STATICS AND MECHANICS OF MATERIALS

Ferdinand P. Beer • E. Russell Johnston, Jr. • John T. DeWolf • David F. Mazurek

STATICS AND MECHANICS OF MATERIALS

Ferdinand P. Beer
Late of Lehigh University

E. Russell Johnston, Jr.
University of Connecticut

John T. DeWolf
University of Connecticut

David F. Mazurek
U.S. Coast Guard Academy

Mc Graw Hill
Connect Learn Succeed™

The McGraw-Hill Companies

STATICS AND MECHANICS OF MATERIALS

This book is printed on acid-free paper.

1 2 3 4 5 6 7 8 9 0 WVR/WVR 1 0 9 8 7 6 5 4 3 2 1 0

ISBN 978-0-07-338015-5
MHID 0-07-338015-6

Vice President & Editor-in-Chief: *Marty Lange*
Vice President EDP & Central Publishing Services: *Kimberly Meriwether-David*
Global Publisher: *Raghothaman Srinivasan*
Senior Sponsoring Editor: *Bill Stenquist*
Director of Development: *Kristine Tibbetts*
Developmental Editor: *Lora Neyens*
Senior Marketing Manager: *Curt Reynolds*
Project Manager: *Melissa M. Leick*
Senior Production Supervisor: *Sherry L. Kane*
Design Coordinator: *Brenda A. Rolwes*
Cover Designer: *Studio Montage, St. Louis, Missouri*
(USE) Cover Image: *Photo of the planetarium structure and roof steel courtesy of Webcor Builders, builder of the new California Academy of Sciences in San Francisco's Golden Gate Park.*
Back Cover Images: *Interior photo by John DeWolf and exterior photos by © Cody Andresen, Arup*
Lead Photo Research Coordinator: *Carrie K. Burger*
Photo Research: *Sabina Dowell*
Compositor: *Aptara®, Inc.*
Typeface: *10.5/12 New Caledonia*
Printer: *World Color Press Inc.*

The photos on the front and back show the California Academy of Sciences Building in San Francisco, both during construction and afterward. This green building utilizes a variety of innovative sustainable-design standards, including a living roof, natural lighting, solar energy generation, and an automated passive ventilation system.

Library of Congress Cataloging-in-Publication Data

Statics and mechanics of materials / Ferdinand P. Beer ... [et al.]. — 1st ed.
p. cm. — (Connect, learn, succeed)
ISBN-13: 978-0-07-338015-5 (alk. paper)
ISBN-10: 0-07-338015-6 (alk. paper)
1. Statics. 2. Materials. 3. Mechanics, Applied. I. Beer, Ferdinand Pierre, 1915–
TA351.S68 2009
620.1—dc22

2009039831

www.mhhe.com

Contents

3 Rigid Bodies: Equivalent Systems of Forces 64

4 Equilibrium of Rigid Bodies 130

7 Distributed Forces: Moments of Inertia of Areas 276

8 Concepts of Stress 300

9 Stress and Strain—Axial Loading 342

13 Shearing Stresses in Beams and Thin-Walled Members 536

14 Transformations of Stress 570

15 Deflection of Beams 604

16 Columns 640

Appendices 675

Preface

OBJECTIVES

The main objective of a basic mechanics course should be to develop in the engineering student the ability to analyze a given problem in a simple and logical manner and to apply to its solution a few fundamental and well-understood principles. This text is designed for a course that combines statics and mechanics of materials—or strength of materials—offered to engineering students in the sophomore year.

GENERAL APPROACH

In this text the study of statics and mechanics of materials is based on the understanding of a few basic concepts and on the use of simplified models. This approach makes it possible to develop all the necessary formulas in a rational and logical manner, and to clearly indicate the conditions under which they can be safely applied to the analysis and design of actual engineering structures and machine components.

Practical Applications Are Introduced Early. One of the characteristics of the approach used in this text is that mechanics of *particles* is clearly separated from the mechanics of *rigid bodies.* This approach makes it possible to consider simple practical applications at an early stage and to postpone the introduction of the more difficult concepts. As an example, statics of particles is treated first (Chap. 2); after the rules of addition and subtraction of vectors are introduced, the principle of equilibrium of a particle is immediately applied to practical situations involving only concurrent forces. The statics of rigid bodies is considered in Chaps. 3 and 4. In Chap. 3, the vector and scalar products of two vectors are introduced and used to define the moment of a force about a point and about an axis. The presentation of these new concepts is followed by a thorough and rigorous discussion of equivalent systems of forces leading, in Chap. 4, to many practical applications involving the equilibrium of rigid bodies under general force systems.

New Concepts Are Introduced in Simple Terms. Since this text is designed for the first course in mechanics, new concepts are presented in simple terms and every step is explained in detail. On the other hand, by discussing the broader aspects of the problems considered and by stressing methods of general applicability, a definite maturity of approach is achieved. For example, the concepts of partial constraints and statical indeterminacy are introduced early and are used throughout.

Fundamental Principles Are Placed in the Context of Simple Applications. The fact that mechanics is essentially a *deductive* science based on a few fundamental principles is stressed. Derivations have been presented in their logical sequence and with all the rigor warranted at this level. However, the learning process being largely *inductive,* simple applications are considered first.

As an example, the statics of particles precedes the statics of rigid bodies, and problems involving internal forces are postponed until Chap. 6. In Chap. 4, equilibrium problems involving only coplanar forces are considered first and solved by ordinary algebra, while problems involving three-dimensional forces and requiring the full use of vector algebra are discussed in the second part of the chapter.

The first four chapters treating mechanics of materials (Chaps. 8, 9, 10, and 11) are devoted to the analysis of the stresses and of the corresponding deformations in various structural members, considering successively axial loading, torsion, and pure bending. Each analysis is based on a few basic concepts, namely, the conditions of equilibrium of the forces exerted on the member, the relations existing between stress and strain in the material, and the conditions imposed by the supports and loading of the member. The study of each type of loading is complemented by a large number of examples, sample problems, and problems to be assigned, all designed to strengthen the students' understanding of the subject.

Free-body Diagrams Are Used Extensively. Throughout the text, free-body diagrams are used to determine external or internal forces. The use of "picture equations" will also help the students understand the superposition of loadings and the resulting stresses and deformations.

Design Concepts Are Discussed Throughout the Text Whenever Appropriate. A discussion of the application of the factor of safety to design can be found in Chap. 8, where the concept of allowable stress design is presented.

A Careful Balance Between SI and U.S. Customary Units Is Consistently Maintained. Because it is essential that students be able to handle effectively both SI metric units and U.S. customary units, half the examples, sample problems, and problems to be assigned have been stated in SI units and half in U.S. customary units. Since a large number of problems are available, instructors can assign problems using each system of units in whatever proportion they find most desirable for their class.

It also should be recognized that using both SI and U.S. customary units entails more than the use of conversion factors. Since the SI system of units is an absolute system based on the units of time, length, and mass, whereas the U.S. customary system is a gravitational system based on the units of time, length, and force, different approaches are required for the solution of many problems. For example, when SI units are used, a body is generally specified by its mass expressed in kilograms; in most problems of statics it will be necessary to determine the weight of the body in newtons, and an additional calculation will be required for this purpose. On the other hand, when U.S. customary units are used, a body is specified by its weight in pounds and, in dynamics problems (such as would be encountered in a follow-on course in dynamics), an additional calculation will be required to determine its mass in slugs (or lb · s^2/ft). The authors, therefore, believe that problem assignments should include both systems of units.

Optional Sections Offer Advanced or Specialty Topics. A number of optional sections have been included. These sections are indicated by asterisks and thus are easily distinguished from those which form the core of the basic first mechanics course. They may be omitted without prejudice to the understanding of the rest of the text.

The material presented in the text and most of the problems require no previous mathematical knowledge beyond algebra, trigonometry, and elementary calculus; all the elements of vector algebra necessary to the understanding of mechanics are carefully presented in Chaps. 2 and 3. In general, a greater emphasis is placed on the correct understanding of the basic mathematical concepts involved than on the nimble manipulation of mathematical formulas. In this connection, it should be mentioned that the determination of the centroids of composite areas precedes the calculation of centroids by integration, thus making it possible to establish the concept of the moment of an area firmly before introducing the use of integration.

CHAPTER ORGANIZATION AND PEDAGOGICAL FEATURES

Each chapter begins with an introductory section setting the purpose and goals of the chapter and describing in simple terms the material to be covered and its application to the solution of engineering problems.

Chapter Lessons. The body of the text has been divided into units, each consisting of one or several theory sections followed by sample problems and a large number of problems to be assigned. Each unit corresponds to a well-defined topic and generally can be covered in one lesson.

Examples and Sample Problems. The theory sections include examples designed to illustrate the material being presented and facilitate its understanding. The sample problems are intended to show some of the applications of the theory to the solution of engineering problems. Since they have been set up in much the same form that students will use in solving the assigned problems, the sample problems serve the double purpose of amplifying the text and demonstrating the type of neat and orderly work that students should cultivate in their own solutions.

Homework Problem Sets. Most of the problems are of a practical nature and should appeal to engineering students. They are primarily designed, however, to illustrate the material presented in the text and help the students understand the basic principles used in engineering mechanics. The problems have been grouped according to the portions of material they illustrate and have been arranged in order of increasing difficulty. Answers to problems are given at the end of the book, except for those with a number set in italics.

Chapter Review and Summary. Each chapter ends with a review and summary of the material covered in the chapter. Notes in the margin have been included to help the students organize their review work, and cross references are provided to help them find the portions of material requiring their special attention.

Review Problems. A set of review problems is included at the end of each chapter. These problems provide students further opportunity to apply the most important concepts introduced in the chapter.

ELECTRONIC TEXTBOOK OPTIONS

Ebooks are an innovative way for students to save money and create a greener environment at the same time. An ebook can save students about half the cost of a traditional textbook and offers unique features like a powerful search engine, highlighting, and the ability to share notes with classmates using ebooks.

McGraw-Hill offers this text as an ebook. To talk about the ebook options, contact your McGraw-Hill sales representative or visit the site www.coursesmart.com to learn more.

ONLINE RESOURCES

A website of instructor resources to accompany the text is available at www.mhhe.com/beerjohnston. Instructors should contact their sales representative to gain full access to these materials.

ACKNOWLEDGMENTS

The authors thank the many companies that provided photographs for this edition. We also wish to recognize the determined efforts and patience of our photo researcher Sabina Dowell.

We are pleased to recognize Dennis Ormand of FineLine Illustrations and Aptara for the artful illustrations which contributed so much to the effectiveness of the text.

Our special thanks go to Professor Dean Updike, of the Department of Mechanical Engineering and Mechanics, Lehigh University, for his patience and cooperation as he checked the solutions and answers of all the problems in this edition.

We also gratefully acknowledge the help, comments, and suggestions offered by the many users of previous editions of books in the Beer & Johnston Engineering Mechanics series.

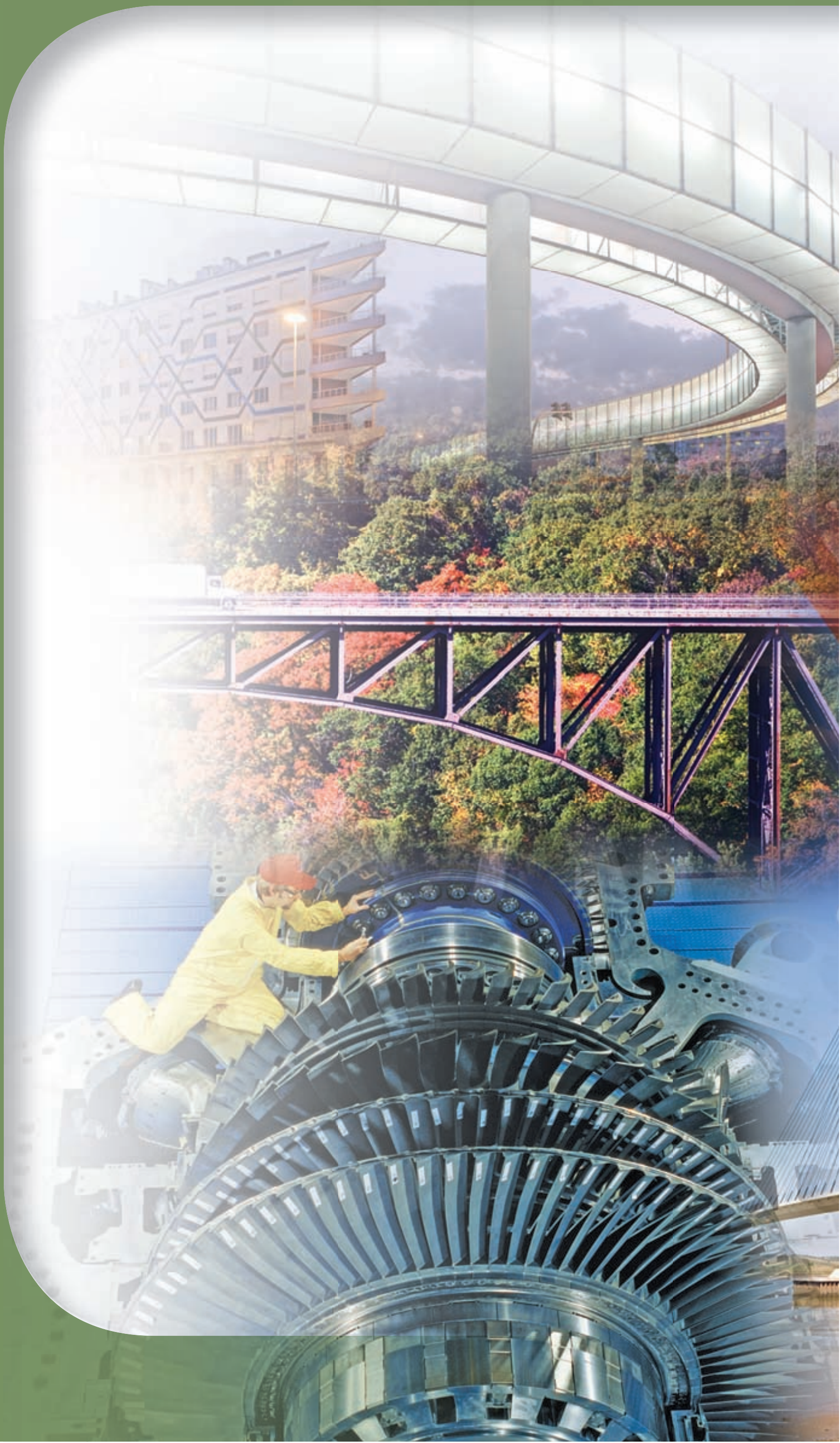
E. Russell Johnston, Jr.
John T. DeWolf
David F. Mazurek

List of Symbols

a	Constant; radius; distance
$\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots$	Forces; reactions at supports and connections
$A, B, C, \ldots$	Points
A	Area
b	Width; distance
c	Constant; distance; radius
C	Centroid
$C_1, C_2, \ldots$	Constants of integration
C_P	Column stability factor
d	Distance; diameter; depth
e	Distance; eccentricity
E	Modulus of elasticity
$\mathbf{F}$	Force; friction force
$F.S.$	Factor of safety
g	Acceleration of gravity
G	Modulus of rigidity; shear modulus
h	Distance; height
H, J, K	Points
$\mathbf{i}, \mathbf{j}, \mathbf{k}$	Unit vectors along coordinate axes
$I, I_x, \ldots$	Moments of inertia
$\bar{I}$	Centroidal moment of inertia
J	Polar moment of inertia
k	Spring constant
K	Stress concentration factor; torsional spring constant
l	Length
L	Length; span
L_e	Effective length
m	Mass
$\mathbf{M}$	Couple
$M, M_x, \ldots$	Bending moment
n	Number; ratio of moduli of elasticity; normal direction
$\mathbf{N}$	Normal component of reaction
O	Origin of coordinates
p	Pressure
$\mathbf{P}$	Force; vector
P_D	Dead load (LRFD)
P_L	Live load (LRFD)
P_U	Ultimate load (LRFD)
q	Shearing force per unit length; shear flow
$\mathbf{Q}$	Force; vector
Q	First moment of area
$\bar{r}$	Centroidal radius of gyration
$\mathbf{r}$	Position vector
r_x, r_y, r_O	Radii of gyration

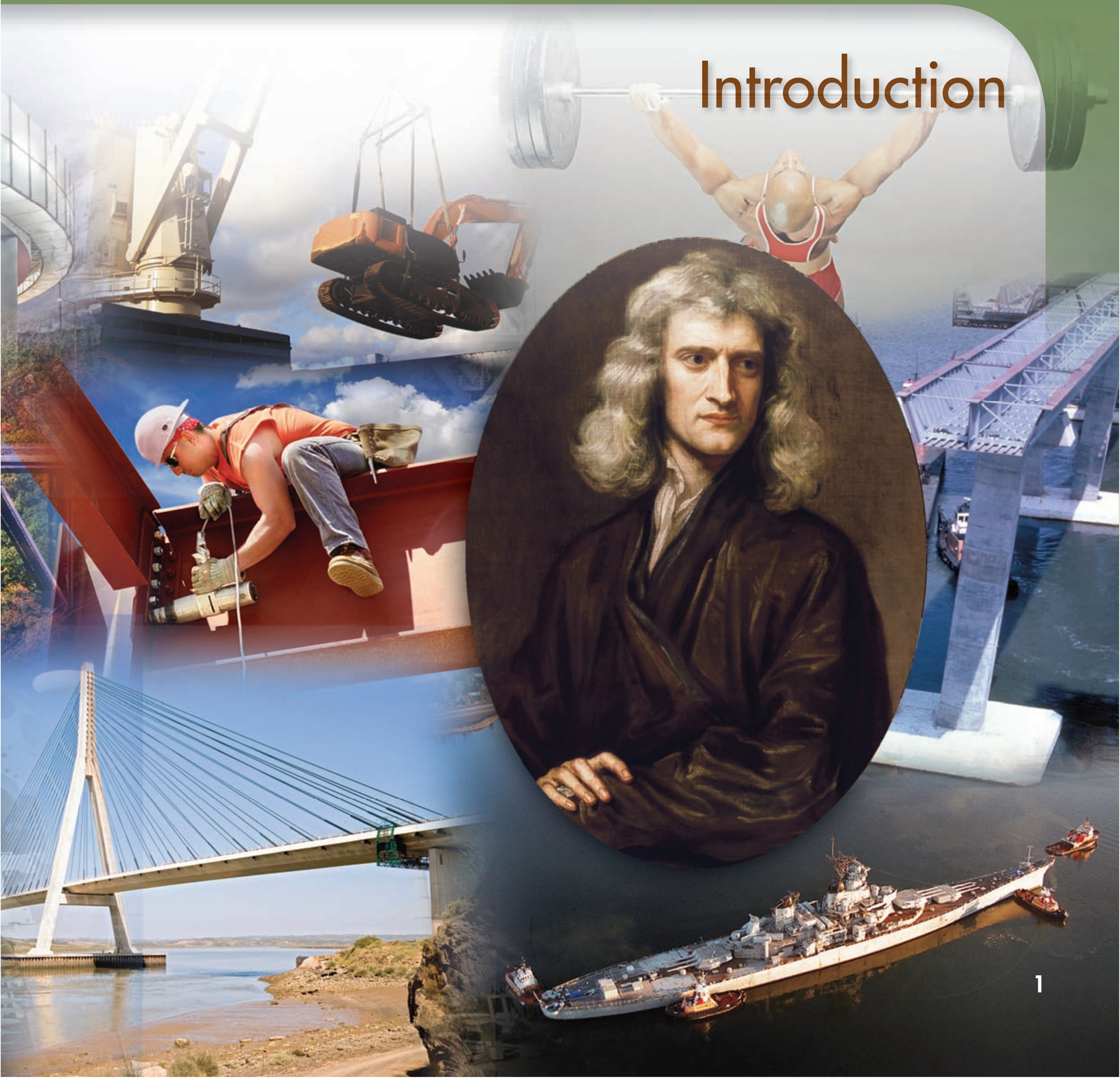
r	Radius; distance; polar coordinate
$\mathbf{R}$	Resultant force; resultant vector; reaction
R	Radius of earth
s	Length
$\mathbf{S}$	Force; vector
S	Elastic section modulus
t	Thickness
$\mathbf{T}$	Force; torque
T	Tension; temperature
u, v	Rectangular coordinates
$\mathbf{V}$	Vector product; shearing force
V	Volume; shear
w	Width; distance; load per unit length
$\mathbf{W}, W$	Weight; load
x, y, z	Rectangular coordinates; distances; displacements; deflections
$\bar{x}, \bar{y}, \bar{z}$	Coordinates of centroid
α, β, γ	Angles
α	Coefficient of thermal expansion; influence coefficient
γ	Shearing strain; specific weight
γ_D	load factor, dead load (LRFD)
γ_L	load factor, live load (LRFD)
δ	Deformation; displacement; elongation
ε	Normal strain
θ	Angle; slope
$\boldsymbol{\lambda}$	Unit vector along a line
μ	Coefficient of friction
ν	Poisson's ratio
ρ	Radius of cuvature; distance; density
σ	Normal stress
τ	Shearing stress
ϕ	Angle; angle of twist; resistance factor

In the latter part of the seventeenth century, Sir Isaac Newton stated the fundamental principles of mechanics, which are the foundation of much of today's engineering.



CHAPTER 1

Introduction



Chapter 1 Introduction

1.1 WHAT IS MECHANICS?

Mechanics can be defined as that science which describes and predicts the conditions of rest or motion of bodies under the action of forces. It is divided into three parts: mechanics of *rigid bodies,* mechanics of *deformable bodies,* and mechanics of *fluids.*

The mechanics of rigid bodies is subdivided into *statics* and *dynamics,* the former dealing with bodies at rest, the latter with bodies in motion. In this part of the study of mechanics, bodies are assumed to be perfectly rigid. Actual structures and machines, however, are never absolutely rigid and deform under the loads to which they are subjected. But these deformations are usually small and do not appreciably affect the conditions of equilibrium or motion of the structure under consideration. They are important, though, as far as the resistance of the structure to failure is concerned and are studied in mechanics of materials, which is a part of the mechanics of deformable bodies. The third division of mechanics, the mechanics of fluids, is subdivided into the study of *incompressible fluids* and of *compressible fluids.* An important subdivision of the study of incompressible fluids is *hydraulics,* which deals with problems involving water.

Mechanics is a physical science, since it deals with the study of physical phenomena. However, some associate mechanics with mathematics, while many consider it as an engineering subject. Both these views are justified in part. Mechanics is the foundation of most engineering sciences and is an indispensable prerequisite to their study. However, it does not have the *empiricism* found in some engineering sciences, i.e., it does not rely on experience or observation alone; by its rigor and the emphasis it places on deductive reasoning, it resembles mathematics. But, again, it is not an *abstract* or even a *pure* science; mechanics is an *applied* science. The purpose of mechanics is to explain and predict physical phenomena and thus to lay the foundations for engineering applications.

1.2 FUNDAMENTAL CONCEPTS AND PRINCIPLES—MECHANICS OF RIGID BODIES

Although the study of mechanics of rigid bodies goes back to the time of Aristotle (384–322 B.C.) and Archimedes (287–212 B.C.), one has to wait until Newton (1642–1727) to find a satisfactory formulation of its fundamental principles. These principles were later expressed in a modified form by d'Alembert, Lagrange, and Hamilton. Their validity remained unchallenged, however, until Einstein formulated his *theory of relativity* (1905). While its limitations have now been recognized, *newtonian mechanics* still remains the basis of today's engineering sciences.

The basic concepts used in mechanics are *space, time, mass,* and *force.* These concepts cannot be truly defined; they should be accepted on the basis of our intuition and experience and used as a mental frame of reference for our study of mechanics.

The concept of *space* is associated with the notion of the position of a point *P.* The position of *P* can be defined by three lengths measured from a certain reference point, or *origin,* in three given directions. These lengths are known as the *coordinates* of *P.*

To define an event, it is not sufficient to indicate its position in space. The *time* of the event should also be given.

The concept of *mass* is used to characterize and compare bodies on the basis of certain fundamental mechanical experiments. Two bodies of the same mass, for example, will be attracted by the earth in the same manner; they will also offer the same resistance to a change in translational motion.

A *force* represents the action of one body on another. It can be exerted by actual contact or at a distance, as in the case of gravitational forces and magnetic forces. A force is characterized by its *point of application*, its *magnitude*, and its *direction;* a force is represented by a *vector* (Sec. 2.3).

In newtonian mechanics, space, time, and mass are absolute concepts, independent of each other. (This is not true in *relativistic mechanics*, where the time of an event depends upon its position, and where the mass of a body varies with its velocity.) On the other hand, the concept of force is not independent of the other three. Indeed, one of the fundamental principles of newtonian mechanics listed below indicates that the resultant force acting on a body is related to the mass of the body and to the manner in which its velocity varies with time.

In the first part of the book, the four basic concepts that we have introduced are used to study the conditions of rest or motion of particles and rigid bodies. By *particle* we mean a very small amount of matter which may be assumed to occupy a single point in space. A *rigid body* is a combination of a large number of particles occupying fixed positions with respect to each other. The study of the mechanics of particles is obviously a prerequisite to that of rigid bodies. Besides, the results obtained for a particle can be used directly in a large number of problems dealing with the conditions of rest or motion of actual bodies.

The study of elementary mechanics rests on six fundamental principles based on experimental evidence.

The Parallelogram Law for the Addition of Forces. This states that two forces acting on a particle may be replaced by a single force, called their *resultant*, obtained by drawing the diagonal of the parallelogram which has sides equal to the given forces (Sec. 2.2).

The Principle of Transmissibility. This states that the conditions of equilibrium or of motion of a rigid body will remain unchanged if a force acting at a given point of the rigid body is replaced by a force of the same magnitude and same direction, but acting at a different point, provided that the two forces have the same line of action (Sec. 3.3).

Newton's Three Fundamental Laws. Formulated by Sir Isaac Newton in the latter part of the seventeenth century, these laws can be stated as follows:

FIRST LAW. If the resultant force acting on a particle is zero, the particle will remain at rest (if originally at rest) or will move with constant speed in a straight line (if originally in motion) (Sec. 2.10).

SECOND LAW. If the resultant force acting on a particle is not zero, the particle will have an acceleration proportional to the magnitude of the resultant and in the direction of this resultant force.

This law can be stated as

$$\mathbf{F} = m\mathbf{a} \tag{1.1}$$

where $\mathbf{F}$, m, and $\mathbf{a}$ represent, respectively, the resultant force acting on the particle, the mass of the particle, and the acceleration of the particle, expressed in a consistent system of units.

THIRD LAW. The forces of action and reaction between bodies in contact have the same magnitude, same line of action, and opposite sense (Sec. 6.1).

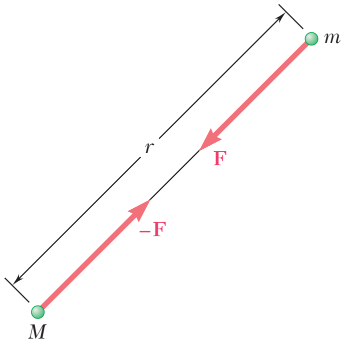


Fig. 1.1

Newton's Law of Gravitation. This states that two particles of mass M and m are mutually attracted with equal and opposite forces $\mathbf{F}$ and $-\mathbf{F}$ (Fig. 1.1) of magnitude F given by the formula

$$F = G\frac{Mm}{r^2} \tag{1.2}$$

where r = distance between the two particles
G = universal constant called the *constant of gravitation*

Newton's law of gravitation introduces the idea of an action exerted at a distance and extends the range of application of Newton's third law: the action $\mathbf{F}$ and the reaction $-\mathbf{F}$ in Fig. 1.1 are equal and opposite, and they have the same line of action.

A particular case of great importance is that of the attraction of the earth on a particle located on its surface. The force $\mathbf{F}$ exerted by the earth on the particle is then defined as the *weight* $\mathbf{W}$ of the particle. Taking M equal to the mass of the earth, m equal to the mass of the particle, and r equal to the radius R of the earth, and introducing the constant

$$g = \frac{GM}{R^2} \tag{1.3}$$

the magnitude W of the weight of a particle of mass m may be expressed as†

$$W = mg \tag{1.4}$$

The value of R in formula (1.3) depends upon the elevation of the point considered; it also depends upon its latitude, since the earth is not truly spherical. The value of g therefore varies with the position of the point considered. As long as the point actually remains on the surface of the earth, it is sufficiently accurate in most engineering computations to assume that g equals 9.81 m/s^2 or 32.2 ft/s^2.

The principles we have just listed will be introduced in the course of our study of mechanics of rigid bodies, covered in Chaps. 2 through 7. The study of the statics of particles carried out in Chap. 2

Photo 1.1 When in earth orbit, people and objects are said to be *weightless* even though the gravitational force acting is approximately 90% of that experienced on the surface of the earth. This apparent contradiction can be resolved in a course on Dynamics when Newton's second law is applied to the motion of particles.

†A more accurate definition of the weight $\mathbf{W}$ should take into account the rotation of the earth.

will be based on the parallelogram law of addition and on Newton's first law alone. The principle of transmissibility will be introduced in Chap. 3 as we begin the study of the statics of rigid bodies, and Newton's third law in Chap. 6 as we analyze the forces exerted on each other by the various members forming a structure.

As noted earlier, the six fundamental principles listed above are based on experimental evidence. Except for Newton's first law and the principle of transmissibility, they are independent principles which cannot be derived mathematically from each other or from any other elementary physical principle. On these principles rests most of the intricate structure of newtonian mechanics. For more than two centuries a tremendous number of problems dealing with the conditions of rest and motion of rigid bodies, deformable bodies, and fluids have been solved by applying these fundamental principles. Many of the solutions obtained could be checked experimentally, thus providing a further verification of the principles from which they were derived. It is only in this century that Newton's mechanics was found at fault, in the study of the motion of atoms and in the study of the motion of certain planets, where it must be supplemented by the theory of relativity. But on the human or engineering scale, where velocities are small compared with the speed of light, Newton's mechanics has yet to be disproved.

1.3 FUNDAMENTAL CONCEPTS—MECHANICS OF DEFORMABLE BODIES

The concepts needed for mechanics of deformable bodies, also referred to as *mechanics of materials,* are necessary for analyzing and designing various machines and load-bearing structures. These concepts involve the determination of *stresses* and *deformations.*

In Chaps. 8 through 16, the analysis of stresses and the corresponding deformations will be developed for structural members subject to axial loading, torsion, and pure bending. This requires the use of basic concepts involving the conditions of equilibrium of forces exerted on the member, the relations existing between stress and deformation in the material, and the conditions imposed by the supports and loading of the member. Subsequent chapters expand on this material, providing a basis for designing both structures that are statically determinant and those that are indeterminant, i.e., structures in which the internal forces cannot be determined from statics alone.

1.4 SYSTEMS OF UNITS

The fundamental concepts introduced in the preceding sections are associated with the so-called *kinetic units,* i.e., the units of *length, time, mass,* and *force*. These units cannot be chosen independently if Eq. (1.1) is to be satisfied. Three of the units may be defined arbitrarily; they are then referred to as *basic units*. The fourth unit, however, must be chosen in accordance with Eq. (1.1) and is referred to as a *derived unit*. Kinetic units selected in this way are said to form a *consistent system of units*.

International System of Units (SI Units†). In this system, the base units are the units of length, mass, and time, and they are called,

†SI stands for *Système International d'Unités* (French).

respectively, the *meter* (m), the *kilogram* (kg), and the *second* (s). All three are arbitrarily defined. The second, which was originally chosen to represent 1/86 400 of the mean solar day, is now defined as the duration of 9 192 631 770 cycles of the radiation corresponding to the transition between two levels of the fundamental state of the cesium-133 atom. The meter, originally defined as one ten-millionth of the distance from the equator to either pole, is now defined as 1 650 763.73 wavelengths of the orange-red light corresponding to a certain transition in an atom of krypton-86. The kilogram, which is approximately equal to the mass of 0.001 m^3 of water, is defined as the mass of a platinum-iridium standard kept at the International Bureau of Weights and Measures at Sèvres, near Paris, France. The unit of force is a derived unit. It is called the *newton* (N) and is defined as the force which gives an acceleration of 1 m/s^2 to a mass of 1 kg (Fig. 1.2). From Eq. (1.1) we write

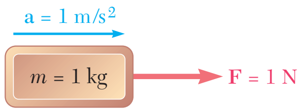


Fig. 1.2

$$1 \text{ N} = (1 \text{ kg})(1 \text{ m/s}^2) = 1 \text{ kg} \cdot \text{m/s}^2 \tag{1.5}$$

The SI units are said to form an *absolute* system of units. This means that the three base units chosen are independent of the location where measurements are made. The meter, the kilogram, and the second may be used anywhere on the earth; they may even be used on another planet. They will always have the same significance.

The *weight* of a body, or the *force of gravity* exerted on that body, should, like any other force, be expressed in newtons. From Eq. (1.4) it follows that the weight of a body of mass 1 kg (Fig. 1.3) is

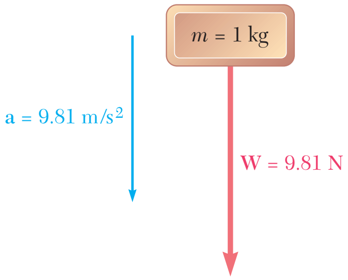


Fig. 1.3

$$\begin{aligned} W &= mg \\ &= (1 \text{ kg})(9.81 \text{ m/s}^2) \\ &= 9.81 \text{ N} \end{aligned}$$

Multiples and submultiples of the fundamental SI units may be obtained through the use of the prefixes defined in Table 1.1. The multiples and submultiples of the units of length, mass, and force most frequently used in engineering are, respectively, the *kilometer* (km) and the *millimeter* (mm); the *megagram*† (Mg) and the *gram* (g); and the *kilonewton* (kN). According to Table 1.1, we have

$$\begin{aligned} 1 \text{ km} &= 1000 \text{ m} \qquad & 1 \text{ mm} &= 0.001 \text{ m} \\ 1 \text{ Mg} &= 1000 \text{ kg} \qquad & 1 \text{ g} &= 0.001 \text{ kg} \end{aligned}$$
$$1 \text{ kN} = 1000 \text{ N}$$

The conversion of these units into meters, kilograms, and newtons, respectively, can be effected by simply moving the decimal point three places to the right or to the left. For example, to convert 3.82 km into meters, one moves the decimal point three places to the right:

$$3.82 \text{ km} = 3820 \text{ m}$$

Similarly, 47.2 mm is converted into meters by moving the decimal point three places to the left:

$$47.2 \text{ mm} = 0.0472 \text{ m}$$

Using scientific notation, one may also write

$$\begin{aligned} 3.82 \text{ km} &= 3.82 \times 10^3 \text{ m} \\ 47.2 \text{ mm} &= 47.2 \times 10^{-3} \text{ m} \end{aligned}$$

†Also known as a *metric ton*.

TABLE 1.1 SI Prefixes

Multiplication Factor	Prefix†	Symbol
$1\ 000\ 000\ 000\ 000 = 10^{12}$	tera	T
$1\ 000\ 000\ 000 = 10^{9}$	giga	G
$1\ 000\ 000 = 10^{6}$	mega	M
$1\ 000 = 10^{3}$	kilo	k
$100 = 10^{2}$	hecto‡	h
$10 = 10^{1}$	deka‡	da
$0.1 = 10^{-1}$	deci‡	d
$0.01 = 10^{-2}$	centi‡	c
$0.001 = 10^{-3}$	milli	m
$0.000\ 001 = 10^{-6}$	micro	μ
$0.000\ 000\ 001 = 10^{-9}$	nano	n
$0.000\ 000\ 000\ 001 = 10^{-12}$	pico	p
$0.000\ 000\ 000\ 000\ 001 = 10^{-15}$	femto	f
$0.000\ 000\ 000\ 000\ 000\ 001 = 10^{-18}$	atto	a

†The first syllable of every prefix is accented so that the prefix will retain its identity. Thus, the preferred pronunciation of kilometer places the accent on the first syllable, not the second.
‡The use of these prefixes should be avoided, except for the measurement of areas and volumes and for the nontechnical use of centimeter, as for body and clothing measurements.

The multiples of the unit of time are the *minute* (min) and the *hour* (h). Since 1 min = 60 s and 1 h = 60 min = 3600 s, these multiples cannot be converted as readily as the others.

By using the appropriate multiple or submultiple of a given unit, one can avoid writing very large or very small numbers. For example, one usually writes 427.2 km rather than 427 200 m, and 2.16 mm rather than 0.002 16 m.†

Units of Area and Volume. The unit of area is the *square meter* (m^2), which represents the area of a square of side 1 m; the unit of volume is the *cubic meter* (m^3), equal to the volume of a cube of side 1 m. In order to avoid exceedingly small or large numerical values in the computation of areas and volumes, one uses systems of subunits obtained by respectively squaring and cubing not only the millimeter but also two intermediate submultiples of the meter, namely, the *decimeter* (dm) and the *centimeter* (cm). Since, by definition,

$$\begin{aligned} 1 \text{ dm} &= 0.1 \text{ m} = 10^{-1} \text{ m} \\ 1 \text{ cm} &= 0.01 \text{ m} = 10^{-2} \text{ m} \\ 1 \text{ mm} &= 0.001 \text{ m} = 10^{-3} \text{ m} \end{aligned}$$

the submultiples of the unit of area are

$$\begin{aligned} 1 \text{ dm}^2 &= (1 \text{ dm})^2 = (10^{-1} \text{ m})^2 = 10^{-2} \text{ m}^2 \\ 1 \text{ cm}^2 &= (1 \text{ cm})^2 = (10^{-2} \text{ m})^2 = 10^{-4} \text{ m}^2 \\ 1 \text{ mm}^2 &= (1 \text{ mm})^2 = (10^{-3} \text{ m})^2 = 10^{-6} \text{ m}^2 \end{aligned}$$

†It should be noted that when more than four digits are used on either side of the decimal point to express a quantity in SI units—as in 427 200 m or 0.002 16 m—spaces, never commas, should be used to separate the digits into groups of three. This is to avoid confusion with the comma used in place of a decimal point, which is the convention in many countries.

and the submultiples of the unit of volume are

$$1\ \text{dm}^3 = (1\ \text{dm})^3 = (10^{-1}\ \text{m})^3 = 10^{-3}\ \text{m}^3$$
$$1\ \text{cm}^3 = (1\ \text{cm})^3 = (10^{-2}\ \text{m})^3 = 10^{-6}\ \text{m}^3$$
$$1\ \text{mm}^3 = (1\ \text{mm})^3 = (10^{-3}\ \text{m})^3 = 10^{-9}\ \text{m}^3$$

It should be noted that when the volume of a liquid is being measured, the cubic decimeter (dm^3) is usually referred to as a *liter* (L).

Other derived SI units used to measure the moment of a force, the work of a force, etc., are shown in Table 1.2. While these units will be introduced in later chapters as they are needed, we should note an important rule at this time: When a derived unit is obtained by dividing a base unit by another base unit, a prefix may be used in the numerator of the derived unit but not in its denominator. For example, the constant k of a spring which stretches 20 mm under a load of 100 N will be expressed as

$$k = \frac{100\ \text{N}}{20\ \text{mm}} = \frac{100\ \text{N}}{0.020\ \text{m}} = 5000\ \text{N/m} \qquad \text{or} \qquad k = 5\ \text{kN/m}$$

but never as $k = 5$ N/mm.

U.S. Customary Units. Most practicing American engineers still commonly use a system in which the base units are the units of length, force, and time. These units are, respectively, the *foot* (ft), the *pound* (lb), and the *second* (s). The second is the same as the corresponding SI unit. The foot is defined as 0.3048 m. The pound is defined as the

TABLE 1.2 Principal SI Units Used in Mechanics

Quantity	Unit	Symbol	Formula
Acceleration	Meter per second squared	. . .	m/s^2
Angle	Radian	rad	†
Angular acceleration	Radian per second squared	. . .	rad/s^2
Angular velocity	Radian per second	. . .	rad/s
Area	Square meter	. . .	m^2
Density	Kilogram per cubic meter	. . .	kg/m^3
Energy	Joule	J	$\text{N} \cdot \text{m}$
Force	Newton	N	$\text{kg} \cdot \text{m/s}^2$
Frequency	Hertz	Hz	s^{-1}
Impulse	Newton-second	. . .	$\text{kg} \cdot \text{m/s}$
Length	Meter	m	‡
Mass	Kilogram	kg	‡
Moment of a force	Newton-meter	. . .	$\text{N} \cdot \text{m}$
Power	Watt	W	J/s
Pressure	Pascal	Pa	N/m^2
Stress	Pascal	Pa	N/m^2
Time	Second	s	‡
Velocity	Meter per second	. . .	m/s
Volume			
Solids	Cubic meter	. . .	m^3
Liquids	Liter	L	10^{-3}m^3
Work	Joule	J	$\text{N} \cdot \text{m}$

†Supplementary unit (1 revolution = 2π rad = 360°).
‡Base unit.

weight of a platinum standard, called the *standard pound,* which is kept at the National Institute of Standards and Technology outside Washington, the mass of which is 0.453 592 43 kg. Since the weight of a body depends upon the earth's gravitational attraction, which varies with location, it is specified that the standard pound should be placed at sea level and at a latitude of 45° to properly define a force of 1 lb. Clearly the U.S. customary units do not form an absolute system of units. Because of their dependence upon the gravitational attraction of the earth, they form a *gravitational* system of units.

While the standard pound also serves as the unit of mass in commercial transactions in the United States, it cannot be so used in engineering computations, since such a unit would not be consistent with the base units defined in the preceding paragraph. Indeed, when acted upon by a force of 1 lb, that is, when subjected to the force of gravity, the standard pound receives the acceleration of gravity, $g = 32.2 \text{ ft/s}^2$ (Fig. 1.4), not the unit acceleration required by Eq. (1.1). The unit of mass consistent with the foot, the pound, and the second is the mass which receives an acceleration of 1 ft/s^2 when a force of 1 lb is applied to it (Fig. 1.5). This unit, sometimes called a *slug,* can be derived from the equation $F = ma$ after substituting 1 lb and 1 ft/s^2 for F and a, respectively. We write

$$F = ma \qquad 1 \text{ lb} = (1 \text{ slug})(1 \text{ ft/s}^2)$$

and obtain

$$1 \text{ slug} = \frac{1 \text{ lb}}{1 \text{ ft/s}^2} = 1 \text{ lb} \cdot \text{s}^2/\text{ft} \tag{1.6}$$

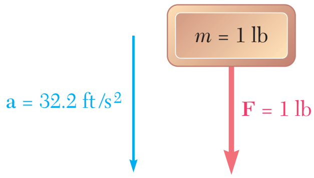


Fig. 1.4

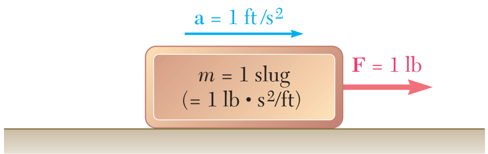


Fig. 1.5

Comparing Figs. 1.4 and 1.5, we conclude that the slug is a mass 32.2 times larger than the mass of the standard pound.

The fact that in the U.S. customary system of units bodies are characterized by their weight in pounds rather than by their mass in slugs will be a convenience in the study of statics, where one constantly deals with weights and other forces and only seldom with masses. However, in the study of dynamics, where forces, masses, and accelerations are involved, the mass m of a body will be expressed in slugs when its weight W is given in pounds. Recalling Eq. (1.4), we write

$$m = \frac{W}{g} \tag{1.7}$$

where g is the acceleration of gravity ($g = 32.2 \text{ ft/s}^2$).

Other U.S. customary units frequently encountered in engineering problems are the *mile* (mi), equal to 5280 ft; the *inch* (in.), equal to $\frac{1}{12}$ ft; and the *kilopound* (kip), equal to a force of 1000 lb. The *ton* is often used to represent a mass of 2000 lb but, like the pound, must be converted into slugs in engineering computations.

The conversion into feet, pounds, and seconds of quantities expressed in other U.S. customary units is generally more involved and requires greater attention than the corresponding operation in SI units. If, for example, the magnitude of a velocity is given as $v = 30$ mi/h, we convert it to ft/s as follows. First we write

$$v = 30 \frac{\text{mi}}{\text{h}}$$

Since we want to get rid of the unit miles and introduce instead the unit feet, we should multiply the right-hand member of the equation by an expression containing miles in the denominator and feet in the numerator. But, since we do not want to change the value of the right-hand member, the expression used should have a value equal to unity. The quotient (5280 ft)/(1 mi) is such an expression. Operating in a similar way to transform the unit hour into seconds, we write

$$v = \left(30\frac{\text{mi}}{\text{h}}\right)\left(\frac{5280\text{ ft}}{1\text{ mi}}\right)\left(\frac{1\text{ h}}{3600\text{ s}}\right)$$

Carrying out the numerical computations and canceling out units which appear in both the numerator and the denominator, we obtain

$$v = 44\frac{\text{ft}}{\text{s}} = 44\text{ ft/s}$$

1.5 CONVERSION FROM ONE SYSTEM OF UNITS TO ANOTHER

There are many instances when an engineer wishes to convert into SI units a numerical result obtained in U.S. customary units or vice versa. Because the unit of time is the same in both systems, only two kinetic base units need be converted. Thus, since all other kinetic units can be derived from these base units, only two conversion factors need be remembered.

Units of Length. By definition the U.S. customary unit of length is

$$\boxed{1\text{ ft} = 0.3048\text{ m}} \tag{1.8}$$

It follows that

$$1\text{ mi} = 5280\text{ ft} = 5280(0.3048\text{ m}) = 1609\text{ m}$$

or

$$1\text{ mi} = 1.609\text{ km} \tag{1.9}$$

Also

$$1\text{ in.} = \tfrac{1}{12}\text{ ft} = \tfrac{1}{12}(0.3048\text{ m}) = 0.0254\text{ m}$$

or

$$1\text{ in.} = 25.4\text{ mm} \tag{1.10}$$

Units of Force. Recalling that the U.S. customary unit of force (pound) is defined as the weight of the standard pound (of mass 0.4536 kg) at sea level and at a latitude of 45° (where $g = 9.807\text{ m/s}^2$) and using Eq. (1.4), we write

$$W = mg$$
$$1\text{ lb} = (0.4536\text{ kg})(9.807\text{ m/s}^2) = 4.448\text{ kg}\cdot\text{m/s}^2$$

or, recalling Eq. (1.5),

$$\boxed{1\text{ lb} = 4.448\text{ N}} \tag{1.11}$$

Units of Mass. The U.S. customary unit of mass (slug) is a derived unit. Thus, using Eqs. (1.6), (1.8), and (1.11), we write

$$1 \text{ slug} = 1 \text{ lb} \cdot \text{s}^2/\text{ft} = \frac{1 \text{ lb}}{1 \text{ ft/s}^2} = \frac{4.448 \text{ N}}{0.3048 \text{ m/s}^2} = 14.59 \text{ N} \cdot \text{s}^2/\text{m}$$

and, recalling Eq. (1.5),

$$\boxed{1 \text{ slug} = 1 \text{ lb} \cdot \text{s}^2/\text{ft} = 14.59 \text{ kg}} \tag{1.12}$$

Although it cannot be used as a consistent unit of mass, we recall that the mass of the standard pound is, by definition,

$$1 \text{ pound mass} = 0.4536 \text{ kg} \tag{1.13}$$

This constant may be used to determine the *mass* in SI units (kilograms) of a body which has been characterized by its *weight* in U.S. customary units (pounds).

To convert a derived U.S. customary unit into SI units, one simply multiplies or divides by the appropriate conversion factors. For example, to convert the moment of a force which was found to be $M = 47$ lb · in. into SI units, we use formulas (1.10) and (1.11) and write

$$\begin{aligned} M = 47 \text{ lb} \cdot \text{in.} &= 47(4.448 \text{ N})(25.4 \text{ mm}) \\ &= 5310 \text{ N} \cdot \text{mm} = 5.31 \text{ N} \cdot \text{m} \end{aligned}$$

The conversion factors given in this section may also be used to convert a numerical result obtained in SI units into U.S. customary units. For example, if the moment of a force was found to be $M = 40$ N · m, we write, following the procedure used in the last paragraph of Sec. 1.4,

$$M = 40 \text{ N} \cdot \text{m} = (40 \text{ N} \cdot \text{m})\left(\frac{1 \text{ lb}}{4.448 \text{ N}}\right)\left(\frac{1 \text{ ft}}{0.3048 \text{ m}}\right)$$

Carrying out the numerical computations and canceling out units which appear in both the numerator and the denominator, we obtain

$$M = 29.5 \text{ lb} \cdot \text{ft}$$

The U.S. customary units most frequently used in mechanics with their SI equivalents are listed in Table 1.3.

1.6 METHOD OF PROBLEM SOLUTION

You should approach a problem in mechanics as you would approach an actual engineering situation. By drawing on your own experience and intuition, you will find it easier to understand and formulate the problem. Once the problem has been clearly stated, however, there is no place in its solution for your particular fancy. Your solution must be based on the fundamental principles of statics and the concepts you will learn in this course. Every step taken must be justified on that basis. Strict rules must be followed, which lead to the solution in an almost automatic fashion, leaving no

TABLE 1.3 U.S. Customary Units and Their SI Equivalents

Quantity	U.S. Customary Unit	SI Equivalent
Acceleration	ft/s^2	$0.3048\ m/s^2$
	$in./s^2$	$0.0254\ m/s^2$
Area	ft^2	$0.0929\ m^2$
	in^2	$645.2\ mm^2$
Energy	ft · lb	1.356 J
Force	kip	4.448 kN
	lb	4.448 N
	oz	0.2780 N
Impulse	lb · s	4.448 N · s
Length	ft	0.3048 m
	in.	25.40 mm
	mi	1.609 km
Mass	oz mass	28.35 g
	lb mass	0.4536 kg
	slug	14.59 kg
	ton	907.2 kg
Moment of a force	lb · ft	1.356 N · m
	lb · in.	0.1130 N · m
Moment of inertia		
Of an area	in^4	$0.4162 \times 10^6\ mm^4$
Of a mass	$lb \cdot ft \cdot s^2$	$1.356\ kg \cdot m^2$
Momentum	lb · s	4.448 kg · m/s
Power	ft · lb/s	1.356 W
	hp	745.7 W
Pressure or stress	lb/ft^2	47.88 Pa
	lb/in^2 (psi)	6.895 kPa
Velocity	ft/s	0.3048 m/s
	in./s	0.0254 m/s
	mi/h (mph)	0.4470 m/s
	mi/h (mph)	1.609 km/h
Volume	ft^3	$0.02832\ m^3$
	in^3	$16.39\ cm^3$
Liquids	gal	3.785 L
	qt	0.9464 L
Work	ft · lb	1.356 J

room for your intuition or "feeling." After an answer has been obtained, it should be checked. Here again, you may call upon your common sense and personal experience. If not completely satisfied with the result obtained, you should carefully check your formulation of the problem, the validity of the methods used for its solution, and the accuracy of your computations.

The *statement* of a problem should be clear and precise. It should contain the given data and indicate what information is required. A neat drawing showing all quantities involved should be included. Separate diagrams should be drawn for all bodies involved, indicating clearly the forces acting on each body. These diagrams are known as *free-body diagrams* and are described in detail in Secs. 2.11 and 4.2.

The *fundamental principles* of mechanics listed in Sec. 1.2 *will be used to write equations* expressing the conditions of rest or motion of the bodies considered. Each equation should be clearly related to one of the free-body diagrams. You will then proceed to solve the problem, observing strictly the usual rules of algebra and recording neatly the various steps taken.

After the answer has been obtained, it should be *carefully checked*. Mistakes in *reasoning* can often be detected by checking the units. For example, to determine the moment of a force of 50 N about a point 0.60 m from its line of action, we would have written (Sec. 3.12)

$$M = Fd = (50\ \text{N})(0.60\ \text{m}) = 30\ \text{N} \cdot \text{m}$$

The unit N · m obtained by multiplying newtons by meters is the correct unit for the moment of a force; if another unit had been obtained, we would have known that some mistake had been made.

Errors in *computation* will usually be found by substituting the numerical values obtained into an equation which has not yet been used and verifying that the equation is satisfied. The importance of correct computations in engineering cannot be overemphasized.

1.7 NUMERICAL ACCURACY

The accuracy of the solution of a problem depends upon two items: (1) the accuracy of the given data and (2) the accuracy of the computations performed.

The solution cannot be more accurate than the less accurate of these two items. For example, if the loading of a bridge is known to be 75,000 lb with a possible error of 100 lb either way, the relative error which measures the degree of accuracy of the data is

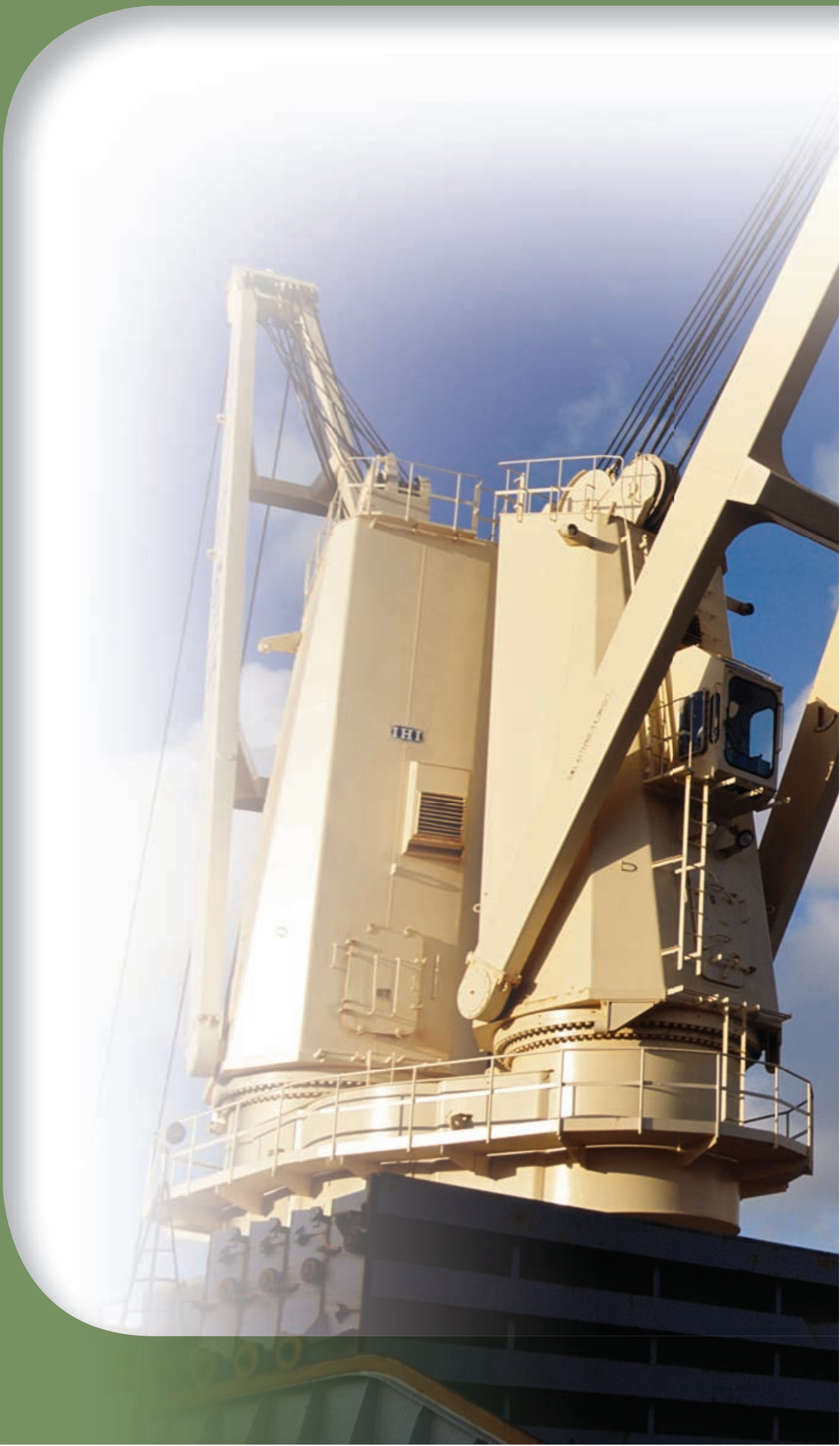
$$\frac{100\ \text{lb}}{75{,}000\ \text{lb}} = 0.0013 = 0.13\ \text{percent}$$

In computing the reaction at one of the bridge supports, it would then be meaningless to record it as 14,322 lb. The accuracy of the solution cannot be greater than 0.13 percent, no matter how accurate the computations are, and the possible error in the answer may be as large as $(0.13/100)(14{,}322\ \text{lb}) \approx 20\ \text{lb}$. The answer should be properly recorded as $14{,}320 \pm 20$ lb.

In engineering problems, the data are seldom known with an accuracy greater than 0.2 percent. It is therefore seldom justified to write the answers to such problems with an accuracy greater than 0.2 percent. A practical rule is to use 4 figures to record numbers beginning with a "1" and 3 figures in all other cases. Unless otherwise indicated, the data given in a problem should be assumed known with a comparable degree of accuracy. A force of 40 lb, for example, should be read 40.0 lb, and a force of 15 lb should be read 15.00 lb.

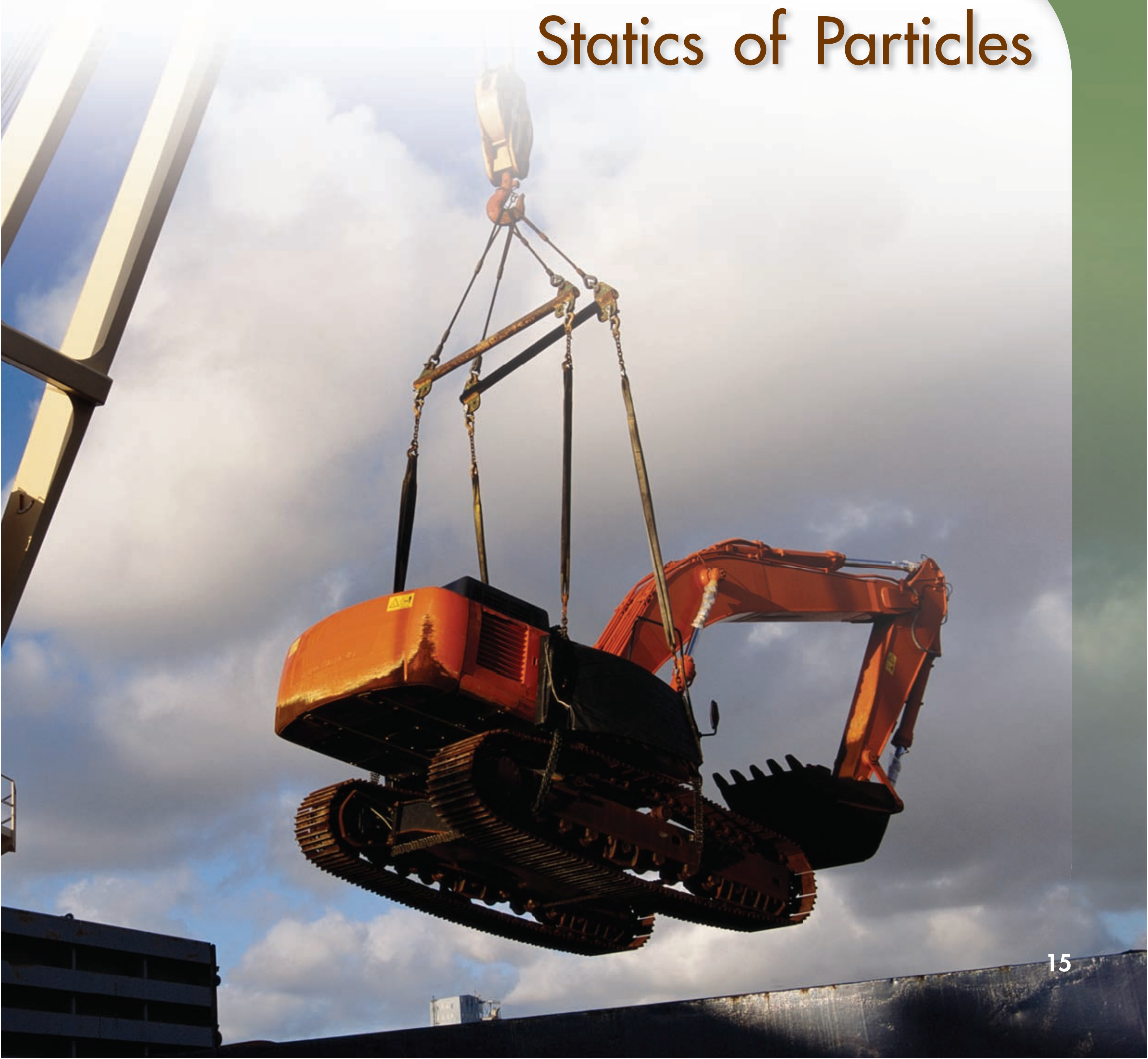
Pocket electronic calculators are widely used by practicing engineers and engineering students. The speed and accuracy of these calculators facilitate the numerical computations in the solution of many problems. However, students should not record more significant figures than can be justified merely because they are easily obtained. As noted above, an accuracy greater than 0.2 percent is seldom necessary or meaningful in the solution of practical engineering problems.

Many engineering problems can be solved by considering the equilibrium of a "particle." In the case of this excavator, which is being loaded onto a ship, a relation between the tensions in the various cables involved can be obtained by considering the equilibrium of the hook to which the cables are attached.



CHAPTER 2

Statics of Particles



2.1 INTRODUCTION

In this chapter you will study the effect of forces acting on particles. First you will learn how to replace two or more forces acting on a given particle by a single force having the same effect as the original forces. This single equivalent force is the *resultant* of the original forces acting on the particle. Later the relations which exist among the various forces acting on a particle in a state of *equilibrium* will be derived and used to determine some of the forces acting on the particle.

The use of the word "particle" does not imply that our study will be limited to that of small corpuscles. What it means is that the size and shape of the bodies under consideration will not significantly affect the solution of the problems treated in this chapter and that all the forces acting on a given body will be assumed to be applied at the same point. Since such an assumption is verified in many practical applications, you will be able to solve a number of engineering problems in this chapter.

The first part of the chapter is devoted to the study of forces contained in a single plane, and the second part to the analysis of forces in three-dimensional space.

FORCES IN A PLANE

2.2 FORCE ON A PARTICLE. RESULTANT OF TWO FORCES

A force represents the action of one body on another and is generally characterized by its *point of application*, its *magnitude*, and its *direction*. Forces acting on a given particle, however, have the same point of application. Each force considered in this chapter will thus be completely defined by its magnitude and direction.

The magnitude of a force is characterized by a certain number of units. As indicated in Chap. 1, the SI units used by engineers to measure the magnitude of a force are the newton (N) and its multiple the kilonewton (kN), equal to 1000 N, while the U.S. customary units used for the same purpose are the pound (lb) and its multiple the kilopound (kip), equal to 1000 lb. The direction of a force is defined by the *line of action* and the *sense* of the force. The line of action is the infinite straight line along which the force acts; it is characterized by the angle it forms with some fixed axis (Fig. 2.1). The force itself is represented by a segment of

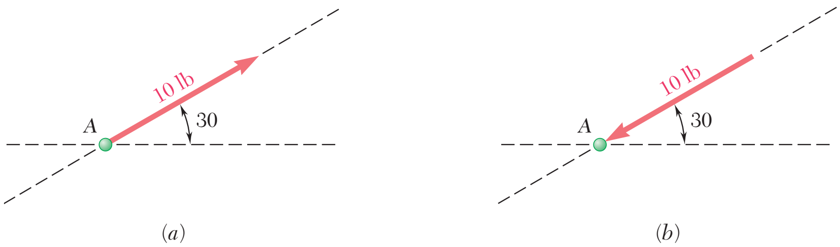


Fig. 2.1

that line; through the use of an appropriate scale, the length of this segment may be chosen to represent the magnitude of the force. Finally, the sense of the force should be indicated by an arrowhead. It is important in defining a force to indicate its sense. Two forces having the same magnitude and the same line of action but different sense, such as the forces shown in Fig. 2.1*a* and *b*, will have directly opposite effects on a particle.

Experimental evidence shows that two forces **P** and **Q** acting on a particle *A* (Fig. 2.2*a*) can be replaced by a single force **R** which has the same effect on the particle (Fig. 2.2*c*). This force is called the *resultant* of the forces **P** and **Q** and can be obtained, as shown in Fig. 2.2*b*, by constructing a parallelogram, using **P** and **Q** as two adjacent sides of the parallelogram. *The diagonal that passes through A represents the resultant.* This method for finding the resultant is known as the *parallelogram law* for the addition of two forces. This law is based on experimental evidence; it cannot be proved or derived mathematically.

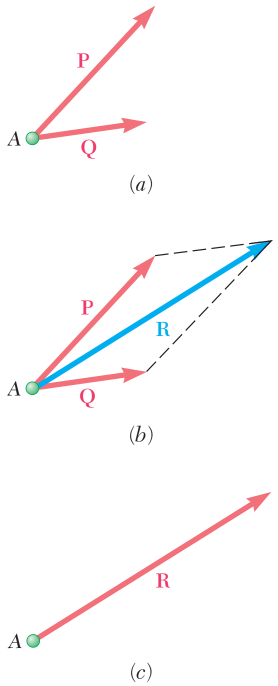


Fig. 2.2

2.3 VECTORS

It appears from the above that forces do not obey the rules of addition defined in ordinary arithmetic or algebra. For example, two forces acting at a right angle to each other, one of 4 lb and the other of 3 lb, add up to a force of 5 lb, *not* to a force of 7 lb. Forces are not the only quantities which follow the parallelogram law of addition. As you will see later, *displacements, velocities, accelerations*, and *momenta* are other examples of physical quantities possessing magnitude and direction that are added according to the parallelogram law. All these quantities can be represented mathematically by *vectors*, while those physical quantities which have magnitude but not direction, such as *volume, mass,* or *energy,* are represented by plain numbers or *scalars.*

Vectors are defined as *mathematical expressions possessing magnitude and direction, which add according to the parallelogram law.* Vectors are represented by arrows in the illustrations and will be distinguished from scalar quantities in this text through the use of boldface type (**P**). In longhand writing, a vector may be denoted by drawing a short arrow above the letter used to represent it ($\vec{P}$) or by underlining the letter ($\underline{P}$). The last method may be preferred since underlining can also be used on a computer. The magnitude of a vector defines the length of the arrow used to represent the vector. In this text, italic type will be used to denote the magnitude of a vector. Thus, the magnitude of the vector **P** will be denoted by *P.*

A vector used to represent a force acting on a given particle has a well-defined point of application, namely, the particle itself. Such a vector is said to be a *fixed,* or *bound,* vector and cannot be moved without modifying the conditions of the problem. Other physical quantities, however, such as couples (see Chap. 3), are represented by vectors that may be freely moved in space; these

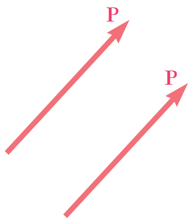


Fig. 2.4

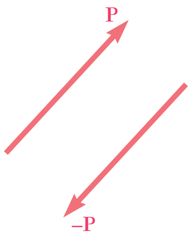


Fig. 2.5

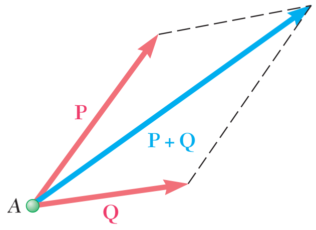


Fig. 2.6

vectors are called *free* vectors. Still other physical quantities, such as forces acting on a rigid body (see Chap. 3), are represented by vectors which can be moved, or slid, along their lines of action; they are known as *sliding* vectors.†

Two vectors which have the same magnitude and the same direction are said to be *equal,* whether or not they also have the same point of application (Fig. 2.4); equal vectors may be denoted by the same letter.

The *negative vector* of a given vector **P** is defined as a vector having the same magnitude as **P** and a direction opposite to that of **P** (Fig. 2.5); the negative of the vector **P** is denoted by $-\mathbf{P}$. The vectors **P** and $-\mathbf{P}$ are commonly referred to as *equal and opposite* vectors. Clearly, we have

$$\mathbf{P} + (-\mathbf{P}) = 0$$

2.4 ADDITION OF VECTORS

We saw in the preceding section that, by definition, vectors add according to the parallelogram law. Thus, the sum of two vectors **P** and **Q** is obtained by attaching the two vectors to the same point A and constructing a parallelogram, using **P** and **Q** as two sides of the parallelogram (Fig. 2.6). The diagonal that passes through A represents the sum of the vectors **P** and **Q**, and this sum is denoted by $\mathbf{P} + \mathbf{Q}$. The fact that the sign + is used to denote both vector and scalar addition should not cause any confusion if vector and scalar quantities are always carefully distinguished. Thus, we should note that the magnitude of the vector $\mathbf{P} + \mathbf{Q}$ is *not,* in general, equal to the sum $P + Q$ of the magnitudes of the vectors **P** and **Q**.

Since the parallelogram constructed on the vectors **P** and **Q** does not depend upon the order in which **P** and **Q** are selected, we conclude that the addition of two vectors is *commutative,* and we write

$$\boxed{\mathbf{P} + \mathbf{Q} = \mathbf{Q} + \mathbf{P}} \tag{2.1}$$

†Some expressions have magnitude and direction but do not add according to the parallelogram law. While these expressions may be represented by arrows, they *cannot* be considered as vectors.

A group of such expressions is the finite rotations of a rigid body. Place a closed book on a table in front of you, so that it lies in the usual fashion, with its front cover up and its binding to the left. Now rotate it through 180° about an axis parallel to the binding (Fig. 2.3*a*); this rotation may be represented by an arrow of length equal to 180 units and oriented as shown. Picking up the book as it lies in its new position, rotate

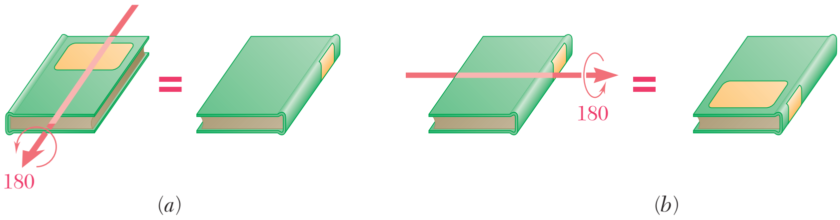


Fig. 2.3 Finite rotations of a rigid body

From the parallelogram law, we can derive an alternative method for determining the sum of two vectors. This method, known as the *triangle rule*, is derived as follows. Consider Fig. 2.6, where the sum of the vectors **P** and **Q** has been determined by the parallelogram law. Since the side of the parallelogram opposite **Q** is equal to **Q** in magnitude and direction, we could draw only half of the parallelogram (Fig. 2.7*a*). The sum of the two vectors can thus be found by *arranging* **P** *and* **Q** *in tip-to-tail fashion and then connecting the tail of* **P** *with the tip of* **Q**. In Fig. 2.7*b*, the other half of the parallelogram is considered, and the same result is obtained. This confirms the fact that vector addition is commutative.

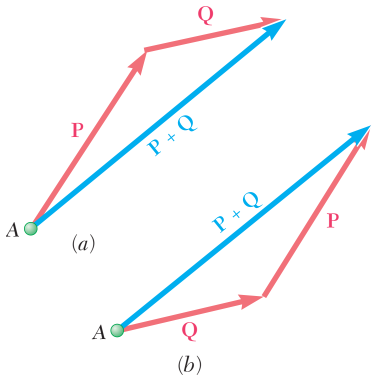


Fig. 2.7

The *subtraction* of a vector is defined as the addition of the corresponding negative vector. Thus, the vector **P** − **Q** representing the difference between the vectors **P** and **Q** is obtained by adding to **P** the negative vector −**Q** (Fig. 2.8). We write

$$\mathbf{P} - \mathbf{Q} = \mathbf{P} + (-\mathbf{Q}) \tag{2.2}$$

Here again we should observe that, while the same sign is used to denote both vector and scalar subtraction, confusion will be avoided if care is taken to distinguish between vector and scalar quantities.

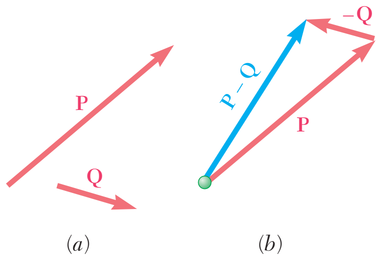


Fig. 2.8

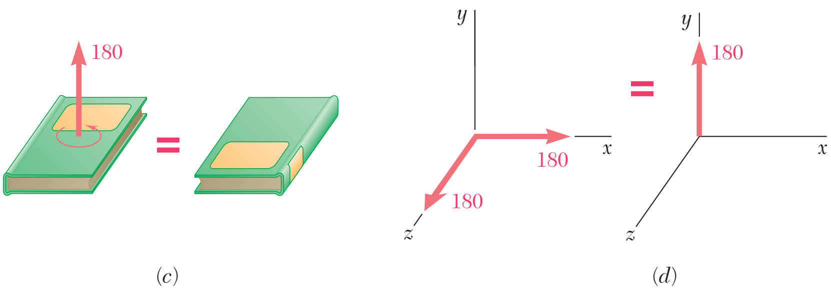
We will now consider the *sum of three or more vectors.* The sum of three vectors **P**, **Q**, and **S** will, *by definition,* be obtained by first adding the vectors **P** and **Q** and then adding the vector **S** to the vector **P** + **Q**. We thus write

$$\mathbf{P} + \mathbf{Q} + \mathbf{S} = (\mathbf{P} + \mathbf{Q}) + \mathbf{S} \tag{2.3}$$

Similarly, the sum of four vectors will be obtained by adding the fourth vector to the sum of the first three. It follows that the sum of any number of vectors can be obtained by applying repeatedly the parallelogram law to successive pairs of vectors until all the given vectors are replaced by a single vector.

---

it now through 180° about a horizontal axis perpendicular to the binding (Fig. 2.3*b*); this second rotation may be represented by an arrow 180 units long and oriented as shown. But the book could have been placed in this final position through a single 180° rotation about a vertical axis (Fig. 2.3*c*). We conclude that the sum of the two 180° rotations represented by arrows directed respectively along the z and x axes is a 180° rotation represented by an arrow directed along the y axis (Fig. 2.3*d*). Clearly, the finite rotations of a rigid body *do not* obey the parallelogram law of addition; therefore, they *cannot* be represented by vectors.



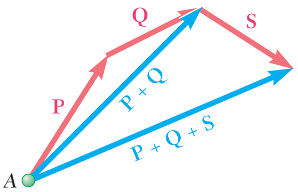


Fig. 2.9

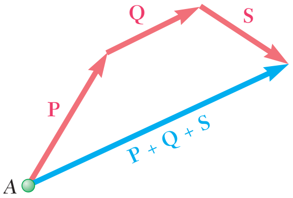


Fig. 2.10

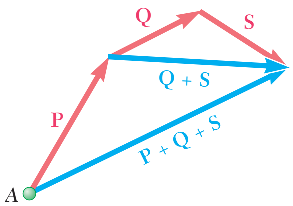


Fig. 2.11

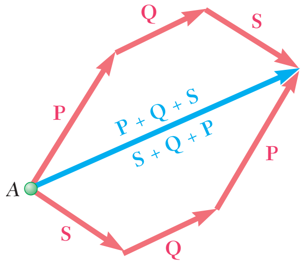


Fig. 2.12

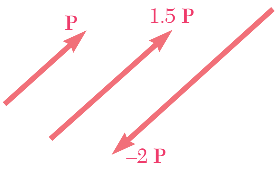


Fig. 2.13

If the given vectors are *coplanar,* i.e., if they are contained in the same plane, their sum can be easily obtained graphically. For this case, the repeated application of the triangle rule is preferred to the application of the parallelogram law. In Fig. 2.9 the sum of three vectors **P**, **Q**, and **S** was obtained in that manner. The triangle rule was first applied to obtain the sum **P** + **Q** of the vectors **P** and **Q**; it was applied again to obtain the sum of the vectors **P** + **Q** and **S**. The determination of the vector **P** + **Q**, however, could have been omitted and the sum of the three vectors could have been obtained directly, as shown in Fig. 2.10, by *arranging the given vectors in tip-to-tail fashion and connecting the tail of the first vector with the tip of the last one.* This is known as the *polygon rule* for the addition of vectors.

We observe that the result obtained would have been unchanged if, as shown in Fig. 2.11, the vectors **Q** and **S** had been replaced by their sum **Q** + **S**. We may thus write

$$\mathbf{P} + \mathbf{Q} + \mathbf{S} = (\mathbf{P} + \mathbf{Q}) + \mathbf{S} = \mathbf{P} + (\mathbf{Q} + \mathbf{S}) \tag{2.4}$$

which expresses the fact that vector addition is *associative.* Recalling that vector addition has also been shown, in the case of two vectors, to be commutative, we write

$$\begin{aligned} \mathbf{P} + \mathbf{Q} + \mathbf{S} &= (\mathbf{P} + \mathbf{Q}) + \mathbf{S} = \mathbf{S} + (\mathbf{P} + \mathbf{Q}) \\ &= \mathbf{S} + (\mathbf{Q} + \mathbf{P}) = \mathbf{S} + \mathbf{Q} + \mathbf{P} \end{aligned} \tag{2.5}$$

This expression, as well as others which may be obtained in the same way, shows that the order in which several vectors are added together is immaterial (Fig. 2.12).

Product of a Scalar and a Vector. Since it is convenient to denote the sum **P** + **P** by 2**P**, the sum **P** + **P** + **P** by 3**P**, and, in general, the sum of n equal vectors **P** by the product n**P**, we will define the product n**P** of a positive integer n and a vector **P** as a vector having the same direction as **P** and the magnitude nP. Extending this definition to include all scalars, and recalling the definition of a negative vector given in Sec. 2.3, we define the product k**P** of a scalar k and a vector **P** as a vector having the same direction as **P** (if k is positive), or a direction opposite to that of **P** (if k is negative), and a magnitude equal to the product of P and of the absolute value of k (Fig. 2.13).

2.5 RESULTANT OF SEVERAL CONCURRENT FORCES

Consider a particle A acted upon by several coplanar forces, i.e., by several forces contained in the same plane (Fig. 2.14*a*). Since the forces considered here all pass through A, they are also said to be *concurrent.* The vectors representing the forces acting on A may be added by the polygon rule (Fig. 2.14*b*). Since the use of the polygon rule is equivalent to the repeated application of the parallelogram law, the vector **R** thus obtained represents the resultant of the given concurrent forces, i.e., the single force which has the same effect on

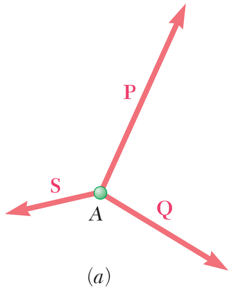


Fig. 2.14

the particle A as the given forces. As indicated above, the order in which the vectors **P**, **Q**, and **S** representing the given forces are added together is immaterial.

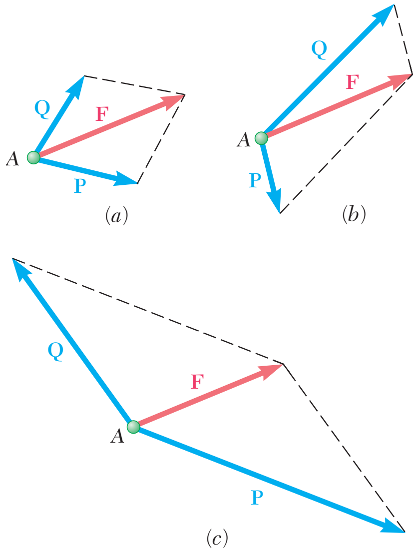


Fig. 2.15

2.6 RESOLUTION OF A FORCE INTO COMPONENTS

We have seen that two or more forces acting on a particle may be replaced by a single force which has the same effect on the particle. Conversely, a single force **F** acting on a particle may be replaced by two or more forces which, together, have the same effect on the particle. These forces are called the *components* of the original force **F**, and the process of substituting them for **F** is called *resolving the force* **F** *into components.*

Clearly, for each force **F** there exist an infinite number of possible sets of components. Sets of *two components* **P** *and* **Q** are the most important as far as practical applications are concerned. But, even then, the number of ways in which a given force **F** may be resolved into two components is unlimited (Fig. 2.15). Two cases are of particular interest:

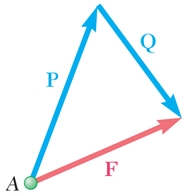


Fig. 2.16

1. *One of the Two Components,* **P**, *Is Known.* The second component, **Q**, is obtained by applying the triangle rule and joining the tip of **P** to the tip of **F** (Fig. 2.16); the magnitude and direction of **Q** are determined graphically or by trigonometry. Once **Q** has been determined, both components **P** and **Q** should be applied at A.
2. *The Line of Action of Each Component Is Known.* The magnitude and sense of the components are obtained by applying the parallelogram law and drawing lines, through the tip of **F**, parallel to the given lines of action (Fig. 2.17). This process leads to two well-defined components, **P** and **Q**, which can be determined graphically or computed trigonometrically by applying the law of sines.

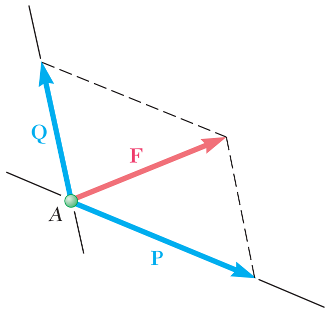
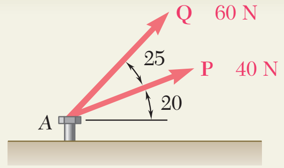


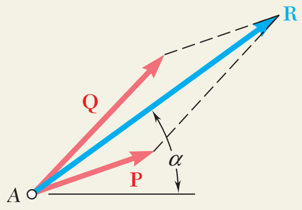
Fig. 2.17

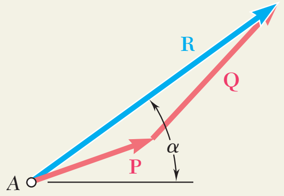
Many other cases can be encountered; for example, the direction of one component may be known, while the magnitude of the other component is to be as small as possible (see Sample Prob. 2.2). In all cases the appropriate triangle or parallelogram which satisfies the given conditions is drawn.

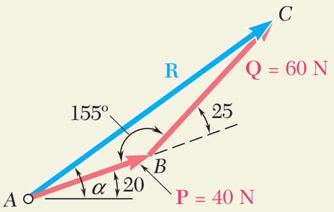


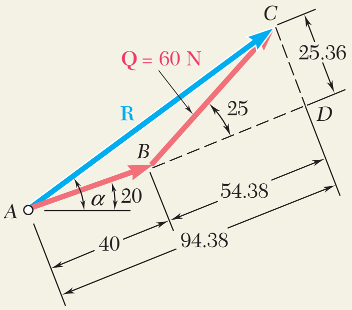
SAMPLE PROBLEM 2.1

The two forces **P** and **Q** act on a bolt A. Determine their resultant.









SOLUTION

Graphical Solution. A parallelogram with sides equal to **P** and **Q** is drawn to scale. The magnitude and direction of the resultant are measured and found to be

$$R = 98 \text{ N} \qquad \alpha = 35° \qquad \mathbf{R} = 98 \text{ N} \measuredangle 35° \quad ◀$$

The triangle rule may also be used. Forces **P** and **Q** are drawn in tip-to-tail fashion. Again the magnitude and direction of the resultant are measured.

$$R = 98 \text{ N} \qquad \alpha = 35° \qquad \mathbf{R} = 98 \text{ N} \measuredangle 35° \quad ◀$$

Trigonometric Solution. The triangle rule is again used; two sides and the included angle are known. We apply the law of cosines.

$$R^2 = P^2 + Q^2 - 2PQ \cos B$$
$$R^2 = (40 \text{ N})^2 + (60 \text{ N})^2 - 2(40 \text{ N})(60 \text{ N}) \cos 155°$$
$$R = 97.73 \text{ N}$$

Now, applying the law of sines, we write

$$\frac{\sin A}{Q} = \frac{\sin B}{R} \qquad \frac{\sin A}{60 \text{ N}} = \frac{\sin 155°}{97.73 \text{ N}} \tag{1}$$

Solving Eq. (1) for sin A, we have

$$\sin A = \frac{(60 \text{ N}) \sin 155°}{97.73 \text{ N}}$$

Using a calculator, we first compute the quotient, then its arc sine, and obtain

$$A = 15.04° \qquad \alpha = 20° + A = 35.04°$$

We use 3 significant figures to record the answer (cf. Sec. 1.7):

$$\mathbf{R} = 97.7 \text{ N} \measuredangle 35.0° \quad ◀$$

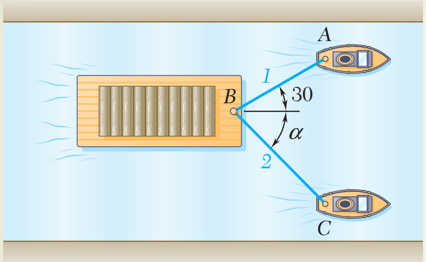
Alternative Trigonometric Solution. We construct the right triangle BCD and compute

$$CD = (60 \text{ N}) \sin 25° = 25.36 \text{ N}$$
$$BD = (60 \text{ N}) \cos 25° = 54.38 \text{ N}$$

Then, using triangle ACD, we obtain

$$\tan A = \frac{25.36 \text{ N}}{94.38 \text{ N}} \qquad A = 15.04°$$
$$R = \frac{25.36}{\sin A} \qquad R = 97.73 \text{ N}$$

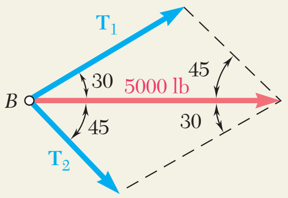
Again, $\alpha = 20° + A = 35.04°$ $\qquad \mathbf{R} = 97.7 \text{ N} \measuredangle 35.0° \quad ◀$



SAMPLE PROBLEM 2.2

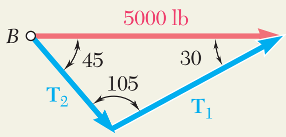
A barge is pulled by two tugboats. If the resultant of the forces exerted by the tugboats is a 5000-lb force directed along the axis of the barge, determine (*a*) the tension in each of the ropes knowing that $\alpha = 45°$, (*b*) the value of α for which the tension in rope *2* is minimum.

SOLUTION



a. Tension for $\alpha = 45°$. ***Graphical Solution.*** The parallelogram law is used; the diagonal (resultant) is known to be equal to 5000 lb and to be directed to the right. The sides are drawn parallel to the ropes. If the drawing is done to scale, we measure

$$T_1 = 3700 \text{ lb} \qquad T_2 = 2600 \text{ lb} \quad ◀$$

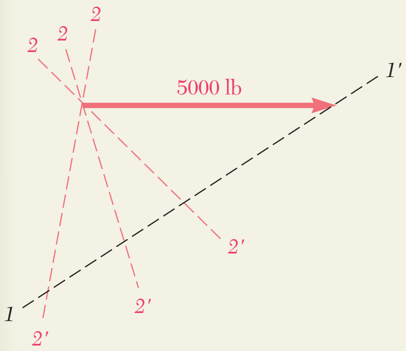


Trigonometric Solution. The triangle rule can be used. We note that the triangle shown represents half of the parallelogram shown above. Using the law of sines, we write

$$\frac{T_1}{\sin 45°} = \frac{T_2}{\sin 30°} = \frac{5000 \text{ lb}}{\sin 105°}$$

With a calculator, we first compute and store the value of the last quotient. Multiplying this value successively by sin 45° and sin 30°, we obtain

$$T_1 = 3660 \text{ lb} \qquad T_2 = 2590 \text{ lb} \quad ◀$$



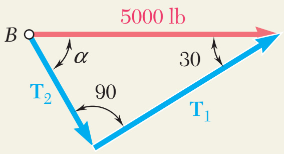
b. Value of α for Minimum T_2. To determine the value of α for which the tension in rope *2* is minimum, the triangle rule is again used. In the sketch shown, line *1-1′* is the known direction of $\mathbf{T}_1$. Several possible directions of $\mathbf{T}_2$ are shown by the lines *2-2′*. We note that the minimum value of T_2 occurs when $\mathbf{T}_1$ and $\mathbf{T}_2$ are perpendicular. The minimum value of T_2 is

$$T_2 = (5000 \text{ lb}) \sin 30° = 2500 \text{ lb}$$

Corresponding values of T_1 and α are

$$T_1 = (5000 \text{ lb}) \cos 30° = 4330 \text{ lb}$$

$$\alpha = 90° - 30° \qquad \alpha = 60° \quad ◀$$



PROBLEMS

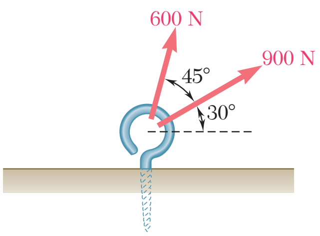


Fig. P2.1

2.1 and 2.2 Determine graphically the magnitude and direction of the resultant of the two forces shown using (*a*) the parallelogram law, (*b*) the triangle rule.

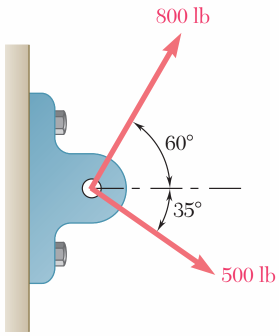


Fig. P2.2

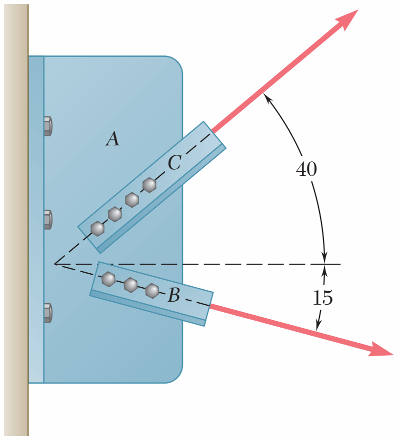


Fig. P2.3 and P2.4

2.3 Two structural members *B* and *C* are bolted to the bracket *A*. Knowing that the tension in member *B* is 6 kN and that the tension in *C* is 10 kN, determine graphically the magnitude and direction of the resultant force acting on the bracket.

2.4 Two structural members *B* and *C* are bolted to the bracket *A*. Knowing that the tension in member *B* is 2500 lb and that the tension in *C* is 2000 lb, determine graphically the magnitude and direction of the resultant force acting on the bracket.

2.5 The force **F** of magnitude 100 lb is to be resolved into two components along the lines *a-a* and *b-b*. Determine by trigonometry the angle α, knowing that the component of **F** along line *a-a* is 70 lb.

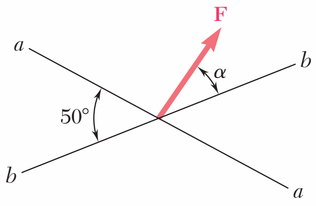


Fig. P2.5 and *P2.6*

2.6 The force **F** of magnitude 800 N is to be resolved into two components along the lines *a-a* and *b-b*. Determine by trigonometry the angle α, knowing that the component of **F** along line *b-b* is 120 N.

2.7 A trolley that moves along a horizontal beam is acted upon by two forces as shown. (*a*) Knowing that $\alpha = 25°$, determine by trigonometry the magnitude of the force **P** so that the resultant force exerted on the trolley is vertical. (*b*) What is the corresponding magnitude of the resultant?

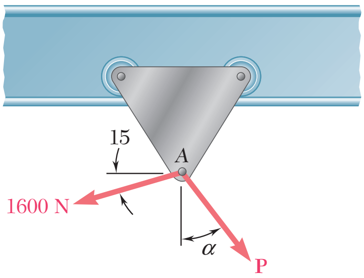


Fig. P2.7 and *P2.11*

2.8 A disabled automobile is pulled by means of two ropes as shown. The tension in *AB* is 500 lb, and the angle α is 25°. Knowing that the resultant of the two forces applied at *A* is directed along the axis of the automobile, determine by trigonometry (*a*) the tension in rope *AC*, (*b*) the magnitude of the resultant of the two forces applied at *A*.

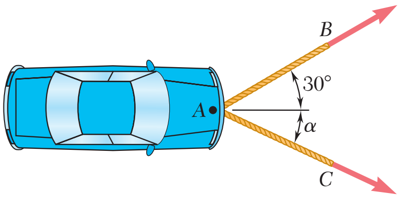


Fig. *P2.8* and P2.10

2.9 Determine by trigonometry the magnitude of the force **P** so that the resultant of the two forces applied at *A* is vertical. What is the corresponding magnitude of the resultant?

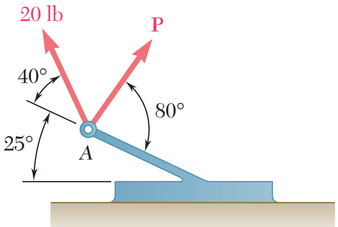


Fig. P2.9 and P2.12

2.10 A disabled automobile is pulled by means of two ropes as shown. Knowing that the tension in rope *AB* is 750 lb, determine by trigonometry the tension in rope *AC* and the value of α so that the resultant force exerted at *A* is a 1200-lb force directed along the axis of the automobile.

2.11 A trolley that moves along a horizontal beam is acted upon by two forces as shown. Determine by trigonometry the magnitude and direction of the force **P** so that the resultant is a vertical force of 2500 N.

2.12 Knowing that $P = 30$ lb, determine by trigonometry the resultant of the two forces applied at point A.

2.13 Solve Prob. 2.1 by trigonometry.

2.14 Solve Prob. 2.4 by trigonometry.

2.15 If the resultant of the two forces exerted on the trolley of Prob. 2.7 is to be vertical, determine (*a*) the value of α for which the magnitude of **P** is minimum, (*b*) the corresponding magnitude of P.

2.7 RECTANGULAR COMPONENTS OF A FORCE. UNIT VECTORS†

In many problems it will be found desirable to resolve a force into two components which are perpendicular to each other. In Fig. 2.18, the force **F** has been resolved into a component $\mathbf{F}_x$ along the x axis and a component $\mathbf{F}_y$ along the y axis. The parallelogram drawn to obtain the two components is a *rectangle,* and $\mathbf{F}_x$ and $\mathbf{F}_y$ are called *rectangular components.*

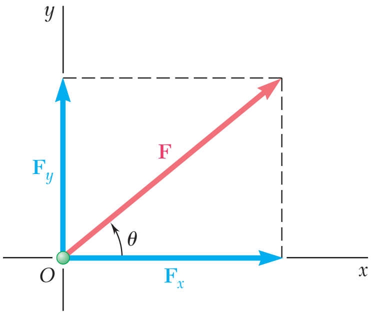


Fig. 2.18

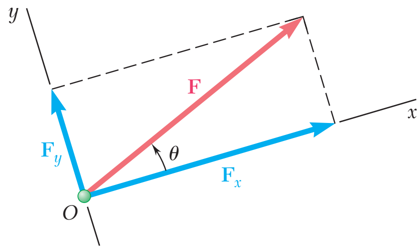


Fig. 2.19

The x and y axes are usually chosen horizontal and vertical, respectively, as in Fig. 2.18; they may, however, be chosen in any two perpendicular directions, as shown in Fig. 2.19. In determining the rectangular components of a force, the student should think of the construction lines shown in Figs. 2.18 and 2.19 as being *parallel* to the x and y axes, rather than *perpendicular* to these axes. This practice will help avoid mistakes in determining *oblique* components as in Sec. 2.6.

†The properties established in Secs. 2.7 and 2.8 may be readily extended to the rectangular components of any vector quantity.

Two vectors of unit magnitude, directed respectively along the positive x and y axes, will be introduced at this point. These vectors are called *unit vectors* and are denoted by $\mathbf{i}$ and $\mathbf{j}$, respectively (Fig. 2.20). Recalling the definition of the product of a scalar and a vector given in Sec. 2.4, we note that the rectangular components $\mathbf{F}_x$ and $\mathbf{F}_y$ of a force $\mathbf{F}$ may be obtained by multiplying respectively the unit vectors $\mathbf{i}$ and $\mathbf{j}$ by appropriate scalars (Fig. 2.21). We write

$$\mathbf{F}_x = F_x\mathbf{i} \qquad \mathbf{F}_y = F_y\mathbf{j} \tag{2.6}$$

and

$$\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} \tag{2.7}$$

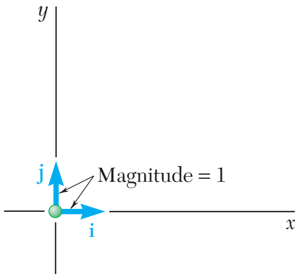


Fig. 2.20

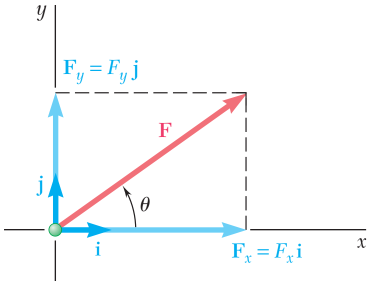


Fig. 2.21

While the scalars F_x and F_y may be positive or negative, depending upon the sense of $\mathbf{F}_x$ and of $\mathbf{F}_y$, their absolute values are respectively equal to the magnitudes of the component forces $\mathbf{F}_x$ and $\mathbf{F}_y$. The scalars F_x and F_y are called the *scalar components* of the force $\mathbf{F}$, while the actual component forces $\mathbf{F}_x$ and $\mathbf{F}_y$ should be referred to as the *vector components* of $\mathbf{F}$. However, when there exists no possibility of confusion, the vector as well as the scalar components of $\mathbf{F}$ may be referred to simply as the *components* of $\mathbf{F}$. We note that the scalar component F_x is positive when the vector component $\mathbf{F}_x$ has the same sense as the unit vector $\mathbf{i}$ (i.e., the same sense as the positive x axis) and is negative when $\mathbf{F}_x$ has the opposite sense. A similar conclusion may be drawn regarding the sign of the scalar component F_y.

Denoting by F the magnitude of the force $\mathbf{F}$ and by θ the angle between $\mathbf{F}$ and the x axis, measured counterclockwise from the positive x axis (Fig. 2.21), we may express the scalar components of $\mathbf{F}$ as follows:

$$F_x = F\cos\theta \qquad F_y = F\sin\theta \tag{2.8}$$

We note that the relations obtained hold for any value of the angle θ from 0° to 360° and that they define the signs as well as the absolute values of the scalar components F_x and F_y.

When a force $\mathbf{F}$ is defined by its rectangular components F_x and F_y (see Fig. 2.21), the angle θ defining its direction can be obtained by writing

$$\tan\theta = \frac{F_y}{F_x} \tag{2.9}$$

The magnitude F of the force can be obtained by applying the Pythagorean theorem and writing

$$F = \sqrt{F_x^2 + F_y^2} \tag{2.10}$$

or by solving for F from one of the formulas in Eqs. (2.8).

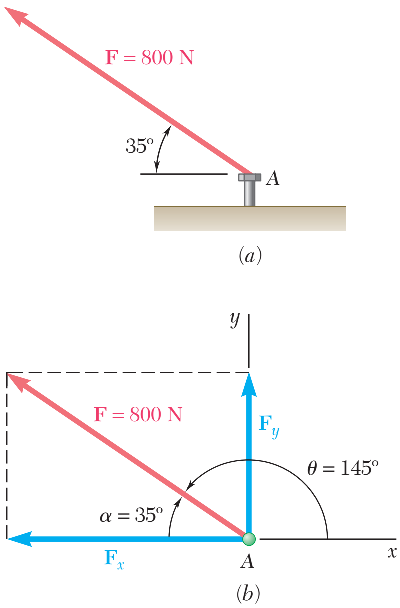


Fig. 2.22

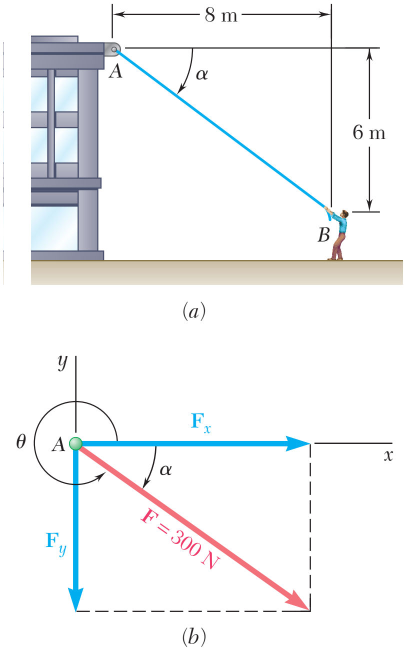


Fig. 2.23

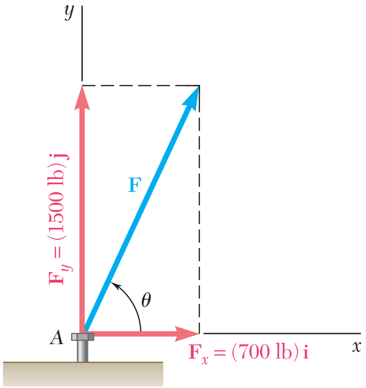


Fig. 2.24

EXAMPLE 2.1 A force of 800 N is exerted on a bolt A as shown in Fig. 2.22a. Determine the horizontal and vertical components of the force.

In order to obtain the correct sign for the scalar components F_x and F_y, the value $180° - 35° = 145°$ should be substituted for θ in Eqs. (2.8). However, it will be found more practical to determine by inspection the signs of F_x and F_y (Fig. 2.22b) and to use the trigonometric functions of the angle $\alpha = 35°$. We write, therefore,

$$F_x = -F \cos \alpha = -(800 \text{ N}) \cos 35° = -655 \text{ N}$$
$$F_y = +F \sin \alpha = +(800 \text{ N}) \sin 35° = +459 \text{ N}$$

The vector components of **F** are thus

$$\mathbf{F}_x = -(655 \text{ N})\mathbf{i} \qquad \mathbf{F}_y = +(459 \text{ N})\mathbf{j}$$

and we may write **F** in the form

$$\mathbf{F} = -(655 \text{ N})\mathbf{i} + (459 \text{ N})\mathbf{j} \;\blacksquare$$

EXAMPLE 2.2 A man pulls with a force of 300 N on a rope attached to a building, as shown in Fig. 2.23a. What are the horizontal and vertical components of the force exerted by the rope at point A?

It is seen from Fig. 2.23b that

$$F_x = +(300 \text{ N}) \cos \alpha \qquad F_y = -(300 \text{ N}) \sin \alpha$$

Observing that $AB = 10$ m, we find from Fig. 2.23a

$$\cos \alpha = \frac{8 \text{ m}}{AB} = \frac{8 \text{ m}}{10 \text{ m}} = \frac{4}{5} \qquad \sin \alpha = \frac{6 \text{ m}}{AB} = \frac{6 \text{ m}}{10 \text{ m}} = \frac{3}{5}$$

We thus obtain

$$F_x = +(300 \text{ N})\tfrac{4}{5} = +240 \text{ N} \qquad F_y = -(300 \text{ N})\tfrac{3}{5} = -180 \text{ N}$$

and write

$$\mathbf{F} = (240 \text{ N})\mathbf{i} - (180 \text{ N})\mathbf{j} \;\blacksquare$$

EXAMPLE 2.3 A force $\mathbf{F} = (700 \text{ lb})\mathbf{i} + (1500 \text{ lb})\mathbf{j}$ is applied to a bolt A. Determine the magnitude of the force and the angle θ it forms with the horizontal.

First we draw a diagram showing the two rectangular components of the force and the angle θ (Fig. 2.24). From Eq. (2.9), we write

$$\tan \theta = \frac{F_y}{F_x} = \frac{1500 \text{ lb}}{700 \text{ lb}}$$

Using a calculator,† we enter 1500 lb and divide by 700 lb; computing the arc tangent of the quotient, we obtain $\theta = 65.0°$. Solving the second formula of Eqs. (2.8) for F, we have

$$F = \frac{F_y}{\sin \theta} = \frac{1500 \text{ lb}}{\sin 65.0°} = 1655 \text{ lb}$$

The last calculation is facilitated if the value of F_y is stored when originally entered; it may then be recalled to be divided by $\sin \theta$. ■

†It is assumed that the calculator used has keys for the computation of trigonometric and inverse trigonometric functions. Some calculators also have keys for the direct conversion of rectangular coordinates into polar coordinates, and vice versa. Such calculators eliminate the need for the computation of trigonometric functions in Examples 2.1, 2.2, and 2.3 and in problems of the same type.

2.8 ADDITION OF FORCES BY SUMMING *X* AND *Y* COMPONENTS

It was seen in Sec. 2.2 that forces should be added according to the parallelogram law. From this law, two other methods, more readily applicable to the *graphical* solution of problems, were derived in Secs. 2.4 and 2.5: the triangle rule for the addition of two forces and the polygon rule for the addition of three or more forces. It was also seen that the force triangle used to define the resultant of two forces could be used to obtain a *trigonometric* solution.

When three or more forces are to be added, no practical trigonometric solution can be obtained from the force polygon which defines the resultant of the forces. In this case, an *analytic* solution of the problem can be obtained by resolving each force into two rectangular components. Consider, for instance, three forces **P**, **Q**, and **S** acting on a particle A (Fig. 2.25*a*). Their resultant **R** is defined by the relation

$$\mathbf{R} = \mathbf{P} + \mathbf{Q} + \mathbf{S} \tag{2.11}$$

Resolving each force into its rectangular components, we write

$$\begin{aligned} R_x\mathbf{i} + R_y\mathbf{j} &= P_x\mathbf{i} + P_y\mathbf{j} + Q_x\mathbf{i} + Q_y\mathbf{j} + S_x\mathbf{i} + S_y\mathbf{j} \\ &= (P_x + Q_x + S_x)\mathbf{i} + (P_y + Q_y + S_y)\mathbf{j} \end{aligned}$$

from which it follows that

$$R_x = P_x + Q_x + S_x \qquad R_y = P_y + Q_y + S_y \tag{2.12}$$

or, for short,

$$R_x = \Sigma F_x \qquad R_y = \Sigma F_y \tag{2.13}$$

We thus conclude that *the scalar components R_x and R_y of the resultant* **R** *of several forces acting on a particle are obtained by adding algebraically the corresponding scalar components of the given forces.*†

In practice, the determination of the resultant **R** is carried out in three steps as illustrated in Fig. 2.25. First the given forces shown in Fig. 2.25*a* are resolved into their x and y components (Fig. 2.25*b*). Adding these components, we obtain the x and y components of **R** (Fig. 2.25*c*). Finally, the resultant $\mathbf{R} = R_x\mathbf{i} + R_y\mathbf{j}$ is determined by applying the parallelogram law (Fig. 2.25*d*). The procedure just described will be carried out most efficiently if the computations are arranged in a table. While it is the only practical analytic method for adding three or more forces, it is also often preferred to the trigonometric solution in the case of the addition of two forces.

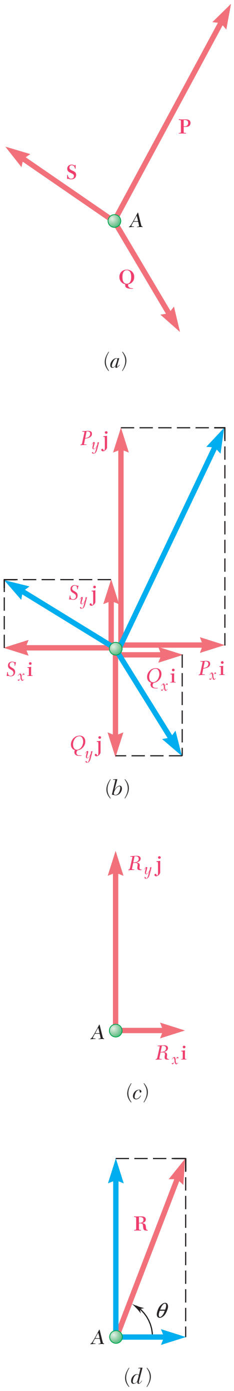
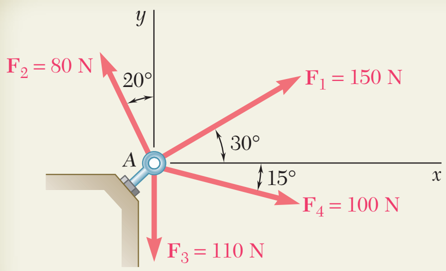


Fig. 2.25

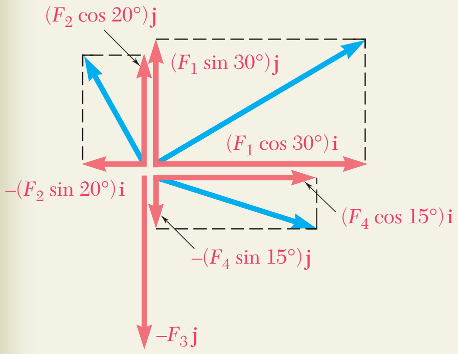
†Clearly, this result also applies to the addition of other vector quantities, such as velocities, accelerations, or momenta.



SAMPLE PROBLEM 2.3

Four forces act on bolt A as shown. Determine the resultant of the forces on the bolt.

SOLUTION



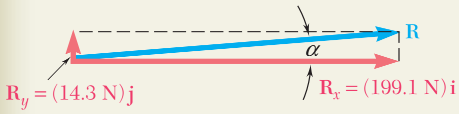
The x and y components of each force are determined by trigonometry as shown and are entered in the table below. According to the convention adopted in Sec. 2.7, the scalar number representing a force component is positive if the force component has the same sense as the corresponding coordinate axis. Thus, x components acting to the right and y components acting upward are represented by positive numbers.

Force	Magnitude, N	x Component, N	y Component, N
$\mathbf{F}_1$	150	+129.9	+75.0
$\mathbf{F}_2$	80	−27.4	+75.2
$\mathbf{F}_3$	110	0	−110.0
$\mathbf{F}_4$	100	+96.6	−25.9
		$R_x = +199.1$	$R_y = +14.3$

Thus, the resultant $\mathbf{R}$ of the four forces is

$$\mathbf{R} = R_x\mathbf{i} + R_y\mathbf{j} \qquad \mathbf{R} = (199.1 \text{ N})\mathbf{i} + (14.3 \text{ N})\mathbf{j} \quad ◀$$

The magnitude and direction of the resultant may now be determined. From the triangle shown, we have



$$\tan\alpha = \frac{R_y}{R_x} = \frac{14.3 \text{ N}}{199.1 \text{ N}} \qquad \alpha = 4.1°$$

$$R = \frac{14.3 \text{ N}}{\sin\alpha} = 199.6 \text{ N} \qquad \mathbf{R} = 199.6 \text{ N} \measuredangle 4.1° \quad ◀$$

With a calculator, the last computation may be facilitated if the value of R_y is stored when originally entered; it may then be recalled to be divided by sin α. (Also see the footnote on p. 28.)

PROBLEMS

2.16 through 2.19 Determine the x and y components of each of the forces shown.

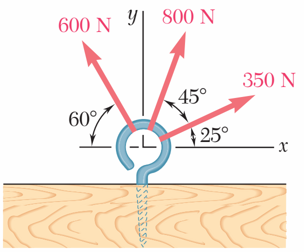


Fig. P2.16

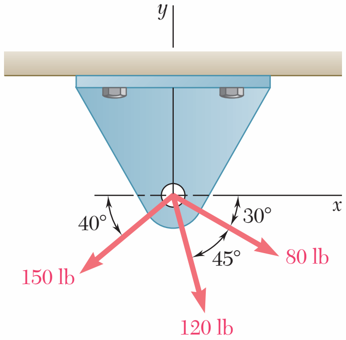


Fig. P2.17

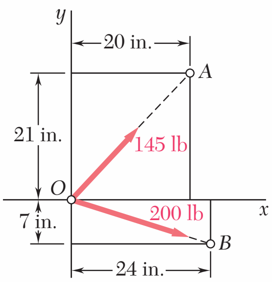


Fig. P2.18

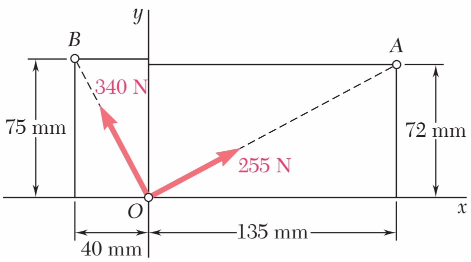


Fig. P2.19

2.20 The tension in the support wire AB is 65 lb. Determine the horizontal and vertical components of the force acting on the pin at A.

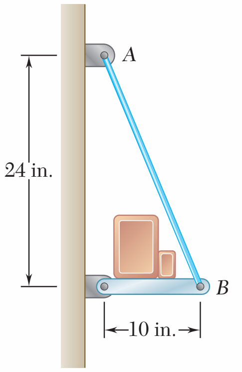


Fig. P2.20

2.21 The hydraulic cylinder GE exerts on member DF a force $\mathbf{P}$ directed along line GE. Knowing that $\mathbf{P}$ must have a 600-N component perpendicular to member DF, determine the magnitude of $\mathbf{P}$ and its component parallel to DF.

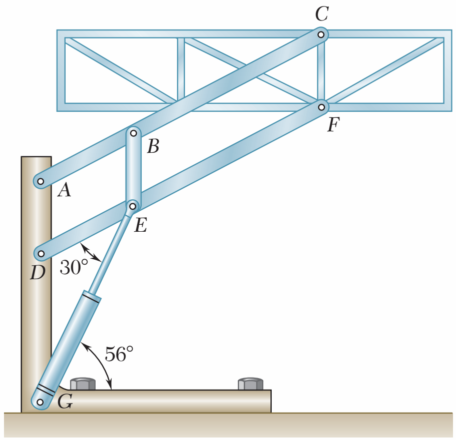


Fig. P2.21

2.22 Cable AC exerts on beam AB a force $\mathbf{P}$ directed along line AC. Knowing that $\mathbf{P}$ must have a 350-lb vertical component, determine (*a*) the magnitude of the force $\mathbf{P}$, (*b*) its horizontal component.

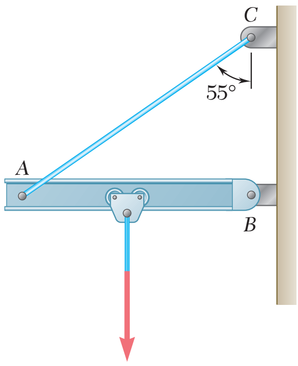


Fig. ***P2.22***

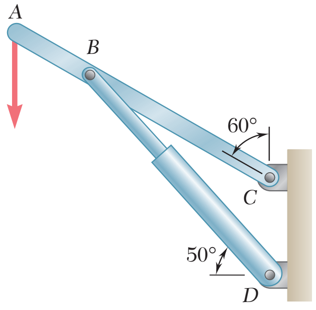


Fig. P2.23

2.23 The hydraulic cylinder BD exerts on member ABC a force $\mathbf{P}$ directed along line BD. Knowing that $\mathbf{P}$ must have a 750-N component perpendicular to member ABC, determine (*a*) the magnitude of the force $\mathbf{P}$, (*b*) its component parallel to ABC.

2.24 Using x and y components, solve Prob. 2.1.

2.25 Using x and y components, solve Prob. 2.2.

2.26 Determine the resultant of the three forces of Prob. 2.17.

2.27 Determine the resultant of the three forces of Prob. 2.19.

2.28 Two cables of known tensions are attached to the top of pylon AB. A third cable AC is used as a guy wire. Determine the tension in AC, knowing that the resultant of the forces exerted at A by the three cables must be vertical.

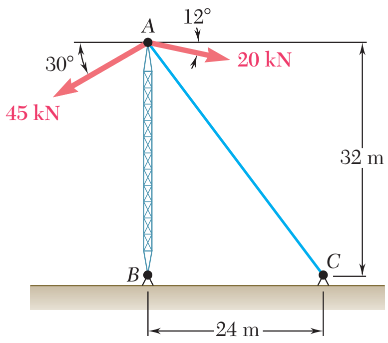


Fig. ***P2.28***

2.29 A hoist trolley is subjected to the three forces shown. Knowing that $\alpha = 40°$, determine (*a*) the magnitude of the force **P** for which the resultant of the three forces is vertical, (*b*) the corresponding magnitude of the resultant.

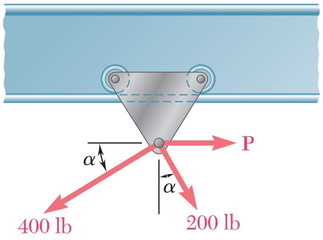


Fig. P2.29 and *P2.30*

2.30 A hoist trolley is subjected to the three forces shown. Knowing that $P = 250$ lb, determine (*a*) the value of the angle α for which the resultant of the three forces is vertical, (*b*) the corresponding magnitude of the resultant.

2.31 A collar that can slide on a vertical rod is subjected to the three forces shown. The direction of the force **F** may be varied. If possible, determine the direction of the force **F** so that the resultant of the three forces is horizontal, knowing that the magnitude of **F** is (*a*) 2.4 kN, (*b*) 1.4 kN.

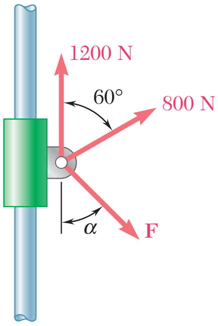


Fig. P2.31

2.9 EQUILIBRIUM OF A PARTICLE

In the preceding sections, we discussed the methods for determining the resultant of several forces acting on a particle. Although it has not occurred in any of the problems considered so far, it is quite possible for the resultant to be zero. In such a case, the net effect of the given forces is zero, and the particle is said to be in equilibrium. We thus have the following definition: *When the resultant of all the forces acting on a particle is zero, the particle is in equilibrium.*

A particle which is acted upon by two forces will be in equilibrium if the two forces have the same magnitude and the same line of action but opposite sense. The resultant of the two forces is then zero. Such a case is shown in Fig. 2.26.

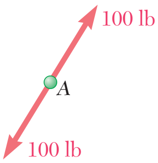


Fig. 2.26

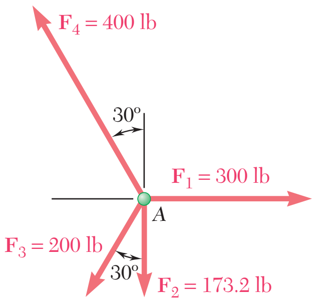


Fig. 2.27

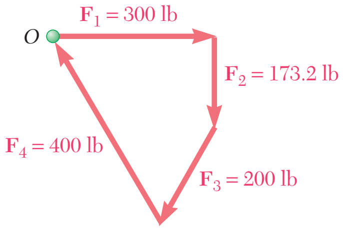


Fig. 2.28

Another case of equilibrium of a particle is represented in Fig. 2.27, where four forces are shown acting on A. In Fig. 2.28, the resultant of the given forces is determined by the polygon rule. Starting from point O with $\mathbf{F}_1$ and arranging the forces in tip-to-tail fashion, we find that the tip of $\mathbf{F}_4$ coincides with the starting point O. Thus the resultant $\mathbf{R}$ of the given system of forces is zero, and the particle is in equilibrium.

The closed polygon drawn in Fig. 2.28 provides a *graphical* expression of the equilibrium of A. To express *algebraically* the conditions for the equilibrium of a particle, we write

$$\mathbf{R} = \Sigma\mathbf{F} = 0 \tag{2.14}$$

Resolving each force $\mathbf{F}$ into rectangular components, we have

$$\Sigma(F_x\mathbf{i} + F_y\mathbf{j}) = 0 \qquad \text{or} \qquad (\Sigma F_x)\mathbf{i} + (\Sigma F_y)\mathbf{j} = 0$$

We conclude that the necessary and sufficient conditions for the equilibrium of a particle are

$$\Sigma F_x = 0 \qquad \Sigma F_y = 0 \tag{2.15}$$

Returning to the particle shown in Fig. 2.27, we check that the equilibrium conditions are satisfied. We write

$$\begin{aligned}
\Sigma F_x &= 300 \text{ lb} - (200 \text{ lb}) \sin 30° - (400 \text{ lb}) \sin 30° \\
&= 300 \text{ lb} - 100 \text{ lb} - 200 \text{ lb} = 0 \\
\Sigma F_y &= -173.2 \text{ lb} - (200 \text{ lb}) \cos 30° + (400 \text{ lb}) \cos 30° \\
&= -173.2 \text{ lb} - 173.2 \text{ lb} + 346.4 \text{ lb} = 0
\end{aligned}$$

2.10 NEWTON'S FIRST LAW OF MOTION

In the latter part of the seventeenth century, Sir Isaac Newton formulated three fundamental laws upon which the science of mechanics is based. The first of these laws can be stated as follows:

If the resultant force acting on a particle is zero, the particle will remain at rest (if originally at rest) or will move with constant speed in a straight line (if originally in motion).

From this law and from the definition of equilibrium given in Sec. 2.9, it is seen that a particle in equilibrium either is at rest or is moving in a straight line with constant speed. In the following section, various problems concerning the equilibrium of a particle will be considered.

2.11 PROBLEMS INVOLVING THE EQUILIBRIUM OF A PARTICLE. FREE-BODY DIAGRAMS

In practice, a problem in engineering mechanics is derived from an actual physical situation. A sketch showing the physical conditions of the problem is known as a *space diagram.*

The methods of analysis discussed in the preceding sections apply to a system of forces acting on a particle. A large number of problems involving actual structures, however, can be reduced to problems concerning the equilibrium of a particle. This is done by choosing a significant particle and drawing a separate diagram showing this particle and all the forces acting on it. Such a diagram is called a *free-body diagram.*

As an example, consider the 75-kg crate shown in the space diagram of Fig. 2.29*a.* This crate was lying between two buildings, and it is now being lifted onto a truck, which will remove it. The crate is supported by a vertical cable, which is joined at A to two ropes which pass over pulleys attached to the buildings at B and C. It is desired to determine the tension in each of the ropes AB and AC.

In order to solve this problem, a free-body diagram showing a particle in equilibrium must be drawn. Since we are interested in the rope tensions, the free-body diagram should include at least one of these tensions or, if possible, both tensions. Point A is seen to be a good free body for this problem. The free-body diagram of point A is shown in Fig. 2.29*b*. It shows point A and the forces exerted on A by the vertical cable and the two ropes. The force exerted by the cable is directed downward, and its magnitude is equal to the weight W of the crate. Recalling Eq. (1.4), we write

$$W = mg = (75 \text{ kg})(9.81 \text{ m/s}^2) = 736 \text{ N}$$

and indicate this value in the free-body diagram. The forces exerted by the two ropes are not known. Since they are respectively equal in magnitude to the tensions in rope AB and rope AC, we denote them by $\mathbf{T}_{AB}$ and $\mathbf{T}_{AC}$ and draw them away from A in the directions shown in the space diagram. No other detail is included in the free-body diagram.

Since point A is in equilibrium, the three forces acting on it must form a closed triangle when drawn in tip-to-tail fashion. This *force triangle* has been drawn in Fig. 2.29*c*. The values T_{AB} and T_{AC} of the tension in the ropes may be found graphically if the triangle is drawn to scale, or they may be found by trigonometry. If the latter method of solution is chosen, we use the law of sines and write

$$\frac{T_{AB}}{\sin 60°} = \frac{T_{AC}}{\sin 40°} = \frac{736 \text{ N}}{\sin 80°}$$

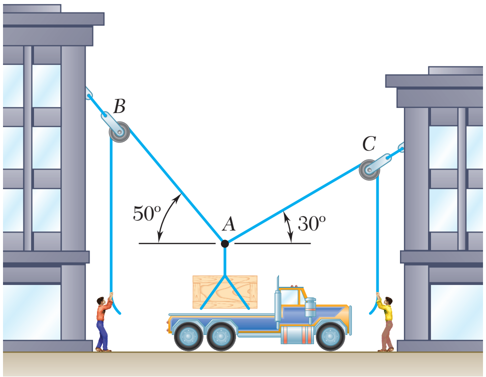
$$T_{AB} = 647 \text{ N} \qquad T_{AC} = 480 \text{ N}$$

When a particle is in *equilibrium under three forces,* the problem can be solved by drawing a force triangle. When a particle is in *equilibrium under more than three forces,* the problem can be solved graphically by drawing a force polygon. If an analytic solution is desired, the *equations of equilibrium* given in Sec. 2.9 should be solved:

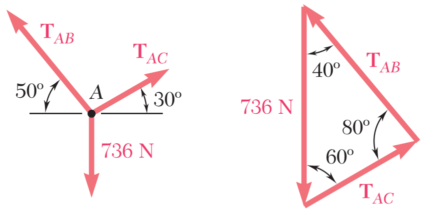
$$\Sigma F_x = 0 \qquad \Sigma F_y = 0 \tag{2.15}$$

These equations can be solved for no more than *two unknowns;* similarly, the force triangle used in the case of equilibrium under three forces can be solved for two unknowns.

The more common types of problems are those in which the two unknowns represent (1) the two components (or the magnitude and direction) of a single force, (2) the magnitudes of two forces, each of known direction. Problems involving the determination of the maximum or minimum value of the magnitude of a force are also encountered (see Probs. 2.40 through 2.45).



(*a*) Space diagram

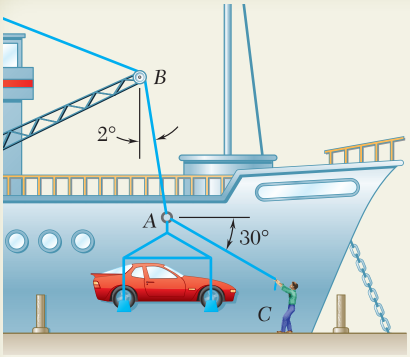


(*b*) Free-body diagram (*c*) Force triangle

Fig. 2.29

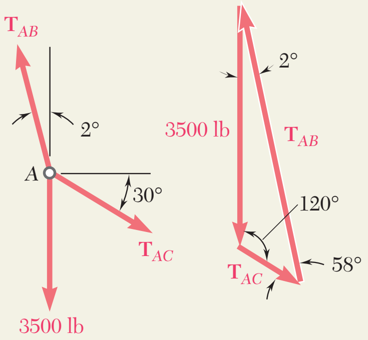
Photo 2.1 As illustrated in the above example, it is possible to determine the tensions in the cables supporting the shaft shown by treating the hook as a particle and then applying the equations of equilibrium to the forces acting on the hook.

SAMPLE PROBLEM 2.4



In a ship-unloading operation, a 3500-lb automobile is supported by a cable. A rope is tied to the cable at A and pulled in order to center the automobile over its intended position. The angle between the cable and the vertical is 2°, while the angle between the rope and the horizontal is 30°. What is the tension in the rope?

SOLUTION



Free-Body Diagram. Point A is chosen as a free body, and the complete free-body diagram is drawn. T_{AB} is the tension in the cable AB, and T_{AC} is the tension in the rope.

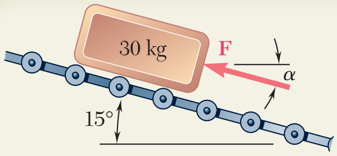
Equilibrium Condition. Since only three forces act on the free body, we draw a force triangle to express that it is in equilibrium. Using the law of sines, we write

$$\frac{T_{AB}}{\sin 120°} = \frac{T_{AC}}{\sin 2°} = \frac{3500 \text{ lb}}{\sin 58°}$$

With a calculator, we first compute and store the value of the last quotient. Multiplying this value successively by sin 120° and sin 2°, we obtain

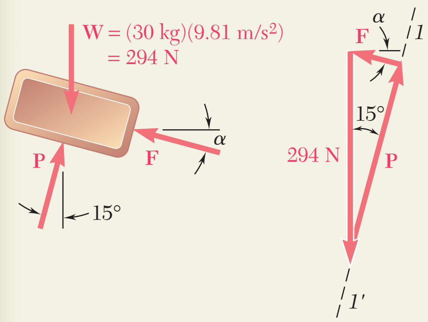
$$T_{AB} = 3570 \text{ lb} \qquad T_{AC} = 144 \text{ lb} \blacktriangleleft$$

SAMPLE PROBLEM 2.5



Determine the magnitude and direction of the smallest force **F** which will maintain the package shown in equilibrium. Note that the force exerted by the rollers on the package is perpendicular to the incline.

SOLUTION

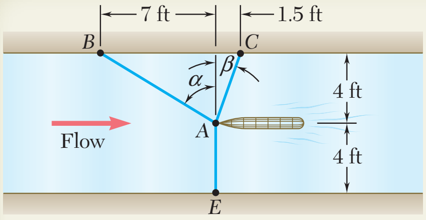


Free-Body Diagram. We choose the package as a free body, assuming that it can be treated as a particle. We draw the corresponding free-body diagram.

Equilibrium Condition. Since only three forces act on the free body, we draw a force triangle to express that it is in equilibrium. Line 1-$1'$ represents the known direction of **P**. In order to obtain the minimum value of the force **F**, we choose the direction of **F** perpendicular to that of **P**. From the geometry of the triangle obtained, we find

$$F = (294 \text{ N}) \sin 15° = 76.1 \text{ N} \qquad \alpha = 15°$$

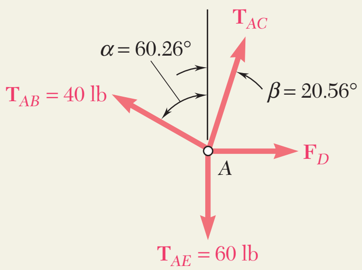
$$\mathbf{F} = 76.1 \text{ N} \measuredangle 15° \blacktriangleleft$$



SAMPLE PROBLEM 2.6

As part of the design of a new sailboat, it is desired to determine the drag force which may be expected at a given speed. To do so, a model of the proposed hull is placed in a test channel and three cables are used to keep its bow on the centerline of the channel. Dynamometer readings indicate that for a given speed, the tension is 40 lb in cable *AB* and 60 lb in cable *AE*. Determine the drag force exerted on the hull and the tension in cable *AC*.

SOLUTION



Determination of the Angles. First, the angles α and β defining the direction of cables *AB* and *AC* are determined. We write

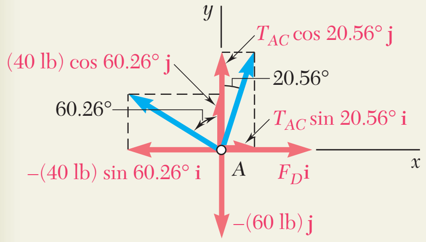
$$\tan \alpha = \frac{7 \text{ ft}}{4 \text{ ft}} = 1.75 \qquad \tan \beta = \frac{1.5 \text{ ft}}{4 \text{ ft}} = 0.375$$

$$\alpha = 60.26° \qquad \beta = 20.56°$$

Free-Body Diagram. Choosing the hull as a free body, we draw the free-body diagram shown. It includes the forces exerted by the three cables on the hull, as well as the drag force $\mathbf{F}_D$ exerted by the flow.

Equilibrium Condition. We express that the hull is in equilibrium by writing that the resultant of all forces is zero:

$$\mathbf{R} = \mathbf{T}_{AB} + \mathbf{T}_{AC} + \mathbf{T}_{AE} + \mathbf{F}_D = 0 \qquad (1)$$



Since more than three forces are involved, we resolve the forces into *x* and *y* components:

$$\begin{aligned}
\mathbf{T}_{AB} &= -(40 \text{ lb}) \sin 60.26°\mathbf{i} + (40 \text{ lb}) \cos 60.26°\mathbf{j} \\
&= -(34.73 \text{ lb})\mathbf{i} + (19.84 \text{ lb})\mathbf{j} \\
\mathbf{T}_{AC} &= T_{AC} \sin 20.56°\mathbf{i} + T_{AC} \cos 20.56°\mathbf{j} \\
&= 0.3512T_{AC}\mathbf{i} + 0.9363T_{AC}\mathbf{j} \\
\mathbf{T}_{AE} &= -(60 \text{ lb})\mathbf{j} \\
\mathbf{F}_D &= F_D\mathbf{i}
\end{aligned}$$

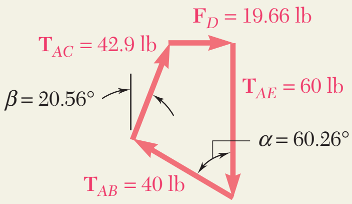
Substituting the expressions obtained into Eq. (1) and factoring the unit vectors **i** and **j**, we have

$$(-34.73 \text{ lb} + 0.3512T_{AC} + F_D)\mathbf{i} + (19.84 \text{ lb} + 0.9363T_{AC} - 60 \text{ lb})\mathbf{j} = 0$$

This equation will be satisfied if, and only if, the coefficients of **i** and **j** are equal to zero. We thus obtain the following two equilibrium equations, which express, respectively, that the sum of the *x* components and the sum of the *y* components of the given forces must be zero.

$$(\Sigma F_x = 0:) \qquad -34.73 \text{ lb} + 0.3512T_{AC} + F_D = 0 \qquad (2)$$

$$(\Sigma F_y = 0:) \qquad 19.84 \text{ lb} + 0.9363T_{AC} - 60 \text{ lb} = 0 \qquad (3)$$



From Eq. (3) we find $\qquad T_{AC} = +42.9 \text{ lb}$ ◀

and, substituting this value into Eq. (2), $\qquad F_D = +19.66 \text{ lb}$ ◀

In drawing the free-body diagram, we assumed a sense for each unknown force. A positive sign in the answer indicates that the assumed sense is correct. The complete force polygon may be drawn to check the results.

PROBLEMS

2.32 through 2.35 Two cables are tied together at C and loaded as shown. Determine the tension in AC and BC.

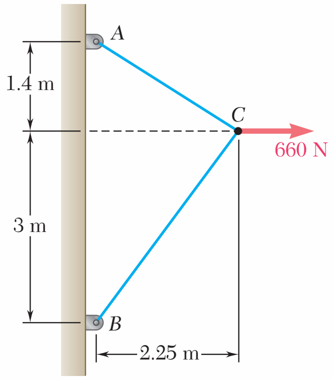


Fig. P2.32

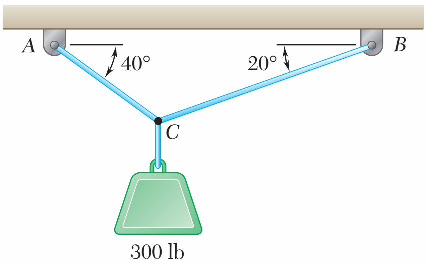


Fig. P2.33

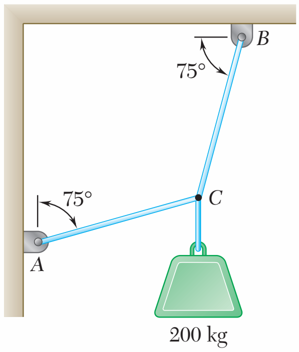


Fig. P2.34

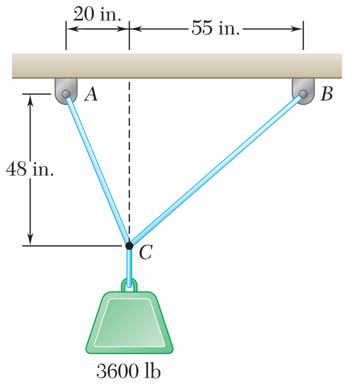


Fig. P2.35

2.36 Two cables are tied together at C and loaded as shown. Knowing that $P = 500$ N and $\alpha = 60°$, determine the tension in AC and BC.

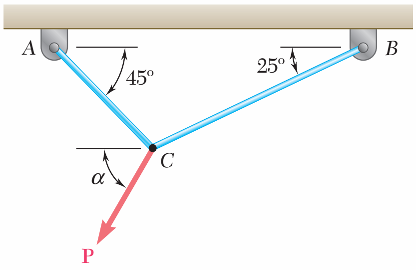


Fig. P2.36

2.37 Two forces of magnitude $T_A = 8$ kips and $T_B = 15$ kips are applied as shown to a welded connection. Knowing that the connection is in equilibrium, determine the magnitudes of the forces T_C and T_D.

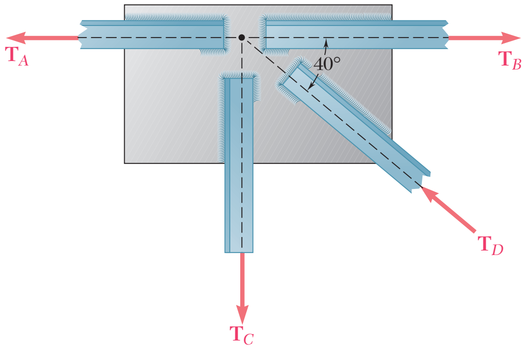


Fig. *P2.37* and P2.38

2.38 Two forces of magnitude $T_A = 6$ kips and $T_C = 9$ kips are applied as shown to a welded connection. Knowing that the connection is in equilibrium, determine the magnitudes of the forces T_B and T_D.

2.39 Two forces of magnitude $T_A = 5000$ N and $T_B = 2500$ N are applied as shown to the connection shown. Knowing that the connection is in equilibrium, determine the magnitudes of the forces T_C and T_D.

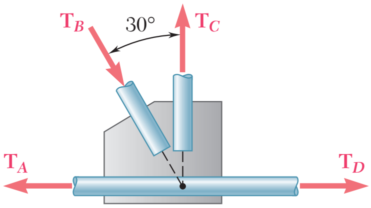


Fig. *P2.39*

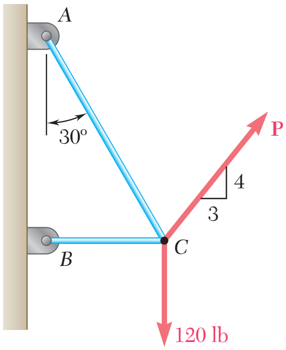


Fig. P2.40

2.40 Determine the range of values of **P** for which both cables remain taut.

2.41 For the cables of Prob. 2.36, it is known that the maximum allowable tension is 600 N in cable AC and 750 N in cable BC. Determine (*a*) the maximum force **P** that can be applied at C, (*b*) the corresponding value of α.

2.42 Two ropes are tied together at C. If the maximum permissible tension in each rope is 2.5 kN, what is the maximum force **F** that can be applied? In what direction must this maximum force act?

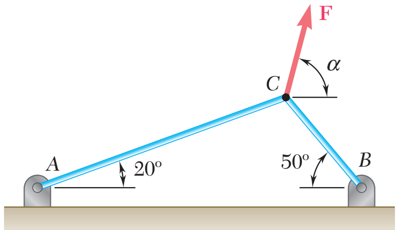


Fig. P2.42

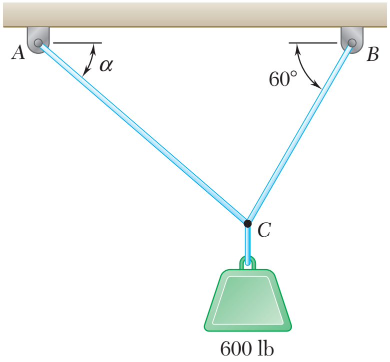


Fig. P2.43 and *P2.44*

2.43 A 600-lb block is supported by two cables *AC* and *BC*. (*a*) For what value of α is the tension in cable *AC* maximum? (*b*) What are the corresponding values of the tension in cables *AC* and *BC*?

2.44 A 600-lb block is supported by two cables *AC* and *BC*. Determine (*a*) the value of α for which the larger of the cable tensions is as small as possible, (*b*) the corresponding values of the tension in cables *AC* and *BC*.

2.45 Two cables are tied together at *C* as shown. Find the value of α for which the tension is as small as possible (*a*) in cable *BC*, (*b*) in both cables simultaneously. In each case determine the tension in both cables.

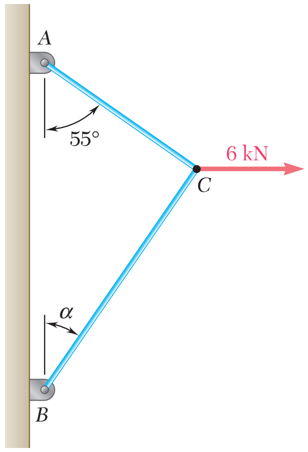


Fig. P2.45

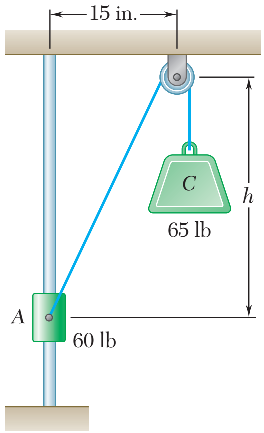


Fig. P2.46

2.46 The 60-lb collar *A* can slide on a frictionless vertical rod and is connected as shown to a 65-lb counterweight *C*. Determine the value of *h* for which the system is in equilibrium.

2.47 The force **P** is applied to a small wheel that rolls on the cable *ACB*. Knowing that the tension in both parts of the cable is 750 N, determine the magnitude and direction of **P**.

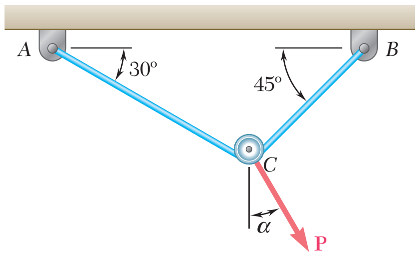


Fig. *P2.47*

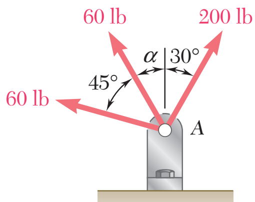


Fig. P2.48

2.48 The directions of the 60-lb forces may vary, but the angle between the forces is always 45°. Determine the value of α for which the resultant of the forces acting at *A* is directed vertically upward.

2.49 A 3.6-m length of steel pipe of mass 300 kg is lifted by a crane cable *CD*. Determine the tension in the cable sling *ACB*, knowing that the length of the sling is (*a*) 4.5 m, (*b*) 6 m.

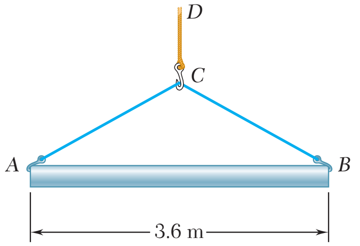


Fig. P2.49

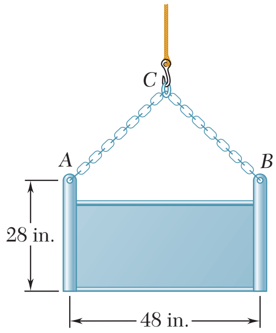


Fig. P2.50

2.50 A movable bin and its contents weigh 700 lb. Determine the shortest chain sling *ACB* that can be used to lift the loaded bin if the tension in the chain is not to exceed 1250 lb.

2.51 A 250-kg crate is supported by several rope-and-pulley arrangements as shown. Determine for each arrangement the tension in the rope. (The tension in the rope is the same on each side of a simple pulley. This can be proved by the methods of Chap. 4.)

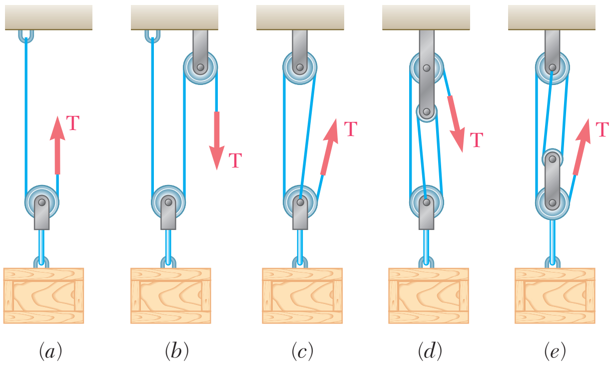


Fig. P2.51

2.52 Solve parts *b* and *d* of Prob. 2.51 assuming that the free end of the rope is attached to the crate.

2.53 A 450-lb crate is to be supported by the rope-and-pulley arrangement shown. Determine the magnitude and direction of the force **F** that should be exerted on the free end of the rope.

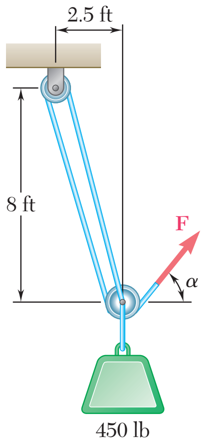


Fig. *P2.53*

2.54 For $W = 800$ N, $P = 200$ N, and $d = 600$ mm, determine the value of h to maintain equilibrium.

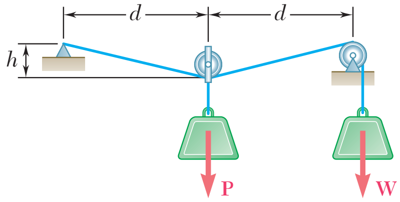


Fig. P2.54

2.55 The collar A can slide freely on the horizontal smooth rod. Determine the magnitude of the force $\mathbf{P}$ required to maintain equilibrium when (*a*) $c = 9$ in., (*b*) $c = 16$ in.

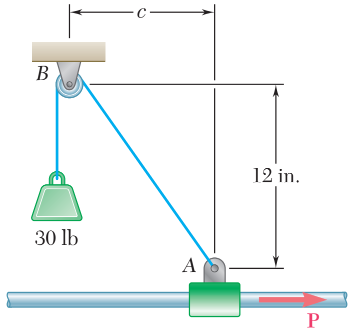


Fig. P2.55

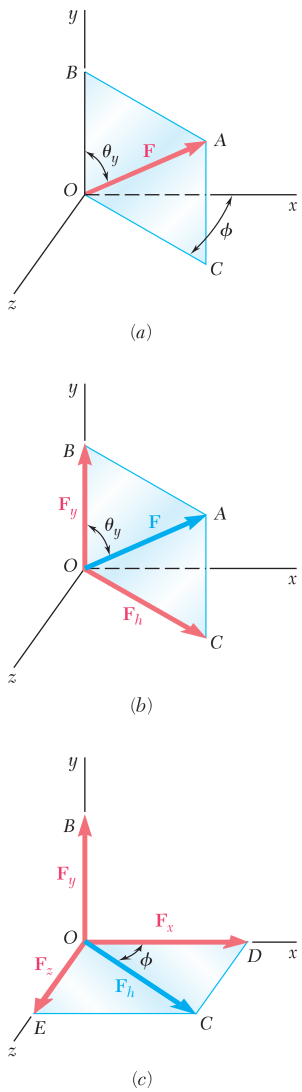


Fig. 2.30

FORCES IN SPACE

2.12 RECTANGULAR COMPONENTS OF A FORCE IN SPACE

The problems considered in the first part of this chapter involved only two dimensions; they could be formulated and solved in a single plane. In this section and in the remaining sections of the chapter, we will discuss problems involving the three dimensions of space.

Consider a force $\mathbf{F}$ acting at the origin O of the system of rectangular coordinates x, y, z. To define the direction of $\mathbf{F}$, we draw the vertical plane $OBAC$ containing $\mathbf{F}$ (Fig. 2.30*a*). This plane passes through the vertical y axis; its orientation is defined by the angle ϕ it forms with the xy plane. The direction of $\mathbf{F}$ within the plane is defined by the angle θ_y that $\mathbf{F}$ forms with the y axis. The force $\mathbf{F}$ may be resolved into a vertical component $\mathbf{F}_y$ and a horizontal component $\mathbf{F}_h$; this operation, shown in Fig. 2.30*b*, is carried out in plane $OBAC$ according to the rules developed in the first part of the chapter. The corresponding scalar components are

$$F_y = F \cos \theta_y \qquad F_h = F \sin \theta_y \tag{2.16}$$

But $\mathbf{F}_h$ may be resolved into two rectangular components $\mathbf{F}_x$ and $\mathbf{F}_z$ along the x and z axes, respectively. This operation, shown in Fig. 2.30*c*, is carried out in the xz plane. We obtain the following expressions for the corresponding scalar components:

$$\begin{aligned} F_x &= F_h \cos \phi = F \sin \theta_y \cos \phi \\ F_z &= F_h \sin \phi = F \sin \theta_y \sin \phi \end{aligned} \tag{2.17}$$

The given force $\mathbf{F}$ has thus been resolved into three rectangular vector components $\mathbf{F}_x$, $\mathbf{F}_y$, $\mathbf{F}_z$, which are directed along the three coordinate axes.

Applying the Pythagorean theorem to the triangles OAB and OCD of Fig. 2.30, we write

$$F^2 = (OA)^2 = (OB)^2 + (BA)^2 = F_y^2 + F_h^2$$
$$F_h^2 = (OC)^2 = (OD)^2 + (DC)^2 = F_x^2 + F_z^2$$

Eliminating F_h^2 from these two equations and solving for F, we obtain the following relation between the magnitude of **F** and its rectangular scalar components:

$$F = \sqrt{F_x^2 + F_y^2 + F_z^2} \tag{2.18}$$

The relationship existing between the force **F** and its three components $\mathbf{F}_x$, $\mathbf{F}_y$, $\mathbf{F}_z$ is more easily visualized if a "box" having $\mathbf{F}_x$, $\mathbf{F}_y$, $\mathbf{F}_z$ for edges is drawn as shown in Fig. 2.31. The force **F** is then represented by the diagonal OA of this box. Figure 2.31*b* shows the right triangle OAB used to derive the first of the formulas (2.16): $F_y = F \cos \theta_y$. In Fig. 2.31*a* and *c*, two other right triangles have also been drawn: OAD and OAE. These triangles are seen to occupy in the box positions comparable with that of triangle OAB. Denoting by θ_x and θ_z, respectively, the angles that **F** forms with the x and z axes, we can derive two formulas similar to $F_y = F \cos \theta_y$. We thus write

$$F_x = F \cos \theta_x \qquad F_y = F \cos \theta_y \qquad F_z = F \cos \theta_z \tag{2.19}$$

The three angles θ_x, θ_y, θ_z define the direction of the force **F**; they are more commonly used for this purpose than the angles θ_y and ϕ introduced at the beginning of this section. The cosines of θ_x, θ_y, θ_z are known as the *direction cosines* of the force **F**.

Introducing the unit vectors **i**, **j**, and **k**, directed respectively along the x, y, and z axes (Fig. 2.32), we can express **F** in the form

$$\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k} \tag{2.20}$$

where the scalar components F_x, F_y, F_z are defined by the relations (2.19).

EXAMPLE 2.4 A force of 500 N forms angles of 60°, 45°, and 120°, respectively, with the x, y, and z axes. Find the components F_x, F_y, and F_z of the force.

Substituting $F = 500$ N, $\theta_x = 60°$, $\theta_y = 45°$, $\theta_z = 120°$ into formulas (2.19), we write

$$F_x = (500 \text{ N}) \cos 60° = +250 \text{ N}$$
$$F_y = (500 \text{ N}) \cos 45° = +354 \text{ N}$$
$$F_z = (500 \text{ N}) \cos 120° = -250 \text{ N}$$

Carrying into Eq. (2.20) the values obtained for the scalar components of **F**, we have

$$\mathbf{F} = (250 \text{ N})\mathbf{i} + (354 \text{ N})\mathbf{j} - (250 \text{ N})\mathbf{k}$$

As in the case of two-dimensional problems, a plus sign indicates that the component has the same sense as the corresponding axis, and a minus sign indicates that it has the opposite sense. ■

The angle a force **F** forms with an axis should be measured from the positive side of the axis and will always be between 0 and 180°. An angle θ_x smaller than 90° (acute) indicates that **F** (assumed attached to O) is on the same side of the yz plane as the positive x axis; cos θ_x and F_x will then be positive. An angle θ_x larger than 90° (obtuse) indicates that **F** is on the other side of the yz plane; cos θ_x and F_x will then be

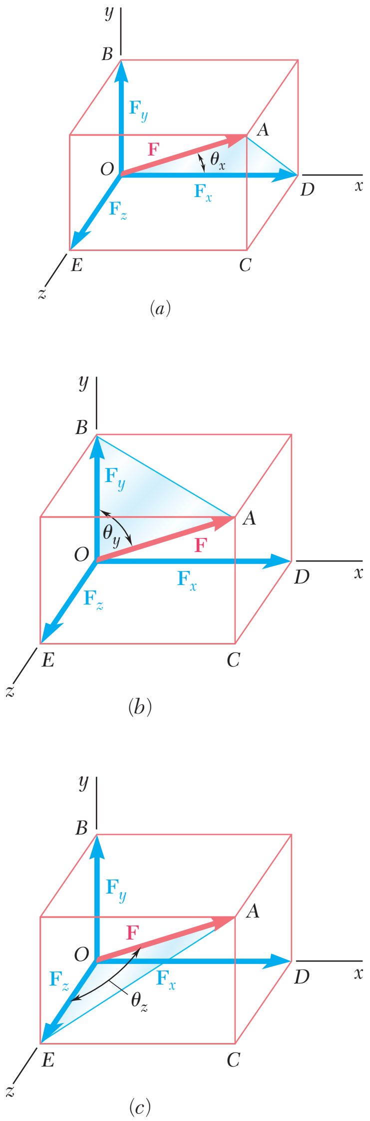


Fig. 2.31

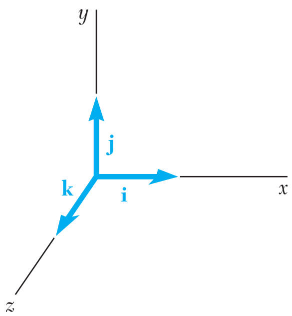


Fig. 2.32

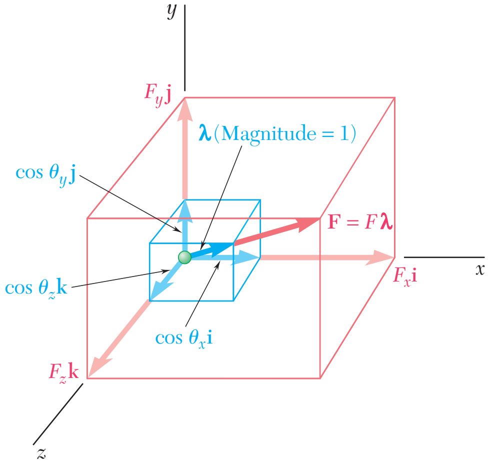


Fig. 2.33

negative. In Example 2.4 the angles θ_x and θ_y are acute, while θ_z is obtuse; consequently, F_x and F_y are positive, while F_z is negative.

Substituting into (2.20) the expressions obtained for F_x, F_y, F_z in (2.19), we write

$$\mathbf{F} = F(\cos\theta_x\mathbf{i} + \cos\theta_y\mathbf{j} + \cos\theta_z\mathbf{k}) \tag{2.21}$$

which shows that the force $\mathbf{F}$ can be expressed as the product of the scalar F and the vector

$$\boldsymbol{\lambda} = \cos\theta_x\mathbf{i} + \cos\theta_y\mathbf{j} + \cos\theta_z\mathbf{k} \tag{2.22}$$

Clearly, the vector $\boldsymbol{\lambda}$ is a vector whose magnitude is equal to 1 and whose direction is the same as that of $\mathbf{F}$ (Fig. 2.33). The vector $\boldsymbol{\lambda}$ is referred to as the *unit vector* along the line of action of $\mathbf{F}$. It follows from (2.22) that the components of the unit vector $\boldsymbol{\lambda}$ are respectively equal to the direction cosines of the line of action of $\mathbf{F}$:

$$\lambda_x = \cos\theta_x \qquad \lambda_y = \cos\theta_y \qquad \lambda_z = \cos\theta_z \tag{2.23}$$

We should observe that the values of the three angles θ_x, θ_y, θ_z are not independent. Recalling that the sum of the squares of the components of a vector is equal to the square of its magnitude, we write

$$\lambda_x^2 + \lambda_y^2 + \lambda_z^2 = 1$$

or, substituting for λ_x, λ_y, λ_z from (2.23),

$$\cos^2\theta_x + \cos^2\theta_y + \cos^2\theta_z = 1 \tag{2.24}$$

In Example 2.4, for instance, once the values $\theta_x = 60°$ and $\theta_y = 45°$ have been selected, the value of θ_z *must* be equal to 60° or 120° in order to satisfy identity (2.24).

When the components F_x, F_y, F_z of a force $\mathbf{F}$ are given, the magnitude F of the force is obtained from (2.18).† The relations (2.19) can then be solved for the direction cosines,

$$\cos\theta_x = \frac{F_x}{F} \qquad \cos\theta_y = \frac{F_y}{F} \qquad \cos\theta_z = \frac{F_z}{F} \tag{2.25}$$

and the angles θ_x, θ_y, θ_z characterizing the direction of $\mathbf{F}$ can be found.

EXAMPLE 2.5 A force $\mathbf{F}$ has the components $F_x = 20$ lb, $F_y = -30$ lb, $F_z = 60$ lb. Determine its magnitude F and the angles θ_x, θ_y, θ_z it forms with the coordinate axes.

From formula (2.18) we obtain†

$$\begin{aligned} F &= \sqrt{F_x^2 + F_y^2 + F_z^2} \\ &= \sqrt{(20\text{ lb})^2 + (-30\text{ lb})^2 + (60\text{ lb})^2} \\ &= \sqrt{4900}\text{ lb} = 70\text{ lb} \end{aligned}$$

†With a calculator programmed to convert rectangular coordinates into polar coordinates, the following procedure will be found more expeditious for computing F: First determine F_h from its two rectangular components F_x and F_z (Fig. 2.30*c*), then determine F from its two rectangular components F_h and F_y (Fig. 2.30*b*). The actual order in which the three components F_x, F_y, F_z are entered is immaterial.

Substituting the values of the components and magnitude of **F** into Eqs. (2.25), we write

$$\cos\theta_x = \frac{F_x}{F} = \frac{20 \text{ lb}}{70 \text{ lb}} \qquad \cos\theta_y = \frac{F_y}{F} = \frac{-30 \text{ lb}}{70 \text{ lb}} \qquad \cos\theta_z = \frac{F_z}{F} = \frac{60 \text{ lb}}{70 \text{ lb}}$$

Calculating successively each quotient and its arc cosine, we obtain

$$\theta_x = 73.4° \qquad \theta_y = 115.4° \qquad \theta_z = 31.0°$$

These computations can be carried out easily with a calculator. ■

2.13 FORCE DEFINED BY ITS MAGNITUDE AND TWO POINTS ON ITS LINE OF ACTION

In many applications, the direction of a force **F** is defined by the coordinates of two points, $M(x_1, y_1, z_1)$ and $N(x_2, y_2, z_2)$, located on its line of action (Fig. 2.34). Consider the vector $\overrightarrow{MN}$ joining M and N

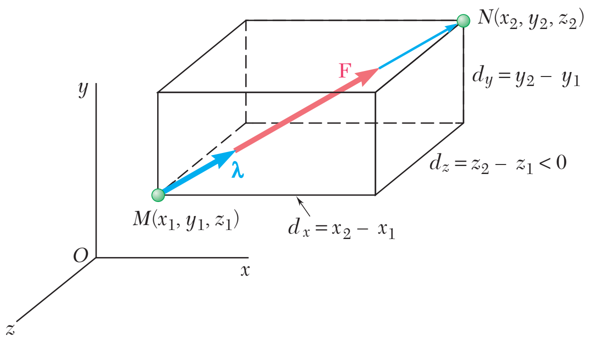


Fig. 2.34

and of the same sense as **F**. Denoting its scalar components by d_x, d_y, d_z, respectively, we write

$$\overrightarrow{MN} = d_x\mathbf{i} + d_y\mathbf{j} + d_z\mathbf{k} \tag{2.26}$$

The unit vector $\boldsymbol{\lambda}$ along the line of action of **F** (i.e., along the line MN) may be obtained by dividing the vector $\overrightarrow{MN}$ by its magnitude MN. Substituting for $\overrightarrow{MN}$ from (2.26) and observing that MN is equal to the distance d from M to N, we write

$$\boldsymbol{\lambda} = \frac{\overrightarrow{MN}}{MN} = \frac{1}{d}(d_x\mathbf{i} + d_y\mathbf{j} + d_z\mathbf{k}) \tag{2.27}$$

Recalling that **F** is equal to the product of F and λ, we have

$$\mathbf{F} = F\boldsymbol{\lambda} = \frac{F}{d}(d_x\mathbf{i} + d_y\mathbf{j} + d_z\mathbf{k}) \tag{2.28}$$

from which it follows that the scalar components of **F** are, respectively,

$$F_x = \frac{Fd_x}{d} \qquad F_y = \frac{Fd_y}{d} \qquad F_z = \frac{Fd_z}{d} \tag{2.29}$$

The relations (2.29) considerably simplify the determination of the components of a force **F** of given magnitude F when the line of action of **F** is defined by two points M and N. Subtracting the coordinates of M from those of N, we first determine the components of the vector $\overrightarrow{MN}$ and the distance d from M to N:

$$d_x = x_2 - x_1 \qquad d_y = y_2 - y_1 \qquad d_z = z_2 - z_1$$
$$d = \sqrt{d_x^2 + d_y^2 + d_z^2}$$

Substituting for F and for d_x, d_y, d_z, and d into the relations (2.29), we obtain the components F_x, F_y, F_z of the force.

