $j = 1, \ldots, k$, is zero for all $\boldsymbol{y} \in U_{\boldsymbol{x}}$. The outer integral about the contour

$\Gamma_1(\boldsymbol{x})$ in equation (5.46) surrounds $\pm\overline{\phi_j}(\boldsymbol{y})^{-1}$ and the integral is calculated from the residues at $\pm\overline{\phi_j}(\boldsymbol{y})^{-1}$ by the Cauchy integral formula.

For each $0 < t < t_0$, there exist contours $\Gamma_1(\gamma(t)) \subset \overline{D}^c$ and $\Gamma_2(\gamma(t)) \subset D$ and neighbourhoods $V_{\gamma(t)}$ of $\gamma(t)$ such that $\Gamma_1(\gamma(t))$ surrounds $\{\overline{\phi_j}(\boldsymbol{y})^{-1}\}_{j=1}^k$ and $\Gamma_2(\gamma(t))$ surrounds $\{\phi_j(\boldsymbol{y})\}_{j=1}^k$ for all $\boldsymbol{y} \in V_{\gamma(t)}$, and the contours do not surround any other poles of the function (5.45) for any $\boldsymbol{y} \in V_{\gamma(t)}$.

To see that this construction is possible, suppose that $\phi_\ell$ is some other distinct solution of the simultaneous equations (5.47) and (5.48) such that $\zeta = \phi_\ell(\gamma(t)) = \phi_1(\gamma(t)) \in D$, say, for some $0 < t < t_0$. Then

$$\langle \gamma(t), s(\zeta)\rangle = \mu_\ell(\zeta) = \mu_1(\zeta)$$

for two eigenvalues $\mu_\ell(z)$ and $\mu_1(z)$ of the matrix $\langle \boldsymbol{A}, s(z)\rangle$, for all $z \in \mathbb{C}$ in a neighbourhood of $\zeta$. Then $\zeta$ must be a branch point of the eigenvalues of the matrix valued function $z \longmapsto \langle \boldsymbol{A}, s(z)\rangle$, that is, $\zeta \in \Xi$. This contradicts our choice of the arc $\gamma$. Hence, all solutions of the simultaneous equations (5.47) and (5.48) have distinct values at each point of $\gamma$. By continuity, for each $0 \le t < t_0$ we can choose a neighbourhood $V_{\gamma(t)}$ of $\gamma(t)$ in which solutions of (5.47) and (5.48) have this property and contours $\Gamma_1(\gamma(t))$ and $\Gamma_2(\gamma(t))$ with the properties described above.

Then the function

$$Res(\boldsymbol{y}) \;=\; \frac{1}{2\pi i}\int_{\Gamma_1(\gamma(t))+\Gamma_2(\gamma(t))} \langle \boldsymbol{y}I - \boldsymbol{A}, s(z)\rangle^{-2} z^{-1}\, dz$$

defined for all $\boldsymbol{y} \in V_{\gamma(t)}$ and $0 \le t < t_0$ agrees on $U_{\boldsymbol{x}} \cap V_{\boldsymbol{x}}$ with the sum $Res(\boldsymbol{y})$ of residues defined above for $\boldsymbol{y} \in U_{\boldsymbol{x}}$. Clearly, $Res(\boldsymbol{y})$ is an analytic function of the two real variables $\boldsymbol{y}$, so by analytic continuation, $Res(\gamma(t)) = 0$ for all $0 \le t < t_0$.

The point $\phi_m(\gamma(t_0)) \in \mathbb{T}$ corresponds to where $\gamma$ crosses the curve $C(A)$ at $t_0$ with $\phi_m(\gamma(t_0))$ the direction of the unit normal. As mentioned above, $\gamma$ may have crossed a curve in $C(A)$ earlier, leading to the appearance of poles of the function (5.45) additional to $\{\phi_j(\boldsymbol{y})\}_{j=1}^k$ for $\boldsymbol{y} \in V_{\gamma(t)}$, but the chosen contours do not surround these.

Because $\gamma$ avoids all intersections, isolated points and cusps, for each $j, m = 1, \dots, k$ with $j \ne m$, we have $\phi_j(\gamma(t_0)) \ne \phi_m(\gamma(t_0))$ and $\phi_j(\gamma(t))$ is bounded away from $\mathbb{T}$ for all $0 \le t \le t_0$ (the unit normal is unique). Any other poles $\phi(\gamma(t_0))$ of (5.45) are not associated with function elements of $\mu$ at which equation (5.25) holds for $\boldsymbol{y} = \gamma(t_0)$. Otherwise, by Proposition 5.17, $\gamma(t_0)$ would lie on the intersection of curves belonging to $C(A)$ with $\phi(\gamma(t_0)) \in \mathbb{T}$, the unit normal to one of the curves.

However, it is impossible that $Res(\gamma(t)) = 0$ for all $0 \le t < t_0$, because the residues diverge at $\phi_m(\gamma(t_0)) \in \mathbb{T}$, but are uniformly bounded at $\phi_j(\gamma(t))$, $0 \le t \le t_0$ for $j \ne m$. This follows from an asymptotic analysis of formula (5.36) as $\boldsymbol{y} \to \gamma(t_0)$ along $\gamma$. The asymptotic analysis is facilitated by the fact that $\mu_j$ and $P_j$ are analytic in a neighbourhood of $\phi_m(\gamma(t_0))$ by

Rellich's theorem. Rather than repeat the calculation here, see [16, Equation (4.24)], and the references there that follow that equation. The original assumption that $Res(\boldsymbol{y}) = 0$ for all $\boldsymbol{y}$ in a neighbourhood $U_{\boldsymbol{x}}$ of $\boldsymbol{x}$ must be *false.*

If $\boldsymbol{x} \in \mathbf{R}(A)^c \cap \overline{\mathbf{R}(A)} = \partial\mathbf{R}(A)$, then by Proposition 5.21, $\boldsymbol{x}$ is an element of $C(A)$, so it only remains to treat the case $\boldsymbol{x} \in C(A)$. In this case, the asymptotic analysis mentioned above ensures that we can actually make $\lim_{\varepsilon\to 0^+}[G_{\boldsymbol{y}+\varepsilon e_0}(\boldsymbol{A}) - G_{\boldsymbol{y}-\varepsilon e_0}(\boldsymbol{A})]$ diverge as $\boldsymbol{y} \to x$ in some direction in $C(A)^c$, namely, from the direction into which the curvature vector points, proving that $\boldsymbol{x} \in \gamma(\boldsymbol{A})$.

We have established the inclusion $\mathbb{R}^2 \setminus \mathbf{R}(A) \subseteq \gamma(\boldsymbol{A})$. □

## 5.4 Simultaneously Triangularisable Matrices

After commuting matrices with real spectra, simultaneously upper triangularisable matrices are the next simplest to study. By factoring out an ideal, they can be considered as if they were commuting [87]. Many conditions guaranteeing simultaneous triangularisability for families of matrices are given in the monograph [90].

The preceding section was concerned with the joint spectrum of a pair of hermitian matrices. It follows from Corollary 5.26 that as soon as the two hermitian matrices $A_1, A_2$ do not commute, then the joint spectrum $\gamma(\boldsymbol{A})$ of $\boldsymbol{A} = (A_1, A_2)$ necessarily has nonempty interior. A key part of the proof of Theorem 5.24 is Rellich's Lemma: the spectral projections and eigenvalues of the matrix $\langle \boldsymbol{A}, s(z)\rangle$ are analytic in $z$ in a neighbourhood of $|z| = 1$.

In the case of a pair of simultaneously triangularisable matrices with real spectra, Rellich's Lemma is no longer applicable. A simple example of $2 \times 2$ upper triangular matrices illustrates the point.

*Example 5.31.* Let $A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$. Then

$$\xi_1 A_1 + \xi_2 A_2 = \begin{pmatrix} 0 & \xi_1 + \xi_2 \\ 0 & \xi_2 \end{pmatrix}$$
$$(\xi_1 A_1 + \xi_2 A_2)^2 = \begin{pmatrix} 0 & \xi_2(\xi_1 + \xi_2) \\ 0 & \xi_2^2 \end{pmatrix}$$
$$(\xi_1 A_1 + \xi_2 A_2)^3 = \begin{pmatrix} 0 & \xi_2^2(\xi_1 + \xi_2) \\ 0 & \xi_2^3 \end{pmatrix}$$
$$\vdots$$
$$(\xi_1 A_1 + \xi_2 A_2)^N = \begin{pmatrix} 0 & \xi_2^{n-1}(\xi_1 + \xi_2) \\ 0 & \xi_2^N \end{pmatrix}$$

It follows that

$$e^{i(\xi_1 A_1+\xi_2 A_2)} = \begin{pmatrix} 0 & (\xi_1+\xi_2)\frac{e^{i\xi_2}-1}{i\xi_2} \\ 0 & e^{i\xi_2} \end{pmatrix}$$

and so

$$\begin{aligned}(2\pi)^{-2}\left(e^{i(\xi_1 A_1+\xi_2 A_2)}\right)\hat{} &= \begin{pmatrix} 0 & (\partial_1+\partial_2)(\delta_0\otimes\chi_{[0,1]}) \\ 0 & \delta_0\otimes\delta_1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \delta_0'\otimes\chi_{[0,1]} \\ 0 & \delta_0\otimes\delta_1 \end{pmatrix} \\ &\qquad + \begin{pmatrix} 0 & \delta_0\otimes(\delta_0-\delta_1) \\ 0 & \delta_0\otimes\delta_1 \end{pmatrix} \end{aligned} \tag{5.51}$$

The presence of the distribution $\delta_0'\otimes\chi_{[0,1]}$ in (5.51) means that the joint spectrum $\gamma(\boldsymbol{A})$ is equal to the subset $\{0\}\times[0,1]$ of $\mathbb{R}^2$. Although the matrices $A_1$ and $A_2$ do not commute, the set $\gamma(\boldsymbol{A})$ has empty interior.

The proof of Theorem 5.24 depends on Rellich's Lemma and, in turn, the assumption that the matrices are hermitian. Let us see where Rellich's lemma fails for the upper triangular matrices considered in Example 5.31 above.

We write $\langle \boldsymbol{A},\zeta\rangle$ for the matrix $A_1\zeta_1+A_2\zeta_2$ for any $\zeta\in\mathbb{C}^2$. Let $s_1(z) = \frac{1}{2}(z+1/z)$ and $s_2(z) = \frac{1}{2i}(z-1/z)$ for all nonzero $z\in\mathbb{C}$. Then

$$\langle \boldsymbol{A}, s(z)\rangle = \begin{pmatrix} 0 & s_1(z)+s_2(z) \\ 0 & s_2(z) \end{pmatrix}$$

and we have the representation $\langle \boldsymbol{A}, s(z)\rangle = 0.P_1(z) + s_2(z)P_2(z)$ for the projections

$$P_1(z) = \begin{pmatrix} 1 & -\frac{s_1(z)}{s_2(z)} - 1 \\ 0 & 0 \end{pmatrix}, \qquad P_2(z) = \begin{pmatrix} 0 & \frac{s_1(z)}{s_2(z)} + 1 \\ 0 & 1 \end{pmatrix}$$

onto the eigenspaces corresponding to the eigenvalues 0 and $s_2(z)$, respectively, of $\langle \boldsymbol{A}, s(z)\rangle$. For any $y\in\mathbb{R}^2$, the matrix $\langle yI-A, s(z)\rangle$ has the representation

$$\langle yI-A, s(z)\rangle = \langle y, s(z)\rangle P_1(z) + \big(\langle y, s(z)\rangle - s_2(z)\big)P_2(z).$$

We are concerned with the residues of the matrix valued function

$$\big(\langle yI-A, s\rangle^2 - y_0^2 I\big)\big(\langle yI-A, s\rangle^2 + y_0^2 I\big)^{-2} \tag{5.52}$$

inside the unit circle for nonzero real numbers $y_0$. According to the spectral representation of $\langle \boldsymbol{A}, s\rangle$, the expression (5.52) is equal to

$$\frac{\langle y,s\rangle^2 - y_0^2}{(\langle y,s\rangle^2+y_0^2)^2}P_1 + \frac{(\langle y,s\rangle - s_2)^2 - y_0^2}{\big((\langle y,s\rangle - s_2)^2 + y_0\big)^2}P_2. \tag{5.53}$$

The poles under consideration are those solutions $z$ of

$$\langle y, s(z)\rangle = \pm i|y_0|, \qquad \langle y, s(z)\rangle - s_2(z) = \pm i|y_0|$$

that lie inside the unit circle. Although the projections $P_1$ and $P_2$ have singularities at $z = \pm 1$ at which points $s_2$ is zero, the function (5.52) is continuous at these points. The line segment joining $(0,0)$ with $(0,1)$ in $\gamma(\boldsymbol{A})$ is produced by the singularities of the spectral projections $P_1, P_2$ on the unit circle.

In this section, we show that this is a general phenomenon, that is, the joint spectrum $\gamma(\boldsymbol{A})$ of two simultaneously upper triangularisable matrices $\boldsymbol{A} = (A_1, A_2)$ with real spectra is contained in a set of line segments joining points belonging to $\sigma(A_1 + iA_2)$ considered as a finite set in $\mathbb{R}^2$. Of course, if the matrices $A_1, A_2$ actually commute, then $\gamma(\boldsymbol{A})$ is a finite set equal to $\sigma(A_1 + iA_2)$.

If $A_1$ and $A_2$ are simultaneously upper triangularisable, then there exists a pair $\boldsymbol{D} = (D_1, D_2)$ of diagonal matrices $D_1, D_2$ with real entries such that

$$p_{\langle \boldsymbol{A}, \xi\rangle} = p_{\langle \boldsymbol{D}, \xi\rangle}, \quad \text{for all } \xi \in \mathbb{R}^2,$$

and $\sigma(D_1 + iD_2) = \sigma(A_1 + iA_2)$, for we may simply put $D_j$ equal to the ordered diagonal entries of $TA_jT^{-1}$, $j = 1, 2$, for any matrix $T$ that puts both $A_1$ and $A_2$ into upper triangular form. An appeal to equations (5.7) and (5.19) establishes that

$$\begin{aligned} \mathcal{W}_{\boldsymbol{A}} = \sum_{k=0}^{N-1} \sum_{j=0}^{N-k-1} \sum_{m=0}^{j} (-1)^{k+m} \binom{j}{m} \frac{1}{(N-1-j+m)!} \\ \times \langle \boldsymbol{A}, \nabla\rangle^k \phi_{N-j-k-1}(\langle \boldsymbol{A}, \nabla\rangle)(\nabla \cdot id)^m \mu_{\boldsymbol{D}}. \end{aligned} \tag{5.54}$$

Here $\mu_{\boldsymbol{D}}$ is the image of the uniform probability measure on the unit sphere $\Sigma_N$ of $\mathbb{C}^N$ by the numerical range map $u \longmapsto ((D_1u, u), (D_2u, u))$, $u \in \Sigma_N$. As argued in the proof of Proposition 5.4, the measure $\mu_{\boldsymbol{D}}$ is just the image of the normalised Lebesgue measure $\sigma$ on the unit $(N-1)$ simplex $\Delta_{N-1} = \{w \in \mathbb{R}^N : w_j \geq 0, \sum_{j=1}^N w_j = 1\}$ in $\mathbb{R}^N$ by the linear map

$$T_{\boldsymbol{D}} w = \sum_{j=1}^{N} \boldsymbol{\lambda}_j w_j, \quad w \in \mathbb{R}^N,$$

where $\boldsymbol{\lambda}_j = (\lambda_{1,j}, \lambda_{2,j})$ for $j = 1, \ldots, N$ and $D_1 = \operatorname{diag}(\lambda_{1,1}, \ldots, \lambda_{1,N})$ and $D_2 = \operatorname{diag}(\lambda_{2,1}, \ldots, \lambda_{2,N})$.

We look at the measure $\mu_{\boldsymbol{D}}$ more closely.

### 5.4.1 Disintegration of Measures

Let $\lambda_N$ be Lebesgue measure on $\mathbb{R}^N$. Lebesgue measure on an $m$-dimensional hyperplane $H_m$ in $\mathbb{R}^N$ is denoted by $\lambda_{H_m}$. Suppose that $T : \mathbb{R}^N \to \mathbb{R}^2$

is a continuous linear map of rank two. Let $\Omega$ be a Borel subset of $\mathbb{R}^N$ with finite $\lambda_N$-measure. Then there exists a *disintegration* of the measure space $(\Omega, \mathcal{B}(\Omega), \lambda_N)$ with respect to the map $T$ and the measure space $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2), \lambda_2)$, that is,

$$\lambda_N \left(E \cap \left(T^{-1}(B)\right)\right) = \int_B \mu_x(E)\, \lambda_2(dx), \tag{5.55}$$

for all $E \in \mathcal{B}(\Omega)$ and $B \in \mathcal{B}(\mathbb{R}^2)$. Here $\mu_x$ is a finite Borel measure for each $x \in \mathbb{R}^2$ and $x \longmapsto \mu_x(E)$, $x \in \mathbb{R}^2$, is a Borel measurable $\lambda_2$-integrable function for each $E \in \mathcal{B}(\Omega)$.

By the Fubini's theorem, the measure valued function $x \longmapsto \mu_x$, $x \in \mathbb{R}^2$, is given by the formula

$$\mu_x(E) = c\lambda_{T^{-1}(x)} \left(T^{-1}(x) \cap E\right), \quad \text{for almost all } x \in \mathbb{R}^2,$$

for each $E \in \mathcal{B}(\Omega)$. Here $\dim(\ker T) = N - 2$ and $\lambda_{T^{-1}(x)}$ is $(N-2)$-dimensional Lebesgue measure on the $(N-2)$-dimensional hyperplane $T^{-1}(x)$. The constant $c$ is calculated as follows. The linear mapping $T_0 = T|(\ker T)^\perp$ is an isomorphism of $(\ker T)^\perp$ in $\mathbb{R}^N$ onto $\mathbb{R}^2$. Then $T_0^{-1}$ maps a rectangle $B$ of area $\lambda_2(B)$ in $\mathbb{R}^2$ into a parallelogram of area $\frac{\lambda_2(B)}{|\det(T_0)|}$ in $(\ker T)^\perp$.

Applying Fubini's theorem to the case that $E$ is a rectangle in an orthonormal basis for $(\ker T) \oplus (\ker T)^\perp$, the equality

$$\mu_x(E) = \frac{\lambda_{T^{-1}(x)} \left(T^{-1}(x) \cap E\right)}{|\det(T_0)|} \tag{5.56}$$

holds a.e., so it follows that $c = \frac{1}{|\det T_0|}$.

### 5.4.2 The Image of Simplicial Measure

The behaviour of $\mu_{\boldsymbol{D}}$, is illustrated by a simple calculation.

*Example 5.32.* Let $\mathbf{a} = (1, -1, -1, 1)$ and $\mathbf{b} = (1, 1, -1, -1)$. If $\Delta_3$ is the 3-simplex, then the substitution $\xi_1 = t_1,\ \xi_2 = t_2,\ \xi_3 = t_3,\ \xi_4 = 1 - t_1 - t_2 - t_3$, maps $\Delta_3$ into the set

$$E_3 = \{t \in \mathbb{R}^3 : t_1 \geq 0,\ t_2 \geq 0,\ t_3 \geq 0,\ t_1 + t_2 + t_3 \leq 1\ \}$$

and the linear map $\xi \longmapsto \sum_{j=1}^4 \xi_j(a_j, b_j)$ becomes the affine map $T_{(\mathbf{a},\mathbf{b})} : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$\begin{aligned} T_{(\mathbf{a},\mathbf{b})}(t_1, t_2, t_3) &= \left(\sum_{j=1}^{3}(a_j - a_4)t_j + a_4, \sum_{j=1}^{3}(b_j - b_4)t_j + b_4\right) \\ &= \begin{pmatrix} 0 & -2 & -2 \\ 2 & 2 & 0 \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{aligned}$$

Hence $T^{-1}_{(\mathbf{a},\mathbf{b})}(x) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} + \operatorname{span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$.

Then $\lambda_{T^{-1}_{(\mathbf{a},\mathbf{b})}(x)}(T^{-1}_{(\mathbf{a},\mathbf{b})}(x) \cap E_3)$ is the length of all $t \in T^{-1}_{(\mathbf{a},\mathbf{b})}(x)$ such that the four inequalities $t_1 \geq 0$, $t_2 \geq 0$, $t_3 \geq 0$ and $t_1 + t_2 + t_3 \leq 1$ are satisfied, that is, $\sqrt{3}$ times the length $|I(x)|$ of the interval $I(x)$ of all $\lambda \in \mathbb{R}$ satisfying

$$\begin{array}{rll} 0 \leq & \frac{1}{2}x_1 + \frac{1}{2}x_2 + \lambda & (t_1 \geq 0) \\ 0 \leq & -\frac{1}{2}x_1 + \frac{1}{2} - \lambda & (t_2 \geq 0) \\ 0 \leq & \lambda & (t_3 \geq 0) \\ \frac{1}{2}x_2 + \frac{1}{2} + \lambda \leq 1 & & (t_1 + t_2 + t_3 \leq 1) \end{array}$$

In the set $T_{(\mathbf{a},\mathbf{b})}E_3 = \{|x_1| \leq 1,\ |x_2| \leq 1\}$ there are 4 regions corresponding to where the inequalities

$$\begin{array}{ll} \lambda \leq \frac{1}{2} - \frac{1}{2}x_1 & (t_2 \geq 0) \\ \lambda \leq \frac{1}{2} - \frac{1}{2}x_2 & (t_1 + t_2 + t_3 \leq 1) \end{array} \tag{5.57}$$

and

$$\begin{array}{rl} -\frac{1}{2}x_1 - \frac{1}{2}x_2 \leq \lambda & (t_1 \geq 0) \\ 0 \leq \lambda & (t_3 \geq 0) \end{array} \tag{5.58}$$

are satisfied, namely

$$\begin{aligned} R_1 &= \{x \in T_{(\mathbf{a},\mathbf{b})}E_3 : x_2 \geq |x_1|\}, \\ R_2 &= \{x \in T_{(\mathbf{a},\mathbf{b})}E_3 : x_1 \geq |x_2|\}, \\ R_3 &= \{x \in T_{(\mathbf{a},\mathbf{b})}E_3 : x_2 \leq |x_1|\}, \\ R_4 &= \{x \in T_{(\mathbf{a},\mathbf{b})}E_3 : x_1 \leq |x_2|\}. \end{aligned}$$

for which

$$\begin{array}{lrl} R_1: & 0 \leq \lambda \leq \frac{1}{2} - \frac{1}{2}x_2, & |I(x)| = \frac{1}{2} - \frac{1}{2}x_2, \\ R_2: & 0 \leq \lambda \leq \frac{1}{2} - \frac{1}{2}x_1, & |I(x)| = \frac{1}{2} - \frac{1}{2}x_1, \\ R_3: & -\frac{1}{2}x_1 - \frac{1}{2}x_2 \leq \lambda \leq \frac{1}{2} - \frac{1}{2}x_1, & |I(x)| = \frac{1}{2} + \frac{1}{2}x_2, \\ R_4: & -\frac{1}{2}x_1 - \frac{1}{2}x_2 \leq \lambda \leq \frac{1}{2} - \frac{1}{2}x_2, & |I(x)| = \frac{1}{2} + \frac{1}{2}x_1. \end{array}$$

The example is generalised in the following statement.

**Proposition 5.33.** *Let $N \geq 3$ and for every $\mathbf{a}, \mathbf{b} \in \mathbb{R}^N$, let $T_{(\mathbf{a},\mathbf{b})} : \mathbb{R}^N \to \mathbb{R}^2$ be the linear map defined by*

$$T_{(\mathbf{a},\mathbf{b})}(\xi) = \left( \sum_{j=1}^{N} a_j \xi_j, \sum_{j=1}^{N} b_j \xi_j \right), \quad \xi \in \mathbb{R}^N.$$

*Suppose that $\mathbf{a}, \mathbf{b} \in \mathbb{R}^N$ and the linear map $T_{(\mathbf{a},\mathbf{b})}$ has rank two on the linear subspace of $\mathbb{R}^N$ spanned by $\Delta_{N-1}$.*

*Then there exists a continuous, homogeneous, piecewise polynomial function* $(x_1, x_2, t) \longmapsto f(x_1, x_2, t)$ *of degree* $N-3$ *such that*

$$t^{N-1}\lambda_{N-1}\left(\Delta_{N-1}\cap\left(T^{-1}_{(t\mathbf{a},t\mathbf{b})}(B)\right)\right)=\int_B f(x_1,x_2,t)\,dx$$

*for every* $B \in \mathcal{B}(T_{(t\mathbf{a},t\mathbf{b})}\Delta_{N-1})$. *The gradient* $x \longmapsto (\nabla_x f)(x,t)$ *of* $f$ *is possibly discontinuous on the union*

$$\bigcup_{1\le j<k\le n}\{t.\mathrm{co}\{(a_j,b_j),(a_k,b_k)\}\}$$

*of line segments.*

*Proof (Sketch).* We just write $T$ for the linear map $T_{(\mathbf{a},\mathbf{b})}$. Then $T^{-1}(x)$ is an $(N-2)$-dimensional hyperplane in $\mathbb{R}^N$ for each $x \in \mathbb{R}^2$. Suppose first that

**Condition 1**. $(a_j, b_j) \neq (a_l, b_l)$ for $j \neq l$ and no three distinct points $(a_{j_1}, b_{j_1})$, $(a_{j_2}, b_{j_2})$, $(a_{j_3}, b_{j_3})$ are colinear.

Let $\Delta^{(k)}_{N-1}$ be the complex of $k$-simplices belonging to $\Delta_{N-1}$. So, if $s$ is a $k$-simplex belonging to $\Delta^{(k)}_{N-1}$, then there exist distinct indices $j_1, \dots, j_{k+1} \in \{1, \dots, n\}$ such that

$$s=\left\{\xi\in\Delta_{N-1}:\sum_{l=1}^{k+1}\xi_{j_l}=1\right\}$$

Under Condition 1, the restriction of $T$ to any 2-simplex $s \in \Delta^{(2)}_{N-1}$ has rank 2, otherwise the image of $s$ by $T$ would be a line or point. Hence, the restriction of $T$ to $s$ is one-to-one. It follows that $T^{-1}(x) \cap s = (T|s)^{-1}\{x\}$ is either empty or a single point.

Because each 2-simplex belonging to $\Delta^{(2)}_{N-1}$ lies in a bounding plane of $\Delta_{N-1}$, every element of $T^{-1}(x) \cap \left(\cup \Delta^{(2)}_{N-1}\right)$ is an extreme point of the convex set $T^{-1}(x) \cap \Delta_{N-1}$ and

$$T^{-1}(x)\cap\Delta_{N-1}=\mathrm{co}\left(T^{-1}(x)\cap\left(\cup\Delta^{(2)}_{N-1}\right)\right)$$

For each $x \in T\Delta_{N-1}$, let $P(x)$ be the set defined by

$$P(x)=\cap\{Ts : s\in\Delta^{(2)}_{N-1},\ T^{-1}(x)\cap s\neq\emptyset\}.$$

Then $x \in P(x)$ and $P(x)$ is either a closed convex polygon in $\mathbb{R}^2$, a line segment or $P(x) = \{x\}$. The convex region $T\Delta_{N-1}$ is the closure of the union of finitely many open polygons $P(x)^\circ$ with $x \in T\Delta_{N-1}$.

Write $\lambda_k$ for $k$-dimensional Hausdorff (surface) measure. According to formulae (5.55) and (5.56), it is enough to show that

**Lemma 5.34.** *Let $a \in T\Delta_{N-1}$. Suppose that the interior $P(a)^\circ$ of $P(a)$ is nonempty. Then the function*

$$p_a : x \longmapsto \lambda_{N-3}(T^{-1}(x) \cap \Delta_{N-1})$$

*is a homogeneous polynomial of degree $N-3$ plus a constant for all $x \in P(a)^\circ$. If $b \in T\Delta_{N-1}$, $P(b)^\circ \neq \emptyset$, $P(a)^\circ \neq P(b)^\circ$ and $P(a) \cap P(b)$ contains a line segment, then the polynomials $p_a$ and $p_b$ have distinct homogeneous parts. Moreover, $p_a = p_b$ on $P(a) \cap P(b)$.*

*Proof.* By Condition 1, $T^{-1}(x)$ intersects each $s \in \Delta^{(2)}_{N-1}$ at most once. Suppose that $\{s_1, \dots, s_l\}$ is an enumeration of $\{s \in \Delta^{(2)}_{N-1} : T^{-1}(a) \cap s \neq \emptyset\}$. Then $\{s \in \Delta^{(2)}_{N-1} : T^{-1}(x) \cap s \neq \emptyset\} = \{s_1, \dots, s_l\}$ for all $x \in P(a)^\circ$.

Define the maps $f_j : P(a)^\circ \to \mathbb{R}^N$ by $T^{-1}(x) \cap s_j = \{f_j(x)\}$, for every $j = 1, \dots, l$ and $x \in P(a)^\circ$. Then $f_j$, $j = 1, \dots, l$ are affine functions and

$$T^{-1}(x) \cap \Delta_{N-1} = \mathrm{co}\left(T^{-1}(x) \cap \left(\cup \Delta^{(2)}_{N-1}\right)\right) = \mathrm{co}\,(f_1(x), \dots, f_l(x))$$

for all $x \in P(a)^\circ$. Furthermore, $T\Delta_{N-1}$ has nonempty interior, so $T^{-1}(x) \cap \Delta_{N-1}$ is a bounded convex subset of an $(N-3)$-dimensional hyperplane in $\mathbb{R}^N$ and

$$\lambda_{N-3}(T^{-1}(x) \cap \Delta_{N-1}) = \lambda_{N-3}(\mathrm{co}\,(\{f_j(x) : j = 1, \dots, l\}))$$

for all $x \in P(a)^\circ$. Each function $(x, t) \longmapsto t f_j(x/t)$, $j = 1, \dots, l$ is linear in $(x, t)$. It follows that $(x, t) \longmapsto t^{N-3}\lambda_{N-3}(T^{-1}(x/t) \cap \Delta_{N-1})$, is homogeneous polynomial of degree $N-3$, so that $p_a(x)$ is a homogeneous polynomial of degree $N-3$ plus a constant for all $x \in P(a)^\circ$. The polynomials $p_a$ and $p_b$ are distinct, because as $x$ moves across $P(a) \cap P(b)$ the point $T^{-1}(x) \cap s$ moves across $\partial s$ into another simplex $s'$. Consequently, the coefficients of the affine functions $f_j : P(a)^\circ \to \mathbb{R}^N$ and $\tilde{f}_{j'} : P(b)^\circ \to \mathbb{R}^N$ that define the extreme points of $T^{-1}(x) \cap \Delta_{N-1}$ change, but $p_a = p_b$ on $P(a) \cap P(b)$. □

If Condition 1 is not satisfied, then by perturbing the coefficients, we obtain the limit of homogeneous polynomials of degree $N-3$. □

### 5.4.3 Joint Spectrum of Triangularisable Matrices

According to Proposition 5.33, if $N \geq 3$, the probability measure $\mu_{\boldsymbol{D}}$ is absolutely continuous with respect to two dimensional Lebesgue measure and it has piecewise polynomial density of degree $N-3$ in regions bounded by the set $\cup\{\mathrm{co}(\{\lambda, \mu\}) : \lambda, \mu \in \sigma(A_1 + iA_2)\ \}$. It follows from formula (5.54) that the distribution $\mathcal{W}_{\boldsymbol{A}}$ is a differential operator of order $N-1$ in $(\nabla, \partial_t)$ acting on the distribution $\mu_{t\boldsymbol{D}}$ at $t = 1$. Regions where $\mu_{\boldsymbol{D}}$ has a polynomial density with respect to Lebesgue measure on $\mathbb{R}^2$ lie outside the support of $\mathcal{W}_{\boldsymbol{A}}$, so that

$$\gamma(\boldsymbol{A}) \subset \bigcup \{ \mathrm{co}(\{\lambda, \mu\}) : \lambda, \mu \in \sigma(A_1 + iA_2) \}.$$

More precisely, we have

**Theorem 5.35.** *Let $A_1, A_2$ be simultaneously triangularisable $(N \times N)$ matrices each with real spectrum and set $\boldsymbol{A} = (A_1, A_2)$. Then $\sigma(\langle \boldsymbol{A}, \xi \rangle) \subset \mathbb{R}$ for all $\xi \in \mathbb{R}^2$ and on identifying $\mathbb{C}$ with $\mathbb{R}^2$, the inclusion*

$$\gamma(\boldsymbol{A}) \subset \bigcup \{ \mathrm{co}(\{\lambda, \mu\}) : \lambda, \mu \in \sigma(A_1 + iA_2) \}$$

*holds.*

*Proof.* Let $\mathcal{D}(A_1) = \mathrm{diag}(\mathbf{a})$, $\mathcal{D}(A_2) = \mathrm{diag}(\mathbf{b})$ be the diagonal matrices corresponding to $A_1$ and $A_2$. If $N \geq 3$ and $\mathbf{a}, \mathbf{b}$ satisfy the conditions of Proposition 5.33, then the argument above works. If $N = 1$, there is nothing to prove. If $N = 2$, then $T_{(\mathbf{a},\mathbf{b})}$ can only have rank one on $\mathrm{span}(\Delta_1)$. If $T_{(\mathbf{a},\mathbf{b})}$ has rank one on $\mathrm{span}(\Delta_{N-1})$, then points of $\sigma(A_1 + iA_2)$ are colinear and from formula (5.54), $\gamma(\boldsymbol{A}) \subset \mathrm{co}(\sigma(A_1 + iA_2))$. □

The inclusion of the line segments $\mathrm{co}(\{\lambda, \mu\})$ for $\lambda, \mu \in \sigma(A_1 + iA_2)$ depends on whether or not the spectral projections of $\langle \boldsymbol{A}, s(z) \rangle$ have singularities on $|z| = 1$.

## 5.5 Systems of Matrices

Let $n$ be an even integer and $\boldsymbol{A} = (A_1, \dots, A_n)$ be an $n$-tuple of $N \times N$ matrices satisfying the spectral condition (5.9). The purpose of this section is to outline a general method for establishing that a point $\boldsymbol{x} \in \mathbb{R}^n$ belongs to the joint spectrum $\gamma(\boldsymbol{A})$. In the case of two hermitian matrices, we have already seen that the geometric condition provided by equation (5.28) ensures that $\boldsymbol{x} \in \mathbb{R}^2$ lies outside joint spectrum $\gamma(\boldsymbol{A})$.

Roughly speaking, the approach of Atiyah, Bott and Gärding [11] is interpreted in the present matrix setting, and we see that the detailed explanation given in Section 5.3 for the fundamental case $n = 2$ may be generalised by using the appropriate tools from algebraic topology. The case for $n$ odd requires additional arguments and is omitted from the present discussion. The presentation of this section is based on the summary of the Herglotz-Petrovsky-Leray formulas [11] given by Y. Berest in [17]. Another brief account is given in [41, Section 12.6].

A general element $x = (x_0, x_1, \dots, x_n)$ of $\mathbb{R}^{n+1}$ will be written as $x = \boldsymbol{x} + x_0 e_0$ with $\boldsymbol{x} = \sum_{j=1}^{n} x_j e_j$. Because $n$ is assumed to be an even integer,

$$\int_{S^{n-1}} s \left( \langle \boldsymbol{x} I - \boldsymbol{A}, s \rangle - x_0 s \right)^{-n} ds = 0$$

and the plane wave decomposition (4.16) for the Cauchy kernel is

$$\begin{aligned} G_x(\boldsymbol{A}) &= \mathcal{W}_{\boldsymbol{A}}(G_x) \\ &= \frac{(n-1)!}{2}\left(\frac{i}{2\pi}\right)^n \operatorname{sgn}(x_0) \\ &\qquad \times \int_{S^{n-1}} \left(\langle \boldsymbol{x}I - \boldsymbol{A}, s\rangle - x_0 s\right)^{-n} ds, \end{aligned} \tag{5.59}$$

for $x \in \mathbb{R}^{n+1}$ with $x_0 \neq 0$. For ease of notation, an element $xI$ of $\mathcal{L}_{(n)}(\mathbb{C}^N)$ for $x \in \mathbb{C}_{(n)}$ will often be written as $x$. Because $x \longmapsto G_x(\boldsymbol{A})$ is actually the monogenic representation of the Weyl calculus $\mathcal{W}_{\boldsymbol{A}}$ off $\mathbb{R}^n$ [19, Definition 27.6], we have

$$\mathcal{W}_{\boldsymbol{A}} = \lim_{\epsilon \to 0+} G_{\boldsymbol{x}+\epsilon e_0}(\boldsymbol{A}) - G_{\boldsymbol{x}-\epsilon e_0}(\boldsymbol{A}) \tag{5.60}$$

in the sense of distributions. Consequently, if the limit on the right hand side of equation (5.60) exists uniformly for all $\boldsymbol{x}$ in a neighbourhood of a point $\boldsymbol{a} \in \mathbb{R}^n$ and is zero there, then $\boldsymbol{a}$ lies outside the support of the matrix valued distribution $\mathcal{W}(\boldsymbol{A})$, that is, $\boldsymbol{a} \in \gamma(\boldsymbol{A})^c$. We shall seek conditions which guarantee that the limit

$$\lim_{\epsilon \to 0+} \int_{S^{n-1}} \left(\langle \boldsymbol{x}I - \boldsymbol{A}, s\rangle - \epsilon s\right)^{-n} + \left(\langle \boldsymbol{x}I - \boldsymbol{A}, s\rangle + \epsilon s\right)^{-n} ds \tag{5.61}$$

exists uniformly and is zero for all elements $\boldsymbol{x}$ of an open subset of $\mathbb{R}^n$.

For the case $n = 2$ considered in Section 5.3, the integral (5.61) was calculated in an elementary manner by converting it into a contour integral and actually computing the residues associated with the spectral representation of the hermitian matrix $\langle \boldsymbol{A}, s\rangle$.

As we move to higher dimensions in this section, we see that it is not really necessary to perform the explicit calculation of residues. Moreover, as seen in Example 5.31, if $\boldsymbol{A} = (A_1, A_2)$ is not a pair of *hermitian* matrices, then the eigenprojections associated with the matrix $\langle \boldsymbol{A}, s\rangle$ may have a singularity at $s_0 \in S^1$ if $z_0 = s_0$ is an exceptional point of the holomorphic matrix valued function $z \longmapsto \langle \boldsymbol{A}, s(z)\rangle$, $z \in \mathbb{C} \setminus \{0\}$, see [65, Theorem II.1.8]. This accounts for the appearance of line segments in $\gamma(\boldsymbol{A})$ in Example 5.31, excluded in the case of hermitian matrices by Rellich's Lemma 5.13. The exceptional point $s_0$ corresponds to a 'double tangent' of the Kippenhahn curve $C(\boldsymbol{A})$, so we exclude such points by examining the behaviour of the characteristic polynomial of the matrix $\langle \boldsymbol{A}, \xi\rangle$ for all $\xi \in \mathbb{R}^n$.

Let

$$\begin{aligned} P^{\boldsymbol{A}}(\zeta_0, \zeta_1, \cdots, \zeta_n) &= \det(\zeta_0 I + \zeta_1 A_1 + \cdots + \zeta_n A_n) \\ &= p_{\langle \boldsymbol{A}, \boldsymbol{\zeta}\rangle}(-\zeta_0), \end{aligned}$$

for all $\zeta \in \mathbb{C}^{n+1}$ with the representation $\zeta = \zeta_0 e_0 + \boldsymbol{\zeta}$, $\boldsymbol{\zeta} \in \mathbb{C}^n$.

Let $\mathbb{RP}^n$ be real $n$-dimensional projective space. Then

$$\Xi(\boldsymbol{A}) = \left\{(\mu : \xi_1 : \cdots : \xi_n) \in \mathbb{RP}^n \;\middle|\; P^{\boldsymbol{A}}(\mu, \xi_1, \cdots, \xi_n) = 0\right\} \tag{5.62}$$

is an algebraic hypersurface. Identifying elements of $\mathbb{RP}^n$ with lines in $\mathbb{R}^{n+1}$, let $\Gamma(\boldsymbol{A})$ denote the open connected component of $\mathbb{R}^{n+1} \setminus \Xi(\boldsymbol{A})$ containing $e_0$. The convex cone $\Gamma(\boldsymbol{A})$ in $\mathbb{R}^{n+1}$ is called the *hyperbolicity cone* of $\boldsymbol{A}$. The trace of the dual cone of $\Gamma(\boldsymbol{A})$ on the set $x_0 = 1$, referred to as the *propagation set* of $\boldsymbol{A}$, is the given by

$$K(\boldsymbol{A}) = \left\{\boldsymbol{x} \in \mathbb{R}^n \;\middle|\; \langle e_0 + \boldsymbol{x}, \xi\rangle \geq 0,\ \forall \xi \in \Gamma(\boldsymbol{A}) \right\}. \tag{5.63}$$

In the case that $n = 2$ and $\boldsymbol{A} = (A_1, A_2)$ is a pair of hermitian matrices, then the result of Kippenhahn mentioned in Section 5.3.1 ensures that the set $K(\boldsymbol{A})$ can be identified with the numerical range of the matrix $A = A_1 + iA_2$.

A *localisation* $P^{\boldsymbol{A}}_\xi$ of $P^{\boldsymbol{A}}$ at $\xi \in \mathbb{R}^{n+1}$, is the lowest nonzero term of the polynomial

$$t \mapsto P^{\boldsymbol{A}}(\xi + t\zeta) = t^{\mu_\xi} P^{\boldsymbol{A}}_\xi(\zeta) + \mathcal{O}(t^{\mu_\xi + 1}) \ , \quad \mu_\xi = \deg\, P^{\boldsymbol{A}}_\xi \ .$$

At this stage, we need to take into account that the homogeneous polynomial $P^{\boldsymbol{A}}$ may not depend on all variables in $\mathbb{C}^{n+1}$. For example, one of the matrices $A_j$ could be the zero matrix.

The *real lineality* $\Lambda(\boldsymbol{A})$ of $\boldsymbol{A}$, is the maximal linear subspace of $\mathbb{R}^{n+1}$ such that the restriction of $P^{\boldsymbol{A}}$ the quotient $\mathbb{R}^{n+1}/\Lambda(\boldsymbol{A})$ is again a polynomial. Then $\Lambda(\boldsymbol{A})$ coincides with the edge of the hyperbolicity cone $\Gamma(\boldsymbol{A})$, so that $\Gamma + \Lambda = \Gamma$, and $K(\boldsymbol{A})$ spans the intersection of its orthogonal complement $\Lambda^\perp(\boldsymbol{A})$ in $\mathbb{R}^{n+1}$ with the plane $x_0 = 1$.

The system $\boldsymbol{A}$ is called *complete* if $\boldsymbol{A}$ has a trivial lineality. In this case, $P^{\boldsymbol{A}}_\xi(\zeta) \equiv P^{\boldsymbol{A}}(\zeta)$ implies $\xi = 0$, the cone $\Gamma(\boldsymbol{A})$ is proper (peaked) in the sense that $\overline{\Gamma(\boldsymbol{A})}$ does not contain any straight lines, and then $K(\boldsymbol{A})$ has a non-empty interior $K^\circ(\boldsymbol{A})$ in $\mathbb{R}^n$.

Let $\boldsymbol{A}$ and $\xi \in \mathbb{R}^{n+1}$ be fixed. Consider the localisation $P^{\boldsymbol{A}}_\xi$ of $P^{\boldsymbol{A}}$ at $\xi$. The *local hyperbolicity cone* and the *local propagation set* of $P^{\boldsymbol{A}}$ at $\xi$ are defined by by setting, respectively,

$$\Gamma_\xi(\boldsymbol{A}) := \Gamma(P^{\boldsymbol{A}}_\xi), \qquad K_\xi(\boldsymbol{A}) := K(P^{\boldsymbol{A}}_\xi) \ .$$

Here the polynomial $P^{\boldsymbol{A}}$ has been replaced by $P^{\boldsymbol{A}}_\xi$ in the definitions (5.62) and (5.63). A similar notation is used for the real lineality $\Lambda(P^{\boldsymbol{A}}_\xi)$ of the polynomial $P^{\boldsymbol{A}}_\xi$.

Clearly, $\Gamma_\xi(\boldsymbol{A}) \supseteq \Gamma(\boldsymbol{A})$ and, hence, $K_\xi(\boldsymbol{A}) \subseteq K(\boldsymbol{A})$ for all $\xi \in \mathbb{R}^{n+1}$. More precisely, the mapping $(\xi, \boldsymbol{A}) \mapsto \Gamma_\xi(\boldsymbol{A})$ (and $(\xi, \boldsymbol{A}) \mapsto K_\xi(\boldsymbol{A})$) is *inner* (resp., *outer*) *continuous* in the sense that $\Gamma_\xi(\boldsymbol{A}) \cap \Gamma_{\tilde{\xi}}(\tilde{\boldsymbol{A}})$ (resp., $K_\xi(\boldsymbol{A}) \cup K_{\tilde{\xi}}(\tilde{\boldsymbol{A}})$) is close to $\varGamma_\xi(\boldsymbol{A})$ (resp., $K_\xi(\boldsymbol{A})$) when $(\tilde{\xi}, \tilde{\boldsymbol{A}})$ is close to $(\xi, \boldsymbol{A})$ with $\xi, \tilde{\xi} \in \mathbb{R}^{n+1}$ and $\boldsymbol{A}, \tilde{\boldsymbol{A}}$ satisfying condition (5.9).

The *wave front surface* $W(\boldsymbol{A})$ of the system $\boldsymbol{A}$ of matrices is generated by the union of local propagation cones:

$$W(\boldsymbol{A}) := \bigcup_{0 \neq \xi \in \mathbb{R}^{n+1}} K_\xi(\boldsymbol{A}) \ . \tag{5.64}$$

The continuity result mentioned above facilitates the generalisation of the results of Section 5.3 to the higher dimensional setting.

When $\boldsymbol{A}$ is complete, that is, $\varLambda(\boldsymbol{A}) = \{0\}$, the wave front surface $W(\boldsymbol{A})$ is a closed semi-algebraic part of the global propagation set $K(\boldsymbol{A})$ containing its boundary. More precisely,

$$\partial K \subseteq W \subseteq K \cap \varXi' \ ,$$

where $\varXi' := \bigcup \varLambda^\perp(P_\xi^{\boldsymbol{A}}) \cap (\{1\} \times \mathbb{R}^n)$, $\xi \in \mathbb{R}^{n+1} \setminus \{0\}$, is a real dual of the hyperbolic hypersurface $\varXi$. If $\varXi$ is regular outside the origin, every nonzero $\xi \in \varXi'$ admits only a one-dimensional space $\varLambda^\perp(P_\xi^{\boldsymbol{A}})$ of real normals, $n_\xi(\boldsymbol{A}) := \dim \varLambda^\perp(P_\xi^{\boldsymbol{A}}) = 1$, the intersection of it with the hyperplane $\{x_0 = 1\}$ being $K_\xi(\boldsymbol{A})$. In that case, we have the equality $W = K \cap \varXi'$. Otherwise, when $\varXi$ has singular points $\xi \neq 0$ with normals of a higher dimension, $n_\xi(\boldsymbol{A}) > 1$, the surface $W$ may be strictly smaller than $K \cap \varXi'$. However, $\operatorname{codim} W(\boldsymbol{A}) = 1$ in any case, since $\xi \in \varLambda(\boldsymbol{A}_\xi)$ and, hence, each $K_\xi(\boldsymbol{A})$ in (5.64) has the property that the set $e_0 + K_\xi(\boldsymbol{A})$ lies in a proper affine hyperplane normal to $\xi \neq 0$. Note also that, unlike $\varXi'$, the set $W(\boldsymbol{A})$ depends on $\boldsymbol{A}$ (outer) continuously.

For incomplete polynomials $P^{\boldsymbol{A}}$ with $\varLambda(\boldsymbol{A}) \neq \{0\}$, the wave front surface $W(\boldsymbol{A})$ equals $K(\boldsymbol{A})$, as $P_\xi^{\boldsymbol{A}}(\zeta) \equiv P^{\boldsymbol{A}}(\zeta)$ for each $\xi \in \varLambda(\boldsymbol{A})$.

The following matrices were considered in Example 5.31.

*Example 5.36.* Let $A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$. Then $\gamma(\boldsymbol{A}) = \{0\} \times [0,1]$. For $\xi \in \mathbb{R}^{n+1}$ and $\zeta \in \mathbb{C}^{n+1}$, we have

$$\begin{aligned} P^{\boldsymbol{A}}(\xi + t\zeta) &= \det\left((\xi_0 + t\zeta_0)I + A_1(\xi_1 + t\zeta_1) + A_2(\xi_2 + t\zeta_2)\right) \\ &= (\xi_0 + t\zeta_0)(\xi_0 + t\zeta_0 + \xi_2 + t\zeta_2) \\ &= \xi_0(\xi_0 + \xi_2) + t(2\zeta_0\xi_0 + \zeta_0\xi_2 + \xi_0\zeta_2) + t^2\zeta_0(\zeta_0 + \zeta_2). \end{aligned}$$

The polynomial $P^{\boldsymbol{A}}(\xi)$ does not depend on the variable $\xi_1$ for $\xi = (\xi_0, \xi_1, \xi_2) \in \mathbb{R}^3$, so it is not complete. The following cases obtain for the localisation $P_\xi^{\boldsymbol{A}}$ of $P^{\boldsymbol{A}}$ at $\xi \in \mathbb{R}^3$, $\xi \neq 0$.

a) $P_\xi^{\boldsymbol{A}}$ has degree zero:

$$\begin{aligned} &\xi_0(\xi_0 + \xi_2) \neq 0, \quad P_\xi^{\boldsymbol{A}}(\zeta) = \xi_0(\xi_0 + \xi_2) \\ &\varXi(P_\xi^{\boldsymbol{A}}) = \emptyset, \qquad \varGamma(P_\xi^{\boldsymbol{A}}) = \mathbb{R}^3, \qquad K(P_\xi^{\boldsymbol{A}}) = \emptyset. \end{aligned}$$

b) $P_\xi^{\boldsymbol{A}}$ has degree one:

$$\begin{aligned}&\xi_0 = 0,\ \xi_2 \neq 0, \quad P_\xi^{\boldsymbol{A}}(\zeta) = \xi_2\zeta_0\\ &\varXi(P_\xi^{\boldsymbol{A}}) = \{\zeta_0 = 0\}, \quad \varGamma(P_\xi^{\boldsymbol{A}}) = \{\zeta_0 > 0\}, \quad K(P_\xi^{\boldsymbol{A}}) = \{(0,0)\}.\end{aligned}$$

$$\begin{aligned}&\xi_0 = -\xi_2 \neq 0, \quad P_\xi^{\boldsymbol{A}}(\zeta) = \xi_2(\zeta_0 + \zeta_2)\\ &\varXi(P_\xi^{\boldsymbol{A}}) = \{\zeta_0 + \zeta_2 = 0\}, \quad \varGamma(P_\xi^{\boldsymbol{A}}) = \{\zeta_0 + \zeta_2 > 0\}, \quad K(P_\xi^{\boldsymbol{A}}) = \{(0,1)\}.\end{aligned}$$

c) $P_\xi^{\boldsymbol{A}}$ has degree two:

$$\begin{aligned}\xi_0 = \xi_2 = 0,\ \xi_1 \neq 0, \quad & P_\xi^{\boldsymbol{A}}(\zeta) = \zeta_0(\zeta_0 + \zeta_2)\xi_1 = P^{\boldsymbol{A}}(\zeta)\xi_1\\ \varXi(P_\xi^{\boldsymbol{A}}) &= \{\zeta_0 = 0\} \cup \{\zeta_0 + \zeta_2 = 0\},\\ \varGamma(P_\xi^{\boldsymbol{A}}) &= \{\zeta_0 + \zeta_2 > 0,\ \zeta_0 > 0\},\\ K(P_\xi^{\boldsymbol{A}}) &= \{(0,\mu) |\ 0 \leq \mu \leq 1\ \}.\end{aligned}$$

Thus, we see that in the last case, $K(P_\xi^{\boldsymbol{A}}) = \gamma(\boldsymbol{A}) = W(\boldsymbol{A})$. We would get the same result for diagonal parts of the matrices $A_1, A_2$. The polynomial $P^{\boldsymbol{A}}$ cannot detect whether or not $A_1$ and $A_2$ are hermitian.

To find an example in which $K(\boldsymbol{A}) \neq \gamma(\boldsymbol{A}) = W(\boldsymbol{A})$, we look at two $(3\times 3)$ upper-triangular matrices.

*Example 5.37.* Let $A_1 = \begin{pmatrix} 1 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}$ and $A_2 = \begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 1\end{pmatrix}$ and $\boldsymbol{A} = (A_1, A_2)$. The simplex in $\mathbb{R}^2$ bounded by the coordinate axes and $x_1 + x_2 = 1$ is denoted by $\varDelta$. A calculation, explicitly given in [43, Example 4.3], shows that

$$(2\pi)^{-2}\left(e^{i\langle\xi,A\rangle}\right)^{\wedge} = \begin{pmatrix} \delta_{(1,0)} & (\partial_1 + \partial_2)(\chi_{[0,1]} \otimes \delta_0) & (\partial_1 + \partial_2)^2\chi_\varDelta\\ 0 & \delta_{(0,0)} & (\partial_1 + \partial_2)(\delta_0 \otimes \chi_{[0,1]})\\ 0 & 0 & \delta_{(0,1)}\end{pmatrix}.$$

The joint spectrum $\gamma(\boldsymbol{A})$ is therefore the *boundary* of the simplex $\varDelta$. In accordance with [11, Example 10.6] and Theorem 5.35 above, the interior points of $\varDelta$ lie in the complement of $\gamma(\boldsymbol{A})$.

For $\xi = (\xi_0, \xi_1, \xi_2) \in \mathbb{R}^3$ and $\zeta = (\zeta_0, \zeta_1, \zeta_2) \in \mathbb{C}^3$, we have

$$\begin{aligned}P^{\boldsymbol{A}}(\xi + t\zeta) &= \det\left((\xi_0 + t\zeta_0)I + A_1(\xi_1 + t\zeta_1) + A_2(\xi_2 + t\zeta_2)\right)\\ &= [\xi_0 + t\zeta_0][(\xi_0 + \xi_1) + t(\zeta_0 + \zeta_1)][(\xi_0 + \xi_2) + t(\zeta_0 + \zeta_2)].\end{aligned}$$

Using the notation $\xi'_j = \xi_0 + \xi_j$ and $\zeta'_j = \zeta_0 + \zeta_j$ for $j = 1, 2$, we obtain

$$P^{\boldsymbol{A}}(\xi + t\zeta) = \xi_0\xi'_1\xi'_2 + (\zeta_0\xi'_1\xi'_2 + \zeta'_1\xi_0\xi'_2 + \zeta'_2\xi_0\xi'_1)t + (\zeta_0\zeta'_1\xi'_2 + \zeta_0\zeta'_2\xi'_1 + \zeta'_2\zeta'_1\xi_0)t^2 + \zeta_0\zeta'_1\zeta'_2 t^3.$$

The following cases hold for the localisation $P^{\boldsymbol{A}}_\xi$ of $P^{\boldsymbol{A}}$ at $\xi \in \mathbb{R}^3$, $\xi \neq 0$.

a) $P^{\boldsymbol{A}}_\xi$ has degree zero:

$$\xi_0\xi'_1\xi'_2 \neq 0, \quad P^{\boldsymbol{A}}_\xi(\zeta) = \xi_0\xi'_1\xi'_2$$
$$\Xi(P^{\boldsymbol{A}}_\xi) = \emptyset, \qquad \Gamma(P^{\boldsymbol{A}}_\xi) = \mathbb{R}^3, \qquad K(P^{\boldsymbol{A}}_\xi) = \emptyset.$$

b) $P^{\boldsymbol{A}}_\xi$ has degree one:

$$\xi_0 = 0,\ \xi_1 \neq 0,\ \xi_2 \neq 0, \quad P^{\boldsymbol{A}}_\xi(\zeta) = \xi_1\xi_2\zeta_0$$
$$\Xi(P^{\boldsymbol{A}}_\xi) = \{\zeta_0 = 0\}, \quad \Gamma(P^{\boldsymbol{A}}_\xi) = \{\zeta_0 > 0\}, \quad K(P^{\boldsymbol{A}}_\xi) = \{(0,0)\}.$$

$$\xi_0 \neq 0,\ \xi'_1 = 0,\ \xi'_2 \neq 0, \quad P^{\boldsymbol{A}}_\xi(\zeta) = \xi_0\xi'_2(\zeta_0 + \zeta_1)$$
$$\Xi(P^{\boldsymbol{A}}_\xi) = \{\zeta_0 + \zeta_1 = 0\}, \quad \Gamma(P^{\boldsymbol{A}}_\xi) = \{\zeta_0 + \zeta_1 > 0\}, \quad K(P^{\boldsymbol{A}}_\xi) = \{(1,0)\}.$$

$$\xi_0 \neq 0,\ \xi'_1 \neq 0,\ \xi'_2 = 0, \quad P^{\boldsymbol{A}}_\xi(\zeta) = \xi_0\xi'_1(\zeta_0 + \zeta_2)$$
$$\Xi(P^{\boldsymbol{A}}_\xi) = \{\zeta_0 + \zeta_2 = 0\}, \quad \Gamma(P^{\boldsymbol{A}}_\xi) = \{\zeta_0 + \zeta_2 > 0\}, \quad K(P^{\boldsymbol{A}}_\xi) = \{(0,1)\}.$$

c) $P^{\boldsymbol{A}}_\xi$ has degree two:

$$\xi_0 = \xi_1 = 0,\ \xi_2 \neq 0, \quad P^{\boldsymbol{A}}_\xi(\zeta) = \zeta_0(\zeta_0 + \zeta_1)\xi_2$$
$$\Xi(P^{\boldsymbol{A}}_\xi) = \{\zeta_0 = 0\} \cup \{\zeta_0 + \zeta_1 = 0\},$$
$$\Gamma(P^{\boldsymbol{A}}_\xi) = \{\zeta_0 + \zeta_1 > 0,\ \zeta_0 > 0\},$$
$$K(P^{\boldsymbol{A}}_\xi) = \{(\mu, 0) |\ 0 \leq \mu \leq 1\ \}.$$

$$\xi_0 = \xi_2 = 0,\ \xi_1 \neq 0, \quad P^{\boldsymbol{A}}_\xi(\zeta) = \zeta_0(\zeta_0 + \zeta_2)\xi_1$$
$$\Xi(P^{\boldsymbol{A}}_\xi) = \{\zeta_0 = 0\} \cup \{\zeta_0 + \zeta_2 = 0\},$$
$$\Gamma(P^{\boldsymbol{A}}_\xi) = \{\zeta_0 + \zeta_2 > 0,\ \zeta_0 > 0\},$$
$$K(P^{\boldsymbol{A}}_\xi) = \{(0, \mu) |\ 0 \leq \mu \leq 1\ \}.$$

$$\begin{aligned}
&\xi_0 \neq 0,\ \xi_1 = \xi_2 = -\xi_0, \quad P_\xi^{\boldsymbol{A}}(\zeta) = \xi_0(\zeta_0+\zeta_1)(\zeta_0+\zeta_2)\\
&\Xi(P_\xi^{\boldsymbol{A}}) = \{\zeta_0+\zeta_1=0\} \cup \{\zeta_0+\zeta_2=0\},\\
&\Gamma(P_\xi^{\boldsymbol{A}}) = \{\zeta_0+\zeta_1>0,\ \zeta_0+\zeta_2>0\},\\
&K(P_\xi^{\boldsymbol{A}}) = \{(\mu, 1-\mu) |\ 0 \le \mu \le 1\ \}.
\end{aligned}$$

Thus, we see that in case c), $\gamma(\boldsymbol{A}) = W(\boldsymbol{A})$. Again we would get the same wave front set $W(\boldsymbol{D})$ for the diagonal parts $\boldsymbol{D}$ of the matrices $A_1, A_2$, but $\gamma(\boldsymbol{D})$ would be the finite set $\{(0,0),\ (0,1),\ (1,0)\} \equiv \sigma(A_1 + iA_2)$ of its extreme points. A similar calculation of $W(\boldsymbol{A})$ can be made for any two upper triangular matrices $A_1$, $A_2$ with real eigenvalues such that $\sigma(A_1+iA_2)$ consists of three distinct points.

First a representation similar to (5.36) is obtained for the limit (5.61).

Let $\overset{\circ}{\mathbb{R}}{}^{n+1} := \mathbb{R}^n \setminus \{0\}$. Given a point $\boldsymbol{x} \in \mathbb{R}^n$, the symbol $X$ stands for a real hyperplane in $\mathbb{R}^{n+1}$ dual to the point $e_0 + \boldsymbol{x} \in \mathbb{R}^{n+1}$ in the remainder of this section. The complex counterpart $X_{\mathbb{C}}$ of $X$ is given by $X_{\mathbb{C}} := \{\, \zeta \in \mathbb{C}^{n+1} \mid \langle e_0 + \boldsymbol{x}, \zeta\rangle = 0\,\}$.

For each $\boldsymbol{x} \in \mathbb{R}^n$, we define a family $\mathfrak{V} = \mathfrak{V}(\boldsymbol{x}, \boldsymbol{A})$ of $C^\infty$-smooth real vector fields $v : \overset{\circ}{\mathbb{R}}{}^{n+1} \to \mathbb{R}^{n+1}$ such that for each $\xi \in \overset{\circ}{\mathbb{R}}{}^{n+1}$:

(i) $v(\xi) \in \Gamma_\xi(\boldsymbol{A}) \cap X$;
(ii) $v(\kappa\xi) = |\kappa|\, v(\xi)$ , $\kappa \in \overset{\circ}{\mathbb{R}}$ ;
(iii) the matrix $(\zeta_0 I + \langle \boldsymbol{A}, \zeta\rangle)$ is invertible in $\mathcal{L}(\mathbb{C}^N)$, when $\zeta = \xi \pm iv(\xi)$.

**Lemma 5.38.** *If $\boldsymbol{x} \notin W(\boldsymbol{A})$, then the family $\mathfrak{V}(\boldsymbol{x}, \boldsymbol{A})$ is not empty, and any two elements of it are homotopic, i.e. may be deformed one into another through a $\mathcal{C}^\infty$-mapping $[0,1] \to \mathfrak{V}$ within the family $\mathfrak{V}$.*

*Proof.* In view of definition (5.64), the point $\boldsymbol{x} \in \mathbb{R}^n$ belongs to $W(\boldsymbol{A})$ if and only if $\langle e_0 + \boldsymbol{x}, \Gamma_\xi\rangle \ge 0$ for at least one point $\xi \in \overset{\circ}{\mathbb{R}}{}^{n+1}$. Hence, $\boldsymbol{x} \notin W(\boldsymbol{A})$ implies the existence of vectors $\eta_\mp$ in $\Gamma_\xi(\boldsymbol{A})$ such that $\langle \eta_-, e_0 + \boldsymbol{x}\rangle < 0$ while $\langle \eta_+, e_0 + \boldsymbol{x}\rangle > 0$, and then, by convexity, $\Gamma_\xi(\boldsymbol{A}) \cap X$ is not empty for any $\xi \in \overset{\circ}{\mathbb{R}}{}^{n+1}$. Further, the homogeneity property $(ii)$ is consistent with $(i)$, since the local cone $\Gamma_\xi(\boldsymbol{A}) = \Gamma(P_\xi)$ depends on the double ray $\overset{\circ}{\mathbb{R}}\,\xi$ only. The condition $(iii)$ is achieved by taking the elements $v(\xi)$, satisfying $(i)$ and $(ii)$ with $|v(\xi)|$ small enough when $|\xi| = 1$. The homotopy of $\mathfrak{V}$ follows essentially from the inner continuity of the mapping $(\xi, \boldsymbol{A}) \mapsto \Gamma_\xi(\boldsymbol{A})$ (see [11], Lemma 6.7). □

Let $\gamma(\xi) = |\xi|$ for $\xi \in \mathbb{R}^{n+1}$. Consider the integration chain

$$\gamma^\sim := \{\, \xi \in \overset{\circ}{\mathbb{R}}{}^{n+1} \mid \gamma(\xi) = 1\,\} \subset \mathbb{R}^{n+1} .$$

We assume $\gamma^\sim$ has the standard orientation.

Following [11], we define the smooth 'complex shift' map for $v \in \mathfrak{V}(\boldsymbol{x}, \boldsymbol{A})$ by

$$\sigma(\boldsymbol{x}, v) : \overset{\circ}{\mathbb{R}^{n+1}} \to \mathbb{C}^{n+1} \ , \quad \xi \mapsto \xi - iv(\xi) \ , \tag{5.65}$$

and consider the image of the chain $\gamma^{\sim}$ in $\mathbb{C}^{n+1}$ under (5.65):

$$\sigma(\boldsymbol{x}, v; \gamma^{\sim}) := \mathrm{Im}_{\sigma}[\, \gamma^{\sim} \,] \subset \mathbb{C}^{n+1} \ .$$

The next step is to associate with the family $\mathfrak{V}(\boldsymbol{x}, \boldsymbol{A})$ certain cycles in the complex projective space and to rewrite the right-hand side of the limit (5.61) as rational integrals over them.

For $x = x_0 e_0 + \boldsymbol{x} \in \mathbb{R}^{n+1}$, let $\varXi_{\mathbb{C}}(x)$ denote the set

$$\{\, \zeta \in X_{\mathbb{C}} \mid ((\zeta_0 I + \langle \boldsymbol{A}, \boldsymbol{\zeta} \rangle) - x_0 \zeta I) \ \text{ is not invertible in } \mathcal{L}_{(n)}(\mathbb{C}^N) \,\}.$$

Here $\zeta \in \mathbb{C}^{n+1}$ has been written as $\zeta = \zeta_0 e_0 + \boldsymbol{\zeta}$ for $\boldsymbol{\zeta} \in \mathbb{C}^n$.

If $U$ is a subset of $\mathbb{R}^{n+1}$, set $\varXi_{\mathbb{C}}(U) = \bigcup_{x \in U} \varXi_{\mathbb{C}}(x)$.

**Lemma 5.39.** *If $\boldsymbol{x} \notin W(\boldsymbol{A})$, then for every $v \in \mathfrak{V}(\boldsymbol{x}, \boldsymbol{A})$ there exists a neighbourhood $U$ of $(0, \boldsymbol{x})$ in $\mathbb{R}^{n+1}$ such that $\sigma(\boldsymbol{x}, v; \gamma^{\sim})$ with a change of orientation at $\langle e_0 + \boldsymbol{x}, \xi \rangle = 0$ is a relative cycle of the pair*

$$\big(\mathbb{C}^{n+1} \setminus \varXi_{\mathbb{C}}(U), X_{\mathbb{C}} \setminus (X_{\mathbb{C}} \cap \varXi_{\mathbb{C}}(U))\big) \, .$$

Let $v \in \mathfrak{V}(\boldsymbol{x}, \boldsymbol{A})$ and let $U$ be an open neighbourhood $(0, \boldsymbol{x})$ in $\mathbb{R}^{n+1}$ satisfying the conditions of Lemma 5.39 and let $\alpha(\boldsymbol{x}, v; \gamma^{\sim})$ be the $(n-1)$-cycle

$$\xi \longmapsto \xi - iv(\xi), \quad \xi \in X \cap \gamma^{\sim}, \tag{5.66}$$

in $X_{\mathbb{C}}$, with the orientation of the boundary of the half sphere

$$\big\{ \xi \in S^n \ \big| \ \langle e_0 + \boldsymbol{x}, \xi \rangle < 0 \big\}.$$

Then $\alpha(\boldsymbol{x}, v; \gamma^{\sim})$ is an $(n-1)$-cycle on $\mathbb{C}^{n+1} \setminus \varXi(U)$ and $X_{\mathbb{C}} \setminus (X_{\mathbb{C}} \cap \varXi_{\mathbb{C}}(U))$. and it can be identified with $\frac{1}{2}$ times the boundary of $\sigma(\boldsymbol{x}, v; \gamma^{\sim})$ with a change of orientation at $\langle e_0 + \boldsymbol{x}, \xi \rangle = 0$.

Since the mapping $\xi \mapsto v(\xi)$ is absolutely homogeneous (of degree 1), it is relevant to project $\alpha(\boldsymbol{x}, v; \gamma^{\sim})$ onto $\mathbb{CP}^n$:

$$\alpha(\boldsymbol{x}, v; \gamma^{\sim})^* := \mathrm{Im}_{\pi}[\, \alpha(\boldsymbol{x}, v; \gamma^{\sim}) \,] \tag{5.67}$$

via the canonical surjection $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ . Let $\varXi^*_{\mathbb{C}}(x)$ stand for the projective image of $\varXi_{\mathbb{C}}(x)$ in $\mathbb{CP}^n$ and for a subset $U$ of $\mathbb{R}^{n+1}$, set $\varXi^*_{\mathbb{C}}(U) = \bigcup_{x \in U} \varXi^*_{\mathbb{C}}(x)$. Similarly, the projective image of $X_{\mathbb{C}}$ in $\mathbb{CP}^n$ is written as $X^*_{\mathbb{C}}$

The homology class $[\alpha^*(\boldsymbol{x})] \in H_{n-1}(\mathbb{CP}^n \setminus \varXi^*_{\mathbb{C}}(U)\,; \mathbb{C})$ is *locally* independent of $\boldsymbol{x} \in \mathbb{R}^n \setminus W(\boldsymbol{A})$.

Let $\pi_0 : \mathbb{CP}^n \to \mathbb{CP}^{n-1}$ be the linear projection

$$(\zeta_0 : \zeta_1 : \cdots : \zeta_n) \longmapsto (\zeta_1 : \cdots : \zeta_n)$$

and set $[\beta^*(\boldsymbol{x})] = \pi_0[\alpha^*(\boldsymbol{x})]$. Then

$$[\beta^*(\boldsymbol{x})] \in H_{n-1}(\mathbb{CP}^{n-1} \setminus \pi_0 \Xi^*_{\mathbb{C}}(U)\,;\,\mathbb{C}).$$

Let $\beta(\boldsymbol{x}, v; \gamma^{\sim}) : \overset{\circ}{\mathbb{R}}{}^n \to \mathbb{C}^n$ be a representative of the cycle $\beta^*(\boldsymbol{x})$. Since the family $\mathfrak{V}(\boldsymbol{x}, \boldsymbol{A})$ is one homotopy class, the homology class $[\beta^*(\boldsymbol{x})]$ in $H_{n-1}(\mathbb{CP}^{n-1} \setminus \pi_0\,(\Xi^*_{\mathbb{C}}(\boldsymbol{x}))\,;\,\mathbb{C})$ does not actually depend on the choice of $v \in \mathfrak{V}(\boldsymbol{x}, \boldsymbol{A})$.