The angles θ_x, θ_y, θ_z that **F** forms with the coordinate axes can then be obtained from Eqs. (2.25). Comparing Eqs. (2.22) and (2.27), we can also write

$$\cos\theta_x = \frac{d_x}{d} \qquad \cos\theta_y = \frac{d_y}{d} \qquad \cos\theta_z = \frac{d_z}{d} \tag{2.30}$$

and determine the angles θ_x, θ_y, θ_z directly from the components and magnitude of the vector $\overrightarrow{MN}$.

2.14 ADDITION OF CONCURRENT FORCES IN SPACE

The resultant **R** of two or more forces in space will be determined by summing their rectangular components. Graphical or trigonometric methods are generally not practical in the case of forces in space.

The method followed here is similar to that used in Sec. 2.8 with coplanar forces. Setting

$$\mathbf{R} = \Sigma\mathbf{F}$$

we resolve each force into its rectangular components and write

$$\begin{aligned} R_x\mathbf{i} + R_y\mathbf{j} + R_z\mathbf{k} &= \Sigma(F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}) \\ &= (\Sigma F_x)\mathbf{i} + (\Sigma F_y)\mathbf{j} + (\Sigma F_z)\mathbf{k} \end{aligned}$$

from which it follows that

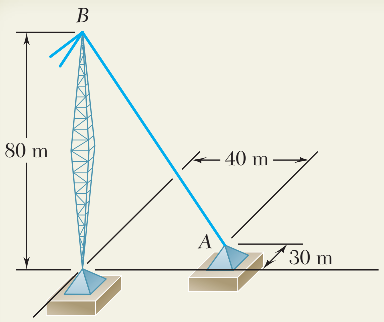
$$R_x = \Sigma F_x \qquad R_y = \Sigma F_y \qquad R_z = \Sigma F_z \tag{2.31}$$

The magnitude of the resultant and the angles θ_x, θ_y, θ_z that the resultant forms with the coordinate axes are obtained using the method discussed in Sec. 2.12. We write

$$R = \sqrt{R_x^2 + R_y^2 + R_z^2} \tag{2.32}$$

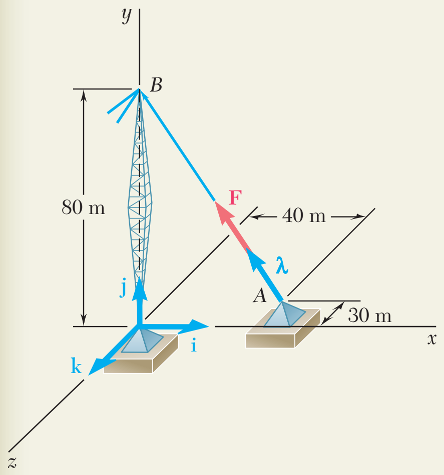
$$\cos\theta_x = \frac{R_x}{R} \qquad \cos\theta_y = \frac{R_y}{R} \qquad \cos\theta_z = \frac{R_z}{R} \tag{2.33}$$

SAMPLE PROBLEM 2.7



A tower guy wire is anchored by means of a bolt at A. The tension in the wire is 2500 N. Determine (*a*) the components F_x, F_y, F_z of the force acting on the bolt, (*b*) the angles θ_x, θ_y, θ_z defining the direction of the force.

SOLUTION



a. Components of the Force. The line of action of the force acting on the bolt passes through A and B, and the force is directed from A to B. The components of the vector $\overrightarrow{AB}$, which has the same direction as the force, are

$$d_x = -40 \text{ m} \qquad d_y = +80 \text{ m} \qquad d_z = +30 \text{ m}$$

The total distance from A to B is

$$AB = d = \sqrt{d_x^2 + d_y^2 + d_z^2} = 94.3 \text{ m}$$

Denoting by $\mathbf{i}, \mathbf{j}, \mathbf{k}$ the unit vectors along the coordinate axes, we have

$$\overrightarrow{AB} = -(40 \text{ m})\mathbf{i} + (80 \text{ m})\mathbf{j} + (30 \text{ m})\mathbf{k}$$

Introducing the unit vector $\boldsymbol{\lambda} = \overrightarrow{AB}/AB$, we write

$$\mathbf{F} = F\boldsymbol{\lambda} = F\frac{\overrightarrow{AB}}{AB} = \frac{2500 \text{ N}}{94.3 \text{ m}}\overrightarrow{AB}$$

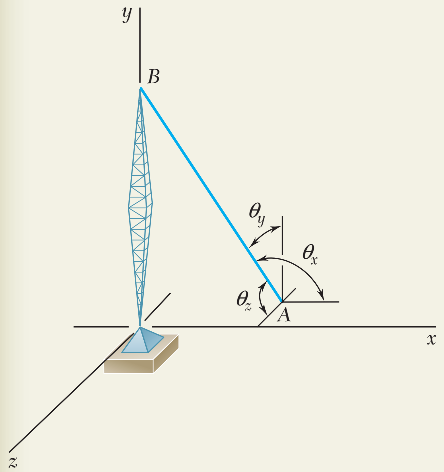
Substituting the expression found for $\overrightarrow{AB}$, we obtain

$$\mathbf{F} = \frac{2500 \text{ N}}{94.3 \text{ m}}[-(40 \text{ m})\mathbf{i} + (80 \text{ m})\mathbf{j} + (30 \text{ m})\mathbf{k}]$$

$$\mathbf{F} = -(1060 \text{ N})\mathbf{i} + (2120 \text{ N})\mathbf{j} + (795 \text{ N})\mathbf{k}$$

The components of $\mathbf{F}$, therefore, are

$$F_x = -1060 \text{ N} \qquad F_y = +2120 \text{ N} \qquad F_z = +795 \text{ N} \quad \blacktriangleleft$$



b. Direction of the Force. Using Eqs. (2.25), we write

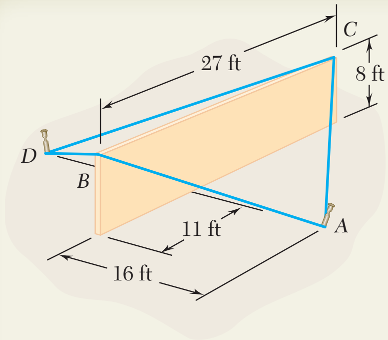
$$\cos\theta_x = \frac{F_x}{F} = \frac{-1060 \text{ N}}{2500 \text{ N}} \qquad \cos\theta_y = \frac{F_y}{F} = \frac{+2120 \text{ N}}{2500 \text{ N}}$$

$$\cos\theta_z = \frac{F_z}{F} = \frac{+795 \text{ N}}{2500 \text{ N}}$$

Calculating successively each quotient and its arc cosine, we obtain

$$\theta_x = 115.1° \qquad \theta_y = 32.0° \qquad \theta_z = 71.5° \quad \blacktriangleleft$$

(*Note.* This result could have been obtained by using the components and magnitude of the vector $\overrightarrow{AB}$ rather than those of the force $\mathbf{F}$.)



SAMPLE PROBLEM 2.8

A wall section of precast concrete is temporarily held by the cables shown. Knowing that the tension is 840 lb in cable AB and 1200 lb in cable AC, determine the magnitude and direction of the resultant of the forces exerted by cables AB and AC on stake A.

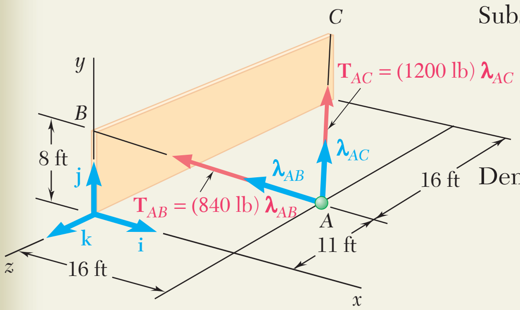
SOLUTION

Components of the Forces. The force exerted by each cable on stake A will be resolved into x, y, and z components. We first determine the components and magnitude of the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$, measuring them from A toward the wall section. Denoting by $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ the unit vectors along the coordinate axes, we write

$$\overrightarrow{AB} = -(16\text{ ft})\mathbf{i} + (8\text{ ft})\mathbf{j} + (11\text{ ft})\mathbf{k} \qquad AB = 21\text{ ft}$$
$$\overrightarrow{AC} = -(16\text{ ft})\mathbf{i} + (8\text{ ft})\mathbf{j} - (16\text{ ft})\mathbf{k} \qquad AC = 24\text{ ft}$$

Denoting by $\boldsymbol{\lambda}_{AB}$ the unit vector along AB, we have

$$\mathbf{T}_{AB} = T_{AB}\boldsymbol{\lambda}_{AB} = T_{AB}\frac{\overrightarrow{AB}}{AB} = \frac{840\text{ lb}}{21\text{ ft}}\overrightarrow{AB}$$



Substituting the expression found for $\overrightarrow{AB}$, we obtain

$$\mathbf{T}_{AB} = \frac{840\text{ lb}}{21\text{ ft}}[-(16\text{ ft})\mathbf{i} + (8\text{ ft})\mathbf{j} + (11\text{ ft})\mathbf{k}]$$
$$\mathbf{T}_{AB} = -(640\text{ lb})\mathbf{i} + (320\text{ lb})\mathbf{j} + (440\text{ lb})\mathbf{k}$$

Denoting by $\boldsymbol{\lambda}_{AC}$ the unit vector along AC, we obtain in a similar way

$$\mathbf{T}_{AC} = T_{AC}\boldsymbol{\lambda}_{AC} = T_{AC}\frac{\overrightarrow{AC}}{AC} = \frac{1200\text{ lb}}{24\text{ ft}}\overrightarrow{AC}$$
$$\mathbf{T}_{AC} = -(800\text{ lb})\mathbf{i} + (400\text{ lb})\mathbf{j} - (800\text{ lb})\mathbf{k}$$

Resultant of the Forces. The resultant $\mathbf{R}$ of the forces exerted by the two cables is

$$\mathbf{R} = \mathbf{T}_{AB} + \mathbf{T}_{AC} = -(1440\text{ lb})\mathbf{i} + (720\text{ lb})\mathbf{j} - (360\text{ lb})\mathbf{k}$$

The magnitude and direction of the resultant are now determined:

$$R = \sqrt{R_x^2 + R_y^2 + R_z^2} = \sqrt{(-1440)^2 + (720)^2 + (-360)^2}$$
$$R = 1650\text{ lb} \quad \blacktriangleleft$$

From Eqs. (2.33) we obtain

$$\cos\theta_x = \frac{R_x}{R} = \frac{-1440\text{ lb}}{1650\text{ lb}} \qquad \cos\theta_y = \frac{R_y}{R} = \frac{+720\text{ lb}}{1650\text{ lb}}$$
$$\cos\theta_z = \frac{R_z}{R} = \frac{-360\text{ lb}}{1650\text{ lb}}$$

Calculating successively each quotient and its arc cosine, we have

$$\theta_x = 150.8° \qquad \theta_y = 64.1° \qquad \theta_z = 102.6° \quad \blacktriangleleft$$

PROBLEMS

2.56 Determine (*a*) the x, y, and z components of the 250-N force, (*b*) the angles θ_x, θ_y, and θ_z that the force forms with the coordinate axes.

2.57 Determine (*a*) the x, y, and z components of the 300-N force, (*b*) the angles θ_x, θ_y, and θ_z that the force forms with the coordinate axes.

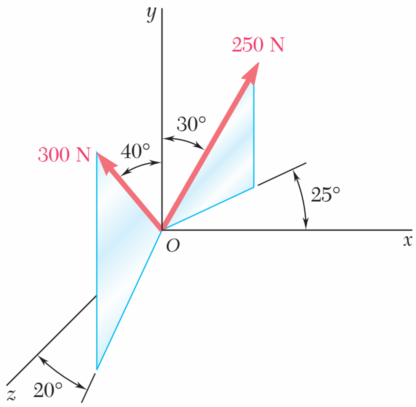


Fig. P2.56 and P2.57

2.58 The angle between the guy wire AB and the mast is 20°. Knowing that the tension in AB is 300 lb, determine (*a*) the x, y, and z components of the force exerted on the boat at B, (*b*) the angles θ_x, θ_y, and θ_z defining the direction of the force exerted at B.

2.59 The angle between the guy wire AC and the mast is 20°. Knowing that the tension in AC is 300 lb, determine (*a*) the x, y, and z components of the force exerted on the boat at C, (*b*) the angles θ_x, θ_y, and θ_z defining the direction of the force exerted at C.

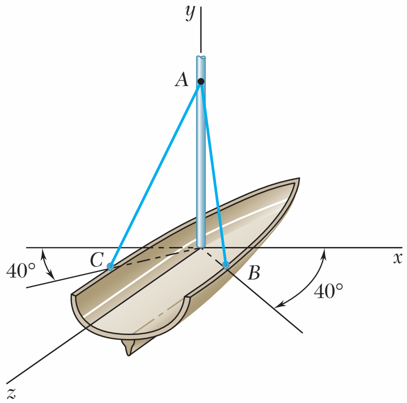


Fig. P2.58 and P2.59

2.60 A gun is aimed at a point A located 20° west of north. Knowing that the barrel of the gun forms an angle of 35° with the horizontal and that the maximum recoil force is 800 N, determine (*a*) the x, y, and z components of the force, (*b*) the angles θ_x, θ_y, and θ_z defining the direction of the recoil force. (Assume that the x, y, and z axes are directed, respectively, east, up, and south.)

2.61 Solve Prob. 2.60, assuming that point A is located 25° north of west and that the barrel of the gun forms an angle of 30° with the horizontal.

2.62 Determine the magnitude and direction of the force $\mathbf{F} = -(240 \text{ lb})\mathbf{i} - (320 \text{ lb})\mathbf{j} + (600 \text{ lb})\mathbf{k}$.

2.63 Determine the magnitude and direction of the force $\mathbf{F} = (690 \text{ lb})\mathbf{i} + (300 \text{ lb})\mathbf{j} - (580 \text{ lb})\mathbf{k}$.

2.64 A force acts at the origin in a direction defined by the angles $\theta_y = 120°$ and $\theta_z = 75°$. It is known that the x component of the force is +40 N. Determine the magnitude of the force and the value of θ_x.

2.65 A 250-lb force acts at the origin in a direction defined by the angles $\theta_x = 65°$ and $\theta_y = 40°$. It is known that the z component of the force is positive. Determine the value of θ_z and the components of the force.

2.66 A force acts at the origin in a direction defined by the angles $\theta_x = 70°$ and $\theta_z = 130°$. Knowing that the y component of the force is +400 lb, determine (*a*) the other components and the magnitude of the force, (*b*) the value of θ_y.

2.67 A force acts at the origin in a direction defined by the angles $\theta_y = 65°$ and $\theta_z = 40°$. Knowing that the x component of the force is −750 N, determine (*a*) the other components and the magnitude of the force, (*b*) the value of θ_x.

2.68 Knowing that the tension in cable *AB* is 900 N, determine the components of the force exerted on the plate at *A*.

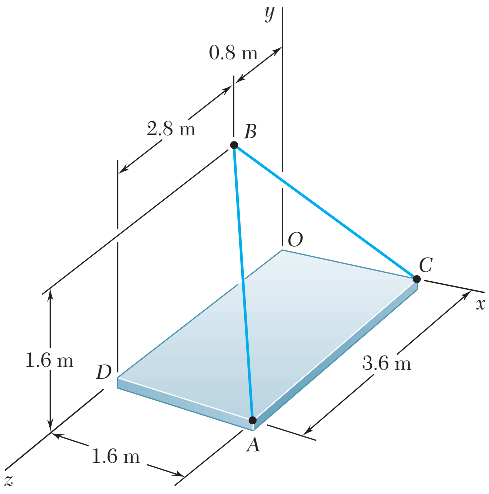


Fig. *P2.68* and P2.69

2.69 Knowing that the tension in cable *BC* is 450 N, determine the components of the force exerted on the plate at *C*.

2.70 Knowing that the tension in cable *AB* is 285 lb, determine the components of the force exerted on the plate at *B*.

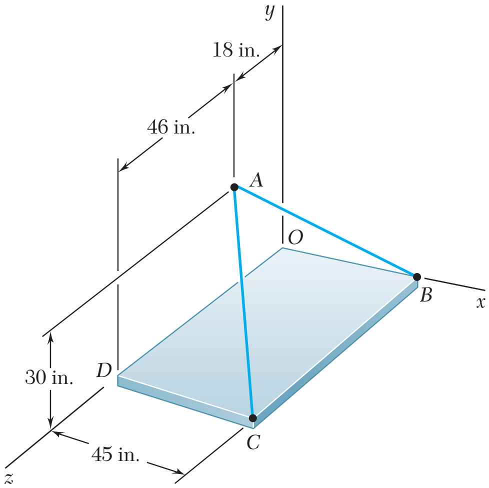


Fig. *P2.70,* P2.71, and P2.73

2.71 Knowing that the tension in cable *AC* is 426 lb, determine the components of the force exerted on the plate at *C*.

2.72 Determine the resultant of the two forces shown.

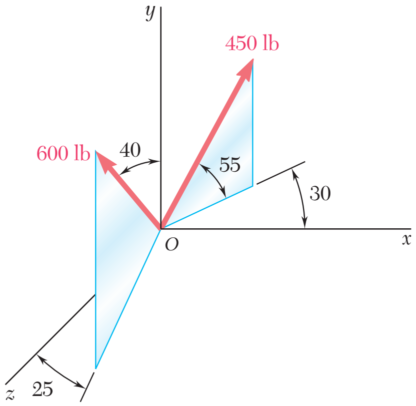


Fig. P2.72

2.73 Knowing that the tension is 285 lb in cable *AB* and 426 lb in cable *AC*, determine the magnitude and direction of the resultant of the forces exerted at *A* by the two cables.

2.74 The angle between each of the springs AB and AC and the post DA is 30°. Knowing that the tension is 50 lb in spring AB and 40 lb in spring AC, determine the magnitude and direction of the resultant of the forces exerted by the springs on the post at A.

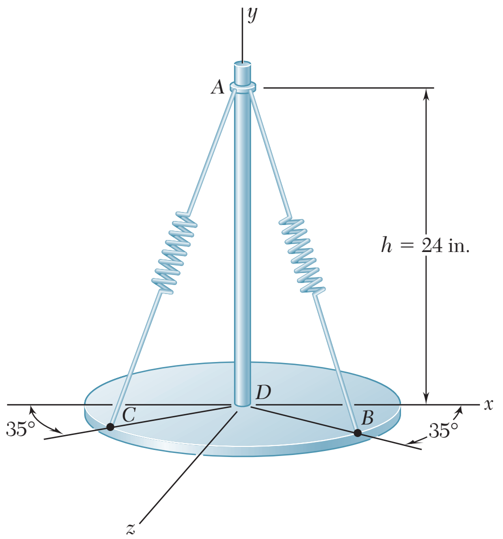


Fig. *P2.74*

2.75 Determine the two possible values of θ_y for a force $\mathbf{F}$, (*a*) if the force forms equal angles with the positive x, y, and z axes, (*b*) if the force forms equal angles with the positive y and z axes and an angle of 45° with the positive x axis.

2.76 Knowing that the tension in AB is 39 kN, determine the required values of the tension in AC and AD so that the resultant of the three forces applied at A is vertical.

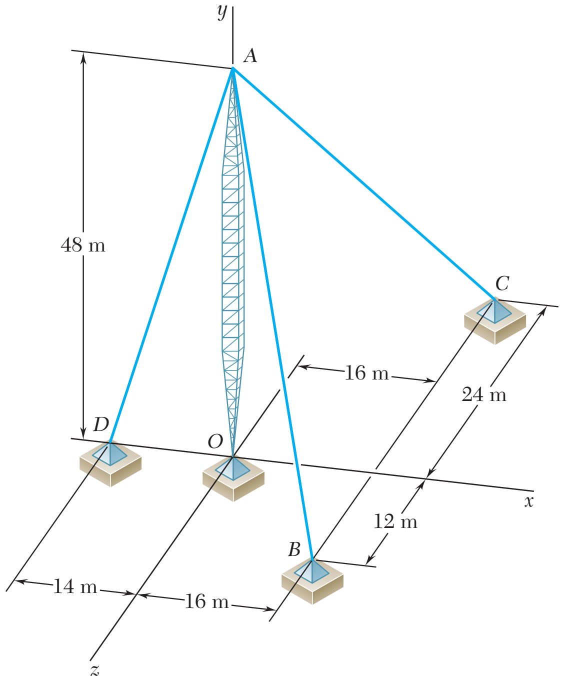


Fig. P2.76 and P2.77

2.77 Knowing that the tension in AC is 28 kN, determine the required values of the tension in AB and AD so that the resultant of the three forces applied at A is vertical.

2.78 The boom OA carries a load $\mathbf{P}$ and is supported by two cables as shown. Knowing that the tension in cable AB is 732 N and that the resultant of the load $\mathbf{P}$ and of the forces exerted at A by the two cables must be directed along OA, determine the tension in cable AC.

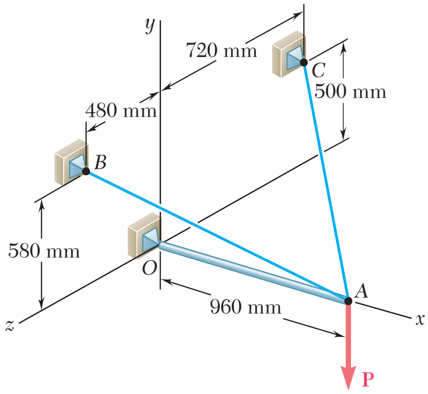


Fig. ***P2.78***

2.79 For the boom and loading of Prob. 2.78, determine the magnitude of the load $\mathbf{P}$.

Photo 2.2 While the tension in the *four* cables supporting the car cannot be found using the *three* equations of (2.34), a relation between the tensions can be obtained by considering the equilibrium of the hook.

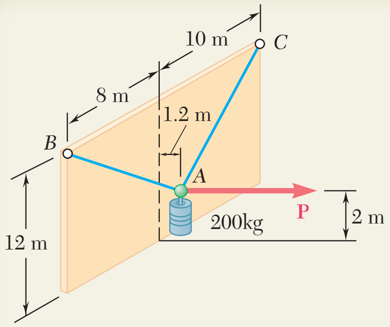
2.15 EQUILIBRIUM OF A PARTICLE IN SPACE

According to the definition given in Sec. 2.9, a particle A is in equilibrium if the resultant of all the forces acting on A is zero. The components R_x, R_y, R_z of the resultant are given by the relations (2.31); expressing that the components of the resultant are zero, we write

$$\Sigma F_x = 0 \qquad \Sigma F_y = 0 \qquad \Sigma F_z = 0 \tag{2.34}$$

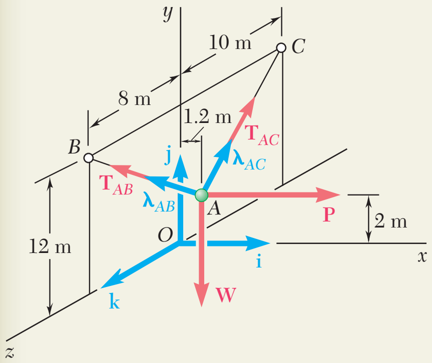
Equations (2.34) represent the necessary and sufficient conditions for the equilibrium of a particle in space. They can be used to solve problems dealing with the equilibrium of a particle involving no more than three unknowns.

To solve such problems, you first should draw a free-body diagram showing the particle in equilibrium and *all* the forces acting on it. You can then write the equations of equilibrium (2.34) and solve them for three unknowns. In the more common types of problems, these unknowns will represent (1) the three components of a single force or (2) the magnitude of three forces, each of known direction.



SAMPLE PROBLEM 2.9

A 200-kg cylinder is hung by means of two cables *AB* and *AC*, which are attached to the top of a vertical wall. A horizontal force **P** perpendicular to the wall holds the cylinder in the position shown. Determine the magnitude of **P** and the tension in each cable.



SOLUTION

Free-body Diagram. Point *A* is chosen as a free body; this point is subjected to four forces, three of which are of unknown magnitude.

Introducing the unit vectors **i**, **j**, **k**, we resolve each force into rectangular components.

$$\mathbf{P} = P\mathbf{i}$$
$$\mathbf{W} = -mg\mathbf{j} = -(200\text{ kg})(9.81\text{ m/s}^2)\mathbf{j} = -(1962\text{ N})\mathbf{j} \qquad (1)$$

In the case of $\mathbf{T}_{AB}$ and $\mathbf{T}_{AC}$, it is necessary first to determine the components and magnitudes of the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Denoting by $\boldsymbol{\lambda}_{AB}$ the unit vector along *AB*, we write

$$\overrightarrow{AB} = -(1.2\text{ m})\mathbf{i} + (10\text{ m})\mathbf{j} + (8\text{ m})\mathbf{k} \qquad AB = 12.862\text{ m}$$
$$\boldsymbol{\lambda}_{AB} = \frac{\overrightarrow{AB}}{12.862\text{ m}} = -0.09330\mathbf{i} + 0.7775\mathbf{j} + 0.6220\mathbf{k}$$
$$\mathbf{T}_{AB} = T_{AB}\boldsymbol{\lambda}_{AB} = -0.09330T_{AB}\mathbf{i} + 0.7775T_{AB}\mathbf{j} + 0.6220T_{AB}\mathbf{k} \qquad (2)$$

Denoting by $\boldsymbol{\lambda}_{AC}$ the unit vector along *AC*, we write in a similar way

$$\overrightarrow{AC} = -(1.2\text{ m})\mathbf{i} + (10\text{ m})\mathbf{j} - (10\text{ m})\mathbf{k} \qquad AC = 14.193\text{ m}$$
$$\boldsymbol{\lambda}_{AC} = \frac{\overrightarrow{AC}}{14.193\text{ m}} = -0.08455\mathbf{i} + 0.7046\mathbf{j} - 0.7046\mathbf{k}$$
$$\mathbf{T}_{AC} = T_{AC}\boldsymbol{\lambda}_{AC} = -0.08455T_{AC}\mathbf{i} + 0.7046T_{AC}\mathbf{j} - 0.7046T_{AC}\mathbf{k} \qquad (3)$$

Equilibrium Condition. Since *A* is in equilibrium, we must have

$$\Sigma\mathbf{F} = 0: \qquad \mathbf{T}_{AB} + \mathbf{T}_{AC} + \mathbf{P} + \mathbf{W} = 0$$

or, substituting from (1), (2), (3) for the forces and factoring **i**, **j**, **k**,

$$(-0.09330T_{AB} - 0.08455T_{AC} + P)\mathbf{i}$$
$$+ (0.7775T_{AB} + 0.7046T_{AC} - 1962\text{ N})\mathbf{j}$$
$$+ (0.6220T_{AB} - 0.7046T_{AC})\mathbf{k} = 0$$

Setting the coefficients of **i**, **j**, **k** equal to zero, we write three scalar equations, which express that the sums of the *x*, *y*, and *z* components of the forces are respectively equal to zero.

$$(\Sigma F_x = 0:) \qquad -0.09330T_{AB} - 0.08455T_{AC} + P = 0$$
$$(\Sigma F_y = 0:) \qquad +0.7775T_{AB} + 0.7046T_{AC} - 1962\text{ N} = 0$$
$$(\Sigma F_z = 0:) \qquad +0.6220T_{AB} - 0.7046T_{AC} = 0$$

Solving these equations, we obtain

$$P = 235\text{ N} \qquad T_{AB} = 1402\text{ N} \qquad T_{AC} = 1238\text{ N} \quad \blacktriangleleft$$

PROBLEMS

2.80 A container is supported by three cables that are attached to a ceiling as shown. Determine the weight W of the container knowing that the tension in cable AB is 6 kN.

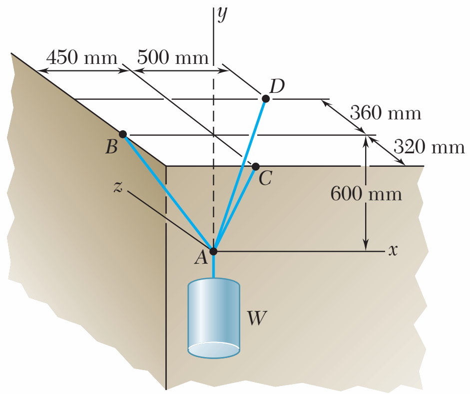


Fig. P2.80, P2.81, and P2.82

2.81 A container is supported by three cables that are attached to a ceiling as shown. Determine the weight W of the container knowing that the tension in cable AD is 4.3 kN.

2.82 A container of weight $W = 9.32$ kN is supported by three cables that are attached to a ceiling as shown. Determine the tension in each cable.

2.83 A load W is supported by three cables as shown. Determine the value of W knowing that the tension in cable BD is 975 lb.

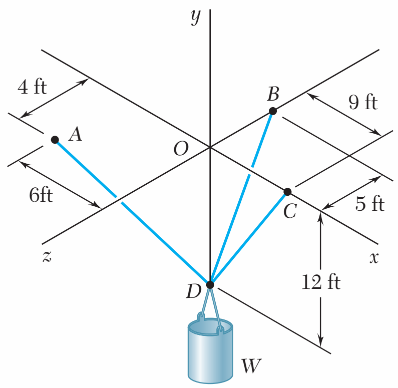


Fig. P2.83, P2.84, and *P2.85*

2.84 A load W is supported by three cables as shown. Determine the value of W knowing that the tension in cable CD is 300 lb.

2.85 A load W of magnitude 555 lb is supported by three cables as shown. Determine the tension in each cable.

2.86 Three wires are connected at point D, which is located 18 in. below the T-shaped pipe support ABC. Determine the tension in each wire when a 180-lb container is suspended from point D as shown.

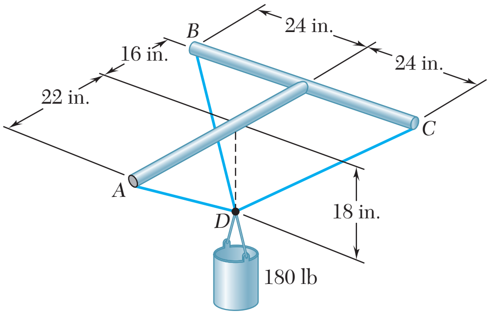


Fig. P2.86

2.87 A triangular plate of weight 18 lb is supported by three wires as shown. Determine the tension in each wire.

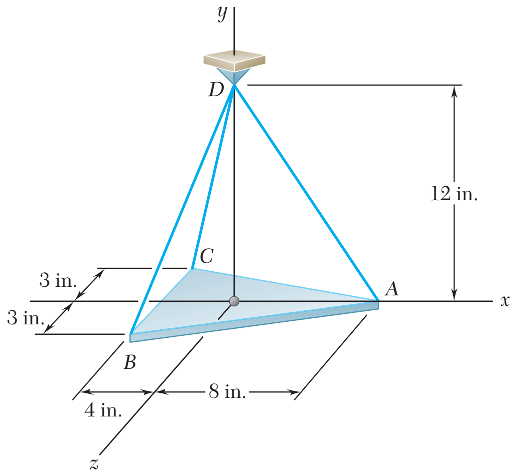


Fig. P2.87

2.88 Three cables are connected at A, where the forces **P** and **Q** are applied as shown. Determine the tension in each of the cables when $P = 0$ and $Q = 36.4$ kN.

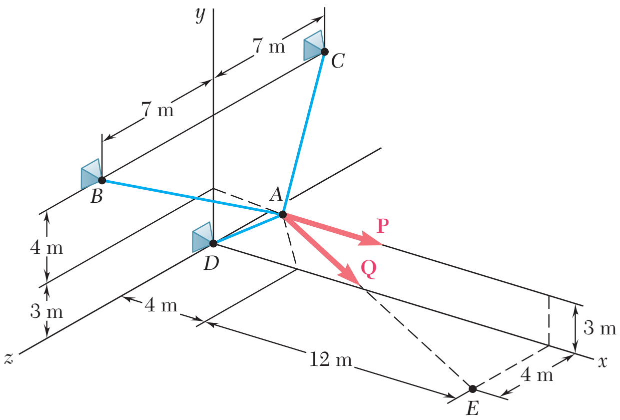


Fig. *P2.88* and P2.89

2.89 Three cables are connected at A, where the forces **P** and **Q** are applied as shown. Knowing that $Q = 36.4$ kN and that the tension in cable AD is zero, determine (*a*) the magnitude and sense of **P**, (*b*) the tension in cables AB and AC.

2.90 In trying to move across a slippery icy surface, a 175-lb man uses two ropes AB and AC. Knowing that the force exerted on the man by the icy surface is perpendicular to that surface, determine the tension in each rope.

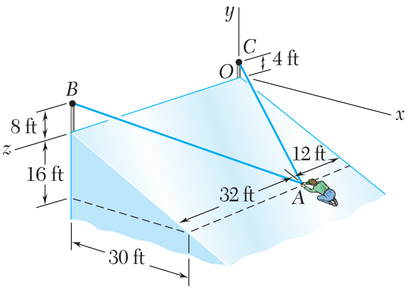


Fig. P2.90

2.91 Solve Prob. 2.90, assuming that a friend is helping the man at A by pulling on him with a force $\mathbf{P} = -(45 \text{ lb})\mathbf{k}$.

2.92 A container of weight $W = 360$ N is supported by cables AB and AC, which are tied to ring A. Knowing that $\mathbf{Q} = 0$, determine (a) the magnitude of the force $\mathbf{P}$ that must be applied to the ring to maintain the container in the position shown, (b) the corresponding values of the tension in cables AB and AC.

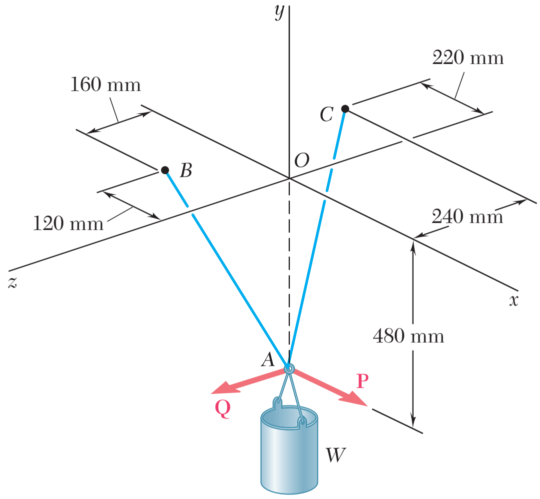


Fig. P2.92 and *P2.94*

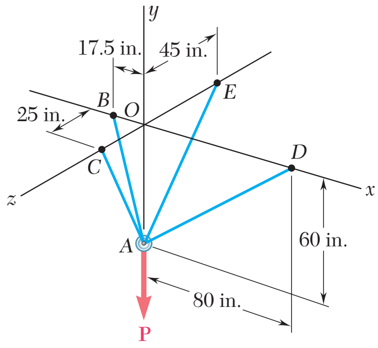


Fig. *P2.96*

2.93 Solve Prob. 2.92 knowing that $\mathbf{Q} = (60 \text{ N})\mathbf{k}$.

2.94 A container is supported by a single cable that passes through a frictionless ring A and is attached to fixed points B and C. Two forces $\mathbf{P} = P\mathbf{i}$ and $\mathbf{Q} = Q\mathbf{k}$ are applied to the ring to maintain the container in the position shown. Knowing that the weight of the container is $W = 660$ N, determine the magnitudes of $\mathbf{P}$ and $\mathbf{Q}$. (*Hint:* The tension must be the same in portions AB and AC of the cable.)

2.95 Determine the weight W of the container of Prob. 2.94 knowing that $P = 478$ N.

2.96 Cable BAC passes through a frictionless ring A and is attached to fixed supports at B and C, while cables AD and AE are both tied to the ring and are attached, respectively, to supports at D and E. Knowing that a 200-lb vertical load $\mathbf{P}$ is applied to ring A, determine the tension in each of the three cables.

2.97 Knowing that the tension in cable AE of Prob. 2.96 is 75 lb, determine (a) the magnitude of the load $\mathbf{P}$, (b) the tension in cables BAC and AD.

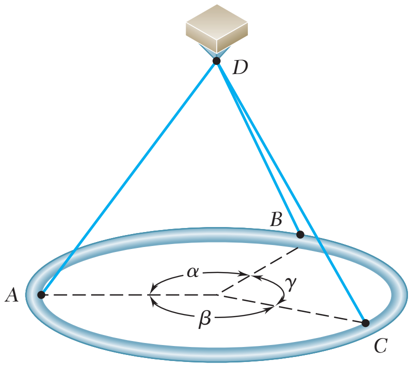


Fig. P2.98

2.98 The uniform circular ring shown has a mass of 20 kg and a diameter of 300 mm. It is supported by three wires each of length 250 mm. If $\alpha = 120°$, $\beta = 150°$, and $\gamma = 90°$, determine the tension in each wire.

2.99 Collar A weighs 5.6 lb and may slide freely on a smooth vertical rod; it is connected to collar B by wire AB. Knowing that the length of wire AB is 18 in., determine the tension in the wire when (*a*) $c = 2$ in., (*b*) $c = 8$ in.

2.100 Solve Prob. 2.99 when (*a*) $c = 14$ in., (*b*) $c = 16$ in.

2.101 Two wires are attached to the top of pole CD. It is known that the force exerted by the pole is vertical and that the 500-lb force applied to point C is horizontal. If the 500-lb force is parallel to the z axis ($\alpha = 90°$), determine the tension in each cable.

2.102 Three cables are connected at D, where an upward force of 30 kN is applied. Determine the tension in each cable.

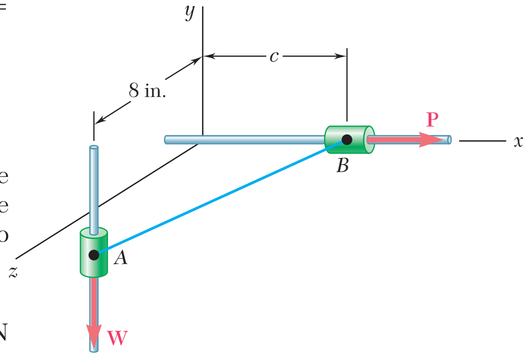


Fig. P2.99

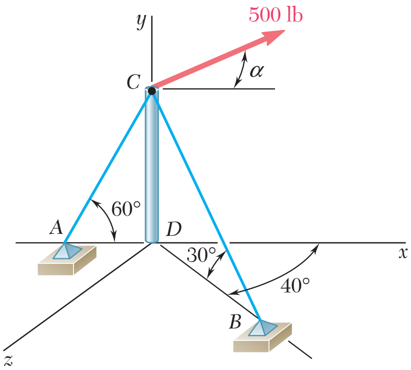


Fig. P2.101

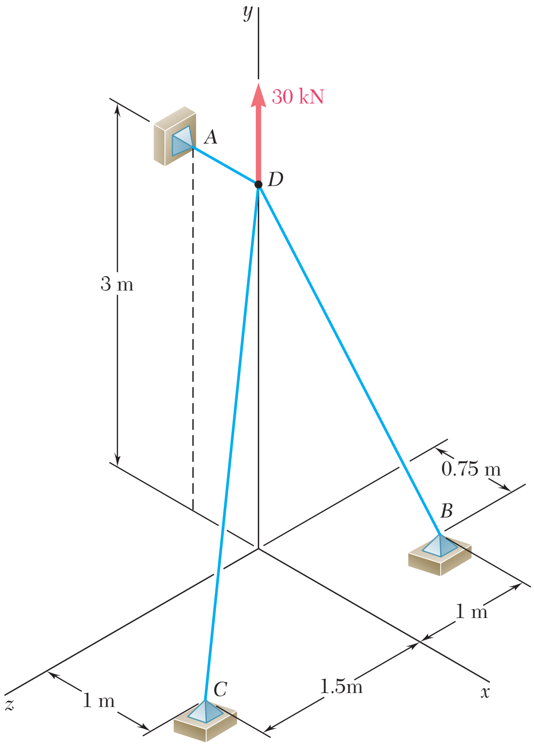


Fig. *P2.102*

2.103 A 6-kg circular plate of 200-mm radius is supported as shown by three wires of length L. Knowing that $\alpha = 30°$, determine the smallest permissible value of the length L if the tension is not to exceed 35 N in any of the wires.

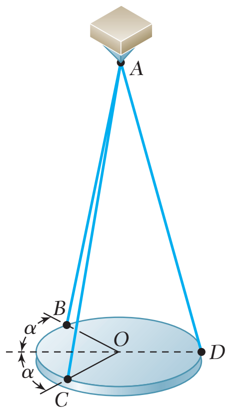


Fig. P2.103

REVIEW AND SUMMARY

In this chapter we have studied the effect of forces on particles, i.e., on bodies of such shape and size that all forces acting on them may be assumed applied at the same point.

Resultant of two forces

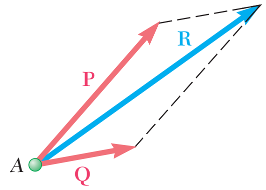


Fig. 2.35

Forces are *vector quantities;* they are characterized by a *point of application,* a *magnitude,* and a *direction,* and they add according to the *parallelogram law* (Fig. 2.35). The magnitude and direction of the resultant **R** of two forces **P** and **Q** can be determined either graphically or by trigonometry, using successively the law of cosines and the law of sines [Sample Prob. 2.1].

Components of a force

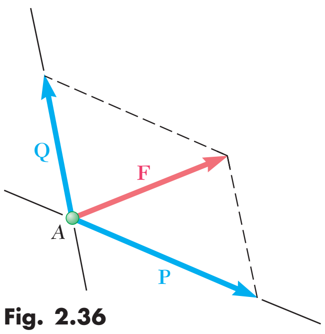


Fig. 2.36

Any given force acting on a particle can be resolved into two or more *components,* i.e., it can be replaced by two or more forces which have the same effect on the particle. A force **F** can be resolved into two components **P** and **Q** by drawing a parallelogram which has **F** for its diagonal; the components **P** and **Q** are then represented by the two adjacent sides of the parallelogram (Fig. 2.36) and can be determined either graphically or by trigonometry [Sec. 2.6].

Rectangular components
Unit vectors

A force **F** is said to have been resolved into two *rectangular components* if its components $\mathbf{F}_x$ and $\mathbf{F}_y$ are perpendicular to each other and are directed along the coordinate axes (Fig. 2.37). Introducing the *unit vectors* **i** and **j** along the x and y axes, respectively, we write [Sec. 2.7]

$$\mathbf{F}_x = F_x\mathbf{i} \qquad \mathbf{F}_y = F_y\mathbf{j} \tag{2.6}$$

and

$$\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} \tag{2.7}$$

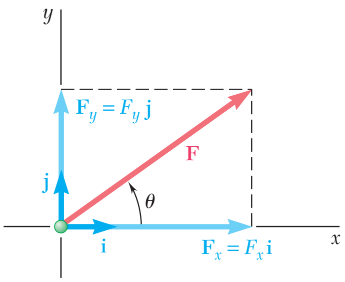


Fig. 2.37

where F_x and F_y are the *scalar components* of **F**. These components, which can be positive or negative, are defined by the relations

$$F_x = F\cos\theta \qquad F_y = F\sin\theta \tag{2.8}$$

When the rectangular components F_x and F_y of a force **F** are given, the angle θ defining the direction of the force can be obtained by writing

$$\tan\theta = \frac{F_y}{F_x} \tag{2.9}$$

The magnitude F of the force can then be obtained by solving one of the equations (2.8) for F or by applying the Pythagorean theorem and writing

$$F = \sqrt{F_x^2 + F_y^2} \tag{2.10}$$

Resultant of several coplanar forces

When *three or more coplanar forces* act on a particle, the rectangular components of their resultant **R** can be obtained by adding algebraically the corresponding components of the given forces [Sec. 2.8]. We have

$$R_x = \Sigma F_x \qquad R_y = \Sigma F_y \tag{2.13}$$

The magnitude and direction of **R** can then be determined from relations similar to Eqs. (2.9) and (2.10) [Sample Prob. 2.3].

Forces in space

A force **F** in *three-dimensional space* can be resolved into rectangular components $\mathbf{F}_x$, $\mathbf{F}_y$, and $\mathbf{F}_z$ [Sec. 2.12]. Denoting by θ_x, θ_y, and θ_z, respectively, the angles that **F** forms with the x, y, and z axes (Fig. 2.38), we have

$$F_x = F \cos \theta_x \qquad F_y = F \cos \theta_y \qquad F_z = F \cos \theta_z \tag{2.19}$$

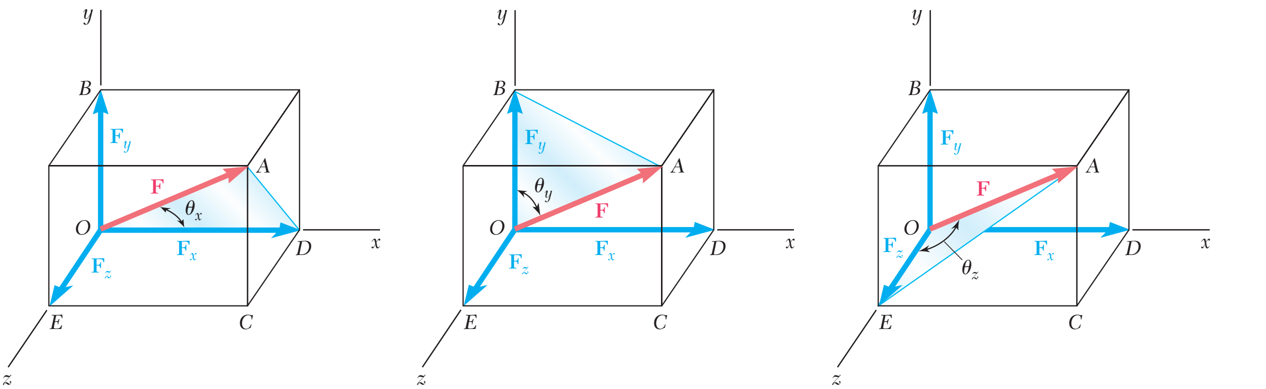


Fig. 2.38 (*a*) (*b*) (*c*)

Direction cosines

The cosines of θ_x, θ_y, θ_z are known as the *direction cosines* of the force **F**. Introducing the unit vectors **i**, **j**, **k** along the coordinate axes, we write

$$\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k} \tag{2.20}$$

or

$$\mathbf{F} = F(\cos \theta_x\mathbf{i} + \cos \theta_y\mathbf{j} + \cos \theta_z\mathbf{k}) \tag{2.21}$$

which shows (Fig. 2.39) that **F** is the product of its magnitude F and the unit vector

$$\boldsymbol{\lambda} = \cos \theta_x\mathbf{i} + \cos \theta_y\mathbf{j} + \cos \theta_z\mathbf{k}$$

Since the magnitude of $\boldsymbol{\lambda}$ is equal to unity, we must have

$$\cos^2 \theta_x + \cos^2 \theta_y + \cos^2 \theta_z = 1 \tag{2.24}$$

When the rectangular components F_x, F_y, F_z of a force **F** are given, the magnitude F of the force is found by writing

$$F = \sqrt{F_x^2 + F_y^2 + F_z^2} \tag{2.18}$$

and the direction cosines of **F** are obtained from Eqs. (2.19). We have

$$\cos \theta_x = \frac{F_x}{F} \qquad \cos \theta_y = \frac{F_y}{F} \qquad \cos \theta_z = \frac{F_z}{F} \tag{2.25}$$

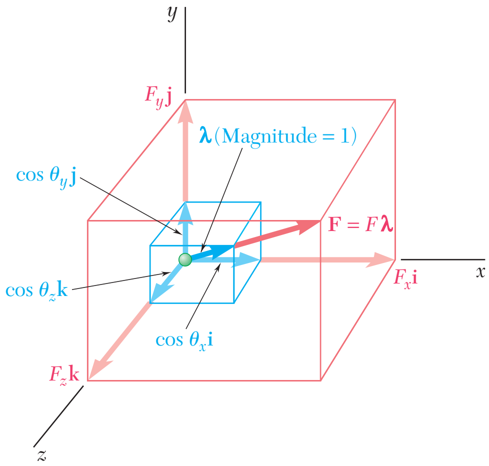


Fig. 2.39

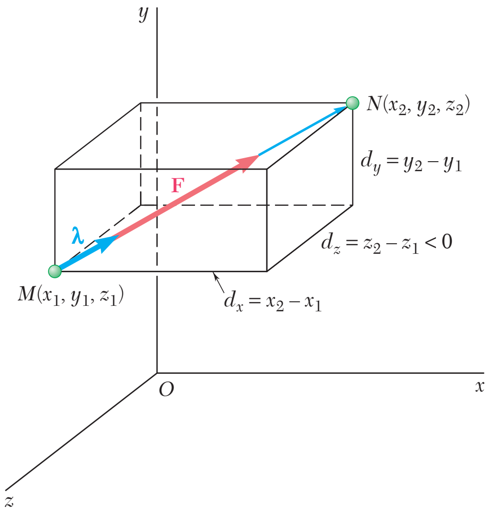


Fig. 2.40

When a force $\mathbf{F}$ is defined in three-dimensional space by its magnitude F and two points M and N on its line of action [Sec. 2.13], its rectangular components can be obtained as follows. We first express the vector $\overrightarrow{MN}$ joining points M and N in terms of its components d_x, d_y, and d_z (Fig. 2.40); we write

$$\overrightarrow{MN} = d_x\mathbf{i} + d_y\mathbf{j} + d_z\mathbf{k} \tag{2.26}$$

We next determine the unit vector $\boldsymbol{\lambda}$ along the line of action of $\mathbf{F}$ by dividing $\overrightarrow{MN}$ by its magnitude $MN = d$:

$$\boldsymbol{\lambda} = \frac{\overrightarrow{MN}}{MN} = \frac{1}{d}(d_x\mathbf{i} + d_y\mathbf{j} + d_z\mathbf{k}) \tag{2.27}$$

Recalling that $\mathbf{F}$ is equal to the product of F and $\boldsymbol{\lambda}$, we have

$$\mathbf{F} = F\boldsymbol{\lambda} = \frac{F}{d}(d_x\mathbf{i} + d_y\mathbf{j} + d_z\mathbf{k}) \tag{2.28}$$

from which it follows [Sample Probs. 2.7 and 2.8] that the scalar components of $\mathbf{F}$ are, respectively,

$$F_x = \frac{Fd_x}{d} \qquad F_y = \frac{Fd_y}{d} \qquad F_z = \frac{Fd_z}{d} \tag{2.29}$$

Resultant of forces in space

When *two or more forces* act on a particle in *three-dimensional space,* the rectangular components of their resultant $\mathbf{R}$ can be obtained by adding algebraically the corresponding components of the given forces [Sec. 2.14]. We have

$$R_x = \Sigma F_x \qquad R_y = \Sigma F_y \qquad R_z = \Sigma F_z \tag{2.31}$$

The magnitude and direction of $\mathbf{R}$ can then be determined from relations similar to Eqs. (2.18) and (2.25) [Sample Prob. 2.8].

Equilibrium of a particle

A particle is said to be in *equilibrium* when the resultant of all the forces acting on it is zero [Sec. 2.9]. The particle will then remain at rest (if originally at rest) or move with constant speed in a straight line (if originally in motion) [Sec. 2.10].

Free-body diagram

To solve a problem involving a particle in equilibrium, one first should draw a *free-body diagram* of the particle showing all the forces acting on it [Sec. 2.11]. If *only three coplanar forces* act on the particle, a *force triangle* may be drawn to express that the particle is in equilibrium. Using graphical methods of trigonometry, this triangle can be solved for no more than two unknowns [Sample Prob. 2.4]. If *more than three coplanar forces* are involved, the equations of equilibrium

$$\Sigma F_x = 0 \qquad \Sigma F_y = 0 \tag{2.15}$$

should be used. These equations can be solved for no more than two unknowns [Sample Prob. 2.6].

Equilibrium in space

When a particle is in *equilibrium in three-dimensional space* [Sec. 2.15], the three equations of equilibrium

$$\Sigma F_x = 0 \qquad \Sigma F_y = 0 \qquad \Sigma F_z = 0 \tag{2.34}$$

should be used. These equations can be solved for no more than three unknowns [Sample Prob. 2.9].

REVIEW PROBLEMS

2.104 A cable loop of length 1.5 m is placed around a crate. Knowing that the mass of the crate is 300 kg, determine the tension in the cable for each of the arrangements shown.

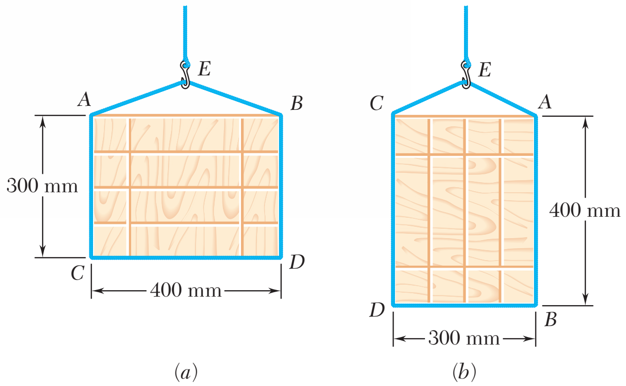


Fig. P2.104

2.105 Knowing that the magnitude of the force **P** is 75 lb, determine the resultant of the three forces applied at *A*.

2.106 Determine the range of values of *P* for which the resultant of the three forces applied at *A* does not exceed 175 lb.

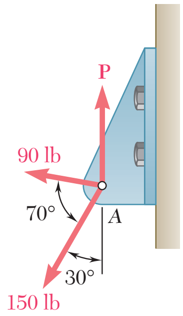


Fig. P2.105 and P2.106

2.107 The directions of the 300-N forces may vary, but the angle between the forces is always 40°. Determine the value of α for which the resultant of the forces acting at *A* is directed parallel to the plane *b*-*b*.

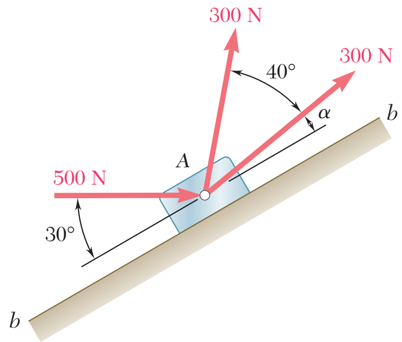


Fig. *P2.107*

2.108 Knowing that $P = 300$ lb, determine the tension in cables *AC* and *BC*.

2.109 Determine the range of values of **P** for which both cables remain taut.

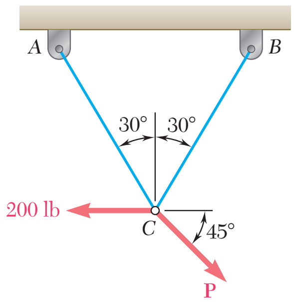


Fig. P2.108 and *P2.109*

2.110 A container is supported by three cables as shown. Determine the weight W of the container knowing that the tension in cable AB is 500 N.

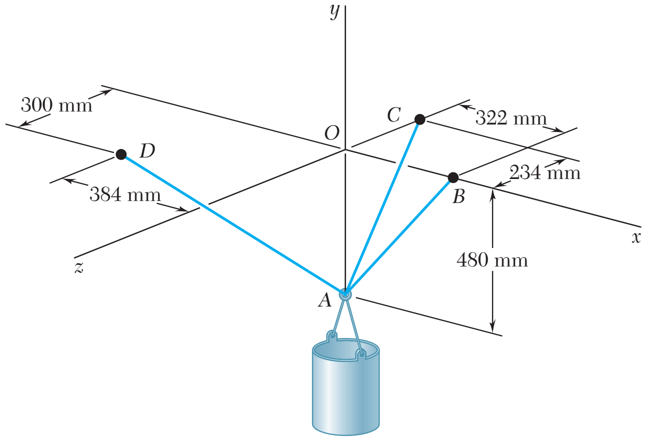


Fig. P2.110

2.111 In Prob. 2.110, determine the angles θ_x, θ_y, and θ_z for the force exerted at D by cable AD.

2.112 A 1200-N force acts at the origin in a direction defined by the angles $\theta_x = 65°$ and $\theta_y = 40°$. It is also known that the z component of the force is positive. Determine the value of θ_z and the components of the force.

2.113 Two cables BG and BH are attached to frame ACD as shown. Knowing that the tension is 540 N in cable BG and 750 N in cable BH, determine the magnitude and direction of the resultant of the forces exerted by the cables on the frame at B.

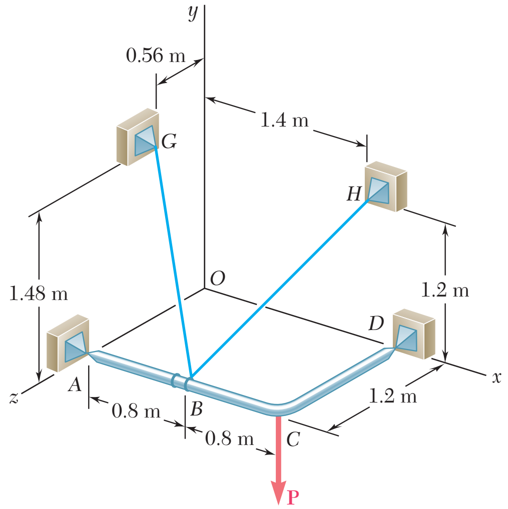


Fig. P2.113

2.114 A crate is supported by three cables as shown. Determine the weight W of the crate knowing that the tension in cable AD is 924 lb.

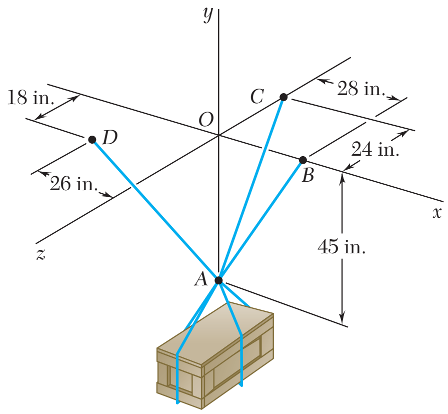


Fig. P2.114

2.115 A triangular steel plate is supported by three wires as shown. Knowing that $a = 6$ in. and that the tension in wire AD is 17 lb, determine the weight of the plate.

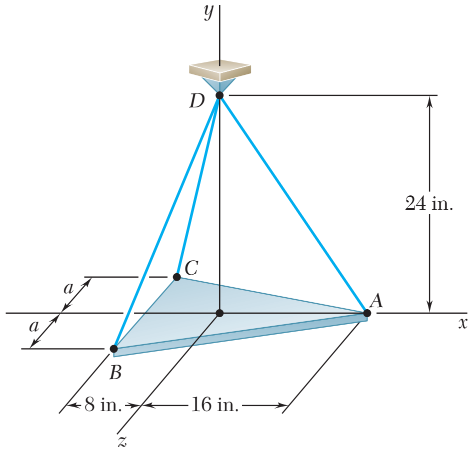
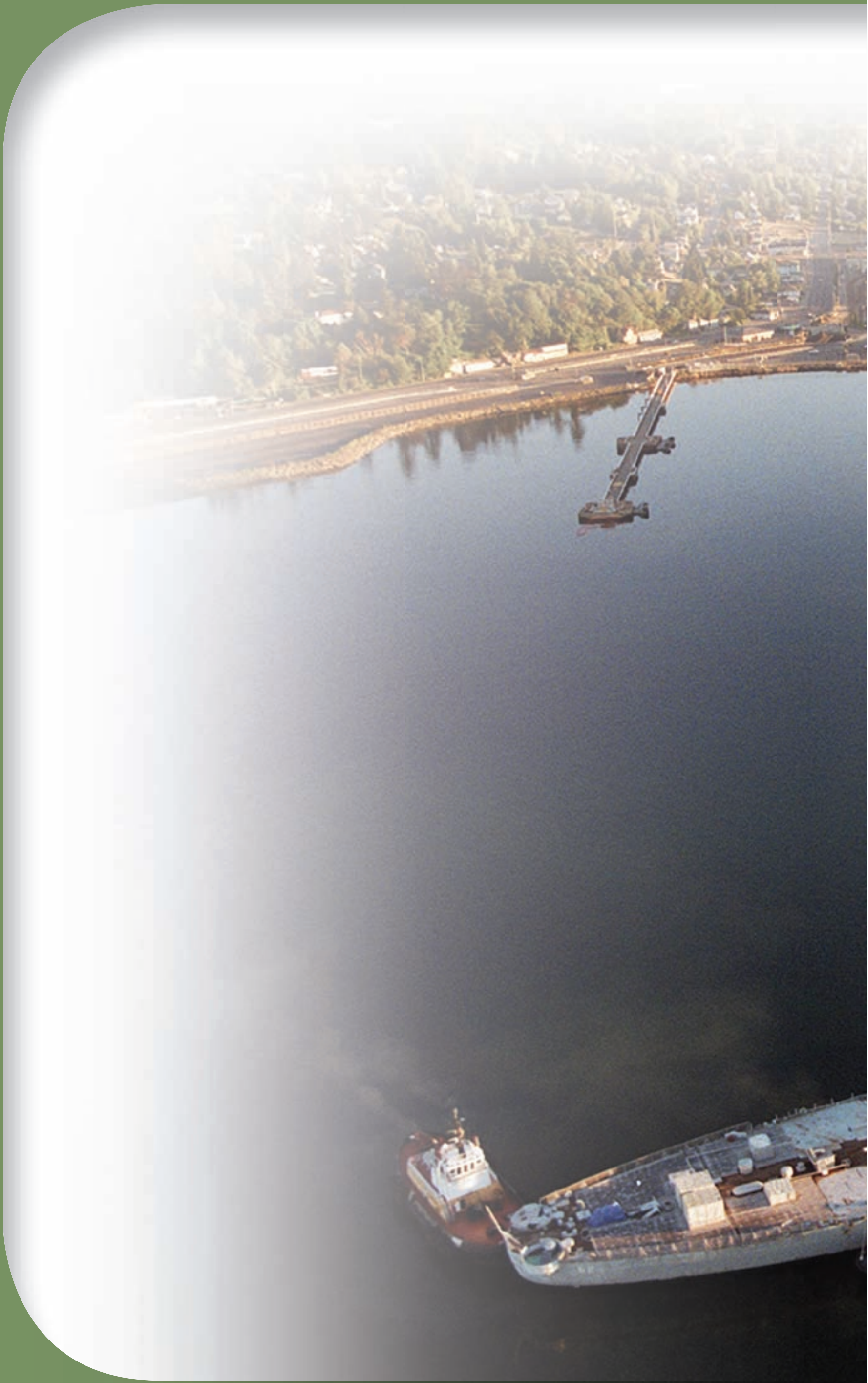


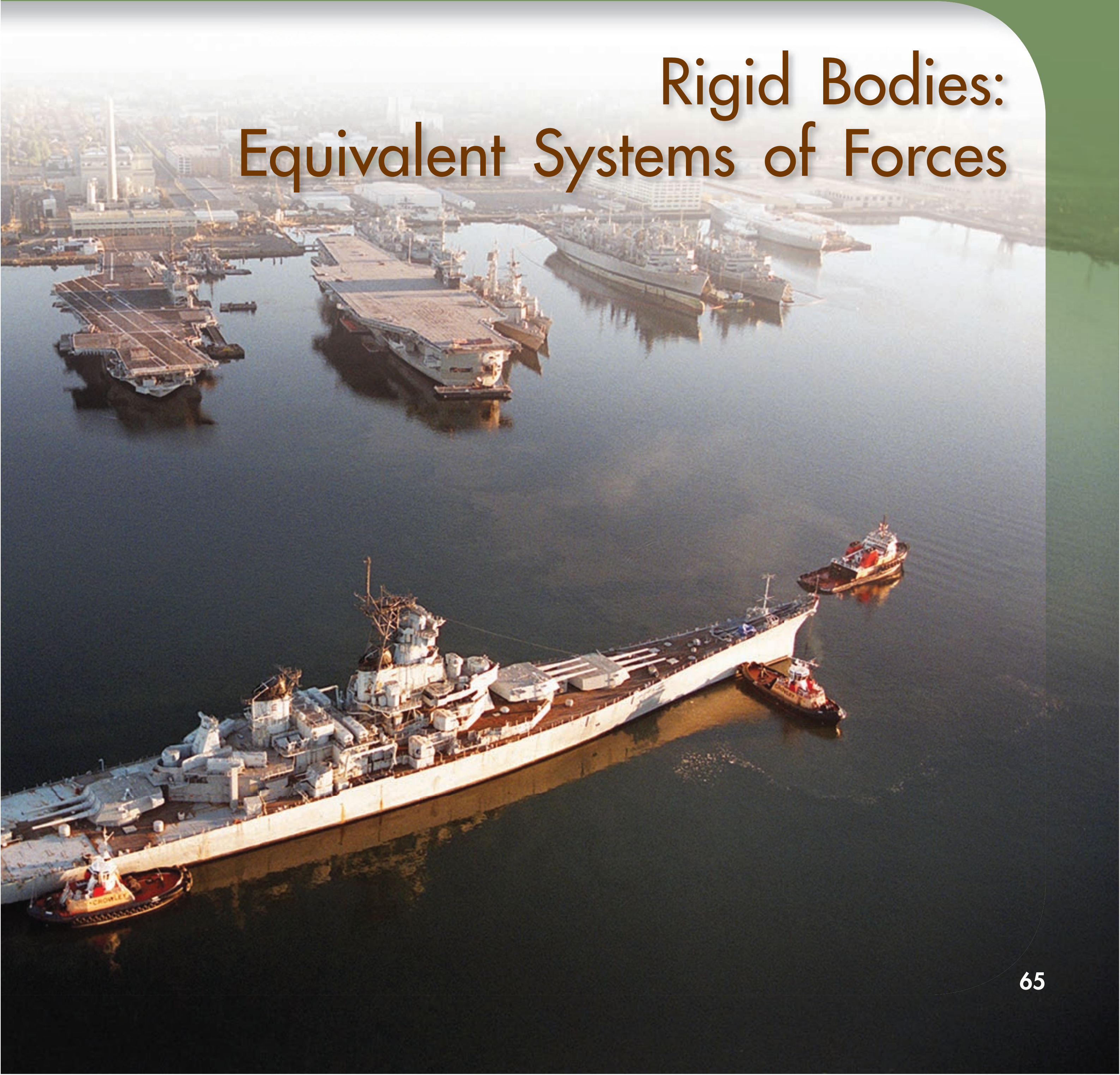
Fig. P2.115

The battleship USS *New Jersey* is maneuvered by four tugboats at Bremerton Naval Shipyard. It will be shown in this chapter that the forces exerted on the ship by the tugboats could be replaced by an equivalent force exerted by a single, more powerful tugboat.



CHAPTER 3

Rigid Bodies: Equivalent Systems of Forces



Chapter 3 Rigid Bodies: Equivalent Systems of Forces

3.1 INTRODUCTION

In the preceding chapter it was assumed that each of the bodies considered could be treated as a single particle. Such a view, however, is not always possible, and a body, in general, should be treated as a combination of a large number of particles. The size of the body will have to be taken into consideration, as well as the fact that forces will act on different particles and thus will have different points of application.

Most of the bodies considered in elementary mechanics are assumed to be *rigid*, a *rigid body* being defined as one which does not deform. Actual structures and machines, however, are never absolutely rigid and deform under the loads to which they are subjected. But these deformations are usually small and do not appreciably affect the conditions of equilibrium or motion of the structure under consideration. They are important, though, as far as the resistance of the structure to failure is concerned and are considered in the study of mechanics of materials.

In this chapter you will study the effect of forces exerted on a rigid body, and you will learn how to replace a given system of forces by a simpler equivalent system. This analysis will rest on the fundamental assumption that the effect of a given force on a rigid body remains unchanged if that force is moved along its line of action (*principle of transmissibility*). It follows that forces acting on a rigid body can be represented by *sliding vectors,* as indicated earlier in Sec. 2.3.

Two important concepts associated with the effect of a force on a rigid body are the *moment of a force about a point* (Sec. 3.6) and the *moment of a force about an axis* (Sec. 3.11). Since the determination of these quantities involves the computation of vector products and scalar products of two vectors, the fundamentals of vector algebra will be introduced in this chapter and applied to the solution of problems involving forces acting on rigid bodies.

Another concept introduced in this chapter is that of a *couple,* i.e., the combination of two forces which have the same magnitude, parallel lines of action, and opposite sense (Sec. 3.12). As you will see, any system of forces acting on a rigid body can be replaced by an equivalent system consisting of one force acting at a given point and one couple. This basic system is called a *force-couple system.* In the case of concurrent, coplanar, or parallel forces, the equivalent force-couple system can be further reduced to a single force, called the *resultant* of the system, or to a single couple, called the *resultant couple* of the system.

3.2 EXTERNAL AND INTERNAL FORCES

Forces acting on rigid bodies can be separated into two groups: (1) *external forces* and (2) *internal forces.*

1. The *external forces* represent the action of other bodies on the rigid body under consideration. They are entirely responsible for the external behavior of the rigid body. They will either cause it to move or ensure that it remains at rest. We shall be concerned only with external forces in this chapter and in Chaps. 4 and 5.

2. The *internal forces* are the forces which hold together the particles forming the rigid body. If the rigid body is structurally composed of several parts, the forces holding the component parts together are also defined as internal forces. Internal forces will be considered in Chaps. 6 and 7.

As an example of external forces, let us consider the forces acting on a disabled truck that three people are pulling forward by means of a rope attached to the front bumper (Fig. 3.1). The external forces acting on the truck are shown in a *free-body diagram* (Fig. 3.2). Let us first consider the *weight* of the truck. Although it embodies the effect of the earth's pull on each of the particles forming the truck, the weight can be represented by the single force **W**. The *point of application* of this force, i.e., the point at which the force acts, is defined as the *center of gravity* of the truck. It will be seen in Chap. 5 how centers of gravity can be determined. The weight **W** tends to make the truck move vertically downward. In fact, it would actually cause the truck to move downward, i.e., to fall, if it were not for the presence of the ground. The ground opposes the downward motion of the truck by means of the reactions $\mathbf{R}_1$ and $\mathbf{R}_2$. These forces are exerted *by* the ground *on* the truck and must therefore be included among the external forces acting on the truck.

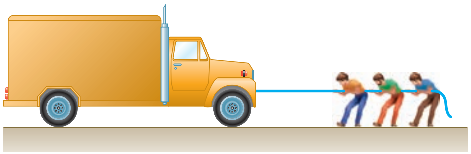


Fig. 3.1

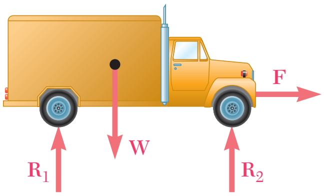


Fig. 3.2

The people pulling on the rope exert the force **F**. The point of application of **F** is on the front bumper. The force **F** tends to make the truck move forward in a straight line and does actually make it move, since no external force opposes this motion. (Rolling resistance has been neglected here for simplicity.) This forward motion of the truck, during which each straight line keeps its original orientation (the floor of the truck remains horizontal, and the walls remain vertical), is known as a *translation*. Other forces might cause the truck to move differently. For example, the force exerted by a jack placed under the front axle would cause the truck to pivot about its rear axle. Such a motion is a *rotation*. It can be concluded, therefore, that each of the *external forces* acting on a *rigid body* can, if unopposed, impart to the rigid body a motion of translation or rotation, or both.

3.3 PRINCIPLE OF TRANSMISSIBILITY. EQUIVALENT FORCES

The *principle of transmissibility* states that the conditions of equilibrium or motion of a rigid body will remain unchanged if a force **F** acting at a given point of the rigid body is replaced by a force **F′** of the same magnitude and same direction, but acting at a different point, *provided that the two forces have the same line of action* (Fig. 3.3). The two forces **F** and **F′** have the same effect on the rigid body and are said to be *equivalent*. This principle, which states that the action of a force may be *transmitted* along its line of action, is based on experimental evidence. It *cannot* be derived from the properties established so far in this text and must therefore be accepted as an experimental law. However, as you will see in Sec. 16.5, the principle of transmissibility can be derived from the study of the dynamics of rigid bodies, but this study requires the introduction of Newton's

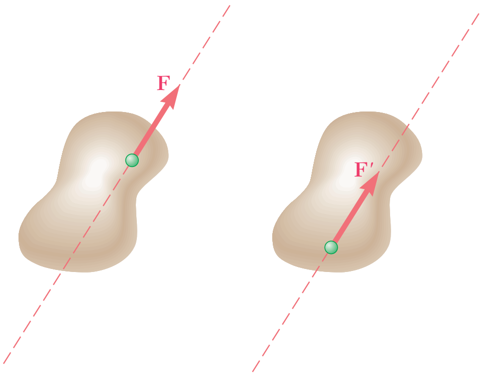


Fig. 3.3

second and third laws and of a number of other concepts as well. Therefore, our study of the statics of rigid bodies will be based on the three principles introduced so far, i.e., the parallelogram law of addition, Newton's first law, and the principle of transmissibility.

It was indicated in Chap. 2 that the forces acting on a particle could be represented by vectors. These vectors had a well-defined point of application, namely, the particle itself, and were therefore fixed, or bound, vectors. In the case of forces acting on a rigid body, however, the point of application of the force does not matter, as long as the line of action remains unchanged. Thus, forces acting on a rigid body must be represented by a different kind of vector, known as a *sliding vector,* since forces may be allowed to slide along their lines of action. We should note that all the properties which will be derived in the following sections for the forces acting on a rigid body will be valid more generally for any system of sliding vectors. In order to keep our presentation more intuitive, however, we will carry it out in terms of physical forces rather than in terms of mathematical sliding vectors.

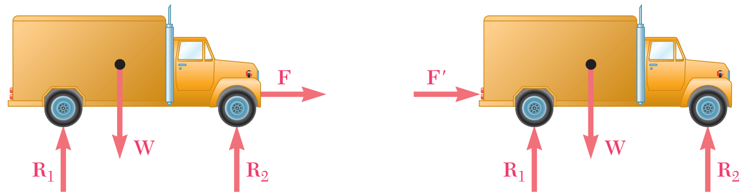


Fig. 3.4

Returning to the example of the truck, we first observe that the line of action of the force $\mathbf{F}$ is a horizontal line passing through both the front and the rear bumpers of the truck (Fig. 3.4). Using the principle of transmissibility, we can therefore replace $\mathbf{F}$ by an *equivalent force* $\mathbf{F}'$ acting on the rear bumper. In other words, the conditions of motion are unaffected, and all the other external forces acting on the truck ($\mathbf{W}$, $\mathbf{R}_1$, $\mathbf{R}_2$) remain unchanged if the people push on the rear bumper instead of pulling on the front bumper.

The principle of transmissibility and the concept of equivalent forces have limitations, however. Consider, for example, a short bar AB acted upon by equal and opposite axial forces $\mathbf{P}_1$ and $\mathbf{P}_2$, as shown in Fig. 3.5*a*. According to the principle of transmissibility, the force $\mathbf{P}_2$ can be replaced by a force $\mathbf{P}'_2$ having the same magnitude, the same direction, and the same line of action but acting at A instead of B (Fig. 3.5*b*). The forces $\mathbf{P}_1$ and $\mathbf{P}'_2$ acting on the same particle

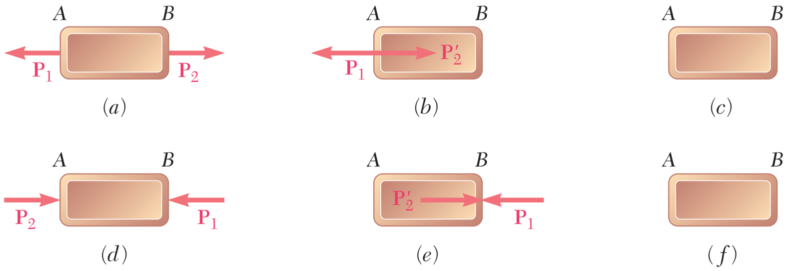


Fig. 3.5

can be added according to the rules of Chap. 2, and, as these forces are equal and opposite, their sum is equal to zero. Thus, in terms of the external behavior of the bar, the original system of forces shown in Fig. 3.5*a* is equivalent to no force at all (Fig. 3.5*c*).

Consider now the two equal and opposite forces $\mathbf{P}_1$ and $\mathbf{P}_2$ acting on the bar *AB* as shown in Fig. 3.5*d*. The force $\mathbf{P}_2$ can be replaced by a force $\mathbf{P}_2'$ having the same magnitude, the same direction, and the same line of action but acting at *B* instead of at *A* (Fig. 3.5*e*). The forces $\mathbf{P}_1$ and $\mathbf{P}_2'$ can then be added, and their sum is again zero (Fig. 3.5*f*). From the point of view of the mechanics of rigid bodies, the systems shown in Fig. 3.5*a* and *d* are thus equivalent. But the *internal forces* and *deformations* produced by the two systems are clearly different. The bar of Fig. 3.5*a* is in *tension* and, if not absolutely rigid, will increase in length slightly; the bar of Fig. 3.5*d* is in *compression* and, if not absolutely rigid, will decrease in length slightly. Thus, while the principle of transmissibility may be used freely to determine the conditions of motion or equilibrium of rigid bodies and to compute the external forces acting on these bodies, it should be avoided, or at least used with care, in determining internal forces and deformations.

3.4 VECTOR PRODUCT OF TWO VECTORS

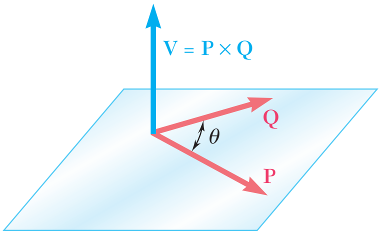
In order to gain a better understanding of the effect of a force on a rigid body, a new concept, the concept of *a moment of a force about a point,* will be introduced at this time. This concept will be more clearly understood, and applied more effectively, if we first add to the mathematical tools at our disposal the *vector product* of two vectors.

The vector product of two vectors **P** and **Q** is defined as the vector **V** which satisfies the following conditions.

1. The line of action of **V** is perpendicular to the plane containing **P** and **Q** (Fig. 3.6*a*).
2. The magnitude of **V** is the product of the magnitudes of **P** and **Q** and of the sine of the angle θ formed by **P** and **Q** (the measure of which will always be 180° or less); we thus have

$$V = PQ \sin \theta \tag{3.1}$$

3. The direction of **V** is obtained from the *right-hand rule*. Close your right hand and hold it so that your fingers are curled in the same sense as the rotation through θ which brings the vector **P** in line with the vector **Q**; your thumb will then indicate the direction of the vector **V** (Fig. 3.6*b*). Note that if **P** and **Q** do not have a common point of application, they should first be redrawn from the same point. The three vectors **P**, **Q**, and **V**—taken in that order—are said to form a *right-handed triad.*†



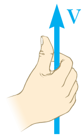


Fig. 3.6

†We should note that the x, y, and z axes used in Chap. 2 form a right-handed system of orthogonal axes and that the unit vectors **i**, **j**, **k** defined in Sec. 2.12 form a right-handed orthogonal triad.

As stated above, the vector **V** satisfying these three conditions (which define it uniquely) is referred to as the vector product of **P** and **Q**; it is represented by the mathematical expression

$$\mathbf{V} = \mathbf{P} \times \mathbf{Q} \tag{3.2}$$

Because of the notation used, the vector product of two vectors **P** and **Q** is also referred to as the *cross product* of **P** and **Q**.

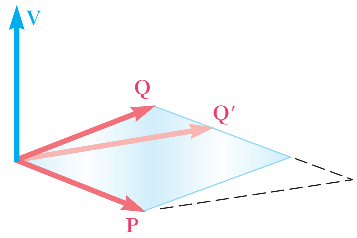


Fig. 3.7

It follows from Eq. (3.1) that, when two vectors **P** and **Q** have either the same direction or opposite directions, their vector product is zero. In the general case when the angle θ formed by the two vectors is neither 0° nor 180°, Eq. (3.1) can be given a simple geometric interpretation: The magnitude V of the vector product of **P** and **Q** is equal to the area of the parallelogram which has **P** and **Q** for sides (Fig. 3.7). The vector product $\mathbf{P} \times \mathbf{Q}$ will therefore remain unchanged if we replace **Q** by a vector **Q′** which is coplanar with **P** and **Q** and such that the line joining the tips of **Q** and **Q′** is parallel to **P**. We write

$$\mathbf{V} = \mathbf{P} \times \mathbf{Q} = \mathbf{P} \times \mathbf{Q}' \tag{3.3}$$

From the third condition used to define the vector product **V** of **P** and **Q**, namely, the condition stating that **P**, **Q**, and **V** must form a right-handed triad, it follows that vector products *are not commutative*, i.e., $\mathbf{Q} \times \mathbf{P}$ is not equal to $\mathbf{P} \times \mathbf{Q}$. Indeed, we can easily check that $\mathbf{Q} \times \mathbf{P}$ is represented by the vector $-\mathbf{V}$, which is equal and opposite to **V**. We thus write

$$\mathbf{Q} \times \mathbf{P} = -(\mathbf{P} \times \mathbf{Q}) \tag{3.4}$$

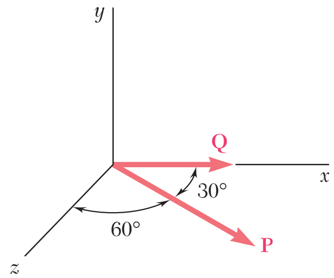


Fig. 3.8

EXAMPLE 3.1 Let us compute the vector product $\mathbf{V} = \mathbf{P} \times \mathbf{Q}$ where the vector **P** is of magnitude 6 and lies in the zx plane at an angle of 30° with the x axis, and where the vector **Q** is of magnitude 4 and lies along the x axis (Fig. 3.8).

It follows immediately from the definition of the vector product that the vector **V** must lie along the y axis and have the magnitude

$$V = PQ \sin \theta = (6)(4) \sin 30° = 12$$

and be directed upward. ■

We saw that the commutative property does not apply to vector products. We may wonder whether the *distributive* property holds, i.e., whether the relation

$$\mathbf{P} \times (\mathbf{Q}_1 + \mathbf{Q}_2) = \mathbf{P} \times \mathbf{Q}_1 + \mathbf{P} \times \mathbf{Q}_2 \tag{3.5}$$

is valid. The answer is *yes*. Many readers are probably willing to accept without formal proof an answer which they intuitively feel is correct. However, since the entire structure of both vector algebra and statics depends upon the relation (3.5), we should take time out to derive it.

We can, without any loss of generality, assume that **P** is directed along the y axis (Fig. 3.9*a*). Denoting by **Q** the sum of $\mathbf{Q}_1$ and $\mathbf{Q}_2$, we drop perpendiculars from the tips of **Q**, $\mathbf{Q}_1$, and $\mathbf{Q}_2$ onto the zx plane, defining in this way the vectors $\mathbf{Q}'$, $\mathbf{Q}'_1$, and $\mathbf{Q}'_2$. These vectors will be referred to, respectively, as the *projections* of **Q**, $\mathbf{Q}_1$, and $\mathbf{Q}_2$ on the zx plane. Recalling the property expressed by Eq. (3.3), we note that the left-hand member of Eq. (3.5) can be replaced by $\mathbf{P} \times \mathbf{Q}'$ and that, similarly, the vector products $\mathbf{P} \times \mathbf{Q}_1$ and $\mathbf{P} \times \mathbf{Q}_2$ can respectively be replaced by $\mathbf{P} \times \mathbf{Q}'_1$ and $\mathbf{P} \times \mathbf{Q}'_2$. Thus, the relation to be proved can be written in the form

$$\mathbf{P} \times \mathbf{Q}' = \mathbf{P} \times \mathbf{Q}'_1 + \mathbf{P} \times \mathbf{Q}'_2 \tag{3.5'}$$

We now observe that $\mathbf{P} \times \mathbf{Q}'$ can be obtained from $\mathbf{Q}'$ by multiplying this vector by the scalar P and rotating it counterclockwise through 90° in the zx plane (Fig. 3.9*b*); the other two vector

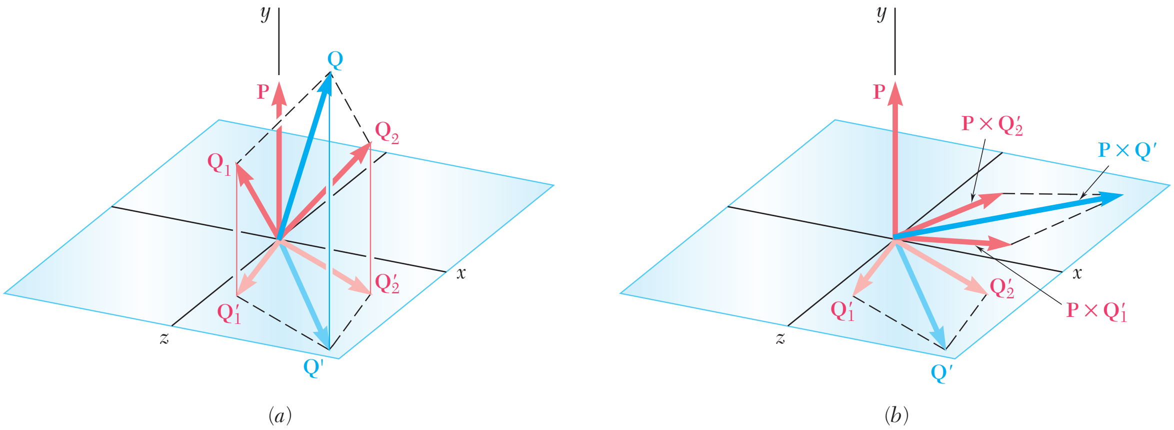


Fig. 3.9

products in (3.5′) can be obtained in the same manner from $\mathbf{Q}'_1$ and $\mathbf{Q}'_2$, respectively. Now, since the projection of a parallelogram onto an arbitrary plane is a parallelogram, the projection $\mathbf{Q}'$ of the sum $\mathbf{Q}$ of $\mathbf{Q}_1$ and $\mathbf{Q}_2$ must be the sum of the projections $\mathbf{Q}'_1$ and $\mathbf{Q}'_2$ of $\mathbf{Q}_1$ and $\mathbf{Q}_2$ on the same plane (Fig. 3.9*a*). This relation between the vectors $\mathbf{Q}'$, $\mathbf{Q}'_1$, and $\mathbf{Q}'_2$ will still hold after the three vectors have been multiplied by the scalar P and rotated through 90° (Fig. 3.9*b*). Thus, the relation (3.5′) has been proved, and we can now be sure that the distributive property holds for vector products.

A third property, the *associative* property, does not apply to vector products; we have in general

$$(\mathbf{P} \times \mathbf{Q}) \times \mathbf{S} \neq \mathbf{P} \times (\mathbf{Q} \times \mathbf{S}) \tag{3.6}$$

3.5 VECTOR PRODUCTS EXPRESSED IN TERMS OF RECTANGULAR COMPONENTS

Let us now determine the vector product of any two of the unit vectors **i**, **j**, and **k**, which were defined in Chap. 2. Consider first the product $\mathbf{i} \times \mathbf{j}$ (Fig. 3.10*a*). Since both vectors have a magnitude equal to 1 and since they are at a right angle to each other, their vector product will also be a unit vector. This unit vector must be **k**, since the vectors **i**, **j**, and **k** are mutually perpendicular and form a right-handed triad. On the other hand, it follows from the right-hand rule given on page 69 that the product $\mathbf{j} \times \mathbf{i}$ will be equal to $-\mathbf{k}$ (Fig. 3.10*b*). Finally, it should be observed that the vector product

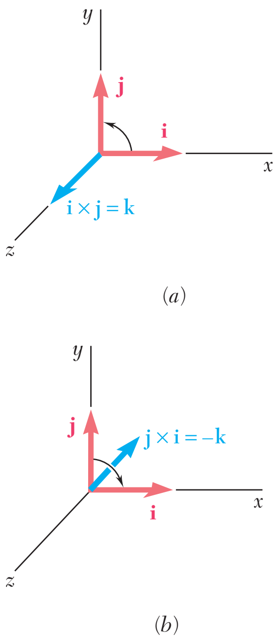


Fig. 3.10

of a unit vector with itself, such as $\mathbf{i} \times \mathbf{i}$, is equal to zero, since both vectors have the same direction. The vector products of the various possible pairs of unit vectors are

$$\begin{array}{lll} \mathbf{i} \times \mathbf{i} = \mathbf{0} & \mathbf{j} \times \mathbf{i} = -\mathbf{k} & \mathbf{k} \times \mathbf{i} = \mathbf{j} \\ \mathbf{i} \times \mathbf{j} = \mathbf{k} & \mathbf{j} \times \mathbf{j} = \mathbf{0} & \mathbf{k} \times \mathbf{j} = -\mathbf{i} \\ \mathbf{i} \times \mathbf{k} = -\mathbf{j} & \mathbf{j} \times \mathbf{k} = \mathbf{i} & \mathbf{k} \times \mathbf{k} = \mathbf{0} \end{array} \tag{3.7}$$

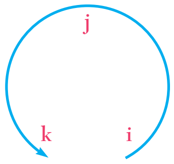


Fig. 3.11

By arranging in a circle and in counterclockwise order the three letters representing the unit vectors (Fig. 3.11), we can simplify the determination of the sign of the vector product of two unit vectors: The product of two unit vectors will be positive if they follow each other in counterclockwise order and will be negative if they follow each other in clockwise order.

We can now easily express the vector product $\mathbf{V}$ of two given vectors $\mathbf{P}$ and $\mathbf{Q}$ in terms of the rectangular components of these vectors. Resolving $\mathbf{P}$ and $\mathbf{Q}$ into components, we first write

$$\mathbf{V} = \mathbf{P} \times \mathbf{Q} = (P_x\mathbf{i} + P_y\mathbf{j} + P_z\mathbf{k}) \times (Q_x\mathbf{i} + Q_y\mathbf{j} + Q_z\mathbf{k})$$

Making use of the distributive property, we express $\mathbf{V}$ as the sum of vector products, such as $P_x\mathbf{i} \times Q_y\mathbf{j}$. Observing that each of the expressions obtained is equal to the vector product of two unit vectors, such as $\mathbf{i} \times \mathbf{j}$, multiplied by the product of two scalars, such as P_xQ_y, and recalling the identities (3.7), we obtain, after factoring out $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$,

$$\mathbf{V} = (P_yQ_z - P_zQ_y)\mathbf{i} + (P_zQ_x - P_xQ_z)\mathbf{j} + (P_xQ_y - P_yQ_x)\mathbf{k} \tag{3.8}$$

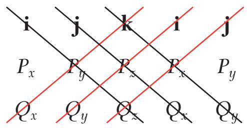
The rectangular components of the vector product $\mathbf{V}$ are thus found to be

$$\begin{aligned} V_x &= P_yQ_z - P_zQ_y \\ V_y &= P_zQ_x - P_xQ_z \\ V_z &= P_xQ_y - P_yQ_x \end{aligned} \tag{3.9}$$

Returning to Eq. (3.8), we observe that its right-hand member represents the expansion of a determinant. The vector product $\mathbf{V}$ can thus be expressed in the following form, which is more easily memorized:†

$$\mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix} \tag{3.10}$$

†Any determinant consisting of three rows and three columns can be evaluated by repeating the first and second columns and forming products along each diagonal line. The sum of the products obtained along the red lines is then subtracted from the sum of the products obtained along the black lines.



3.6 MOMENT OF A FORCE ABOUT A POINT

Let us now consider a force **F** acting on a rigid body (Fig. 3.12*a*). As we know, the force **F** is represented by a vector which defines its magnitude and direction. However, the effect of the force on the rigid body depends also upon its point of application A. The position of A can be conveniently defined by the vector **r** which joins the fixed reference point O with A; this vector is known as the *position vector* of A.† The position vector **r** and the force **F** define the plane shown in Fig. 3.12*a*.

We will define the *moment of* **F** *about* O as the vector product of **r** and **F**:

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} \tag{3.11}$$

According to the definition of the vector product given in Sec. 3.4, the moment $\mathbf{M}_O$ must be perpendicular to the plane containing O and the force **F**. The sense of $\mathbf{M}_O$ is defined by the sense of the rotation which will bring the vector **r** in line with the vector **F**; this rotation will be observed as *counterclockwise* by an observer located at the tip of $\mathbf{M}_O$. Another way of defining the sense of $\mathbf{M}_O$ is furnished by a variation of the right-hand rule: Close your right hand and hold it so that your fingers are curled in the sense of the rotation that **F** would impart to the rigid body about a fixed axis directed along the line of action of $\mathbf{M}_O$; your thumb will indicate the sense of the moment $\mathbf{M}_O$ (Fig. 3.12*b*).

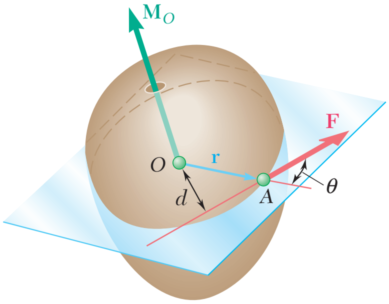
Finally, denoting by θ the angle between the lines of action of the position vector **r** and the force **F**, we find that the magnitude of the moment of **F** about O is

$$M_O = rF \sin\theta = Fd \tag{3.12}$$

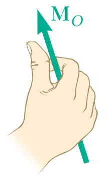
where d represents the perpendicular distance from O to the line of action of **F**. Since the tendency of a force **F** to make a rigid body rotate about a fixed axis perpendicular to the force depends upon the distance of **F** from that axis as well as upon the magnitude of **F**, we note that *the magnitude of* $\mathbf{M}_O$ *measures the tendency of the force* **F** *to make the rigid body rotate about a fixed axis directed along* $\mathbf{M}_O$.

In the SI system of units, where a force is expressed in newtons (N) and a distance in meters (m), the moment of a force is expressed in newton-meters (N · m). In the U.S. customary system of units, where a force is expressed in pounds and a distance in feet or inches, the moment of a force is expressed in lb · ft or lb · in.

We can observe that although the moment $\mathbf{M}_O$ of a force about a point depends upon the magnitude, the line of action, and the sense of the force, it does *not* depend upon the actual position of the point of application of the force along its line of action. Conversely, the moment $\mathbf{M}_O$ of a force **F** does not characterize the position of the point of application of **F**.



(*a*)



(*b*)

Fig. 3.12

†We can easily verify that position vectors obey the law of vector addition and, thus, are truly vectors. Consider, for example, the position vectors **r** and **r**′ of A with respect to two reference points O and O' and the position vector **s** of O with respect to O' (Fig. 3.40*a*, Sec. 3.16). We verify that the position vector $\mathbf{r}' = \overrightarrow{O'A}$ can be obtained from the position vectors $\mathbf{s} = \overrightarrow{O'O}$ and $\mathbf{r} = \overrightarrow{OA}$ by applying the triangle rule for the addition of vectors.

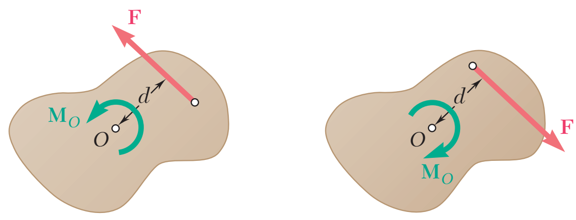
However, as it will be seen presently, the moment $\mathbf{M}_O$ of a force $\mathbf{F}$ of given magnitude and direction *completely defines the line of action of* $\mathbf{F}$. Indeed, the line of action of $\mathbf{F}$ must lie in a plane through O perpendicular to the moment $\mathbf{M}_O$; its distance d from O must be equal to the quotient M_O/F of the magnitudes of $\mathbf{M}_O$ and $\mathbf{F}$; and the sense of $\mathbf{M}_O$ determines whether the line of action of $\mathbf{F}$ is to be drawn on one side or the other of the point O.

We recall from Sec. 3.3 that the principle of transmissibility states that two forces $\mathbf{F}$ and $\mathbf{F}'$ are equivalent (i.e., have the same effect on a rigid body) if they have the same magnitude, same direction, and same line of action. This principle can now be restated as follows: *Two forces* $\mathbf{F}$ *and* $\mathbf{F}'$ *are equivalent if, and only if, they are equal* (i.e., have the same magnitude and same direction) *and have equal moments about a given point* O. The necessary and sufficient conditions for two forces $\mathbf{F}$ and $\mathbf{F}'$ to be equivalent are thus

$$\mathbf{F} = \mathbf{F}' \qquad \text{and} \qquad \mathbf{M}_O = \mathbf{M}'_O \tag{3.13}$$

We should observe that it follows from this statement that if the relations (3.13) hold for a given point O, they will hold for any other point.

Problems Involving Only Two Dimensions. Many applications deal with two-dimensional structures, i.e., structures which have length and breadth but only negligible depth and which are subjected to forces contained in the plane of the structure. Two-dimensional structures and the forces acting on them can be readily represented on a sheet of paper or on a blackboard. Their analysis is therefore considerably simpler than that of three-dimensional structures and forces.



(*a*) $\mathbf{M}_O = +Fd$ (*b*) $\mathbf{M}_O = -Fd$

Fig. 3.13

Consider, for example, a rigid slab acted upon by a force $\mathbf{F}$ (Fig. 3.13). The moment of $\mathbf{F}$ about a point O chosen in the plane of the figure is represented by a vector $\mathbf{M}_O$ perpendicular to that plane and of magnitude Fd. In the case of Fig. 3.13*a* the vector $\mathbf{M}_O$ points *out of* the paper, while in the case of Fig. 3.13*b* it points *into* the paper. As we look at the figure, we observe in the first case that $\mathbf{F}$ tends to rotate the slab counterclockwise and in the second case that it tends to rotate the slab clockwise. Therefore, it is natural to refer to the sense of the moment of $\mathbf{F}$ about O in Fig. 3.13*a* as counterclockwise ↺, and in Fig. 3.13*b* as clockwise ↻.

Since the moment of a force $\mathbf{F}$ acting in the plane of the figure must be perpendicular to that plane, we need only specify the *magnitude* and the *sense* of the moment of $\mathbf{F}$ about O. This can be done by assigning to the magnitude M_O of the moment a positive or negative sign according to whether the vector $\mathbf{M}_O$ points out of or into the paper.

3.7 VARIGNON'S THEOREM

The distributive property of vector products can be used to determine the moment of the resultant of several *concurrent forces*. If several forces $\mathbf{F}_1$, $\mathbf{F}_2$, . . . are applied at the same point A (Fig. 3.14), and if we denote by $\mathbf{r}$ the position vector of A, it follows immediately from Eq. (3.5) of Sec. 3.4 that

$$\mathbf{r} \times (\mathbf{F}_1 + \mathbf{F}_2 + \cdots) = \mathbf{r} \times \mathbf{F}_1 + \mathbf{r} \times \mathbf{F}_2 + \cdots \tag{3.14}$$

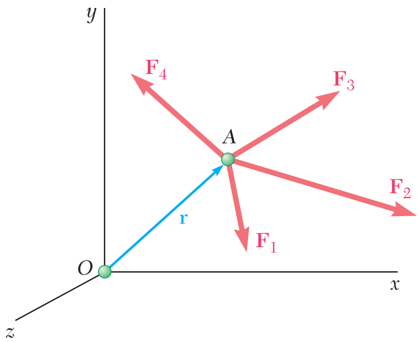


Fig. 3.14

In words, *the moment about a given point O of the resultant of several concurrent forces is equal to the sum of the moments of the various forces about the same point O*. This property, which was originally established by the French mathematician Varignon (1654–1722) long before the introduction of vector algebra, is known as *Varignon's theorem.*

The relation (3.14) makes it possible to replace the direct determination of the moment of a force $\mathbf{F}$ by the determination of the moments of two or more component forces. As you will see in the next section, $\mathbf{F}$ will generally be resolved into components parallel to the coordinate axes. However, it may be more expeditious in some instances to resolve $\mathbf{F}$ into components which are not parallel to the coordinate axes (see Sample Prob. 3.3).

3.8 RECTANGULAR COMPONENTS OF THE MOMENT OF A FORCE

In general, the determination of the moment of a force in space will be considerably simplified if the force and the position vector of its point of application are resolved into rectangular x, y, and z components. Consider, for example, the moment $\mathbf{M}_O$ about O of a force $\mathbf{F}$ whose components are F_x, F_y, and F_z and which is applied at a point A of coordinates x, y, and z (Fig. 3.15). Observing that the components of the position vector $\mathbf{r}$ are respectively equal to the coordinates x, y, and z of the point A, we write

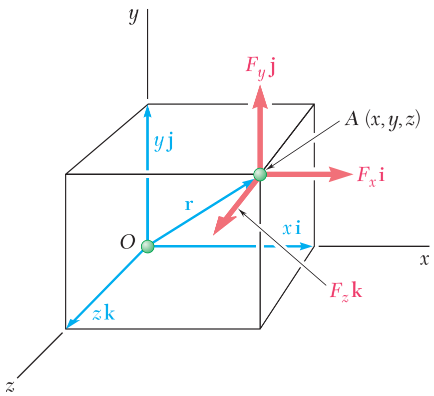


Fig. 3.15

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{3.15}$$
$$\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k} \tag{3.16}$$

Substituting for $\mathbf{r}$ and $\mathbf{F}$ from (3.15) and (3.16) into

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} \tag{3.11}$$

and recalling the results obtained in Sec. 3.5, we write the moment $\mathbf{M}_O$ of $\mathbf{F}$ about O in the form

$$\mathbf{M}_O = M_x\mathbf{i} + M_y\mathbf{j} + M_z\mathbf{k} \tag{3.17}$$

where the components M_x, M_y, and M_z are defined by the relations

$$\begin{aligned} M_x &= yF_z - zF_y \\ M_y &= zF_x - xF_z \\ M_z &= xF_y - yF_x \end{aligned} \tag{3.18}$$

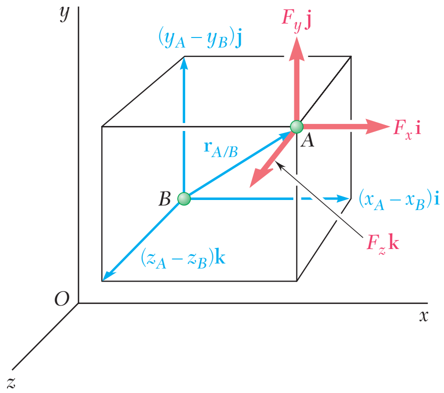


Fig. 3.16

As you will see in Sec. 3.11, the scalar components M_x, M_y, and M_z of the moment $\mathbf{M}_O$ measure the tendency of the force $\mathbf{F}$ to impart to a rigid body a motion of rotation about the x, y, and z axes, respectively. Substituting from (3.18) into (3.17), we can also write $\mathbf{M}_O$ in the form of the determinant

$$\mathbf{M}_O = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix} \tag{3.19}$$

To compute the moment $\mathbf{M}_B$ about an arbitrary point B of a force $\mathbf{F}$ applied at A (Fig. 3.16), we must replace the position vector $\mathbf{r}$ in Eq. (3.11) by a vector drawn from B to A. This vector is the *position vector of A relative to B* and will be denoted by $\mathbf{r}_{A/B}$. Observing that $\mathbf{r}_{A/B}$ can be obtained by subtracting $\mathbf{r}_B$ from $\mathbf{r}_A$, we write

$$\mathbf{M}_B = \mathbf{r}_{A/B} \times \mathbf{F} = (\mathbf{r}_A - \mathbf{r}_B) \times \mathbf{F} \tag{3.20}$$

or, using the determinant form,

$$\mathbf{M}_B = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_{A/B} & y_{A/B} & z_{A/B} \\ F_x & F_y & F_z \end{vmatrix} \tag{3.21}$$

where $x_{A/B}$, $y_{A/B}$, and $z_{A/B}$ denote the components of the vector $\mathbf{r}_{A/B}$:

$$x_{A/B} = x_A - x_B \qquad y_{A/B} = y_A - y_B \qquad z_{A/B} = z_A - z_B$$

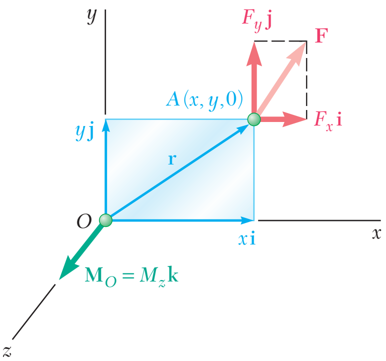


Fig. 3.17

In the case of *problems involving only two dimensions*, the force $\mathbf{F}$ can be assumed to lie in the xy plane (Fig. 3.17). Setting $z = 0$ and $F_z = 0$ in Eq. (3.19), we obtain

$$\mathbf{M}_O = (xF_y - yF_x)\mathbf{k}$$

We verify that the moment of $\mathbf{F}$ about O is perpendicular to the plane of the figure and that it is completely defined by the scalar

$$M_O = M_z = xF_y - yF_x \tag{3.22}$$

As noted earlier, a positive value for M_O indicates that the vector $\mathbf{M}_O$ points out of the paper (the force $\mathbf{F}$ tends to rotate the body counterclockwise about O), and a negative value indicates that the vector $\mathbf{M}_O$ points into the paper (the force $\mathbf{F}$ tends to rotate the body clockwise about O).

To compute the moment about $B(x_B, y_B)$ of a force lying in the xy plane and applied at $A(x_A, y_A)$ (Fig. 3.18), we set $z_{A/B} = 0$ and $F_z = 0$ in the relations (3.21) and note that the vector $\mathbf{M}_B$ is perpendicular to the xy plane and is defined in magnitude and sense by the scalar

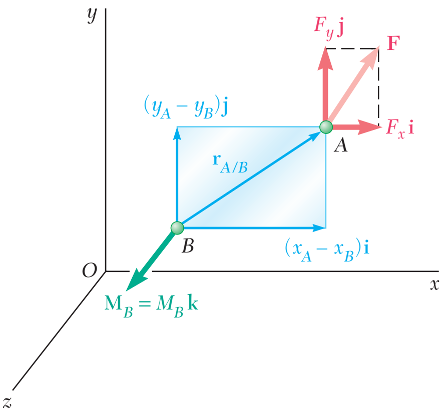
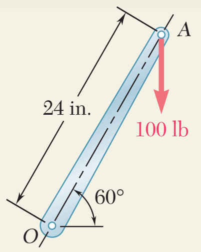


Fig. 3.18

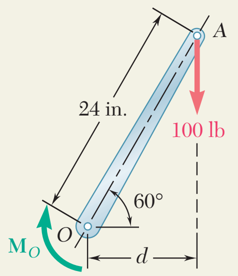
$$M_B = (x_A - x_B)F_y - (y_A - y_B)F_x \tag{3.23}$$



SAMPLE PROBLEM 3.1

A 100-lb vertical force is applied to the end of a lever which is attached to a shaft at O. Determine (a) the moment of the 100-lb force about O; (b) the horizontal force applied at A which creates the same moment about O; (c) the smallest force applied at A which creates the same moment about O; (d) how far from the shaft a 240-lb vertical force must act to create the same moment about O; (e) whether any one of the forces obtained in parts b, c, and d is equivalent to the original force.

SOLUTION



a. Moment about O. The perpendicular distance from O to the line of action of the 100-lb force is

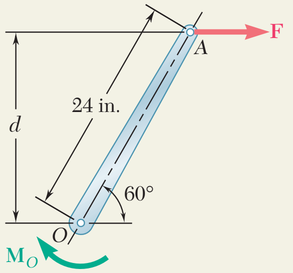
$$d = (24 \text{ in.}) \cos 60° = 12 \text{ in.}$$

The magnitude of the moment about O of the 100-lb force is

$$M_O = Fd = (100 \text{ lb})(12 \text{ in.}) = 1200 \text{ lb} \cdot \text{in.}$$

Since the force tends to rotate the lever clockwise about O, the moment will be represented by a vector $\mathbf{M}_O$ perpendicular to the plane of the figure and pointing *into* the paper. We express this fact by writing

$$\mathbf{M}_O = 1200 \text{ lb} \cdot \text{in.} \circlearrowright \quad ◀$$

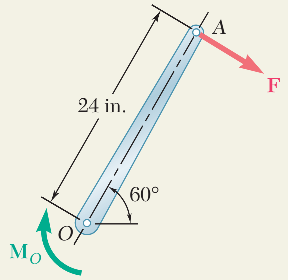


b. Horizontal Force. In this case, we have

$$d = (24 \text{ in.}) \sin 60° = 20.8 \text{ in.}$$

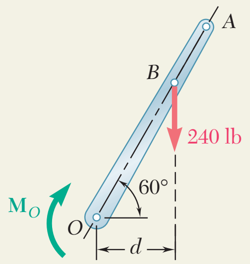
Since the moment about O must be 1200 lb · in., we write

$$M_O = Fd$$
$$1200 \text{ lb} \cdot \text{in.} = F(20.8 \text{ in.})$$
$$F = 57.7 \text{ lb} \qquad \mathbf{F} = 57.7 \text{ lb} \rightarrow \quad ◀$$



c. Smallest Force. Since $M_O = Fd$, the smallest value of F occurs when d is maximum. We choose the force perpendicular to OA and note that $d = 24$ in.; thus,

$$M_O = Fd$$
$$1200 \text{ lb} \cdot \text{in.} = F(24 \text{ in.})$$
$$F = 50 \text{ lb} \qquad \mathbf{F} = 50 \text{ lb} \angle 30° \quad ◀$$

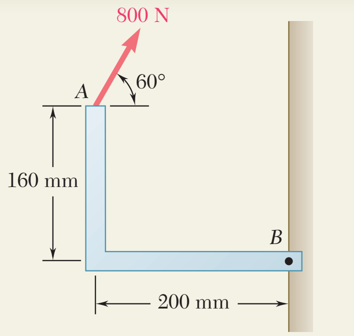


d. 240-lb Vertical Force. In this case $M_O = Fd$ yields

$$1200 \text{ lb} \cdot \text{in.} = (240 \text{ lb})d \qquad d = 5 \text{ in.}$$

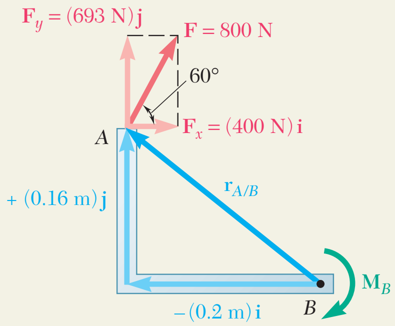
but $\quad OB \cos 60° = d \qquad OB = 10 \text{ in.} \quad ◀$

e. None of the forces considered in parts b, c, and d is equivalent to the original 100-lb force. Although they have the same moment about O, they have different x and y components. In other words, although each force tends to rotate the shaft in the same manner, each causes the lever to pull on the shaft in a different way.



SAMPLE PROBLEM 3.2

A force of 800 N acts on a bracket as shown. Determine the moment of the force about B.



SOLUTION

The moment $\mathbf{M}_B$ of the force $\mathbf{F}$ about B is obtained by forming the vector product

$$\mathbf{M}_B = \mathbf{r}_{A/B} \times \mathbf{F}$$

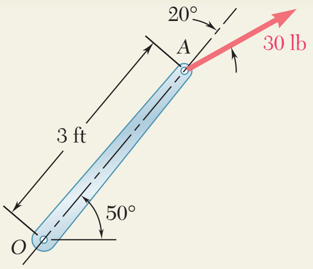
where $\mathbf{r}_{A/B}$ is the vector drawn from B to A. Resolving $\mathbf{r}_{A/B}$ and $\mathbf{F}$ into rectangular components, we have

$$\begin{aligned}\mathbf{r}_{A/B} &= -(0.2\text{ m})\mathbf{i} + (0.16\text{ m})\mathbf{j}\\ \mathbf{F} &= (800\text{ N})\cos 60°\mathbf{i} + (800\text{ N})\sin 60°\mathbf{j}\\ &= (400\text{ N})\mathbf{i} + (693\text{ N})\mathbf{j}\end{aligned}$$

Recalling the relations (3.7) for the cross products of unit vectors (Sec. 3.5), we obtain

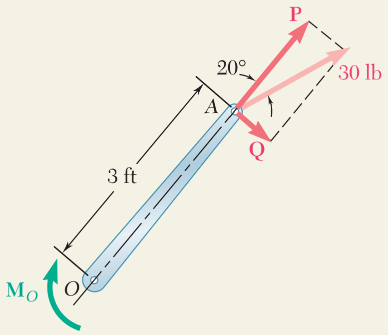
$$\begin{aligned}\mathbf{M}_B = \mathbf{r}_{A/B} \times \mathbf{F} &= [-(0.2\text{ m})\mathbf{i} + (0.16\text{ m})\mathbf{j}] \times [(400\text{ N})\mathbf{i} + (693\text{ N})\mathbf{j}]\\ &= -(138.6\text{ N}\cdot\text{m})\mathbf{k} - (64.0\text{ N}\cdot\text{m})\mathbf{k}\\ &= -(202.6\text{ N}\cdot\text{m})\mathbf{k} \qquad \mathbf{M}_B = 203\text{ N}\cdot\text{m}\ \curvearrowright \quad ◀\end{aligned}$$

The moment $\mathbf{M}_B$ is a vector perpendicular to the plane of the figure and pointing *into* the paper.



SAMPLE PROBLEM 3.3

A 30-lb force acts on the end of the 3-ft lever as shown. Determine the moment of the force about O.



SOLUTION

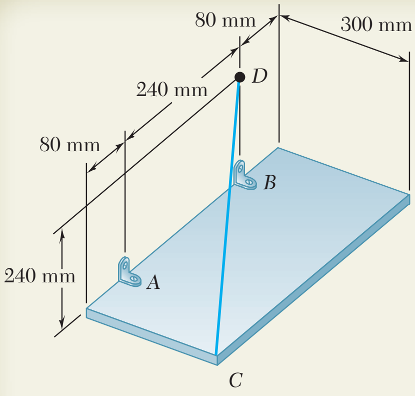
The force is replaced by two components, one component $\mathbf{P}$ in the direction of OA and one component $\mathbf{Q}$ perpendicular to OA. Since O is on the line of action of $\mathbf{P}$, the moment of $\mathbf{P}$ about O is zero and the moment of the 30-lb force reduces to the moment of $\mathbf{Q}$, which is clockwise and, thus, is represented by a negative scalar.

$$\begin{aligned}Q &= (30\text{ lb})\sin 20° = 10.26\text{ lb}\\ M_O &= -Q(3\text{ ft}) = -(10.26\text{ lb})(3\text{ ft}) = -30.8\text{ lb}\cdot\text{ft}\end{aligned}$$

Since the value obtained for the scalar M_O is negative, the moment $\mathbf{M}_O$ points *into* the paper. We write

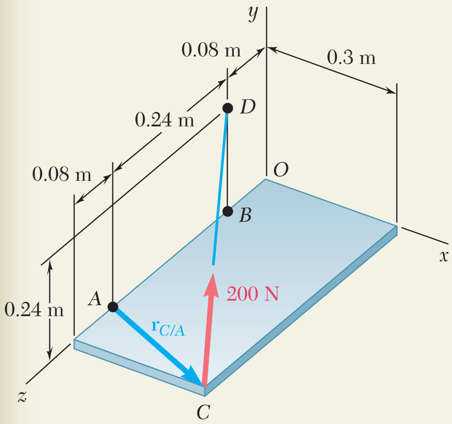
$$\mathbf{M}_O = 30.8\text{ lb}\cdot\text{ft}\ \curvearrowright \quad ◀$$

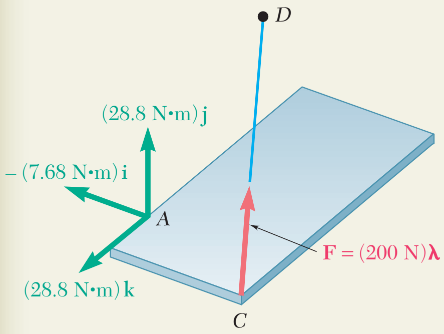
SAMPLE PROBLEM 3.4



A rectangular plate is supported by brackets at A and B and by a wire CD. Knowing that the tension in the wire is 200 N, determine the moment about A of the force exerted by the wire on point C.

SOLUTION





The moment $\mathbf{M}_A$ about A of the force $\mathbf{F}$ exerted by the wire on point C is obtained by forming the vector product

$$\mathbf{M}_A = \mathbf{r}_{C/A} \times \mathbf{F} \tag{1}$$

where $\mathbf{r}_{C/A}$ is the vector drawn from A to C,

$$\mathbf{r}_{C/A} = \overrightarrow{AC} = (0.3 \text{ m})\mathbf{i} + (0.08 \text{ m})\mathbf{k} \tag{2}$$

and $\mathbf{F}$ is the 200-N force directed along CD. Introducing the unit vector $\boldsymbol{\lambda} = \overrightarrow{CD}/CD$, we write

$$\mathbf{F} = F\boldsymbol{\lambda} = (200 \text{ N})\frac{\overrightarrow{CD}}{CD} \tag{3}$$

Resolving the vector $\overrightarrow{CD}$ into rectangular components, we have

$$\overrightarrow{CD} = -(0.3 \text{ m})\mathbf{i} + (0.24 \text{ m})\mathbf{j} - (0.32 \text{ m})\mathbf{k} \qquad CD = 0.50 \text{ m}$$

Substituting into (3), we obtain

$$\mathbf{F} = \frac{200 \text{ N}}{0.50 \text{ m}}[-(0.3 \text{ m})\mathbf{i} + (0.24 \text{ m})\mathbf{j} - (0.32 \text{ m})\mathbf{k}]$$
$$= -(120 \text{ N})\mathbf{i} + (96 \text{ N})\mathbf{j} - (128 \text{ N})\mathbf{k} \tag{4}$$

Substituting for $\mathbf{r}_{C/A}$ and $\mathbf{F}$ from (2) and (4) into (1) and recalling the relations (3.7) of Sec. 3.5, we obtain

$$\mathbf{M}_A = \mathbf{r}_{C/A} \times \mathbf{F} = (0.3\mathbf{i} + 0.08\mathbf{k}) \times (-120\mathbf{i} + 96\mathbf{j} - 128\mathbf{k})$$
$$= (0.3)(96)\mathbf{k} + (0.3)(-128)(-\mathbf{j}) + (0.08)(-120)\mathbf{j} + (0.08)(96)(-\mathbf{i})$$
$$\mathbf{M}_A = -(7.68 \text{ N} \cdot \text{m})\mathbf{i} + (28.8 \text{ N} \cdot \text{m})\mathbf{j} + (28.8 \text{ N} \cdot \text{m})\mathbf{k} \quad \blacktriangleleft$$

Alternative Solution. As indicated in Sec. 3.8, the moment $\mathbf{M}_A$ can be expressed in the form of a determinant:

$$\mathbf{M}_A = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_C - x_A & y_C - y_A & z_C - z_A \\ F_x & F_y & F_z \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.3 & 0 & 0.08 \\ -120 & 96 & -128 \end{vmatrix}$$

$$\mathbf{M}_A = -(7.68 \text{ N} \cdot \text{m})\mathbf{i} + (28.8 \text{ N} \cdot \text{m})\mathbf{j} + (28.8 \text{ N} \cdot \text{m})\mathbf{k} \quad \blacktriangleleft$$

PROBLEMS

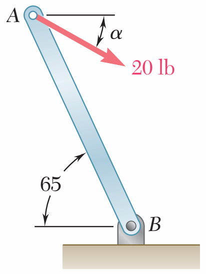


Fig. P3.1 and P3.2

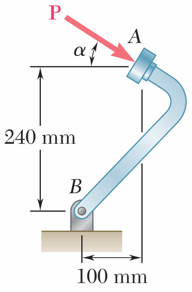


Fig. P3.3 and *P3.4*

3.1 A 20-lb force is applied to the control rod *AB* as shown. Knowing that the length of the rod is 9 in. and that $\alpha = 25°$, determine the moment of the force about point *B* by resolving the force into horizontal and vertical components.

3.2 A 20-lb force is applied to the control rod *AB* as shown. Knowing that the length of the rod is 9 in. and that the moment of the force about *B* is 120 lb · in. clockwise, determine the value of α.

3.3 For the brake pedal shown, determine the magnitude and direction of the smallest force **P** that has a 104-N · m clockwise moment about *B*.

3.4 A force **P** is applied to the brake pedal at *A*. Knowing that $P =$ 450 N and $\alpha = 30°$, determine the moment of **P** about *B*.

3.5 A 450-N force is applied at *A* as shown. Determine (*a*) the moment of the 450-N force about *D*, (*b*) the smallest force applied at *B* that creates the same moment about *D*.

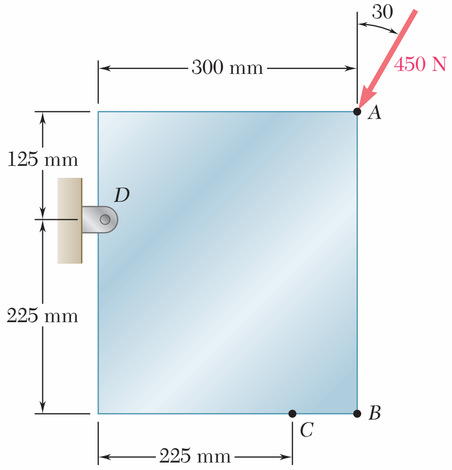


Fig. P3.5 and P3.6

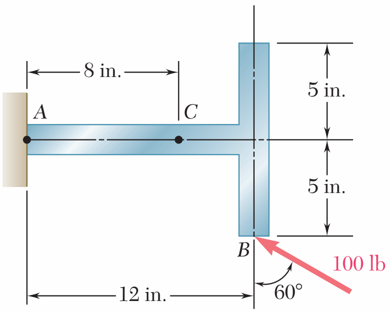


Fig. P3.7 and *P3.8*

3.6 A 450-N force is applied at *A* as shown. Determine (*a*) the moment of the 450-N force about *D*, (*b*) the magnitude and sense of the horizontal force applied at *C* that creates the same moment about *D*, (*c*) the smallest force applied at *C* that creates the same moment about *D*.

3.7 Compute the moment of the 100-lb force about *A*, (*a*) by using the definition of the moment of a force, (*b*) by resolving the force into horizontal and vertical components, (*c*) by resolving the force into components along *AB* and in the direction perpendicular to *AB*.

3.8 Determine the moment of the 100-lb force about *C*.

3.9 and 3.10 It is known that the connecting rod AB exerts on the crank BC a 2.5-kN force directed down to the left along the centerline AB. Determine the moment of that force about C.

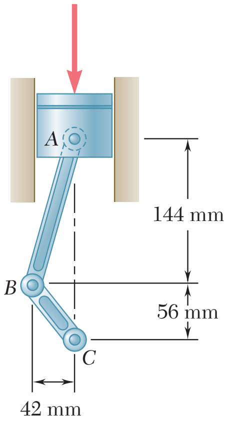


Fig. P3.9

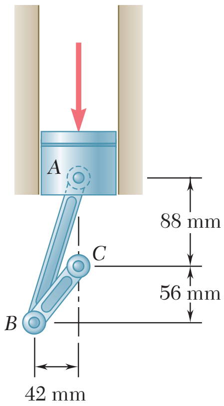


Fig. P3.10

3.11 Rod AB is held in place by the cord AC. Knowing that the tension in the cord is 300 lb and that $c = 18$ in., determine the moment about B of the force exerted by the cord at point A by resolving that force into horizontal and vertical components applied (*a*) at point A, (*b*) at point C.

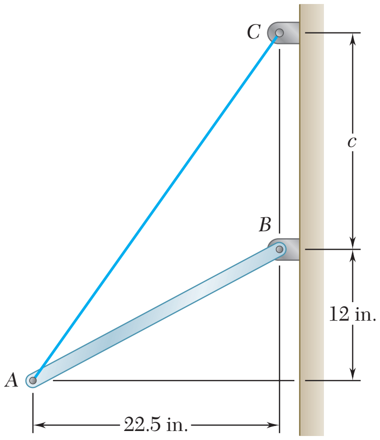


Fig. *P3.11* and P3.12

3.12 Rod AB is held in place by the cord AC. Knowing that $c = 42$ in. and that the moment about B of the force exerted by the cord at point A is 700 lb · ft, determine the tension in the cord.

3.13 Determine the moment about the origin of coordinates O of the force $\mathbf{F} = 4\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ that acts at a point A. Assume that the position of A is (*a*) $\mathbf{r} = \mathbf{i} + 5\mathbf{j} + 6\mathbf{k}$, (*b*) $\mathbf{r} = 6\mathbf{i} + \mathbf{j} + 3\mathbf{k}$, (*c*) $\mathbf{r} = 5\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}$.

3.14 Determine the moment about the origin of coordinates O of the force $\mathbf{F} = -\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$ that acts at a point A. Assume that the position of A is (*a*) $\mathbf{r} = 2\mathbf{i} - 4\mathbf{j} + \mathbf{k}$, (*b*) $\mathbf{r} = 4\mathbf{i} + 6\mathbf{j} + 10\mathbf{k}$, (*c*) $\mathbf{r} = -3\mathbf{i} + 9\mathbf{j} + 15\mathbf{k}$.

3.15 The line of action of the force **P** of magnitude 420 lb passes through the two points *A* and *B* as shown. Compute the moment of **P** about *O* using the position vector (*a*) of point *A*, (*b*) of point *B*.

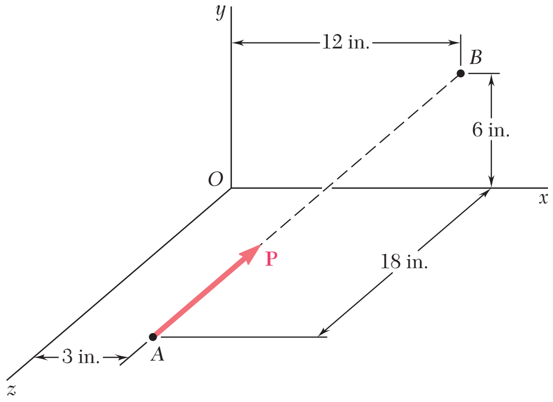


Fig. P3.15

3.16 A force **P** of magnitude 200 N acts along the diagonal *BC* of the bent plate shown. Determine the moment of **P** about point *E*.

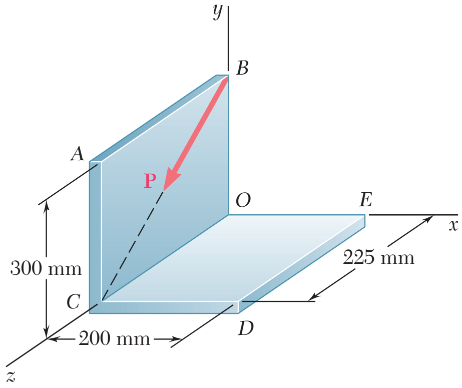


Fig. P3.16

3.17 Knowing that the tension in cable *AB* is 1800 lb, determine the moment of the force exerted on the plate at *A* about (*a*) the origin of coordinates *O*, (*b*) corner *D*.

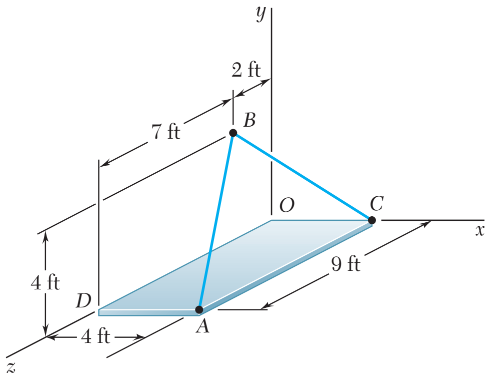


Fig. P3.17 and P3.18

3.18 Knowing that the tension in cable *BC* is 900 lb, determine the moment of the force exerted on the plate at *C* about (*a*) the origin of coordinates *O*, (*b*) corner *D*.

3.19 A 200-N force is applied as shown to the bracket *ABC*. Determine the moment of the force about *A*.

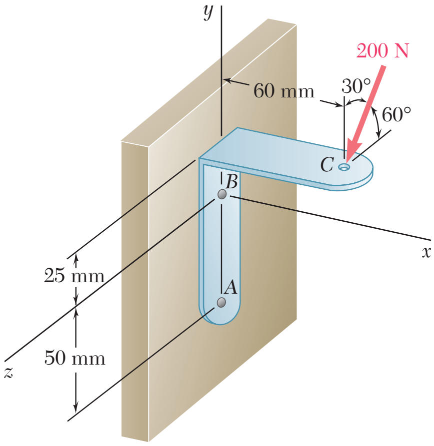


Fig. P3.19

3.20 A small boat hangs from two davits, one of which is shown in the figure. The tension in line *ABAD* is 82 lb. Determine the moment about *C* of the resultant force $\mathbf{R}_A$ exerted on the davit at *A*.

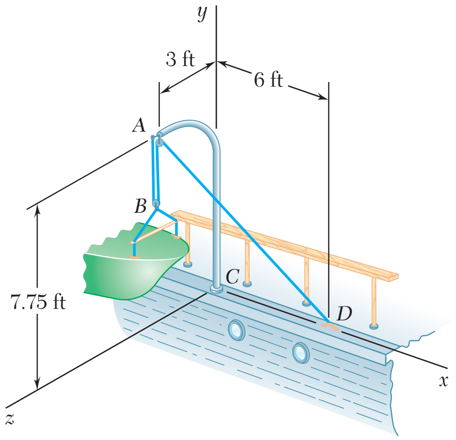


Fig. P3.20

3.21 In Prob. 3.15, determine the perpendicular distance from the line of action of **P** to the origin *O*.

3.22 In Prob. 3.16, determine the perpendicular distance from the line of action of **P** to point *E*.

3.23 In Prob. 3.20, determine the perpendicular distance from the point *C* to the portion *AD* of line *ABAD*.

3.24 In Sample Prob. 3.4, determine the perpendicular distance from point *A* to wire *CD*.

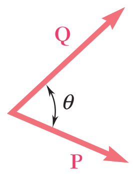


Fig. 3.19

3.9 SCALAR PRODUCT OF TWO VECTORS

The *scalar product* of two vectors **P** and **Q** is defined as the product of the magnitudes of **P** and **Q** and of the cosine of the angle θ formed by **P** and **Q** (Fig. 3.19). The scalar product of **P** and **Q** is denoted by $\mathbf{P} \cdot \mathbf{Q}$. We write therefore

$$\mathbf{P} \cdot \mathbf{Q} = PQ \cos \theta \tag{3.24}$$

Note that the expression just defined is not a vector but a *scalar*, which explains the name *scalar product*; because of the notation used, $\mathbf{P} \cdot \mathbf{Q}$ is also referred to as the *dot product* of the vectors **P** and **Q**.

It follows from its very definition that the scalar product of two vectors is *commutative*, i.e., that

$$\mathbf{P} \cdot \mathbf{Q} = \mathbf{Q} \cdot \mathbf{P} \tag{3.25}$$

To prove that the scalar product is also *distributive*, we must prove the relation

$$\mathbf{P} \cdot (\mathbf{Q}_1 + \mathbf{Q}_2) = \mathbf{P} \cdot \mathbf{Q}_1 + \mathbf{P} \cdot \mathbf{Q}_2 \tag{3.26}$$

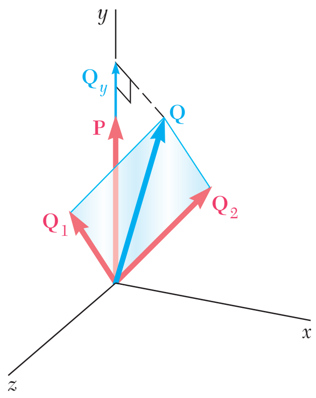


Fig. 3.20

We can, without any loss of generality, assume that **P** is directed along the y axis (Fig. 3.20). Denoting by **Q** the sum of $\mathbf{Q}_1$ and $\mathbf{Q}_2$ and by θ_y the angle **Q** forms with the y axis, we express the left-hand member of (3.26) as follows:

$$\mathbf{P} \cdot (\mathbf{Q}_1 + \mathbf{Q}_2) = \mathbf{P} \cdot \mathbf{Q} = PQ \cos \theta_y = PQ_y \tag{3.27}$$

where Q_y is the y component of **Q**. We can, in a similar way, express the right-hand member of (3.26) as

$$\mathbf{P} \cdot \mathbf{Q}_1 + \mathbf{P} \cdot \mathbf{Q}_2 = P(Q_1)_y + P(Q_2)_y \tag{3.28}$$

Since **Q** is the sum of $\mathbf{Q}_1$ and $\mathbf{Q}_2$, its y component must be equal to the sum of the y components of $\mathbf{Q}_1$ and $\mathbf{Q}_2$. Thus, the expressions obtained in (3.27) and (3.28) are equal, and the relation (3.26) has been proved.

As far as the third property—the associative property—is concerned, we note that this property cannot apply to scalar products. Indeed, $(\mathbf{P} \cdot \mathbf{Q}) \cdot \mathbf{S}$ has no meaning since $\mathbf{P} \cdot \mathbf{Q}$ is not a vector but a scalar.

The scalar product of two vectors **P** and **Q** can be expressed in terms of their rectangular components. Resolving **P** and **Q** into components, we first write

$$\mathbf{P} \cdot \mathbf{Q} = (P_x\mathbf{i} + P_y\mathbf{j} + P_z\mathbf{k}) \cdot (Q_x\mathbf{i} + Q_y\mathbf{j} + Q_z\mathbf{k})$$

Making use of the distributive property, we express $\mathbf{P} \cdot \mathbf{Q}$ as the sum of scalar products, such as $P_x\mathbf{i} \cdot Q_x\mathbf{i}$ and $P_x\mathbf{i} \cdot Q_y\mathbf{j}$. However, from the definition of the scalar product it follows that the scalar products of the unit vectors are either zero or one.

$$\begin{array}{lll} \mathbf{i} \cdot \mathbf{i} = 1 & \mathbf{j} \cdot \mathbf{j} = 1 & \mathbf{k} \cdot \mathbf{k} = 1 \\ \mathbf{i} \cdot \mathbf{j} = 0 & \mathbf{j} \cdot \mathbf{k} = 0 & \mathbf{k} \cdot \mathbf{i} = 0 \end{array} \tag{3.29}$$

Thus, the expression obtained for $\mathbf{P} \cdot \mathbf{Q}$ reduces to

$$\mathbf{P} \cdot \mathbf{Q} = P_x Q_x + P_y Q_y + P_z Q_z \tag{3.30}$$

In the particular case when $\mathbf{P}$ and $\mathbf{Q}$ are equal, we note that

$$\mathbf{P} \cdot \mathbf{P} = P_x^2 + P_y^2 + P_z^2 = P^2 \tag{3.31}$$

Applications

1. *Angle formed by two given vectors.* Let two vectors be given in terms of their components:

$$\mathbf{P} = P_x\mathbf{i} + P_y\mathbf{j} + P_z\mathbf{k}$$
$$\mathbf{Q} = Q_x\mathbf{i} + Q_y\mathbf{j} + Q_z\mathbf{k}$$

To determine the angle formed by the two vectors, we equate the expressions obtained in (3.24) and (3.30) for their scalar product and write

$$PQ \cos \theta = P_x Q_x + P_y Q_y + P_z Q_z$$

Solving for $\cos \theta$, we have

$$\cos \theta = \frac{P_x Q_x + P_y Q_y + P_z Q_z}{PQ} \tag{3.32}$$

2. *Projection of a vector on a given axis.* Consider a vector $\mathbf{P}$ forming an angle θ with an axis, or directed line, OL (Fig. 3.21). The *projection of* $\mathbf{P}$ *on the axis* OL is defined as the scalar

$$P_{OL} = P \cos \theta \tag{3.33}$$

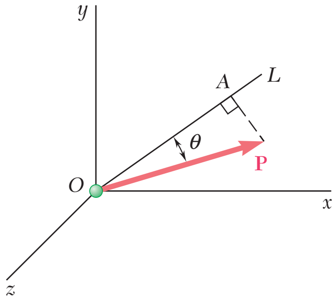


Fig. 3.21

We note that the projection P_{OL} is equal in absolute value to the length of the segment OA; it will be positive if OA has the same sense as the axis OL, that is, if θ is acute, and negative otherwise. If $\mathbf{P}$ and OL are at a right angle, the projection of $\mathbf{P}$ on OL is zero.

Consider now a vector $\mathbf{Q}$ directed along OL and of the same sense as OL (Fig. 3.22). The scalar product of $\mathbf{P}$ and $\mathbf{Q}$ can be expressed as

$$\mathbf{P} \cdot \mathbf{Q} = PQ \cos \theta = P_{OL} Q \tag{3.34}$$

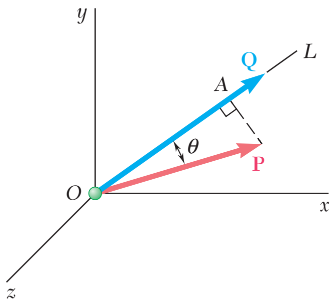


Fig. 3.22

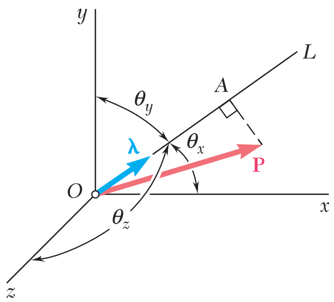


Fig. 3.23

from which it follows that

$$P_{OL} = \frac{\mathbf{P} \cdot \mathbf{Q}}{Q} = \frac{P_x Q_x + P_y Q_y + P_z Q_z}{Q} \tag{3.35}$$

In the particular case when the vector selected along OL is the unit vector $\boldsymbol{\lambda}$ (Fig. 3.23), we write

$$\boxed{P_{OL} = \mathbf{P} \cdot \boldsymbol{\lambda}} \tag{3.36}$$

Resolving $\mathbf{P}$ and $\boldsymbol{\lambda}$ into rectangular components and recalling from Sec. 2.12 that the components of $\boldsymbol{\lambda}$ along the coordinate axes are respectively equal to the direction cosines of OL, we express the projection of $\mathbf{P}$ on OL as

$$P_{OL} = P_x \cos \theta_x + P_y \cos \theta_y + P_z \cos \theta_z \tag{3.37}$$

where θ_x, θ_y, and θ_z denote the angles that the axis OL forms with the coordinate axes.

3.10 MIXED TRIPLE PRODUCT OF THREE VECTORS

We define the *mixed triple product* of the three vectors $\mathbf{S}$, $\mathbf{P}$, and $\mathbf{Q}$ as the scalar expression

$$\boxed{\mathbf{S} \cdot (\mathbf{P} \times \mathbf{Q})} \tag{3.38}$$

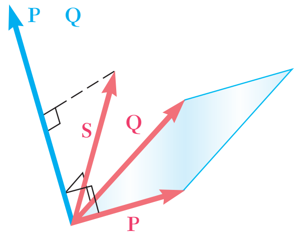


Fig. 3.24

obtained by forming the scalar product of $\mathbf{S}$ with the vector product of $\mathbf{P}$ and $\mathbf{Q}$.

A simple geometrical interpretation can be given for the mixed triple product of $\mathbf{S}$, $\mathbf{P}$, and $\mathbf{Q}$ (Fig. 3.24). We first recall from Sec. 3.4 that the vector $\mathbf{P} \times \mathbf{Q}$ is perpendicular to the plane containing $\mathbf{P}$ and $\mathbf{Q}$ and that its magnitude is equal to the area of the parallelogram which has $\mathbf{P}$ and $\mathbf{Q}$ for sides. On the other hand, Eq. (3.34) indicates that the scalar product of $\mathbf{S}$ and $\mathbf{P} \times \mathbf{Q}$ can be obtained by multiplying the magnitude of $\mathbf{P} \times \mathbf{Q}$ (i.e., the area of the parallelogram defined by $\mathbf{P}$ and $\mathbf{Q}$) by the projection of $\mathbf{S}$ on the vector $\mathbf{P} \times \mathbf{Q}$ (i.e., by the projection of $\mathbf{S}$ on the normal to the plane containing the parallelogram). The mixed triple product is thus equal, in absolute value, to the volume of the parallelepiped having the vectors $\mathbf{S}$, $\mathbf{P}$, and $\mathbf{Q}$ for sides (Fig. 3.25). We note that the sign of the mixed triple product will be positive if $\mathbf{S}$, $\mathbf{P}$, and $\mathbf{Q}$ form a right-handed triad and negative if they form a left-handed triad [that is, $\mathbf{S} \cdot (\mathbf{P} \times \mathbf{Q})$ will be negative if the rotation which brings $\mathbf{P}$ into line with $\mathbf{Q}$ is observed as clockwise from the tip of $\mathbf{S}$]. The mixed triple product will be zero if $\mathbf{S}$, $\mathbf{P}$, and $\mathbf{Q}$ are coplanar.

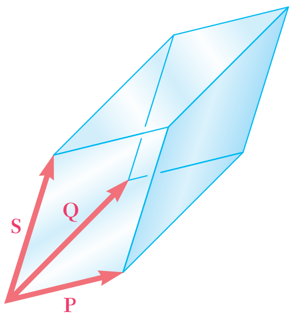


Fig. 3.25

Since the parallelepiped defined in the preceding paragraph is independent of the order in which the three vectors are taken, the six mixed triple products which can be formed with **S**, **P**, and **Q** will all have the same absolute value, although not the same sign. It is easily shown that

$$\begin{aligned}\mathbf{S}\cdot(\mathbf{P}\times\mathbf{Q}) &= \mathbf{P}\cdot(\mathbf{Q}\times\mathbf{S}) = \mathbf{Q}\cdot(\mathbf{S}\times\mathbf{P})\\ &= -\mathbf{S}\cdot(\mathbf{Q}\times\mathbf{P}) = -\mathbf{P}\cdot(\mathbf{S}\times\mathbf{Q}) = -\mathbf{Q}\cdot(\mathbf{P}\times\mathbf{S})\end{aligned} \tag{3.39}$$

Arranging in a circle and in counterclockwise order the letters representing the three vectors (Fig. 3.26), we observe that the sign of the mixed triple product remains unchanged if the vectors are permuted in such a way that they are still read in counterclockwise order. Such a permutation is said to be a *circular permutation*. It also follows from Eq. (3.39) and from the commutative property of scalar products that the mixed triple product of **S**, **P**, and **Q** can be defined equally well as $\mathbf{S}\cdot(\mathbf{P}\times\mathbf{Q})$ or $(\mathbf{S}\times\mathbf{P})\cdot\mathbf{Q}$.

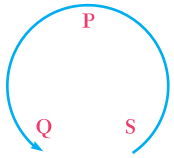


Fig. 3.26

The mixed triple product of the vectors **S**, **P**, and **Q** can be expressed in terms of the rectangular components of these vectors. Denoting $\mathbf{P}\times\mathbf{Q}$ by **V** and using formula (3.30) to express the scalar product of **S** and **V**, we write

$$\mathbf{S}\cdot(\mathbf{P}\times\mathbf{Q}) = \mathbf{S}\cdot\mathbf{V} = S_xV_x + S_yV_y + S_zV_z$$

Substituting from the relations (3.9) for the components of **V**, we obtain

$$\begin{aligned}\mathbf{S}\cdot(\mathbf{P}\times\mathbf{Q}) = S_x(P_yQ_z - P_zQ_y) &+ S_y(P_zQ_x - P_xQ_z)\\ &+ S_z(P_xQ_y - P_yQ_x)\end{aligned} \tag{3.40}$$

This expression can be written in a more compact form if we observe that it represents the expansion of a determinant:

$$\mathbf{S}\cdot(\mathbf{P}\times\mathbf{Q}) = \begin{vmatrix} S_x & S_y & S_z \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix} \tag{3.41}$$

By applying the rules governing the permutation of rows in a determinant, we could easily verify the relations (3.39) which were derived earlier from geometrical considerations.

3.11 MOMENT OF A FORCE ABOUT A GIVEN AXIS

Now that we have further increased our knowledge of vector algebra, we can introduce a new concept, the concept of *moment of a force about an axis*. Consider again a force **F** acting on a rigid body and the moment $\mathbf{M}_O$ of that force about O (Fig. 3.27). Let OL be

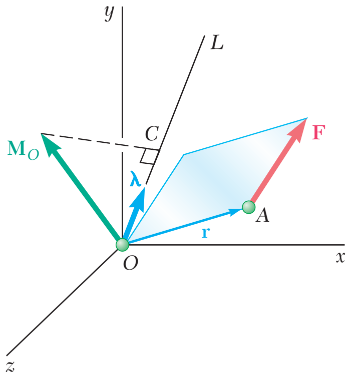


Fig. 3.27

an axis through O; *we define the moment M_{OL} of* **F** *about OL as the projection OC of the moment* $\mathbf{M}_O$ *onto the axis OL.* Denoting by $\boldsymbol{\lambda}$ the unit vector along OL and recalling from Secs. 3.9 and 3.6, respectively, the expressions (3.36) and (3.11) obtained for the projection of a vector on a given axis and for the moment $\mathbf{M}_O$ of a force **F**, we write

$$M_{OL} = \boldsymbol{\lambda} \cdot \mathbf{M}_O = \boldsymbol{\lambda} \cdot (\mathbf{r} \times \mathbf{F}) \tag{3.42}$$

which shows that the moment M_{OL} of **F** about the axis OL is the scalar obtained by forming the mixed triple product of $\boldsymbol{\lambda}$, **r**, and **F**. Expressing M_{OL} in the form of a determinant, we write

$$M_{OL} = \begin{vmatrix} \lambda_x & \lambda_y & \lambda_z \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix} \tag{3.43}$$

where $\lambda x, \lambda y, \lambda z$ = direction cosines of axis OL
x, y, z = coordinates of point of application of **F**
F_x, F_y, F_z = components of force **F**

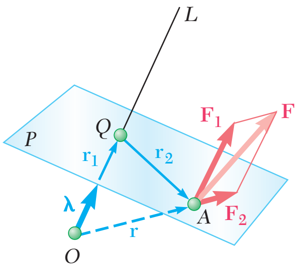


Fig. 3.28

The physical significance of the moment M_{OL} of a force **F** about a fixed axis OL becomes more apparent if we resolve **F** into two rectangular components $\mathbf{F}_1$ and $\mathbf{F}_2$, with $\mathbf{F}_1$ parallel to OL and $\mathbf{F}_2$ lying in a plane P perpendicular to OL (Fig. 3.28). Resolving **r** similarly into two components $\mathbf{r}_1$ and $\mathbf{r}_2$ and substituting for **F** and **r** into (3.42), we write

$$\begin{aligned} M_{OL} &= \boldsymbol{\lambda} \cdot [(\mathbf{r}_1 + \mathbf{r}_2) \times (\mathbf{F}_1 + \mathbf{F}_2)] \\ &= \boldsymbol{\lambda} \cdot (\mathbf{r}_1 \times \mathbf{F}_1) + \boldsymbol{\lambda} \cdot (\mathbf{r}_1 \times \mathbf{F}_2) + \boldsymbol{\lambda} \cdot (\mathbf{r}_2 \times \mathbf{F}_1) + \boldsymbol{\lambda} \cdot (\mathbf{r}_2 \times \mathbf{F}_2) \end{aligned}$$

Noting that all of the mixed triple products except the last one are equal to zero, since they involve vectors which are coplanar when drawn from a common origin (Sec. 3.10), we have

$$M_{OL} = \boldsymbol{\lambda} \cdot (\mathbf{r}_2 \times \mathbf{F}_2) \tag{3.44}$$

The vector product $\mathbf{r}_2 \times \mathbf{F}_2$ is perpendicular to the plane P and represents the moment of the component $\mathbf{F}_2$ of **F** about the point Q where OL intersects P. Therefore, the scalar M_{OL}, which will be positive if $\mathbf{r}_2 \times \mathbf{F}_2$ and OL have the same sense and negative otherwise, measures the tendency of $\mathbf{F}_2$ to make the rigid body rotate about the fixed axis OL. Since the other component $\mathbf{F}_1$ of **F** does not tend to make the body rotate about OL, we conclude that *the moment M_{OL} of* **F** *about OL measures the tendency of the force* **F** *to impart to the rigid body a motion of rotation about the fixed axis OL.*

It follows from the definition of the moment of a force about an axis that the moment of **F** about a coordinate axis is equal to the component of $\mathbf{M}_O$ along that axis. Substituting successively each of the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ for $\boldsymbol{\lambda}$ in (3.42), we observe that the expressions thus obtained for the *moments of* $\mathbf{F}$ *about the coordinate axes* are respectively equal to the expressions obtained in Sec. 3.8 for the components of the moment $\mathbf{M}_O$ of $\mathbf{F}$ about O:

$$\begin{aligned} M_x &= yF_z - zF_y \\ M_y &= zF_x - xF_z \\ M_z &= xF_y - yF_x \end{aligned} \tag{3.18}$$

We observe that just as the components F_x, F_y, and F_z of a force $\mathbf{F}$ acting on a rigid body measure, respectively, the tendency of $\mathbf{F}$ to move the rigid body in the x, y, and z directions, the moments M_x, M_y, and M_z of $\mathbf{F}$ about the coordinate axes measure the tendency of $\mathbf{F}$ to impart to the rigid body a motion of rotation about the x, y, and z axes, respectively.

More generally, the moment of a force $\mathbf{F}$ applied at A about an axis which does not pass through the origin is obtained by choosing an arbitrary point B on the axis (Fig. 3.29) and determining the projection on the axis BL of the moment $\mathbf{M}_B$ of $\mathbf{F}$ about B. We write

$$M_{BL} = \boldsymbol{\lambda} \cdot \mathbf{M}_B = \boldsymbol{\lambda} \cdot (\mathbf{r}_{A/B} \times \mathbf{F}) \tag{3.45}$$

where $\mathbf{r}_{A/B} = \mathbf{r}_A - \mathbf{r}_B$ represents the vector drawn from B to A. Expressing M_{BL} in the form of a determinant, we have

$$M_{BL} = \begin{vmatrix} \lambda_x & \lambda_y & \lambda_z \\ x_{A/B} & y_{A/B} & z_{A/B} \\ F_x & F_y & F_z \end{vmatrix} \tag{3.46}$$

where $\lambda_x, \lambda_y, \lambda_z$ = direction cosines of axis BL
$x_{A/B} = x_A - x_B \qquad y_{A/B} = y_A - y_B \qquad z_{A/B} = z_A - z_B$
F_x, F_y, F_z = components of force $\mathbf{F}$

It should be noted that the result obtained is independent of the choice of the point B on the given axis. Indeed, denoting by M_{CL} the result obtained with a different point C, we have

$$\begin{aligned} M_{CL} &= \boldsymbol{\lambda} \cdot [(\mathbf{r}_A - \mathbf{r}_C) \times \mathbf{F}] \\ &= \boldsymbol{\lambda} \cdot [(\mathbf{r}_A - \mathbf{r}_B) \times \mathbf{F}] + \boldsymbol{\lambda} \cdot [(\mathbf{r}_B - \mathbf{r}_C) \times \mathbf{F}] \end{aligned}$$

But, since the vectors $\boldsymbol{\lambda}$ and $\mathbf{r}_B - \mathbf{r}_C$ lie in the same line, the volume of the parallelepiped having the vectors $\boldsymbol{\lambda}$, $\mathbf{r}_B - \mathbf{r}_C$, and $\mathbf{F}$ for its sides is zero, as is the mixed triple product of these three vectors (Sec. 3.10). The expression obtained for M_{CL} thus reduces to its first term, which is the expression used earlier to define M_{BL}. In addition, it follows from Sec. 3.6 that, when computing the moment of $\mathbf{F}$ about the given axis, A can be any point on the line of action of $\mathbf{F}$.

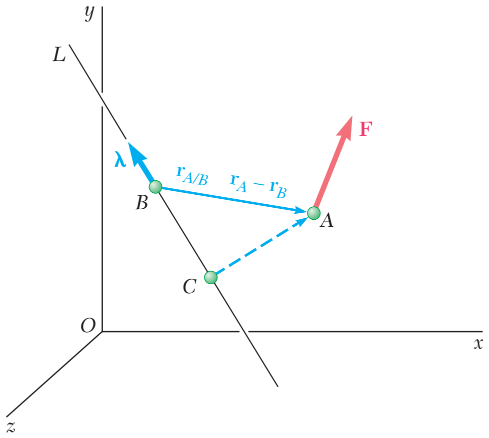
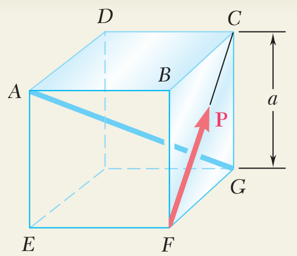


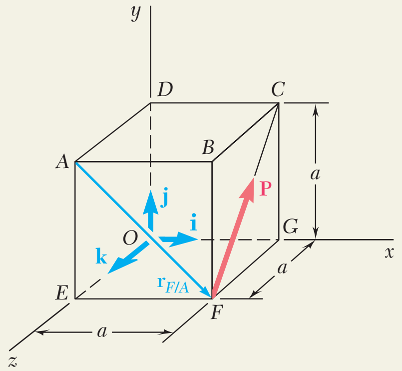
Fig. 3.29

SAMPLE PROBLEM 3.5



A cube of side a is acted upon by a force $\mathbf{P}$ as shown. Determine the moment of $\mathbf{P}$ (*a*) about A, (*b*) about the edge AB, (*c*) about the diagonal AG of the cube. (*d*) Using the result of part *c*, determine the perpendicular distance between AG and FC.

SOLUTION



a. Moment about A. Choosing x, y, and z axes as shown, we resolve into rectangular components the force $\mathbf{P}$ and the vector $\mathbf{r}_{F/A} = \overrightarrow{AF}$ drawn from A to the point of application F of $\mathbf{P}$.

$$\mathbf{r}_{F/A} = a\mathbf{i} - a\mathbf{j} = a(\mathbf{i} - \mathbf{j})$$
$$\mathbf{P} = (P/\sqrt{2})\mathbf{j} - (P/\sqrt{2})\mathbf{k} = (P/\sqrt{2})(\mathbf{j} - \mathbf{k})$$

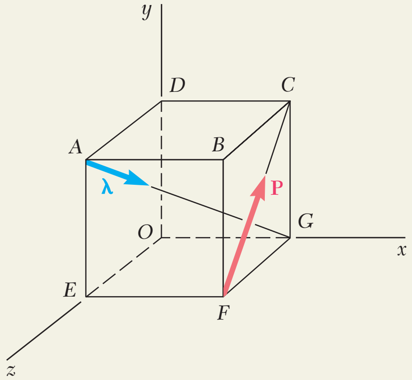
The moment of $\mathbf{P}$ about A is

$$\mathbf{M}_A = \mathbf{r}_{F/A} \times \mathbf{P} = a(\mathbf{i} - \mathbf{j}) \times (P/\sqrt{2})(\mathbf{j} - \mathbf{k})$$
$$\mathbf{M}_A = (aP/\sqrt{2})(\mathbf{i} + \mathbf{j} + \mathbf{k}) \quad ◀$$

b. Moment about AB. Projecting $\mathbf{M}_A$ on AB, we write

$$M_{AB} = \mathbf{i} \cdot \mathbf{M}_A = \mathbf{i} \cdot (aP/\sqrt{2})(\mathbf{i} + \mathbf{j} + \mathbf{k})$$
$$M_{AB} = aP/\sqrt{2} \quad ◀$$

We verify that, since AB is parallel to the x axis, M_{AB} is also the x component of the moment $\mathbf{M}_A$.

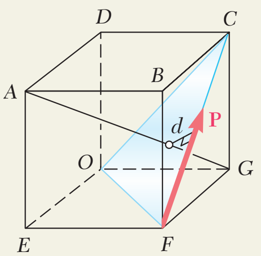


c. Moment about Diagonal AG. The moment of $\mathbf{P}$ about AG is obtained by projecting $\mathbf{M}_A$ on AG. Denoting by $\boldsymbol{\lambda}$ the unit vector along AG, we have

$$\boldsymbol{\lambda} = \frac{\overrightarrow{AG}}{AG} = \frac{a\mathbf{i} - a\mathbf{j} - a\mathbf{k}}{a\sqrt{3}} = (1/\sqrt{3})(\mathbf{i} - \mathbf{j} - \mathbf{k})$$
$$M_{AG} = \boldsymbol{\lambda} \cdot \mathbf{M}_A = (1/\sqrt{3})(\mathbf{i} - \mathbf{j} - \mathbf{k}) \cdot (aP/\sqrt{2})(\mathbf{i} + \mathbf{j} + \mathbf{k})$$
$$M_{AG} = (aP/\sqrt{6})(1 - 1 - 1) \qquad M_{AG} = -aP/\sqrt{6} \quad ◀$$

Alternative Method. The moment of $\mathbf{P}$ about AG can also be expressed in the form of a determinant:

$$M_{AG} = \begin{vmatrix} \lambda_x & \lambda_y & \lambda_z \\ x_{F/A} & y_{F/A} & z_{F/A} \\ F_x & F_y & F_z \end{vmatrix} = \begin{vmatrix} 1/\sqrt{3} & -1/\sqrt{3} & -1/\sqrt{3} \\ a & -a & 0 \\ 0 & P/\sqrt{2} & -P/\sqrt{2} \end{vmatrix} = -aP/\sqrt{6}$$



d. Perpendicular Distance between AG and FC. We first observe that $\mathbf{P}$ is perpendicular to the diagonal AG. This can be checked by forming the scalar product $\mathbf{P} \cdot \boldsymbol{\lambda}$ and verifying that it is zero:

$$\mathbf{P} \cdot \boldsymbol{\lambda} = (P/\sqrt{2})(\mathbf{j} - \mathbf{k}) \cdot (1/\sqrt{3})(\mathbf{i} - \mathbf{j} - \mathbf{k}) = (P\sqrt{6})(0 - 1 + 1) = 0$$

The moment M_{AG} can then be expressed as $-Pd$, where d is the perpendicular distance from AG to FC. (The negative sign is used since the rotation imparted to the cube by $\mathbf{P}$ appears as clockwise to an observer at G.) Recalling the value found for M_{AG} in part *c*,

$$M_{AG} = -Pd = -aP/\sqrt{6} \qquad d = a/\sqrt{6} \quad ◀$$

PROBLEMS

3.25 Given the vectors $\mathbf{P} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$, $\mathbf{Q} = 3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$, and $\mathbf{S} = -4\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, compute the scalar products $\mathbf{P} \cdot \mathbf{Q}$, $\mathbf{P} \cdot \mathbf{S}$, and $\mathbf{Q} \cdot \mathbf{S}$.

3.26 Form the scalar product $\mathbf{P}_1 \cdot \mathbf{P}_2$, and use the result obtained to prove the identity $\cos(\theta_1 - \theta_2) = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2$.

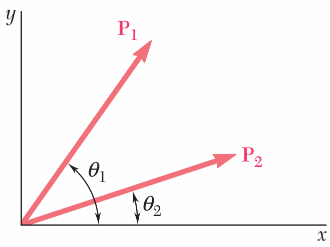


Fig. P3.26

3.27 Knowing that the tension in cable *BC* is 1400 N, determine (*a*) the angle between cable *BC* and the boom *AB*, (*b*) the projection on *AB* of the force exerted by cable *BC* at point *B*.

3.28 Knowing that the tension in cable *BD* is 900 N, determine (*a*) the angle between cable *BD* and the boom *AB*, (*b*) the projection on *AB* of the force exerted by cable *BD* at point *B*.

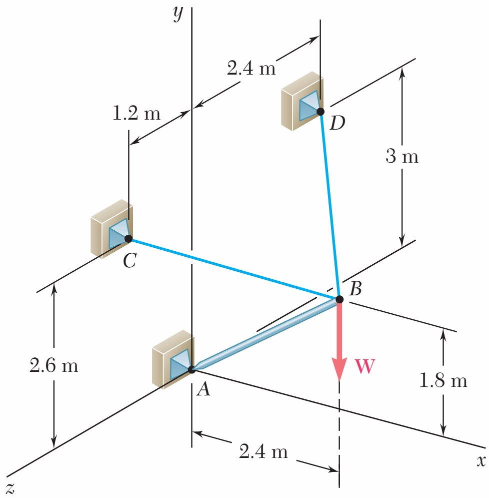


Fig. P3.27 and P3.28

3.29 Three cables are used to support a container as shown. Determine the angle formed by cables *AB* and *AD*.

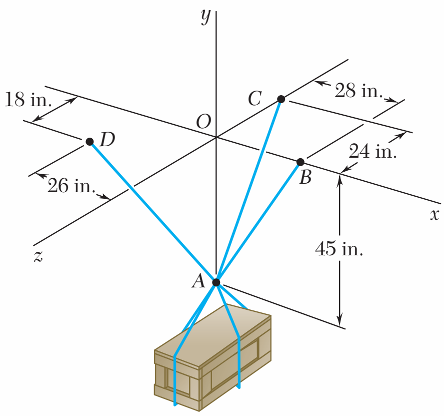


Fig. P3.29 and *P3.30*

3.30 Three cables are used to support a container as shown. Determine the angle formed by cables *AC* and *AD*.

3.31 The 500-mm tube *AB* can slide along a horizontal rod. The ends *A* and *B* of the tube are connected by elastic cords to the fixed point *C*. For the position corresponding to $x = 275$ mm, determine the angle formed by the two cords, (*a*) using Eq. (3.32), (*b*) applying the law of cosines to triangle *ABC*.