Let $\boldsymbol{\omega}(\zeta)$ be the Kronecker $(n-1)$-form

$$\boldsymbol{\omega}(\zeta) := \sum_{k=1}^{n} (-1)^{k-1}\, \zeta^k\, d\zeta^1 \wedge \cdots \wedge d\zeta^{k-1} \wedge d\zeta^{k+1} \wedge \cdots \wedge d\zeta^n\ . \tag{5.68}$$

**Proposition 5.40.** *Let $\boldsymbol{x} \notin W(\boldsymbol{A})$ and $v \in \mathfrak{V}(\boldsymbol{x}, \boldsymbol{A})$. Then the limit* (5.61) *is equal to*

$$\int\limits_{\beta(\boldsymbol{x},v;\gamma^{\sim})} \langle \boldsymbol{x}I - \boldsymbol{A}, \zeta\rangle^{-n}\, \boldsymbol{\omega}(\zeta). \tag{5.69}$$

*Proof (Sketch).* By Stoke's theorem, there exists an open neighbourhood $U_\epsilon$ of $(0, \boldsymbol{x})$ in $\mathbb{R}^{n+1}$ such that the difference between

$$\int\limits_{S^{n-1}} (\langle \boldsymbol{x}I - \boldsymbol{A}, s\rangle + \epsilon s)^{-n} + (\langle \boldsymbol{x}I - \boldsymbol{A}, s\rangle - \epsilon s)^{-n}\, ds$$

and

$$\frac{1}{2} \int\limits_{\beta(\boldsymbol{x},v;\gamma^{\sim})} (\langle \boldsymbol{x}I - \boldsymbol{A}, \zeta\rangle + \epsilon\zeta)^{-n} + (\langle \boldsymbol{x}I - \boldsymbol{A}, \zeta\rangle - \epsilon\zeta)^{-n}\, \omega(\zeta)$$

can be written as

$$\frac{1}{2} \int\limits_{\beta_\epsilon(\boldsymbol{x},v;\gamma^{\sim})} (\langle \boldsymbol{x}I - \boldsymbol{A}, \zeta\rangle + \epsilon\zeta)^{-n} + (\langle \boldsymbol{x}I - \boldsymbol{A}, \zeta\rangle - \epsilon\zeta)^{-n}\, \omega(\zeta) \tag{5.70}$$

for a properly oriented $\beta_\epsilon \in H_{n-1}(\mathbb{CP}^{n-1} \setminus \pi_0 \Xi^*_{\mathbb{C}}(U_\epsilon)\,;\,\mathbb{C})$. As $\epsilon \to 0$, the integral (5.70) converges to zero. □

Notice that for upper-triangular matrices, $W(\boldsymbol{A})$ consists of $C(A)$ plus 'double tangents' inside $K(\boldsymbol{A})$, see Example 5.36 above.

The integrand in the right-hand side of (5.69) has the form $F(\zeta)\,\boldsymbol{\omega}$, where $F(\zeta)$ is a rational function in $\zeta$ homogeneous of degrees $-n$. Such a differential form is invariant under coordinate changes $\zeta_j \to f(\zeta)\,\zeta_j$ and, hence, is a pull-back of a differential form on $\mathbb{CP}^{n-1}$ under the canonical projection $\pi : \mathbb{C}^n \setminus \{0\} \to \mathbb{CP}^{n-1}$.

Letting $\zeta = (\zeta_1 : \ldots : \zeta_n)$ denote homogeneous coordinates in $\mathbb{CP}^{n-1}$, we get from (5.67) and (5.69)

$$\int_{\beta^*(\boldsymbol{x})} \langle \boldsymbol{x}I - \boldsymbol{A}, \zeta\rangle^{-n}\, \boldsymbol{\omega}(\zeta) \tag{5.71}$$

where $\beta^*(\boldsymbol{x})$ is an absolute Petrovsky cycle induced by $\sigma(\boldsymbol{x})$, and $\boldsymbol{\omega}(\zeta)$ is the Kronecker form on $\mathbb{CP}^{n-1}$ given (in terms of homogeneous coordinates) by (5.68).

Moreover, corresponding to the point $\boldsymbol{x} \in \mathbb{R}^n \setminus W(\boldsymbol{A})$, from equation (5.60) we have

$$\mathcal{W}_{\boldsymbol{A}}(\boldsymbol{x}) = (-1)^{n/2} \frac{(n-1)!}{4(2\pi)^n} \int_{\beta^*(\boldsymbol{x})} \langle \boldsymbol{x}I - \boldsymbol{A}, \zeta\rangle^{-n}\, \boldsymbol{\omega}(\zeta) \tag{5.72}$$

The integral (5.72) depends *only* on the homology class $[\beta^*(\boldsymbol{x})]$ of the cycle $\beta^*(\boldsymbol{x})$ in $H_{n-1}(\mathbb{CP}^{n-1} \setminus \pi_0 \Xi^*_{\mathbb{C}}(\boldsymbol{x})\,; \mathbb{C})$, since its integrand is a closed form of highest degree holomorphic on $\mathbb{CP}^{n-1} \setminus \pi_0 \Xi^*_{\mathbb{C}}(\boldsymbol{x})$.

**Theorem 5.41.** *Let $n$ be a nonzero even integer, $\boldsymbol{A}$ an $n$-tuple of $(N \times N)$ matrices satisfying the spectral condition (5.9). If $\boldsymbol{a} \in \mathbb{R}^n \setminus W(\boldsymbol{A})$ and for all points $\boldsymbol{x}$ in a neighbourhood $U$ of $\boldsymbol{a}$ in $\mathbb{R}^n$ we have*

$$\Xi^*_{\mathbb{C}}(\boldsymbol{x}) \subset \mathbb{RP}^n, \tag{5.73}$$

*then $\boldsymbol{a} \in \gamma(\boldsymbol{A})^c$.*

*Proof.* If condition (5.73) holds for $\boldsymbol{x} \in \mathbb{R}^n$, then $[\beta^*(\boldsymbol{x})] = 0$ in

$$H_{n-1}(\mathbb{CP}^{n-1} \setminus \pi_0\left(\Xi^*_{\mathbb{C}}(\boldsymbol{x})\right); \mathbb{C}),$$

so $\mathcal{W}_{\boldsymbol{A}}(\boldsymbol{x}) = 0$ and $\boldsymbol{x} \in \gamma(\boldsymbol{A})^c$, by equation (5.73). □

In view of the examples considered so far, it is plausible to propose the following

**Conjecture**. *Let $n$ be a nonzero even integer, $N \geq n+1$ an integer and $\boldsymbol{A}$ an $n$-tuple of $(N \times N)$ matrices satisfying the spectral condition (5.9). Then the set $\gamma(\boldsymbol{A}) \cap W(\boldsymbol{A})^c$ is the complement in $\mathbb{R}^n \setminus W(\boldsymbol{A})$ of the set of all points in a neighbourhood $U$ of which*

$$[\beta^*(\boldsymbol{x})] = 0 \tag{5.74}$$

*for all $\boldsymbol{x} \in U$.*

It follows from formula (5.23) that a point $\boldsymbol{a}$ belongs to $\mathbb{R}^n \setminus (W(\boldsymbol{A}) \cup \gamma(\boldsymbol{A}))$ whenever equation (5.74) holds in a neighbourhood of $\boldsymbol{a}$. In Section 5.3, we saw that equation (5.73) is necessarily satisfied in a neighbourhood of points outside $\gamma(\boldsymbol{A})$ for the case of hermitian matrices and $n = 2$, see equation (5.28). It then follows that (5.74) holds for points belonging to $W(\boldsymbol{A})^c$, but outside $\gamma(\boldsymbol{A})$.

The conjecture can be paraphrased by saying that condition (5.74) determines all open parts of the complement of the joint spectrum $\gamma(\boldsymbol{A})$ of $\boldsymbol{A}$, because $W(\boldsymbol{A})$ has codimension one in $\mathbb{R}^n$ if $\boldsymbol{A}$ is complete. In higher dimensions, the necessity of condition (5.74) in certain special cases is not so straightforward and relies on deep results in algebraic topology [12]. For systems $\boldsymbol{A}$ of upper triangularisable matrices with real spectra, conditions (5.73) and (5.74) are valid outside the wave front surface $W(\boldsymbol{A})$.

# 6

# The Monogenic Calculus for Sectorial Operators

Up until this point, only a functional calculus for systems of matrices or, more generally, bounded linear operators $A_1, \dots, A_n$ acting on a Banach space $X$ has been considered. Many of the techniques can be generalised to an $n$-tuple $\boldsymbol{A} = (A_1, \dots, A_n)$ of densely defined operators such that for each $\xi \in \mathbb{R}^n$, the operator $\sum_{j=1}^n \xi_j A_j$ is densely defined and closable and its closure $\overline{\sum_{j=1}^n \xi_j A_j}$ has real spectrum. Well-known examples from quantum mechanics show that care must be exercised in forming the sums of unbounded operators.

Rather than assume that the operator $\overline{\sum_{j=1}^n \xi_j A_j}$ has real spectrum, we will suppose that $X$ is a Banach space, the spectrum is contained in a double sector of the complex plane centred at zero and containing the real axis, and certain resolvent estimates for $\overline{\sum_{j=1}^n \xi_j A_j}$ are satisfied near the boundary of the sector. Of course, this also extends the class of bounded linear operators or matrices for which the monogenic calculus is applicable. Under these assumptions, we can form functions $f(A_1, \dots, A_n) \in \mathcal{L}(X)$ of the operators $A_1, \dots, A_n$ in the case that $f$ is monogenic in a sector in $\mathbb{R}^{n+1}$ and has decay at zero and infinity, so that the associated Riesz-Dunford integral converges. For functions of this type, there is an associated bounded holomorphic function $\tilde{f}$ defined in a corresponding sector in $\mathbb{C}^n$ such that $\tilde{f}$ and $f$ are equal on $\mathbb{R}^n \setminus \{0\}$, so it make sense to set $\tilde{f}(A_1, \dots, A_n) = f(A_1, \dots, A_n)$.

In the case that $X = \mathcal{H}$ is a Hilbert space, additional 'square function' estimates ensure that we can form functions $f(A_1, \dots, A_n) \in \mathcal{L}(\mathcal{H})$ of the commuting operators $A_1, \dots, A_n$ in the case that $f$ is an $H^\infty$-function defined in a sector in $\mathbb{C}^n$.

The significance of this result comes from problems in real-variable harmonic analysis. In the one dimensional case, the momentum operator $P = \frac{1}{i}\frac{\mathrm{d}}{\mathrm{d}x}$ is selfadjoint in $L^2(\mathbb{R})$. The bounded linear operator $\mathrm{sgn}(P)$ is defined by the functional calculus for selfadjoint operators. But $\mathrm{sgn}(P)$ is also the operator of convolution with respect to the distribution $\xi \longmapsto \mathrm{sgn}\hat{\ }(-\xi)$ because of the identity $(\mathrm{sgn}(P)u)\hat{\ }(\xi) = \mathrm{sgn}(\xi)\hat{u}(\xi)$ for $u \in L^2(\mathbb{R})$, that is, the operator $\mathrm{sgn}(P)$ is just the Hilbert transform

B. Jefferies: LNM 1843, pp. 123–155, 2004.

$$(\mathrm{sgn}(P)u)(x) = 2i\mathrm{p.v.}\int_{-\infty}^{\infty} \frac{1}{x-y} u(y)\,dy, \quad u \in L^2(\mathbb{R}).$$

So we have the connection between a functional calculus for a single operator and singular convolution operators.

In the case that $A_1, \dots, A_n$ are the (commuting) directional derivatives on a Lipschitz surface $\Sigma$, acting in $L^2(\Sigma)$, for a certain function $f$ satisfying the assumptions, the bounded linear operator $f(A_1, \dots, A_n) : L^2(\Sigma) \to L^2(\Sigma)$ is the Cauchy integral operator on the Lipschitz surface. The boundedness of the Cauchy integral operator is an important step in the solution of irregular boundary value problems [66].

There are now many proofs of the boundedness of the Cauchy integral operator and other convolution operators on a Lipschitz surface (see for example [72] for a proof using Fourier theory of monogenic functions). The related ideas in this chapter appeal to the general techniques introduced in Chapter 4 for forming functions of noncommuting operators.

In the earlier chapters, the assumption that the spectrum $\sigma(\sum_{j=1}^{n} \xi_j A_j)$ is real for all $\xi \in \mathbb{R}^n$ meant that any real-analytic function defined in a neighbourhood of the joint spectrum $\gamma(\boldsymbol{A})$ had a unique monogenic extension to a neighbourhood of $\gamma(\boldsymbol{A})$ in $\mathbb{R}^{n+1}$. Without this assumption, some work needs to be done to find those real-analytic functions defined on $\mathbb{R}^n \setminus \{0\}$ that have a monogenic extension to a suitable sector containing the monogenic spectrum $\gamma(\boldsymbol{A})$. This is considered in Sections 6.3 and 6.4 below.

## 6.1 The $H^\infty$-Functional Calculus for a Single Operator

We first set down the known results concerning the $H^\infty$-functional calculus for a single linear operator $A : \mathcal{D}(A) \to \mathcal{H}$ acting in a Hilbert space $\mathcal{H}$. The domain $\mathcal{D}(A)$ of $A$ is a dense linear subspace of $\mathcal{H}$. When we come to consider an $n$-tuple $\boldsymbol{A} = (A_1, \dots, A_n)$ of linear operators acting in $\mathcal{H}$, we can consider $A = A_1 e_1 + \cdots + A_n e_n$ as a single operator acting in $\mathcal{H}_{(n)}$, so a functional calculus for $\boldsymbol{A}$ produces a functional calculus for $A$.

For any $0 < \mu < \frac{\pi}{2}$, set

$$S_{\mu+}(\mathbb{C}) = \{z \in \mathbb{C} : |\arg z| \le \mu\} \cup \{0\}, \quad S_{\mu-}(\mathbb{C}) = -S_{\mu+}(\mathbb{C}) \tag{6.1}$$

$$S^\circ_{\mu+}(\mathbb{C}) = \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \mu\}, \quad S^\circ_{\mu-}(\mathbb{C}) = -S^\circ_{\mu+}(\mathbb{C}) \tag{6.2}$$

$$S_{\mu}(\mathbb{C}) = S_{\mu+}(\mathbb{C}) \cup S_{\mu-}(\mathbb{C}), \tag{6.3}$$

$$S^\circ_{\mu}(\mathbb{C}) = S^\circ_{\mu+}(\mathbb{C}) \cup S^\circ_{\mu-}(\mathbb{C}). \tag{6.4}$$

Let $0 \le \omega < \frac{\pi}{2}$. An operator $A$ acting in $\mathcal{H}$ is said to be of *type* $\omega_+$ if its spectrum $\sigma(A)$ is contained in the sector $S_{\omega+}(\mathbb{C})$ and for each $\mu > \omega$, there exits $C_\mu > 0$ such that the bound

$$\|(zI - A)^{-1}\| \le C_\mu |z|^{-1} \tag{6.5}$$

holds for all $z \notin S_{\mu+}(\mathbb{C})$. Another way of saying this is that for each $\mu > \omega$ the operator $-A$ generates a bounded holomorphic semigroup $e^{-zA}$ for all $z \in S^\circ_{(\pi/2-\mu)+}(\mathbb{C})$ [65, pp. 490–491].

Similarly, $A$ is of *type* $\omega$ if $\sigma(A) \subset S_\omega(\mathbb{C})$ and for each $\mu > \omega$, there exists $C_\mu > 0$ such that

$$\|(zI - A)^{-1}\| \le C_\mu |z|^{-1}, \quad z \notin S_\mu(\mathbb{C}). \tag{6.6}$$

For any $0 < \mu < \frac{\pi}{2}$ let $H^\infty(S^\circ_{\mu+}(\mathbb{C}))$ denote the Banach algebra of uniformly bounded holomorphic functions defined on $S^\circ_{\mu+}(\mathbb{C})$ under pointwise multiplication and equipped with the supremum norm over $S^\circ_{\mu+}(\mathbb{C})$.

The operator $A$ has an $H^\infty$*-functional calculus* over $S_{\omega+}(\mathbb{C})$ if for every $\mu > \omega$, there exists $C_\mu > 0$ and an algebra homomorphism $f \longmapsto f(A)$ from $H^\infty(S^\circ_{\mu+}(\mathbb{C}))$ into $\mathcal{L}(\mathcal{H})$ that agrees with the usual definition of polynomials of $A$ and resolvent operators and is such that the bound

$$\|f(A)\| \le C_\mu \|f\|_\infty, \quad f \in H^\infty(S^\circ_{\mu+}(\mathbb{C}))$$

holds. A similar definition holds for the closed double sectors $S_\omega(\mathbb{C})$.

Now suppose that $A$ is a one-to-one operator of type $\omega+$. Then $A$ is a closed operator with dense domain and range . Given $\omega < \mu < \pi/2$ and a function $\psi \in H^\infty(S^\circ_{\mu+}(\mathbb{C}))$ for which there exists $C, s > 0$ such that

$$|\psi(z)| \le C \frac{|z|^s}{1 + |z|^{2s}}, \quad z \in S^\circ_{\mu+}(\mathbb{C}), \tag{6.7}$$

the operator $\psi(A) \in \mathcal{L}(\mathcal{H})$ is defined by the Riesz-Dunford calculus

$$\psi(A) = \frac{1}{2\pi i} \int_C (\zeta I - A)^{-1} \psi(\zeta)\, d\zeta \tag{6.8}$$

for a suitably chosen contour $C$ inside $S^\circ_{\mu+}(\mathbb{C}) \cup \{0\}$, but surrounding $S_{\omega+}(\mathbb{C}) \setminus \{0\}$. The integral (6.8) converges because of the decay of $\psi$ at infinity and zero and the resolvent estimate (6.6). If $t > 0$ and $\psi_t$ is defined by the formula $\psi_t(z) = \psi(tz)$ for all $z \in S_{\omega+}(\mathbb{C})$, then $\psi_t(A)$ is similarly defined by the Cauchy integral formula (6.8).

An alternative method for defining $\psi(A)$ in the case that $\psi$ satisfies (6.7) and $A$ satisfies (6.6) for all $z \in S^\circ_{\mu+}(\mathbb{C})$ is given by the formula

$$\psi(A) = \frac{i}{2\pi} \int_{H_\theta} e^{-\zeta A} \hat{\psi}(i\zeta)\, d\zeta, \tag{6.9}$$

where $0 < \theta < \frac{\pi}{2} - \mu$, $H_\theta$ is the contour

$$\{z \in \mathbb{C} : \Re z \ge 0,\ \Re z = |\Im z| \tan\theta\}$$

in $S^\circ_{(\pi/2-\mu)+}(\mathbb{C})$ and $\hat{\psi}$ is the Fourier transform of $\psi\chi_{[0,\infty)}$.

The possibility of extending the definition of the mapping $\psi \longmapsto \psi(A)$ given by (6.8) to all of $H^\infty(S^\circ_{\mu+}(\mathbb{C}))$ for $\mu > \omega$ depends on the validity of *square function estimates.* The following result is from [71, Theorem 6.2.2].

**Theorem 6.1.** *Suppose that $A$ is a one-to-one operator of type $\omega_+$ in $\mathcal{H}$. Then $A$ has a bounded $H^\infty$-functional calculus if and only if for every $\mu > \omega$, there exists $c_\mu > 0$ such that $A$ and its adjoint $A^*$ satisfy the square function estimates*

$$\int_0^\infty \|\psi_t(A)u\|^2 \frac{dt}{t} \le c_\mu \|u\|^2, \quad u \in H, \tag{6.10}$$

$$\int_0^\infty \|\psi_t(A^*)u\|^2 \frac{dt}{t} \le c_\mu \|u\|^2, \quad u \in H, \tag{6.11}$$

*for some (resp. every) function $\psi \in H^\infty(S^\circ_{\mu+}(\mathbb{C}))$ satisfying (6.7) and*

$$\int_0^\infty \psi^3(t)\frac{dt}{t} = \int_0^\infty \psi^3(-t)\frac{dt}{t} = 1.$$

Suppose that the square function estimates (6.10) and (6.11) are valid. Then for any $b \in H^\infty(S^\circ_{\mu+}(\mathbb{C}))$ the bounded linear operator $b(A)$ is defined by the formula

$$(b(A)u, v) = \int_0^\infty ((b\psi_t)(A)\psi_t(A)u, \psi_t(A)^*v)\frac{dt}{t} \tag{6.12}$$

for all $u, v \in \mathcal{H}$. Then

$$\begin{aligned}|(b(A)u, v)| &\le \sup_{t>0} \|(b\psi_t)(A)\| \left\{\int_0^\infty \|\psi_t(A)u\|^2 \frac{dt}{t}\right\}^{\frac{1}{2}} \left\{\int_0^\infty \|\psi_t(A^*)u\|^2 \frac{dt}{t}\right\}^{\frac{1}{2}} \\ &\le C\|b\|_\infty \|u\|\,\|v\|, \quad u, v \in H,\end{aligned}$$

and the mapping $b \longmapsto b(A)$ from $H^\infty(S^\circ_{\mu+}(\mathbb{C}))$ into $\mathcal{L}(\mathcal{H})$ has the required properties [71].

## 6.2 The Cauchy Kernel for $n$ Sectorial Operators

The idea of using the analogue of formula (6.9) in higher dimensions is pursued in [71]. Rather, we are aiming to use the higher-dimensional analogue of (6.8), namely

$$\psi(\boldsymbol{A}) = \int_{\partial\Omega} G_x(\boldsymbol{A})n(x)\tilde{\psi}(x)\,d\mu(x) \tag{6.13}$$

for a suitable function $\psi$ holomorphic in higher-dimensional sector and a suitable subset $\Omega$ of $\mathbb{R}^{n+1}$. The function $\tilde{\psi}$ is a monogenic function canonically associated with $\psi$. This association is discussed in Sections 6.3 and 6.4 below.

The difficulty, as usual, is the definition of the Cauchy kernel $x \longmapsto G_x(\boldsymbol{A})$. If we take the equation

$$G_x(\boldsymbol{A}) = \frac{(n-1)!}{2}\left(\frac{i}{2\pi}\right)^n \operatorname{sgn}(x_0)^{n-1} \times \int_{S^{n-1}} (e_0 + is)\left(\langle \boldsymbol{x}I - \boldsymbol{A}, s\rangle - x_0 sI\right)^{-n} ds. \quad (6.14)$$

obtained from the plane wave decomposition of the Cauchy kernel as the definition of $G_x(\boldsymbol{A})$, then the convergence of the integral

$$\int_{S^{n-1}} (e_0 + is)\left(\langle \boldsymbol{x}I - \boldsymbol{A}, s\rangle - x_0 sI\right)^{-n} ds$$

for particular values of $x = x_0 e_0 + \boldsymbol{x} \in \mathbb{R}^{n+1}$ is at issue. Now

$$\left(\langle \boldsymbol{x}I - \boldsymbol{A}, s\rangle - x_0 sI\right)^{-1} = \left(\langle \boldsymbol{x}I - \boldsymbol{A}, s\rangle + x_0 sI\right)\left(\langle \boldsymbol{x}I - \boldsymbol{A}, s\rangle^2 + x_0^2 I\right)^{-1}$$

if $0 \notin \sigma\left(\langle \boldsymbol{x}I - \boldsymbol{A}, s\rangle^2 + x_0^2\right)$. Thus, we need to ensure the appropriate uniform operator bounds for

$$\left(\langle \boldsymbol{x}I - \boldsymbol{A}, s\rangle^2 + x_0^2 I\right)^{-1}, \quad s \in S^{n-1}$$

as $x = x_0 e_0 + \boldsymbol{x}$ ranges over a subset of $\mathbb{R}^{n+1}$.

In the case that $\sigma(\langle \boldsymbol{A}, \xi\rangle) \subset \mathbb{R}$ and $(\lambda I - \langle \boldsymbol{A}, \xi\rangle)^{-1}$ is suitably bounded for all $\xi \in S^{n-1}$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then $G_{x_0 e_0 + \boldsymbol{x}}(\boldsymbol{A})$ is defined for all $x_0 \neq 0$. In the case that $\sigma(\langle \boldsymbol{A}, \xi\rangle)$ is contained in a fixed sector for every $\xi \in S^{n-1}$, the following assumption is made.

The set of $s \in S^{n-1}$ with nonzero coordinates $s_j$ for every $j = 1, \dots, n$ is denoted by $S_0^{n-1}$. Then $S_0^{n-1}$ is a dense open subset of $S^{n-1}$ with full surface measure.

**Definition 6.2.** Let $\boldsymbol{A} = (A_1, \dots, A_n)$ be an $n$-tuple of closed, densely defined linear operators $A_j : \mathcal{D}(A_j) \to X$ acting in a Banach space $X$ such that $\cap_{j=1}^n \mathcal{D}(A_j)$ is dense in $X$ and let $0 \le \omega < \frac{\pi}{2}$. Then $\boldsymbol{A}$ is said to be *uniformly of type* $\omega$ if $\sigma(\langle A, s\rangle) \subset S_\omega(\mathbb{C})$ for all $s \in S_0^{n-1}$ and for each $\nu > \omega$, there exists $C_\nu > 0$ such that

$$\|(zI - \langle \boldsymbol{A}, s\rangle)^{-1}\| \le C_\mu |z|^{-1}, \quad z \notin S_\mu^\circ(\mathbb{C}), \quad s \in S_0^{n-1}. \quad (6.15)$$

It follows that $s \longmapsto \langle A, s\rangle$ is continuous on $S_0^{n-1}$ in the sense of strong resolvent convergence [65, Theorem VIII.1.5]. The subset $S_0^{n-1}$ of $S^{n-1}$ is used here simply because $\cap_{j=1}^n \mathcal{D}(A_j)$ may be strictly contained in $\mathcal{D}(A_k)$ for $k = 1, \dots, n$.

*Remark 6.3.* By taking adjoints in the case that $X$ is Hilbert space, we see that $n$-tuple of operator $\boldsymbol{A}$ is uniformly of type $\omega$ if and only if its adjoint $(\boldsymbol{A})^* = (A_1^*, \dots, A_n^*)$ is uniformly of type $\omega$.

*Example 6.4.* i) Let $\boldsymbol{A}$ be an $n$-tuple of selfadjoint (possibly unbounded) operators acting on a Hilbert space. Let $\mu > 0$. If $\langle \boldsymbol{A}, \xi\rangle$ is selfadjoint for $\xi \in \mathbb{R}^n$, then $\sigma(\langle \boldsymbol{A}, \xi\rangle) \subset \mathbb{R}$ and by the functional calculus for selfadjoint operators we have

$$\begin{aligned}\|(zI - \langle \boldsymbol{A}, \xi\rangle)^{-1}\| &= \sup_{x\in\sigma(\langle \boldsymbol{A},\xi\rangle)} |z-x|^{-1} \\ &\leq \frac{1}{|\Im z|} \leq \frac{1}{\sin\mu}\frac{1}{|z|}, \quad \text{for all } z \notin S^\circ_\mu(\mathbb{C}),\ \xi \in \mathbb{R}^n.\end{aligned}$$

Hence, $\boldsymbol{A}$ is uniformly of type $\omega$ provided that $\langle A, s\rangle$ is selfadjoint for all $s \in S_0^{n-1}$.

ii) Let $T_1, \ldots, T_n$ be $n$ *commuting* selfadjoint operators acting on a Hilbert space $\mathcal{H}$. Then $\boldsymbol{T} = (T_1, \ldots, T_n)$ has a joint spectral measure $P : \gamma(\boldsymbol{T}) \to \mathcal{L}(\mathcal{H})$ and $\langle T, s\rangle$ is selfadjoint for all $s \in S^{n-1}$. Let $0 < \omega < \mu < \pi/2$ and let $\theta : \gamma(\boldsymbol{T}) \to [-\omega, \omega]$ be a Borel measurable function. Then

$$\boldsymbol{A} = \left(\int_{\gamma(\boldsymbol{T})} \lambda_1 e^{i\theta(\lambda)} dP(\lambda), \ldots, \int_{\gamma(\boldsymbol{T})} \lambda_n e^{i\theta(\lambda)} dP(\lambda)\right)$$

is uniformly of type $\omega$, because

$$\langle \boldsymbol{A}, \xi\rangle = \int_{\gamma(\boldsymbol{T})} e^{i\theta(\lambda)} \langle \lambda, \xi\rangle\, dP(\lambda)$$

is a normal operator for each $\xi \in \mathbb{R}^n$, $\sigma(\langle \boldsymbol{A}, \xi\rangle) \subset S_\omega(\mathbb{C})$ for all $\xi \in \mathbb{R}^n$ and by the functional calculus for normal operators, we have

$$\begin{aligned}\|(zI - \langle \boldsymbol{A}, \xi\rangle)^{-1}\| &= \sup_{\zeta\in\sigma(\langle \boldsymbol{A},\xi\rangle)} |z-\zeta|^{-1} \\ &\leq \frac{1}{\operatorname{dist}(z, S_\omega(\mathbb{C}))} \\ &\leq \frac{1}{\sin(\mu-\omega)}\frac{1}{|z|}, \quad \text{for all } z \notin S^\circ_\mu(\mathbb{C}),\ \xi \in \mathbb{R}^n.\end{aligned}$$

iii) In the following example, we have $n$ commuting operators, each of which is of type $\omega > 0$, but generally, not selfadjoint.

Let $\Sigma = \{x : x = s + g(s)e_0,\ s \in \mathbb{R}^n\}$ be a Lipschitz surface. Let

$$A_j u = (e_0 - Dg)^{-1} \frac{\partial}{\partial s_j} u(s + g(s)e_0)$$

for $u \in W_2^1(\Sigma)$, $j = 1, \ldots, n$. Then $\langle A, \xi\rangle$ corresponds to a directional derivative towards $\xi \in S^{n-1}$, so we obtain the required estimates uniformly in $\xi \in S^{n-1}$.

Now suppose that equation (6.15) is satisfied and let $z = \langle \boldsymbol{x}, s\rangle + ix_0$. Then $z \notin S^\circ_\mu(\mathbb{C})$ means that $|\arg z| \geq \mu$ for $-\frac{\pi}{2} \leq \arg z \leq \frac{\pi}{2}$ or $\pi - \arg z \geq \mu$ for $\frac{\pi}{2} \leq \arg z \leq \pi$ or $\pi + \arg z \geq \mu$ for $-\pi \leq \arg z \leq -\frac{\pi}{2}$. Hence, for $x_0 > 0$, we have $x_0 \geq \tan\mu|\langle \boldsymbol{x}, s\rangle|$ and for $x_0 < 0$ we have $x_0 \leq -\tan\mu|\langle \boldsymbol{x}, s\rangle|$.

**Definition 6.5.** The sector $\{x_0e_0 + \boldsymbol{x} \in \mathbb{R}^{n+1} : |x_0| \leq \tan\mu|\boldsymbol{x}|\}$ in $\mathbb{R}^{n+1}$ is denoted by $S_\mu(\mathbb{R}^{n+1})$. Let

$$N^+_\mu = \{x \in \mathbb{R}^{n+1} : x = x_0e_0 + \boldsymbol{x},\ x_0 \geq \tan\mu|\boldsymbol{x}|\ \}, \tag{6.16}$$

$$N^-_\mu = \{x \in \mathbb{R}^{n+1} : x = x_0e_0 + \boldsymbol{x},\ x_0 \leq -\tan\mu|\boldsymbol{x}|\ \} = -N^+_\mu, \tag{6.17}$$

$$N_\mu = N^+_\mu \cup N^-_\mu. \tag{6.18}$$

Then $N_\mu$ is the complement in $\mathbb{R}^{n+1}$ of the interior $S^\circ_\mu(\mathbb{R}^{n+1})$ of $S_\mu(\mathbb{R}^{n+1})$.

Note that if $x_0e_0 + \boldsymbol{x} \in N_\mu$, then $z = \langle \boldsymbol{x}, s\rangle + ix_0 \notin S_\mu(\mathbb{C})$ for every $s \in S^{n-1}$ because either

$$x_0 \geq \tan\mu|y| \geq \tan\mu|\langle \boldsymbol{x}, s\rangle|$$

or

$$x_0 \leq -\tan\mu|y| \leq -\tan\mu|\langle \boldsymbol{x}, s\rangle|.$$

**Lemma 6.6.** *Let $\omega < \mu < \pi/2$. Suppose that the $n$-tuple $\boldsymbol{A}$ of linear operators is uniformly of type $\omega$. Then for all $x_0e_0 + \boldsymbol{x} \in N_\mu$, the integral*

$$\int_{S^{n-1}} \left\| (\langle \boldsymbol{x}I - \boldsymbol{A}, s\rangle - x_0 sI)^{-n} \right\|_{\mathcal{L}_{(n)}(\mathcal{H})} ds$$

*converges and satisfies the bound*

$$\int_{S^{n-1}} \left\| (\langle \boldsymbol{x}I - \boldsymbol{A}, s\rangle - x_0 sI)^{-n} \right\|_{\mathcal{L}_{(n)}(\mathcal{H})} ds \leq \frac{C'_\mu}{|x_0|^n}.$$

*Proof.* For every $x_0e_0 + \boldsymbol{x} \in N_\mu$, we have $z = \langle \boldsymbol{x}, s\rangle \pm ix_0 \notin S_\mu(\mathbb{C})$ so that the operator $(\langle \boldsymbol{x}, s\rangle \pm ix_0)I - \langle \boldsymbol{A}, s\rangle$ is invertible and the bound

$$\left\| ((\langle \boldsymbol{x}, s\rangle \pm ix_0)I - \langle \boldsymbol{A}, s\rangle)^{-1} \right\|_{\mathcal{L}(\mathcal{H})} \leq \frac{C_\mu}{\sqrt{\langle \boldsymbol{x}, s\rangle^2 + x_0^2}}$$

holds. Now

$$(\langle \boldsymbol{x}I - \boldsymbol{A}, s\rangle - x_0 sI)^{-1} = (\langle \boldsymbol{x}I - \boldsymbol{A}, s\rangle + x_0 sI) \left(\langle \boldsymbol{x}I - \boldsymbol{A}, s\rangle^2 + x_0^2 I\right)^{-1}$$

where $\left(\langle \boldsymbol{x}I - \boldsymbol{A}, s\rangle^2 + x_0^2 I\right)^{-1}$ is equal to

$$((\langle \boldsymbol{x}, s\rangle + ix_0)I - \langle \boldsymbol{A}, s\rangle)^{-1} ((\langle \boldsymbol{x}, s\rangle - ix_0)I - \langle \boldsymbol{A}, s\rangle)^{-1}.$$

Writing $(\langle \boldsymbol{x}I - \boldsymbol{A}, s\rangle + x_0 sI) = ((\langle \boldsymbol{x}, s\rangle + ix_0)I - \langle \boldsymbol{A}, s\rangle) - ix_0 I + x_0 sI$, we obtain

$$
\begin{aligned}
(\langle \boldsymbol{x}I - \boldsymbol{A}, s\rangle - x_0 sI)^{-1} = ((\langle \boldsymbol{x}, s\rangle - ix_0)I - \langle \boldsymbol{A}, s\rangle)^{-1} \\
-ix_0(e_0 + is)\left(\langle \boldsymbol{x}I - \boldsymbol{A}, s\rangle^2 + x_0^2 I\right)^{-1}
\end{aligned}
$$

so that

$$
\begin{aligned}
\left\|(\langle \boldsymbol{x}I - \boldsymbol{A}, s\rangle - x_0 sI)^{-1}\right\|_{\mathcal{L}_{(n)}(\mathcal{H})} &\leq \frac{C_\mu}{\sqrt{\langle \boldsymbol{x}, s\rangle^2 + x_0^2}} + \frac{2|x_0|C_\mu^2}{\langle \boldsymbol{x}, s\rangle^2 + x_0^2} \\
&\leq \frac{C_\mu + 2C_\mu^2}{|x_0|},
\end{aligned}
$$

from which the stated bound follows. □

Thus, $x_0e_0 + \boldsymbol{x} \longmapsto G_{x_0e_0+\boldsymbol{x}}(\boldsymbol{A})$ is defined by equation (6.14) for all $x_0e_0 + \boldsymbol{x} \in N_\mu$ with $\omega < \mu < \pi/2$. Standard arguments ensure that $x_0e_0 + \boldsymbol{x} \longmapsto G_{x_0e_0+\boldsymbol{x}}(\boldsymbol{A})$ is both left and right monogenic. If we denote by $\gamma(\boldsymbol{A}) \subset \mathbb{R}^{n+1}$ the set of all singularities of the function $x_0e_0 + \boldsymbol{x} \longmapsto G_{x_0e_0+\boldsymbol{x}}(\boldsymbol{A})$, then

$$
\gamma(\boldsymbol{A}) \subseteq S_\mu(\mathbb{R}^{n+1}).
$$

## 6.3 Monogenic and Holomorphic Functions in Sectors

In view of Lemma 6.6 and the representation (6.14) for the Cauchy kernel, it is apparent that functions $\psi(\boldsymbol{A})$ of the $n$-tuple $\boldsymbol{A}$ uniformly of type $\omega$ can be defined by formula (6.13) for left monogenic functions with decay at zero and infinity in the sector $S_\mu(\mathbb{R}^{n+1})$, for some $\omega < \mu < \pi/2$.