3.32 Solve Prob. 3.31 for the position corresponding to $x = 100$ mm.

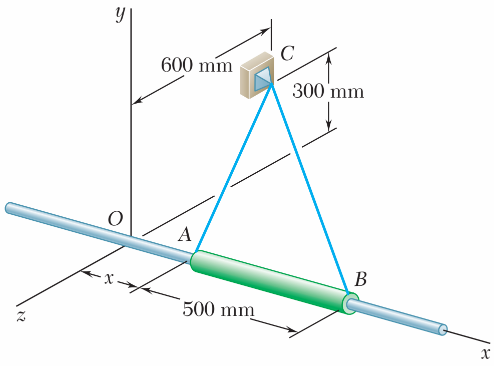


Fig. P3.31

3.33 Given the vectors $\mathbf{P} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$, $\mathbf{Q} = 2\mathbf{i} + \mathbf{j}$, and $\mathbf{S} = \mathbf{i}$, compute $\mathbf{P} \cdot (\mathbf{Q} \times \mathbf{S})$, $(\mathbf{P} \times \mathbf{Q}) \cdot \mathbf{S}$, and $(\mathbf{S} \times \mathbf{Q}) \cdot \mathbf{P}$.

3.34 Given the vectors $\mathbf{P} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$, $\mathbf{Q} = -\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$, and $\mathbf{S} = -3\mathbf{i} - \mathbf{j} + S_z\mathbf{k}$, determine the value of S_z for which the three vectors are coplanar.

3.35 The jib crane is oriented so that the boom DA is parallel to the x axis. At the instant shown, the tension in cable AB is 13 kN. Determine the moment about each of the coordinate axes of the force exerted on A by the cable AB.

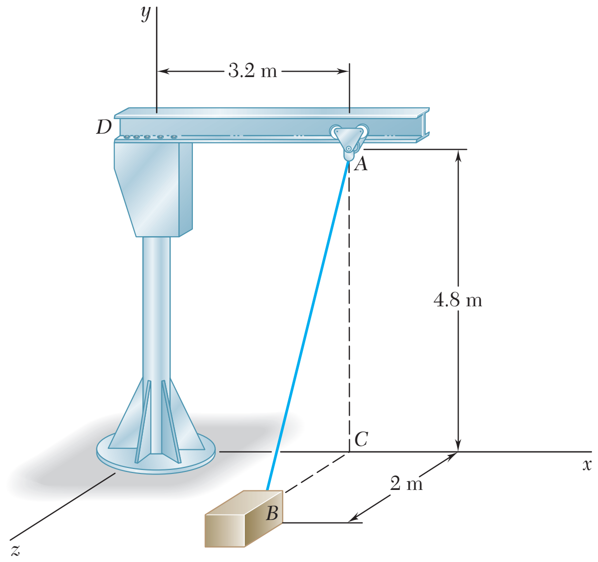


Fig. P3.35 and *P3.36*

3.36 The jib crane is oriented so that the boom DA is parallel to the x axis. Determine the maximum permissible tension in the cable AB if the absolute values of the moments about the coordinate axes of the force exerted on A must be as follows: $|M_x| \leq 10$ kN · m, $|M_y| \leq 6$ kN · m, and $|M_z| \leq 16$ kN · m.

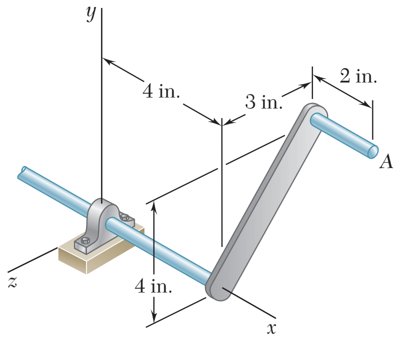


Fig. P3.37 and *P3.38*

3.37 The primary purpose of the crank shown is to produce a moment about the x axis. Show that a single force acting at A and having moment M_x different from zero about the x axis must also have a moment different from zero about at least one of the other coordinate axes.

3.38 A single force **F** of unknown magnitude and direction acts at point A of the crank shown. Determine the moment M_x of **F** about the x axis knowing that $M_y = +180$ lb · in. and $M_z = -320$ lb · in.

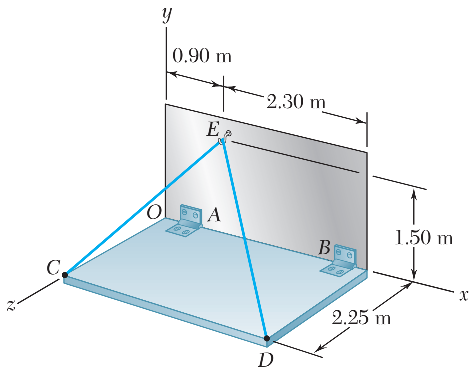


Fig. P3.39

3.39 The rectangular platform is hinged at A and B and supported by a cable that passes over a frictionless hook at E. Knowing that the tension in the cable is 1349 N, determine the moment about each of the coordinate axes of the force exerted by the cable at C.

3.40 For the platform of Prob. 3.39, determine the moment about each of the coordinate axes of the force exerted by the cable at D.

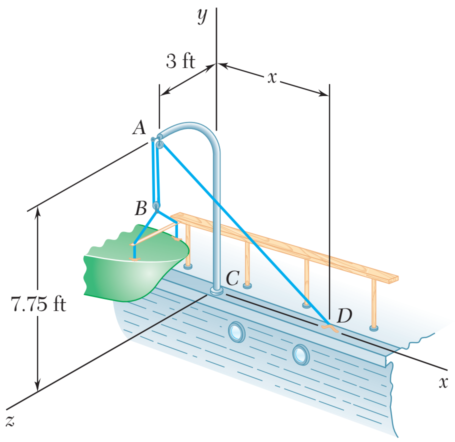


Fig. P3.41

3.41 A small boat hangs from two davits, one of which is shown in the figure. It is known that the moment about the z axis of the resultant force $\mathbf{R}_A$ exerted on the davit at A must not exceed 279 lb · ft in absolute value. Determine the largest allowable tension in the line $ABAD$ when $x = 6$ ft.

3.42 For the davit of Prob. 3.41, determine the largest allowable distance x when the tension in the line $ABAD$ is 60 lb.

3.43 A force **P** of magnitude 25 lb acts on a bent rod as shown. Determine the moment of **P** about (*a*) a line joining points *C* and *F*, (*b*) a line joining points *O* and *C*.

3.44 A force **P** of magnitude 25 lb acts on a bent rod as shown. Determine the moment of **P** about (*a*) a line joining points *A* and *C*, (*b*) a line joining points *A* and *D*.

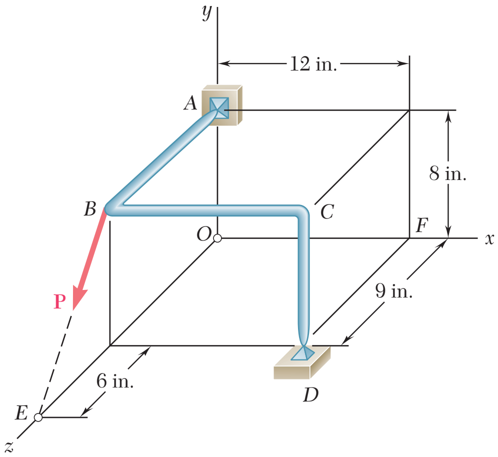


Fig. P3.43 and P3.44

3.45 Two rods are welded together to form a T-shaped lever that is acted upon by a 650-N force as shown. Determine the moment of the force about rod *AB*.

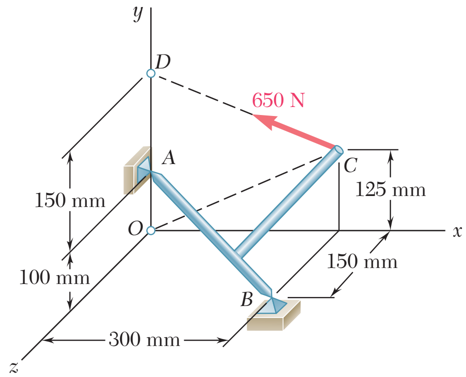


Fig. *P3.45*

3.46 The rectangular plate *ABCD* is held by hinges along its edge *AD* and by the wire *BE*. Knowing that the tension in the wire is 546 N, determine the moment about *AD* of the force exerted by the wire at point *B*.

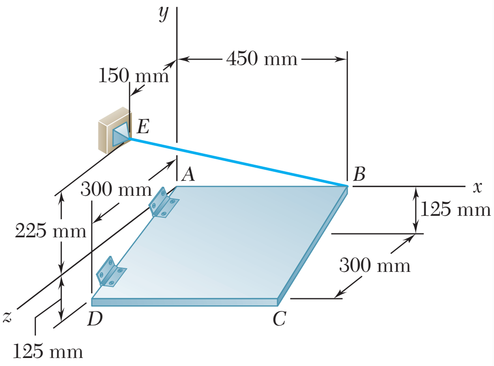


Fig. P3.46

3.47 The 23-in. vertical rod *CD* is welded to the midpoint *C* of the 50-in. rod *AB*. Determine the moment about *AB* of the 235-lb force **P**.

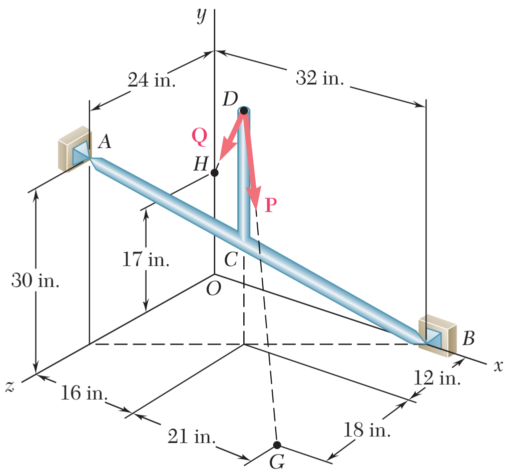


Fig. *P3.47* and P3.48

3.48 The 23-in. vertical rod *CD* is welded to the midpoint *C* of the 50-in. rod *AB*. Determine the moment about *AB* of the 174-lb force **Q**.

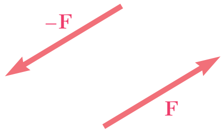


Fig. 3.30

3.12 MOMENT OF A COUPLE

Two forces **F** *and* $-\mathbf{F}$ *having the same magnitude, parallel lines of action, and opposite sense are said to form a couple* (Fig. 3.30). Clearly, the sum of the components of the two forces in any direction is zero. The sum of the moments of the two forces about a given point, however, is not zero. While the two forces will not translate the body on which they act, they will tend to make it rotate.

Denoting by $\mathbf{r}_A$ and $\mathbf{r}_B$, respectively, the position vectors of the points of application of **F** and $-\mathbf{F}$ (Fig. 3.31), we find that the sum of the moments of the two forces about O is

$$\mathbf{r}_A \times \mathbf{F} + \mathbf{r}_B \times (-\mathbf{F}) = (\mathbf{r}_A - \mathbf{r}_B) \times \mathbf{F}$$

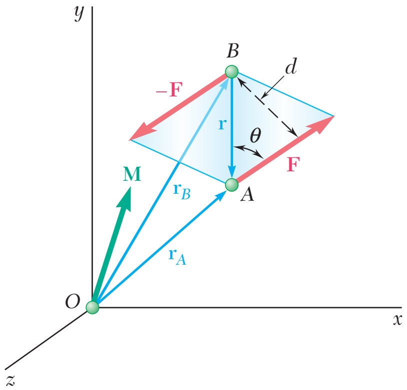


Fig. 3.31

Setting $\mathbf{r}_A - \mathbf{r}_B = \mathbf{r}$, where $\mathbf{r}$ is the vector joining the points of application of the two forces, we conclude that the sum of the moments of **F** and $-\mathbf{F}$ about O is represented by the vector

$$\boxed{\mathbf{M} = \mathbf{r} \times \mathbf{F}} \tag{3.47}$$

The vector **M** is called the *moment of the couple;* it is a vector perpendicular to the plane containing the two forces, and its magnitude is

$$\boxed{M = rF \sin \theta = Fd} \tag{3.48}$$

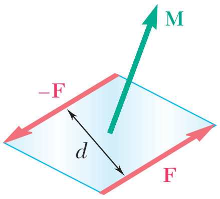


Fig. 3.32

where d is the perpendicular distance between the lines of action of **F** and $-\mathbf{F}$. The sense of **M** is defined by the right-hand rule.

Since the vector $\mathbf{r}$ in (3.47) is independent of the choice of the origin O of the coordinate axes, we note that the same result would have been obtained if the moments of **F** and $-\mathbf{F}$ had been computed about a different point O'. Thus, the moment **M** of a couple is a *free vector* (Sec. 2.3) which can be applied at any point (Fig. 3.32).

From the definition of the moment of a couple, it also follows that two couples, one consisting of the forces $\mathbf{F}_1$ and $-\mathbf{F}_1$, the other of the forces $\mathbf{F}_2$ and $-\mathbf{F}_2$ (Fig. 3.33), will have equal moments if

$$F_1 d_1 = F_2 d_2 \tag{3.49}$$

and if the two couples lie in parallel planes (or in the same plane) and have the same sense.

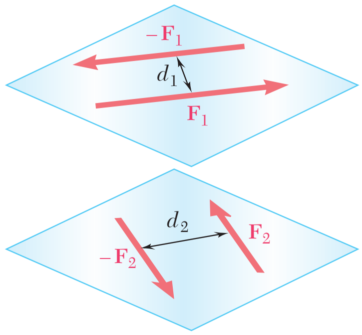


Fig. 3.33

3.13 EQUIVALENT COUPLES

Figure 3.34 shows three couples which act successively on the same rectangular box. As seen in the preceding section, the only motion a couple can impart to a rigid body is a rotation. Since each of the three couples shown has the same moment **M** (same direction and same magnitude $M = 120$ lb · in.), we can expect the three couples to have the same effect on the box.

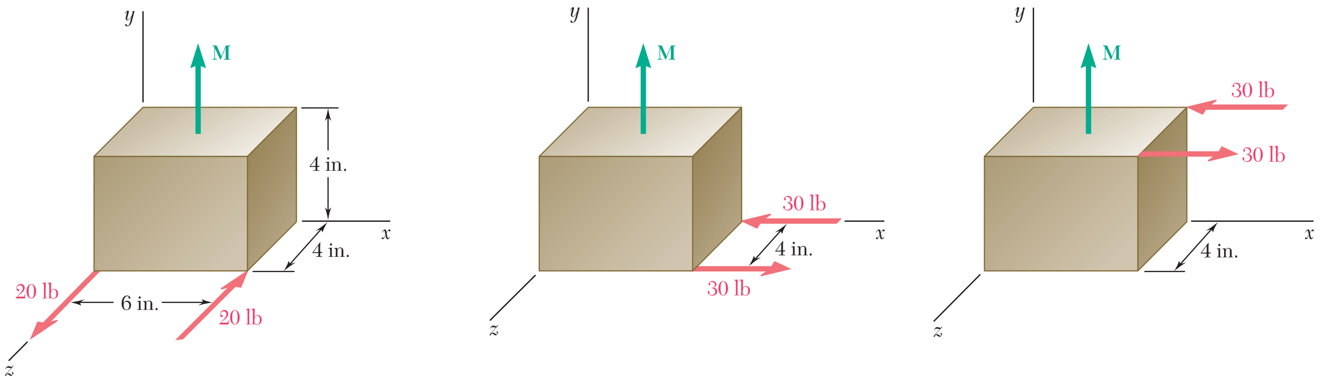


Fig. 3.34 (a) (b) (c)

As reasonable as this conclusion appears, we should not accept it hastily. While intuitive feeling is of great help in the study of mechanics, it should not be accepted as a substitute for logical reasoning. Before stating that two systems (or groups) of forces have the same effect on a rigid body, we should prove that fact on the basis of the experimental evidence introduced so far. This evidence consists of the parallelogram law for the addition of two forces (Sec. 2.2) and the principle of transmissibility (Sec. 3.3). Therefore, we will state that *two systems of forces are equivalent* (i.e., they have the same effect on a rigid body) *if we can transform one of them into the other by means of one or several of the following operations*: (1) replacing two forces acting on the same particle by their resultant; (2) resolving a force into two components; (3) canceling two equal and opposite forces acting on the same particle; (4) attaching to the same particle two equal and opposite forces; (5) moving a force along its line of action. Each of these operations is easily justified on the basis of the parallelogram law or the principle of transmissibility.

Let us now prove that *two couples having the same moment* **M** *are equivalent*. First consider two couples contained in the same plane, and assume that this plane coincides with the plane of the figure (Fig. 3.35). The first couple consists of the forces $\mathbf{F}_1$ and $-\mathbf{F}_1$ of magnitude F_1, which are located at a distance d_1 from each other (Fig. 3.35*a*), and the second couple consists of the forces $\mathbf{F}_2$ and $-\mathbf{F}_2$ of magnitude F_2, which are located at a distance d_2 from each other (Fig. 3.35*d*). Since the two couples have the same moment **M**, which is perpendicular to the plane of the figure, they must have the same sense (assumed here to be counterclockwise), and the relation

$$F_1 d_1 = F_2 d_2 \tag{3.49}$$

must be satisfied. To prove that they are equivalent, we shall show that the first couple can be transformed into the second by means of the operations listed above.

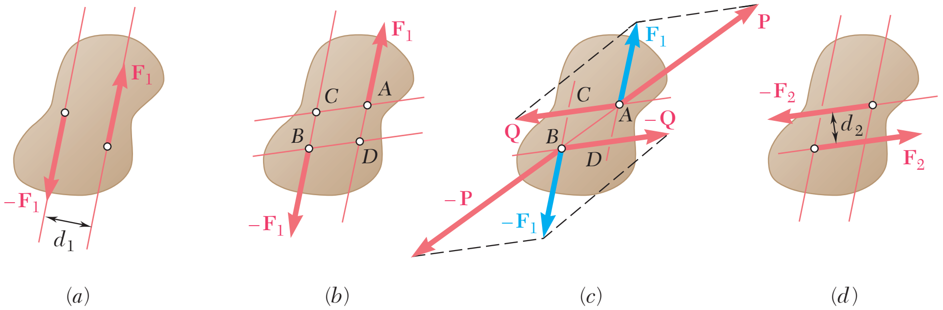


Fig. 3.35

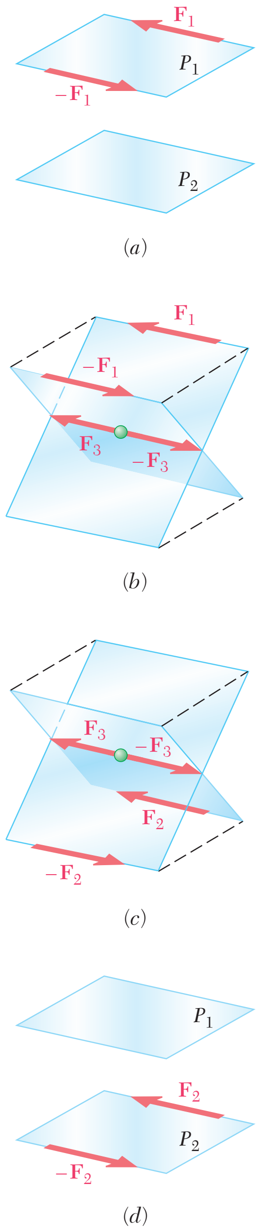


Fig. 3.36

Denoting by A, B, C, and D the points of intersection of the lines of action of the two couples, we first slide the forces $\mathbf{F}_1$ and $-\mathbf{F}_1$ until they are attached, respectively, at A and B, as shown in Fig. 3.35*b*. The force $\mathbf{F}_1$ is then resolved into a component $\mathbf{P}$ along line AB and a component $\mathbf{Q}$ along AC (Fig. 3.35*c*); similarly, the force $-\mathbf{F}_1$ is resolved into $-\mathbf{P}$ along AB and $-\mathbf{Q}$ along BD. The forces $\mathbf{P}$ and $-\mathbf{P}$ have the same magnitude, the same line of action, and opposite sense; they can be moved along their common line of action until they are applied at the same point and may then be canceled. Thus the couple formed by $\mathbf{F}_1$ and $-\mathbf{F}_1$ reduces to a couple consisting of $\mathbf{Q}$ and $-\mathbf{Q}$.

We will now show that the forces $\mathbf{Q}$ and $-\mathbf{Q}$ are respectively equal to the forces $-\mathbf{F}_2$ and $\mathbf{F}_2$. The moment of the couple formed by $\mathbf{Q}$ and $-\mathbf{Q}$ can be obtained by computing the moment of $\mathbf{Q}$ about B; similarly, the moment of the couple formed by $\mathbf{F}_1$ and $-\mathbf{F}_1$ is the moment of $\mathbf{F}_1$ about B. But, by Varignon's theorem, the moment of $\mathbf{F}_1$ is equal to the sum of the moments of its components $\mathbf{P}$ and $\mathbf{Q}$. Since the moment of $\mathbf{P}$ about B is zero, the moment of the couple formed by $\mathbf{Q}$ and $-\mathbf{Q}$ must be equal to the moment of the couple formed by $\mathbf{F}_1$ and $-\mathbf{F}_1$. Recalling (3.49), we write

$$Qd_2 = F_1d_1 = F_2d_2 \qquad \text{and} \qquad Q = F_2$$

Thus the forces $\mathbf{Q}$ and $-\mathbf{Q}$ are respectively equal to the forces $-\mathbf{F}_2$ and $\mathbf{F}_2$, and the couple of Fig. 3.35*a* is equivalent to the couple of Fig. 3.35*d*.

Next consider two couples contained in parallel planes P_1 and P_2; we will prove that they are equivalent if they have the same moment. In view of the foregoing, we can assume that the couples consist of forces of the same magnitude F acting along parallel lines (Fig. 3.36*a* and *d*). We propose to show that the couple contained in plane P_1 can be transformed into the couple contained in plane P_2 by means of the standard operations listed above.

Let us consider the two planes defined respectively by the lines of action of $\mathbf{F}_1$ and $-\mathbf{F}_2$ and by those of $-\mathbf{F}_1$ and $\mathbf{F}_2$ (Fig. 3.36*b*). At a point on their line of intersection we attach two forces $\mathbf{F}_3$ and $-\mathbf{F}_3$, respectively equal to $\mathbf{F}_1$ and $-\mathbf{F}_1$. The couple formed by $\mathbf{F}_1$ and $-\mathbf{F}_3$ can be replaced by a couple consisting of $\mathbf{F}_3$ and $-\mathbf{F}_2$ (Fig. 3.36*c*), since both couples clearly have the same moment and are contained in the same plane. Similarly, the couple formed by $-\mathbf{F}_1$ and $\mathbf{F}_3$ can be replaced by a couple consisting of $-\mathbf{F}_3$ and $\mathbf{F}_2$. Canceling the two equal and opposite forces $\mathbf{F}_3$ and $-\mathbf{F}_3$, we obtain the desired couple in plane P_2 (Fig. 3.36*d*). Thus, we conclude that two couples having the same moment **M** are equivalent, whether they are contained in the same plane or in parallel planes.

The property we have just established is very important for the correct understanding of the mechanics of rigid bodies. It indicates that when a couple acts on a rigid body, it does not matter where the two forces forming the couple act or what magnitude and direction they have. The only thing which counts is the *moment* of the couple (magnitude and direction). Couples with the same moment will have the same effect on the rigid body.

3.14 ADDITION OF COUPLES

Consider two intersecting planes P_1 and P_2 and two couples acting respectively in P_1 and P_2. We can, without any loss of generality, assume that the couple in P_1 consists of two forces $\mathbf{F}_1$ and $-\mathbf{F}_1$ perpendicular to the line of intersection of the two planes and acting respectively at A and B (Fig. 3.37*a*). Similarly, we assume that the couple in P_2 consists of two forces $\mathbf{F}_2$ and $-\mathbf{F}_2$ perpendicular to AB and acting, respectively, at A and B. It is clear that the resultant $\mathbf{R}$ of $\mathbf{F}_1$ and $\mathbf{F}_2$ and the resultant $-\mathbf{R}$ of $-\mathbf{F}_1$ and $-\mathbf{F}_2$ form a couple. Denoting by $\mathbf{r}$ the vector joining B to A and recalling the definition of the moment of a couple (Sec. 3.12), we express the moment $\mathbf{M}$ of the resulting couple as follows:

$$\mathbf{M} = \mathbf{r} \times \mathbf{R} = \mathbf{r} \times (\mathbf{F}_1 + \mathbf{F}_2)$$

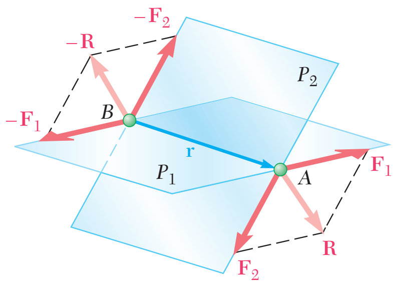
and, by Varignon's theorem,

$$\mathbf{M} = \mathbf{r} \times \mathbf{F}_1 + \mathbf{r} \times \mathbf{F}_2$$

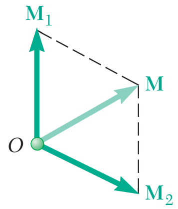
But the first term in the expression obtained represents the moment $\mathbf{M}_1$ of the couple in P_1, and the second term represents the moment $\mathbf{M}_2$ of the couple in P_2. We have

$$\mathbf{M} = \mathbf{M}_1 + \mathbf{M}_2 \tag{3.50}$$

and we conclude that the sum of two couples of moments $\mathbf{M}_1$ and $\mathbf{M}_2$ is a couple of moment $\mathbf{M}$ equal to the vector sum of $\mathbf{M}_1$ and $\mathbf{M}_2$ (Fig. 3.37*b*).



(*a*)



(*b*)

Fig. 3.37

3.15 COUPLES CAN BE REPRESENTED BY VECTORS

As we saw in Sec. 3.13, couples which have the same moment, whether they act in the same plane or in parallel planes, are equivalent. There is therefore no need to draw the actual forces forming a given couple in order to define its effect on a rigid body (Fig. 3.38*a*). It is sufficient to draw an arrow equal in magnitude and direction to the moment **M** of the couple (Fig. 3.38*b*). On the other hand, we saw in Sec. 3.14 that the sum of two couples is itself a couple and that the moment **M** of the resultant couple can be obtained by forming the vector sum of the moments $\mathbf{M}_1$ and $\mathbf{M}_2$ of the given couples. Thus, couples obey the law of addition of vectors, and the arrow used in Fig. 3.38*b* to represent the couple defined in Fig. 3.38*a* can truly be considered a vector.

The vector representing a couple is called a *couple vector*. Note that, in Fig. 3.38, a red arrow is used to distinguish the couple vector, *which represents the couple itself*, from the *moment* of the couple, which was represented by a green arrow in earlier figures. Also note that the symbol ↰ is added to this red arrow to avoid any confusion with vectors representing forces. A couple vector, like the moment of a couple, is a free vector. Its point of application, therefore, can be chosen at the origin of the system of coordinates, if so desired (Fig. 3.38*c*). Furthermore, the couple vector **M** can be resolved into component vectors $\mathbf{M}_x$, $\mathbf{M}_y$, and $\mathbf{M}_z$, which are directed along the coordinate axes (Fig. 3.38*d*). These component vectors represent couples acting, respectively, in the *yz*, *zx*, and *xy* planes.

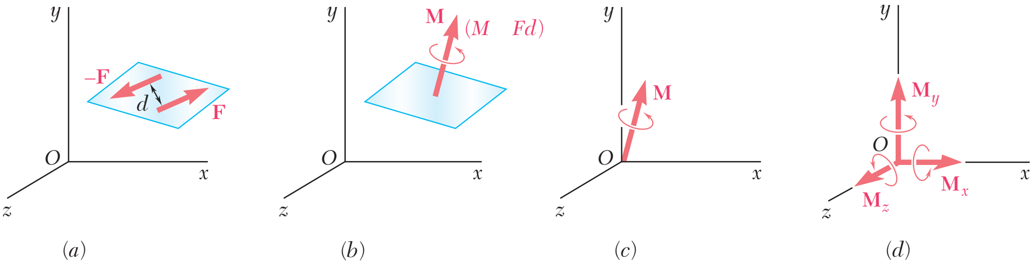


Fig. 3.38

3.16 RESOLUTION OF A GIVEN FORCE INTO A FORCE AT *O* AND A COUPLE

Consider a force **F** acting on a rigid body at a point *A* defined by the position vector **r** (Fig. 3.39*a*). Suppose that for some reason we would rather have the force act at point *O*. While we can move **F** along its line of action (principle of transmissibility), we cannot move it to a point *O* which does not lie on the original line of action without modifying the action of **F** on the rigid body.

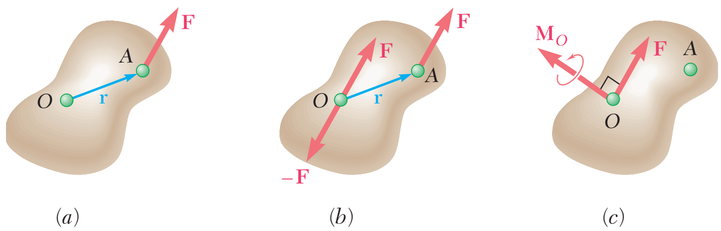


Fig. 3.39

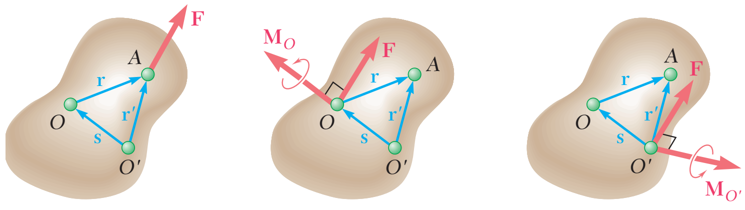
We can, however, attach two forces at point *O*, one equal to **F** and the other equal to $-\mathbf{F}$, without modifying the action of the original force on the rigid body (Fig. 3.39*b*). As a result of this transformation, a force **F** is now applied at *O*; the other two forces form a couple of moment $\mathbf{M}_O = \mathbf{r} \times \mathbf{F}$. Thus, *any force* **F** *acting on a rigid body can be moved to an arbitrary point O provided that a couple is added whose moment is equal to the moment of* **F** *about O*. The couple tends to impart to the rigid body the same rotational motion about O that the force **F** tended to produce before it was transferred to O. The couple is represented by a couple vector $\mathbf{M}_O$ perpendicular to the plane containing **r** and **F**. Since $\mathbf{M}_O$ is a free vector, it may be applied anywhere; for convenience, however, the couple vector is usually attached at O, together with **F**, and the combination obtained is referred to as a *force-couple system* (Fig. 3.39*c*).

If the force **F** had been moved from A to a different point O' (Fig. 3.40*a* and *c*), the moment $\mathbf{M}_{O'} = \mathbf{r}' \times \mathbf{F}$ of **F** about O' should have been computed, and a new force-couple system, consisting of **F** and of the couple vector $\mathbf{M}_{O'}$, would have been attached at O'. The relation existing between the moments of **F** about O and O' is obtained by writing

$$\mathbf{M}_{O'} = \mathbf{r}' \times \mathbf{F} = (\mathbf{r} + \mathbf{s}) \times \mathbf{F} = \mathbf{r} \times \mathbf{F} + \mathbf{s} \times \mathbf{F}$$

$$\boxed{\mathbf{M}_{O'} = \mathbf{M}_O + \mathbf{s} \times \mathbf{F}} \tag{3.51}$$

where **s** is the vector joining O' to O. Thus, the moment $\mathbf{M}_{O'}$ of **F** about O' is obtained by adding to the moment $\mathbf{M}_O$ of **F** about O the vector product $\mathbf{s} \times \mathbf{F}$ representing the moment about O' of the force **F** applied at O.

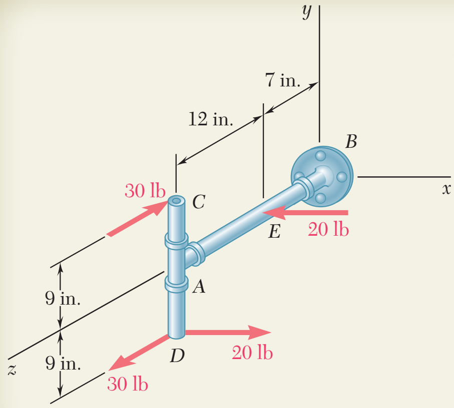


(*a*) (*b*) (*c*)

Fig. 3.40

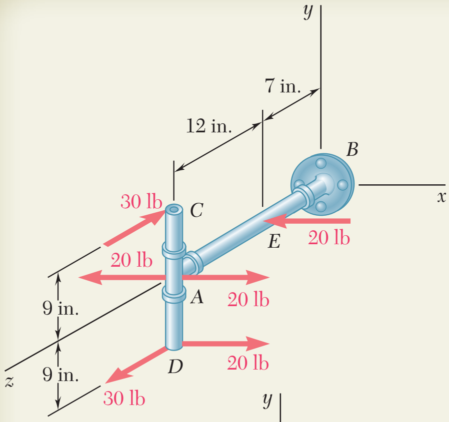
This result could also have been established by observing that, in order to transfer to O' the force-couple system attached at O (Fig. 3.40*b* and *c*), the couple vector $\mathbf{M}_O$ can be freely moved to O'; to move the force **F** from O to O', however, it is necessary to add to **F** a couple vector whose moment is equal to the moment about O' of the force **F** applied at O. Thus, the couple vector $\mathbf{M}_{O'}$ must be the sum of $\mathbf{M}_O$ and the vector $\mathbf{s} \times \mathbf{F}$.

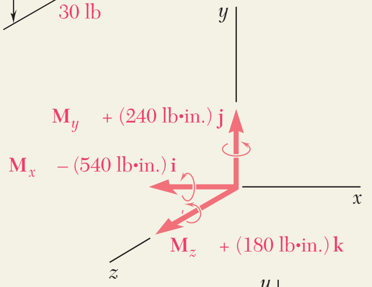
As noted above, the force-couple system obtained by transferring a force **F** from a point A to a point O consists of **F** and a couple vector $\mathbf{M}_O$ perpendicular to **F**. Conversely, any force-couple system consisting of a force **F** and a couple vector $\mathbf{M}_O$ which are *mutually perpendicular* can be replaced by a single equivalent force. This is done by moving the force **F** in the plane perpendicular to $\mathbf{M}_O$ until its moment about O is equal to the moment of the couple to be eliminated.



SAMPLE PROBLEM 3.6

Determine the components of the single couple equivalent to the two couples shown.





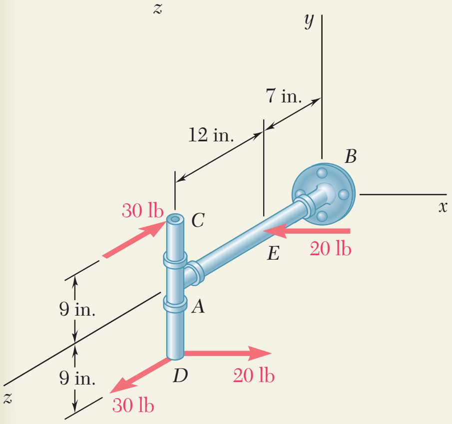
SOLUTION

Our computations will be simplified if we attach two equal and opposite 20-lb forces at A. This enables us to replace the original 20-lb-force couple by two new 20-lb-force couples, one of which lies in the zx plane and the other in a plane parallel to the xy plane. The three couples shown in the adjoining sketch can be represented by three couple vectors $\mathbf{M}_x$, $\mathbf{M}_y$, and $\mathbf{M}_z$ directed along the coordinate axes. The corresponding moments are

$$M_x = -(30 \text{ lb})(18 \text{ in.}) = -540 \text{ lb} \cdot \text{in.}$$
$$M_y = +(20 \text{ lb})(12 \text{ in.}) = +240 \text{ lb} \cdot \text{in.}$$
$$M_z = +(20 \text{ lb})(9 \text{ in.}) = +180 \text{ lb} \cdot \text{in.}$$

These three moments represent the components of the single couple $\mathbf{M}$ equivalent to the two given couples. We write

$$\mathbf{M} = -(540 \text{ lb} \cdot \text{in.})\mathbf{i} + (240 \text{ lb} \cdot \text{in.})\mathbf{j} + (180 \text{ lb} \cdot \text{in.})\mathbf{k} \quad \blacktriangleleft$$



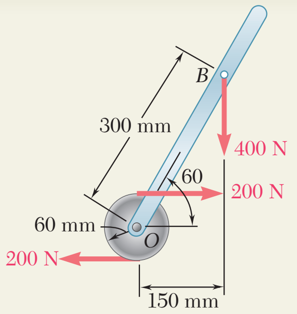
Alternative Solution. The components of the equivalent single couple $\mathbf{M}$ can also be obtained by computing the sum of the moments of the four given forces about an arbitrary point. Selecting point D, we write

$$\mathbf{M} = \mathbf{M}_D = (18 \text{ in.})\mathbf{j} \times (-30 \text{ lb})\mathbf{k} + [(9 \text{ in.})\mathbf{j} - (12 \text{ in.})\mathbf{k}] \times (-20 \text{ lb})\mathbf{i}$$

and, after computing the various cross products,

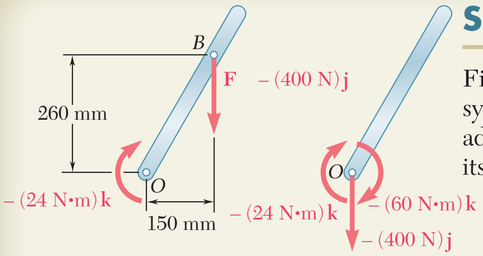
$$\mathbf{M} = -(540 \text{ lb} \cdot \text{in.})\mathbf{i} + (240 \text{ lb} \cdot \text{in.})\mathbf{j} + (180 \text{ lb} \cdot \text{in.})\mathbf{k} \quad \blacktriangleleft$$

SAMPLE PROBLEM 3.7



Replace the couple and force shown by an equivalent single force applied to the lever. Determine the distance from the shaft to the point of application of this equivalent force.

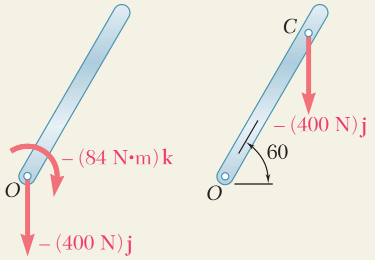
SOLUTION



First, the given force and couple are replaced by an equivalent force-couple system at O. We move the force $\mathbf{F} = -(400\text{ N})\mathbf{j}$ to O and at the same time add a couple of moment $\mathbf{M}_O$ equal to the moment about O of the force in its original position.

$$\mathbf{M}_O = \overrightarrow{OB} \times \mathbf{F} = [(0.150\,\text{m})\mathbf{i} + (0.260\,\text{m})\mathbf{j}] \times (-400\,\text{N})\mathbf{j}$$
$$= -(60\text{ N}\cdot\text{m})\mathbf{k}$$

This couple is added to the couple of moment $-(24\text{ N}\cdot\text{m})\mathbf{k}$ formed by the two 200-N forces, and a couple of moment $-(84\text{ N}\cdot\text{m})\mathbf{k}$ is obtained. This last couple can be eliminated by applying $\mathbf{F}$ at a point C chosen in such a way that

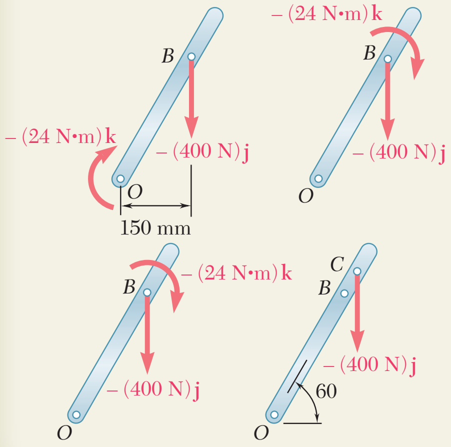


$$-(84\,\text{N}\cdot\text{m})\mathbf{k} = \overrightarrow{OC} \times \mathbf{F}$$
$$= [(OC)\cos 60°\mathbf{i} + (OC)\sin 60°\mathbf{j}] \times (-400\,\text{N})\mathbf{j}$$
$$= -(OC)\cos 60°(400\,\text{N})\mathbf{k}$$

We conclude that

$$(OC)\cos 60° = 0.210\text{ m} = 210\text{ mm} \qquad OC = 420\text{ mm} \quad ◀$$

Alternative Solution. Since the effect of a couple does not depend on its location, the couple of moment $-(24\text{ N}\cdot\text{m})\mathbf{k}$ can be moved to B; we thus obtain a force-couple system at B. The couple can now be eliminated by applying $\mathbf{F}$ at a point C chosen in such a way that



$$-(24\,\text{N}\cdot\text{m})\mathbf{k} = \overrightarrow{BC} \times \mathbf{F}$$
$$= -(BC)\cos 60°(400\,\text{N})\mathbf{k}$$

We conclude that

$$(BC)\cos 60° = 0.060\text{ m} = 60\text{ mm} \qquad BC = 120\text{ mm}$$
$$OC = OB + BC = 300\text{ mm} + 120\text{ mm} \qquad OC = 420\text{ mm} \quad ◀$$

PROBLEMS

3.49 A couple formed by two 975-N forces is applied to the pulley assembly shown. Determine an equivalent couple that is formed by (*a*) vertical forces acting at *A* and *C*, (*b*) the smallest possible forces acting at *B* and *D*, (*c*) the smallest possible forces that can be attached to the assembly.

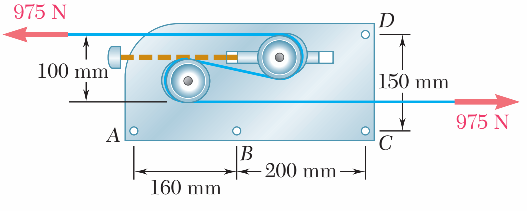


Fig. P3.49

3.50 Four 1-in.-diameter pegs are attached to a board as shown. Two strings are passed around the pegs and pulled with forces of magnitude $P = 20$ lb and $Q = 35$ lb. Determine the resultant couple acting on the board.

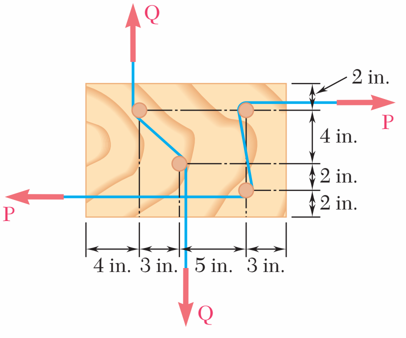


Fig. P3.50

3.51 Two 80-N forces are applied as shown to the corners *B* and *D* of a rectangular plate. (*a*) Determine the moment of the couple formed by the two forces by resolving each force into horizontal and vertical components and adding the moments of the two resulting couples. (*b*) Use the result obtained to determine the perpendicular distance between lines *BE* and *DF*.

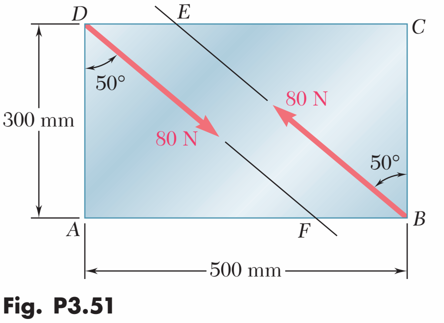


Fig. P3.51

3.52 A piece of plywood in which several holes are being drilled successively has been secured to a workbench by means of two nails. Knowing that the drill exerts a 12 N · m couple on the piece of plywood, determine the magnitude of the resulting forces applied to the nails if they are located (*a*) at *A* and *B*, (*b*) at *B* and *C*, (*c*) at *A* and *C*.

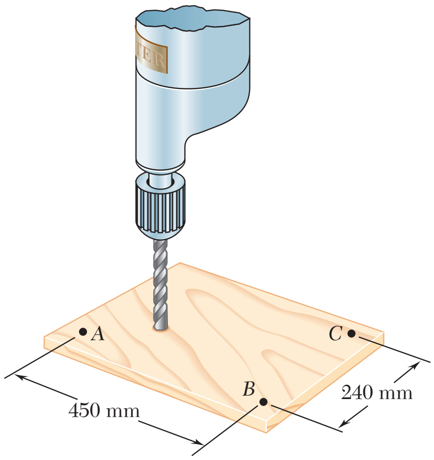


Fig. P3.52

3.53 Four $1\frac{1}{2}$-in.-diameter pegs are attached to a board as shown. Two strings are passed around the pegs and pulled with the forces indicated. (*a*) Determine the resultant couple acting on the board. (*b*) If only one string is used, around which pegs should it pass and in what directions should it be pulled to create the same couple with the minimum tension in the string? (*c*) What is the value of that minimum tension?

3.54 Four pegs of the same diameter are attached to a board as shown. Two strings are passed around the pegs and pulled with the forces indicated. Determine the diameter of the pegs knowing that the resultant couple applied to the board is 1132.5 lb · in. counterclockwise.

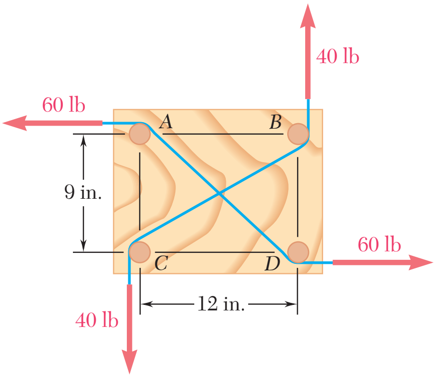


Fig. P3.53 and P3.54

3.55 The axles and drive shaft of a rear-wheel drive automobile are acted upon by the three couples shown. Replace these three couples by a single equivalent couple.

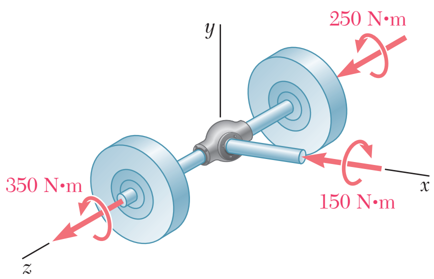


Fig. *P3.55*

3.56 Two shafts for a speed-reducer unit are subjected to couples of magnitude $M_1 = 12$ lb · ft and $M_2 = 5$ lb · ft. Replace the two couples by a single equivalent couple, specifying its magnitude and the direction of its axis.

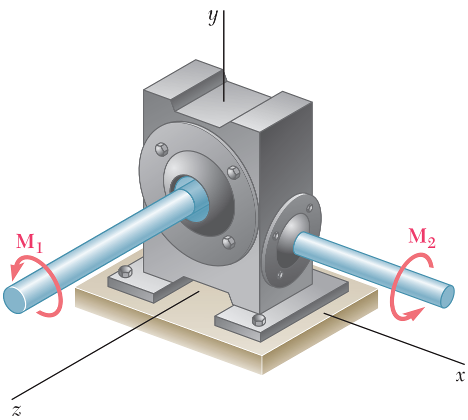


Fig. P3.56

3.57 Replace the two couples shown by a single equivalent couple, specifying its magnitude and the direction of its axis.

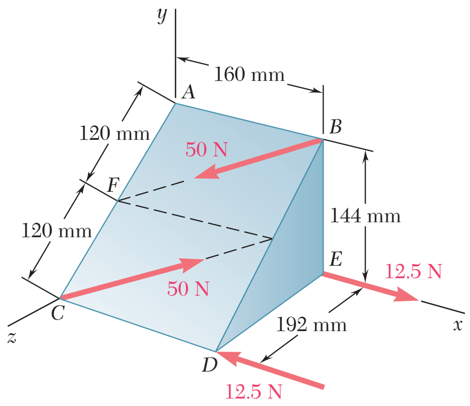


Fig. P3.57

3.58 Solve Prob. 3.57 assuming that two 10-N vertical forces have been added, one acting upward at C and the other downward at B.

3.59 Shafts A and B connect the gear box to the wheel assemblies of a tractor, and shaft C connects it to the engine. Shafts A and B lie in the vertical yz plane, while shaft C is directed along the x axis. Replace the couples applied to the shafts by a single equivalent couple, specifying its magnitude and the direction of its axis.

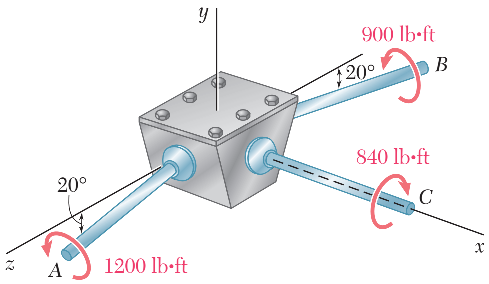


Fig. P3.59

3.60 $\mathbf{M}_1$ and $\mathbf{M}_2$ represent couples that are contained in the planes ABC and ACD, respectively. Assuming that $M_1 = M_2 = M$, determine a single couple equivalent to the two given couples.

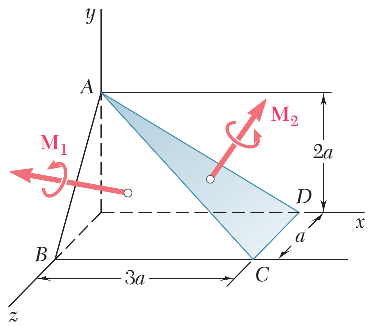


Fig. P3.60

3.61 A 60-lb vertical force **P** is applied at A to the bracket shown, which is held by screws at B and C. (*a*) Replace **P** by an equivalent force-couple system at B. (*b*) Find the two horizontal forces at B and C that are equivalent to the couple obtained in part *a*.

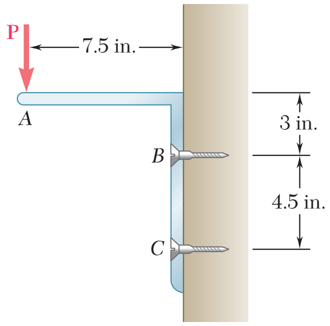


Fig. P3.61

3.62 The force and couple shown are to be replaced by an equivalent single force. Determine the required value of α so that the line of action of the single equivalent force will pass through point B.

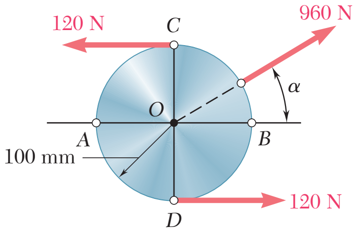


Fig. *P3.62* and P3.63

3.63 Knowing that $\alpha = 60°$, replace the force and couple shown by a single force applied at a point located (*a*) on line AB, (*b*) on line CD. In each case determine the distance from the center O to the point of application of the force.

3.64 A 260-lb force is applied at A to the rolled-steel section shown. Replace that force by an equivalent force-couple system at the center C of the section.

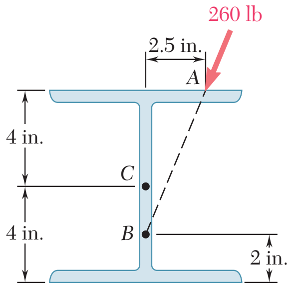


Fig. *P3.64*

3.65 Force **P** has a magnitude of 300 N and is applied at A in a direction perpendicular to the handle ($\alpha = 0$). Assuming $\beta = 30°$, replace force **P** by (*a*) an equivalent force-couple system at B, (*b*) an equivalent system formed by two parallel forces applied at B and C.

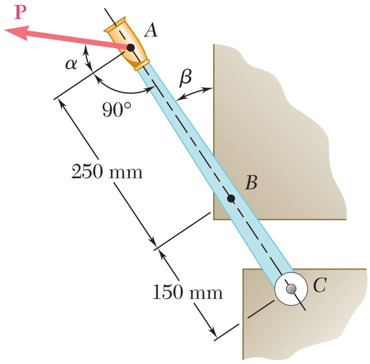


Fig. P3.65

3.66 A force and couple act as shown on a square plate of side $a = 25$ in. Knowing that $P = 60$ lb, $Q = 40$ lb, and $\alpha = 50°$, replace the given force and couple by a single force applied at a point located (*a*) on line AB, (*b*) on line AC. In each case determine the distance from A to the point of application of the force.

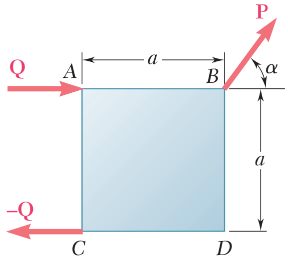


Fig. P3.66

3.67 Replace the 250-kN force **P** by an equivalent force-couple system at G.

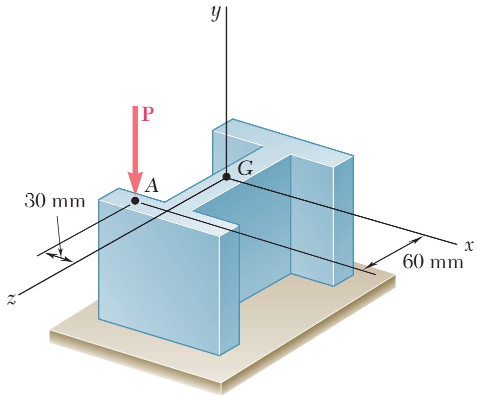


Fig. P3.67

3.68 A 4-kip force is applied on the outside face of the flange of a steel channel. Determine the components of the force and couple at G that are equivalent to the 4-kip load.

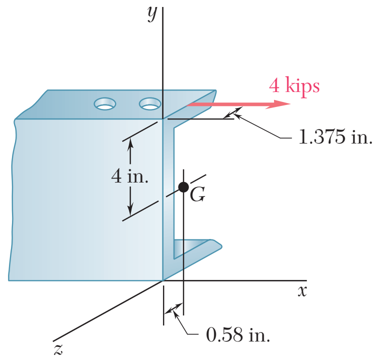


Fig. P3.68

3.69 The 12-ft boom AB has a fixed end A, and the tension in cable BC is 570 lb. Replace the force that the cable exerts at B by an equivalent force-couple system at A.

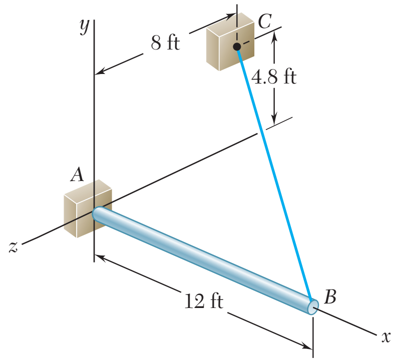


Fig. *P3.69*

3.70 Replace the 150-N force by an equivalent force-couple system at A.

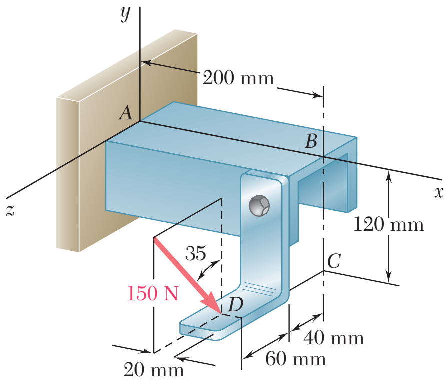


Fig. P3.70

3.71 The jib crane shown is orientated so that its boom AD is parallel to the x axis and is used to move a heavy crate. Knowing that the tension in cable AB is 2.6 kips, replace the force exerted by the cable at A by an equivalent force-couple system at the center O of the base of the crane.

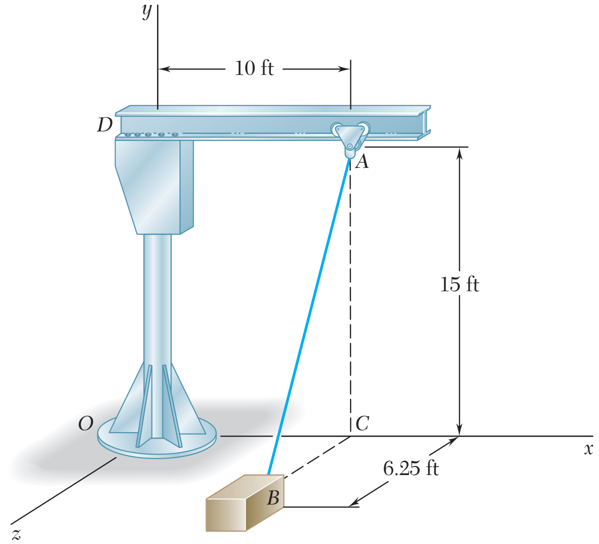


Fig. P3.71

3.72 A 200-N force is applied as shown on the bracket ABC. Determine the components of the force and couple at A that are equivalent to this force.

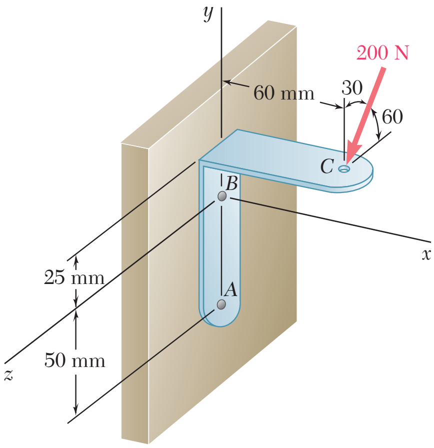


Fig. P3.72

3.17 REDUCTION OF A SYSTEM OF FORCES TO ONE FORCE AND ONE COUPLE

Consider a system of forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, . . . , acting on a rigid body at the points A_1, A_2, A_3, . . . , *defined by the position vectors* $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, $\boldsymbol{r}_3$, *etc.* (Fig. 3.41*a*). As seen in the preceding section, $\mathbf{F}_1$ can be moved from A_1 to a given point O if a couple of moment $\mathbf{M}_1$ equal to the moment $\mathbf{r}_1 \times \mathbf{F}_1$ of $\mathbf{F}_1$ about O is added to the original system of forces. Repeating this procedure with $\mathbf{F}_2$, $\mathbf{F}_3$, . . . , we obtain the

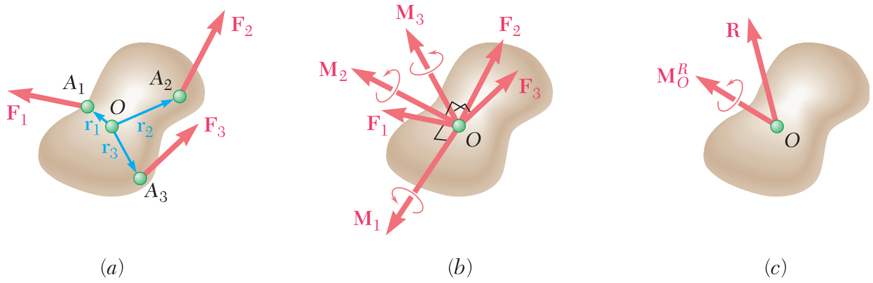


Fig. 3.41

system shown in Fig. 3.41*b*, which consists of the original forces, now acting at O, and the added couple vectors. Since the forces are now concurrent, they can be added vectorially and replaced by their resultant $\mathbf{R}$. Similarly, the couple vectors $\mathbf{M}_1$, $\mathbf{M}_2$, $\mathbf{M}_3$, . . . , can be added vectorially and replaced by a single couple vector $\mathbf{M}_O^R$. Any system of forces, however complex, can thus be reduced to an *equivalent force-couple system acting at a given point O* (Fig. 3.41*c*). We should note that while each of the couple vectors $\mathbf{M}_1$, $\mathbf{M}_2$, $\mathbf{M}_3$, . . . , in Fig. 3.41*b* is perpendicular to its corresponding force, the resultant force $\mathbf{R}$ and the resultant couple vector $\mathbf{M}_O^R$ in Fig. 3.41*c* will not, in general, be perpendicular to each other.

The equivalent force-couple system is defined by the equations

$$\mathbf{R} = \Sigma\mathbf{F} \qquad \mathbf{M}_O^R = \Sigma\mathbf{M}_O = \Sigma(\mathbf{r} \times \mathbf{F}) \tag{3.52}$$

which express that the force $\mathbf{R}$ is obtained by adding all the forces of the system, while the moment of the resultant couple vector $\mathbf{M}_O^R$, called the *moment resultant* of the system, is obtained by adding the moments about O of all the forces of the system.

Once a given system of forces has been reduced to a force and a couple at a point O, it can easily be reduced to a force and a couple at another point O'. While the resultant force $\mathbf{R}$ will remain unchanged, the new moment resultant $\mathbf{M}_{O'}^R$ will be equal to the sum of $\mathbf{M}_O^R$ and the moment about O' of the force $\mathbf{R}$ attached at O (Fig. 3.42). We have

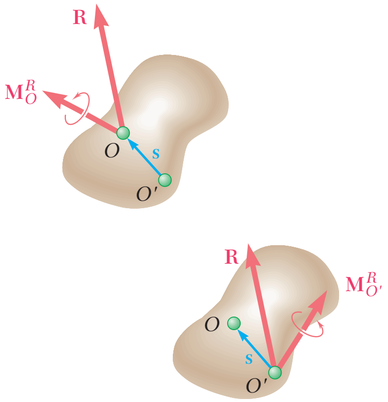


Fig. 3.42

$$\mathbf{M}_{O'}^R = \mathbf{M}_O^R + \mathbf{s} \times \mathbf{R} \tag{3.53}$$

In practice, the reduction of a given system of forces to a single force $\mathbf{R}$ at O and a couple vector $\mathbf{M}_O^R$ will be carried out in terms of components. Resolving each position vector $\mathbf{r}$ and each force $\mathbf{F}$ of the system into rectangular components, we write

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{3.54}$$
$$\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k} \tag{3.55}$$

Substituting for $\mathbf{r}$ and $\mathbf{F}$ in (3.52) and factoring out the unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, we obtain $\mathbf{R}$ and $\mathbf{M}_O^R$ in the form

$$\mathbf{R} = R_x\mathbf{i} + R_y\mathbf{j} + R_z\mathbf{k} \qquad \mathbf{M}_O^R = M_x^R\mathbf{i} + M_y^R\mathbf{j} + M_z^R\mathbf{k} \tag{3.56}$$

The components R_x, R_y, R_z represent, respectively, the sums of the x, y, and z components of the given forces and measure the tendency of the system to impart to the rigid body a motion of translation in the x, y, or z direction. Similarly, the components M_x^R, M_y^R, M_z^R represent, respectively, the sum of the moments of the given forces about the x, y, and z axes and measure the tendency of the system to impart to the rigid body a motion of rotation about the x, y, or z axis.

If the magnitude and direction of the force $\mathbf{R}$ are desired, they can be obtained from the components R_x, R_y, R_z by means of the relations (2.18) and (2.19) of Sec. 2.12; similar computations will yield the magnitude and direction of the couple vector $\mathbf{M}_O^R$.

3.18 EQUIVALENT SYSTEMS OF FORCES

We saw in the preceding section that any system of forces acting on a rigid body can be reduced to a force-couple system at a given point O. This equivalent force-couple system characterizes completely the effect of the given force system on the rigid body. *Two systems of forces are equivalent, therefore, if they can be reduced to the same force-couple system at a given point O.* Recalling that the force-couple system at O is defined by the relations (3.52), we state that *two systems of forces*, $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, . . . , and $\mathbf{F}'_1$, $\mathbf{F}'_2$, $\mathbf{F}'_3$, . . . , *which act on the same rigid body are equivalent if, and only if, the sums of the forces and the sums of the moments about a given point O of the forces of the two systems are, respectively, equal*. Expressed mathematically, the necessary and sufficient conditions for the two systems of forces to be equivalent are

$$\Sigma\mathbf{F} = \Sigma\mathbf{F}' \qquad \text{and} \qquad \Sigma\mathbf{M}_O = \Sigma\mathbf{M}'_O \tag{3.57}$$

Note that to prove that two systems of forces are equivalent, the second of the relations (3.57) must be established with respect to *only one point O*. It will hold, however, with respect to *any point* if the two systems are equivalent.

Resolving the forces and moments in (3.57) into their rectangular components, we can express the necessary and sufficient conditions for the equivalence of two systems of forces acting on a rigid body as follows:

$$\begin{aligned} \Sigma F_x &= \Sigma F'_x & \Sigma F_y &= \Sigma F'_y & \Sigma F_z &= \Sigma F'_z \\ \Sigma M_x &= \Sigma M'_x & \Sigma M_y &= \Sigma M'_y & \Sigma M_z &= \Sigma M'_z \end{aligned} \tag{3.58}$$

These equations have a simple physical significance. They express that two systems of forces are equivalent if they tend to impart to the rigid body (1) the same translation in the x, y, and z directions, respectively, and (2) the same rotation about the x, y, and z axes, respectively.

3.19 EQUIPOLLENT SYSTEMS OF VECTORS

In general, when two systems of vectors satisfy Eqs. (3.57) or (3.58), i.e., when their resultants and their moment resultants about an arbitrary point O are respectively equal, the two systems are said to be *equipollent*. The result established in the preceding section can thus be restated as follows: *If two systems of forces acting on a rigid body are equipollent, they are also equivalent.*

It is important to note that this statement does not apply to *any* system of vectors. Consider, for example, a system of forces acting on a set of independent particles which do *not* form a rigid body. A different system of forces acting on the same particles may happen to be equipollent to the first one; i.e., it may have the same resultant and the same moment resultant. Yet, since different forces will now act on the various particles, their effects on these particles will be different; the two systems of forces, while equipollent, are *not equivalent*.

3.20 FURTHER REDUCTION OF A SYSTEM OF FORCES

We saw in Sec. 3.17 that any given system of forces acting on a rigid body can be reduced to an equivalent force-couple system at O consisting of a force $\mathbf{R}$ equal to the sum of the forces of the system and a couple vector $\mathbf{M}_O^R$ of moment equal to the moment resultant of the system.

When $\mathbf{R} = 0$, the force-couple system reduces to the couple vector $\mathbf{M}_O^R$. The given system of forces can then be reduced to a single couple, called the *resultant couple* of the system.

Let us now investigate the conditions under which a given system of forces can be reduced to a single force. It follows from Sec. 3.16 that the force-couple system at O can be replaced by a single force $\mathbf{R}$ acting along a new line of action if $\mathbf{R}$ and $\mathbf{M}_O^R$ are mutually perpendicular. The systems of forces which can be reduced to a single force, or *resultant*, are therefore the systems for which the force $\mathbf{R}$ and the couple vector $\mathbf{M}_O^R$ are mutually perpendicular. While this condition *is generally not satisfied* by systems of forces in space, it *will be satisfied* by systems consisting of (1) concurrent forces, (2) coplanar forces, or (3) parallel forces. These three cases will be discussed separately.

1. *Concurrent forces* are applied at the same point and can therefore be added directly to obtain their resultant $\mathbf{R}$. Thus, they

always reduce to a single force. Concurrent forces were discussed in detail in Chap. 2.

2. *Coplanar forces* act in the same plane, which may be assumed to be the plane of the figure (Fig. 3.43*a*). The sum **R** of the forces of the system will also lie in the plane of the figure, while the moment of each force about O, and thus the moment resultant $\mathbf{M}_O^R$, will be perpendicular to that plane. The force-couple system at O consists, therefore, of a force **R** and a couple vector $\mathbf{M}_O^R$ which are mutually perpendicular (Fig. 3.43*b*).† They can be reduced to a single force **R** by moving **R** in the plane of the figure until its moment about O becomes equal to $\mathbf{M}_O^R$. The distance from O to the line of action of **R** is $d = M_O^R/R$ (Fig. 3.43*c*).

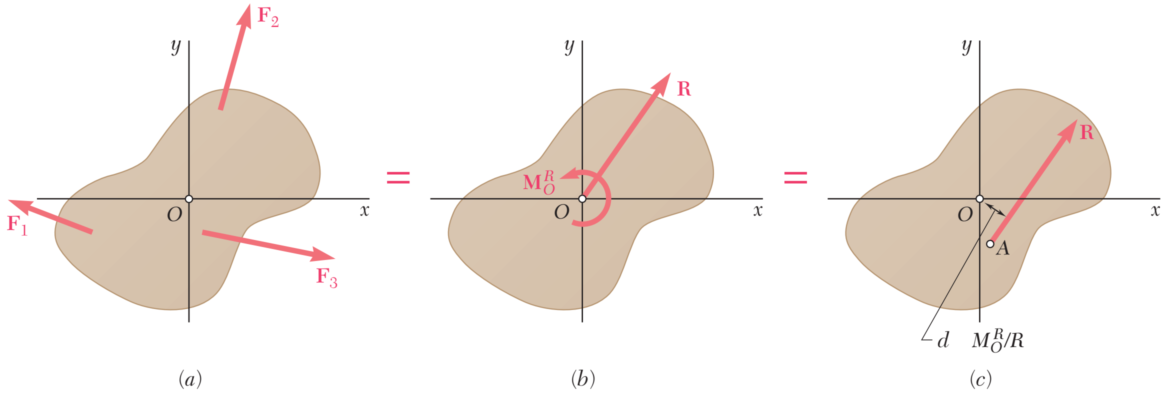


Fig. 3.43

As noted in Sec. 3.17, the reduction of a system of forces is considerably simplified if the forces are resolved into rectangular components. The force-couple system at O is then characterized by the components (Fig. 3.44*a*)

$$R_x = \Sigma F_x \qquad R_y = \Sigma F_y \qquad M_z^R = M_O^R = \Sigma M_O \tag{3.59}$$

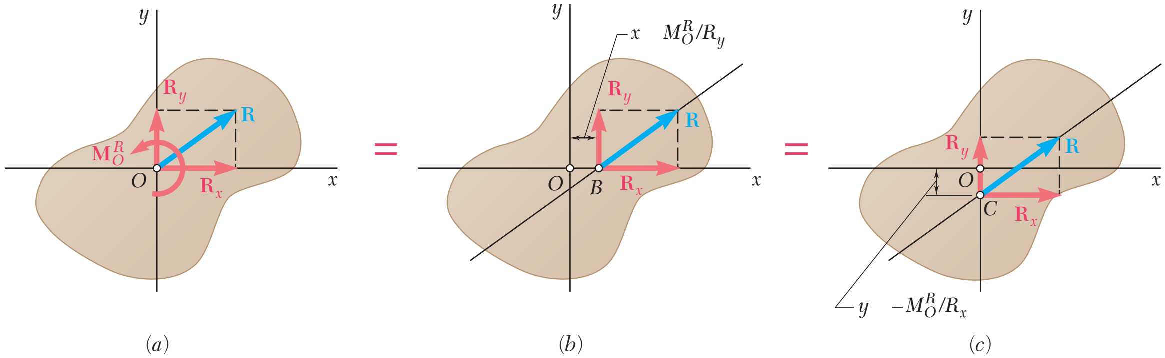


Fig. 3.44

†Since the couple vector $\mathbf{M}_O^R$ is perpendicular to the plane of the figure, it has been represented by the symbol ↺. A counterclockwise couple ↺ represents a vector pointing out of the paper, and a clockwise couple ↻ represents a vector pointing into the paper.

To reduce the system to a single force **R**, we express that the moment of **R** about O must be equal to $\mathbf{M}_O^R$. Denoting by x and y the coordinates of the point of application of the resultant and recalling formula (3.22) of Sec. 3.8, we write

$$xR_y - yR_x = M_O^R$$

which represents the equation of the line of action of **R**. We can also determine directly the x and y intercepts of the line of action of the resultant by noting that $\mathbf{M}_O^R$ must be equal to the moment about O of the y component of **R** when **R** is attached at B (Fig. 3.44*b*) and to the moment of its x component when **R** is attached at C (Fig. 3.44*c*).

3. *Parallel forces* have parallel lines of action and may or may not have the same sense. Assuming here that the forces are parallel to the y axis (Fig. 3.45*a*), we note that their sum **R** will also be parallel to the y axis. On the other hand, since the moment of a given force must be perpendicular to that force, the moment about O of each force of the system, and thus the moment resultant $\mathbf{M}_O^R$, will lie in the zx plane. The force-couple system at O consists, therefore,

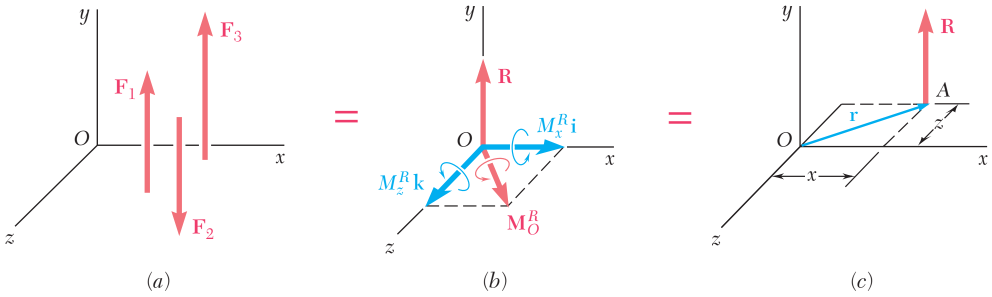


Fig. 3.45

of a force **R** and a couple vector $\mathbf{M}_O^R$ which are mutually perpendicular (Fig. 3.45*b*). They can be reduced to a single force **R** (Fig. 3.45*c*) or, if $\mathbf{R} = 0$, to a single couple of moment $\mathbf{M}_O^R$.

In practice, the force-couple system at O will be characterized by the components

$$R_y = \Sigma F_y \qquad M_x^R = \Sigma M_x \qquad M_z^R = \Sigma M_z \tag{3.60}$$

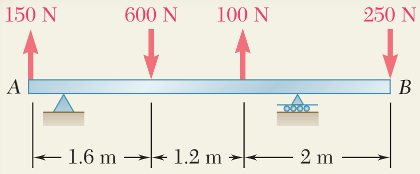
The reduction of the system to a single force can be carried out by moving **R** to a new point of application $A(x, 0, z)$ chosen so that the moment of **R** about O is equal to $\mathbf{M}_O^R$. We write

$$\mathbf{r} \times \mathbf{R} = \mathbf{M}_O^R$$
$$(x\mathbf{i} + z\mathbf{k}) \times R_y\mathbf{j} = M_x^R\mathbf{i} + M_z^R\mathbf{k}$$

By computing the vector products and equating the coefficients of the corresponding unit vectors in both members of the equation, we obtain two scalar equations which define the coordinates of A:

$$-zR_y = M_x^R \qquad xR_y = M_z^R$$

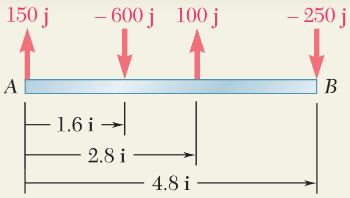
These equations express that the moments of **R** about the x and z axes must, respectively, be equal to M_x^R and M_z^R.

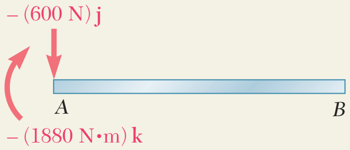


SAMPLE PROBLEM 3.8

A 4.80-m-long beam is subjected to the forces shown. Reduce the given system of forces to (*a*) an equivalent force-couple system at *A*, (*b*) an equivalent force-couple system at *B*, (*c*) a single force or resultant.

Note. Since the reactions at the supports are not included in the given system of forces, the given system will not maintain the beam in equilibrium.





SOLUTION

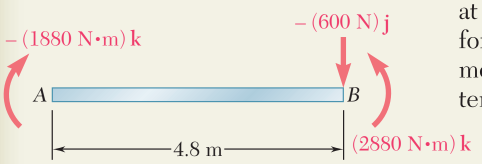
***a.* Force-Couple System at *A*.** The force-couple system at *A* equivalent to the given system of forces consists of a force **R** and a couple $\mathbf{M}_A^R$ defined as follows:

$$\begin{aligned}
\mathbf{R} &= \Sigma\mathbf{F} \\
&= (150\ \mathrm{N})\mathbf{j} - (600\ \mathrm{N})\mathbf{j} + (100\ \mathrm{N})\mathbf{j} - (250\ \mathrm{N})\mathbf{j} = -(600\ \mathrm{N})\mathbf{j} \\
\mathbf{M}_A^R &= \Sigma(\mathbf{r} \times \mathbf{F}) \\
&= (1.6\mathbf{i}) \times (-600\mathbf{j}) + (2.8\mathbf{i}) \times (100\mathbf{j}) + (4.8\mathbf{i}) \times (-250\mathbf{j}) \\
&= -(1880\ \mathrm{N}\cdot\mathrm{m})\mathbf{k}
\end{aligned}$$

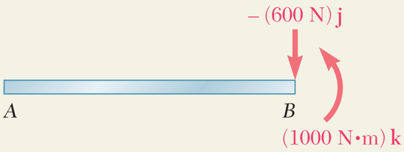
The equivalent force-couple system at *A* is thus

$$\mathbf{R} = 600\ \mathrm{N}\downarrow \qquad \mathbf{M}_A^R = 1880\ \mathrm{N}\cdot\mathrm{m}\ \circlearrowright \quad \blacktriangleleft$$

***b.* Force-Couple System at *B*.** We propose to find a force-couple system at *B* equivalent to the force-couple system at *A* determined in part *a*. The force **R** is unchanged, but a new couple $\mathbf{M}_B^R$ must be determined, the moment of which is equal to the moment about *B* of the force-couple system determined in part *a*. Thus, we have



$$\begin{aligned}
\mathbf{M}_B^R &= \mathbf{M}_A^R + \overrightarrow{BA} \times \mathbf{R} \\
&= -(1880\ \mathrm{N}\cdot\mathrm{m})\mathbf{k} + (-4.8\ \mathrm{m})\mathbf{i} \times (-600\ \mathrm{N})\mathbf{j} \\
&= -(1880\ \mathrm{N}\cdot\mathrm{m})\mathbf{k} + (2880\ \mathrm{N}\cdot\mathrm{m})\mathbf{k} = +(1000\ \mathrm{N}\cdot\mathrm{m})\mathbf{k}
\end{aligned}$$

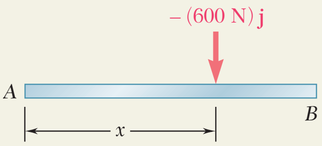


The equivalent force-couple system at *B* is thus

$$\mathbf{R} = 600\ \mathrm{N}\downarrow \qquad \mathbf{M}_B^R = 1000\ \mathrm{N}\cdot\mathrm{m}\ \circlearrowleft \quad \blacktriangleleft$$

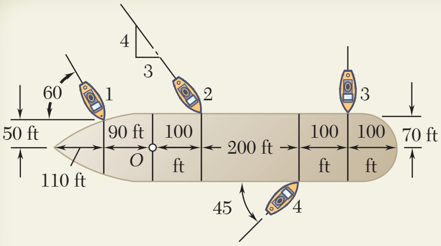
***c.* Single Force or Resultant.** The resultant of the given system of forces is equal to **R**, and its point of application must be such that the moment of **R** about *A* is equal to $\mathbf{M}_A^R$. We write

$$\begin{aligned}
\mathbf{r} \times \mathbf{R} &= \mathbf{M}_A^R \\
x\mathbf{i} \times (-600\ \mathrm{N})\mathbf{j} &= -(1880\ \mathrm{N}\cdot\mathrm{m})\mathbf{k} \\
-x(600\ \mathrm{N})\mathbf{k} &= -(1880\ \mathrm{N}\cdot\mathrm{m})\mathbf{k}
\end{aligned}$$



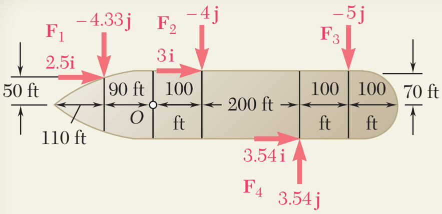
and conclude that $x = 3.13$ m. Thus, the single force equivalent to the given system is defined as

$$\mathbf{R} = 600\ \mathrm{N}\downarrow \qquad x = 3.13\ \mathrm{m} \quad \blacktriangleleft$$



SAMPLE PROBLEM 3.9

Four tugboats are used to bring an ocean liner to its pier. Each tugboat exerts a 5000-lb force in the direction shown. Determine (*a*) the equivalent force-couple system at the foremast *O*, (*b*) the point on the hull where a single, more powerful tugboat should push to produce the same effect as the original four tugboats.

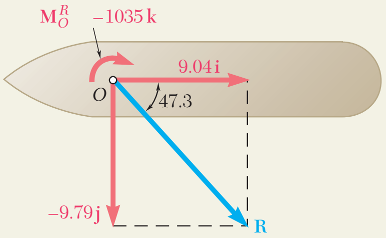


SOLUTION

a. Force-Couple System at O. Each of the given forces is resolved into components in the diagram shown (kip units are used). The force-couple system at *O* equivalent to the given system of forces consists of a force **R** and a couple $\mathbf{M}_O^R$ defined as follows:

$$\begin{aligned}\mathbf{R} &= \Sigma\mathbf{F}\\ &= (2.50\mathbf{i} - 4.33\mathbf{j}) + (3.00\mathbf{i} - 4.00\mathbf{j}) + (-5.00\mathbf{j}) + (3.54\mathbf{i} + 3.54\mathbf{j})\\ &= 9.04\mathbf{i} - 9.79\mathbf{j}\end{aligned}$$

$$\begin{aligned}\mathbf{M}_O^R &= \Sigma(\mathbf{r} \times \mathbf{F})\\ &= (-90\mathbf{i} + 50\mathbf{j}) \times (2.50\mathbf{i} - 4.33\mathbf{j})\\ &\quad + (100\mathbf{i} + 70\mathbf{j}) \times (3.00\mathbf{i} - 4.00\mathbf{j})\\ &\quad + (400\mathbf{i} + 70\mathbf{j}) \times (-5.00\mathbf{j})\\ &\quad + (300\mathbf{i} - 70\mathbf{j}) \times (3.54\mathbf{i} + 3.54\mathbf{j})\\ &= (390 - 125 - 400 - 210 - 2000 + 1062 + 248)\mathbf{k}\\ &= -1035\mathbf{k}\end{aligned}$$

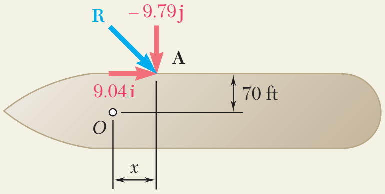


The equivalent force-couple system at *O* is thus

$$\mathbf{R} = (9.04 \text{ kips})\mathbf{i} - (9.79 \text{ kips})\mathbf{j} \qquad \mathbf{M}_O^R = -(1035 \text{ kip} \cdot \text{ft})\mathbf{k}$$

or $\quad \mathbf{R} = 13.33 \text{ kips} \measuredangle 47.3° \qquad \mathbf{M}_O^R = 1035 \text{ kip} \cdot \text{ft} \curvearrowright$ ◀

Remark. Since all the forces are contained in the plane of the figure, we could have expected the sum of their moments to be perpendicular to that plane. Note that the moment of each force component could have been obtained directly from the diagram by first forming the product of its magnitude and perpendicular distance to *O* and then assigning to this product a positive or a negative sign depending upon the sense of the moment.



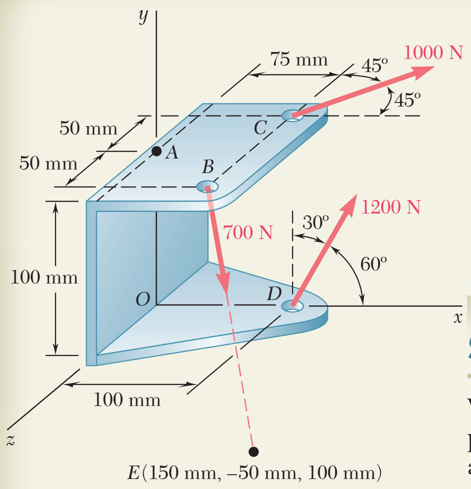
b. Single Tugboat. The force exerted by a single tugboat must be equal to **R**, and its point of application *A* must be such that the moment of **R** about *O* is equal to $\mathbf{M}_O^R$. Observing that the position vector of *A* is

$$\mathbf{r} = x\mathbf{i} + 70\mathbf{j}$$

we write

$$\begin{aligned}\mathbf{r} \times \mathbf{R} &= \mathbf{M}_O^R\\ (x\mathbf{i} + 70\mathbf{j}) \times (9.04\mathbf{i} - 9.79\mathbf{j}) &= -1035\mathbf{k}\\ -x(9.79)\mathbf{k} - 633\mathbf{k} &= -1035\mathbf{k} \qquad x = 41.1 \text{ ft} \; ◀\end{aligned}$$

SAMPLE PROBLEM 3.10



Three cables are attached to a bracket as shown. Replace the forces exerted by the cables with an equivalent force-couple system at A.

SOLUTION

We first determine the relative position vectors drawn from point A to the points of application of the various forces and resolve the forces into rectangular components. Observing that $\mathbf{F}_B = (700\text{ N})\boldsymbol{\lambda}_{BE}$ where

$$\boldsymbol{\lambda}_{BE} = \frac{\overrightarrow{BE}}{BE} = \frac{75\mathbf{i} - 150\mathbf{j} + 50\mathbf{k}}{175}$$

we have, using meters and newtons,

$$\begin{aligned}
\mathbf{r}_{B/A} &= \overrightarrow{AB} = 0.075\mathbf{i} + 0.050\mathbf{k} & \mathbf{F}_B &= 300\mathbf{i} - 600\mathbf{j} + 200\mathbf{k}\\
\mathbf{r}_{C/A} &= \overrightarrow{AC} = 0.075\mathbf{i} - 0.050\mathbf{k} & \mathbf{F}_C &= 707\mathbf{i} \qquad\qquad - 707\mathbf{k}\\
\mathbf{r}_{D/A} &= \overrightarrow{AD} = 0.100\mathbf{i} - 0.100\mathbf{j} & \mathbf{F}_D &= 600\mathbf{i} + 1039\mathbf{j}
\end{aligned}$$

The force-couple system at A equivalent to the given forces consists of a force $\mathbf{R} = \Sigma\mathbf{F}$ and a couple $\mathbf{M}_A^R = \Sigma(\mathbf{r} \times \mathbf{F})$. The force $\mathbf{R}$ is readily obtained by adding respectively the x, y, and z components of the forces:

$$\mathbf{R} = \Sigma\mathbf{F} = (1607\text{ N})\mathbf{i} + (439\text{ N})\mathbf{j} - (507\text{ N})\mathbf{k} \quad ◀$$

The computation of $\mathbf{M}_A^R$ will be facilitated if we express the moments of the forces in the form of determinants (Sec. 3.8):

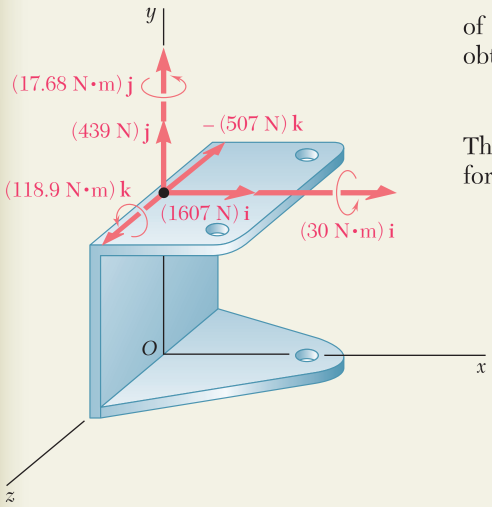
$$\mathbf{r}_{B/A} \times \mathbf{F}_B = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.075 & 0 & 0.050 \\ 300 & -600 & 200 \end{vmatrix} = 30\mathbf{i} \qquad\qquad -45\mathbf{k}$$

$$\mathbf{r}_{C/A} \times \mathbf{F}_C = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.075 & 0 & -0.050 \\ 707 & 0 & -707 \end{vmatrix} = \qquad 17.68\mathbf{j}$$

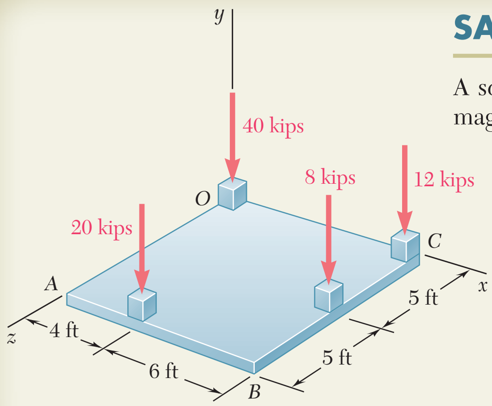
$$\mathbf{r}_{D/A} \times \mathbf{F}_D = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.100 & -0.100 & 0 \\ 600 & 1039 & 0 \end{vmatrix} = \qquad\qquad 163.9\mathbf{k}$$

Adding the expressions obtained, we have

$$\mathbf{M}_A^R = \Sigma(\mathbf{r} \times \mathbf{F}) = (30\text{ N}\cdot\text{m})\mathbf{i} + (17.68\text{ N}\cdot\text{m})\mathbf{j} + (118.9\text{ N}\cdot\text{m})\mathbf{k} \quad ◀$$



The rectangular components of the force $\mathbf{R}$ and the couple $\mathbf{M}_A^R$ are shown in the adjoining sketch.



SAMPLE PROBLEM 3.11

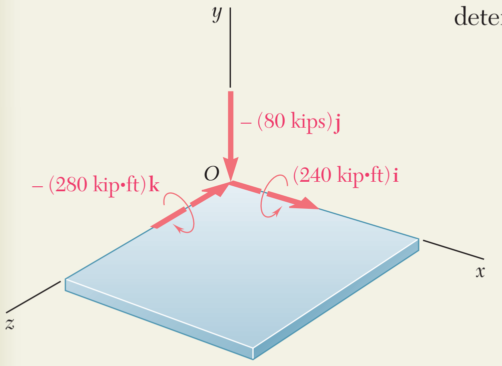
A square foundation mat supports the four columns shown. Determine the magnitude and point of application of the resultant of the four loads.

SOLUTION

We first reduce the given system of forces to a force-couple system at the origin O of the coordinate system. This force-couple system consists of a force $\mathbf{R}$ and a couple vector $\mathbf{M}_O^R$ defined as follows:

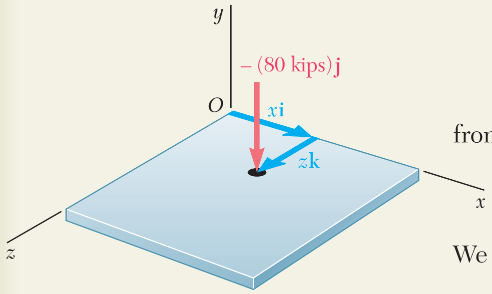
$$\mathbf{R} = \Sigma\mathbf{F} \qquad \mathbf{M}_O^R = \Sigma(\mathbf{r} \times \mathbf{F})$$

The position vectors of the points of application of the various forces are determined, and the computations are arranged in tabular form.



r, ft	F, kips	r × F, kip · ft
0	$-40\mathbf{j}$	0
$10\mathbf{i}$	$-12\mathbf{j}$	$-120\mathbf{k}$
$10\mathbf{i} + 5\mathbf{k}$	$-8\mathbf{j}$	$40\mathbf{i} - 80\mathbf{k}$
$4\mathbf{i} + 10\mathbf{k}$	$-20\mathbf{j}$	$200\mathbf{i} - 80\mathbf{k}$
	$\mathbf{R} = -80\mathbf{j}$	$\mathbf{M}_O^R = 240\mathbf{i} - 280\mathbf{k}$

Since the force $\mathbf{R}$ and the couple vector $\mathbf{M}_O^R$ are mutually perpendicular, the force-couple system obtained can be reduced further to a single force $\mathbf{R}$. The new point of application of $\mathbf{R}$ will be selected in the plane of the mat and in such a way that the moment of $\mathbf{R}$ about O will be equal to $\mathbf{M}_O^R$. Denoting by $\mathbf{r}$ the position vector of the desired point of application, and by x and z its coordinates, we write



$$\mathbf{r} \times \mathbf{R} = \mathbf{M}_O^R$$
$$(x\mathbf{i} + z\mathbf{k}) \times (-80\mathbf{j}) = 240\mathbf{i} - 280\mathbf{k}$$
$$-80x\mathbf{k} + 80z\mathbf{i} = 240\mathbf{i} - 280\mathbf{k}$$

from which it follows that

$$-80x = -280 \qquad 80z = 240$$
$$x = 3.50 \text{ ft} \qquad z = 3.00 \text{ ft}$$

We conclude that the resultant of the given system of forces is

$$\mathbf{R} = 80 \text{ kips} \downarrow \qquad \text{at } x = 3.50 \text{ ft}, z = 3.00 \text{ ft} \quad \blacktriangleleft$$

PROBLEMS

3.73 A 12-ft beam is loaded in the various ways represented in the figure. Find two loadings that are equivalent.

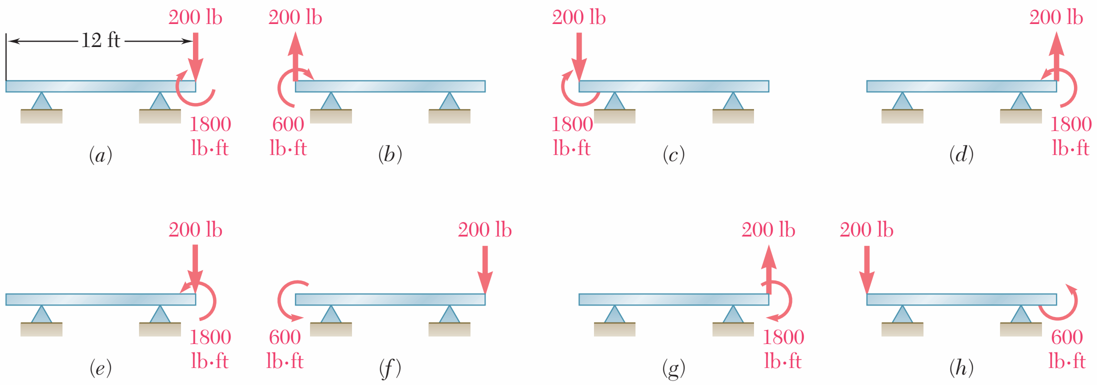


Fig. P3.73

3.74 A 12-ft beam is loaded as shown. Determine the loading of Prob. 3.73 that is equivalent to this loading.

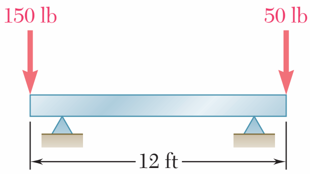


Fig. P3.74

3.75 By driving the truck shown over a scale, it was determined that the loads on the front and rear axles are, respectively, 18 kN and 12 kN when the truck is empty. Determine (*a*) the location of the center of gravity of the truck, (*b*) the weight and location of the center of gravity of the heaviest load that can be carried by the truck if the load on each axle is not to exceed 40 kN.

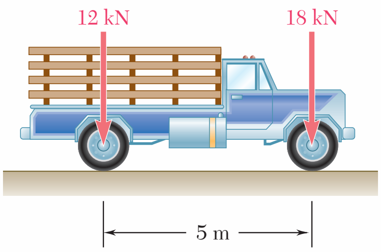


Fig. P3.75

3.76 Four packages are transported at constant speed from *A* to *B* by the conveyor. At the instant shown, determine the resultant of the loading and the location of its line of action.

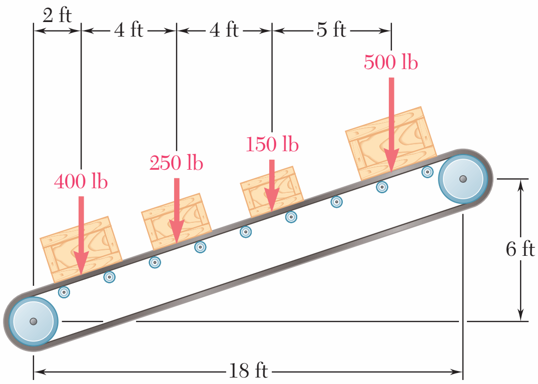


Fig. P3.76

3.77 Determine the distance from point *A* to the line of action of the resultant of the three forces shown when (*a*) $a = 1$ m, (*b*) $a = 1.5$ m, (*c*) $a = 2.5$ m.

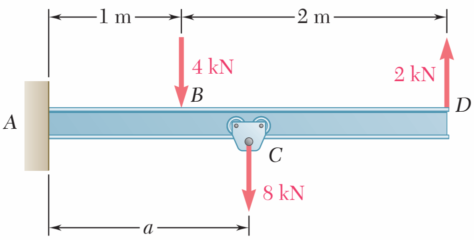


Fig. P3.77

3.78 Two parallel forces **P** and **Q** are applied at the ends of a beam AB of length L. Find the distance x from A to the line of action of their resultant. Check the formula obtained by assuming L = 200 mm and (*a*) P = 50 N down, Q = 150 N down; (*b*) P = 50 N down, Q = 150 N up.

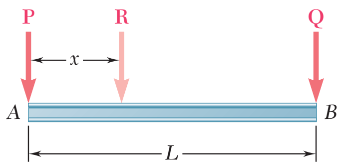


Fig. P3.78

3.79 Three forces act as shown on a traffic-signal pole. Determine (*a*) the equivalent force-couple system at A, (*b*) the resultant of the system and the point of intersection of its line of action with the pole.

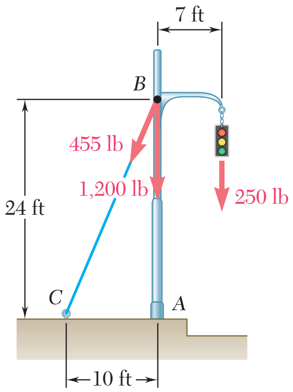


Fig. P3.79

3.80 Four forces act on a 700 × 375 mm plate as shown. (*a*) Find the resultant of these forces. (*b*) Locate the two points where the line of action of the resultant intersects the edge of the plate.

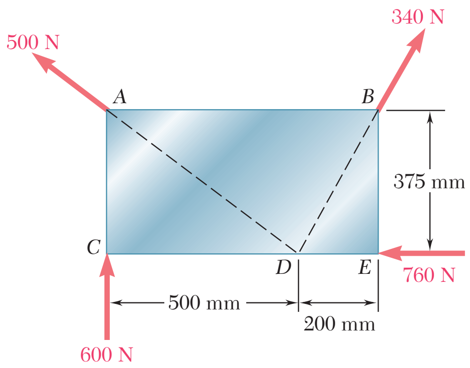


Fig. P3.80

3.81 The three forces shown and a couple of magnitude M = 80 lb · in. are applied to an angle bracket. (*a*) Find the resultant of this system of forces. (*b*) Locate the points where the line of action of the resultant intersects line AB and line BC.

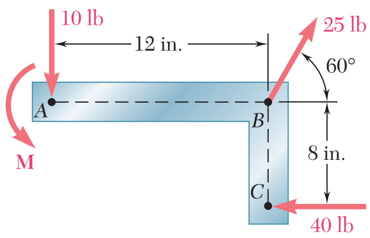


Fig. P3.81

3.82 A bracket is subjected to the system of forces and couples shown. Find the resultant of the system and the point of intersection of its line of action with (*a*) line AB, (*b*) line BC, (*c*) line CD.

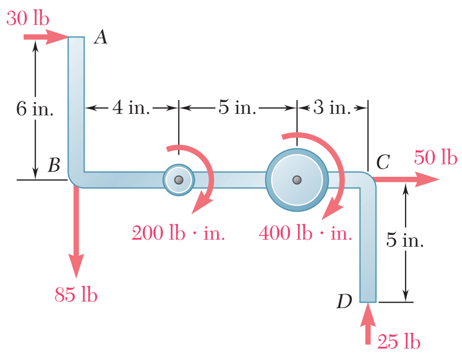


Fig. P3.82

3.83 The roof of a building frame is subjected to the wind loading shown. Determine (*a*) the equivalent force-couple system at *D*, (*b*) the resultant of the loading and its line of action.

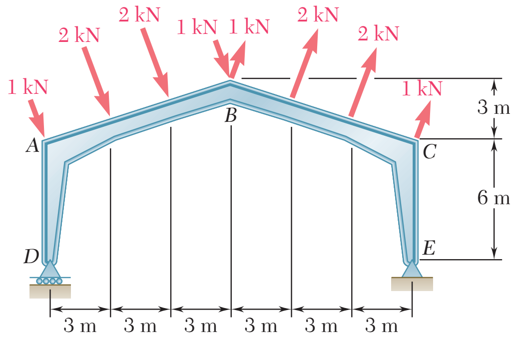


Fig. P3.83

3.84 Two cables exert forces of 90 kN each on a truss of weight W = 200 kN. Find the resultant force acting on the truss and the point of intersection of its line of action with line AB.

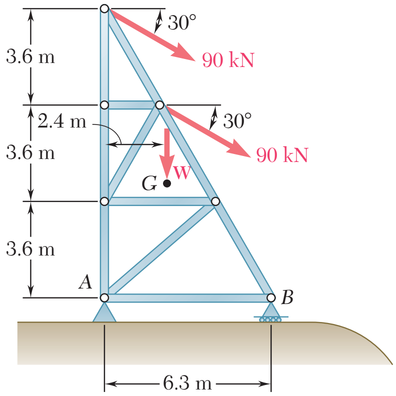


Fig. P3.84

3.85 Two forces are applied to the vertical post as shown. Determine the force and couple at *O* equivalent to the two forces.

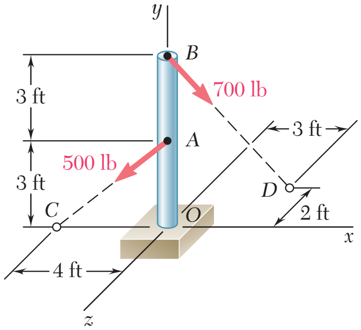


Fig. P3.85

3.86 In order to move a 70.6-kg crate, two men push on it while two other men pull on it by means of ropes. The force exerted by man *A* is 600 N and that exerted by man *B* is 200 N; both forces are horizontal. Man *C* pulls with a force equal to 320 N and man *D* with a force equal to 480 N. Both cables form an angle of 30° with the vertical. Determine the resultant of all the forces acting on the crate.

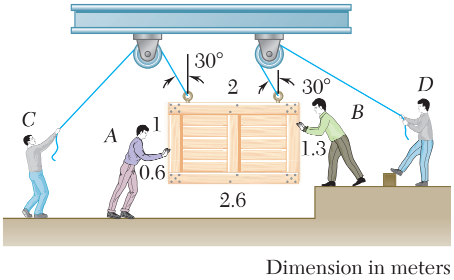


Fig. *P3.86*

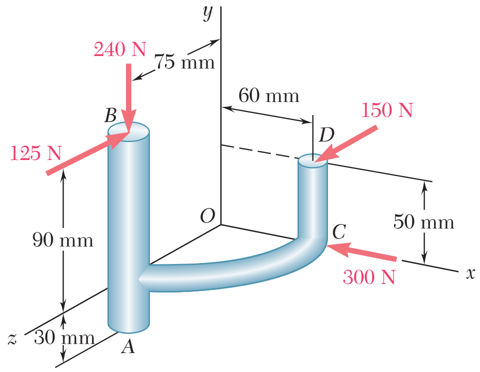


Fig. P3.87

3.87 The machine component is subject to the forces shown, each of which is parallel to one of the coordinate axes. Replace these forces by an equivalent force-couple system at A.

3.88 In drilling a hole in a wall, a man applies a vertical 30-lb force at B on the brace and bit, while pushing at C with a 10-lb force. The brace lies in the horizontal xz plane. (a) Determine the other components of the total force that should be exerted at C if the bit is not to be bent about the y and z axes (i.e., if the system of forces applied on the brace is to have zero moment about both the y and z axes). (b) Reduce the 30-lb force and the total force at C to an equivalent force and couple at A.

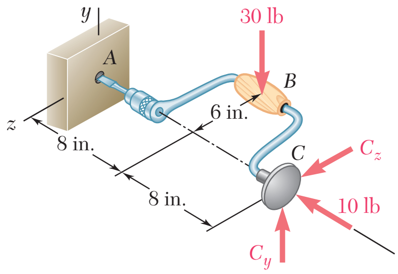


Fig. *P3.88*

3.89 In order to unscrew the tapped faucet A, a plumber uses two pipe wrenches as shown. By exerting a 40-lb force on each wrench, at a distance of 10 in. from the axis of the pipe and in a direction perpendicular to the pipe and to the wrench, the plumber prevents the pipe from rotating, and thus avoids loosening or further tightening the joint between the pipe and the tapped elbow C. Determine (a) the angle θ that the wrench at A should form with the vertical if elbow C is not to rotate about the vertical, (b) the force-couple system at C equivalent to the two 40-lb forces when this condition is satisfied.

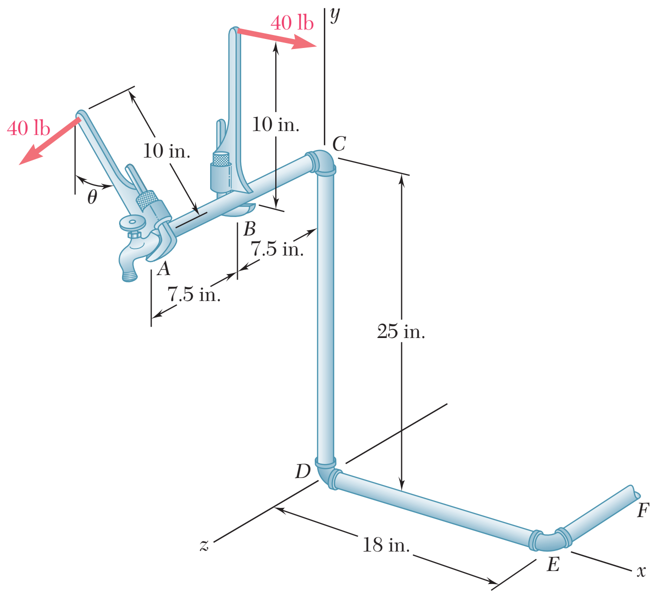


Fig. P3.89

3.90 Assuming $\theta = 60°$ in Prob. 3.89, replace the two 40-lb forces by an equivalent force-couple system at D and determine whether the plumber's action tends to tighten or loosen the joint between (a) pipe CD and elbow D, (b) elbow D and pipe DE. Assume all the threads to be right-handed.

3.91 A rectangular concrete foundation mat supports four column loads as shown. Determine the magnitude and point of application of the resultant of the four loads.

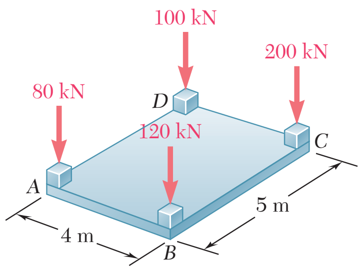


Fig. P3.91

3.92 A concrete foundation mat in the shape of a regular hexagon of 10-ft sides supports four column loads as shown. Determine the magnitude and point of application of the resultant of the four loads.

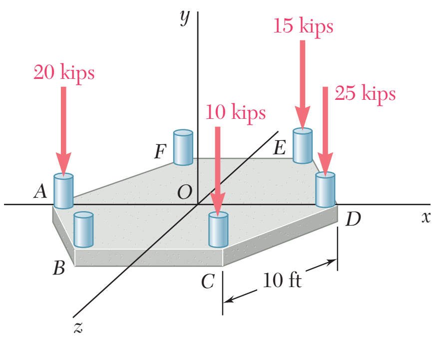


Fig. P3.92 and *P3.93*

3.93 Determine the magnitudes of the additional loads that must be applied at B and F if the resultant of all six loads is to pass through the center of the mat.

3.94 In Prob. 3.91, determine the magnitude and point of application of the smallest additional load that must be applied to the foundation mat if the resultant of the five loads is to pass through the center of the mat.

3.95 Four horizontal forces act on a vertical quarter-circular plate of radius 250 mm. Determine the magnitude and point of application of the resultant of the four forces if $P = 40$ N.

3.96 Determine the magnitude of the force **P** for which the resultant of the four forces acts on the rim of the plate.

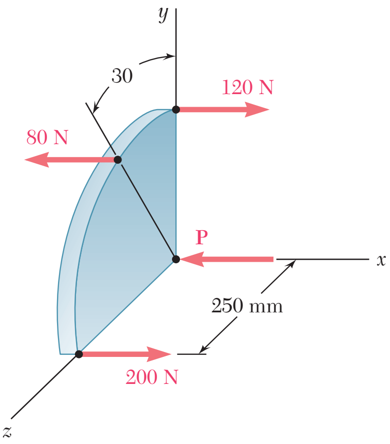


Fig. P3.95 and P3.96

REVIEW AND SUMMARY

Principle of transmissibility

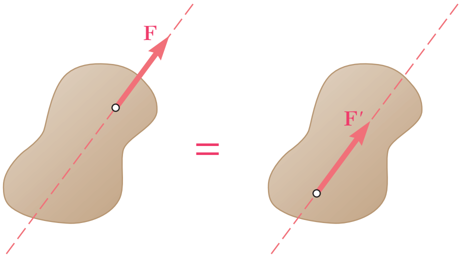


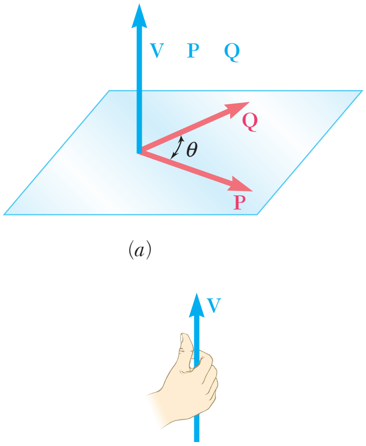
Fig. 3.46

In this chapter we studied the effect of forces exerted on a rigid body. We first learned to distinguish between *external* and *internal* forces [Sec. 3.2] and saw that, according to the *principle of transmissibility*, the effect of an external force on a rigid body remains unchanged if that force is moved along its line of action [Sec. 3.3]. In other words, two forces **F** and **F′** acting on a rigid body at two different points have the same effect on that body if they have the same magnitude, same direction, and same one of action (Fig. 3.46). Two such forces are said to be *equivalent*.

Vector product of two vectors

Before proceeding with the discussion of *equivalent systems of forces*, we introduced the concept of the *vector product of two vectors* [Sec. 3.4]. The vector product

$$\mathbf{V} = \mathbf{P} \times \mathbf{Q}$$



(*a*)

(*b*)

Fig. 3.47

of the vectors **P** and **Q** was defined as a vector perpendicular to the plane containing **P** and **Q** (Fig. 3.47), of magnitude

$$V = PQ \sin \theta \qquad (3.1)$$

and directed in such a way that a person located at the tip of **V** will observe as counterclockwise the rotation through θ which brings the vector **P** in line with the vector **Q**. The three vectors **P**, **Q**, and **V**—taken in that order—are said to form a *right-handed triad*. It follows that the vector products $\mathbf{Q} \times \mathbf{P}$ and $\mathbf{P} \times \mathbf{Q}$ are represented by equal and opposite vectors. We have

$$\mathbf{Q} \times \mathbf{P} = -(\mathbf{P} \times \mathbf{Q}) \qquad (3.4)$$

It also follows from the definition of the vector product of two vectors that the vector products of the unit vectors **i**, **j**, and **k** are

$$\mathbf{i} \times \mathbf{i} = 0 \qquad \mathbf{i} \times \mathbf{j} = \mathbf{k} \qquad \mathbf{j} \times \mathbf{i} = -\mathbf{k}$$

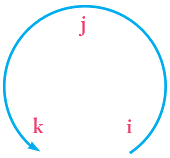


Fig. 3.48

and so on. The sign of the vector product of two unit vectors can be obtained by arranging in a circle and in counterclockwise order the three letters representing the unit vectors (Fig. 3.48): The vector product of two unit vectors will be positive if they follow each other in counterclockwise order and negative if they follow each other in clockwise order.

Rectangular components of vector product

The *rectangular components of the vector product* **V** of two vectors **P** and **Q** were expressed [Sec. 3.5] as

$$\begin{aligned} V_x &= P_y Q_z - P_z Q_y \\ V_y &= P_z Q_x - P_x Q_z \\ V_z &= P_x Q_y - P_y Q_x \end{aligned} \qquad (3.9)$$

Using a determinant, we also wrote

$$\mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix} \tag{3.10}$$

Moment of a force about a point

The *moment of a force* **F** *about a point O* was defined [Sec. 3.6] as the vector product

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} \tag{3.11}$$

where **r** is the *position vector* drawn from O to the point of application A of the force **F** (Fig. 3.49). Denoting by θ the angle between the lines of action of **r** and **F**, we found that the magnitude of the moment of **F** about O can be expressed as

$$M_O = rF \sin \theta = Fd \tag{3.12}$$

where d represents the perpendicular distance from O to the line of action of **F**.

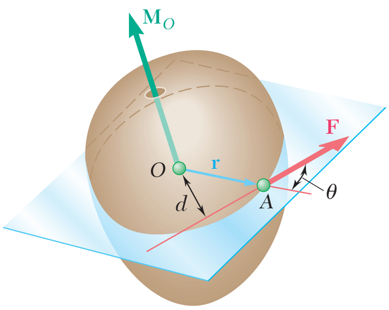


Fig. 3.49

Rectangular components of moment

The *rectangular components of the moment* $\mathbf{M}_O$ *of a force* **F** were expressed [Sec. 3.8] as

$$\begin{aligned} M_x &= yF_z - zF_y \\ M_y &= zF_x - xF_z \\ M_z &= xF_y - yF_x \end{aligned} \tag{3.18}$$

where x, y, z are the components of the position vector **r** (Fig. 3.50). Using a determinant form, we also wrote

$$\mathbf{M}_O = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix} \tag{3.19}$$

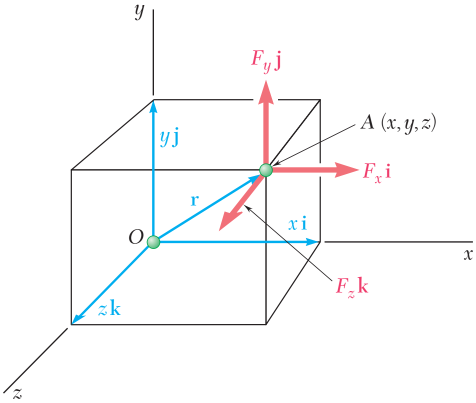


Fig. 3.50

In the more general case of the moment about an arbitrary point B of a force **F** applied at A, we had

$$\mathbf{M}_B = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_{A/B} & y_{A/B} & z_{A/B} \\ F_x & F_y & F_z \end{vmatrix} \tag{3.21}$$

where $x_{A/B}$, $y_{A/B}$, and $z_{A/B}$ denote the components of the vector $\mathbf{r}_{A/B}$:

$$x_{A/B} = x_A - x_B \qquad y_{A/B} = y_A - y_B \qquad z_{A/B} = z_A - z_B$$

In the case of *problems involving only two dimensions*, the force **F** can be assumed to lie in the xy plane. Its moment $\mathbf{M}_B$ about a point B in the same plane is perpendicular to that plane (Fig. 3.51) and is completely defined by the scalar

$$M_B = (x_A - x_B)F_y - (y_A - y_B)F_x \tag{3.23}$$

Various methods for the computation of the moment of a force about a point were illustrated in Sample Probs. 3.1 through 3.4.

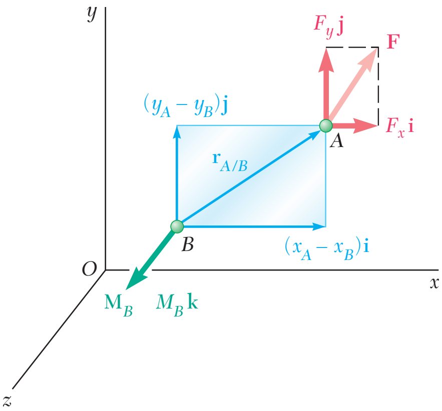


Fig. 3.51

Scalar product of two vectors

The *scalar product* of two vectors **P** and **Q** [Sec. 3.9] was denoted by $\mathbf{P} \cdot \mathbf{Q}$ and was defined as the scalar quantity

$$\mathbf{P} \cdot \mathbf{Q} = PQ \cos \theta \tag{3.24}$$

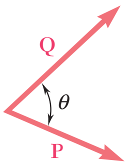


Fig. 3.52

where θ is the angle between **P** and **Q** (Fig. 3.52). By expressing the scalar product of **P** and **Q** in terms of the rectangular components of the two vectors, we determined that

$$\mathbf{P} \cdot \mathbf{Q} = P_xQ_x + P_yQ_y + P_zQ_z \qquad (3.30)$$

Mixed triple product of three vectors

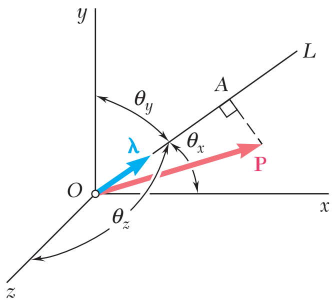


Fig. 3.53

The *projection of a vector* **P** *on an axis OL* (Fig. 3.53) can be obtained by forming the scalar product of **P** and the unit vector $\boldsymbol{\lambda}$ along *OL*. We have

$$P_{OL} = \mathbf{P} \cdot \boldsymbol{\lambda} \qquad (3.36)$$

or, using rectangular components,

$$P_{OL} = P_x \cos\theta_x + P_y \cos\theta_y + P_z \cos\theta_z \qquad (3.37)$$

where θ_x, θ_y, and θ_z denote the angles that the axis *OL* forms with the coordinate axes.

Mixed triple product of three vectors

The *mixed triple product* of the three vectors **S**, **P**, and **Q** was defined as the scalar expression

$$\mathbf{S} \cdot (\mathbf{P} \times \mathbf{Q}) \qquad (3.38)$$

obtained by forming the scalar product of **S** with the vector product of **P** and **Q** [Sec. 3.10]. It was shown that

$$\mathbf{S} \cdot (\mathbf{P} \times \mathbf{Q}) = \begin{vmatrix} S_x & S_y & S_z \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix} \qquad (3.41)$$

where the elements of the determinant are the rectangular components of the three vectors.

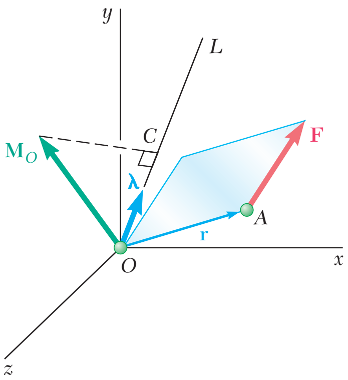


Fig. 3.54

Moment of a force about an axis

The *moment of a force* **F** *about an axis OL* [Sec. 3.11] was defined as the projection *OC* on *OL* of the moment $\mathbf{M}_O$ of the force **F** (Fig. 3.54), i.e., as the mixed triple product of the unit vector $\boldsymbol{\lambda}$, the position vector **r**, and the force **F**:

$$M_{OL} = \boldsymbol{\lambda} \cdot \mathbf{M}_O = \boldsymbol{\lambda} \cdot (\mathbf{r} \times \mathbf{F}) \qquad (3.42)$$

Using the determinant form for the mixed triple product, we have

$$M_{OL} = \begin{vmatrix} \lambda_x & \lambda_y & \lambda_z \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix} \qquad (3.43)$$

where $\lambda_x, \lambda_y, \lambda_z$ = direction cosines of axis *OL*
x, y, z = components of **r**
F_x, F_y, F_z = components of **F**

An example of the determination of the moment of a force about a skew axis was given in Sample Prob. 3.5.

Couples

Two forces **F** *and* −**F** *having the same magnitude, parallel lines of action, and opposite sense are said to form a couple* [Sec. 3.12]. It was shown that the moment of a couple is independent of the point about which it is computed; it is a vector **M** perpendicular to the plane of the couple and equal in magnitude to the product of the common magnitude F of the forces and the perpendicular distance d between their lines of action (Fig. 3.55).

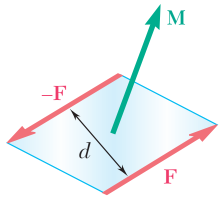


Fig. 3.55

Two couples having the same moment **M** are *equivalent*, i.e., they have the same effect on a given rigid body [Sec. 3.13]. The sum of two couples is itself a couple [Sec. 3.14], and the moment **M** of the resultant couple can be obtained by adding vectorially the moments $\mathbf{M}_1$ and $\mathbf{M}_2$ of the original couples [Sample Prob. 3.6]. It follows that a couple can be represented by a vector, called a *couple vector*, equal in magnitude and direction to the moment **M** of the couple [Sec. 3.15]. A couple vector is a *free vector* which can be attached to the origin O if so desired and resolved into components (Fig. 3.56).

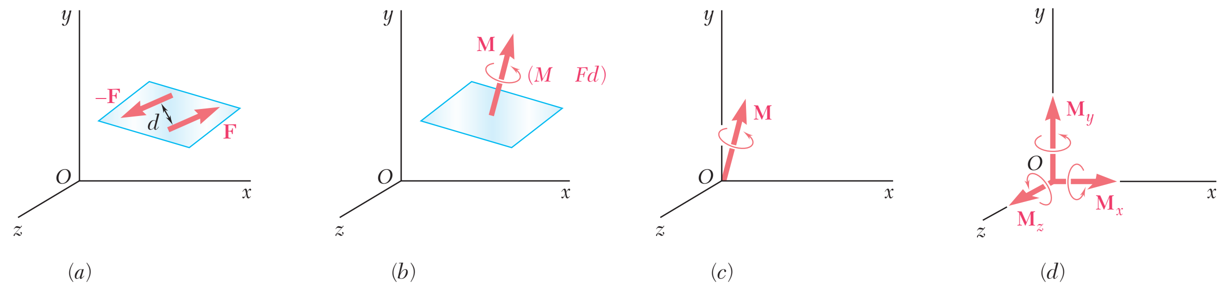


Fig. 3.56

Force-couple system

Any force **F** acting at a point A of a rigid body can be replaced by a *force-couple system* at an arbitrary point O, consisting of the force **F** applied at O and a couple of moment $\mathbf{M}_O$ equal to the moment about O of the force **F** in its original position [Sec. 3.16]; it should be noted that the force **F** and the couple vector $\mathbf{M}_O$ are always perpendicular to each other (Fig. 3.57).

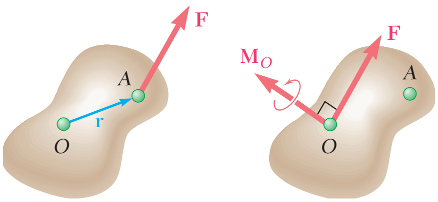


Fig. 3.57

Reduction of a system of forces to a force-couple system

It follows [Sec. 3.17] that *any system of forces can be reduced to a force-couple system at a given point* O by first replacing each of the forces of the system by an equivalent force-couple system at O

(Fig. 3.58) and then adding all the forces and all the couples determined in this manner to obtain a resultant force **R** and a resultant couple vector $\mathbf{M}_O^R$ [Sample Probs. 3.8 through 3.11]. Note that, in general, the resultant **R** and the couple vector $\mathbf{M}_O^R$ will not be perpendicular to each other.

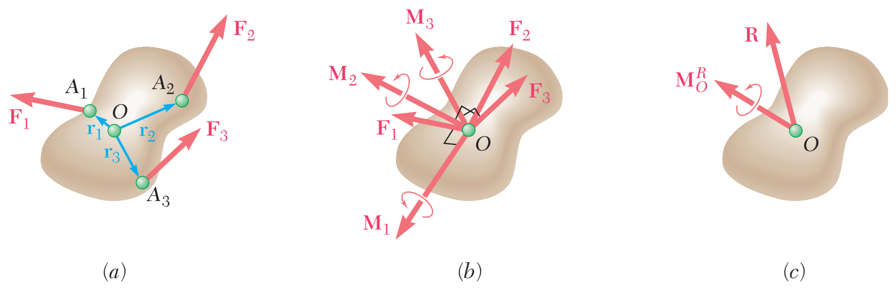


Fig. 3.58

Equivalent systems of forces

We concluded from the above [Sec. 3.18] that, as far as rigid bodies are concerned, *two systems of forces,* $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, . . . *and* $\mathbf{F}'_1$, $\mathbf{F}'_2$, $\mathbf{F}'_3$, . . . , *are equivalent if, and only if,*

$$\Sigma\mathbf{F} = \Sigma\mathbf{F}' \qquad \text{and} \qquad \Sigma\mathbf{M}_O = \Sigma\mathbf{M}'_O \tag{3.57}$$

Further reduction of a system of forces

If the resultant force **R** and the resultant couple vector $\mathbf{M}_O^R$ are perpendicular to each other, the force-couple system at O can be further reduced to a single resultant force [Sec. 3.20]. This will be the case for systems consisting either of (*a*) concurrent forces (cf. Chap. 2), (*b*) coplanar forces [Sample Probs. 3.8 and 3.9], or (*c*) parallel forces [Sample Prob. 3.11]. If the resultant **R** and the couple vector $\mathbf{M}_O^R$ are *not* perpendicular to each other, the system *cannot* be reduced to a single force.

REVIEW PROBLEMS

3.97 A force **P** of magnitude 520 lb acts on the frame shown at point E. Determine the moment of **P** (*a*) about point D, (*b*) about a line joining points O and D.

3.98 A force **P** acts on the frame shown at point E. Knowing that the absolute value of the moment of **P** about a line joining points F and B is 300 lb · ft, determine the magnitude of the force **P**.

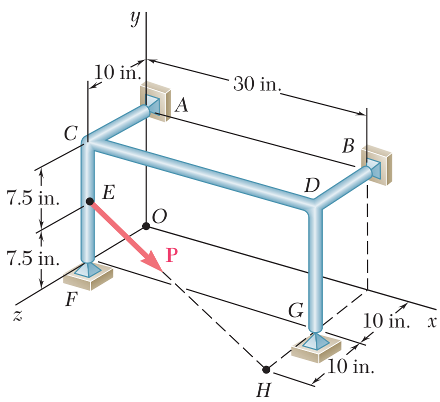


Fig. P3.97 and *P3.98*

3.99 A crane is oriented so that the end of the 25-m boom AO lies in the yz plane. At the instant shown the tension in cable AB is 4 kN. Determine the moment about each of the coordinate axes of the force exerted on A by cable AB.

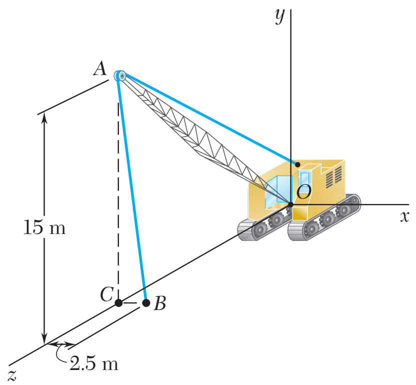


Fig. P3.99 and *P3.100*

3.100 The 25-m crane boom AO lies in the yz plane. Determine the maximum permissible tension in cable AB if the absolute value of the moments about the coordinate axes of the force exerted on A by cable AB must be as follows: $|M_x| \leq 60$ kN · m, $|M_y| \leq 12$ kN · m, and $|M_z| \leq 8$ kN · m.

3.101 A single force **P** acts at C in a direction perpendicular to the handle BC of the crank shown. Determine the moment M_x of **P** about the x axis when $\theta = 65°$ knowing that $M_y = -15$ N · m and $M_z = -36$ N · m.

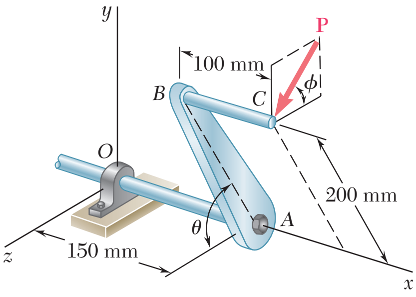


Fig. P3.101

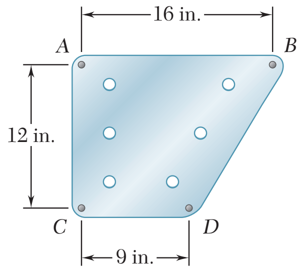


Fig. P3.102

3.102 A multiple-drilling machine is used to drill simultaneously six holes in the steel plate shown. Each drill exerts a clockwise couple of magnitude 40 lb · in. on the plate. Determine an equivalent couple formed by the smallest possible forces acting (*a*) at *A* and *C*, (*b*) at *A* and *D*, (*c*) on the plate.

3.103 A 500-N force is applied to a bent plate as shown. Determine (*a*) an equivalent force-couple system at *B*, (*b*) an equivalent system formed by a vertical force at *A* and a force at *B*.

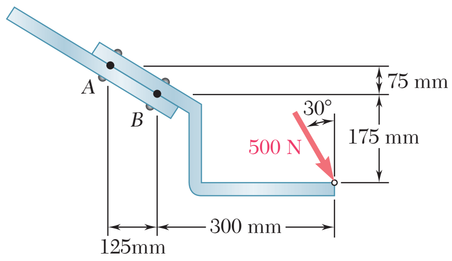


Fig. P3.103

3.104 A 100-kN load is applied eccentrically to the column shown. Determine the components of the force and couple at *G* that are equivalent to the 100-kN load.

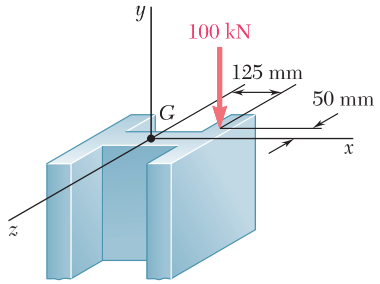


Fig. *P3.104*

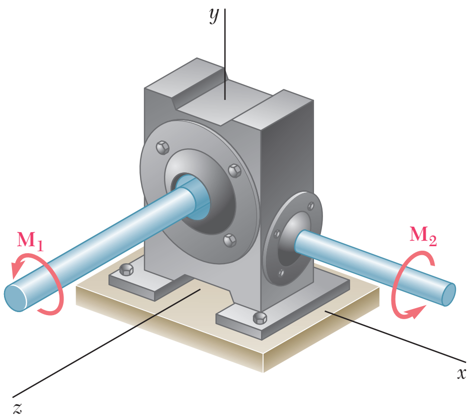


Fig. P3.105

3.105 The speed-reducer unit shown weighs 75 lb, and its center of gravity is located on the *y* axis. Show that the weight of the unit and the two couples acting on it, of magnitude $M_1 = 20$ lb · ft and $M_2 = 4$ lb · ft, respectively, can be replaced by a single equivalent force and determine (*a*) the magnitude and direction of that force, (*b*) the point where its line of action intersects the floor.

3.106 For the truss and loading shown, determine the resultant of the loads and the distance from point *A* to its line of action.

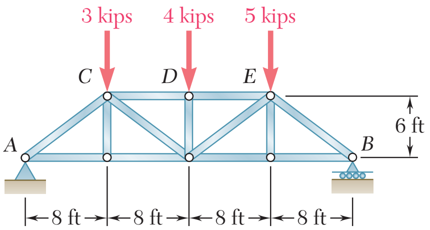


Fig. P3.106

3.107 A force **P** of given magnitude P is applied to the edge of a semicircular plate of radius a as shown. (a) Replace **P** by an equivalent force-couple system at point D obtained by drawing the perpendicular from B to the x axis. (b) Determine the value of θ for which the moment of the equivalent force-couple system at D is maximum.

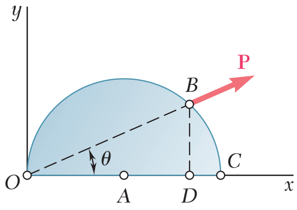


Fig. P3.107

3.108 A concrete foundation mat of 5-m radius supports four equally spaced columns, each of which is located 4 m from the center of the mat. Determine the magnitude and point of application of the smallest additional load that must be applied to the foundation mat if the resultant of the five loads is to pass through the center of the mat.

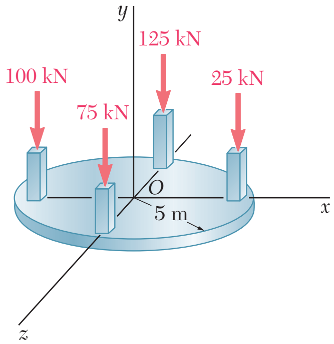
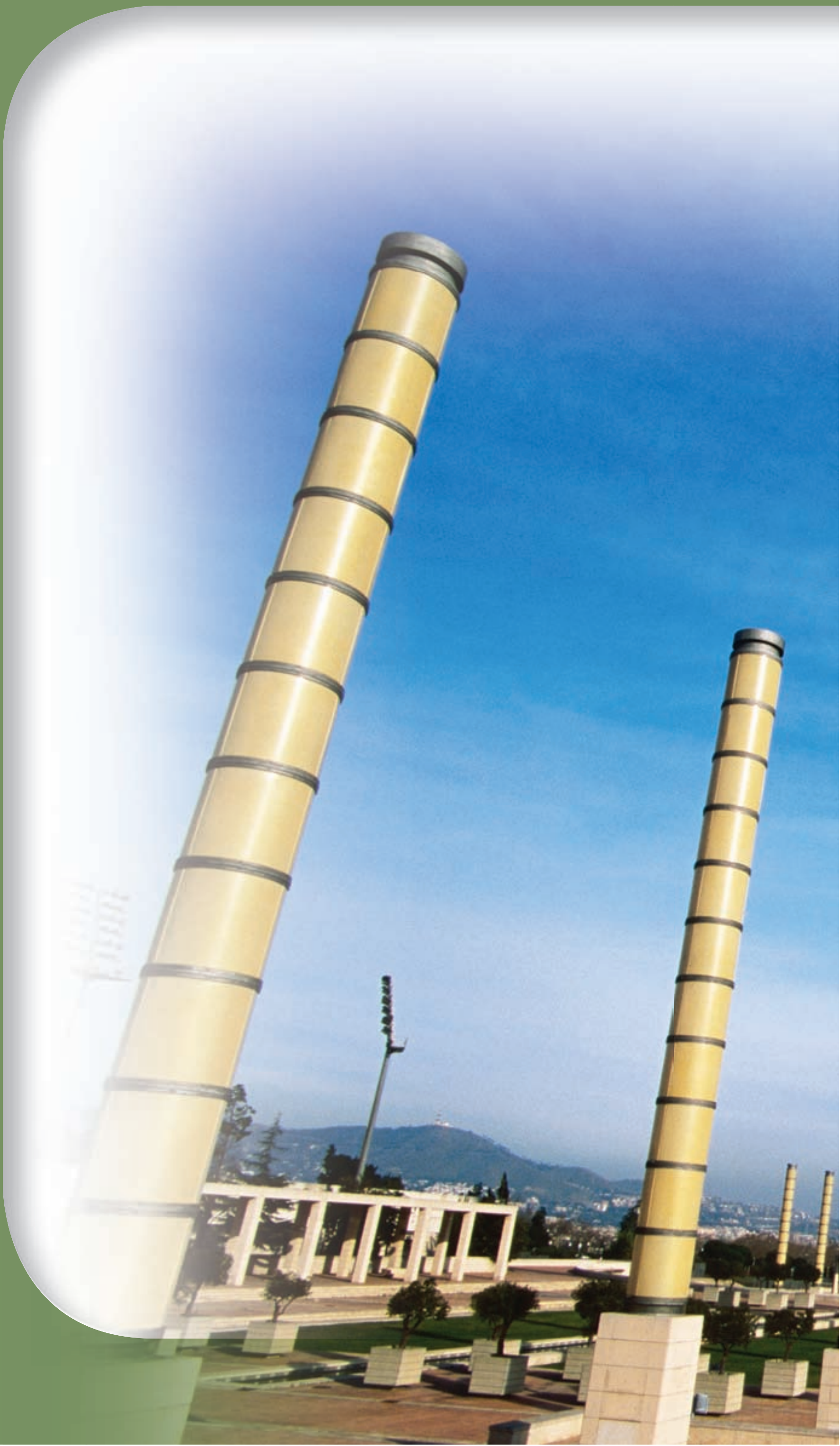


Fig. P3.108

This telecommunications tower, constructed in the heart of the Barcelona Olympic complex to broadcast the 1992 games, was designed to remain in equilibrium under the vertical force of gravity and the lateral forces exerted by wind.



CHAPTER 4

Equilibrium of Rigid Bodies

4.1 INTRODUCTION

We saw in the preceding chapter that the external forces acting on a rigid body can be reduced to a force-couple system at some arbitrary point O. When the force and the couple are both equal to zero, the external forces form a system equivalent to zero, and the rigid body is said to be in *equilibrium.*

The necessary and sufficient conditions for the equilibrium of a rigid body, therefore, can be obtained by setting $\mathbf{R}$ and $\mathbf{M}_O^R$ equal to zero in the relations (3.52) of Sec. 3.17:

$$\Sigma \mathbf{F} = 0 \qquad \Sigma \mathbf{M}_O = \Sigma(\mathbf{r} \times \mathbf{F}) = 0 \tag{4.1}$$

Resolving each force and each moment into its rectangular components, we can express the necessary and sufficient conditions for the equilibrium of a rigid body with the following six scalar equations:

$$\Sigma F_x = 0 \qquad \Sigma F_y = 0 \qquad \Sigma F_z = 0 \tag{4.2}$$
$$\Sigma M_x = 0 \qquad \Sigma M_y = 0 \qquad \Sigma M_z = 0 \tag{4.3}$$

The equations obtained can be used to determine unknown forces applied to the rigid body or unknown reactions exerted on it by its supports. We note that Eqs. (4.2) express the fact that the components of the external forces in the x, y, and z directions are balanced; Eqs. (4.3) express the fact that the moments of the external forces about the x, y, and z axes are balanced. Therefore, for a rigid body in equilibrium, the system of the external forces will impart no translational or rotational motion to the body considered.

In order to write the equations of equilibrium for a rigid body, it is essential to first identify all of the forces acting on that body and then to draw the corresponding *free-body diagram.* In this chapter we first consider the equilibrium of *two-dimensional structures* subjected to forces contained in their planes and learn how to draw their free-body diagrams. In addition to the forces *applied* to a structure, the *reactions* exerted on the structure by its supports will be considered. A specific reaction will be associated with each type of support. You will learn how to determine whether the structure is properly supported, so that you can know in advance whether the equations of equilibrium can be solved for the unknown forces and reactions.

Later in the chapter, the equilibrium of three-dimensional structures will be considered, and the same kind of analysis will be given to these structures and their supports. This will be followed with a discussion of equilibrium of rigid bodies supported on surfaces in which friction acts to restrain motion of one surface with respect to the other.

4.2 FREE-BODY DIAGRAM

In solving a problem concerning the equilibrium of a rigid body, it is essential to consider *all* of the forces acting on the body; it is equally important to exclude any force which is not directly applied to the body. Omitting a force or adding an extraneous one would destroy the conditions of equilibrium. Therefore, the first step in the solution of the problem should be to draw a *free-body diagram* of the rigid body under consideration. Free-body diagrams have already been used on many occasions in Chap. 2. However, in view of their importance to the solution of equilibrium problems, we summarize here the various steps which must be followed in drawing a free-body diagram.

Photo 4.1 A free-body diagram of the tractor shown would include all of the external forces acting on the tractor: the weight of the tractor, the weight of the load in the bucket, and the forces exerted by the ground on the tires.

1. A clear decision should be made regarding the choice of the free body to be used. This body is then detached from the ground and is separated from all other bodies. The contour of the body thus isolated is sketched.
2. All external forces should be indicated on the free-body diagram. These forces represent the actions exerted *on* the free body *by* the ground and *by* the bodies which have been detached; they should be applied at the various points where the free body was supported by the ground or was connected to the other bodies. The *weight* of the free body should also be included among the external forces, since it represents the attraction exerted by the earth on the various particles forming the free body. As will be seen in Chap. 5, the weight should be applied at the center of gravity of the body. When the free body is made of several parts, the forces the various parts exert on each other should *not* be included among the external forces. These forces are internal forces as far as the free body is concerned.
3. The magnitudes and directions of the *known external forces* should be clearly marked on the free-body diagram. When indicating the directions of these forces, it must be remembered that the forces shown on the free-body diagram must be those which are exerted *on,* and not *by,* the free body. Known external forces generally include the *weight* of the free body and *forces applied* for a given purpose.
4. *Unknown external forces* usually consist of the *reactions,* through which the ground and other bodies oppose a possible motion of the free body. The reactions constrain the free body to remain in the same position, and, for that reason, are sometimes called *constraining forces.* Reactions are exerted at the points where the free body is *supported by* or *connected to* other bodies and should be clearly indicated. Reactions are discussed in detail in Secs. 4.3 and 4.8.
5. The free-body diagram should also include dimensions, since these may be needed in the computation of moments of forces. Any other detail, however, should be omitted.

Photo 4.2 In Chap. 6, we will discuss how to determine the internal forces in structures made of several connected pieces, such as the forces in the members that support the bucket of the tractor of Photo 4.1.

Photo 4.3 As the link of the awning window opening mechanism is extended, the force it exerts on the slider results in a normal force being applied to the rod, which causes the window to open.

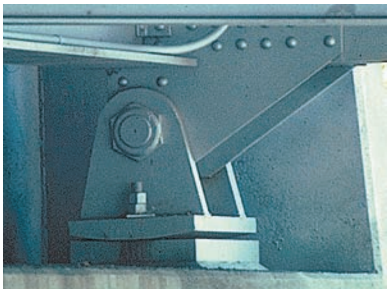


Photo 4.4 The abutment-mounted rocker bearing shown is used to support the roadway of a bridge.

Photo 4.5 Shown is the rocker expansion bearing of a plate girder bridge. The convex surface of the rocker allows the support of the girder to move horizontally.

EQUILIBRIUM IN TWO DIMENSIONS

4.3 REACTIONS AT SUPPORTS AND CONNECTIONS FOR A TWO-DIMENSIONAL STRUCTURE

In the first part of this chapter, the equilibrium of a two-dimensional structure is considered; i.e., it is assumed that the structure being analyzed and the forces applied to it are contained in the same plane. Clearly, the reactions needed to maintain the structure in the same position will also be contained in this plane.

The reactions exerted on a two-dimensional structure can be divided into three groups corresponding to three types of *supports,* or *connections:*

1. *Reactions Equivalent to a Force with Known Line of Action.* Supports and connections causing reactions of this type include *rollers, rockers, frictionless surfaces, short links and cables, collars on frictionless rods,* and *frictionless pins in slots.* Each of these supports and connections can prevent motion in one direction only. They are shown in Fig. 4.1, together with the reactions they produce. Each of these reactions involves *one unknown,* namely, the magnitude of the reaction; this magnitude should be denoted by an appropriate letter. The line of action of the reaction is known and should be indicated clearly in the free-body diagram. The sense of the reaction must be as shown in Fig. 4.1 for the cases of a frictionless surface (toward the free body) or a cable (away from the free body). The reaction can be directed either way in the case of double-track rollers, links, collars on rods, and pins in slots. Single-track rollers and rockers are generally assumed to be reversible, and thus the corresponding reactions can also be directed either way.

2. *Reactions Equivalent to a Force of Unknown Direction and Magnitude.* Supports and connections causing reactions of this type include *frictionless pins in fitted holes, hinges,* and *rough surfaces.* They can prevent translation of the free body in all directions, but they cannot prevent the body from rotating about the connection. Reactions of this group involve *two unknowns* and are usually represented by their x and y components. In the case of a rough surface, the component normal to the surface must be directed away from the surface.

3. *Reactions Equivalent to a Force and a Couple.* These reactions are caused by *fixed supports,* which oppose any motion of the free body and thus constrain it completely. Fixed supports actually produce forces over the entire surface of contact; these forces, however, form a system which can be reduced to a force and a couple. Reactions of this group involve *three unknowns,* consisting usually of the two components of the force and the moment of the couple.

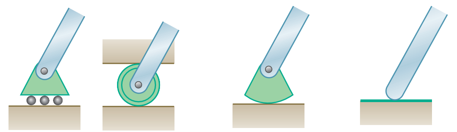
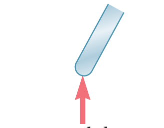
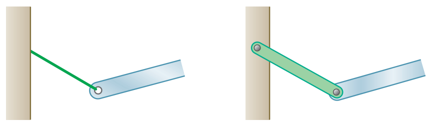
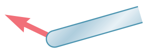
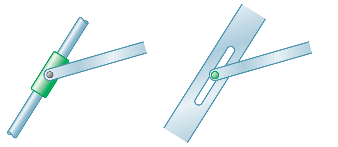
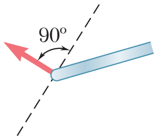
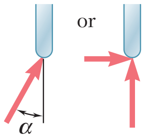
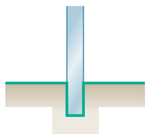
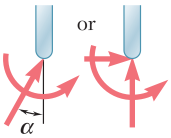
Support or Connection	Reaction	Number of Unknowns
 Rollers, Rocker, Frictionless surface	 Force with known line of action	1
 Short cable, Short link	 Force with known line of action	1
 Collar on frictionless rod, Frictionless pin in slot	 Force with known line of action	1
 Frictionless pin or hinge, Rough surface	 Force of unknown direction	2
 Fixed support	 Force and couple	3

Fig. 4.1 Reactions at supports and connections.

When the sense of an unknown force or couple is not readily apparent, no attempt should be made to determine it. Instead, the sense of the force or couple should be arbitrarily assumed; the sign of the answer obtained will indicate whether the assumption is correct or not.

4.4 EQUILIBRIUM OF A RIGID BODY IN TWO DIMENSIONS

The conditions stated in Sec. 4.1 for the equilibrium of a rigid body become considerably simpler for the case of a two-dimensional structure. Choosing the x and y axes to be in the plane of the structure, we have

$$F_z = 0 \qquad M_x = M_y = 0 \qquad M_z = M_O$$

for each of the forces applied to the structure. Thus, the six equations of equilibrium derived in Sec. 4.1 reduce to

$$\Sigma F_x = 0 \qquad \Sigma F_y = 0 \qquad \Sigma M_O = 0 \tag{4.4}$$

and to three trivial identities, $0 = 0$. Since $\Sigma M_O = 0$ must be satisfied regardless of the choice of the origin O, we can write the equations of equilibrium for a two-dimensional structure in the more general form

$$\Sigma F_x = 0 \qquad \Sigma F_y = 0 \qquad \Sigma M_A = 0 \tag{4.5}$$

where A is any point in the plane of the structure. The three equations obtained can be solved for no more than *three unknowns.*

We saw in the preceding section that unknown forces include reactions and that the number of unknowns corresponding to a given reaction depends upon the type of support or connection causing that reaction. Referring to Sec. 4.3, we observe that the equilibrium equations (4.5) can be used to determine the reactions associated with two rollers and one cable, one fixed support, or one roller and one pin in a fitted hole, etc.

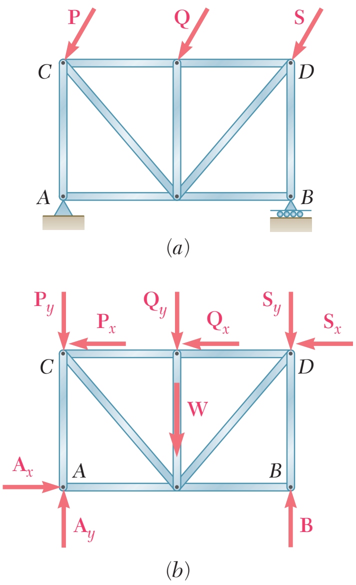


Fig. 4.2

Consider Fig. 4.2*a*, in which the truss shown is subjected to the given forces **P**, **Q**, and **S**. The truss is held in place by a pin at A and a roller at B. The pin prevents point A from moving by exerting on the truss a force which can be resolved into the components $\mathbf{A}_x$ and $\mathbf{A}_y$; the roller keeps the truss from rotating about A by exerting the vertical force **B**. The free-body diagram of the truss is shown in Fig. 4.2*b*; it includes the reactions $\mathbf{A}_x$, $\mathbf{A}_y$, and **B** as well as the applied forces **P**, **Q**, **S** and the weight **W** of the truss. Expressing that the sum of the moments about A of all of the forces shown in Fig. 4.2*b* is zero, we write the equation $\Sigma M_A = 0$, which can be used to determine the magnitude B since it does not contain A_x or A_y. Next, expressing that the sum of the x components and the sum of the y components of the forces are zero, we write the equations $\Sigma F_x = 0$ and $\Sigma F_y = 0$, from which we can obtain the components A_x and A_y, respectively.

An additional equation could be obtained by expressing that the sum of the moments of the external forces about a point other than A is zero. We could write, for instance, $\Sigma M_B = 0$. Such a statement, however, does not contain any new information, since it has already been established that the system of the forces shown in Fig. 4.2*b* is equivalent to zero. The additional equation *is not independent* and cannot be used to determine a fourth unknown. It will be useful, however, for checking the solution obtained from the original three equations of equilibrium.

While the three equations of equilibrium cannot be *augmented* by additional equations, any of them can be *replaced* by another equation. Therefore, an alternative system of equations of equilibrium is

$$\Sigma F_x = 0 \qquad \Sigma M_A = 0 \qquad \Sigma M_B = 0 \tag{4.6}$$

where the second point about which the moments are summed (in this case, point B) cannot lie on the line parallel to the y axis that passes through point A (Fig. 4.2*b*). These equations are sufficient conditions for the equilibrium of the truss. The first two equations indicate that the external forces must reduce to a single vertical force at A. Since the third equation requires that the moment of this force be zero about a point B which is not on its line of action, the force must be zero, and the rigid body is in equilibrium.

A third possible set of equations of equilibrium is

$$\Sigma M_A = 0 \qquad \Sigma M_B = 0 \qquad \Sigma M_C = 0 \tag{4.7}$$

where the points A, B, and C do not lie in a straight line (Fig. 4.2*b*). The first equation requires that the external forces reduce to a single force at A; the second equation requires that this force pass through B; and the third equation requires that it pass through C. Since the points A, B, C do not lie in a straight line, the force must be zero, and the rigid body is in equilibrium.

The equation $\Sigma M_A = 0$, which expresses that the sum of the moments of the forces about pin A is zero, possesses a more definite physical meaning than either of the other two equations in (4.7). These two equations express a similar idea of balance, but with respect to points about which the rigid body is not actually hinged. They are, however, as useful as the first equation, and our choice of equilibrium equations should not be unduly influenced by the physical meaning of these equations. Indeed, it will be desirable in practice to choose equations of equilibrium containing only one unknown, since this eliminates the necessity of solving simultaneous equations. Equations containing only one unknown can be obtained by summing moments about the point of intersection of the lines of action of two unknown forces or, if these forces are parallel, by summing components in a direction perpendicular to their common direction. For example, in Fig. 4.3, in which the truss shown is held by rollers at A and B and a short link at D, the reactions at A and B can be eliminated by summing x components. The reactions at A and D will be eliminated by summing moments about C, and the reactions at B and D by summing moments about D. The equations obtained are

$$\Sigma F_x = 0 \qquad \Sigma M_C = 0 \qquad \Sigma M_D = 0$$

Each of these equations contains only one unknown.

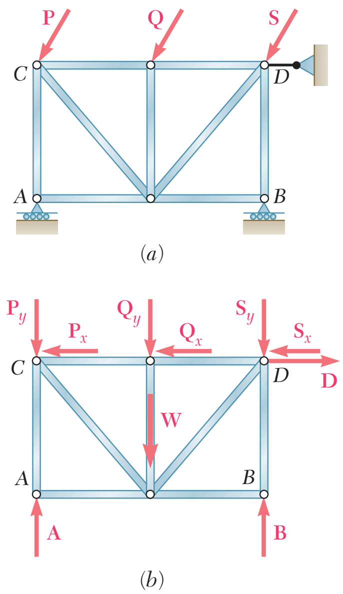


Fig. 4.3

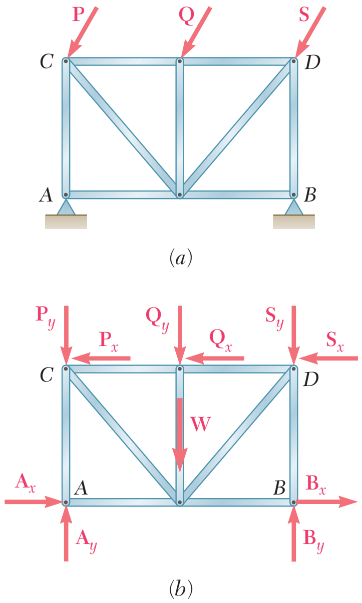


Fig. 4.4 Statically indeterminate reactions.

4.5 STATICALLY INDETERMINATE REACTIONS. PARTIAL CONSTRAINTS

In the two examples considered in the preceding section (Figs. 4.2 and 4.3), the types of supports used were such that the rigid body could not possibly move under the given loads or under any other loading conditions. In such cases, the rigid body is said to be *completely constrained.* We also recall that the reactions corresponding to these supports involved *three unknowns* and could be determined by solving the three equations of equilibrium. When such a situation exists, the reactions are said to be *statically determinate.*

Consider Fig. 4.4*a*, in which the truss shown is held by pins at A and B. These supports provide more constraints than are necessary to keep the truss from moving under the given loads or under any other loading conditions. We also note from the free-body diagram of Fig. 4.4*b* that the corresponding reactions involve *four unknowns.* Since, as was pointed out in Sec. 4.4, only three independent equilibrium equations are available, there are *more unknowns than equations;* thus, all of the unknowns cannot be determined. While the equations $\Sigma M_A = 0$ and $\Sigma M_B = 0$ yield the vertical components B_y and A_y, respectively, the equation $\Sigma F_x = 0$ gives only the sum $A_x + B_x$ of the horizontal components of the reactions at A and B. The components A_x and B_x are said to be *statically indeterminate.* They could be determined by considering the deformations produced in the truss by the given loading, but this method is beyond the scope of statics and belongs to the study of mechanics of materials.

The supports used to hold the truss shown in Fig. 4.5*a* consist of rollers at A and B. Clearly, the constraints provided by these supports are not sufficient to keep the truss from moving. While any vertical motion is prevented, the truss is free to move horizontally. The truss is said to be *partially constrained.*† Turning our attention to Fig. 4.5*b*, we note that the reactions at A and B involve only *two unknowns.* Since three equations of equilibrium must still be satisfied, there are *fewer unknowns than equations,* and, in general, one of the equilibrium equations will not be satisfied. While the equations $\Sigma M_A = 0$ and $\Sigma M_B = 0$ can be satisfied by a proper choice of reactions at A and B, the equation $\Sigma F_x = 0$ will not be satisfied unless the sum of the horizontal components of the applied forces happens to be zero. We thus observe that the equlibrium of the truss of Fig. 4.5 cannot be maintained under general loading conditions.

It appears from the above that if a rigid body is to be completely constrained and if the reactions at its supports are to be statically determinate, *there must be as many unknowns as there are equations of equilibrium.* When this condition is *not* satisfied, we can be certain that either the rigid body is not completely constrained or that the reactions at its supports are not statically determinate; it is also possible that the rigid body is not completely constrained *and* that the reactions are statically indeterminate.

We should note however that, while *necessary,* the above condition is *not sufficient.* In other words, the fact that the number of

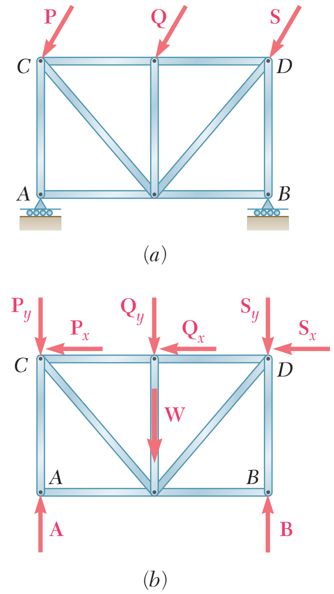


Fig. 4.5 Partial constraints.

†Partially constrained bodies are often referred to as *unstable.* However, to avoid confusion between this type of instability, due to insufficient constraints, and the type of instability considered in Chap. 16, which relates to the behavior of columns, we shall restrict the use of the words *stable* and *unstable* to the latter case.

unknowns is equal to the number of equations is no guarantee that the body is completely constrained or that the reactions at its supports are statically determinate. Consider Fig. 4.6*a*, in which the truss shown is held by rollers at A, B, and E. While there are three unknown reactions, **A**, **B**, and **E** (Fig. 4.6*b*), the equation $\Sigma F_x = 0$ will not be satisfied unless the sum of the horizontal components of the applied forces happens to be zero. Although there are a sufficient number of constraints, these constraints are not properly arranged, and the truss is free to move horizontally. We say that the truss is *improperly constrained.* Since only two equilibrium equations are left for determining three unknowns, the reactions will be statically indeterminate. Thus, improper constraints also produce static indeterminacy.

Another example of improper constraints—and of static indeterminacy—is provided by the truss shown in Fig. 4.7. This truss is held by a pin at A and by rollers at B and C, which altogether involve four unknowns. Since only three independent equilibrium equations are available, the reactions at the supports are statically indeterminate. On the other hand, we note that the equation $\Sigma M_A = 0$ cannot be satisfied under general loading conditions, since the lines of action of the reactions **B** and **C** pass through A. We conclude that the truss can rotate about A and that it is improperly constrained.†

The examples of Figs. 4.6 and 4.7 lead us to conclude that *a rigid body is improperly constrained whenever the supports,* even though they may provide a sufficient number of reactions, *are arranged in such a way that the reactions must be either concurrent or parallel.*‡

In summary, to be sure that a two-dimensional rigid body is completely constrained and that the reactions at its supports are statically determinate, we should verify that the reactions involve three—and only three—unknowns and that the supports are arranged in such a way that they do not require the reactions to be either concurrent or parallel.

Supports involving statically indeterminate reactions should be used with care in the *design* of structures and only with a full knowledge of the problems they may cause. On the other hand, the *analysis* of structures possessing statically indeterminate reactions often can be partially carried out by the methods of statics. In the case of the truss of Fig. 4.4, for example, the vertical components of the reactions at A and B were obtained from the equilibrium equations.

For obvious reasons, supports producing partial or improper constraints should be avoided in the design of stationary structures. However, a partially or improperly constrained structure will not necessarily collapse; under particular loading conditions, equilibrium can be maintained. For example, the trusses of Figs. 4.5 and 4.6 will be in equilibrium if the applied forces **P**, **Q**, and **S** are vertical. Besides, structures which are designed to move *should* be only partially constrained. A railroad car, for instance, would be of little use if it were completely constrained by having its brakes applied permanently.

†Rotation of the truss about A requires some "play" in the supports at B and C. In practice such play will always exist. In addition, we note that if the play is kept small, the displacements of the rollers B and C and, thus, the distances from A to the lines of action of the reactions **B** and **C** will also be small. The equation $\Sigma M_A = 0$ then requires that **B** and **C** be very large, a situation which can result in the failure of the supports at B and C.

‡Because this situation arises from an inadequate arrangement or *geometry* of the supports, it is often referred to as *geometric instability.*

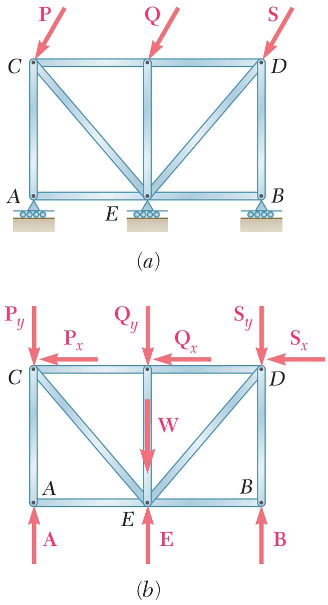


Fig. 4.6 Improper constraints.

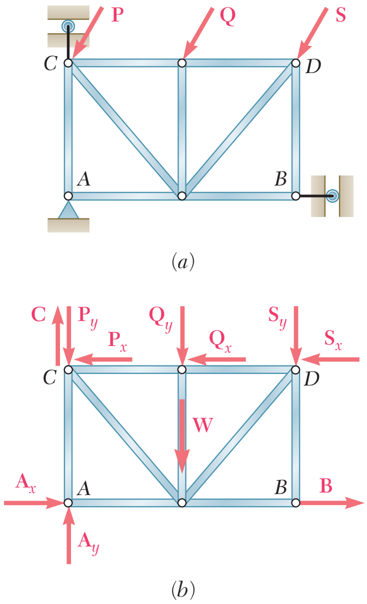
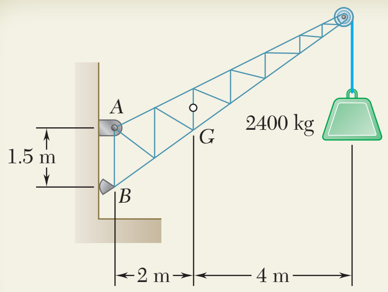
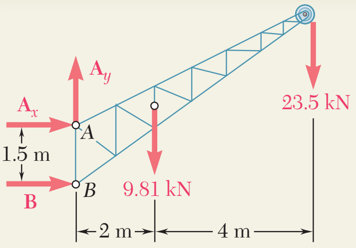


Fig. 4.7 Improper constraints.



SAMPLE PROBLEM 4.1

A fixed crane has a mass of 1000 kg and is used to lift a 2400-kg crate. It is held in place by a pin at A and a rocker at B. The center of gravity of the crane is located at G. Determine the components of the reactions at A and B.