In Chapter 3, we were able to form functions $\psi(\boldsymbol{A})$ of $\boldsymbol{A}$ whenever the $n$-tuple of bounded linear operators $\boldsymbol{A}$ satisfies the spectral reality condition (4.10) and $\psi$ is real-analytic in a neighbourhood of the monogenic spectrum $\gamma(\boldsymbol{A})$ of $\boldsymbol{A}$, simply by taking the two-sided monogenic extension $\tilde{\psi}$ of $\psi$ to a neighbourhood of $\gamma(\boldsymbol{A})$ in $\mathbb{R}^{n+1}$.

In this section, we show that the restriction of a monogenic function $f$ defined in a sector in $\mathbb{R}^{n+1}$ to $\mathbb{R}^n$ has an analytic continuation $\tilde{f}$ to a corresponding sector in $\mathbb{C}^n$. Of course, if the restriction of $f$ to $\mathbb{R}^n$ takes values in $\mathbb{C}$, then so does its analytic continuation. The analytic continuation is achieved via the Cauchy integral formula and if $f$ is uniformly bounded on subsectors, then so is $\tilde{f}$.

The correspondence between bounded monogenic functions and bounded holomorphic functions in sectors is completed in the next section where a formula deriving $f$ from $\tilde{f}$ is obtained by Fourier analysis.

### 6.3.1 Joint Spectral Theory in the Algebra $\mathbb{C}_{(n)}$

Let $\zeta = (\zeta_1, \ldots, \zeta_n)$ be a vector belonging to $\mathbb{C}^n$. There are two ways to examine how $\zeta$ acts on the Clifford algebra $\mathbb{C}_{(n)}$. The first, and most natural, is to consider it as an element $(\zeta_1 e_1 + \cdots + \zeta_n e_n)$ of the algebra $\mathbb{C}_{(n)}$. The other point of view is to consider $\zeta$ as a commuting $n$-tuple of multiplication operators $\zeta_j$, $j = 1, \ldots, n$, acting on $\mathbb{C}_{(n)}$ and this provides the link between holomorphic functions defined on subsets of $\mathbb{C}^n$ and monogenic functions defined on corresponding subsets of $\mathbb{R}^{n+1}$.

The complex spectrum $\sigma(i\zeta)$ of the element $i\zeta = i(\zeta_1 e_1 + \cdots + \zeta_n e_n)$ of the algebra $\mathbb{C}_{(n)}$ is

$$\sigma(i\zeta) = \{\lambda \in \mathbb{C} : (\lambda e_0 - i\zeta) \text{ does not have an inverse in } \mathbb{C}_{(n)}\}.$$

Following [73, Section 5.2], we check that

$$(\lambda e_0 + i\zeta)(\lambda e_0 - i\zeta) = \lambda^2 e_0 - i^2 \zeta^2 = (\lambda^2 - |\zeta|_{\mathbb{C}}^2) e_0,$$

where $|\zeta|_{\mathbb{C}}^2 = \sum_{j=1}^n \zeta_j^2$. So, for all $\lambda \in \mathbb{C}$ for which, $\lambda \neq \pm|\zeta|_{\mathbb{C}}$, the element $(\lambda e_0 - i\zeta)$ of the algebra $\mathbb{C}_{(n)}$ is invertible and

$$(\lambda e_0 - i\zeta)^{-1} = \frac{\lambda e_0 + i\zeta}{\lambda^2 - |\zeta|_{\mathbb{C}}^2}.$$

If $|\zeta|_{\mathbb{C}}^2 \neq 0$, the two square roots of $|\zeta|_{\mathbb{C}}^2$ are written as $\pm|\zeta|_{\mathbb{C}}$ and $|\zeta|_{\mathbb{C}} = 0$ for $|\zeta|_{\mathbb{C}}^2 = 0$. Hence $\sigma(i\zeta) = \{\pm|\zeta|_{\mathbb{C}}\}$. When $|\zeta|_{\mathbb{C}}^2 \neq 0$, the spectral projections

$$\chi_{\pm}(\zeta) = \frac{1}{2}\left(e_0 + \frac{i\zeta}{\pm|\zeta|_{\mathbb{C}}}\right) \tag{6.19}$$

are associated with each element $\pm|\zeta|_{\mathbb{C}}$ of the spectrum $\sigma(i\zeta)$ and $i\zeta$ has the spectral representation $i\zeta = |\zeta|_{\mathbb{C}}\chi_+(\zeta) + (-|\zeta|_{\mathbb{C}})\chi_-(\zeta)$. Henceforth, we use the symbol $|\zeta|_{\mathbb{C}}$ to denote the positive square root of $|\zeta|_{\mathbb{C}}^2$ in the case that $|\zeta|_{\mathbb{C}}^2 \notin (-\infty, 0]$. Because the function $\zeta \longmapsto |\zeta|_{\mathbb{C}}^2$ is homogeneous of degree two, $\zeta \longmapsto |\zeta|_{\mathbb{C}}$ is homogeneous of degree one.

On the other hand, according to the point of view mentioned in Chapter 4, the *monogenic spectrum* $\gamma(\zeta)$ of the commuting $n$-tuple $\zeta \in \mathbb{C}^n$ should be the set of singularities of the Cauchy kernel $x \longmapsto G_x(\zeta)$ in the algebra $\mathbb{C}_{(n)}$. Although $G_x(\zeta)$ is defined by formula (1.7) only for $\zeta \in \mathbb{R}^n$ and $x \neq \zeta$, a natural choice for the Cauchy kernel for $\zeta \in \mathbb{C}^n$ is to take the maximal holomorphic extension $\zeta \longmapsto G_x(\zeta)$ of formula (1.7) for $\zeta \in \mathbb{C}^n$, that is,

$$G_x(\zeta) = \frac{1}{\Sigma_n} \frac{\overline{x} + \zeta}{|x - \zeta|_{\mathbb{C}}^{n+1}}, \tag{6.20}$$

$$\text{for all } x \in \mathbb{R}^{n+1}, \text{ with } \begin{cases} |x - \zeta|_{\mathbb{C}}^2 \notin (-\infty, 0], & n \text{ even}, \\ |x - \zeta|_{\mathbb{C}}^2 \neq 0, & n \text{ odd}. \end{cases}$$

Here $|x - \zeta|_{\mathbb{C}}^2 = x_0^2 + \sum_{j=1}^n (x_j - \zeta_j)^2$ and $|x - \zeta|_{\mathbb{C}}$ is the positive square root of $|x - \zeta|_{\mathbb{C}}^2$, coinciding with the holomorphic extension of the modulus function $\xi \longmapsto |x - \xi|$, $\xi \in \mathbb{R}^n \setminus \{x\}$ in the case that $x \in \mathbb{R}^n$. There is a discontinuity in the function $(x, \zeta) \longmapsto |x - \zeta|_{\mathbb{C}}$ on the set

$$\{(x,\zeta) \in \mathbb{R}^{n+1} \times \mathbb{C}^n : |x-\zeta|_{\mathbb{C}}^2 \in (-\infty, 0]\,\}.$$

The analogous reasoning for multiplication by $x \in \mathbb{R}^{n+1}$ in the algebra $\mathbb{C}_{(n)}$ just gives us the Cauchy kernel (1.7), so that $\gamma(x) = \{x\}$, as expected.

*Remark 6.7.* If $\zeta = (\zeta_1, \ldots, \zeta_n)$ satisfies the conditions of Definition 6.2, then there exists $\theta \in [-\omega, \omega]$ and $x \in \mathbb{R}^n$ such that $\zeta = e^{i\theta}x$. To see this, write $\zeta = \alpha + i\beta$ for $\alpha, \beta \in \mathbb{R}^n$. If $|\langle \beta, \xi\rangle| \leq |\langle \alpha, \xi\rangle| \tan\omega$ for all $\xi \in \mathbb{R}^n$, then $\alpha^\perp \subset \beta^\perp$, so that $\beta \in \operatorname{span}\{\alpha\}$.

In this case, the plane wave formula (6.14) with $A = \zeta$ and equation (6.20) agree by analytic continuation, at least for $\omega \in N_\nu$ with $\nu > |\theta|$.

Given $\zeta \in \mathbb{C}^n$, if singularities of (6.20) occur at $x \in \mathbb{R}^{n+1}$, then $|x-\zeta|_{\mathbb{C}}^2 \in (-\infty, 0]$, otherwise we can simply take the positive square root of $|x-\zeta|_{\mathbb{C}}^2$ in formula (6.20) to obtain a monogenic function of $x$. To determine this set, write $\zeta = \xi + i\eta$ for $\xi, \eta \in \mathbb{R}^n$ and $x = x_0 e_0 + \boldsymbol{x}$ for $x_0 \in \mathbb{R}$ and $\boldsymbol{x} \in \mathbb{R}^n$. Then

$$\begin{aligned} |x-\zeta|_{\mathbb{C}}^2 &= x_0^2 + \sum_{j=1}^n (x_j - \zeta_j)^2 \\ &= x_0^2 + \sum_{j=1}^n (x_j - \xi_j - i\eta_j)^2 \\ &= x_0^2 + |\boldsymbol{x} - \xi|^2 - |\eta|^2 - 2i\langle \boldsymbol{x} - \xi, \eta\rangle. \end{aligned} \tag{6.21}$$

Thus, $|x-\zeta|_{\mathbb{C}}^2$ belongs to $(-\infty, 0]$ if and only if $x$ lies in the intersection of the hyperplane $\langle \boldsymbol{x} - \xi, \eta\rangle = 0$ passing through $\xi$ and with normal $\eta$, and the ball $x_0^2 + |\boldsymbol{x} - \xi|^2 \leq |\eta|^2$ centred at $\xi$ with radius $|\eta|$. If $n$ is even, then

$$\gamma(\zeta) = \{x = x_0 e_0 + \boldsymbol{x} \in \mathbb{R}^{n+1} : \langle \boldsymbol{x} - \xi, \eta\rangle = 0,\ x_0^2 + |\boldsymbol{x} - \xi|^2 \leq |\eta|^2\ \}. \tag{6.22}$$

and if $n$ is odd, then

$$\gamma(\zeta) = \{x = x_0 e_0 + \boldsymbol{x} \in \mathbb{R}^{n+1} : \langle \boldsymbol{x} - \xi, \eta\rangle = 0,\ x_0^2 + |\boldsymbol{x} - \xi|^2 = |\eta|^2\ \}. \tag{6.23}$$

In particular, if $\Im(\zeta) = 0$, then $\gamma(\zeta) = \{\zeta\} \subset \mathbb{R}^n$.

*Remark 6.8.* The distinction between $n$ odd and even is reminiscent of the support of the fundamental solution of the wave equation in even and odd dimensions.

Off $\gamma(\zeta)$, the function $x \longmapsto G_x(\zeta)$ is clearly two-sided monogenic, so the Cauchy integral formula gives

$$\tilde{f}(\zeta) = \int_{\partial\Omega} G_x(\zeta)\boldsymbol{n}(x) f(x)\, d\mu(x) \tag{6.24}$$

for a bounded open neighbourhood $\Omega$ of $\gamma(\zeta)$ with smooth oriented boundary $\partial\Omega$, outward unit normal $\boldsymbol{n}(x)$ at $x \in \partial\Omega$ and surface measure $\mu$. The function $f$ is assumed to be left monogenic in a neighbourhood of $\overline{\Omega}$, but $\zeta \longmapsto \tilde{f}(\zeta)$ is a holomorphic $\mathbb{C}_{(n)}$-valued function as the closed set $\gamma(\zeta)$ varies inside $\Omega$. Moreover, $\tilde{f}$ equals $f$ on $\Omega \cap \mathbb{R}^n$ by the usual Cauchy integral formula (7.15) of Clifford analysis, so if $f$ is, say, the monogenic extension of a polynomial $p : \mathbb{C}^n \to \mathbb{C}$ restricted to $\mathbb{R}^n$, then $\tilde{f}(\zeta) = p(\zeta)$, as expected. In this way, for each left monogenic function $f$ defined in a neighbourhood of $\gamma(\zeta)$, in a natural way we associate a holomorphic function $\tilde{f}$ defined in a neighbourhood of $\zeta$.

It is clear that if $\zeta = \xi + i\eta$ lies in a sector in $\mathbb{C}^n$, say, $|\eta| \leq |\xi| \tan\nu$, then the monogenic spectrum $\gamma(\zeta)$ lies in a corresponding sector in $\mathbb{R}^{n+1}$. Before we make the correspondence between sectors in $\mathbb{C}^n$ and $\mathbb{R}^{n+1}$ more precise, we need a simple geometric lemma.

**Lemma 6.9.** *Let $\zeta \in \mathbb{C}^n \setminus \{0\}$, $\zeta = \xi + i\eta$, $\xi, \eta \in \mathbb{R}^n$ and $0 < \theta < \pi/2$. The cone*

$$H_\theta^+ = \{x_0 e_0 + \boldsymbol{x} \in \mathbb{R}^{n+1} : x_0 > 0,\ x_0 = |\boldsymbol{x}| \tan\theta\ \} \tag{6.25}$$

*is tangential to the boundary of $\gamma(\zeta)$ if and only if*

$$|\eta|^2 = \sin^2\theta(|\xi|^2 + \tan^2\theta |P_\eta \xi|^2). \tag{6.26}$$

*Here $P_\eta : u \longmapsto \langle u, \eta\rangle \eta / |\eta|^2$, $u \in \mathbb{R}^n$, is the projection operator onto span$\{\eta\}$.*

*Proof.* Let $x_0 e_0 + \boldsymbol{x} \in H_\theta^+$ be the point where $H_\theta^+$ intersects $\gamma(\zeta)$ tangentially. The inward unit normal to $H_\theta^+$ at $x_0 e_0 + \boldsymbol{x}$ is given by

$$n = \sin\theta(\cot\theta e_0 - \boldsymbol{x}/|\boldsymbol{x}|). \tag{6.27}$$

According to equations (6.22) and (6.23), $x_0 e_0 + (\boldsymbol{x} - \xi) \in \eta^\perp$ lies along the normal to the boundary of $\gamma(\zeta)$ at $x_0 e_0 + \boldsymbol{x}$. The notation $\eta^\perp$ means all vectors orthogonal to $0.e_0 + \eta$ in $\mathbb{R}^{n+1}$.

By the tangency condition, the intersection of the tangent plane $n^\perp$ of $H_\theta^+$ with the subspace $\eta^\perp$ must be tangential to $\gamma(\zeta)$ in $\eta^\perp$. Hence, the projection $P_{\eta^\perp} n$ of $n$ onto $\eta^\perp$ must be normal to the boundary of $\gamma(\zeta)$ at $x_0 e_0 + \boldsymbol{x}$ too, so $x_0 e_0 + (\boldsymbol{x} - \xi) = \lambda P_{\eta^\perp} n$ for some $\lambda > 0$. From equation (6.27), we obtain

$$\frac{\boldsymbol{x} - \xi}{x_0} = -\tan\theta P_{\eta^\perp} \frac{\boldsymbol{x}}{|\boldsymbol{x}|}.$$

On the other hand, $x_0 = |\boldsymbol{x}| \tan\theta$, so that $\boldsymbol{x} - \xi = -\tan^2\theta P_{\eta^\perp} \boldsymbol{x}$ and

$$\boldsymbol{x} = (I + \tan^2\theta P_{\eta^\perp})^{-1} \xi.$$

Because $x_0^2 + |\boldsymbol{x} - \xi|^2 = |\eta|^2$, we obtain

$$|\eta|^2 = \tan^2\theta\big(|(I+\tan^2\theta P_{\eta^\perp})^{-1}\xi|^2 \\ + \tan^2\theta|P_{\eta^\perp}(I+\tan^2\theta P_{\eta^\perp})^{-1})\xi|^2\big).$$

A calculation shows that $(I+\tan^2\theta P_{\eta^\perp})^{-1} = P_\eta + \cos^2\theta P_{\eta^\perp}$ so that

$$\begin{aligned} |\eta|^2 &= \tan^2\theta\big(|P_\eta\xi + \cos^2\theta P_{\eta^\perp}\xi|^2 + \sin^2\theta\cos^2\theta|P_{\eta^\perp}\xi|^2\big) \\ &= \tan^2\theta\big(|\cos^2\theta\xi + \sin^2\theta P_\eta\xi|^2 + \sin^2\theta\cos^2\theta|\xi - P_\eta\xi|^2)\big) \\ &= \sin^2\theta(|\xi|^2 + \tan^2\theta|P_\eta\xi|^2). \qquad \square \end{aligned}$$

**Proposition 6.10.** *Let* $\zeta \in \mathbb{C}^n\backslash\{0\}$ *and* $0<\omega<\pi/2$. *Then* $\gamma(\zeta) \subset S_\omega(\mathbb{R}^{n+1})$ *if and only if*

$$|\zeta|^2_{\mathbb{C}} \neq (-\infty, 0] \quad \text{and} \quad |\Im(\zeta)| \leq \Re(|\zeta|_{\mathbb{C}})\tan\omega. \tag{6.28}$$

*Proof.* The statement is trivially valid if $\zeta \in \mathbb{R}^n \setminus \{0\}$, so suppose that $\Im(\zeta) \neq 0$. Then the monogenic spectrum $\gamma(\zeta)$ of $\zeta$ given by (6.22) is a subset of $S_\omega(\mathbb{R}^{n+1})$ if and only if there exists $0<\theta\leq\omega$ such that the cone

$$H_\theta^+ = \{x_0e_0 + \boldsymbol{x} \in \mathbb{R}^{n+1} : x_0 > 0,\ x_0 = |x|\tan\theta\ \}$$

is tangential to the boundary of $\gamma(\zeta)$. According to Lemma 6.9, $H_\theta^+$ is tangential to the boundary of $\gamma(\zeta)$ for all $\zeta = \xi + i\eta$ with $\xi, \eta \in \mathbb{R}^n$, satisfying equation (6.26).

To relate condition (6.26) to the inequality (6.28), suppose that $m = m_0e_0 + \boldsymbol{m}$ is the unit vector normal to $H_\theta$ such that $\boldsymbol{m}$ lies in the direction of $\eta$. Hence, $m_0 = \cot\theta|\boldsymbol{m}|$, $\tan\theta = |\boldsymbol{m}|/m_0$ and $P_\eta\xi = \langle\xi, \boldsymbol{m}\rangle\boldsymbol{m}/|\boldsymbol{m}|^2$. Then equation (6.26) becomes

$$\eta = \sin\theta(m_0^2|\xi|^2 + \langle\xi, \boldsymbol{m}\rangle^2)^{1/2}\frac{\boldsymbol{m}}{|\boldsymbol{m}|m_0}.$$

But $|m_0e_0 + \boldsymbol{m}| = 1$, so $(\cot^2\theta + 1)|\boldsymbol{m}|^2 = 1$. We have $|\boldsymbol{m}| = \sin\theta$ and

$$\eta = (m_0^2|\xi|^2 + \langle\xi, \boldsymbol{m}\rangle^2)^{1/2}\frac{\boldsymbol{m}}{m_0}. \tag{6.29}$$

As mentioned in [73, p. 67], the set of all $\zeta = \xi + i\eta$ with $\eta \neq 0$ satisfying (6.29) is equal to the set of all $\zeta = \xi + i\eta$ with $\eta \neq 0$ satisfying

$$|\zeta|^2_{\mathbb{C}} \neq (-\infty, 0] \quad \text{and} \quad \eta = \Re(|\zeta|_{\mathbb{C}})\frac{\boldsymbol{m}}{m_0}.$$

Because $|\boldsymbol{m}|/m_0 = \tan\theta \leq \tan\omega$, we obtain the desired equivalence by letting $\boldsymbol{m}$ vary over all directions in $\mathbb{R}^n$ and taking all $0<\theta\leq\omega$. $\square$

**Definition 6.11.** For each $0<\omega<\pi/2$, let $S_\omega(\mathbb{C}^n)$ denote the set of all $\zeta \in \mathbb{C}^n$ satisfying condition (6.28) and let $S_\omega^\circ(\mathbb{C}^n)$ be its interior.

**Corollary 6.12.** *Let $f : S^\circ_\omega(\mathbb{R}^{n+1}) \to \mathbb{C}_{(n)}$ be a left monogenic function and suppose that $\tilde{f} : S^\circ_\omega(\mathbb{C}^n) \to \mathbb{C}_{(n)}$ is defined by formula* (6.24) *for every $\zeta \in S^\circ_\omega(\mathbb{C}^n)$ with $\Omega$ chosen such that $\gamma(\zeta) \subset \Omega \subset \overline{\Omega} \subset S^\circ_\omega(\mathbb{R}^{n+1})$.*

*Then $\zeta \longmapsto \tilde{f}(\zeta)$, $\zeta \in S^\circ_\omega(\mathbb{C}^n)$, is a holomorphic $\mathbb{C}_{(n)}$-valued function equal to $f$ on $\mathbb{R}^n \setminus \{0\}$.*

*If $K$ is a compact subset of $S^\circ_\omega(\mathbb{C}^n)$ and $\Omega$ is a bounded open neighbourhood of $\cup_{\zeta\in K}\gamma(\zeta)$ with smooth oriented boundary $\partial\Omega$ such that $\overline{\Omega} \subset S^\circ_\omega(\mathbb{R}^{n+1})$, then there exists $C_{K,\Omega} > 0$, independent of $f$, such that*

$$\sup_{\zeta\in K} |\tilde{f}(\zeta)| \le C_{K,\Omega} \sup_{\omega\in\partial\Omega} |f(\omega)|.$$

**Corollary 6.13.** *Let $f : S^\circ_\omega(\mathbb{R}^{n+1}) \to \mathbb{C}_{(n)}$ be a left monogenic function such that the restriction $\tilde{f}$ of $f$ to $\mathbb{R}^n \setminus \{0\}$ takes values in $\mathbb{C}$. Then $\tilde{f}$ is the restriction to $\mathbb{R}^n \setminus \{0\}$ of a holomorphic function defined on $S^\circ_\omega(\mathbb{C}^n)$.*

The sectors $S_\omega(\mathbb{C}^n) \subset \mathbb{C}^n$ and $S_\omega(\mathbb{R}^{n+1}) \subset \mathbb{R}^{n+1}$ are dual to each other in the sense that the mapping

$$(x,\zeta) \longmapsto G_x(\zeta), \quad x \in \mathbb{R}^{n+1} \setminus S_\omega(\mathbb{R}^{n+1}),\ \zeta \in S^\circ_\omega(\mathbb{C}^n)$$

is two-sided monogenic in $x$ and holomorphic in $\zeta$.

The sector $S_\omega(\mathbb{C}^n)$ arose in [72] as the set of $\zeta \in \mathbb{C}^n$ for which the exponential functions

$$e_+(x,\zeta) = e^{i\langle \boldsymbol{x},\zeta\rangle} e^{-x_0|\zeta|_{\mathbb{C}}} \chi_+(\zeta), \quad x = x_0 e_0 + \boldsymbol{x}, \tag{6.30}$$

have decay at infinity for all $x \in \mathbb{R}^{n+1}$ with $\langle x, m\rangle > 0$ and all unit vectors $m = m_0 e_0 + \boldsymbol{m} \in \mathbb{R}^{n+1}$ satisfying $m_0 \ge \cot\omega|\boldsymbol{m}|$. The projection $\chi_+$ is defined by equation (6.19).

### 6.3.2 Plane Wave Decompositions

As observed in [72], the plane wave decomposition of the Cauchy kernel arises naturally in the context of spectral theory in $\mathbb{C}_{(n)}$.

Let $0 < \mu < \pi/2$. To every function bounded holomorphic function $B \in H^\infty(S^\circ_{\mu+}(\mathbb{C}))$ and $\zeta \in S^\circ_\mu(\mathbb{C}^n)$, the element $b(\zeta) = B\{i\zeta\}$ of $\mathbb{C}_{(n)}$ is defined by the functional calculus for the element $i\zeta = i(\zeta_1 e_1 + \cdots + \zeta_n e_n)$ of the algebra $\mathbb{C}_{(n)}$ by the formula

$$b(\zeta) = B\{i\zeta\} = B(|\zeta|_{\mathbb{C}})\chi_+(\zeta),$$

so that $b \in H^\infty(S^\circ_\mu(\mathbb{C}^n))$. Here we consider $B(\zeta)$ to be zero if $\Re\zeta < 0$.

The inverse Fourier transform $\Phi$ of $B$ is given by

$$\Phi(z) = \frac{1}{2\pi}\int_0^\infty B(r)e^{irz}\,dr$$

for $z \in \mathbb{C}$ with $\Im z > 0$. For the inverse Fourier transform $\phi$ of $b$ we have

$$\begin{aligned}
\phi(x) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} B(|\xi|) e_+(x,\xi)\, d\xi \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} B(|\xi|) \chi_+(\xi) e^{i\langle \boldsymbol{x},\xi\rangle - x_0|\xi|}\, d\xi \\
&= \frac{1}{2(2\pi)^n} \int_{\mathbb{R}^n} B(|\xi|) \left( e_0 + \frac{i\xi}{|\xi|} \right) e^{i\langle \boldsymbol{x},\xi\rangle - x_0|\xi|}\, d\xi \\
&= \frac{1}{2(2\pi)^n} \int_{S^{n-1}} (e_0 + i\tau) \left( \int_0^\infty B(r) e^{i\langle \boldsymbol{x},\tau\rangle r - x_0 r} r^{n-1}\, dr \right) d\tau \\
&= \frac{1}{2(2\pi i)^{n-1}} \int_{S^{n-1}} (e_0 + i\tau) \Phi^{(n-1)}(\langle \boldsymbol{x},\tau\rangle + i x_0) d\tau
\end{aligned}$$

where $\Phi^{(n-1)}$ is the $(n-1)$'st derivative of $\Phi$. Note that

$$\begin{aligned}
(e_0 + i\tau) \int_0^\infty g(r) e^{ir(x+iy)}\, dr &= \int_0^\infty g(r)(e_0 + i\tau) e^{ir(x+iy)}\, dr \\
&= \int_0^\infty g(r)(e_0 + i\tau) e^{ir(x-y\tau)}\, dr \\
&= (e_0 + i\tau) \int_0^\infty g(r) e^{ir(x-y\tau)}\, dr,
\end{aligned}$$

so that $(e_0 + i\tau)\Phi^{(n-1)}(\langle \boldsymbol{x},\tau\rangle + ix_0) = (e_0 + i\tau)\Phi^{(n-1)}(\langle \boldsymbol{x},\tau\rangle - x_0\tau)$ for all $\tau \in S^{n-1}$ and $x_0 > 0$. Hence,

$$\phi(x) = \frac{1}{2(2\pi i)^{n-1}} \int_{S^{n-1}} (e_0 + i\tau) \Phi^{(n-1)}(\langle \boldsymbol{x},\tau\rangle - x_0\tau)\, d\tau.$$

This is the plane wave decomposition for the monogenic function $\phi$. Although such functions $\phi$ are not generally bounded on $S^\circ_\mu(\mathbb{R}^{n+1})$, they do satisfy the estimate

$$|\phi(x)| \le \frac{C_\mu}{|x|^n}, \quad x \in S^\circ_\mu(\mathbb{R}^{n+1}).$$

For the monogenic function $B$ defined by $B(z) = 1$ for $\Re z > 0$ and $B(z) = 0$ for $\Re z < 0$, we have

$$\begin{aligned}
\phi(x) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e_+(x,\xi)\, d\xi \\
&= \frac{1}{\Sigma_n} \frac{\overline{x}}{|x|^{n+1}}
\end{aligned}$$

for $x = x_0 e_0 + \boldsymbol{x}$ and $x_0 > 0$ and we obtain the plane wave decomposition

$$\frac{1}{\Sigma_n} \frac{\overline{x}}{|x|^{n+1}} = \frac{(n-1)!}{2} \left( \frac{i}{2\pi} \right)^n \int_{S^{n-1}} (e_0 + is) \left( \langle \boldsymbol{x}, s\rangle - x_0 s \right)^{-n} ds.$$

of the Cauchy kernel in the case $x_0 > 0$ stated in Proposition 3.4 above. A similar argument works for $x_0 < 0$.

### 6.3.3 Bounded Monogenic Functions in a Sector

Let $0 < \omega < \pi/2$. In this section we suppose that $\omega < \nu < \pi/2$ and $f : S^\circ_\nu(\mathbb{R}^{n+1}) \to \mathbb{C}_{(n)}$ is a left monogenic function that is uniformly bounded in $S^\circ_\nu(\mathbb{R}^{n+1})$. Denote the supremum of $|f(x)|$ for $x \in S^\circ_\nu(\mathbb{R}^{n+1})$ by $\|f\|_{\nu,\infty}$.

According to Corollary 6.12, there exists a holomorphic function $\tilde{f} : S^\circ_{\nu'}(\mathbb{C}^n) \to \mathbb{C}_{(n)}$ for $0 < \nu' < \nu$ coinciding with $f$ on $\mathbb{R}^n$. By analytic continuation, $\tilde{f}$ takes its values in the subspace spanned by the range of $f$ on $\mathbb{R}^n$. We are aiming to bound the uniform norm of $\tilde{f}$ on $S^\circ_{\nu'}(\mathbb{C}^n)$ in terms of a uniform bound for $f$ on $S^\circ_\nu(\mathbb{R}^{n+1})$.

We can easily find such a bound on a smaller sector in $S_\nu(\mathbb{C}^n)$. Let $0 < \nu' < \pi/2$ and let $\tilde{S}_{\nu'}(\mathbb{C}^n)$ denote the set of all $\zeta \in \mathbb{C}^n$ such that $|\eta| \le \sin\nu'|\xi|$ for $\zeta = \xi + i\eta$ with $\xi, \eta \in \mathbb{R}^n$. Then for each $\zeta \in \tilde{S}_{\nu'}(\mathbb{C}^n)$, the closed ball of radius $|\eta|$ centred at $\xi$ is contained in $S_{\nu'}(\mathbb{R}^{n+1})$. For this to be true, necessarily $|\eta| < |\xi|$, which does not hold for all $\zeta \in S_{\nu'}(\mathbb{C}^n)$.

Now let $0 < \nu' < \theta < \nu$. If $\zeta \in \tilde{S}_{\nu'}(\mathbb{C}^n)$, then the closed ball $B_{\zeta,\delta}$ of radius $|\eta|(1+\delta)$ in $\mathbb{R}^{n+1}$ centred at $\xi$ strictly contains $\gamma(\zeta)$ and is contained in $S_\theta(\mathbb{R}^{n+1}) \setminus \{0\} \subset S^\circ_\nu(\mathbb{R}^{n+1})$, provided that $0 < \delta \le \sin\theta/\sin\nu' - 1$, it follows that

$$\tilde{f}(\zeta) = \int_{\partial B_{\zeta,\delta}} G_x(\zeta)\boldsymbol{n}(x)f(x)\,d\mu(x).$$

Then $|\tilde{f}(\zeta)| \le 2^{n/2}\|f\|_{\nu,\infty}\int_{\partial B_{\zeta,\delta}} |G_x(\zeta)|\,d\mu(x)$, so we need to estimate $|G_x(\zeta)|$ for $x \in \partial B_{\zeta,\delta}$ and $\zeta \in \tilde{S}_{\nu'}(\mathbb{C}^n)$. To this end,

$$\frac{\overline{x}+\zeta}{|x-\zeta|_{\mathbb{C}}^{n+1}} = \frac{1}{|\zeta|^n}\frac{\overline{x}/|\zeta| + \zeta/|\zeta|}{|x/|\zeta| - \zeta/|\zeta||_{\mathbb{C}}^{n+1}}.$$

Every $x \in \partial B_{\zeta,\delta}$ can be written as $x = \xi + |\eta|(1+\delta)\alpha$ for $\alpha \in S^n$ so that $x/|\zeta| = \xi/|\zeta| + \alpha|\eta|(1+\delta)/|\zeta|$. It turns out that the numbers $|\zeta|$ and $|\xi|$ are comparable for $\zeta \in S_{\nu'}(\mathbb{C}^n)$ and $|\eta|$ is dominated by a $|\zeta|$. Because $\mu(B_{\zeta,\delta}) = O(|\zeta|^n)$ as $\zeta$ goes to infinity or zero, we obtain a uniform bound on $\int_{\partial B_{\zeta,\delta}} |G_x(\zeta)|\,d\mu(x)$ for $\zeta \in \tilde{S}_{\nu'}(\mathbb{C}^n)$. The bound depends only on $\nu'$, $\delta$ and $n$.

If we want to extend this bound from the sector $\tilde{S}_{\nu'}(\mathbb{C}^n)$ to all of $S_{\nu'}(\mathbb{C}^n)$, then we need to take into account the geometry of the situation, that is, the difficulty with fitting a ball into a sector in $\mathbb{R}^{n+1}$. The obvious remedy is to replace the ball $B_{\zeta,\delta}$ by a suitable disk about $\gamma(\zeta)$. We now work out the details of the approach outlined above for this case.

Given $\zeta \in S_{\nu'}(\mathbb{C}^n)$, write $\zeta = \xi + i\eta$ for $\xi \in \mathbb{R}^n$ and $\eta \in \mathbb{R}^n$. Suppose that $\eta \ne 0$. For each $\delta > 0$, let $D_{\zeta,\delta}$ denote the right hypercylinder in $\mathbb{R}^{n+1} \equiv \mathbb{R}\times\mathbb{R}^n$ centred at $(0,\xi)$ with radius $|\eta|(1+\delta)$ and bounded by the hyperplanes

$$P_\pm = \{x \in \mathbb{R}^{n+1} : \langle x - (\xi \pm \delta\eta/2), \eta\rangle = 0\}.$$

Then the monogenic spectrum $\gamma(\zeta)$ of $\zeta$ is contained in $D_{\zeta,\delta}$ and $\mu(\partial D_{\zeta,\delta}) = O(|\eta|^n)$ as $\zeta$ goes to zero or infinity in $S_{\nu'}(\mathbb{C}^n)$, for if $v_n$ is the $n$-volume of the unit ball $\{x \in \mathbb{R}^n : |x| \le 1\}$ in $\mathbb{R}^n$ and $\Sigma_{n-1}$ is the $(n-1)$-volume of the hypersphere $\{x \in \mathbb{R}^n : |x| = 1\}$ in $\mathbb{R}^n$, then

$$\mu(\partial D_{\zeta,\delta}) = 2v_n|\eta|^n(1+\delta)^n + \Sigma_{n-1}|\eta|^{n-1}(1+\delta)^{n-1}\delta|\eta|. \tag{6.31}$$

Of course, $D_{\zeta,\delta} \to \{\xi\}$ as $\eta \to 0$.