SOLUTION

Free-Body Diagram. A free-body diagram of the crane is drawn. By multiplying the masses of the crane and of the crate by $g = 9.81 \text{ m/s}^2$, we obtain the corresponding weights, that is, 9810 N or 9.81 kN, and 23 500 N or 23.5 kN. The reaction at pin A is a force of unknown direction; it is represented by its components $\mathbf{A}_x$ and $\mathbf{A}_y$. The reaction at the rocker B is perpendicular to the rocker surface; thus, it is horizontal. We assume that $\mathbf{A}_x$, $\mathbf{A}_y$, and $\mathbf{B}$ act in the directions shown.

Determination of *B*. We express that the sum of the moments of all external forces about point A is zero. The equation obtained will contain neither A_x nor A_y, since the moments of $\mathbf{A}_x$ and $\mathbf{A}_y$ about A are zero. Multiplying the magnitude of each force by its perpendicular distance from A, we write

$$+\circlearrowleft \Sigma M_A = 0: \qquad +B(1.5 \text{ m}) - (9.81 \text{ kN})(2 \text{ m}) - (23.5 \text{ kN})(6 \text{ m}) = 0$$
$$B = +107.1 \text{ kN} \qquad \mathbf{B} = 107.1 \text{ kN} \rightarrow \quad ◀$$

Since the result is positive, the reaction is directed as assumed.

Determination of A_x. The magnitude of $\mathbf{A}_x$ is determined by expressing that the sum of the horizontal components of all external forces is zero.

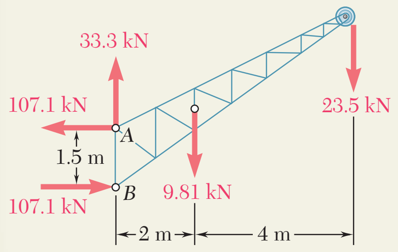
$$\overset{+}{\rightarrow}\Sigma F_x = 0: \qquad A_x + B = 0$$
$$A_x + 107.1 \text{ kN} = 0$$
$$A_x = -107.1 \text{ kN} \qquad \mathbf{A}_x = 107.1 \text{ kN} \leftarrow \quad ◀$$

Since the result is negative, the sense of $\mathbf{A}_x$ is opposite to that assumed originally.

Determination of A_y. The sum of the vertical components must also equal zero.

$$+\uparrow\Sigma F_y = 0: \qquad A_y - 9.81 \text{ kN} - 23.5 \text{ kN} = 0$$
$$A_y = +33.3 \text{ kN} \qquad \mathbf{A}_y = 33.3 \text{ kN} \uparrow \quad ◀$$

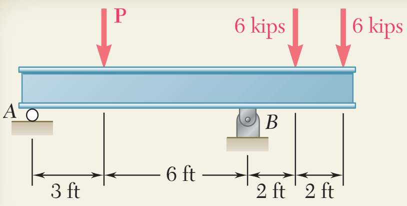
Adding vectorially the components $\mathbf{A}_x$ and $\mathbf{A}_y$, we find that the reaction at A is 112.2 kN $\nwarrow$17.3°.



Check. The values obtained for the reactions can be checked by recalling that the sum of the moments of all of the external forces about any point must be zero. For example, considering point B, we write

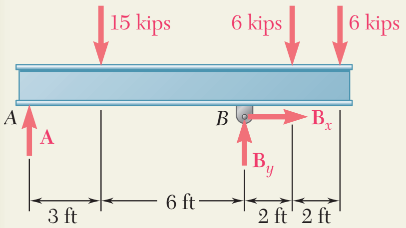
$$+\circlearrowleft \Sigma M_B = -(9.81 \text{ kN})(2 \text{ m}) - (23.5 \text{ kN})(6 \text{ m}) + (107.1 \text{ kN})(1.5 \text{ m}) = 0$$

SAMPLE PROBLEM 4.2



Three loads are applied to a beam as shown. The beam is supported by a roller at A and by a pin at B. Neglecting the weight of the beam, determine the reactions at A and B when $P = 15$ kips.

SOLUTION



Free-Body Diagram. A free-body diagram of the beam is drawn. The reaction at A is vertical and is denoted by **A**. The reaction at B is represented by components $\mathbf{B}_x$ and $\mathbf{B}_y$. Each component is assumed to act in the direction shown.

Equilibrium Equations. We write the following three equilibrium equations and solve for the reactions indicated:

$$\overset{+}{\rightarrow}\Sigma F_x = 0: \qquad B_x = 0 \qquad \mathbf{B}_x = 0 \quad \blacktriangleleft$$

$$+\circlearrowleft\Sigma M_A = 0:$$
$$-(15 \text{ kips})(3 \text{ ft}) + B_y(9 \text{ ft}) - (6 \text{ kips})(11 \text{ ft}) - (6 \text{ kips})(13 \text{ ft}) = 0$$
$$B_y = +21.0 \text{ kips} \qquad \mathbf{B}_y = 21.0 \text{ kips}\uparrow \quad \blacktriangleleft$$

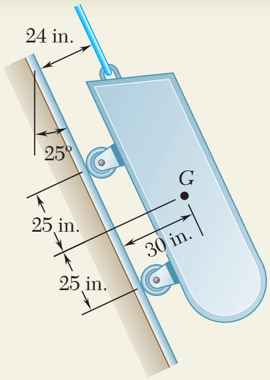
$$+\circlearrowleft\Sigma M_B = 0:$$
$$-A(9 \text{ ft}) + (15 \text{ kips})(6 \text{ ft}) - (6 \text{ kips})(2 \text{ ft}) - (6 \text{ kips})(4 \text{ ft}) = 0$$
$$A = +6.00 \text{ kips} \qquad \mathbf{A} = 6.00 \text{ kips}\uparrow \quad \blacktriangleleft$$

Check. The results are checked by adding the vertical components of all of the external forces:

$$+\uparrow\Sigma F_y = +6.00 \text{ kips} - 15 \text{ kips} + 21.0 \text{ kips} - 6 \text{ kips} - 6 \text{ kips} = 0$$

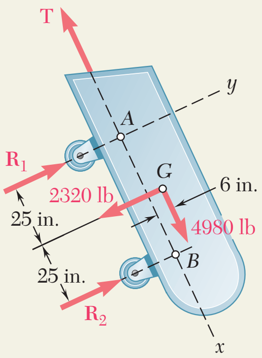
Remark. In this problem the reactions at both A and B are vertical; however, these reactions are vertical for different reasons. At A, the beam is supported by a roller; hence the reaction cannot have any horizontal component. At B, the horizontal component of the reaction is zero because it must satisfy the equilibrium equation $\Sigma F_x = 0$ and because none of the other forces acting on the beam has a horizontal component.

We could have noticed at first glance that the reaction at B was vertical and dispensed with the horizontal component $\mathbf{B}_x$. This, however, is a bad practice. In following it, we would run the risk of forgetting the component $\mathbf{B}_x$ when the loading conditions require such a component (i.e., when a horizontal load is included). Also, the component $\mathbf{B}_x$ was found to be zero by using and solving an equilibrium equation, $\Sigma F_x = 0$. By setting $\mathbf{B}_x$ equal to zero immediately, we might not realize that we actually make use of this equation and thus might lose track of the number of equations available for solving the problem.



SAMPLE PROBLEM 4.3

A loading car is at rest on a track forming an angle of 25° with the vertical. The gross weight of the car and its load is 5500 lb, and it is applied at a point 30 in. from the track, halfway between the two axles. The car is held by a cable attached 24 in. from the track. Determine the tension in the cable and the reaction at each pair of wheels.



SOLUTION

Free-Body Diagram. A free-body diagram of the car is drawn. The reaction at each wheel is perpendicular to the track, and the tension force **T** is parallel to the track. For convenience, we choose the x axis parallel to the track and the y axis perpendicular to the track. The 5500-lb weight is then resolved into x and y components.

$$W_x = +(5500 \text{ lb}) \cos 25° = +4980 \text{ lb}$$
$$W_y = -(5500 \text{ lb}) \sin 25° = -2320 \text{ lb}$$

Equilibrium Equations. We take moments about A to eliminate **T** and $\mathbf{R}_1$ from the computation.

$$+\circlearrowleft \Sigma M_A = 0: \quad -(2320 \text{ lb})(25 \text{ in.}) - (4980 \text{ lb})(6 \text{ in.}) + R_2(50 \text{ in.}) = 0$$
$$R_2 = +1758 \text{ lb} \qquad \mathbf{R}_2 = 1758 \text{ lb} \nearrow \quad \blacktriangleleft$$

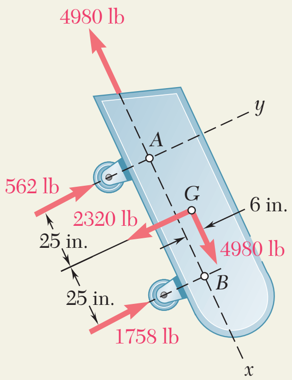
Now, taking moments about B to eliminate **T** and $\mathbf{R}_2$ from the computation, we write

$$+\circlearrowleft \Sigma M_B = 0: \quad (2320 \text{ lb})(25 \text{ in.}) - (4980 \text{ lb})(6 \text{ in.}) - R_1(50 \text{ in.}) = 0$$
$$R_1 = +562 \text{ lb} \qquad \mathbf{R}_1 = +562 \text{ lb} \nearrow \quad \blacktriangleleft$$

The value of T is found by writing

$$\searrow + \Sigma F_x = 0: \quad +4980 \text{ lb} - T = 0$$
$$T = +4980 \text{ lb} \qquad \mathbf{T} = 4980 \text{ lb} \nwarrow \quad \blacktriangleleft$$

The computed values of the reactions are shown in the adjacent sketch.

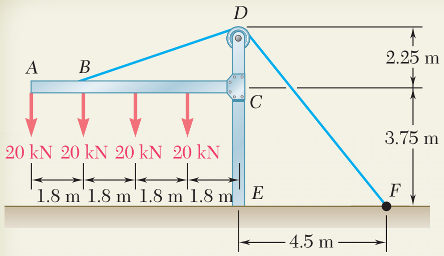


Check. The computations are verified by writing

$$\nearrow + \Sigma F_y = +562 \text{ lb} + 1758 \text{ lb} - 2320 \text{ lb} = 0$$

The solution could also have been checked by computing moments about any point other than A or B.

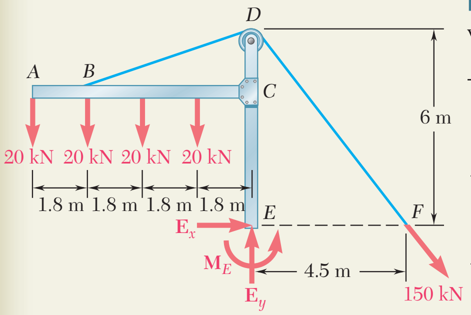
SAMPLE PROBLEM 4.4



The frame shown supports part of the roof of a small building. Knowing that the tension in the cable is 150 kN, determine the reaction at the fixed end E.

SOLUTION

Free-Body Diagram. A free-body diagram of the frame and of the cable BDF is drawn. The reaction at the fixed end E is represented by the force components $\mathbf{E}_x$ and $\mathbf{E}_y$ and the couple $\mathbf{M}_E$. The other forces acting on the free body are the four 20-kN loads and the 150-kN force exerted at end F of the cable.



Equilibrium Equations. Noting that $DF = \sqrt{(4.5\text{ m})^2 + (6\text{ m})^2} = 7.5$ m, we write

$$\overset{+}{\rightarrow}\Sigma F_x = 0: \qquad E_x + \frac{4.5}{7.5}(150\text{ kN}) = 0$$

$$E_x = -90.0\text{ kN} \qquad \mathbf{E}_x = 90.0\text{ kN} \leftarrow \blacktriangleleft$$

$$+\uparrow\Sigma F_y = 0: \qquad E_y - 4(20\text{ kN}) - \frac{6}{7.5}(150\text{ kN}) = 0$$

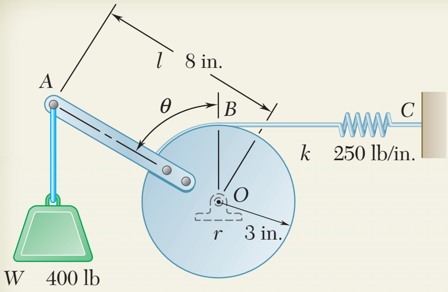
$$E_y = +200\text{ kN} \qquad \mathbf{E}_y = 200\text{ kN} \uparrow \blacktriangleleft$$

$$+\circlearrowleft\Sigma M_E = 0: \qquad (20\text{ kN})(7.2\text{ m}) + (20\text{ kN})(5.4\text{ m}) + (20\text{ kN})(3.6\text{ m})$$

$$+ (20\text{ kN})(1.8\text{ m}) - \frac{6}{7.5}(150\text{ kN})(4.5\text{ m}) + M_E = 0$$

$$M_E = +180.0\text{ kN}\cdot\text{m} \qquad \mathbf{M}_E = 180.0\text{ kN}\cdot\text{m} \circlearrowleft \blacktriangleleft$$

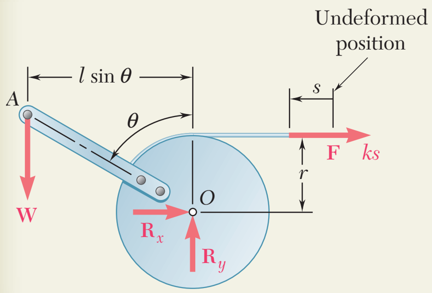
SAMPLE PROBLEM 4.5



A 400-lb weight is attached at A to the lever shown. The constant of the spring BC is $k = 250$ lb/in., and the spring is unstretched when $\theta = 0$. Determine the position of equilibrium.

SOLUTION

Free-Body Diagram. We draw a free-body diagram of the lever and cylinder. Denoting by s the deflection of the spring from its undeformed position, and noting that $s = r\theta$, we have $F = ks = kr\theta$.



Equilibrium Equation. Summing the moments of $\mathbf{W}$ and $\mathbf{F}$ about O, we write

$$+\circlearrowleft\Sigma M_O = 0: \qquad Wl\sin\theta - r(kr\theta) = 0 \qquad \sin\theta = \frac{kr^2}{Wl}\theta$$

Substituting the given data, we obtain

$$\sin\theta = \frac{(250\text{ lb/in.})(3\text{ in.})^2}{(400\text{ lb})(8\text{ in.})}\theta \qquad \sin\theta = 0.703\,\theta$$

Solving by trial and error, we find $\qquad \theta = 0 \qquad \theta = 80.3° \blacktriangleleft$

PROBLEMS

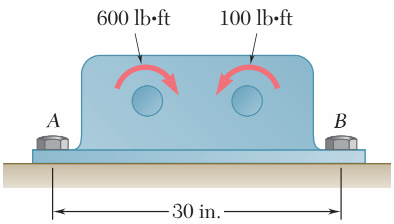


Fig. P4.1

4.1 Two external shafts of a gearbox carry torques as shown. Determine the vertical components of the forces that must be exerted by the bolts at A and B to maintain the gearbox in equilibrium.

4.2 A 2800-kg forklift truck is used to lift a 1500-kg crate. Determine the reaction at each of the two (a) front wheels A, (b) rear wheels B.

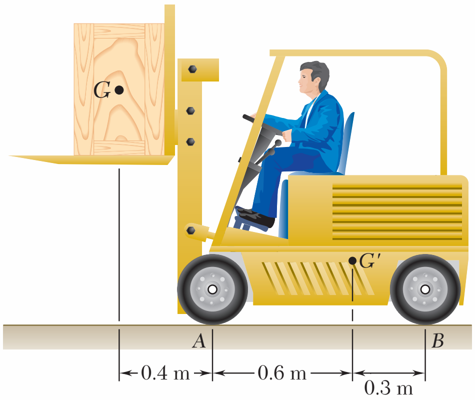


Fig. P4.2

4.3 A gardener uses a 12-lb wheelbarrow to transport a 50-lb bag of fertilizer. What force must the gardener exert on each handle?

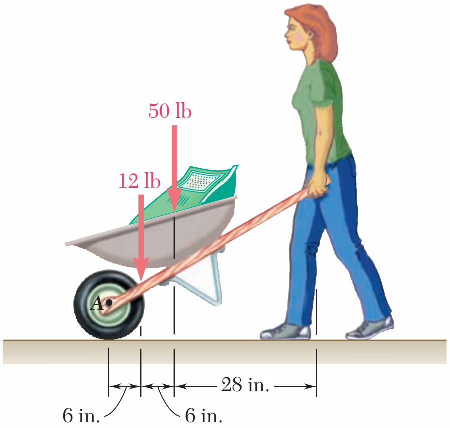


Fig. P4.3

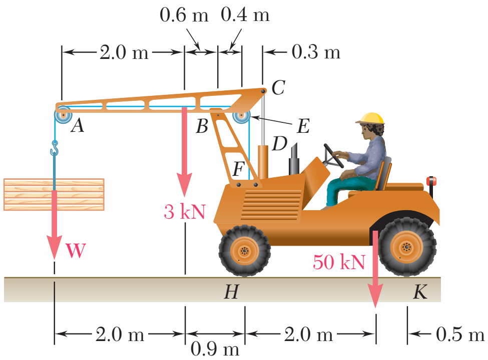


Fig. P4.4

4.4 A load of lumber of weight $W = 25$ kN is being raised as shown by a mobile crane. Knowing that the tension is 25 kN in all portions of cable AEF and that the weight of boom ABC is 3 kN, determine (a) the tension in rod CD, (b) the reaction at pin B.

4.5 Three loads are applied as shown to a light beam supported by cables attached at B and D. Neglecting the weight of the beam, determine the range of values of Q for which neither cable becomes slack when $P = 0$.

4.6 Three loads are applied as shown to a light beam supported by cables attached at B and D. Knowing that the maximum allowable tension in each cable is 12 kN and neglecting the weight of the beam, determine the range of values of Q for which the loading is safe when $P = 5$ kN.

4.7 The 10-ft beam AB rests upon, but is not attached, to supports at C and D. Neglecting the weight of the beam, determine the range of values of P for which the beam will remain in equilibrium.

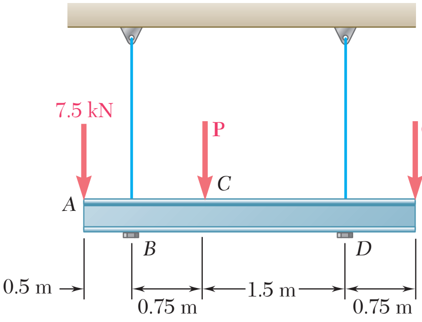


Fig. P4.5 and P4.6

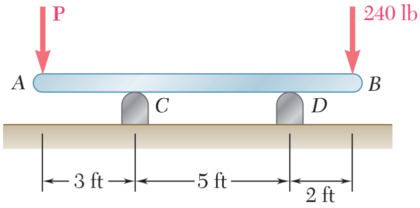


Fig. P4.7

4.8 For the beam of Sample Prob. 4.2, determine the range of values of P for which the beam will be safe knowing that the maximum allowable value for each of the reactions is 25 kips and that the reaction at A must be directed upward.

4.9 The 40-ft boom AB weighs 2 kips; the distance from the axle A to the center of gravity G of the boom is 20 ft. For the position shown, determine the tension T in the cable and the reaction at A.

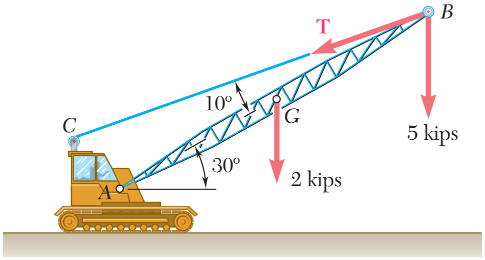


Fig. P4.9

4.10 The ladder AB, of length L and weight W, can be raised by the cable BC. Determine the tension T required to raise end B just off the floor (*a*) in terms of W and θ, (*b*) if $h = 8$ ft, $L = 10$ ft, and $W = 35$ lb.

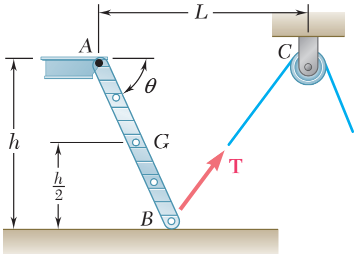


Fig. P4.10

4.11 Neglecting the radius of the pulley, determine the tension in cable ABD and the reaction at the support C.

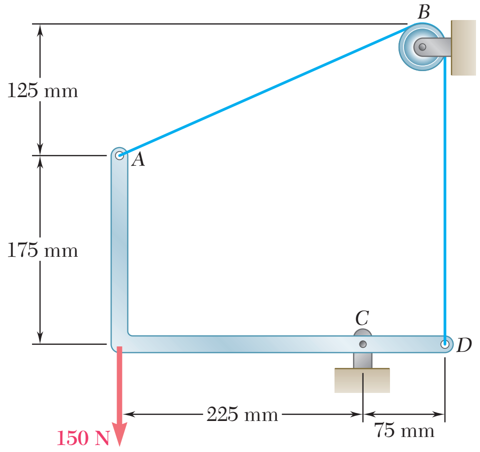


Fig. P4.11

4.12 The lever AB is hinged at C and attached to a control cable at A. If the lever is subjected at B to a 500-N horizontal force, determine (*a*) the tension in the cable, (*b*) the reaction at C.

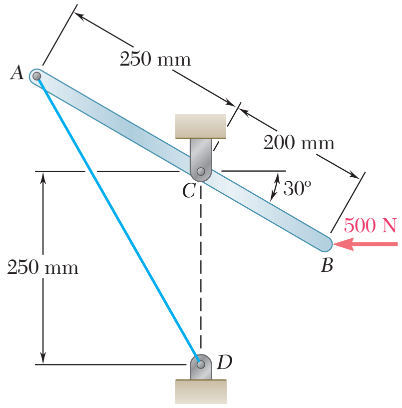


Fig. P4.12

4.13 Determine the reactions at A and B when $\alpha = 60°$.

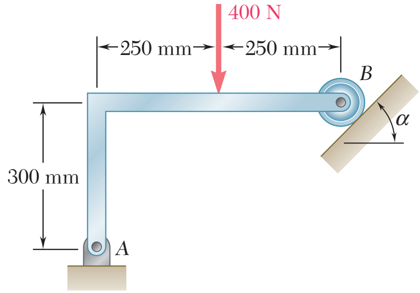


Fig. *P4.13*

4.14 The required tension in cable AB is 300 lb. Determine (*a*) the vertical force **P** that must be applied to the pedal, (*b*) the corresponding reaction at C.

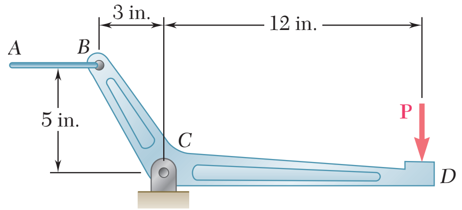


Fig. P4.14 and P4.15

4.15 Determine the maximum tension that can be developed in cable AB if the maximum allowable magnitude of the reaction at C is 650 lb.

4.16 A truss may be supported in three different ways as shown. In each one, determine the reactions at the supports.

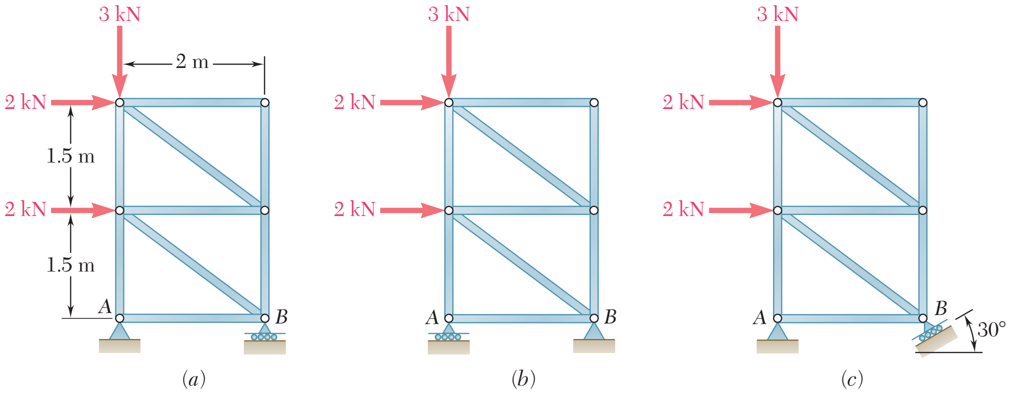


Fig. P4.16

4.17 A light bar AD is suspended from a cable BE and supports a 20-kg block at C. The extremities A and D of the bar are in contact with frictionless, vertical walls. Determine the tension in cable BE and the reactions A and D.

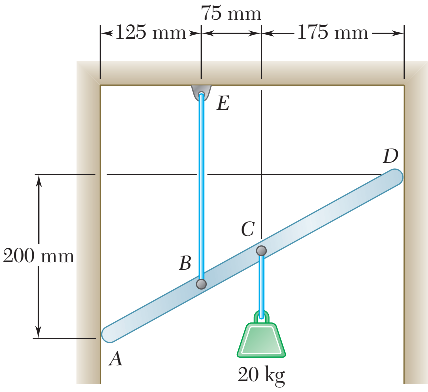


Fig. P4.17

4.18 A light rod, supported by rollers at B, C, and D, is subjected to an 800-N force applied at A. If $\beta = 0$, determine (*a*) the reactions at B, C, and D, (*b*) the rollers that can be safely removed for this loading.

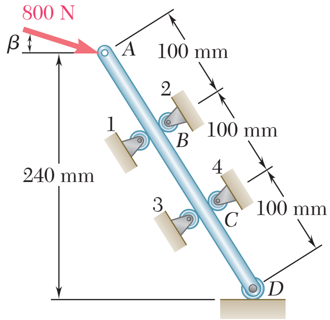


Fig. P4.18

4.19 A 160-lb overhead garage door consists of a uniform rectangular panel AC, 84 in. long, supported by the cable AE attached at the middle of the upper edge of the door and by two sets of frictionless rollers at A and B. Each set consists of two rollers located on either side of the door. The rollers A are free to move in horizontal channels, while the rollers B are guided by vertical channels. If the door is held in the position for which $BD = 42$ in., determine (*a*) the tension in cable AE, (*b*) the reaction at each of the four rollers.

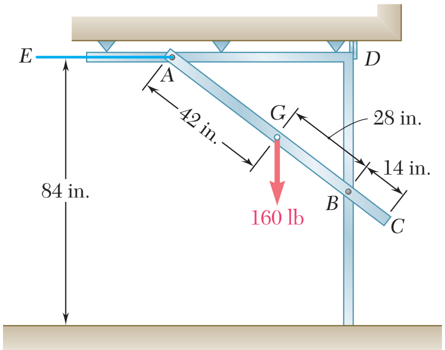


Fig. P4.19

4.20 In Prob. 4.19, determine the distance BD for which the tension in cable AE is equal to 600 lb.

4.21 A 150-kg telephone pole is used to support the ends of two wires as shown. The tension in the wire to the left is 400 N, and, at the point of support, the wire forms an angle of 10° with the horizontal. (*a*) If the tension T_2 is zero, determine the reaction at the base A. (*b*) Determine the largest and smallest allowable tension T_2 if the magnitude of the couple at A may not exceed 900 N · m.

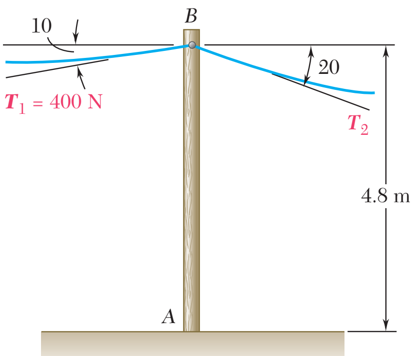


Fig. *P4.21*

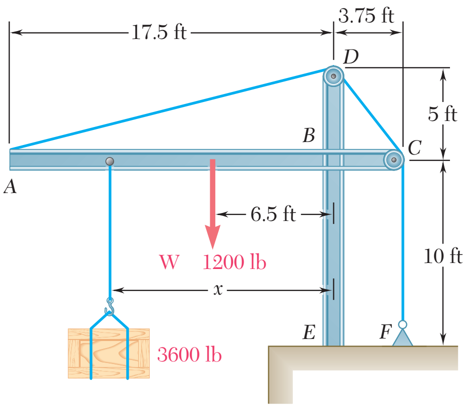


Fig. P4.22

4.22 The rig shown consists of a 1200-lb horizontal member *ABC* and a vertical member *DBE* welded together at *B*. The rig is being used to raise a 3600-lb crate at a distance $x = 12$ ft from the vertical member *DBE*. If the tension in the cable is 4 kips, determine the reaction at *E*, assuming that the cable is (*a*) anchored at *F* as shown in the figure, (*b*) attached to the vertical member at a point located 1 ft above *E*.

4.23 For the rig and crate of Prob. 4.22, and assuming that the cable is anchored at *F* as shown, determine (*a*) the required tension in cable *ADCF* if the maximum value of the couple at *E* as *x* varies from 1.5 to 17.5 ft is to be as small as possible, (*b*) the corresponding maximum value of the couple.

4.24 A traffic-signal pole may be supported in the three ways shown; in part *c*, the tension in cable *BC* is to be 1950 N. Determine the reactions for each type of support.

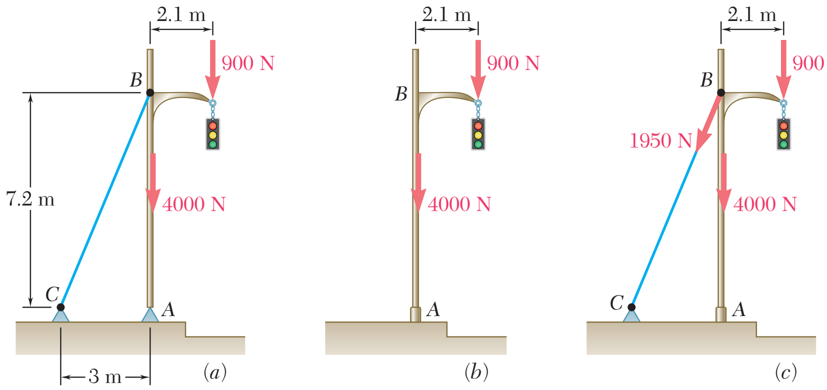


Fig. P4.24

4.25 A truss may be supported in eight different ways as shown. All connections consist of frictionless pins, rollers, and short links. In each case, determine whether (*a*) the truss is completely, partially, or improperly constrained, (*b*) the reactions are statically determinate or indeterminate, (*c*) the equilibrium of the truss is maintained in the position shown. Also, wherever possible, compute the reactions, assuming that the magnitude of the force P is 12 kips.

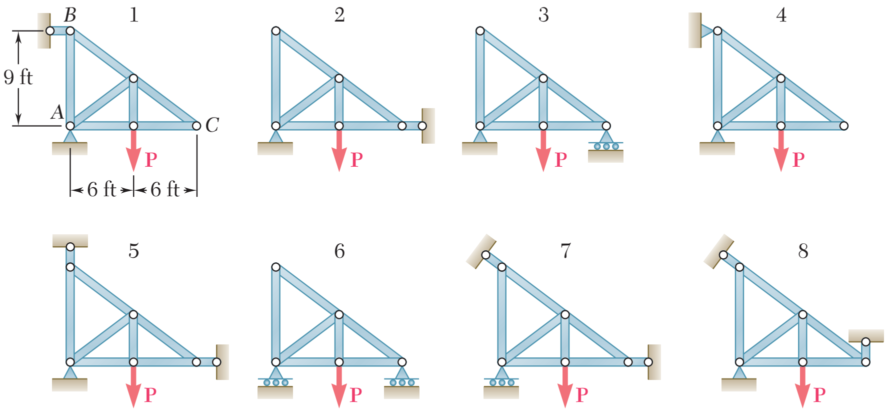


Fig. P4.25

4.26 Nine identical rectangular plates, 500 × 750 mm, and each of mass $m = 40$ kg, are held in a vertical plane as shown. All connections consist of frictionless pins, rollers, and short links. For each case, answer the questions listed in Prob. 4.25, and wherever possible, compute the reactions.

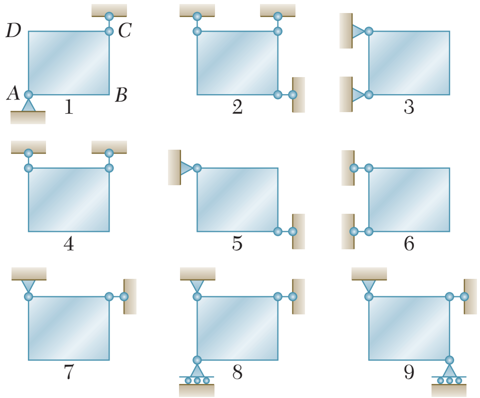


Fig. P4.26

4.6 EQUILIBRIUM OF A TWO-FORCE BODY

A particular case of equilibrium which is of considerable interest is that of a rigid body subjected to two forces. Such a body is commonly called a *two-force body.* It will be shown that *if a two-force body is in equilibrium, the two forces must have the same magnitude, the same line of action, and opposite sense.*

Consider a corner plate subjected to two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ acting at A and B, respectively (Fig. 4.8*a*). If the plate is to be in equilibrium, the sum of the moments of $\mathbf{F}_1$ and $\mathbf{F}_2$ about any axis must be zero. First, we sum moments about A. Since the moment of $\mathbf{F}_1$ is obviously zero, the moment of $\mathbf{F}_2$ must also be zero and the line of action of $\mathbf{F}_2$ must pass through A (Fig. 4.8*b*). Summing moments about B, we prove similarly that the line of action of $\mathbf{F}_1$ must pass through B (Fig. 4.8*c*). Therefore, both forces have the same line of action (line AB). From either of the equations $\Sigma F_x = 0$ and $\Sigma F_y = 0$ it is seen that they must also have the same magnitude but opposite sense.

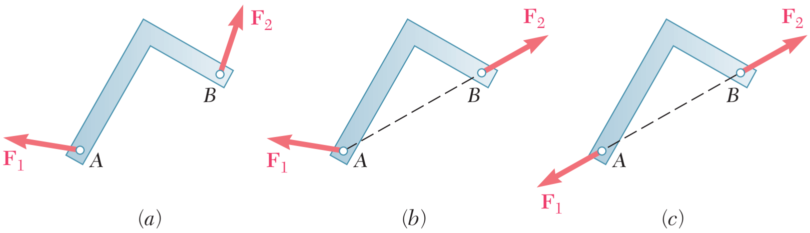


Fig. 4.8

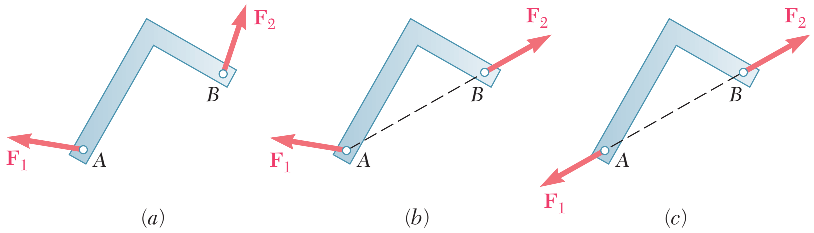


Fig. 4.8 (*repeated*)

If several forces act at two points A and B, the forces acting at A can be replaced by their resultant $\mathbf{F}_1$ and those acting at B can be replaced by their resultant $\mathbf{F}_2$. Thus a two-force body can be more generally defined as *a rigid body subjected to forces acting at only two points.* The resultants $\mathbf{F}_1$ and $\mathbf{F}_2$ then must have the same line of action, the same magnitude, and opposite sense (Fig. 4.8).

In the study of structures, frames, and machines, you will see how the recognition of two-force bodies simplifies the solution of certain problems.

4.7 EQUILIBRIUM OF A THREE-FORCE BODY

Another case of equilibrium that is of great interest is that of a *three-force body,* i.e., a rigid body subjected to three forces or, more generally, *a rigid body subjected to forces acting at only three points.* Consider a rigid body subjected to a system of forces which can be reduced to three forces $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$ acting at A, B, and C, respectively (Fig. 4.9*a*). It will be shown that if the body is in equilibrium, *the lines of action of the three forces must be either concurrent or parallel.*

Since the rigid body is in equilibrium, the sum of the moments of $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$ about any axis must be zero. Assuming that the lines of action of $\mathbf{F}_1$ and $\mathbf{F}_2$ intersect and denoting their point of intersection by D, we sum moments about D (Fig. 4.9*b*). Since the moments of $\mathbf{F}_1$ and $\mathbf{F}_2$ about D are zero, the moment of $\mathbf{F}_3$ about D must also be zero, and the line of action of $\mathbf{F}_3$ must pass through D (Fig. 4.9*c*). Therefore, the three lines of action are concurrent. The only exception occurs when none of the lines intersect; the lines of action are then parallel.

Although problems concerning three-force bodies can be solved by the general methods of Secs. 4.3 to 4.5, the property just established can be used to solve them either graphically or mathematically from simple trigonometric or geometric relations.

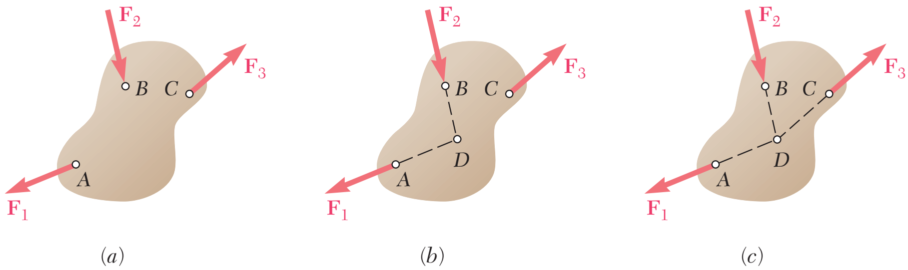
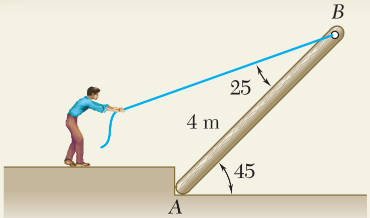


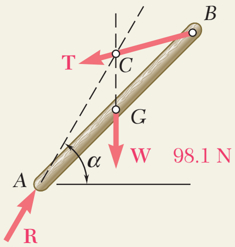
Fig. 4.9



SAMPLE PROBLEM 4.6

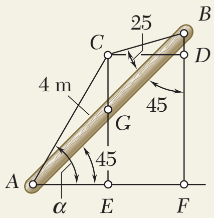
A man raises a 10-kg joist, of length 4 m, by pulling on a rope. Find the tension T in the rope and the reaction at A.

SOLUTION



Free-Body Diagram. The joist is a three-force body, since it is acted upon by three forces: its weight **W**, the force **T** exerted by the rope, and the reaction **R** of the ground at A. We note that

$$W = mg = (10 \text{ kg})(9.81 \text{ m/s}^2) = 98.1 \text{ N}$$



Three-Force Body. Since the joist is a three-force body, the forces acting on it must be concurrent. The reaction **R**, therefore, will pass through the point of intersection C of the lines of action of the weight **W** and the tension force **T**. This fact will be used to determine the angle α that **R** forms with the horizontal.

Drawing the vertical BF through B and the horizontal CD through C, we note that

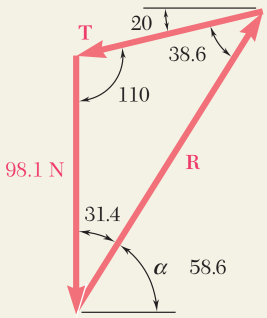
$$\begin{aligned}
AF &= BF = (AB)\cos 45° = (4 \text{ m})\cos 45° = 2.828 \text{ m}\\
CD &= EF = AE = \tfrac{1}{2}(AF) = 1.414 \text{ m}\\
BD &= (CD)\cot(45° + 25°) = (1.414 \text{ m})\tan 20° = 0.515 \text{ m}\\
CE &= DF = BF - BD = 2.828 \text{ m} - 0.515 \text{ m} = 2.313 \text{ m}
\end{aligned}$$

We write

$$\tan\alpha = \frac{CE}{AE} = \frac{2.313 \text{ m}}{1.414 \text{ m}} = 1.636$$

$\alpha = 58.6°$ ◀

We now know the direction of all the forces acting on the joist.



Force Triangle. A force triangle is drawn as shown, and its interior angles are computed from the known directions of the forces. Using the law of sines, we write

$$\frac{T}{\sin 31.4°} = \frac{R}{\sin 110°} = \frac{98.1 \text{ N}}{\sin 38.6°}$$

$T = 81.9 \text{ N}$ ◀

$\mathbf{R} = 147.8 \text{ N} \measuredangle 58.6°$ ◀

PROBLEMS

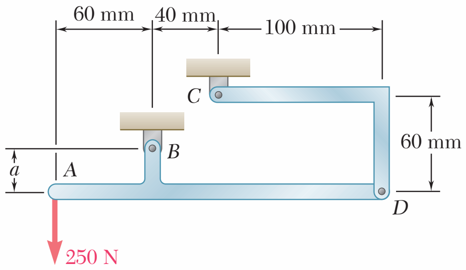


Fig. P4.27

4.27 Determine the reactions at B and C when $a = 30$ mm.

4.28 The spanner shown is used to rotate a shaft. A pin fits in a hole at A, while a flat, frictionless surface rests against the shaft at B. If a 300-N force **P** is exerted on the spanner at D, find the reactions at A and B.

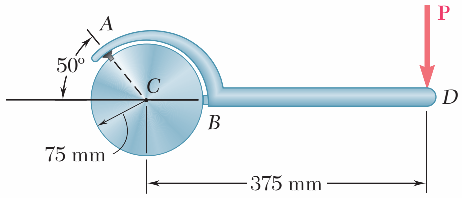


Fig. P4.28

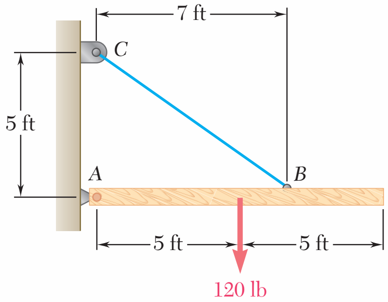


Fig. P4.29

4.29 A 10-ft wooden beam weighing 120 lb is supported by a pin and bracket at A and by cable BC. Find the reaction at A and the tension in the cable.

4.30 A T-shaped bracket supports a 300-N load as shown. Determine the reactions at A and C when (*a*) $\alpha = 90°$, (*b*) $\alpha = 45°$.

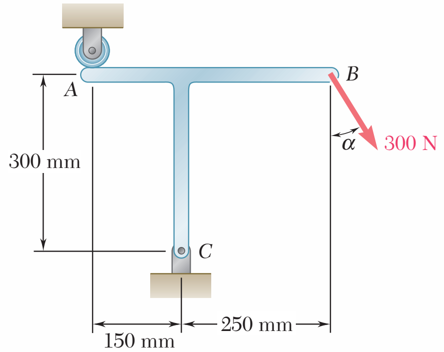


Fig. *P4.30*

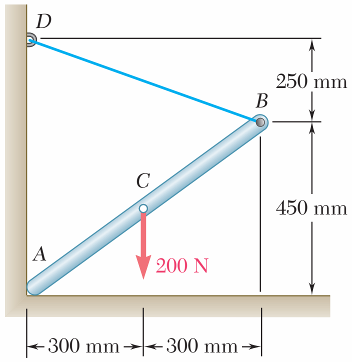


Fig. P4.31

4.31 One end of a rod AB rests in the corner A, and the other is attached to cord BD. If the rod supports a 200-N load at its midpoint C, find the reaction at A and the tension in the cord.

4.32 Using the method of Sec. 4.7, solve Prob. 4.12.

4.33 Using the method of Sec. 4.7, solve Prob. 4.13.

4.34 Using the method of Sec. 4.7, solve Prob. 4.14.

4.35 Using the method of Sec. 4.7, solve Prob. 4.15.

4.36 Determine the reactions at A and E when $\alpha = 0$.

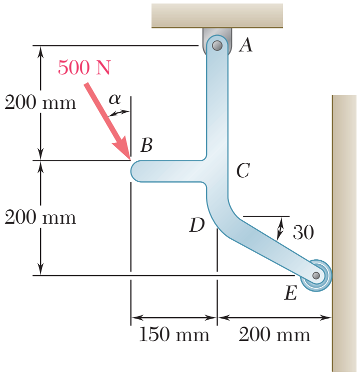


Fig. *P4.36* and P4.37

4.37 Determine (*a*) the value of α for which the reaction at A is vertical, (*b*) the corresponding reactions at A and E.

4.38 Determine the reactions at A and B when $\alpha = 90°$.

4.39 Determine the reactions at A and B when $\alpha = 30°$.

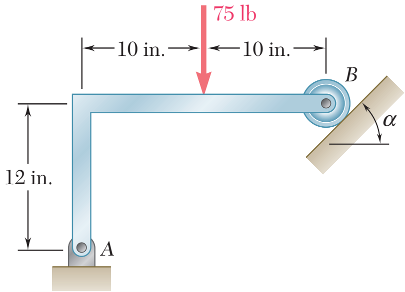


Fig. P4.38 and *P4.39*

4.40 A slender rod BC of length L and weight W is held by two cables as shown. Knowing that cable AB is horizontal and that the rod forms an angle of 40° with the horizontal, determine (*a*) the angle θ that cable CD forms with the horizontal, (*b*) the tension in each cable.

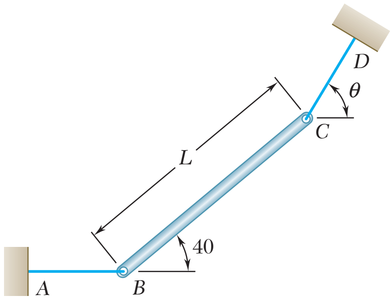


Fig. P4.40

4.41 A slender rod AB of length L and weight W is attached to a collar at A and rests on a small wheel at C. Neglecting the effect of friction and the weight of the collar, determine the angle θ corresponding to equilibrium.

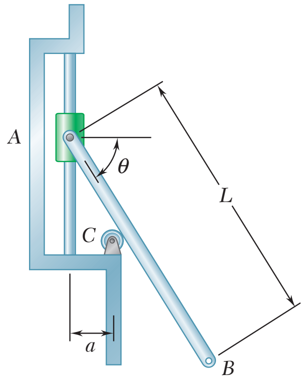


Fig. *P4.41*

4.42 Determine the reactions at A and B when $a = 7.5$ in.

4.43 Determine the value of a for which the magnitude of the reaction **B** is equal to 200 lb.

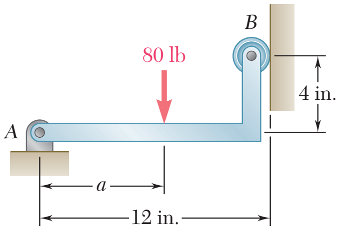


Fig. P4.42 and P4.43

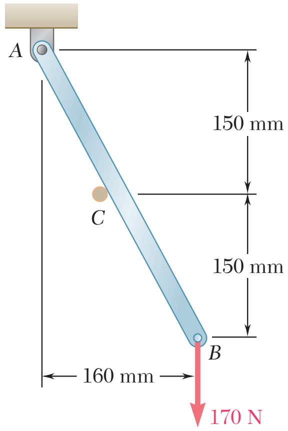


Fig. P4.44

4.44 Rod AB is supported by a pin and bracket at A and rests against a frictionless peg at C. Determine the reactions at A and C when a 170-N vertical force is applied at B.

4.45 Solve Prob. 4.44 assuming that the 170-N force applied at B is horizontal and directed to the left.

4.46 A uniform plate girder weighing 6000 lb is held in a horizontal position by two crane cables. Determine the angle α and the tension in each cable.

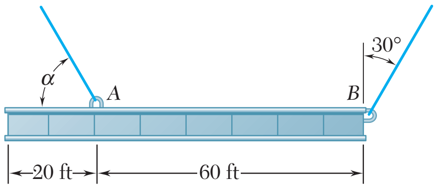


Fig. P4.46

4.47 A 12-ft ladder, weighing 40 lb, leans against a frictionless vertical wall. The lower end of the ladder rests on rough ground, 4 ft away from the wall. Determine the reactions at both ends.

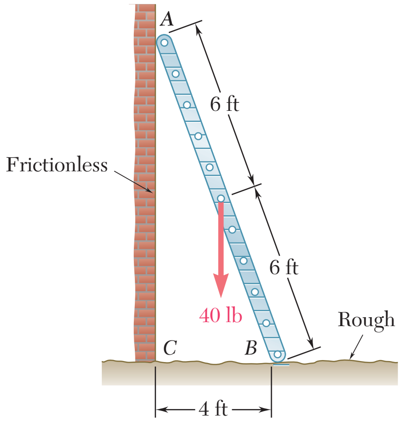


Fig. P4.47

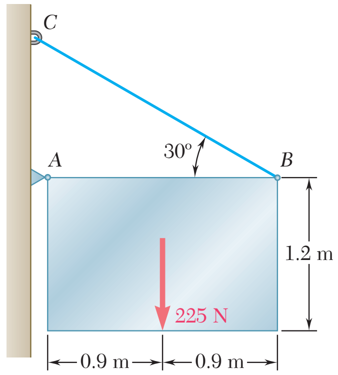


Fig. P4.48

4.48 A 225-N sign is supported by a pin and bracket at A and by a cable BC. Determine the reaction at A and the tension in the cable.

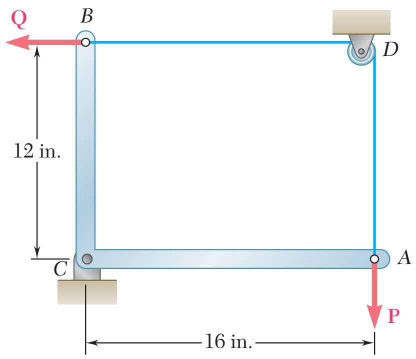


Fig. P4.49

4.49 The L-shaped member ACB is supported by a pin and bracket at C and by an inextensible cord attached at A and B and passing over a frictionless pulley at D. The tension may be assumed to be the same in portions AD and BD of the cord. If the magnitudes of the forces applied at A and B are, respectively, $P = 25$ lb and $Q = 0$, determine (*a*) the tension in the cord, (*b*) the reaction at C.

4.50 For the L-shaped member of Prob. 4.49, (*a*) express the tension T in the cord in terms of the magnitudes P and Q of the forces applied at A and B, (*b*) assuming $Q = 40$ lb, find the smallest allowable value of P if the equilibrium is to be maintained.

EQUILIBRIUM IN THREE DIMENSIONS

4.8 EQUILIBRIUM OF A RIGID BODY IN THREE DIMENSIONS

We saw in Sec. 4.1 that six scalar equations are required to express the conditions for the equilibrium of a rigid body in the general three-dimensional case:

$$\Sigma F_x = 0 \qquad \Sigma F_y = 0 \qquad \Sigma F_z = 0 \tag{4.2}$$
$$\Sigma M_x = 0 \qquad \Sigma M_y = 0 \qquad \Sigma M_z = 0 \tag{4.3}$$

These equations can be solved for no more than *six unknowns,* which generally will represent reactions at supports or connections.

In most problems the scalar equations (4.2) and (4.3) will be more conveniently obtained if we first express in vector form the conditions for the equilibrium of the rigid body considered. We write

$$\Sigma \mathbf{F} = 0 \qquad \Sigma \mathbf{M}_O = \Sigma(\mathbf{r} \times \mathbf{F}) = 0 \tag{4.1}$$

and express the forces **F** and position vectors **r** in terms of scalar components and unit vectors. Next, we compute all vector products, either by direct calculation or by means of determinants (see Sec. 3.8). We observe that as many as three unknown reaction components may be eliminated from these computations through a judicious choice of the point O. By equating to zero the coefficients of the unit vectors in each of the two relations (4.1), we obtain the desired scalar equations.†

4.9 REACTIONS AT SUPPORTS AND CONNECTIONS FOR A THREE-DIMENSIONAL STRUCTURE

The reactions on a three-dimensional structure range from the single force of known direction exerted by a frictionless surface to the force-couple system exerted by a fixed support. Consequently, in problems involving the equilibrium of a three-dimensional structure, there can be between one and six unknowns associated with the reaction at each support or connection. Various types of supports and

†In some problems, it will be found convenient to eliminate the reactions at two points A and B from the solution by writing the equilibrium equation $\Sigma M_{AB} = 0$, which involves the determination of the moments of the forces about the axis AB joining points A and B (see Sample Prob. 4.10).

Photo 4.6 Universal joints, easily seen on the drive shafts of rear-wheel-drive cars and trucks, allow rotational motion to be transferred between two noncollinear shafts.

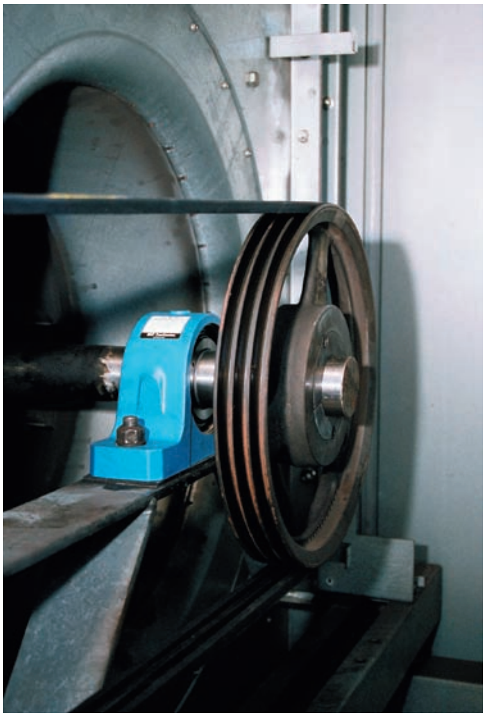


Photo 4.7 The pillow block bearing shown supports the shaft of a fan used in an industrial facility.

connections are shown in Fig. 4.10 with their corresponding reactions. A simple way of determining the type of reaction corresponding to a given support or connection and the number of unknowns involved is to find which of the six fundamental motions (translation in the x, y, and z directions and rotation about the x, y, and z axes) are allowed and which motions are prevented.

Ball supports, frictionless surfaces, and cables, for example, prevent translation in one direction only and thus exert a single force whose line of action is known; each of these supports involves one unknown, namely, the magnitude of the reaction. Rollers on rough surfaces and wheels on rails prevent translation in two directions; the corresponding reactions consist of two unknown force components. Rough surfaces in direct contact and ball-and-socket supports prevent translation in three directions; these supports involve three unknown force components.

Some supports and connections can prevent rotation as well as translation; the corresponding reactions include couples as well as forces. For example, the reaction at a fixed support, which prevents any motion (rotation as well as translation), consists of three unknown forces and three unknown couples. A universal joint, which is designed to allow rotation about two axes, will exert a reaction consisting of three unknown force components and one unknown couple.

Other supports and connections are primarily intended to prevent translation; their design, however, is such that they also prevent some rotations. The corresponding reactions consist essentially of force components but *may* also include couples. One group of supports of this type includes hinges and bearings designed to support radial loads only (for example, journal bearings, roller bearings). The corresponding reactions consist of two force components but may also include two couples. Another group includes pin-and-bracket supports, hinges, and bearings designed to support an axial thrust as well as a radial load (for example, ball bearings). The corresponding reactions consist of three force components but may include two couples. However, these supports will not exert any appreciable couples under normal conditions of use. Therefore, *only* force components should be included in their analysis *unless* it is found that couples are necessary to maintain the equilibrium of the rigid body, or unless the support is known to have been specifically designed to exert a couple (see Probs. 4.71 and 4.72).

If the reactions involve more than six unknowns, there are more unknowns than equations, and some of the reactions are *statically indeterminate.* If the reactions involve fewer than six unknowns, there are more equations than unknowns, and some of the equations of equilibrium cannot be satisfied under general loading conditions; the rigid body is only *partially constrained.* Under the particular loading conditions corresponding to a given problem, however, the extra equations often reduce to trivial identities, such as $0 = 0$, and can be disregarded; although only partially constrained, the rigid body remains in equilibrium (see Sample Probs. 4.7 and 4.8). Even with six or more unknowns, it is possible that some equations of equilibrium will not be satisfied. This can occur when the reactions associated with the given supports either are parallel or intersect the same line; the rigid body is then *improperly constrained.*

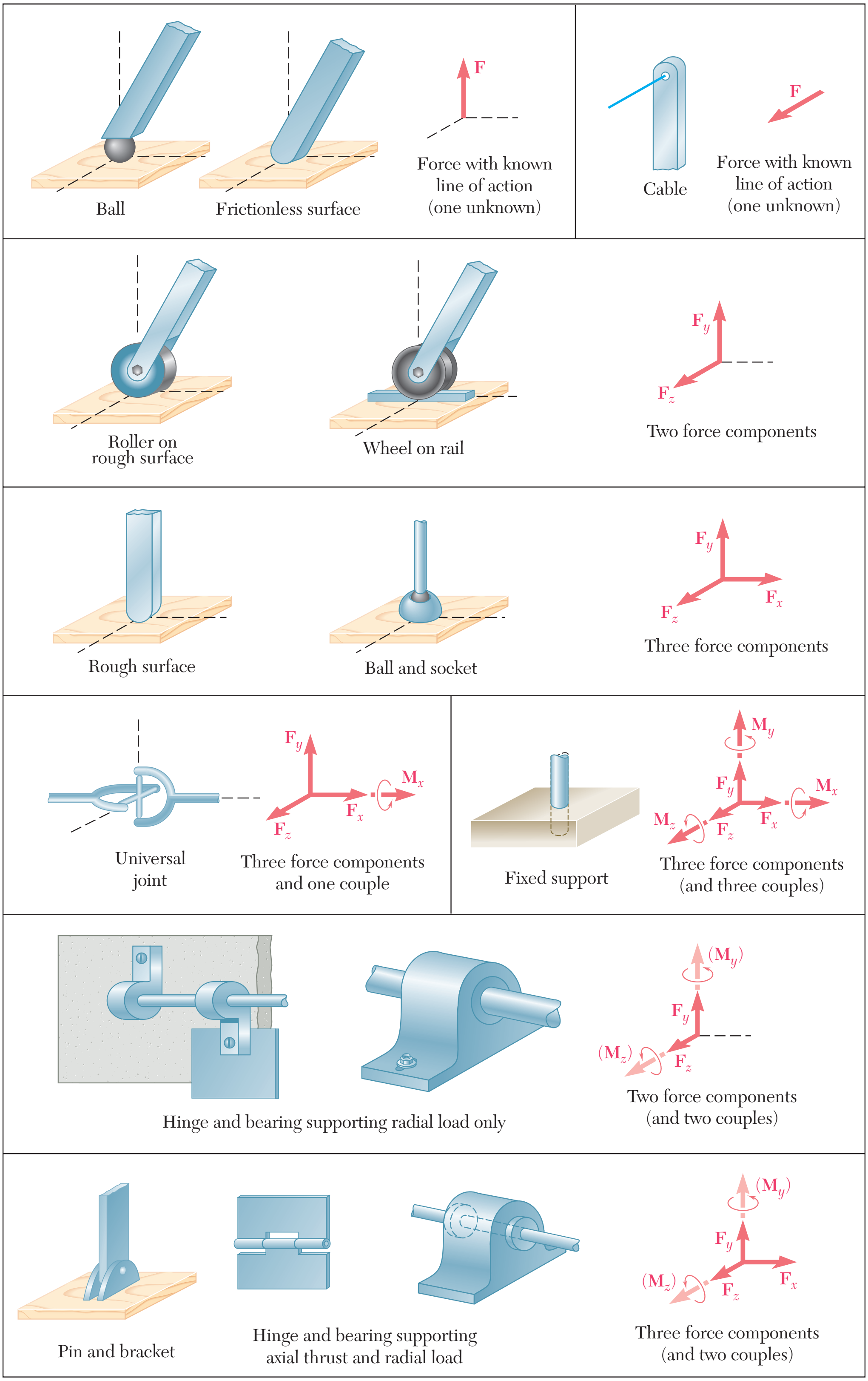
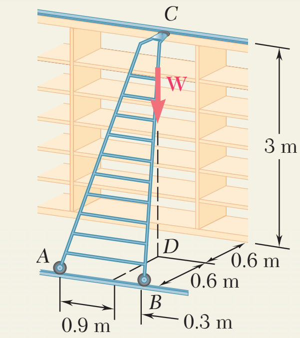


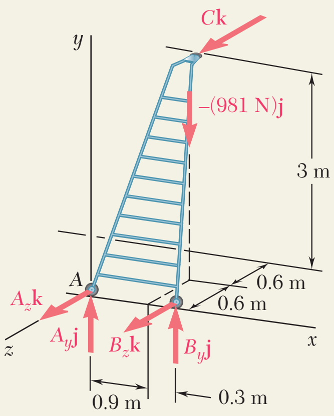
Fig. 4.10 Reactions at supports and connections.

SAMPLE PROBLEM 4.7



A 20-kg ladder used to reach high shelves in a storeroom is supported by two flanged wheels A and B mounted on a rail and by an unflanged wheel C resting against a rail fixed to the wall. An 80-kg man stands on the ladder and leans to the right. The line of action of the combined weight **W** of the man and ladder intersects the floor at point D. Determine the reactions at A, B, and C.

SOLUTION



Free-Body Diagram. A free-body diagram of the ladder is drawn. The forces involved are the combined weight of the man and ladder,

$$\mathbf{W} = -mg\mathbf{j} = -(80 \text{ kg} + 20 \text{ kg})(9.81 \text{ m/s}^2)\mathbf{j} = -(981 \text{ N})\mathbf{j}$$

and five unknown reaction components, two at each flanged wheel and one at the unflanged wheel. The ladder is thus only partially constrained; it is free to roll along the rails. It is, however, in equilibrium under the given load since the equation $\Sigma F_x = 0$ is satisfied.

Equilibrium Equations. We express that the forces acting on the ladder form a system equivalent to zero:

$$\Sigma\mathbf{F} = 0: \quad A_y\mathbf{j} + A_z\mathbf{k} + B_y\mathbf{j} + B_z\mathbf{k} - (981 \text{ N})\mathbf{j} + C\mathbf{k} = 0$$
$$(A_y + B_y - 981 \text{ N})\mathbf{j} + (A_z + B_z + C)\mathbf{k} = 0 \qquad (1)$$
$$\Sigma\mathbf{M}_A = \Sigma(\mathbf{r} \times \mathbf{F}) = 0: \quad 1.2\mathbf{i} \times (B_y\mathbf{j} + B_z\mathbf{k}) + (0.9\mathbf{i} - 0.6\mathbf{k}) \times (-981\mathbf{j})$$
$$+ (0.6\mathbf{i} + 3\mathbf{j} - 1.2\mathbf{k}) \times C\mathbf{k} = 0$$

Computing the vector products, we have†

$$1.2B_y\mathbf{k} - 1.2B_z\mathbf{j} - 882.9\mathbf{k} - 588.6\mathbf{i} - 0.6C\mathbf{j} + 3C\mathbf{i} = 0$$
$$(3C - 588.6)\mathbf{i} - (1.2B_z + 0.6C)\mathbf{j} + (1.2B_y - 882.9)\mathbf{k} = 0 \qquad (2)$$

Setting the coefficients of **i**, **j**, **k** equal to zero in Eq. (2), we obtain the following three scalar equations, which express that the sum of the moments about each coordinate axis must be zero:

$$3C - 588.6 = 0 \qquad C = +196.2 \text{ N}$$
$$1.2B_z + 0.6C = 0 \qquad B_z = -98.1 \text{ N}$$
$$1.2B_y - 882.9 = 0 \qquad B_y = +736 \text{ N}$$

The reactions at B and C are therefore

$$\mathbf{B} = +(736 \text{ N})\mathbf{j} - (98.1 \text{ N})\mathbf{k} \qquad \mathbf{C} = +(196.2 \text{ N})\mathbf{k} \quad ◀$$

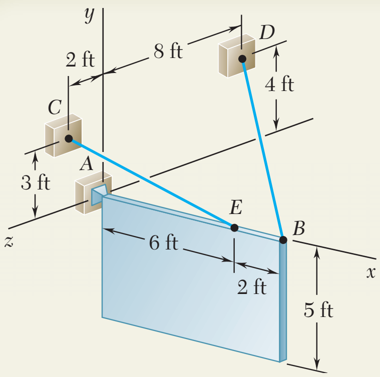
Setting the coefficients of **j** and **k** equal to zero in Eq. (1), we obtain two scalar equations expressing that the sums of the components in the y and z directions are zero. Substituting for B_y, B_z, and C the values obtained above, we write

$$A_y + B_y - 981 = 0 \qquad A_y + 736 - 981 = 0 \qquad A_y = +245 \text{ N}$$
$$A_z + B_z + C = 0 \qquad A_z - 98.1 + 196.2 = 0 \qquad A_z = -98.1 \text{ N}$$

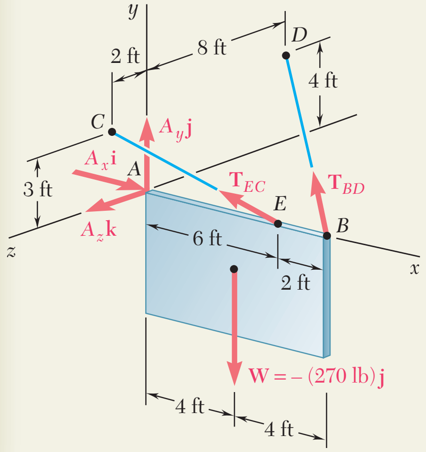
We conclude that the reaction at A is $\quad \mathbf{A} = +(245 \text{ N})\mathbf{j} - (98.1 \text{ N})\mathbf{k} \quad ◀$

†The moments in this sample problem and in Sample Probs. 4.8 and 4.9 can also be expressed in the form of determinants (see Sample Prob. 3.10).

SAMPLE PROBLEM 4.8



A 5 × 8-ft sign of uniform density weighs 270 lb and is supported by a ball-and-socket joint at A and by two cables. Determine the tension in each cable and the reaction at A.



SOLUTION

Free-Body Diagram. A free-body diagram of the sign is drawn. The forces acting on the free body are the weight $\mathbf{W} = -(270 \text{ lb})\mathbf{j}$ and the reactions at A, B, and E. The reaction at A is a force of unknown direction and is represented by three unknown components. Since the directions of the forces exerted by the cables are known, these forces involve only one unknown each, namely, the magnitudes T_{BD} and T_{EC}. Since there are only five unknowns, the sign is partially constrained. It can rotate freely about the x axis; it is, however, in equilibrium under the given loading, since the equation $\Sigma M_x = 0$ is satisfied.

The components of the forces $\mathbf{T}_{BD}$ and $\mathbf{T}_{EC}$ can be expressed in terms of the unknown magnitudes T_{BD} and T_{EC} by writing

$$\overrightarrow{BD} = -(8 \text{ ft})\mathbf{i} + (4 \text{ ft})\mathbf{j} - (8 \text{ ft})\mathbf{k} \qquad BD = 12 \text{ ft}$$
$$\overrightarrow{EC} = -(6 \text{ ft})\mathbf{i} + (3 \text{ ft})\mathbf{j} + (2 \text{ ft})\mathbf{k} \qquad EC = 7 \text{ ft}$$
$$\mathbf{T}_{BD} = T_{BD}\left(\frac{\overrightarrow{BD}}{BD}\right) = T_{BD}(-\tfrac{2}{3}\mathbf{i} + \tfrac{1}{3}\mathbf{j} - \tfrac{2}{3}\mathbf{k})$$
$$\mathbf{T}_{EC} = T_{EC}\left(\frac{\overrightarrow{EC}}{EC}\right) = T_{EC}(-\tfrac{6}{7}\mathbf{i} + \tfrac{3}{7}\mathbf{j} - \tfrac{2}{7}\mathbf{k})$$

Equilibrium Equations. We express that the forces acting on the sign form a system equivalent to zero:

$$\Sigma\mathbf{F} = 0: \quad A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k} + \mathbf{T}_{BD} + \mathbf{T}_{EC} - (270 \text{ lb})\mathbf{j} = 0$$
$$(A_x - \tfrac{2}{3}T_{BD} - \tfrac{6}{7}T_{EC})\mathbf{i} + (A_y + \tfrac{1}{3}T_{BD} + \tfrac{3}{7}T_{EC} - 270 \text{ lb})\mathbf{j} + (A_z - \tfrac{2}{3}T_{BD} + \tfrac{2}{7}T_{EC})\mathbf{k} = 0 \qquad (1)$$

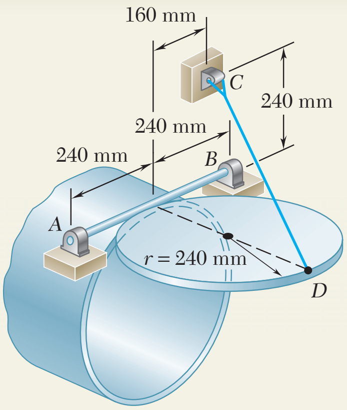
$$\Sigma\mathbf{M}_A = \Sigma(\mathbf{r} \times \mathbf{F}) = 0:$$
$$(8 \text{ ft})\mathbf{i} \times T_{BD}(-\tfrac{2}{3}\mathbf{i} + \tfrac{1}{3}\mathbf{j} - \tfrac{2}{3}\mathbf{k}) + (6 \text{ ft})\mathbf{i} \times T_{EC}(-\tfrac{6}{7}\mathbf{i} + \tfrac{3}{7}\mathbf{j} + \tfrac{2}{7}\mathbf{k}) + (4 \text{ ft})\mathbf{i} \times (-270 \text{ lb})\mathbf{j} = 0$$
$$(2.667T_{BD} + 2.571T_{EC} - 1080 \text{ lb})\mathbf{k} + (5.333T_{BD} - 1.714T_{EC})\mathbf{j} = 0 \qquad (2)$$

Setting the coefficients of $\mathbf{j}$ and $\mathbf{k}$ equal to zero in Eq. (2), we obtain two scalar equations which can be solved for T_{BD} and T_{EC}:

$$T_{BD} = 101.3 \text{ lb} \qquad T_{EC} = 315 \text{ lb}$$

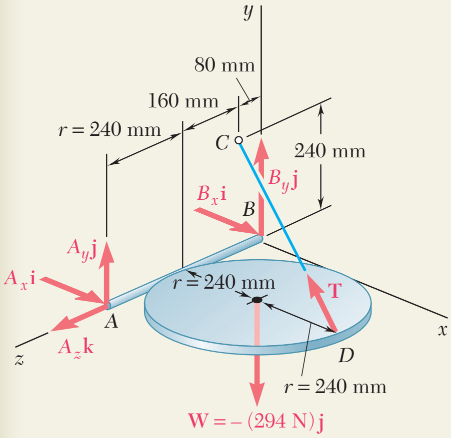
Setting the coefficients of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ equal to zero in Eq. (1), we obtain three more equations, which yield the components of $\mathbf{A}$. We have

$$\mathbf{A} = +(338 \text{ lb})\mathbf{i} + (101.2 \text{ lb})\mathbf{j} - (22.5 \text{ lb})\mathbf{k}$$



SAMPLE PROBLEM 4.9

A uniform pipe cover of radius r = 240 mm and mass 30 kg is held in a horizontal position by the cable CD. Assuming that the bearing at B does not exert any axial thrust, determine the tension in the cable and the reactions at A and B.



SOLUTION

Free-Body Diagram. A free-body diagram is drawn with the coordinate axes shown. The forces acting on the free body are the weight of the cover,

$$\mathbf{W} = -mg\mathbf{j} = -(30 \text{ kg})(9.81 \text{ m/s}^2)\mathbf{j} = -(294 \text{ N})\mathbf{j}$$

and reactions involving six unknowns, namely, the magnitude of the force $\mathbf{T}$ exerted by the cable, three force components at hinge A, and two at hinge B. The components of $\mathbf{T}$ are expressed in terms of the unknown magnitude T by resolving the vector $\overrightarrow{DC}$ into rectangular components and writing

$$\overrightarrow{DC} = -(480 \text{ mm})\mathbf{i} + (240 \text{ mm})\mathbf{j} - (160 \text{ mm})\mathbf{k} \qquad DC = 560 \text{ mm}$$

$$\mathbf{T} = T\frac{\overrightarrow{DC}}{DC} = -\tfrac{6}{7}T\mathbf{i} + \tfrac{3}{7}T\mathbf{j} - \tfrac{2}{7}T\mathbf{k}$$

Equilibrium Equations. We express that the forces acting on the pipe cover form a system equivalent to zero:

$$\Sigma\mathbf{F} = 0: \qquad A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k} + B_x\mathbf{i} + B_y\mathbf{j} + \mathbf{T} - (294 \text{ N})\mathbf{j} = 0$$

$$(A_x + B_x - \tfrac{6}{7}T)\mathbf{i} + (A_y + B_y + \tfrac{3}{7}T - 294 \text{ N})\mathbf{j} + (A_z - \tfrac{2}{7}T)\mathbf{k} = 0 \qquad (1)$$

$$\Sigma\mathbf{M}_B = \Sigma(\mathbf{r} \times \mathbf{F}) = 0:$$

$$2r\mathbf{k} \times (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k})$$
$$+ (2r\mathbf{i} + r\mathbf{k}) \times (-\tfrac{6}{7}T\mathbf{i} + \tfrac{3}{7}T\mathbf{j} - \tfrac{2}{7}T\mathbf{k})$$
$$+ (r\mathbf{i} + r\mathbf{k}) \times (-294 \text{ N})\mathbf{j} = 0$$

$$(-2A_y - \tfrac{3}{7}T + 294 \text{ N})r\mathbf{i} + (2A_x - \tfrac{2}{7}T)r\mathbf{j} + (\tfrac{6}{7}T - 294 \text{ N})r\mathbf{k} = 0 \qquad (2)$$

Setting the coefficients of the unit vectors equal to zero in Eq. (2), we write three scalar equations, which yield

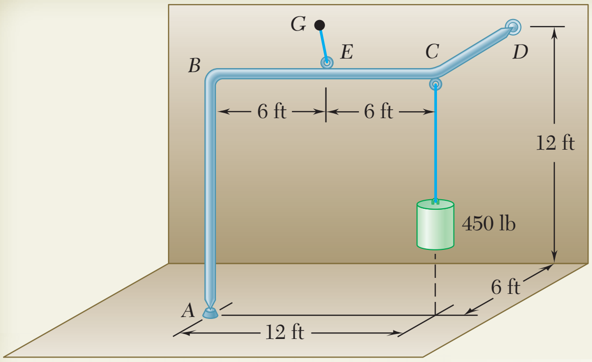
$$A_x = +49.0 \text{ N} \qquad A_y = +73.5 \text{ N} \qquad T = 343 \text{ N} \quad \blacktriangleleft$$

Setting the coefficients of the unit vectors equal to zero in Eq. (1), we obtain three more scalar equations. After substituting the values of T, A_x, and A_y into these equations, we obtain

$$A_z = +98.0 \text{ N} \qquad B_x = +245 \text{ N} \qquad B_y = +73.5 \text{ N}$$

The reactions at A and B are therefore

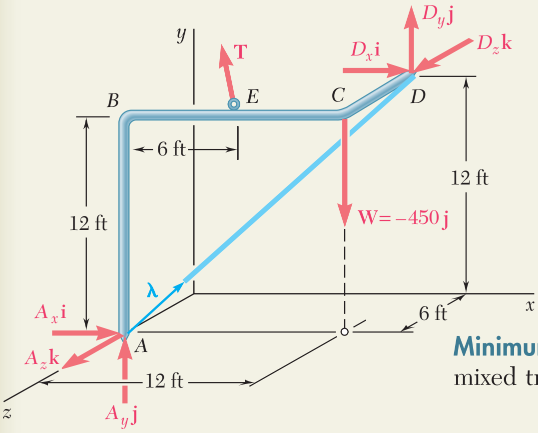
$$\mathbf{A} = +(49.0 \text{ N})\mathbf{i} + (73.5 \text{ N})\mathbf{j} + (98.0 \text{ N})\mathbf{k} \quad \blacktriangleleft$$
$$\mathbf{B} = +(245 \text{ N})\mathbf{i} + (73.5 \text{ N})\mathbf{j} \quad \blacktriangleleft$$



SAMPLE PROBLEM 4.10

A 450-lb load hangs from the corner C of a rigid piece of pipe $ABCD$ which has been bent as shown. The pipe is supported by the ball-and-socket joints A and D, which are fastened, respectively, to the floor and to a vertical wall, and by a cable attached at the midpoint E of the portion BC of the pipe and at a point G on the wall. Determine (*a*) where G should be located if the tension in the cable is to be minimum, (*b*) the corresponding minimum value of the tension.

SOLUTION



Free-Body Diagram. The free-body diagram of the pipe includes the load $\mathbf{W} = (-450 \text{ lb})\mathbf{j}$, the reactions at A and D, and the force $\mathbf{T}$ exerted by the cable. To eliminate the reactions at A and D from the computations, we express that the sum of the moments of the forces about AD is zero. Denoting by $\boldsymbol{\lambda}$ the unit vector along AD, we write

$$\Sigma M_{AD} = 0: \qquad \boldsymbol{\lambda} \cdot (\overrightarrow{AE} \times \mathbf{T}) + \boldsymbol{\lambda} \cdot (\overrightarrow{AC} \times \mathbf{W}) = 0 \qquad (1)$$

The second term in Eq. (1) can be computed as follows:

$$\overrightarrow{AC} \times \mathbf{W} = (12\mathbf{i} + 12\mathbf{j}) \times (-450\mathbf{j}) = -5400\mathbf{k}$$

$$\boldsymbol{\lambda} = \frac{\overrightarrow{AD}}{AD} = \frac{12\mathbf{i} + 12\mathbf{j} - 6\mathbf{k}}{18} = \tfrac{2}{3}\mathbf{i} + \tfrac{2}{3}\mathbf{j} - \tfrac{1}{3}\mathbf{k}$$

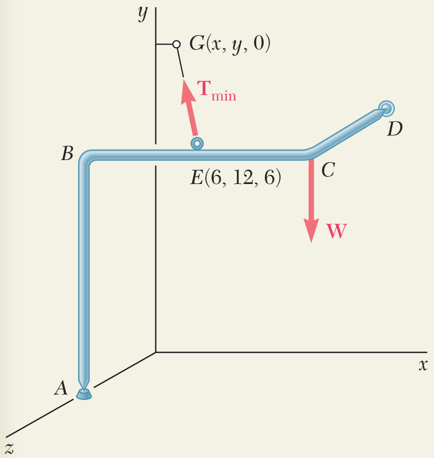
$$\boldsymbol{\lambda} \cdot (\overrightarrow{AC} \times \mathbf{W}) = (\tfrac{2}{3}\mathbf{i} + \tfrac{2}{3}\mathbf{j} - \tfrac{1}{3}\mathbf{k}) \cdot (-5400\mathbf{k}) = +1800$$

Substituting the value obtained into Eq. (1), we write

$$\boldsymbol{\lambda} \cdot (\overrightarrow{AE} \times \mathbf{T}) = -1800 \text{ lb} \cdot \text{ft} \qquad (2)$$

Minimum Value of Tension. Recalling the commutative property for mixed triple products, we rewrite Eq. (2) in the form

$$\mathbf{T} \cdot (\boldsymbol{\lambda} \times \overrightarrow{AE}) = -1800 \text{ lb} \cdot \text{ft} \qquad (3)$$



which shows that the projection of $\mathbf{T}$ on the vector $\boldsymbol{\lambda} \times \overrightarrow{AE}$ is a constant. It follows that $\mathbf{T}$ is minimum when parallel to the vector

$$\boldsymbol{\lambda} \times \overrightarrow{AE} = (\tfrac{2}{3}\mathbf{i} + \tfrac{2}{3}\mathbf{j} - \tfrac{1}{3}\mathbf{k}) \times (6\mathbf{i} + 12\mathbf{j}) = 4\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$$

Since the corresponding unit vector is $\tfrac{2}{3}\mathbf{i} - \tfrac{1}{3}\mathbf{j} + \tfrac{2}{3}\mathbf{k}$, we write

$$\mathbf{T}_{\text{min}} = T(\tfrac{2}{3}\mathbf{i} - \tfrac{1}{3}\mathbf{j} + \tfrac{2}{3}\mathbf{k}) \qquad (4)$$

Substituting for $\mathbf{T}$ and $\boldsymbol{\lambda} \times \overrightarrow{AE}$ in Eq. (3) and computing the dot products, we obtain $6T = -1800$ and, thus, $T = -300$. Carrying this value into (4), we obtain

$$\mathbf{T}_{\text{min}} = -200\mathbf{i} + 100\mathbf{j} - 200\mathbf{k} \qquad T_{\text{min}} = 300 \text{ lb} \quad ◀$$

Location of G. Since the vector $\overrightarrow{EG}$ and the force $\mathbf{T}_{\text{min}}$ have the same direction, their components must be proportional. Denoting the coordinates of G by x, y, 0, we write

$$\frac{x-6}{-200} = \frac{y-12}{+100} = \frac{0-6}{-200} \qquad x = 0 \qquad y = 15 \text{ ft} \quad ◀$$

PROBLEMS

4.51 Two transmission belts pass over a double-sheaved pulley that is attached to an axle supported by bearings at A and D. The radius of the inner sheave is 125 mm and the radius of the outer sheave is 250 mm. Knowing that when the system is at rest, the tension is 90 N in both portions of belt B and 150 N in both portions of belt C, determine the reactions at A and D. Assume that the bearing at D does not exert any axial thrust.

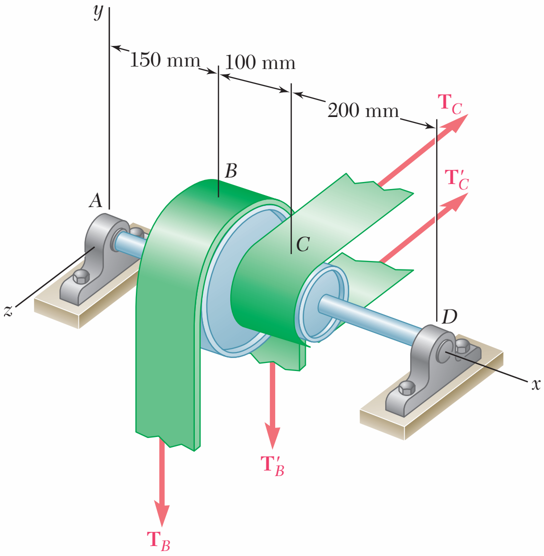


Fig. P4.51

4.52 Solve Prob. 4.51, assuming that the pulley rotates at a constant rate and that $T_B = 104$ N, $T'_B = 84$ N, and $T_C = 175$ N.

4.53 A 4 × 8 ft sheet of plywood weighing 40 lb has been temporarily propped against column CD. It rests at A and B on small wooden blocks and against protruding nails. Neglecting friction at all the surfaces of contact, determine the reactions at A, B and C.

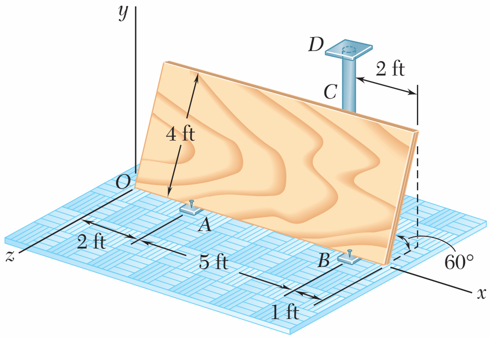


Fig. P4.53

4.54 A small wrench is used to raise a 120-lb load. Find (*a*) the magnitude of the vertical force **P** that should be applied at *C* to maintain equilibrium in the position shown, (*b*) the reactions at *A* and *B*, assuming that the bearing at *B* does not exert any axial thrust.

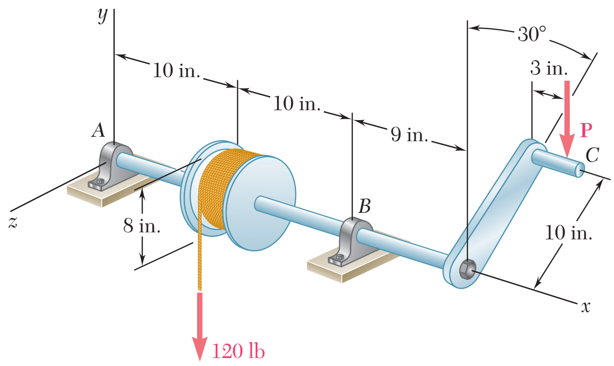


Fig. P4.54

4.55 A 200-mm lever and a 240-mm-diameter pulley are welded to the axle *BE* that is supported by bearings at *C* and *D*. If a 720-N vertical load is applied at *A* when the lever is horizontal, determine (*a*) the tension in the cord, (*b*) the reactions at *C* and *D*. Assume that the bearing at *D* does not exert any axial thrust.

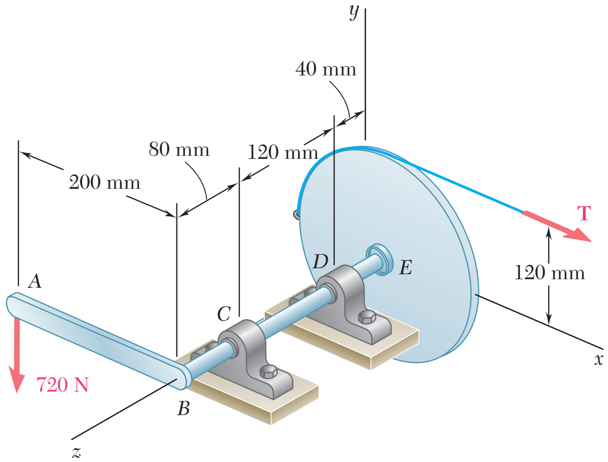


Fig. P4.55

4.56 Solve Prob. 4.55 assuming that the axle has been rotated clockwise in its bearings by 30° and that the 720-N load remains vertical.

4.57 The rectangular plate shown weighs 80 lb and is supported by three wires. Determine the tension in each wire.

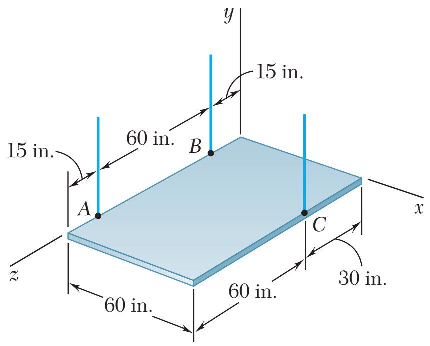


Fig. P4.57

4.58 A load *W* is to be placed on the 80-lb plate of Prob. 4.57. Determine the magnitude of *W* and the point where it should be placed if the tension is to be 60 lb in each of the three wires.

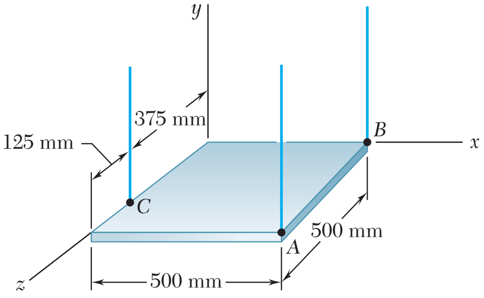


Fig. P4.59

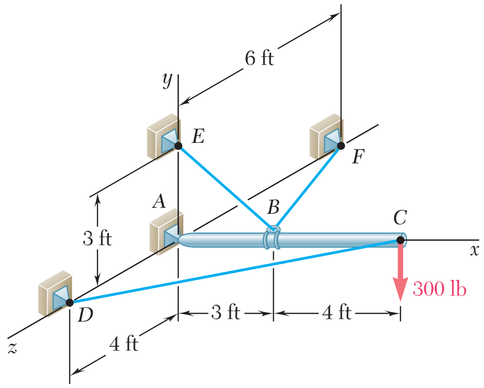


Fig. *P4.63*

4.59 The 20-kg square plate is supported by the three wires shown. Determine the tension in each wire.

4.60 Determine the mass and location of the smallest block that should be placed on the 20-kg plate of Prob. 4.59 if the tensions in the three wires are to be equal.

4.61 The 12-ft boom *AB* is acted upon by the 850-lb force shown. Determine (*a*) the tension in each cable, (*b*) the reaction of the ball and socket at *A*.

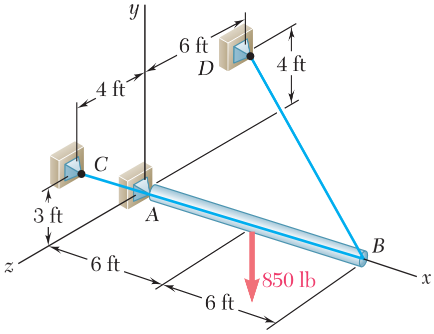


Fig. P4.61

4.62 Solve Prob. 4.61 assuming that the 850-lb load is applied at point *B*.

4.63 A 7-ft boom is held by a ball and socket at *A* and by two cables *EBF* and *DC*; cable *EBF* passes around a frictionless pulley at *B*. Determine the tension in each cable.

4.64 A 300-kg crate hangs from a cable that passes over a pulley *B* and is attached to a support at *H*. The 100-kg boom *AB* is supported by a ball and socket at *A* and by two cables *DE* and *DF*. The center of gravity of the boom is located at *G*. Determine (*a*) the tension in cables *DE* and *DF*, (*b*) the reaction at *A*.

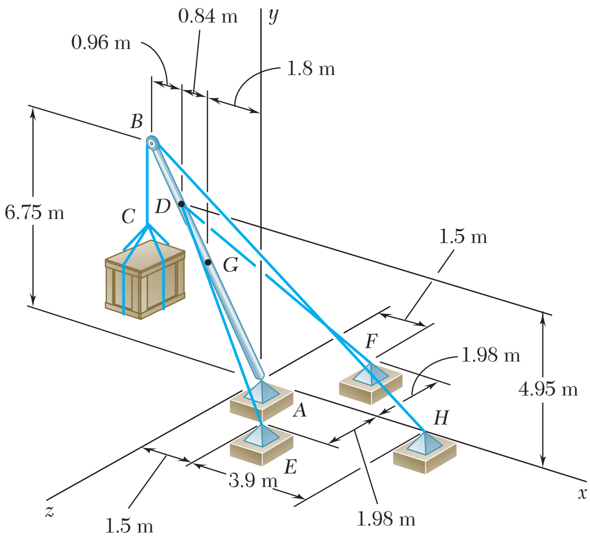


Fig. P4.64

4.65 The horizontal platform $ABCD$ weighs 60 lb and supports a 240-lb load at its center. The platform is normally held in position by hinges at A and B and by braces CE and DE. If brace DE is removed, determine the reactions at the hinges and the force exerted by the remaining brace CE. The hinge at A does not exert any axial thrust.

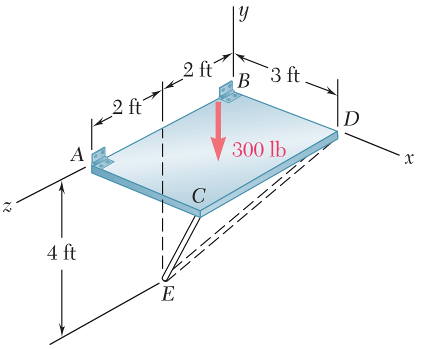


Fig. P4.65

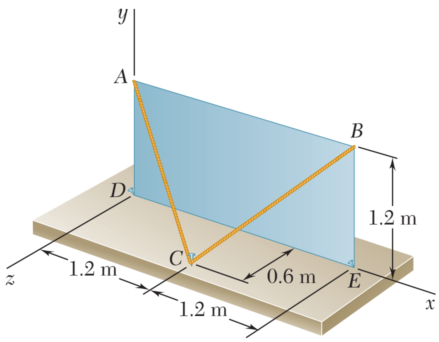


Fig. P4.66

4.66 A 1.2 × 2.4-m sheet of plywood is temporarily held by nails at D and E and by two wooden braces nailed at A, B and C. Wind is blowing on the hidden face of the plywood sheet, and it is assumed that its effect may be represented by a force $P\mathbf{k}$ applied at the center of the sheet. Knowing that each brace becomes unsafe with respect to buckling when subjected to a 1.8-kN axial force, determine (*a*) the maximum allowable value of the magnitude of P of the wind force, (*b*) the corresponding value of the z component of the reaction at E. Assume that the nails are loose and do not exert any couple.

4.67 A 3 × 4-ft plate weighs 150 lb and is supported by hinges at A and B. It is held in the position shown by the 2-ft chain CD. Assuming that the hinge at A does not exert any axial thrust, determine the tension in the chain and the reactions at A and B.

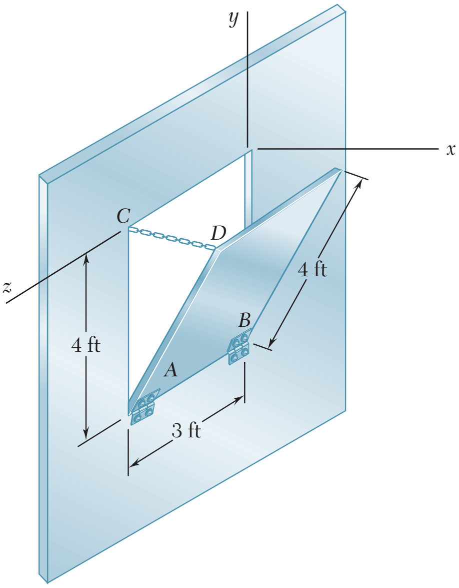


Fig. P4.67

4.68 The lid of a roof scuttle weighs 75 lb. It is hinged at corners A and B and maintained in the desired position by a rod CD pivoted at C; a pin at end D of the rod fits into one of several holes drilled in the edge of the lid. For $\alpha = 50°$, determine (*a*) the magnitude of the force exerted by rod CD, (*b*) the reactions at the hinges. Assume that the hinge at B does not exert any axial thrust.

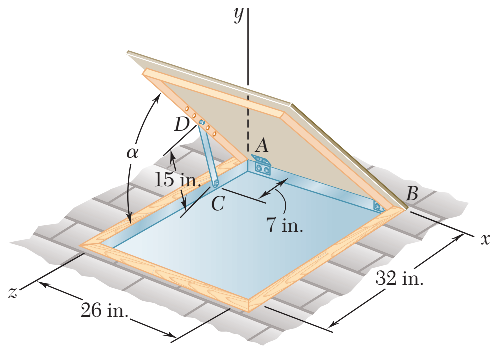


Fig. P4.68

4.69 A 10-kg storm window measuring 900 × 1500 mm is held by hinges at *A* and *B*. In the position shown, it is held away from the side of the house by a 600-mm stick *CD*. Assuming that the hinge at *A* does not exert any axial thrust, determine the magnitude of the force exerted by the stick and the components of the reactions *A* and *B*.

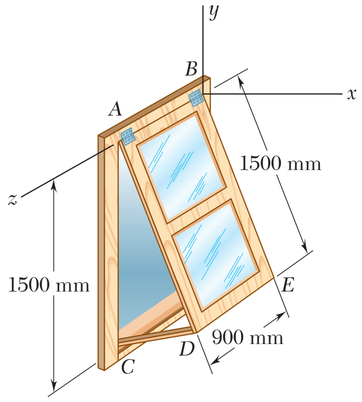


Fig. *P4.69*

4.70 A 20-kg door is made self-closing by hanging a 15-kg counterweight from a cable attached at *C*. The door is held open by a force **P** applied at the knob *D* in a direction perpendicular to the door. Determine the magnitude of **P** and the components of the reactions *A* and *B* when $\theta = 90°$. It is assumed that the hinge at *A* does not exert any axial thrust.

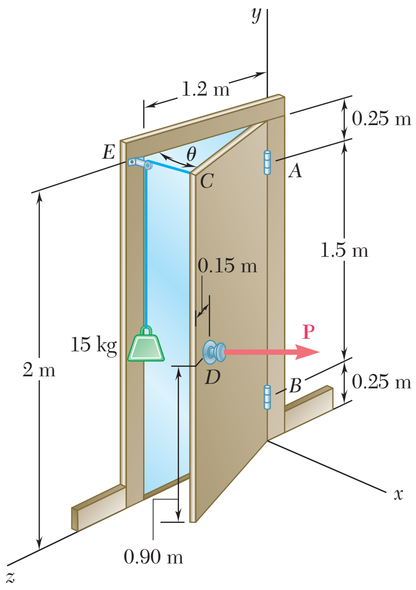


Fig. P4.70

4.71 Solve Prob. 4.65 assuming that the hinge at *A* has been removed and that the hinge at *B* can exert couples about the axes parallel to the *x* and *y* axes, respectively.

4.72 Solve Prob. 4.69 assuming that the hinge at *A* has been removed.

4.73 The rigid L-shaped member *ABC* is supported by a ball and socket at *A* and three cables. Determine the tension in each cable and the reaction at *A* caused by the 500-lb load applied at *G*.

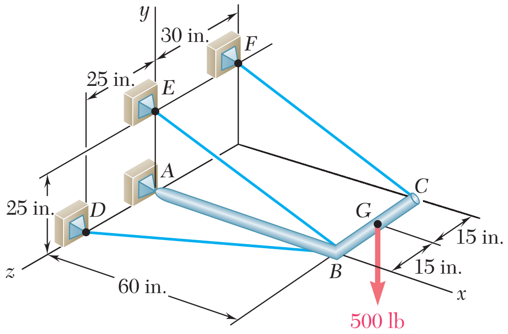


Fig. P4.73

4.74 Three rods are welded together to form the "corner" shown. The corner is supported by three smooth eyebolts. Determine the reactions at *A*, *B*, and *C* when $P = 1.2$ kN, $a = 300$ mm, $b = 200$ mm, and $c = 250$ mm.

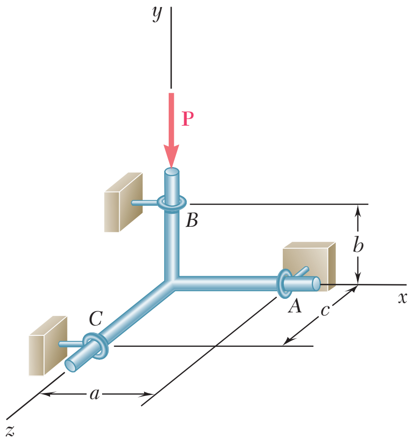


Fig. P4.74

FRICTION

4.10 FRICTION FORCES

In the preceding sections, it was assumed that surfaces in contact were either *frictionless* or *rough*. If they were frictionless, the force each surface exerted on the other was normal to the surfaces and the two surfaces could move freely with respect to each other. If they were rough, it was assumed that tangential forces could develop to prevent the motion of one surface with respect to the other.

This view was a simplified one. Actually, no perfectly frictionless surface exists. When two surfaces are in contact, tangential forces, called *friction forces,* will always develop if one attempts to move one surface with respect to the other. On the other hand, these friction forces are limited in magnitude and will not prevent motion if sufficiently large forces are applied. The distinction between frictionless and rough surfaces is thus a matter of degree. This will be seen more clearly in the following sections, which are devoted to the study of friction and of its applications to common engineering situations.

There are two types of friction: *dry friction,* sometimes called *Coulomb friction,* and *fluid friction*. Fluid friction develops between layers of fluid moving at different velocities. Fluid friction is of great importance in problems involving the flow of fluids through pipes and orifices or dealing with bodies immersed in moving fluids. It is also basic in the analysis of the motion of *lubricated mechanisms*. Such problems are considered in texts on fluid mechanics. The present study is limited to dry friction, i.e., to problems involving rigid bodies which are in contact along *nonlubricated* surfaces.

4.11 THE LAWS OF DRY FRICTION. COEFFICIENTS OF FRICTION

The laws of dry friction are exemplified by the following experiment. A block of weight **W** is placed on a horizontal plane surface (Fig. 4.11*a*). The forces acting on the block are its weight **W** and the reaction of the surface. Since the weight has no horizontal component,

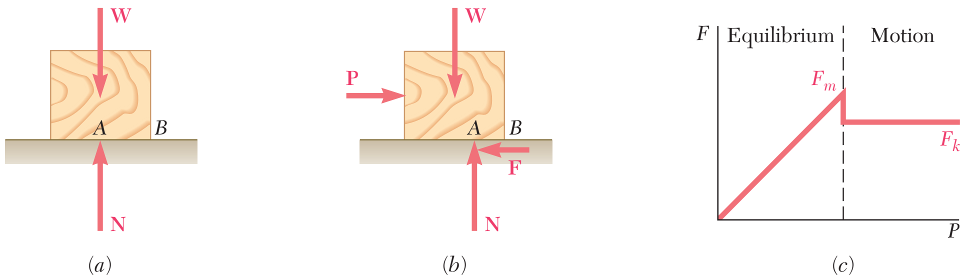


Fig. 4.11

the reaction of the surface also has no horizontal component; the reaction is therefore *normal* to the surface and is represented by **N** in Fig. 4.11*a*. Suppose, now, that a horizontal force **P** is applied to the block (Fig. 4.11*b*). If **P** is small, the block will not move; some other horizontal force must therefore exist, which balances **P**. This other force is the *static-friction force* **F**, which is actually the resultant of a great number of forces acting over the entire surface of contact between the block and the plane. The nature of these forces is not known exactly, but it is generally assumed that these forces are due to the irregularities of the surfaces in contact and, to a certain extent, to molecular attraction.

If the force **P** is increased, the friction force **F** also increases, continuing to oppose **P**, until its magnitude reaches a certain *maximum value* F_m (Fig. 4.11*c*). If **P** is further increased, the friction force cannot balance it any more and the block starts sliding.† As soon as the block has been set in motion, the magnitude of **F** drops from F_m to a lower value F_k. This is because there is less interpenetration between the irregularities of the surfaces in contact when these surfaces move with respect to each other. From then on, the block keeps sliding with increasing velocity while the friction force, denoted by $\mathbf{F}_k$ and called the *kinetic-friction force,* remains approximately constant.

Experimental evidence shows that the maximum value F_m of the static-friction force is proportional to the normal component N of the reaction of the surface. We have

$$F_m = \mu_s N \tag{4.8}$$

where μ_s is a constant called the *coefficient of static friction*. Similarly, the magnitude F_k of the kinetic-friction force may be put in the form

$$F_k = \mu_k N \tag{4.9}$$

where μ_k is a constant called the *coefficient of kinetic friction*. The coefficients of friction μ_s and μ_k do not depend upon the area of the surfaces in contact. Both coefficients, however, depend strongly on the *nature* of the surfaces in contact. Since they also depend upon the exact condition of the surfaces, their value is

†It should be noted that, as the magnitude F of the friction force increases from 0 to F_m, the point of application A of the resultant **N** of the normal forces of contact moves to the right, so that the couples formed, respectively, by **P** and **F** and by **W** and **N** remain balanced. If **N** reaches B before F reaches its maximum value F_m, the block will tip about B before it can start sliding (see Probs. 4.85 through 4.88).

seldom known with an accuracy greater than 5 percent. Approximate values of coefficients of static friction for various dry surfaces are given in Table 4.1. The corresponding values of the coefficient of kinetic friction would be about 25 percent smaller. Since coefficients of friction are dimensionless quantities, the values given in Table 4.1 can be used with both SI units and U.S. customary units.

TABLE 4.1 Approximate Values of Coefficient of Static Friction for Dry Surfaces

Metal on metal	0.15–0.60
Metal on wood	0.20–0.60
Metal on stone	0.30–0.70
Metal on leather	0.30–0.60
Wood on wood	0.25–0.50
Wood on leather	0.25–0.50
Stone on stone	0.40–0.70
Earth on earth	0.20–1.00
Rubber on concrete	0.60–0.90

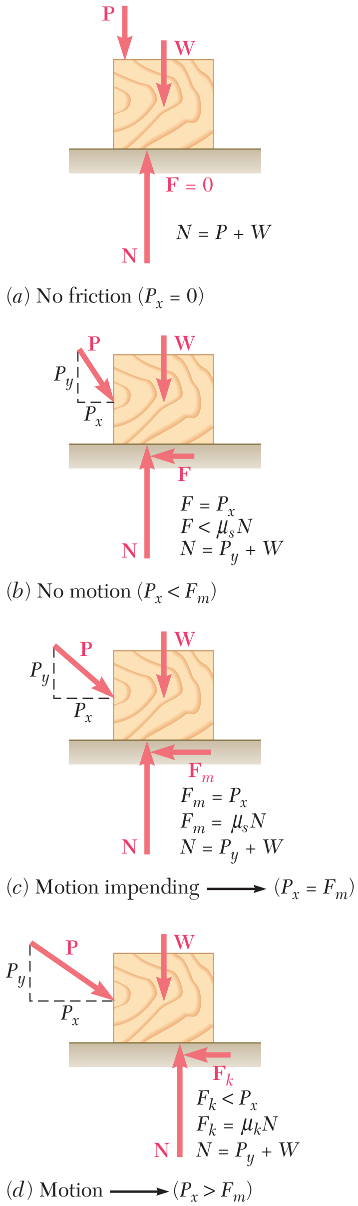


Fig. 4.12

From the description given above, it appears that four different situations can occur when a rigid body is in contact with a horizontal surface:

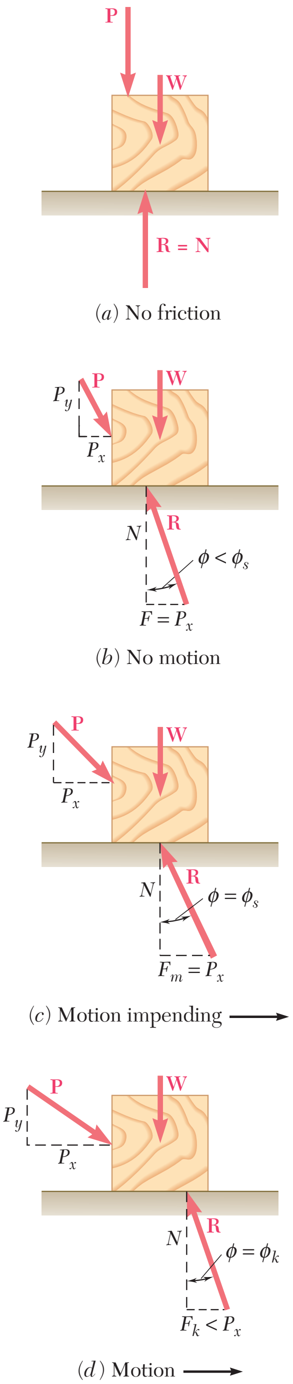
1. The forces applied to the body do not tend to move it along the surface of contact; there is no friction force (Fig. 4.12*a*).
2. The applied forces tend to move the body along the surface of contact but are not large enough to set it in motion. The friction force **F** which has developed can be found by solving the equations of equilibrium for the body. Since there is no evidence that **F** has reached its maximum value, the equation $F_m = \mu_s N$ *cannot be used* to determine the friction force (Fig. 4.12*b*).
3. The applied forces are such that the body is just about to slide. We say that *motion is impending*. The friction force **F** has reached its maximum value F_m and, together with the normal force **N**, balances the applied forces. Both the equations of equilibrium and the equation $F_m = \mu_s N$ *can be used*. We also note that the friction force has a sense opposite to the sense of impending motion (Fig. 4.12*c*).
4. The body is sliding under the action of the applied forces, and the equations of equilibrium do not apply any more. However, **F** is now equal to $\mathbf{F}_k$, and the equation $F_k = \mu_k N$ may be used. The sense of $\mathbf{F}_k$ is opposite to the sense of motion (Fig. 4.12*d*).

4.12 ANGLES OF FRICTION

It is sometimes convenient to replace the normal force **N** and the friction force **F** by their resultant **R**. Let us consider again a block of weight **W** resting on a horizontal plane surface. If no horizontal force is applied to the block, the resultant **R** reduces to the normal force **N** (Fig. 4.13*a*). However, if the applied force **P** has a horizontal component $\mathbf{P}_x$ which tends to move the block, the force **R** will have a horizontal component **F** and, thus, will form an angle ϕ with the normal to the surface (Fig. 4.13*b*). If $\mathbf{P}_x$ is increased until motion becomes impending, the angle between **R** and the vertical grows and reaches a maximum value (Fig. 4.13*c*). This value is called the *angle of static friction* and is denoted by ϕ_s. From the geometry of Fig. 4.13*c*, we note that

$$\tan \phi_s = \frac{F_m}{N} = \frac{\mu_s N}{N}$$

$$\boxed{\tan \phi_s = \mu_s} \tag{4.10}$$



(*a*) No friction

(*b*) No motion

(*c*) Motion impending

(*d*) Motion

Fig. 4.13

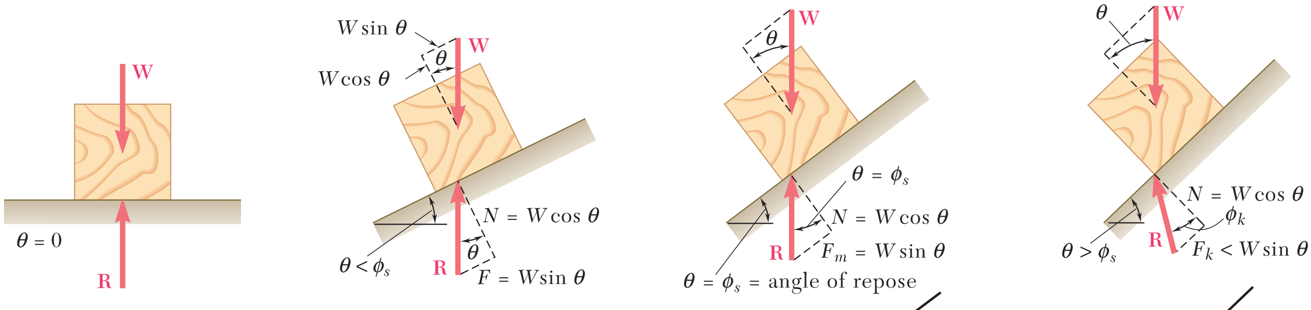
If motion actually takes place, the magnitude of the friction force drops to F_k; similarly, the angle ϕ between **R** and **N** drops to a lower value ϕ_k, called the *angle of kinetic friction* (Fig. 4.13*d*). From the geometry of Fig. 4.13*d*, we write

$$\tan \phi_k = \frac{F_k}{N} = \frac{\mu_k N}{N}$$

$$\boxed{\tan \phi_k = \mu_k} \tag{4.11}$$

Another example will show how the angle of friction can be used to advantage in the analysis of certain types of problems. Consider a block resting on a board and subjected to no other force than its weight **W** and the reaction **R** of the board. The board can be given any desired inclination. If the board is horizontal, the force **R** exerted by the board on the block is perpendicular to the board and balances the weight **W** (Fig. 4.14*a*). If the board is given a small angle of inclination θ, the force **R** will deviate from the perpendicular to the board by the angle θ and will keep balancing **W** (Fig. 4.14*b*); it will then have a normal component **N** of magnitude $N = W \cos \theta$ and a tangential component **F** of magnitude $F = W \sin \theta$.

If we keep increasing the angle of inclination, motion will soon become impending. At that time, the angle between **R** and the normal will have reached its maximum value ϕ_s (Fig. 4.14*c*). The value of the angle of inclination corresponding to impending motion is called the *angle of repose*. Clearly, the angle of repose is equal to the angle of static friction ϕ_s. If the angle of inclination θ is further increased, motion starts and the angle between **R** and the normal drops to the lower value ϕ_k (Fig. 4.14*d*). The reaction **R** is not vertical any more, and the forces acting on the block are unbalanced.



(*a*) No friction (*b*) No motion (*c*) Motion impending (*d*) Motion

Fig. 4.14

4.13 PROBLEMS INVOLVING DRY FRICTION

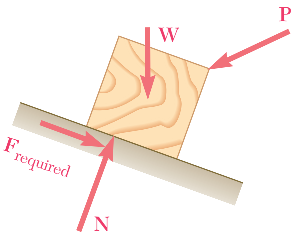
Photo 4.8 The coefficient of static friction between a package and the inclined conveyer belt must be sufficiently large to enable the package to be transported without slipping.

Problems involving dry friction are found in many engineering applications. Some deal with simple situations such as the block sliding on a plane described in the preceding sections. Others involve more complicated situations as in Sample Prob. 4.13; many deal with the stability of rigid bodies in accelerated motion and are studied in dynamics. Also, a number of common machines and mechanisms can be analyzed by applying the laws of dry friction.

The *methods* which should be used to solve problems involving dry friction are the same that were used in the preceding chapters. If a problem involves only a motion of translation, with no possible rotation, the body under consideration can usually be treated as a particle, and the methods of Chap. 2 used. If the problem involves a possible rotation, the body must be considered as a rigid body.

If the body considered is acted upon by more than three forces (including the reactions at the surfaces of contact), the reaction at each surface will be represented by its components **N** and **F** and the problem will be solved from the equations of equilibrium. If only three forces act on the body under consideration, it may be more convenient to represent each reaction by the single force **R** and to solve the problem by drawing a force triangle.

Most problems involving friction fall into one of the following *three groups:* In the *first group* of problems, all applied forces are given and the coefficients of friction are known; we are to determine whether the body considered will remain at rest or slide. The friction force **F** *required to maintain equilibrium* is unknown (its magnitude is *not* equal to $\mu_s N$) and should be determined, together with the normal force **N**, by drawing a free-body diagram and *solving the equations of equilibrium* (Fig. 4.15*a*). The value found for the magnitude F of the friction force is then compared with the maximum value $F_m = \mu_s N$. If F is smaller than or equal to F_m, the body remains at rest. If the value found for F is larger than F_m, equilibrium cannot be maintained and motion takes place; the actual magnitude of the friction force is then $F_k = \mu_k N$.



(*a*)

Fig. 4.15a

In problems of the *second group,* all applied forces are given and the motion is known to be impending; we are to determine the value of the coefficient of static friction. Here again, we determine the friction force and the normal force by drawing a free-body diagram and solving the equations of equilibrium (Fig. 4.15*b*). Since we know that the value found for F is the maximum value F_m, the coefficient of friction may be found by writing and solving the equation $F_m = \mu_s N$.

In problems of the *third group,* the coefficient of static friction is given, and it is known that the motion is impending in a given direction; we are to determine the magnitude or the direction of one of the applied forces. The friction force should be shown in the free-body diagram with a *sense opposite to that of the impending motion* and with a magnitude $F_m = \mu_s N$ (Fig. 4.15*c*). The equations of equilibrium can then be written, and the desired force determined.

As noted above, when only three forces are involved, it may be more convenient to represent the reaction of the surface by a single force **R** and to solve the problem by drawing a force triangle. Such a solution is used in Sample Prob. 4.12.

When two bodies A and B are in contact (Fig. 4.16*a*), the forces of friction exerted, respectively, by A on B and by B on A are equal and opposite (Newton's third law). In drawing the free-body diagram of one of the bodies, it is important to include the appropriate friction force with its correct sense. The following rule should then be observed: *The sense of the friction force acting on A is opposite to that of the motion (or impending motion) of A as observed from B* (Fig. 4.16*b*).† The sense of the friction force acting on B is determined in a similar way (Fig. 4.16*c*). Note that the motion of A as observed from B is a *relative motion*. For example, if body A is fixed and body B moves, body A will have a relative motion with respect to B. Also, if both B and A are moving down but B is moving faster than A, body A will be observed, from B, to be moving up.

†It is therefore *the same as that of the motion of B as observed from A.*

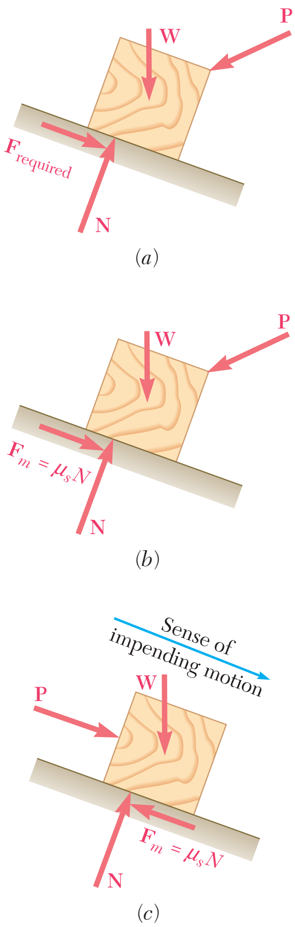


Fig. 4.15

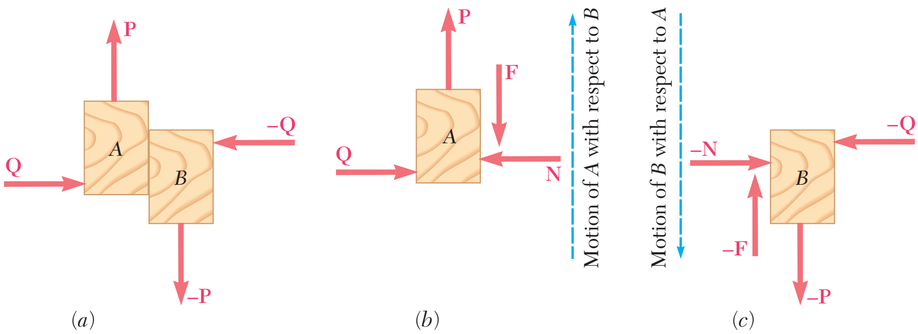
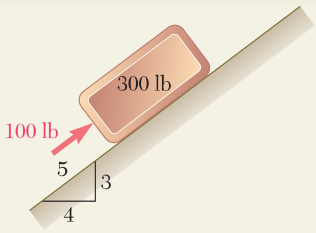


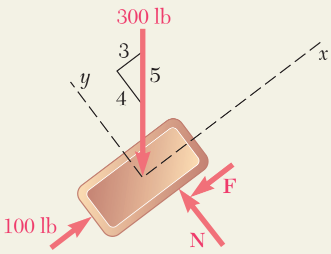
Fig. 4.16



SAMPLE PROBLEM 4.11

A 100-lb force acts as shown on a 300-lb block placed on an inclined plane. The coefficients of friction between the block and the plane are $\mu_s = 0.25$ and $\mu_k = 0.20$. Determine whether the block is in equilibrium, and find the value of the friction force.

SOLUTION



Force Required for Equilibrium. We first determine the value of the friction force *required to maintain equilibrium*. Assuming that **F** is directed down and to the left, we draw the free-body diagram of the block and write

$$+\nearrow \Sigma F_x = 0: \qquad 100 \text{ lb} - \tfrac{3}{5}(300 \text{ lb}) - F = 0$$
$$F = -80 \text{ lb} \qquad \mathbf{F} = 80 \text{ lb} \nearrow$$

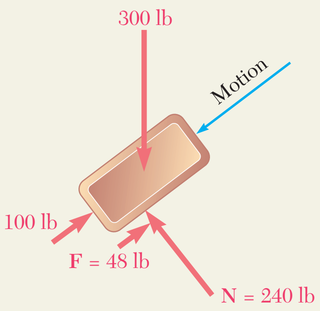
$$+\nwarrow \Sigma F_y = 0: \qquad N - \tfrac{4}{5}(300 \text{ lb}) = 0$$
$$N = +240 \text{ lb} \qquad \mathbf{N} = 240 \text{ lb} \nwarrow$$

The force **F** required to maintain equilibrium is an 80-lb force directed up and to the right; the tendency of the block is thus to move down the plane.

Maximum Friction Force. The magnitude of the maximum friction force which may be developed is

$$F_m = \mu_s N \qquad F_m = 0.25(240 \text{ lb}) = 60 \text{ lb}$$

Since the value of the force required to maintain equilibrium (80 lb) is larger than the maximum value which may be obtained (60 lb), equilibrium will not be maintained and *the block will slide down the plane.*



Actual Value of Friction Force. The magnitude of the actual friction force is obtained as follows:

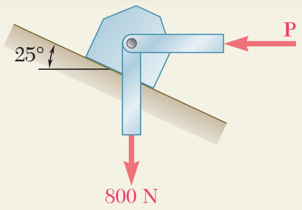
$$F_{\text{actual}} = F_k = \mu_k N$$
$$= 0.20(240 \text{ lb}) = 48 \text{ lb}$$

The sense of this force is opposite to the sense of motion; the force is thus directed up and to the right:

$$\mathbf{F}_{\text{actual}} = 48 \text{ lb} \nearrow \quad \blacktriangleleft$$

It should be noted that the forces acting on the block are not balanced; the resultant is

$$\tfrac{3}{5}(300 \text{ lb}) - 100 \text{ lb} - 48 \text{ lb} = 32 \text{ lb} \swarrow$$



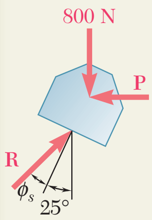
SAMPLE PROBLEM 4.12

A support block is acted upon by two forces as shown. Knowing that the coefficients of friction between the block and the incline are $\mu_s = 0.35$ and $\mu_k = 0.25$, determine the force **P** required (*a*) to start the block moving up the incline, (*b*) to keep it moving up, (*c*) to prevent it from sliding down.

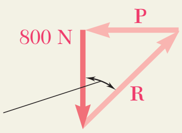
SOLUTION

Free-Body Diagram. For each part of the problem we draw a free-body diagram of the block and a force triangle including the 800-N vertical force, the horizontal force **P**, and the force **R** exerted on the block by the incline. The direction of **R** must be determined in each separate case. We note that since **P** is perpendicular to the 800-N force, the force triangle is a right triangle, which can easily be solved for **P**. In most other problems, however, the force triangle will be an oblique triangle and should be solved by applying the law of sines.

a. Force *P* to Start Block Moving Up

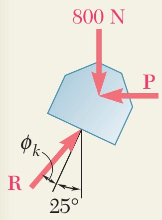


$\tan \phi_s = \mu_s$
$= 0.35$
$\phi_s = 19.29°$
$25° + 19.29° = 44.29°$

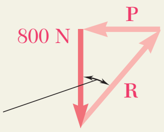


$P = (800 \text{ N}) \tan 44.29°$ $\qquad$ $\mathbf{P} = 780 \text{ N} \leftarrow$ ◄

b. Force *P* to Keep Block Moving Up

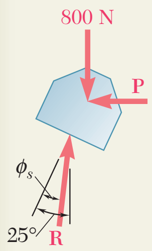


$\tan \phi_k = \mu_k$
$= 0.25$
$\phi_k = 14.04°$
$25° + 14.04° = 39.04°$

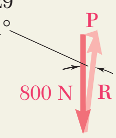


$P = (800 \text{ N}) \tan 39.04°$ $\qquad$ $\mathbf{P} = 649 \text{ N} \leftarrow$ ◄

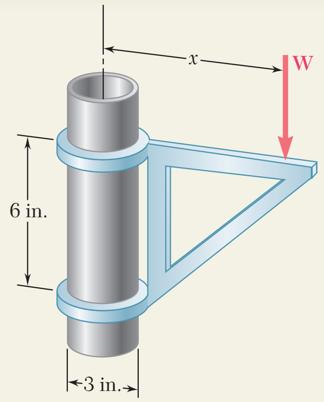
c. Force *P* to Prevent Block from Sliding Down



$\phi_s = 19.29°$
$25° - 19.29° = 5.71°$

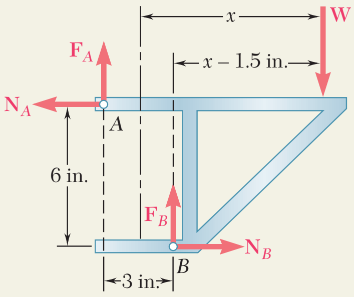


$P = (800 \text{ N}) \tan 5.71°$ $\qquad$ $\mathbf{P} = 80.0 \text{ N} \leftarrow$ ◄



SAMPLE PROBLEM 4.13

The movable bracket shown may be placed at any height on the 3-in.-diameter pipe. If the coefficient of static friction between the pipe and bracket is 0.25, determine the minimum distance x at which the load **W** can be supported. Neglect the weight of the bracket.



SOLUTION

Free-Body Diagram. We draw the free-body diagram of the bracket. When **W** is placed at the minimum distance x from the axis of the pipe, the bracket is just about to slip, and the forces of friction at A and B have reached their maximum values:

$$F_A = \mu_s N_A = 0.25\ N_A$$
$$F_B = \mu_s N_B = 0.25\ N_B$$

Equilibrium Equations

$$\overset{+}{\rightarrow}\Sigma F_x = 0: \quad N_B - N_A = 0$$
$$N_B = N_A$$

$$+\uparrow\Sigma F_y = 0: \quad F_A + F_B - W = 0$$
$$0.25N_A + 0.25N_B = W$$

And, since N_B has been found equal to N_A,

$$0.50N_A = W$$
$$N_A = 2W$$

$$+\circlearrowleft \Sigma M_B = 0: \quad N_A(6\text{ in.}) - F_A(3\text{ in.}) - W(x - 1.5\text{ in.}) = 0$$
$$6N_A - 3(0.25N_A) - Wx + 1.5W = 0$$
$$6(2W) - 0.75(2W) - Wx + 1.5W = 0$$

Dividing through by W and solving for x,

$$x = 12\text{ in.} \blacktriangleleft$$

PROBLEMS

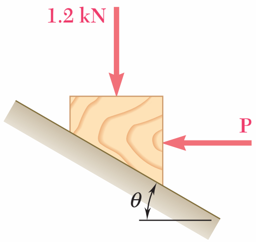


Fig. P4.75

4.75 The coefficients of friction between the block and the incline are $\mu_s = 0.35$ and $\mu_k = 0.25$. Determine whether the block is in equilibrium, and find the magnitude and direction of the friction force when $\theta = 25°$ and $P = 750$ N.

4.76 Solve Prob. 4.75 when $\theta = 30°$ and $P = 150$ N.

4.77 The coefficients of friction between the 50-lb block and the incline are $\mu_s = 0.40$ and $\mu_k = 0.30$. Determine whether the block is in equilibrium, and find the magnitude and direction of the friction force when $P = 120$ lb.

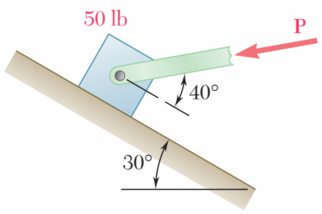


Fig. P4.77

4.78 Solve Prob. 4.77 assuming that $P = 80$ lb.

4.79 A support block is acted upon by the two forces shown. Determine the magnitude of **P** required to start the block up the plane.

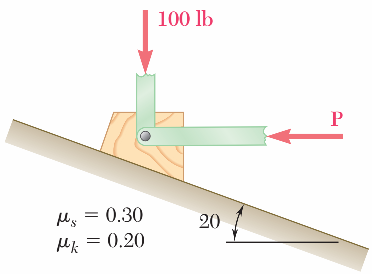


Fig. *P4.79* and P4.80

4.80 Determine the smallest magnitude of the force **P** that will prevent the support block from sliding down the plane.

4.81 Denoting by ϕ_s the angle of static friction between the block and the plane, determine the magnitude and direction of the smallest force **P** that will cause the block to move up the plane.

4.82 A block of mass $m = 20$ kg rests on a rough plane as shown. Knowing that $\alpha = 25°$ and $\mu_s = 0.20$, determine the magnitude and direction of the smallest force **P** required (*a*) to start the block up the plane, (*b*) to prevent the block from moving down the plane.

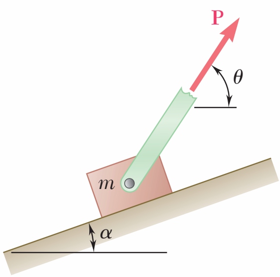


Fig. P4.81 and P4.82

4.83 The coefficients of friction between the block and the rail are $\mu_s = 0.30$ and $\mu_k = 0.25$. Knowing that $\theta = 65°$, determine the smallest value of P required (*a*) to start the block up the rail, (*b*) to keep it from moving down.

4.84 The coefficients of friction between the block and the rail are $\mu_s = 0.30$ and $\mu_k = 0.25$. Find the magnitude and direction of the smallest force **P** required (*a*) to start the block up the rail, (*b*) to keep it from moving down.

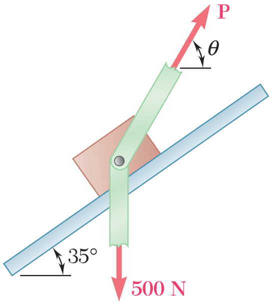


Fig. P4.83 and *P4.84*

4.85 A 60-kg cabinet is mounted on casters that can be locked to prevent their rotation. The coefficient of static friction between the floor and each caster is 0.35. If $h = 600$ mm, determine the magnitude of the force **P** required to move the cabinet to the right (*a*) if all the casters are locked, (*b*) if the casters at B are locked and the casters at A are free to rotate, (*c*) if the casters at A are locked and the casters at B are free to rotate.

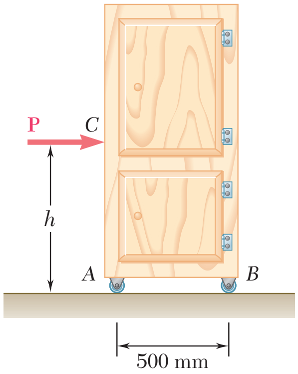


Fig. P4.85 and *P4.86*

4.86 A 60-kg cabinet is mounted on casters that can be locked to prevent their rotation. The coefficient of static friction between the floor and each caster is 0.35. Assuming that the casters at both A and B are locked, determine (*a*) the force **P** required to move the cabinet to the right, (*b*) the largest allowable value of h if the cabinet is not to tip over.

4.87 A packing crate of mass 40 kg must be moved to the left along the floor without tipping. Knowing that the coefficient of static friction between the crate and the floor is 0.35, determine (*a*) the largest allowable value of α, (*b*) the corresponding magnitude of the force **P**.

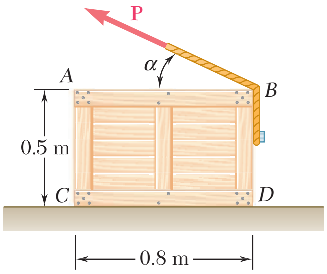


Fig. P4.87 and P4.88

4.88 A packing crate of mass 40 kg is pulled by a rope as shown. The coefficient of static friction between the crate and the floor is 0.35. If $\alpha = 40°$, determine (*a*) the magnitude of the force **P** required to move the crate, (*b*) whether the crate will slide or tip.

4.89 A 180-lb sliding door is mounted on a horizontal rail as shown. The coefficients of static friction between the rail and the door at A and B are 0.20 and 0.30, respectively. Determine the horizontal force that must be applied to the handle C in order to move the door to the left.

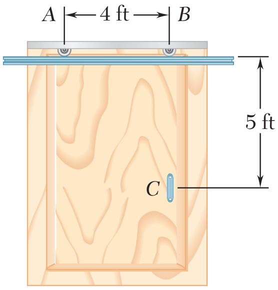


Fig. *P4.89*

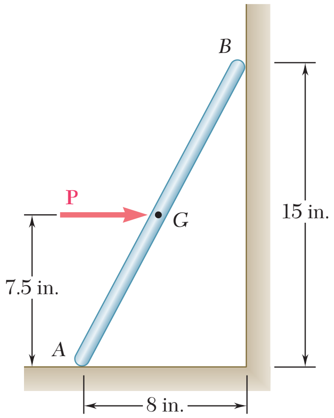


Fig. P4.91

4.90 Solve Prob. 4.89 assuming that the door is to be moved to the right.

4.91 The 10-lb uniform rod AB is held in the position shown by the force $\mathbf{P}$. Knowing that the coefficient of friction is 0.20 at A and B, determine the smallest value of P for which equilibrium is maintained.

4.92 In Prob. 4.91, determine the largest value of $\mathbf{P}$ for which equilibrium is maintained.

4.93 The end A of a slender, uniform rod of length L and weight W bears on the horizontal surface, while its end B is supported by a cord BC. Knowing that the coefficients of friction are $\mu_s = 0.30$ and $\mu_k = 0.25$, determine (*a*) the maximum value of θ for which equilibrium is maintained, (*b*) the corresponding value of the tension in the cord.

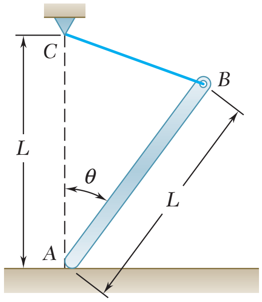


Fig. P4.93

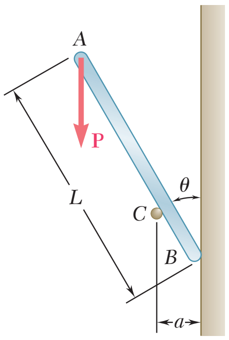


Fig. P4.95

4.94 Determine whether the rod of Prob. 4.93 is in equilibrium when $\theta = 30°$, and find the magnitude and direction of the friction force exerted on the rod at A.

4.95 A slender rod of length L is lodged between peg C and the vertical wall and supports a load $\mathbf{P}$ at end A. Knowing that $L = 12.5a$, $\theta = 30°$, and that the coefficients of friction are $\mu_s = 0.20$ and $\mu_k = 0.15$ at C and zero at B, determine whether the rod is in equilibrium.

4.96 Solve Prob. 4.95 assuming that $L = 6a$, $\theta = 30°$, and that the coefficients of friction are $\mu_s = 0.20$ and $\mu_k = 0.15$ at B and zero at C.

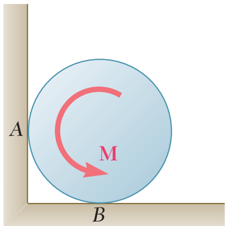


Fig. P4.97 and P4.98

4.97 Find the magnitude of the largest couple $\mathbf{M}$ that can be applied to the cylinder if it is not to spin. The cylinder has a weight W and a radius r, and the coefficient of static friction μ_s is the same at A and B.

4.98 The cylinder has a weight W and a radius r. Express in terms of W and r the magnitude of the largest couple $\mathbf{M}$ that can be applied to the cylinder if it is not to spin, assuming that the coefficient of static friction is to be (*a*) zero at A and 0.35 at B, (*b*) 0.28 at A and 0.35 at B.

REVIEW AND SUMMARY

Equilibrium equations

This chapter was devoted to the study of the *equilibrium of rigid bodies*, i.e., to the situation when the external forces acting on a rigid body *form a system equivalent to zero* [Sec. 4.1]. We then have

$$\Sigma \mathbf{F} = 0 \qquad \Sigma \mathbf{M}_O = \Sigma(\mathbf{r} \times \mathbf{F}) = 0 \tag{4.1}$$

Resolving each force and each moment into its rectangular components, we can express the necessary and sufficient conditions for the equilibrium of a rigid body with the following six scalar equations:

$$\Sigma F_x = 0 \qquad \Sigma F_y = 0 \qquad \Sigma F_z = 0 \tag{4.2}$$
$$\Sigma M_x = 0 \qquad \Sigma M_y = 0 \qquad \Sigma M_z = 0 \tag{4.3}$$

These equations can be used to determine unknown forces applied to the rigid body or unknown reactions exerted by its supports.

Free-body diagram

When solving a problem involving the equilibrium of a rigid body, it is essential to consider *all* of the forces acting on the body. Therefore, the first step in the solution of the problem should be to draw a *free-body diagram* showing the body under consideration and all of the unknown as well as known forces acting on it [Sec. 4.2].

Equilibrium of a two-dimensional structure

In the first part of the chapter, we considered the *equilibrium of a two-dimensional structure;* i.e., we assumed that the structure considered and the forces applied to it were contained in the same plane. We saw that each of the reactions exerted on the structure by its supports could involve one, two, or three unknowns, depending upon the type of support [Sec. 4.3].

In the case of a two-dimensional structure, Eqs. (4.1), or Eqs. (4.2) and (4.3), reduce to *three equilibrium equations,* namely

$$\Sigma F_x = 0 \qquad \Sigma F_y = 0 \qquad \Sigma M_A = 0 \tag{4.5}$$

where A is an arbitrary point in the plane of the structure [Sec. 4.4]. These equations can be used to solve for three unknowns. While the three equilibrium equations (4.5) cannot be *augmented* with additional equations, any of them can be *replaced* by another equation. Therefore, we can write alternative sets of equilibrium equations, such as

$$\Sigma F_x = 0 \qquad \Sigma M_A = 0 \qquad \Sigma M_B = 0 \tag{4.6}$$

where point B is chosen in such a way that the line AB is not parallel to the y axis, or

$$\Sigma M_A = 0 \qquad \Sigma M_B = 0 \qquad \Sigma M_C = 0 \tag{4.7}$$

where the points A, B, and C do not lie in a straight line.

Statical indeterminacy

Partial constraints

Improper constraints

Since any set of equilibrium equations can be solved for only three unknowns, the reactions at the supports of a rigid two-dimensional structure cannot be completely determined if they involve *more than three unknowns;* they are said to be *statically indeterminate* [Sec. 4.5]. On the other hand, if the reactions involve *fewer than three unknowns,* equilibrium will not be maintained under general loading conditions; the structure is said to be *partially constrained.* The fact that the reactions involve exactly three unknowns is no guarantee that the equilibrium equations can be solved for all three unknowns. If the supports are arranged in such a way that the reactions are *either concurrent or parallel,* the reactions are statically indeterminate, and the structure is said to be *improperly constrained.*

Two-force body

Three-force body

Two particular cases of equilibrium of a rigid body were given special attention. In Sec. 4.6, a *two-force body* was defined as a rigid body subjected to forces at only two points, and it was shown that the resultants $\mathbf{F}_1$ and $\mathbf{F}_2$ of these forces must have the *same magnitude, the same line of action, and opposite sense* (Fig. 4.17), a property which will simplify the solution of certain problems in later chapters. In Sec. 4.7, a *three-force body* was defined as a rigid body subjected to forces at only three points, and it was shown that the resultants $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$ of these forces must be *either concurrent* (Fig. 4.18) *or parallel.* This property provides us with an alternative approach to the solution of problems involving a three-force body [Sample Prob. 4.6].

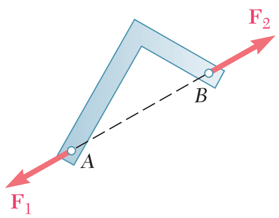


Fig. 4.17

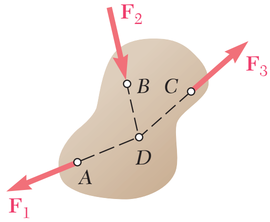


Fig. 4.18

Equilibrium of a three-dimensional body

In the second part of the chapter, we considered the *equilibrium of a three-dimensional body* and saw that each of the reactions exerted on the body by its supports could involve between one and six unknowns, depending upon the type of support [Sec. 4.8].

In the general case of the equilibrium of a three-dimensional body, all of the six scalar equilibrium equations (4.2) and (4.3) listed at the beginning of this review should be used and solved for *six unknowns* [Sec. 4.9]. In most problems, however, these equations will be more conveniently obtained if we first write

$$\Sigma \mathbf{F} = 0 \qquad \Sigma \mathbf{M}_O = \Sigma(\mathbf{r} \times \mathbf{F}) = 0 \tag{4.1}$$

and express the forces $\mathbf{F}$ and position vectors $\mathbf{r}$ in terms of scalar components and unit vectors. The vector products can then be computed either directly or by means of determinants, and the desired scalar equations obtained by equating to zero the coefficients of the unit vectors [Sample Probs. 4.7 through 4.9].

We noted that as many as three unknown reaction components may be eliminated from the computation of $\Sigma\mathbf{M}_O$ in the second of the relations (4.1) through a judicious choice of point O. Also, the reactions at two points A and B can be eliminated from the solution of some problems by writing the equation $\Sigma M_{AB} = 0$, which involves the computation of the moments of the forces about an axis AB joining points A and B [Sample Prob. 4.10].

If the reactions involve more than six unknowns, some of the reactions are *statically indeterminate;* if they involve fewer than six unknowns, the rigid body is only *partially constrained.* Even with six or more unknowns, the rigid body will be *improperly constrained* if the reactions associated with the given supports either are parallel or intersect the same line.

Static and kinetic friction

The last part of this chapter was devoted to the study of *dry friction,* i.e., to problems involving rigid bodies which are in contact along *nonlubricated surfaces*.

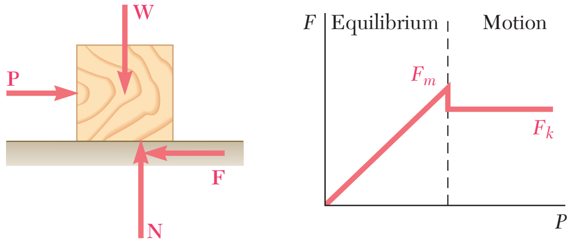


Fig. 4.19

Applying a horizontal force **P** to a block resting on a horizontal surface [Sec. 4.11], we note that the block at first does not move. This shows that a *friction force* **F** must have developed to balance **P** (Fig. 4.19). As the magnitude of **P** is increased, the magnitude of **F** also increases until it reaches a maximum value F_m. If **P** is further increased, the block starts sliding and the magnitude of **F** drops from F_m to a lower value F_k. Experimental evidence shows that F_m and F_k are proportional to the normal component N of the reaction of the surface. We have

$$F_m = \mu_s N \qquad F_k = \mu_k N \tag{4.8, 4.9}$$

where μ_s and μ_k are called, respectively, the *coefficient of static friction* and the *coefficient of kinetic friction*. These coefficients depend on the nature and the condition of the surfaces in contact. Approximate values of the coefficients of static friction were given in Table 4.1.

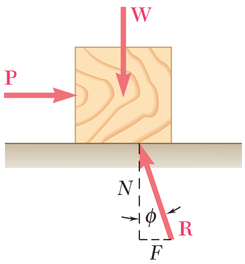


Fig. 4.20

Angles of friction

It is sometimes convenient to replace the normal force **N** and the friction force **F** by their resultant **R** (Fig. 4.20). As the friction force increases and reaches its maximum value $F_m = \mu_s N$, the angle ϕ that **R** forms with the normal to the surface increases and reaches a maximum value ϕ_s, called the *angle of static friction*. If motion actually takes place, the magnitude of **F** drops to F_k; similarly the angle ϕ

drops to a lower value ϕ_k, called the *angle of kinetic friction*. As shown in Sec. 4.12, we have

$$\tan \phi_s = \mu_s \qquad \tan \phi_k = \mu_k \qquad (4.10, 4.11)$$

Problems involving friction

When solving equilibrium problems involving friction, we should keep in mind that the magnitude F of the friction force is equal to $F_m = \mu_s N$ *only if the body is about to slide* [Sec. 4.13]. *If motion is not impending*, F and N should be considered as *independent unknowns* to be determined from the equilibrium equations (Fig. 4.21*a*). We

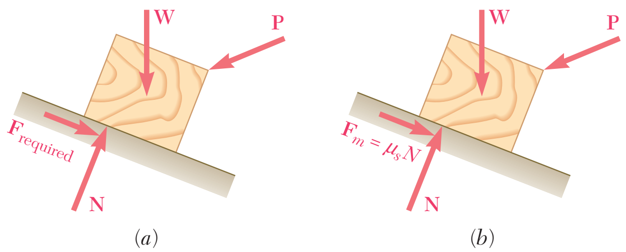


Fig. 4.21

should also check that the value of F required to maintain equilibrium is not larger than F_m; if it is, the body would move and the magnitude of the friction force would be $F_k = \mu_k N$ [Sample Prob. 4.11]. On the other hand, *if motion is known to be impending*, F has reached its maximum value $F_m = \mu_s N$ (Fig. 4.21*b*), and this expression may be substituted for F in the equilibrium equations [Sample Prob. 4.13]. When only three forces are involved in a free-body diagram, including the reaction **R** of the surface in contact with the body, it is usually more convenient to solve the problem by drawing a force triangle [Sample Prob. 4.12].

When a problem involves the analysis of the forces exerted on each other by *two bodies A and B*, it is important to show the friction forces with their correct sense. The correct sense for the friction force exerted by B on A, for instance, is opposite to that of the *relative motion* (or impending motion) of A with respect to B [Fig. 4.22].

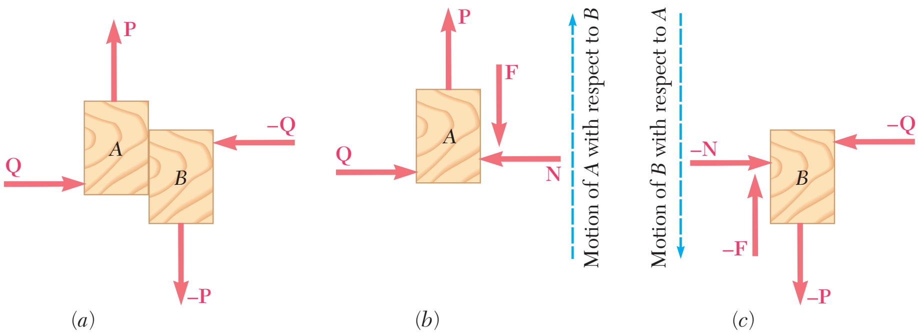


Fig. 4.22

REVIEW PROBLEMS

4.99 The maximum allowable value for each of the reactions is 150 kN, and the reaction at A must be directed upward. Neglecting the weight of the beam, determine the range of values of P for which the beam is safe.

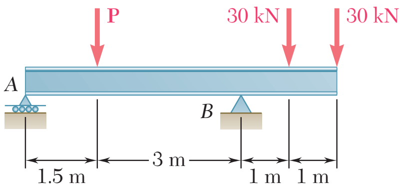


Fig. P4.99

4.100 Determine the reactions at A and B for the loading shown.

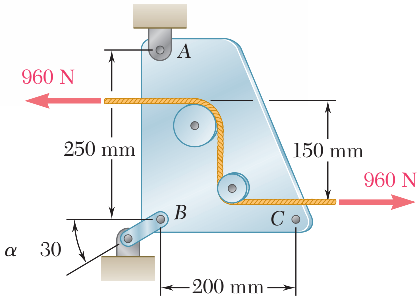


Fig. *P4.100*

4.101 The light bar AD is attached to collars B and C that can move freely on vertical rods. Knowing that the surface at A is smooth, determine the reactions at A, B, and C (*a*) if $\alpha = 60°$, (*b*) if $\alpha = 90°$.

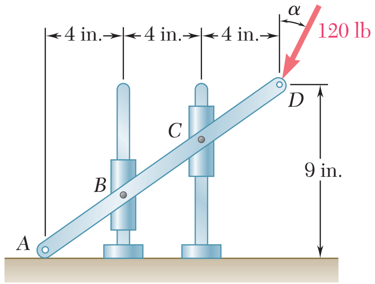


Fig. P4.101

4.102 A movable bracket is held at rest by a cable attached at C and by frictionless rollers at A and B. For the loading shown, determine (*a*) the tension in the cable, (*b*) the reactions at A and B.

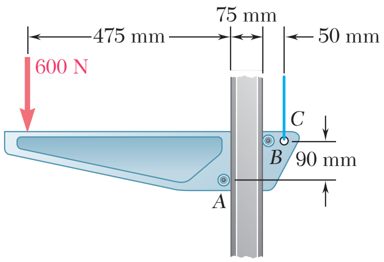


Fig. P4.102

4.103 The 300-lb beam AB carries a 500-lb load at B. The beam is held by a fixed support at A and by the cable CD that is attached to the counterweight W. (*a*) If $W = 1300$ lb, determine the reaction at A. (*b*) Determine the range of values of W for which the magnitude of the couple at A does not exceed 1500 lb · ft.

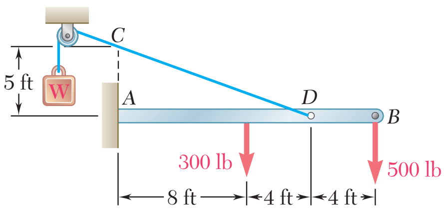


Fig. *P4.103*

4.104 A 100-kg roller, of diameter 500 mm, is used on a lawn. Determine the force **F** required to make it roll over a 50-mm obstruction (*a*) if the roller is pushed as shown, (*b*) if the roller is pulled as shown.

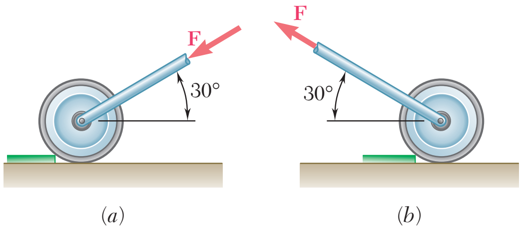


Fig. P4.104

4.105 The overhead transmission shaft *AE* is driven at a constant speed by an electric motor connected by a flat belt to pulley *B*. Pulley *C* may be used to drive a machine tool located directly below *C*, while pulley *D* drives a parallel shaft located at the same height as *AE*. Knowing that $T_B + T'_B = 36$ lb, $T_C = 40$ lb, $T'_C = 16$ lb, $T_D = 0$, and $T'_D = 0$, determine (*a*) the tension in each portion of the belt driving pulley *B*, (*b*) the reactions at the bearings *A* and *E* caused by the tension in the belts.

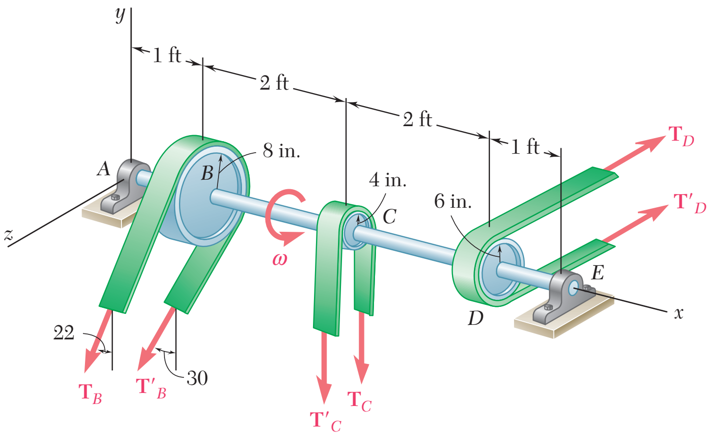


Fig. P4.105

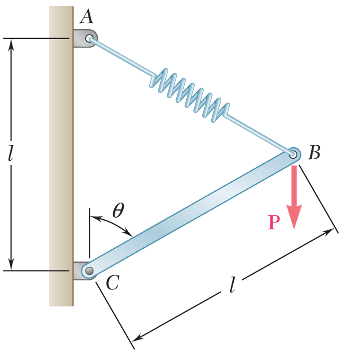


Fig. *P4.106*

4.106 A vertical load **P** is applied at end *B* of rod *BC*. The constant of the spring is *k* and the spring is unstretched when $\theta = 60°$. (*a*) Neglecting the weight of the rod, express the angle θ corresponding to the equilibrium position in terms of *P*, *k*, and *l*. (*b*) Determine the values of θ corresponding to equilibrium if $P = \frac{1}{4}kl$.

4.107 A force **P** is applied to a bent rod *AD* that may be supported in four different ways as shown. In each case determine the reactions at the supports.

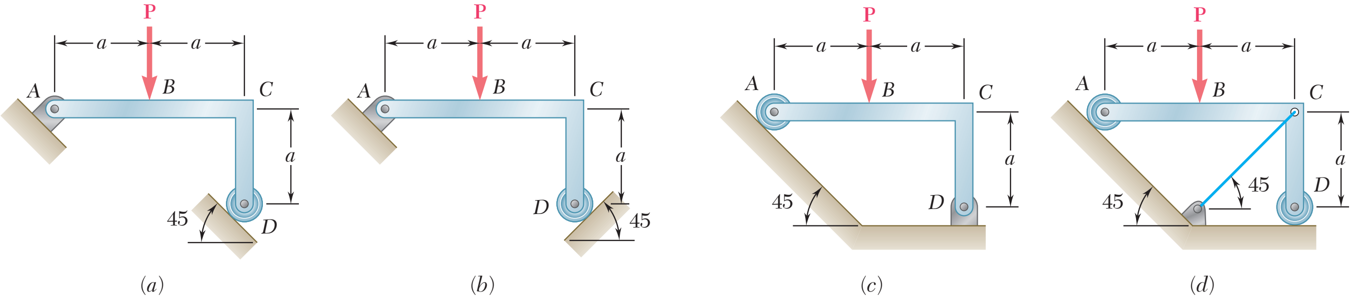


Fig. P4.107

4.108 A 500-lb marquee, 8 × 10 ft, is held in a horizontal position by two horizontal hinges at *A* and *B* and by a cable *CD* attached to a point *D* located 5 ft directly above *B*. Determine the tension in the cable and the components of the reactions at the hinges.

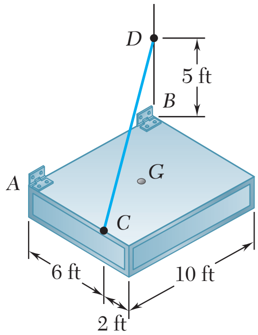


Fig. *P4.108*

4.109 The 10-kg block is attached to link *AB* and rests on a conveyor belt that is moving to the left. Knowing that the coefficients of friction between the block and the belt are $\mu_s = 0.30$ and $\mu_k = 0.25$ and neglecting the weight of the link, determine (*a*) the force in link *AB*, (*b*) the horizontal force **P** that should be applied to the belt to maintain its motion.

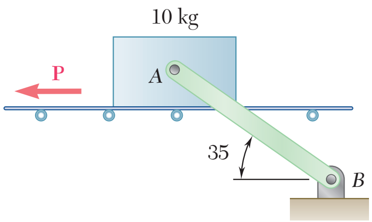


Fig. P4.109

4.110 A 10-ft uniform plank of weight 45 lb rests on two joists as shown. The coefficient of static friction between the joists and the plank is 0.40. (*a*) Determine the magnitude of the horizontal force **P** required to move the plank. (*b*) Solve part *a* assuming that a single nail driven into joist *A* prevents motion of the plank along joist *A*.

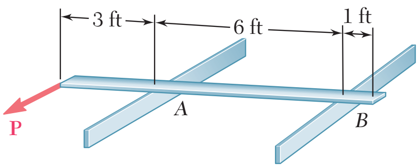
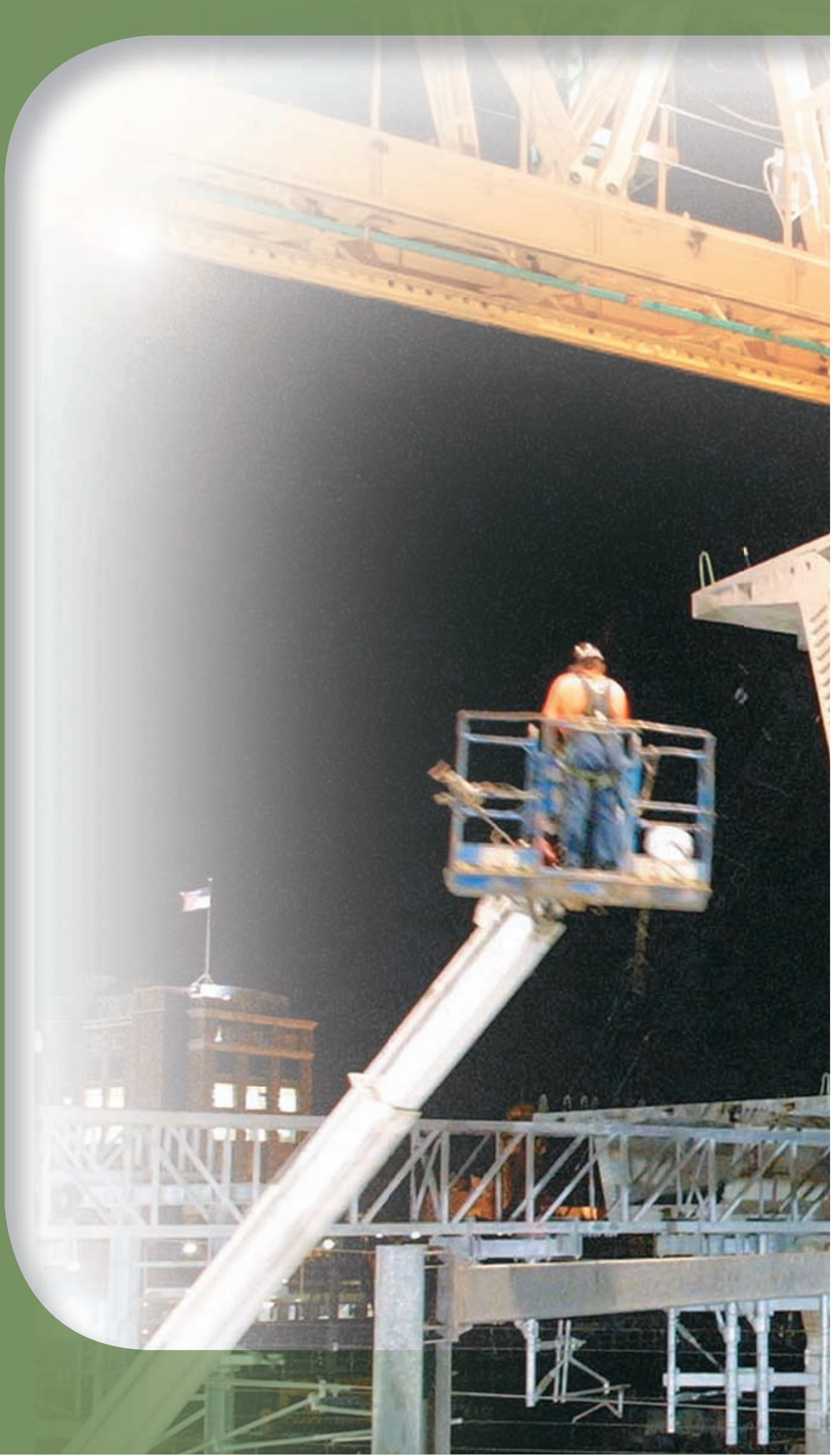


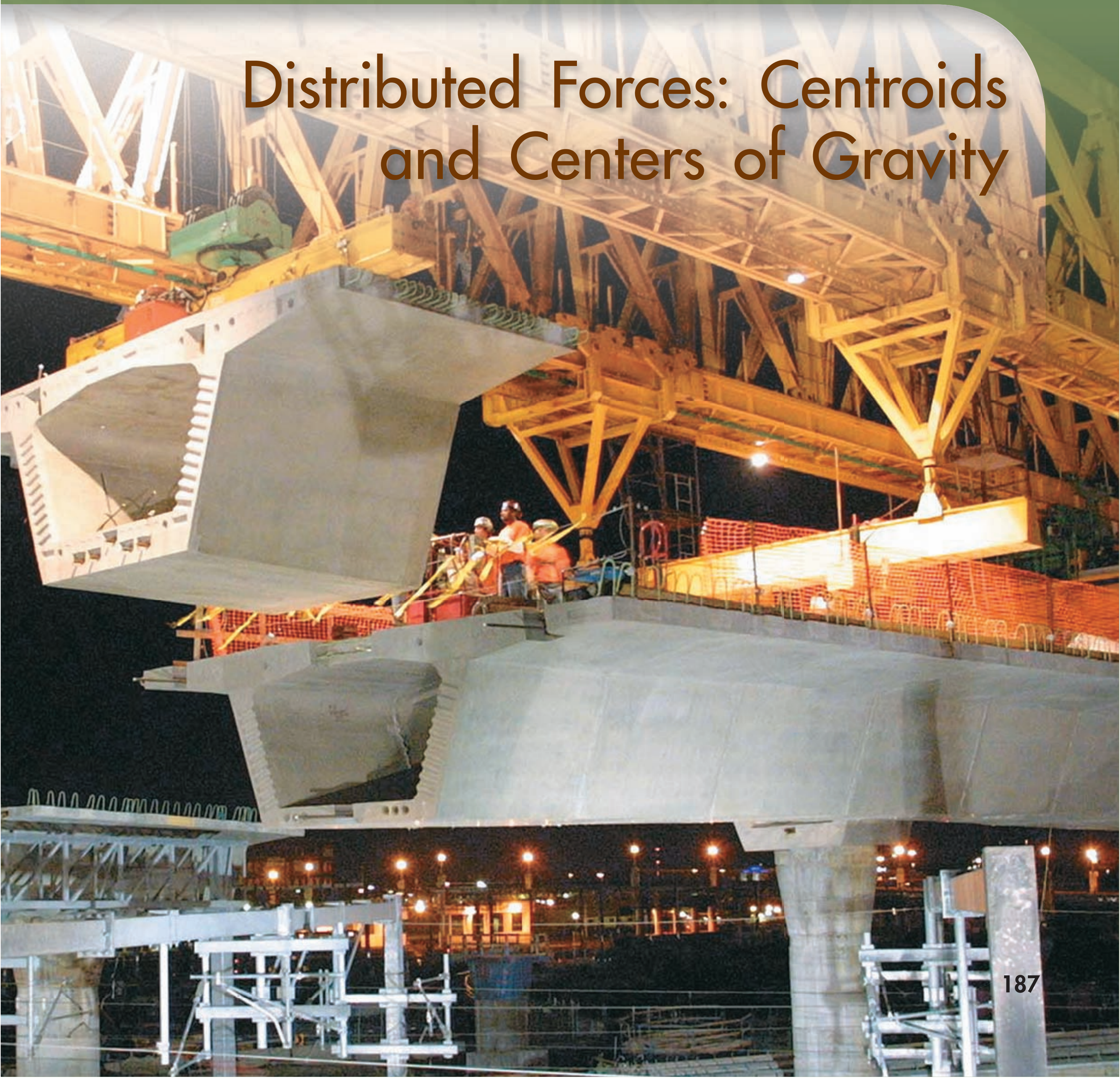
Fig. P4.110

A precast section of roadway for a new interchange on Interstate 93 is shown being lowered from a gantry crane. In this chapter we will introduce the concept of the centroid of an area; in later chapters the relation between the location of the centroid and the behavior of the roadway under loading will be established.



CHAPTER 5

Distributed Forces: Centroids and Centers of Gravity



Chapter 5 Distributed Forces: Centroids and Centers of Gravity

Photo 5.1 The precise balancing of the components of a mobile requires an understanding of centers of gravity and centroids, the main topics of this chapter.

5.1 INTRODUCTION

We have assumed so far that the attraction exerted by the earth on a rigid body could be represented by a single force **W**. This force, called the force of gravity or the weight of the body, was to be applied at the *center of gravity* of the body (Sec. 3.2). Actually, the earth exerts a force on each of the particles forming the body. The action of the earth on a rigid body should thus be represented by a large number of small forces distributed over the entire body. You will learn in this chapter, however, that all of these small forces can be replaced by a single equivalent force **W**. You will also learn how to determine the center of gravity, i.e., the point of application of the resultant **W**, for bodies of various shapes.

In the first part of the chapter, two-dimensional bodies, such as flat plates and wires contained in a given plane, are considered. Two concepts closely associated with the determination of the center of gravity of a plate or a wire are introduced: the concept of the *centroid* of an area or a line and the concept of the *first moment* of an area or a line with respect to a given axis.

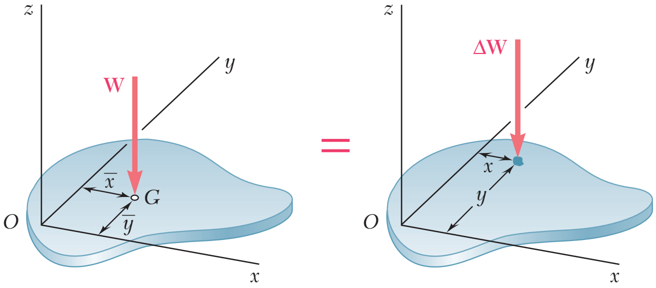
You will also learn that the computation of the area of a surface of revolution or of the volume of a body of revolution is directly related to the determination of the centroid of the line or area used to generate that surface or body of revolution (Theorems of Pappus-Guldinus). And, as is shown in Sec. 5.8, the determination of the centroid of an area simplifies the analysis of beams subjected to distributed loads.

In the last part of the chapter, you will learn how to determine the center of gravity of a three-dimensional body as well as the centroid of a volume and the first moments of that volume with respect to the coordinate planes.

AREAS AND LINES

5.2 CENTER OF GRAVITY OF A TWO-DIMENSIONAL BODY

Let us first consider a flat horizontal plate (Fig. 5.1). We can divide the plate into n small elements. The coordinates of the first element



$$\Sigma M_y: \quad \bar{x}W = \Sigma x\,\Delta W$$
$$\Sigma M_x: \quad \bar{y}W = \Sigma y\,\Delta W$$

Fig. 5.1 Center of gravity of a plate.

are denoted by x_1 and y_1, those of the second element by x_2 and y_2, etc. The forces exerted by the earth on the elements of plate will be denoted, respectively, by $\Delta\mathbf{W}_1$, $\Delta\mathbf{W}_2$, ... , $\Delta\mathbf{W}_n$. These forces or weights are directed toward the center of the earth; however, for all practical purposes they can be assumed to be parallel. Their resultant is therefore a single force in the same direction. The magnitude W of this force is obtained by adding the magnitudes of the elemental weights.

$$\Sigma F_z: \qquad W = \Delta W_1 + \Delta W_2 + \cdots + \Delta W_n$$

To obtain the coordinates $\bar{x}$ and $\bar{y}$ of the point G where the resultant $\mathbf{W}$ should be applied, we write that the moments of $\mathbf{W}$ about the y and x axes are equal to the sum of the corresponding moments of the elemental weights,

$$\begin{aligned} \Sigma M_y: \qquad \bar{x}W &= x_1\,\Delta W_1 + x_2\,\Delta W_2 + \cdots + x_n\,\Delta W_n \\ \Sigma M_x: \qquad \bar{y}W &= y_1\,\Delta W_1 + y_2\,\Delta W_2 + \cdots + y_n\,\Delta W_n \end{aligned} \tag{5.1}$$

If we now increase the number of elements into which the plate is divided and simultaneously decrease the size of each element, we obtain in the limit the following expressions:

$$W = \int dW \qquad \bar{x}\,W = \int x\,dW \qquad \bar{y}\,W = \int y\,dW \tag{5.2}$$

These equations define the weight $\mathbf{W}$ and the coordinates $\bar{x}$ and $\bar{y}$ of the center of gravity G of a flat plate. The same equations can be derived for a wire lying in the xy plane (Fig. 5.2). We note that the center of gravity G of a wire is usually not located on the wire.

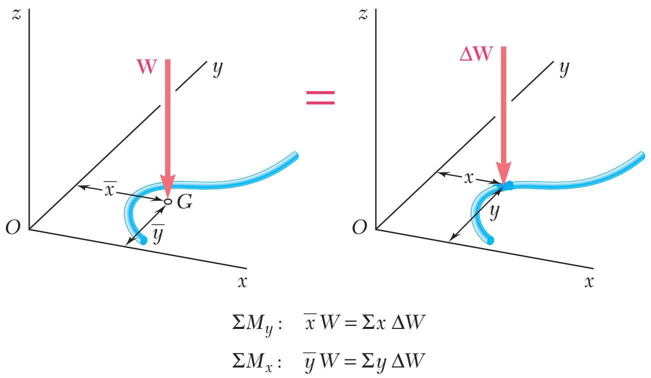


Fig. 5.2 Center of gravity of a wire.

5.3 CENTROIDS OF AREAS AND LINES

In the case of a flat homogeneous plate of uniform thickness, the magnitude ΔW of the weight of an element of the plate can be expressed as

$$\Delta W = \gamma t \, \Delta A$$

where γ = specific weight (weight per unit volume) of the material
t = thickness of the plate
ΔA = area of the element

Similarly, we can express the magnitude W of the weight of the entire plate as

$$W = \gamma t A$$

where A is the total area of the plate.

If U.S. customary units are used, the specific weight γ should be expressed in lb/ft^3, the thickness t in feet, and the areas ΔA and A in square feet. We observe that ΔW and W will then be expressed in pounds. If SI units are used, γ should be expressed in N/m^3, t in meters, and the areas ΔA and A in square meters; the weights ΔW and W will then be expressed in newtons.†

Substituting for ΔW and W in the moment equations (5.1) and dividing throughout by γt, we obtain

$$\Sigma M_y: \quad \bar{x}A = x_1 \, \Delta A_1 + x_2 \, \Delta A_2 + \cdots + x_n \, \Delta A_n$$
$$\Sigma M_x: \quad \bar{y}A = y_1 \, \Delta A_1 + y_2 \, \Delta A_2 + \cdots + y_n \, \Delta A_n$$

If we increase the number of elements into which the area A is divided and simultaneously decrease the size of each element, we obtain in the limit

$$\bar{x}A = \int x \, dA \qquad \bar{y}A = \int y \, dA \tag{5.3}$$

These equations define the coordinates $\bar{x}$ and $\bar{y}$ of the center of gravity of a homogeneous plate. The point whose coordinates are $\bar{x}$ and $\bar{y}$ is also known as the *centroid C of the area A* of the plate (Fig. 5.3). If the plate is not homogeneous, these equations cannot be used to determine the center of gravity of the plate; they still define, however, the centroid of the area.

In the case of a homogeneous wire of uniform cross section, the magnitude ΔW of the weight of an element of wire can be expressed as

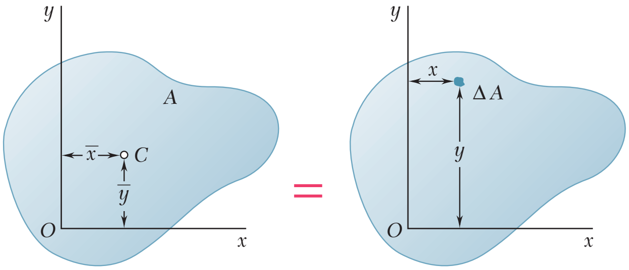
$$\Delta W = \gamma a \, \Delta L$$

where γ = specific weight of the material
a = cross-sectional area of the wire
ΔL = length of the element

†It should be noted that in the SI system of units a given material is generally characterized by its density ρ (mass per unit volume) rather than by its specific weight γ. The specific weight of the material can then be obtained from the relation

$$\gamma = \rho g$$

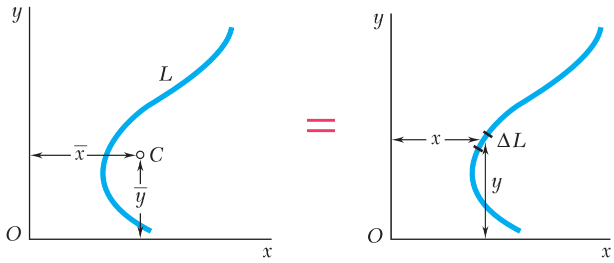
where g = 9.81 m/s^2. Since ρ is expressed in kg/m^3, we observe that γ will be expressed in (kg/m^3)(m/s^2), that is, in N/m^3.



ΣM_y: $\bar{x}A = \Sigma x\,\Delta A$

ΣM_x: $\bar{y}A = \Sigma y\,\Delta A$

Fig. 5.3 Centroid of an area.



ΣM_y: $\bar{x}L = \Sigma x\,\Delta L$

ΣM_x: $\bar{y}L = \Sigma y\,\Delta L$

Fig. 5.4 Centroid of a line.

The center of gravity of the wire then coincides with the *centroid C of the line L* defining the shape of the wire (Fig. 5.4). The coordinates $\bar{x}$ and $\bar{y}$ of the centroid of the line L are obtained from the equations

$$\bar{x}L = \int x\,dL \qquad \bar{y}L = \int y\,dL \tag{5.4}$$

5.4 FIRST MOMENTS OF AREAS AND LINES

The integral $\int x\,dA$ in Eqs. (5.3) of the preceding section is known as the *first moment of the area A with respect to the y axis* and is denoted by Q_y. Similarly, the integral $\int y\,dA$ defines the *first moment of A with respect to the x axis* and is denoted by Q_x. We write

$$Q_y = \int x\,dA \qquad Q_x = \int y\,dA \tag{5.5}$$

Comparing Eqs. (5.3) with Eqs. (5.5), we note that the first moments of the area A can be expressed as the products of the area and the coordinates of its centroid:

$$Q_y = \bar{x}A \qquad Q_x = \bar{y}A \tag{5.6}$$

It follows from Eqs. (5.6) that the coordinates of the centroid of an area can be obtained by dividing the first moments of that area by the area itself. The first moments of the area are also useful in mechanics of materials for determining the shearing stresses in beams under transverse loadings. Finally, we observe from Eqs. (5.6) that if the centroid of an area is located on a coordinate axis, the first moment of the area with respect to that axis is zero. Conversely, if the first moment of an area with respect to a coordinate axis is zero, then the centroid of the area is located on that axis.

Relations similar to Eqs. (5.5) and (5.6) can be used to define the first moments of a line with respect to the coordinate axes and to express these moments as the products of the length L of the line and the coordinates $\bar{x}$ and $\bar{y}$ of its centroid.

An area A is said to be *symmetric with respect to an axis BB′* if for every point P of the area there exists a point P' of the same area such that the line PP' is perpendicular to BB' and is divided into two equal parts by that axis (Fig. 5.5*a*). A line L is said to be symmetric with respect to an axis BB' if it satisfies similar conditions. When an area A or a line L possesses an axis of symmetry BB', its first moment with respect to BB' is zero, and its centroid is located on that axis. For example, in the case of the area A of Fig. 5.5*b*, which is symmetric with respect to the y axis, we observe that for every element of area dA of abscissa x there exists an element dA' of equal area and with abscissa $-x$. It follows that the integral in the first of Eqs. (5.5) is zero and, thus, that $Q_y = 0$. It also follows from the first of the relations (5.3) that $\bar{x} = 0$. Thus, if an area A or a line L possesses an axis of symmetry, its centroid C is located on that axis.

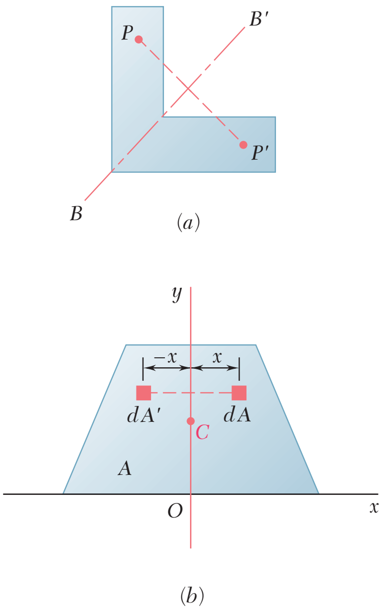


Fig. 5.5

We further note that if an area or line possesses two axes of symmetry, its centroid C must be located at the intersection of the two axes (Fig. 5.6). This property enables us to determine immediately the centroid of areas such as circles, ellipses, squares, rectangles, equilateral triangles, or other symmetric figures as well as the centroid of lines in the shape of the circumference of a circle, the perimeter of a square, etc.

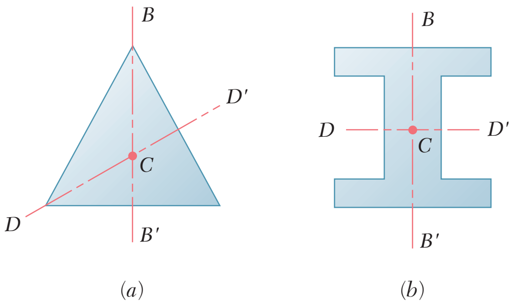


Fig. 5.6

An area A is said to be *symmetric with respect to a center O* if for every element of area dA of coordinates x and y there exists an element dA' of equal area with coordinates $-x$ and $-y$ (Fig. 5.7). It then follows that the integrals in Eqs. (5.5) are both zero and that $Q_x = Q_y = 0$. It also follows from Eqs. (5.3) that $\bar{x} = \bar{y} = 0$, that is, that the centroid of the area coincides with its center of symmetry O. Similarly, if a line possesses a center of symmetry O, the centroid of the line will coincide with the center O.

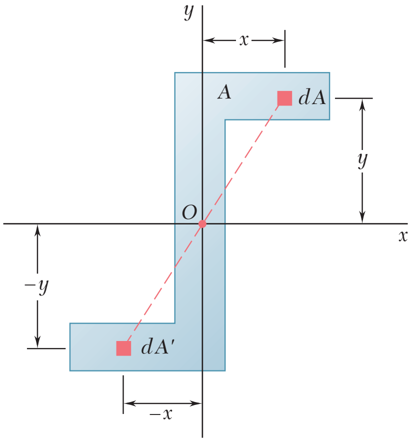


Fig. 5.7

It should be noted that a figure possessing a center of symmetry does not necessarily possess an axis of symmetry (Fig. 5.7), while a figure possessing two axes of symmetry does not necessarily possess a center of symmetry (Fig. 5.6*a*). However, if a figure possesses two axes of symmetry at a right angle to each other, the point of intersection of these axes is a center of symmetry (Fig. 5.6*b*).

Determining the centroids of unsymmetrical areas and lines and of areas and lines possessing only one axis of symmetry will be discussed in Secs. 5.6 and 5.7. Centroids of common shapes of areas and lines are shown in Fig. 5.8A and B.

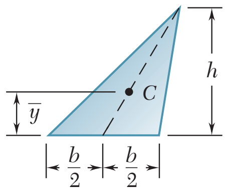
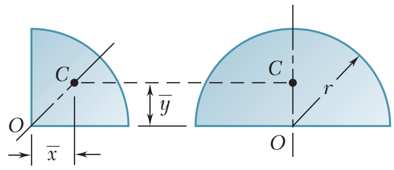
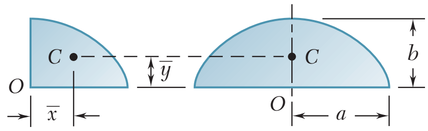
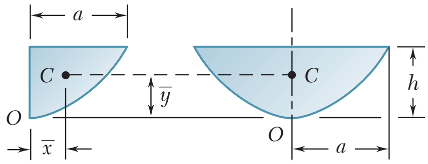
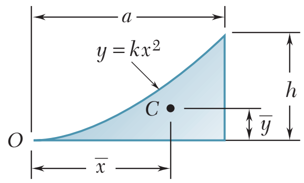
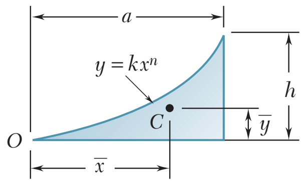
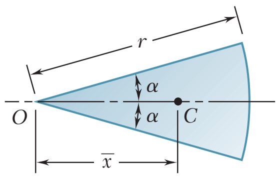
Shape		$\bar{x}$	$\bar{y}$	Area
Triangular area			$\frac{h}{3}$	$\frac{bh}{2}$
Quarter-circular area		$\frac{4r}{3\pi}$	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{4}$
Semicircular area		0	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{2}$
Quarter-elliptical area		$\frac{4a}{3\pi}$	$\frac{4b}{3\pi}$	$\frac{\pi ab}{4}$
Semielliptical area		0	$\frac{4b}{3\pi}$	$\frac{\pi ab}{2}$
Semiparabolic area		$\frac{3a}{8}$	$\frac{3h}{5}$	$\frac{2ah}{3}$
Parabolic area		0	$\frac{3h}{5}$	$\frac{4ah}{3}$
Parabolic spandrel		$\frac{3a}{4}$	$\frac{3h}{10}$	$\frac{ah}{3}$
General spandrel		$\frac{n+1}{n+2}a$	$\frac{n+1}{4n+2}h$	$\frac{ah}{n+1}$
Circular sector		$\frac{2r\sin\alpha}{3\alpha}$	0	αr^2

Fig. 5.8A Centroids of common shapes of areas.

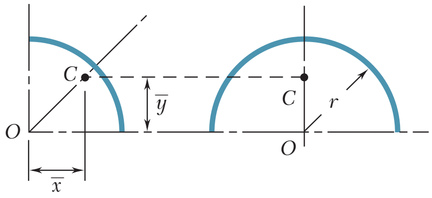
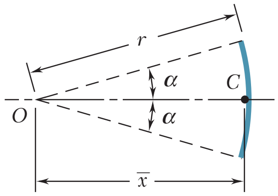
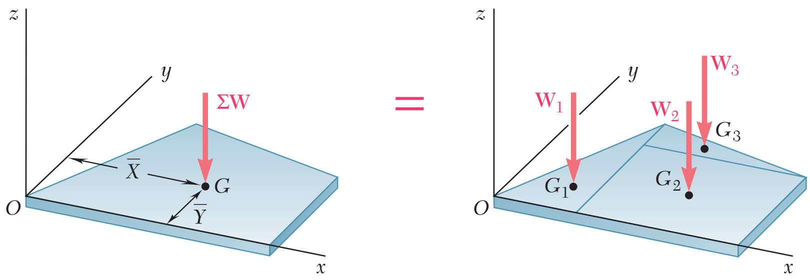
Shape		$\bar{x}$	$\bar{y}$	Length
Quarter-circular arc		$\frac{2r}{\pi}$	$\frac{2r}{\pi}$	$\frac{\pi r}{2}$
Semicircular arc		0	$\frac{2r}{\pi}$	πr
Arc of circle		$\frac{r \sin \alpha}{\alpha}$	0	$2\alpha r$

Fig. 5.8B Centroids of common shapes of lines.

5.5 COMPOSITE PLATES AND WIRES

In many instances, a flat plate can be divided into rectangles, triangles, or the other common shapes shown in Fig. 5.8A. The abscissa $\bar{X}$ of its center of gravity G can be determined from the abscissas $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n$ of the centers of gravity of the various parts by expressing that the moment of the weight of the whole plate about the y axis is equal to the sum of the moments of the weights of the various parts about the same axis (Fig. 5.9). The ordinate $\bar{Y}$ of the center of gravity of the plate is found in a similar way by equating moments about the x axis. We write

$$
\begin{aligned}
\Sigma M_y:&\quad \bar{X}(W_1 + W_2 + \cdots + W_n) = \bar{x}_1 W_1 + \bar{x}_2 W_2 + \cdots + \bar{x}_n W_n \\
\Sigma M_x:&\quad \bar{Y}(W_1 + W_2 + \cdots + W_n) = \bar{y}_1 W_1 + \bar{y}_2 W_2 + \cdots + \bar{y}_n W_n
\end{aligned}
$$



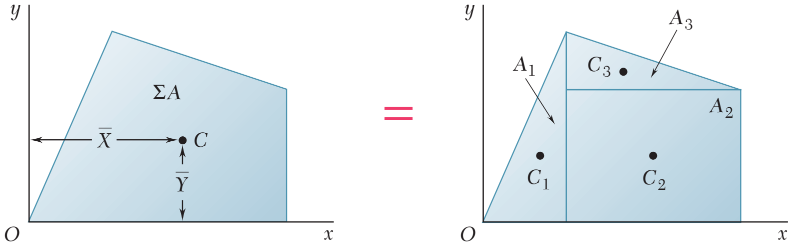
$$
\begin{aligned}
\Sigma M_y:&\quad \bar{X}\,\Sigma W = \Sigma \bar{x}\, W \\
\Sigma M_x:&\quad \bar{Y}\,\Sigma W = \Sigma \bar{y}\, W
\end{aligned}
$$

Fig. 5.9 Center of gravity of a composite plate.

or, for short,

$$\bar{X}\Sigma W = \Sigma \bar{x} W \qquad \bar{Y}\Sigma W = \Sigma \bar{y} W \tag{5.7}$$

These equations can be solved for the coordinates $\bar{X}$ and $\bar{Y}$ of the center of gravity of the plate.



$$Q_y = \bar{X}\Sigma A = \Sigma \bar{x} A$$
$$Q_x = \bar{Y}\Sigma A = \Sigma \bar{y} A$$

Fig. 5.10 Centroid of a composite area.

If the plate is homogeneous and of uniform thickness, the center of gravity coincides with the centroid C of its area. The abscissa $\bar{X}$ of the centroid of the area can be determined by noting that the first moment Q_y of the composite area with respect to the y axis can be expressed both as the product of $\bar{X}$ and the total area and as the sum of the first moments of the elementary areas with respect to the y axis (Fig. 5.10). The ordinate $\bar{Y}$ of the centroid is found in a similar way by considering the first moment Q_x of the composite area. We have

$$Q_y = \bar{X}(A_1 + A_2 + \cdots + A_n) = \bar{x}_1 A_1 + \bar{x}_2 A_2 + \cdots + \bar{x}_n A_n$$
$$Q_x = \bar{Y}(A_1 + A_2 + \cdots + A_n) = \bar{y}_1 A_1 + \bar{y}_2 A_2 + \cdots + \bar{y}_n A_n$$

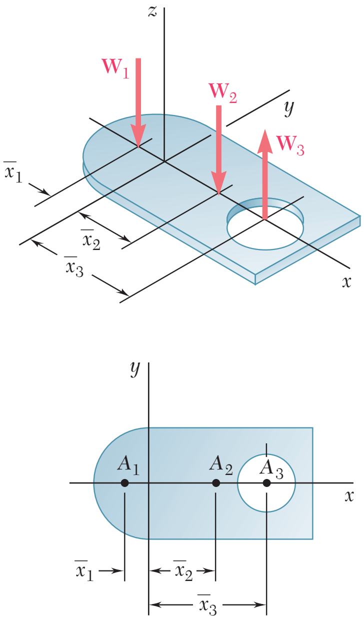
or, for short,

$$Q_y = \bar{X}\Sigma A = \Sigma \bar{x} A \qquad Q_x = \bar{Y}\Sigma A = \Sigma \bar{y} A \tag{5.8}$$

These equations yield the first moments of the composite area, or they can be used to obtain the coordinates $\bar{X}$ and $\bar{Y}$ of its centroid.

Care should be taken to assign the appropriate sign to the moment of each area. First moments of areas, like moments of forces, can be positive or negative. For example, an area whose centroid is located to the left of the y axis will have a negative first moment with respect to that axis. Also, the area of a hole should be assigned a negative sign (Fig. 5.11).

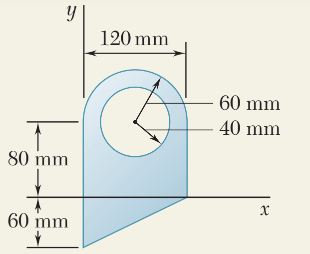
Similarly, it is possible in many cases to determine the center of gravity of a composite wire or the centroid of a composite line by dividing the wire or line into simpler elements (see Sample Prob. 5.2).



	$\bar{x}$	A	$\bar{x}A$
A_1 Semicircle	–	+	–
A_2 Full rectangle	+	+	+
A_3 Circular hole	+	–	–

Fig. 5.11

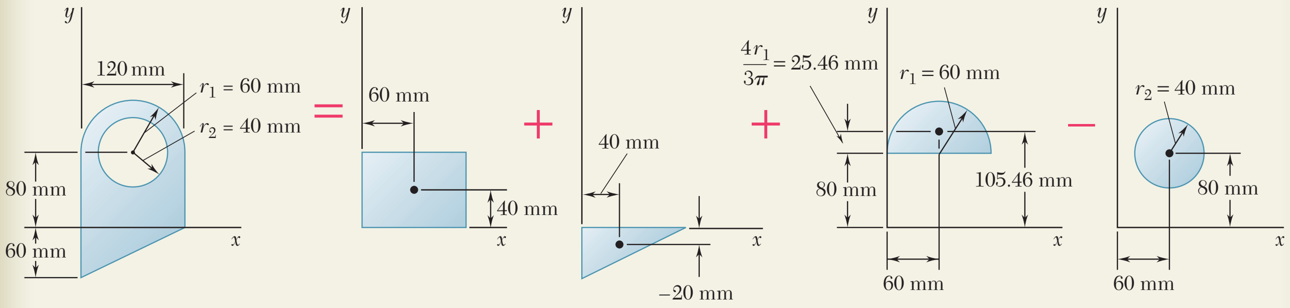
SAMPLE PROBLEM 5.1



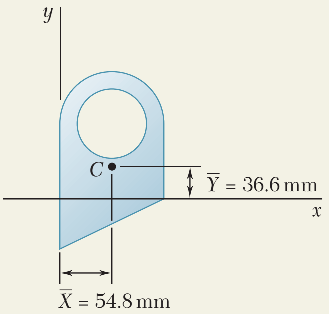
For the plane area shown, determine (*a*) the first moments with respect to the x and y axes, (*b*) the location of the centroid.

SOLUTION

Components of Area. The area is obtained by adding a rectangle, a triangle, and a semicircle and by then subtracting a circle. Using the coordinate axes shown, the area and the coordinates of the centroid of each of the component areas are determined and entered in the table below. The area of the circle is indicated as negative, since it is to be subtracted from the other areas. We note that the coordinate $\bar{y}$ of the centroid of the triangle is negative for the axes shown. The first moments of the component areas with respect to the coordinate axes are computed and entered in the table.



Component	A, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}A$, mm^3	$\bar{y}A$, mm^3
Rectangle	$(120)(80) = 9.6 \times 10^3$	60	40	$+576 \times 10^3$	$+384 \times 10^3$
Triangle	$\frac{1}{2}(120)(60) = 3.6 \times 10^3$	40	-20	$+144 \times 10^3$	-72×10^3
Semicircle	$\frac{1}{2}\pi(60)^2 = 5.655 \times 10^3$	60	105.46	$+339.3 \times 10^3$	$+596.4 \times 10^3$
Circle	$-\pi(40)^2 = -5.027 \times 10^3$	60	80	-301.6×10^3	-402.2×10^3
	$\Sigma A = 13.828 \times 10^3$			$\Sigma\bar{x}A = +757.7 \times 10^3$	$\Sigma\bar{y}A = +506.2 \times 10^3$



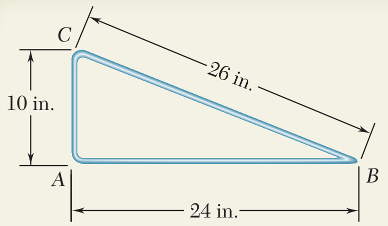
a. First Moments of the Area. Using Eqs. (5.8), we write

$$Q_x = \Sigma\bar{y}A = 506.2 \times 10^3 \text{ mm}^3 \qquad Q_x = 506 \times 10^3 \text{ mm}^3 \blacktriangleleft$$
$$Q_y = \Sigma\bar{x}A = 757.7 \times 10^3 \text{ mm}^3 \qquad Q_y = 758 \times 10^3 \text{ mm}^3 \blacktriangleleft$$

b. Location of Centroid. Substituting the values given in the table into the equations defining the centroid of a composite area, we obtain

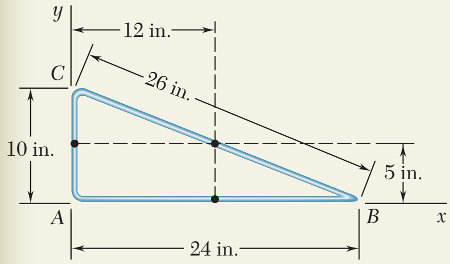
$$\bar{X}\Sigma A = \Sigma\bar{x}A: \qquad \bar{X}(13.828 \times 10^3 \text{ mm}^2) = 757.7 \times 10^3 \text{ mm}^3$$
$$\bar{X} = 54.8 \text{ mm} \blacktriangleleft$$

$$\bar{Y}\Sigma A = \Sigma\bar{y}A: \qquad \bar{Y}(13.828 \times 10^3 \text{ mm}^2) = 506.2 \times 10^3 \text{ mm}^3$$
$$\bar{Y} = 36.6 \text{ mm} \blacktriangleleft$$



SAMPLE PROBLEM 5.2

The figure shown is made from a piece of thin, homogeneous wire. Determine the location of its center of gravity.



SOLUTION

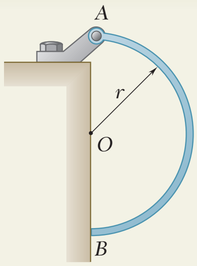
Since the figure is formed of homogeneous wire, its center of gravity coincides with the centroid of the corresponding line. Therefore, that centroid will be determined. Choosing the coordinate axes shown, with origin at *A*, we determine the coordinates of the centroid of each line segment and compute the first moments with respect to the coordinate axes.

Segment	L, in.	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{x}L$, in^2	$\bar{y}L$, in^2
AB	24	12	0	288	0
BC	26	12	5	312	130
CA	10	0	5	0	50
	$\Sigma L = 60$			$\Sigma\bar{x}L = 600$	$\Sigma\bar{y}L = 180$

Substituting the values obtained from the table into the equations defining the centroid of a composite line, we obtain

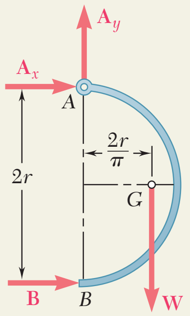
$$\bar{X}\Sigma L = \Sigma\bar{x}L: \qquad \bar{X}(60 \text{ in.}) = 600 \text{ in}^2 \qquad \bar{X} = 10 \text{ in.} \blacktriangleleft$$

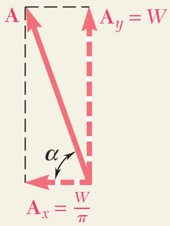
$$\bar{Y}\Sigma L = \Sigma\bar{y}L: \qquad \bar{Y}(60 \text{ in.}) = 180 \text{ in}^2 \qquad \bar{Y} = 3 \text{ in.} \blacktriangleleft$$



SAMPLE PROBLEM 5.3

A uniform semicircular rod of weight W and radius r is attached to a pin at A and rests against a frictionless surface at B. Determine the reactions at A and B.





SOLUTION

Free-Body Diagram. A free-body diagram of the rod is drawn. The forces acting on the rod are its weight **W**, which is applied at the center of gravity G (whose position is obtained from Fig. 5.8B); a reaction at A, represented by its components $\mathbf{A}_x$ and $\mathbf{A}_y$; and a horizontal reaction at B.

Equilibrium Equations

$$+\circlearrowleft \Sigma M_A = 0: \qquad B(2r) - W\left(\frac{2r}{\pi}\right) = 0$$

$$B = +\frac{W}{\pi} \qquad\qquad \mathbf{B} = \frac{W}{\pi} \rightarrow \quad \blacktriangleleft$$

$$\xrightarrow{+} \Sigma F_x = 0: \qquad A_x + B = 0$$

$$A_x = -B = -\frac{W}{\pi} \qquad \mathbf{A}_x = \frac{W}{\pi} \leftarrow$$

$$+\uparrow \Sigma F_y = 0: \qquad A_y - W = 0 \qquad \mathbf{A}_y = W \uparrow$$

Adding the two components of the reaction at A:

$$A = \left[W^2 + \left(\frac{W}{\pi}\right)^2\right]^{1/2} \qquad\qquad A = W\left(1 + \frac{1}{\pi^2}\right)^{1/2} \quad \blacktriangleleft$$

$$\tan\alpha = \frac{W}{W/\pi} = \pi \qquad\qquad \alpha = \tan^{-1}\pi \quad \blacktriangleleft$$

The answers can also be expressed as follows:

$$\mathbf{A} = 1.049W \measuredangle 72.3° \qquad \mathbf{B} = 0.318W \rightarrow \quad \blacktriangleleft$$

PROBLEMS

5.1 through 5.8 Locate the centroid of the plane area shown.

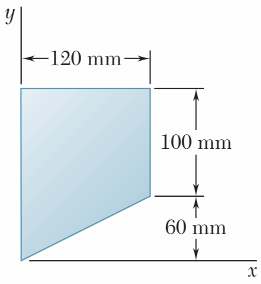


Fig. P5.1

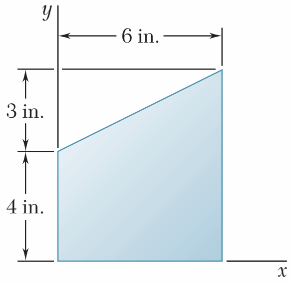


Fig. P5.2

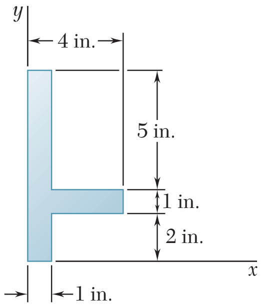


Fig. P5.3

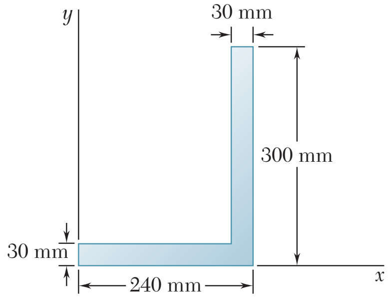


Fig. *P5.4*

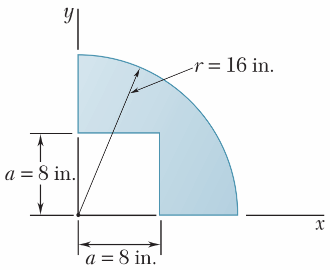


Fig. P5.5

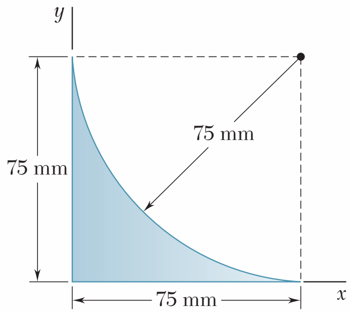


Fig. P5.6

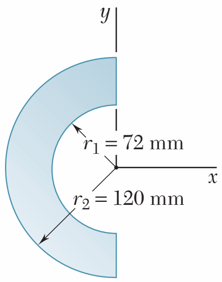


Fig. *P5.7*

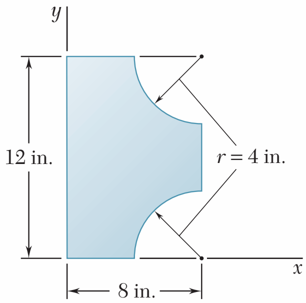


Fig. *P5.8*

5.9 through 5.12 Locate the centroid of the plane area shown.

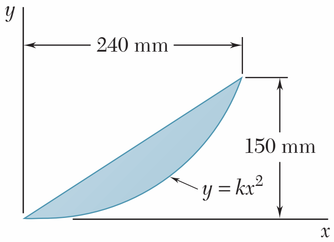


Fig. P5.9

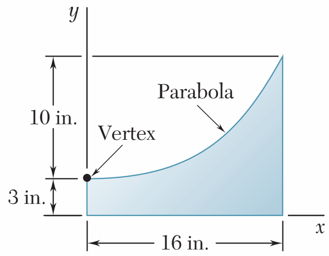


Fig. P5.10

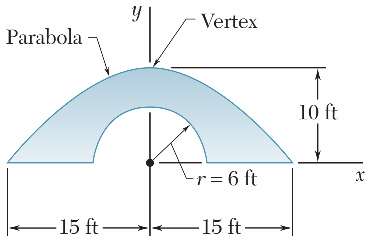


Fig. P5.11

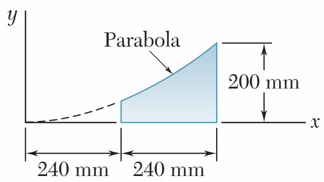


Fig. P5.12

5.13 and 5.14 The horizontal x axis is drawn through the centroid C of the area shown and divides it into two component areas A_1 and A_2. Determine the first moment of each component area with respect to the x axis and explain the results obtained.

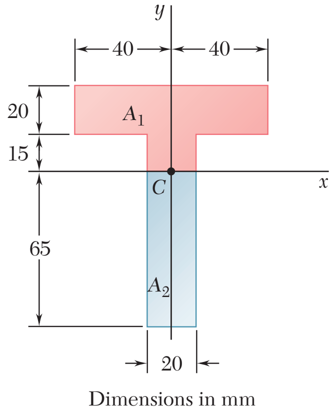


Fig. P5.13

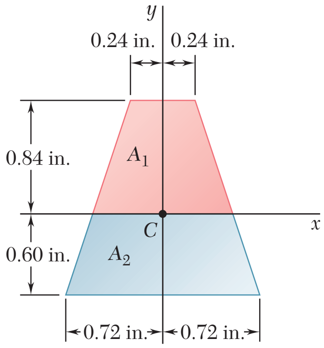


Fig. P5.14

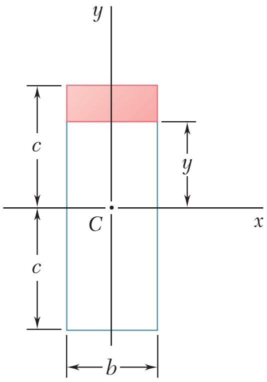


Fig. *P5.15*

5.15 The first moment of the shaded area with respect to the x axis is denoted by Q_x. (*a*) Express Q_x in terms of b, c, and the distance y from the base of the shaded area to the x axis. (*b*) For what value of y is Q_x maximum, and what is that maximum value?

5.16 A built-up beam has been constructed by nailing together seven planks as shown. The nails are equally spaced along the beam, and the beam supports a vertical load. As will be shown in Chapter 13, the shearing forces exerted on the nails at A and B are proportional to the first moments with respect to the centroidal x axis of the red-shaded areas shown, respectively, in parts a and b of the figure. Knowing that the force exerted on the nail at A is 120 N, determine the force exerted on the nail at B.

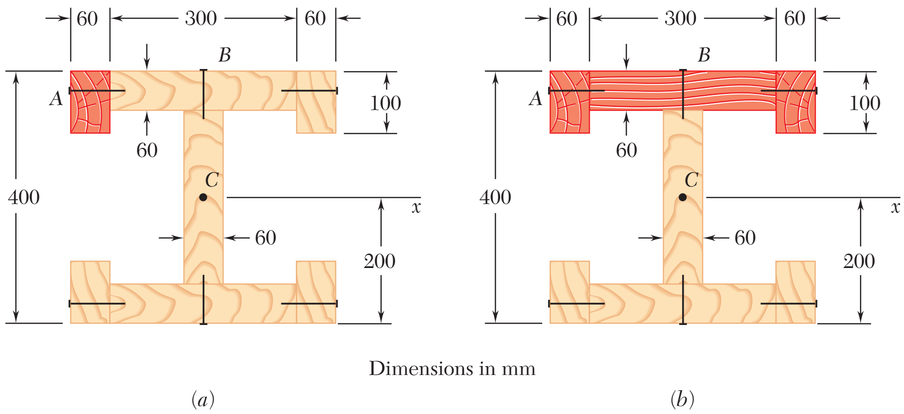


Fig. *P5.16*

5.17 through 5.20 A thin homogeneous wire is bent to form the perimeter of the figure indicated. Locate the center of gravity of the wire figure thus formed.

5.17 Fig. P5.1.
5.18 Fig. P5.2.
5.19 Fig. P5.4.
5.20 Fig. P5.8.

5.21 The homogeneous wire *ABCD* is bent as shown and is attached to a hinge at *C*. Determine the length *L* that results in portion *BCD* of the wire being horizontal.

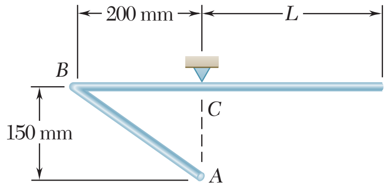


Fig. P5.21 and *P5.22*

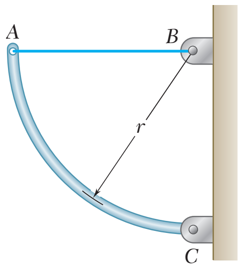


Fig. P5.23

5.22 The homogeneous wire *ABCD* is bent as shown and is attached to a hinge at *C*. Determine the length *L* that results in portion *AB* of the wire being horizontal.

5.23 A uniform circular rod of weight 8 lb and radius 10 in. is attached to a pin at *C* and to the cable *AB*. Determine (*a*) the tension in the cable, (*b*) the reaction at *C*.

5.24 Knowing that the object shown is formed of a thin homogeneous wire, determine the angle α for which the center of gravity of the object is located at the origin *O*.

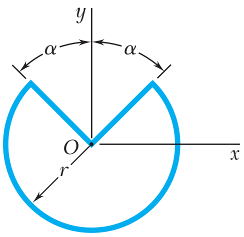


Fig. *P5.24*

5.6 DETERMINATION OF CENTROIDS BY INTEGRATION

The centroid of an area bounded by analytical curves (i.e., curves defined by algebraic equations) is usually determined by evaluating the integrals in Eqs. (5.3) of Sec. 5.3:

$$\bar{x}A = \int x\,dA \qquad \bar{y}A = \int y\,dA \tag{5.3}$$

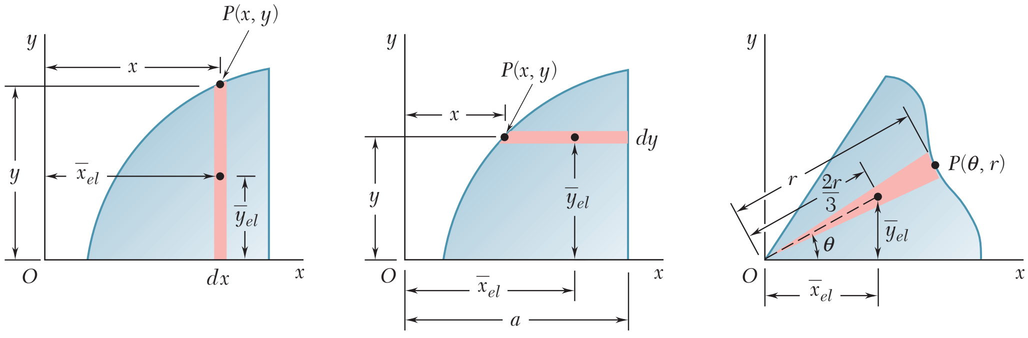
If the element of area dA is a small rectangle of sides dx and dy, the evaluation of each of these integrals requires a *double integration* with respect to x and y. A double integration is also necessary if polar coordinates are used for which dA is a small element of sides dr and $r\,d\theta$.

In most cases, however, it is possible to determine the coordinates of the centroid of an area by performing a single integration. This is achieved by choosing dA to be a thin rectangle or strip or a thin sector or pie-shaped element (Fig. 5.12); the centroid of the thin rectangle is located at its center, and the centroid of the thin sector is located at a distance $\frac{2}{3}r$ from its vertex (as it is for a triangle). The coordinates of the centroid of the area under consideration are then obtained by expressing that the first moment of the entire area with respect to each of the coordinate axes is equal to the sum (or integral) of the corresponding moments of the elements of area. Denoting by $\bar{x}_{el}$ and $\bar{y}_{el}$ the coordinates of the centroid of the element dA, we write

$$Q_y = \bar{x}A = \int \bar{x}_{el}\,dA \qquad Q_x = \bar{y}A = \int \bar{y}_{el}\,dA \tag{5.9}$$

If the area A is not already known, it can also be computed from these elements.

The coordinates $\bar{x}_{el}$ and $\bar{y}_{el}$ of the centroid of the element of area dA should be expressed in terms of the coordinates of a point located on the curve bounding the area under consideration. Also, the area of the element dA should be expressed in terms of the coordinates of that point and the appropriate differentials. This has been done in Fig. 5.12 for three common types of elements; the pie-shaped element of part c should be used when the equation of the curve bounding the area is given in polar coordinates. The appropriate expressions should be substituted into formulas (5.9), and the equation of the bounding curve should be used to express one of the coordinates in terms of the other. The integration is thus reduced to a single integration. Once the area has been determined and the integrals in Eqs. (5.9) have been evaluated, these equations can be solved for the coordinates $\bar{x}$ and $\bar{y}$ of the centroid of the area.



$\bar{x}_{el} = x$ $\quad$ $\bar{y}_{el} = y/2$ $\quad$ $dA = y\,dx$

(a)

$\bar{x}_{el} = \dfrac{a+x}{2}$ $\quad$ $\bar{y}_{el} = y$ $\quad$ $dA = (a-x)\,dy$

(b)

$\bar{x}_{el} = \dfrac{2r}{3}\cos\theta$ $\quad$ $\bar{y}_{el} = \dfrac{2r}{3}\sin\theta$ $\quad$ $dA = \dfrac{1}{2}r^2\,d\theta$

(c)

Fig. 5.12 Centroids and areas of differential elements.

When a line is defined by an algebraic equation, its centroid can be determined by evaluating the integrals in Eqs. (5.4) of Sec. 5.3:

$$\bar{x}L = \int x\, dL \qquad \bar{y}L = \int y\, dL \tag{5.4}$$

The differential length dL should be replaced by one of the following expressions depending upon which coordinate, x, y, or θ, is chosen as the independent variable in the equation used to define the line (these expressions can be derived using the Pythagorean theorem):

$$dL = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad dL = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$dL = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

After the equation of the line has been used to express one of the coordinates in terms of the other, the integration can be performed, and Eqs. (5.4) can be solved for the coordinates $\bar{x}$ and $\bar{y}$ of the centroid of the line.

5.7 THEOREMS OF PAPPUS-GULDINUS

These theorems, which were first formulated by the Greek geometer Pappus during the third century A.D. and later restated by the Swiss mathematician Guldinus, or Guldin, (1577–1643) deal with surfaces and bodies of revolution.

A *surface of revolution* is a surface which can be generated by rotating a plane curve about a fixed axis. For example (Fig. 5.13), the

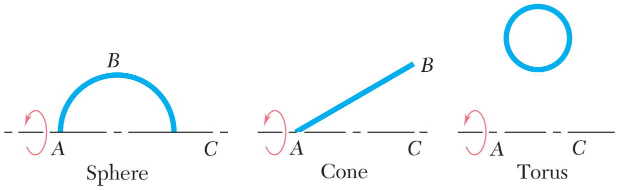


Fig. 5.13

surface of a sphere can be obtained by rotating a semicircular arc *ABC* about the diameter *AC*, the surface of a cone can be produced by rotating a straight line *AB* about an axis *AC*, and the surface of a torus or ring can be generated by rotating the circumference of a circle about a nonintersecting axis. A *body of revolution* is a body which can be generated by rotating a plane area about a fixed axis. As shown in Fig. 5.14, a sphere, a cone, and a torus can each be generated by rotating the appropriate shape about the indicated axis.

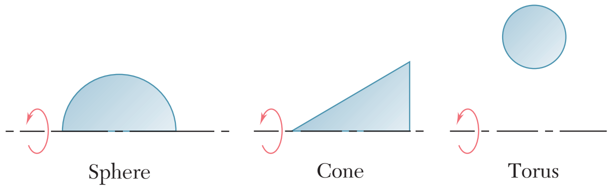


Fig. 5.14

Photo 5.2 The storage tanks shown are all bodies of revolution. Thus, their surface areas and volumes can be determined using the theorems of Pappus-Guldinus.

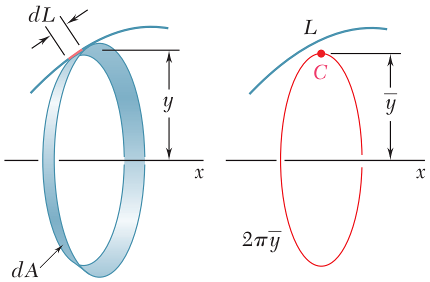


Fig. 5.15

THEOREM I. *The area of a surface of revolution is equal to the length of the generating curve times the distance traveled by the centroid of the curve while the surface is being generated.*

Proof. Consider an element dL of the line L (Fig. 5.15), which is revolved about the x axis. The area dA generated by the element dL is equal to $2\pi y\, dL$. Thus, the entire area generated by L is $A = \int 2\pi y\, dL$. Recalling that we found in Sec. 5.3 that the integral $\int y\, dL$ is equal to $\bar{y}L$, we therefore have

$$A = 2\pi\bar{y}L \tag{5.10}$$

where $2\pi\bar{y}$ is the distance traveled by the centroid of L (Fig. 5.15). It should be noted that the generating curve must not cross the axis about which it is rotated; if it did, the two sections on either side of the axis would generate areas having opposite signs, and the theorem would not apply.

THEOREM II. *The volume of a body of revolution is equal to the generating area times the distance traveled by the centroid of the area while the body is being generated.*

Proof. Consider an element dA of the area A which is revolved about the x axis (Fig. 5.16). The volume dV generated by the element dA is equal to $2\pi y\, dA$. Thus, the entire volume generated by A is $V = \int 2\pi y\, dA$, and since the integral $\int y\, dA$ is equal to $\bar{y}A$ (Sec. 5.3), we have

$$V = 2\pi\bar{y}A \tag{5.11}$$

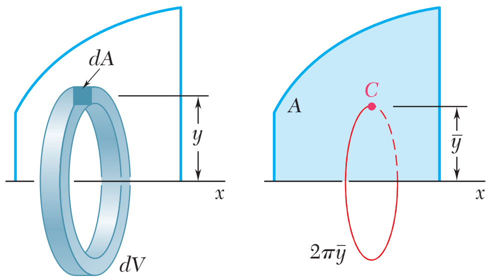
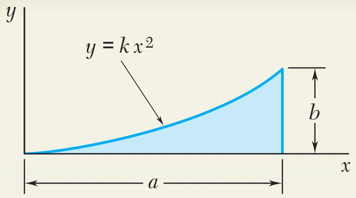


Fig. 5.16

where $2\pi\bar{y}$ is the distance traveled by the centroid of A. Again, it should be noted that the theorem does not apply if the axis of rotation intersects the generating area.

The theorems of Pappus-Guldinus offer a simple way to compute the areas of surfaces of revolution and the volumes of bodies of revolution. Conversely, they can also be used to determine the centroid of a plane curve when the area of the surface generated by the curve is known or to determine the centroid of a plane area when the volume of the body generated by the area is known (see Sample Prob. 5.8).

SAMPLE PROBLEM 5.4

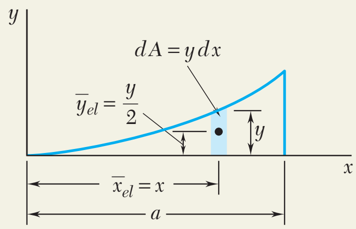


Determine by direct integration the location of the centroid of a parabolic spandrel.

SOLUTION

Determination of the Constant k. The value of k is determined by substituting $x = a$ and $y = b$ into the given equation. We have $b = ka^2$ or $k = b/a^2$. The equation of the curve is thus

$$y = \frac{b}{a^2}x^2 \qquad \text{or} \qquad x = \frac{a}{b^{1/2}}y^{1/2}$$



Vertical Differential Element. We choose the differential element shown and find the total area of the figure.

$$A = \int dA = \int y\,dx = \int_0^a \frac{b}{a^2}x^2\,dx = \left[\frac{b}{a^2}\frac{x^3}{3}\right]_0^a = \frac{ab}{3}$$

The first moment of the differential element with respect to the y axis is $\bar{x}_{el}\,dA$; hence, the first moment of the entire area with respect to this axis is

$$Q_y = \int \bar{x}_{el}\,dA = \int xy\,dx = \int_0^a x\left(\frac{b}{a^2}x^2\right)dx = \left[\frac{b}{a^2}\frac{x^4}{4}\right]_0^a = \frac{a^2b}{4}$$

Since $Q_y = \bar{x}A$, we have

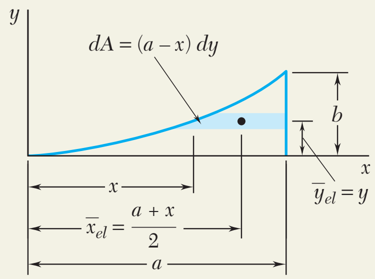
$$\bar{x}A = \int \bar{x}_{el}\,dA \qquad \bar{x}\frac{ab}{3} = \frac{a^2b}{4} \qquad \bar{x} = \tfrac{3}{4}a \quad \blacktriangleleft$$

Likewise, the first moment of the differential element with respect to the x axis is $\bar{y}_{el}\,dA$, and the first moment of the entire area is

$$Q_x = \int \bar{y}_{el}\,dA = \int \frac{y}{2}y\,dx = \int_0^a \frac{1}{2}\left(\frac{b}{a^2}x^2\right)^2 dx = \left[\frac{b^2}{2a^4}\frac{x^5}{5}\right]_0^a = \frac{ab^2}{10}$$

Since $Q_x = \bar{y}A$, we have

$$\bar{y}A = \int \bar{y}_{el}\,dA \qquad \bar{y}\frac{ab}{3} = \frac{ab^2}{10} \qquad \bar{y} = \tfrac{3}{10}b \quad \blacktriangleleft$$

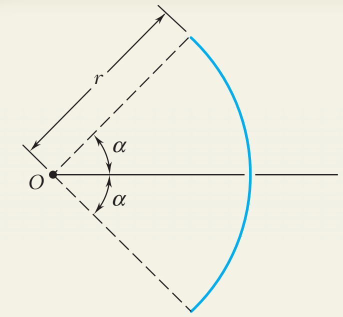


Horizontal Differential Element. The same results can be obtained by considering a horizontal element. The first moments of the area are

$$Q_y = \int \bar{x}_{el}\,dA = \int \frac{a+x}{2}(a-x)\,dy = \int_0^b \frac{a^2 - x^2}{2}dy$$
$$= \frac{1}{2}\int_0^b \left(a^2 - \frac{a^2}{b}y\right)dy = \frac{a^2b}{4}$$
$$Q_x = \int \bar{y}_{el}\,dA = \int y(a-x)\,dy = \int y\left(a - \frac{a}{b^{1/2}}y^{1/2}\right)dy$$
$$= \int_0^b \left(ay - \frac{a}{b^{1/2}}y^{3/2}\right)dy = \frac{ab^2}{10}$$

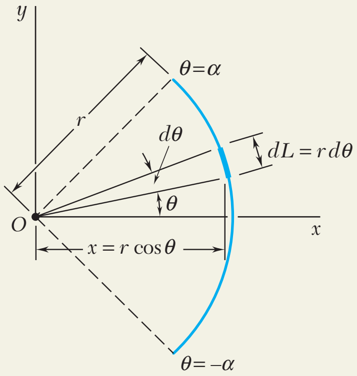
To determine $\bar{x}$ and $\bar{y}$, the expressions obtained are again substituted into the equations defining the centroid of the area.

SAMPLE PROBLEM 5.5



Determine the location of the centroid of the arc of circle shown.

SOLUTION



Since the arc is symmetrical with respect to the x axis, $\bar{y} = 0$. A differential element is chosen as shown, and the length of the arc is determined by integration.

$$L = \int dL = \int_{-\alpha}^{\alpha} r\,d\theta = r\int_{-\alpha}^{\alpha} d\theta = 2r\alpha$$

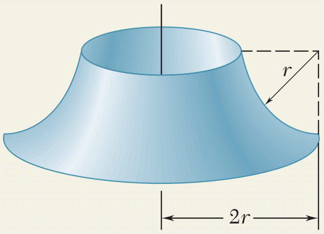
The first moment of the arc with respect to the y axis is

$$Q_y = \int x\,dL = \int_{-\alpha}^{\alpha} (r\cos\theta)(r\,d\theta) = r^2\int_{-\alpha}^{\alpha} \cos\theta\,d\theta$$
$$= r^2[\sin\theta]_{-\alpha}^{\alpha} = 2r^2\sin\alpha$$

Since $Q_y = \bar{x}L$, we write

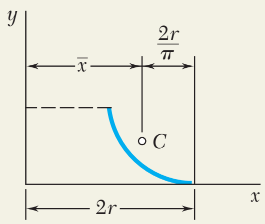
$$\bar{x}(2r\alpha) = 2r^2\sin\alpha \qquad \bar{x} = \frac{r\sin\alpha}{\alpha} \quad ◀$$

SAMPLE PROBLEM 5.6



Determine the area of the surface of revolution shown, which is obtained by rotating a quarter-circular arc about a vertical axis.

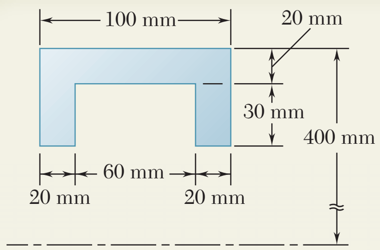
SOLUTION

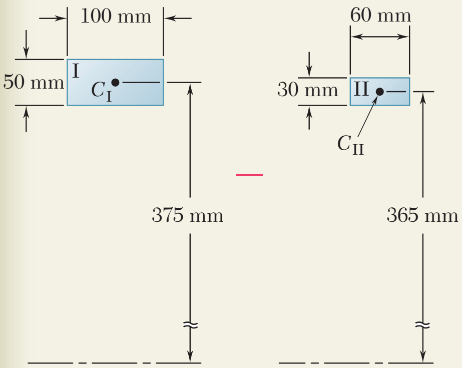


According to Theorem I of Pappus-Guldinus, the area generated is equal to the product of the length of the arc and the distance traveled by its centroid. Referring to Fig. 5.8B, we have

$$\bar{x} = 2r - \frac{2r}{\pi} = 2r\left(1 - \frac{1}{\pi}\right)$$
$$A = 2\pi\bar{x}L = 2\pi\left[2r\left(1 - \frac{1}{\pi}\right)\right]\left(\frac{\pi r}{2}\right)$$

$$A = 2\pi r^2(\pi - 1) \quad ◀$$





SAMPLE PROBLEM 5.7

The outside diameter of a pulley is 0.8 m, and the cross section of its rim is as shown. Knowing that the pulley is made of steel and that the density of steel is $\rho = 7.85 \times 10^3$ kg/m^3, determine the mass and the weight of the rim.

SOLUTION

The volume of the rim can be found by applying Theorem II of Pappus-Guldinus, which states that the volume equals the product of the given cross-sectional area and the distance traveled by its centroid in one complete revolution. However, the volume can be more easily determined if we observe that the cross section can be formed from rectangle I, whose area is positive, and rectangle II, whose area is negative.

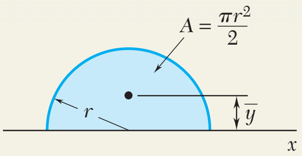
	Area, mm^2	$\bar{y}$, mm	Distance Traveled by C, mm	Volume, mm^3
I	$+5000$	375	$2\pi(375) = 2356$	$(5000)(2356) = 11.78 \times 10^6$
II	-1800	365	$2\pi(365) = 2293$	$(-1800)(2293) = -4.13 \times 10^6$
				Volume of rim $= 7.65 \times 10^6$

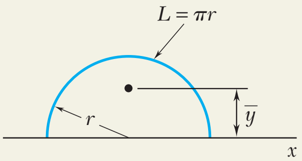
Since 1 mm $= 10^{-3}$ m, we have 1 mm$^3 = (10^{-3}\text{ m})^3 = 10^{-9}$ m^3, and we obtain $V = 7.65 \times 10^6$ mm$^3 = (7.65 \times 10^6)(10^{-9}\text{ m}^3) = 7.65 \times 10^{-3}$ m^3.

$$m = \rho V = (7.85 \times 10^3 \text{ kg/m}^3)(7.65 \times 10^{-3} \text{ m}^3) \qquad m = 60.0 \text{ kg} \blacktriangleleft$$
$$W = mg = (60.0 \text{ kg})(9.81 \text{ m/s}^2) = 589 \text{ kg} \cdot \text{m/s}^2 \qquad W = 589 \text{ N} \blacktriangleleft$$

SAMPLE PROBLEM 5.8

Using the theorems of Pappus-Guldinus, determine (*a*) the centroid of a semicircular area, (*b*) the centroid of a semicircular arc. We recall that the volume and the surface area of a sphere are $\frac{4}{3}\pi r^3$ and $4\pi r^2$, respectively.





SOLUTION

The volume of a sphere is equal to the product of the area of a semicircle and the distance traveled by the centroid of the semicircle in one revolution about the x axis.

$$V = 2\pi\bar{y}A \qquad \tfrac{4}{3}\pi r^3 = 2\pi\bar{y}(\tfrac{1}{2}\pi r^2) \qquad \bar{y} = \frac{4r}{3\pi} \blacktriangleleft$$

Likewise, the area of a sphere is equal to the product of the length of the generating semicircle and the distance traveled by its centroid in one revolution.

$$A = 2\pi\bar{y}L \qquad 4\pi r^2 = 2\pi\bar{y}(\pi r) \qquad \bar{y} = \frac{2r}{\pi} \blacktriangleleft$$

PROBLEMS

5.25 through 5.28 Determine by direct integration the centroid of the area shown.

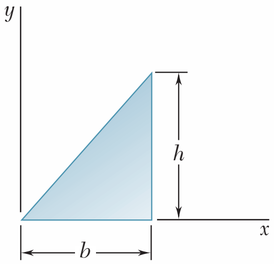


Fig. P5.25

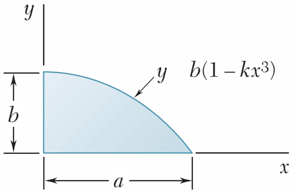


Fig. P5.26

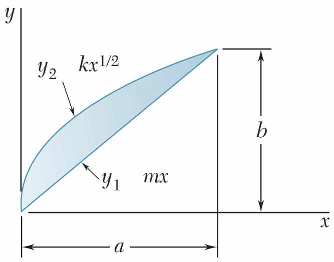


Fig. *P5.27*

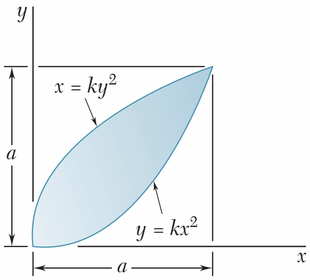


Fig. *P5.28*

5.29 through 5.32 Derive by direct integration the expressions for $\bar{x}$ and $\bar{y}$ given in Fig. 5.8A for

5.29 A general spandrel ($y = kx^n$)
5.30 A quarter-elliptical area
5.31 A semicircular area
5.32 A semiparabolic area

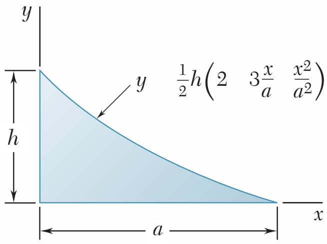


Fig. P5.33 and P5.34

5.33 Determine by direct integration the x coordinate of the centroid of the area shown.

5.34 Determine by direct integration the y coordinate of the centroid of the area shown.

5.35 Determine the centroid of the area shown when $a = 4$ in.

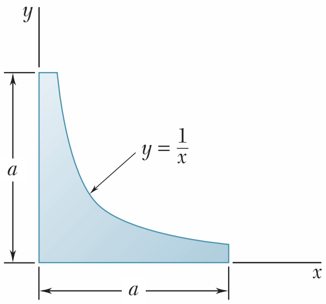


Fig. P5.35 and P5.36

5.36 Determine the centroid of the area shown in terms of a.

5.37 Determine the volume of the solid obtained by rotating the trapezoid of Prob. 5.2 about (a) the x axis, (b) the y axis.

5.38 Determine the volume of the solid obtained by rotating the area of Prob. 5.4 about (a) the x axis, (b) the y axis.

5.39 Determine the volume of the solid obtained by rotating the semiparabolic area shown about (a) the y axis, (b) the x axis.

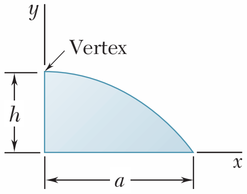


Fig. P5.39

5.40 Determine the surface area and the volume of the half-torus shown.

5.41 A spherical pressure vessel has an inside diameter of 0.8 m. Determine (*a*) the volume of liquefied propane required to fill the vessel to a depth of 0.6 m, (*b*) the corresponding mass of the liquefied propane. (Density of liquefied propane = 580 kg/m^3.)

5.42 For the pressure vessel of Prob. 5.41, determine the area of the surface in contact with the liquefied propane.

5.43 A spherical dish is formed by passing a horizontal plane through a spherical shell of radius R. Knowing that R = 10 in. and ϕ = 60°, determine the area of the inside surface of the dish.

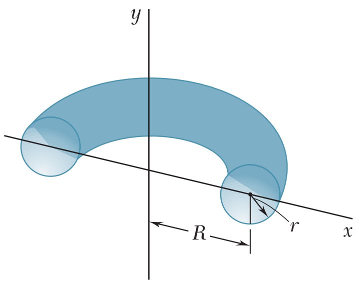


Fig. *P5.40*

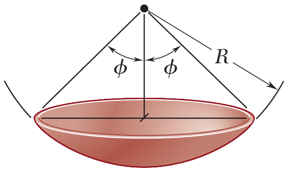


Fig. P5.43

5.44 Determine the volume and weight of water required to completely fill the spherical dish of Prob. 5.43. (Specific weight of water = 62.4 lb/ft^3.)

5.45 Determine the volume and weight of the solid brass knob shown. (Specific weight of brass = 0.306 lb/in^3.)

5.46 Determine the total surface area of the solid brass knob shown.

5.47 Determine the volume and total surface area of the body shown.

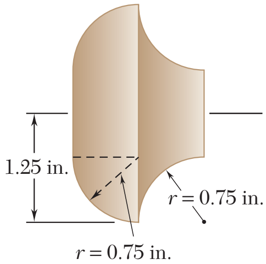


Fig. P5.45 and *P5.46*

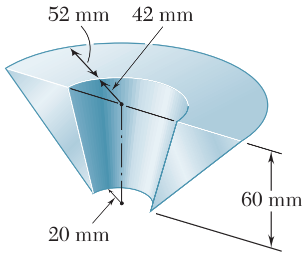


Fig. *P5.47*

5.48 Determine the volume of the steel collar obtained by rotating the shaded area shown about the vertical axis AA'.

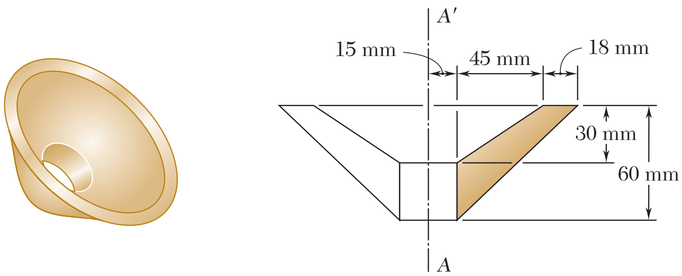


Fig. P5.48

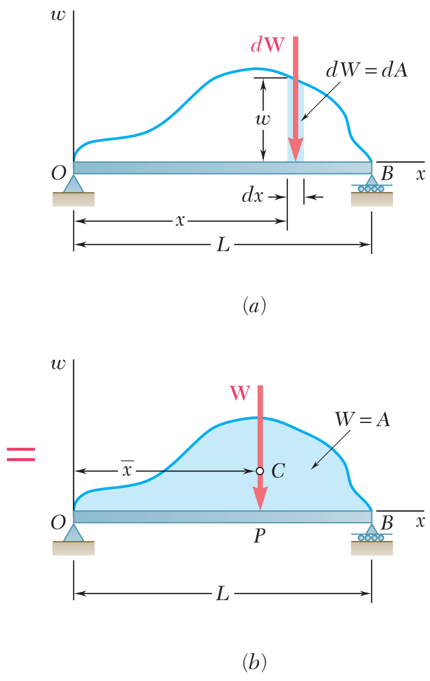


Fig. 5.17

Photo 5.3 The roofs of the buildings shown must be able to support not only the total weight of the snow but also the nonsymmetric distributed loads resulting from drifting of the snow.

*5.8 DISTRIBUTED LOADS ON BEAMS

The concept of the centroid of an area can be used to solve other problems besides those dealing with the weights of flat plates. Consider, for example, a beam supporting a *distributed load;* this load may consist of the weight of materials supported directly or indirectly by the beam, or it may be caused by wind or hydrostatic pressure. The distributed load can be represented by plotting the load w supported per unit length (Fig. 5.17); this load is expressed in N/m or in lb/ft. The magnitude of the force exerted on an element of beam of length dx is $dW = w\,dx$, and the total load supported by the beam is

$$W = \int_0^L w\,dx$$

We observe that the product $w\,dx$ is equal in magnitude to the element of area dA shown in Fig. 5.17*a*. The load W is thus equal in magnitude to the total area A under the load curve:

$$W = \int dA = A$$

We now determine where a *single concentrated load* **W**, of the same magnitude W as the total distributed load, should be applied on the beam if it is to produce the same reactions at the supports (Fig. 5.17*b*). However, this concentrated load **W**, which represents the resultant of the given distributed loading, is equivalent to the loading only when considering the free-body diagram of the entire beam. The point of application P of the equivalent concentrated load **W** is obtained by expressing that the moment of **W** about point O is equal to the sum of the moments of the elemental loads $d\mathbf{W}$ about O:

$$(OP)W = \int x\,dW$$

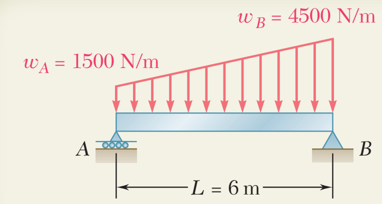
or, since $dW = w\,dx = dA$ and $W = A$,

$$(OP)A = \int_0^L x\,dA \tag{5.12}$$

Since the integral represents the first moment with respect to the w axis of the area under the load curve, it can be replaced by the product $\bar{x}A$. We therefore have $OP = \bar{x}$, where $\bar{x}$ is the distance from the w axis to the centroid C of the area A (this is *not* the centroid of the beam).

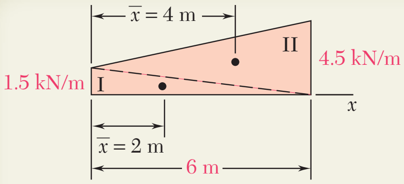
A distributed load on a beam can thus be replaced by a concentrated load; the magnitude of this single load is equal to the area under the load curve, and its line of action passes through the centroid of that area. It should be noted, however, that the concentrated load is equivalent to the given loading only as far as external forces are concerned. It can be used to determine reactions but should not be used to compute internal forces and deflections.

SAMPLE PROBLEM 5.9



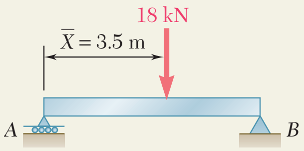
A beam supports a distributed load as shown. (*a*) Determine the equivalent concentrated load. (*b*) Determine the reactions at the supports.

SOLUTION



a. Equivalent Concentrated Load. The magnitude of the resultant of the load is equal to the area under the load curve, and the line of action of the resultant passes through the centroid of the same area. We divide the area under the load curve into two triangles and construct the table below. To simplify the computations and tabulation, the given loads per unit length have been converted into kN/m.

Component	*A*, kN	$\bar{x}$, m	$\bar{x}A$, kN · m
Triangle I	4.5	2	9
Triangle II	13.5	4	54
	$\Sigma A = 18.0$		$\Sigma \bar{x}A = 63$



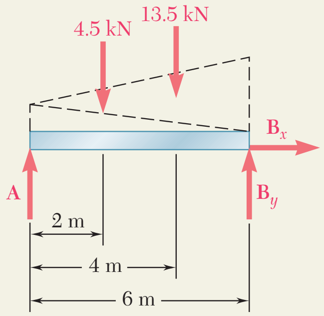
Thus, $\bar{X}\Sigma A = \Sigma \bar{x}A$: $\quad \bar{X}(18 \text{ kN}) = 63 \text{ kN} \cdot \text{m} \quad \bar{X} = 3.5 \text{ m}$

The equivalent concentrated load is

$$\mathbf{W} = 18 \text{ kN}\downarrow \quad ◀$$

and its line of action is located at a distance

$$\bar{X} = 3.5 \text{ m to the right of } A \quad ◀$$



b. Reactions. The reaction at *A* is vertical and is denoted by **A**; the reaction at *B* is represented by its components $\mathbf{B}_x$ and $\mathbf{B}_y$. The given load can be considered to be the sum of two triangular loads as shown. The resultant of each triangular load is equal to the area of the triangle and acts at its centroid. We write the following equilibrium equations for the free body shown:

$$\overset{+}{\rightarrow}\Sigma F_x = 0: \qquad \mathbf{B}_x = 0 \quad ◀$$

$$+\circlearrowleft \Sigma M_A = 0: \qquad -(4.5 \text{ kN})(2 \text{ m}) - (13.5 \text{ kN})(4 \text{ m}) + B_y(6 \text{ m}) = 0$$

$$\mathbf{B}_y = 10.5 \text{ kN}\uparrow \quad ◀$$

$$+\circlearrowleft \Sigma M_B = 0: \qquad +(4.5 \text{ kN})(4 \text{ m}) + (13.5 \text{ kN})(2 \text{ m}) - A(6 \text{ m}) = 0$$

$$\mathbf{A} = 7.5 \text{ kN}\uparrow \quad ◀$$

Alternative Solution. The given distributed load can be replaced by its resultant, which was found in part *a*. The reactions can be determined by writing the equilibrium equations $\Sigma F_x = 0$, $\Sigma M_A = 0$, and $\Sigma M_B = 0$. We again obtain

$$\mathbf{B}_x = 0 \qquad \mathbf{B}_y = 10.5 \text{ kN}\uparrow \qquad \mathbf{A} = 7.5 \text{ kN}\uparrow \quad ◀$$

PROBLEMS

5.49 and 5.50 Determine the magnitude and location of the resultant of the distributed load shown. Also calculate the reactions at A and B.

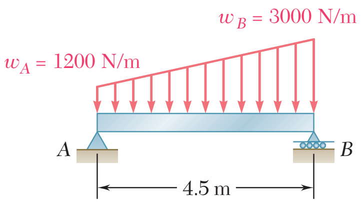


Fig. P5.49

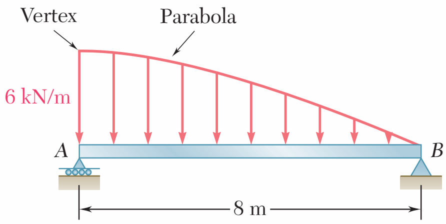


Fig. P5.50

5.51 through 5.56 Determine the reactions at the beam supports for the given loading.

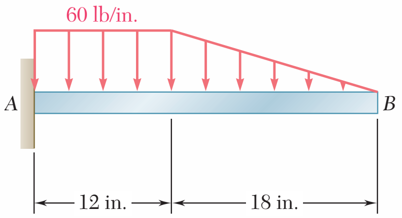


Fig. P5.51

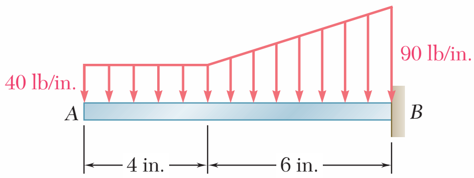


Fig. P5.52

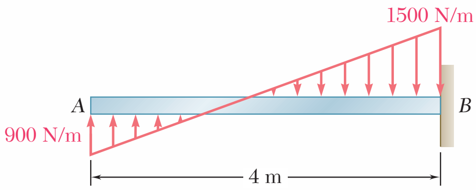


Fig. P5.53

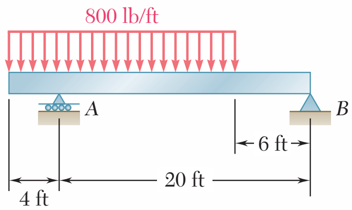


Fig. P5.54

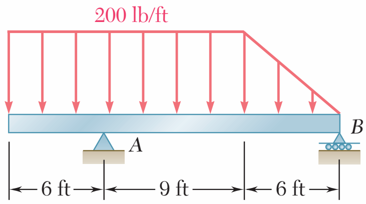


Fig. P5.55

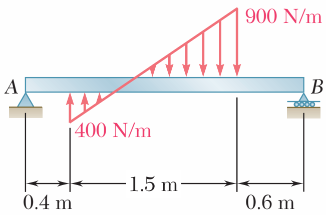


Fig. P5.56

VOLUMES

*5.9 CENTER OF GRAVITY OF A THREE-DIMENSIONAL BODY. CENTROID OF A VOLUME

The *center of gravity* G of a three-dimensional body is obtained by dividing the body into small elements and by then expressing that the weight $\mathbf{W}$ of the body acting at G is equivalent to the system of distributed forces $\Delta\mathbf{W}$ representing the weights of the small elements. Choosing the y axis to be vertical with positive sense upward (Fig. 5.18) and denoting by $\bar{\mathbf{r}}$ the position vector of G, we write that $\mathbf{W}$ is equal to the sum of the elemental weights $\Delta\mathbf{W}$, and its moment about O is equal to the sum of the moments about O of the elemental weights:

$$\Sigma\mathbf{F}: \qquad -W\mathbf{j} = \Sigma(-\Delta W\mathbf{j})$$
$$\Sigma\mathbf{M}_O: \qquad \bar{\mathbf{r}} \times (-W\mathbf{j}) = \Sigma[\mathbf{r} \times (-\Delta W\mathbf{j})] \qquad (5.13)$$

Rewriting the last equation in the form

$$\bar{\mathbf{r}}W \times (-\mathbf{j}) = (\Sigma\mathbf{r}\,\Delta W) \times (-\mathbf{j}) \qquad (5.14)$$

we observe that the weight $\mathbf{W}$ of the body is equivalent to the system of the elemental weights $\Delta\mathbf{W}$ if the following conditions are satisfied:

$$W = \Sigma\,\Delta W \qquad \bar{\mathbf{r}}W = \Sigma\mathbf{r}\,\Delta W$$

Increasing the number of elements and simultaneously decreasing the size of each element, we obtain in the limit

$$W = \int dW \qquad \bar{\mathbf{r}}W = \int \mathbf{r}\,dW \qquad (5.15)$$

We note that the relations obtained are independent of the orientation of the body. For example, if the body and the coordinate axes were rotated so that the z axis pointed upward, the unit vector $-\mathbf{j}$ would be replaced by $-\mathbf{k}$ in Eqs. (5.13) and (5.14), but the relations (5.15) would remain unchanged. Resolving the vectors $\bar{\mathbf{r}}$ and $\mathbf{r}$ into rectangular components, we note that the second of the relations (5.15) is equivalent to the three scalar equations

$$\bar{x}W = \int x\,dW \qquad \bar{y}W = \int y\,dW \qquad \bar{z}W = \int z\,dW \qquad (5.16)$$

If the body is made of a homogeneous material of specific weight γ, the magnitude dW of the weight of an infinitesimal element can be expressed in terms of the volume dV of the element, and the magnitude W of the total weight can be expressed in terms of the total volume V. We write

$$dW = \gamma\,dV \qquad W = \gamma V$$

Substituting for dW and W in the second of the relations (5.15), we write

$$\bar{\mathbf{r}}V = \int \mathbf{r}\,dV \qquad (5.17)$$

or, in scalar form,

$$\bar{x}V = \int x\,dV \qquad \bar{y}V = \int y\,dV \qquad \bar{z}V = \int z\,dV \qquad (5.18)$$

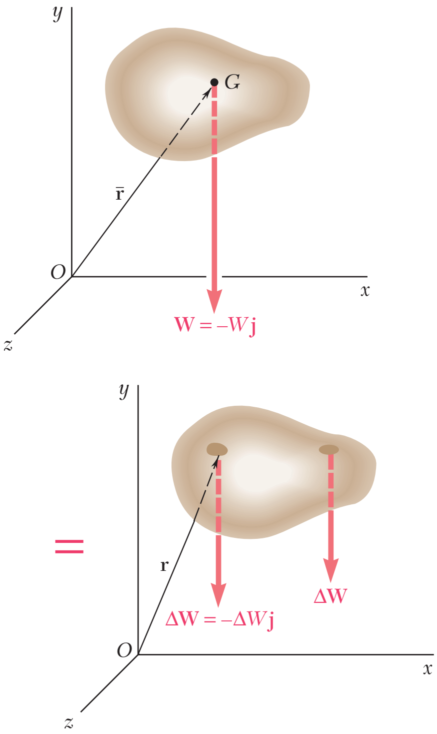


Fig. 5.18

Photo 5.4 To predict the flight characteristics of the modified Boeing 747 when used to transport a space shuttle, the center of gravity of each craft had to be determined.

The point whose coordinates are $\bar{x}$, $\bar{y}$, $\bar{z}$ is also known as the *centroid C of the volume V* of the body. If the body is not homogeneous, Eqs. (5.18) cannot be used to determine the center of gravity of the body; however, Eqs. (5.18) still define the centroid of the volume.

The integral $\int x\,dV$ is known as the *first moment of the volume with respect to the yz plane*. Similarly, the integrals $\int y\,dV$ and $\int z\,dV$ define the first moments of the volume with respect to the zx plane and the xy plane, respectively. It is seen from Eqs. (5.18) that if the centroid of a volume is located in a coordinate plane, the first moment of the volume with respect to that plane is zero.

A volume is said to be symmetrical with respect to a given plane if for every point P of the volume there exists a point P' of the same volume, such that the line PP' is perpendicular to the given plane and is bisected by that plane. The plane is said to be a *plane of symmetry* for the given volume. When a volume V possesses a plane of symmetry, the first moment of V with respect to that plane is zero, and the centroid of the volume is located in the plane of symmetry. When a volume possesses two planes of symmetry, the centroid of the volume is located on the line of intersection of the two planes. Finally, when a volume possesses three planes of symmetry which intersect at a well-defined point (i.e., not along a common line), the point of intersection of the three planes coincides with the centroid of the volume. This property enables us to determine immediately the locations of the centroids of spheres, ellipsoids, cubes, rectangular parallelepipeds, etc.

The centroids of unsymmetrical volumes or of volumes possessing only one or two planes of symmetry should be determined by integration.† The centroids of several common volumes are shown in Fig. 5.19. It should be observed that in general the centroid of a volume of revolution *does not coincide* with the centroid of its cross section. Thus, the centroid of a hemisphere is different from that of a semicircular area, and the centroid of a cone is different from that of a triangle.

*5.10 COMPOSITE BODIES

If a body can be divided into several of the common shapes shown in Fig. 5.19, its center of gravity G can be determined by expressing that the moment about O of its total weight is equal to the sum of the moments about O of the weights of the various component parts. Proceeding as in Sec. 5.9, we obtain the following equations defining the coordinates $\bar{X}$, $\bar{Y}$, $\bar{Z}$ of the center of gravity G.

$$\bar{X}\Sigma W = \Sigma\bar{x}W \qquad \bar{Y}\Sigma W = \Sigma\bar{y}W \qquad \bar{Z}\Sigma W = \Sigma\bar{z}W \tag{5.19}$$

If the body is made of a homogeneous material, its center of gravity coincides with the centroid of its volume, and we obtain:

$$\bar{X}\Sigma V = \Sigma\bar{x}V \qquad \bar{Y}\Sigma V = \Sigma\bar{y}V \qquad \bar{Z}\Sigma V = \Sigma\bar{z}V \tag{5.20}$$

†For the determination of centroids of volumes by integration, see Ferdinand P. Beer, E. Russell Johnston, Jr., David F. Mazurek, and Elliot R. Eisenberg, *Vector Mechanics for Engineers,* 9th ed., McGraw-Hill, New York, 2010, sec. 5.12.

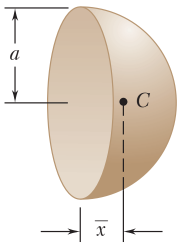
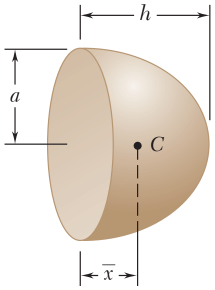
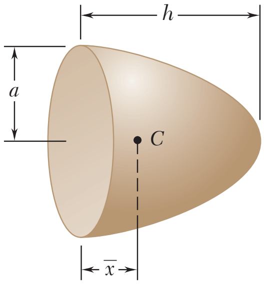
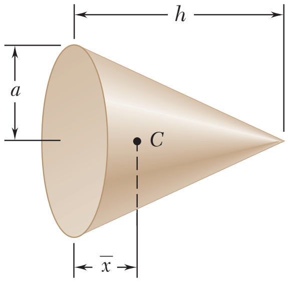
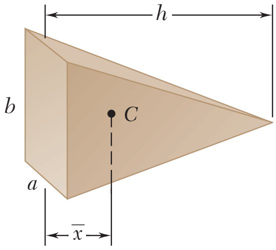
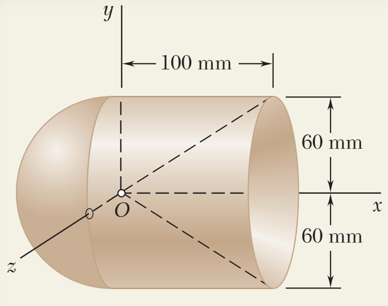
Shape		$\overline{x}$	Volume
Hemisphere		$\frac{3a}{8}$	$\frac{2}{3}\pi a^3$
Semiellipsoid of revolution		$\frac{3h}{8}$	$\frac{2}{3}\pi a^2 h$
Paraboloid of revolution		$\frac{h}{3}$	$\frac{1}{2}\pi a^2 h$
Cone		$\frac{h}{4}$	$\frac{1}{3}\pi a^2 h$
Pyramid		$\frac{h}{4}$	$\frac{1}{3}abh$

Fig. 5.19 Centroids of common shapes and volumes.

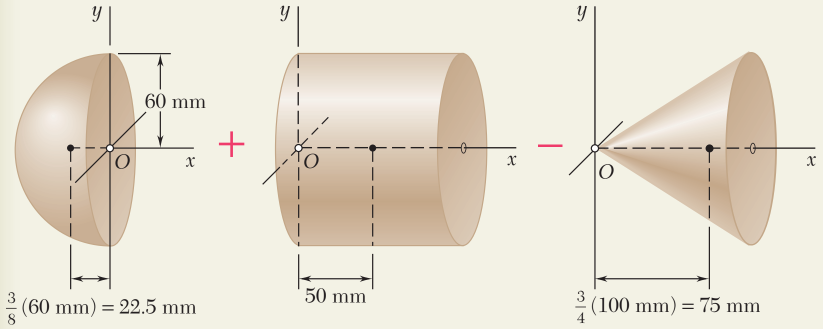


SAMPLE PROBLEM 5.10

Determine the location of the center of gravity of the homogeneous body of revolution shown, which was obtained by joining a hemisphere and a cylinder and carving out a cone.

SOLUTION

Because of symmetry, the center of gravity lies on the x axis. As shown in the figure below, the body can be obtained by adding a hemisphere to a cylinder and then subtracting a cone. The volume and the abscissa of the centroid of each of these components are obtained from Fig. 5.19 and are entered in the table below. The total volume of the body and the first moment of its volume with respect to the yz plane are then determined.

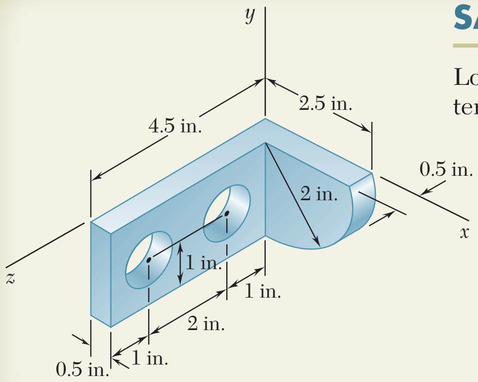


Component	Volume, mm^3	$\bar{x}$, mm	$\bar{x}V$, mm^4
Hemisphere	$\frac{1}{2}\frac{4\pi}{3}(60)^3 = 0.4524 \times 10^6$	-22.5	-10.18×10^6
Cylinder	$\pi(60)^2(100) = 1.1310 \times 10^6$	$+50$	$+56.55 \times 10^6$
Cone	$-\frac{\pi}{3}(60)^2(100) = -0.3770 \times 10^6$	$+75$	-28.28×10^6
	$\Sigma V = 1.206 \times 10^6$		$\Sigma\bar{x}V = +18.09 \times 10^6$

Thus,

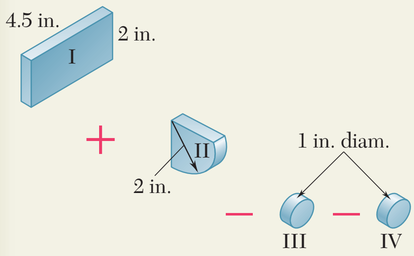
$$\bar{X}\Sigma V = \Sigma\bar{x}V: \qquad \bar{X}(1.206 \times 10^6 \text{ mm}^3) = 18.09 \times 10^6 \text{ mm}^4$$

$$\bar{X} = 15 \text{ mm}$$



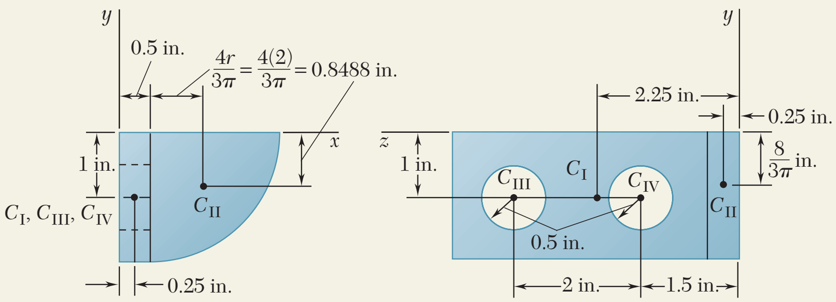
SAMPLE PROBLEM 5.11

Locate the center of gravity of the steel machine element shown. The diameter of each hole is 1 in.



SOLUTION

The machine element can be obtained by adding a rectangular parallelepiped (I) to a quarter cylinder (II) and then subtracting two 1-in.-diameter cylinders (III and IV). The volume and the coordinates of the centroid of each component are determined and are entered in the table below. Using the data in the table, we then determine the total volume and the moments of the volume with respect to each of the coordinate planes.



	V, in^3	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{z}$, in.	$\bar{x}V$, in^4	$\bar{y}V$, in^4	$\bar{z}V$, in^4
I	$(4.5)(2)(0.5) = 4.5$	0.25	-1	2.25	1.125	-4.5	10.125
II	$\frac{1}{4}\pi(2)^2(0.5) = 1.571$	1.3488	-0.8488	0.25	2.119	-1.333	0.393
III	$-\pi(0.5)^2(0.5) = -0.3927$	0.25	-1	3.5	-0.098	0.393	-1.374
IV	$-\pi(0.5)^2(0.5) = -0.3927$	0.25	-1	1.5	-0.098	0.393	-0.589
	$\Sigma V = 5.286$				$\Sigma\bar{x}V = 3.048$	$\Sigma\bar{y}V = -5.047$	$\Sigma\bar{z}V = 8.555$

Thus,

$\bar{X}\Sigma V = \Sigma\bar{x}V$: $\quad \bar{X}(5.286\text{ in}^3) = 3.048\text{ in}^4 \qquad \bar{X} = 0.577\text{ in.}$ ◀

$\bar{Y}\Sigma V = \Sigma\bar{y}V$: $\quad \bar{Y}(5.286\text{ in}^3) = -5.047\text{ in}^4 \qquad \bar{Y} = -0.955\text{ in.}$ ◀

$\bar{Z}\Sigma V = \Sigma\bar{z}V$: $\quad \bar{Z}(5.286\text{ in}^3) = 8.555\text{ in}^4 \qquad \bar{Z} = 1.618\text{ in.}$ ◀

PROBLEMS

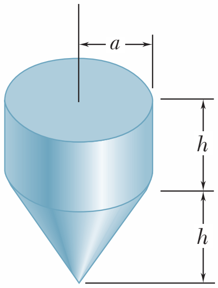


Fig. P5.57

5.57 A cone and a cylinder of the same radius a and height h are attached as shown. Determine the location of the centroid of the composite body.

5.58 Determine the y coordinate of the centroid of the body shown when (a) $b = \frac{1}{3}h$, (b) $b = \frac{1}{2}h$.

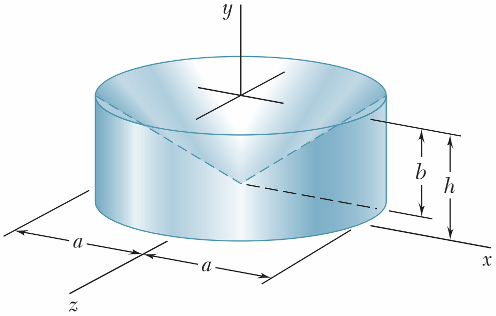


Fig. P5.58

5.59 A hemisphere and a cylinder are placed together as shown. Determine the ratio h/r for which the centroid of the composite body is located in the plane between the hemisphere and the cylinder.

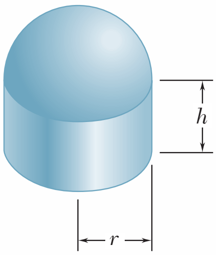


Fig. P5.59

5.60 Determine the location of the center of gravity of the parabolic reflector shown, which is formed by machining a rectangular block so that the curved surface is a paraboloid of revolution of base radius a and height h.

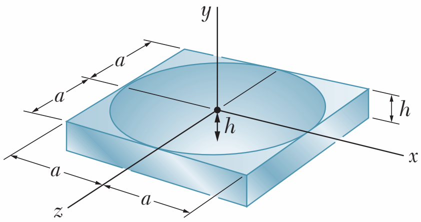


Fig. P5.60

5.61 For the machine element shown, locate the x coordinate of the center of gravity.

5.62 For the machine element shown, locate the y coordinate of the center of gravity.

5.63 For the machine element shown, locate the x coordinate of the center of gravity.

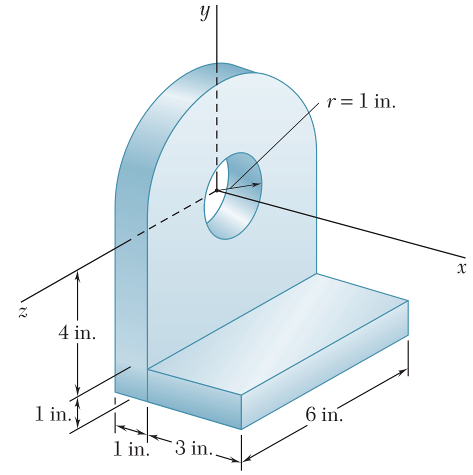


Fig. P5.61 and *P5.62*

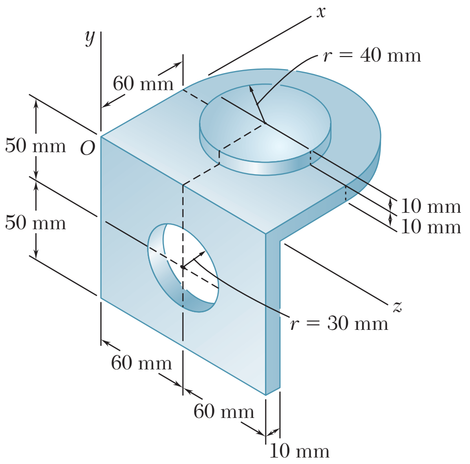


Fig. P5.63 and *P5.64*

5.64 For the machine element shown, locate the y coordinate of the center of gravity.

5.65 A wastebasket, designed to fit in the corner of a room, is 400 mm high and has a base in the shape of a quarter circle of radius 250 mm. Locate the center of gravity of the wastebasket, knowing that it is made of sheet metal of uniform thickness.

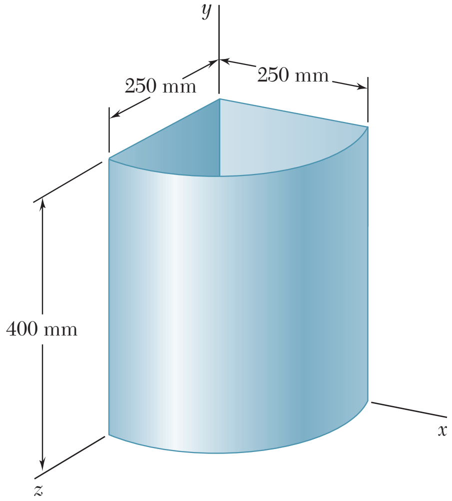


Fig. P5.65

5.66 through *5.68* Locate the center of gravity of the sheet-metal form shown.

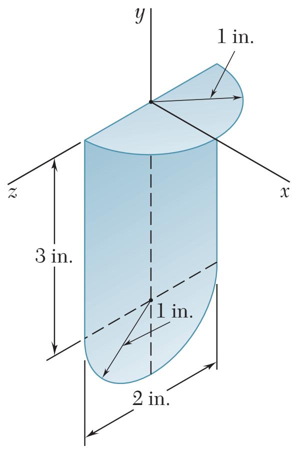


Fig. P5.66

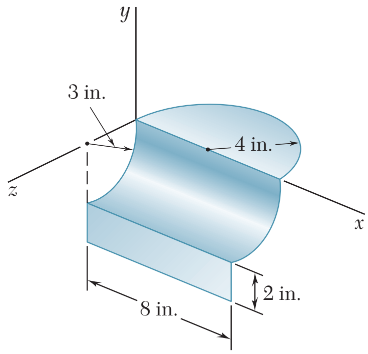


Fig. *P5.67*

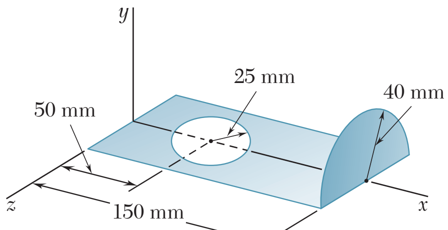


Fig. *P5.68*

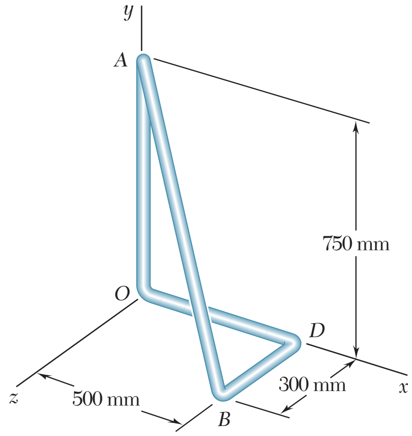


Fig. P5.69

5.69 and 5.70 Locate the center of gravity of the figure shown, knowing that it is made of thin brass rods of uniform diameter.

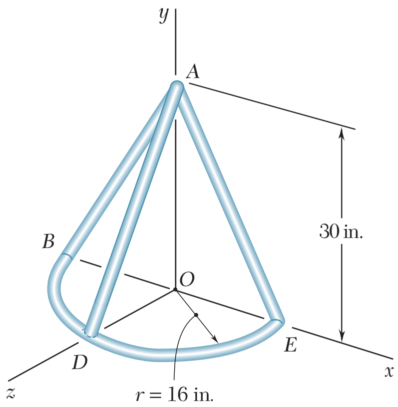


Fig. P5.70

5.71 Three brass plates are brazed to a steel pipe to form the flagpole base shown. Knowing that the pipe has a wall thickness of 0.25 in. and that each plate is 0.2 in. thick, determine the location of the center of gravity of the base. (Specific weights: brass = 0.306 lb/in^3, steel = 0.284 lb/in^3.)

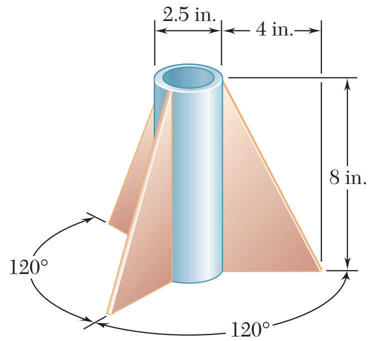


Fig. P5.71

5.72 A brass collar, of length 50 mm, is mounted on an aluminum rod of length 80 mm. Locate the center of gravity of the composite body. (Densities: brass = 8470 kg/m^3, aluminum = 2800 kg/m^3.)

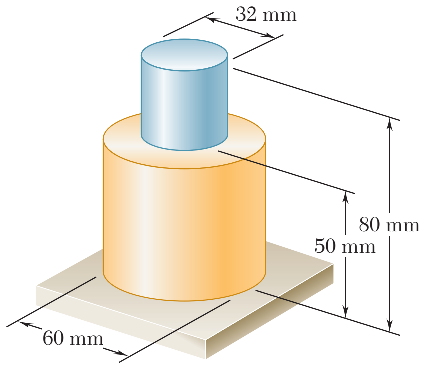


Fig. P5.72

REVIEW AND SUMMARY

This chapter was devoted chiefly to the determination of the *center of gravity* of a rigid body, i.e., to the determination of the point G where a single force $\mathbf{W}$, called the *weight* of the body, can be applied to represent the effect of the earth's attraction on the body.

Center of gravity of a two-dimensional body

In the first part of the chapter, we considered *two-dimensional bodies,* such as flat plates and wires contained in the xy plane. By adding force components in the vertical z direction and moments about the horizontal y and x axes [Sec. 5.2], we derived the relations

$$W = \int dW \qquad \bar{x}W = \int x\,dW \qquad \bar{y}W = \int y\,dW \tag{5.2}$$

which define the weight of the body and the coordinates $\bar{x}$ and $\bar{y}$ of its center of gravity.

Centroid of an area or line

In the case of a *homogeneous flat plate of uniform thickness* [Sec. 5.3], the center of gravity G of the plate coincides with the *centroid C of the area A* of the plate, the coordinates of which are defined by the relations

$$\bar{x}A = \int x\,dA \qquad \bar{y}A = \int y\,dA \tag{5.3}$$

Similarly, the determination of the center of gravity of a *homogeneous wire of uniform cross section* contained in a plane reduces to the determination of the *centroid C of the line L* representing the wire; we have

$$\bar{x}L = \int x\,dL \qquad \bar{y}L = \int y\,dL \tag{5.4}$$

First moments

The integrals in Eqs. (5.3) are referred to as the *first moments* of the area A with respect to the y and x axes and are denoted by Q_y and Q_x, respectively [Sec. 5.4]. We have

$$Q_y = \bar{x}A \qquad Q_x = \bar{y}A \tag{5.6}$$

The first moments of a line can be defined in a similar way.

Properties of symmetry

The determination of the centroid C of an area or line is simplified when the area or line possesses certain *properties of symmetry*. If the area or line is symmetric with respect to an axis, its centroid C

lies on that axis; if it is symmetric with respect to two axes, C is located at the intersection of the two axes; if it is symmetric with respect to a center O, C coincides with O.

Center of gravity of a composite body

The *areas and the centroids of various common shapes* are tabulated in Fig. 5.8. When a flat plate can be divided into several of these shapes, the coordinates $\overline{X}$ and $\overline{Y}$ of its center of gravity G can be determined from the coordinates $\bar{x}_1, \bar{x}_2, \ldots$ and $\bar{y}_1, \bar{y}_2, \ldots$ of the centers of gravity $G_1, G_2, \ldots$ of the various parts [Sec. 5.5]. Equating moments about the y and x axes, respectively (Fig. 5.20), we have

$$\overline{X}\Sigma W = \Sigma\bar{x}W \qquad \overline{Y}\Sigma W = \Sigma\bar{y}W \tag{5.7}$$

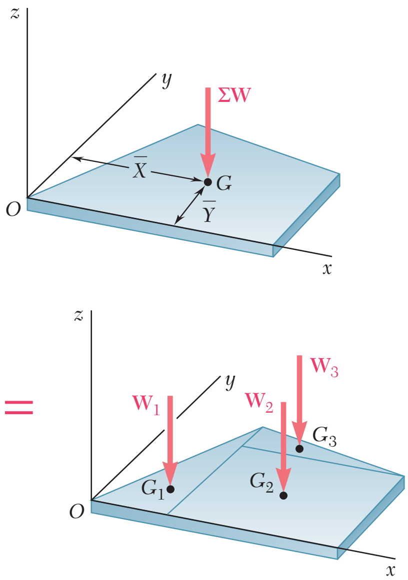


Fig. 5.20

If the plate is homogeneous and of uniform thickness, its center of gravity coincides with the centroid C of the area of the plate, and Eqs. (5.7) reduce to

$$Q_y = \overline{X}\Sigma A = \Sigma\bar{x}A \qquad Q_x = \overline{Y}\Sigma A = \Sigma\bar{y}A \tag{5.8}$$

These equations yield the first moments of the composite area, or they can be solved for the coordinates $\overline{X}$ and $\overline{Y}$ of its centroid [Sample Prob. 5.1]. The determination of the center of gravity of a composite wire is carried out in a similar fashion [Sample Prob. 5.2].

Determination of centroid by integration

When an area is bounded by analytical curves, the coordinates of its centroid can be determined by *integration* [Sec. 5.6]. This can be done by evaluating either the double integrals in Eqs. (5.3) or a *single integral* which uses one of the thin rectangular or pie-shaped elements of area shown in Fig. 5.12. Denoting by $\bar{x}_{el}$ and $\bar{y}_{el}$ the coordinates of the centroid of the element dA, we have

$$Q_y = \bar{x}A = \int \bar{x}_{el}\, dA \qquad Q_x = \bar{y}A = \int \bar{y}_{el}\, dA \tag{5.9}$$

It is advantageous to use the same element of area to compute both of the first moments Q_y and Q_x; the same element can also be used to determine the area A [Sample Prob. 5.4].

Theorems of Pappus-Guldinus

The *theorems of Pappus-Guldinus* relate the determination of the area of a surface of revolution or the volume of a body of revolution to the determination of the centroid of the generating curve or area [Sec. 5.7]. The area A of the surface generated by rotating a curve of length L about a fixed axis (Fig. 5.21*a*) is

$$A = 2\pi\bar{y}L \tag{5.10}$$

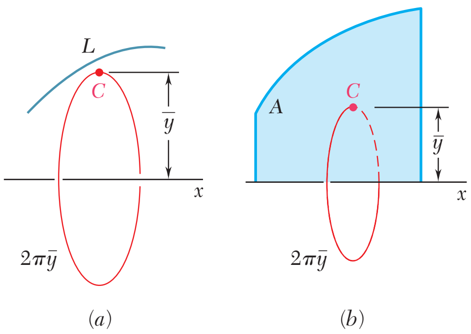


Fig. 5.21

where $\bar{y}$ represents the distance from the centroid C of the curve to the fixed axis. Similarly, the volume V of the body generated by rotating an area A about a fixed axis (Fig. 5.21*b*) is

$$V = 2\pi\bar{y}A \tag{5.11}$$

where $\bar{y}$ represents the distance from the centroid C of the area to the fixed axis.

The concept of centroid of an area can also be used to solve problems other than those dealing with the weight of flat plates. For example, to determine the reactions at the supports of a beam [Sec. 5.8], we can replace a *distributed load* w by a concentrated load **W** equal in magnitude to the area A under the load curve and passing through the centroid C of that area (Fig. 5.22).

Distributed loads

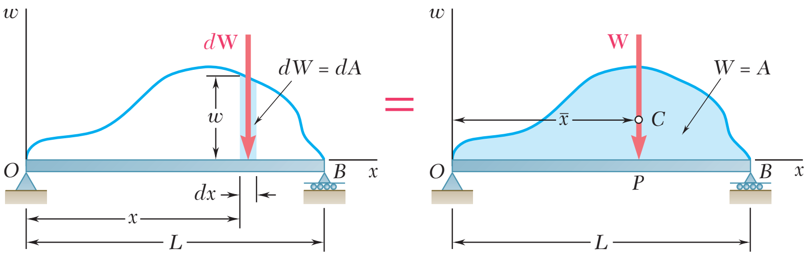


Fig. 5.22

Center of gravity of a three-dimensional body

The last part of the chapter was devoted to the determination of the *center of gravity G of a three-dimensional body*. The coordinates $\bar{x}$, $\bar{y}$, $\bar{z}$ of G were defined by the relations

$$\bar{x}W = \int x\,dW \qquad \bar{y}W = \int y\,dW \qquad \bar{z}W = \int z\,dW \tag{5.16}$$

Centroid of a volume

In the case of a *homogeneous body*, the center of gravity G coincides with the *centroid C of the volume V* of the body; the coordinates of C are defined by the relations

$$\bar{x}V = \int x\,dV \qquad \bar{y}V = \int y\,dV \qquad \bar{z}V = \int z\,dV \tag{5.18}$$

If the volume possesses a *plane of symmetry*, its centroid C will lie in that plane; if it possesses two planes of symmetry, C will be located on the line of intersection of the two planes; if it possesses three planes of symmetry which intersect at only one point, C will coincide with that point [Sec. 5.9].

Center of gravity of a composite body

The *volumes and centroids of various common three-dimensional shapes* are tabulated in Fig. 5.19. When a body can be divided into several of these shapes, the coordinates $\bar{X}$, $\bar{Y}$, $\bar{Z}$ of its center of gravity G can be determined from the corresponding coordinates of the centers of gravity of its various parts [Sec. 5.10]. We have

$$\bar{X}\Sigma W = \Sigma\bar{x}W \qquad \bar{Y}\Sigma W = \Sigma\bar{y}W \qquad \bar{Z}\Sigma W = \Sigma\bar{z}W \tag{5.19}$$

If the body is made of a homogeneous material, its center of gravity coincides with the centroid C of its volume, and we write [Sample Probs. 5.10 and 5.11]

$$\bar{X}\Sigma V = \Sigma\bar{x}V \qquad \bar{Y}\Sigma V = \Sigma\bar{y}V \qquad \bar{Z}\Sigma V = \Sigma\bar{z}V \tag{5.20}$$

REVIEW PROBLEMS

5.73 and *5.74* Locate the centroid of the plane area shown.

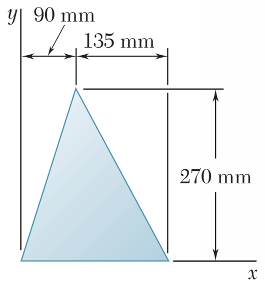


Fig. P5.73

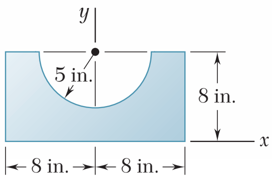


Fig. *P5.74*

5.75 A thin homogenous wire is bent to form the perimeter of the plane area of Prob. 5.73. Locate the center of gravity of the wire figure thus formed.

5.76 Knowing that the figure shown is formed of a thin homogeneous wire, determine the length l of portion CE of the wire for which the center of gravity of the figure is located at point C when (*a*) $\theta = 15°$, (*b*) $\theta = 60°$.

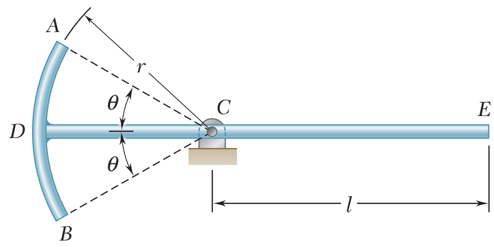


Fig. P5.76

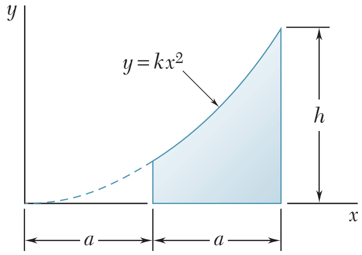


Fig. P5.77

5.77 Determine by direct integration the centroid of the area shown.

5.78 Determine by direct integration the x coordinate of the centroid of the area shown.

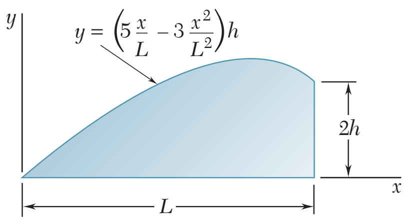


Fig. *P5.78*

5.79 Determine the volume of the body shown.

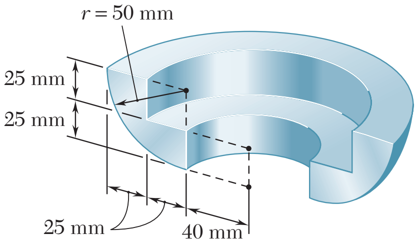


Fig. P5.79 and *P5.80*

5.80 Determine the total surface area of the body shown.

5.81 Determine the reactions at the beam supports for the given loading when $w_0 = 450$ lb/ft.

5.82 Determine (*a*) the distributed load w_0 at the end C of the beam ABC for which the reaction at C is zero, (*b*) the corresponding reaction at B.

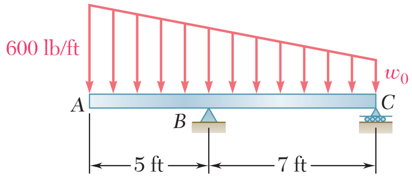


Fig. P5.81 and *P5.82*

5.83 Determine the center of gravity of the machine element shown.

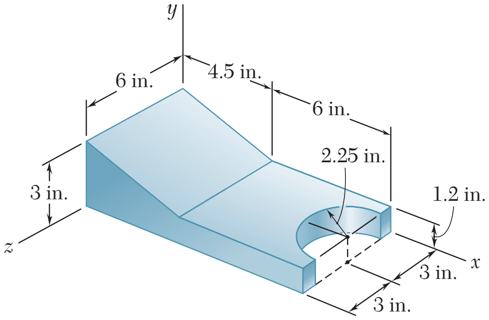


Fig. P5.83

5.84 A regular pyramid 300 mm high, with a square base of side 250 mm, is made of wood. Its four triangular faces are covered with steel sheets 1 mm thick. Locate the center of gravity of the composite body. (Densities: steel = 7850 kg/m^3, wood = 500 kg/m^3.)

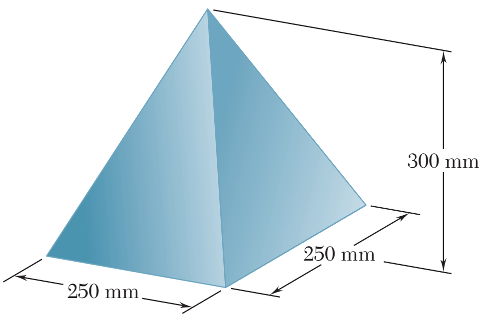


Fig. P5.84

Trusses, such as this Pratt-style cantilever arch bridge in New York State, provide both a practical and an economical solution to many engineering problems.

CHAPTER 6

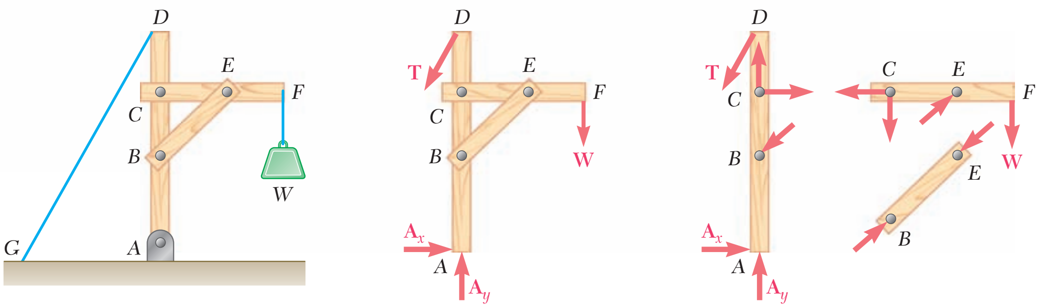
Analysis of Structures

Chapter 6 Analysis of Structures

6.1 INTRODUCTION

The problems considered in the preceding chapters concerned the equilibrium of a single rigid body, and all the forces involved were external to the rigid body. We now consider problems dealing with the equilibrium of structures made of several connected parts. These problems call for the determination not only of the external forces acting on the structure but also of the forces which hold together the various parts of the structure. From the point of view of the structure as a whole, these forces are *internal forces*.

Consider, for example, the crane shown in Fig. 6.1*a*, which carries a load W. The crane consists of three beams AD, CF, and BE connected by frictionless pins; it is supported by a pin at A and by a cable DG. The free-body diagram of the crane has been drawn in Fig. 6.1*b*. The external forces, which are shown in the diagram, include the weight $\mathbf{W}$, the two components $\mathbf{A}_x$ and $\mathbf{A}_y$ of the reaction at A, and the force $\mathbf{T}$ exerted by the cable at D. The internal forces holding the various parts of the crane together do not appear in the diagram. If, however, the crane is dismembered and if a free-body diagram is drawn for each of its component parts, the forces holding the three beams together will also be represented, since these forces are external forces from the point of view of each component part (Fig. 6.1*c*).



(*a*) (*b*) (*c*)

Fig. 6.1

It will be noted that the force exerted at B by member BE on member AD has been represented as equal and opposite to the force exerted at the same point by member AD on member BE; the force exerted at E by BE on CF is shown equal and opposite to the force exerted by CF on BE; and the components of the force exerted at C by CF on AD are shown equal and opposite to the components of the force exerted by AD on CF. This is in conformity with Newton's third law, which states that *the forces of action and reaction between bodies in contact have the same magnitude, same line of action, and opposite sense*. As pointed out in Chap. 1, this law, which is based on experimental evidence, is one of the six fundamental principles of elementary mechanics, and its application is essential to the solution of problems involving connected bodies.

In this chapter, three broad categories of engineering structures will be considered:

1. *Trusses,* which are designed to support loads and are usually stationary, fully constrained structures. Trusses consist exclusively of straight members connected at joints located at the ends of each member. Members of a truss, therefore, are *two-force members,* i.e., members acted upon by two equal and opposite forces directed along the member.
2. *Frames,* which are also designed to support loads and are also usually stationary, fully constrained structures. However, like the crane of Fig. 6.1, frames always contain at least one *multiforce member,* i.e., a member acted upon by three or more forces which, in general, are not directed along the member.
3. *Machines,* which are designed to transmit and modify forces and are structures containing moving parts. Machines, like frames, always contain at least one multiforce member.

Photo 6.1 Shown is a pin-jointed connection on the approach span to the San Francisco–Oakland Bay Bridge.

TRUSSES

6.2 DEFINITION OF A TRUSS

The truss is one of the major types of engineering structures. It provides both a practical and an economical solution to many engineering situations, especially in the design of bridges and buildings. A typical truss is shown in Fig. 6.2*a*. A truss consists of straight members connected at joints. Truss members are connected at their extremities only; thus no member is continuous through a joint. In Fig. 6.2*a*, for example, there is no member *AB*; there are instead two distinct members *AD* and *DB*. Most actual structures are made of several trusses joined together to form a space framework. Each truss is designed to carry those loads which act in its plane and thus may be treated as a two-dimensional structure.

In general, the members of a truss are slender and can support little lateral load; all loads, therefore, must be applied to the various joints, and not to the members themselves. When a concentrated load is to be applied between two joints, or when a distributed load is to be supported by the truss, as in the case of a bridge truss, a floor system must be provided which, through the use of stringers and floor beams, transmits the load to the joints (Fig. 6.3).

The weights of the members of the truss are also assumed to be applied to the joints, half of the weight of each member being applied to each of the two joints the member connects. Although the members are actually joined together by means of welded, bolted, or riveted connections, it is customary to assume that the members are pinned together; therefore, the forces acting at each end of a member reduce to a single force and no couple. Thus, the only forces assumed to be applied to a truss member are a single

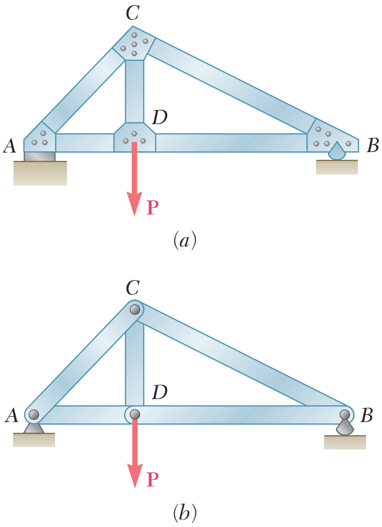


Fig. 6.2

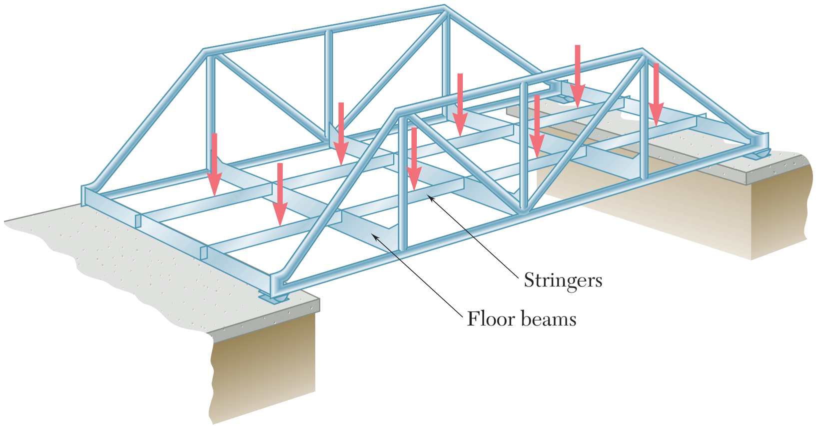


Fig. 6.3

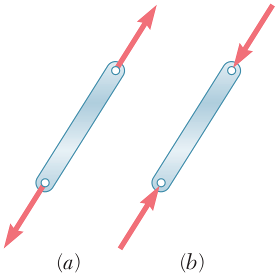


Fig. 6.4

force at each end of the member. Each member can then be treated as a two-force member, and the entire truss can be considered as a group of pins and two-force members (Fig. 6.2*b*). An individual member can be acted upon as shown in either of the two sketches of Fig. 6.4. In Fig. 6.4*a*, the forces tend to pull the member apart, and the member is in tension; in Fig. 6.4*b*, the forces tend to compress the member, and the member is in compression. A number of typical trusses are shown in Fig. 6.5.

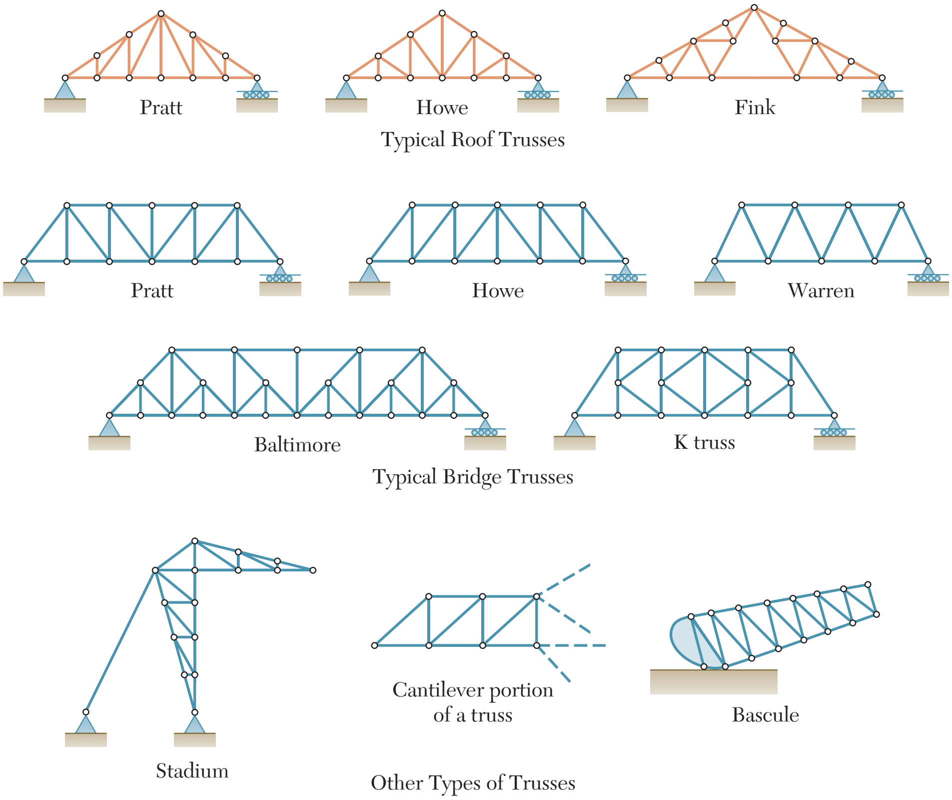


Fig. 6.5

6.3 SIMPLE TRUSSES

Consider the truss of Fig. 6.6*a*, which is made of four members connected by pins at A, B, C, and D. If a load is applied at B, the truss will greatly deform, completely losing its original shape. In contrast, the truss of Fig. 6.6*b*, which is made of three members connected by pins at A, B, and C, will deform only slightly under a load applied at B. The only possible deformation for this truss is one involving small changes in the length of its members. The truss of Fig. 6.6*b* is said to be a *rigid truss*, the term rigid being used here to indicate that the truss *will not collapse*.

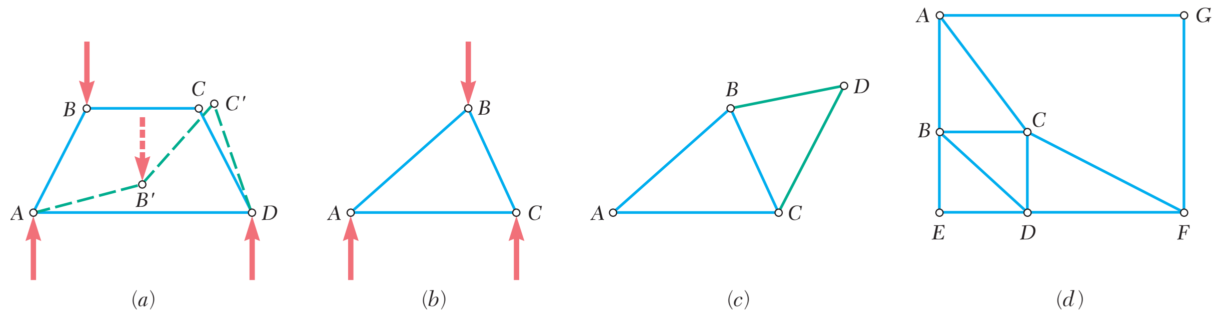


Fig. 6.6

As shown in Fig. 6.6*c*, a larger rigid truss can be obtained by adding two members BD and CD to the basic triangular truss of Fig. 6.6*b*. This procedure can be repeated as many times as desired, and the resulting truss will be rigid if each time two new members are added, they are attached to two existing joints and connected at a new joint.† A truss which can be constructed in this manner is called a *simple truss*.

It should be noted that a simple truss is not necessarily made only of triangles. The truss of Fig. 6.6*d*, for example, is a simple truss which was constructed from triangle ABC by adding successively the joints D, E, F, and G. On the other hand, rigid trusses are not always simple trusses, even when they appear to be made of triangles. The Fink and Baltimore trusses shown in Fig. 6.5, for instance, are not simple trusses, since they cannot be constructed from a single triangle in the manner described above. All the other trusses shown in Fig. 6.5 are simple trusses, as may be easily checked. (For the K truss, start with one of the central triangles.)

Returning to Fig. 6.6, we note that the basic triangular truss of Fig. 6.6*b* has three members and three joints. The truss of Fig. 6.6*c* has two more members and one more joint, i.e., five members and four joints altogether. Observing that every time two new members are added, the number of joints is increased by one, we find that in a simple truss the total number of members is $m = 2n - 3$, where n is the total number of joints.

Photo 6.2 Two K-trusses were used as the main components of the movable bridge shown which moved above a large stockpile of ore. The bucket below the trusses picked up ore and redeposited it until the ore was thoroughly mixed. The ore was then sent to the mill for processing into steel.

†The three joints must not be in a straight line.

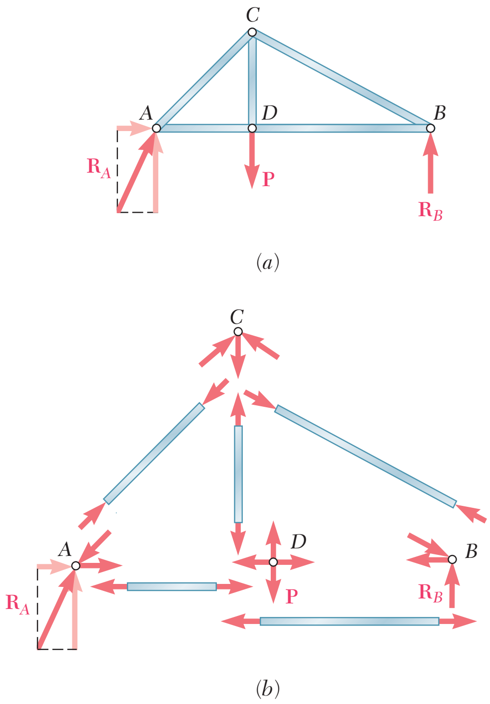


Fig. 6.7

Photo 6.3 Because roof trusses, such as those shown, require support only at their ends, it is possible to construct buildings with large, unobstructed floor areas.

6.4 ANALYSIS OF TRUSSES BY THE METHOD OF JOINTS

We saw in Sec. 6.2 that a truss can be considered as a group of pins and two-force members. The truss of Fig. 6.2, whose free-body diagram is shown in Fig. 6.7*a*, can thus be dismembered, and a free-body diagram can be drawn for each pin and each member (Fig. 6.7*b*). Each member is acted upon by two forces, one at each end; these forces have the same magnitude, same line of action, and opposite sense (Sec. 4.6). Furthermore, Newton's third law indicates that the forces of action and reaction between a member and a pin are equal and opposite. Therefore, the forces exerted by a member on the two pins it connects must be directed along that member and be equal and opposite. The common magnitude of the forces exerted by a member on the two pins it connects is commonly referred to as the *force in the member* considered, even though this quantity is actually a scalar. Since the lines of action of all the internal forces in a truss are known, the analysis of a truss reduces to computing the forces in its various members and to determining whether each of its members is in tension or in compression.

Since the entire truss is in equilibrium, each pin must be in equilibrium. The fact that a pin is in equilibrium can be expressed by drawing its free-body diagram and writing two equilibrium equations (Sec. 2.9). If the truss contains n pins, there will, therefore, be $2n$ equations available, which can be solved for $2n$ unknowns. In the case of a simple truss, we have $m = 2n - 3$, that is, $2n = m + 3$, and the number of unknowns which can be determined from the free-body diagrams of the pins is thus $m + 3$. This means that the forces in all the members, the two components of the reaction $\mathbf{R}_A$, and the reaction $\mathbf{R}_B$ can be found by considering the free-body diagrams of the pins.

The fact that the entire truss is a rigid body in equilibrium can be used to write three more equations involving the forces shown in the free-body diagram of Fig. 6.7*a*. Since they do not contain any new information, these equations are not independent of the equations associated with the free-body diagrams of the pins. Nevertheless, they can be used to determine the components of the reactions at the supports. The arrangement of pins and members in a simple truss is such that it will then always be possible to find a joint involving only two unknown forces. These forces can be determined by the methods of Sec. 2.11 and their values transferred to the adjacent joints and treated as known quantities at these joints. This procedure can be repeated until all the unknown forces have been determined.

As an example, the truss of Fig. 6.7 will be analyzed by considering the equilibrium of each pin successively, starting with a joint at which only two forces are unknown. In the truss considered, all pins are subjected to at least three unknown forces. Therefore, the reactions at the supports must first be determined by considering the entire truss as a free body and using the equations of equilibrium of a rigid body. We find in this way that $\mathbf{R}_A$ is vertical and determine the magnitudes of $\mathbf{R}_A$ and $\mathbf{R}_B$.

The number of unknown forces at joint A is thus reduced to two, and these forces can be determined by considering the equilibrium of pin A. The reaction $\mathbf{R}_A$ and the forces $\mathbf{F}_{AC}$ and $\mathbf{F}_{AD}$ exerted on pin A by members AC and AD, respectively, must form a force

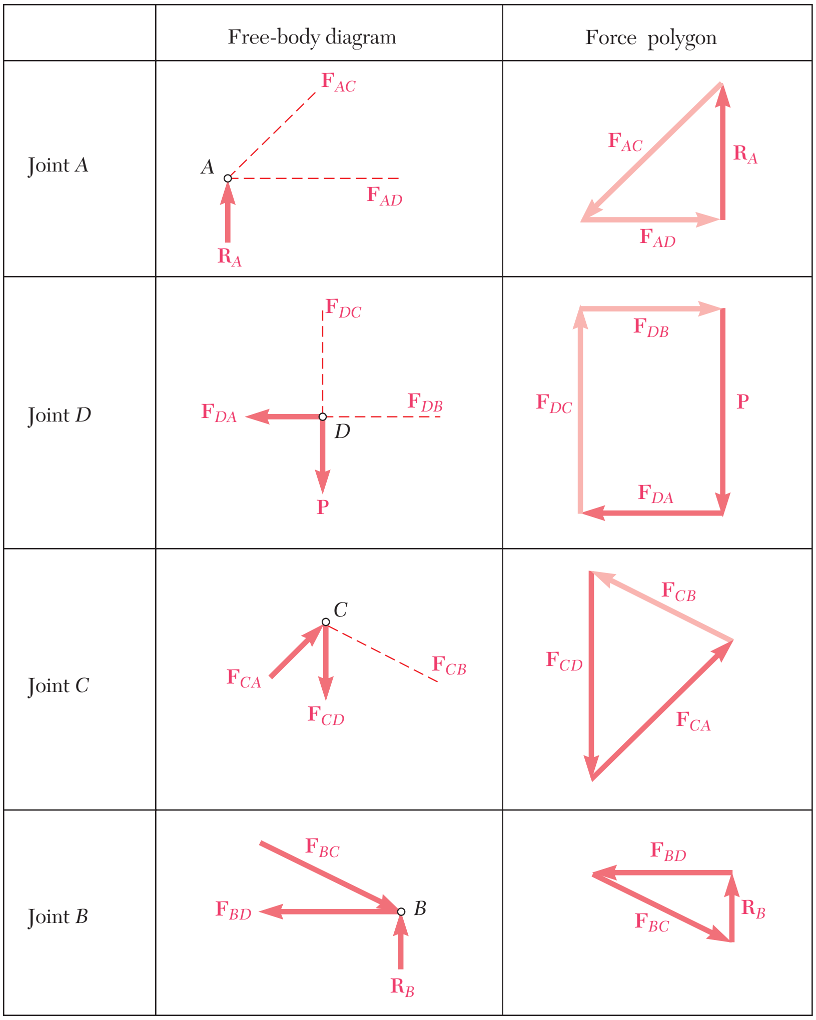


Fig. 6.8

triangle. First we draw $\mathbf{R}_A$ (Fig. 6.8); noting that $\mathbf{F}_{AC}$ and $\mathbf{F}_{AD}$ are directed along AC and AD, respectively, we complete the triangle and determine the magnitude and sense of $\mathbf{F}_{AC}$ and $\mathbf{F}_{AD}$. The magnitudes F_{AC} and F_{AD} represent the forces in members AC and AD, respectively. Since $\mathbf{F}_{AC}$ is directed down and to the left, that is, *toward* joint A, member AC pushes on pin A and is in compression. Since $\mathbf{F}_{AD}$ is directed *away* from joint A, member AD pulls on pin A and is in tension.

We can now proceed to joint D, where only two forces, $\mathbf{F}_{DC}$ and $\mathbf{F}_{DB}$, are still unknown. The other forces are the load $\mathbf{P}$, which is given, and the force $\mathbf{F}_{DA}$ exerted on the pin by member AD. As indicated above, this force is equal and opposite to the force $\mathbf{F}_{AD}$ exerted by the same member on pin A. We can draw the force polygon corresponding to joint D, as shown in Fig. 6.8, and determine the forces $\mathbf{F}_{DC}$ and $\mathbf{F}_{DB}$ from that polygon. However, when more than three forces are involved, it is usually more convenient to solve the equations of equilibrium $\Sigma F_x = 0$ and $\Sigma F_y = 0$ for the two unknown forces. Since both of these forces are found to be directed away from joint D, members DC and DB pull on the pin and are in tension.

Next, joint C is considered; its free-body diagram is shown in Fig. 6.8. It is noted that both $\mathbf{F}_{CD}$ and $\mathbf{F}_{CA}$ are known from the analysis of the preceding joints and that only $\mathbf{F}_{CB}$ is unknown. Since the equilibrium of each pin provides sufficient information to determine two unknowns, a check of our analysis is obtained at this joint. The force triangle is drawn, and the magnitude and sense of $\mathbf{F}_{CB}$ are determined. Since $\mathbf{F}_{CB}$ is directed toward joint C, member CB pushes on pin C and is in compression. The check is obtained by verifying that the force $\mathbf{F}_{CB}$ and member CB are parallel.

At joint B, all of the forces are known. Since the corresponding pin is in equilibrium, the force triangle must close and an additional check of the analysis is obtained.

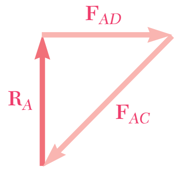


Fig. 6.9

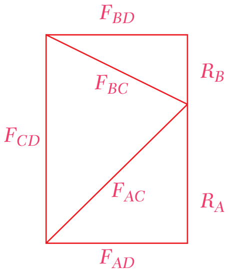


Fig. 6.10

It should be noted that the force polygons shown in Fig. 6.8 are not unique. Each of them could be replaced by an alternative configuration. For example, the force triangle corresponding to joint A could be drawn as shown in Fig. 6.9. The triangle actually shown in Fig. 6.8 was obtained by drawing the three forces $\mathbf{R}_A$, $\mathbf{F}_{AC}$, and $\mathbf{F}_{AD}$ in tip-to-tail fashion in the order in which their lines of action are encountered when moving clockwise around joint A. The other force polygons in Fig. 6.8, having been drawn in the same way, can be made to fit into a single diagram, as shown in Fig. 6.10. Such a diagram, known as *Maxwell's diagram,* greatly facilitates the *graphical analysis* of truss problems.

6.5 JOINTS UNDER SPECIAL LOADING CONDITIONS

Consider Fig. 6.11*a*, in which the joint shown connects four members lying in two intersecting straight lines. The free-body diagram of Fig. 6.11*b* shows that pin A is subjected to two pairs of directly opposite forces. The corresponding force polygon, therefore, must be a parallelogram (Fig. 6.11*c*), and *the forces in opposite members must be equal*.

Consider next Fig. 6.12*a*, in which the joint shown connects three members and supports a load $\mathbf{P}$. Two of the members lie in the same line, and the load $\mathbf{P}$ acts along the third member. The free-body diagram of pin A and the corresponding force polygon will be as shown in Fig. 6.11*b* and *c*, with $\mathbf{F}_{AE}$ replaced by the load $\mathbf{P}$. Thus, *the forces in the two opposite members must be equal, and the force in the other member must equal P*. A particular case of special interest is shown in Fig. 6.12*b*. Since, in this case, no external load is applied to the joint, we have $P = 0$, and the force in member AC is zero. Member AC is said to be a *zero-force member*.

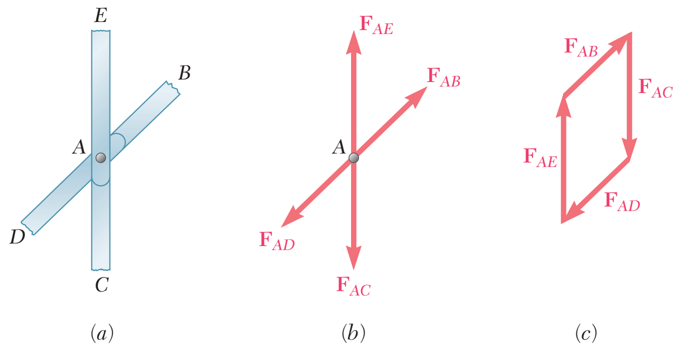


Fig. 6.11

Consider now a joint connecting two members only. From Sec. 2.9, we know that a particle which is acted upon by two forces will be in equilibrium if the two forces have the same magnitude, same line of action, and opposite sense. In the case of the joint of Fig. 6.13*a*, which connects two members *AB* and *AD* lying in the same line, *the forces in the two members must be equal* for pin *A* to be in equilibrium. In the case of the joint of Fig. 6.13*b*, pin *A* cannot be in equilibrium unless the forces in both members are zero. Members connected as shown in Fig. 6.13*b*, therefore, must be *zero-force members*.

Spotting the joints which are under the special loading conditions listed above will expedite the analysis of a truss. Consider, for example, a Howe truss loaded as shown in Fig. 6.14. All of the members represented by green lines will be recognized as zero-force members. Joint *C* connects three members, two of which lie in the same line and is not subjected to any external load; member *BC* is thus a zero-force member. Applying the same reasoning to joint *K*, we find that member *JK* is also a zero-force member. But joint *J* is now in the same situation as joints *C* and *K*, and member *IJ* must be a zero-force member. The examination of joints *C*, *J*, and *K* also shows that the forces in members *AC* and *CE* are equal, that the forces in members *HJ* and *JL* are equal, and that the forces in members *IK* and *KL* are equal. Turning our attention to joint *I*, where the 20-kN load and member *HI* are collinear, we note that the force in member *HI* is 20 kN (tension) and that the forces in members *GI* and *IK* are equal. Hence, the forces in members *GI*, *IK*, and *KL* are equal.

Note that the conditions described above do not apply to joints *B* and *D* in Fig. 6.14, and it would be wrong to assume that the force in member *DE* is 25 kN or that the forces in members *AB* and *BD* are equal. The forces in these members and in all the remaining members should be found by carrying out the analysis of joints *A*, *B*, *D*, *E*, *F*, *G*, *H*, and *L* in the usual manner. Thus, until you have become thoroughly familiar with the conditions under which the rules established in this section can be applied, you should draw the free-body diagrams of all the pins and write the corresponding equilibrium equations (or draw the corresponding force polygons) whether or not the joints being considered are under one of the special loading conditions described above.

A final remark concerning zero-force members: These members are not useless. For example, although the zero-force members of Fig. 6.14 do not carry any loads under the loading conditions shown, the same members would probably carry loads if the loading conditions were changed. Besides, even in the case considered, these members are needed to support the weight of the truss and to maintain the truss in the desired shape.

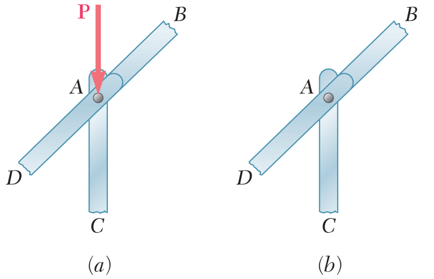


Fig. 6.12

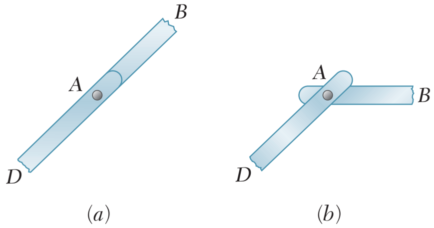


Fig. 6.13

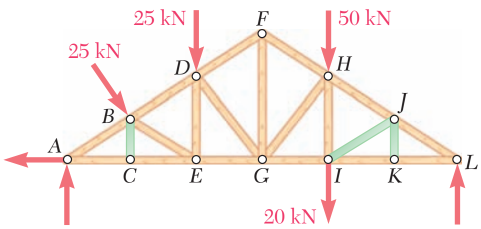
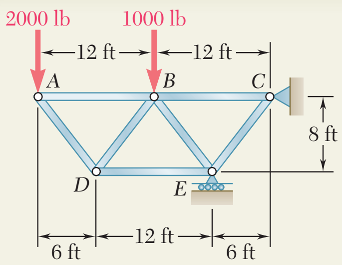
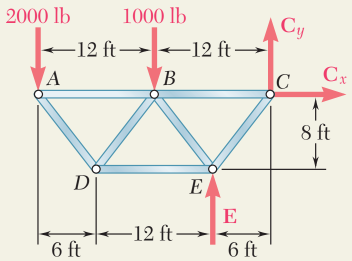


Fig. 6.14



SAMPLE PROBLEM 6.1

Using the method of joints, determine the force in each member of the truss shown.



SOLUTION

Free-Body: Entire Truss. A free-body diagram of the entire truss is drawn; external forces acting on this free body consist of the applied loads and the reactions at C and E. We write the following equilibrium equations.

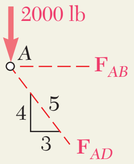
$+\circlearrowleft \Sigma M_C = 0$: $\quad (2000 \text{ lb})(24 \text{ ft}) + (1000 \text{ lb})(12 \text{ ft}) - E(6 \text{ ft}) = 0$

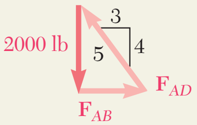
$E = +10{,}000 \text{ lb} \qquad\qquad \mathbf{E} = 10{,}000 \text{ lb}\uparrow$

$\xrightarrow{+} \Sigma F_x = 0$: $\qquad\qquad \mathbf{C}_x = 0$

$+\uparrow \Sigma F_y = 0$: $\quad -2000 \text{ lb} - 1000 \text{ lb} + 10{,}000 \text{ lb} + C_y = 0$

$C_y = -7000 \text{ lb} \qquad\qquad \mathbf{C}_y = 7000 \text{ lb}\downarrow$



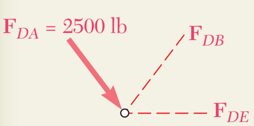


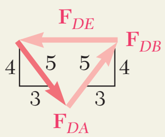
Free-Body: Joint *A*. This joint is subjected to only two unknown forces, namely, the forces exerted by members AB and AD. A force triangle is used to determine $\mathbf{F}_{AB}$ and $\mathbf{F}_{AD}$. We note that member AB pulls on the joint and thus is in tension and that member AD pushes on the joint and thus is in compression. The magnitudes of the two forces are obtained from the proportion

$$\frac{2000 \text{ lb}}{4} = \frac{F_{AB}}{3} = \frac{F_{AD}}{5}$$

$F_{AB} = 1500 \text{ lb } T$ ◀

$F_{AD} = 2500 \text{ lb } C$ ◀

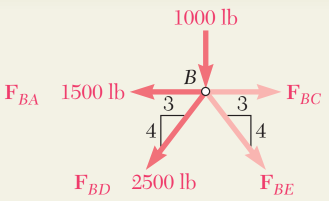




Free-Body: Joint *D*. Since the force exerted by member AD has been determined, only two unknown forces are now involved at this joint. Again, a force triangle is used to determine the unknown forces in members DB and DE.

$F_{DB} = F_{DA} \qquad\qquad F_{DB} = 2500 \text{ lb } T$ ◀

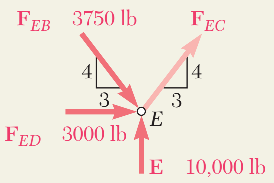
$F_{DE} = 2(\frac{3}{5})F_{DA} \qquad\qquad F_{DE} = 3000 \text{ lb } C$ ◀



Free-Body: Joint *B*. Since more than three forces act at this joint, we determine the two unknown forces $\mathbf{F}_{BC}$ and $\mathbf{F}_{BE}$ by solving the equilibrium equations $\Sigma F_x = 0$ and $\Sigma F_y = 0$. We arbitrarily assume that both unknown forces act away from the joint, i.e., that the members are in tension. The positive value obtained for F_{BC} indicates that our assumption was correct; member *BC* is in tension. The negative value of F_{BE} indicates that our assumption was wrong; member *BE* is in compression.

$$+\uparrow\Sigma F_y = 0: \qquad -1000 - \tfrac{4}{5}(2500) - \tfrac{4}{5}F_{BE} = 0$$
$$F_{BE} = -3750 \text{ lb} \qquad F_{BE} = 3750 \text{ lb } C \quad ◀$$

$$\overset{+}{\rightarrow}\Sigma F_x = 0: \qquad F_{BC} - 1500 - \tfrac{3}{5}(2500) - \tfrac{3}{5}(3750) = 0$$
$$F_{BC} = +5250 \text{ lb} \qquad F_{BC} = 5250 \text{ lb } T \quad ◀$$

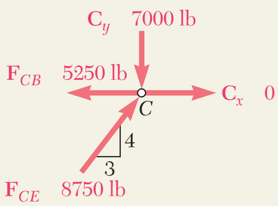


Free-Body: Joint *E*. The unknown force $\mathbf{F}_{EC}$ is assumed to act away from the joint. Summing *x* components, we write

$$\overset{+}{\rightarrow}\Sigma F_x = 0: \qquad \tfrac{3}{5}F_{EC} + 3000 + \tfrac{3}{5}(3750) = 0$$
$$F_{EC} = -8750 \text{ lb} \qquad F_{EC} = 8750 \text{ lb } C \quad ◀$$

Summing *y* components, we obtain a check of our computations:

$$\begin{aligned} +\uparrow\Sigma F_y &= 10{,}000 - \tfrac{4}{5}(3750) - \tfrac{4}{5}(8750) \\ &= 10{,}000 - 3000 - 7000 = 0 \qquad \text{(checks)} \end{aligned}$$



Free-Body: Joint *C*. Using the computed values of $\mathbf{F}_{CB}$ and $\mathbf{F}_{CE}$, we can determine the reactions $\mathbf{C}_x$ and $\mathbf{C}_y$ by considering the equilibrium of this joint. Since these reactions have already been determined from the equilibrium of the entire truss, we will obtain two checks of our computations. We can also simply use the computed values of all forces acting on the joint (forces in members and reactions) and check that the joint is in equilibrium:

$$\overset{+}{\rightarrow}\Sigma F_x = -5250 + \tfrac{3}{5}(8750) = -5250 + 5250 = 0 \qquad \text{(checks)}$$
$$+\uparrow\Sigma F_y = -7000 + \tfrac{4}{5}(8750) = -7000 + 7000 = 0 \qquad \text{(checks)}$$

PROBLEMS

6.1 through 6.18 Using the method of joints, determine the force in each member of the truss shown. State whether each member is in tension or compression.

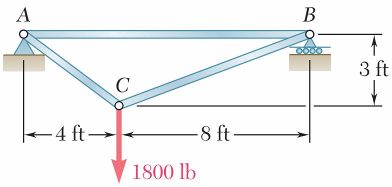


Fig. P6.1

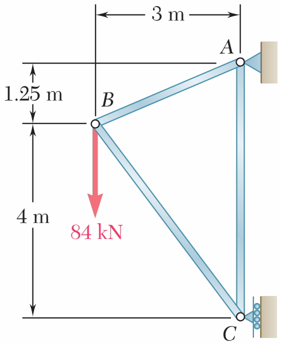


Fig. P6.2

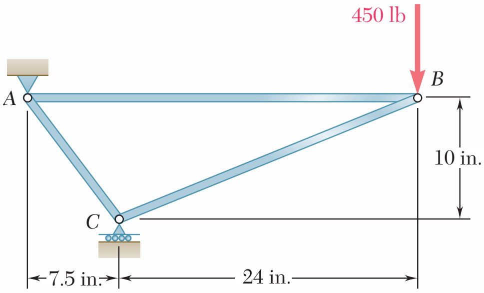


Fig. P6.3

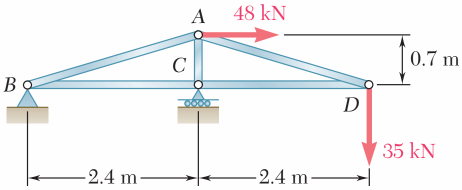


Fig. P6.4

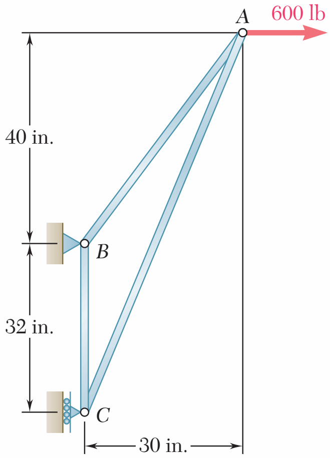


Fig. ***P6.5***

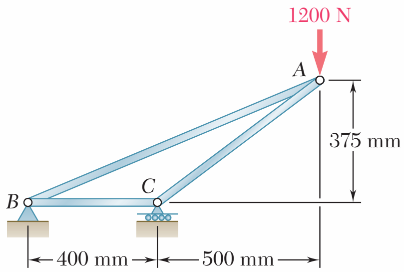


Fig. P6.6

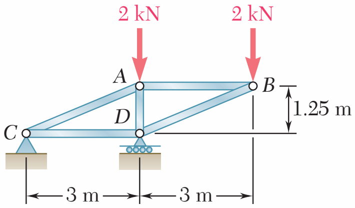


Fig. ***P6.7***

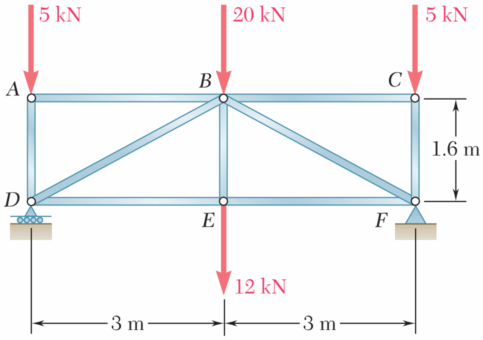


Fig. P6.8

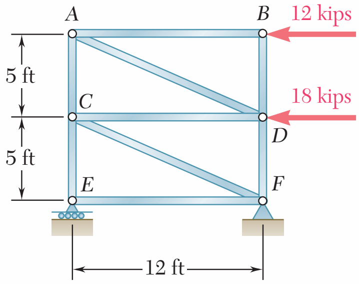


Fig. P6.9

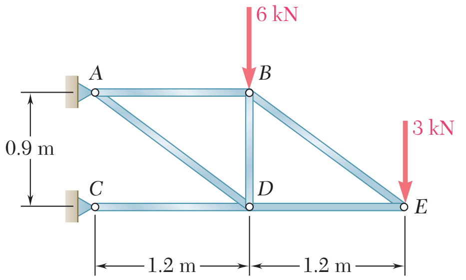


Fig. P6.10

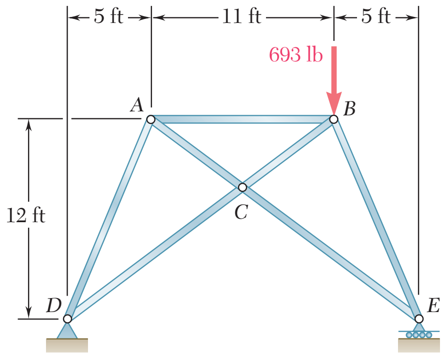


Fig. P6.11

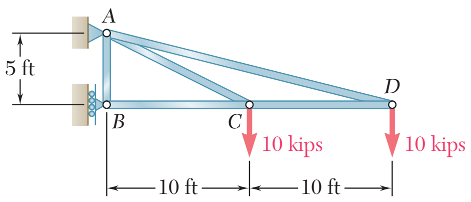


Fig. P6.12

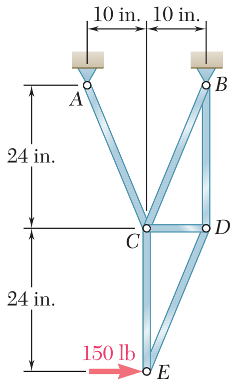


Fig. P6.13

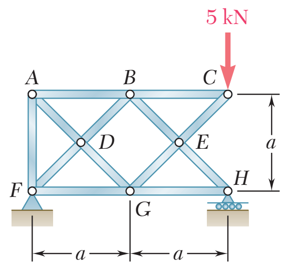


Fig. P6.14

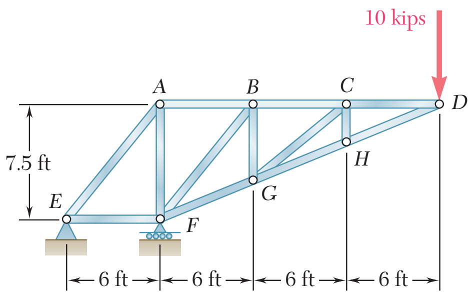


Fig. P6.15

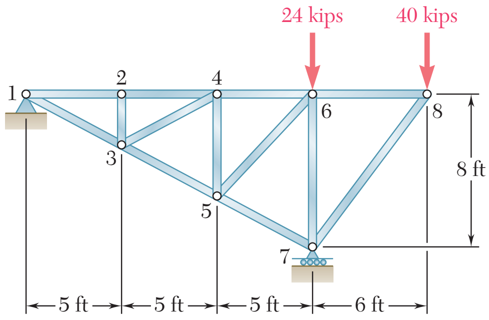


Fig. P6.16

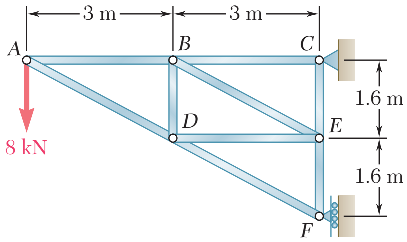


Fig. P6.17

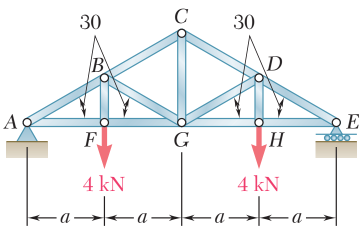


Fig. P6.18

6.19 Determine whether the trusses given as Probs. 6.17, 6.21, and 6.23 are simple trusses.

6.20 Determine whether the trusses given as Probs. 6.12, 6.14, 6.22, and 6.24 are simple trusses.

6.21 through 6.24 Determine the zero-force members in the truss shown for the given loading.

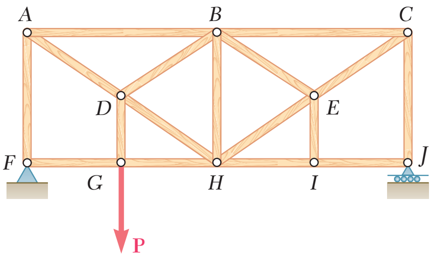


Fig. P6.21

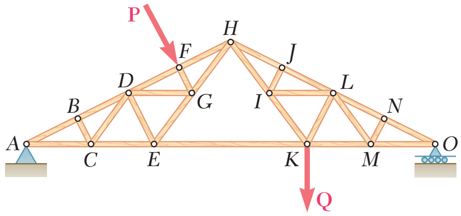


Fig. *P6.22*

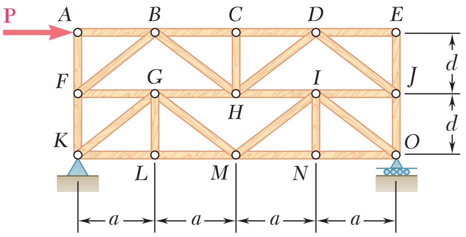


Fig. *P6.23*

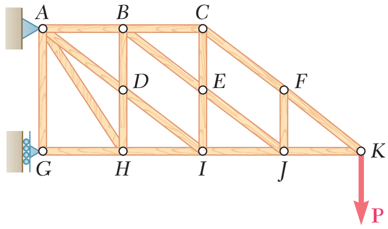


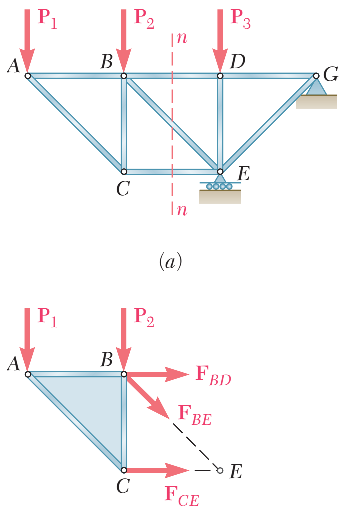
Fig. P6.24

6.6 ANALYSIS OF TRUSSES BY THE METHOD OF SECTIONS

The method of joints is most effective when the forces in all the members of a truss are to be determined. If, however, the force in only one member or the forces in a very few members are desired, another method, the method of sections, is more efficient.