**Lemma 6.14.** *For each $\delta > 0$, there exists $\epsilon_\delta > 0$ depending only on $\delta$ and $n$ such that $||x-\zeta|_{\mathbb{C}}| \ge \epsilon_\delta|\zeta|$ for all $x \in \partial D_{\zeta,\delta}$ and $\zeta \in \mathbb{C}^n$ with $\Im\zeta \neq 0$.*

*Proof.* Because $D_{t\zeta,\delta} = tD_{\zeta,\delta}$ for all $t > 0$ and the functions $|\cdot|$ and $|\cdot|_{\mathbb{C}}$ are homogeneous of degree one, it is enough to prove the statement for all $\zeta \in \mathbb{C}^n$ with $|\zeta| = 1$.

Let $S(\mathbb{C}^n)$ be the set of all $\zeta \in \mathbb{C}^n$ with $|\zeta| = 1$. Given $\zeta \in S(\mathbb{C}^n)$ with $\zeta = \xi + i\eta$ for $\xi \in \mathbb{R}^n$ and $\eta \in \mathbb{R}^n \setminus \{0\}$, every element of $\mathbb{R}^{n+1}$ belonging to the intersection of $\partial D_{\zeta,\delta}$ with the hyperplane $P_+$ can be parametrised as $x(\zeta, r, \alpha) = \xi + \delta\eta/2 + |\eta|(1+\delta)rT_\eta\alpha$ for $0 \le r \le 1$ and $\alpha$ belonging to the unit sphere $S^{n-1}$ in $\mathbb{R}^n$. Here $T_\eta$ is a rotation mapping $\mathbb{R}^n$ onto the $n$-dimensional subspace $\{\eta\}^\perp$ of $\mathbb{R}^{n+1}$ such that $\eta \longmapsto T_\eta$ is continuous on $S^{n-1}$. Then $x(\zeta, r, \alpha) \notin \gamma(\zeta)$, so $|x(\zeta, r, \alpha) - \zeta|_{\mathbb{C}}$ is nonzero. The function $(\zeta, r, \alpha) \longmapsto ||x(\zeta, r, \alpha) - \zeta|_{\mathbb{C}}|$ is positive and continuous on the compact set $S(\mathbb{C}^n) \times [0,1] \times S^{n-1}$ and so

$$\inf\{||x(\zeta, r, \alpha) - \zeta|_{\mathbb{C}}| : (\zeta, r, \alpha) \in S(\mathbb{C}^n) \times [0,1] \times S^{n-1}\}$$

is a strictly positive number. A similar argument works for the other face intersecting $P_-$.

On the third face, the parametrisation $x(\zeta, t, \alpha) = \xi + \delta\eta t/2 + |\eta|(1+\delta)T_\eta\alpha$ for $-1 \le t \le 1$ and $\alpha$ belonging to the unit sphere $S^{n-1}$ is used. The required number $\epsilon_\delta$ is the minimum of these three numbers. □

*Remark 6.15.* The expression $||x(\zeta, r, \alpha) - \zeta|_{\mathbb{C}}^2|^2$ is quartic in the $3n+1$ real variables $\xi$, $\eta$, $\gamma$ and $r$, if we take $\gamma$ to represent $T_\eta\alpha$. Estimating $\epsilon_\delta$ is likely to be unrewarding.

**Lemma 6.16.** *For each $\delta > 0$, let $\epsilon_\delta > 0$ be the number given in* Lemma 6.14. *Then*

$$\int_{\partial D_{\zeta,\delta}} |G_x(\zeta)|\, d\mu(x) \le \frac{3(1+\delta/2)(1+\delta)^n(2v_n + \Sigma_{n-1}\delta)}{\Sigma_n \epsilon_\delta^{n+1}}$$

*for all $\zeta \in \mathbb{C}^n$ with $\Im\zeta \neq 0$.*

*Proof.* If $x \in \partial D_{\zeta,\delta}$ and $\zeta \in \mathbb{C}^n$ with $\Im\zeta \neq 0$, then

$$\begin{aligned}
|G_x(\zeta)| &\le \frac{1}{\Sigma_n|\zeta|^n} \frac{\left|\overline{x}/|\zeta| + \zeta/|\zeta|\right|}{\left||x/|\zeta| - \zeta/|\zeta||_{\mathbb{C}}^{n+1}\right|} \\
&\le \frac{1}{\Sigma_n|\zeta|^n} \frac{|\overline{x}|/|\zeta| + 1}{||x/|\zeta| - \zeta/|\zeta||_{\mathbb{C}}|^{n+1}} \\
&\le \frac{1}{\Sigma_n|\zeta|^n} \frac{|\overline{x}|/|\zeta| + 1}{\epsilon_\delta^{n+1}}.
\end{aligned}$$

For $x \in \partial D_{\zeta,\delta}$, we have $|\overline{x}|/|\zeta| \le 2 + 3\delta/2$ and from equation (6.31), we obtain

$$\frac{\mu(\partial D_{\zeta,\delta})}{|\zeta|^n} \le 2v_n(1+\delta)^n + \Sigma_{n-1}(1+\delta)^{n-1}\delta.$$

Combining these estimates gives the stated inequality. □

**Theorem 6.17.** *Let* $0 < \nu < \pi/2$ *and let* $f : S^\circ_\nu(\mathbb{R}^{n+1}) \to \mathbb{C}_{(n)}$ *be a uniformly bounded left monogenic function. Suppose that for every* $0 < \omega < \nu$ *the holomorphic function* $\tilde{f} : S^\circ_\omega(\mathbb{C}^n) \to \mathbb{C}_{(n)}$ *is defined by formula* (6.24) *for every* $\zeta \in S^\circ_\omega(\mathbb{C}^n)$*, with the open set* $\Omega$ *chosen such that* $\gamma(\zeta) \subset \Omega \subset \overline{\Omega} \subset S^\circ_\omega(\mathbb{R}^{n+1})$*.*

*Then for every* $0 < \nu' < \nu$*, the function* $\zeta \longmapsto \tilde{f}(\zeta)$*,* $\zeta \in S^\circ_{\nu'}(\mathbb{C}^n)$*, is a well-defined holomorphic* $\mathbb{C}_{(n)}$*-valued function equal to* $f$ *on* $\mathbb{R}^n \setminus \{0\}$*. It is uniformly bounded in* $S^\circ_{\nu'}(\mathbb{C}^n)$ *by* $\|f\|_{\nu,\infty}$ *times a constant* $C$ *depending only on* $n$*,* $\nu'$ *and* $\nu$*.*

*Proof.* That $\tilde{f}$ is well-defined by formula (6.24) follows by analytic continuation.

Let $0 < \nu' < \theta < \nu$. According to Proposition 6.10, $\gamma(\zeta) \subset S_{\nu'}(\mathbb{R}^{n+1})$ for all $\zeta \in S_{\nu'}(\mathbb{C}^n)$. We can choose $\delta > 0$ such that $D_{\zeta,\delta} \subset S_\theta(\mathbb{R}^{n+1})$ for all $\zeta \in S_{\nu'}(\mathbb{C}^n)$. To see this, suppose that $\zeta = \xi + i\eta \in S_{\nu'}(\mathbb{C}^n)$ and that the cone $H^+_\theta$ given in the proof of Proposition 6.10 is tangential to one of the faces of $D_{\zeta,\delta_\zeta}$ normal to $\eta \ne 0$ and the other face is contained in $S_\theta(\mathbb{R}^{n+1})$. According to equation (6.26), $\delta_\zeta > 0$ satisfies one of the quadratic equations

$$(1+\delta_\zeta)^2|\eta|^2 = \sin^2\theta(|\xi \pm \delta_\zeta\eta/2|^2 + \tan^2\theta\langle \xi \pm \delta_\zeta\eta/2, \hat{\eta}\rangle^2)$$

with $\hat{\eta} = \eta/|\eta|$. Then $\delta = \inf\{\delta_\zeta : \zeta \in S_{\nu'}(\mathbb{C}^n),\ |\zeta| = 1\}$ is the required positive number because $\delta_{t\zeta} = \delta_\zeta$ for all $t > 0$ and $\zeta \in S_{\nu'}(\mathbb{C}^n)$ with $\eta \ne 0$. The infimum is attained when $|\xi|/|\eta|$ is bounded away from zero.

By Cauchy's Theorem in Clifford analysis, we have

$$\tilde{f}(\zeta) = \int_{\partial D_{\zeta,\delta}} G_x(\zeta)\boldsymbol{n}(x)f(x)\,d\mu(x).$$

Although the boundary $\partial D_{\zeta,\delta}$ is not smooth, the edges can be smoothed out to obtain the given representation. Then by Lemma 6.14,

$$\begin{aligned}|\tilde{f}(\zeta)| &\le 2^{n/2}\|f\|_{\nu,\infty}\int_{\partial D_{\zeta,\delta}} |G_x(\zeta)|\,d\mu(x)\\ &\le 3.2^{n/2}\|f\|_{\nu,\infty}\frac{(1+\delta/2)(1+\delta)^n(2\Sigma_n + \Sigma_{n-1}\delta)}{\Sigma_n\epsilon_\delta^{n+1}}\end{aligned}$$

for all $\zeta \in S_{\nu'}(\mathbb{C}^n)$. The positive numbers $\delta$ and $\epsilon_\delta$ depend only on $n$, $\nu'$ and $\nu' < \theta < \nu$. □

## 6.4 Bounded Holomorphic Functions in Sectors

To obtain holomorphic functions from monogenic functions, we show that the mapping $f \longmapsto \tilde{f}$ given by the Cauchy integral formula (6.24) maps the space of all left monogenic functions which are uniformly bounded on every sector $S^{\circ}_{\nu'}(\mathbb{R}^{n+1})$ in $\mathbb{R}^{n+1}$ with $0 < \nu' < \nu$ *onto* the space of all holomorphic functions which are uniformly bounded on every sector $S^{\circ}_{\nu'}(\mathbb{C}^n)$ in $\mathbb{C}^n$ with $0 < \nu' < \nu$. The sector $S^{\circ}_{\nu'}(\mathbb{R}^{n+1})$ in $\mathbb{R}^{n+1}$ is understood in the sense of Definition 6.5 and the sector $S^{\circ}_{\nu'}(\mathbb{C}^n)$ in $\mathbb{C}^n$ is given in Definition 6.11.

To show that the mapping $f \longmapsto \tilde{f}$ is onto in this sense, we construct the inverse map and show that for every $0 < \nu' < \nu'' < \nu$, there exists $C_{\nu',\nu''} > 0$, such that for every holomorphic function $\tilde{f}$ uniformly bounded over the subsector $S^{\circ}_{\nu'}(\mathbb{C}^n)$, the supremum of $|\tilde{f}|$ on $S^{\circ}_{\nu'}(\mathbb{C}^n)$ is bounded by $C_{\nu',\nu''}$ times the supremum of $|f|$ over the subsector $S^{\circ}_{\nu''}(\mathbb{R}^{n+1})$.

Here we appeal to the Fourier theory of monogenic functions exposed in [72]. As mentioned in Subsection 6.3.1, the sector $S_\nu(\mathbb{C}^n)$ arose in [72] as the set of $\zeta \in \mathbb{C}^n$ for which the exponential functions (6.30) have decay at infinity for all $x \in \mathbb{R}^{n+1}$ with $\langle x, m\rangle > 0$ and all unit vectors $m = m_0 e_0 + \boldsymbol{m} \in \mathbb{R}^{n+1}$ satisfying $m_0 \geq \cot\nu|\boldsymbol{m}|$. We exploit this property to construct a left monogenic function $f : S^{\circ}_{\nu}(\mathbb{R}^{n+1}) \to \mathbb{C}_{(n)}$ bounded on subsectors from a holomorphic function $\tilde{f} : S^{\circ}_{\nu}(\mathbb{C}^n) \to \mathbb{C}_{(n)}$ bounded on subsectors. Before doing so, we recall some facts about the sectors $S^{\circ}_{\nu}(\mathbb{C}^n)$ from [72, Section 4].

### 6.4.1 Sectors in $\mathbb{C}^n$

For each unit vector $m \in \mathbb{R}^{n+1}$ with $m = m_0 e_0 + \boldsymbol{m}$ satisfying $m_0 > 0$ and $\boldsymbol{m} \in \mathbb{R}^n$, the real $n$-dimensional manifold $m(\mathbb{C}^n)$ in $\mathbb{C}^n$ is defined as the set of all nonzero $\zeta = \xi + i\eta \in \mathbb{C}^n$ such that $\xi, \eta \in \mathbb{R}^n$ satisfy equation (6.26), or equivalently, equation (6.29).

According to the proof of Proposition 6.10, the manifold $m(\mathbb{C}^n)$ is the collection of all $\zeta = \xi + i\eta \in \mathbb{C}^n$ such that $\eta \in \mathbb{R}^n$ lies in the direction of $\boldsymbol{m}$ and the monogenic spectrum $\gamma(\zeta)$ of $\zeta$ is tangential to the cone $H^+_\theta$ given by (6.25) with $\tan\theta = m_0/|\boldsymbol{m}|$ . Manifolds associated with distinct unit vectors $m$ are disjoint. Moreover, for $0 < \omega < \pi/2$, the sector $S_\omega(\mathbb{C}^n)$ of all $\zeta \in \mathbb{C}^n$ satisfying condition (6.28) is the disjoint union of the manifolds $m(\mathbb{C}^n)$ for all unit vectors $m \in \mathbb{R}^{n+1}$ with $m = m_0 e_0 + \boldsymbol{m}$ and $m_0 \geq \cot\omega|\boldsymbol{m}|$, including $\{0\}$ as well. Its interior $S^{\circ}_{\omega}(\mathbb{C}^n)$ is the union of all such manifolds $m(\mathbb{C}^n)$ with $m_0 > \cot\omega|\boldsymbol{m}|$. For the vector $m = e_0$, we have $e_0(\mathbb{C}^n) = \mathbb{R}^n \setminus \{0\}$.

Let $m \in \mathbb{R}^{n+1}$ be a unit vector with $m = m_0 e_0 + \boldsymbol{m}$ satisfying $m_0 > 0$. For all $\zeta = \xi + i\eta \in m(\mathbb{C}^n)$ with $\xi, \eta \in \mathbb{R}^n$, the quantities $|\xi|$, $|\zeta|$, $\Re(|\zeta|_{\mathbb{C}})$ and $||\zeta|_{\mathbb{C}}|$ are equivalent:

$$\Re|\zeta|_{\mathbb{C}} \leq |\xi| \leq \frac{\Re|\zeta|_{\mathbb{C}}}{m_0} \quad \text{and} \tag{6.32}$$

$$\Re|\zeta|_{\mathbb{C}} \leq ||\zeta|_{\mathbb{C}}| \leq \frac{\Re|\zeta|_{\mathbb{C}}}{m_0} \leq |\zeta| \leq \frac{\sqrt{1+|\boldsymbol{m}|^2}\,\Re|\zeta|_{\mathbb{C}}}{m_0}, \quad \zeta \in m(\mathbb{C}^n). \tag{6.33}$$

The Jacobian $\det(\partial\zeta_j/\partial\xi_k)$ of the parametrization $\xi \longmapsto \xi + i\eta$ of $m(\mathbb{C}^n)$ given by formula (6.29) satisfies the bound

$$\left|\det\left(\frac{\partial\zeta_j}{\partial\xi_k}\right)\right| \le \frac{1}{m_0}. \tag{6.34}$$

The pullback of a differential form $\omega$ on $\mathbb{C}^n$ via the embedding of $m(\mathbb{C}^n)$ in $\mathbb{C}^n$ is denoted by the same symbol. In particular, integration with respect to the complex $n$-form $d\zeta_1 \wedge \cdots \wedge d\zeta_n$ on $m(\mathbb{C}^n)$ is equivalent to integration with respect to surface measure on $m(\mathbb{C}^n)$. The symbol $|d\zeta_1 \wedge \cdots \wedge d\zeta_n|$ is used to denote the image of the measure $\left|\det\left(\frac{\partial\zeta_j}{\partial\xi_k}\right)\right| d\xi$ with respect to the parametrization $\xi \longmapsto \xi + i\eta$ of $m(\mathbb{C}^n)$, that is, the surface measure $d\mu$ of the $n$-dimensional real manifold $m(\mathbb{C}^n)$.

Besides the exponential function $e_+(x,\zeta)$ defined by formula (6.30), the function

$$e_-(x,\zeta) = e^{i\langle \boldsymbol{x},\zeta\rangle} e^{x_0|\zeta|_{\mathbb{C}}}\chi_-(\zeta), \quad x = x_0e_0+\boldsymbol{x},\ \zeta \in \mathbb{C}^n,\ |\zeta|^2_{\mathbb{C}} \notin (-\infty, 0] \tag{6.35}$$

is also important. Then the functions $(x,\zeta) \longmapsto e_\pm(x,\zeta)$ are left monogenic in $x \in \mathbb{R}^{n+1}$ and holomorphic in $\zeta \in \mathbb{C}^n$. Let $m \in \mathbb{R}^{n+1}$ be a unit vector with $m = m_0e_0 + \boldsymbol{m}$ satisfying $m_0 \ge \cot\nu|\boldsymbol{m}|$. Then for $x = x_0e_0 + \boldsymbol{x} \in \mathbb{R}^{n+1}$ and $\zeta = \xi + i\eta \in \mathbb{C}^n$, the bounds

$$\begin{aligned} |e_+(x,\zeta)| &= e^{-\langle \boldsymbol{x},\eta\rangle - x_0\Re|\zeta|_{\mathbb{C}}}|\chi_+(\zeta)| \\ &\le \frac{\sec\nu}{\sqrt{2}} e^{-\langle x,m\rangle\Re|\zeta|_{\mathbb{C}}/m_0}, \quad \zeta \in m(\mathbb{C}^n), \end{aligned} \tag{6.36}$$

$$\begin{aligned} |e_-(x,\zeta)| &= e^{-\langle \boldsymbol{x},\eta\rangle + x_0\Re|\zeta|_{\mathbb{C}}}|\chi_-(\zeta)| \\ &\le \frac{\sec\nu}{\sqrt{2}} e^{\langle x,m\rangle\Re|\zeta|_{\mathbb{C}}/m_0}, \quad \zeta \in \overline{m}(\mathbb{C}^n), \end{aligned} \tag{6.37}$$

are valid.

The set of $x = (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$ with $x_0 > 0$ is written as $\mathbb{R}^{n+1}_+$ and for $x_0 < 0$, as $\mathbb{R}^{n+1}_-$. For $0 < \nu < \pi/2$, let

$$\begin{aligned} C^+_\nu(\mathbb{R}^{n+1}) &= \{x \in \mathbb{R}^{n+1} : x = x_0e_0 + \boldsymbol{x},\ x_0 > -\tan\nu|\boldsymbol{x}|\ \}, \\ C^-_\nu(\mathbb{R}^{n+1}) &= \{x \in \mathbb{R}^{n+1} : x = x_0e_0 + \boldsymbol{x},\ x_0 < \tan\nu|\boldsymbol{x}|\ \} = -C^+_\nu(\mathbb{R}^{n+1}). \end{aligned}$$

Then $S^\circ_\nu(\mathbb{R}^{n+1}) = C^+_\nu(\mathbb{R}^{n+1}) \cap C^-_\nu(\mathbb{R}^{n+1})$. Given a unit vector $m = m_0e_0 + \boldsymbol{m} \in \mathbb{R}^{n+1}$, let $H_m$ denote the half-space $\{x \in \mathbb{R}^{n+1} : \langle x, m\rangle > 0\}$. We also note here that

$$\begin{aligned} C^+_\nu(\mathbb{R}^{n+1}) &= \bigcup\{H_m : m \in S^n,\ m = m_0e_0 + \boldsymbol{m},\ m_0 > \cot\nu|\boldsymbol{m}|\ \}, \\ C^-_\nu(\mathbb{R}^{n+1}) &= \bigcup\{-H_m : m \in S^n,\ m = m_0e_0 + \boldsymbol{m},\ m_0 > \cot\nu|\boldsymbol{m}|\ \}. \end{aligned}$$

### 6.4.2 Fourier Analysis in Sectors

For each $\zeta \in \mathbb{C}^n$ such that $|\zeta|_{\mathbb{C}}^2 \notin (-\infty, 0]$, set $\psi(\zeta) = \chi_+(\zeta)|\zeta|_{\mathbb{C}} e^{-|\zeta|_{\mathbb{C}}}$ and $\psi_t(\zeta) = \psi(t\zeta)$ for $t > 0$. The function $\chi_+$ is defined by equation (6.19).

Suppose that $b : S_\nu^\circ(\mathbb{C}^n) \to \mathbb{C}_{(n)}$ is a uniformly bounded holomorphic function. Then for each $t > 0$, the product $b.\psi_t$ is a bounded holomorphic function with exponential decay at infinity in $S_\nu^\circ(\mathbb{C}^n)$. Hence, the Fourier transform

$$(b.\psi_t)\hat{}(\xi) = \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} b(x)\psi_t(x)\, dx$$

converges for all $\xi \in \mathbb{R}^n$.

**Lemma 6.18.** *Let $b : S_\nu^\circ(\mathbb{C}^n) \to \mathbb{C}_{(n)}$ be a uniformly bounded holomorphic function. Then for each $t > 0$, the Fourier transform $(b.\psi_t)\hat{}: \mathbb{R}^n \to \mathbb{C}_{(n)}$ has a left monogenic extension to $C_\nu^-(\mathbb{R}^{n+1})$ (denoted by the same symbol).*

*Moreover, for every $0 < \nu' < \nu$, there exists $B_{\nu'} > 0$, such that for every uniformly bounded holomorphic function $b : S_\nu^\circ(\mathbb{C}^n) \to \mathbb{C}_{(n)}$, the bound*

$$|(b.\psi_t)\hat{}(x)| \le B_{\nu'} \|b\|_\infty \frac{|x|/t}{|x|^n(1 + |x|^2/t^2)}, \quad x \in C_{\nu'}^-(\mathbb{R}^{n+1}), \tag{6.38}$$

*holds for all $t > 0$.*

*Proof.* Let $m = m_0 e_0 + \boldsymbol{m} \in \mathbb{R}^{n+1}$ satisfying $m_0 > \cot\nu|\boldsymbol{m}|$. Set

$$f_m(x) = \int_{m(\mathbb{C}^n)} e_+(-x,\zeta) b(\zeta)\psi_t(\zeta)\, d\zeta_1 \wedge \cdots \wedge d\zeta_n \tag{6.39}$$

for all $x \in \mathbb{R}^{n+1}$ such that $\langle x, m\rangle < 0$. Then $f_m$ is left monogenic in the set $\{x \in \mathbb{R}^{n+1} : \langle x, m\rangle < 0\}$ because $e_+(-x,\zeta)$ is left monogenic in $x$ and has exponential decay in $\zeta$ for $\langle x, m\rangle < 0$, according to the bound (6.36). By dominated convergence, $\lim_{x_0 \to 0-} f_{e_0}(x_0 e_0 + \boldsymbol{x}) = (b.\psi_t)\hat{}(\boldsymbol{x})$, so $f_{e_0}$ is the left monogenic extension of $(b.\psi_t)\hat{}$ to $\mathbb{R}_-^{n+1}$.

For $x \in \mathbb{R}^{n+1}$ fixed, $m \longmapsto f_m(x)$ is constant on the set of unit vectors $m = m_0 e_0 + \boldsymbol{m} \in \mathbb{R}^{n+1}$ satisfying $m_0 > \cot\nu|\boldsymbol{m}|$ and $\langle x, m\rangle < 0$, see [73, p.70]. It follows that $f_m$ is the unique extension of $(b.\psi_t)\hat{}$ from $\mathbb{R}_-^{n+1}$ to all of $\{x \in \mathbb{R}^{n+1} : \langle x, m\rangle < 0\}$. Because $C_\nu^-(\mathbb{R}^{n+1})$ is the union of these sets for all unit vectors $m = m_0 e_0 + \boldsymbol{m} \in \mathbb{R}^{n+1}$ satisfying $m_0 > \cot\nu|\boldsymbol{m}|$, the Fourier transform $(b.\psi_t)\hat{}$ has a left monogenic extension to $C_\nu^-(\mathbb{R}^{n+1})$ given by formula (6.39). Denote this left monogenic extension by $(b.\psi_t)\hat{}$ as well.

To check the bound (6.38), we note that

$$\begin{aligned}
|(b.\psi_t)\hat{}(x)| &\leq \int_{m(\mathbb{C}^n)} |e_+(-x,\zeta)|b(\zeta)\psi_t(\zeta)|\,|d\zeta_1 \wedge \cdots \wedge d\zeta_n| \\
&\leq \frac{t\|b\|_\infty}{\sqrt{2}\cos\nu} \int_{m(\mathbb{C}^n)} e^{\langle x,m\rangle \Re|\zeta|_{\mathbb{C}}/m_0} ||\zeta|_{\mathbb{C}}| e^{-t\Re|\zeta|_{\mathbb{C}}}\,|d\zeta_1 \wedge \cdots \wedge d\zeta_n| \\
&\leq \frac{t\|b\|_\infty}{\sqrt{2}m_0^2\cos\nu} \int_{\mathbb{R}^n} e^{(\langle x,m\rangle - tm_0)|\xi|}|\xi|\,d\xi \\
&= \|b\|_\infty \frac{n!\Sigma_{n-1}}{\sqrt{2}m_0^2\cos\nu} \frac{t}{(-\langle x,m\rangle + tm_0)^{n+1}}
\end{aligned}$$

for all $x \in C_\nu^-(\mathbb{R}^{n+1})$ and all unit vectors $m = m_0e_0 + \boldsymbol{m} \in \mathbb{R}^{n+1}$ satisfying $m_0 > \cot\nu|\boldsymbol{m}|$ and $\langle x, m\rangle < 0$. Here we have appealed to the bounds (6.32) and (6.33) on the manifolds $m(\mathbb{C}^n)$ and the bound (6.34) for the Jacobian of the parametrization $\xi \longmapsto \xi + i\eta$ of $m(\mathbb{C}^n)$.

Now let $0 < \nu' < \nu$. There exists $a_{\nu'} > 0$, such that for any $x \in C_{\nu'}^-(\mathbb{R}^{n+1})$, we can choose a unit vector $m = m_0e_0 + \boldsymbol{m} \in \mathbb{R}^{n+1}$ satisfying $m_0 > \cot\nu|\boldsymbol{m}|$ and $-\langle x, m\rangle \geq a_{\nu'}|x|$ with $a_{\nu'}$ independent of $x$ and $m$. Then

$$\begin{aligned}
\frac{t}{(a_{\nu'}|x| + tm_0)^{n+1}} &\leq \frac{A_{\nu'}}{|x|^n} \frac{|x|^n/t^n}{(|x|/t+1)^{n+1}} \\
&\leq \frac{A_{\nu'}}{|x|^n} \frac{|x|/t}{1+|x|^2/t^2}
\end{aligned}$$

and we obtain the bound (6.38). □

We also need some bounds on the denominator of the Cauchy kernel (6.20).

**Proposition 6.19.** *i) The bound*

$$||x - \zeta|_{\mathbb{C}}| \geq |x|(1-\kappa^{-1})$$

*holds for all $\kappa \geq 1$, $x \in \mathbb{R}^{n+1}$ and $\zeta = \xi + i\eta \in \mathbb{C}^n$ with $|x| \geq \kappa(|\xi| + |\eta|)$.*

*ii) The bound*

$$||x - \zeta|_{\mathbb{C}}| \leq 2|x|$$

*holds for all $x \in \mathbb{R}^{n+1}$ with $|x| \geq |\zeta|$.*

*iii) Let $0 < \nu < \pi/2$. The bound*

$$||\zeta - x|_{\mathbb{C}}| \geq \frac{\cos\nu}{(1+\sin^2\nu)^{1/2}}(1-\kappa^{-1})|\zeta|$$

*holds for all $\kappa > 1$, $x \in \mathbb{R}^{n+1}$ and $\zeta \in S_\nu^\circ(\mathbb{C}^n)$ such that*

$$|\zeta| \geq \kappa\left((1+2\sqrt{2})\frac{(1+\sin^2\nu)}{\cos^2\nu}\right)|x|.$$

*iv) For every $0 < \nu < \theta < \pi/2$, there exists $\epsilon_{\nu,\theta} > 0$ such that*

$$||x - \zeta|_{\mathbb{C}}| > \epsilon_{\nu,\theta}|x|$$

*for all $\zeta \in S_\nu^\circ(\mathbb{C}^n)$ and $x \in \mathbb{R}^{n+1} \setminus S_\theta^\circ(\mathbb{R}^{n+1})$.*

*v) For every* $0 < \nu < \theta < \pi/2$*, there exists* $\epsilon'_{\nu,\theta} > 0$ *such that*

$$||x - \zeta|_{\mathbb{C}}| > \epsilon'_{\nu,\theta}|\zeta|$$

*for all* $\zeta \in S^\circ_\nu(\mathbb{C}^n)$ *and* $x \in \mathbb{R}^{n+1} \setminus S^\circ_\theta(\mathbb{R}^{n+1})$.

*Proof.* i) Let $x = x_0e_0 + \boldsymbol{x} \in \mathbb{R}^{n+1}$ and $\zeta = \xi + i\eta \in \mathbb{C}^n$. Then

$$\begin{aligned} \left||x - \zeta|^2_{\mathbb{C}}\right|^2 &= \left|x_0^2 + |\boldsymbol{x} - \xi|^2 - |\eta|^2 - 2i\langle \boldsymbol{x} - \xi, \eta\rangle\right|^2 \\ &= (x_0^2 + |\boldsymbol{x} - \xi|^2 - |\eta|^2)^2 + 4\langle \boldsymbol{x} - \xi, \eta\rangle^2 \\ &\geq (x_0^2 + |\boldsymbol{x} - \xi|^2 - |\eta|^2)^2 \\ &= |x|^4(|x/|x| - \xi/|x||^2 - |\eta|^2/|x|^2)^2, \end{aligned}$$

where we have identified $\mathbb{R}^n$ with the subspace $\{0\} \times \mathbb{R}^n$ of $\mathbb{R}^{n+1}$. Now $|x/|x| - \xi/|x|| \geq 1 - |\xi|/|x|$ for $|x| \geq |\xi|$, so in this case, we have

$$\begin{aligned} \left||x - \zeta|^2_{\mathbb{C}}\right|^2 &\geq |x|^4((1 - |\xi|/|x|)^2 - |\eta|^2/|x|^2)^2 \\ &= |x|^4 \left(1 - \frac{|\xi| + |\eta|}{|x|}\right)^2 \left(1 - \frac{|\xi| - |\eta|}{|x|}\right)^2 \\ &\geq |x|^4 \left(1 - \frac{|\xi| + |\eta|}{|x|}\right)^4 \end{aligned}$$

Hence, $||x - \zeta|_{\mathbb{C}}| \geq |x|(1 - \kappa^{-1})$ for all $x \in \mathbb{R}^{n+1}$ with $|x| \geq \kappa(|\xi| + |\eta|)$ and all $\kappa \geq 1$.

ii) On the other hand, if $x \in \mathbb{R}^{n+1}$, $\zeta = \xi + i\eta \in \mathbb{C}^n$ and $|x| \geq \beta(|\xi|^2 + |\eta|^2)^{1/2}$ for $\beta \geq 1$, then

$$\begin{aligned} \left||x - \zeta|^2_{\mathbb{C}}\right|^2 &\leq |x|^4(|x/|x| - \xi/|x||^2 - |\eta|^2/|x|^2)^2 + 4|\boldsymbol{x} - \xi|^2|\eta|^2 \\ &\leq |x|^4((1 + |\xi|/|x|)^2 + |\eta|^2/|x|^2)^2 \\ &\leq |x|^4 \left(1 + \frac{2|\xi|}{|x|} + \frac{1}{\beta^2}\right)^2 \\ &\leq 2^4|x|^4. \end{aligned}$$

Hence, $||x - \zeta|_{\mathbb{C}}| \leq 2|x|$ for all $x \in \mathbb{R}^{n+1}$ with $|x| \geq (|\xi|^2 + |\eta|^2)^{1/2} = |\zeta|$.

iii) Here we are looking at the limiting behaviour of $|\zeta - x|_{\mathbb{C}}$ as $|\zeta| \to \infty$ in $S^\circ_\nu(\mathbb{C}^n)$. Let $m = m_0e_0 + \boldsymbol{m}$ be a unit vector in $\mathbb{R}^{n+1}$ such that $m_0 > \cot \nu|\boldsymbol{m}|$ and let $m(\mathbb{C}^n)$ be the real manifold defined by equation (6.29) in $\mathbb{C}^n$. Then as noted above, $S^\circ_\nu(\mathbb{C}^n)$ is the union of all such manifolds and $||\zeta|_{\mathbb{C}}|$ and $|\zeta|$ are comparable on $m(\mathbb{C}^n)$. Note that by continuity,

$$||\zeta/|\zeta| - x/|\zeta||_{\mathbb{C}}| \approx \frac{||\zeta|_{\mathbb{C}}|}{|\zeta|} \geq \frac{m_0}{(1 + |\boldsymbol{m}|^2)^{1/2}}$$

on $m(\mathbb{C}^n)$ as $|\zeta| \to \infty$. Indeed, suppose that $\zeta \in m(\mathbb{C}^n)$ and $|\zeta| = 1$. Then

$$\begin{aligned}
||\zeta - x|_{\mathbb{C}} - |\zeta|_{\mathbb{C}}| &= \frac{||\zeta - x|_{\mathbb{C}}^2 - |\zeta|_{\mathbb{C}}^2|}{||\zeta - x|_{\mathbb{C}} + |\zeta|_{\mathbb{C}}|} \\
&\le \frac{||\zeta - x|_{\mathbb{C}}^2 - |\zeta|_{\mathbb{C}}^2|}{\Re|\zeta|_{\mathbb{C}}} \\
&\le \frac{(1+|\boldsymbol{m}|^2)^{1/2}}{m_0}||\zeta - x|_{\mathbb{C}}^2 - |\zeta|_{\mathbb{C}}^2|
\end{aligned}$$

by the estimates (6.33). If $x = x_0e_0 + \boldsymbol{x}$ satisfies $|x| \le 1$ and $\zeta = \xi + i\eta \in \mathbb{C}^n$ has norm one, we get

$$\begin{aligned}
||\zeta - x|_{\mathbb{C}}^2 - |\zeta|_{\mathbb{C}}^2| &= |x_0^2 + |\boldsymbol{x} - \xi|^2 - |\xi|^2 - 2i\langle \eta, \boldsymbol{x}\rangle| \\
&\le x_0^2 + ||\xi - \boldsymbol{x}| - |\xi||.||\xi - \boldsymbol{x}| + |\xi|| + 2|\eta||\boldsymbol{x}| \\
&\le x_0^2 + |\boldsymbol{x}|.(2|\xi| + |\boldsymbol{x}|) + 2|\eta||\boldsymbol{x}| \\
&\le x_0^2 + |\boldsymbol{x}|^2 + 2(|\xi| + |\eta|)|\boldsymbol{x}| \\
&\le (1 + 2|\xi| + 2|\eta|)|x| \\
&\le (1 + 2\sqrt{2})|x|,
\end{aligned}$$

so that $||\zeta - x|_{\mathbb{C}} - |\zeta|_{\mathbb{C}}| \le (1+2\sqrt{2})(1+|\boldsymbol{m}|^2)^{1/2}|x|/m_0$.