Assume, for example, that we want to determine the force in member *BD* of the truss shown in Fig. 6.15*a*. To do this, we must determine the force with which member *BD* acts on either joint *B* or joint *D*. If we were to use the method of joints, we would choose either joint *B* or joint *D* as a free body. However, we can also choose as a free body a larger portion of the truss, composed of several joints and members, provided that the desired force is one of the external forces acting on that portion. If, in addition, the portion of the truss is chosen so that there is a total of only three unknown forces acting upon it, the desired force can be obtained by solving the equations of equilibrium for this portion of the truss. In practice, the portion of the truss to be utilized is obtained by *passing a section* through three members of the truss, one of which is the desired member, i.e., by drawing a line which divides the truss into two completely separate parts but does not intersect more than three members. Either of the two portions of the truss obtained after the intersected members have been removed can then be used as a free body.†

In Fig. 6.15*a*, the section *nn* has been passed through members *BD*, *BE*, and *CE*, and the portion *ABC* of the truss is chosen as the



(*b*)

Fig. 6.15

†In the analysis of certain trusses, sections are passed which intersect more than three members; the forces in one, or possibly two, of the intersected members may be obtained if equilibrium equations can be found, each of which involves only one unknown (see Probs. 6.41 through 6.43).

free body (Fig. 6.15*b*). The forces acting on the free body are the loads $\mathbf{P}_1$ and $\mathbf{P}_2$ at points A and B, respectively, and the three unknown forces $\mathbf{F}_{BD}$, $\mathbf{F}_{BE}$, and $\mathbf{F}_{CE}$. Since it is not known whether the members removed were in tension or compression, the three forces have been arbitrarily drawn away from the free body as if the members were in tension.

The fact that the rigid body ABC is in equilibrium can be expressed by writing three equations which can be solved for the three unknown forces. If only the force $\mathbf{F}_{BD}$ is desired, we need write only one equation, provided that the equation does not contain the other unknowns. Thus, the equation $\Sigma M_E = 0$ yields the value of the magnitude F_{BD} of the force $\mathbf{F}_{BD}$ (Fig. 6.15). A positive sign in the answer will indicate that our original assumption regarding the sense of $\mathbf{F}_{BD}$ was correct and that member BD is in tension; a negative sign will indicate that our assumption was incorrect and that BD is in compression.

On the other hand, if only the force $\mathbf{F}_{CE}$ is desired, an equation which does not involve $\mathbf{F}_{BD}$ or $\mathbf{F}_{BE}$ should be written; the appropriate equation is $\Sigma M_B = 0$. Again a positive sign for the magnitude F_{CE} of the desired force indicates a correct assumption, that is, tension; and a negative sign indicates an incorrect assumption, that is, compression.

If only the force $\mathbf{F}_{BE}$ is desired, the appropriate equation is $\Sigma F_y = 0$. Whether the member is in tension or compression is again determined from the sign of the answer.

When the force in only one member is determined, no independent check of the computation is available. However, when all the unknown forces acting on the free body are determined, the computations can be checked by writing an additional equation. For instance, if $\mathbf{F}_{BD}$, $\mathbf{F}_{BE}$, and $\mathbf{F}_{CE}$ are determined as indicated above, the computation can be checked by verifying that $\Sigma F_x = 0$.

*6.7 TRUSSES MADE OF SEVERAL SIMPLE TRUSSES

Consider two simple trusses ABC and DEF. If they are connected by three bars BD, BE, and CE as shown in Fig. 6.16*a*, they will form together a rigid truss $ABDF$. The trusses ABC and DEF can also be combined into a single rigid truss by joining joints B and D into a single joint B and by connecting joints C and E by a bar CE (Fig. 6.16*b*). The truss thus obtained is known as a *Fink truss*. It should be noted that the trusses of Fig. 6.16*a* and *b* are *not* simple trusses; they cannot be constructed from a triangular truss by adding successive pairs of members as prescribed in Sec. 6.3. They are rigid trusses, however, as we can check by comparing the systems of connections used to hold the simple trusses ABC and DEF together (three bars in Fig. 6.16*a*, one pin and one bar in Fig. 6.16*b*) with the systems of supports

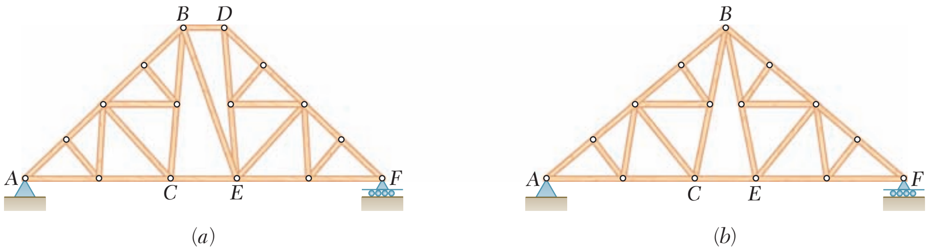
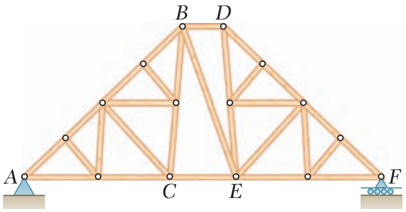


Fig. 6.16

discussed in Secs. 4.4 and 4.5. Trusses made of several simple trusses rigidly connected are known as *compound trusses*.

In a compound truss the number of members m and the number of joints n are still related by the formula $m = 2n - 3$. This can be verified by observing that, if a compound truss is supported by a frictionless pin and a roller (involving three unknown reactions), the total number of unknowns is $m + 3$, and this number must be equal to the number $2n$ of equations obtained by expressing that the n pins are in equilibrium; it follows that $m = 2n - 3$. Compound trusses supported by a pin and a roller, or by an equivalent system of supports, are *statically determinate, rigid,* and *completely constrained*. This means that all of the unknown reactions and the forces in all the members can be determined by the methods of statics and that the truss will neither collapse nor move. The forces in the members, however, cannot all be determined by the method of joints, except by solving a large number of simultaneous equations. In the case of the compound truss of Fig. 6.16*a*, for example, it is more efficient to pass a section through members BD, BE, and CE to determine the forces in these members.



(*a*)

Fig. 6.16 (*repeated*)

Suppose, now, that the simple trusses ABC and DEF are connected by *four* bars BD, BE, CD, and CE (Fig. 6.17). The number of members m is now larger than $2n - 3$; the truss obtained is *overrigid,* and one of the four members BD, BE, CD, or CE is said to be *redundant*. If the truss is supported by a pin at A and a roller at F, the total number of unknowns is $m + 3$. Since $m > 2n - 3$, the number $m + 3$ of unknowns is now larger than the number $2n$ of available independent equations; the truss is *statically indeterminate*.

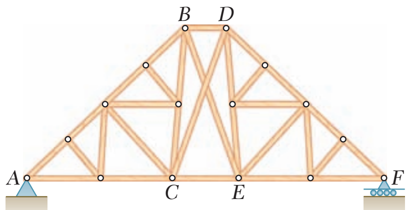


Fig. 6.17

Finally, let us assume that the two simple trusses ABC and DEF are joined by a pin as shown in Fig. 6.18*a*. The number of members m is smaller than $2n - 3$. If the truss is supported by a pin at A and a roller at F, the total number of unknowns is $m + 3$. Since $m < 2n - 3$, the number $m + 3$ of unknowns is now smaller than the number $2n$ of equilibrium equations which should be satisfied; the truss is *nonrigid* and will collapse under its own weight. However, if two pins are used to support it, the truss becomes *rigid* and will not collapse (Fig. 6.18*b*). We note that the total number of unknowns is now $m + 4$ and is equal to the number $2n$ of equations. More generally, if the reactions at the supports involve r unknowns, the condition for a compound truss to be statically determinate, rigid, and completely constrained is $m + r = 2n$. However, while necessary, this condition is not sufficient for the equilibrium of a structure which ceases to be rigid when detached from its supports (see Sec. 6.10).

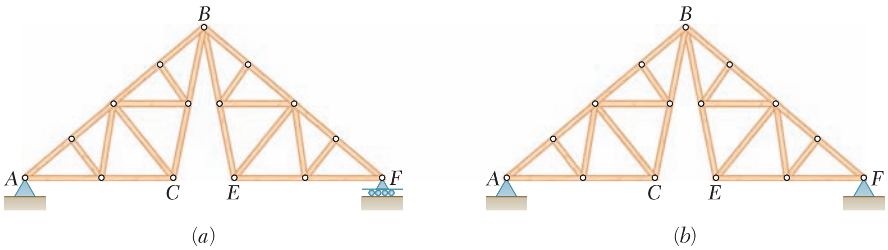
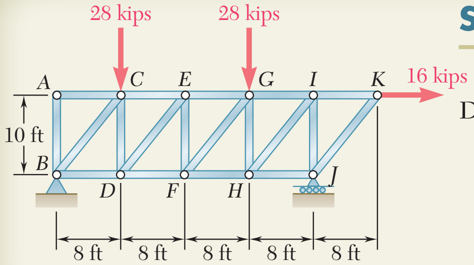
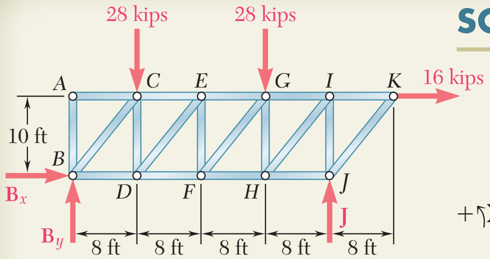


Fig. 6.18



SAMPLE PROBLEM 6.2

Determine the force in members *EF* and *GI* of the truss shown.



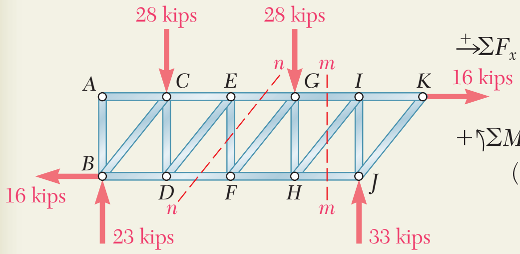
SOLUTION

Free-Body: Entire Truss. A free-body diagram of the entire truss is drawn; external forces acting on this free body consist of the applied loads and the reactions at *B* and *J*. We write the following equilibrium equations.

$+\circlearrowleft \Sigma M_B = 0$:

$$-(28 \text{ kips})(8 \text{ ft}) - (28 \text{ kips})(24 \text{ ft}) - (16 \text{ kips})(10 \text{ ft}) + J(32 \text{ ft}) = 0$$

$$J = +33 \text{ kips} \qquad \mathbf{J} = 33 \text{ kips} \uparrow$$



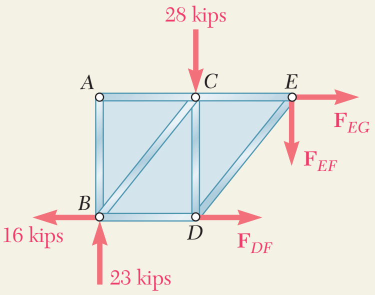
$\overset{+}{\rightarrow} \Sigma F_x = 0$: $\qquad B_x + 16 \text{ kips} = 0$

$$B_x = -16 \text{ kips} \qquad \mathbf{B}_x = 16 \text{ kips} \leftarrow$$

$+\circlearrowleft \Sigma M_J = 0$:

$$(28 \text{ kips})(24 \text{ ft}) + (28 \text{ kips})(8 \text{ ft}) - (16 \text{ kips})(10 \text{ ft}) - B_y(32 \text{ ft}) = 0$$

$$B_y = +23 \text{ kips} \qquad \mathbf{B}_y = 23 \text{ kips} \uparrow$$



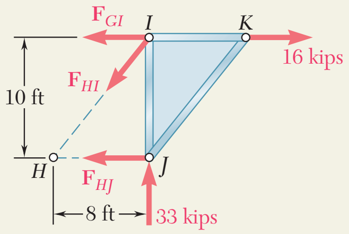
Force in Member *EF*. Section *nn* is passed through the truss so that it intersects member *EF* and only two additional members. After the intersected members have been removed, the left-hand portion of the truss is chosen as a free body. Three unknowns are involved; to eliminate the two horizontal forces, we write

$+\uparrow \Sigma F_y = 0$: $\qquad +23 \text{ kips} - 28 \text{ kips} - F_{EF} = 0$

$$F_{EF} = -5 \text{ kips}$$

The sense of $\mathbf{F}_{EF}$ was chosen assuming member *EF* to be in tension; the negative sign obtained indicates that the member is in compression.

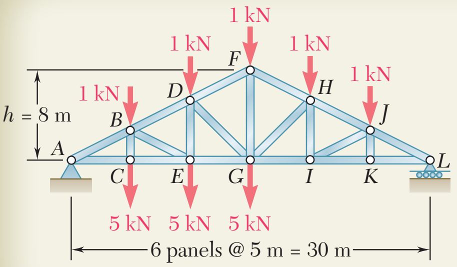
$$F_{EF} = 5 \text{ kips } C \quad ◀$$



Force in Member *GI*. Section *mm* is passed through the truss so that it intersects member *GI* and only two additional members. After the intersected members have been removed, we choose the right-hand portion of the truss as a free body. Three unknown forces are again involved; to eliminate the two forces passing through point *H*, we write

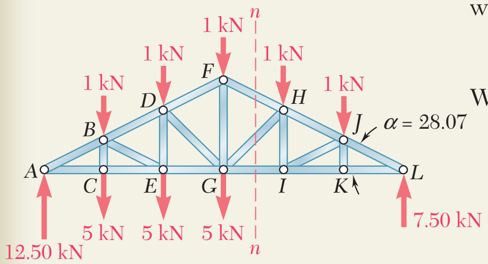
$+\circlearrowleft \Sigma M_H = 0$: $\qquad (33 \text{ kips})(8 \text{ ft}) - (16 \text{ kips})(10 \text{ ft}) + F_{GI}(10 \text{ ft}) = 0$

$$F_{GI} = -10.4 \text{ kips} \qquad F_{GI} = 10.4 \text{ kips } C \quad ◀$$



SAMPLE PROBLEM 6.3

Determine the force in members FH, GH, and GI of the roof truss shown.



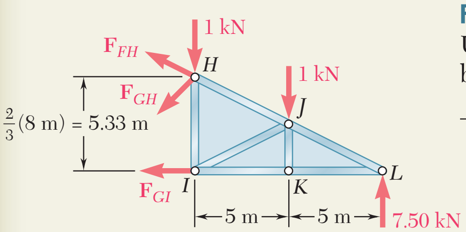
SOLUTION

Free Body: Entire Truss. From the free-body diagram of the entire truss, we find the reactions at A and L:

$$\mathbf{A} = 12.50 \text{ kN}\uparrow \qquad \mathbf{L} = 7.50 \text{ kN}\uparrow$$

We note that

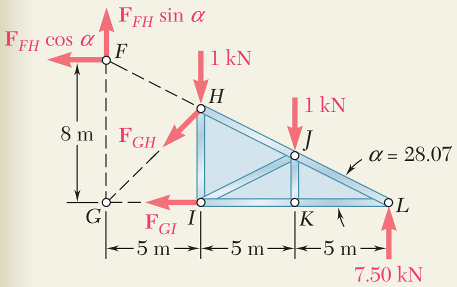
$$\tan\alpha = \frac{FG}{GL} = \frac{8 \text{ m}}{15 \text{ m}} = 0.5333 \qquad \alpha = 28.07°$$



Force in Member *GI*. Section nn is passed through the truss as shown. Using the portion HLI of the truss as a free body, the value of F_{GI} is obtained by writing

$$+\circlearrowleft \Sigma M_H = 0: \qquad (7.50 \text{ kN})(10 \text{ m}) - (1 \text{ kN})(5 \text{ m}) - F_{GI}(5.33 \text{ m}) = 0$$

$$F_{GI} = +13.13 \text{ kN} \qquad F_{GI} = 13.13 \text{ kN } T \blacktriangleleft$$

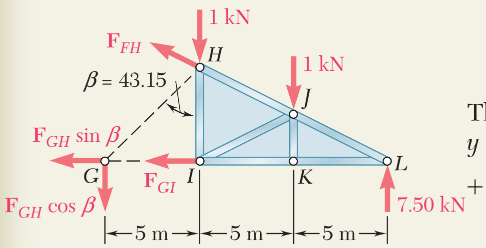


Force in Member *FH*. The value of F_{FH} is obtained from the equation $\Sigma M_G = 0$. We move $\mathbf{F}_{FH}$ along its line of action until it acts at point F, where it is resolved into its x and y components. The moment of $\mathbf{F}_{FH}$ with respect to point G is now equal to $(F_{FH}\cos\alpha)(8 \text{ m})$.

$$+\circlearrowleft \Sigma M_G = 0:$$

$$(7.50 \text{ kN})(15 \text{ m}) - (1 \text{ kN})(10 \text{ m}) - (1 \text{ kN})(5 \text{ m}) + (F_{FH}\cos\alpha)(8 \text{ m}) = 0$$

$$F_{FH} = -13.81 \text{ kN} \qquad F_{FH} = 13.81 \text{ kN } C \blacktriangleleft$$



Force in Member *GH*. We first note that

$$\tan\beta = \frac{GI}{HI} = \frac{5 \text{ m}}{\frac{2}{3}(8 \text{ m})} = 0.9375 \qquad \beta = 43.15°$$

The value of F_{GH} is then determined by resolving the force $\mathbf{F}_{GH}$ into x and y components at point G and solving the equation $\Sigma M_L = 0$.

$$+\circlearrowleft \Sigma M_L = 0: \qquad (1 \text{ kN})(10 \text{ m}) + (1 \text{ kN})(5 \text{ m}) + (F_{GH}\cos\beta)(15 \text{ m}) = 0$$

$$F_{GH} = -1.371 \text{ kN} \qquad F_{GH} = 1.371 \text{ kN } C \blacktriangleleft$$

PROBLEMS

6.25 Determine the force in members *BD* and *CD* of the truss shown.

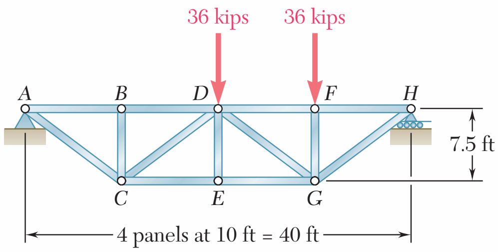


Fig. P6.25 and P6.26

6.26 Determine the force in members *DF* and *DG* of the truss shown.

6.27 Determine the force in members *FG* and *FH* of the truss shown when $P = 35$ kN.

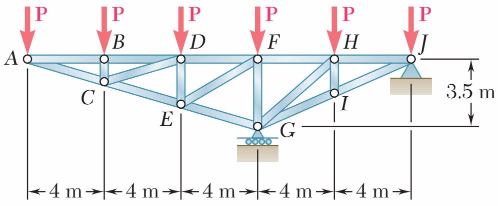


Fig. P6.27 and P6.28

6.28 Determine the force in members *EF* and *EG* of the truss shown when $P = 35$ kN.

6.29 Determine the force in members *DE* and *DF* of the truss shown when $P = 20$ kips.

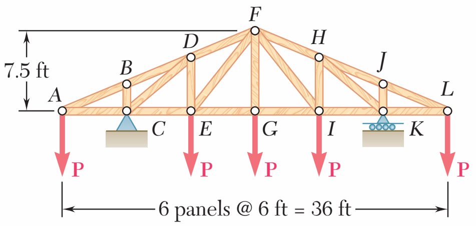


Fig. P6.29 and *P6.30*

6.30 Determine the force in members *EG* and *EF* of the truss shown when $P = 20$ kips.

6.31 Determine the force in members *DF* and *DE* of the truss shown.

6.32 Determine the force in members *CD* and *CE* of the truss shown.

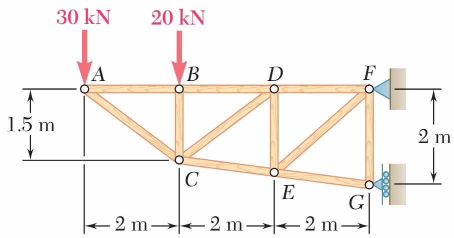


Fig. P6.31 and *P6.32*

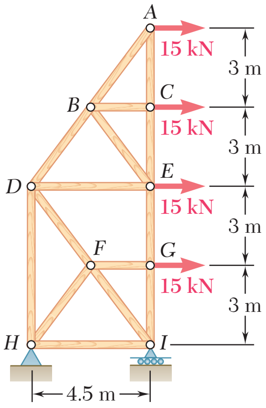


Fig. P6.33 and P6.34

6.33 Determine the force in members *BD* and *DE* of the truss shown.

6.34 Determine the force in members *FH* and *DH* of the truss shown.

6.35 Determine the force in members *FH*, *GH*, and *GI* of the stadium truss shown.

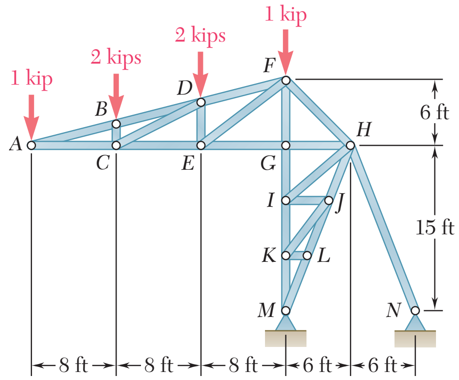


Fig. P6.35 and *P6.36*

6.36 Determine the force in members *DF*, *DE*, and *CE* of the stadium truss shown.

6.37 Determine the force in members *CE*, *DE*, and *DF* of the truss shown.

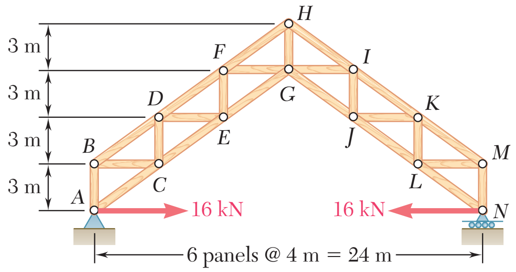


Fig. P6.37 and *P6.38*

6.38 Determine the force in members *GI*, *GJ*, and *HI* of the truss shown.

6.39 Determine the force in members *AD*, *CD*, and *CE* of the truss shown.

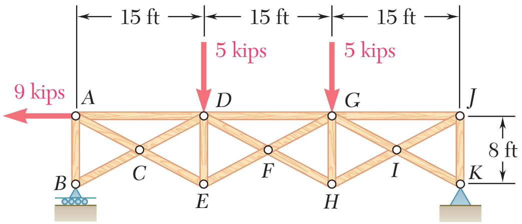


Fig. P6.39 and P6.40

6.40 Determine the force in members *DG*, *FG*, and *FH* of the truss shown.

6.41 Determine the force in member *GJ* of the truss shown. (*Hint:* Use section *a-a*.)

6.42 Determine the force in members *AB* and *KL* of the truss shown. (*Hint:* Use section *a-a*.)

6.43 Determine the force in members *DG* and *FH* of the truss shown. (*Hint:* Use section *a-a*.)

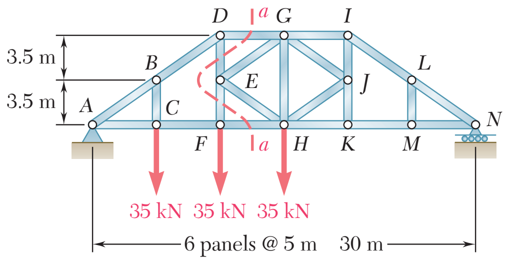


Fig. *P6.43*

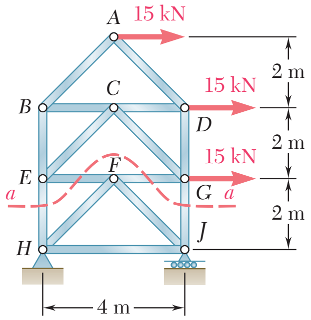


Fig. P6.41

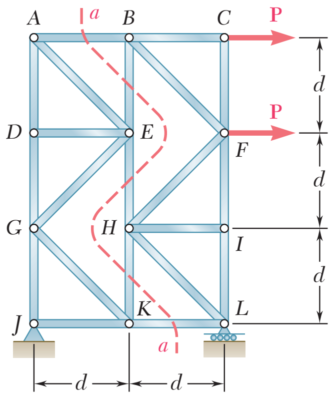


Fig. P6.42

6.44 The diagonal members in the center panels of the truss shown are very slender and can act only in tension; such members are known as *counters*. Determine the force in member *DE* and in the counters that are acting under the given loading.

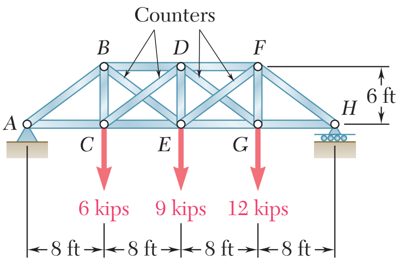


Fig. P6.44

6.45 Solve Prob. 6.44 assuming that the 6-kip load has been removed.

6.46 Solve Prob. 6.44 assuming that the 9-kip load has been removed.

6.47 and 6.48 Classify each of the given structures as completely, partially, or improperly constrained; if completely constrained, further classify as determinate or indeterminate. All members can act both in tension and in compression.

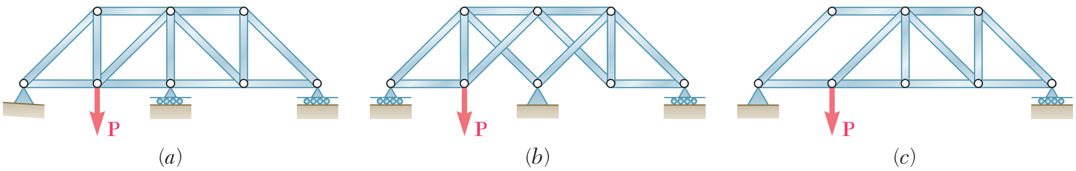


Fig. P6.47

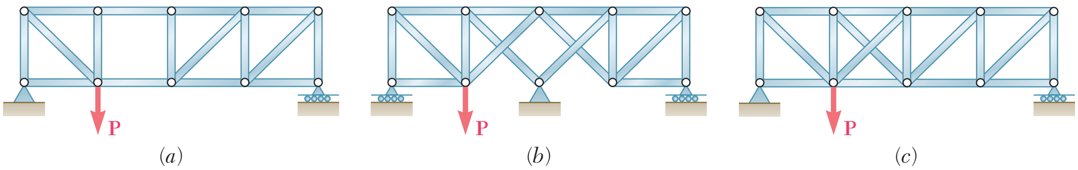


Fig. P6.48

FRAMES AND MACHINES

6.8 STRUCTURES CONTAINING MULTIFORCE MEMBERS

Under trusses, we have considered structures consisting entirely of pins and straight two-force members. The forces acting on the two-force members were known to be directed along the members themselves. We now consider structures in which at least one of the members is a *multiforce* member, i.e., a member acted upon by three or more forces. These forces will generally not be directed along the members on which they act; their direction is unknown, and they should be represented therefore by two unknown components.

Frames and machines are structures containing multiforce members. *Frames* are designed to support loads and are usually stationary, fully constrained structures. *Machines* are designed to transmit and modify forces; they may or may not be stationary and will always contain moving parts.

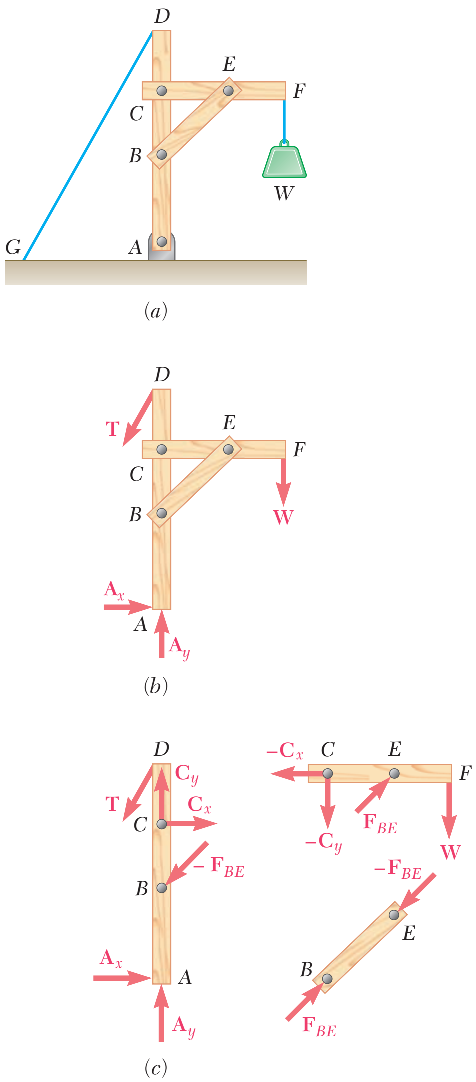


Fig. 6.19

6.9 ANALYSIS OF A FRAME

As a first example of analysis of a frame, the crane described in Sec. 6.1, which carries a given load W (Fig. 6.19*a*), will again be considered. The free-body diagram of the entire frame is shown in Fig. 6.19*b*. This diagram can be used to determine the external forces acting on the frame. Summing moments about A, we first determine the force $\mathbf{T}$ exerted by the cable; summing x and y components, we then determine the components $\mathbf{A}_x$ and $\mathbf{A}_y$ of the reaction at the pin A.

In order to determine the internal forces holding the various parts of a frame together, we must dismember the frame and draw a free-body diagram for each of its component parts (Fig. 6.19*c*). First, the two-force members should be considered. In this frame, member BE is the only two-force member. The forces acting at each end of this member must have the same magnitude, same line of action, and opposite sense (Sec. 4.6). They are therefore directed along BE and will be denoted, respectively, by $\mathbf{F}_{BE}$ and $-\mathbf{F}_{BE}$. Their sense will be arbitrarily assumed as shown in Fig. 6.19*c*; later the sign obtained for the common magnitude F_{BE} of the two forces will confirm or deny this assumption.

Next, we consider the multiforce members, i.e., the members which are acted upon by three or more forces. According to Newton's third law, the force exerted at B by member BE on member AD must be equal and opposite to the force $\mathbf{F}_{BE}$ exerted by AD on BE. Similarly, the force exerted at E by member BE on member CF must be equal and opposite to the force $-\mathbf{F}_{BE}$ exerted by CF on BE. Thus the forces that the two-force member BE exerts on AD and CF are equal to $-\mathbf{F}_{BE}$ and $\mathbf{F}_{BE}$, respectively; they have the same magnitude F_{BE} and opposite sense and should be directed as shown in Fig. 6.19*c*.

At C two multiforce members are connected. Since neither the direction nor the magnitude of the forces acting at C is known, these forces will be represented by their x and y components. The components $\mathbf{C}_x$ and $\mathbf{C}_y$ of the force acting on member *AD* will be arbitrarily directed to the right and upward. Since, according to Newton's third law, the forces exerted by member *CF* on *AD* and by member *AD* on *CF* are equal and opposite, the components of the force acting on member *CF must* be directed to the left and downward; they will be denoted, respectively, by $-\mathbf{C}_x$ and $-\mathbf{C}_y$. Whether the force $\mathbf{C}_x$ is actually directed to the right and the force $-\mathbf{C}_x$ is actually directed to the left will be determined later from the sign of their common magnitude C_x, a plus sign indicating that the assumption made was correct and a minus sign that it was wrong. The free-body diagrams of the multiforce members are completed by showing the external forces acting at *A*, *D*, and *F*.†

The internal forces can now be determined by considering the free-body diagram of either of the two multiforce members. Choosing the free-body diagram of *CF*, for example, we write the equations $\Sigma M_C = 0$, $\Sigma M_E = 0$, and $\Sigma F_x = 0$, which yield the values of the magnitudes F_{BE}, C_y, and C_x, respectively. These values can be checked by verifying that member *AD* is also in equilibrium.

It should be noted that the pins in Fig. 6.19 were assumed to form an integral part of one of the two members they connected and so it was not necessary to show their free-body diagram. This assumption can always be used to simplify the analysis of frames and machines. When a pin connects three or more members, however, or when a pin connects a support and two or more members, or when a load is applied to a pin, a clear decision must be made in choosing the member to which the pin will be assumed to belong. (If multiforce members are involved, the pin should be attached to one of these members.) The various forces exerted on the pin should then be clearly identified. This is illustrated in Sample Prob. 6.6.

6.10 FRAMES WHICH CEASE TO BE RIGID WHEN DETACHED FROM THEIR SUPPORTS

The crane analyzed in Sec. 6.9 was so constructed that it could keep the same shape without the help of its supports; it was therefore considered as a rigid body. Many frames, however, will collapse if detached from their supports; such frames cannot be considered as rigid bodies. Consider, for example, the frame shown in Fig. 6.20*a*,

†It is not strictly necessary to use a minus sign to distinguish the force exerted by one member on another from the equal and opposite force exerted by the second member on the first since the two forces belong to different free-body diagrams and thus cannot easily be confused. In the Sample Problems, the same symbol is used to represent equal and opposite forces which are applied to different free bodies. It should be noted that, under these conditions, the sign obtained for a given force component will not directly relate the sense of that component to the sense of the corresponding coordinate axis. Rather, a positive sign will indicate that *the sense assumed for that component in the free-body diagram* is correct, and a negative sign will indicate that it is wrong.

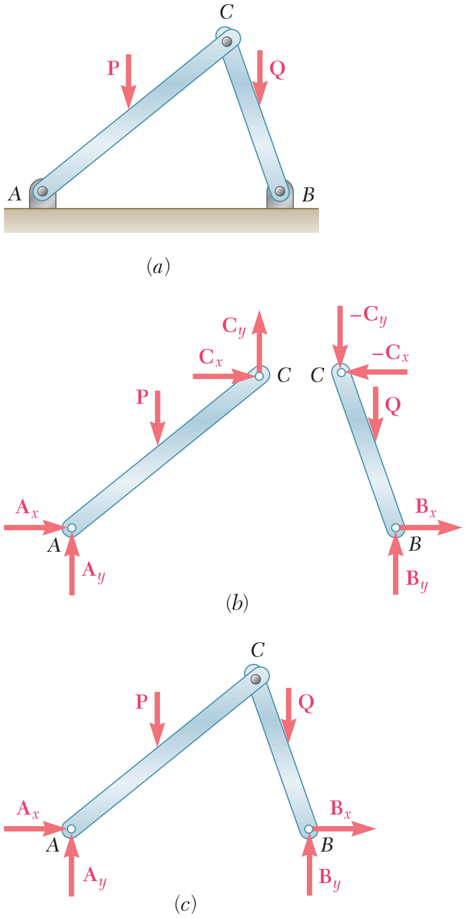


Fig. 6.20

which consists of two members AC and CB carrying loads $\mathbf{P}$ and $\mathbf{Q}$, respectively, at their midpoints; the members are supported by pins at A and B and are connected by a pin at C. If detached from its supports, this frame will not maintain its shape; it should therefore be considered as made of *two distinct rigid parts* AC and CB.

The equations $\Sigma F_x = 0$, $\Sigma F_y = 0$, $\Sigma M = 0$ (about any given point) express the conditions for the *equilibrium of a rigid body* (Chap. 4); we should use them, therefore, in connection with the free-body diagrams of rigid bodies, namely, the free-body diagrams of members AC and CB (Fig. 6.20*b*). Since these members are multiforce members, and since pins are used at the supports and at the connection, the reactions at A and B and the forces at C will each be represented by two components. In accordance with Newton's third law, the components of the force exerted by CB on AC and the components of the force exerted by AC on CB will be represented by vectors of the same magnitude and opposite sense; thus, if the first pair of components consists of $\mathbf{C}_x$ and $\mathbf{C}_y$, the second pair will be represented by $-\mathbf{C}_x$ and $-\mathbf{C}_y$. We note that four unknown force components act on free body AC, while only three independent equations can be used to express that the body is in equilibrium; similarly, four unknowns, but only three equations, are associated with CB. However, only six different unknowns are involved in the analysis of the two members, and altogether six equations are available to express that the members are in equilibrium. Writing $\Sigma M_A = 0$ for free body AC and $\Sigma M_B = 0$ for CB, we obtain two simultaneous equations which may be solved for the common magnitude C_x of the components $\mathbf{C}_x$ and $-\mathbf{C}_x$, and for the common magnitude C_y of the components $\mathbf{C}_y$ and $-\mathbf{C}_y$. We then write $\Sigma F_x = 0$ and $\Sigma F_y = 0$ for each of the two free bodies, obtaining, successively, the magnitudes A_x, A_y, B_x, and B_y.

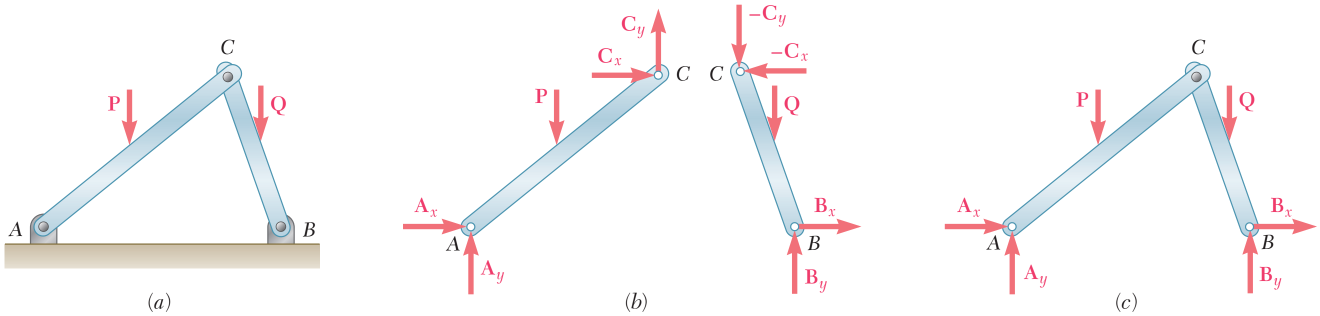


Fig. 6.20 (*repeated*)

It can now be observed that since the equations of equilibrium $\Sigma F_x = 0$, $\Sigma F_y = 0$, and $\Sigma M = 0$ (about any given point) are satisfied by the forces acting on free body AC, and since they are also satisfied by the forces acting on free body CB, they must be satisfied when the forces acting on the two free bodies are considered simultaneously. Since the internal forces at C cancel each other, we find that the equations of equilibrium must be satisfied by the external forces shown on the free-body diagram of the frame ACB itself (Fig. 6.20c) although the frame is not a rigid body. These equations can be used to determine some of the components of the reactions at A and B. We will also find, however, that *the reactions cannot be completely determined from the free-body diagram of the whole frame*. It is thus necessary to dismember the frame and to consider the free-body diagrams of its component parts (Fig. 6.20b), even when we are interested in determining external reactions only. This is because the equilibrium equations obtained for free body ACB *are necessary conditions* for the equilibrium of a nonrigid structure, *but are not sufficient conditions*.

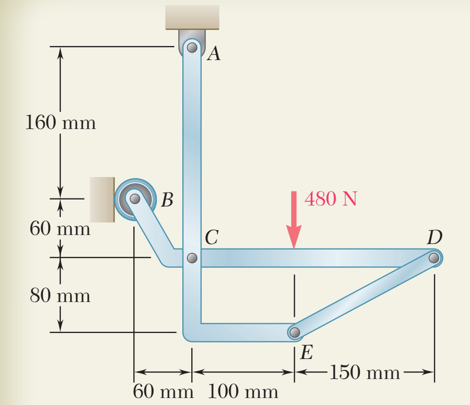
The method of solution outlined in the second paragraph of this section involved simultaneous equations. A more efficient method is now presented, which utilizes the free body ACB as well as the free bodies AC and CB. Writing $\Sigma M_A = 0$ and $\Sigma M_B = 0$ for free body ACB, we obtain B_y and A_y. Writing $\Sigma M_C = 0$, $\Sigma F_x = 0$, and $\Sigma F_y = 0$ for free body AC, we obtain, successively, A_x, C_x, and C_y. Finally, writing $\Sigma F_x = 0$ for ACB, we obtain B_x.

We noted above that the analysis of the frame of Fig. 6.20 involves six unknown force components and six independent equilibrium equations. (The equilibrium equations for the whole frame were obtained from the original six equations and, therefore, are not independent.) Moreover, we checked that all unknowns could be actually determined and that all equations could be satisfied. The frame considered is *statically determinate and rigid*.† In general, to determine whether a structure is statically determinate and rigid, we should draw a free-body diagram for each of its component parts and count the reactions and internal forces involved. We should also determine the number of independent equilibrium equations (excluding equations expressing the equilibrium of the whole structure or of groups of component parts already analyzed). If there are more unknowns than equations, the structure is *statically indeterminate*. If there are fewer unknowns than equations, the structure is *nonrigid*. If there are as many unknowns as equations, *and if all the unknowns can be determined and all the equations satisfied* under general loading conditions, the structure is *statically determinate and rigid*. If, however, due to an *improper arrangement* of members and supports, all the unknowns cannot be determined and all the equations cannot be satisfied, the structure is *statically indeterminate and nonrigid*.

†The word "rigid" is used here to indicate that the frame will maintain its shape as long as it remains attached to its supports.

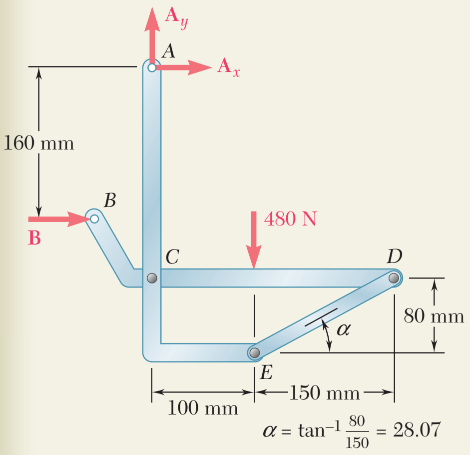
SAMPLE PROBLEM 6.4

In the frame shown, members *ACE* and *BCD* are connected by a pin at *C* and by the link *DE*. For the loading shown, determine the force in link *DE* and the components of the force exerted at *C* on member *BCD*.



SOLUTION

Free Body: Entire Frame. Since the external reactions involve only three unknowns, we compute the reactions by considering the free-body diagram of the entire frame.



$+\uparrow\Sigma F_y = 0$: $\quad A_y - 480\text{ N} = 0 \qquad A_y = +480\text{ N} \qquad \mathbf{A}_y = 480\text{ N}\uparrow$

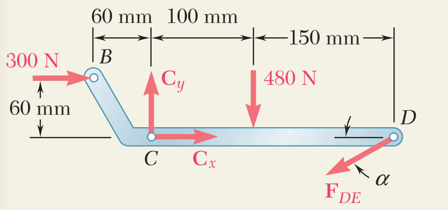
$+\circlearrowleft\Sigma M_A = 0$: $\quad -(480\text{ N})(100\text{ mm}) + B(160\text{ mm}) = 0$

$B = +300\text{ N} \qquad \mathbf{B} = 300\text{ N}\rightarrow$

$\xrightarrow{+}\Sigma F_x = 0$: $\quad B + A_x = 0$

$300\text{ N} + A_x = 0 \qquad A_x = -300\text{ N} \qquad \mathbf{A}_x = 300\text{ N}\leftarrow$

Members. We now dismember the frame. Since only two members are connected at *C*, the components of the unknown forces acting on *ACE* and *BCD* are, respectively, equal and opposite and are assumed directed as shown. We assume that link *DE* is in tension and exerts equal and opposite forces at *D* and *E*, directed as shown.



Free Body: Member BCD. Using the free body *BCD*, we write

$+\circlearrowright\Sigma M_C = 0$:

$$(F_{DE}\sin\alpha)(250\text{ mm}) + (300\text{ N})(80\text{ mm}) + (480\text{ N})(100\text{ mm}) = 0$$

$$F_{DE} = -561\text{ N} \qquad F_{DE} = 561\text{ N } C$$

$\xrightarrow{+}\Sigma F_x = 0$: $\quad C_x - F_{DE}\cos\alpha + 300\text{ N} = 0$

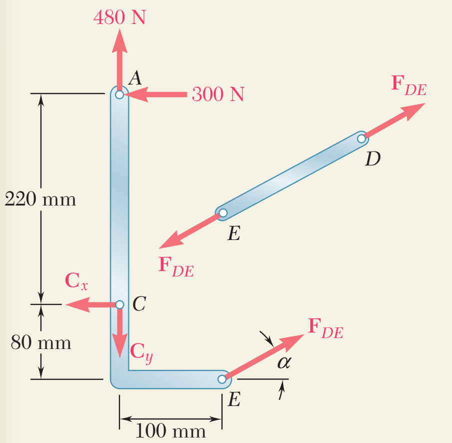
$$C_x - (-561\text{ N})\cos 28.07° + 300\text{ N} = 0 \qquad C_x = -795\text{ N}$$

$+\uparrow\Sigma F_y = 0$: $\quad C_y - F_{DE}\sin\alpha - 480\text{ N} = 0$

$$C_y - (-561\text{ N})\sin 28.07° - 480\text{ N} = 0 \qquad C_y = +216\text{ N}$$

From the signs obtained for C_x and C_y we conclude that the force components $\mathbf{C}_x$ and $\mathbf{C}_y$ exerted on member *BCD* are directed, respectively, to the left and up. We have

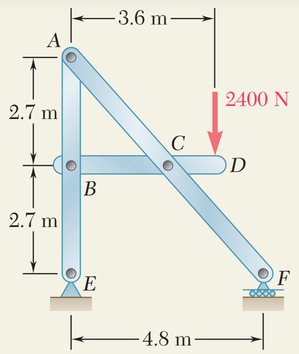
$$\mathbf{C}_x = 795\text{ N}\leftarrow,\ \mathbf{C}_y = 216\text{ N}\uparrow$$



Free Body: Member ACE (Check). The computations are checked by considering the free body *ACE*. For example,

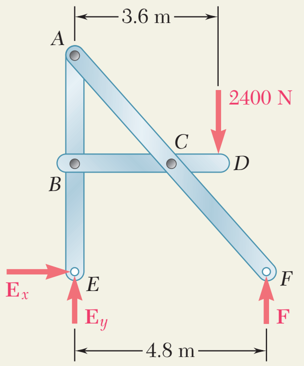
$$+\circlearrowleft\Sigma M_A = (F_{DE}\cos\alpha)(300\text{ mm}) + (F_{DE}\sin\alpha)(100\text{ mm}) - C_x(220\text{ mm})$$
$$= (-561\cos\alpha)(300) + (-561\sin\alpha)(100) - (-795)(220) = 0$$

SAMPLE PROBLEM 6.5



Determine the components of the forces acting on each member of the frame shown.

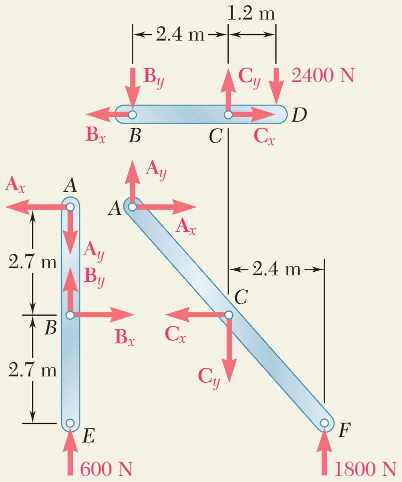
SOLUTION



Free Body: Entire Frame. Since the external reactions involve only three unknowns, we compute the reactions by considering the free-body diagram of the entire frame.

$$+\circlearrowleft \Sigma M_E = 0: \quad -(2400 \text{ N})(3.6 \text{ m}) + F(4.8 \text{ m}) = 0$$
$$F = +1800 \text{ N} \qquad \mathbf{F} = 1800 \text{ N}\uparrow$$
$$+\uparrow \Sigma F_y = 0: \quad -2400 \text{ N} + 1800 \text{ N} + E_y = 0$$
$$E_y = +600 \text{ N} \qquad \mathbf{E}_y = 600 \text{ N}\uparrow$$
$$\xrightarrow{+} \Sigma F_x = 0: \qquad \mathbf{E}_x = 0$$

Members. The frame is now dismembered; since only two members are connected at each joint, equal and opposite components are shown on each member at each joint.



Free Body: Member BCD

$$+\circlearrowleft \Sigma M_B = 0: \quad -(2400 \text{ N})(3.6 \text{ m}) + C_y(2.4 \text{ m}) = 0 \qquad C_y = +3600 \text{ N}$$
$$+\circlearrowleft \Sigma M_C = 0: \quad -(2400 \text{ N})(1.2 \text{ m}) + B_y(2.4 \text{ m}) = 0 \qquad B_y = +1200 \text{ N}$$
$$\xrightarrow{+} \Sigma F_x = 0: \quad -B_x + C_x = 0$$

We note that neither B_x nor C_x can be obtained by considering only member *BCD*. The positive values obtained for B_y and C_y indicate that the force components $\mathbf{B}_y$ and $\mathbf{C}_y$ are directed as assumed.

Free Body: Member ABE

$$+\circlearrowleft \Sigma M_A = 0: \quad B_x(2.7 \text{ m}) = 0 \qquad B_x = 0$$
$$\xrightarrow{+} \Sigma F_x = 0: \quad +B_x - A_x = 0 \qquad A_x = 0$$
$$+\uparrow \Sigma F_y = 0: \quad -A_y + B_y + 600 \text{ N} = 0$$
$$-A_y + 1200 \text{ N} + 600 \text{ N} = 0 \qquad A_y = +1800 \text{ N}$$

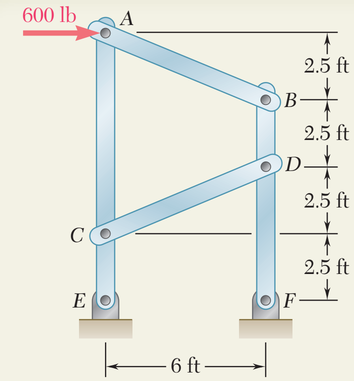
Free Body: Member BCD. Returning now to member *BCD*, we write

$$\xrightarrow{+} \Sigma F_x = 0: \quad -B_x + C_x = 0 \qquad 0 + C_x = 0 \qquad C_x = 0$$

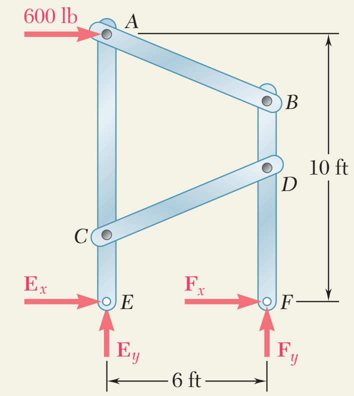
Free Body: Member ACF (Check). All unknown components have now been found; to check the results, we verify that member *ACF* is in equilibrium.

$$+\circlearrowleft \Sigma M_C = (1800 \text{ N})(2.4 \text{ m}) - A_y(2.4 \text{ m}) - A_x(2.7 \text{ m})$$
$$= (1800 \text{ N})(2.4 \text{ m}) - (1800 \text{ N})(2.4 \text{ m}) - 0 = 0 \qquad \text{(checks)}$$

SAMPLE PROBLEM 6.6



A 600-lb horizontal force is applied to pin A of the frame shown. Determine the forces acting on the two vertical members of the frame.

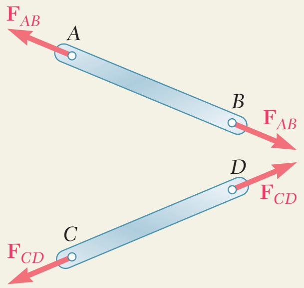


SOLUTION

Free Body: Entire Frame. The entire frame is chosen as a free body; although the reactions involve four unknowns, $\mathbf{E}_y$ and $\mathbf{F}_y$ may be determined by writing

$$+\circlearrowleft \Sigma M_E = 0: \quad -(600 \text{ lb})(10 \text{ ft}) + F_y(6 \text{ ft}) = 0$$
$$F_y = +1000 \text{ lb} \qquad \mathbf{F}_y = 1000 \text{ lb}\uparrow \blacktriangleleft$$
$$+\uparrow \Sigma F_y = 0: \quad E_y + F_y = 0$$
$$E_y = -1000 \text{ lb} \qquad \mathbf{E}_y = 1000 \text{ lb}\downarrow \blacktriangleleft$$

Members. The equations of equilibrium of the entire frame are not sufficient to determine $\mathbf{E}_x$ and $\mathbf{F}_x$. The free-body diagrams of the various members must now be considered in order to proceed with the solution. In dismembering the frame, we will assume that pin A is attached to the multiforce member ACE and, thus, that the 600-lb force is applied to that member. We also note that AB and CD are two-force members.



Free Body: Member ACE

$$+\uparrow \Sigma F_y = 0: \quad -\tfrac{5}{13}F_{AB} + \tfrac{5}{13}F_{CD} - 1000 \text{ lb} = 0$$
$$+\circlearrowleft \Sigma M_E = 0: \quad -(600 \text{ lb})(10 \text{ ft}) - (\tfrac{12}{13}F_{AB})(10 \text{ ft}) - (\tfrac{12}{13}F_{CD})(2.5 \text{ ft}) = 0$$

Solving these equations simultaneously, we find

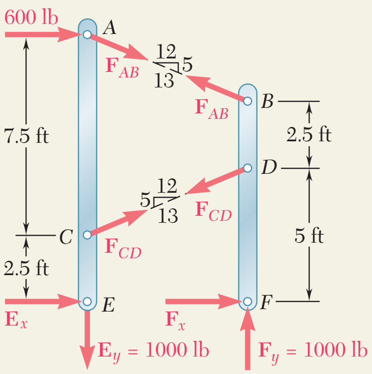
$$F_{AB} = -1040 \text{ lb} \qquad F_{CD} = +1560 \text{ lb} \quad \blacktriangleleft$$

The signs obtained indicate that the sense assumed for F_{CD} was correct and the sense for F_{AB} incorrect. Summing now x components,

$$\xrightarrow{+} \Sigma F_x = 0: \quad 600 \text{ lb} + \tfrac{12}{13}(-1040 \text{ lb}) + \tfrac{12}{13}(+1560 \text{ lb}) + E_x = 0$$
$$E_x = -1080 \text{ lb} \qquad \mathbf{E}_x = 1080 \text{ lb}\leftarrow \blacktriangleleft$$

Free Body: Entire Frame. Since $\mathbf{E}_x$ has been determined, we can return to the free-body diagram of the entire frame and write

$$\xrightarrow{+} \Sigma F_x = 0: \quad 600 \text{ lb} - 1080 \text{ lb} + F_x = 0$$
$$F_x = +480 \text{ lb} \qquad \mathbf{F}_x = 480 \text{ lb}\rightarrow \blacktriangleleft$$



Free Body: Member BDF (Check). We can check our computations by verifying that the equation $\Sigma M_B = 0$ is satisfied by the forces acting on member BDF.

$$\begin{aligned} +\circlearrowleft \Sigma M_B &= -\left(\tfrac{12}{13}F_{CD}\right)(2.5 \text{ ft}) + (F_x)(7.5 \text{ ft}) \\ &= -\tfrac{12}{13}(1560 \text{ lb})(2.5 \text{ ft}) + (480 \text{ lb})(7.5 \text{ ft}) \\ &= -3600 \text{ lb} \cdot \text{ft} + 3600 \text{ lb} \cdot \text{ft} = 0 \qquad \text{(checks)} \end{aligned}$$

PROBLEMS

6.49 through 6.51 Determine the force in member *BD* and the components of the reaction at *C*.

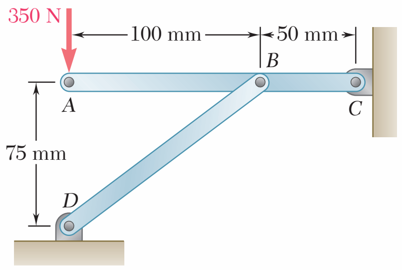


Fig. P6.49

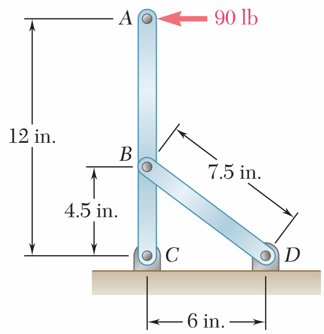


Fig. P6.50

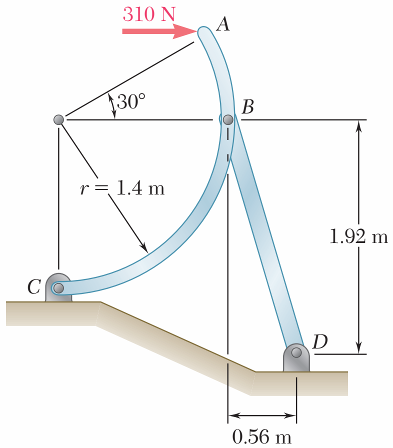


Fig. P6.51

6.52 Determine the components of all the forces acting on member *ABCD* of the assembly shown.

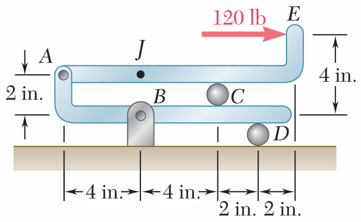


Fig. P6.52

6.53 Determine the components of all the forces acting on member *ABCD* when $\theta = 0$.

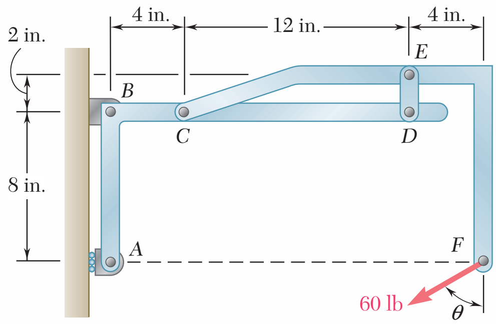


Fig. P6.53 and *P6.54*

6.54 Determine the components of all the forces acting on member *ABCD* when $\theta = 90°$.

6.55 An aircraft tow bar is positioned by means of a single hydraulic cylinder *CD* that is connected to two identical arm-and-wheel units *DEF*. The entire tow bar has a mass of 200 kg, and its center of gravity is located at *G*. For the position shown, determine (*a*) the force exerted by the cylinder on bracket *C*, (*b*) the force exerted on each arm by the pin at *E*.

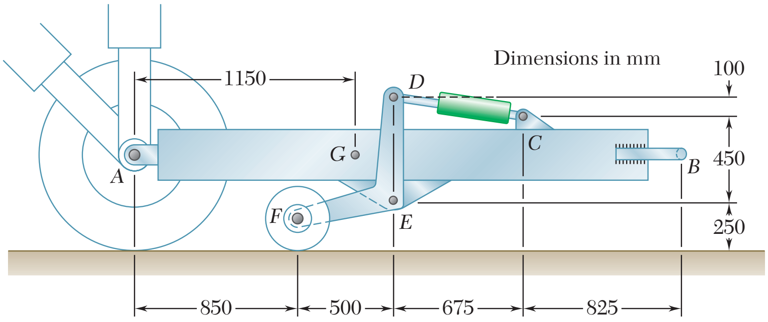


Fig. P6.55

6.56 Solve Prob. 6.55, assuming that a 70-kg mechanic is standing on the tow bar at point *B*.

6.57 Knowing that $P = 90$ lb and $Q = 60$ lb, determine the components of all the forces acting on member *BCDE* of the assembly shown.

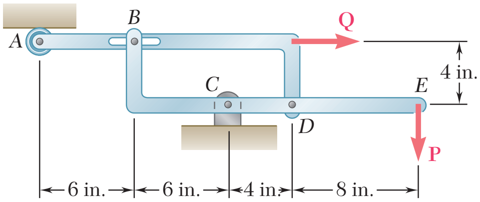


Fig. P6.57

6.58 The marine crane shown is used in offshore drilling operations. Determine (*a*) the force in link *CD*, (*b*) the force in the brace *AC*, (*c*) the force exerted at *A* on the boom *AB*.

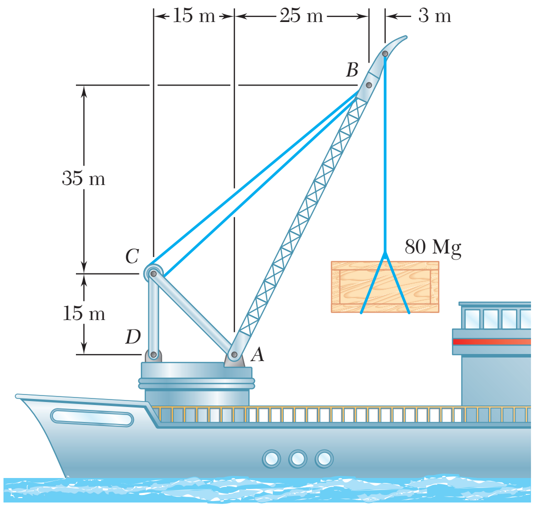


Fig. P6.58

6.59 Determine the components of the reactions at D and E if the frame is loaded by a clockwise couple of magnitude 150 N · m applied (a) at point A, (b) at point B.

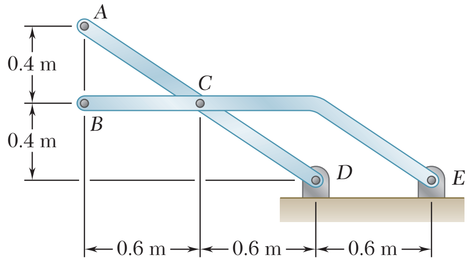


Fig. P6.59

6.60 Determine the components of the force exerted at B on member BE (a) if the 200-lb load is applied as shown, (b) if the 200-lb load is moved along its line of action and is applied at point F.

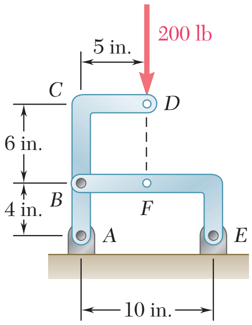


Fig. P6.60

6.61 Determine all of the forces exerted on member AI if the frame is loaded by a clockwise couple of magnitude 180 lb · ft applied (a) at point D, (b) at point E.

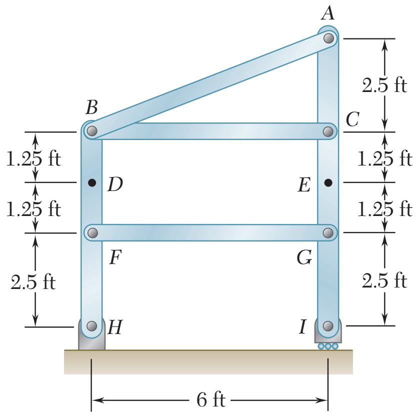


Fig. P6.61 and P6.62

6.62 Determine all of the forces exerted on member AI if the frame is loaded by a 48-lb force directed horizontally to the right and applied (a) at point D, (b) at point E.

6.63 The hydraulic cylinder CF, which partially controls the position of rod DE, has been locked in the position shown. Knowing that $\theta = 60°$, determine (*a*) the force **P** for which the tension in link AB is 410 N, (*b*) the corresponding force exerted on member BCD at point C.

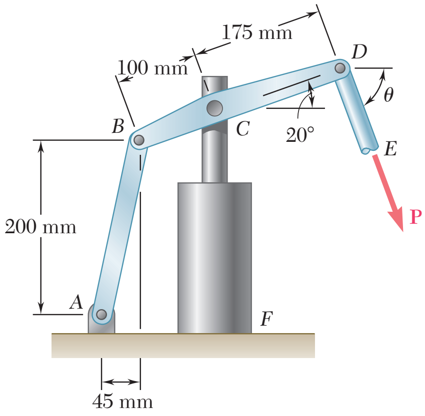


Fig. ***P6.63*** **and P6.64**

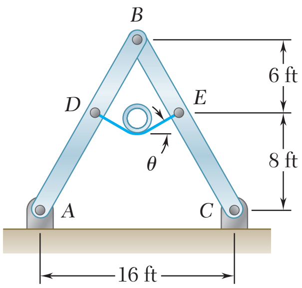


Fig. P6.65

6.64 The hydraulic cylinder CF, which partially controls the position of rod DE, has been locked in the position shown. Knowing that $P = 400$ N and $\theta = 75°$, determine (*a*) the force in link AB, (*b*) the corresponding force exerted on member BCD at point C.

6.65 A pipe weights 40 lb/ft and is supported every 30 ft by the small frame shown. Knowing that $\theta = 30°$, determine the components of the reactions and the components of the force exerted at B on member AB.

6.66 A 2-ft diameter pipe is supported every 16 ft by the small frame shown. Knowing that the combined weight of the pipe and its contents is 300 lb/ft and neglecting the effect of friction, determine the components (*a*) of the reaction at E, (*b*) of the force exerted at C on member CDE.

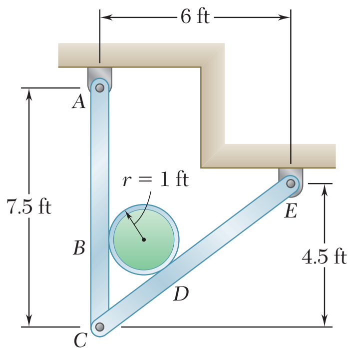


Fig. P6.66

6.67 Knowing that each pulley has a radius of 250 mm, determine the components of the reactions at D and E.

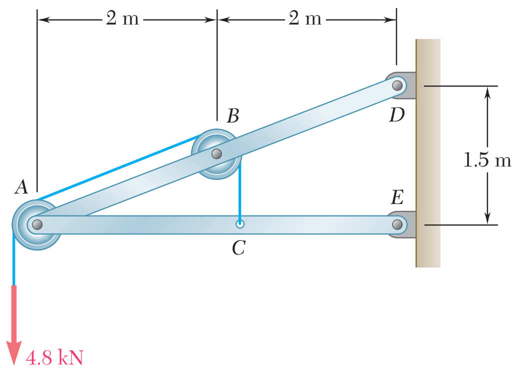


Fig. P6.67

6.68 Knowing that the pulley has a radius of 75 mm, determine the components of the reactions at A and B.

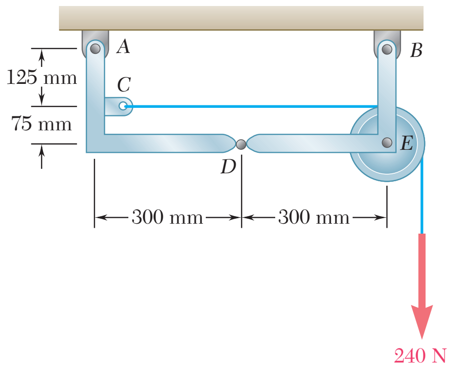


Fig. P6.68

6.69 The cab and motor units of the front-end loader shown are connected by a vertical pin located 60 in. behind the cab wheels. The distance from C to D is 30 in. The center of gravity of the 50-kip motor unit is located at G_m, while the centers of gravity of the 18-kip cab and 16-kip load are located, respectively, at G_c and G_l. Knowing that the machine is at rest with its brakes released, determine (*a*) the reactions at each of the four wheels, (*b*) the forces exerted on the motor unit at C and D.

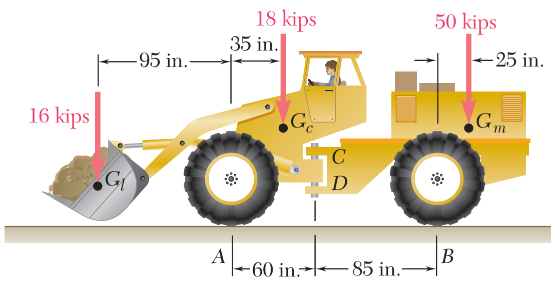


Fig. P6.69

6.70 Solve Prob. 6.69, assuming that the 16-kip load has been removed.

6.71 The tractor and scraper units shown are connected by a vertical pin located 0.6 m behind the tractor wheels. The distance from C to D is 0.75 m. The center of gravity of the 10-Mg tractor unit is located at G_t. The scraper unit and the load have a total mass of 50 Mg and a combined center of gravity located at G_s. Knowing that the machine is at rest, with its brakes released, determine (*a*) the reactions at each of the four wheels, (*b*) the forces exerted on the tractor unit at C and D.

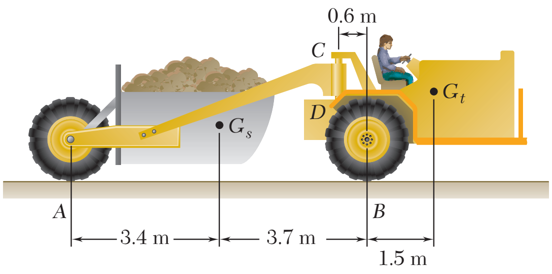


Fig. P6.71

6.72 The 1000-kg trailer is attached to a 1250-kg automobile by a ball-and-socket trailer hitch at D. Determine (*a*) the reactions at each of the six wheels when the automobile and trailer are at rest, (*b*) the additional load on each of the automobile wheels due to the trailer.

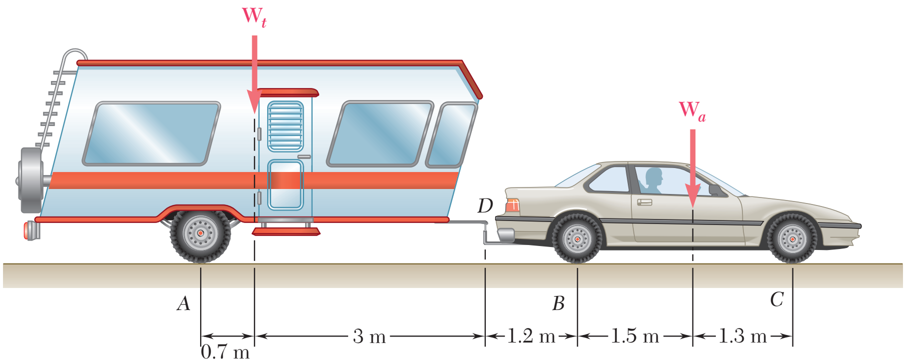


Fig. P6.72

6.11 MACHINES

Machines are structures designed to transmit and modify forces. Whether they are simple tools or include complicated mechanisms, their main purpose is to transform *input forces* into *output forces*. Consider, for example, a pair of cutting pliers used to cut a wire (Fig. 6.21*a*). If we apply two equal and opposite forces **P** and −**P** on their handles, they will exert two equal and opposite forces **Q** and −**Q** on the wire (Fig. 6.21*b*).

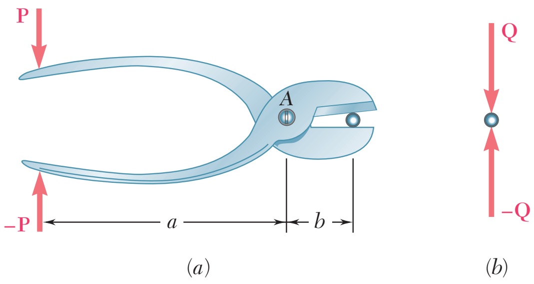


Fig. 6.21

To determine the magnitude Q of the output forces when the magnitude P of the input forces is known (or, conversely, to determine P when Q is known), we draw a free-body diagram of the pliers *alone,* showing the input forces $\mathbf{P}$ and $-\mathbf{P}$ and the *reactions* $-\mathbf{Q}$ and $\mathbf{Q}$ that the wire exerts on the pliers (Fig. 6.22). However, since a pair of pliers forms a nonrigid structure, we must use one of the component parts as a free body in order to determine the unknown forces. Considering Fig. 6.23*a*, for example, and taking moments about A, we obtain the relation $Pa = Qb$, which defines the magnitude Q in terms of P or P in terms of Q. The same free-body diagram can be used to determine the components of the internal force at A; we find $A_x = 0$ and $A_y = P + Q$.

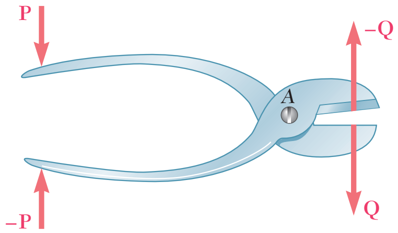


Fig. 6.22

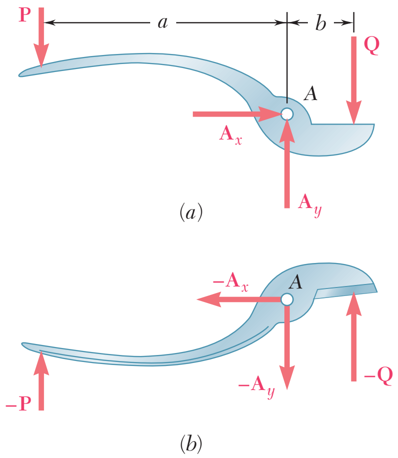
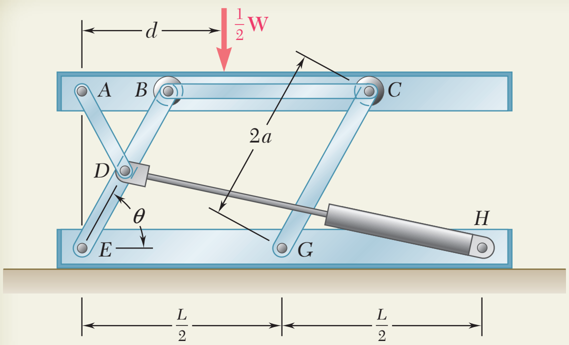


Fig. 6.23

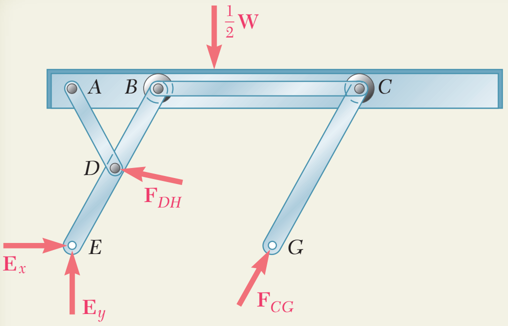
In the case of more complicated machines, it generally will be necessary to use several free-body diagrams and, possibly, to solve simultaneous equations involving various internal forces. The free bodies should be chosen to include the input forces and the reactions to the output forces, and the total number of unknown force components involved should not exceed the number of available independent equations. It is advisable, before attempting to solve a problem, to determine whether the structure considered is determinate. There is no point, however, in discussing the rigidity of a machine, since a machine includes moving parts and thus *must* be nonrigid.

Photo 6.4 The lamp shown can be placed in many positions. By considering various free bodies, the force in the springs and the internal forces at the joints can be determined.



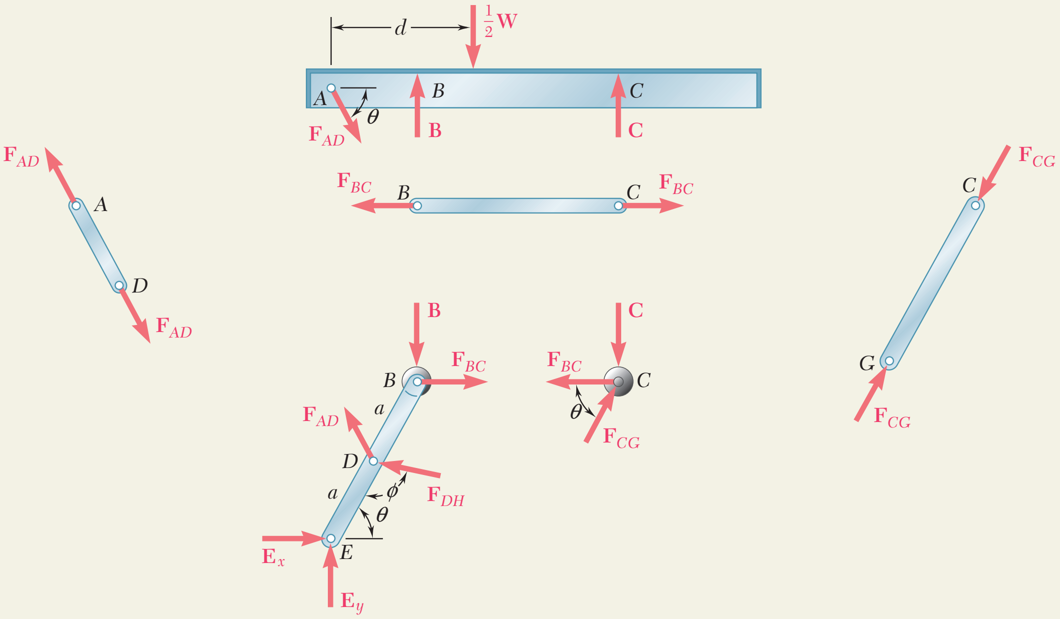
SAMPLE PROBLEM 6.7

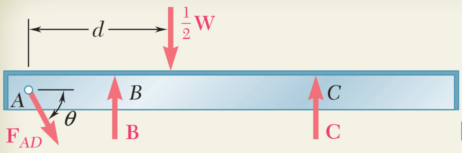
A hydraulic-lift table is used to raise a 1000-kg crate. It consists of a platform and two identical linkages on which hydraulic cylinders exert equal forces. (Only one linkage and one cylinder are shown.) Members *EDB* and *CG* are each of length $2a$, and member *AD* is pinned to the midpoint of *EDB*. If the crate is placed on the table, so that half of its weight is supported by the system shown, determine the force exerted by each cylinder in raising the crate for $\theta = 60°$, $a = 0.70$ m, and $L = 3.20$ m. Show that the result obtained is independent of the distance d.



SOLUTION

The machine considered consists of the platform and of the linkage. Its free-body diagram includes an input force $\mathbf{F}_{DH}$ exerted by the cylinder, the weight $\frac{1}{2}\mathbf{W}$, equal and opposite to the output force, and reactions at *E* and *G* that we assume to be directed as shown. Since more than three unknowns are involved, this diagram will not be used. The mechanism is dismembered and a free-body diagram is drawn for each of its component parts. We note that *AD*, *BC*, and *CG* are two-force members. We already assumed member *CG* to be in compression; we now assume that *AD* and *BC* are in tension and direct as shown the forces exerted on them. Equal and opposite vectors will be used to represent the forces exerted by the two-force members on the platform, on member *BDE*, and on roller *C*.

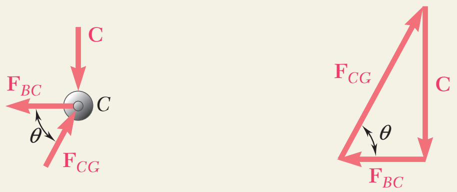


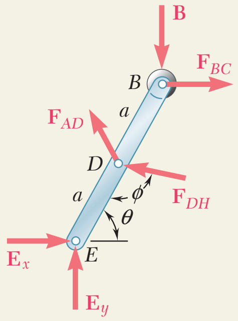


Free Body: Platform *ABC*.

$$\overset{+}{\rightarrow}\Sigma F_x = 0: \qquad F_{AD}\cos\theta = 0 \qquad F_{AD} = 0$$
$$+\uparrow\Sigma F_y = 0: \qquad B + C - \tfrac{1}{2}W = 0 \qquad B + C = \tfrac{1}{2}W \tag{1}$$

Free Body: Roller *C*. We draw a force triangle and obtain $F_{BC} = C\cot\theta$.





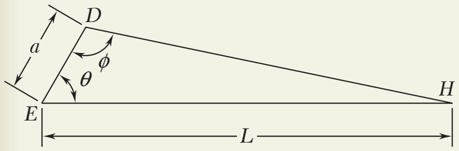
Free Body: Member *BDE*. Recalling that $F_{AD} = 0$,

$$+\circlearrowleft\Sigma M_E = 0: \qquad F_{DH}\cos(\phi - 90°)a - B(2a\cos\theta) - F_{BC}(2a\sin\theta) = 0$$
$$F_{DH}a\sin\phi - B(2a\cos\theta) - (C\cot\theta)(2a\sin\theta) = 0$$
$$F_{DH}\sin\phi - 2(B + C)\cos\theta = 0$$

Recalling Eq. (1), we have

$$F_{DH} = W\frac{\cos\theta}{\sin\phi} \tag{2}$$

and we observe that the result obtained is independent of d. ◀



Applying first the law of sines to triangle EDH, we write

$$\frac{\sin\phi}{EH} = \frac{\sin\theta}{DH} \qquad \sin\phi = \frac{EH}{DH}\sin\theta \tag{3}$$

Using now the law of cosines, we have

$$(DH)^2 = a^2 + L^2 - 2aL\cos\theta$$
$$= (0.70)^2 + (3.20)^2 - 2(0.70)(3.20)\cos 60°$$
$$(DH)^2 = 8.49 \qquad DH = 2.91\text{ m}$$

We also note that

$$W = mg = (1000\text{ kg})(9.81\text{ m/s}^2) = 9810\text{ N} = 9.81\text{ kN}$$

Substituting for $\sin\phi$ from (3) into (2) and using the numerical data, we write

$$F_{DH} = W\frac{DH}{EH}\cot\theta = (9.81\text{ kN})\frac{2.91\text{ m}}{3.20\text{ m}}\cot 60°$$

$$F_{DH} = 5.15\text{ kN}$$ ◀

PROBLEMS

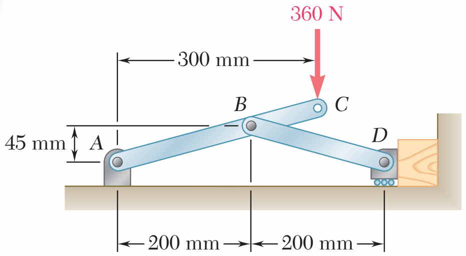


Fig. P6.73

6.73 A 360-N force is applied to the toggle vise at C. Determine (*a*) the horizontal force exerted on the block at D, (*b*) the force exerted on member ABC at B.

6.74 The control rod CE passes through a horizontal hole in the body of the toggle clamp shown. Determine (*a*) the force **Q** required to hold the clamp in equilibrium, (*b*) the corresponding force in link BD.

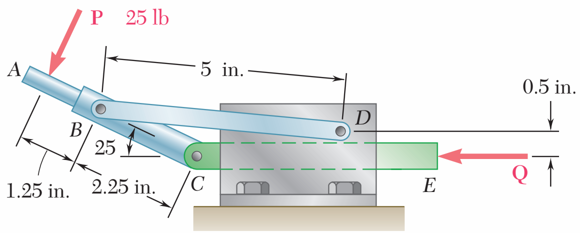


Fig. P6.74

6.75 The shear shown is used to cut and trim electronic-circuit-board laminates. For the position shown, determine (*a*) the vertical component of the force exerted on the shearing blade at D, (*b*) the reaction at C.

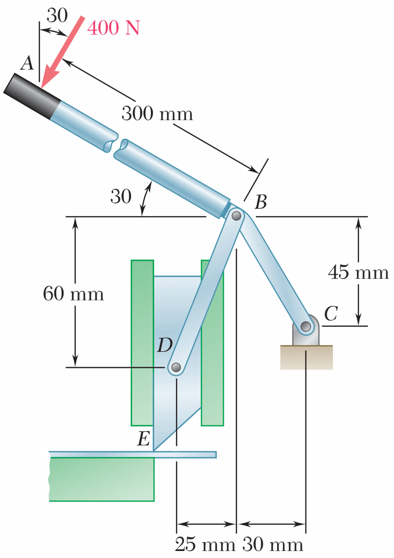


Fig. P6.75

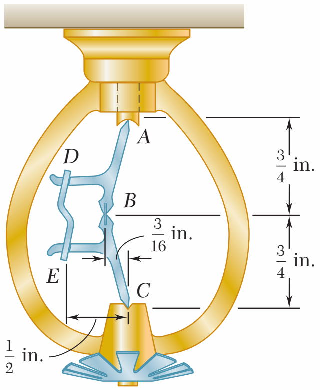


Fig. P6.76

6.76 Water pressure in the supply system exerts a downward force of 30 lb on the vertical plug at A. Determine the tension in the fusible link DE and the force exerted on member BCE at B.

6.77 A 9-m length of railroad rail of mass 40 kg/m is lifted by the tongs shown. Determine the forces exerted at D and F on tong BDF.

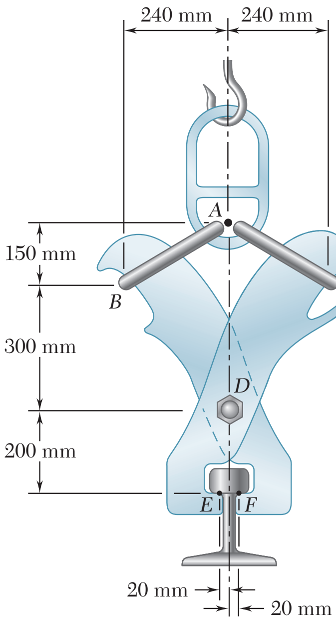


Fig. P6.77

6.78 A steel ingot weighing 8000 lb is lifted by a pair of tongs as shown. Determine the forces exerted at C and E on the tong BCE.

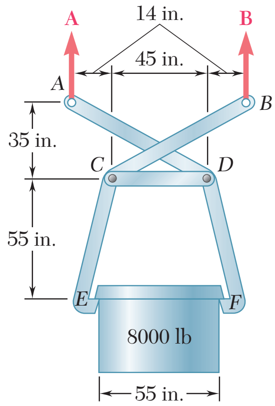


Fig. P6.78

6.79 If the toggle shown is added to the tongs of Prob. 6.78 and the load is lifted by applying a single force at G, determine the forces exerted at C and E on the tong BCE.

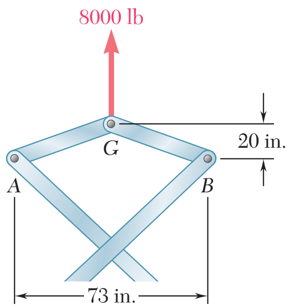


Fig. P6.79

6.80 The gear-pulling assembly shown consists of a crosshead CF, two grip arms ABC and FGH, two links BD and EG, and a threaded center rod JK. Knowing that the center rod JK must exert a 4800-N force on the vertical shaft KL in order to start the removal of the gear, determine all the forces acting on grip arm ABC. Assume that the rounded ends of the crosshead are smooth and exert horizontal forces on the grip arms.

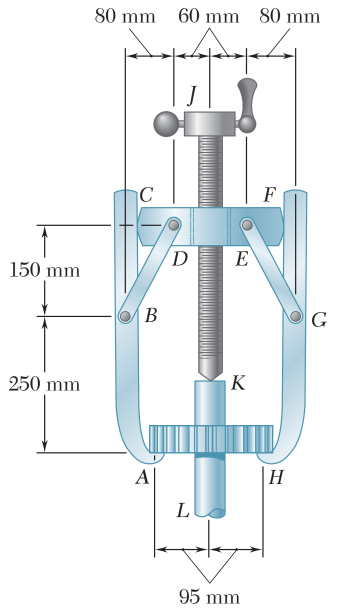


Fig. P6.80

6.81 A force **P** of magnitude 2.4 kN is applied to the piston of the engine system shown. For each of the two positions shown, determine the couple **M** required to hold the system in equilibrium.

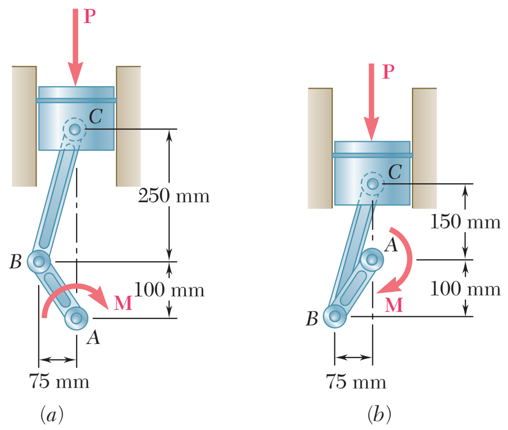


Fig. P6.81 and P6.82

6.82 A couple **M** of magnitude 315 N · m is applied to the crank of the engine system shown. For each of the two positions shown, determine the force **P** required to hold the system in equilibrium.

6.83 and *6.84* Two rods are connected by a frictionless collar B. Knowing that the magnitude of the couple $\mathbf{M}_A$ is 500 lb · in., determine (*a*) the couple $\mathbf{M}_C$ required for equilibrium, (*b*) the corresponding components of the reaction at C.

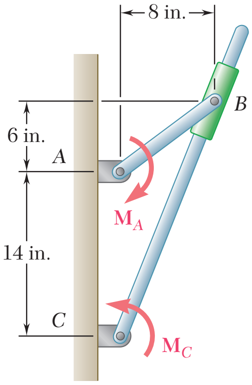


Fig. P6.83

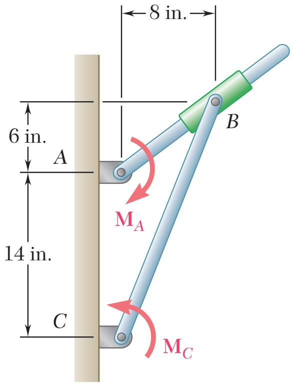


Fig. *P6.84*

6.85 Two 300-N forces are applied to the handles of the pliers as shown. Determine (*a*) the magnitude of the forces exerted on the rod, (*b*) the force exerted by the pin at A on portion AB of the pliers.

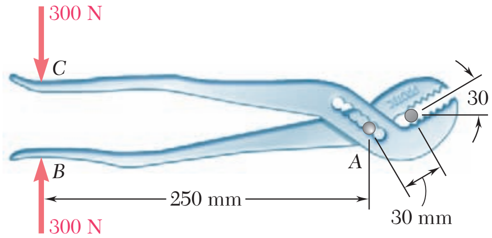


Fig. P6.85

6.86 In using the bolt cutter shown, a worker applies two 100-lb forces to the handles. Determine the magnitude of the forces exerted by the cutter on the bolt.

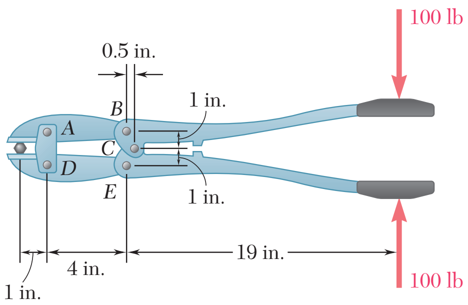


Fig. P6.86

6.87 The upper blade and lower handle of the compound-lever shears are pin connected to the main element *ABE* at *A* and *B*, respectively, and to the short link *CD* at *C* and *D*, respectively. Determine the forces exerted on a twig when two 120-N forces are applied to the handles.

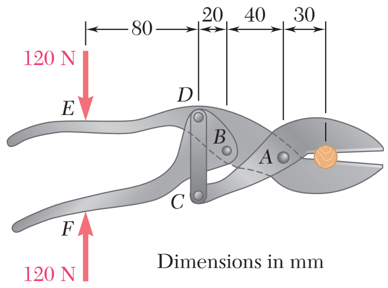


Fig. *P6.87*

6.88 A hand-operated hydraulic cylinder has been designed for use where space is severely limited. Determine the magnitude of the force exerted on the piston at *D* when two 90-lb forces are applied as shown.

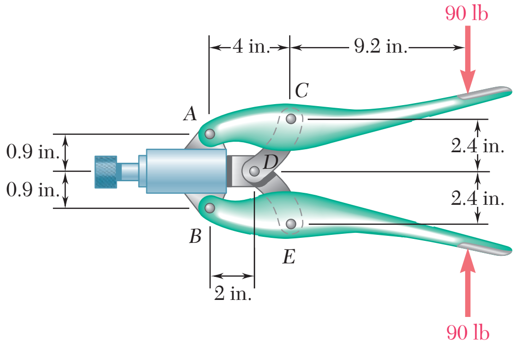


Fig. P6.88

6.89 A shelf is held horizontally by a self-locking brace that consists of two parts *EDC* and *CDB* hinged at *C* and bearing against each other at *D*. If the shelf is 10 in. wide and weighs 24 lb, determine the force **P** required to release the brace. (*Hint:* To release the brace, the forces of contact at *D* must be zero.)

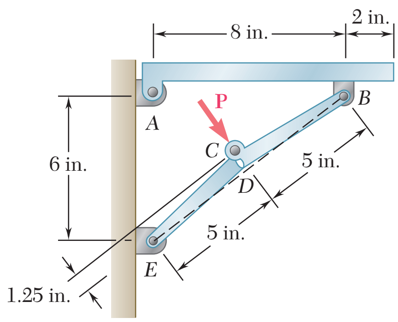


Fig. P6.89

6.90 Since the brace shown must remain in position even when the magnitude of **P** is very small, a single safety spring is attached at D and E. The spring DE has a constant of 50 lb/in. and an unstretched length of 7 in. Knowing that l = 10 in. and that the magnitude of **P** is 800 lb, determine the force **Q** required to release the brace.

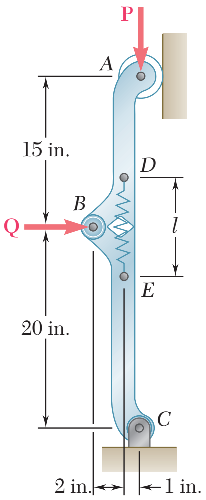


Fig. P6.90

6.91 and 6.92 Determine the force **P** that must be applied to the toggle CDE to maintain bracket ABC in the position shown.

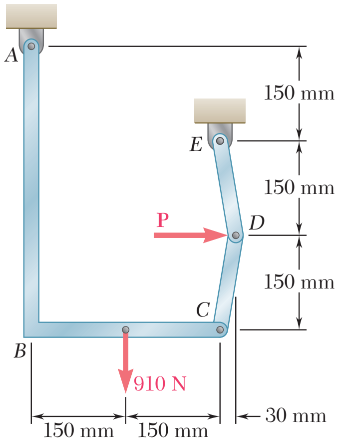


Fig. P6.91

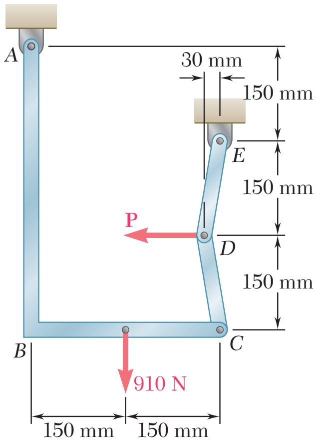


Fig. P6.92

6.93 In the boring rig shown, the center of gravity of the 3000-kg tower is located at point G. For the position shown, determine the force exerted by the hydraulic cylinder AB.

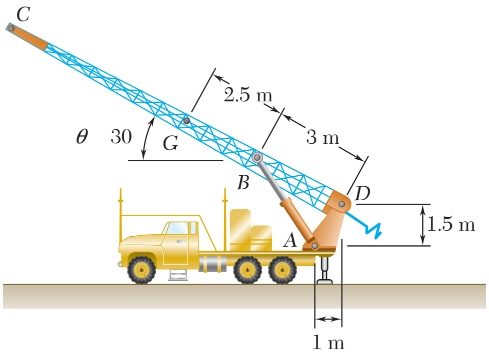


Fig. P6.93

6.94 The action of the backhoe bucket is controlled by the three hydraulic cylinders shown. Determine the force exerted by each cylinder in supporting the 3000-lb load shown.

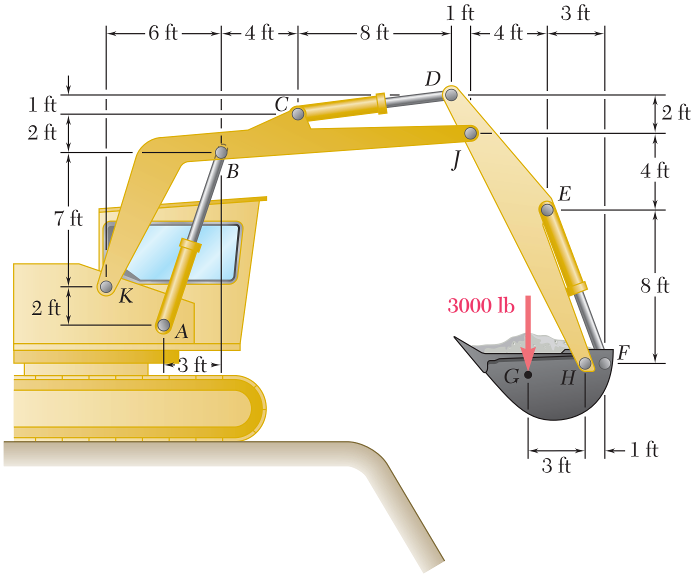


Fig. P6.94

6.95 The motion of the backhoe bucket is controlled by the hydraulic cylinders *AB*, *DE*, and *FI*. Determine the force exerted by each cylinder in supporting the 7.5-kN load shown.

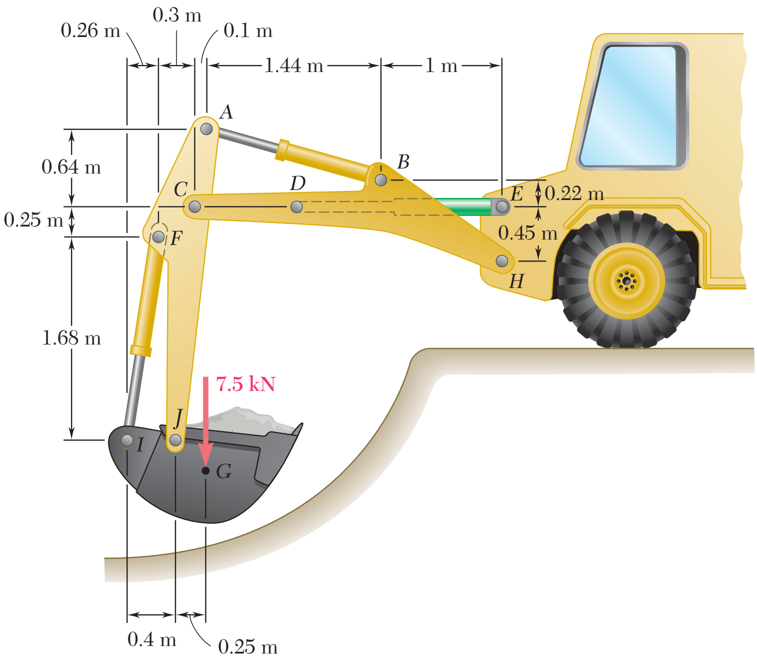


Fig. P6.95

6.96 The elevation of the platform is controlled by two identical mechanisms, only one of which is shown. A load of 1200 lb is applied to the mechanism shown. Knowing that the pin at *C* can transmit only a horizontal force, determine (*a*) the force in link *BE*, (*b*) the components of the force exerted by the hydraulic cylinder on *H*.

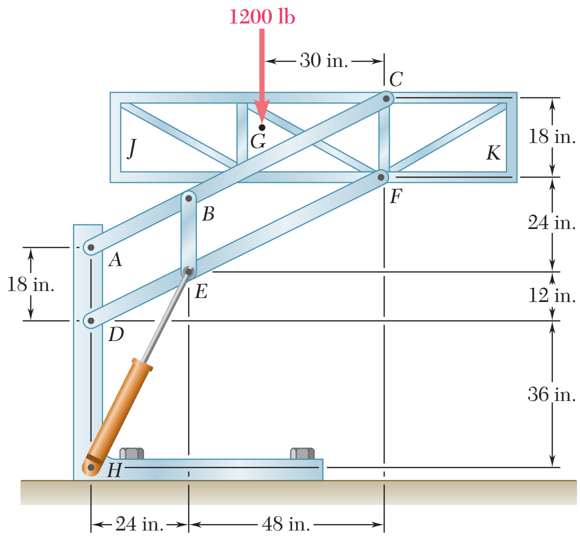


Fig. P6.96

REVIEW AND SUMMARY

In this chapter you learned to determine the *internal forces* holding together the various parts of a structure.

Analysis of trusses

The first half of the chapter was devoted to the analysis of *trusses*, i.e., to the analysis of structures consisting of *straight members connected at their extremities only*. The members being slender and unable to support lateral loads, all the loads must be applied at the joints; a truss may thus be assumed to consist of *pins and two-force members* [Sec. 6.2].

Simple trusses

A truss is said to be *rigid* if it is designed in such a way that it will not greatly deform or collapse under a small load. A triangular truss consisting of three members connected at three joints is clearly a rigid truss (Fig. 6.24*a*) and so will be the truss obtained by adding two new members to the first one and connecting them at a new joint (Fig. 6.24*b*). Trusses obtained by repeating this procedure are called *simple trusses*. We may check that in a simple truss the total number of members is $m = 2n - 3$, where n is the total number of joints [Sec. 6.3].

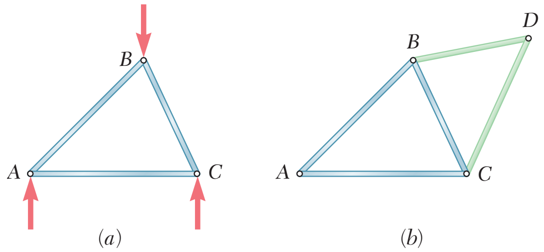


Fig. 6.24

Method of joints

The forces in the various members of a simple truss can be determined by the *method of joints* [Sec. 6.4]. First, the reactions at the supports can be obtained by considering the entire truss as a free body. The free-body diagram of each pin is then drawn, showing the forces exerted on the pin by the members or supports it connects. Since the members are straight two-force members, the force exerted by a member on the pin is directed along that member, and only the magnitude of the force is unknown. It is always possible in the case of a simple truss to draw the free-body diagrams of the pins in such an order that only two unknown forces are included in each diagram. These forces can be obtained from the corresponding two equilibrium equations or—if only three forces are involved—from the corresponding force triangle. If the force exerted by a member on a pin is directed toward that pin, the member is in *compression;*

if it is directed away from the pin, the member is in *tension* [Sample Prob. 6.1]. The analysis of a truss is sometimes expedited by first recognizing *joints under special loading conditions* [Sec. 6.5].

Method of sections

The *method of sections* is usually preferred to the method of joints when the force in only one member—or very few members—of a truss is desired [Sec. 6.6]. To determine the force in member BD of the truss of Fig. 6.25*a*, for example, we *pass a section* through members BD, BE, and CE, remove these members, and use the portion ABC of the truss as a free body (Fig. 6.25*b*). Writing $\Sigma M_E = 0$, we determine the magnitude of the force $\mathbf{F}_{BD}$, which represents the force in member BD. A positive sign indicates that the member is in *tension;* a negative sign indicates that it is in *compression* [Sample Probs. 6.2 and 6.3].

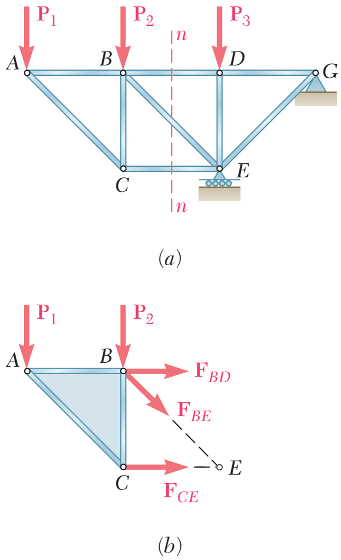


Fig. 6.25

Compound trusses

The method of sections is particularly useful in the analysis of *compound trusses,* i.e., trusses which cannot be constructed from the basic triangular truss of Fig. 6.24*a* but which can be obtained by rigidly connecting several simple trusses [Sec. 6.7]. If the component trusses have been properly connected (e.g., one pin and one link, or three nonconcurrent and nonparallel links) and if the resulting structure is properly supported (e.g., one pin and one roller), the compound truss is *statically determinate, rigid, and completely constrained*. The following necessary—but not sufficient—condition is then satisfied: $m + r = 2n$, where m is the number of members, r is the number of unknowns representing the reactions at the supports, and n is the number of joints.

Frames and machines

The second part of the chapter was devoted to the analysis of *frames and machines*. Frames and machines are structures which contain *multiforce members,* i.e., members acted upon by three or more forces. Frames are designed to support loads and are usually stationary, fully constrained structures. Machines are designed to transmit or modify forces and always contain moving parts [Sec. 6.8].

Analysis of a frame

To *analyze a frame,* we first consider the *entire frame as a free body* and write three equilibrium equations [Sec. 6.9]. If the frame remains rigid when detached from its supports, the reactions involve only three unknowns and may be determined from these equations [Sample Probs. 6.4 and 6.5]. On the other hand, if the frame ceases to be rigid when detached from its supports, the reactions involve more than three unknowns and cannot be completely determined from the equilibrium equations of the frame [Sec. 6.10; Sample Prob. 6.6].

Multiforce members

We then *dismember the frame* and identify the various members as either two-force members or multiforce members; pins are assumed to form an integral part of one of the members they connect. We draw the free-body diagram of each of the multiforce members, noting that when two multiforce members are connected to the same two-force member, they are acted upon by that member with *equal and opposite forces of unknown magnitude but known direction*. When two multiforce members are connected by a pin, they exert on each other *equal and opposite forces of unknown direction,* which should be represented by *two unknown components*. The equilibrium equations obtained from the free-body diagrams of the multiforce members can then be solved for the various internal forces [Sample Probs. 6.4 and 6.5]. The equilibrium equations can also be used to complete the determination of the reactions at the supports [Sample Prob. 6.6]. Actually, if the frame is *statically determinate and rigid,* the free-body diagrams of the multiforce members could provide as many equations as there are unknown forces (including the reactions) [Sec. 6.10]. However, as suggested above, it is advisable to first consider the free-body diagram of the entire frame to minimize the number of equations that must be solved simultaneously.

Analysis of a machine

To *analyze a machine,* we dismember it and, following the same procedure as for a frame, draw the free-body diagram of each of the multiforce members. The corresponding equilibrium equations yield the *output forces* exerted by the machine in terms of the *input forces* applied to it as well as the *internal forces* at the various connections [Sec. 6.11; Sample Prob. 6.7].

REVIEW PROBLEMS

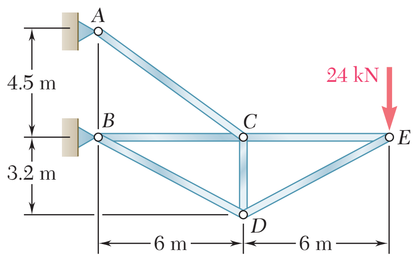


Fig. *P6.97*

6.97 Using the method of joints, determine the force in each member of the truss shown.

6.98 Determine the force in each member of the truss shown.

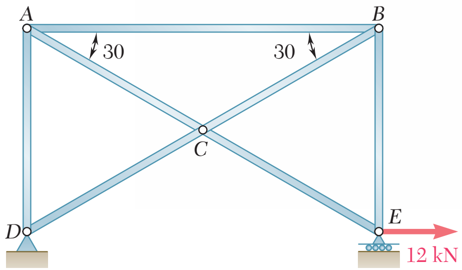


Fig. *P6.98*

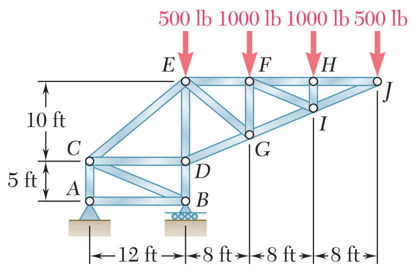


Fig. P6.99 and P6.100

6.99 Determine the force in members EF, FG, and GI of the truss shown.

6.100 Determine the force in members CE, CD, and CB of the truss shown.

6.101 The low-bed trailer shown is designed so that the rear end of the bed can be lowered to ground level in order to facilitate the loading of equipment or wrecked vehicles. A 1400-kg vehicle has been hauled to the position shown by a winch; the trailer is then returned to a traveling position where $\alpha = 0$ and both AB and BE are horizontal. Considering only the weight of the disabled automobile, determine the force that must be exerted by the hydraulic cylinder to maintain a position with $\alpha = 0$.

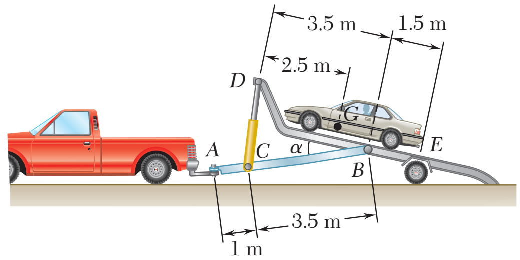


Fig. P6.101

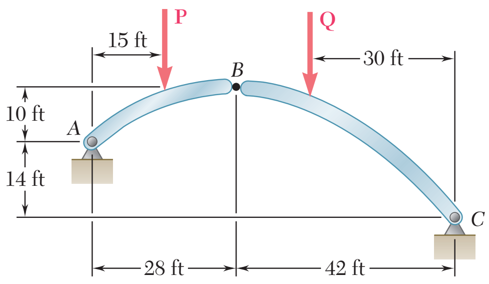


Fig. *P6.102*

6.102 The axis of the three-hinged arch ABC is a parabola with vertex at B. Knowing that $P = 109.2$ kips and $Q = 72.8$ kips, determine (*a*) the components of the reaction at C, (*b*) the components of the force exerted at B on segment AB.

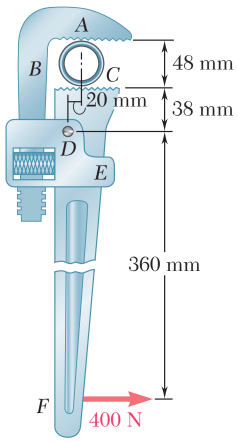


Fig. P6.103

6.103 A 48-mm-diameter pipe is gripped by the Stillson wrench shown. Portions AB and DE of the wrench are rigidly attached to each other, and portion CF is connected by a pin at D. Assuming that no slipping occurs between the pipe and the wrench, determine the components of the forces exerted on the pipe at A and C.

6.104 The compound-lever pruning shears shown can be adjusted by placing pin A at various ratchet positions on blade ACE. Knowing that 292-lb vertical forces are required to complete the pruning of a twig, determine the magnitude P of the forces that must be applied to the handles when the shears are adjusted as shown.

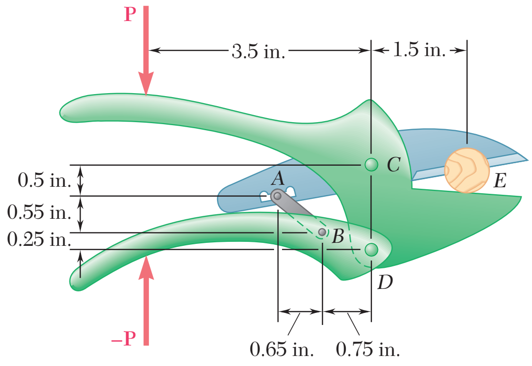


Fig. P6.104

6.105 Determine the couple **M** that must be applied to the crank CD to hold the mechanism in equilibrium. The block at D is pinned to the crank CD and is free to slide in a slot cut in member AB.

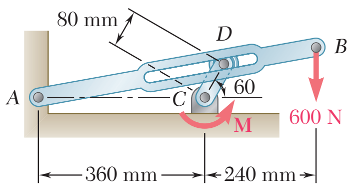


Fig. P6.105

6.106 An automobile front-wheel assembly supports 750 lb. Determine the force exerted by the spring and the components of the forces exerted on the frame at points A and D.

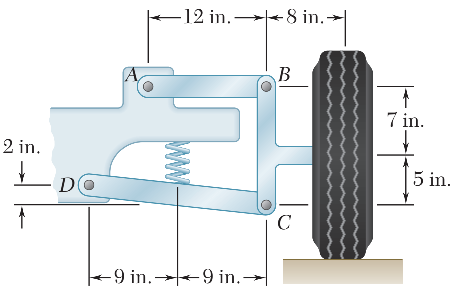


Fig. P6.106

6.107 For the bevel-gear system shown, determine the required value of α if the ratio of M_B to M_A is to be three.

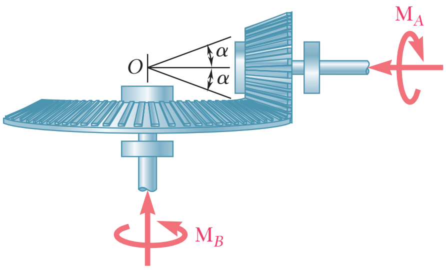


Fig. P6.107

6.108 A 400-kg block may be supported by a small frame in each of the four ways shown. The diameter of the pulley is 250 mm. For each case, determine (*a*) the force components and the couple representing the reaction at A, (*b*) the force exerted at D on the vertical member.

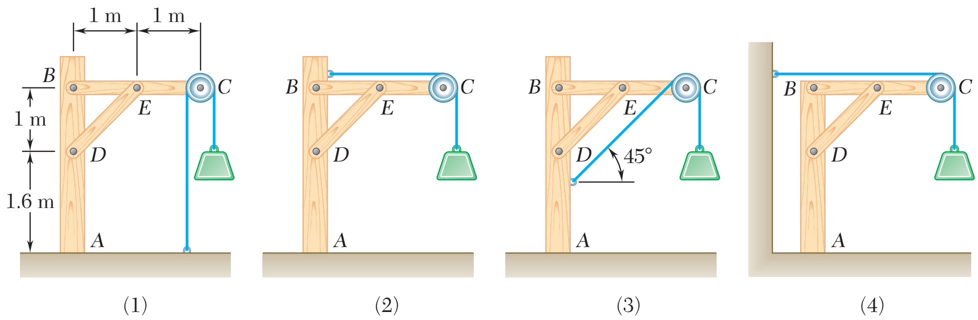
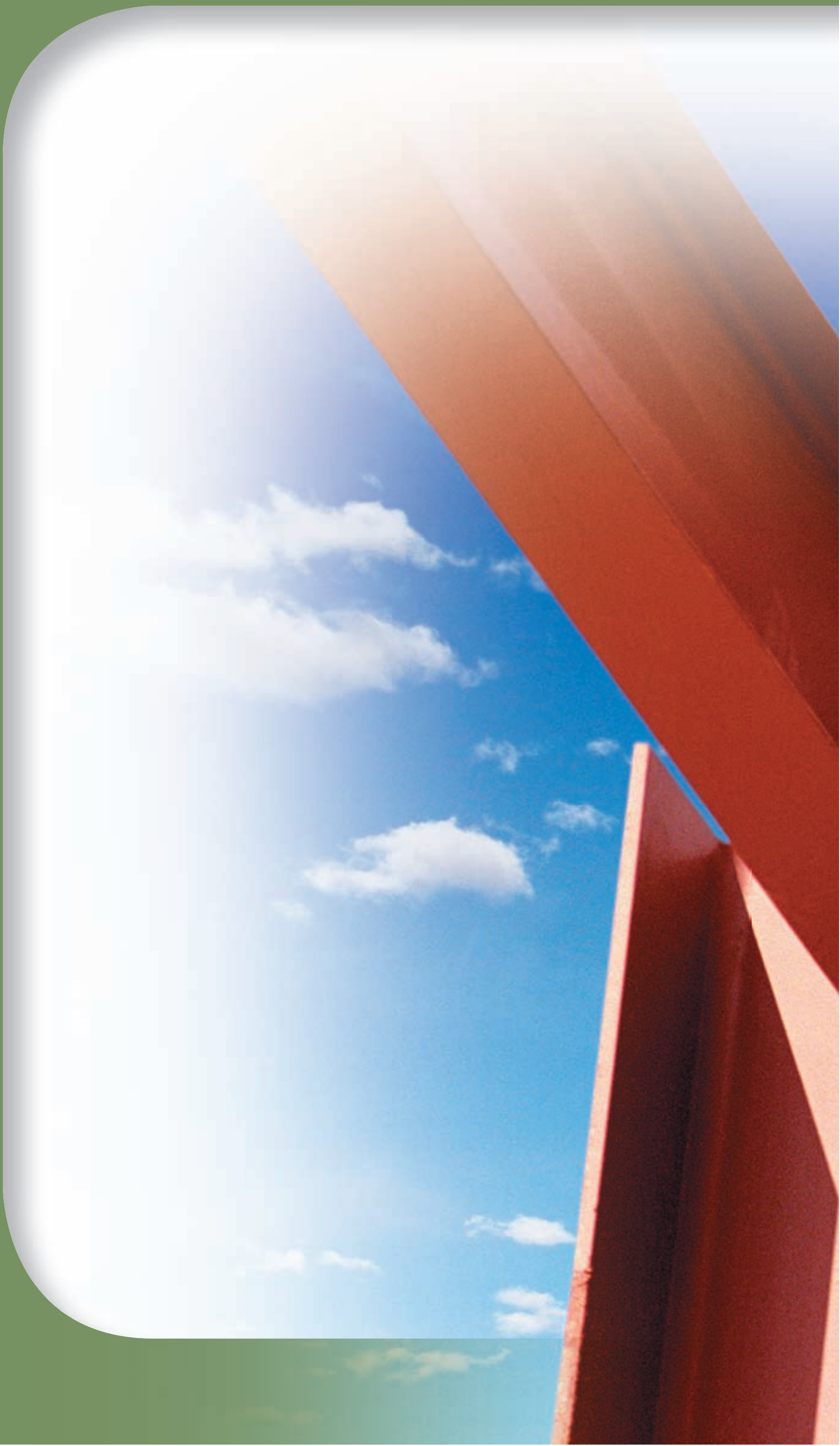


Fig. P6.108

The strength of structural members used in the construction of buildings depends to a large extent on the properties of their cross sections. This includes the second moments of area, or moments of inertia, of these cross sections.



CHAPTER 7

Distributed Forces: Moments of Inertia of Areas

7.1 INTRODUCTION

In Chap. 5, we analyzed various systems of forces distributed over an area or volume. The three main types of forces considered were (1) weights of homogeneous plates of uniform thickness (Secs. 5.3 through 5.6), (2) distributed loads on beams (Sec. 5.8), and (3) weights of homogeneous three-dimensional bodies (Secs. 5.9 and 5.10). In the case of homogeneous plates, the magnitude ΔW of the weight of an element of a plate was proportional to the area ΔA of the element. For distributed loads on beams, the magnitude ΔW of each elemental weight was represented by an element of area $\Delta A = \Delta W$ under the load curve. In the case of homogeneous three-dimensional bodies, the magnitude ΔW of the weight of an element of the body was proportional to the volume ΔV of the element. Thus, in all cases considered in Chap. 5, the distributed forces were proportional to the elemental areas or volumes associated with them. The resultant of these forces, therefore, could be obtained by summing the corresponding areas or volumes, and the moment of the resultant about any given axis could be determined by computing the first moments of the areas or volumes about that axis.

In this chapter, we consider distributed forces $\Delta\mathbf{F}$ whose magnitudes depend not only upon the elements of area ΔA on which these forces act but also upon the distance from ΔA to some given axis. More precisely, the magnitude of the force per unit area $\Delta F/\Delta A$ is assumed to vary linearly with the distance to the axis. As indicated in the next section, forces of this type are found in the study of the bending of beams. Assuming that the elemental forces involved are distributed over an area A and vary linearly with the distance y to the x axis, it will be shown that while the magnitude of their resultant $\mathbf{R}$ depends upon the first moment $Q_x = \int y\,dA$ of the area A, the location of the point where $\mathbf{R}$ is applied depends upon the *second moment,* or *moment of inertia,* $I_x = \int y^2\,dA$ of the same area with respect to the x axis. You will learn to compute the moments of inertia of various areas with respect to given x and y axes. Also introduced in this chapter is the *polar moment of inertia* $J_O = \int r^2\,dA$ of an area, where r is the distance from the element of area dA to the point O. To facilitate your computations, a relation will be established between the moment of inertia I_x of an area A with respect to a given x axis and the moment of inertia $I_{x'}$ of the same area with respect to the parallel centroidal x' axis (parallel-axis theorem).

7.2 SECOND MOMENT, OR MOMENT OF INERTIA, OF AN AREA

In this chapter, we consider distributed forces $\Delta\mathbf{F}$ whose magnitudes ΔF are proportional to the elements of area ΔA on which the forces act and at the same time vary linearly with the distance from ΔA to a given axis.

Consider, for example, a beam of uniform cross section which is subjected to two equal and opposite couples applied at each end of the beam. Such a beam is said to be in *pure bending,* and it is shown in mechanics of materials that the internal forces in any section of the beam are distributed forces whose magnitudes $\Delta F = ky\,\Delta A$ vary linearly with the distance y between the element of area

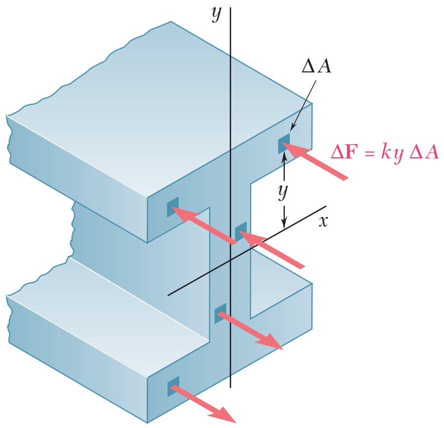


Fig. 7.1

ΔA and an axis passing through the centroid of the section. This axis, represented by the x axis in Fig. 7.1, is known as the *neutral axis* of the section. The forces on one side of the neutral axis are forces of compression, while those on the other side are forces of tension; on the neutral axis itself the forces are zero.

The magnitude of the resultant **R** of the elemental forces **ΔF** which act over the entire section is

$$R = \int ky\,dA = k\int y\,dA$$

The last integral obtained is recognized as the *first moment* Q_x of the section about the x axis; it is equal to $\bar{y}A$ and is thus equal to zero, since the centroid of the section is located on the x axis. The system of the forces **ΔF** thus reduces to a couple. The magnitude M of this couple (bending moment) must be equal to the sum of the moments $\Delta M_x = y\,\Delta F = ky^2\,\Delta A$ of the elemental forces. Integrating over the entire section, we obtain

$$M = \int ky^2\,dA = k\int y^2\,dA$$

The last integral is known as the *second moment,* or *moment of inertia,*† of the beam section with respect to the x axis and is denoted by I_x. It is obtained by multiplying each element of area dA by the *square of its distance* from the x axis and integrating over the beam section. Since each product $y^2\,dA$ is positive, regardless of the sign of y, or zero (if y is zero), the integral I_x will always be positive.

7.3 DETERMINATION OF THE MOMENT OF INERTIA OF AN AREA BY INTEGRATION

We defined in the preceding section the second moment, or moment of inertia, of an area A with respect to the x axis. Defining in a similar

†The term *second moment* is more proper than the term *moment of inertia* since, logically, the latter should be used only to denote integrals of mass. In engineering practice, however, moment of inertia is used in connection with areas as well as masses.

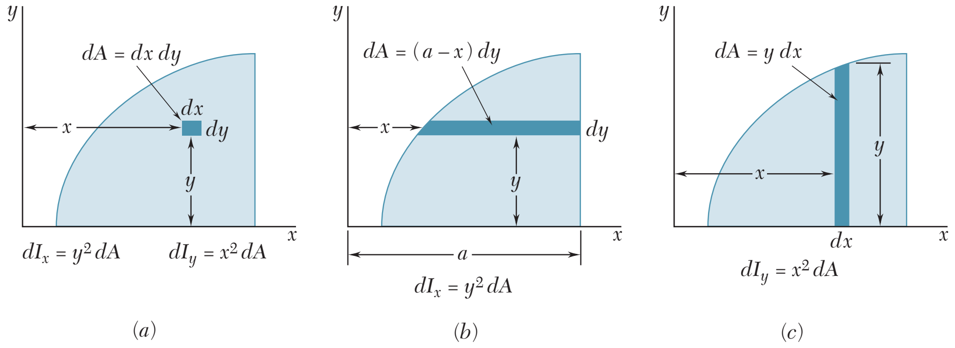


Fig. 7.2

way the moment of inertia I_y of the area A with respect to the y axis, we write (Fig. 7.2*a*)

$$I_x = \int y^2\,dA \qquad I_y = \int x^2\,dA \tag{7.1}$$

These integrals, known as the *rectangular moments of inertia* of the area A, can be more easily evaluated if we choose dA to be a thin strip parallel to one of the coordinate axes. To compute I_x, the strip is chosen parallel to the x axis, so that all of the points of the strip are at the same distance y from the x axis (Fig. 7.2*b*); the moment of inertia dI_x of the strip is then obtained by multiplying the area dA of the strip by y^2. To compute I_y, the strip is chosen parallel to the y axis so that all of the points of the strip are at the same distance x from the y axis (Fig. 7.2*c*); the moment of inertia dI_y of the strip is $x^2\,dA$.

Moment of Inertia of a Rectangular Area. As an example, let us determine the moment of inertia of a rectangle with respect to its base (Fig. 7.3). Dividing the rectangle into strips parallel to the x axis, we obtain

$$dA = b\,dy \qquad dI_x = y^2 b\,dy$$

$$I_x = \int_0^h by^2\,dy = \tfrac{1}{3}bh^3 \tag{7.2}$$

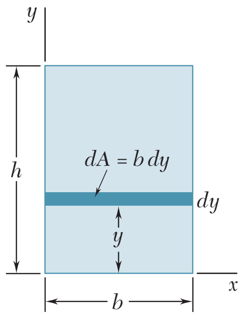


Fig. 7.3

Computing I_x and I_y Using the Same Elemental Strips. The formula just derived can be used to determine the moment of inertia dI_x with respect to the x axis of a rectangular strip which is parallel to the y axis, such as the strip shown in Fig. 7.2*c*. Setting $b = dx$ and $h = y$ in formula (7.2), we write

$$dI_x = \tfrac{1}{3}y^3\,dx$$

On the other hand, we have

$$dI_y = x^2\,dA = x^2y\,dx$$

The same element can thus be used to compute the moments of inertia I_x and I_y of a given area (Fig. 7.4).

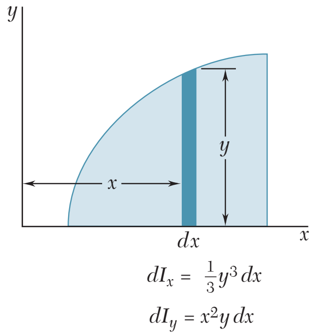


Fig. 7.4

7.4 POLAR MOMENT OF INERTIA

An integral of great importance in problems concerning the torsion of cylindrical shafts and in problems dealing with the rotation of slabs is

$$J_O = \int \rho^2\,dA \tag{7.3}$$

where ρ is the distance from O to the element of area dA (Fig. 7.5). This integral is the *polar moment of inertia* of the area A with respect to *the "pole"* O.

The polar moment of inertia of a given area can be computed from the rectangular moments of inertia I_x and I_y of the area if these quantities are already known. Indeed, noting that $\rho^2 = x^2 + y^2$, we write

$$J_O = \int \rho^2\,dA = \int (x^2 + y^2)\,dA = \int y^2\,dA + \int x^2\,dA$$

that is,

$$J_O = I_x + I_y \tag{7.4}$$

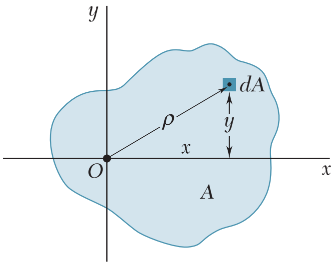


Fig. 7.5

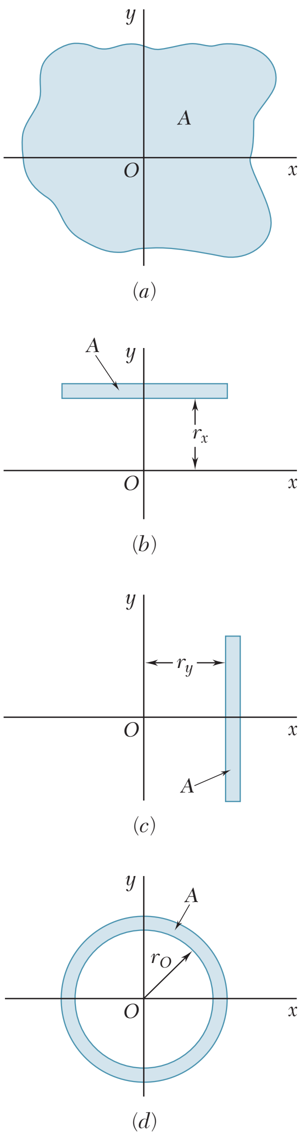


Fig. 7.6

7.5 RADIUS OF GYRATION OF AN AREA

Consider an area A which has a moment of inertia I_x with respect to the x axis (Fig. 7.6*a*). Let us imagine that we concentrate this area into a thin strip parallel to the x axis (Fig. 7.6*b*). If the area A, thus concentrated, is to have the same moment of inertia with respect to the x axis, the strip should be placed at a distance r_x from the x axis, where r_x is defined by the relation

$$I_x = r_x^2 A$$

Solving for r_x, we write

$$r_x = \sqrt{\frac{I_x}{A}} \tag{7.5}$$

The distance r_x is referred to as the *radius of gyration* of the area with respect to the x axis. In a similar way, we can define the radii of gyration r_y and r_O (Fig. 7.6*c* and *d*); we write

$$I_y = r_y^2 A \qquad r_y = \sqrt{\frac{I_y}{A}} \tag{7.6}$$

$$J_O = r_O^2 A \qquad r_O = \sqrt{\frac{J_O}{A}} \tag{7.7}$$

If we rewrite Eq. (7.4) in terms of the radii of gyration, we find that

$$r_O^2 = r_x^2 + r_y^2 \tag{7.8}$$

EXAMPLE 7.1 For the rectangle shown in Fig. 7.7, let us compute the radius of gyration r_x with respect to its base. Using formulas (7.5) and (7.2), we write

$$r_x^2 = \frac{I_x}{A} = \frac{\frac{1}{3}bh^3}{bh} = \frac{h^2}{3} \qquad\qquad r_x = \frac{h}{\sqrt{3}}$$

The radius of gyration r_x of the rectangle is shown in Fig. 7.7. It should not be confused with the ordinate $\bar{y} = h/2$ of the centroid of the area. While r_x depends upon the *second moment*, or moment of inertia, of the area, the ordinate $\bar{y}$ is related to the *first moment* of the area. ■

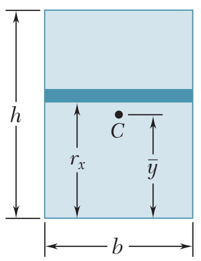
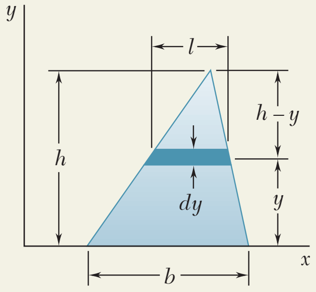


Fig. 7.7

SAMPLE PROBLEM 7.1

Determine the moment of inertia of a triangle with respect to its base.



SOLUTION

A triangle of base b and height h is drawn; the x axis is chosen to coincide with the base. A differential strip parallel to the x axis is chosen to be dA. Since all portions of the strip are at the same distance from the x axis, we write

$$dI_x = y^2\,dA \qquad dA = l\,dy$$

Using similar triangles, we have

$$\frac{l}{b} = \frac{h-y}{h} \qquad l = b\frac{h-y}{h} \qquad dA = b\frac{h-y}{h}dy$$

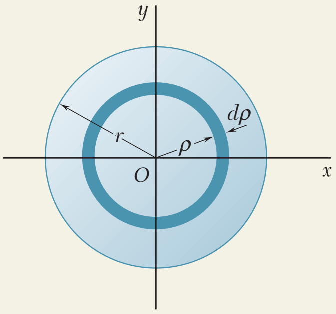
Integrating dI_x from $y = 0$ to $y = h$, we obtain

$$I_x = \int y^2\,dA = \int_0^h y^2 b\frac{h-y}{h}dy = \frac{b}{h}\int_0^h (hy^2 - y^3)\,dy$$

$$= \frac{b}{h}\left[h\frac{y^3}{3} - \frac{y^4}{4}\right]_0^h \qquad I_x = \frac{bh^3}{12} \blacktriangleleft$$

SAMPLE PROBLEM 7.2

(a) Determine the centroidal polar moment of inertia of a circular area by direct integration. (b) Using the result of part a, determine the moment of inertia of a circular area with respect to a diameter.



SOLUTION

a. Polar Moment of Inertia. An annular differential element of area is chosen to be dA. Since all portions of the differential area are at the same distance from the origin, we write

$$dJ_O = \rho^2\,dA \qquad dA = 2\pi\rho\,d\rho$$

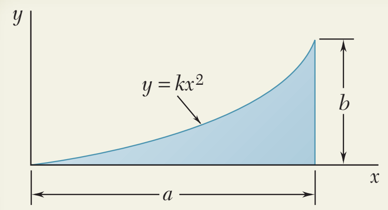
$$J_O = \int dJ_O = \int_0^r \rho^2(2\pi\rho\,d\rho) = 2\pi\int_0^r \rho^3\,d\rho$$

$$J_O = \frac{\pi}{2}r^4 \blacktriangleleft$$

b. Moment of Inertia with Respect to a Diameter. Because of the symmetry of the circular area, we have $I_x = I_y$. We then write

$$J_O = I_x + I_y = 2I_x \qquad \frac{\pi}{2}r^4 = 2I_x \qquad I_{\text{diameter}} = I_x = \frac{\pi}{4}r^4 \blacktriangleleft$$

SAMPLE PROBLEM 7.3



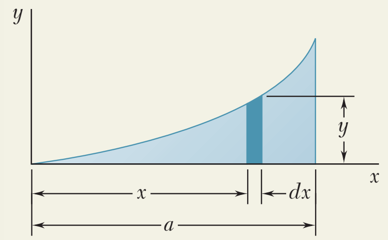
(a) Determine the moment of inertia of the shaded area shown with respect to each of the coordinate axes. (Properties of this area were considered in Sample Prob. 5.4.) (b) Using the results of part a, determine the radius of gyration of the shaded area with respect to each of the coordinate axes.

SOLUTION

Referring to Sample Prob. 5.4, we obtain the following expressions for the equation of the curve and the total area:

$$y = \frac{b}{a^2}x^2 \qquad A = \tfrac{1}{3}ab$$

Moment of Inertia I_x. A vertical differential element of area is chosen to be dA. Since all portions of this element are *not* at the same distance from the x axis, we must treat the element as a thin rectangle. The moment of inertia of the element with respect to the x axis is then



$$dI_x = \tfrac{1}{3}y^3\,dx = \frac{1}{3}\left(\frac{b}{a^2}x^2\right)^3 dx = \frac{1}{3}\frac{b^3}{a^6}x^6\,dx$$

$$I_x = \int dI_x = \int_0^a \frac{1}{3}\frac{b^3}{a^6}x^6\,dx = \left[\frac{1}{3}\frac{b^3}{a^6}\frac{x^7}{7}\right]_0^a$$

$$I_x = \frac{ab^3}{21} \quad ◀$$

Moment of Inertia I_y. The same vertical differential element of area is used. Since all portions of the element are at the same distance from the y axis, we write

$$dI_y = x^2\,dA = x^2(y\,dx) = x^2\left(\frac{b}{a^2}x^2\right)dx = \frac{b}{a^2}x^4\,dx$$

$$I_y = \int dI_y = \int_0^a \frac{b}{a^2}x^4\,dx = \left[\frac{b}{a^2}\frac{x^5}{5}\right]_0^a$$

$$I_y = \frac{a^3b}{5} \quad ◀$$

Radii of Gyration r_x and r_y. We have, by definition,

$$r_x^2 = \frac{I_x}{A} = \frac{ab^3/21}{ab/3} = \frac{b^2}{7} \qquad r_x = \sqrt{\tfrac{1}{7}}\,b \quad ◀$$

and

$$r_y^2 = \frac{I_y}{A} = \frac{a^3b/5}{ab/3} = \tfrac{3}{5}a^2 \qquad r_y = \sqrt{\tfrac{3}{5}}\,a \quad ◀$$

PROBLEMS

7.1 through 7.4 Determine by direct integration the moment of inertia of the shaded area with respect to the y axis.

7.5 through *7.8* Determine by direct integration the moment of inertia of the shaded area with respect to the x axis.

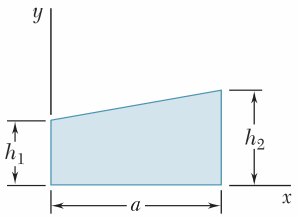


Fig. P7.1 and P7.5

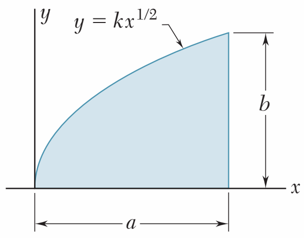


Fig. P7.2 and P7.6

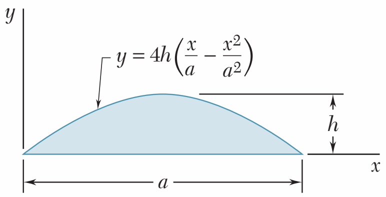


Fig. P7.3 and *P7.7*

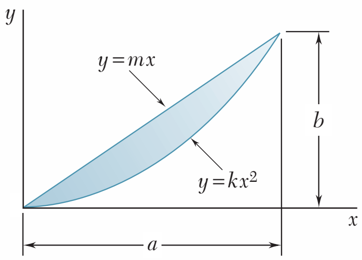


Fig. P7.4 and *P7.8*

7.9 through 7.12 Determine the moment of inertia and radius of gyration of the shaded area shown with respect to the x axis.

7.13 through *7.16* Determine the moment of inertia and radius of gyration of the shaded area shown with respect to the y axis.

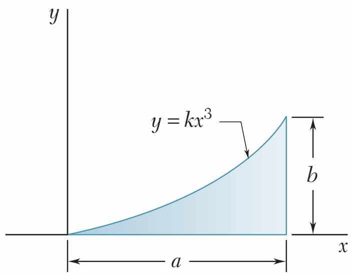


Fig. P7.9 and P7.13

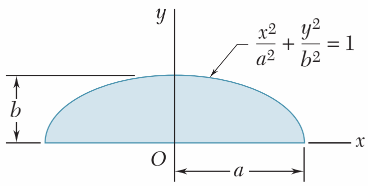


Fig. P7.10 and P7.14

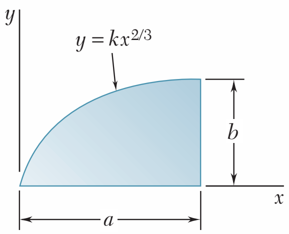


Fig. P7.11 and *P7.15*

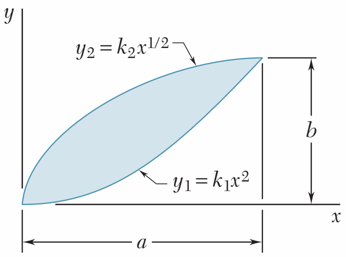


Fig. P7.12 and *P7.16*

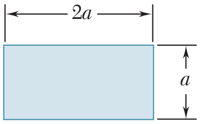


Fig. P7.17 and P7.18

7.17 Determine the polar moment of inertia and the polar radius of gyration of the rectangle shown with respect to the midpoint of one of its (*a*) longer sides, (*b*) shorter sides.

7.18 Determine the polar moment of inertia and the polar radius of gyration of the rectangle shown with respect to one of its corners.

7.19 Determine the polar moment of inertia and the polar radius of gyration of the trapezoid shown with respect to point *P*.

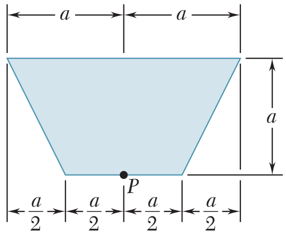


Fig. *P7.19*

7.20 Determine the polar moment of inertia and the polar radius of gyration of the semielliptical area of Prob. 7.10 with respect to *O*.

7.21 (*a*) Determine by direct integration the polar moment of inertia of the annular area shown with respect to point *O*. (*b*) Using the result of part *a*, determine the moment of inertia of the given area with respect to the *x* axis.

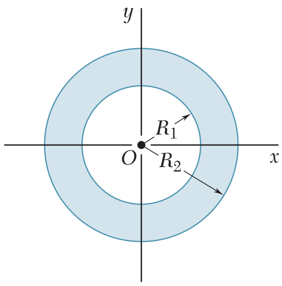


Fig. P7.21 and *P7.22*

7.22 (*a*) Show that the polar radius of gyration r_O of the annular area shown is approximately equal to the mean radius $R_m = (R_1 + R_2)/2$ for small values of the thickness $t = R_2 - R_1$. (*b*) Determine the percentage error introduced by using R_m in place of r_O for the following values of t/R_m: 1, $\frac{1}{2}$, and $\frac{1}{10}$.

7.23 Determine the moment of inertia of the shaded area with respect to the *x* axis.

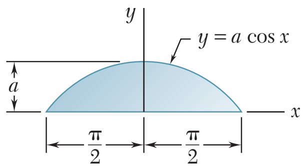


Fig. P7.23 and P7.24

7.24 Determine the moment of inertia of the shaded area with respect to the *y* axis.

7.6 PARALLEL-AXIS THEOREM

Consider the moment of inertia $I_{AA'}$ of an area A with respect to an axis AA' (Fig. 7.8). Denoting by y the distance from an element of area dA to AA', we write

$$I_{AA'} = \int y^2\, dA$$

Let us now draw through the centroid C of the area an axis BB' parallel to AA'; this axis is called a *centroidal axis*. Denoting by y'

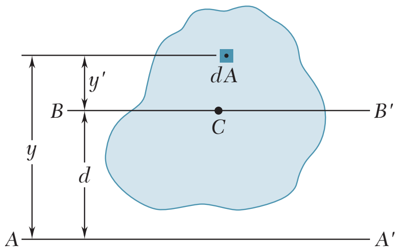


Fig. 7.8

the distance from the element dA to BB', we write $y = y' + d$, where d is the distance between the axes AA' and BB'. Substituting for y in the above integral, we write

$$\begin{aligned} I_{AA'} &= \int y^2\, dA = \int (y' + d)^2\, dA \\ &= \int y'^2\, dA + 2d \int y'\, dA + d^2 \int dA \end{aligned}$$

The first integral represents the moment of inertia $\bar{I}_{BB'}$ of the area with respect to the centroidal axis BB'. The second integral represents the first moment of the area with respect to BB'; since the centroid C of the area is located on that axis, the second integral must be zero. Finally, we observe that the last integral is equal to the total area A. Therefore, we have

$$I_{AA'} = \bar{I}_{BB'} + Ad^2 \qquad (7.9)$$

This formula expresses that the moment of inertia $I_{AA'}$ of an area with respect to any given axis AA' is equal to the moment of inertia $\bar{I}_{BB'}$ of the area with respect to a centroidal axis BB' parallel to AA' *plus* the product of the area A and the square of the distance d between the two axes. This theorem is known as the *parallel-axis theorem*. Substituting $r_{AA'}^2 A$ for $I_{AA'}$ and $\bar{r}_{BB'}^2 A$ for $\bar{I}_{BB'}$, the theorem can also be expressed as

$$r_{AA'}^2 = \bar{r}_{BB'}^2 + d^2 \qquad (7.10)$$

A similar theorem can be used to relate the polar moment of inertia J_O of an area about a point O to the polar moment of inertia $\bar{J}_C$ of the same area about its centroid C. Denoting by d the distance between O and C, we write

$$J_O = \bar{J}_C + Ad^2 \qquad \text{or} \qquad r_O^2 = \bar{r}_C^2 + d^2 \qquad (7.11)$$

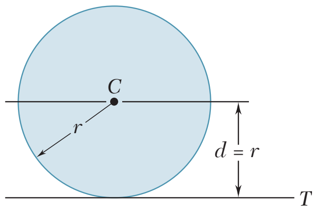


Fig. 7.9

EXAMPLE 7.2 As an application of the parallel-axis theorem, let us determine the moment of inertia I_T of a circular area with respect to a line tangent to the circle (Fig. 7.9). We found in Sample Prob. 7.2 that the moment of inertia of a circular area about a centroidal axis is $\bar{I} = \frac{1}{4}\pi r^4$. We can write, therefore,

$$I_T = \bar{I} + Ad^2 = \tfrac{1}{4}\pi r^4 + (\pi r^2)r^2 = \tfrac{5}{4}\pi r^4 \quad ■$$

EXAMPLE 7.3 The parallel-axis theorem can also be used to determine the centroidal moment of inertia of an area when the moment of inertia of the area with respect to a parallel axis is known. Consider, for instance, a triangular area (Fig. 7.10). We found in Sample Prob. 7.1 that the moment of inertia of a triangle with respect to its base AA' is equal to $\frac{1}{12}bh^3$. Using the parallel-axis theorem, we write

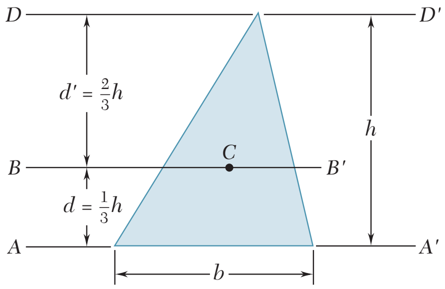


Fig. 7.10

$$I_{AA'} = \bar{I}_{BB'} + Ad^2$$
$$\bar{I}_{BB'} = I_{AA'} - Ad^2 = \tfrac{1}{12}bh^3 - \tfrac{1}{2}bh(\tfrac{1}{3}h)^2 = \tfrac{1}{36}bh^3$$

It should be observed that the product Ad^2 was *subtracted* from the given moment of inertia in order to obtain the centroidal moment of inertia of the triangle. Note that this product is *added* when transferring *from* a centroidal axis to a parallel axis, but it should be *subtracted* when transferring *to* a centroidal axis. In other words, the moment of inertia of an area is always smaller with respect to a centroidal axis than with respect to any parallel axis.

Returning to Fig. 7.10, we observe that the moment of inertia of the triangle with respect to the line DD' (which is drawn through a vertex) can be obtained by writing

$$I_{DD'} = \bar{I}_{BB'} + Ad'^2 = \tfrac{1}{36}bh^3 + \tfrac{1}{2}bh(\tfrac{2}{3}h)^2 = \tfrac{1}{4}bh^3$$

Note that $I_{DD'}$ could not have been obtained directly from $I_{AA'}$. The parallel-axis theorem can be applied only if one of the two parallel axes passes through the centroid of the area. ■

7.7 MOMENTS OF INERTIA OF COMPOSITE AREAS

Consider a composite area A made of several component areas A_1, A_2, A_3, . . . Since the integral representing the moment of inertia of A can be subdivided into integrals evaluated over A_1, A_2, A_3, . . . , the moment of inertia of A with respect to a given axis is obtained by adding the moments of inertia of the areas A_1, A_2, A_3, . . . , with respect to the same axis. The moment of inertia of an area consisting of several of the common shapes shown in Fig. 7.11 can thus be obtained by using the formulas given in that figure. Before adding the moments of inertia of the component areas, however, the parallel-axis theorem may have to be used to transfer each moment of inertia to the desired axis. This is shown in Sample Probs. 7.4 and 7.5.

Photo 7.1 Appendix B tabulates data for a small sample of the rolled-steel shapes that are readily available. Shown above are two examples of wide-flange shapes that are commonly used in the construction of buildings.

The properties of the cross sections of various structural shapes are given in App. B. As noted in Sec. 7.2, the moment of inertia of a beam section about its neutral axis is closely related to the computation of the bending moment in that section of the beam. The

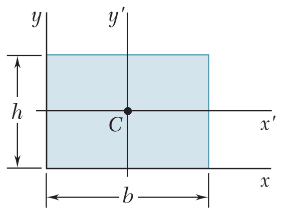
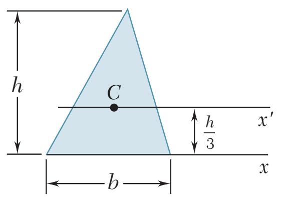
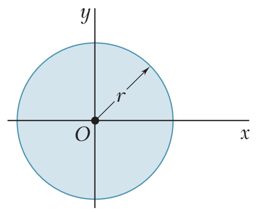
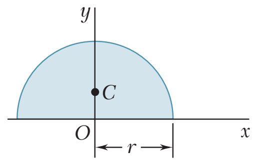
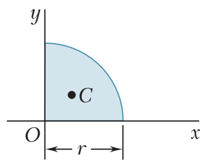
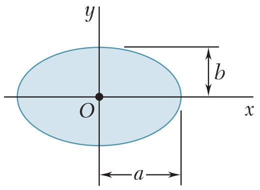
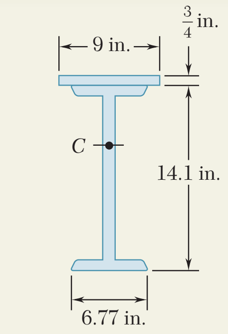
Rectangle		$\bar{I}_{x'} = \frac{1}{12}bh^3$ $\bar{I}_{y'} = \frac{1}{12}b^3h$ $I_x = \frac{1}{3}bh^3$ $I_y = \frac{1}{3}b^3h$ $J_C = \frac{1}{12}bh(b^2 + h^2)$
Triangle		$\bar{I}_{x'} = \frac{1}{36}bh^3$ $I_x = \frac{1}{12}bh^3$
Circle		$\bar{I}_x = \bar{I}_y = \frac{1}{4}\pi r^4$ $J_O = \frac{1}{2}\pi r^4$
Semicircle		$I_x = I_y = \frac{1}{8}\pi r^4$ $J_O = \frac{1}{4}\pi r^4$
Quarter circle		$I_x = I_y = \frac{1}{16}\pi r^4$ $J_O = \frac{1}{8}\pi r^4$
Ellipse		$\bar{I}_x = \frac{1}{4}\pi ab^3$ $\bar{I}_y = \frac{1}{4}\pi a^3b$ $J_O = \frac{1}{4}\pi ab(a^2 + b^2)$

Fig. 7.11 Moments of inertia of common geometric shapes.

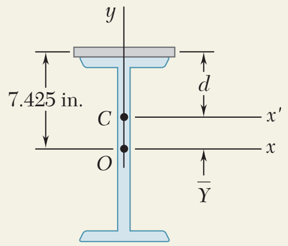
determination of moments of inertia is thus a prerequisite to the analysis and design of structural members.

It should be noted that the radius of gyration of a composite area is *not* equal to the sum of the radii of gyration of the component areas. In order to determine the radius of gyration of a composite area, it is first necessary to compute the moment of inertia of the composite area.



SAMPLE PROBLEM 7.4

The strength of a W14 × 38 rolled-steel beam is increased by attaching a 9 × $\frac{3}{4}$-in. plate to its upper flange as shown. Determine the moment of inertia and the radius of gyration of the composite section with respect to an axis which is parallel to the plate and passes through the centroid C of the section.



SOLUTION

The origin O of the coordinates is placed at the centroid of the wide-flange shape, and the distance $\overline{Y}$ to the centroid of the composite section is computed using the methods of Chap. 5. The area of the wide-flange shape is found by referring to App. B. The area and the y coordinate of the centroid of the plate are

$$A = (9 \text{ in.})(0.75 \text{ in.}) = 6.75 \text{ in}^2$$
$$\bar{y} = \tfrac{1}{2}(14.1 \text{ in.}) + \tfrac{1}{2}(0.75 \text{ in.}) = 7.425 \text{ in.}$$

Section	Area, in^2	$\bar{y}$, in.	$\bar{y}A$, in^3
Plate	6.75	7.425	50.12
Wide-flange shape	11.2	0	0
	$\Sigma A = 17.95$		$\Sigma \bar{y}A = 50.12$

$$\overline{Y}\Sigma A = \Sigma \bar{y}A \qquad \overline{Y}(17.95) = 50.12 \qquad \overline{Y} = 2.792 \text{ in.}$$

Moment of Inertia. The parallel-axis theorem is used to determine the moments of inertia of the wide-flange shape and the plate with respect to the x' axis. This axis is a centroidal axis for the composite section but *not* for either of the elements considered separately. The value of $\bar{I}_x$ for the wide-flange shape is obtained from App. B.

For the wide-flange shape,

$$I_{x'} = \bar{I}_x + A\overline{Y}^2 = 385 + (11.2)(2.792)^2 = 472.3 \text{ in}^4$$

For the plate,

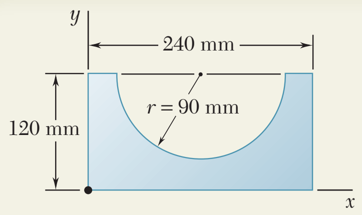
$$I_{x'} = \bar{I}_x + Ad^2 = (\tfrac{1}{12})(9)(\tfrac{3}{4})^3 + (6.75)(7.425 - 2.792)^2 = 145.2 \text{ in}^4$$

For the composite area,

$$I_{x'} = 472.3 + 145.2 = 617.5 \text{ in}^4 \qquad I_{x'} = 618 \text{ in}^4 \blacktriangleleft$$

Radius of Gyration. We have

$$r_{x'}^2 = \frac{I_{x'}}{A} = \frac{617.5 \text{ in}^4}{17.95 \text{ in}^2} \qquad r_{x'} = 5.87 \text{ in.} \blacktriangleleft$$

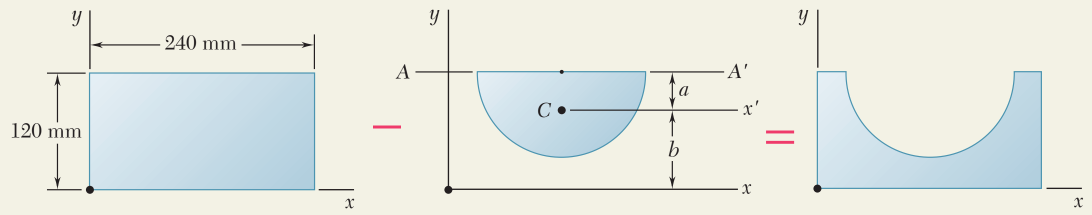


SAMPLE PROBLEM 7.5

Determine the moment of inertia of the shaded area with respect to the x axis.

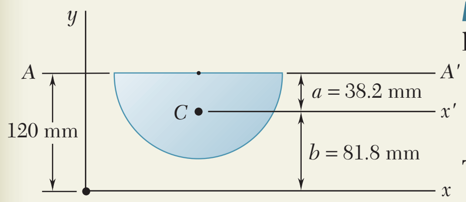
SOLUTION

The given area can be obtained by subtracting a half circle from a rectangle. The moments of inertia of the rectangle and the half circle will be computed separately.



Moment of Inertia of Rectangle. Referring to Fig. 7.11, we obtain

$$I_x = \tfrac{1}{3}bh^3 = \tfrac{1}{3}(240 \text{ mm})(120 \text{ mm})^3 = 138.2 \times 10^6 \text{ mm}^4$$



Moment of Inertia of Half Circle. Referring to Fig. 5.8, we determine the location of the centroid C of the half circle with respect to diameter AA'.

$$a = \frac{4r}{3\pi} = \frac{(4)(90 \text{ mm})}{3\pi} = 38.2 \text{ mm}$$

The distance b from the centroid C to the x axis is

$$b = 120 \text{ mm} - a = 120 \text{ mm} - 38.2 \text{ mm} = 81.8 \text{ mm}$$

Referring now to Fig. 7.11, we compute the moment of inertia of the half circle with respect to diameter AA'; we also compute the area of the half circle.

$$I_{AA'} = \tfrac{1}{8}\pi r^4 = \tfrac{1}{8}\pi(90 \text{ mm})^4 = 25.76 \times 10^6 \text{ mm}^4$$
$$A = \tfrac{1}{2}\pi r^2 = \tfrac{1}{2}\pi(90 \text{ mm})^2 = 12.72 \times 10^3 \text{ mm}^2$$

Using the parallel-axis theorem, we obtain the value of $\bar{I}_{x'}$:

$$I_{AA'} = \bar{I}_{x'} + Aa^2$$
$$25.76 \times 10^6 \text{ mm}^4 = \bar{I}_{x'} + (12.72 \times 10^3 \text{ mm}^2)(38.2 \text{ mm})^2$$
$$\bar{I}_{x'} = 7.20 \times 10^6 \text{ mm}^4$$

Again using the parallel-axis theorem, we obtain the value of I_x:

$$I_x = \bar{I}_{x'} + Ab^2 = 7.20 \times 10^6 \text{ mm}^4 + (12.72 \times 10^3 \text{ mm}^2)(81.8 \text{ mm})^2$$
$$= 92.3 \times 10^6 \text{ mm}^4$$

Moment of Inertia of Given Area. Subtracting the moment of inertia of the half circle from that of the rectangle, we obtain

$$I_x = 138.2 \times 10^6 \text{ mm}^4 - 92.3 \times 10^6 \text{ mm}^4$$

$$I_x = 45.9 \times 10^6 \text{ mm}^4 \quad ◀$$

PROBLEMS

7.25 through *7.28* Determine the moment of inertia and the radius of gyration of the shaded area with respect to the x axis.

***7.29* through 7.32** Determine the moment of inertia and the radius of gyration of the shaded area with respect to the y axis.

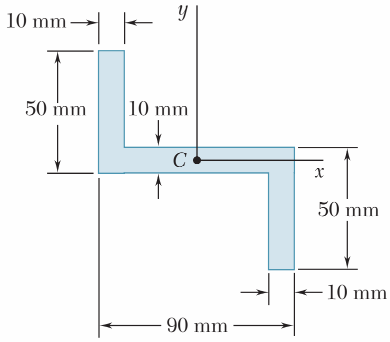


Fig. P7.25 and *P7.29*

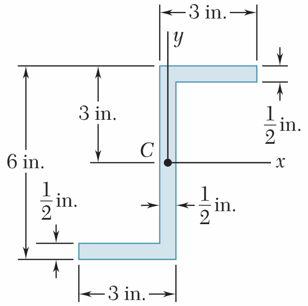


Fig. P7.26 and P7.30

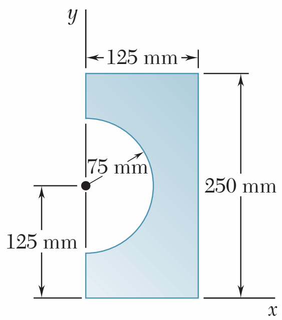


Fig. P7.27 and P7.31

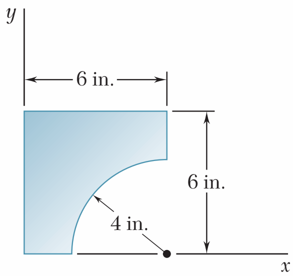


Fig. *P7.28* and P7.32

7.33 Determine the shaded area and its moment of inertia with respect to a centroidal axis parallel to AA', knowing that its moments of inertia with respect to AA' and BB' are, respectively, 2.2×10^6 mm^4 and 4×10^6 mm^4, and that $d_1 = 25$ mm and $d_2 = 10$ mm.

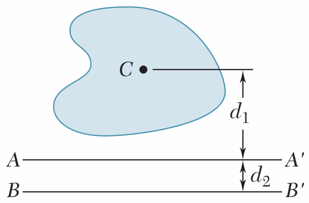


Fig. P7.33 and *P7.34*

7.34 Knowing that the shaded area is equal to 6000 mm^2 and that its moment of inertia with respect to AA' is 18×10^6 mm^4, determine its moment of inertia with respect to BB' for $d_1 = 50$ mm and $d_2 = 10$ mm.

7.35 and 7.36 Determine the moments of inertia $\bar{I}_x$ and $\bar{I}_y$ of the area shown with respect to centroidal axes that are respectively parallel and perpendicular to the side AB.

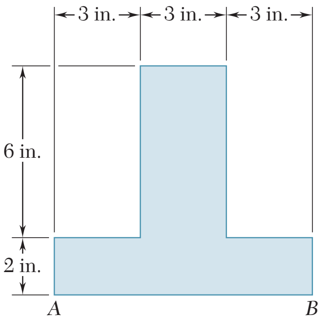


Fig. P7.35

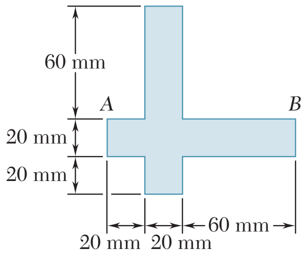


Fig. P7.36

7.37 Determine the moments of inertia $\bar{I}_x$ and $\bar{I}_y$ of the area shown with respect to centroidal axes that are respectively parallel and perpendicular to the side AB.

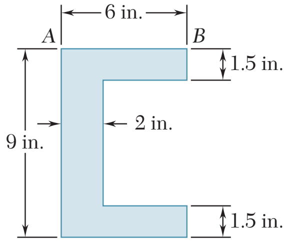


Fig. *P7.37* and P7.38

7.38 Determine the centroidal polar moment of inertia of the area shown.

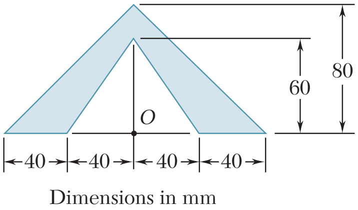


Fig. P7.39

7.39 and 7.40 Determine the polar moment of inertia of the area shown with respect to (*a*) point O, (*b*) the centroid of the area.

7.41 Two W8 × 31 rolled sections can be welded at A and B in either of the two ways shown. For each arrangement, determine the moment of inertia of the section with respect to the horizontal centroidal axis.