Hence, for all nonzero $\zeta \in m(\mathbb{C}^n)$ and $x \in \mathbb{R}^{n+1}$ satisfying $|x| \le |\zeta|$, we have

$$\begin{aligned}
||\zeta/|\zeta| - x/|\zeta||_{\mathbb{C}}| &\ge \frac{||\zeta|_{\mathbb{C}}|}{|\zeta|} - (1+2\sqrt{2})\frac{(1+|\boldsymbol{m}|^2)^{1/2}}{m_0}\frac{|x|}{|\zeta|} \\
&\ge \frac{m_0}{(1+|\boldsymbol{m}|^2)^{1/2}}\left(1 - (1+2\sqrt{2})\frac{(1+|\boldsymbol{m}|^2)}{m_0^2}\frac{|x|}{|\zeta|}\right).
\end{aligned}$$

on appealing to the estimates (6.33) again. Now $|\boldsymbol{m}|^2(1+\cot^2\nu) \le 1 = m_0^2 + |\boldsymbol{m}|^2 \le m_0^2(1+\tan^2\nu)$, so $m_0 > \cos\nu$ and $|\boldsymbol{m}| < \sin\nu$ and the inequality iii) follows.

iv) Let $0 < \nu < \theta < \pi/2$. Then there exists $\epsilon_{\nu,\theta} > 0$ such that

$$||x - \zeta|_{\mathbb{C}}| > \epsilon_{\nu,\theta}$$

for all $\zeta \in S_\nu^\circ(\mathbb{C}^n)$ and unit vectors $x \in \mathbb{R}^{n+1} \setminus S_\theta^\circ(\mathbb{R}^{n+1})$. To see this, let $m = m_0e_0 + \boldsymbol{m}$ be a unit vector in $\mathbb{R}^{n+1}$ such that $m_0 > \cot\nu|\boldsymbol{m}|$ and suppose that $\kappa > 1$ and $\zeta \in m(\mathbb{C}^n)$ satisfies $|\zeta| > r = \kappa(1+2\sqrt{2})(1+\sin^2\nu)/\cos^2\nu$. Then by iii), we have

$$||x - \zeta|_{\mathbb{C}}| > \frac{\cos\nu}{(1+\sin^2\nu)^{1/2}}(1-\kappa^{-1})|\zeta| > \frac{r\cos\nu}{(1+\sin^2\nu)^{1/2}}(1-\kappa^{-1}).$$

On the other hand, according to Proposition 6.10, the function

$$(\zeta, x) \longmapsto ||x - \zeta|_{\mathbb{C}}|$$

is positive and continuous on the compact set

$$(\{|\zeta| \le r\} \cap S_\nu(\mathbb{C}^n)) \times (S^n \cap (\mathbb{R}^{n+1} \setminus S^\circ_\theta(\mathbb{R}^{n+1})),$$

so it must be bounded below there. Then $\epsilon_{\nu,\theta}$ is the minimum of these two lower bounds. It follows that $||x - \zeta|_{\mathbb{C}}| > \epsilon_{\nu,\theta}|x|$ for all $\zeta \in S^\circ_\nu(\mathbb{C}^n)$ and $x \in \mathbb{R}^{n+1} \setminus S^\circ_\theta(\mathbb{R}^{n+1})$.

v) It suffices to prove the result for $|\zeta| = 1$. According to i), if $\kappa > 1$ and $|x| > \sqrt{2}\kappa$, then $||x - \zeta|_{\mathbb{C}}| > |x|(1 - \kappa^{-1}) > \sqrt{2}(\kappa - 1)$ for all $\zeta \in \mathbb{C}^n$ with $|\zeta| = 1$. On the other hand the function

$$(\zeta, x) \longmapsto ||x - \zeta|_{\mathbb{C}}|$$

is positive and continuous on the compact set

$$(\{|\zeta| = 1\} \cap S_\nu(\mathbb{C}^n)) \times (\{|x| \le \sqrt{2}\kappa\} \cap (\mathbb{R}^{n+1} \setminus S^\circ_\theta(\mathbb{R}^{n+1})),$$

so it must be bounded below there. Then $\epsilon'_{\nu,\theta}$ is the minimum of these two lower bounds. □

**Lemma 6.20.** *The function* $(x, y) \longmapsto \hat{\psi}_t(x - y)$, $x, y \in \mathbb{R}^n$, $t > 0$, *is the restriction to* $\mathbb{R}^n \times \mathbb{R}^n$ *of a function* $(\zeta, y) \longmapsto \hat{\psi}_t(\zeta - y)$ *which is holomorphic in* $\zeta \in \mathbb{C}^n$ *and two-sided monogenic for all* $y \in \mathbb{R}^{n+1}$ *with* $y + te_0 \notin \gamma(\zeta)$. *Moreover,*

$$\hat{\psi}_t(\zeta - y) = -\frac{(2\pi)^n}{\Sigma_n}\left(\frac{te_0}{|y - \zeta + te_0|^{n+1}_{\mathbb{C}}} - (n+1)\frac{(\overline{y} + \zeta + te_0)t^2}{|y - \zeta + te_0|^{n+3}_{\mathbb{C}}}\right) \quad (6.40)$$

*Proof.* Let $k(x) = \frac{1}{\Sigma_n}\frac{\overline{x}}{|x|^{n+1}}$ for all nonzero $x \in \mathbb{R}^{n+1}$, so that $k(x - y)$, $x \ne y$, is the Cauchy kernel $G_x(y)$. We first note that

$$k(x + te_0) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e_+(x, \xi)\chi_+(\xi)e^{-t|\xi|}\, d\xi$$

for all $x \in \mathbb{R}^{n+1}_+$ and $t > 0$, so

$$\hat{\psi}_t(x - y) = \int_{\mathbb{R}^n} e_+(-(x - y), \xi)\psi_t(\xi)\, d\xi = -(2\pi)^n t\frac{\partial}{\partial t}k(y - x + te_0)$$

for all $x, y \in \mathbb{R}^n$ and $t > 0$. Holomorphically extending in $x$ and monogenically extending in $y$ gives the expression (6.40), defined for all $y \in \mathbb{R}^{n+1}$ and $\zeta \in \mathbb{C}^n$ with $y + te_0 \notin \gamma(\zeta)$. □

**Lemma 6.21.** *Let* $b : S^\circ_\nu(\mathbb{C}^n) \to \mathbb{C}_{(n)}$ *be a uniformly bounded holomorphic function. Then for each* $t > 0$ *and* $0 < \nu' < \nu$, *the Fourier transform* $(b.\psi_t^2)\hat{}$ *has a holomorphic extension to* $S^\circ_{\nu'}(\mathbb{C}^n)$ *(denoted by the same symbol) given by*

$$(b.\psi_t^2)\hat{}(\zeta) = \frac{1}{(2\pi)^n}\int_{G_\theta} \hat{\psi}_t(\zeta - y)\boldsymbol{n}(y)(b.\psi_t)\hat{}(y)\, d\mu(y), \quad \zeta \in S^\circ_{\nu'}(\mathbb{C}^n), \quad (6.41)$$

*where* $\nu' < \theta < \nu$ *and*

$$G_\theta = \{y \in \mathbb{R}^{n+1} : y = x_0 e_0 + \boldsymbol{y},\ x_0 = \tan\theta |\boldsymbol{y}|\ \}. \tag{6.42}$$

*Proof.* First, suppose that $\zeta = x \in \mathbb{R}^n$. Then $y \longmapsto \hat{\psi}_t(x-y)$ is uniformly bounded and two-sided monogenic in $\mathbb{R}^{n+1}_+$ and $(b.\psi_t)\hat{}$ is left monogenic in $C^-_\nu(\mathbb{R}^{n+1})$ by Lemma 6.18.

According to the bound (6.38) we have

$$\int_{G_\theta} |(b.\psi_t)\hat{}(y)|\, d\mu(y) \le C\|b\|_\infty \int_0^\infty \frac{r/t}{(1+\sec^2\theta r^2/t^2)} \frac{dr}{r} < \infty,$$

so that the integral (6.41) converges for all $0 \le \theta < \nu$. The convolution formula

$$(b.\psi_t^2)\hat{}(x) = \frac{1}{(2\pi)^n} (\hat{\psi}_t * \widehat{b.\psi_t})(x), \quad x \in \mathbb{R}^n,$$

and Cauchy's theorem in Clifford analysis [19, Corollary 9.3] now gives the representation (6.41). Then we can holomorphically extend the integral and the equality (6.41) for all $\zeta$ in $S^\circ_{\nu'}(\mathbb{C}^n)$.

We need to check that for $\zeta$ in a fixed compact subset of $S^\circ_{\nu'}(\mathbb{C}^n)$, the function $y \longmapsto \hat{\psi}_t(\zeta - y)$ is uniformly bounded for all $y \in G_\theta$. For $y \in G_\theta$ and $|y|$ large, this follows from formula (6.40) and the estimate Proposition 6.19 iv). For $|y|$ small, we note that for each $t > 0$, the positive continuous function $(y,\zeta) \longmapsto |y - \zeta + te_0|_{\mathbb{C}}$ is necessarily bounded below on compact subsets of $G_\theta \times S^\circ_{\nu'}(\mathbb{C}^n)$. □

**Lemma 6.22.** *Let* $b : S^\circ_\nu(\mathbb{C}^n) \to \mathbb{C}_{(n)}$ *be a uniformly bounded holomorphic function. Then for each* $t > 0$ *and* $0 < \nu' < \nu$*, the restriction of* $b.\psi_t^2$ *to* $\mathbb{R}^n$ *has a left monogenic extension* $b._\ell\psi_t^2$ *to* $S^\circ_{\nu'}(\mathbb{R}^{n+1})$*. Moreover,*

$$|(b._\ell\psi_t^2)(x)| = \begin{cases} O((t|x|)^{1/2}), & \text{as } t|x| \to 0 \text{ in } S^\circ_{\nu'}(\mathbb{R}^{n+1}),\\ O((t|x|)^{-n}), & \text{as } t|x| \to \infty \text{ in } S^\circ_{\nu'}(\mathbb{R}^{n+1}). \end{cases} \tag{6.43}$$

*The order of convergence is uniform for* $\|b\|_\infty \le 1$.

*Proof.* For each $x \in S^\circ_{\nu'}(\mathbb{R}^{n+1}) = C^+_{\nu'}(\mathbb{R}^{n+1}) \cap C^-_{\nu'}(\mathbb{R}^{n+1})$ set

$$(b._\ell\psi_t^2)_+(x) = \frac{1}{(2\pi)^n} \int_{m(\mathbb{C}^n)} e_+(x,\zeta)(b.\psi_t^2)\hat{}(\zeta)\, d\zeta_1 \wedge \cdots \wedge d\zeta_n, \tag{6.44}$$

$$(b._\ell\psi_t^2)_-(x) = \frac{1}{(2\pi)^n} \int_{m'(\mathbb{C}^n)} e_-(x,\zeta)(b.\psi_t^2)\hat{}(\zeta)\, d\zeta_1 \wedge \cdots \wedge d\zeta_n, \tag{6.45}$$

$$(b._\ell\psi_t^2) = (b._\ell\psi_t^2)_+ + (b._\ell\psi_t^2)_- \ . \tag{6.46}$$

Here $m = m_0 e_0 + \boldsymbol{m} \in \mathbb{R}^{n+1}$ and $m' = m'_0 e_0 + \boldsymbol{m}' \in \mathbb{R}^{n+1}$ are two unit vectors satisfying $m_0 > \cot\nu' |\boldsymbol{m}|$ and $m'_0 > \cot\nu' |\boldsymbol{m}'|$ and $\langle x, m\rangle > 0$ and $\langle x, m'\rangle < 0$. As mentioned in Section 4.1 above, the sector $S^\circ_{\nu'}(\mathbb{C}^n)$ is the disjoint union of all manifolds $l(\mathbb{C}^n)$ for all unit vectors $l \in \mathbb{R}^{n+1}$ satisfying $l_0 > \cot \nu' |\boldsymbol{l}|$. According to Lemma 6.21, $(b.\psi_t^2)\hat{}$ is defined on $S^\circ_{\nu'}(\mathbb{C}^n)$. Once we establish the absolute convergence of the integrals (6.44) and (6.45), the argument of [73, p. 70] shows that the right hand sides of equations (6.44) and (6.45) are independent of the choice of the unit vectors $m, m'$, so that $(b._\ell\psi_t^2)_\pm$ and, hence, $(b._\ell\psi_t^2)$ are well defined functions on $S^\circ_{\nu'}(\mathbb{R}^{n+1})$. Because the functions $e_\pm(\cdot, \zeta)$ are left monogenic for each $\zeta \in \mathbb{C}^n$, the functions $(b._\ell\psi_t^2)_\pm$ and $(b._\ell\psi_t^2)$ are left monogenic functions defined on $S^\circ_{\nu'}(\mathbb{R}^{n+1})$.

We first see that the right hand side of equation (6.44) converges for all $x \in C^+_{\nu'}(\mathbb{R}^{n+1})$. The integral $\int_{m(\mathbb{C}^n)} e_+(x,\zeta)(b.\psi_t^2)\hat{}(\zeta)\, d\zeta_1 \wedge \cdots \wedge d\zeta_n$ is estimated for $0 < \nu' < \theta < \nu$ and $x \in C^+_{\nu'}(\mathbb{R}^{n+1})$ by

$$
\begin{aligned}
&\int_{m(\mathbb{C}^n)} |e_+(x,\zeta)|.|(b.\psi_t^2)\hat{}(\zeta)||d\zeta_1 \wedge \cdots \wedge d\zeta_n| \\
&\le \int_{m(\mathbb{C}^n)} \int_{G_\theta} |e_+(x,\zeta)|.|\hat{\psi}_t(\zeta - y)||(b.\psi_t)\hat{}(y)|\, d\mu(y) |d\zeta_1 \wedge \cdots \wedge d\zeta_n| \\
&= \int_{m(\mathbb{C}^n)} \int_{G_\theta} |e_+(x,\zeta)|.|\hat{\psi}_{t/|\zeta|}(\zeta/|\zeta| - y/|\zeta|)| \\
&\qquad\qquad \times |(b.\psi_t)\hat{}(y)|\, d\mu(y) \frac{|d\zeta_1 \wedge \cdots \wedge d\zeta_n|}{|\zeta|^n} \\
&= \int_{m(\mathbb{C}^n)} \int_{G_\theta} |e_+(tx,\zeta)|.|\hat{\psi}_{1/|\zeta|}(\zeta/|\zeta| - y/(t|\zeta|))| \\
&\qquad\qquad \times |(b.\psi_t)\hat{}(y)|\, d\mu(y) \frac{|d\zeta_1 \wedge \cdots \wedge d\zeta_n|}{|\zeta|^n} \\
&\le C'_\nu \|b\|_\infty \int_{m(\mathbb{C}^n)} \int_{G_\theta} |e_+(tx,\zeta)|.|\hat{\psi}_{1/|\zeta|}(\zeta/|\zeta| - y/|\zeta|)| \\
&\qquad\qquad \times \frac{|y|}{1+|y|^2} \frac{d\mu(y)}{|y|^n} \frac{|d\zeta_1 \wedge \cdots \wedge d\zeta_n|}{|\zeta|^n}.
\end{aligned}
$$

Here we have used the explicit formula (6.40), estimate (6.38) and the fact that both measures

$$\frac{|d\zeta_1 \wedge \cdots \wedge d\zeta_n|}{|\zeta|^n}, \quad \frac{d\mu(y)}{|y|^n}$$

are invariant under dilations. It remains to estimate the function

$$(y,\zeta) \longmapsto |\hat{\psi}_{1/|\zeta|}(\zeta/|\zeta| - y/|\zeta|)|, \quad y \in G_\theta,\ \zeta \in m(\mathbb{C}^n),$$

which is independent of $t > 0$.

We now show that $|\hat{\psi}_{1/|\zeta|}(\zeta/|\zeta| - y/|\zeta|)| = O(|\zeta|^{-1})$, uniformly in $y \in G_\theta$ as $|\zeta| \to \infty$ for $\zeta \in S_{\nu'}(\mathbb{C}^n)$, and $|\hat{\psi}_{1/|\zeta|}(\zeta/|\zeta| - y/|\zeta|)| = O(|\zeta|^n)$, uniformly in $y \in G_\theta$ as $|\zeta| \to 0$ for $\zeta \in S_{\nu'}(\mathbb{C}^n)$.

Let $\kappa > 1$. Then by Proposition 6.19 i), for $\zeta' \in m(\mathbb{C}^n)$ with $|\zeta'| = 1$ and all $x \in \mathbb{R}^{n+1}$ with $|x| > \sqrt{2}\kappa$, we have $||x - \zeta'|_{\mathbb{C}}| > |x|(1 - \kappa^{-1})$. Set

$x = (y + e_0)/|\zeta|$ for $y \in G_\theta$ and $\zeta \in m(\mathbb{C}^n)$. Comparison with formula (6.40) shows that

$$|\hat{\psi}_{1/|\zeta|}(\zeta/|\zeta| - y/|\zeta|)| \le \frac{(2\pi)^n}{\Sigma_n 2^{(n+1)/2}|\zeta|}\left(\frac{1}{(\kappa-1)^{n+1}} + \frac{\sqrt{2}\kappa(n+1)/|\zeta|}{(\kappa-1)^{n+3}}\right) \tag{6.47}$$

if $|x| > \sqrt{2}\kappa$.

According to Proposition 6.19 v), there exists $\epsilon'_{\nu,\theta}$ such that $|x - \zeta'|_{\mathbb{C}} > \epsilon'_{\nu',\theta}$ for all $\zeta' \in m(\mathbb{C}^n)$ with $|\zeta'| = 1$ and all $x \in \mathbb{R}^{n+1} \setminus S^\circ_\theta(\mathbb{R}^{n+1})$, so if $x = (y + e_0)/|\zeta|$ for $y \in G_\theta$ and $|x| \le \sqrt{2}\kappa$, we have

$$|\hat{\psi}_{1/|\zeta|}(\zeta/|\zeta| - y/|\zeta|)| \le \frac{(2\pi)^n}{\Sigma_n|\zeta|}\left(\frac{1}{(\epsilon'_{\nu',\theta})^{n+1}} + (n+1)\frac{(\sqrt{2}\kappa+1)/|\zeta|}{(\epsilon'_{\nu',\theta})^{n+3}}\right). \tag{6.48}$$

Hence, there exists $C > 0$ such that

$$|\hat{\psi}_{1/|\zeta|}(\zeta/|\zeta| - y/|\zeta|)| \le \frac{C}{|\zeta|}$$

for all $y \in G_\theta$ and $\zeta \in m(\mathbb{C}^n)$ with $|\zeta| \ge 1$.

On the other hand, $\hat{\psi}_{1/|\zeta|}(\zeta/|\zeta| - y/|\zeta|) = |\zeta|^n \hat{\psi}_1(\zeta - y)$. Let $x = y + e_0$. According to Proposition 6.19 iv), there exists $\epsilon_{\nu,\theta}$ such that $|x - \zeta|_{\mathbb{C}} > \epsilon_{\nu',\theta}$ for all $\zeta \in m(\mathbb{C}^n)$. For $a > 0$ and $x \in G_\theta$ such that $|x| \le a$, we have

$$|\hat{\psi}_1(\zeta - y)| \le \frac{(2\pi)^n}{\Sigma_n}\left(\frac{e_0}{(\epsilon_{\nu',\theta})^{n+1}} + (n+1)\frac{1+a}{(\epsilon_{\nu',\theta})^{n+3}}\right)$$

for all $\zeta \in \{|\zeta| \le 1\} \cap m(\mathbb{C}^n)$. Now let $\kappa > 1$ and suppose that $a = \sqrt{2}\kappa$. Then by Proposition 6.19 i), for $\zeta \in m(\mathbb{C}^n)$ with $|\zeta| \le 1$ and all $x \in \mathbb{R}^{n+1}$ with $|x| > \sqrt{2}\kappa$, we have $||x - \zeta'|_{\mathbb{C}}| > |x|(1 - \kappa^{-1})$, so that

$$|\hat{\psi}_1(\zeta - y)| \le \frac{(2\pi)^n}{\Sigma_n 2^{(n+1)/2}}\left(\frac{e_0}{(\kappa-1)^{n+1}} + \frac{\sqrt{2}\kappa(n+1)}{(\kappa-1)^{n+3}}\right).$$

We have shown that $|\hat{\psi}_{1/|\zeta|}(\zeta/|\zeta| - y/|\zeta|)| = O(|\zeta|^{-1})$, uniformly in $y \in G_\theta$ as $|\zeta| \to \infty$ for $\zeta \in S_{\nu'}(\mathbb{C}^n)$, and $|\hat{\psi}_{1/|\zeta|}(\zeta/|\zeta| - y/|\zeta|)| = O(|\zeta|^n)$, uniformly in $y \in G_\theta$ as $|\zeta| \to 0$ for $\zeta \in S_{\nu'}(\mathbb{C}^n)$. In particular,

$$\int_{m(\mathbb{C}^n)} \sup_{y \in G_\theta} |\hat{\psi}_{1/|\zeta|}(\zeta/|\zeta| - y/|\zeta|)| \frac{|d\zeta_1 \wedge \cdots \wedge d\zeta_n|}{|\zeta|^n} < \infty. \tag{6.49}$$

Hence, $|(b._\ell \psi_t^2)_+(x)|$ is bounded by

$$
\begin{aligned}
&C''_\nu \|b\|_\infty \int_{m(\mathbb{C}^n)} |e_+(tx,\zeta)| \\
&\qquad \times \sup_{y\in G_\theta} |\hat{\psi}_{1/|\zeta|}(\zeta/|\zeta| - y/|\zeta|)| \, \frac{|d\zeta_1\wedge\cdots\wedge d\zeta_n|}{|\zeta|^n} \\
&\le C''_\nu \|b\|_\infty \sup_{y\in G_\theta,\zeta\in m(\mathbb{C}^n)} \{|\zeta|^{-n} |\hat{\psi}_{1/|\zeta|}(\zeta/|\zeta| - y/|\zeta|)|\} \\
&\qquad \times \int_{m(\mathbb{C}^n)} |e_+(tx,\zeta)| \, |d\zeta_1\wedge\cdots\wedge d\zeta_n| \\
&\le C''_\nu \|b\|_\infty \sup_{y\in G_\theta,\zeta\in m(\mathbb{C}^n)} \{|\zeta|^{-n} |\hat{\psi}_{1/|\zeta|}(\zeta/|\zeta| - y/|\zeta|)|\} \int_{\mathbb{R}^n} e^{-t\langle x,m\rangle|\xi|} \, d\xi \\
&\le \frac{C'''_\nu \|b\|_\infty}{t^n \langle x,m\rangle^n} \sup_{y\in G_\theta,\zeta\in m(\mathbb{C}^n)} \{|\zeta|^{-n} |\hat{\psi}_{1/|\zeta|}(\zeta/|\zeta| - y/|\zeta|)|\}.
\end{aligned}
$$

Hence we obtain decay as $t|x| \to \infty$ in $S^\circ_{\nu'}(\mathbb{R}^{n+1})$. To see this, let $\nu''$ satisfy $\nu' < \nu'' < \theta$. Then for all $x \in S^\circ_{\nu'}(\mathbb{R}^{n+1})$ we can find a unit vector $m = m_0 e_0 + \boldsymbol{m} \in \mathbb{R}^{n+1}$ satisfying $m_0 > \cot\nu''|\boldsymbol{m}|$ such that $\langle x,m\rangle > a|x|$, where $a$ depends only on $\nu'$ and $\nu''$.

We now estimate the convergence of $(b._\ell \psi_t^2)_+(x)$ as $tx \to 0$ in $S^\circ_{\nu'}(\mathbb{R}^{n+1})$. Set

$$
V_\theta(\zeta) = \sup_{y\in G_\theta} |\hat{\psi}_{1/|\zeta|}(\zeta/|\zeta| - y/|\zeta|)|, \quad \zeta \in S_{\nu'}(\mathbb{C}^n).
$$

Then as shown above, $V_\theta(\zeta) = O(|\zeta|^{-1})$ as $|\zeta| \to \infty$ for $\zeta \in S_{\nu'}(\mathbb{C}^n)$, and $V_\theta(\zeta) = O(|\zeta|^n)$ as $|\zeta| \to 0$ for $\zeta \in S_{\nu'}(\mathbb{C}^n)$.

Because

$$
(b._\ell \psi_t^2)_+(0) = \frac{1}{(2\pi)^n} \int_{m(\mathbb{C}^n)} \chi_+(\zeta)(b.\psi_t^2)\hat{\ }(\zeta) \, d\zeta_1\wedge\cdots\wedge d\zeta_n,
$$

the number $|(b._\ell\psi_t^2)_+(x) - (b._\ell\psi_t^2)_+(0)|$ is estimated by

$$
C''_\nu \|b\|_\infty \int_{m(\mathbb{C}^n)} |e_+(tx,\zeta) - \chi_+(\zeta)| V_\theta(\zeta) \, \frac{|d\zeta_1\wedge\cdots\wedge d\zeta_n|}{|\zeta|^n}.
$$

For notational simplicity, replace $tx$ by $x$. Then for $|x| \le 1$, we have

$$
\begin{aligned}
&\int_{m(\mathbb{C}^n)\cap\{|\zeta|\ge|x|^{-1/2}\}} |e_+(x,\zeta) - \chi_+(\zeta)| V_\theta(\zeta) \, \frac{|d\zeta_1\wedge\cdots\wedge d\zeta_n|}{|\zeta|^n} \\
&\qquad \le C \int_{m(\mathbb{C}^n)\cap\{|\zeta|\ge|x|^{-1/2}\}} V_\theta(\zeta) \, \frac{|d\zeta_1\wedge\cdots\wedge d\zeta_n|}{|\zeta|^n} \\
&\qquad \le C' \int_{m(\mathbb{C}^n)\cap\{|\zeta|\ge|x|^{-1/2}\}} \frac{|d\zeta_1\wedge\cdots\wedge d\zeta_n|}{|\zeta|^{n+1}} \\
&\qquad \le C'' \int_{|x|^{-1/2}}^\infty \frac{dr}{r^2} = C'''|x|^{1/2}.
\end{aligned}
$$

On the other hand,

$$\begin{aligned}\int_{m(\mathbb{C}^n)\cap\{|\zeta|\le|x|^{-1/2}\}} &|e_+(x,\zeta)-\chi_+(\zeta)|V_\theta(\zeta)\,\frac{|d\zeta_1\wedge\cdots\wedge d\zeta_n|}{|\zeta|^n}\\ &\le |x|^{1/2}\sup_{|\zeta|\le|x|^{-1/2},\zeta\in m(\mathbb{C}^n)}\frac{|e_+(x,\zeta)-\chi_+(\zeta)|}{|x|^{1/2}}\\ &\qquad\times\int_{m(\mathbb{C}^n)} V_\theta(\zeta)\,\frac{|d\zeta_1\wedge\cdots\wedge d\zeta_n|}{|\zeta|^n}.\end{aligned}$$

Because $\langle x, m\rangle$ is comparable to $|x|$ for $x\in S^\circ_{\nu'}(\mathbb{R}^{n+1})$ and $|\zeta|$ is comparable to $\Re|\zeta|_{\mathbb{C}}$ for $\zeta\in m(\mathbb{C}^n)$, there exists $a>0$ and $C>0$ such that

$$\begin{aligned}\sup_{|\zeta|\le|x|^{-1/2},\zeta\in m(\mathbb{C}^n)}\frac{|e_+(x,\zeta)-\chi_+(\zeta)|}{|x|^{1/2}} &\le C\sup_{|\zeta|\le|x|^{-1/2},\zeta\in m(\mathbb{C}^n)}\frac{1-e^{-a|x||\zeta|}}{|x|^{1/2}}\\ &= C\frac{1-e^{-a|x|^{1/2}}}{|x|^{1/2}}\le aC.\end{aligned}$$

It follows that $|(b._\ell\psi_t^2)_+(x)-(b._\ell\psi_t^2)_+(0)|$ is $O((t|x|)^{1/2})$ as $t|x|\to 0$ in $S_{\nu'}(\mathbb{R}^{n+1})$.

If in the integral representation (6.44), $x\in\mathbb{R}^n$ and $m_0\to e_0$, $m_0'\to e_0$, then we obtain

$$\begin{aligned}b._\ell\psi_t^2(x) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle}\chi_+(\xi)(b.\psi_t^2)\hat{}\,(\xi)\,d\xi\\ &\quad+\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle}\chi_-(\xi)(b.\psi_t^2)\hat{}\,(\xi)\,d\xi=(b.\psi_t^2)(x),\end{aligned}$$

as expected, because $\chi_+(\xi)+\chi_-(\xi)=1$ for all $\xi\in\mathbb{R}^n\setminus\{0\}$.

Now write the second integral in (6.44) as $(b._\ell\psi_t^2)_-$. Then

$$(b._\ell\psi_t^2)_+(0)+(b._\ell\psi_t^2)_-(0)=(b.\psi_t^2)(0)=0,$$

and as above, we obtain $|(b._\ell\psi_t^2)_-(x)-(b._\ell\psi_t^2)_-(0)|$ is $O((t|x|)^{1/2})$ as $t|x|\to 0$ in $S_{\nu'}(\mathbb{R}^{n+1})$ and $|(b._\ell\psi_t^2)_-(x)|$ is $O((t|x|)^{-n})$ as $t|x|\to\infty$ in $S_{\nu'}(\mathbb{R}^{n+1})$.

Because, $(b._\ell\psi_t^2)(x)=(b._\ell\psi_t^2)_+(x)+(b._\ell\psi_t^2)_-(x)$ for all $x\in S_{\nu'}(\mathbb{R}^{n+1})$, it follows that $|(b._\ell\psi_t^2)(x)|$ is $O((t|x|)^{1/2})$ as $t|x|\to 0$ in $S_{\nu'}(\mathbb{R}^{n+1})$ and $|(b._\ell\psi_t^2)(x)|$ is $O((t|x|)^{-n})$ as $t|x|\to\infty$ in $S_{\nu'}(\mathbb{R}^{n+1})$. All constants we have calculated are proportional to the supremum norm $\|b\|_\infty$ of $b$ on $S_\nu(\mathbb{C}^n)$, so the convergence is uniform for $\|b\|_\infty\le 1$. □

Because $b._\ell\psi_t^2$ has decay $O(t|x|)$ as $t|x|\to 0$ in $\mathbb{R}^n$ and $b._\ell\psi_t^2$ has exponential decay as $t|x|\to\infty$ in $\mathbb{R}^n$, the estimate (6.43) may not be the best possible.

**Theorem 6.23.** *Let $0 < \nu < \pi/2$. If $b : S^\circ_\nu(\mathbb{C}^n) \to \mathbb{C}_{(n)}$ is a uniformly bounded holomorphic function, then there exists a left monogenic function function $f : S^\circ_\nu(\mathbb{R}^{n+1}) \to \mathbb{C}_{(n)}$, uniformly bounded on subsectors of $S^\circ_\nu(\mathbb{R}^{n+1})$, such that $b = \tilde{f}$ is represented by the Cauchy integral formula* (6.24)*. Moreover, $f$ is the left monogenic extension to $S^\circ_\nu(\mathbb{R}^{n+1})$ of the restriction of $b$ to $\mathbb{R}^n \setminus \{0\}$.*

*Proof.* For each $\zeta \in \mathbb{C}^n$ such that $|\zeta|^2_{\mathbb{C}} \notin (-\infty, 0]$, set $\phi(\zeta) = \chi_-(\zeta)|\zeta|_{\mathbb{C}} e^{-|\zeta|_{\mathbb{C}}}$ and $\phi_t(\zeta) = \phi(t\zeta)$ for $t > 0$. A similar argument to the proof of Lemma 6.22 shows that we may substitute $\phi$ for $\psi$ and the same statement holds. The bound (6.38) now holds for $x \in C^+_{\nu'}$ when $\phi$ is substituted for $\psi$, because in formula (6.39) for $(b\phi_t)\hat{}$, the expression $e_+(-x, \zeta)$ is replaced by $e_-(-x, \zeta)$ for all $\langle x, m\rangle > 0$ and $\zeta \in m(\mathbb{C}^n)$. Set

$$f(x) = 4\int_0^\infty \left((b.{}_\ell\psi_t^2)(x) + (b.{}_\ell\phi_t^2)(x)\right)\frac{dt}{t}. \tag{6.50}$$

Then according to Lemma 6.22 and its analogue when $\psi$ is replaced by $\phi$, the decay estimates (6.43) ensure that the integral converges absolutely for all $x \in S^\circ_\nu(\mathbb{R}^{n+1})$ and $f$ is a left monogenic function in $S^\circ_\nu(\mathbb{R}^{n+1})$, because $b.{}_\ell\psi_t^2$ and $b.{}_\ell\phi_t^2$ are both left monogenic functions there. If $x \in \mathbb{R}^n \setminus \{0\}$, then we have

$$\begin{aligned} f(x) &= 4b(x)\int_0^\infty \left(\psi_t^2(x) + \phi_t^2(x)\right)\frac{dt}{t} \\ &= 4b(x)\int_0^\infty \left(\chi_+(x) + \chi_-(x)\right)(t|x|)^2 e^{-2t|x|}\frac{dt}{t} \\ &= 4b(x)\int_0^\infty te^{-2t}dt \\ &= b(x). \end{aligned}$$

Here we have used the facts that $\chi_\pm(x)$ are projections and $\chi_+(x)+\chi_-(x) = 1$ for each nonzero $x \in \mathbb{R}^n$. The uniformity of the decay estimates (6.43) for $\|b\|_\infty \le 1$ ensure that for every $0 < \nu' < \nu$, there exists $C_{\nu'} > 0$ independent of $b$, such that the bound $|f(x)| \le C_{\nu'}\|b\|_\infty$ holds for all $x \in S^\circ_{\nu'}(\mathbb{R}^{n+1})$. This complete the proof of Theorem 6.23. □

*Remark 6.24.* Formula (6.50) represents the inverse mapping of $f \longmapsto \tilde{f}$ defined by the Cauchy integral formula (6.24) with $b = \tilde{f}$. Another way of looking at (6.50) is to set

$$\Psi(z) = 2(\chi_{\Re(z)>0} z e^{-z} + \chi_{\Re(z)<0} z e^{z}), \quad z \in S^\circ_\nu(\mathbb{C}).$$

Then $\int_0^\infty \Psi^2(t)t^{-1}\,dt = \int_0^\infty \Psi^2(-t)t^{-1}\,dt = 1$. As noted in Section 6.3.1, the spectral projections $\chi_\pm(\zeta)$ are associated with multiplication by $i\zeta$ on the Clifford algebra $\mathbb{C}_{(n)}$. Let $\tilde{\Psi}$ be the function of $\zeta \in \mathbb{C}^n$ defined by the functional calculus

$$\tilde{\Psi}(\zeta) = \Psi\{i\zeta\} = \Psi(|\zeta|_{\mathbb{C}})\chi_+(\zeta) + \Psi(-|\zeta|_{\mathbb{C}})\chi_-(\zeta)$$

for multiplication by $i\zeta$ and set $\tilde{\Psi}_t(\zeta) = \tilde{\Psi}(t\zeta)$ for $t > 0$. Then formula (6.50) may be written as

$$f = \int_0^\infty b._\ell \tilde{\Psi}_t^2 \, \frac{dt}{t}.$$

We have shown that by multiplying a bounded holomorphic function $b$ in a sector by a suitable holomorphic function $\tilde{\Psi}_t$ with decay at zero and infinity, the product $b.\tilde{\Psi}_t^2$ may be extended from $\mathbb{R}^n \setminus \{0\}$ to a left monogenic function $b._\ell\tilde{\Psi}_t^2$ for each $t > 0$ (cf. [73, p. 65]). Then we can integrate out the scaling factor $t$.