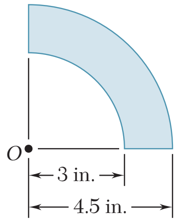


Fig. P7.40

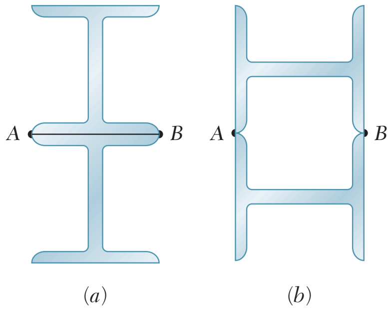


Fig. P7.41

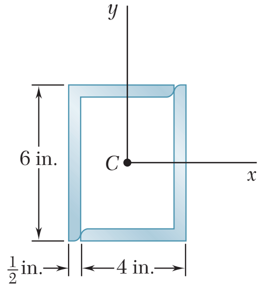


Fig. P7.42

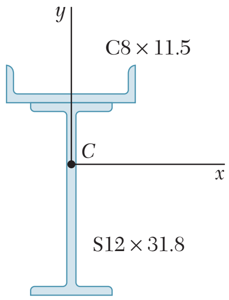


Fig. P7.45

7.42 Two 6 × 4 × $\frac{1}{2}$-in. angles are welded together to form the section shown. Determine the moments of inertia and the radii of gyration of the section with respect to the centroidal axes shown.

7.43 Two channels and two plates are used to form the column section shown. For $b = 200$ mm, determine the moments of inertia and the radii of gyration of the combined section with respect to the centroidal axes.

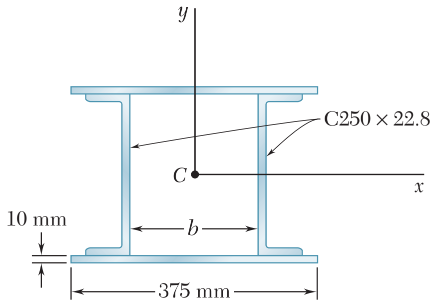


Fig. P7.43

7.44 In Prob. 7.43, determine the distance b for which the centroidal moments of inertia $\bar{I}_x$ and $\bar{I}_y$ of the column section are equal.

7.45 The strength of the rolled S section shown is increased by welding a channel to its upper flange. Determine the moments of inertia of the combined section with respect to its centroidal x and y axes.

7.46 A channel and a plate are welded together as shown to form a section that is symmetrical with respect to the y axis. Determine the moments of inertia of the section with respect to its centroidal x and y axes.

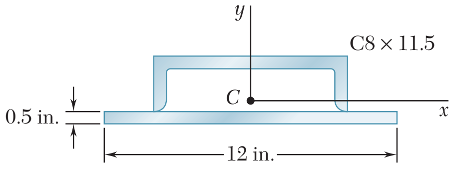


Fig. P7.46

7.47 Two L102 × 102 × 12.7-mm angles are welded to a 12-mm steel plate as shown. For $b = 250$ mm, determine the moments of inertia of the combined section with respect to centroidal axes that are respectively parallel and perpendicular to the plate.

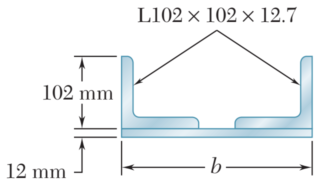


Fig. P7.47

7.48 Solve Prob. 7.47 assuming that $b = 300$ mm.

REVIEW AND SUMMARY

In this chapter, we discussed the determination of the resultant **R** of forces $\Delta\mathbf{F}$ distributed over a plane area A when the magnitudes of these forces are proportional to both the areas ΔA of the elements on which they act and the distances y from these elements to a given x axis; we thus had $\Delta F = ky\,\Delta A$. We found that the magnitude of the resultant **R** is proportional to the first moment $Q_x = \int y\,dA$ of the area A, while the moment of **R** about the x axis is proportional to the *second moment*, or *moment of inertia*, $I_x = \int y^2\,dA$ of A with respect to the same axis [Sec. 7.2].

Rectangular moments of inertia

The *rectangular moments of inertia I_x and I_y of an area* [Sec. 7.3] were obtained by evaluating the integrals

$$I_x = \int y^2\,dA \qquad I_y = \int x^2\,dA \tag{7.1}$$

These computations can be reduced to single integrations by choosing dA to be a thin strip parallel to one of the coordinate axes. We also recall that it is possible to compute I_x and I_y from the same elemental strip (Fig. 7.12) using the formula for the moment of inertia of a rectangular area [Sample Prob. 7.3].

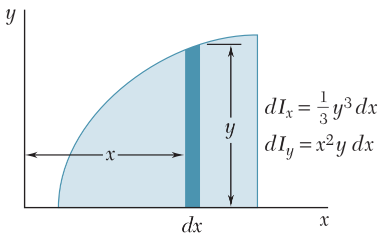


Fig. 7.12

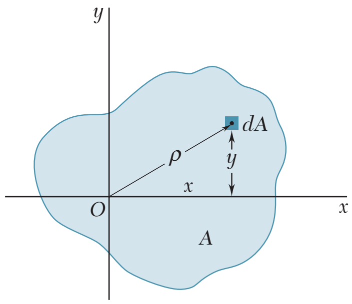


Fig. 7.13

Polar moment of inertia

The *polar moment of inertia of an area* A with respect to the pole O [Sec. 7.4] was defined as

$$J_O = \int \rho^2\,dA \tag{7.3}$$

where r is the distance from O to the element of area dA (Fig. 7.13). Observing that $\rho^2 = x^2 + y^2$, we established the relation

$$J_O = I_x + I_y \tag{7.4}$$

Radius of gyration

The *radius of gyration of an area* A with respect to the x axis [Sec. 7.5] was defined as the distance r_x, where $I_x = r_x^2 A$. With similar definitions for the radii of gyration of A with respect to the y axis and with respect to O, we had

$$r_x = \sqrt{\frac{I_x}{A}} \qquad r_y = \sqrt{\frac{I_y}{A}} \qquad r_O = \sqrt{\frac{J_O}{A}} \tag{7.5–7.7}$$

Parallel-axis theorem

The parallel-axis theorem was presented in Sec. 7.6. It states that the moment of inertia $I_{AA'}$ of an area with respect to any given axis AA' (Fig. 7.14) is equal to the moment of inertia $\bar{I}_{BB'}$ of the area with respect to the centroidal axis BB' that is parallel to AA' *plus* the product of the area A and the square of the distance d between the two axes:

$$I_{AA'} = \bar{I}_{BB'} + Ad^2 \tag{7.9}$$

This formula can also be used to determine the moment of inertia $\bar{I}_{BB'}$ of an area with respect to a centroidal axis BB' when its moment of inertia $I_{AA'}$ with respect to a parallel axis AA' is known. In this case, however, the product Ad^2 should be *subtracted* from the known moment of inertia $I_{AA'}$.

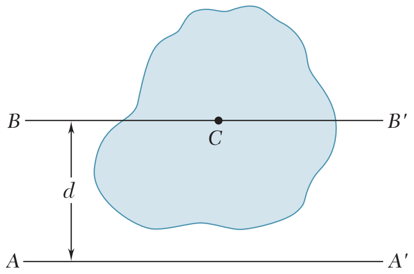


Fig. 7.14

A similar relation holds between the polar moment of inertia J_O of an area about a point O and the polar moment of inertia $\bar{J}_C$ of the same area about its centroid C. Letting d be the distance between O and C, we have

$$J_O = \bar{J}_C + Ad^2 \tag{7.11}$$

Composite areas

The parallel-axis theorem can be used very effectively to compute the *moment of inertia of a composite area* with respect to a given axis [Sec. 7.7]. Considering each component area separately, we first compute the moment of inertia of each area with respect to its centroidal axis, using the data provided in Fig. 7.11 and App. B whenever possible. The parallel-axis theorem is then applied to determine the moment of inertia of each component area with respect to the desired axis, and the various values obtained are added [Sample Probs. 7.4 and 7.5].

REVIEW PROBLEMS

7.49 Determine by direct integration the moment of inertia of the shaded area with respect to the y axis.

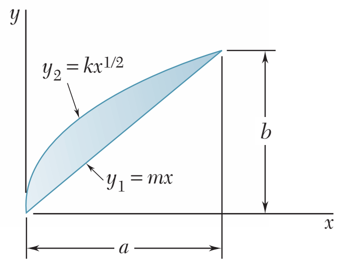


Fig. P7.49 and *P7.50*

7.50 Determine by direct integration the moment of inertia of the shaded area with respect to the x axis.

7.51 Determine the moment of inertia and radius of gyration of the shaded area shown with respect to the x axis.

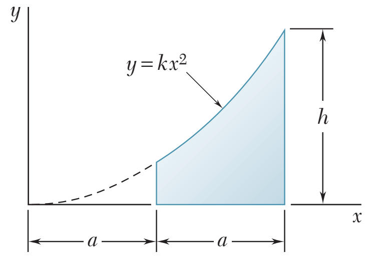


Fig. P7.51 and *P7.52*

7.52 Determine the moment of inertia and radius of gyration of the shaded area shown with respect to the y axis.

7.53 Determine the polar moment of inertia and the polar radius of gyration of an equilateral triangle of side a with respect to one of its vertices.

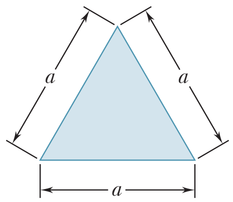


Fig. P7.53

7.54 Determine the moments of inertia of the shaded area shown with respect to the x and y axes when $a = 20$ mm.

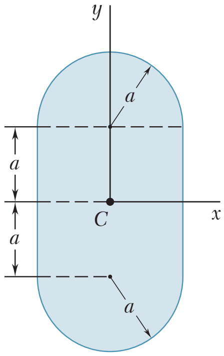


Fig. P7.54

7.55 (a) Determine I_x and I_y if $b = 10$ in. (b) Determine the dimension b for which $I_x = I_y$.

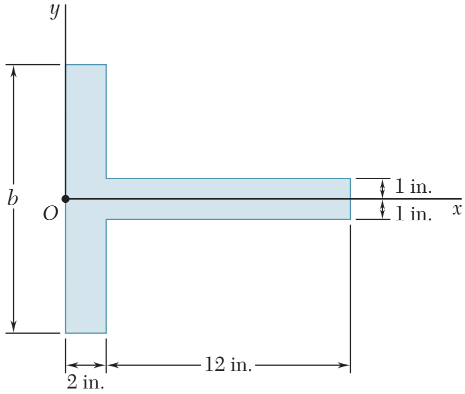


Fig. P7.55

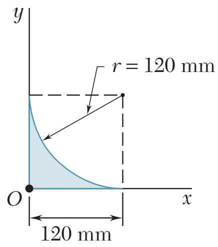


Fig. P7.56

7.56 Determine the moment of inertia of the shaded area shown with respect to the y axis.

7.57 The shaded area is equal to 5000 mm^2. Determine its centroidal moments of inertia $\bar{I}_x$ and $\bar{I}_y$, knowing that $\bar{I}_y = 2\bar{I}_x$ and that the polar moment of inertia of the area about point A is $J_A = 22.5 \times 10^6$ mm^4.

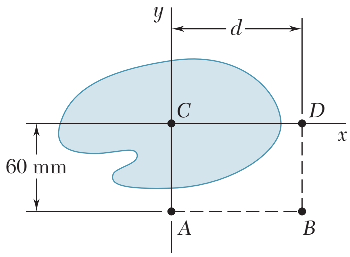


Fig. *P7.57*

7.58 Determine the polar moment of inertia and the polar radius of gyration of the shaded area shown with respect to its centroid C.

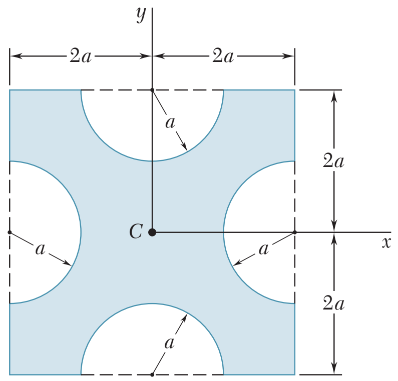


Fig. P7.58

7.59 Determine the polar moment of inertia of the area shown with respect to (*a*) point O, (*b*) the centroid of the area.

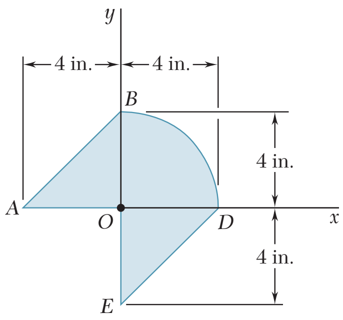


Fig. *P7.59*

7.60 Three 1-in. steel plates are bolted to four L6 × 6 × 1-in. angles to form the column whose cross section is shown. Determine the moments of inertia and the radii of gyration of the section with respect to centroidal axes that are respectively parallel and perpendicular to the flanges.

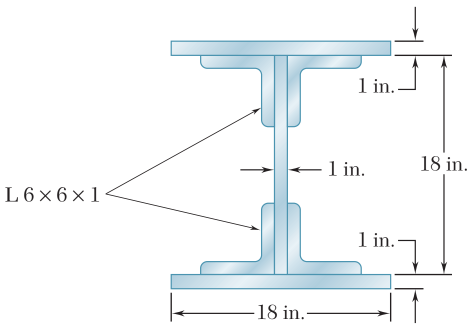


Fig. P7.60

This chapter is devoted to the study of the stresses occurring in many of the elements contained in these excavators, such as two-force members, axles, bolts, and pins.

CHAPTER 8

Concept of Stress

Chapter 8 Concept of Stress

8.1 INTRODUCTION

The main objective of the study of the mechanics of materials is to provide the future engineer with the means of analyzing and designing various machines and load-bearing structures.

Both the analysis and the design of a given structure involve the determination of *stresses* and *deformations*. This chapter is devoted to the concept of *stress*.

Section 8.2 will introduce you to the concept of *stress* in a member of a structure, and you will be shown how that stress can be determined from the *force* in the member. You will consider successively the *normal stresses* in a member under axial loading (Sec. 8.3), the *shearing stresses* caused by the application of equal and opposite transverse forces (Sec. 8.4), and the *bearing stresses* created by bolts and pins in the members they connect (Sec. 8.5). These various concepts will be applied in Sec. 8.6 to the determination of the stresses in the members of the simple structure. Engineering design will be discussed in Sec. 8.7.

In Sec. 8.8, where a two-force member under axial loading is considered again, it will be observed that the stresses on an *oblique* plane include both *normal* and *shearing* stresses, while in Sec. 8.9 you will note that *six components* are required to describe the state of stress at a point in a body under the most general loading conditions.

Finally, Sec. 8.10 will be devoted to the determination from test specimens of the *ultimate strength* of a given material and to the use of a *factor of safety* in the computation of the *allowable load* for a structural component made of that material.

8.2 STRESSES IN THE MEMBERS OF A STRUCTURE

The force per unit area, or intensity of the forces distributed over a given section, is called the *stress* on that section. When the stress is perpendicular to the cross-section, it is denoted by the Greek letter σ (sigma). The stress in a member of cross-sectional area A subjected to an axial load **P** (Fig. 8.1) is therefore obtained by dividing the magnitude P of the load by the area A:

$$\sigma = \frac{P}{A} \tag{8.1}$$

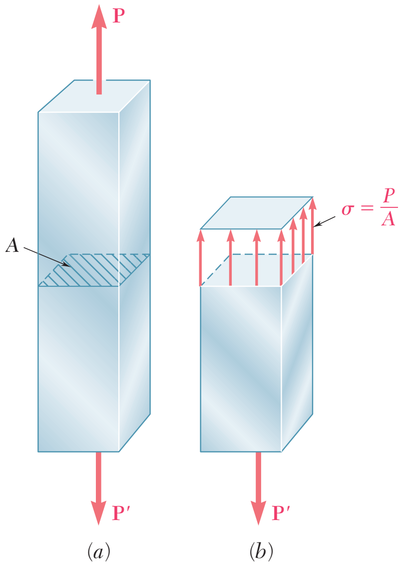


Fig. 8.1

A positive sign will be used to indicate a tensile stress (member in tension) and a negative sign to indicate a compressive stress (member in compression).

Since SI metric units are used in this discussion, with P expressed in newtons (N) and A in square meters (m^2), the stress σ will be expressed in N/m^2. This unit is called a *pascal* (Pa). However, one finds that the pascal is an exceedingly small quantity and that, in practice, multiples of this unit must be used, namely, the kilopascal (kPa), the megapascal (MPa), and the gigapascal (GPa). We have

$$1 \text{ kPa} = 10^3 \text{ Pa} = 10^3 \text{ N/m}^2$$
$$1 \text{ MPa} = 10^6 \text{ Pa} = 10^6 \text{ N/m}^2$$
$$1 \text{ GPa} = 10^9 \text{ Pa} = 10^9 \text{ N/m}^2$$

When U.S. customary units are used, the force P is usually expressed in pounds (lb) or kilopounds (kip), and the cross-sectional area A in square inches (in^2). The stress σ will then be expressed in pounds per square inch (psi) or kilopounds per square inch (ksi).†

8.3 AXIAL LOADING. NORMAL STRESS

The member shown in Fig. 8.1 in the preceding section is subject to forces **P** and **P′** applied at the ends. The forces are directed along the axis of the member, and we say that the member is under *axial loading*. An actual example of structural members under axial loading is provided by the members of the bridge truss shown in Photo 8.1.

Photo 8.1 This bridge truss consists of two-force members that may be in tension or in compression.

As shown in Fig. 8.1*b*, the internal force and the corresponding stress are perpendicular to the axis of the member; the corresponding stress is described as a *normal stress*. Thus, formula (8.1) gives us the *normal stress in a member under axial loading*:

$$\sigma = \frac{P}{A} \tag{8.1}$$

We should also note that, in formula (8.1), σ is obtained by dividing the magnitude P of the resultant of the internal forces distributed over the cross section by the area A of the cross section; it represents, therefore, the *average value* of the stress over the cross section, rather than the stress at a specific point of the cross section.

To define the stress at a given point Q of the cross section, we should consider a small area ΔA (Fig. 8.2). Dividing the magnitude

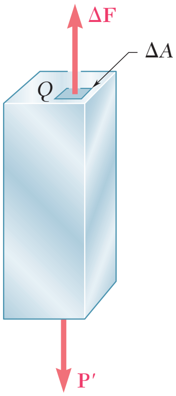


Fig. 8.2

†The principal SI and U.S. customary units used in mechanics for stresses are listed in tables inside the front cover of this book. From this table, we note that 1 psi is approximately equal to 7 kPa, and 1 ksi is approximately equal to 7 MPa.

of $\Delta\mathbf{F}$ by ΔA, we obtain the average value of the stress over ΔA. Letting ΔA approach zero, we obtain the stress at point Q:

$$\sigma = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A} \tag{8.2}$$

In general, the value obtained for the stress σ at a given point Q of the section is different from the value of the average stress given by formula (8.1), and σ is found to vary across the section. In a slender rod subjected to equal and opposite concentrated loads $\mathbf{P}$ and $\mathbf{P}'$ (Fig. 8.3*a*), this variation is small in a section away from the points of application of the concentrated loads (Fig. 8.3*c*), but it is quite noticeable in the neighborhood of these points (Fig. 8.3*b* and *d*).

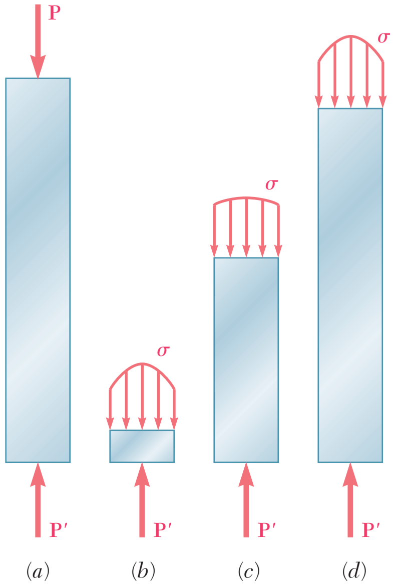


Fig. 8.3

It follows from Eq. (8.2) that the magnitude of the resultant of the distributed internal forces is

$$\int dF = \int_A \sigma\, dA$$

But the conditions of equilibrium of each of the portions of rod shown in Fig. 8.3 require that this magnitude be equal to the magnitude P of the concentrated loads. We have, therefore,

$$P = \int dF = \int_A \sigma\, dA \tag{8.3}$$

which means that the volume under each of the stress surfaces in Fig. 8.3 must be equal to the magnitude P of the loads. This, however, is the only information that we can derive from our knowledge of statics, regarding the distribution of normal stresses in the various sections of the rod. The actual distribution of stresses in any given section is *statically indeterminate*. To learn more about this distribution, it is necessary to consider the deformations resulting from the particular mode of application of the loads at the ends of the rod. This will be discussed further in Chap. 9.

In practice, it will be assumed that the distribution of normal stresses in an axially loaded member is uniform, except in the immediate vicinity of the points of application of the loads. The value σ of the stress is then equal to σ_{ave} and can be obtained from formula (8.1). However, we should realize that, when we assume a uniform distribution of stresses in the section, i.e., when we assume that the internal forces are uniformly distributed across the section, it follows from elementary statics that the resultant $\mathbf{P}$ of the internal forces must be applied at the centroid C of the section (Fig. 8.4). This means that *a uniform distribution of stress is possible only if the line of action of the concentrated loads* $\mathbf{P}$ *and* $\mathbf{P}'$ *passes through the centroid of the section considered* (Fig. 8.5). This type of loading is called *centric loading* and will be assumed to take place in all straight two-force members found in trusses and pin-connected structures. However, if a two-force member is loaded axially, but *eccentrically* as shown in Fig. 8.6*a*, we find from the conditions of equilibrium of the portion of the member shown in Fig. 8.6*b* that the internal forces in a given section must be equivalent to a force

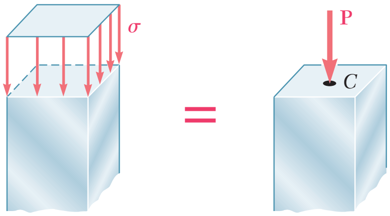


Fig. 8.4

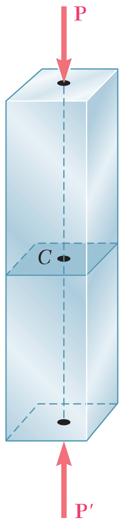


Fig. 8.5

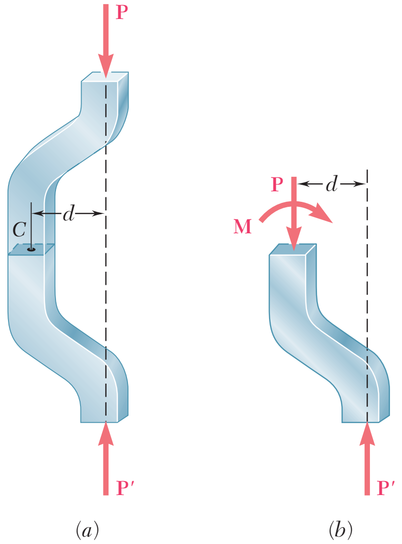


Fig. 8.6

P applied at the centroid of the section and a couple **M** of moment $M = Pd$. The distribution of forces—and, thus, the corresponding distribution of stresses—*cannot be uniform*. Nor can the distribution of stresses be symmetric as shown in Fig. 8.3. This point will be discussed in detail in Chap. 11.

8.4 SHEARING STRESS

The internal forces and the corresponding stresses discussed in Secs. 8.2 and 8.3 were normal to the section considered. A very different type of stress is obtained when transverse forces **P** and **P**′ are applied to a member AB (Fig. 8.7). Passing a section at C between the points of application of the two forces (Fig. 8.8*a*), we obtain the diagram of portion AC shown in Fig. 8.8*b*. We conclude that internal forces must exist in the plane of the section, and that their resultant is equal to **P**. These elementary internal forces are called *shearing forces,* and the magnitude P of their resultant is the *shear* in the section. Dividing the shear P by the area A of the cross section, we obtain the *average shearing stress* in the section. Denoting the shearing stress by the Greek letter τ (tau), we write

$$\tau_{\text{ave}} = \frac{P}{A} \tag{8.4}$$

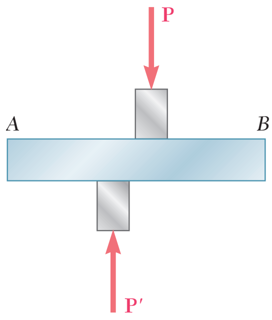


Fig. 8.7

It should be emphasized that the value obtained is an average value of the shearing stress over the entire section. Contrary to what we said earlier for normal stresses, the distribution of shearing stresses across the section *cannot* be assumed uniform. As you will see in Chap. 13, the actual value τ of the shearing stress varies from zero at the surface of the member to a maximum value $\tau_{\max}$ that may be much larger than the average value τ_{ave}.

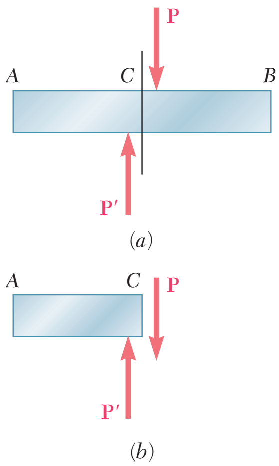


Fig. 8.8

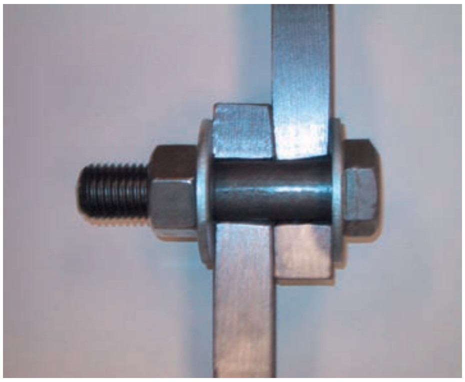


Photo 8.2 Cutaway view of a connection with a bolt in shear.

Shearing stresses are commonly found in bolts, pins, and rivets used to connect various structural members and machine components (Photo 8.2). Consider the two plates A and B, which are connected by a bolt CD (Fig. 8.9). If the plates are subjected to tension forces of magnitude F, stresses will develop in the section of bolt corresponding to the plane EE'. Drawing the diagrams of the bolt and of the portion located above the plane EE' (Fig. 8.10), we conclude that the shear P in the section is equal to F. The average shearing stress in the section is obtained, according to formula (8.4), by dividing the shear $P = F$ by the area A of the cross section:

$$\tau_{\text{ave}} = \frac{P}{A} = \frac{F}{A} \tag{8.5}$$

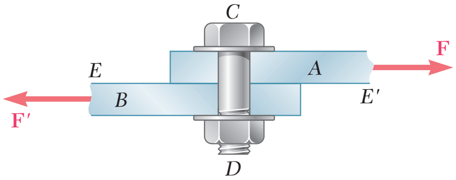


Fig. 8.9

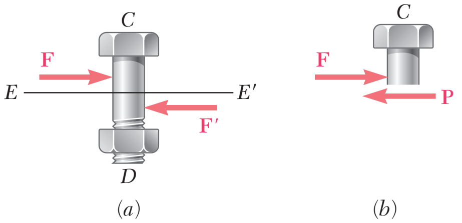


Fig. 8.10

The bolt we have just considered is said to be in *single shear*. Different loading situations may arise, however. For example, if splice plates C and D are used to connect plates A and B (Fig. 8.11), shear will take place in bolt HJ in each of the two planes KK' and LL' (and similarly in bolt EG). The bolts are said to be in *double shear*. To determine the average shearing stress in each plane, we draw free-body diagrams of bolt HJ and of the portion of bolt located between the two planes (Fig. 8.12). Observing that the shear P in each of the sections is $P = F/2$, we conclude that the average shearing stress is

$$\tau_{\text{ave}} = \frac{P}{A} = \frac{F/2}{A} = \frac{F}{2A} \tag{8.6}$$

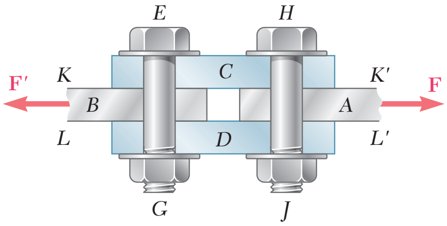


Fig. 8.11

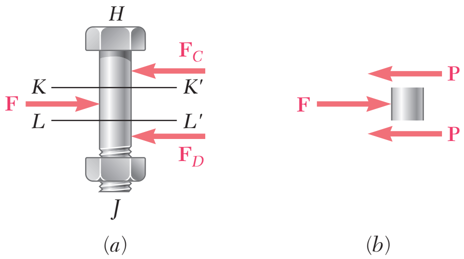


Fig. 8.12

8.5 BEARING STRESS IN CONNECTIONS

Bolts, pins, and rivets create stresses in the members they connect along the *bearing surface*, or surface of contact. For example, consider again the two plates A and B connected by a bolt CD that we have discussed in the preceding section (Fig. 8.9). The bolt exerts on plate A a force $\mathbf{P}$ equal and opposite to the force $\mathbf{F}$ exerted by the plate on the bolt (Fig. 8.13). The force $\mathbf{P}$ represents the resultant of elementary forces distributed on the inside surface of a half-cylinder of diameter d and of length t equal to the thickness of the plate. Since the distribution of these forces—and of the corresponding stresses—is quite complicated, one uses in practice an average nominal value σ_b of the stress, called the *bearing stress*, obtained by dividing the load P by the area of the rectangle representing the projection of the bolt on the plate section (Fig. 8.14). Since this area is equal to td, where t is the plate thickness and d the diameter of the bolt, we have

$$\sigma_b = \frac{P}{A} = \frac{P}{td} \tag{8.7}$$

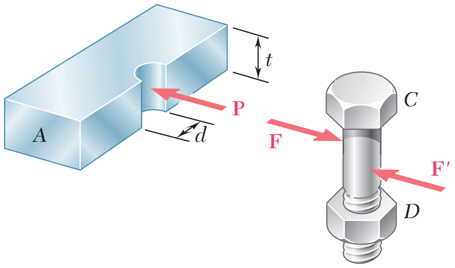


Fig. 8.13

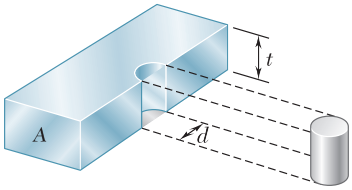


Fig. 8.14

8.6 APPLICATION TO THE ANALYSIS OF A SIMPLE STRUCTURE

We are now in a position to determine the stresses in the members and connections of simple two-dimensional structure and, thus, to design such a structure.

The structure shown in Fig. 8.15 was designed to support a 30-kN load. It consists of a boom AB with a 30 × 50-mm rectangular cross section and a rod BC with a 20-mm-diameter circular cross section. The boom and the rod are connected by a pin at B and are supported by pins and brackets at A and C, respectively.

We first use the basic methods of statics to find the reactions and then the internal forces in the members. We start by drawing a *free-body diagram* of the structure by detaching it from its supports at A and C, and showing the reactions that these supports exert on

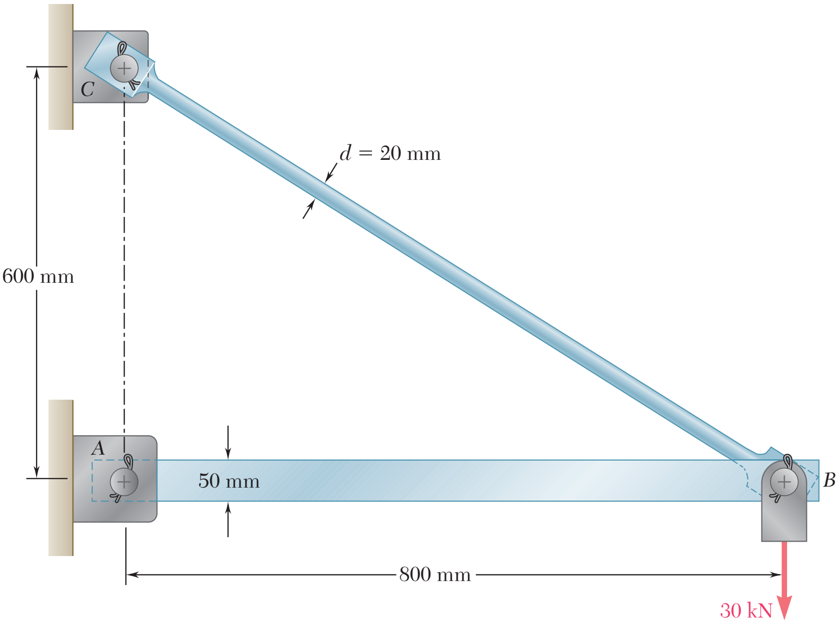


Fig. 8.15

the structure (Fig. 8.16). The reactions are represented by two components $\mathbf{A}_x$ and $\mathbf{A}_y$ at A, and $\mathbf{C}_x$ and $\mathbf{C}_y$ at C. We write the following three equilibrium equations:

$$+\circlearrowleft \Sigma M_C = 0: \quad A_x(0.6 \text{ m}) - (30 \text{ kN})(0.8 \text{ m}) = 0$$
$$A_x = +40 \text{ kN} \tag{8.8}$$
$$\xrightarrow{+} \Sigma F_x = 0: \quad A_x + C_x = 0$$
$$C_x = -A_x \qquad C_x = -40 \text{ kN} \tag{8.9}$$
$$+\uparrow \Sigma F_y = 0: \quad A_y + C_y - 30 \text{ kN} = 0$$
$$A_y + C_y = +30 \text{ kN} \tag{8.10}$$

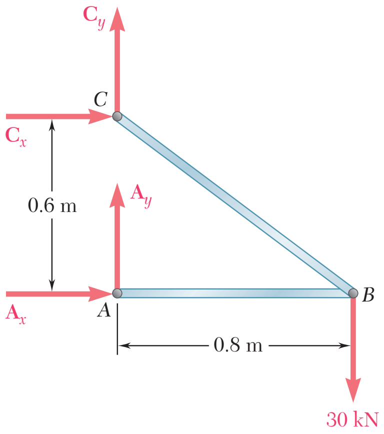


Fig. 8.16

We have found two of the four unknowns. We must now dismember the structure. Considering the free-body diagram of the boom AB (Fig. 8.17), we write the following equilibrium equation:

$$+\circlearrowleft \Sigma M_B = 0: \quad -A_y(0.8 \text{ m}) = 0 \qquad A_y = 0 \tag{8.11}$$

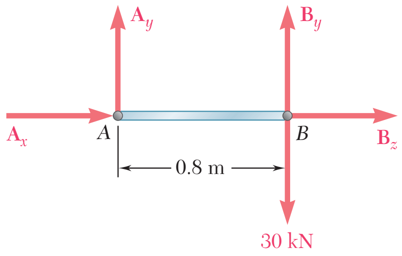


Fig. 8.17

Substituting for A_y from (8.11) into (8.10), we obtain $C_y = +30$ kN. Expressing the results obtained for the reactions at A and C in vector form, we have

$$\mathbf{A} = 40 \text{ kN} \rightarrow, \mathbf{C}_x = 40 \text{ kN} \leftarrow, \mathbf{C}_y = 30 \text{ kN} \uparrow$$

We note that the reaction at A is directed along the axis of the boom AB and causes compression in that member. Observing that the components C_x and C_y of the reaction at C are, respectively, proportional to the horizontal and vertical components of the distance from B to C, we conclude that the reaction at C is equal to 50 kN, is directed along the axis of the rod BC, and causes tension in that member.

These results could have been anticipated by recognizing that AB and BC are two-force members, i.e., members that are subjected to forces at only two points, these points being A and B for member AB, and B and C for member BC. Indeed, for a two-force member the lines of action of the resultants of the forces acting at each of the two points are equal and opposite and pass through both points. Using this property, we could have obtained a simpler solution by considering the free-body diagram of pin B. The forces on pin B are the forces $\mathbf{F}_{AB}$ and $\mathbf{F}_{BC}$ exerted, respectively, by members AB and BC, and the 30-kN load (Fig. 8.18*a*). We can express that pin B is in equilibrium by drawing the corresponding force triangle (Fig. 8.18*b*).

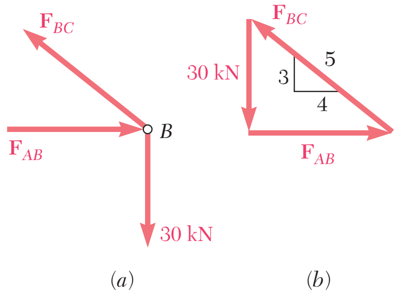


Fig. 8.18

Since the force $\mathbf{F}_{BC}$ is directed along member BC, its slope is the same as that of BC, namely, 3/4. We can, therefore, write the proportion

$$\frac{F_{AB}}{4} = \frac{F_{BC}}{5} = \frac{30 \text{ kN}}{3}$$

from which we obtain

$$F_{AB} = 40 \text{ kN} \qquad F_{BC} = 50 \text{ kN}$$

The forces $\mathbf{F}'_{AB}$ and $\mathbf{F}'_{BC}$ exerted by pin B, respectively, on boom AB and rod BC are equal and opposite to $\mathbf{F}_{AB}$ and $\mathbf{F}_{BC}$ (Fig. 8.19).

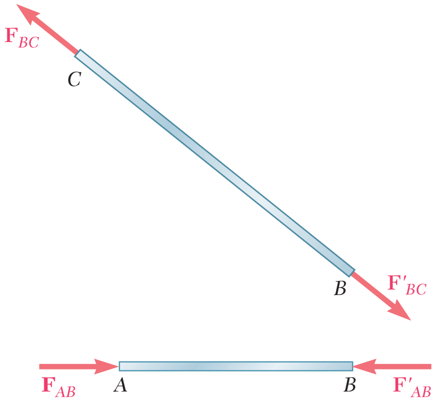


Fig. 8.19

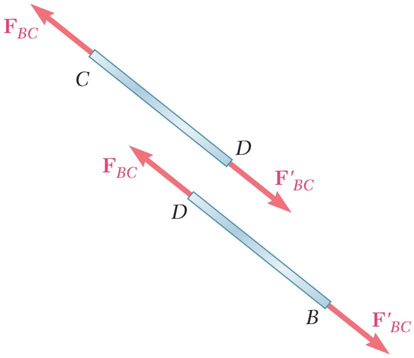


Fig. 8.20

Knowing the forces at the ends of each of the members, we can now determine the internal forces in these members. Passing a section at some arbitrary point D of rod BC, we obtain two portions BD and CD (Fig. 8.20). Since 50-kN forces must be applied at D to both portions of the rod to keep them in equilibrium, we conclude that an internal force of 50 kN is produced in rod BC when a 30-kN load is applied at B. We further check from the directions of the forces $\mathbf{F}_{BC}$ and $\mathbf{F}'_{BC}$ in Fig. 8.20 that the rod is in tension. A similar procedure would enable us to determine that the internal force in boom AB is 40 kN and that the boom is in compression.

We now determine the stresses in the members and connections. As shown in Fig. 8.21, the 20-mm-diameter rod BC has flat ends of 20×40-mm-rectangular cross section, while boom AB has a 30×50-mm rectangular cross section and is fitted with a clevis at end B. Both members are connected at B by a pin from which the 30-kN load is suspended by means of a U-shaped bracket. Boom AB is supported at A by a pin fitted into a double bracket, while rod BC is connected at C to a single bracket. All pins are 25 mm in diameter.

a. Determination of the Normal Stress in Boom *AB* and Rod *BC*. The force in rod BC is $F_{BC} = 50$ kN (tension). Recalling that the diameter of the rod is 20 mm, we use Eq. (8.1) to determine the stress created in the rod by the given loading. We have

$$P = F_{BC} = +50 \text{ kN} = +50 \times 10^3 \text{ N}$$

$$A = \pi r^2 = \pi\left(\frac{20 \text{ mm}}{2}\right)^2 = \pi(10 \times 10^{-3} \text{ m})^2 = 314 \times 10^{-6} \text{ m}^2$$

$$\sigma_{BC} = \frac{P}{A} = \frac{+50 \times 10^3 \text{ N}}{314 \times 10^{-6} \text{ m}^2} = +159 \times 10^6 \text{ Pa} = +159 \text{ MPa}$$

However, the flat parts of the rod are also under tension and at the narrowest section, where a hole is located, we have

$$A = (20 \text{ mm})(40 \text{ mm} - 25 \text{ mm}) = 300 \times 10^{-6} \text{ m}^2$$

The corresponding average value of the stress, therefore, is

$$(\sigma_{BC})_{\text{end}} = \frac{P}{A} = \frac{50 \times 10^3 \text{ N}}{300 \times 10^{-6} \text{ m}^2} = 167 \text{ MPa}$$

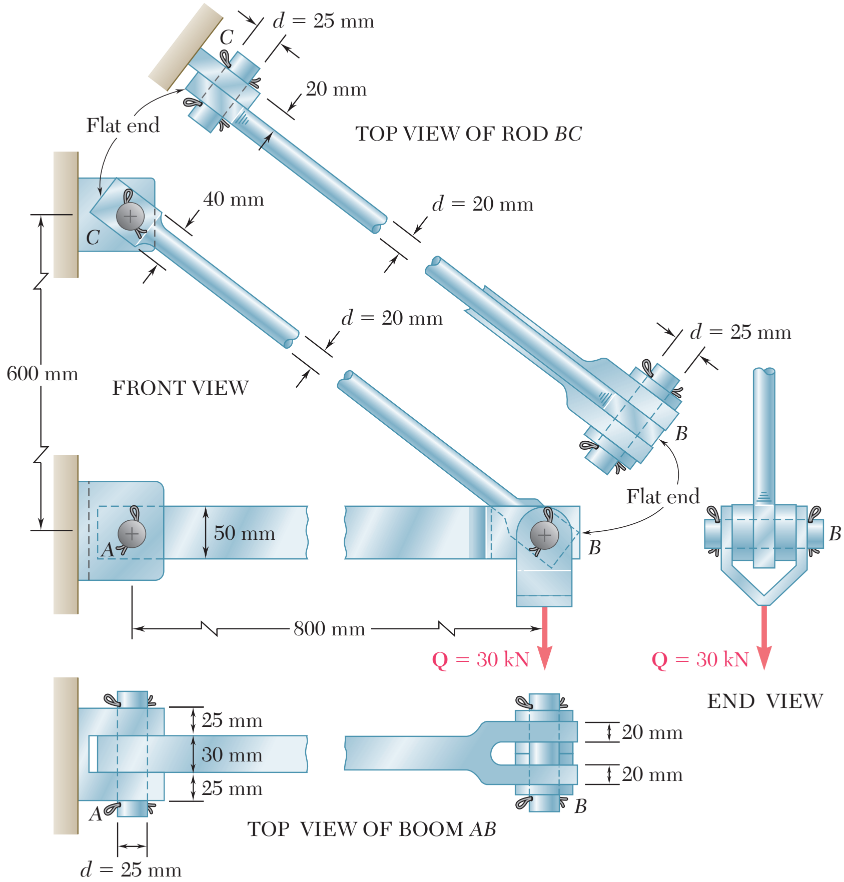


Fig. 8.21

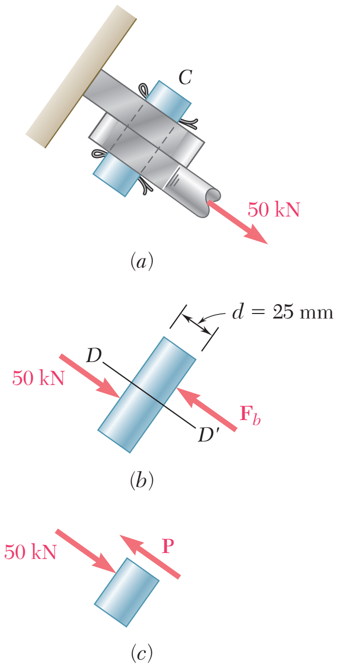


Fig. 8.22

Note that this is an *average value;* close to the hole, the stress will actually reach a much larger value, as you will see in Sec. 9.15. It is clear that, under an increasing load, the rod will fail near one of the holes rather than in its cylindrical portion; its design, therefore, could be improved by increasing the width or the thickness of the flat ends of the rod.

Turning now our attention to boom *AB*, we recall that the force in the boom is $F_{AB} = 40$ kN (compression). Since the area of the boom's rectangular cross section is $A = 30 \text{ mm} \times 50 \text{ mm} = 1.5 \times 10^{-3} \text{ m}^2$, the average value of the normal stress in the main part of the rod, between pins *A* and *B*, is

$$\sigma_{AB} = -\frac{40 \times 10^3 \text{ N}}{1.5 \times 10^{-3} \text{ m}^2} = -26.7 \times 10^6 \text{ Pa} = -26.7 \text{ MPa}$$

Note that the sections of minimum area at *A* and *B* are not under stress, since the boom is in compression, and, therefore, *pushes* on the pins (instead of *pulling* on the pins as rod *BC* does).

b. Determination of the Shearing Stress in Various Connections. To determine the shearing stress in a connection such as a bolt, pin, or rivet, we first clearly show the forces exerted by the various members it connects. Thus, in the case of pin *C* of our example (Fig. 8.22*a*), we draw Fig. 8.22*b*, showing the 50-kN force exerted

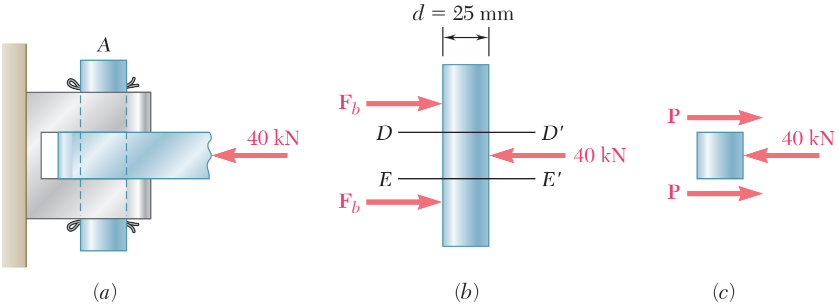


Fig. 8.23

by member BC on the pin, and the equal and opposite force exerted by the bracket. Drawing now the diagram of the portion of the pin located below the plane DD' where shearing stresses occur (Fig. 8.22c), we conclude that the shear in that plane is $P = 50$ kN. Since the cross-sectional area of the pin is

$$A = \pi r^2 = \pi\left(\frac{25 \text{ mm}}{2}\right)^2 = \pi(12.5 \times 10^{-3} \text{ m})^2 = 491 \times 10^{-6} \text{ m}^2$$

we find that the average value of the shearing stress in the pin at C is

$$\tau_{\text{ave}} = \frac{P}{A} = \frac{50 \times 10^3 \text{ N}}{491 \times 10^{-6} \text{ m}^2} = 102 \text{ MPa}$$

Considering now the pin at A (Fig. 8.23), we note that it is in double shear. Drawing the free-body diagrams of the pin and of the portion of pin located between the planes DD' and EE' where shearing stresses occur, we conclude that $P = 20$ kN and that

$$\tau_{\text{ave}} = \frac{P}{A} = \frac{20 \text{ kN}}{491 \times 10^{-6} \text{ m}^2} = 40.7 \text{ MPa}$$

Considering the pin at B (Fig. 8.24a), we note that the pin may be divided into five portions which are acted upon by forces exerted by the boom, rod, and bracket. Considering successively the portions DE (Fig. 8.24b) and DG (Fig. 8.24c), we conclude that the shear in section E is $P_E = 15$ kN, while the shear in section G is $P_G = 25$ kN. Since the loading of the pin is symmetric, we conclude that the maximum value of the shear in pin B is $P_G = 25$ kN, and that the largest shearing stresses occur in sections G and H, where

$$\tau_{\text{ave}} = \frac{P_G}{A} = \frac{25 \text{ kN}}{491 \times 10^{-6} \text{ m}^2} = 50.9 \text{ MPa}$$

c. Determination of the Bearing Stresses. To determine the nominal bearing stress at A in member AB, we use formula (8.7) of Sec. 8.5. From Fig. 8.21, we have $t = 30$ mm and $d = 25$ mm. Recalling that $P = F_{AB} = 40$ kN, we have

$$\sigma_b = \frac{P}{td} = \frac{40 \text{ kN}}{(30 \text{ mm})(25 \text{ mm})} = 53.3 \text{ MPa}$$

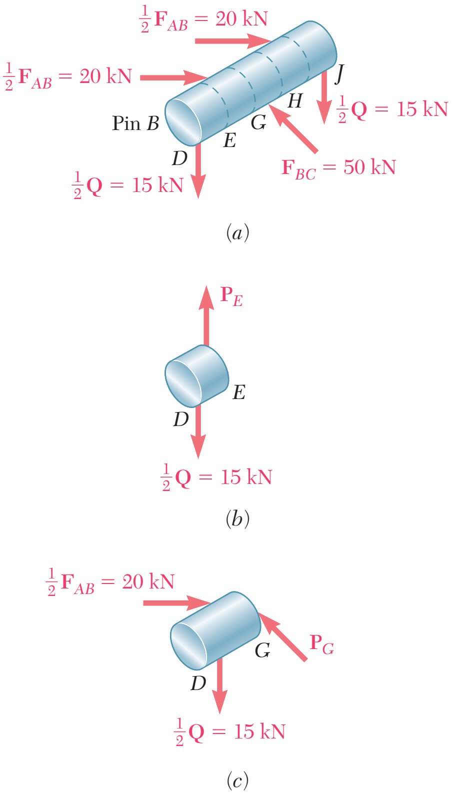


Fig. 8.24

To obtain the bearing stress in the bracket at A, we use $t = 2(25 \text{ mm}) = 50 \text{ mm}$ and $d = 25 \text{ mm}$:

$$\sigma_b = \frac{P}{td} = \frac{40 \text{ kN}}{(50 \text{ mm})(25 \text{ mm})} = 32.0 \text{ MPa}$$

The bearing stresses at B in member AB, at B and C in member BC, and in the bracket at C are found in a similar way.

8.7 DESIGN

Considering again the structure of Fig. 8.15, let us assume that rod BC is made of a steel with a maximum allowable stress $\sigma_{\text{all}} = 165$ MPa. Can rod BC safely support the load to which it will be subjected? The magnitude of the force F_{BC} in the rod was found earlier to be 50 kN and the stress σ_{BC} was found to be 159 MPa. Since the value obtained is smaller than the value σ_{all} of the allowable stress in the steel used, we conclude that rod BC can safely support the load to which it will be subjected. We should also determine whether the deformations produced by the given loading are acceptable. The study of deformations under axial loads will be the subject of Chap. 9. An additional consideration required for members in compression involves the *stability* of the member, i.e., its ability to support a given load without experiencing a sudden change in configuration. This will be discussed in Chap. 16.

The engineer's role is not limited to the analysis of existing structures and machines subjected to given loading conditions. Of even greater importance to the engineer is the *design* of new structures and machines, that is, the selection of appropriate components to perform a given task. As an example of design, let us return to the structure of Fig. 8.15, and assume that aluminum with an allowable stress $\sigma_{\text{all}} = 100$ MPa is to be used. Since the force in rod BC will still be $P = F_{BC} = 50$ kN under the given loading, we must have, from Eq. (8.1),

$$\sigma_{\text{all}} = \frac{P}{A} \qquad A = \frac{P}{\sigma_{\text{all}}} = \frac{50 \times 10^3 \text{ N}}{100 \times 10^6 \text{ Pa}} = 500 \times 10^{-6} \text{ m}^2$$

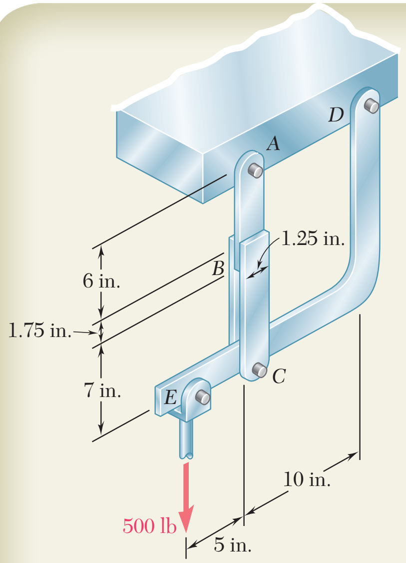
and, since $A = \pi r^2$,

$$r = \sqrt{\frac{A}{\pi}} = \sqrt{\frac{500 \times 10^{-6} \text{ m}^2}{\pi}} = 12.62 \times 10^{-3} \text{ m} = 12.62 \text{ mm}$$

$$d = 2r = 25.2 \text{ mm}$$

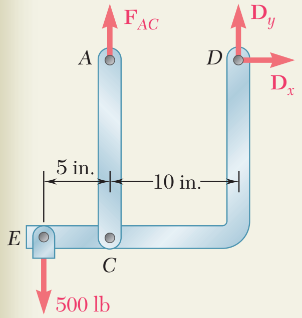
We conclude that an aluminum rod 26 mm or more in diameter will be adequate.

SAMPLE PROBLEM 8.1



In the hanger shown, the upper portion of link ABC is $\frac{3}{8}$ in. thick and the lower portions are each $\frac{1}{4}$ in. thick. Epoxy resin is used to bond the upper and lower portions together at B. The pin at A is of $\frac{3}{8}$-in. diameter while a $\frac{1}{4}$-in.-diameter pin is used at C. Determine (a) the shearing stress in pin A, (b) the shearing stress in pin C, (c) the largest normal stress in link ABC, (d) the average shearing stress on the bonded surfaces at B, (e) the bearing stress in the link at C.

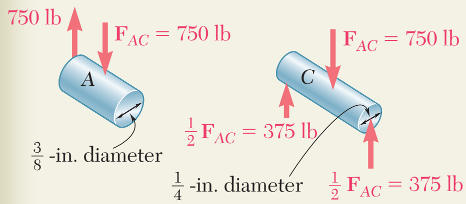
SOLUTION



Free Body: Entire Hanger. Since the link ABC is a two-force member, the reaction at A is vertical; the reaction at D is represented by its components $\mathbf{D}_x$ and $\mathbf{D}_y$. We write

$$+\circlearrowleft \Sigma M_D = 0: \qquad (500 \text{ lb})(15 \text{ in.}) - F_{AC}(10 \text{ in.}) = 0$$

$$F_{AC} = +750 \text{ lb} \qquad F_{AC} = 750 \text{ lb} \quad \textit{tension}$$

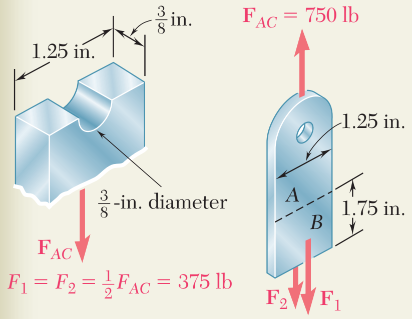


a. Shearing Stress in Pin *A*. Since this $\frac{3}{8}$-in.-diameter pin is in single shear, we write

$$\tau_A = \frac{F_{AC}}{A} = \frac{750 \text{ lb}}{\frac{1}{4}\pi(0.375 \text{ in.})^2} \qquad \tau_A = 6790 \text{ psi} \blacktriangleleft$$

b. Shearing Stress in Pin *C*. Since this $\frac{1}{4}$-in.-diameter pin is in double shear, we write

$$\tau_C = \frac{\frac{1}{2}F_{AC}}{A} = \frac{375 \text{ lb}}{\frac{1}{4}\pi(0.25 \text{ in.})^2} \qquad \tau_C = 7640 \text{ psi} \blacktriangleleft$$

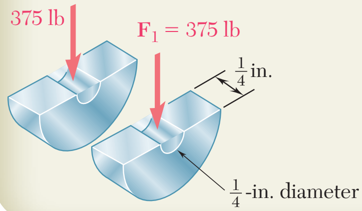


c. Largest Normal Stress in Link *ABC*. The largest stress is found where the area is smallest; this occurs at the cross section at A where the $\frac{3}{8}$-in. hole is located. We have

$$\sigma_A = \frac{F_{AC}}{A_{\text{net}}} = \frac{750 \text{ lb}}{(\frac{3}{8} \text{ in.})(1.25 \text{ in.} - 0.375 \text{ in.})} = \frac{750 \text{ lb}}{0.328 \text{ in}^2} \qquad \sigma_A = 2290 \text{ psi} \blacktriangleleft$$

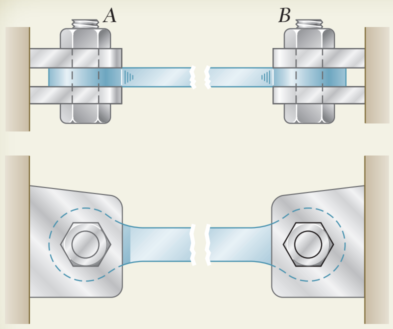
d. Average Shearing Stress at *B*. We note that bonding exists on both sides of the upper portion of the link and that the shear force on each side is $F_1 = (750 \text{ lb})/2 = 375$ lb. The average shearing stress on each surface is thus

$$\tau_B = \frac{F_1}{A} = \frac{375 \text{ lb}}{(1.25 \text{ in.})(1.75 \text{ in.})} \qquad \tau_B = 171.4 \text{ psi} \blacktriangleleft$$



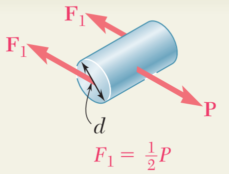
e. Bearing Stress in Link at *C*. For each portion of the link, $F_1 = 375$ lb and the nominal bearing area is $(0.25 \text{ in.})(0.25 \text{ in.}) = 0.0625 \text{ in}^2$.

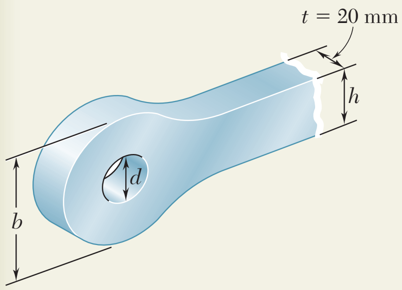
$$\sigma_b = \frac{F_1}{A} = \frac{375 \text{ lb}}{0.0625 \text{ in}^2} \qquad \sigma_b = 6000 \text{ psi} \blacktriangleleft$$

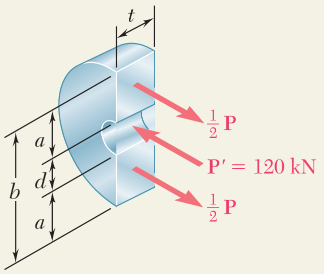


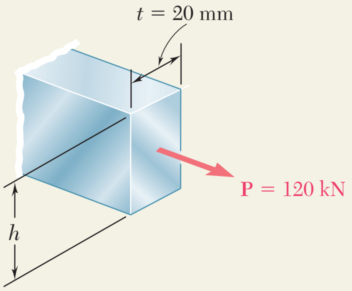
SAMPLE PROBLEM 8.2

The steel tie bar shown is to be designed to carry a tension force of magnitude $P = 120$ kN when bolted between double brackets at A and B. The bar will be fabricated from 20-mm-thick plate stock. For the grade of steel to be used, the maximum allowable stresses are: $\sigma = 175$ MPa, $\tau = 100$ MPa, $\sigma_b = 350$ MPa. Design the tie bar by determining the required values of (a) the diameter d of the bolt, (b) the dimension b at each end of the bar, (c) the dimension h of the bar.









SOLUTION

a. Diameter of the Bolt. Since the bolt is in double shear, $F_1 = \frac{1}{2}P = 60$ kN.

$$\tau = \frac{F_1}{A} = \frac{60 \text{ kN}}{\frac{1}{4}\pi\, d^2} \qquad 100 \text{ MPa} = \frac{60 \text{ kN}}{\frac{1}{4}\pi\, d^2} \qquad d = 27.6 \text{ mm}$$

We will use $d = 28$ mm ◀

At this point we check the bearing stress between the 20-mm-thick plate and the 28-mm-diameter bolt.

$$\tau_b = \frac{P}{td} = \frac{120 \text{ kN}}{(0.020 \text{ m})(0.028 \text{ m})} = 214 \text{ MPa} < 350 \text{ MPa} \qquad \text{OK}$$

b. Dimension b at Each End of the Bar. We consider one of the end portions of the bar. Recalling that the thickness of the steel plate is $t = 20$ mm and that the average tensile stress must not exceed 175 MPa, we write

$$\sigma = \frac{\frac{1}{2}P}{ta} \qquad 175 \text{ MPa} = \frac{60 \text{ kN}}{(0.02 \text{ m})a} \qquad a = 17.14 \text{ mm}$$

$$b = d + 2a = 28 \text{ mm} + 2(17.14 \text{ mm}) \qquad b = 62.3 \text{ mm} ◀$$

c. Dimension h of the Bar. Recalling that the thickness of the steel plate is $t = 20$ mm, we have

$$\sigma = \frac{P}{th} \qquad 175 \text{ MPa} = \frac{120 \text{ kN}}{(0.020 \text{ m})h} \qquad h = 34.3 \text{ mm}$$

We will use $h = 35$ mm ◀

PROBLEMS

8.1 Two solid cylindrical rods AB and BC are welded together at B and loaded as shown. Knowing that $d_1 = 50$ mm and $d_2 = 30$ mm, find the average normal stress at the midsection of (a) rod AB, (b) rod BC.

8.2 Two solid cylindrical rods AB and BC are welded together at B and loaded as shown. Knowing that the average normal stress must not exceed 140 MPa in either rod, determine the smallest allowable values of d_1 and d_2.

8.3 Two solid cylindrical rods AB and BC are welded together at B and loaded as shown. Determine the average normal stress at the midsection of (a) rod AB, (b) rod BC.

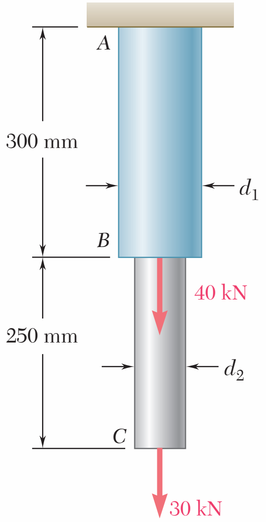


Fig. P8.1 and P8.2

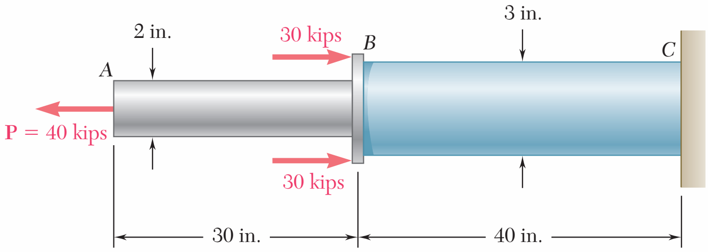


Fig. P8.3

8.4 In Prob. 8.3, determine the magnitude of the force **P** for which the tensile stress in rod AB has the same magnitude as the compressive stress in rod BC.

8.5 Link BD consists of a single bar 30 mm wide and 12 mm thick. Knowing that each pin has a 10-mm diameter, determine the maximum value of the average normal stress in link BD if (a) $\theta = 0°$, (b) $\theta = 90°$.

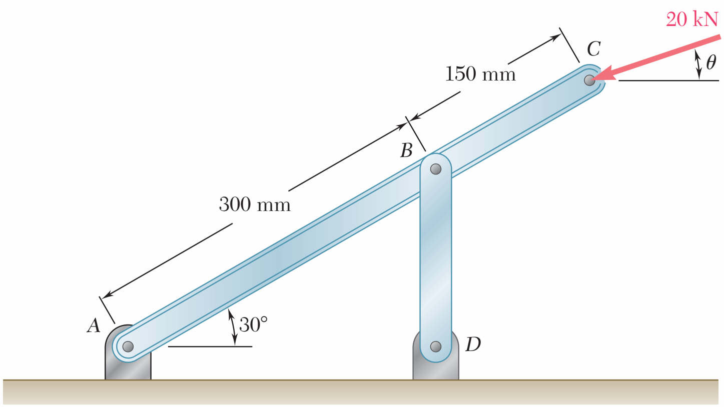


Fig. *P8.5*

8.6 Knowing that the central portion of the link BD has a uniform cross-sectional area of 800 mm^2, determine the magnitude of the load **P** for which the normal stress in that portion of BD is 50 MPa.

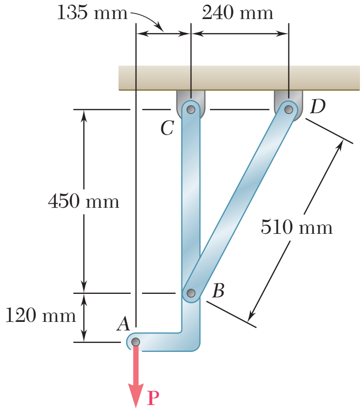


Fig. P8.6

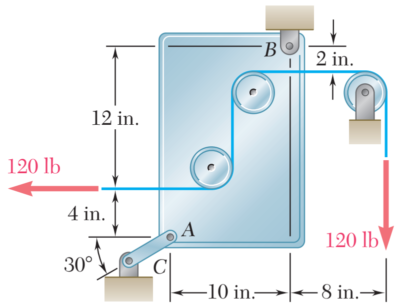


Fig. P8.7

8.7 Link AC has a uniform rectangular cross section $\frac{1}{8}$ in. thick and 1 in. wide. Determine the normal stress in the central portion of the link.

8.8 Two horizontal 5-kip forces are applied to pin B of the assembly shown. Knowing that a pin of 0.8-in. diameter is used at each connection, determine the maximum value of the average normal stress (*a*) in link AB, (*b*) in link BC.

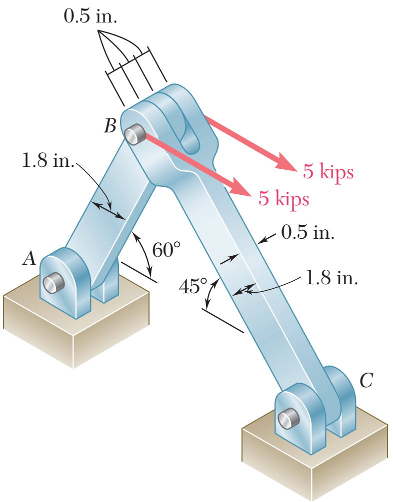


Fig. *P8.8*

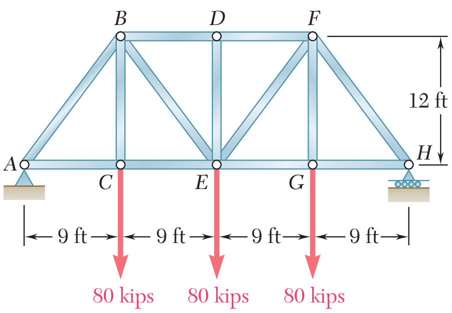


Fig. P8.9 and P8.10

8.9 For the Pratt bridge truss and loading shown, determine the average normal stress in member BE, knowing that the cross-sectional area of that member is 5.87 in^2.

8.10 Knowing that the average normal stress in member CE of the Pratt bridge truss shown must not exceed 21 ksi for the given loading, determine the cross-sectional area of the member that will yield the most economical and safe design. Assume that both ends of the member will be adequately reinforced.

8.11 A couple **M** of magnitude 1500 N · m is applied to the crank of an engine. For the position shown, determine (*a*) the force **P** required to hold the engine system in equilibrium, (*b*) the average normal stress in the connecting rod *BC*, which has a 450-mm^2 uniform cross section.

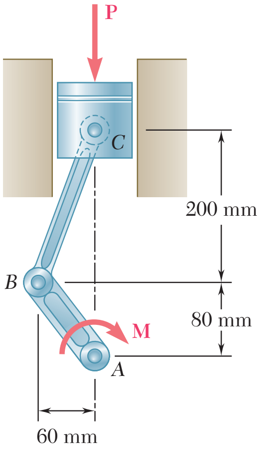


Fig. P8.11

8.12 Two hydraulic cylinders are used to control the position of the robotic arm *ABC*. Knowing that the control rods attached at *A* and *D* each have a 20-mm diameter and happen to be parallel in the position shown, determine the average normal stress in (*a*) member *AE*, (*b*) member *DG*.

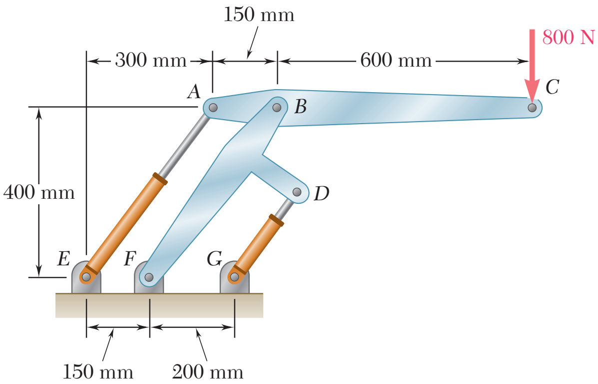


Fig. P8.12

8.13 The wooden members *A* and *B* are to be joined by plywood splice plates that will be fully glued on the surfaces in contact. As part of the design of the joint, and knowing that the clearance between the ends of the members is to be 8 mm, determine the smallest allowable length *L* if the average shearing stress in the glue is not to exceed 800 kPa.

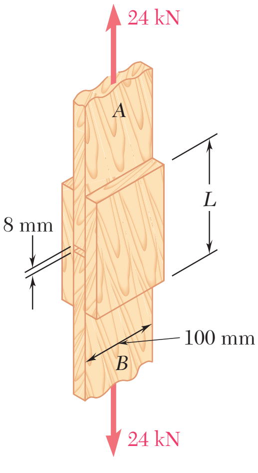


Fig. *P8.13*

8.14 Determine the diameter of the largest circular hole that can be punched into a sheet of polystyrene 6 mm thick, knowing that the force exerted by the punch is 45 kN and that a 55-MPa average shearing stress is required to cause the material to fail.

8.15 Two wooden planks, each $\frac{7}{8}$ in. thick and 6 in. wide, are joined by the glued mortise joint shown. Knowing that the joint will fail when the average shearing stress in the glue reaches 120 psi, determine the smallest allowable length *d* of the cuts if the joint is to withstand an axial load of magnitude $P = 1200$ lb.

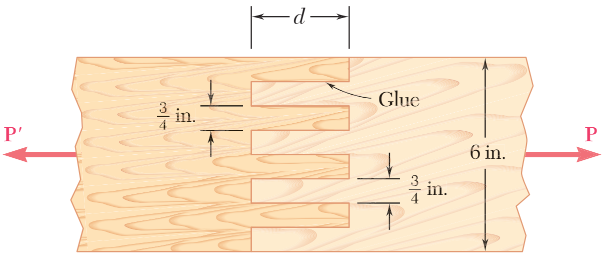


Fig. *P8.15*

8.16 A load **P** is applied to a steel rod supported as shown by an aluminum plate into which a 0.6-in.-diameter hole has been drilled. Knowing that the shearing stress must not exceed 18 ksi in the steel rod and 10 ksi in the aluminum plate, determine the largest load **P** that can be applied to the rod.

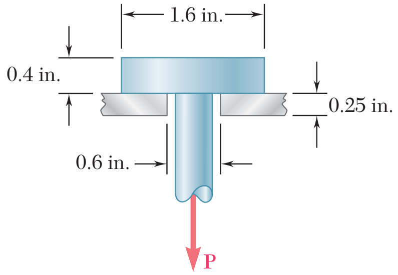


Fig. P8.16

8.17 An axial load **P** is supported by a short W250 × 67 column of cross-sectional area $A = 8580 \text{ mm}^2$ and is distributed to a concrete foundation by a square plate as shown. Knowing that the average normal stress in the column must not exceed 150 MPa and that the bearing stress on the concrete foundation must not exceed 12.5 MPa, determine the side a of the plate that will provide the most economical and safe design.

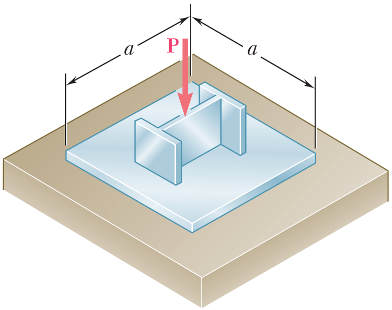


Fig. P8.17

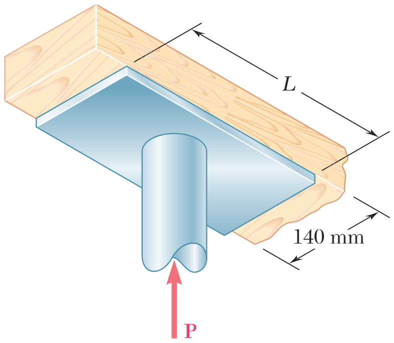


Fig. P8.18

8.18 The axial force in the column supporting the timber beam shown is $P = 75$ kN. Determine the smallest allowable length L of the bearing plate if the bearing stress in the timber is not to exceed 3.0 MPa.

8.19 Three wooden planks are fastened together by a series of bolts to form a column. The diameter of each bolt is $\frac{1}{2}$ in. and the inner diameter of each washer is $\frac{5}{8}$ in., which is slightly larger than the diameter of the holes in the planks. Determine the smallest allowable outer diameter d of the washers, knowing that the average normal stress in the bolts is 5 ksi and that the bearing stress between the washers and the planks must not exceed 1.2 ksi.

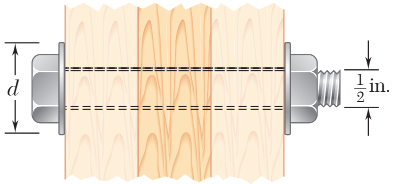


Fig. *P8.19*

8.20 Link AB, of width $b = 2$ in. and thickness $t = \frac{1}{4}$ in., is used to support the end of a horizontal beam. Knowing that the average normal stress in the link is -20 ksi and that the average shearing stress in each of the two pins is 12 ksi, determine (*a*) the diameter d of the pins, (*b*) the average bearing stress in the link.

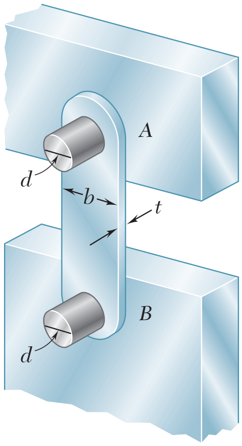


Fig. P8.20

8.21 For the assembly and loading of Prob. 8.8, determine (*a*) the average shearing stress in the pin at A, (*b*) the average bearing stress at A in member AB.

8.22 The hydraulic cylinder CF, which partially controls the position of rod DE, has been locked in the position shown. Member BD is $\frac{5}{8}$ in. thick and is connected to the vertical rod by a $\frac{3}{8}$-in.-diameter bolt. Determine (*a*) the average shearing stress in the bolt, (*b*) the bearing stress at C in member BD.

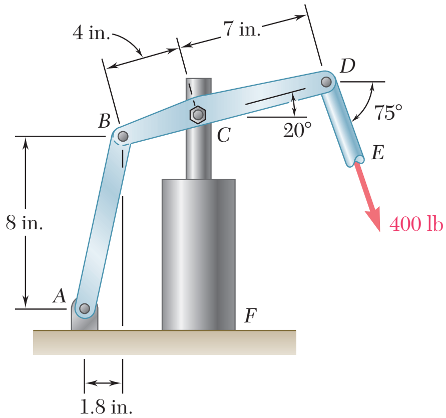


Fig. P8.22

8.23 Knowing that $\theta = 40°$ and $P = 9$ kN, determine (*a*) the smallest allowable diameter of the pin at B if the average shearing stress in the pin is to not exceed 120 MPa, (*b*) the corresponding average bearing stress in member AB at B, (*c*) the corresponding average bearing stress in each of the support brackets at B.

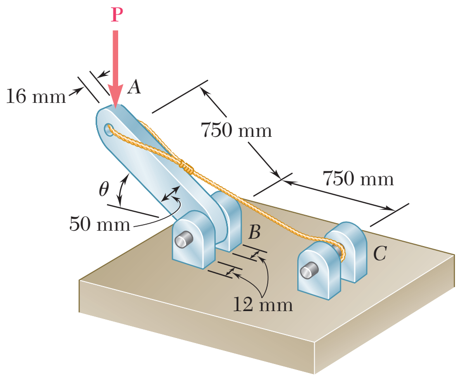


Fig. *P8.23* and P8.24

8.24 Determine the largest load **P** that can be applied at A when $\theta = 60°$, knowing that the average shearing stress in the 10-mm-diameter pin at B must not exceed 120 MPa and that the average bearing stress in member AB and in the bracket at B must not exceed 90 MPa.

8.8 STRESS ON AN OBLIQUE PLANE UNDER AXIAL LOADING

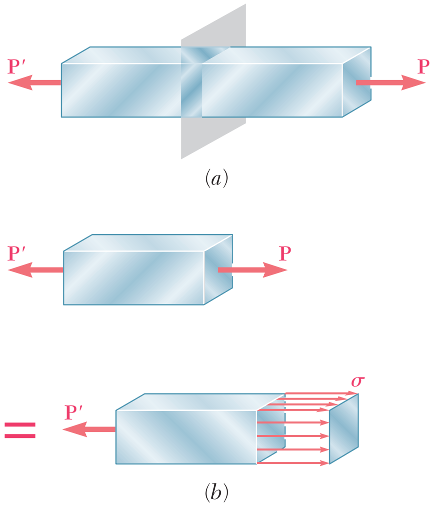


Fig. 8.25

In the preceding sections, axial forces exerted on a two-force member (Fig. 8.25*a*) were found to cause normal stresses in that member (Fig. 8.25*b*), while transverse forces exerted on bolts and pins (Fig 8.26*a*) were found to cause shearing stresses in those connections (Fig. 8.26*b*). The reason such a relation was observed between axial forces and normal stresses on the one hand and transverse forces and shearing stresses on the other was because stresses were being determined only on planes perpendicular to the axis of the member or connection. As you will see in this section, axial forces cause both normal and shearing stresses on planes which are not perpendicular to the axis of the member. Similarly, transverse forces exerted on a bolt or a pin cause both normal and shearing stresses on planes which are not perpendicular to the axis of the bolt or pin.

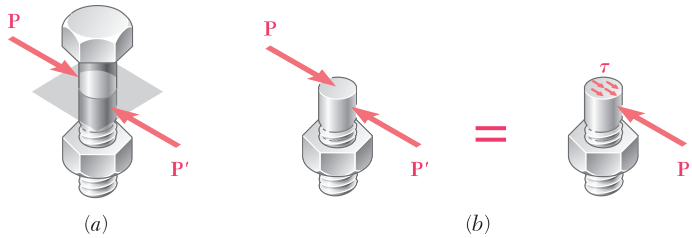


Fig. 8.26

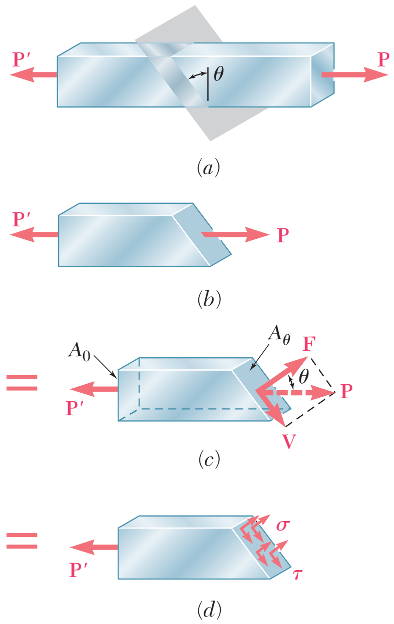


Fig. 8.27

Consider the two-force member of Fig. 8.25, which is subjected to axial forces **P** and **P′**. If we pass a section forming an angle θ with a normal plane (Fig. 8.27*a*) and draw the free-body diagram of the portion of member located to the left of that section (Fig. 8.27*b*), we find from the equilibrium conditions of the free body that the distributed forces acting on the section must be equivalent to the force **P**.

Resolving **P** into components **F** and **V**, respectively normal and tangential to the section (Fig. 8.27*c*), we have

$$F = P\cos\theta \qquad V = P\sin\theta \tag{8.12}$$

The force **F** represents the resultant of normal forces distributed over the section, and the force **V** the resultant of shearing forces (Fig. 8.27*d*). The average values of the corresponding normal and shearing stresses are obtained by dividing, respectively, F and V by the area A_θ of the section:

$$\sigma = \frac{F}{A_\theta} \qquad \tau = \frac{V}{A_\theta} \tag{8.13}$$

Substituting for F and V from (8.12) into (8.13), and observing from Fig. 8.27*c* that $A_0 = A_\theta \cos\theta$, or $A_\theta = A_0/\cos\theta$, where A_0

denotes the area of a section perpendicular to the axis of the member, we obtain

$$\sigma = \frac{P\cos\theta}{A_0/\cos\theta} \qquad \tau = \frac{P\sin\theta}{A_0/\cos\theta}$$

or

$$\sigma = \frac{P}{A_0}\cos^2\theta \qquad \tau = \frac{P}{A_0}\sin\theta\cos\theta \tag{8.14}$$

We note from the first of Eqs. (8.14) that the normal stress σ is maximum when $\theta = 0°$, i.e., when the plane of the section is perpendicular to the axis of the member, and that it approaches zero as θ approaches 90°. We check that the value of σ when $\theta = 0°$ is

$$\sigma_m = \frac{P}{A_0} \tag{8.15}$$

as we found earlier in Sec. 8.2. The second of Eqs. (8.14) shows that the shearing stress τ is zero for $\theta = 0°$ and $\theta = 90°$, and that for $\theta = 45°$ it reaches its maximum value

$$\tau_m = \frac{P}{A_0}\sin 45°\cos 45° = \frac{P}{2A_0} \tag{8.16}$$

The first of Eqs. (8.14) indicates that, when $\theta = 45°$, the normal stress σ' is also equal to $P/2A_0$:

$$\sigma' = \frac{P}{A_0}\cos^2 45° = \frac{P}{2A_0} \tag{8.17}$$

The results obtained in Eqs. (8.15), (8.16), and (8.17) are shown graphically in Fig. 8.28. We note that the same loading may produce either a normal stress $\sigma_m = P/A_0$ and no shearing stress (Fig. 8.28*b*), or a normal and a shearing stress of the same magnitude $\sigma' = \tau_m = P/2A_0$ (Fig. 8.28 *c* and *d*), depending upon the orientation of the section.

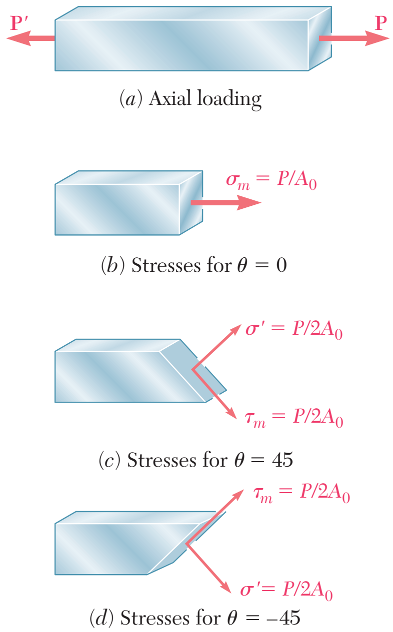


Fig. 8.28

8.9 STRESS UNDER GENERAL LOADING CONDITIONS. COMPONENTS OF STRESS

The examples of the previous sections were limited to members under axial loading and connections under transverse loading. Most structural members and machine components are under more involved loading conditions.

Consider a body subjected to several loads $\mathbf{P}_1$, $\mathbf{P}_2$, etc. (Fig. 8.29). To understand the stress condition created by these loads at some point Q within the body, we shall first pass a section through Q, using a plane parallel to the yz plane. The portion of the body to the left of the section is subjected to some of the original loads and to normal and shearing forces distributed over the section. We shall denote by $\Delta\mathbf{F}^x$ and $\Delta\mathbf{V}^x$, respectively, the normal and the shearing

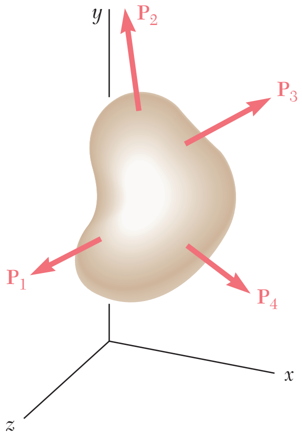


Fig. 8.29

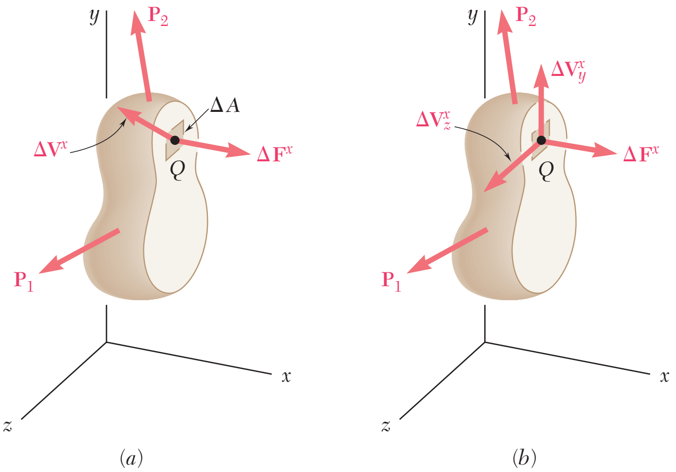


Fig. 8.30

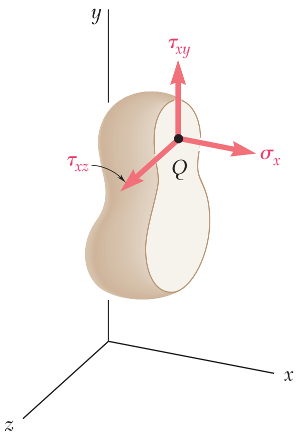


Fig. 8.31

forces acting on a small area ΔA surrounding point Q (Fig. 8.30*a*). Note that the superscript x is used to indicate that the forces $\Delta \mathbf{F}^x$ and $\Delta \mathbf{V}^x$ act on a surface perpendicular to the x axis. While the normal force $\Delta \mathbf{F}^x$ has a well-defined direction, the shearing force $\Delta \mathbf{V}^x$ may have any direction in the plane of the section. We therefore resolve $\Delta \mathbf{V}^x$ into two component forces, $\Delta \mathbf{V}_y^x$ and $\Delta \mathbf{V}_z^x$, in directions parallel to the y and z axes, respectively (Fig. 8.30*b*). Dividing now the magnitude of each force by the area ΔA, and letting ΔA approach zero, we define the three stress components shown in Fig. 8.31:

$$\sigma_x = \lim_{\Delta A \to 0} \frac{\Delta F^x}{\Delta A} \qquad \tau_{xy} = \lim_{\Delta A \to 0} \frac{\Delta V_y^x}{\Delta A} \qquad \tau_{xz} = \lim_{\Delta A \to 0} \frac{\Delta V_z^x}{\Delta A} \tag{8.18}$$

We note that the first subscript in σ_x, τ_{xy}, and τ_{xz} is used to indicate that the stresses under consideration are exerted *on a surface perpendicular to the x axis*. The second subscript in τ_{xy} and τ_{xz} identifies *the direction of the component*. The normal stress σ_x is positive if the corresponding arrow points in the positive x direction, i.e., if the body is in tension, and negative otherwise. Similarly, the shearing stress components τ_{xy} and τ_{xz} are positive if the corresponding arrows point, respectively, in the positive y and z directions.

The above analysis may also be carried out by considering the portion of body located to the right of the vertical plane through Q (Fig. 8.32). The same magnitudes, but opposite directions, are obtained for the normal and shearing forces $\Delta \mathbf{F}^x$, $\Delta \mathbf{V}_y^x$, and $\Delta \mathbf{V}_z^x$. Therefore, the same values are also obtained for the corresponding stress components, but since the section in Fig. 8.32 now faces the *negative x axis*, a positive sign for σ_x will indicate that the corresponding arrow points *in the negative x direction*. Similarly, positive signs for τ_{xy} and τ_{xz} will indicate that the corresponding arrows point, respectively, in the negative y and z directions, as shown in Fig. 8.32.

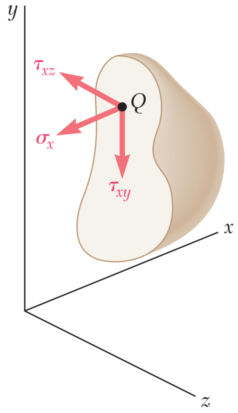


Fig. 8.32

Passing a section through Q parallel to the zx plane, we define in the same manner the stress components, σ_y, τ_{yz}, and τ_{yx}. Finally, a section through Q parallel to the xy plane yields the components σ_z, τ_{zx}, and τ_{zy}.

To facilitate the visualization of the stress condition at point Q, we shall consider a small cube of side a centered at Q and the stresses exerted on each of the six faces of the cube (Fig. 8.33). The stress components shown in the figure are σ_x, σ_y, and σ_z, which represent the normal stress on faces respectively perpendicular to the x, y, and z axes, and the six shearing stress components τ_{xy}, τ_{xz}, etc. We recall that, according to the definition of the shearing stress components, τ_{xy} represents the y component of the shearing stress exerted on the face perpendicular to the x axis, while τ_{yx} represents the x component of the shearing stress exerted on the face perpendicular to the y axis. Note that only three faces of the cube are actually visible in Fig. 8.33, and that equal and opposite stress components act on the hidden faces. While the stresses acting on the faces of the cube differ slightly from the stresses at Q, the error involved is small and vanishes as side a of the cube approaches zero.

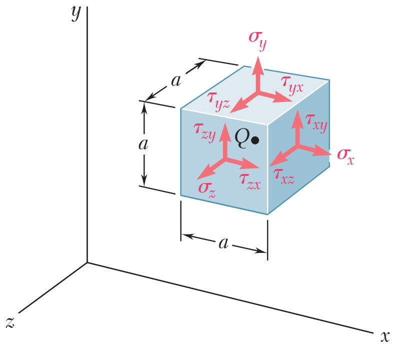


Fig. 8.33

Important relations among the shearing stress components will now be derived. Let us consider the free-body diagram of the small cube centered at point Q (Fig. 8.34). The normal and shearing forces acting on the various faces of the cube are obtained by multiplying the corresponding stress components by the area ΔA of each face. We first write the following three equilibrium equations:

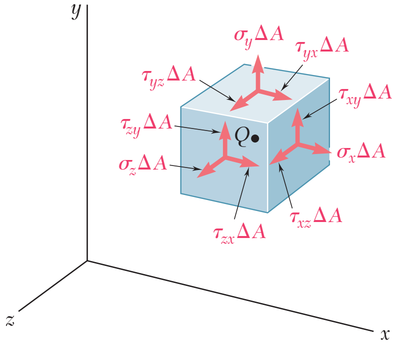


Fig. 8.34

$$\Sigma F_x = 0 \qquad \Sigma F_y = 0 \qquad \Sigma F_z = 0 \tag{8.19}$$

Since forces equal and opposite to the forces actually shown in Fig. 8.34 are acting on the hidden faces of the cube, it is clear that Eqs. (8.19) are satisfied. Considering now the moments of the forces about axes x', y', and z' drawn from Q in directions respectively parallel to the x, y, and z axes, we write the three additional equations

$$\Sigma M_{x'} = 0 \qquad \Sigma M_{y'} = 0 \qquad \Sigma M_{z'} = 0 \tag{8.20}$$

Using a projection on the $x'y'$ plane (Fig. 8.35), we note that the only forces with moments about the z axis different from zero are the shearing forces. These forces form two couples, one of counterclockwise (positive) moment $(\tau_{xy}\,\Delta A)a$, the other of clockwise (negative) moment $-(\tau_{xy}\,\Delta A)a$. The last of the three Eqs. (8.20) yields, therefore,

$$+\circlearrowleft \Sigma M_z = 0: \qquad (\tau_{xy}\,\Delta A)a - (\tau_{yx}\,\Delta A)a = 0$$

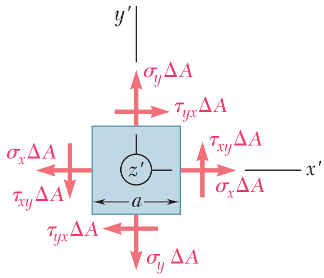


Fig. 8.35

from which we conclude that

$$\boxed{\tau_{xy} = \tau_{yx}} \tag{8.21}$$

The relation obtained shows that the y component of the shearing stress exerted on a face perpendicular to the x axis is equal to the

x component of the shearing stress exerted on a face perpendicular to the y axis. From the remaining two equations (8.20), we derive in a similar manner the relations

$$\tau_{yz} = \tau_{zy} \qquad \tau_{zx} = \tau_{xz} \tag{8.22}$$

We conclude from Eqs. (8.21) and (8.22) that only six stress components are required to define the condition of stress at a given point Q, instead of nine as originally assumed. These six components are σ_x, σ_y, σ_z, τ_{xy}, τ_{yz}, and τ_{zx}. We also note that, at a given point, *shear cannot take place in one plane only;* an equal shearing stress must be exerted on another plane perpendicular to the first one. For example, considering again the bolt of Fig. 8.26 and a small cube at the center Q of the bolt (Fig. 8.36*a*), we find that shearing stresses of equal magnitude must be exerted on the two horizontal faces of the cube and on the two faces that are perpendicular to the forces **P** and **P′** (Fig. 8.36*b*).

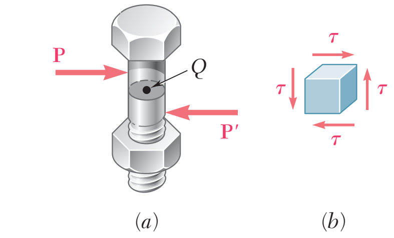


Fig. 8.36

Before concluding our discussion of stress components, let us consider again the case of a member under axial loading. If we consider a small cube with faces respectively parallel to the faces of the member and recall the results obtained in Sec. 8.8, we find that the conditions of stress in the member may be described as shown in Fig. 8.37*a*; the only stresses are normal stresses σ_x exerted on the faces of the cube which are perpendicular to the x axis. However, if the small cube is rotated by 45° about the z axis so that its new orientation matches the orientation of the sections considered in Fig. 8.28*c* and *d*, we conclude that normal and shearing stresses of equal magnitude are exerted on four faces of the cube (Fig. 8.37*b*). We thus observe that the same loading condition may lead to different interpretations of the stress situation at a given point, depending upon the orientation of the element considered. More will be said about this in Chap. 14.

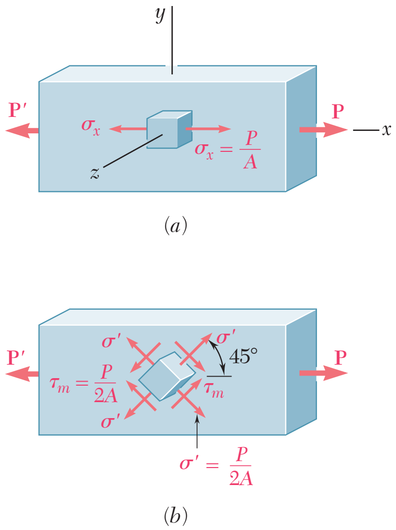


Fig. 8.37

8.10 DESIGN CONSIDERATIONS

In the preceding sections you learned to determine the stresses in rods, bolts, and pins under simple loading conditions. In later chapters you will learn to determine stresses in more complex situations. In engineering applications, however, the determination of stresses is seldom an end in itself. Rather, the knowledge of stresses is used by engineers to assist in their most important task, namely, the design of structures and machines that will safely and economically perform a specified function.

a. Determination of the Ultimate Strength of a Material. An important element to be considered by a designer is how the material that has been selected will behave under a load. For a given material, this is determined by performing specific tests on prepared samples of the material. For example, a test specimen of steel may be prepared and placed in a laboratory testing machine to be subjected to a known centric axial tensile force, as described in Sec. 9.3. As the magnitude of the force is increased, various changes in the specimen are measured, for example, changes in its length and its diameter.

Eventually the largest force which may be applied to the specimen is reached, and the specimen either breaks or begins to carry less load. This largest force is called the *ultimate load* for the test specimen and is denoted by P_U. Since the applied load is centric, we may divide the ultimate load by the original cross-sectional area of the rod to obtain the *ultimate normal stress* of the material used. This stress, also known as the *ultimate strength in tension* of the material, is

$$\sigma_U = \frac{P_U}{A} \tag{8.23}$$

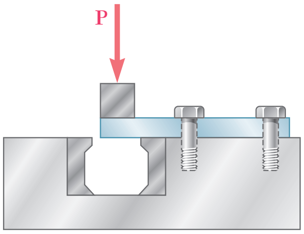


Fig. 8.38

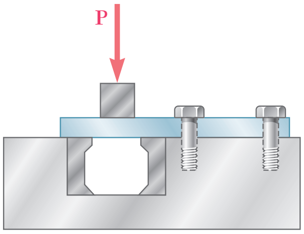


Fig. 8.39

Several test procedures are available to determine the *ultimate shearing stress*, or *ultimate strength in shear*, of a material. The one most commonly used involves the twisting of a circular tube (Sec. 10.5). A more direct, if less accurate, procedure consists in clamping a rectangular or round bar in a shear tool (Fig. 8.38) and applying an increasing load P until the ultimate load P_U for single shear is obtained. If the free end of the specimen rests on both of the hardened dies (Fig. 8.39), the ultimate load for double shear is obtained. In either case, the ultimate shearing stress τ_U is obtained by dividing the ultimate load by the total area over which shear has taken place. We recall that, in the case of single shear, this area is the cross-sectional area A of the specimen, while in double shear it is equal to twice the cross-sectional area.

b. Allowable Load and Allowable Stress. Factor of Safety. The maximum load that a structural member or a machine component will be allowed to carry under normal conditions of utilization is considerably smaller than the ultimate load. This smaller load is referred to as the *allowable load* and, sometimes, as the *working load* or *design load*. Thus, only a fraction of the ultimate-load capacity of the member is utilized when the allowable load is applied. The remaining portion of the load-carrying capacity of the member is kept in reserve to assure its safe performance. The ratio of the ultimate load to the allowable load is used to define the *factor of safety*.† We have

$$\text{Factor of safety} = F.S. = \frac{\text{ultimate load}}{\text{allowable load}} \tag{8.24}$$

An alternative definition of the factor of safety is based on the use of stresses:

$$\text{Factor of safety} = F.S. = \frac{\text{ultimate stress}}{\text{allowable stress}} \tag{8.25}$$

The two expressions given for the factor of safety in Eqs. (8.24) and (8.25) are identical when a linear relationship exists between the load and the stress. In most engineering applications, however, this relationship ceases to be linear as the load approaches its ultimate value, and the factor of safety obtained from Eq. (8.25) does not provide a

†In some fields of engineering, notably aeronautical engineering, the *margin of safety* is used in place of the factor of safety. The margin of safety is defined as the factor of safety minus one; that is, margin of safety = $F.S. - 1.00$.

true assessment of the safety of a given design. Nevertheless, the *allowable-stress method* of design, based on the use of Eq. (8.25), is widely used.

c. Selection of an Appropriate Factor of Safety. The selection of the factor of safety to be used for various applications is one of the most important engineering tasks. On the one hand, if a factor of safety is chosen too small, the possibility of failure becomes unacceptably large; on the other hand, if a factor of safety is chosen unnecessarily large, the result is an uneconomical or nonfunctional design. The choice of the factor of safety that is appropriate for a given design application requires engineering judgment based on many considerations, such as the following:

1. *Variations that may occur in the properties of the member under consideration*. The composition, strength, and dimensions of the member are all subject to small variations during manufacture. In addition, material properties may be altered and residual stresses introduced through heating or deformation that may occur during manufacture, storage, transportation, or construction.
2. *The number of loadings that may be expected during the life of the structure or machine*. For most materials the ultimate stress decreases as the number of load applications is increased. This phenomenon is known as *fatigue* and, if ignored, may result in sudden failure (see Sec. 9.6).
3. *The type of loadings that are planned for in the design, or that may occur in the future*. Very few loadings are known with complete accuracy—most design loadings are engineering estimates. In addition, future alterations or changes in usage may introduce changes in the actual loading. Larger factors of safety are also required for dynamic, cyclic, or impulsive loadings.
4. *The type of failure that may occur*. Brittle materials fail suddenly, usually with no prior indication that collapse is imminent. On the other hand, ductile materials, such as structural steel, normally undergo a substantial deformation called *yielding* before failing, thus providing a warning that overloading exists. However, most buckling or stability failures are sudden, whether the material is brittle or not. When the possibility of sudden failure exists, a larger factor of safety should be used than when failure is preceded by obvious warning signs.
5. *Uncertainty due to methods of analysis*. All design methods are based on certain simplifying assumptions which result in calculated stresses being approximations of actual stresses.
6. *Deterioration that may occur in the future because of poor maintenance or because of unpreventable natural causes*. A larger factor of safety is necessary in locations where conditions such as corrosion and decay are difficult to control or even to discover.
7. *The importance of a given member to the integrity of the whole structure*. Bracing and secondary members may in many cases be designed with a factor of safety lower than that used for primary members.

In addition to the above considerations, there is the additional consideration concerning the risk to life and property that a failure would produce. Where a failure would produce no risk to life and only minimal risk to property, the use of a smaller factor of safety can be considered. Finally, there is the practical consideration that, unless a careful design with a nonexcessive factor of safety is used, a structure or machine might not perform its design function. For example, high factors of safety may have an unacceptable effect on the weight of an aircraft.

For the majority of structural and machine applications, factors of safety are specified by design specifications or building codes written by committees of experienced engineers working with professional societies, with industries, or with federal, state, or city agencies. Examples of such design specifications and building codes are

1. *Steel:* American Institute of Steel Construction, Specification for Structural Steel Buildings
2. *Concrete:* American Concrete Institute, Building Code Requirement for Structural Concrete
3. *Timber:* American Forest and Paper Association, National Design Specification for Wood Construction
4. *Highway bridges:* American Association of State Highway Officials, Standard Specifications for Highway Bridges

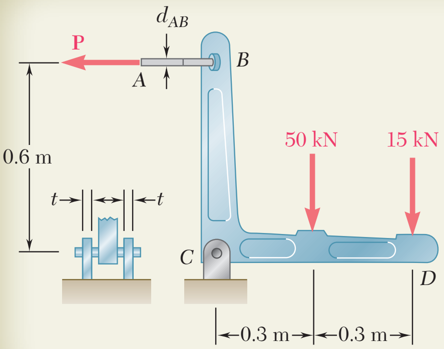
***d. Load and Resistance Factor Design.** As we saw above, the allowable-stress method requires that all the uncertainties associated with the design of a structure or machine element be grouped into a single factor of safety. An alternative method of design, which is gaining acceptance chiefly among structural engineers, makes it possible through the use of three different factors to distinguish between the uncertainties associated with the structure itself and those associated with the load it is designed to support. This method, referred to as *Load and Resistance Factor Design (LRFD),* further allows the designer to distinguish between uncertainties associated with the *live load,* P_L, that is, with the load to be supported by the structure, and the *dead load,* P_D, that is, with the weight of the portion of structure contributing to the total load.

When this method of design is used, the *ultimate load,* P_U, of the structure, that is, the load at which the structure ceases to be useful, should first be determined. The proposed design is then acceptable if the following inequality is satisfied:

$$\gamma_D P_D + \gamma_L P_L \leq \phi P_U \tag{8.26}$$

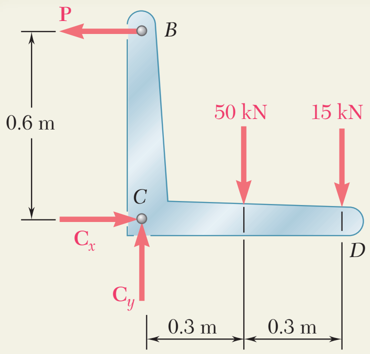
The coefficient ϕ is referred to as the *resistance factor;* it accounts for the uncertainties associated with the structure itself and will normally be less than 1. The coefficients γ_D and γ_L are referred to as the *load factors;* they account for the uncertainties associated, respectively, with the dead and live load and will normally be greater than 1, with γ_L generally larger than γ_D. The allowable-stress method of design will be used in this text.

SAMPLE PROBLEM 8.3



Two forces are applied to the bracket *BCD* as shown. (*a*) Knowing that the control rod *AB* is to be made of a steel having an ultimate normal stress of 600 MPa, determine the diameter of the rod for which the factor of safety with respect to failure will be 3.3. (*b*) The pin at *C* is to be made of a steel having an ultimate shearing stress of 350 MPa. Determine the diameter of the pin *C* for which the factor of safety with respect to shear will also be 3.3. (*c*) Determine the required thickness of the bracket supports at *C* knowing that the allowable bearing stress of the steel used is 300 MPa.

SOLUTION



Free Body: Entire Bracket. The reaction at *C* is represented by its components C_x and C_y.

$+\circlearrowleft \Sigma M_C = 0{:}\quad P(0.6\text{ m}) - (50\text{ kN})(0.3\text{ m}) - (15\text{ kN})(0.6\text{ m}) = 0 \qquad P = 40\text{ kN}$

$\Sigma F_x = 0{:} \qquad C_x = 40\text{ k}$

$\Sigma F_y = 0{:} \qquad C_y = 65\text{ kN} \qquad C = \sqrt{C_x^2 + C_y^2} = 76.3\text{ kN}$

a. Control Rod *AB*. Since the factor of safety is to be 3.3, the allowable stress is

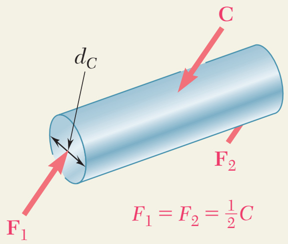
$$\sigma_{\text{all}} = \frac{\sigma_U}{F.S.} = \frac{600\text{ MPa}}{3.3} = 181.8\text{ MPa}$$

For $P = 40$ kN the cross-sectional area required is

$$A_{\text{req}} = \frac{P}{\sigma_{\text{all}}} = \frac{40\text{ kN}}{181.8\text{ MPa}} = 220 \times 10^{-6}\text{ m}^2$$

$$A_{\text{req}} = \frac{\pi}{4} d_{AB}^2 = 220 \times 10^{-6}\text{ m}^2 \qquad d_{AB} = 16.74\text{ mm} \quad ◀$$

b. Shear in Pin *C*. For a factor of safety of 3.3, we have



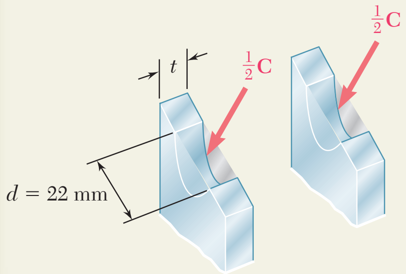
$$\tau_{\text{all}} = \frac{\tau_U}{F.S.} = \frac{350\text{ MPa}}{3.3} = 106.1\text{ MPa}$$

Since the pin is in double shear, we write

$$A_{\text{req}} = \frac{C/2}{\tau_{\text{all}}} = \frac{(76.3\text{ kN})/2}{106.1\text{ MPa}} = 360\text{ mm}^2$$

$$A_{\text{req}} = \frac{\pi}{4} d_C^2 = 360\text{ mm}^2 \qquad d_C = 21.4\text{ mm} \qquad \text{Use: } d_C = 22\text{ mm} \quad ◀$$

The next larger size pin available is of 22-mm diameter and should be used.

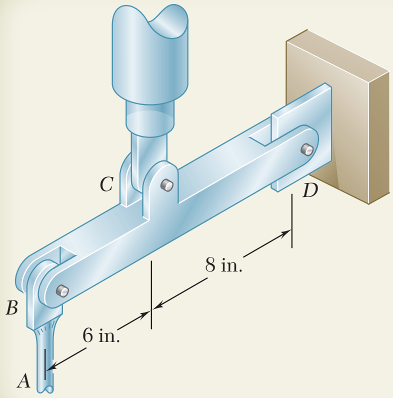


c. Bearing at *C*. Using $d = 22$ mm, the nominal bearing area of each bracket is $22t$. Since the force carried by each bracket is $C/2$ and the allowable bearing stress is 300 MPa, we write

$$A_{\text{req}} = \frac{C/2}{\sigma_{\text{all}}} = \frac{(76.3\text{ kN})/2}{300\text{ MPa}} = 127.2\text{ mm}^2$$

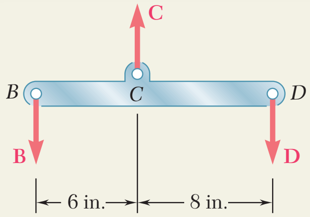
Thus $22t = 127.2 \qquad t = 5.78\text{ mm} \qquad \text{Use: } t = 6\text{ mm}$ ◀

SAMPLE PROBLEM 8.4



The rigid beam BCD is attached by bolts to a control rod at B, to a hydraulic cylinder at C, and to a fixed support at D. The diameters of the bolts used are: $d_B = d_D = \frac{3}{8}$ in., $d_C = \frac{1}{2}$ in. Each bolt acts in double shear and is made from a steel for which the ultimate shearing stress is $\tau_U = 40$ ksi. The control rod AB has a diameter $d_A = \frac{7}{16}$ in. and is made of a steel for which the ultimate tensile stress is $\sigma_U = 60$ ksi. If the minimum factor of safety is to be 3.0 for the entire unit, determine the largest upward force which may be applied by the hydraulic cylinder at C.

SOLUTION



The factor of safety with respect to failure must be 3.0 or more in each of the three bolts and in the control rod. These four independent criteria will be considered separately.

Free Body: Beam *BCD*. We first determine the force at C in terms of the force at B and in terms of the force at D.

$$+\circlearrowleft \Sigma M_D = 0: \qquad B(14 \text{ in.}) - C(8 \text{ in.}) = 0 \qquad C = 1.750B \qquad (1)$$

$$+\circlearrowleft \Sigma M_B = 0: \qquad -D(14 \text{ in.}) + C(6 \text{ in.}) = 0 \qquad C = 2.33D \qquad (2)$$

Control Rod. For a factor of safety of 3.0 we have

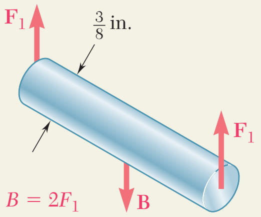
$$\sigma_{\text{all}} = \frac{\sigma_U}{F.S.} = \frac{60 \text{ ksi}}{3.0} = 20 \text{ ksi}$$

The allowable force in the control rod is

$$B = \sigma_{\text{all}}(A) = (20 \text{ ksi})\,\tfrac{1}{4}\pi\,(\tfrac{7}{16} \text{ in.})^2 = 3.01 \text{ kips}$$

Using Eq. (1) we find the largest permitted value of C:

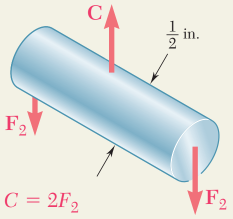
$$C = 1.750B = 1.750(3.01 \text{ kips}) \qquad C = 5.27 \text{ kips} \blacktriangleleft$$



Bolt at *B*. $\tau_{\text{all}} = \tau_U/F.S. = (40 \text{ ksi})/3 = 13.33$ ksi. Since the bolt is in double shear, the allowable magnitude of the force **B** exerted on the bolt is

$$B = 2F_1 = 2(\tau_{\text{all}} A) = 2(13.33 \text{ ksi})(\tfrac{1}{4}\pi)(\tfrac{3}{8} \text{ in.})^2 = 2.94 \text{ kips}$$

From Eq. (1): $\qquad C = 1.750B = 1.750(2.94 \text{ kips}) \qquad C = 5.15 \text{ kips} \blacktriangleleft$



Bolt at *D*. Since this bolt is the same as bolt B, the allowable force is $D = B = 2.94$ kips. From Eq. (2):

$$C = 2.33D = 2.33(2.94 \text{ kips}) \qquad C = 6.85 \text{ kips} \blacktriangleleft$$

Bolt at *C*. We again have $\tau_{\text{all}} = 13.33$ ksi and write

$$C = 2F_2 = 2(\tau_{\text{all}} A) = 2(13.33 \text{ ksi})(\tfrac{1}{4}\pi)(\tfrac{1}{2} \text{ in.})^2 \qquad C = 5.23 \text{ kips} \blacktriangleleft$$

Summary. We have found separately four maximum allowable values of the force C. In order to satisfy all these criteria, we must choose the smallest value, namely: $\qquad C = 5.15 \text{ kips} \blacktriangleleft$

PROBLEMS

8.25 Two wooden members of 3×6-in. uniform rectangular cross section are joined by the simple glued scarf splice shown. Knowing that $P = 2400$ lb, determine the normal and shearing stresses in the glued splice.

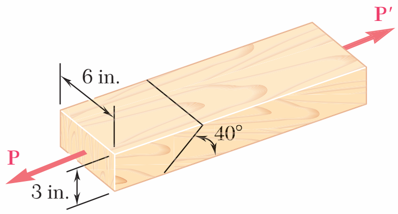


Fig. P8.25 and P8.26

8.26 Two wooden members of 3×6-in. uniform rectangular cross section are joined by the simple glued scarf splice shown. Knowing that the maximum allowable shearing stress in the glued splice is 90 psi, determine (*a*) the largest load **P** that can be safely applied, (*b*) the corresponding tensile stress in the splice.

8.27 The 6-kN load **P** is supported by two wooden members of 75×125-mm uniform cross section that are joined by the simple glued scarf splice shown. Determine the normal and shearing stresses in the glued splice.

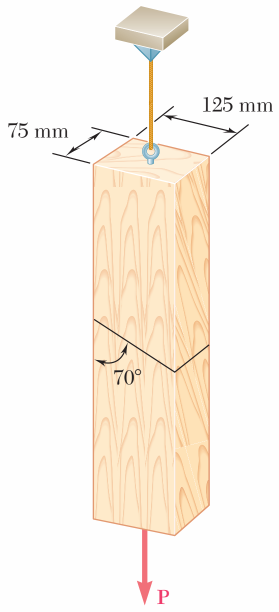


Fig. P8.27 and P8.28

8.28 Two wooden members of 75×125-mm uniform cross section are joined by the simple glued scarf splice shown. Knowing that the maximum allowable tensile stress in the glued splice is 500 kPa, determine (*a*) the largest load **P** that can be safely supported, (*b*) the corresponding shearing stress in the splice.

8.29 A 240-kip load **P** is applied to the granite block shown. Determine the resulting maximum value of (*a*) the normal stress, (*b*) the shearing stress. Specify the orientation of the plane on which each of these maximum values occurs.

8.30 A centric load **P** is applied to the granite block shown. Knowing that the resulting maximum value of the shearing stress in the block is 2.5 ksi, determine (*a*) the magnitude of **P**, (*b*) the orientation of the surface on which the maximum shearing stress occurs, (*c*) the normal stress exerted on that surface, (*d*) the maximum value of the normal stress in the block.

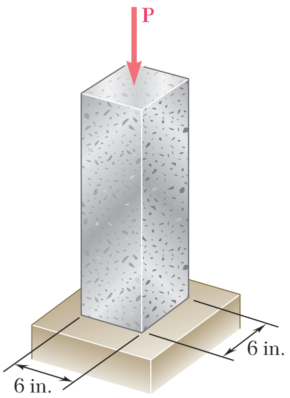


Fig. *P8.29* and P8.30

8.31 A steel pipe of 300-mm outer diameter is fabricated from 6-mm-thick plate by welding along a helix that forms an angle of 25° with a plane perpendicular to the axis of the pipe. Knowing that a 250-kN axial force **P** is applied to the pipe, determine the normal and shearing stresses in directions respectively normal and tangential to the weld.

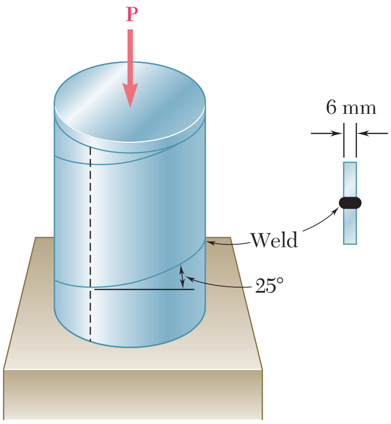


Fig. P8.31 and *P8.32*

8.32 A steel pipe of 300-mm outer diameter is fabricated from 6-mm-thick plate by welding along a helix that forms an angle of 25° with a plane perpendicular to the axis of the pipe. Knowing that the maximum allowable normal and shearing stresses in the directions respectively normal and tangential to the weld are $\sigma = 50$ MPa and $\tau = 30$ MPa, determine the magnitude *P* of the largest axial force that can be applied to the pipe.

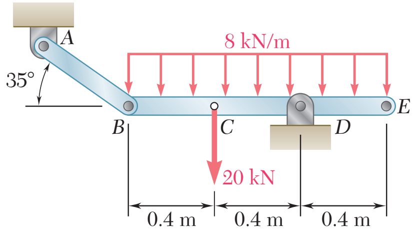


Fig. P8.33

8.33 Link *AB* is to be made of a steel for which the ultimate normal stress is 450 MPa. Determine the cross-sectional area for *AB* for which the factor of safety will be 3.50. Assume that the link will be adequately reinforced around the pins at *A* and *B*.

8.34 Member *ABC*, which is supported by a pin and bracket at *C* and a cable *BD*, was designed to support the 4-kip load **P** as shown. Knowing that the ultimate load for cable *BD* is 25 kips, determine the factor of safety with respect to cable failure.

8.35 Knowing that the ultimate load for cable *BD* is 25 kips and that a factor of safety of 3.2 with respect to cable failure is required, determine the magnitude of the largest force **P** that can be safely applied as shown to member *ABC*.

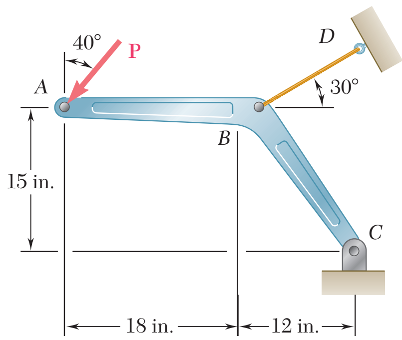


Fig. P8.34 and P8.35

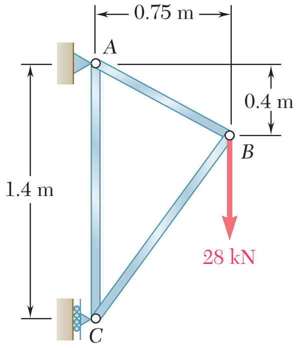


Fig. P8.36

8.36 Members AB and AC of the truss shown consist of bars of square cross section made of the same alloy. It is known that a 20-mm-square bar of the same alloy was tested to failure and that an ultimate load of 120 kN was recorded. If a factor of safety of 3.2 is to be achieved for both bars, determine the required dimensions of the cross section of (*a*) bar AB, (*b*) bar AC.

8.37 Three $\frac{3}{4}$-in.-diameter steel bolts are to be used to attach the steel plate shown to a wooden beam. Knowing that the plate will support a 24-kip load and that the ultimate shearing stress for the steel used is 52 ksi, determine the factor of safety for this design.

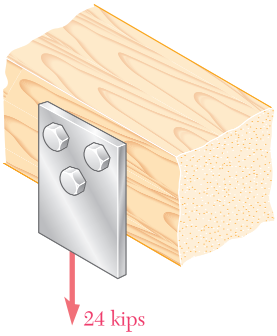


Fig. *P8.37*

8.38 Two plates, each 3 mm thick, are used to splice a plastic strip as shown. Knowing that the ultimate shearing stress of the bonding between the surfaces is 900 kPa, determine the factor of safety with respect to shear when $P = 1500$ N.

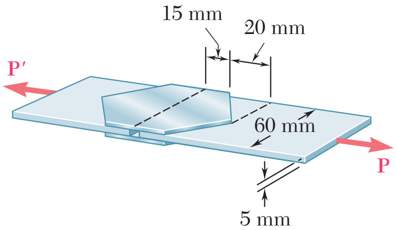


Fig. P8.38

8.39 Two wooden members of 3.5×5.5-in. uniform rectangular cross section are joined by the simple glued scarf splice shown. Knowing that the maximum allowable shearing stress in the glued splice is 75 psi, determine the largest axial load **P** that can be safely applied.

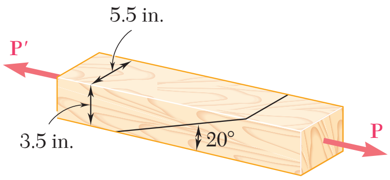


Fig. P8.39

8.40 A load **P** is supported as shown by a steel pin that has been inserted in a short wooden member hanging from the ceiling. The ultimate strength of the wood used is 60 MPa in tension and 7.5 MPa in shear, while the ultimate strength of the steel is 150 MPa in shear. Knowing that the diameter of the pin is $d = 16$ mm and that the magnitude of the load is $P = 20$ kN, determine (*a*) the factor of safety for the pin, (*b*) the required values of b and c if the factor of safety for the wooden member is to be the same as that found in part *a* for the pin.

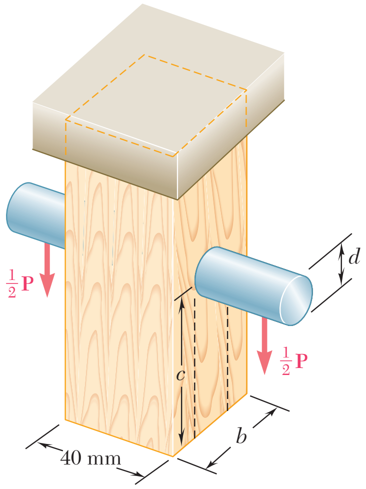


Fig. *P8.40*

8.41 A steel plate $\frac{5}{16}$ in. thick is embedded in a horizontal concrete slab and is used to anchor a high-strength vertical cable as shown. The diameter of the hole in the plate is $\frac{3}{4}$ in., the ultimate strength of the steel used is 36 ksi, and the ultimate bonding stress between plate and concrete is 300 psi. Knowing that a factor of safety of 3.60 is desired when $P = 2.5$ kips, determine (*a*) the required width a of the plate, (*b*) the minimum depth b to which a plate of that width should be embedded in the concrete slab. (Neglect the normal stresses between the concrete and the lower end of the plate.)

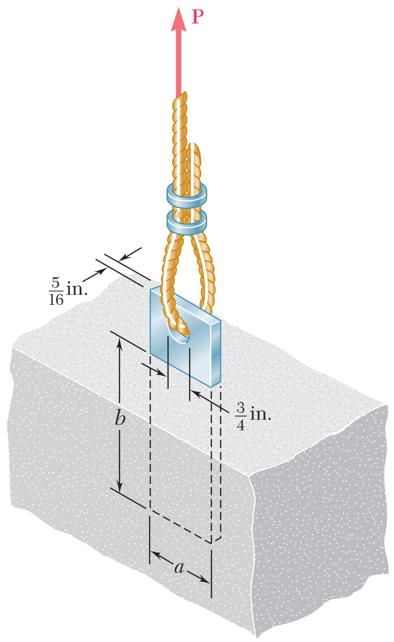


Fig. P8.41

8.42 Determine the factor of safety for the cable anchor in Prob. 8.41 when $P = 3$ kips, knowing that $a = 2$ in. and $b = 7.5$ in.

8.43 In the structure shown, an 8-mm-diameter pin is used at A and 12-mm-diameter pins are used at B and D. Knowing that the ultimate shearing stress is 100 MPa at all connections and the ultimate normal stress is 250 MPa in each of the two links joining B and D, determine the allowable load **P** if an overall factor of safety of 3.0 is desired.

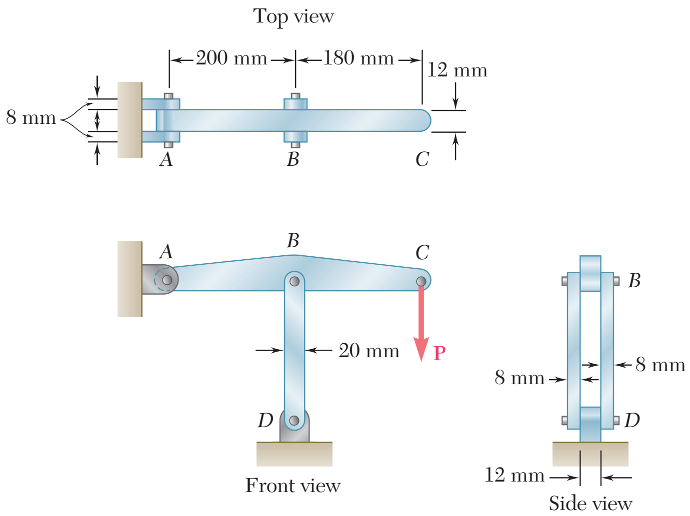


Fig. *P8.43* and P8.44

8.44 In an alternative design for the structure of Prob. 8.43, a pin of 10-mm-diameter is to be used at A. Assuming that all other specifications remain unchanged, determine the allowable load **P** if an overall factor of safety of 3.0 is desired.

8.45 Link AC is made of a steel with a 65-ksi ultimate normal stress and has a $\frac{1}{4} \times \frac{1}{2}$-in. uniform rectangular cross section. It is connected to a support at A and to member BCD at C by $\frac{3}{8}$-in.-diameter pins, while member BCD is connected to its support at B by a $\frac{5}{16}$-in.-diameter pin; all of the pins are made of a steel with a 25-ksi ultimate shearing stress and are in single shear. Knowing that a factor of safety of 3.25 is desired, determine the largest load **P** that can be applied at D. Note that link AC is not reinforced around the pin holes.

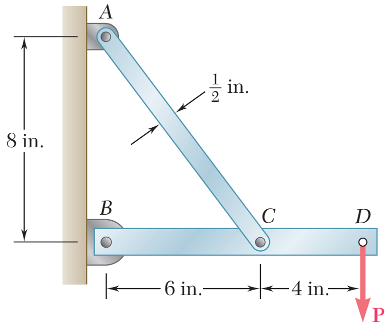


Fig. *P8.45*

8.46 Solve Prob. 8.45 assuming that the structure has been redesigned to use $\frac{5}{16}$-in.-diameter pins at A and C as well as at B and that no other change has been made.

8.47 Each of the two vertical links CF connecting the two horizontal members AD and EG has a 10×40-mm uniform rectangular cross section and is made of a steel with an ultimate strength in tension of 400 MPa, while each of the pins at C and F has a 20-mm diameter and is made of a steel with an ultimate strength in shear of 150 MPa. Determine the overall factor of safety for the links CF and the pins connecting them to the horizontal members.

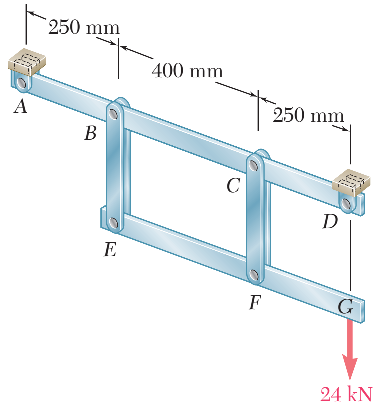


Fig. P8.47

8.48 Solve Prob. 8.47 assuming that the pins at C and F have been replaced by pins with a 30-mm diameter.

REVIEW AND SUMMARY

This chapter was devoted to the concept of stress and to an introduction to the methods used for the analysis and design of machines and load-bearing structures.

Axial loading. Normal stress

The concept of *stress* was first introduced in Sec. 8.2 by considering a two-force member under an *axial loading*. The *normal stress* in that member was obtained by dividing the magnitude P of the load by the cross-sectional area A of the member (Fig. 8.40). We wrote

$$\sigma = \frac{P}{A} \tag{8.1}$$

As noted in Sec. 8.3, the value of σ obtained from Eq. (8.1) represents the *average stress* over the section rather than the stress at a specific point Q of the section. Considering a small area ΔA surrounding Q and the magnitude ΔF of the force exerted on ΔA, we defined the stress at point Q as

$$\sigma = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A} \tag{8.2}$$

In general, the value obtained for the stress σ at point Q is different from the value of the average stress given by formula (8.1) and is found to vary across the section. However, this variation is small in any section away from the points of application of the loads. In practice, therefore, the distribution of the normal stresses in an axially loaded member is assumed to be *uniform,* except in the immediate vicinity of the points of application of the loads.

However, for the distribution of stresses to be uniform in a given section, it is necessary that the line of action of the loads **P** and **P′** pass through the centroid C of the section. Such a loading is called a *centric* axial loading. In the case of an *eccentric* axial loading, the distribution of stresses is *not* uniform. Stresses in members subjected to an eccentric axial loading will be discussed in Chap 11.

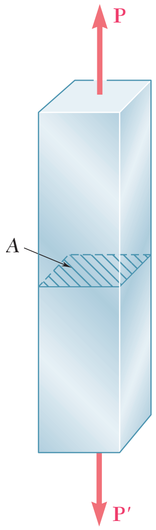


Fig. 8.40

Transverse forces. Shearing stress

When equal and opposite *transverse forces* **P** and **P′** of magnitude P are applied to a member AB (Fig. 8.41), *shearing stresses* τ are created over any section located between the points of application of the two forces [Sec 8.4]. These stresses vary greatly across the section and their distribution *cannot* be assumed uniform. However dividing the magnitude P—referred to as the *shear* in the section—by the cross-sectional area A, we defined the *average shearing stress* over the section:

$$\tau_{\text{ave}} = \frac{P}{A} \tag{8.4}$$

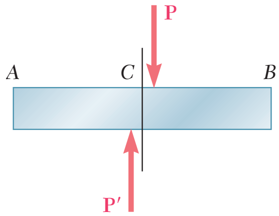


Fig. 8.41

Single and double shear

Shearing stresses are found in bolts, pins, or rivets connecting two structural members or machine components. For example, in the case of bolt CD (Fig. 8.42), which is in *single shear,* we wrote

$$\tau_{\text{ave}} = \frac{P}{A} = \frac{F}{A} \tag{8.5}$$

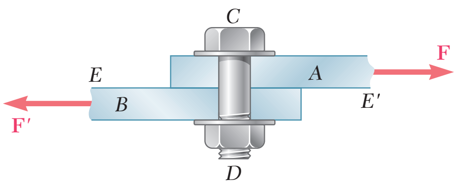


Fig. 8.42

while, in the case of bolts EG and HJ (Fig. 8.43), which are both in *double shear,* we had

$$\tau_{\text{ave}} = \frac{P}{A} = \frac{F/2}{A} = \frac{F}{2A} \tag{8.6}$$

Bearing stress

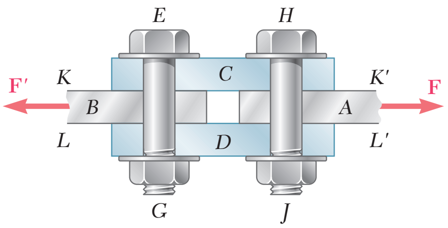


Fig. 8.43

Bolts, pins, and rivets also create stresses in the members they connect, along the *bearing surface,* or surface of contact [Sec. 8.5]. The bolt CD of Fig. 8.42, for example, creates stresses on the semicylindrical surface of plate A with which it is in contact (Fig. 8.44). Since the distribution of these stresses is quite complicated, one uses in practice an average nominal value σ_b of the stress, called *bearing stress,* obtained by dividing the load P by the area of the rectangle representing the projection of the bolt on the plate section. Denoting by t the thickness of the plate and by d the diameter of the bolt, we wrote

$$\sigma_b = \frac{P}{A} = \frac{P}{td} \tag{8.7}$$

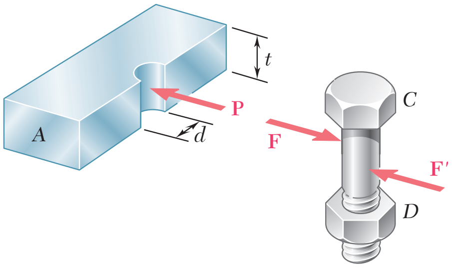


Fig. 8.44

In Sec. 8.6, we applied the concept introduced in the previous sections to the analysis of a simple structure consisting of two pin-connected members supporting a given load. We determined successively the normal stresses in the two members, paying special attention to their narrowest sections, the shearing stresses in the various pins, and the bearing stress at each connection.

Design

Section 8.7 was devoted to a short discussion of the *design* of structures and machines.

Stresses on an oblique section

In Sec. 8.8, we considered the stresses created on an *oblique section* in a two-force member under axial loading. We found that both *normal* and *shearing* stresses occurred in such a situation. Denoting by θ the angle formed by the section with a normal plane (Fig. 8.45*a*) and by A_0 the area of a section perpendicular to the axis of the member, we derived the following expressions for the normal stress σ and the shearing stress τ on the oblique section:

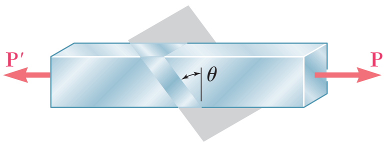


Fig. 8.45

$$\sigma = \frac{P}{A_0}\cos^2\theta \qquad \tau = \frac{P}{A_0}\sin\theta\cos\theta \tag{8.14}$$

We observed from these formulas that the normal stress is maximum and equal to $\sigma_m = P/A_0$ for $\theta = 0°$, while the shearing stress is maximum and equal to $\tau_m = P/2A_0$ for $\theta = 45°$. We also noted that $\tau = 0$ when $\theta = 0°$, while $\sigma = P/2A_0$ when $\theta = 45°$.

Stress under general loading

Next, we discussed the state of stress at a point Q in a body under the most general loading condition [Sec. 8.9]. Considering a small cube centered at Q (Fig. 8.46), we denoted by σ_x the normal stress exerted on a face of the cube perpendicular to the x axis, and by τ_{xy} and τ_{xz}, respectively, the y and z components of the shearing stress exerted on the same face of the cube. Repeating this procedure for the other two faces of the cube and observing that $\tau_{xy} = \tau_{yx}$, $\tau_{yz} = \tau_{zy}$, and $\tau_{zx} = \tau_{xz}$, we concluded that *six stress components* are required to define the state of stress at a given point Q, namely, σ_x, σ_y, σ_z, τ_{xy}, τ_{yz}, and τ_{zx}.

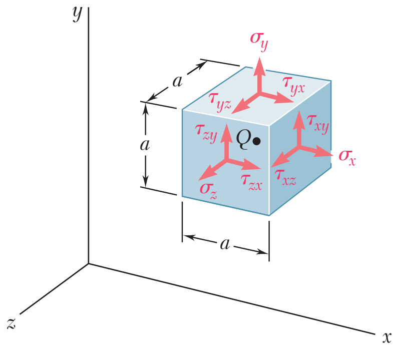


Fig. 8.46

Factor of safety

Section 8.10 was devoted to a discussion of the various concepts used in the design of engineering structures. The *ultimate load* of a given structural member or machine component is the load at which the member or component is expected to fail; it is computed from the *ultimate stress* or *ultimate strength* of the material used, as determined by a laboratory test on a specimen of that material. The ultimate load should be considerably larger than the *allowable load*, i.e., the load that the member or component will be allowed to carry under normal conditions. The ratio of the ultimate load to the allowable load is defined as the *factor of safety:*

$$\text{Factor of safety} = F.S. = \frac{\text{ultimate load}}{\text{allowable load}} \qquad (8.24)$$

The determination of the factor of safety that should be used in the design of a given structure depends upon a number of considerations, some of which were listed in this section.

Load and Resistance Factor Design

Section 8.10 ended with the discussion of an alternative approach to design, known as *Load and Resistance Factor Design,* which allows the engineer to distinguish between the uncertainties associated with the structure and those associated with the load.

REVIEW PROBLEMS

8.49 A 40-kN axial load is applied to a short wooden post that is supported by a concrete footing resting on undisturbed soil. Determine (*a*) the maximum bearing stress on the concrete footing, (*b*) the size of the footing for which the average bearing stress in the soil is 145 kPa.

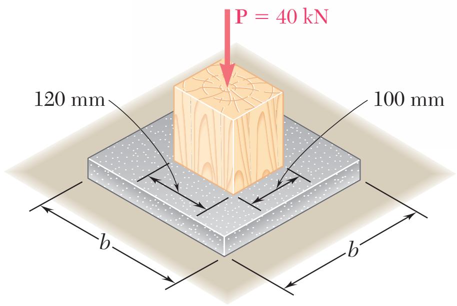


Fig. P8.49

8.50 The frame shown consists of four wooden members, *ABC*, *DEF*, *BE*, and *CF*. Knowing that each member has a 2 × 4-in. rectangular cross section and that each pin has a $\frac{1}{2}$-in. diameter, determine the maximum value of the average normal stress (*a*) in member *BE*, (*b*) in member *CF*.

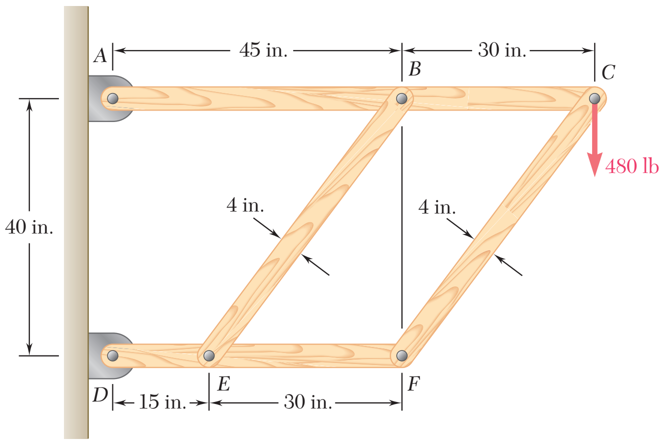


Fig. *P8.50*

8.51 Two steel plates are to be held together by means of $\frac{1}{4}$-in.-diameter high-strength steel bolts fitting snugly inside cylindrical brass spacers. Knowing that the average normal stress must not exceed 30 ksi in the bolts and 18 ksi in the spacers, determine the outer diameter of the spacers that yields the most economical and safe design.

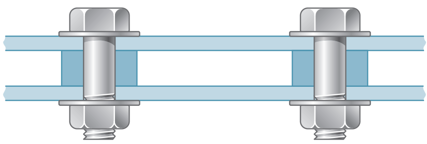


Fig. P8.51

8.52 When the force **P** reached 8 kN, the wooden specimen shown failed in shear along the surface indicated by the dashed line. Determine the average shearing stress along that surface at the time of failure.

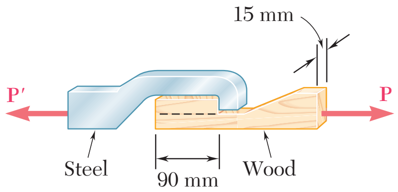


Fig. *P8.52*

8.53 Knowing that link *DE* is 1 in. wide and $\frac{1}{8}$ in. thick, determine the normal stress in the central portion of that link when (*a*) $\theta = 0$, (*b*) $\theta = 90°$.

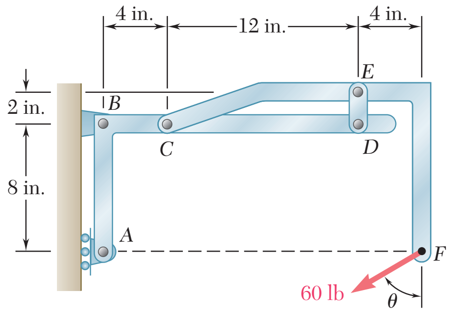


Fig. P8.53

8.54 Two wooden planks, each 12 mm thick and 225 mm wide, are joined by the dry mortise joint shown. Knowing that the wood used shears off along its grain when the average shearing stress reaches 8 MPa, determine the magnitude *P* of the axial load that will cause the joint to fail.

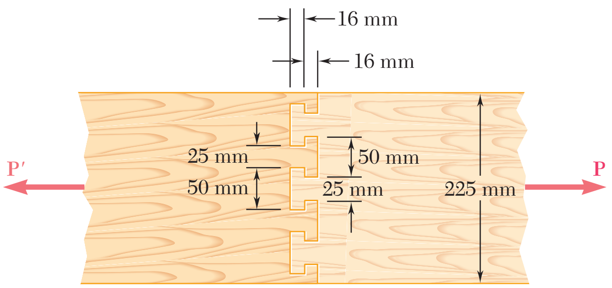


Fig. P8.54

8.55 Two identical linkage-and-hydraulic-cylinder systems control the position of the forks of a fork-lift truck. The load supported by the one system shown is 1500 lb. Knowing that the thickness of member *BD* is $\frac{5}{8}$ in., determine (*a*) the average shearing stress in the $\frac{1}{2}$-in.-diameter pin at *B*, (*b*) the bearing stress at *B* in member *BD*.

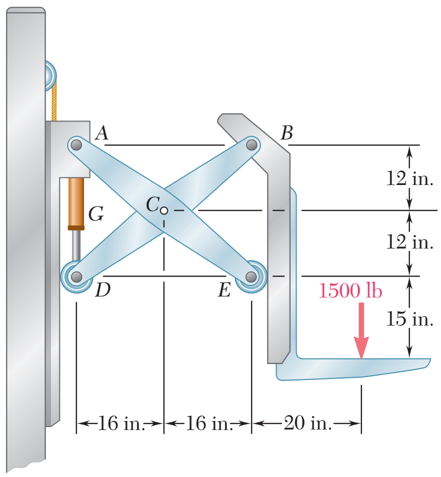


Fig. P8.55

8.56 A $\frac{1}{2}$-in.-diameter steel rod AB is fitted to a round hole near end C of the wooden member CD. For the loading shown, determine (*a*) the maximum average normal stress in the wood, (*b*) the distance b for which the average shearing stress is 90 psi on the surfaces indicated by the dashed lines, (*c*) the average bearing stress on the wood.

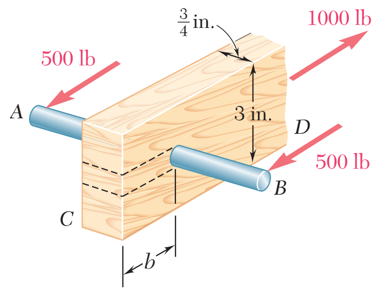


Fig. P8.56

8.57 A steel loop $ABCD$ of length 1.2 m and of 10-mm diameter is placed as shown around a 24-mm-diameter aluminum rod AC. Cables BE and DF, each of 12-mm diameter, are used to apply the load **Q**. Knowing that the ultimate strength of the aluminum used for the rod is 260 MPa and that the ultimate strength of the steel used for the loop and the cables is 480 MPa, determine the largest load **Q** that can be applied if an overall factor of safety of 3 is desired.

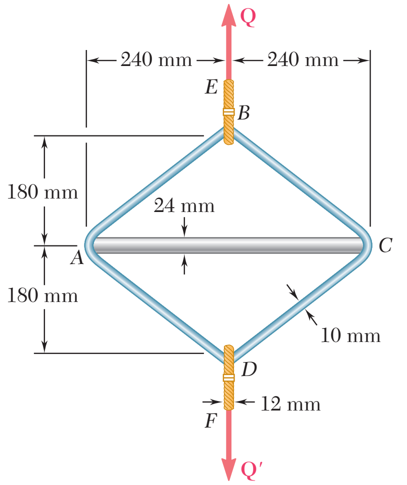


Fig. P8.57

8.58 Two wooden members of 75 × 125-mm uniform rectangular cross section are joined by the simple glued joint shown. Knowing that $P = 3.6$ kN and that the ultimate strength of the glue is 1.1 MPa in tension and 1.4 MPa in shear, determine the factor of safety.

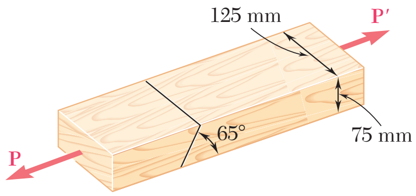


Fig. P8.58

8.59 Link BC is 6 mm thick, has a width $w = 25$ mm, and is made of a steel with a 480-MPa ultimate strength in tension. What was the safety factor used if the structure shown was designed to support a 16-kN load **P**?

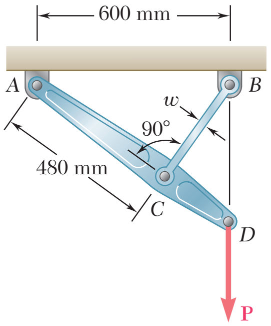


Fig. *P8.59*

8.60 The two portions of member AB are glued together along a plane forming an angle θ with the horizontal. Knowing that the ultimate stress for the glued joint is 2.5 ksi in tension and 1.3 ksi in shear, determine the range of values of θ for which the factor of safety of the members is at least 3.0.

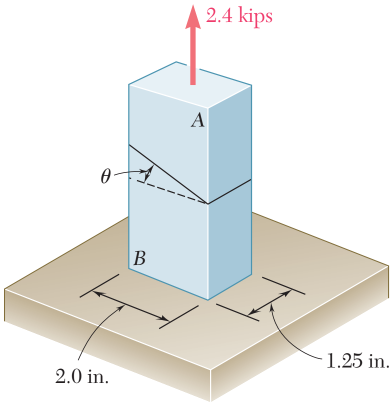
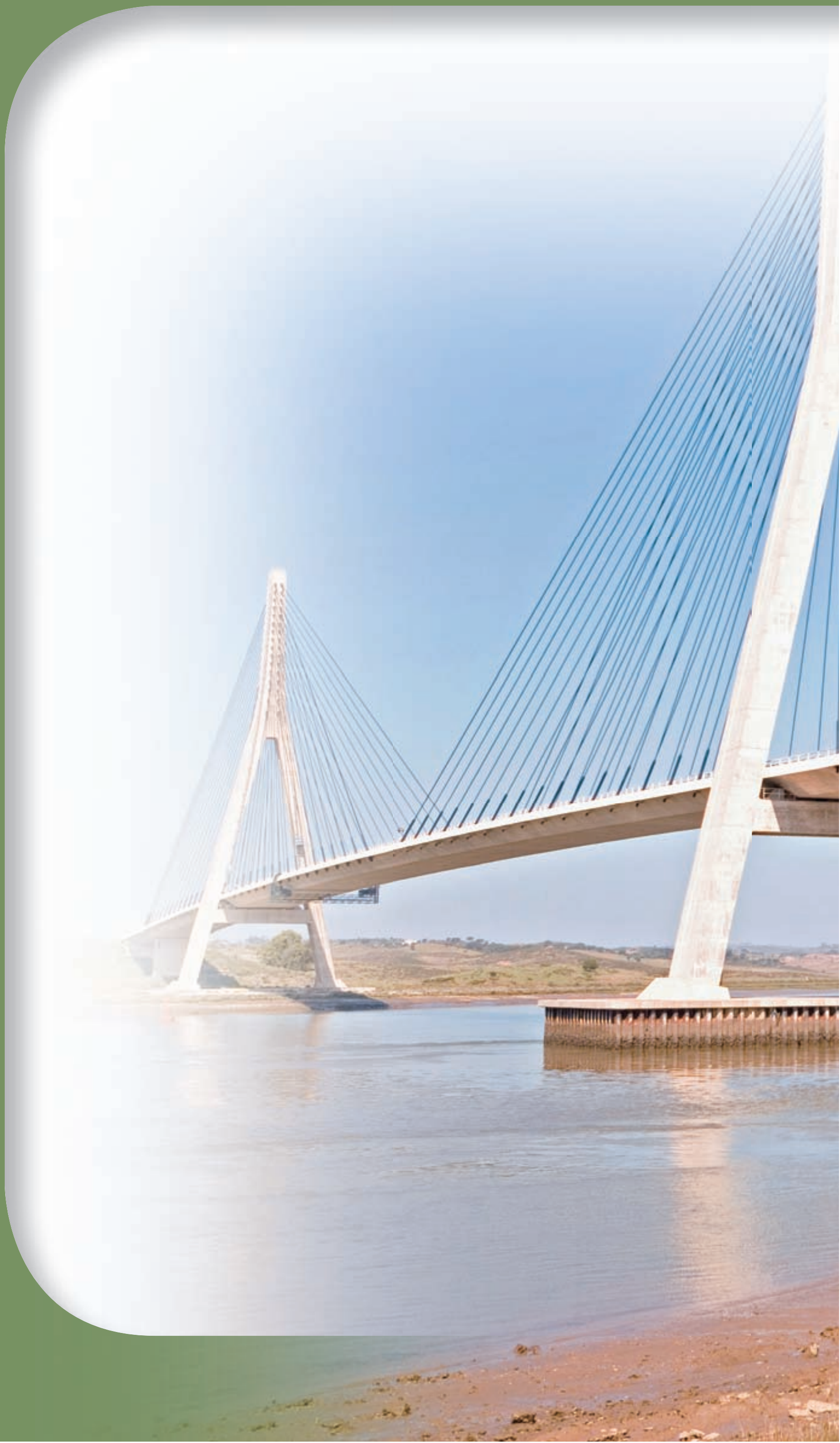


Fig. P8.60

This chapter is devoted to the study of deformations occurring in structural components subjected to axial loading. The change in length of the diagonal stays was carefully accounted for in the design of this cable-stayed bridge.



CHAPTER 9

Stress and Strain—Axial Loading

9.1 INTRODUCTION

In Chap. 8 we analyzed the stresses created in various members and connections by the loads applied to a structure or machine. We also learned to design simple members and connections so that they would not fail under specified loading conditions. Another important aspect of the analysis and design of structures relates to the *deformations* caused by the loads applied to a structure. Clearly, it is important to avoid deformations so large that they may prevent the structure from fulfilling the purpose for which it was intended. But the analysis of deformations may also help us in the determination of stresses. Indeed, it is not always possible to determine the forces in the members of a structure by applying only the principles of statics. This is because statics is based on the assumption of undeformable, rigid structures. By considering engineering structures as *deformable* and analyzing the deformations in their various members, it will be possible for us to compute forces that are *statically indeterminate,* i.e., indeterminate within the framework of statics. Also, as we indicated in Sec. 8.3, the distribution of stresses in a given member is statically indeterminate, even when the force in that member is known. To determine the actual distribution of stresses within a member, it is thus necessary to analyze the deformations that take place in that member. In this chapter, you will consider the deformations of a structural member such as a rod, bar, or plate under *axial loading*.

First, the *normal strain* ϵ in a member will be defined as the *deformation of the member per unit length*. Plotting the stress σ versus the strain ϵ as the load applied to the member is increased will yield a *stress-strain diagram* for the material used. From such a diagram we can determine some important properties of the material, such as its *modulus of elasticity,* and whether the material is *ductile* or *brittle* (Secs. 9.2 to 9.4).

From the stress-strain diagram, we can also determine whether the strains in the specimen will disappear after the load has been removed—in which case the material is said to behave *elastically*—or whether *a permanent set* or *plastic deformation* will result (Sec. 9.5).

Section 9.6 is devoted to the phenomenon of *fatigue,* which causes structural or machine components to fail after a very large number of repeated loadings, even though the stresses remain in the elastic range.

The first part of the chapter ends with Sec. 9.7, which is devoted to the determination of the deformation of various types of members under various conditions of axial loading.

In Secs. 9.8 and 9.9, *statically indeterminate problems* will be considered, i.e., problems in which the reactions and the internal forces *cannot* be determined from statics alone. The equilibrium equations derived from the free-body diagram of the member under consideration must be complemented by relations involving deformations; these relations will be obtained from the geometry of the problem.

In Secs. 9.10 to 9.13, additional constants associated with isotropic materials—i.e., materials with mechanical characteristics independent of direction—will be introduced. They include *Poisson's ratio,* which relates lateral and axial strain, and the *modulus of rigidity,* which relates the components of the shearing stress and shearing strain. Stress-strain relationships for an isotropic material under a multi-axial loading will also be derived.

In the text material described so far, stresses are assumed to be uniformly distributed in any given cross section; they are also assumed to remain within the elastic range. The validity of the first assumption is discussed in Sec. 9.14, while *stress concentrations* near circular holes and fillets in flat bars are considered in Sec. 9.15.

9.2 NORMAL STRAIN UNDER AXIAL LOADING

Let us consider a rod BC, of length L and uniform cross-sectional area A, which is suspended from B (Fig. 9.1*a*). If we apply a load **P** to end C, the rod elongates (Fig. 9.1*b*). Plotting the magnitude P of the load against the deformation δ (Greek letter delta), we obtain a certain load-deformation diagram (Fig. 9.2). While this diagram contains information useful to the analysis of the rod under consideration, it cannot be used directly to predict the deformation of a rod of the same material but of different dimensions. Indeed, we observe that, if a deformation δ is produced in rod BC by a load **P**, a load 2**P** is required to cause the same deformation in a rod $B'C'$ of the same length L, but of cross-sectional area $2A$ (Fig. 9.3). We note that, in both cases, the value of the stress is the same: $\sigma = P/A$. On the other hand, a load **P** applied to a rod $B''C''$, of the same cross-sectional area A, but of length $2L$, causes a deformation 2δ in that rod (Fig. 9.4), i.e., a deformation twice as large as the deformation δ it produces in rod BC. But in both cases the ratio of the deformation over the length of the rod is the same; it is equal to δ/L. This observation brings us to introduce the concept of *strain:* We define the *normal strain* in a rod under axial loading as the *deformation per unit length* of that rod. Denoting the normal strain by ϵ (Greek letter epsilon), we write

$$\epsilon = \frac{\delta}{L} \tag{9.1}$$

Plotting the stress $\sigma = P/A$ against the strain $\epsilon = \delta/L$, we obtain a curve that is characteristic of the properties of the material

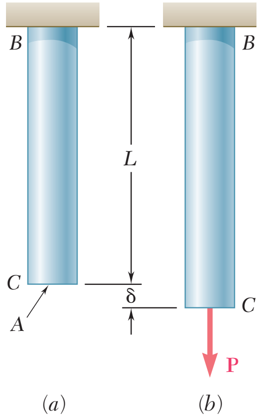


Fig. 9.1

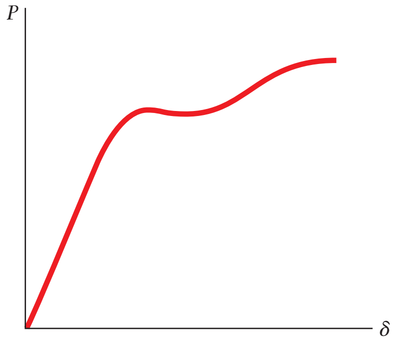


Fig. 9.2

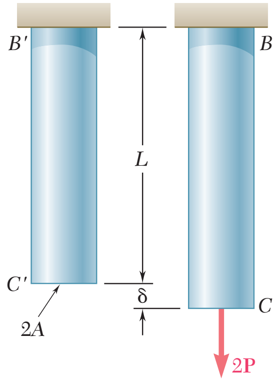


Fig. 9.3

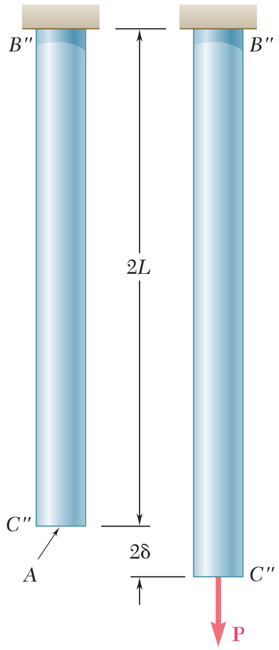


Fig. 9.4

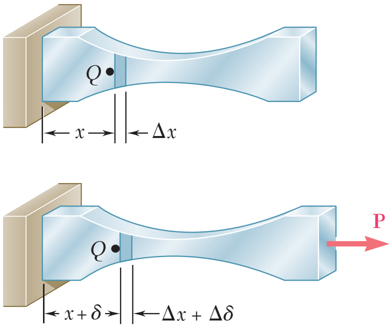


Fig. 9.5

and does not depend upon the dimensions of the particular specimen used. This curve is called a *stress-strain diagram* and will be discussed in detail in Sec. 9.3.

Since the rod BC considered in the preceding discussion had a uniform cross section of area A, the normal stress σ could be assumed to have a constant value P/A throughout the rod. Thus, it was appropriate to define the strain ϵ as the ratio of the total deformation δ over the total length L of the rod. In the case of a member of variable cross-sectional area A, however, the normal stress $\sigma = P/A$ varies along the member, and it is necessary to define the strain at a given point Q by considering a small element of undeformed length Δx (Fig. 9.5). Denoting by $\Delta\delta$ the deformation of the element under the given loading, we define the *normal strain at point* Q as

$$\epsilon = \lim_{\Delta x \to 0} \frac{\Delta\delta}{\Delta x} = \frac{d\delta}{dx} \tag{9.2}$$

Since deformation and length are expressed in the same units, the normal strain ϵ obtained by dividing δ by L (or $d\delta$ by dx) is a *dimensionless quantity*. Thus, the same numerical value is obtained for the normal strain in a given member, whether SI metric units or U.S. customary units are used. Consider, for instance, a bar of length $L = 0.600$ m and uniform cross section, which undergoes a deformation $\delta = 150 \times 10^{-6}$ m. The corresponding strain is

$$\epsilon = \frac{\delta}{L} = \frac{150 \times 10^{-6}\text{ m}}{0.600\text{ m}} = 250 \times 10^{-6}\text{ m/m} = 250 \times 10^{-6}$$

Note that the deformation could have been expressed in micrometers: $\delta = 150\ \mu$m. We would then have written

$$\epsilon = \frac{\delta}{L} = \frac{150\ \mu\text{m}}{0.600\text{ m}} = 250\ \mu\text{m/m} = 250\ \mu$$

and read the answer as "250 micros." If U.S. customary units are used, the length and deformation of the same bar are, respectively, $L = 23.6$ in. and $\delta = 5.91 \times 10^{-3}$ in. The corresponding strain is

$$\epsilon = \frac{\delta}{L} = \frac{5.91 \times 10^{-3}\text{ in.}}{23.6\text{ in.}} = 250 \times 10^{-6}\text{ in./in.}$$

which is the same value that we found using SI units. It is customary, however, when lengths and deformations are expressed in inches or microinches (μin.), to keep the original units in the expression obtained for the strain. Thus, in our example, the strain would be recorded as $\epsilon = 250 \times 10^{-6}$ in./in. or, alternatively, as $\epsilon = 250\ \mu$in./in.

9.3 STRESS-STRAIN DIAGRAM

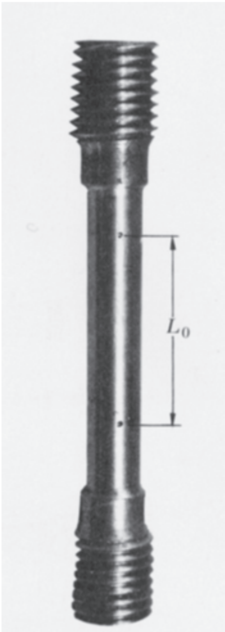


Photo 9.1 Typical tensile-test specimen.

We saw in Sec. 9.2 that the diagram representing the relation between stress and strain in a given material is an important characteristic of the material. To obtain the stress-strain diagram of a material, one usually conducts a *tensile test* on a specimen of the material. One type of specimen commonly used is shown in Photo 9.1. The cross-sectional

Photo 9.2 This machine is used to test tensile-test specimens, such as those shown in this chapter.

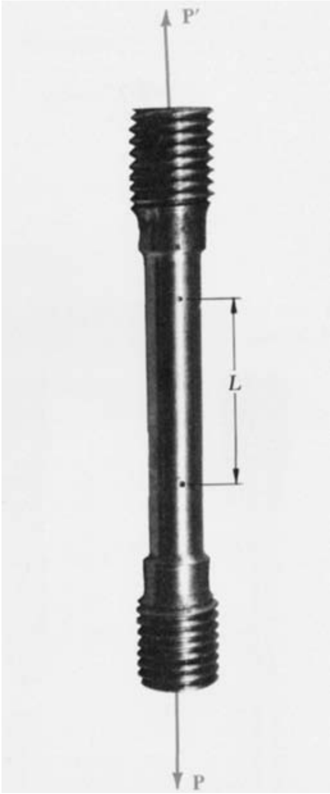


Photo 9.3 Test specimen with tensile load.

area of the cylindrical central portion of the specimen has been accurately determined and two gage marks have been inscribed on that portion at a distance L_0 from each other. The distance L_0 is known as the *gage length* of the specimen.

The test specimen is then placed in a testing machine (Photo 9.2), which is used to apply a centric load **P**. As the load **P** increases, the distance L between the two gage marks also increases (Photo 9.3). The distance L is measured with a dial gage, and the elongation $\delta = L - L_0$ is recorded for each value of P. A second dial gage is often used simultaneously to measure and record the change in diameter of the specimen. From each pair of readings P and δ, the stress σ is computed by dividing P by the original cross-sectional area A_0 of the specimen, and the strain ϵ is computed by dividing the elongation δ by the original distance L_0 between the two gage marks. The stress-strain diagram may then be obtained by plotting ϵ as an abscissa and σ as an ordinate.