It is plausible that similar techniques could be applied to domains other than sectors and other Hardy spaces of functions by using decompositions of functions other than in terms of trigonometric functions. In this context the work of M. Mitrea [78] on Clifford wavelets is especially relevant.

## 6.5 The Monogenic Calculus for $n$ Sectorial Operators

Let $0 < \omega < \pi/2$ and let $\boldsymbol{A}$ be an $n$-tuple of operators uniformly of type $\omega$ in the Hilbert space $\mathcal{H}$. The elements of $\boldsymbol{A}$ do not necessarily commute with each other, so there is some ambiguity deciding what is actually a functional calculus for $\boldsymbol{A}$. In the case that $\boldsymbol{A}$ consisted of bounded linear operators, we decided in Chapter 4 that for any polynomial $p : \mathbb{C} \to \mathbb{C}$ and any vector $\xi \in \mathbb{R}^n$, the function $x \longmapsto p(\langle x, \xi\rangle)$, $x \in \mathbb{R}^n$, should be associated with the bounded linear operator $p(\langle \boldsymbol{A}, \xi\rangle)$ defined in the obvious way as a polynomial $p$ of the single operator $\langle \boldsymbol{A}, \xi\rangle$.

In the present context, we have domain difficulties in forming polynomials of the unbounded operator $\langle \boldsymbol{A}, \xi\rangle$. We can circumvent these by noticing that the unbounded linear operator $\langle \boldsymbol{A}, \xi\rangle$ and the resolvent operators $(\lambda I - \langle \boldsymbol{A}, \xi\rangle)^{-1}$ commute and

$$\langle \boldsymbol{A}, \xi\rangle(\lambda I - \langle \boldsymbol{A}, \xi\rangle)^{-1} = \lambda(\lambda I - \langle \boldsymbol{A}, \xi\rangle)^{-1} - I$$

is bounded. Thus, if $p$ is a polynomial of degree $m = 1, 2, \ldots$, then the function $x \longmapsto p(\langle x, \xi\rangle)(\lambda I - \langle x, \xi\rangle)^{-m}$, $x \in \mathbb{R}^n$ is bounded as $|x| \to \infty$ and we can expect a reasonable symmetric functional calculus to assign the bounded linear operator

$$p(\langle \boldsymbol{A}, \xi\rangle)(\lambda I - \langle \boldsymbol{A}, \xi\rangle)^{-m}$$

to this function.

Suppose that $0 < \omega < \mu < \pi/2$ and $f$ is a left monogenic function defined on the sector $S_\mu^\circ(\mathbb{R}^{n+1})$ such that for every $0 < \nu < \mu$ there exists $C_\nu > 0$ such that

$$|f(x)| \le C_\nu \frac{|x|^s}{(1 + |x|^{2s})}, \quad x \in S_\nu^\circ(\mathbb{R}^{n+1}). \tag{6.51}$$

Then by Lemma 6.6 and equation (6.14), the bound

$$\|G_x(\boldsymbol{A})\|.|f(x)| \le C'_\nu \frac{|x|^s}{|x_0|^n(1+|x|^{2s})}, \quad x = x_0e_0 + \boldsymbol{x},$$

holds for all $x \in S^\circ_\nu(\mathbb{R}^{n+1}) \cap N_{\nu'}$ with $\omega < \nu' < \nu$.

Now if $\omega < \theta < \mu$ and

$$H_\theta = \{x \in \mathbb{R}^{n+1} : x = x_0e_0 + \boldsymbol{x},\ x_0/|\boldsymbol{x}| = \tan\theta\},$$

then $\|G_x(\boldsymbol{A})\|.|f(x)| = O(1/|\boldsymbol{x}|^{n-s})$ as $x \to 0$ in $H_\theta$. Hence, the function $x \longmapsto G_x(\boldsymbol{A})\boldsymbol{n}(x)f(x)$ is locally integrable at zero with respect to $n$-dimensional Lebesgue measure on $\pm H_\theta$. Similarly,

$$\|G_x(\boldsymbol{A})\|.|f(x)| = O(1/|\boldsymbol{x}|^{n+s})$$

as $|x| \to \infty$ in $H_\theta$, so $x \longmapsto G_x(\boldsymbol{A})\boldsymbol{n}(x)f(x)$ is integrable with respect to $n$-dimensional Lebesgue measure on $\pm H_\theta$.

Therefore, we define

$$f(\boldsymbol{A}) = \int_{H_\theta} G_x(\boldsymbol{A})\boldsymbol{n}(x)f(x)\,d\mu(x). \tag{6.52}$$

If $\psi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{C}$ is a real-analytic function whose two-sided monogenic extension $\psi_m$ to $S^\circ_\mu(\mathbb{R}^{n+1})$ exists and satisfies the bound (6.51) for all $0 < \nu < \mu$, then $\psi_m(\boldsymbol{A})$ is written just as $\psi(\boldsymbol{A})$.

**Theorem 6.25.** *Let $\boldsymbol{A} = (A_1, \dots, A_n)$ be an $n$-tuple of densely defined commuting linear operators $A_j : \mathcal{D}(A_j) \to \mathcal{H}$ acting in a Hilbert space $\mathcal{H}$ such that $\cap_{j=1}^n \mathcal{D}(A_j)$ is dense in $\mathcal{H}$. Suppose that $0 \le \omega < \frac{\pi}{2}$ and $\boldsymbol{A}$ is uniformly of type $\omega$.*

*If $T = i(A_1e_1 + \cdots + A_ne_n)$ is a one-to-one operator of type $\omega$ acting in $\mathcal{H}_{(n)}$ and $T$ has an $H^\infty$-functional calculus, then the $n$-tuple $\boldsymbol{A}$ has a bounded $H^\infty$-functional calculus on $S^\circ_\nu(\mathbb{C}^n)$ for any $\omega < \nu < \pi/2$, that is, there exists a homomorphism $b \longmapsto b(\boldsymbol{A})$, $b \in H^\infty(S^\circ_\nu(\mathbb{C}^n))$, from $H^\infty(S^\circ_\nu(\mathbb{C}^n))$ into $\mathcal{L}_{(n)}(\mathcal{H}_{(n)})$ and there exists $C_\nu > 0$ such that*

$$\|b(\boldsymbol{A})\| \le C_\nu\|b\|_\infty \quad \text{for all} \quad b \in H^\infty(S^\circ_\nu(\mathbb{C}^n)).$$

*Moreover, if $f$ is the unique two-sided monogenic function defined on the sector $S^\circ_\nu(\mathbb{R}^{n+1})$ such that $b = \tilde{f}$, as in Theorem 6.23, and $f$ satisfies the bound (6.51), then $b(\boldsymbol{A}) = f(\boldsymbol{A})$ is given by formula (6.52).*

*Proof.* By assumption, the operator $T$ has an $H^\infty$-functional calculus, so there exists a function $\psi \in H^\infty(S^\circ_\nu(\mathbb{C}))$ satisfying conditions (6.7), (6.10) and (6.11). Our aim is to define $b(\boldsymbol{A})$ for $b \in H^\infty(S^\circ_\nu(\mathbb{C}^n))$, by the formula

$$(b(\boldsymbol{A})u, v) = \int_0^\infty \left((b\phi_t)(\boldsymbol{A})\psi_t(T)u, \psi_t(T)^*v\right)\frac{dt}{t} \tag{6.53}$$

for all $u, v \in H_{(n)}$. The function $\phi : S_\nu^\circ(\mathbb{C}^n) \to \mathbb{C}$ is constructed from $\psi$ by setting

$$\phi(\zeta) = \psi^2\{i\zeta\} = \psi^2(|\zeta|_{\mathbb{C}})\chi_+(\zeta) + \psi^2(-|\zeta|_{\mathbb{C}})\chi_-(\zeta),$$

for all $\zeta \in S_\nu^\circ(\mathbb{C}^n)$. Then for the choice of the function $\psi$ described in the proof of Theorem 6.23, the holomorphic function $\phi_t$ defined for $t > 0$ by $\phi_t(\zeta) = \phi(t\zeta)$ has the property that

$$\boldsymbol{x} \longmapsto b(\boldsymbol{x})\phi_t(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^n \setminus \{0\}, \tag{6.54}$$

has a left (and right) monogenic extension $b \cdot_\ell \phi_t$ to $S_\nu^\circ(\mathbb{R}^{n+1})$ with decay at zero and infinity, see Lemma 6.22. Hence $(b\phi_t)(\boldsymbol{A}) := (b \cdot_\ell \phi_t)(\boldsymbol{A})$ is defined by formula (6.52) and satisfies

$$\sup_{t>0} \|(b \cdot_\ell \phi_t)(\boldsymbol{A})\| \le C\|b\|_\infty \,.$$

By normalising $\psi$ so that

$$\int_0^\infty \psi^4(t)\,\frac{dt}{t} = 1,$$

the desired functional calculus $b \longmapsto b(\boldsymbol{A})$, $b \in H^\infty(S_\nu^\circ(\mathbb{C}^n))$, is obtained. $\square$

# 7

# Feynman's Operational Calculus

The Weyl functional calculus considered in the preceding chapters assumed symmetric operator orderings in operator products. This chapter is concerned with other choices of operator orderings achieved by applying an idea of R. Feynman. Time indices are attached to each operator, formal calculations are applied to the resulting operator valued functions, treated as if they commuted, and then time-ordering is restored later with the assumption that operators attached to earlier times act first. We shall outline this idea in the present chapter with an additional ingredient of a family of probability measures which index possible choices of operator ordering.

## 7.1 Operants for the Weyl Calculus

The term *spectral theory* has been used somewhat loosely so far. In the case that $\boldsymbol{A} = (A_1, \dots, A_n)$ is a commuting $n$-tuple of bounded linear operators acting on a Banach space $X$, such that the spectrum $\sigma(A_j)$ of each operator $A_j$ is contained in the real axis for every $j = 1, \dots, n$, then it is known that the joint spectrum $\gamma(\boldsymbol{A}) \subset \mathbb{R}^n$ of $\boldsymbol{A}$ coincides with alternative notions of joint spectra [76].

In the case that $\boldsymbol{A} = (A_1, \dots, A_n)$ is of Paley-Wiener type $(s, r)$ for some $s \geq 0$ and $r > 0$, then according to Theorem 4.8, the joint spectrum $\gamma(\boldsymbol{A})$ is identical to the support of the Weyl functional calculus $\mathcal{W}_{\boldsymbol{A}}$.

In this section, the joint spectrum $\gamma(\boldsymbol{A})$ is identified with the *Gelfand spectrum* of a certain commutative Banach algebra, along the lines of [83], [4], in the case that $\boldsymbol{A} = (A_1, \dots, A_n)$ consists of bounded selfadjoint operators; that this is possible under more general conditions is unknown, but the algebra of *operants* introduced by E. Nelson [83] serves to justify the term 'spectral theory' as used throughout these notes. The more general concept of an *operating algebra* has been studied by E. Albrecht [4] and is well-suited to the present context.

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In the following, let $\mathfrak{R}$ be a complex Banach algebra with unit $1_{\mathfrak{R}}$, let $\mathfrak{U}$ be a commutative Banach algebra with unit $1_{\mathfrak{U}}$, let $E$ be a linear subspace of $\mathfrak{R}$ and let $T : \mathfrak{U} \to \mathfrak{R}$ a continuous linear mapping. The set of all permutations on $n$ elements is denoted by $\mathfrak{S}_n$.

**Definition 7.1.** A commutative Banach algebra $\mathfrak{U}$ with unit $1_{\mathfrak{U}}$ is called a *$T$-operating algebra with respect to $E$ and* $\mathfrak{R}$ if there exists a linear mapping $\tilde{} : E \to \mathfrak{U}$ such that the following three conditions are satisfied:

(7.1) The subalgebra of $\mathfrak{U}$ generated by $1_{\mathfrak{U}}$ and the range of $\tilde{}$ is dense in $\mathfrak{U}$.
(7.2) $T(1_{\mathfrak{U}}) = 1_{\mathfrak{R}}$.
(7.3) For each $n = 1, 2, \ldots$, the equality

$$T(\tilde{x}_1 \cdots \tilde{x}_n \tilde{a}) = aT(\tilde{x}_1 \cdots \tilde{x}_n) = T(\tilde{x}_1 \cdots \tilde{x}_n)a$$

holds if $a \in E$ commutes with $x_1, \ldots, x_n \in \mathrm{span}(\{1_{\mathfrak{R}}, E\})$.

The Banach algebra $\mathfrak{U}$ is called a *faithfully $T$-operating algebra with respect to $E$ and* $\mathfrak{R}$ if, in addition to conditions (7.1)-(7.3), the following condition is satisfied:

(7.4) If $\alpha$ is an element of $\mathfrak{U}$ such that $T(\alpha\beta) = 0$ for all $\beta \in \mathfrak{U}$, then $\alpha = 0$.

If the linear map in condition (7.3) is given by

$$T(\tilde{x}_1 \cdots \tilde{x}_n) = \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} x_{\pi(1)} \cdots x_{\pi(n)},$$

then $\mathfrak{U}$ is called a *symmetric* operating algebra with respect to $E$ and $\mathfrak{R}$

The following properties of the maps $T$ and $\tilde{}$ are immediate, see [4, p. 14].

(a) The mapping $\tilde{}$ in Definition 7.1 is injective and the identity

$$T \circ \tilde{} = id_E$$

holds.
(b) If $\tilde{}$ is continuous with $\|\tilde{}\| \leq 1$ and if $\|T\| \leq 1$, then $\tilde{}$ is a linear isometry.
(c) The set $\mathfrak{N} = \{\alpha \in \mathfrak{U} : T(\alpha\beta) = 0 \text{ for all } \beta \in \mathfrak{U}\}$ is a closed ideal in $\mathfrak{U}$. The algebra $\mathfrak{U}_0 \equiv \mathfrak{U}/\mathfrak{N}$, endowed with the natural mapping induced by $\tilde{} : E \to \mathfrak{U}$ and $T : \mathfrak{U} \to \mathfrak{R}$, is then a faithfully operating algebra with respect to $E$ and $\mathfrak{R}$.
(d) If $a \in E$ commutes with all $x \in E$, then $T(\tilde{a}\beta) = aT(\beta) = T(\beta)a$ for all $\beta \in \mathfrak{U}$.
(e) If $1_{\mathfrak{U}} \in E$ and if $\mathfrak{U}$ is a faithfully operating algebra with respect to $E$ and $\mathfrak{R}$, then $\tilde{1}_{\mathfrak{R}} = 1_{\mathfrak{U}}$.

The symmetric tensor algebra over $E$ is denoted by $S_0(E)$. The element of $S_0(E)$ corresponding to $x \in E$ is denoted by $\hat{x}$. If $x_1, \ldots, x_n \in E$ and $n \in \mathbb{N}$, then the linear map $T_0 : S_0(E) \to \mathfrak{R}$ is defined by

$$T_0(\hat{x}_1 \cdots \hat{x}_n) = \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} x_{\pi(1)} \cdots x_{\pi(n)}. \tag{7.1}$$

**Proposition 7.2 ([4, Proposition 3.3 (a)]).** *Any operating algebra $\mathfrak{U}$ with respect to $E$ and $\mathfrak{R}$ is isometrically isomorphic to the completion of a quotient algebra $S_{\mathfrak{U}}$ of $S_0(E)$ endowed with some algebra norm.*

*Proof.* Let $\mathfrak{U}$ be an operating algebra with respect to $E$ and $\mathfrak{R}$. By the universal property of the symmetric tensor algebra, there exists a unital algebraic homomorphism $h_{\mathfrak{U}} : S_0(E) \to \mathfrak{U}$ such that $h_{\mathfrak{U}}(\hat{x}) = \tilde{x}$ for all $x \in E$. Set $p_{\mathfrak{U}}(\alpha)$ equal to $\|h_{\mathfrak{U}}(\alpha)\|_{\mathfrak{U}}$ for $\alpha \in S_0(E)$. Then $p_{\mathfrak{U}}$ is a sub-multiplicative seminorm on $S_0(E)$ and there corresponds an algebra norm on the quotient algebra $S_{\mathfrak{U}} = S_0(E)/p_{\mathfrak{U}}^{-1}(0)$. Endowed with this norm, $S_{\mathfrak{U}}$ is isometrically isomorphic to the normed subalgebra of $\mathfrak{U}$ algebraically generated by $1_{\mathfrak{U}}$ and the range of the linear map $\tilde{}$. Hence, by (7.1), the completion of $S_{\mathfrak{U}}$ is isometrically isomorphic to $\mathfrak{U}$. □

Conversely, suppose that $T : S_0(E) \to \mathfrak{R}$ is a linear mapping satisfying

(7.2′) $T(1) = 1_{\mathfrak{R}}$ and
(7.3′) for each $n = 1, 2, \ldots$, the equality

$$T(\hat{x}_1 \cdots \hat{x}_n \hat{a}) = aT(\hat{x}_1 \cdots \hat{x}_n) = T(\hat{x}_1 \cdots \hat{x}_n)a$$

holds if $a \in E$ commutes with $x_1, \ldots, x_n \in \text{span}(\{1, E\})$.

If $p$ is a submultiplicative seminorm on $S_0(E)$ such that $T$ is $p$-continuous, then $T$ induces a continuous linear map $T_{\mathfrak{U}_p}$ on the completion $\mathfrak{U}_p$ of $S_0(E)/p^{-1}(0)$ with respect to the norm induced by $p$. Then $\mathfrak{U}_p$ is, in a natural way, a $T_{\mathfrak{U}_p}$-operating algebra with respect to $E$ and $\mathfrak{R}$.

The *complete symmetric tensor algebra* $S(E)$ is the completion of $S_0(E)$ with respect to the algebra norm $\|\cdot\|_\pi$ defined by

$$\|\alpha\|_\pi = \inf \left\{ |a| + \sum_{j=1}^{n} \|x_{j,1}\|_{\mathfrak{R}} \cdots \|x_{j,m_j}\|_{\mathfrak{R}} \right\}.$$

The infimum is taken over all representations of $\alpha \in S_0(E)$ of the form

$$\alpha = a + \sum_{j=1}^{n} \hat{x}_{j,1} \cdots \hat{x}_{j,m_j}$$

with $a \in \mathbb{C}$, $n, m_1, \ldots, m_n \in \mathbb{N}$, and $x_{j,k} \in E$ for all $j = 1, \ldots, n$ and $k = 1, \ldots, m_j$. The canonical maps $\hat{}: E \to S_0(E)$ and $T_0 : S_0(E) \to \mathfrak{R}$ induce the maps $\tilde{} : E \to S(E)$ and $T : S(E) \to \mathfrak{R}$ by continuity.

Another example where $E$ consists of the linear subspace of $\mathcal{L}(L^p([0,1]))$ consisting of multiplication operators plus Volterra integral operators is considered in [4, Proposition 3.11]. In this context, it is more natural to take a submultiplicative seminorm $\nu$ on $S_0(E)$ which is different from the projective norm $\|\cdot\|_\pi$ to obtain a symmetric operating algebra.

As noted by Nelson [83, Theorem 1], the functional calculus obtained from the complete symmetric tensor algebra $S(E)$ gives nothing of interest, by contrast with

**Definition 7.3.** Let $S(E)$ be the complete symmetric tensor algebra over $E$ and suppose that $T : S(E) \to \mathfrak{R}$ is the continuous linear map induced by the linear map $T_0$ defined in equation (7.1).

Let $\mathfrak{N}(E) = \{\alpha \in S(E) : T(\alpha\beta) = 0 \text{ for all } \beta \in S(E)\}$. The algebra $\mathfrak{U}(E)$ of *operants of* $E$ is the quotient algebra $\mathfrak{U}(E) = S(E)/\mathfrak{N}(E)$ of $S(E)$ with $\mathfrak{N}(E)$.

As noted in (c) above, it follows that $\mathfrak{U}(E)$ is a faithfully operating algebra with respect to $E$ and $\mathfrak{R}$. Also, by (b), the map $\tilde{} : E \to \mathfrak{U}(E)$ is a linear isometry.

For an $n$-tuple $\boldsymbol{A}$ of selfadjoint operators $(A_1, \dots, A_n)$ acting in a Hilbert space $H$, E. Nelson [83, Theorem 8] showed that the joint spectrum of $\tilde{A}_1, \dots, \tilde{A}_n$ in the algebra of operants $\mathfrak{U}(E)$ is the support of the Weyl functional calculus for $\boldsymbol{A}$. Here $E$ is the linear span of $id_H$ and $A_1, \dots, A_n$ and $\mathfrak{R} = \mathcal{L}(H)$. This actually follows from general facts about (non-analytic) functional calculi of operators [4, Theorem 5.10]. The essential point is that $\tilde{A}_1, \dots, \tilde{A}_n$ are also hermitian elements of the Banach algebra $\mathfrak{U}$ of operants over $E$ [4, Corollary 5.8].

Another point of contact of these notes with the work of E. Albrecht [4] is the noncommutative Shilov idempotent theorem [4, Theorem 4.1]. A version of this for $n$ operators satisfying the spectral reality condition (4.10) is proved in Theorem 4.27 using techniques of Clifford analysis.

It would be interesting to know whether or not the joint spectrum $\gamma(\boldsymbol{A})$ of an $n$-tuple $\boldsymbol{A} = (A_1, \dots, A_n)$ of bounded linear operators acting on a Banach space is equal to the the joint spectrum of $\tilde{A}_1, \dots, \tilde{A}_n$ in the algebra of operants $\mathfrak{U}(E)$ just under the spectral reality condition (4.10).

Other candidates for linear maps $T : S_0(E) \to \mathfrak{R}$ satisfying (7.2′) and (7.3′) arise from Feynman's operational calculus considered in Section 7.2 below. It is not known whether or not the associated joint spectrum $\gamma_{\boldsymbol{\mu}}(\boldsymbol{A})$ can be identified with the joint spectrum of $\tilde{A}_1, \dots, \tilde{A}_n$ in some algebra of operants.

## 7.2 Feynman's $\boldsymbol{\mu}$-Operational Calculus for $n$ Operators

The motivation for studying Feynman's $\boldsymbol{\mu}$-operational calculus comes from perturbation series expansions for Wiener and Feynman integrals [61, Chapter 15]. More generally, 'disentangling' formal expressions like

$$\exp\left\{-t\alpha + \int_0^t \beta_1(s)\mu_1(ds) + \cdots + \int_0^t \beta_n(s)\mu_n(ds)\right\} \tag{7.2}$$

leads to the solution of operator equations involving the 'time-ordering measures' $\mu_1, \ldots, \mu_n$ which represent the possibility that the perturbations $\beta_1, \ldots, \beta_n$ can be switched on and off at varying times [61, Chapter 19]. From a mathematical point of view, it would be better to represent the terms in the exponent of the formal expression (7.2) in some commutative Banach algebra along the lines of the preceding section. In this section, a commutative Banach algebra much larger than would be desirable is used to examine the combinatorial aspects of a functional calculus for $n$ bounded linear operators determined by $n$ 'time-ordering measures' $\mu_1, \ldots, \mu_n$.

We first define two commutative Banach algebras $\mathbb{A}$ and $\mathbb{D}$ for our functional calculus. The Banach algebra $\mathbb{D}$ plays the role of the algebra of operants in Section 7.1 and $\mathbb{A}$ is the initial domain of the functional calculus defined by 'disentangling' expressions in $\mathbb{D}$ determined by the 'time-ordering measures' $\mu_1, \ldots, \mu_n$. In accordance with the theme of these notes, the functional calculus is expanded from $\mathbb{A}$ to a far richer domain of functions defined in a neighbourhood of the *spectrum*.

For a positive integer $n$ and positive numbers $r_1, \ldots, r_n$, let $\mathbb{A}(r_1, \ldots, r_n)$ or, more briefly $\mathbb{A}$, be the space of complex-valued functions $(z_1, \ldots, z_n) \longmapsto f(z_1, \ldots, z_n)$ of $n$ complex variables, which are analytic at $(0, \ldots, 0)$, and are such that their power series expansion

$$f(z_1, \ldots, z_n) = \sum_{m_1, \ldots, m_n = 0}^{\infty} c_{m_1, \ldots, m_n} z_1^{m_1} \cdots z_n^{m_n} \tag{7.3}$$

converges absolutely, at least on the closed polydisk $|z_1| \leq r_1, \ldots, |z_n| \leq r_n$. Note that any entire function of $n$ complex variables belongs to $\mathbb{A}(r_1, \ldots, r_n)$ for all positive numbers $r_1, \ldots, r_n$. For $f \in \mathbb{A}(r_1, \ldots, r_n)$ given by (7.3), we let

$$\|f\| = \|f\|_{\mathbb{A}(r_1, \ldots, r_n)} := \sum_{m_1, \ldots, m_n = 0}^{\infty} |c_{m_1, \ldots, m_n}| \, r_1^{m_1} \cdots r_n^{m_n}. \tag{7.4}$$

The function on $\mathbb{A}(r_1, \ldots, r_n)$ defined by (7.4) makes $\mathbb{A}(r_1, \ldots, r_n)$ into a commutative Banach algebra under pointwise operations.

We turn next to the Banach algebra $\mathbb{D}$. Let $X$ be a Banach space and let $A_1, \ldots, A_n$ be nonzero bounded linear operators $X$. Except for the numbers $\|A_1\|, \ldots, \|A_n\|$, which will serve as weights, we ignore for the present the nature of $A_1, \ldots, A_n$ as operators and introduce a commutative Banach algebra consisting of 'analytic functions' $f(\tilde{A}_1, \ldots, \tilde{A}_n)$, where $\tilde{A}_1, \ldots, \tilde{A}_n$ are treated as purely formal commuting objects.

Consider the collection $\mathbb{D} = \mathbb{D}(A_1, \ldots, A_n)$ of all expressions of the form

$$f(\tilde{A}_1, \ldots, \tilde{A}_n) = \sum_{m_1, \ldots, m_n = 0}^{\infty} c_{m_1, \ldots, m_n} \tilde{A}_1^{m_1} \cdots \tilde{A}_n^{m_n} \tag{7.5}$$

where $c_{m_1,\ldots,m_n} \in \mathbb{C}$ for all $m_1, \ldots, m_n = 0, 1, \ldots,$ and

$$\begin{aligned}\|f(\tilde{A}_1, \ldots, \tilde{A}_n)\| &= \|f(\tilde{A}_1, \ldots, \tilde{A}_n)\|_{\mathbb{D}(A_1,\ldots,A_n)} \\ &:= \sum_{m_1,\ldots,m_n=0}^{\infty} |c_{m_1,\ldots,m_n}| \|A_1\|^{m_1} \cdots \|A_n\|^{m_n} < \infty. \end{aligned} \tag{7.6}$$

The function on $\mathbb{D}(A_1, \ldots, A_n)$ defined by (7.6) makes $\mathbb{D}(A_1, \ldots, A_n)$ into a commutative Banach algebra under pointwise operations. In fact, if we take $r_j = \|A_j\|$ for $j = 1, \ldots, n$, then $\mathbb{D}(A_1, \ldots, A_n)$ is obtained from $\mathbb{A}(r_1, \ldots, r_n)$ simply by renaming the indeterminates; hence, $\mathbb{D}$ and $\mathbb{A}$ are isometrically isomorphic in a natural way as Banach algebras.

We refer to $\mathbb{D}(A_1, \ldots, A_n)$ as the *disentangling algebra* associated with the $n$-tuple $(A_1, \ldots, A_n)$ of bounded linear operators acting on $X$.

Let $A_1, \ldots, A_n$ be nonzero operators from $\mathcal{L}(X)$ and let $\mu_1, \ldots, \mu_n$ be continuous probability measures defined at least on $\mathcal{B}[0,1]$, the Borel class of $[0,1]$. The idea is to replace the operators $A_1, \ldots, A_n$ with the elements $\tilde{A}_1, \ldots, \tilde{A}_n$ from $\mathbb{D}$ and then form the desired function of $\tilde{A}_1, \ldots, \tilde{A}_n$. Still working in $\mathbb{D}$, we time order the expression for the function and then pass back to $\mathcal{L}(X)$ simply by removing the tildes.

Given nonnegative integers $m_1, \ldots, m_n$, we let $m = m_1 + \cdots + m_n$ and

$$P^{m_1,\ldots,m_n}(z_1, \ldots, z_n) = z_1^{m_1} \cdots z_n^{m_n}. \tag{7.7}$$

We are now ready to define the disentangling map $\mathcal{T}_{\mu_1,\ldots,\mu_n}$ which will return us from our commutative framework to the noncommutative setting of $\mathcal{L}(X)$. For $j = 1, \ldots, n$ and all $s \in [0,1]$, we take $A_j(s) = A_j$ and, for $i = 1, \ldots, m$, we define

$$C_i(s) := \begin{cases} A_1(s) \text{ if } i \in \{1, \ldots, m_1\}, \\ A_2(s) \text{ if } i \in \{m_1 + 1, \ldots, m_1 + m_2\}, \\ \quad \vdots \qquad\qquad\qquad \vdots \\ A_n(s) \text{ if } i \in \{m_1 + \cdots + m_{n-1} + 1, \ldots, m\}. \end{cases} \tag{7.8}$$

For each $m = 0, 1, \ldots,$ let $S_m$ denote the set of all permutations of the integers $\{1, \ldots, m\}$, and given $\pi \in S_m$, we let

$$\varDelta_m(\pi) = \{(s_1, \ldots, s_m) \in [0,1]^m : 0 < s_{\pi(1)} < \cdots < s_{\pi(m)} < 1\}.$$

**Definition 7.4.** $\mathcal{T}_{\mu_1,\ldots,\mu_n}\left(P^{m_1,\ldots,m_n}(\tilde{A}_1, \ldots, \tilde{A}_n)\right) :=$

$$\sum_{\pi \in S_m} \int_{\varDelta_m(\pi)} C_{\pi(m)}(s_{\pi(m)}) \cdots C_{\pi(1)}(s_{\pi(1)}) (\mu_1^{m_1} \times \cdots \times \mu_n^{m_n})(ds_1, \ldots, ds_n). \tag{7.9}$$

Then, for $f(\tilde{A}_1, \ldots, \tilde{A}_n) \in \mathbb{D}(A_1, \ldots, A_n)$ given by

$$f(\tilde{A}_1, \ldots, \tilde{A}_n) = \sum_{m_1, \ldots, m_n = 0}^{\infty} c_{m_1, \ldots, m_n} \tilde{A}_1^{m_1} \cdots \tilde{A}_n^{m_n}, \tag{7.10}$$

we set $\mathcal{T}_{\mu_1, \ldots, \mu_n}\big(f(\tilde{A}_1, \ldots, \tilde{A}_n)\big)$ equal to

$$\sum_{m_1, \ldots, m_n = 0}^{\infty} c_{m_1, \ldots, m_n} \mathcal{T}_{\mu_1, \ldots, \mu_n} \left(P^{m_1, \ldots, m_n}(\tilde{A}_1, \ldots, \tilde{A}_n)\right). \tag{7.11}$$

In the commutative setting, the right-hand side of (7.9) gives us what we would expect [48, Proposition 2.2].

As is usual, we shall write the operator $\mathcal{T}_{\mu_1, \ldots, \mu_n} x$ in place of $\mathcal{T}_{\mu_1, \ldots, \mu_n}(x)$ for an element $x$ of $\mathbb{D}(A_1, \ldots, A_n)$. The mapping

$$f \longmapsto \mathcal{T}_{\mu_1, \ldots, \mu_n}\big(f(\tilde{A}_1, \ldots, \tilde{A}_n)\big), \quad f \in \mathbb{A}(\|A_1\|, \ldots, \|A_n\|)$$

into $\mathcal{L}(X)$ is a *functional calculus* in the loose sense used in these notes. Here we are identifying $\mathbb{A}(\|A_1\|, \ldots, \|A_n\|)$ with $\mathbb{D}(A_1, \ldots, A_n)$.

We shall sometimes write the bounded linear operator

$$\mathcal{T}_{\mu_1, \ldots, \mu_n}\big(f(\tilde{A}_1, \ldots, \tilde{A}_n)\big)$$

as $f_{\mu_1, \ldots, \mu_n}(A_1, \ldots, A_n)$. In particular,

$$P^{m_1 \ldots m_n}_{\mu_1, \ldots, \mu_n}(\boldsymbol{A}) = \mathcal{T}_{\mu_1, \ldots, \mu_n} \left(P^{m_1, \ldots, m_n}(\tilde{A}_1, \ldots, \tilde{A}_n)\right). \tag{7.12}$$

The following exponential estimate similar to Definition 2.2 for the Weyl functional calculus was used in [50] to obtain a larger domain for the functional calculus. Under this underlying assumption, many of the arguments for the Weyl calculus also hold for Feynman's operational calculus.

**Definition 7.5.** Let $A_1, \ldots, A_n$ be bounded linear operators acting on a Banach space $X$. Let $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$ be an $n$-tuple of continuous probability measures on $\mathcal{B}[0,1]$ and let

$$\mathcal{T}_{\mu_1, \ldots, \mu_n} : \mathbb{D}(A_1, \ldots, A_n) \to \mathcal{L}(X)$$

be the disentangling map defined in Definition 7.4. If there exists $C, r, s \geq 0$ such that

$$\|\mathcal{T}_{\mu_1, \ldots, \mu_n}(e^{i\langle \zeta, \tilde{A} \rangle})\|_{\mathcal{L}(X)} \leq C(1 + |\zeta|)^s e^{r|\Im \zeta|}, \quad \text{for all } \zeta \in \mathbb{C}^n, \tag{7.13}$$

then the $n$-tuple $\boldsymbol{A} = (A_1, \ldots, A_n)$ of operators is said to be of *Paley-Wiener type* $(s, r, \boldsymbol{\mu})$.

As might be expected, bounded selfadjoint operators exhibit stability under disentangling, facilitating the passage to unbounded operators required by quantum theory [61].

**Theorem 7.6.** *An $n$-tuple $\boldsymbol{A} = (A_1, \dots, A_n)$ of bounded selfadjoint operators acting on a Hilbert space $H$ is of Paley-Wiener type $(0, r, \boldsymbol{\mu})$ with $r = \left(\|A_1\|^2 + \cdots + \|A_n\|^2\right)^{1/2}$, for any $n$-tuple $\boldsymbol{\mu} = (\mu_1, \dots, \mu_n)$ of continuous probability measures on $\mathcal{B}[0,1]$.*

*Proof.* To see this, note that for each $x \in \mathbb{D}(A_1, \dots, A_n)$, the mapping $(\mu_1, \dots, \mu_n) \longmapsto \mathcal{T}_{\mu_1,\dots,\mu_n} x$ is continuous for the uniform operator topology of $\mathcal{L}(H)$ and weak convergence of measures [62, Theorem 3.1]. The $n$-tuple $(\mu_1, \dots, \mu_n)$ can be approximated weakly by $n$-tuples of continuous measures with alternating support for which formula [50, Equation (2.8)] applies. □

The difference between the Weyl calculus considered in Chapter 4 and Feynman's $\boldsymbol{\mu}$-operational calculus is best illustrated by a simple example involving Pauli matrices, see Example 4.1 for comparison.

Let $S(\mu)$ denote the support of a measure $\mu$. If $A$ and $B$ are two subsets of $\mathbb{R}$, we write $A \le B$ if $a \le b$ for all $a \in A$ and $b \in B$.

*Example 7.7.* Suppose that $\mu$ and $\nu$ are Borel probability measures on $[0,1]$ such that $S(\mu) \le S(\nu)$. The operator norms of the Pauli matrices $\sigma_1$ and $\sigma_3$ are equal to one, so by [49, Corollary 4.4], if $f \in \mathbb{A}(1,1)$ has a power series expansion $f(z_1, z_2) = \sum_{m_1,m_2=0}^{\infty} c_{m_1,m_2} z_1^{m_1} z_2^{m_2}$, then we have

$$f_{\mu,\nu}(\sigma_1, \sigma_3) = \sum_{m_1,m_2=0}^{\infty} c_{m_1,m_2} \sigma_3^{m_2} \sigma_1^{m_1}. \tag{7.14}$$

This is in accordance with the idea that $\sigma_1$ always acts *before* $\sigma_3$, because the support of the time-ordering measure $\mu$ associated with $\sigma_1$ lies to the *left* of the support of the time-ordering measure $\nu$ associated with $\sigma_3$ in the time interval $[0,1]$.

Let $Z_0$ be the nonnegative even integers, and let $Z_1$ be the nonnegative odd integers. The matrices $\sigma_1, \sigma_3$ satisfy $\sigma_1^2 = \sigma_3^2 = Id$, so the sum (7.14) becomes

$$\begin{aligned} f_{\mu,\nu}(\sigma_1, \sigma_3) &= \sum_{(m_1,m_2)\in Z_0\times Z_0} c_{m_1,m_2} Id + \sum_{(m_1,m_2)\in Z_1\times Z_0} c_{m_1,m_2}\sigma_1 \\ &\quad + \sum_{(m_1,m_2)\in Z_0\times Z_1} c_{m_1,m_2}\sigma_3 + \sum_{(m_1,m_2)\in Z_1\times Z_1} c_{m_1,m_2}\sigma_3\sigma_1 \\ &= \frac{1}{4}\left(f(1,1) + f(-1,1) + f(1,-1) + f(-1,-1)\right) Id \\ &\quad + \frac{1}{4}\big(f(1,1) - f(-1,1) + f(1,-1) - f(-1,-1)\big)\sigma_1 \\ &\quad + \frac{1}{4}\big(f(1,1) + f(-1,1) - f(1,-1) - f(-1,-1)\big)\sigma_3 \end{aligned}$$

$$+\frac{1}{4}\big(f(1,1) - f(-1,1) - f(1,-1) + f(-1,-1)\big)\sigma_3\sigma_1$$
$$= \frac{1}{2}\sum_{j,k=0}^{1} f((-1)^j,(-1)^k)P_{j,k} \ ; \tag{7.15}$$

the matrices $P_{j,k}, j,k = 0,1$ in (7.15) are projections given by

$$P_{0,0} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad P_{0,1} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \quad P_{1,0} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \quad P_{1,1} = \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}.$$

Note that the right-hand side of equation (7.15) makes sense if $f$ is any mapping from

$$\gamma_{\mu,\nu}(\sigma_1,\sigma_3) := \{-1,1\} \times \{-1,1\} = \sigma(\sigma_1) \times \sigma(\sigma_3)$$

into $\mathbb{C}$. The optimal domain of the functional calculus associated with the pair $(\mu,\nu)$ of probabilities, and the pair $(\sigma_1,\sigma_3)$ of $2\times 2$ matrices is therefore the set of *all* functions from $\gamma_{\mu,\nu}(\sigma_1,\sigma_3)$ to $\mathbb{C}$, rather than the much smaller set $\mathbb{A}(1,1)$. The functional calculus $f \mapsto f_{\mu,\nu}(\sigma_1,\sigma_3)$, $f \in \mathbb{A}(1,1)$, is the restriction to $\mathbb{A}(1,1)$ of the matrix valued distribution

$$\Phi^{(\boldsymbol{\mu},\nu)}_{(\sigma_1,\sigma_3)} = \frac{1}{2}\sum_{j,k=0}^{1} \delta_{((-1)^j,(-1)^k)}P_{j,k} \tag{7.16}$$

of order zero, whose support is the set $\gamma_{\mu,\nu}(\sigma_1,\sigma_3)$. Here $\delta_a$ is the unit point mass at $a \in \mathbb{C}^2$. Explicit calculations of this sort cannot usually be done. Because of the absence of symmetry, $\Phi^{(\boldsymbol{\mu},\nu)}_{(\sigma_1,\sigma_3)}(f)$ need not be an hermitian matrix for real valued functions $f$. By contrast, it follows from Example 4.1 that the support of $\Phi^{(\boldsymbol{\mu},\boldsymbol{\mu})}_{(\sigma_1,\sigma_3)} = \mathcal{W}_{(\sigma_1,\sigma_3)}$ is the closed unit disk centred at zero.

## 7.3 The $\boldsymbol{\mu}$-Monogenic Calculus for $n$ Operators

The purpose of the remainder of this work is to identify the support $\gamma_{\boldsymbol{\mu}}(\boldsymbol{A})$ of the $\boldsymbol{\mu}$-functional calculus for an $n$-tuple $\boldsymbol{A}$ of bounded operators on $X$ satisfying (7.13) with the set of singularities of a certain monogenic function taking values in the Banach module $\mathcal{L}_{(n)}(X_{(n)})$ – the Cauchy kernel $G_{\boldsymbol{\mu}}(\,\cdot\,,\boldsymbol{A})$ associated with $\boldsymbol{A}$ and $\boldsymbol{\mu}$. The argument is analogous to that of Chapter 4 for the Weyl functional calculus.

### The $\boldsymbol{\mu}$-Cauchy Kernel for an $n$-tuple of Operators

Let $\boldsymbol{\mu} = (\mu_1,\dots,\mu_n)$ be an $n$-tuple of continuous probability measures acting on $[0,1]$. Let $\boldsymbol{A} = (A_1,\dots,A_n)$ be an $n$-tuple of bounded operators of Paley-Wiener type $(s,r,\boldsymbol{\mu})$ acting on a Banach space $X$.

The nonempty compact subset $\gamma_{\boldsymbol{\mu}}(\boldsymbol{A})$ of $\mathbb{R}^n$ and Feynman's $\boldsymbol{\mu}$-functional calculus $\mathcal{F}_{\boldsymbol{\mu},\boldsymbol{A}} : f \longmapsto f_{\mu_1,\dots,\mu_n}(\boldsymbol{A})$, $f \in C^\infty(\gamma_{\boldsymbol{\mu}}(\boldsymbol{A}))$ is defined as in Chapter 2. Here the $\mathcal{L}(X)$-valued distribution $\mathcal{F}_{\boldsymbol{\mu},\boldsymbol{A}}$ is the Fourier transform of $\xi \longmapsto (2\pi)^{-n}\mathcal{T}_{\mu_1,\dots,\mu_n}\big(e^{i\langle\xi,\tilde{A}\rangle}\big)$, $\xi \in \mathbb{R}^n$. The bound (7.13) and the Paley-Wiener theorem Proposition 2.1 shows that $\mathcal{F}_{\boldsymbol{\mu},\boldsymbol{A}}$ has compact support $\gamma_{\boldsymbol{\mu}}(\boldsymbol{A})$.

The set $\mathbb{R}^n$ is identified with the subspace $\{x \in \mathbb{R}^{n+1} : x_0 = 0\}$ of $\mathbb{R}^{n+1}$. The algebraic tensor product $\mathcal{F}_{\boldsymbol{\mu},\boldsymbol{A}} \otimes I_{(n)} : C^\infty(V)_{(n)} \to \mathcal{L}_{(n)}(H_{(n)})$ of $\mathcal{F}_{\boldsymbol{\mu},\boldsymbol{A}}$ with the identity operator $I_{(n)}$ on $\mathbb{F}_{(n)}$ is also denoted just by $\mathcal{F}_{\boldsymbol{\mu},\boldsymbol{A}}$. Here $V$ is an open neighborhood of $\gamma_{\boldsymbol{\mu}}(\boldsymbol{A})$ in $\mathbb{R}^{n+1}$ and $C^\infty(V)_{(n)}$ is the locally convex module obtained by tensoring the locally convex space $C^\infty(V)$ with $\mathbb{F}_{(n)}$, as mentioned in Chapter 3. The mapping $\mathcal{F}_{\boldsymbol{\mu},\boldsymbol{A}} : C^\infty(V) \to \mathcal{L}(H)$ is defined by applying $\mathcal{F}_{\boldsymbol{\mu},\boldsymbol{A}}$ to the restriction of functions $f \in C^\infty(V)$ to the open subset $V \cap \mathbb{R}^n$ of $\mathbb{R}^n$. The map $\mathcal{F}_{\boldsymbol{\mu},\boldsymbol{A}} : C^\infty(V)_{(n)} \to \mathcal{L}_{(n)}(H_{(n)})$ is a right module homomorphism. The symbols $\mathcal{F}_{\boldsymbol{\mu},\boldsymbol{A}}(f)$ and $f_{\mu_1,\dots,\mu_n}(\boldsymbol{A})$ are used interchangeably.

The support $\gamma_{\boldsymbol{\mu}}(\boldsymbol{A}) := \operatorname{supp}\mathcal{F}_{\boldsymbol{\mu},\boldsymbol{A}}$ of the distribution $\mathcal{F}_{\boldsymbol{\mu},\boldsymbol{A}}$, is a nonempty compact subset of $\mathbb{R}^n$ (independently of the particular meaning attached to it above). Let $U$ be an open neighborhood of $\gamma_{\boldsymbol{\mu}}(\boldsymbol{A}) := \operatorname{supp}\mathcal{F}_{\boldsymbol{\mu},\boldsymbol{A}}$ in $\mathbb{R}^n$ and suppose that the function $f : U \to \mathbb{C}$ is analytic. Let $\tilde{f}$ be a left monogenic extension of $f$ to an open neighborhood of $U$ in $\mathbb{R}^{n+1}$. Then according to the definition of $(\tilde{f})_{\boldsymbol{\mu}}(\boldsymbol{A})$, the equality $(\tilde{f})_{\boldsymbol{\mu}}(\boldsymbol{A}) = f_{\mu_1,\dots,\mu_n}(\boldsymbol{A}) \otimes I_{(n)}$ is valid. The Weierstrass convergence theorem for monogenic functions [19, Theorem 9.11] ensures that the Feynman calculus $\mathcal{F}_{\boldsymbol{\mu},\boldsymbol{A}} : M(\gamma_{\boldsymbol{\mu}}(\boldsymbol{A}), \mathbb{F}_{(n)}) \to \mathcal{L}_{(n)}\big(H_{(n)}\big)$ is a continuous module homomorphism.

Suppose that $f$ is an analytic $\mathbb{F}$-valued function defined on an open neighborhood of zero in $\mathbb{R}^n$ and the Taylor series of $f$ is given by (3.3). Then the unique monogenic extension $\tilde{f}$ of $f$ is given by (3.4).

Let $V^{l_1\dots l_k}$ be monogenic polynomials defined by (3.5). For any ordered set $(l_1,\dots,l_k)$ of $k$ integers belonging to $\{1,\dots,n\}$, set

$$V^{l_1\dots l_k}_{\mu_1,\dots,\mu_n}(\boldsymbol{A}) = \mathcal{T}_{\mu_1,\dots,\mu_n}(V^{l_1\dots l_k}(\tilde{A}_1,\dots,\tilde{A}_n)). \tag{7.17}$$

The polynomial $V^{l_1\dots l_k}(\tilde{A}_1,\dots,\tilde{A}_n)$ is understood to be formed in the disentangling algebra $\mathbb{D}(A_1,\dots,A_n)$ by replacing the monogenic functions $\mathbf{z}_j$ in (3.5) by $\tilde{A}_j$ for each $j = 1,\dots,n$. Then we have

$$V^{l_1\dots l_k}(\tilde{A}) = \frac{1}{k!}\sum_{\sigma_1,\dots,\sigma_k} \tilde{A}_{\sigma_1}\cdots\tilde{A}_{\sigma_k},$$

where the sum is over all distinguishable permutations of $(l_1,\dots,l_k)$. Suppose that for each $j = 1,\dots,n$, the index $j$ appears exactly $k_j = 0,\dots,n$ times in the $k$-tuple $(l_1,\dots,l_k)$. Then $k = k_1 + \cdots + k_n$ and there are $\frac{k!}{k_1!\cdots k_n!}$ distinguishable permutations of $(l_1,\dots,l_k)$. It follows that

$$V^{l_1\dots l_k}(\tilde{A}) = \frac{1}{k_1!\cdots k_n!}\tilde{A}_1^{k_1}\cdots\tilde{A}_n^{k_n}, \tag{7.18}$$

$$V^{l_1\dots l_k}_{\mu_1,\dots,\mu_n}(\boldsymbol{A}) = \frac{1}{k_1!\cdots k_n!}P^{k_1\dots k_n}_{\mu_1,\dots,\mu_n}(\boldsymbol{A}). \tag{7.19}$$

The operators $P^{k_1\dots k_n}_{\mu_1,\dots,\mu_n}(\boldsymbol{A})$ are given by formula (7.12).

The equality

$$f_{\mu_1,\dots,\mu_n}(\boldsymbol{A}) = \sum_{k=0}^{\infty}\left(\sum_{(l_1,\dots,l_k)} a_{l_1\dots l_k}V^{l_1\dots l_k}_{\mu_1,\dots,\mu_n}(\boldsymbol{A})\right) \tag{7.20}$$

holds if (3.3) converges in a suitable neighborhood of $\gamma_{\boldsymbol{\mu}}(\boldsymbol{A})$.

The monogenic expansion of a function about a point need not converge over all of $\gamma_{\boldsymbol{\mu}}(\boldsymbol{A})$, in which case the *Cauchy integral formula* is useful, as for the Riesz-Dunford functional calculus for a single operator. Moreover, when the Feynman functional calculus for $\boldsymbol{A}$ is not defined, the Cauchy integral formula can be used to define functions of the $n$-tuple $\boldsymbol{A}$, see Chapter 4 for the case of the Weyl calculus.

For any $\omega \in \mathbb{R}^{n+1}$ not belonging to $\gamma_{\boldsymbol{\mu}}(\boldsymbol{A})$, there exists an open neighborhood $U_\omega$ of $\gamma_{\boldsymbol{\mu}}(\boldsymbol{A})$ in $\mathbb{R}^{n+1}$ such that the $\mathbb{F}_{(n)}$-valued function

$$x \longmapsto G(\omega, x) = \frac{1}{\Sigma_n}\frac{\overline{\omega - x}}{|\omega - x|^{n+1}},$$

for each $x \neq \omega$ belongs to $C^\infty(U_\omega)_{(n)}$. Then $G_{\boldsymbol{\mu}}(\omega, \boldsymbol{A}) := \mathcal{F}_{\boldsymbol{\mu},\boldsymbol{A}}(G(\omega,\cdot))$ may be viewed as an element of $\mathcal{L}_{(n)}(X_{(n)})$ for each $\omega \in \mathbb{R}^{n+1}\setminus\gamma_{\boldsymbol{\mu}}(\boldsymbol{A})$.

According to Proposition 4.2 and Theorem 4.4, the following analogues of Corollaries 4.3 and 4.5 are valid.

**Proposition 7.8.** *The $\mathcal{L}_{(n)}\left(X_{(n)}\right)$-valued function $\omega \longmapsto G_{\boldsymbol{\mu}}(\omega, \boldsymbol{A})$ is left and right monogenic in $\mathbb{R}^{n+1}\setminus\gamma_{\boldsymbol{\mu}}(\boldsymbol{A})$.*

**Proposition 7.9.** *Let $\Omega$ be a bounded open neighborhood of $\gamma_{\boldsymbol{\mu}}(\boldsymbol{A})$ in $\mathbb{R}^{n+1}$ with smooth boundary $\partial\Omega$ and exterior unit normal $n(\omega)$ defined for all $\omega \in \partial\Omega$. Let $\mu$ be the surface measure of $\Omega$.*

*Suppose that $f$ is left monogenic and $g$ is right monogenic in a neighborhood of the closure $\overline{\Omega} = \Omega\cup\partial\Omega$ of $\Omega$. Then*

$$\begin{aligned} f_{\mu_1,\dots,\mu_n}(\boldsymbol{A}) &= \int_{\partial\Omega} G_{\boldsymbol{\mu}}(\omega, \boldsymbol{A})n(\omega)f(\omega)\,d\mu(\omega), \\ g_{\mu_1,\dots,\mu_n}(\boldsymbol{A}) &= \int_{\partial\Omega} g(\omega)n(\omega)G_{\boldsymbol{\mu}}(\omega, \boldsymbol{A})\,d\mu(\omega). \end{aligned} \tag{7.21}$$

### The $\boldsymbol{\mu}$-Monogenic Spectrum

Let $\boldsymbol{A}$ be an $n$-tuple of bounded operators of Paley-Wiener type $(s, r, \boldsymbol{\mu})$ acting on a Banach space $X$. If $|\omega| > r$, then the monogenic power series expansion

(4.8) of $G_\omega(x)$ converges normally for all $x \in \mathbb{R}^{n+1}$ in the closed ball $B_r(0)$ of radius $r$ centred at zero and also in $C^\infty(B_r(0))$. According to the Paley-Wiener theorem [50, Proposition 3.3], the support $\gamma_{\boldsymbol{\mu}}(\boldsymbol{A})$ of $\mathcal{F}_{\boldsymbol{\mu},\boldsymbol{A}}$ is contained in $B_r(0) \cap \big(\{0\} \times \mathbb{R}^n\big)$.

It follows from formulas (7.20) and the continuity of $\mathcal{F}_{\boldsymbol{\mu},\boldsymbol{A}}$ on $C^\infty(B_r(0))$ that

$$G_{\boldsymbol{\mu}}(\omega, \boldsymbol{A}) = \mathcal{F}_{\boldsymbol{\mu},\boldsymbol{A}}\big(G(\omega,\,\cdot\,)\big) = \sum_{k=0}^{\infty}\left(\sum_{(l_1,\dots,l_k)} W_{l_1\dots l_k}(\omega) V^{l_1\dots l_k}_{\mu_1,\dots,\mu_n}(\boldsymbol{A})\right) \quad (7.22)$$

for all $\omega \in \mathbb{R}^{n+1}$ such that $|\omega| > \max\{r, (1+\sqrt{2})\|\sum_{j=1}^n A_j e_j\|\}$. Lemma 4.7 ensures that the sum (7.22)converges in $\mathcal{L}_{(n)}(X_{(n)})$.

We know from Proposition 7.8 that the function defined by formula (7.22) for all $|\omega| > \max\{r, (1+\sqrt{2})\|\sum_{j=1}^n A_j e_j\|\}$ is actually the restriction of an $\mathcal{L}_{(n)}(X_{(n)})$-valued function monogenic in $\mathbb{R}^{n+1} \setminus \gamma_{\boldsymbol{\mu}}(\boldsymbol{A})$. The question remains as to whether there is a larger open set on which this function has a monogenic extension.

The spectrum of a single operator $T$ is the set of 'singularities' of the resolvent function $\lambda \longmapsto (\lambda I - T)^{-1}$. Similarly, the *$\boldsymbol{\mu}$-monogenic spectrum* $\mathrm{sp}_{\boldsymbol{\mu}}(\boldsymbol{A})$ of the $n$-tuple $\boldsymbol{A}$ of bounded operators of Paley-Wiener type $(s, r, \boldsymbol{\mu})$ is the complement of the largest open set $U \subset \mathbb{R}^{n+1}$ in which the function $\omega \longmapsto G_{\boldsymbol{\mu}}(\omega, \boldsymbol{A})$ is the restriction of a monogenic function with domain $U$. The proofs of the following statements follow the case for the Weyl calculus when the measure $\mu_1, \dots, \mu_n$ are identical and all possible choices of operator products are equally weighted.

**Theorem 7.10.** *Let $\boldsymbol{A}$ be an $n$-tuple of bounded operators of Paley-Wiener type $(s, r, \boldsymbol{\mu})$ acting on a Banach space $X$. Then* $\mathrm{sp}_{\boldsymbol{\mu}}(\boldsymbol{A}) = \gamma_{\boldsymbol{\mu}}(\boldsymbol{A})$.

*Proof.* We have established in Proposition 7.8 that $\mathrm{sp}_{\boldsymbol{\mu}}(\boldsymbol{A}) \subseteq \gamma_{\boldsymbol{\mu}}(\boldsymbol{A})$. Let $x \in \mathrm{sp}_{\boldsymbol{\mu}}(\boldsymbol{A})^c$, let $U \subset \mathrm{sp}_{\boldsymbol{\mu}}(\boldsymbol{A})^c$ be an open neighborhood of $x$ in $\mathbb{R}^n$ and suppose that $\phi$ is a smooth function with compact support in $U$.

Let $x \in X$ and $x^* \in X^*$. A comparison with [19, Definition 27.6] shows that the $\mathbb{F}_{(n)}$-valued monogenic function $\omega \longmapsto \langle G_{\boldsymbol{\mu}}(\omega, \boldsymbol{A})x, x^*\rangle$, $\omega \in \mathbb{R}^{n+1} \setminus \operatorname{supp} \mathcal{F}_{\boldsymbol{\mu},\boldsymbol{A}}$, is actually the monogenic representation of the distribution $\langle \mathcal{F}_{\boldsymbol{\mu},\boldsymbol{A}} x, x^*\rangle : f \longmapsto \langle \mathcal{F}_{\boldsymbol{\mu},\boldsymbol{A}}(f)x, x^*\rangle$, for all smooth $f$ defined in an open neighborhood of $\gamma_{\boldsymbol{\mu}}(\boldsymbol{A})$. Then $\langle \mathcal{F}_{\boldsymbol{\mu},\boldsymbol{A}} x, x^*\rangle(G(\omega,\,\cdot\,)) = \langle G_{\boldsymbol{\mu}}(\omega, \boldsymbol{A})x, x^*\rangle$ and by Theorem 3.3 and $\langle \mathcal{F}_{\boldsymbol{\mu},\boldsymbol{A}} x, x^*\rangle(\phi)$ equals

$$\lim_{y_0 \to 0+} \int_U [\langle G_{\boldsymbol{\mu}}(y + y_0 e_0, \boldsymbol{A})x, x^*\rangle - \langle G_{\boldsymbol{\mu}}(y - y_0 e_0, \boldsymbol{A})x, x^*\rangle]\phi(y)\, dy.$$

Because $\omega \longmapsto G_{\boldsymbol{\mu}}(\omega, \boldsymbol{A})$ is monogenic (hence continuous) for all $\omega$ in $U$, the limit is zero, that is, $\langle \mathcal{F}_{\boldsymbol{\mu},\boldsymbol{A}} x, x^*\rangle(\phi) = 0$ for all $x \in X$ and $x^* \in X^*$ and all smooth $\phi$ supported by $U$. Hence $x \in \gamma_{\boldsymbol{\mu}}(\boldsymbol{A})^c$, as was to be proved. □

**Corollary 7.11.** *Let $\boldsymbol{A}$ be an $n$-tuple of bounded operators of Paley-Wiener type $(s, r', \boldsymbol{\mu})$ acting on a Banach space $X$. Set $r = \|\sum_{j=1}^n A_j e_j\|$. Then $r_\mu(\boldsymbol{A}) \le r$ and*

$$\mathrm{sp}_{\boldsymbol{\mu}}(\boldsymbol{A}) \subseteq \left( \prod_{j=1}^n [-\|A_j\|, \|A_j\|] \right) \subset B_r(0).$$

*Proof.* Apply [50, Proposition 3.7] and invoke the equality $\mathrm{sp}_{\boldsymbol{\mu}}(\boldsymbol{A}) = \gamma_{\boldsymbol{\mu}}(\boldsymbol{A})$. □

This estimate for the spectral radius $r_\mu(\boldsymbol{A})$ is better by a factor of $\sqrt{2}-1$ than the one obtained from Lemma 4.7.

*Remark 7.12.* As for the Weyl calculus, the Cauchy kernel $\omega \longmapsto G_{\boldsymbol{\mu}}(\omega, \boldsymbol{A})$ is the monogenic representation of the distribution $\mathcal{F}_{\boldsymbol{\mu},\boldsymbol{A}}$ off $\gamma_{\boldsymbol{\mu}}(\boldsymbol{A})$ – the distribution $\mathcal{F}_{\boldsymbol{\mu},\boldsymbol{A}}$ represents the 'boundary values' on $\{0\} \times \mathbb{R}^n$ of the monogenic function

$$\omega \longmapsto G_{\boldsymbol{\mu}}(\omega, \boldsymbol{A}), \quad \omega \in \mathbb{R}^{n+1} \setminus (\{0\} \times \gamma_{\boldsymbol{\mu}}(\boldsymbol{A})).$$

In the final part of these notes, we examine what can be said if a Paley-Wiener estimate (7.13) fails for an $n$-tuple $\boldsymbol{A}$ of bounded operators. Rather than a $C^\infty$-functional calculus like $\mathcal{F}_{\boldsymbol{\mu},\boldsymbol{A}}$, we could hope for a functional calculus defined for functions of $n$ real variables analytic in a neighborhood of a nonempty compact set $\gamma_{\boldsymbol{\mu}}(\boldsymbol{A})$. At least this is the case for the Weyl functional calculus when all the measures $\mu_j$, $j = 1, \dots, n$, are equal, see Chapter 4.

Let $\boldsymbol{A}$ be any $n$-tuple of bounded operators acting on a Banach space $X$. Let $V^{l_1 \dots l_k}_{\mu_1,\dots,\mu_n}(\boldsymbol{A})$ be defined as in equation (7.17). Suppose that we set

$$G_{\boldsymbol{\mu}}(\omega, \boldsymbol{A}) = \sum_{k=0}^\infty \left( \sum_{(l_1,\dots,l_k)} W_{l_1 \dots l_k}(\omega) V^{l_1 \dots l_k}_{\mu_1,\dots,\mu_n}(\boldsymbol{A}) \right) \tag{7.23}$$

for all $\omega \in \mathbb{R}^{n+1}$ such that $|\omega| > (1+\sqrt{2})\|\sum_{j=1}^n A_j e_j\|$. The sum converges in $\mathcal{L}_{(n)}(X_{(n)})$ because $\sum_{k=0}^\infty \sum_{(l_1,\dots,l_k)} |W_{l_1\dots l_k}(\omega)| \, \|V^{l_1 \dots l_k}_{\mu_1,\dots,\mu_n}(\boldsymbol{A})\|$ converges uniformly for $|\omega| \ge R, \omega \in \mathbb{R}^{n+1}$, for each $R > (1+\sqrt{2})\|\sum_{j=1}^n A_j e_j\|$ by Lemma 4.7.

Each function $W_{l_1\dots l_k}$ is monogenic, so equation (7.23) defines a monogenic $\mathcal{L}_{(n)}(X_{(n)})$-valued function for all $\omega \in \mathbb{R}^{n+1}$ such that $|\omega| > (1+\sqrt{2})\|\sum_{j=1}^n A_j e_j\|$. It follows that the representation (7.21) is valid provided that the set $\Omega$ in the statement of Proposition 7.9 contains the closed unit ball of radius $(1+\sqrt{2})\|\sum_{j=1}^n A_j e_j\|$ centered at zero in $\mathbb{R}^{n+1}$. However, this case is of little interest, because $f_{\mu_1,\dots,\mu_n}(\boldsymbol{A})$ can also be expressed by the sum (7.20).

Although (7.23) makes sense for any $n$-tuple of bounded operators, the problem remains of enlarging the domain of definition of the monogenic function defined by (7.23) to be as large as possible in a unique way, such as in the case when the natural domain is connected or at least has no bounded component.

The Paley-Wiener condition (7.13) guarantees the existence of a nonempty compact subset $\gamma_{\boldsymbol{\mu}}(\boldsymbol{A})$ of $\mathbb{R}^n$ such that the function $\omega \longmapsto G_{\boldsymbol{\mu}}(\omega, \boldsymbol{A})$ has a monogenic extension to all of $\mathbb{R}^{n+1} \setminus (\{0\} \times \gamma_{\boldsymbol{\mu}}(\boldsymbol{A}))$. If all the measures $\mu_j$, $j = 1, \dots, n$, are equal, then as described in Chapter 4, it suffices to assume the spectral reality condition (4.10).

A general principle of operator theory is that *resolvent estimates* are equivalent to *exponential estimates* via the Laplace transform and its inverse

$$\begin{aligned}(\lambda I - T)^{-1} &= \int_0^\infty e^{-\lambda t} e^{tT}\, dt,\\ e^{tT} &= \frac{1}{2\pi i}\int_C (\lambda I - T)^{-1} e^{t\lambda}\, d\lambda\end{aligned}$$

for a suitable contour $C$. Let $\xi \in \mathbb{R}^n$ be a unit vector. In our setting, the Paley-Wiener estimate (7.13) implies the resolvent estimate

$$\begin{aligned}\|\mathcal{T}_{\boldsymbol{\mu}}(\lambda - i\langle \tilde{A}, \xi\rangle)^{-1}\|_{\mathcal{L}(X)} &= \left\| \int_0^\infty e^{-\lambda t} \mathcal{T}_{\boldsymbol{\mu}} e^{it\langle \tilde{A}, \xi\rangle}\, dt \right\|_{\mathcal{L}(X)}\\ &\le \int_0^\infty e^{-t\Re\lambda} \left\| \mathcal{T}_{\boldsymbol{\mu}} e^{it\langle \tilde{A}, \xi\rangle} \right\|_{\mathcal{L}(X)} dt\\ &\le C \int_0^\infty e^{-t\Re\lambda}(1+t)^s\, dt\\ &= O\left((\Re\lambda)^{-s-1}\right) \quad \text{as} \quad \Re\lambda \to 0+ .\end{aligned}$$

A similar formula holds for $\Re\lambda < 0$.

The function $\lambda \longmapsto \mathcal{T}_{\boldsymbol{\mu}}(\lambda - i\langle \tilde{A}, \xi\rangle)^{-1}$ therefore has a unique analytic continuation from the set $\left\{\lambda \in \mathbb{C} \setminus (i\mathbb{R}) : |\lambda| > \left(\sum_{j=1}^n \|A_j\|^2\right)^{1/2}\right\}$ to all of $\mathbb{C} \setminus (i\mathbb{R})$. We adopt this conclusion as a *definition* in case the exponential bound (7.13) fails. The closed disk of radius $r > 0$ in $\mathbb{C}$ centered at zero is written as $D_r$.

**Definition 7.13.** Let $A_1, \dots, A_n$ be bounded linear operators acting on a Banach space $X$. Set $r = \left(\sum_{j=1}^n \|A_j\|^2\right)^{1/2}$. Let $\boldsymbol{\mu} = (\mu_1, \dots, \mu_n)$ be an $n$-tuple of continuous probability measures on $\mathcal{B}[0,1]$ and let $\mathcal{T}_{\mu_1,\dots,\mu_n} : \mathbb{D}(A_1, \dots, A_n) \to \mathcal{L}(X)$ be the disentangling map defined in Definition 7.4. We say that the $n$-tuple $\boldsymbol{A} = (A_1, \dots, A_n)$ has *real* $\boldsymbol{\mu}$*-joint spectrum* if for each $\xi \in \mathbb{R}^n$ with $|\xi| = 1$, the function

$$\lambda \longmapsto \mathcal{T}_{\mu_1,\dots,\mu_n}(\lambda - \langle \tilde{A}, \xi\rangle)^{-1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},\ |\lambda| > r, \tag{7.24}$$

is the restriction to $\mathbb{C} \setminus (D_r \cup \mathbb{R})$ of an analytic function on $\mathbb{C} \setminus \mathbb{R}$.

In view of the preceding discussion, we immediately have the following examples.

*Example 7.14.* (i) Suppose that $\boldsymbol{A}$ is of Paley-Wiener type $(s, r, \boldsymbol{\mu})$. For $\xi \in \mathbb{R}^n$ and $\lambda \in \mathbb{C}$ with $\Im\lambda \neq 0$, set

$$R_{\boldsymbol{\mu}}(\lambda, \langle \tilde{A}, \xi \rangle) = -i \int_0^\infty e^{it\lambda} \mathcal{T}_{\boldsymbol{\mu}} e^{-it\langle \tilde{A}, \xi \rangle} \, dt, \quad \Im\lambda > 0,$$
$$R_{\boldsymbol{\mu}}(\lambda, \langle \tilde{A}, \xi \rangle) = i \int_0^\infty e^{-it\lambda} \mathcal{T}_{\boldsymbol{\mu}} e^{it\langle \tilde{A}, \xi \rangle} \, dt, \quad \Im\lambda < 0.$$

Then $\lambda \longmapsto R_{\boldsymbol{\mu}}(\lambda, \langle \tilde{A}, \xi \rangle)$, $\lambda \in \mathbb{C} \backslash \mathbb{R}$, is an analytic continuation of the function (7.24). In the case that $\Im\lambda > r$, the equality

$$(\lambda - \langle \tilde{A}, \xi \rangle)^{-1} = -i \int_0^\infty e^{it\lambda} e^{-it\langle \tilde{A}, \xi \rangle} \, dt$$

holds in the disentangling algebra $\mathbb{D}(A_1, \ldots, A_n)$, so

$$\mathcal{T}_{\boldsymbol{\mu}}(\lambda - \langle \tilde{A}, \xi \rangle)^{-1} = -i\mathcal{T}_{\boldsymbol{\mu}} \left( \int_0^\infty e^{it\lambda} e^{-it\langle \tilde{A}, \xi \rangle} \, dt \right) = R_{\boldsymbol{\mu}}(\lambda, \langle \tilde{A}, \xi \rangle).$$

Similar equalities hold for $\Im\lambda < -r$.

(ii) In case $\mu_1 = \cdots = \mu_n$, we drop the subscript $\boldsymbol{\mu}$ and obtain the equality

$$\mathcal{T}(\lambda - \langle \tilde{A}, \xi \rangle)^{-1} = (\lambda I - \langle A, \xi \rangle)^{-1}, \quad |\lambda| > r,$$

from [49, Remark 5.6]. Then $\boldsymbol{A}$ has real joint spectrum if and only if the spectrum $\sigma(\langle A, \xi \rangle)$ of the bounded linear operator $\langle A, \xi \rangle$ is real for every $\xi \in \mathbb{R}^n$. The analytic continuation $R(\,\cdot\,, \langle \tilde{A}, \xi \rangle)$ of the function (7.24) is given by $R(\lambda, \langle \tilde{A}, \xi \rangle) = (\lambda I - \langle A, \xi \rangle)^{-1}$, for all $\lambda \in \mathbb{C} \setminus \sigma(\langle A, \xi \rangle)$.

When $\boldsymbol{A}$ has real $\boldsymbol{\mu}$-joint spectrum, it remains to define the real $\boldsymbol{\mu}$-joint spectrum $\gamma_{\boldsymbol{\mu}}(\boldsymbol{A})$ of $\boldsymbol{A}$ and define a functional calculus for functions real analytic in a neighbourhood of $\gamma_{\boldsymbol{\mu}}(\boldsymbol{A})$ in $\mathbb{R}^n$. This can be done along lines similar to the reasoning in Chapter 4 using the plane wave decomposition (4.16) of the Cauchy kernel $G(\omega, x)$.

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# Index of Notation

# Index