Stress-strain diagrams of various materials vary widely, and different tensile tests conducted on the same material may yield different results, depending upon the temperature of the specimen and the speed of loading. It is possible, however, to distinguish some common characteristics among the stress-strain diagrams of various groups of materials and to divide materials into two broad categories

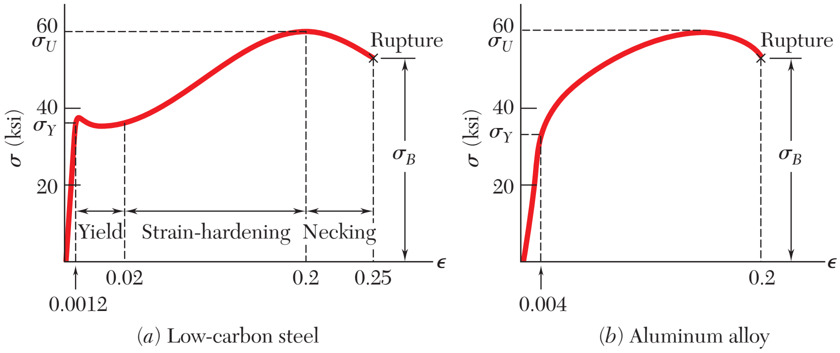


Fig. 9.6 Stress-strain diagrams of two typical ductile materials.

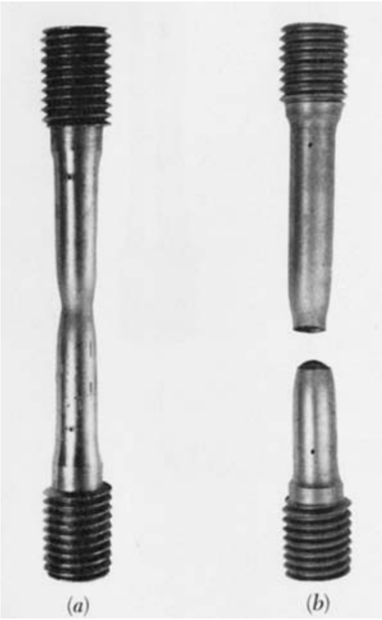


Photo 9.4 Tested specimen of a ductile material.

on the basis of these characteristics, namely, the *ductile* materials and the *brittle* materials.

Ductile materials, which comprise structural steel, as well as many alloys of other metals, are characterized by their ability to *yield* at normal temperatures. As the specimen is subjected to an increasing load, its length first increases linearly with the load and at a very slow rate. Thus, the initial portion of the stress-strain diagram is a straight line with a steep slope (Fig. 9.6). However, after a critical value σ_Y of the stress has been reached, the specimen undergoes a large deformation with a relatively small increase in the applied load. This deformation is caused by slippage of the material along oblique surfaces and is due, therefore, primarily to shearing stresses. As we can note from the stress-strain diagrams of two typical ductile materials (Fig. 9.6), the elongation of the specimen after it has started to yield can be 200 times as large as its deformation before yield. After a certain maximum value of the load has been reached, the diameter of a portion of the specimen begins to decrease because of local instability (Photo 9.4*a*). This phenomenon is known as *necking*. After necking has begun, somewhat lower loads are sufficient to keep the specimen elongating further, until it finally ruptures (Photo 9.4*b*). We note that rupture occurs along a cone-shaped surface that forms an angle of approximately 45° with the original surface of the specimen. This indicates that shear is primarily responsible for the failure of ductile materials, and confirms the fact that, under an axial load, shearing stresses are largest on surfaces forming an angle of 45° with the load (cf. Sec. 8.8). The stress σ_Y at which yield is initiated is called the *yield strength* of the material, the stress σ_U corresponding to the maximum load applied to the specimen is known as the *ultimate strength*, and the stress σ_B corresponding to rupture is called the *breaking strength*.

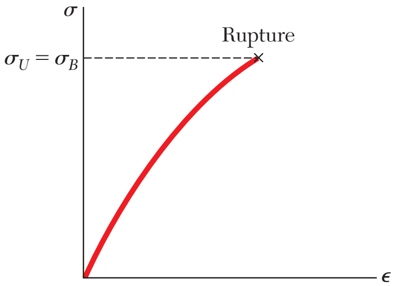


Fig. 9.7 Stress-strain diagrams for a typical brittle material.

Brittle materials, which comprise cast iron, glass, and stone, are characterized by the fact that rupture occurs without any noticeable prior change in the rate of elongation (Fig. 9.7). Thus, for brittle materials, there is no difference between the ultimate strength and the breaking strength. Also, the strain at the time of rupture is much smaller for brittle than for ductile materials. From Photo 9.5, we note the absence of any necking of the specimen in the case of a brittle material, and observe that rupture occurs along a surface perpendicular to the load. We conclude from this observation that normal stresses are primarily responsible for the failure of brittle materials.†

The stress-strain diagrams of Fig. 9.6 show that structural steel and aluminum, while both ductile, have different yield characteristics. In the case of structural steel (Fig. 9.6*a*), the stress remains constant over a large range of values of the strain after the onset of yield. Later the stress must be increased to keep elongating the specimen, until the maximum value σ_U has been reached. This is due to a property of the material known as strain-hardening. The yield strength of structural steel can be determined during the tensile test by watching the load shown on the display of the testing machine. After increasing steadily, the load is observed to suddenly drop to a slightly lower value, which is maintained for a certain period while the specimen keeps elongating. In a very carefully conducted test, one may be able to distinguish between the *upper yield point,* which corresponds to the load reached just before yield starts, and the *lower yield point,* which corresponds to the load required to maintain yield. Since the upper yield point is transient, the lower yield point should be used to determine the yield strength of the material.

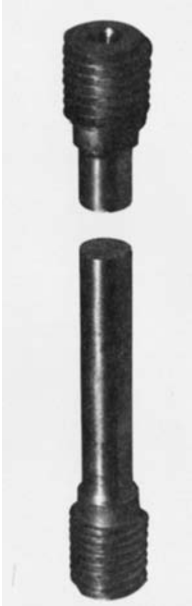


Photo 9.5 Tested specimen of a brittle material.

In the case of aluminum (Fig. 9.6*b*) and of many other ductile materials, the onset of yield is not characterized by a horizontal portion of the stress-strain curve. Instead, the stress keeps increasing—although not linearly—until the ultimate strength is reached. Necking then begins, leading eventually to rupture. For such materials, the yield strength σ_Y can be defined by the offset method. The yield strength at 0.2% offset, for example, is obtained by drawing through the point of the horizontal axis of abscissa $\epsilon = 0.2\%$ (or $\epsilon = 0.002$), a line parallel to the initial straight-line portion of the stress-strain diagram (Fig. 9.8). The stress σ_Y corresponding to the point Y obtained in this fashion is defined as the yield strength at 0.2% offset.

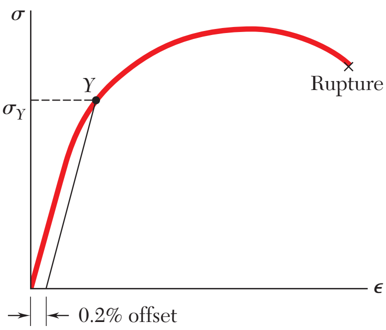


Fig. 9.8 Determination of yield strength by offset method.

A standard measure of the ductility of a material is its *percent elongation,* which is defined as

$$\text{Percent elongation} = 100\,\frac{L_B - L_0}{L_0}$$

where L_0 and L_B denote, respectively, the initial length of the tensile test specimen and its final length at rupture. The specified minimum elongation for a 2-in. gage length for commonly used steels with yield strengths up to 50 ksi is 21%. We note that this means that the average strain at rupture should be at least 0.21 in./in.

†The tensile tests described in this section were assumed to be conducted at normal temperatures. However, a material that is ductile at normal temperatures may display the characteristics of a brittle material at very low temperatures, while a normally brittle material may behave in a ductile fashion at very high temperatures. At temperatures other than normal, therefore, one should refer to *a material in a ductile state* or to *a material in a brittle state,* rather than to a ductile or brittle material.

Another measure of ductility which is sometimes used is the *percent reduction in area,* defined as

$$\text{Percent reduction in area} = 100 \frac{A_0 - A_B}{A_0}$$

where A_0 and A_B denote, respectively, the initial cross-sectional area of the specimen and its minimum cross-sectional area at rupture. For structural steel, percent reductions in area of 60 to 70 percent are common.

Thus far, we have discussed only tensile tests. If a specimen made of a ductile material were loaded in compression instead of tension, the stress-strain curve obtained would be essentially the same through its initial straight-line portion and through the beginning of the portion corresponding to yield and strain-hardening. Particularly noteworthy is the fact that for a given steel, the yield strength is the same in both tension and compression. For larger values of the strain, the tension and compression stress-strain curves diverge, and it should be noted that necking cannot occur in compression. For most brittle materials, one finds that the ultimate strength in compression is much larger than the ultimate strength in tension. This is due to the presence of flaws, such as microscopic cracks or cavities, which tend to weaken the material in tension, while not appreciably affecting its resistance to compressive failure.

An example of brittle material with different properties in tension and compression is provided by *concrete,* whose stress-strain diagram is shown in Fig. 9.9. On the tension side of the diagram, we first observe a linear elastic range in which the strain is proportional to the stress. After the yield point has been reached, the strain increases faster than the stress until rupture occurs. The behavior of the material in compression is different. First, the linear elastic range is significantly larger. Second, rupture does not occur as the stress reaches its maximum value. Instead, the stress decreases in magnitude while the strain keeps increasing until rupture occurs. Note that the modulus of elasticity, which is represented by the slope of the stress-strain curve in its linear portion, is the same in tension and compression. This is true of most brittle materials.

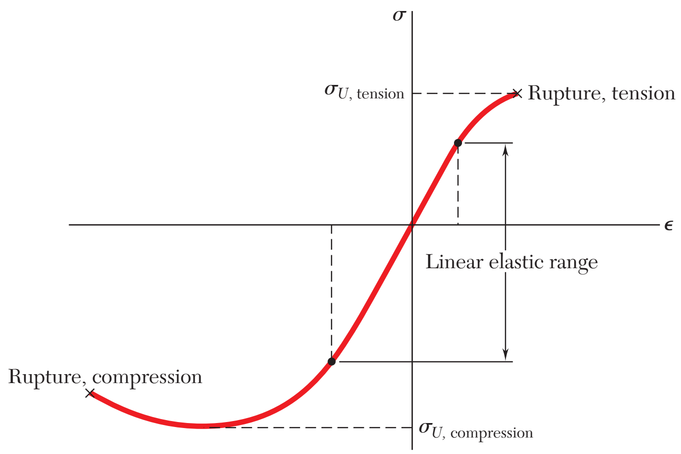


Fig. 9.9 Stress-strain diagram for concrete.

9.4 HOOKE'S LAW. MODULUS OF ELASTICITY

Most engineering structures are designed to undergo relatively small deformations, involving only the straight-line portion of the corresponding stress-strain diagram. For that initial portion of the diagram (Fig. 9.6), the stress σ is directly proportional to the strain ϵ, and we can write

$$\sigma = E\,\epsilon \tag{9.3}$$

This relation is known as *Hooke's law,* after the English mathematician Robert Hooke (1635–1703). The coefficient E is called the *modulus of elasticity* of the material involved, or also *Young's modulus*, after the English scientist Thomas Young (1773–1829). Since the strain ϵ is a dimensionless quantity, the modulus E is expressed in the same units as the stress σ, namely in pascals or one of its multiples if SI units are used, and in psi or ksi if U.S. customary units are used.

The largest value of the stress for which Hooke's law can be used for a given material is known as the *proportional limit* of that material. In the case of ductile materials possessing a well-defined yield point, as in Fig. 9.6*a*, the proportional limit almost coincides with the yield point. For other materials, the proportional limit cannot be defined as easily, since it is difficult to determine with accuracy the value of the stress σ for which the relation between σ and ϵ ceases to be linear. But from this very difficulty we can conclude for such materials that using Hooke's law for values of the stress slightly larger than the actual proportional limit will not result in any significant error.

Some of the physical properties of structural metals, such as strength, ductility, and corrosion resistance, can be greatly affected by alloying, heat treatment, and the manufacturing process used. For example, we note from the stress-strain diagrams of pure iron and of three different grades of steel (Fig. 9.10) that large variations in the yield strength, ultimate strength, and final strain (ductility) exist among these four metals. All of them, however, possess the same modulus of elasticity; in other words, their "stiffness," or ability to resist a deformation within the linear range, is the same. Therefore, if a high-strength steel is substituted for a lower-strength steel in a given structure, and if all dimensions are kept the same, the structure will have an increased load-carrying capacity, but its stiffness will remain unchanged.

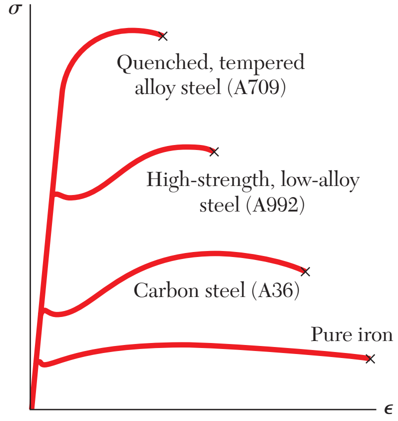


Fig. 9.10 Stress-strain diagrams for iron and different grades of steel.

For each of the materials considered so far, the relation between normal stress and normal strain, $\sigma = E\epsilon$, is independent of the direction of loading. This is because the mechanical properties of each material, including its modulus of elasticity E, are independent of the direction considered. Such materials are said to be *isotropic*. Materials whose properties depend upon the direction considered are said to be *anisotropic*.

An important class of anisotropic materials consists of *fiber-reinforced composite materials*. These composite materials are obtained by embedding fibers of a strong, stiff material into a weaker, softer material, referred to as a *matrix*. Typical materials used as fibers are graphite, glass, and polymers, while various types of resins are used as a matrix. Figure 9.11 shows a layer, or *lamina*, of a composite

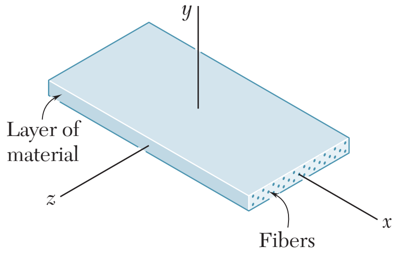


Fig. 9.11 Layer of fiber-reinforced composite material.

material consisting of a large number of parallel fibers embedded in a matrix. An axial load applied to the lamina along the x axis, that is, in a direction parallel to the fibers, will create a normal stress σ_x in the lamina and a corresponding normal strain ϵ_x which will satisfy Hooke's law as the load is increased and as long as the elastic limit of the lamina is not exceeded. Similarly, an axial load applied along the y axis, that is, in a direction perpendicular to the lamina, will create a normal stress σ_y and a normal strain ϵ_y satisfying Hooke's law, and an axial load applied along the z axis will create a normal stress σ_z and a normal strain ϵ_z which again satisfy Hooke's law. However, the moduli of elasticity E_x, E_y, and E_z corresponding, respectively, to each of the above loadings will be different. Because the fibers are parallel to the x axis, the lamina will offer a much stronger resistance to a loading directed along the x axis than to a loading directed along the y or z axis, and E_x will be much larger than either E_y or E_z.

A flat *laminate* is obtained by superposing a number of layers or laminas. If the laminate is to be subjected only to an axial load causing tension, the fibers in all layers should have the same orientation as the load in order to obtain the greatest possible strength. But if the laminate may be in compression, the matrix material may not be sufficiently strong to prevent the fibers from kinking or buckling. The lateral stability of the laminate may then be increased by positioning some of the layers so that their fibers will be perpendicular to the load. Positioning some layers so that their fibers are oriented at 30°, 45°, or 60° to the load may also be used to increase the resistance of the laminate to in-plane shear.

*9.5 ELASTIC VERSUS PLASTIC BEHAVIOR OF A MATERIAL

If the strains caused in a test specimen by the application of a given load disappear when the load is removed, the material is said to behave *elastically*. The largest value of the stress for which the material behaves elastically is called the *elastic limit* of the material.

If the material has a well-defined yield point as in Fig. 9.6*a*, the elastic limit, the proportional limit (Sec. 9.4), and the yield point are essentially equal. In other words, the material behaves elastically and linearly as long as the stress is kept below the yield point. If the yield point is reached, however, yield takes place as described in Sec. 9.3 and, when the load is removed, the stress and strain decrease in a linear fashion, along a line CD parallel to the straight-line portion AB of the loading curve (Fig. 9.12). The fact that ϵ does not return to zero after the load has been removed indicates that a *permanent set* or *plastic deformation* of the material has taken place. For most materials, the plastic deformation depends not only upon the maximum value reached by the stress, but also upon the time elapsed before the load is removed. The stress-dependent part of the plastic deformation is referred to as *slip*, and the time-dependent part—which is also influenced by the temperature—as *creep*.

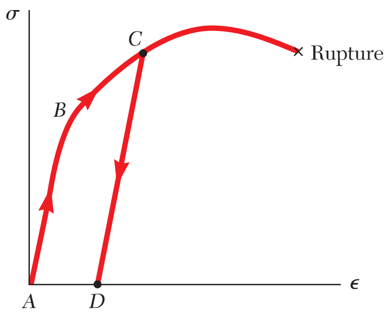


Fig. 9.12

When a material does not possess a well-defined yield point, the elastic limit cannot be determined with precision. However, assuming the elastic limit equal to the yield strength as defined by the offset method (Sec. 9.3) results in only a small error. Indeed, referring to Fig. 9.8, we note that the straight line used to determine point Y also represents the unloading curve after a maximum stress σ_Y has been reached. While the material does not behave truly elastically, the resulting plastic strain is as small as the selected offset.

If, after being loaded and unloaded (Fig. 9.13), the test specimen is loaded again, the new loading curve will closely follow the earlier unloading curve until it almost reaches point C; it will then bend to the right and connect with the curved portion of the original stress-strain diagram. We note that the straight-line portion of the new loading curve is longer than the corresponding portion of the initial one. Thus, the proportional limit and the elastic limit have increased as a result of the strain-hardening that occurred during the earlier loading of the specimen. However, since the point of rupture R remains unchanged, the ductility of the specimen, which should now be measured from point D, has decreased.

We have assumed in our discussion that the specimen was loaded twice in the same direction, i.e., that both loads were tensile loads. Let us now consider the case when the second load is applied in a direction opposite to that of the first one.

We assume that the material is mild steel, for which the yield strength is the same in tension and in compression. The initial load is tensile and is applied until point C has been reached on the stress-strain diagram (Fig. 9.14). After unloading (point D), a compressive load is applied, causing the material to reach point H, where the stress is equal to $-\sigma_Y$. We note that portion DH of the stress-strain diagram is curved and does not show any clearly defined yield point. This is referred to as the *Bauschinger effect*. As the compressive load is maintained, the material yields along line HJ.

If the load is removed after point J has been reached, the stress returns to zero along line JK, and we note that the slope of JK is equal to the modulus of elasticity E. The resulting permanent set AK may be positive, negative, or zero, depending upon the lengths of the segments BC and HJ. If a tensile load is applied again to the test specimen, the portion of the stress-strain diagram beginning at K (dashed line) will curve up and to the right until the yield stress σ_Y has been reached.

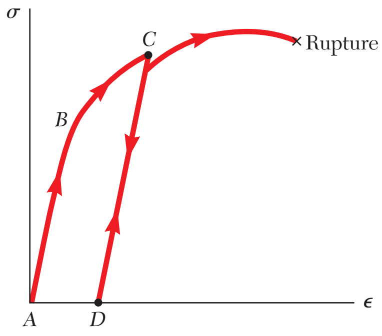


Fig. 9.13

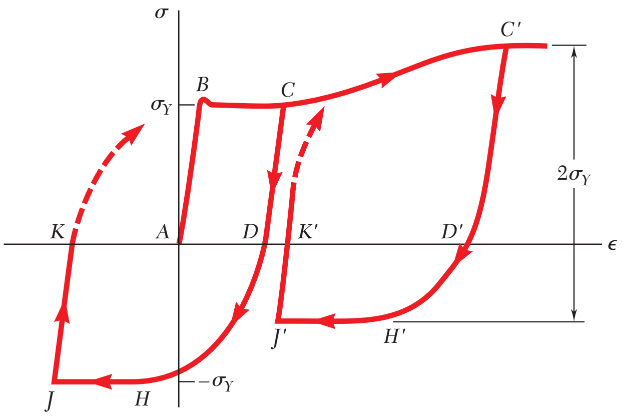


Fig. 9.14

If the initial loading is large enough to cause strain-hardening of the material (point C'), unloading takes place along line $C'D'$. As the reverse load is applied, the stress becomes compressive, reaching its maximum value at H' and maintaining it as the material yields along line $H'J'$. We note that while the maximum value of the compressive stress is less than σ_Y, the total change in stress between C' and H' is still equal to $2\sigma_Y$.

If point K or K' coincides with the origin A of the diagram, the permanent set is equal to zero, and the specimen may appear to have returned to its original condition. However, internal changes will have taken place and, while the same loading sequence may be repeated, the specimen will rupture without any warning after relatively few repetitions. This indicates that the excessive plastic deformations to which the specimen was subjected have caused a radical change in the characteristics of the material. Reverse loadings into the plastic range, therefore, are seldom allowed, and only under carefully controlled conditions. Such situations occur in the straightening of damaged material and in the final alignment of a structure or machine.

*9.6 REPEATED LOADINGS. FATIGUE

In the preceding sections we have considered the behavior of a test specimen subjected to an axial loading. We recall that, if the maximum stress in the specimen does not exceed the elastic limit of the material, the specimen returns to its initial condition when the load is removed. You might conclude that a given loading may be repeated many times, provided that the stresses remain in the elastic range. Such a conclusion is correct for loadings repeated a few dozen or even a few hundred times. However, as you will see, it is not correct when loadings are repeated thousands or millions of times. In such cases, rupture will occur at a stress much lower than the static breaking strength; this phenomenon is known as *fatigue*. A fatigue failure is of a brittle nature, even for materials that are normally ductile.

Fatigue must be considered in the design of all structural and machine components that are subjected to repeated or to fluctuating loads. The number of loading cycles that may be expected during the useful life of a component varies greatly. For example, a beam supporting an industrial crane may be loaded as many as two million times in 25 years (about 300 loadings per working day), an automobile crankshaft will be loaded about half a billion times if the automobile is driven 200,000 miles, and an individual turbine blade may be loaded several hundred billion times during its lifetime.

Some loadings are of a fluctuating nature. For example, the passage of traffic over a bridge will cause stress levels that will fluctuate about the stress level due to the weight of the bridge. A more severe condition occurs when a complete reversal of the load occurs during the loading cycle. The stresses in the axle of a railroad car, for example, are completely reversed after each half-revolution of the wheel.

The number of loading cycles required to cause the failure of a specimen through repeated successive loadings and reverse loadings may be determined experimentally for any given maximum stress level. If a series of tests is conducted, using different maximum stress levels, the resulting data may be plotted as a σ-n curve. For each test, the maximum stress σ is plotted as an ordinate and the number of cycles n as an abscissa; because of the large number of cycles required for rupture, the cycles n are plotted on a logarithmic scale.

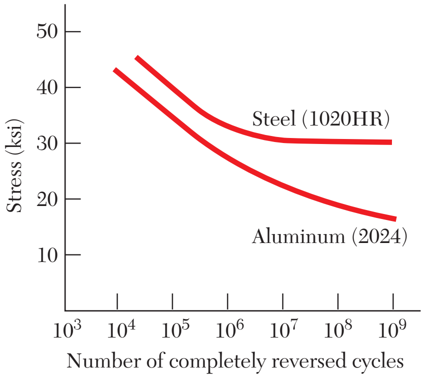


Fig. 9.15

A typical σ-n curve for steel is shown in Fig. 9.15. We note that, if the applied maximum stress is high, relatively few cycles are required to cause rupture. As the magnitude of the maximum stress is reduced, the number of cycles required to cause rupture increases, until a stress, known as the *endurance limit,* is reached. The endurance limit is the stress for which failure does not occur, even for an indefinitely large number of loading cycles. For a low-carbon steel, such as structural steel, the endurance limit is about one-half of the ultimate strength of the steel.

For nonferrous metals, such as aluminum and copper, a typical σ-n curve (Fig. 9.15) shows that the stress at failure continues to decrease as the number of loading cycles is increased. For such metals, one defines the *fatigue limit* as the stress corresponding to failure after a specified number of loading cycles, such as 500 million.

Examination of test specimens, of shafts, of springs, and of other components that have failed in fatigue shows that the failure was initiated at a microscopic crack or at some similar imperfection. At each loading, the crack was very slightly enlarged. During successive loading cycles, the crack propagated through the material until the amount of undamaged material was insufficient to carry the maximum load, and an abrupt, brittle failure occurred. Because fatigue failure may be initiated at any crack or imperfection, the surface condition of a specimen has an important effect on the value of the endurance limit obtained in testing. The endurance limit for machined and polished specimens is higher than for rolled or forged components, or for components that are corroded. In applications in or near seawater, or in other applications where corrosion is expected, a reduction of up to 50% in the endurance limit can be expected.

9.7 DEFORMATIONS OF MEMBERS UNDER AXIAL LOADING

Consider a homogeneous rod BC of length L and uniform cross section of area A subjected to a centric axial load $\mathbf{P}$ (Fig. 9.16). If the resulting axial stress $\sigma = P/A$ does not exceed the proportional limit of the material, we may apply Hooke's law and write

$$\sigma = E\epsilon \tag{9.3}$$

from which it follows that

$$\epsilon = \frac{\sigma}{E} = \frac{P}{AE} \tag{9.4}$$

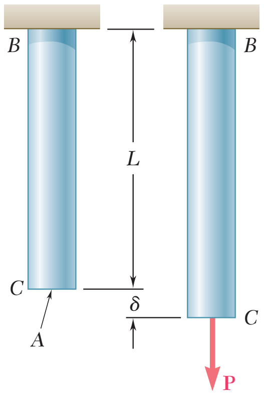


Fig. 9.16

Recalling that the strain ϵ was defined in Sec. 9.2 as $\epsilon = \delta/L$, we have

$$\delta = \epsilon L \tag{9.5}$$

and, substituting for ϵ from (9.4) into (9.5):

$$\boxed{\delta = \frac{PL}{AE}} \tag{9.6}$$

Equation (9.6) may be used only if the rod is homogeneous (constant E), has a uniform cross section of area A, and is loaded at its ends. If the rod is loaded at other points, or if it consists of several portions of various cross sections and possibly of different materials, we must divide it into component parts that satisfy individually the required conditions for the application of formula (9.6). Denoting, respectively, by P_i, L_i, A_i, and E_i the internal force, length, cross-sectional area, and modulus of elasticity corresponding to part i, we express the deformation of the entire rod as

$$\boxed{\delta = \sum_i \frac{P_i L_i}{A_i E_i}} \tag{9.7}$$

We recall from Sec. 9.2 that, in the case of a rod of variable cross section (Fig. 9.5), the strain ϵ depends upon the position of the point Q where it is computed and is defined as $\epsilon = d\delta/dx$. Solving for $d\delta$ and substituting for ϵ from Eq. (9.4), we express the deformation of an element of length dx as

$$d\delta = \epsilon\, dx = \frac{P\, dx}{AE}$$

The total deformation δ of the rod is obtained by integrating this expression over the length L of the rod:

$$\boxed{\delta = \int_0^L \frac{P\, dx}{AE}} \tag{9.8}$$

Formula (9.8) should be used in place of (9.6), not only when the cross-sectional area A is a function of x, but also when the internal force P depends upon x, as is the case for a rod hanging under its own weight.

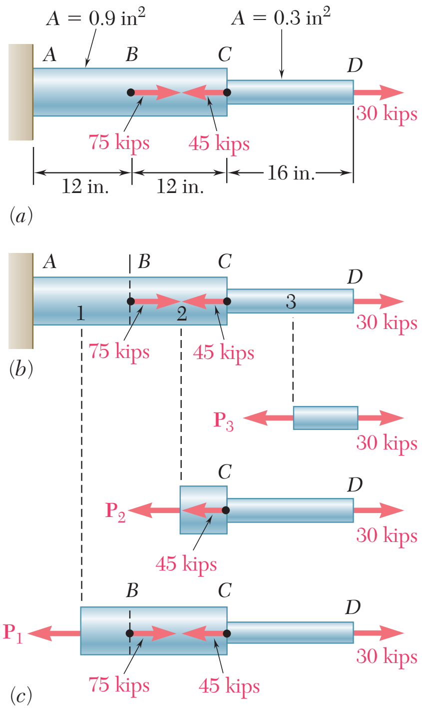


Fig. 9.17

EXAMPLE 9.1 Determine the deformation of the steel rod shown in Fig. 9.17*a* under the given loads ($E = 29 \times 10^6$ psi).

We divide the rod into three component parts shown in Fig. 9.17*b* and write

$$L_1 = L_2 = 12 \text{ in.} \qquad L_3 = 16 \text{ in.}$$
$$A_1 = A_2 = 0.9 \text{ in}^2 \qquad A_3 = 0.3 \text{ in}^2$$

To find the internal forces P_1, P_2, and P_3, we must pass sections through each of the component parts, drawing each time the free-body diagram of the portion of rod located to the right of the section (Fig. 9.17*c*). Expressing that each of the free bodies is in equilibrium, we obtain successively

$$P_1 = 60 \text{ kips} = 60 \times 10^3 \text{ lb}$$
$$P_2 = -15 \text{ kips} = -15 \times 10^3 \text{ lb}$$
$$P_3 = 30 \text{ kips} = 30 \times 10^3 \text{ lb}$$

Carrying the values obtained into Eq. (9.7), we have

$$\begin{aligned}\delta &= \sum_i \frac{P_i L_i}{A_i E_i} = \frac{1}{E}\left(\frac{P_1 L_1}{A_1} + \frac{P_2 L_2}{A_2} + \frac{P_3 L_3}{A_3}\right)\\ &= \frac{1}{29 \times 10^6}\left[\frac{(60 \times 10^3)(12)}{0.9}\right.\\ &\quad \left. + \frac{(-15 \times 10^3)(12)}{0.9} + \frac{(30 \times 10^3)(16)}{0.3}\right]\\ \delta &= \frac{2.20 \times 10^6}{29 \times 10^6} = 75.9 \times 10^{-3} \text{ in.} \end{aligned}$$

The rod BC of Fig. 9.16, which was used to derive formula (9.6), and the rod AD of Fig. 9.17, which has just been discussed in Example 9.1, both had one end attached to a fixed support. In each case, therefore, the deformation δ of the rod was equal to the displacement of its free end. When both ends of a rod move, however, the deformation of the rod is measured by the *relative displacement* of one end of the rod with respect to the other. Consider, for instance, the assembly shown in Fig. 9.18*a*, which consists of three elastic bars of length L connected by a rigid pin at A. If a load $\mathbf{P}$ is applied at B (Fig. 9.18*b*), each of the three bars will deform. Since the bars AC and AC' are attached to fixed supports at C and C', their common deformation is measured by the displacement δ_A of point A. On the other hand, since both ends of bar AB move, the deformation of AB is measured by the difference between the displacements δ_A and δ_B of points A and B, i.e., by the relative displacement of B with respect to A. Denoting this relative displacement by $\delta_{B/A}$, we write

$$\delta_{B/A} = \delta_B - \delta_A = \frac{PL}{AE} \tag{9.9}$$

where A is the cross-sectional area of AB and E is its modulus of elasticity.

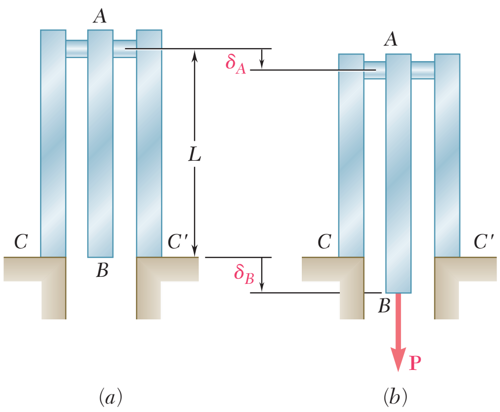
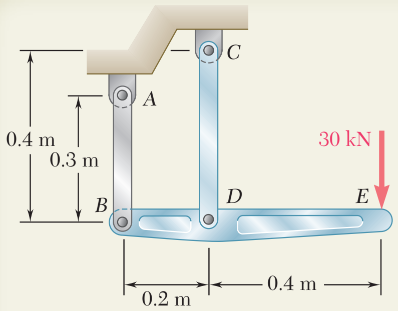


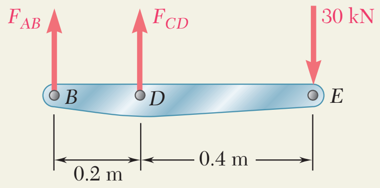
Fig. 9.18

SAMPLE PROBLEM 9.1



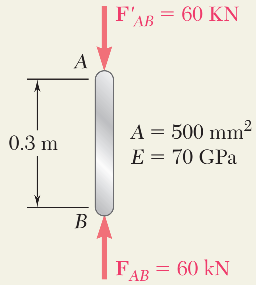
The rigid bar BDE is supported by two links AB and CD. Link AB is made of aluminum ($E = 70$ GPa) and has a cross-sectional area of 500 mm²; link CD is made of steel ($E = 200$ GPa) and has a cross-sectional area of 600 mm². For the 30-kN force shown, determine the deflection (*a*) of B, (*b*) of D, (*c*) of E.

SOLUTION



Free Body: Bar *BDE*

$$+\circlearrowleft \Sigma M_B = 0: \qquad -(30\text{ kN})(0.6\text{ m}) + F_{CD}(0.2\text{ m}) = 0$$
$$F_{CD} = +90\text{ kN} \qquad F_{CD} = 90\text{ kN} \quad \textit{tension}$$
$$+\circlearrowleft \Sigma M_D = 0: \qquad -(30\text{ kN})(0.4\text{ m}) - F_{AB}(0.2\text{ m}) = 0$$
$$F_{AB} = -60\text{ kN} \qquad F_{AB} = 60\text{ kN} \quad \textit{compression}$$

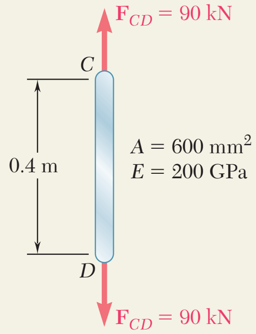


a. Deflection of *B*. Since the internal force in link AB is compressive, we have $P = -60$ kN

$$\delta_B = \frac{PL}{AE} = \frac{(-60 \times 10^3\text{ N})(0.3\text{ m})}{(500 \times 10^{-6}\text{ m}^2)(70 \times 10^9\text{ Pa})} = -514 \times 10^{-6}\text{ m}$$

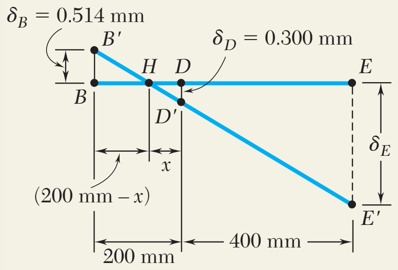
The negative sign indicates a contraction of member AB, and, thus, an upward deflection of end B:

$$\delta_B = 0.514\text{ mm} \uparrow \quad ◀$$



b. Deflection of *D*. Since in rod CD, $P = 90$ kN, we write

$$\delta_D = \frac{PL}{AE} = \frac{(90 \times 10^3\text{ N})(0.4\text{ m})}{(600 \times 10^{-6}\text{ m}^2)(200 \times 10^9\text{ Pa})}$$
$$= 300 \times 10^{-6}\text{ m} \qquad \delta_D = 0.300\text{ mm} \downarrow \quad ◀$$

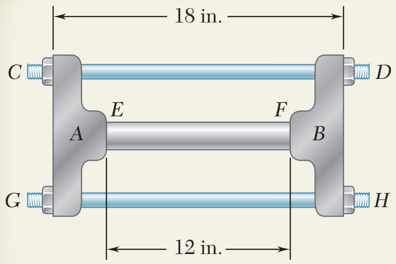


c. Deflection of *E*. We denote by B' and D' the displaced positions of points B and D. Since the bar BDE is rigid, points B', D', and E' lie in a straight line and we write

$$\frac{BB'}{DD'} = \frac{BH}{HD} \qquad \frac{0.514\text{ mm}}{0.300\text{ mm}} = \frac{(200\text{ mm}) - x}{x} \qquad x = 73.7\text{ mm}$$

$$\frac{EE'}{DD'} = \frac{HE}{HD} \qquad \frac{\delta_E}{0.300\text{ mm}} = \frac{(400\text{ mm}) + (73.7\text{ mm})}{73.7\text{ mm}}$$

$$\delta_E = 1.928\text{ mm} \downarrow \quad ◀$$



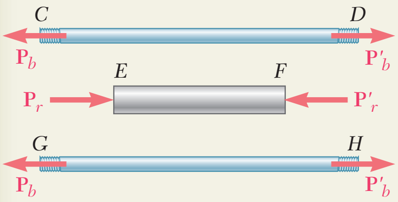
SAMPLE PROBLEM 9.2

The rigid castings A and B are connected by two $\frac{3}{4}$-in.-diameter steel bolts CD and GH and are in contact with the ends of a 1.5-in.-diameter aluminum rod EF. Each bolt is single-threaded with a pitch of 0.1 in., and after being snugly fitted, the nuts at D and H are both tightened one-quarter of a turn. Knowing that E is 29×10^6 psi for steel and 10.6×10^6 psi for aluminum, determine the normal stress in the rod.

SOLUTION

Deformations

Bolts CD and GH. Tightening the nuts causes tension in the bolts. Because of symmetry, both are subjected to the same internal force P_b and undergo the same deformation δ_b. We have



$$\delta_b = +\frac{P_b L_b}{A_b E_b} = +\frac{P_b(18 \text{ in.})}{\frac{1}{4}\pi(0.75 \text{ in.})^2(29 \times 10^6 \text{ psi})} = +1.405 \times 10^{-6} P_b \qquad (1)$$

Rod EF. The rod is in compression. Denoting by P_r the magnitude of the force in the rod and by δ_r the deformation of the rod, we write

$$\delta_r = -\frac{P_r L_r}{A_r E_r} = -\frac{P_r(12 \text{ in.})}{\frac{1}{4}\pi(1.5 \text{ in.})^2(10.6 \times 10^6 \text{ psi})} = -0.6406 \times 10^{-6} P_r \qquad (2)$$

Displacement of D Relative to B. Tightening the nuts one-quarter of a turn causes ends D and H of the bolts to undergo a displacement of $\frac{1}{4}(0.1 \text{ in.})$ *relative to casting* B. Considering end D, we write

$$\delta_{D/B} = \tfrac{1}{4}(0.1 \text{ in.}) = 0.025 \text{ in.} \qquad (3)$$

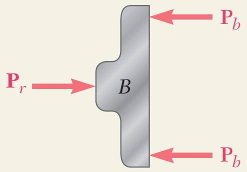
But $\delta_{D/B} = \delta_D - \delta_B$, where δ_D and δ_B represent the displacements of D and B. If we assume that casting A is held in a fixed position while the nuts at D and H are being tightened, these displacements are equal to the deformations of the bolts and of the rod, respectively. We have, therefore,

$$\delta_{D/B} = \delta_b - \delta_r \qquad (4)$$

Substituting from (1), (2), and (3) into (4), we obtain

$$0.025 \text{ in.} = 1.405 \times 10^{-6} P_b + 0.6406 \times 10^{-6} P_r \qquad (5)$$

Free Body: Casting B



$$\overset{+}{\rightarrow}\Sigma F = 0: \qquad P_r - 2P_b = 0 \qquad P_r = 2P_b \qquad (6)$$

Forces in Bolts and Rod Substituting for P_r from (6) into (5), we have

$$0.025 \text{ in.} = 1.405 \times 10^{-6} P_b + 0.6406 \times 10^{-6}(2P_b)$$
$$P_b = 9.307 \times 10^3 \text{ lb} = 9.307 \text{ kips}$$
$$P_r = 2P_b = 2(9.307 \text{ kips}) = 18.61 \text{ kips}$$

Stress in Rod

$$\sigma_r = \frac{P_r}{A_r} = \frac{18.61 \text{ kips}}{\frac{1}{4}\pi(1.5 \text{ in.})^2} \qquad \sigma_r = 10.53 \text{ ksi} \blacktriangleleft$$

PROBLEMS

9.1 A 4.8-ft-long steel wire of $\frac{1}{4}$-in. diameter is subjected to a 750-lb tensile load. Knowing that $E = 29 \times 10^6$ psi, determine (*a*) the elongation of the wire, (*b*) the corresponding normal stress.

9.2 Two gage marks are placed exactly 250 mm apart on a 12-mm-diameter aluminum rod with $E = 73$ GPa and an ultimate strength of 140 MPa. Knowing that the distance between the gage marks is 250.28 mm after a load is applied, determine (*a*) the stress in the rod, (*b*) the factor of safety.

9.3 A nylon thread is subjected to a 2-lb tensile load. Knowing that $E = 0.7 \times 10^6$ psi and that the length of the thread increases by 1.1%, determine (*a*) the diameter of the thread, (*b*) the stress in the thread.

9.4 A 9-m length of 6-mm-diameter steel wire is to be used in a hanger. It is noted that the wire stretches 18 mm when a tensile force **P** is applied. Knowing that $E = 200$ GPa, determine (*a*) the magnitude of the force **P**, (*b*) the corresponding normal stress in the wire.

9.5 A steel rod is 2.2 m long and must not stretch more than 1.2 mm when an 8.5-kN load is applied to it. Knowing that $E = 200$ GPa, determine (*a*) the smallest diameter rod that should be used, (*b*) the corresponding normal stress caused by the load.

9.6 A control rod made of yellow brass must not stretch more than $\frac{1}{8}$ in. when the tension in the wire is 800 lb. Knowing that $E = 15 \times 10^6$ psi and that the maximum allowable normal stress is 32 ksi, determine (*a*) the smallest diameter that can be selected for the rod, (*b*) the corresponding maximum length of the rod.

9.7 An aluminum pipe must not stretch more than 0.05 in. when it is subjected to a tensile load. Knowing that $E = 10.1 \times 10^6$ psi and that the allowable tensile strength is 14 ksi, determine (*a*) the maximum allowable length of the pipe, (*b*) the required area of the pipe if the tensile load is 127.5 kips.

9.8 A cast-iron tube is used to support a compressive load. Knowing that $E = 69$ GPa and that the maximum allowable change in length is 0.025%, determine (*a*) the maximum normal stress in the tube, (*b*) the minimum wall thickness for a load of 7.2 kN if the outside diameter of the tube is 50 mm.

9.9 A block of 10-in. length and 1.8×1.6-in. cross section is to support a centric compressive load **P**. The material to be used is a bronze for which $E = 14 \times 10^6$ psi. Determine the largest load that can be applied, knowing that the normal stress must not exceed 18 ksi and that the decrease in length of the block should be at most 0.12% of its original length.

9.10 A 9-kN tensile load will be applied to a 50-m length of steel wire with $E = 200$ GPa. Determine the smallest-diameter wire that can be used knowing that the normal stress must not exceed 150 MPa and that the increase in the length of the wire should be at most 25 mm.

9.11 The 4-mm-diameter cable BC is made of a steel with $E = 200$ GPa. Knowing that the maximum stress in the cable must not exceed 190 MPa and that the elongation of the cable must not exceed 6 mm, find the maximum load **P** that can be applied as shown.

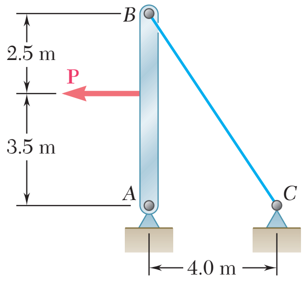


Fig. P9.11

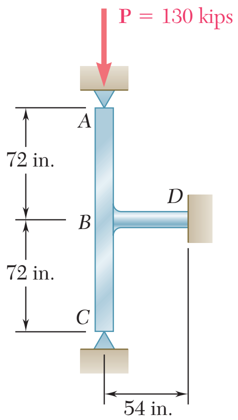


Fig. P9.12

9.12 Rod BD is made of steel ($E = 29 \times 10^6$ psi) and is used to brace the axially compressed member ABC. The maximum force that can be developed in member BD is $0.02P$. If the stress must not exceed 18 ksi and the maximum change in length of BD must not exceed 0.001 times the length of ABC, determine the smallest-diameter rod that can be used for member BD.

9.13 The specimen shown is made from a 1-in.-diameter cylindrical steel rod with two 1.5-in.-outer-diameter sleeves bonded to the rod as shown. Knowing that $E = 29 \times 10^6$ psi, determine (*a*) the load **P** so that the total deformation is 0.002 in., (*b*) the corresponding deformation of the central portion BC.

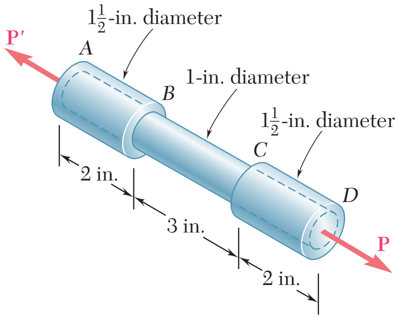


Fig. P9.13

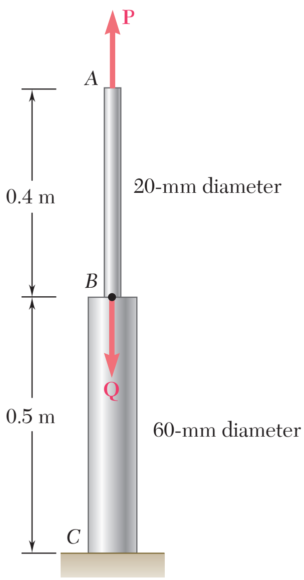


Fig. P9.14 and P9.15

9.14 Both portions of the rod ABC are made of an aluminum for which $E = 70$ GPa. Knowing that the magnitude of **P** is 4 kN, determine (*a*) the value of **Q** so that the deflection at A is zero, (*b*) the corresponding deflection of B.

9.15 The rod ABC is made of an aluminum for which $E = 70$ GPa. Knowing that $P = 6$ kN and $Q = 42$ kN, determine the deflection of (*a*) point A, (*b*) point B.

9.16 Two solid cylindrical rods are joined at B and loaded as shown. Rod AB is made of steel ($E = 29 \times 10^6$ psi), and rod BC of brass ($E = 15 \times 10^6$ psi). Determine (*a*) the total deformation of the composite rod ABC, (*b*) the deflection of point B.

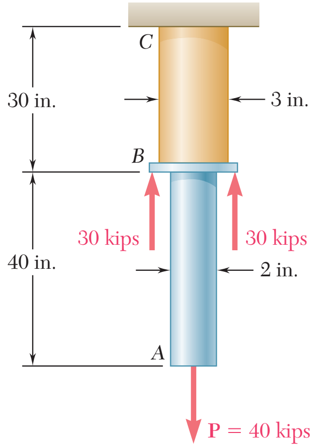


Fig. *P9.16*

9.17 A $\frac{1}{8}$-in.-thick hollow polystyrene cylinder ($E = 0.45 \times 10^6$ psi) and a rigid circular plate (only part of which is shown) are used to support a 10-in.-long steel rod AB ($E = 29 \times 10^6$ psi) of $\frac{1}{4}$-in. diameter. If an 800-lb load **P** is applied at B, determine (*a*) the elongation of rod AB, (*b*) the deflection of point B, (*c*) the average normal stress in rod AB.

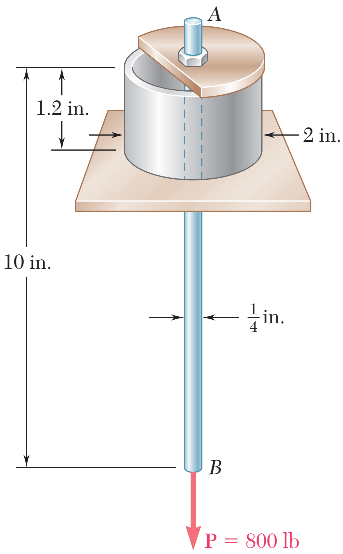


Fig. P9.17

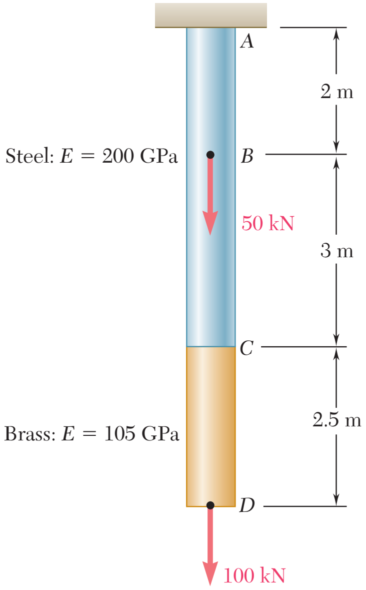


Fig. P9.18

9.18 The 36-mm-diameter steel rod ABC and a brass rod CD of the same diameter are joined at point C to form the 7.5-m rod $ABCD$. For the loading shown and neglecting the weight of the rod, determine the deflection of (*a*) point C, (*b*) point D.

9.19 The steel frame ($E = 200$ GPa) shown has a diagonal brace BD with an area of 1920 mm^2. Determine the largest allowable load $\mathbf{P}$ if the change in length of member BD is not to exceed 1.6 mm.

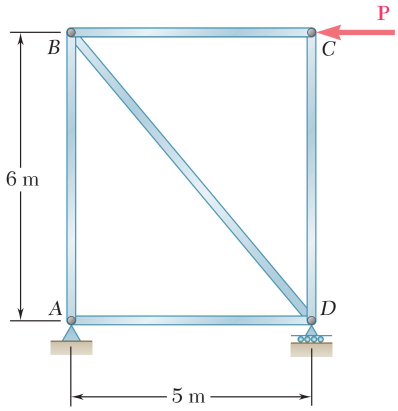


Fig. P9.19

9.20 For the steel truss ($E = 29 \times 10^6$ psi) and loading shown, determine the deformations of members AB and AD, knowing that their cross-sectional areas are 4.0 in^2 and 2.8 in^2, respectively.

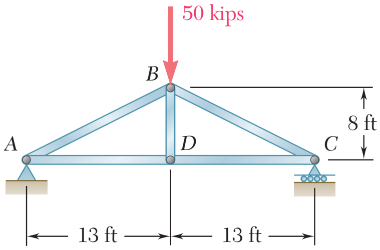


Fig. P9.20

9.21 Members AB and BC are made of steel ($E = 29 \times 10^6$ psi) with cross-sectional areas of 0.80 in^2 and 0.64 in^2, respectively. For the loading shown, determine the elongation of (a) member AB, (b) member BC.

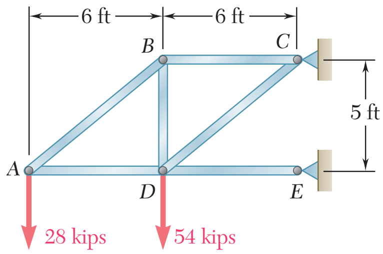


Fig. P9.21

9.22 Members ABC and DEF are joined with steel links ($E = 200$ GPa). Each of the links is made of a pair of 25×35-mm plates. Determine the change in length of (a) member BE, (b) member CF.

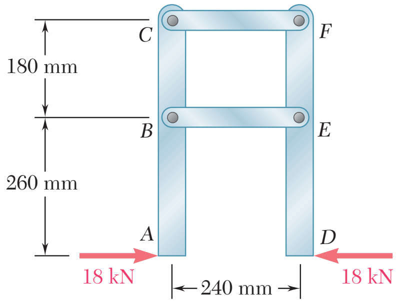


Fig. *P9.22*

9.23 Each of the links AB and CD is made of aluminum ($E = 75$ GPa) and has a cross-sectional area of 125 mm^2. Knowing that they support the rigid member BC, determine the deflection of point E.

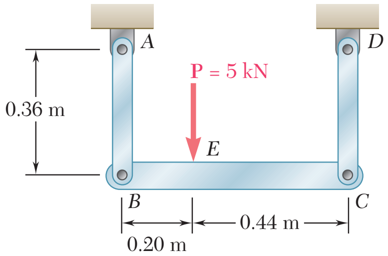


Fig. P9.23

9.24 Link BD is made of brass ($E = 15 \times 10^6$ psi) and has a cross-sectional area of 0.40 in^2. Link CE is made of aluminum ($E = 10.4 \times 10^6$ psi) and has a cross-sectional area of 0.50 in^2. Determine the maximum force $\mathbf{P}$ that can be applied vertically at point A if the deflection of A is not to exceed 0.014 in.

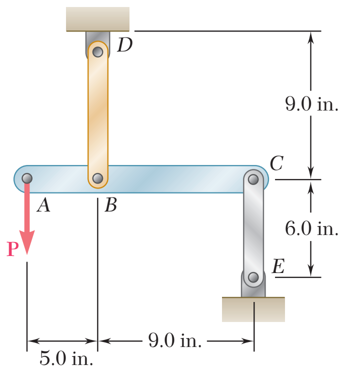


Fig. *P9.24*

9.8 STATICALLY INDETERMINATE PROBLEMS

In the problems considered in the preceding section, we could always use free-body diagrams and equilibrium equations to determine the internal forces produced in the various portions of a member under given loading conditions. The values obtained for the internal forces were then entered into Eq. (9.7) or (9.8) to obtain the deformation of the member.

There are many problems, however, in which the internal forces cannot be determined from statics alone. In fact, in most of these problems the reactions themselves—which are external forces—cannot be determined by simply drawing a free-body diagram of the member and writing the corresponding equilibrium equations. The equilibrium equations must be complemented by relations involving deformations obtained by considering the geometry of the problem. Because statics is not sufficient to determine either the reactions or the internal forces, problems of this type are said to be *statically indeterminate*. The following examples will show how to handle this type of problem.

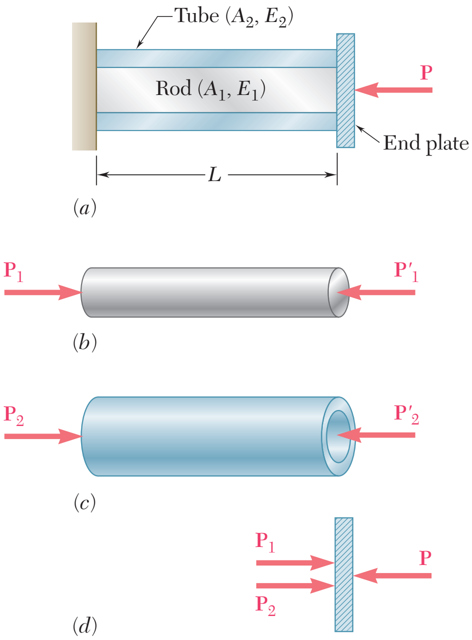


Fig. 9.19

EXAMPLE 9.2 A rod of length L, cross-sectional area A_1, and modulus of elasticity E_1, has been placed inside a tube of the same length L, but of cross-sectional area A_2 and modulus of elasticity E_2 (Fig. 9.19*a*). What is the deformation of the rod and tube when a force **P** is exerted on a rigid end plate as shown?

Denoting by P_1 and P_2, respectively, the axial forces in the rod and in the tube, we draw free-body diagrams of all three elements (Fig. 9.19*b*, *c*, *d*). Only the last of the diagrams yields any significant information, namely:

$$P_1 + P_2 = P \tag{9.10}$$

Clearly, one equation is not sufficient to determine the two unknown internal forces P_1 and P_2. The problem is statically indeterminate.

However, the geometry of the problem shows that the deformations δ_1 and δ_2 of the rod and tube must be equal. Recalling Eq. (9.6), we write

$$\delta_1 = \frac{P_1 L}{A_1 E_1} \qquad \delta_2 = \frac{P_2 L}{A_2 E_2} \tag{9.11}$$

Equating the deformations δ_1 and δ_2, we obtain:

$$\frac{P_1}{A_1 E_1} = \frac{P_2}{A_2 E_2} \tag{9.12}$$

Equations (9.10) and (9.12) can be solved simultaneously for P_1 and P_2:

$$P_1 = \frac{A_1 E_1 P}{A_1 E_1 + A_2 E_2} \qquad P_2 = \frac{A_2 E_2 P}{A_1 E_1 + A_2 E_2}$$

Either of Eqs. (9.11) can then be used to determine the common deformation of the rod and tube. ■

EXAMPLE 9.3 A bar AB of length L and uniform cross section is attached to rigid supports at A and B before being loaded. What are the stresses in portions AC and BC due to the application of a load P at point C (Fig. 9.20*a*)?

Drawing the free-body diagram of the bar (Fig. 9.20*b*), we obtain the equilibrium equation

$$R_A + R_B = P \tag{9.13}$$

Since this equation is not sufficient to determine the two unknown reactions R_A and R_B, the problem is statically indeterminate.

However, the reactions may be determined if we observe from the geometry that the total elongation δ of the bar must be zero. Denoting by δ_1 and δ_2, respectively, the elongations of the portions AC and BC, we write

$$\delta = \delta_1 + \delta_2 = 0$$

or, expressing δ_1 and δ_2 in terms of the corresponding internal forces P_1 and P_2:

$$\delta = \frac{P_1L_1}{AE} + \frac{P_2L_2}{AE} = 0 \tag{9.14}$$

But we note from the free-body diagrams shown respectively in parts b and c of Fig. 9.21 that $P_1 = R_A$ and $P_2 = -R_B$. Carrying these values into (9.14), we write

$$R_AL_1 - R_BL_2 = 0 \tag{9.15}$$

Equations (9.13) and (9.15) can be solved simultaneously for R_A and R_B; we obtain $R_A = PL_2/L$ and $R_B = PL_1/L$. The desired stresses σ_1 in AC and σ_2 in BC are obtained by dividing, respectively, $P_1 = R_A$ and $P_2 = -R_B$ by the cross-sectional area of the bar:

$$\sigma_1 = \frac{PL_2}{AL} \qquad \sigma_2 = -\frac{PL_1}{AL} \quad ■$$

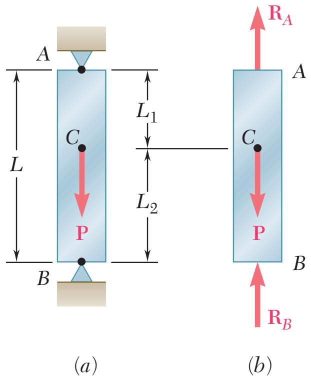


Fig. 9.20

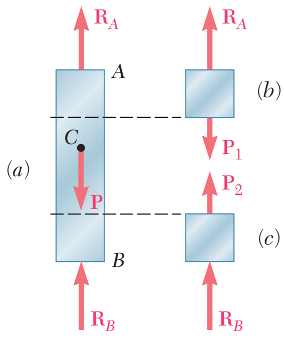


Fig. 9.21

Superposition Method. We observe that a structure is statically indeterminate whenever it is held by more supports than are required to maintain its equilibrium. This results in more unknown reactions than available equilibrium equations. It is often found convenient to designate one of the reactions as *redundant* and to eliminate the corresponding support. Since the stated conditions of the problem cannot be arbitrarily changed, the redundant reaction must be maintained in the solution. But it will be treated as an *unknown load* that, together with the other loads, must produce deformations that are compatible with the original constraints. The actual solution of the problem is carried out by considering separately the deformations caused by the given loads and by the redundant reaction, and by adding—or *superposing*—the results obtained.†

†The general conditions under which the combined effect of several loads can be obtained in this way are discussed in Sec. 9.11.

EXAMPLE 9.4 Determine the reactions at A and B for the steel bar and loading shown in Fig. 9.22, assuming a close fit at both supports before the loads are applied.

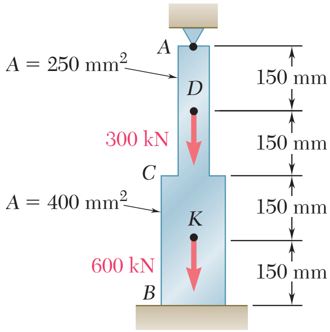


Fig. 9.22

We consider the reaction at B as redundant and release the bar from that support. The reaction $\mathbf{R}_B$ is now considered as an unknown load (Fig. 9.23*a*) and will be determined from the condition that the deformation δ of the rod must be equal to zero. The solution is carried out by considering separately the deformation δ_L caused by the given loads (Fig. 9.23*b*) and the deformation δ_R due to the redundant reaction $\mathbf{R}_B$ (Fig. 9.23*c*).

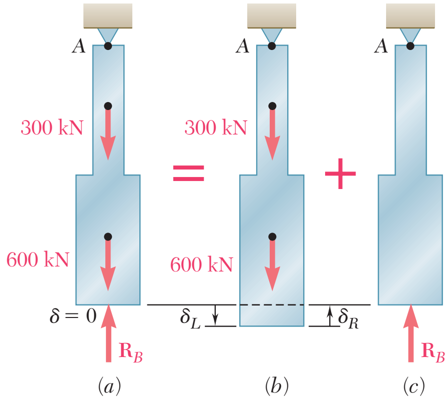


Fig. 9.23

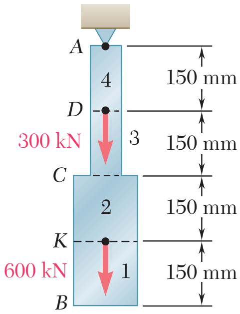


Fig. 9.24

The deformation δ_L is obtained from Eq. (9.7) after the bar has been divided into four portions, as shown in Fig. 9.24.
Following the same procedure as in Example 9.1, we write

$$P_1 = 0 \qquad P_2 = P_3 = 600 \times 10^3\ \mathrm{N} \qquad P_4 = 900 \times 10^3\ \mathrm{N}$$
$$A_1 = A_2 = 400 \times 10^{-6}\ \mathrm{m}^2 \qquad A_3 = A_4 = 250 \times 10^{-6}\ \mathrm{m}^2$$
$$L_1 = L_2 = L_3 = L_4 = 0.150\ \mathrm{m}$$

Substituting these values into Eq. (9.7), we obtain

$$\delta_L = \sum_{i=1}^{4} \frac{P_i L_i}{A_i E} = \left(0 + \frac{600 \times 10^3\ \mathrm{N}}{400 \times 10^{-6}\ \mathrm{m}^2} + \frac{600 \times 10^3\ \mathrm{N}}{250 \times 10^{-6}\ \mathrm{m}^2} + \frac{900 \times 10^3\ \mathrm{N}}{250 \times 10^{-6}\ \mathrm{m}^2}\right)\frac{0.150\ \mathrm{m}}{E}$$

$$\delta_L = \frac{1.125 \times 10^9}{E} \tag{9.16}$$

Considering now the deformation δ_R due to the redundant reaction R_B, we divide the bar into two portions, as shown in Fig. 9.25, and write

$$P_1 = P_2 = -R_B$$
$$A_1 = 400 \times 10^{-6}\,\text{m}^2 \qquad A_2 = 250 \times 10^{-6}\,\text{m}^2$$
$$L_1 = L_2 = 0.300\ \text{m}$$

Substituting these values into Eq. (9.7), we obtain

$$\delta_R = \frac{P_1 L_1}{A_1 E} + \frac{P_2 L_2}{A_2 E} = -\frac{(1.95 \times 10^3) R_B}{E} \tag{9.17}$$

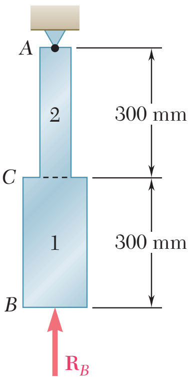


Fig. 9.25

Expressing that the total deformation δ of the bar must be zero, we write

$$\delta = \delta_L + \delta_R = 0 \tag{9.18}$$

and, substituting for δ_L and δ_R from (9.16) and (9.17) into (9.18),

$$\delta = \frac{1.125 \times 10^9}{E} - \frac{(1.95 \times 10^3) R_B}{E} = 0$$

Solving for R_B, we have

$$R_B = 577 \times 10^3\,\text{N} = 577\ \text{kN}$$

The reaction R_A at the upper support is obtained from the free-body diagram of the bar (Fig. 9.26). We write

$$+\uparrow \Sigma F_y = 0: \qquad R_A - 300\ \text{kN} - 600\ \text{kN} + R_B = 0$$
$$R_A = 900\ \text{kN} - R_B = 900\ \text{kN} - 577\ \text{kN} = 323\ \text{kN}$$

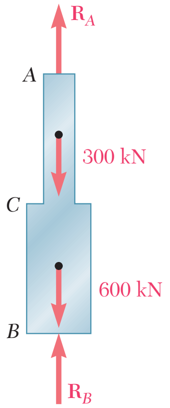


Fig. 9.26

Once the reactions have been determined, the stresses and strains in the bar can easily be obtained. It should be noted that, while the total deformation of the bar is zero, each of its component parts *does deform* under the given loading and restraining conditions. ■

EXAMPLE 9.5 Determine the reactions at A and B for the steel bar and loading of Example 9.4, assuming now that a 4.50-mm clearance exists between the bar and the ground before the loads are applied (Fig. 9.27). Assume $E = 200$ GPa.

We follow the same procedure as in Example 9.4. Considering the reaction at B as redundant, we compute the deformations δ_L and δ_R caused, respectively, by the given loads and by the redundant reaction $\mathbf{R}_B$. However, in this case the total deformation is not zero, but $\delta = 4.5$ mm. We write therefore

$$\delta = \delta_L + \delta_R = 4.5 \times 10^{-3}\ \text{m} \tag{9.19}$$

Substituting for δ_L and δ_R from (9.16) and (9.17) into (9.19), and recalling that $E = 200\ \text{GPa} = 200 \times 10^9$ Pa, we have

$$\delta = \frac{1.125 \times 10^9}{200 \times 10^9} - \frac{(1.95 \times 10^3) R_B}{200 \times 10^9} = 4.5 \times 10^{-3}\ \text{m}$$

Solving for R_B, we obtain

$$R_B = 115.4 \times 10^3\ \text{N} = 115.4\ \text{kN}$$

The reaction at A is obtained from the free-body diagram of the bar (Fig. 9.27):

$$+\uparrow \Sigma F_y = 0: \qquad R_A - 300\ \text{kN} - 600\ \text{kN} + R_B = 0$$
$$R_A = 900\ \text{kN} - R_B = 900\ \text{kN} - 115.4\ \text{kN} = 785\ \text{kN} \quad ■$$

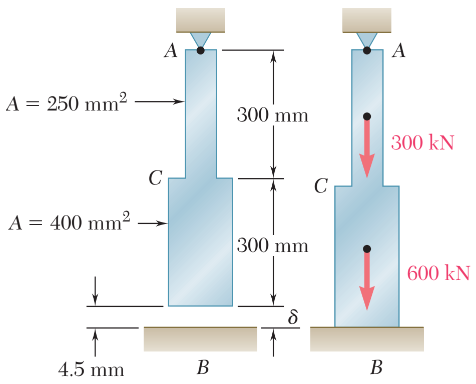


Fig. 9.27

9.9 PROBLEMS INVOLVING TEMPERATURE CHANGES

All of the members and structures that we have considered so far were assumed to remain at the same temperature while they were being loaded. We are now going to consider various situations involving changes in temperature.

Let us first consider a homogeneous rod AB of uniform cross section, which rests freely on a smooth horizontal surface (Fig. 9.28*a*). If the temperature of the rod is raised by ΔT, we observe that the rod elongates by an amount δ_T which is proportional to both the temperature change ΔT and the length L of the rod (Fig. 9.28*b*). We have

$$\delta_T = \alpha(\Delta T)L \tag{9.20}$$

where α is a constant characteristic of the material, called the *coefficient of thermal expansion*. Since δ_T and L are both expressed in

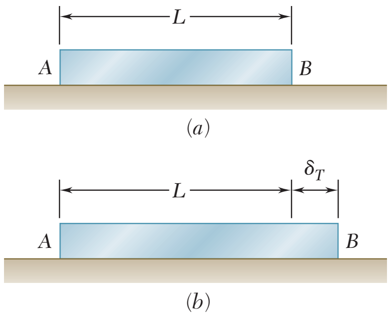


Fig. 9.28

units of length, α represents a quantity *per degree C*, or *per degree F*, depending whether the temperature change is expressed in degrees Celsius or in degrees Fahrenheit.

With the deformation δ_T must be associated a strain $\epsilon_T = \delta_T/L$. Recalling Eq. (9.20), we conclude that

$$\epsilon_T = \alpha \, \Delta T \tag{9.21}$$

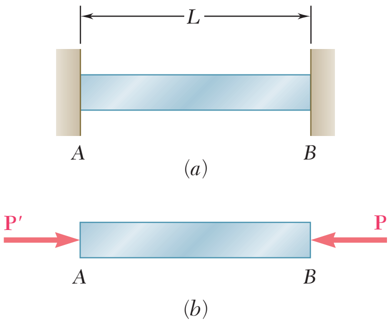


Fig. 9.29

The strain ϵ_T is referred to as a *thermal strain*, since it is caused by the change in temperature of the rod. In the case we are considering here, there is *no stress associated with the strain* ϵ_T.

Let us now assume that the same rod AB of length L is placed between two fixed supports at a distance L from each other (Fig. 9.29*a*). Again, there is neither stress nor strain in this initial condition. If we raise the temperature by ΔT, the rod cannot elongate because of the restraints imposed on its ends; the elongation δ_T of the rod is thus zero. Since the rod is homogeneous and of uniform cross section, the strain ϵ_T at any point is $\epsilon_T = \delta_T/L$ and, thus, also zero. However, the supports will exert equal and opposite forces **P** and **P**′ on the rod after the temperature has been raised, to keep it from elongating (Fig. 9.29*b*). It thus follows that a state of stress (with no corresponding strain) is created in the rod.

As we prepare to determine the stress σ created by the temperature change ΔT, we observe that the problem we have to solve is statically indeterminate. Therefore, we should first compute the magnitude P

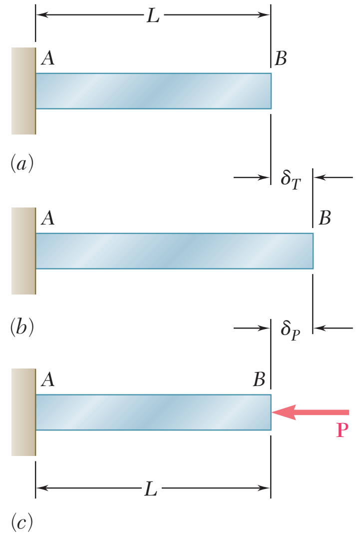


Fig. 9.30

of the reactions at the supports from the condition that the elongation of the rod is zero. Using the superposition method described in Sec. 9.8, we detach the rod from its support B (Fig. 9.30*a*) and let it elongate freely as it undergoes the temperature change ΔT (Fig. 9.30*b*). According to formula (9.20), the corresponding elongation is

$$\delta_T = \alpha(\Delta T)L$$

Applying now to end B the force **P** representing the redundant reaction, and recalling formula (9.6), we obtain a second deformation (Fig. 9.30*c*)

$$\delta_P = \frac{PL}{AE}$$

Expressing that the total deformation δ must be zero, we have

$$\delta = \delta_T + \delta_P = \alpha(\Delta T)L + \frac{PL}{AE} = 0$$

from which we conclude that

$$P = -AE\alpha(\Delta T)$$

and that the stress in the rod due to the temperature change ΔT is

$$\sigma = \frac{P}{A} = -E\alpha(\Delta T) \tag{9.22}$$

It should be kept in mind that the result we have obtained here and our earlier remark regarding the absence of any strain in the rod *apply only in the case of a homogeneous rod of uniform cross section*. Any other problem involving a restrained structure undergoing a change in temperature must be analyzed on its own merits. However, the same general approach can be used, i.e., we can consider separately the deformation due to the temperature change and the deformation due to the redundant reaction and superpose the solutions obtained.

EXAMPLE 9.6 Determine the values of the stress in portions AC and CB of the steel bar shown (Fig. 9.31) when the temperature of the bar is $-50°F$, knowing that a close fit exists at both of the rigid supports when the temperature is $+75°F$. Use the values $E = 29 \times 10^6$ psi and $\alpha = 6.5 \times 10^6/°F$ for steel.

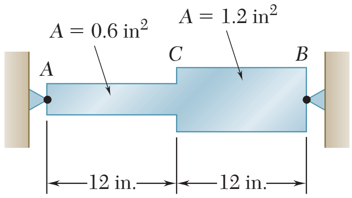


Fig. 9.31

We first determine the reactions at the supports. Since the problem is statically indeterminate, we detach the bar from its support at B and let it undergo the temperature change

$$\Delta T = (-50°F) - (75°F) = -125°F$$

The corresponding deformation (Fig. 9.32b) is

$$\begin{aligned}\delta_T &= \alpha(\Delta T)L = (6.5 \times 10^{-6}/°F)(-125°F)(24 \text{ in.})\\ &= -19.50 \times 10^{-3} \text{ in.}\end{aligned}$$

Applying now the unknown force $\mathbf{R}_B$ at end B (Fig. 9.32c), we use Eq. (9.7) to express the corresponding deformation δ_R. Substituting

$$L_1 = L_2 = 12 \text{ in.}$$
$$A_1 = 0.6 \text{ in}^2 \qquad A_2 = 1.2 \text{ in}^2$$
$$P_1 = P_2 = R_B \qquad E = 29 \times 10^6 \text{ psi}$$

into Eq. (9.7), we write

$$\begin{aligned}\delta_R &= \frac{P_1L_1}{A_1E} + \frac{P_2L_2}{A_2E}\\ &= \frac{R_B}{29 \times 10^6 \text{ psi}}\left(\frac{12 \text{ in.}}{0.6 \text{ in}^2} + \frac{12 \text{ in.}}{1.2 \text{ in}^2}\right)\\ &= (1.0345 \times 10^{-6} \text{ in./lb})R_B\end{aligned}$$

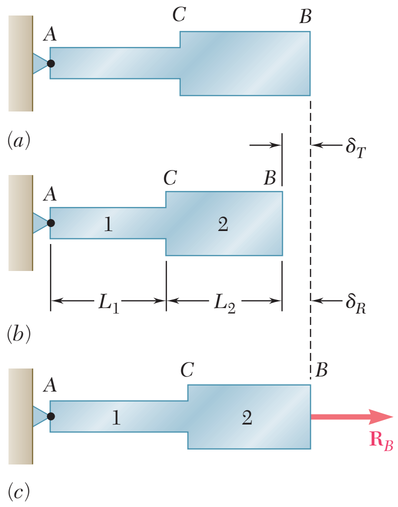


Fig. 9.32

Expressing that the total deformation of the bar must be zero as a result of the imposed constraints, we write

$$\delta = \delta_T + \delta_R = 0$$
$$= -19.50 \times 10^{-3} \text{ in.} + (1.0345 \times 10^{-6} \text{ in./lb})R_B = 0$$

from which we obtain

$$R_B = 18.85 \times 10^3 \text{ lb} = 18.85 \text{ kips}$$

The reaction at A is equal and opposite.

Noting that the forces in the two portions of the bar are $P_1 = P_2 = 18.85$ kips, we obtain the following values of the stress in portions AC and CB of the bar:

$$\sigma_1 = \frac{P_1}{A_1} = \frac{18.85 \text{ kips}}{0.6 \text{ in}^2} = +31.42 \text{ ksi}$$
$$\sigma_2 = \frac{P_2}{A_2} = \frac{18.85 \text{ kips}}{1.2 \text{ in}^2} = +15.71 \text{ ksi}$$

We cannot emphasize too strongly the fact that, while the *total deformation* of the bar must be zero, the deformations of the portions AC and CB *are not zero*. A solution of the problem based on the assumption that these deformations are zero would therefore be wrong. Neither can the values of the strain in AC or CB be assumed equal to zero. To amplify this point, let us determine the strain ϵ_{AC} in portion AC of the bar. The strain ϵ_{AC} can be divided into two component parts; one is the thermal strain ϵ_T produced in the unrestrained bar by the temperature change ΔT (Fig. 9.32*b*). From Eq. (9.21) we write

$$\epsilon_T = \alpha\, \Delta T = (6.5 \times 10^{-6}/°\text{F})(-125°\text{F})$$
$$= -812.5 \times 10^{-6} \text{ in./in.}$$

The other component of ϵ_{AC} is associated with the stress σ_1 due to the force $\mathbf{R}_B$ applied to the bar (Fig. 9.32*c*). From Hooke's law, we express this component of the strain as

$$\frac{\sigma_1}{E} = \frac{+31.42 \times 10^3 \text{ psi}}{29 \times 10^6 \text{ psi}} = +1083.4 \times 10^{-6} \text{ in./in.}$$

Adding the two components of the strain in AC, we obtain

$$\epsilon_{AC} = \epsilon_T + \frac{\sigma_1}{E} = -812.5 \times 10^{-6} + 1083.4 \times 10^{-6}$$
$$= +271 \times 10^{-6} \text{ in./in.}$$

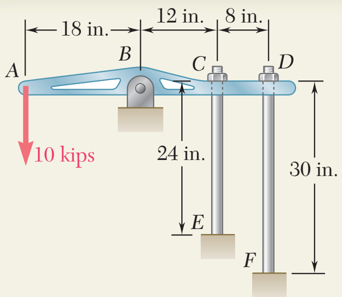
A similar computation yields the strain in portion CB of the bar:

$$\epsilon_{CB} = \epsilon_T + \frac{\sigma_2}{E} = -812.5 \times 10^{-6} + 541.7 \times 10^{-6}$$
$$= -271 \times 10^{-6} \text{ in./in.}$$

The deformations δ_{AC} and δ_{CB} of the two portions of the bar are expressed respectively as

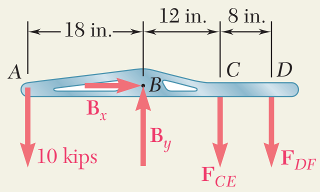
$$\delta_{AC} = \epsilon_{AC}(AC) = (+271 \times 10^{-6})(12 \text{ in.})$$
$$= +3.25 \times 10^{-3} \text{ in.}$$
$$\delta_{CB} = \epsilon_{CB}(CB) = (-271 \times 10^{-6})(12 \text{ in.})$$
$$= -3.25 \times 10^{-3} \text{ in.}$$

We thus check that, while the sum $\delta = \delta_{AC} + \delta_{CB}$ of the two deformations is zero, neither of the deformations is zero. ■



SAMPLE PROBLEM 9.3

The $\frac{1}{2}$-in.-diameter rod CE and the $\frac{3}{4}$-in.-diameter rod DF are attached to the rigid bar $ABCD$ as shown. Knowing that the rods are made of aluminum and using $E = 10.6 \times 10^6$ psi, determine (*a*) the force in each rod caused by the loading shown, (*b*) the corresponding deflection of point A.

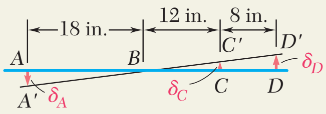


SOLUTION

Statics. Considering the free body of bar $ABCD$, we note that the reaction at B and the forces exerted by the rods are indeterminate. However, using statics, we may write

$$+\circlearrowleft \Sigma M_B = 0: \qquad (10 \text{ kips})(18 \text{ in.}) - F_{CE}(12 \text{ in.}) - F_{DF}(20 \text{ in.}) = 0$$

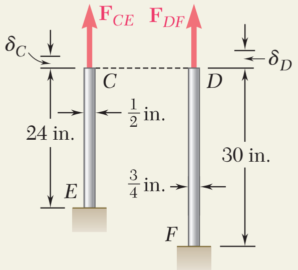
$$12F_{CE} + 20F_{DF} = 180 \qquad (1)$$



Geometry. After application of the 10-kip load, the position of the bar is $A'BC'D'$. From the similar triangles BAA', BCC', and BDD' we have

$$\frac{\delta_C}{12 \text{ in.}} = \frac{\delta_D}{20 \text{ in.}} \qquad \delta_C = 0.6\delta_D \qquad (2)$$

$$\frac{\delta_A}{18 \text{ in.}} = \frac{\delta_D}{20 \text{ in.}} \qquad \delta_A = 0.9\delta_D \qquad (3)$$



Deformations. Using Eq. (9.6), we have

$$\delta_C = \frac{F_{CE}L_{CE}}{A_{CE}E} \qquad \delta_D = \frac{F_{DF}L_{DF}}{A_{DF}E}$$

Substituting for δ_C and δ_D into (2), we write

$$\delta_C = 0.6\delta_D \qquad \frac{F_{CE}L_{CE}}{A_{CE}E} = 0.6\frac{F_{DF}L_{DF}}{A_{DF}E}$$

$$F_{CE} = 0.6\frac{L_{DF}}{L_{CE}}\frac{A_{CE}}{A_{DF}}F_{DF} = 0.6\left(\frac{30 \text{ in.}}{24 \text{ in.}}\right)\left[\frac{\frac{1}{4}\pi(\frac{1}{2} \text{ in.})^2}{\frac{1}{4}\pi(\frac{3}{4} \text{ in.})^2}\right]F_{DF} \qquad F_{CE} = 0.333F_{DF}$$

Force in Each Rod. Substituting for F_{CE} into (1) and recalling that all forces have been expressed in kips, we have

$$12(0.333F_{DF}) + 20F_{DF} = 180 \qquad F_{DF} = 7.50 \text{ kips} \quad \blacktriangleleft$$

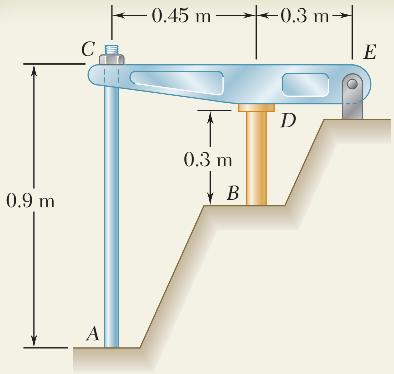
$$F_{CE} = 0.333F_{DF} = 0.333(7.50 \text{ kips}) \qquad F_{CE} = 2.50 \text{ kips} \quad \blacktriangleleft$$

Deflections. The deflection of point D is

$$\delta_D = \frac{F_{DF}L_{DF}}{A_{DF}E} = \frac{(7.50 \times 10^3 \text{ lb})(30 \text{ in.})}{\frac{1}{4}\pi(\frac{3}{4} \text{ in.})^2(10.6 \times 10^6 \text{ psi})} \qquad \delta_D = 48.0 \times 10^{-3} \text{ in.}$$

Using (3), we write

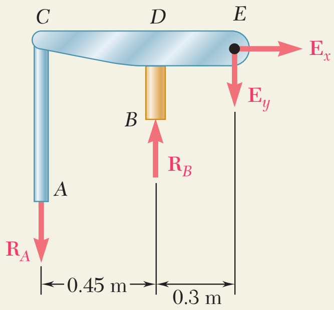
$$\delta_A = 0.9\delta_D = 0.9(48.0 \times 10^{-3} \text{ in.}) \qquad \delta_A = 43.2 \times 10^{-3} \text{ in.} \quad \blacktriangleleft$$



SAMPLE PROBLEM 9.4

The rigid bar *CDE* is attached to a pin support at *E* and rests on the 30-mm-diameter brass cylinder *BD*. A 22-mm-diameter steel rod *AC* passes through a hole in the bar and is secured by a nut which is snugly fitted when the temperature of the entire assembly is 20°C. The temperature of the brass cylinder is then raised to 50°C while the steel rod remains at 20°C. Assuming that no stresses were present before the temperature change, determine the stress in the cylinder.

Rod *AC*: Steel	Cylinder *BD*: Brass
$E = 200$ GPa	$E = 105$ GPa
$\alpha = 11.7 \times 10^{-6}/°\text{C}$	$\alpha = 20.9 \times 10^{-6}/°\text{C}$



SOLUTION

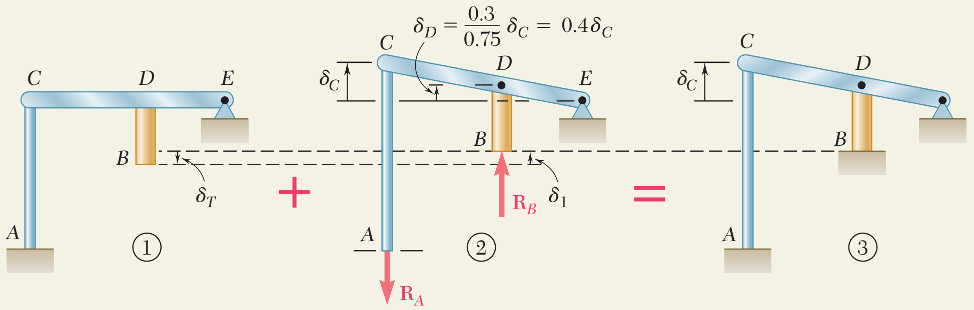
Statics. Considering the free body of the entire assembly, we write

$$+\circlearrowleft \Sigma M_E = 0: \qquad R_A(0.75 \text{ m}) - R_B(0.3 \text{ m}) = 0 \qquad R_A = 0.4R_B \qquad (1)$$

Deformations. We use the method of superposition, considering $\mathbf{R}_B$ as redundant. With the support at *B* removed, the temperature rise of the cylinder causes point *B* to move down through δ_T. The reaction $\mathbf{R}_B$ must cause a deflection δ_1 equal to δ_T so that the final deflection of *B* will be zero (Fig. 3).

Deflection δ_T. Because of a temperature rise of $50° - 20° = 30°\text{C}$, the length of the brass cylinder increases by δ_T.

$$\delta_T = L(\Delta T)\alpha = (0.3 \text{ m})(30°\text{C})(20.9 \times 10^{-6}/°\text{C}) = 188.1 \times 10^{-6} \text{ m} \downarrow$$



Deflection δ_1. We note that $\delta_D = 0.4\delta_C$ and $\delta_1 = \delta_D + \delta_{B/D}$.

$$\delta_C = \frac{R_A L}{AE} = \frac{R_A(0.9 \text{ m})}{\frac{1}{4}\pi(0.022 \text{ m})^2(200 \text{ GPa})} = 11.84 \times 10^{-9} R_A \uparrow$$

$$\delta_D = 0.40\delta_C = 0.4(11.84 \times 10^{-9} R_A) = 4.74 \times 10^{-9} R_A \uparrow$$

$$\delta_{B/D} = \frac{R_B L}{AE} = \frac{R_B(0.3 \text{ m})}{\frac{1}{4}\pi(0.03 \text{ m})^2(105 \text{ GPa})} = 4.04 \times 10^{-9} R_B \uparrow$$

We recall from (1) that $R_A = 0.4R_B$ and write

$$\delta_1 = \delta_D + \delta_{B/D} = [4.74(0.4R_B) + 4.04R_B]10^{-9} = 5.94 \times 10^{-9} R_B \uparrow$$

But $\delta_T = \delta_1$: $\qquad 188.1 \times 10^{-6} \text{ m} = 5.94 \times 10^{-9} R_B \qquad R_B = 31.7 \text{ kN}$

Stress in Cylinder: $\sigma_B = \dfrac{R_B}{A} = \dfrac{31.7 \text{ kN}}{\frac{1}{4}\pi(0.03\text{m})^2} \qquad \sigma_B = 44.8 \text{ MPa}$ ◀

PROBLEMS

9.25 An axial force of 60 kN is applied to the assembly shown by means of rigid end plates. Determine (*a*) the normal stress in the brass shell, (*b*) the corresponding deformation of the assembly.

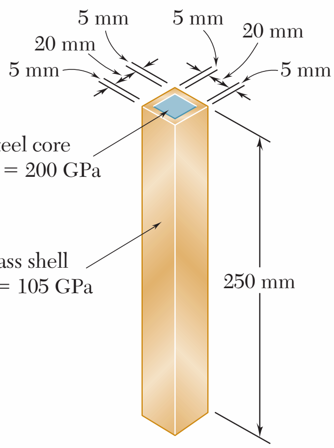


Fig. P9.25 and P9.26

9.26 The length of the assembly decreases by 0.15 mm when an axial force is applied by means of rigid end plates. Determine (*a*) the magnitude of the applied force, (*b*) the corresponding stress in the steel core.

9.27 The 4.5-ft concrete post is reinforced with six steel bars, each with a $1\frac{1}{8}$-in. diameter. Knowing that $E_s = 29 \times 10^6$ psi and $E_c = 4.2 \times 10^6$ psi, determine the normal stresses in the steel and in the concrete when a 350-kip axial centric force **P** is applied to the post.

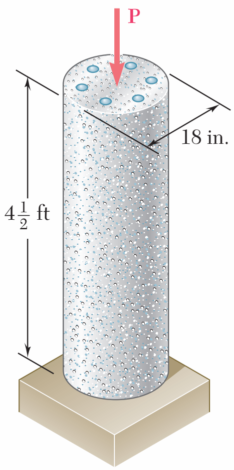


Fig. P9.27

9.28 For the concrete post of Prob. 9.27, determine the maximum centric force that can be applied if the allowable normal stress is 20 ksi in the steel and 2.4 ksi in the concrete.

9.29 Three steel rods ($E = 29 \times 10^6$ psi) support an 8.5-kip load **P**. Each of the rods *AB* and *CD* has a 0.32-in^2 cross-sectional area and rod *EF* has a 1-in^2 cross-sectional area. Neglecting the deformation of rod *BED*, determine (*a*) the change in length of rod *EF*, (*b*) the stress in each rod.

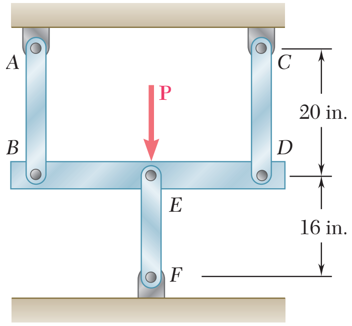


Fig. *P9.29*

9.30 Two cylindrical rods, one of steel and the other of brass, are joined at *C* and restrained by rigid supports at *A* and *E*. For the loading shown and knowing that $E_s = 200$ GPa and $E_b = 105$ GPa, determine (*a*) the reactions at *A* and *E*, (*b*) the deflection of point *C*.

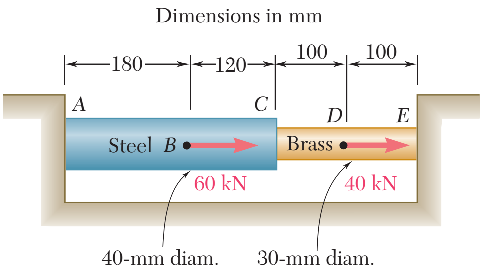


Fig. P9.30

9.31 Solve Prob. 9.30 assuming that rod *AC* is made of brass and rod *CE* is made of steel.

9.32 Two cylindrical rods, *CD* made of steel ($E = 29 \times 10^6$ psi) and *AC* made of aluminum ($E = 10.4 \times 10^6$ psi), are joined at *C* and restrained by rigid supports at *A* and *D*. Determine (*a*) the reactions at *A* and *D*, (*b*) the deflection of point *C*.

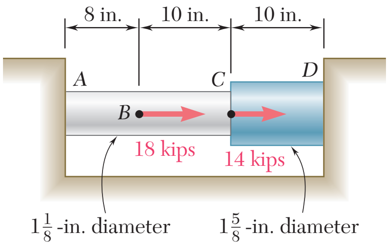


Fig. P9.32

9.33 Three wires are used to suspend the plate shown. Aluminum wires of $\frac{1}{8}$-in. diameter are used at *A* and *B* while a steel wire of $\frac{1}{12}$-in. diameter is used at *C*. Knowing that the allowable stress for aluminum ($E = 10.4 \times 10^6$ psi) is 14 ksi and that the allowable stress for steel ($E = 29 \times 10^6$ psi) is 18 ksi, determine the maximum load **P** that can be applied.

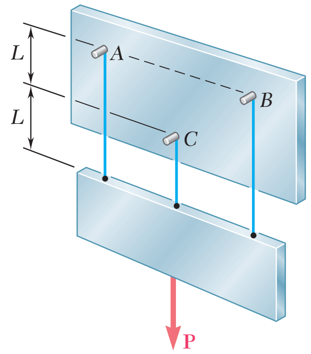


Fig. P9.33

9.34 The rigid bar AD is supported by two steel wires of $\frac{1}{16}$-in. diameter ($E = 29 \times 10^6$ psi) and a pin and bracket at D. Knowing that the wires were initially taut, determine (*a*) the additional tension in each wire when a 220-lb load **P** is applied at D, (*b*) the corresponding deflection of point D.

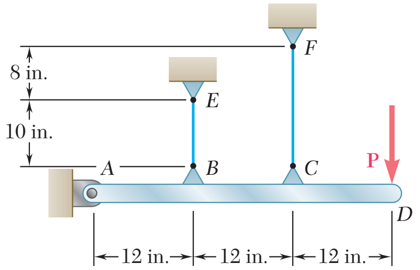


Fig. P9.34

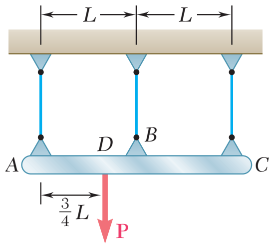


Fig. P9.35

9.35 The rigid rod ABC is suspended from three wires of the same material. The cross-sectional area of the wire at B is equal to half of the cross-sectional area of the wires at A and C. Determine the tension in each wire caused by the load **P**.

9.36 The rigid bar $ABCD$ is suspended from four identical wires. Determine the tension in each wire caused by the load **P**.

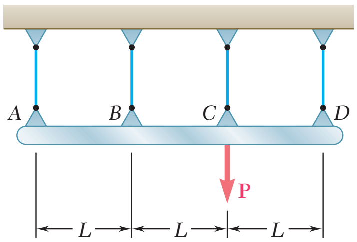


Fig. P9.36

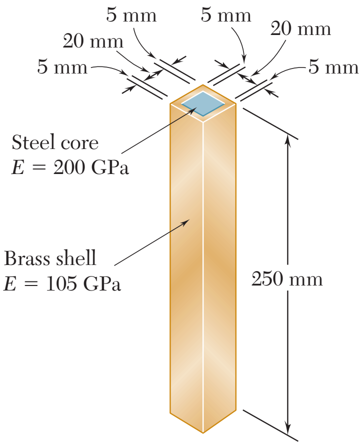


Fig. P9.37

9.37 The brass shell ($\alpha_b = 20.9 \times 10^{-6}/°C$) is fully bonded to the steel core ($\alpha_s = 11.7 \times 10^{-6}/°C$). Determine the largest allowable increase in temperature if the stress in the steel core is not to exceed 55 MPa.

9.38 The assembly shown consists of an aluminum shell ($E_a = 70$ GPa, $\alpha_a = 23.6 \times 10^{-6}/°C$) fully bonded to a steel core ($E_s = 200$ GPa, $\alpha_s = 11.7 \times 10^{-6}/°C$) and is unstressed at a temperature of 20°C. Considering only axial deformations, determine the stress in the aluminum shell when the temperature reaches 180°C.

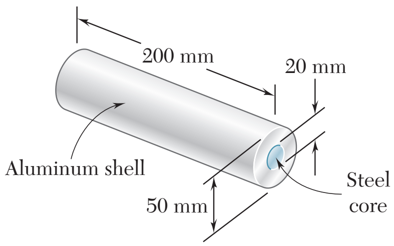


Fig. P9.38

9.39 A 4-ft concrete post is reinforced by four steel bars, each of $\frac{3}{4}$-in. diameter. Knowing that $E_s = 29 \times 10^6$ psi, $\alpha_s = 6.5 \times 10^{-6}$/°F and $E_c = 3.6 \times 10^6$ psi and $\alpha_c = 5.5 \times 10^{-6}$/°F, determine the normal stresses induced in the steel and in the concrete by a temperature rise of 80°F.

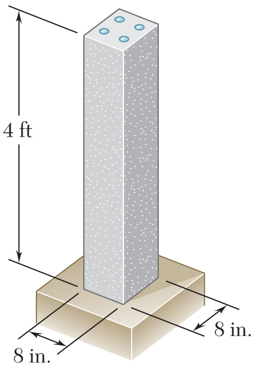


Fig. P9.39

9.40 The steel rails for a railroad track ($E_s = 29 \times 10^6$ psi, $\alpha_s = 6.5 \times 10^{-6}$/°F) were laid out at a temperature of 30°F. Determine the normal stress in the rails when the temperature reaches 125°F assuming that the rails (*a*) are welded to form a continuous track, (*b*) are 39 ft long with $\frac{1}{4}$-in. gaps between them.

9.41 A rod consisting of two cylindrical portions *AB* and *BC* is restrained at both ends. Portion *AB* is made of brass ($E_b = 105$ GPa, $\alpha_b = 20.9 \times 10^{-6}$/°C) and portion *BC* is made of aluminum ($E_a = 72$ GPa, $\alpha_a = 23.9 \times 10^{-6}$/°C). Knowing that the rod is initially unstressed, determine (*a*) the normal stresses induced in portions *AB* and *BC* by a temperature rise of 42°C, (*b*) the corresponding deflection of point *B*.

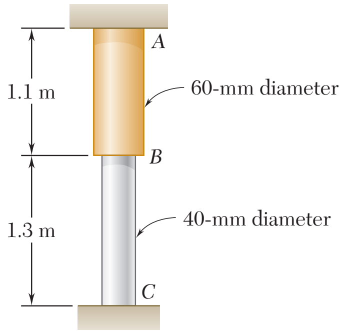


Fig. P9.41

9.42 A rod consisting of two cylindrical portions *AB* and *BC* is restrained at both ends. Portion *AB* is made of steel ($E_s = 29 \times 10^6$ psi, $\alpha_s = 6.5 \times 10^{-6}$/°F) and portion *BC* is made of brass ($E_b = 15 \times 10^6$ psi, $\alpha_b = 10.4 \times 10^{-6}$/°F). Knowing that the rod is initially unstressed, determine (*a*) the normal stresses induced in portions *AB* and *BC* by a temperature rise of 65°F, (*b*) the corresponding deflection of point *B*.

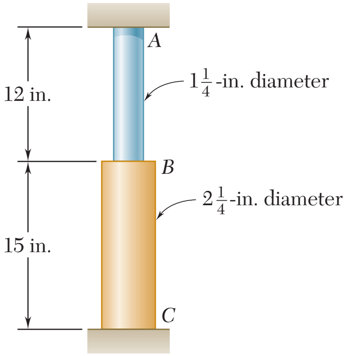
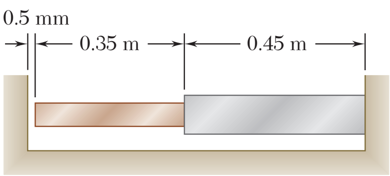


Fig. P9.42

9.43 For the rod of Prob. 9.42, determine the maximum allowable temperature change if the stress in the steel portion *AB* is not to exceed 18 ksi and if the stress in the brass portion *BC* is not to exceed 7 ksi.

9.44 Determine (*a*) the compressive force in the bars shown after a temperature rise of 96°C, (*b*) the corresponding change in length of the bronze bar.

9.45 Knowing that a 0.5-mm gap exists when the temperature is 20°C, determine (*a*) the temperature at which the normal stress in the aluminum bar will be equal to –90 MPa, (*b*) the corresponding exact length of the aluminum bar.



Bronze	Aluminum
$A = 1500$ mm^2	$A = 1800$ mm^2
$E = 105$ GPa	$E = 73$ GPa
$\alpha = 21.6 \times 10^{-6}$/°C	$\alpha = 23.2 \times 10^{-6}$/°C

Fig. P9.44 and *P9.45*

9.46 At room temperature (70°F) a 0.02-in. gap exists between the ends of the rods shown. At a later time when the temperature has reached 320°F, determine (*a*) the normal stress in the aluminum rod, (*b*) the change in length of the aluminum rod.

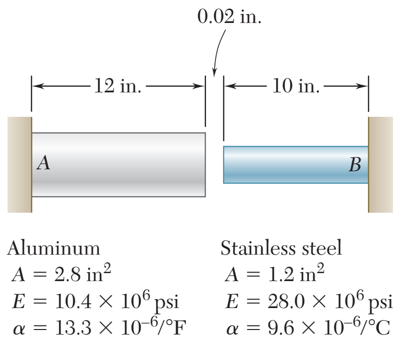


Fig. P9.46

9.47 A brass link ($E_b = 15 \times 10^6$ psi, $\alpha_b = 10.4 \times 10^{-6}/°\text{F}$) and a steel rod ($E_s = 29 \times 10^6$ psi, $\alpha_s = 6.5 \times 10^{-6}/°\text{F}$) have the dimensions shown at a temperature of 65°F. The steel rod is cooled until it fits freely into the link. The temperature of the whole assembly is then raised to 100°F. Determine (*a*) the final normal stress in the steel rod, (*b*) the final length of the steel rod.

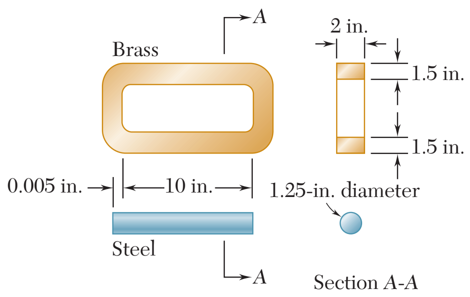


Fig. P9.47

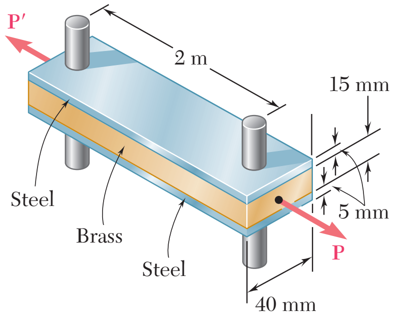


Fig. P9.48

9.48 Two steel bars ($E_s = 200$ GPa and $\alpha_s = 11.7 \times 10^{-6}/°\text{C}$) are used to reinforce a brass bar ($E_b = 105$ GPa, $\alpha_b = 20.9 \times 10^{-6}/°\text{C}$) that is subjected to a load $P = 25$ kN. When the steel bars were fabricated, the distance between the centers of the holes that were to fit on the pins was made 0.5 mm smaller than the 2 m needed. The steel bars were then placed in an oven to increase their length so that they would just fit on the pins. Following fabrication, the temperature in the steel bars dropped back to room temperature. Determine (*a*) the increase in temperature that was required to fit the steel bars on the pins, (*b*) the stress in the brass bar after the load is applied to it.

9.10 POISSON'S RATIO

We saw in the earlier part of this chapter that, when a homogeneous slender bar is axially loaded, the resulting stress and strain satisfy Hooke's law as long as the elastic limit of the material is not exceeded. Assuming that the load **P** is directed along the x axis (Fig. 9.33*a*), we have $\sigma_x = P/A$, where A is the cross-sectional area of the bar, and, from Hooke's law,

$$\epsilon_x = \sigma_x/E \tag{9.23}$$

where E is the modulus of elasticity of the material.

We also note that the normal stresses on faces respectively perpendicular to the y and z axes are zero: $\sigma_y = \sigma_z = 0$ (Fig. 9.33*b*). It would be tempting to conclude that the corresponding strains ϵ_y and ϵ_z are also zero. This, however, *is not the case*. In all engineering materials, the elongation produced by an axial tensile force **P** in the direction of the force is accompanied by a contraction in any transverse direction (Fig. 9.34). In this section and the following sections (Secs. 9.11 through 9.13), all materials considered will be assumed to be both *homogeneous* and *isotropic,* i.e., their mechanical properties will be assumed independent of both *position* and *direction*. It follows that the strain must have the same value for any transverse direction. Therefore, for the loading shown in Fig. 9.33 we must have $\epsilon_y = \epsilon_z$. This common value is referred to as the *lateral strain*. An important constant for a given material is its *Poisson's ratio,* named after the French mathematician Siméon Denis Poisson (1781–1840) and denoted by the Greek letter ν (nu). It is defined as

$$\boxed{\nu = -\frac{\text{lateral strain}}{\text{axial strain}}} \tag{9.24}$$

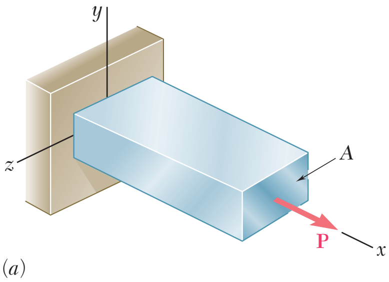
or

$$\nu = -\frac{\epsilon_y}{\epsilon_x} = -\frac{\epsilon_z}{\epsilon_x} \tag{9.25}$$

for the loading condition represented in Fig. 9.33. Note the use of a minus sign in the above equations to obtain a positive value for v, the axial and lateral strains having opposite signs for all engineering materials.† Solving Eq. (9.25) for ϵ_y and ϵ_z, and recalling (9.23), we write the following relations, which fully describe the condition of strain under an axial load applied in a direction parallel to the x axis:

$$\boxed{\epsilon_x = \frac{\sigma_x}{E}} \qquad \boxed{\epsilon_y = \epsilon_z = -\frac{\nu\sigma_x}{E}} \tag{9.26}$$

†However, some experimental materials, such as polymer foams, expand laterally when stretched. Since the axial and lateral strains have then the same sign, the Poisson's ratio of these materials is negative. (*See* Roderic Lakes, "Foam Structures with a Negative Poisson's Ratio," *Science*, 27 February 1987, Volume 235, pp. 1038–1040.)



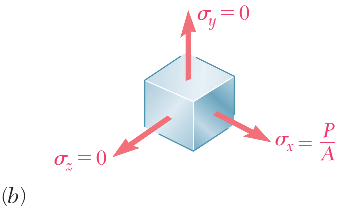


Fig. 9.33

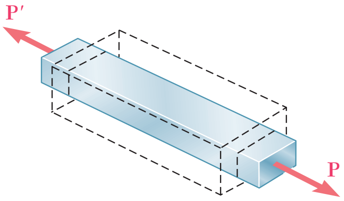


Fig. 9.34

EXAMPLE 9.7 A 500-mm-long, 16-mm-diameter rod made of a homogenous, isotropic material is observed to increase in length by 300 μm, and to decrease in diameter by 2.4 μm when subjected to an axial 12-kN load. Determine the modulus of elasticity and Poisson's ratio of the material.

The cross-sectional area of the rod is

$$A = \pi r^2 = \pi(8 \times 10^{-3}\ \text{m})^2 = 201 \times 10^{-6}\ \text{m}^2$$

Choosing the x axis along the axis of the rod (Fig. 9.35), we write

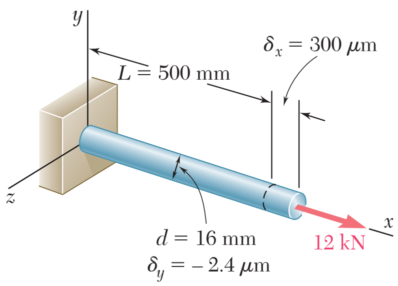


Fig. 9.35

$$\sigma_x = \frac{P}{A} = \frac{12 \times 10^3\ \text{N}}{201 \times 10^{-6}\ \text{m}^2} = 59.7\ \text{MPa}$$

$$\epsilon_x = \frac{\delta_x}{L} = \frac{300\ \mu\text{m}}{500\ \text{mm}} = 600 \times 10^{-6}$$

$$\epsilon_y = \frac{\delta_y}{d} = \frac{-2.4\ \mu\text{m}}{16\ \text{mm}} = -150 \times 10^{-6}$$

From Hooke's law, $\sigma_x = E\epsilon_x$, we obtain

$$E = \frac{\sigma_x}{\epsilon_x} = \frac{59.7\ \text{MPa}}{600 \times 10^{-6}} = 99.5\ \text{GPa}$$

and, from Eq. (9.25),

$$\nu = -\frac{\epsilon_y}{\epsilon_x} = -\frac{-150 \times 10^{-6}}{600 \times 10^{-6}} = 0.25 \ \blacksquare$$

9.11 MULTIAXIAL LOADING. GENERALIZED HOOKE'S LAW

All the examples considered so far in this chapter have dealt with slender members subjected to axial loads, i.e., to forces directed along a single axis. Choosing this axis as the x axis, and denoting by P the internal force at a given location, the corresponding stress components were found to be $\sigma_x = P/A$, $\sigma_y = 0$, and $\sigma_z = 0$.

Let us now consider structural elements subjected to loads acting in the directions of the three coordinate axes and producing normal stresses σ_x, σ_y, and σ_z which are all different from zero (Fig. 9.36). This condition is referred to as a *multiaxial loading*. Note that this is not the general stress condition described in Sec. 8.9, since no shearing stresses are included among the stresses shown in Fig. 9.36.

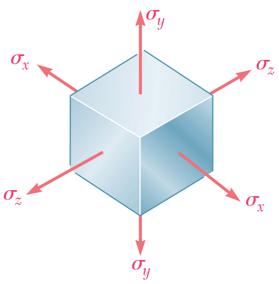


Fig. 9.36

Consider an element of an isotropic material in the shape of a cube (Fig. 9.37*a*). We can assume the side of the cube to be equal to unity, since it is always possible to select the side of the cube as a unit of length. Under the given multiaxial loading, the element will deform into a *rectangular parallelepiped* of sides equal, respectively, to $1 + \epsilon_x$, $1 + \epsilon_y$, and $1 + \epsilon_z$, where ϵ_x, ϵ_y, and ϵ_z denote the values of the normal strain in the directions of the three coordinate axes (Fig. 9.37*b*). You should note that, as a result of the deformations of the other elements of the material, the element under consideration could also undergo a translation, but we are concerned here only with the *actual deformation* of the element, and not with any possible superimposed rigid-body displacement.

In order to express the strain components ϵ_x, ϵ_y, ϵ_z in terms of the stress components σ_x, σ_y, σ_z, we will consider separately the effect of each stress component and combine the results obtained. The approach we propose here will be used repeatedly in this text, and is based on the *principle of superposition*. This principle states that the effect of a given combined loading on a structure can be obtained by *determining separately the effects of the various loads and combining the results obtained*, provided that the following conditions are satisfied:

1. Each effect is linearly related to the load that produces it.
2. The deformation resulting from any given load is small and does not affect the conditions of application of the other loads.

In the case of a multiaxial loading, the first condition will be satisfied if the stresses do not exceed the proportional limit of the material, and the second condition will also be satisfied if the stress on any given face does not cause deformations of the other faces that are large enough to affect the computation of the stresses on those faces.

Considering first the effect of the stress component σ_x, we recall from Sec. 9.10 that σ_x causes a strain equal to σ_x/E in the x direction, and strains equal to $-\nu\sigma_x/E$ in each of the y and z directions. Similarly, the stress component σ_y, if applied separately, will cause a strain σ_y/E in the y direction and strains $-\nu\sigma_y/E$ in the other two directions. Finally, the stress component σ_z causes a strain σ_z/E in the z direction and strains $-\nu\sigma_z/E$ in the x and y directions. Combining the results obtained, we conclude that the components of strain corresponding to the given multiaxial loading are

$$
\begin{aligned}
\epsilon_x &= +\frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E} \\
\epsilon_y &= -\frac{\nu\sigma_x}{E} + \frac{\sigma_y}{E} - \frac{\nu\sigma_z}{E} \\
\epsilon_z &= -\frac{\nu\sigma_x}{E} - \frac{\nu\sigma_y}{E} + \frac{\sigma_z}{E}
\end{aligned}
\tag{9.27}
$$

The relations (9.27) are referred to as the *generalized Hooke's law for the multiaxial loading of a homogeneous isotropic material*. As we indicated earlier, the results obtained are valid only as long as the stresses do not exceed the proportional limit and as long as the deformations involved remain small. We also recall that a positive value for a stress component signifies tension, and a negative value compression. Similarly, a positive value for a strain component indicates expansion in the corresponding direction, and a negative value contraction.

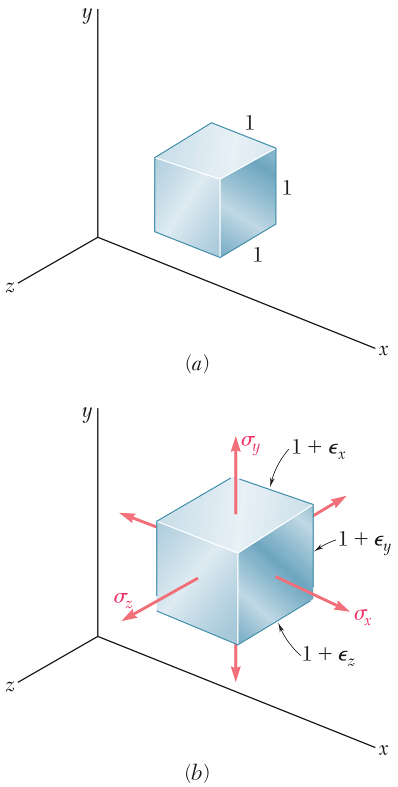


Fig. 9.37

EXAMPLE 9.8 The steel block shown (Fig. 9.38) is subjected to a uniform pressure on all its faces. Knowing that the change in length of edge AB is -1.2×10^{-3} in., determine (*a*) the change in length of the other two edges, (*b*) the pressure p applied to the faces of the block. Assume $E = 29 \times 10^6$ psi and $\nu = 0.29$.

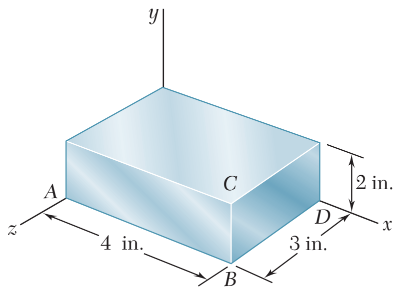


Fig. 9.38

(a) Change in Length of Other Edges. Substituting $\sigma_x = \sigma_y = \sigma_z = -p$ into the relations (9.27), we find that the three strain components have the common value

$$\epsilon_x = \epsilon_y = \epsilon_z = -\frac{p}{E}(1 - 2\nu) \tag{9.28}$$

Since

$$\begin{aligned}\epsilon_x &= \delta_x/AB = (-1.2 \times 10^{-3}\text{ in.})/(4\text{ in.})\\ &= -300 \times 10^{-6}\text{ in./in.}\end{aligned}$$

we obtain

$$\epsilon_y = \epsilon_z = \epsilon_x = -300 \times 10^{-6}\text{ in./in.}$$

from which it follows that

$$\begin{aligned}\delta_y &= \epsilon_y(BC) = (-300 \times 10^{-6})(2\text{ in.}) = -600 \times 10^{-6}\text{ in.}\\ \delta_z &= \epsilon_z(BD) = (-300 \times 10^{-6})(3\text{ in.}) = -900 \times 10^{-6}\text{ in.}\end{aligned}$$

(b) Pressure. Solving Eq. (9.28) for p, we write

$$p = -\frac{E\epsilon_x}{1 - 2\nu} = -\frac{(29 \times 10^6\text{ psi})(-300 \times 10^{-6})}{1 - 0.58}$$

$$p = 20.7\text{ ksi}$$

9.12 SHEARING STRAIN

When we derived in Sec. 9.11 the relations (9.27) between normal stresses and normal strains in a homogeneous isotropic material, we assumed that no shearing stresses were involved. In the more general stress situation represented in Fig. 9.39, shearing stresses τ_{xy}, τ_{yz}, and τ_{zx} will be present (as well, of course, as the corresponding shearing stresses τ_{yx}, τ_{zy}, and τ_{xz}). These stresses have no direct effect on the normal strains and, as long as all the deformations involved remain small, they will not affect the derivation nor the validity of the relations (9.27). The shearing stresses, however, will tend to deform a cubic element of material into an *oblique* parallelepiped.

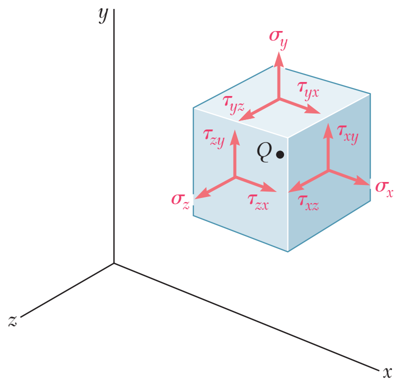


Fig. 9.39

Consider first a cubic element of side one (Fig. 9.40) subjected to no other stresses than the shearing stresses τ_{xy} and τ_{yx} applied to faces of the element respectively perpendicular to the x and y axes. (We recall from Sec. 8.9 that $\tau_{xy} = \tau_{yx}$.) The element is observed to deform into a rhomboid of sides equal to one (Fig. 9.41). Two of the angles formed by the four faces under stress are reduced from $\frac{\pi}{2}$ to $\frac{\pi}{2} - \gamma_{xy}$, while the other two are increased from $\frac{\pi}{2}$ to $\frac{\pi}{2} + \gamma_{xy}$. The small angle γ_{xy} (expressed in radians) defines the *shearing strain* corresponding to the x and y directions. When the deformation involves a *reduction* of the angle formed by the two faces oriented respectively toward the positive x and y axes (as shown in Fig. 9.41), the shearing strain γ_{xy} is said to be *positive;* otherwise, it is said to be negative.

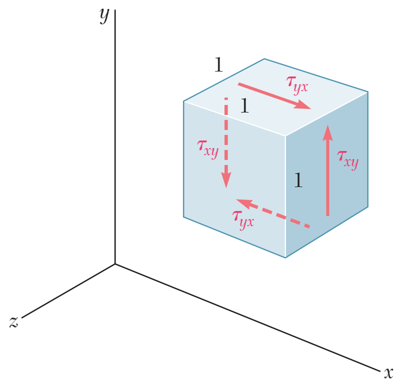


Fig. 9.40

We should note that, as a result of the deformations of the other elements of the material, the element under consideration can also undergo an overall rotation. However, as was the case in our study of normal strains, we are concerned here only with the *actual deformation* of the element, and not with any possible superimposed rigid-body displacement.†

Plotting successive values of τ_{xy} against the corresponding values of γ_{xy}, we obtain the shearing stress-strain diagram for the material under consideration. This can be accomplished by carrying out a torsion test, as you will see in Chap. 10. The diagram obtained is similar to the normal stress-strain diagram obtained for the same material from the tensile test described earlier in this chapter. However, the values obtained for the yield strength, ultimate strength, etc., of a given material are only about half as large in shear as they are in tension. As was the case for normal stresses and strains, the initial portion of the shearing stress-strain diagram is a straight line. For values of the shearing stress that do not exceed the proportional limit in shear, we can therefore write for any homogeneous isotropic material,

$$\boxed{\tau_{xy} = G\gamma_{xy}} \tag{9.28}$$

This relation is known as *Hooke's law for shearing stress and strain*, and the constant G is called the *modulus of rigidity* or *shear modulus* of the material. Since the strain γ_{xy} was defined as an angle in radians, it is dimensionless, and the modulus G is expressed in the same units as τ_{xy}, that is, in pascals or in psi. The modulus of rigidity G of any given material is less than one-half, but more than one-third of the modulus of elasticity E of that material.

Considering now a small element of material subjected to shearing stresses τ_{yz} and τ_{zy} (Fig. 9.44*a*), we define the shearing strain γ_{yz} as the change in the angle formed by the faces under stress. The shearing strain γ_{zx} is defined in a similar way by considering an element subjected to shearing stresses τ_{zx} and τ_{xz} (Fig. 9.44*b*). For values of the stress that do not exceed the proportional limit, we can write the two additional relations

$$\boxed{\tau_{yz} = G\gamma_{yz}} \qquad \boxed{\tau_{zx} = G\gamma_{zx}} \tag{9.29}$$

where the constant G is the same as in Eq. (9.28).

†In defining the strain γ_{xy}, some authors arbitrarily assume that the actual deformation of the element is accompanied by a rigid-body rotation such that the horizontal faces of the element do not rotate. The strain γ_{xy} is then represented by the angle through which the other two faces have rotated (Fig. 9.42). Others assume a rigid-body rotation such that the horizontal faces rotate through $\frac{1}{2}\gamma_{xy}$ counterclockwise and the vertical faces through $\frac{1}{2}\gamma_{xy}$ clockwise (Fig. 9.43). Since both assumptions are unnecessary and may lead to confusion, we prefer in this text to associate the shearing strain γ_{xy} with the *change in the angle* formed by the two faces, rather than with the *rotation of a given face* under restrictive conditions.

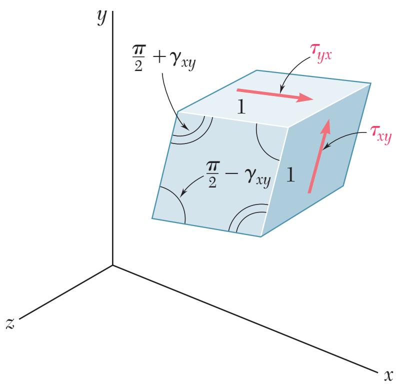


Fig. 9.41

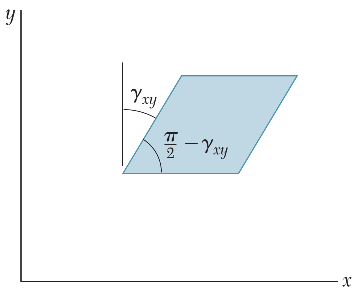


Fig. 9.42

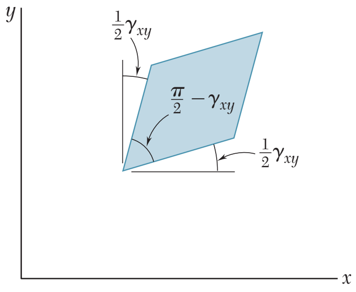


Fig. 9.43

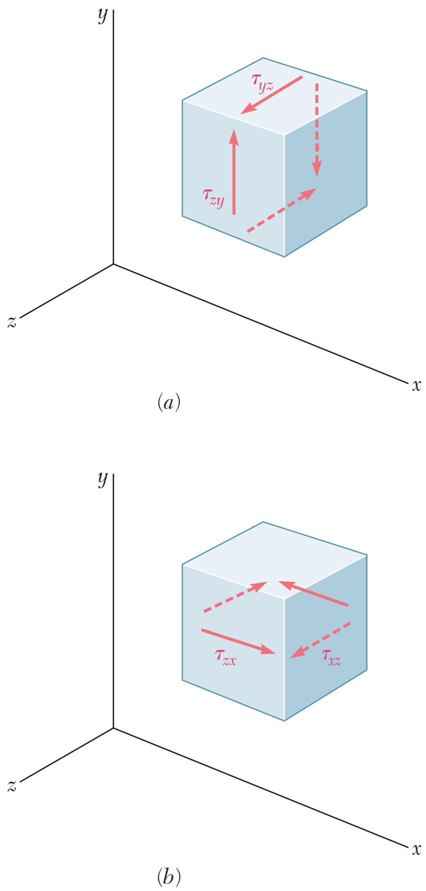


Fig. 9.44

For the general stress condition represented in Fig. 9.39, and as long as none of the stresses involved exceeds the corresponding proportional limit, we can apply the principle of superposition and combine the results obtained in this section and in Sec. 9.11. We obtain the following group of equations representing the generalized Hooke's law for a homogeneous isotropic material under the most general stress condition.

$$
\begin{aligned}
\epsilon_x &= +\frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E} \\
\epsilon_y &= -\frac{\nu\sigma_x}{E} + \frac{\sigma_y}{E} - \frac{\nu\sigma_z}{E} \\
\epsilon_z &= -\frac{\nu\sigma_x}{E} - \frac{\nu\sigma_y}{E} + \frac{\sigma_z}{E} \\
\gamma_{xy} &= \frac{\tau_{xy}}{G} \qquad \gamma_{yz} = \frac{\tau_{yz}}{G} \qquad \gamma_{zx} = \frac{\tau_{zx}}{G}
\end{aligned}
\tag{9.30}
$$

An examination of Eqs. (9.30) might lead us to believe that three distinct constants, E, ν, and G, must first be determined experimentally, if we are to predict the deformations caused in a given material by an arbitrary combination of stresses. Actually, only two of these constants need be determined experimentally for any given material. As you will see in the next section, the third constant can then be obtained through a very simple computation.

EXAMPLE 9.9 A rectangular block of a material with a modulus of rigidity $G = 90$ ksi is bonded to two rigid horizontal plates. The lower plate is fixed, while the upper plate is subjected to a horizontal force **P** (Fig. 9.45). Knowing that the upper plate moves through 0.04 in. under the action of the force, determine (*a*) the average shearing strain in the material, (*b*) the force **P** exerted on the upper plate.

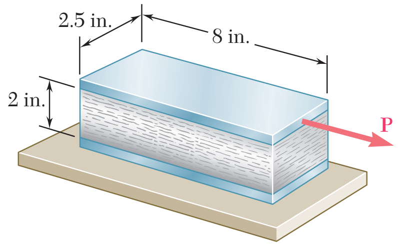


Fig. 9.45

(a) Shearing Strain. We select coordinate axes centered at the midpoint C of edge AB and directed as shown (Fig. 9.46). According to its definition, the shearing strain γ_{xy} is equal to the angle formed by the vertical and the line CF joining the midpoints of edges AB and DE. Noting that this is a very small angle and recalling that it should be expressed in radians, we write

$$\gamma_{xy} \approx \tan \gamma_{xy} = \frac{0.04 \text{ in.}}{2 \text{ in.}} \qquad \gamma_{xy} = 0.020 \text{ rad}$$

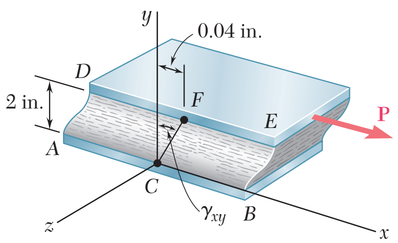


Fig. 9.46

(b) Force Exerted on Upper Plate. We first determine the shearing stress τ_{xy} in the material. Using Hooke's law for shearing stress and strain, we have

$$\tau_{xy} = G\gamma_{xy} = (90 \times 10^3 \text{ psi})(0.020 \text{ rad}) = 1800 \text{ psi}$$

The force exerted on the upper plate is thus

$$P = \tau_{xy} A = (1800 \text{ psi})(8 \text{ in.})(2.5 \text{ in.}) = 36.0 \times 10^3 \text{ lb}$$

$$P = 36.0 \text{ kips} \blacksquare$$

*9.13 FURTHER DISCUSSION OF DEFORMATIONS UNDER AXIAL LOADING. RELATION AMONG E, ν, AND G

We saw in Sec. 9.10 that a slender bar subjected to an axial tensile load **P** directed along the x axis will elongate in the x direction and contract in both of the transverse y and z directions. If ϵ_x denotes the axial strain, the lateral strain is expressed as $\epsilon_y = \epsilon_z = -\nu\epsilon_x$, where ν is Poisson's ratio. Thus, an element in the shape of a cube of side equal to one and oriented as shown in Fig. 9.47*a* will deform into a rectangular parallelepiped of sides $1 + \epsilon_x$, $1 - \nu\epsilon_x$, and $1 - \nu\epsilon_x$. (Note that only one face of the element is shown in the figure.) On the other hand, if the element is oriented at 45° to the axis of the load (Fig. 9.47*b*), the face shown in the figure is observed to deform into a rhombus. We conclude that the axial load **P** causes in this element a shearing strain γ' equal to the amount by which each of the angles shown in Fig. 9.47*b* increases or decreases.

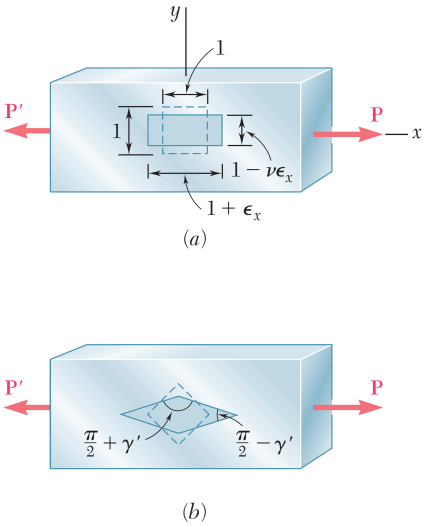


Fig. 9.47

The fact that shearing strains, as well as normal strains, result from an axial loading should not come to us as a surprise, since we already observed at the end of Sec. 8.9 that an axial load **P** causes normal and shearing stresses of equal magnitude on four of the faces of an element oriented at 45° to the axis of the member. This was illustrated in Fig. 8.37, which, for convenience, has been repeated here. It was also shown in Sec. 8.8 that the shearing stress is maximum on a plane forming an angle of 45° with the axis of the load. It follows from Hooke's law for shearing stress and strain that the shearing strain γ' associated with the element of Fig. 9.47*b* is also maximum: $\gamma' = \gamma_m$.

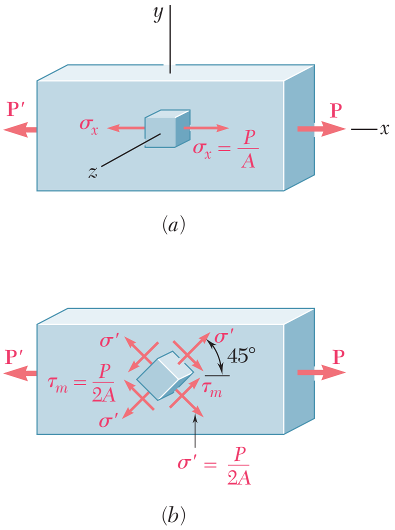


Fig. 8.37 *(repeated)*

We will now derive a relation between the maximum shearing strain $\gamma' = \gamma_m$ associated with the element of Fig. 9.47*b* and the normal strain ϵ_x in the direction of the load. Let us consider for this purpose the prismatic element obtained by intersecting the cubic element of Fig. 9.47*a* by a diagonal plane (Fig. 9.48*a* and *b*). Referring to Fig. 9.47*a*, we conclude that this new element will deform into the element shown in Fig. 9.48*c*, which has horizontal and vertical sides respectively equal to $1 + \epsilon_x$ and $1 - \nu\epsilon_x$. But the angle formed by the oblique and horizontal faces of the element of Fig. 9.48*b* is precisely half of one of the right angles of the cubic element considered in Fig. 9.47*b*. The angle β into which this angle deforms must therefore be equal to half of $\pi/2 - \gamma_m$. We write

$$\beta = \frac{\pi}{4} - \frac{\gamma_m}{2}$$

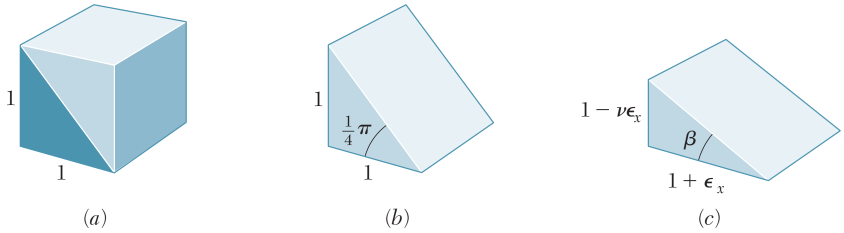


Fig. 9.48

Applying the formula for the tangent of the difference of two angles, we obtain

$$\tan\beta = \frac{\tan\dfrac{\pi}{4} - \tan\dfrac{\gamma_m}{2}}{1 + \tan\dfrac{\pi}{4}\tan\dfrac{\gamma_m}{2}} = \frac{1 - \tan\dfrac{\gamma_m}{2}}{1 + \tan\dfrac{\gamma_m}{2}}$$

or, since $\gamma_m/2$ is a very small angle,

$$\tan\beta = \frac{1 - \dfrac{\gamma_m}{2}}{1 + \dfrac{\gamma_m}{2}} \tag{9.31}$$

But, from Fig. 9.48*c*, we observe that

$$\tan\beta = \frac{1 - \nu\epsilon_x}{1 + \epsilon_x} \tag{9.32}$$

Equating the right-hand members of (9.31) and (9.32), and solving for γ_m, we write

$$\gamma_m = \frac{(1 + \nu)\epsilon_x}{1 + \dfrac{1 - \nu}{2}\epsilon_x}$$

Since $\epsilon_x \ll 1$, the denominator in the expression obtained can be assumed equal to one; we have, therefore,

$$\gamma_m = (1 + \nu)\epsilon_x \tag{9.33}$$

which is the desired relation between the maximum shearing strain γ_m and the axial strain ϵ_x.

To obtain a relation among the constants E, ν, and G, we recall that, by Hooke's law, $\gamma_m = \tau_m/G$, and that, for an axial loading, $\epsilon_x = \sigma_x/E$. Equation (9.33) can therefore be written as

$$\frac{\tau_m}{G} = (1 + \nu)\frac{\sigma_x}{E}$$

or

$$\frac{E}{G} = (1 + \nu)\frac{\sigma_x}{\tau_m} \tag{9.34}$$

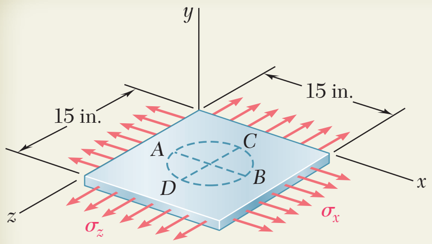
We now recall from Fig. 8.37 that $\sigma_x = P/A$ and $\tau_m = P/2A$, where A is the cross-sectional area of the member. It thus follows that $\sigma_x/\tau_m = 2$. Substituting this value into (9.34) and dividing both members by 2, we obtain the relation

$$\boxed{\frac{E}{2G} = 1 + \nu} \tag{9.35}$$

which can be used to determine one of the constants E, ν, or G from the other two. For example, solving Eq. (9.35) for G, we write

$$G = \frac{E}{2(1 + \nu)} \tag{9.35$'$}$$

SAMPLE PROBLEM 9.5



A circle of diameter $d = 9$ in. is scribed on an unstressed aluminum plate of thickness $t = \frac{3}{4}$ in. Forces acting in the plane of the plate later cause normal stresses $\sigma_x = 12$ ksi and $\sigma_z = 20$ ksi. For $E = 10 \times 10^6$ psi and $\nu = \frac{1}{3}$, determine the change in (a) the length of diameter AB, (b) the length of diameter CD, (c) the thickness of the plate.

SOLUTION

Hooke's Law. We note that $\sigma_y = 0$. Using Eqs. (9.27), we find the strain in each of the coordinate directions.

$$\epsilon_x = +\frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E}$$

$$= \frac{1}{10 \times 10^6 \text{ psi}}\left[(12 \text{ ksi}) - 0 - \frac{1}{3}(20 \text{ ksi})\right] = +0.533 \times 10^{-3} \text{ in./in.}$$

$$\epsilon_y = -\frac{\nu\sigma_x}{E} + \frac{\sigma_y}{E} - \frac{\nu\sigma_z}{E}$$

$$= \frac{1}{10 \times 10^6 \text{ psi}}\left[-\frac{1}{3}(12 \text{ ksi}) + 0 - \frac{1}{3}(20 \text{ ksi})\right] = -1.067 \times 10^{-3} \text{ in./in.}$$

$$\epsilon_z = -\frac{\nu\sigma_x}{E} - \frac{\nu\sigma_y}{E} + \frac{\sigma_z}{E}$$

$$= \frac{1}{10 \times 10^6 \text{ psi}}\left[-\frac{1}{3}(12 \text{ ksi}) - 0 + (20 \text{ ksi})\right] = +1.600 \times 10^{-3} \text{ in./in.}$$

a. Diameter *AB*. The change in length is $\delta_{B/A} = \epsilon_x d$.

$$\delta_{B/A} = \epsilon_x d = (+0.533 \times 10^{-3} \text{ in./in.})(9 \text{ in.})$$

$$\delta_{B/A} = +4.8 \times 10^{-3} \text{ in.} \quad \blacktriangleleft$$

b. Diameter *CD*.

$$\delta_{C/D} = \epsilon_z d = (+1.600 \times 10^{-3} \text{ in./in.})(9 \text{ in.})$$

$$\delta_{C/D} = +14.4 \times 10^{-3} \text{ in.} \quad \blacktriangleleft$$

c. Thickness. Recalling that $t = \frac{3}{4}$ in., we have

$$\delta_t = \epsilon_y t = (-1.067 \times 10^{-3} \text{ in./in.})(\tfrac{3}{4} \text{ in.})$$

$$\delta_t = -0.800 \times 10^{-3} \text{ in.} \quad \blacktriangleleft$$

PROBLEMS

9.49 In a standard tensile test a steel rod of $\frac{7}{8}$-in. diameter is subjected to a tension force of 17 kips. Knowing that $\nu = 0.3$ and $E = 29 \times 10^6$ psi, determine (*a*) the elongation of the rod in an 8-in. gage length, (*b*) the change in diameter of the rod.

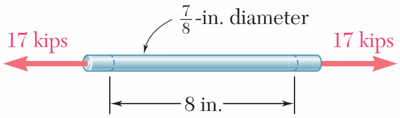


Fig. *P9.49*

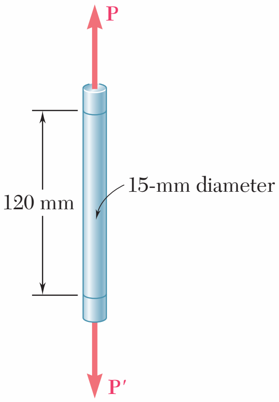


Fig. P9.50

9.50 A standard tension test is used to determine the properties of an experimental plastic. The test specimen is a 15-mm-diameter rod, and it is subjected to a 3.5-kN tensile force. Knowing that an elongation of 11 mm and a decrease in diameter of 0.62 mm are observed in a 120-mm gage length, determine the modulus of elasticity, the modulus of rigidity, and Poisson's ratio of the material.

9.51 A 2-m length of an aluminum pipe of 240-mm outer diameter and 10-mm wall thickness is used as a short column and carries a centric axial load of 640 kN. Knowing that $E = 73$ GPa and $\nu = 0.33$, determine (*a*) the change in length of the pipe, (*b*) the change in its outer diameter, (*c*) the change in its wall thickness.

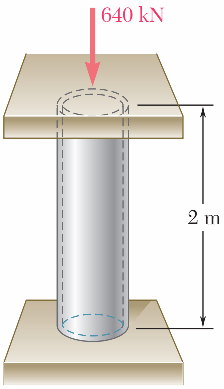


Fig. *P9.51*

9.52 The change in diameter of a large steel bolt is carefully measured as the nut is tightened. Knowing that $E = 200$ GPa and $\nu = 0.29$, determine the internal force in the bolt if the diameter is observed to decrease by 13 μm.

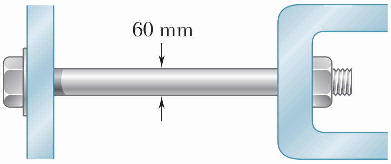


Fig. P9.52

9.53 An aluminum plate (E = 74 GPa, ν = 0.33) is subjected to a centric axial load that causes a normal stress σ. Knowing that, before loading, a line of slope 2:1 is scribed on the plate, determine the slope of the line when σ = 125 MPa.

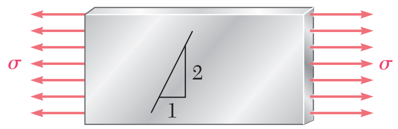


Fig. P9.53

9.54 A 600-lb tensile load is applied to a test coupon made from $\frac{1}{16}$-in. flat steel plate ($E = 29 \times 10^6$ psi, ν = 0.30). Determine the resulting change (a) in the 2-in. gage length, (b) in the width of portion AB of the test coupon, (c) in the thickness of portion AB, (d) in the cross-sectional area of portion AB.

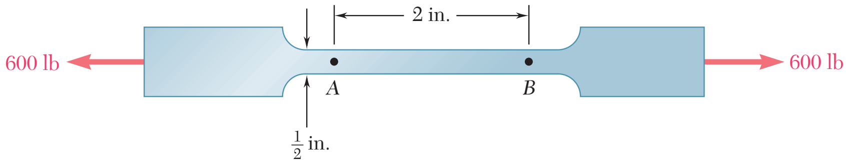


Fig. P9.54

9.55 The aluminum rod AD is fitted with a jacket that is used to apply a hydrostatic pressure of 6000 psi to the 12-in. portion BC of the rod. Knowing that $E = 10.1 \times 10^6$ psi and ν = 0.36, determine (a) the change in the total length AD, (b) the change in diameter at the middle of the rod.

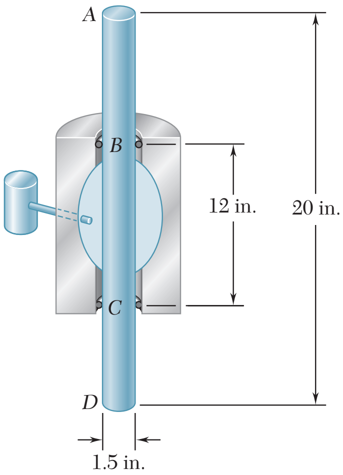


Fig. P9.55

9.56 For the rod of Prob. 9.55, determine the forces that should be applied to the ends A and D of the rod (a) if the axial strain in portion BC of the rod is to remain zero as the hydrostatic pressure is applied, (b) if the total length AD of the rod is to remain unchanged.

9.57 A 20-mm square has been scribed on the side of a large steel pressure vessel. After pressurization, the biaxial stress condition of the square is as shown. Using the data available in App. A, for structural steel, determine the percent change in the slope of diagonal DB due to the pressurization of the vessel.

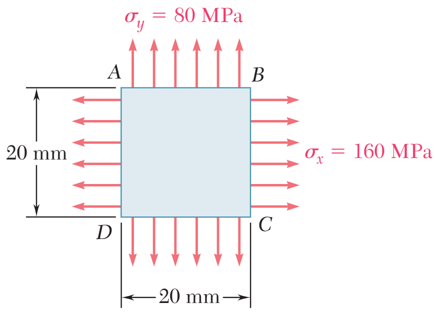


Fig. P9.57

9.58 A fabric used in air-inflated structures is subjected to a biaxial loading that results in normal stresses $\sigma_x = 120$ MPa and $\sigma_z = 160$ MPa. Knowing that the properties of the fabric can be approximated as $E = 87$ GPa and $\nu = 0.34$, determine the change in length of (*a*) side *AB*, (*b*) side *BC*, (*c*) diagonal *AC*.

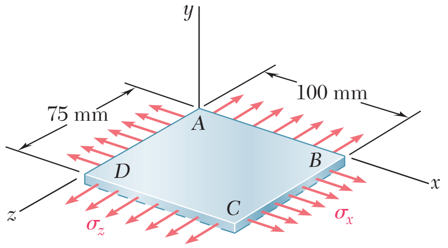


Fig. P9.58

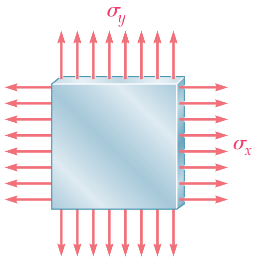


Fig. P9.59

9.59 In many situations it is known that the normal stress in a given direction is zero. For example, $\sigma_z = 0$ in the case of the thin plate shown. For this case, which is known as *plane stress,* show that if the strains ϵ_x and ϵ_y have been determined experimentally, we can express σ_x, σ_y and ϵ_z as follows:

$$\sigma_x = E\frac{\epsilon_x + \nu\epsilon_y}{1 - \nu^2}$$

$$\sigma_y = E\frac{\epsilon_y + \nu\epsilon_x}{1 - \nu^2}$$

$$\epsilon_z = -\frac{\nu}{1 - \nu}(\epsilon_x + \epsilon_y)$$

9.60 In many situations physical constraints prevent strain from occurring in a given direction. For example, $\epsilon_z = 0$ in the case shown, where longitudinal movement of the long prism is prevented at every point. Plane sections perpendicular to the longitudinal axis remain plane and the same distance apart. Show that for this situation, which is known as *plane strain*, we can express σ_z, ϵ_x, and ϵ_y as follows:

$$\sigma_z = \nu(\sigma_x + \sigma_y)$$

$$\epsilon_x = \frac{1}{E}[(1 - \nu^2)\sigma_x - \nu(1 + \nu)\sigma_y]$$

$$\epsilon_y = \frac{1}{E}[(1 - \nu^2)\sigma_y - \nu(1 + \nu)\sigma_x]$$

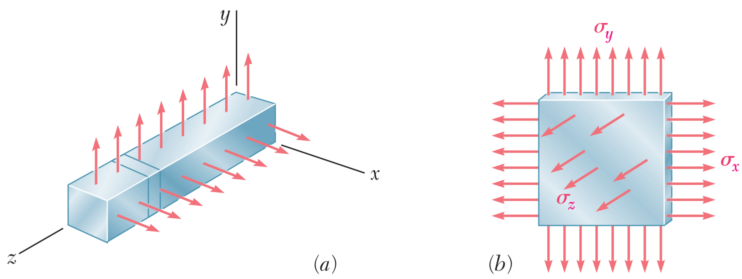


Fig. P9.60

9.61 The plastic block shown is bonded to a rigid support and to a vertical plate to which a 240-kN load **P** is applied. Knowing that for the plastic used $G = 1050$ MPa, determine the deflection of the plate.

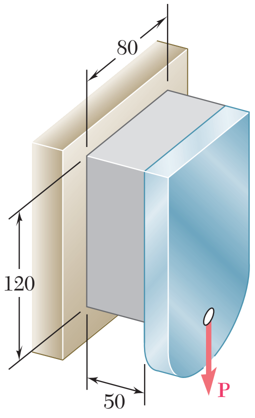


Fig. P9.61

9.62 A vibration isolation unit consists of two blocks of hard rubber bonded to a plate AB and to rigid supports as shown. Knowing that a force of magnitude $P = 6$ kips causes a deflection $\delta = \frac{1}{16}$ in. of plate AB, determine the modulus of rigidity of the rubber used.

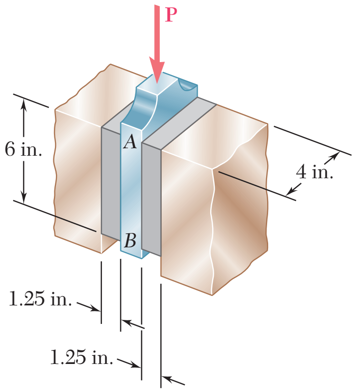


Fig. *P9.62* and P9.63

9.63 A vibration isolation unit consists of two blocks of hard rubber with a modulus of rigidity $G = 2.75$ ksi bonded to a plate AB and to rigid supports as shown. Denoting by P the magnitude of the force applied to the plate and by δ the corresponding deflection, determine the effective spring constant, $k = P/\delta$, of the system.

9.64 An elastomeric bearing ($G = 0.9$ MPa) is used to support a bridge girder as shown to provide flexibility during earthquakes. The beam must not displace more than 10 mm when a 22-kN lateral load is applied as shown. Knowing that the maximum allowable shearing stress is 420 kPa, determine (*a*) the smallest allowable dimension b, (*b*) the smallest required thickness a.

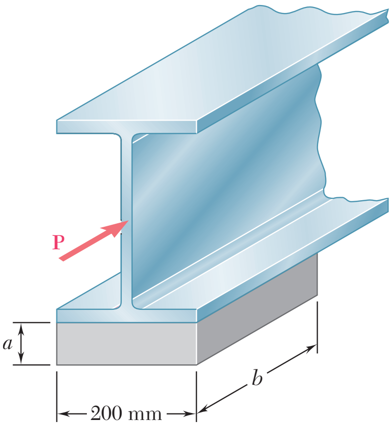


Fig. P9.64

9.14 STRESS AND STRAIN DISTRIBUTION UNDER AXIAL LOADING. SAINT-VENANT'S PRINCIPLE

We have assumed so far that, in an axially loaded member, the normal stresses are uniformly distributed in any section perpendicular to the axis of the member. As we saw in Sec. 8.3, such an assumption may be quite in error in the immediate vicinity of the points of application of the loads. However, the determination of the actual stresses in a given section of the member requires the solution of a statically indeterminate problem.

In Sec. 9.8, you saw that statically indeterminate problems involving the determination of *forces* can be solved by considering the *deformations* caused by these forces. It is thus reasonable to conclude that the determination of the *stresses* in a member requires the analysis

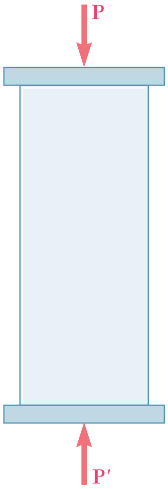
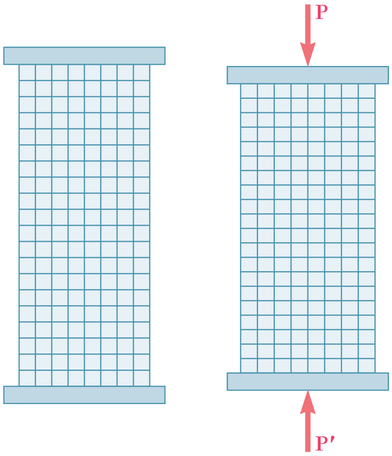


Fig. 9.49



(*a*) (*b*)

Fig. 9.50

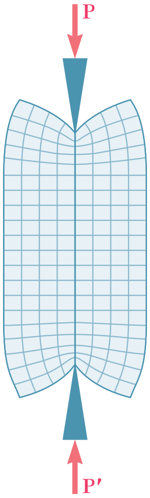


Fig. 9.51

of the *strains* produced by the stresses in the member. This is essentially the approach found in advanced textbooks, where the mathematical theory of elasticity is used to determine the distribution of stresses corresponding to various modes of application of the loads at the ends of the member. Given the more limited mathematical means at our disposal, our analysis of stresses will be restricted to the particular case when two rigid plates are used to transmit the loads to a member made of a homogeneous isotropic material (Fig. 9.49).

If the loads are applied at the center of each plate,† the plates will move toward each other without rotating, causing the member to get shorter, while increasing in width and thickness. It is reasonable to assume that the member will remain straight, that plane sections will remain plane, and that all elements of the member will deform in the same way, since such an assumption is clearly compatible with the given end conditions. This is illustrated in Fig. 9.50, which shows a rubber model before and after loading.‡ Now, if all elements deform in the same way, the distribution of strains throughout the member must be uniform. In other words, the axial strain ϵ_y and the lateral strain $\epsilon_x = -\nu\epsilon_y$ are constant. But, if the stresses do not exceed the proportional limit, Hooke's law applies and we may write $\sigma_y = E\epsilon_y$, from which it follows that the normal stress σ_y is also constant. Thus, the distribution of stresses is uniform throughout the member and, at any point,

$$\sigma_y = (\sigma_y)_{\text{ave}} = \frac{P}{A}$$

On the other hand, if the loads are concentrated, as illustrated in Fig. 9.51, the elements in the immediate vicinity of the points of application of the loads are subjected to very large stresses, while other elements near the ends of the member are unaffected by the loading. This may be verified by observing that strong deformations, and thus large strains and large stresses, occur near the points of application of the loads, while no deformation takes place at the corners. As we consider elements farther and farther from the ends, however, we note a progressive equalization of the deformations involved, and thus a more nearly uniform distribution of the strains and stresses across a section of the member. This is further illustrated in Fig. 9.52, which shows the result of the calculation by advanced mathematical methods of the distribution of stresses across various sections of a thin rectangular plate subjected to concentrated loads. We note that at a distance b from either end, where b is the width of the plate, the stress distribution is nearly uniform across the section, and the value of the stress σ_y at any point of that section can be assumed equal to the average value P/A. Thus, at a distance equal to, or greater than, the width of the member, the distribution of stresses across a given section is the same, whether the member is loaded as shown in Fig. 9.49 or Fig. 9.51. In other words, except in the immediate vicinity of the points of application of the loads, the

†More precisely, the common line of action of the loads should pass through the centroid of the cross section (cf. Sec. 8.3).

‡Note that for long, slender members, another configuration is possible, and indeed will prevail, if the load is sufficiently large; the member *buckles* and assumes a curved shape. This will be discussed in Chap. 16.

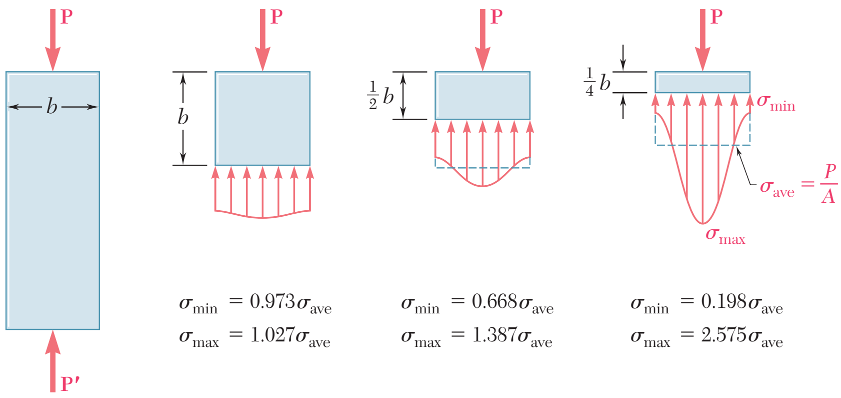


Fig. 9.52

stress distribution may be assumed independent of the actual mode of application of the loads. This statement, which applies not only to axial loadings, but to practically any type of load, is known as *Saint-Venant's principle,* after the French mathematician and engineer Adhémar Barré de Saint-Venant (1797–1886).

While Saint-Venant's principle makes it possible to replace a given loading by a simpler one for the purpose of computing the stresses in a structural member, you should keep in mind two important points when applying this principle:

1. The actual loading and the loading used to compute the stresses must be *statically equivalent.*
2. Stresses cannot be computed in this manner in the immediate vicinity of the points of application of the loads. Advanced theoretical or experimental methods must be used to determine the distribution of stresses in these areas.

You should also observe that the plates used to obtain a uniform stress distribution in the member of Fig. 9.50 must allow the member to freely expand laterally. Thus, the plates cannot be rigidly attached to the member; you must assume them to be just in contact with the member, and smooth enough not to impede the lateral expansion of the member. While such end conditions can actually be achieved for a member in compression, they cannot be physically realized in the case of a member in tension. It does not matter, however, whether or not an actual fixture can be realized and used to load a member so that the distribution of stresses in the member is uniform. The important thing is to *imagine a model* that will allow such a distribution of stresses, and to keep this model in mind so that you may later compare it with the actual loading conditions.

9.15 STRESS CONCENTRATIONS

As you saw in the preceding section, the stresses near the points of application of concentrated loads can reach values much larger than the average value of the stress in the member. When a structural member contains a discontinuity, such as a hole or a sudden change in cross section, high localized stresses can also occur near the discontinuity.

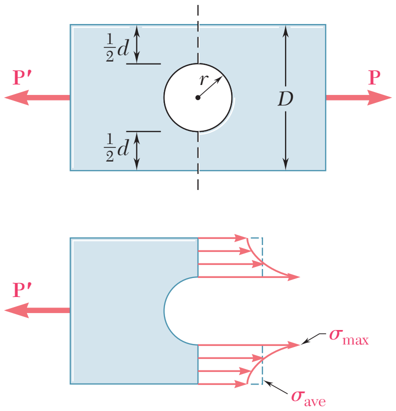


Fig. 9.53 Stress distribution near circular hole in flat bar under axial loading.

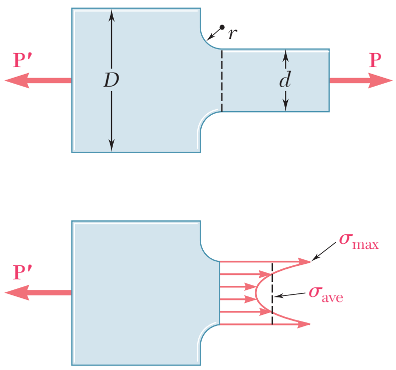


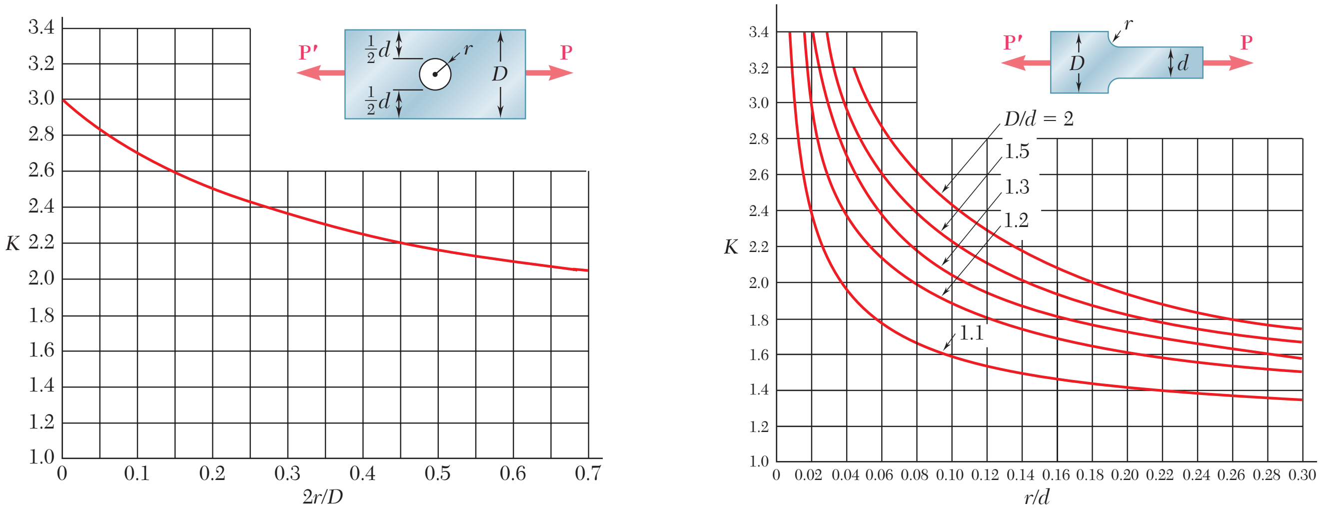
Fig. 9.54 Stress distribution near fillets in flat bar under axial loading.

Figures 9.53 and 9.54 show the distribution of stresses in critical sections corresponding to two such situations. Figure 9.53 refers to a flat bar with a *circular hole* and shows the stress distribution in a section passing through the center of the hole. Figure 9.54 refers to a flat bar consisting of two portions of different widths connected by *fillets;* it shows the stress distribution in the narrowest part of the connection, where the highest stresses occur.

These results were obtained experimentally through the use of a photoelastic method. Fortunately for the engineer who has to design a given member and cannot afford to carry out such an analysis, the results obtained are independent of the size of the member and of the material used; they depend only upon the ratios of the geometric parameters involved, i.e., upon the ratio r/d in the case of a circular hole, and upon the ratios r/d and D/d in the case of fillets. Furthermore, the designer is more interested in the *maximum value* of the stress in a given section than in the actual distribution of stresses in that section, since his main concern is to determine *whether* the allowable stress will be exceeded under a given loading, and not *where* this value will be exceeded. For this reason, one defines the ratio

$$K = \frac{\sigma_{max}}{\sigma_{ave}} \tag{9.36}$$

of the maximum stress over the average stress computed in the critical (narrowest) section of the discontinuity. This ratio is referred to as the *stress-concentration factor* of the given discontinuity. Stress-concentration factors can be computed once and for all in terms of the ratios of the geometric parameters involved, and the results obtained can be expressed in the form of tables or of graphs, as shown in Fig. 9.55. To determine the maximum stress occurring near a discontinuity in a given member subjected to a given axial load P, the designer needs only to compute the average stress $\sigma_{ave} = P/A$ in the critical section and multiply the result obtained by the appropriate value of the stress-concentration factor K. You should note, however, that this procedure is valid only as long as σ_{max} does not exceed the proportional limit of the material, since the values of K plotted in Fig. 9.55 were obtained by assuming a linear relation between stress and strain.



(*a*) Flat bars with holes (*b*) Flat bars with fillets

Fig. 9.55 Stress concentration factors for flat bars under axial loading†

Note that the average stress must be computed across the narrowest section: $\sigma_{ave} = P/td$, where t is the thickness of the bar.

EXAMPLE 9.10 Determine the largest axial load **P** that can be safely supported by a flat steel bar consisting of two portions, both 10 mm thick and, respectively, 40 and 60 mm wide, connected by fillets of radius $r = 8$ mm. Assume an allowable normal stress of 165 MPa.

We first compute the ratios

$$\frac{D}{d} = \frac{60 \text{ mm}}{40 \text{ mm}} = 1.50 \qquad \frac{r}{d} = \frac{8 \text{ mm}}{40 \text{ mm}} = 0.20$$

Using the curve in Fig. 9.55*b* corresponding to $D/d = 1.50$, we find that the value of the stress-concentration factor corresponding to $r/d = 0.20$ is

$$K = 1.82$$

Carrying this value into Eq. (9.36) and solving for σ_{ave}, we have

$$\sigma_{ave} = \frac{\sigma_{max}}{1.82}$$

But σ_{max} cannot exceed the allowable stress $\sigma_{all} = 165$ MPa. Substituting this value for σ_{max}, we find that the average stress in the narrower portion ($d = 40$ mm) of the bar should not exceed the value

$$\sigma_{ave} = \frac{165 \text{ MPa}}{1.82} = 90.7 \text{ MPa}$$

Recalling that $\sigma_{ave} = P/A$, we have

$$P = A\sigma_{ave} = (40 \text{ mm})(10 \text{ mm})(90.7 \text{ MPa}) = 36.3 \times 10^3 \text{ N}$$
$$P = 36.3 \text{ kN} \ \blacksquare$$

†W. D. Pilkey, *Peterson's Stress Concentration Factors*, 2nd ed., John Wiley & Sons, New York, 1997.

PROBLEMS

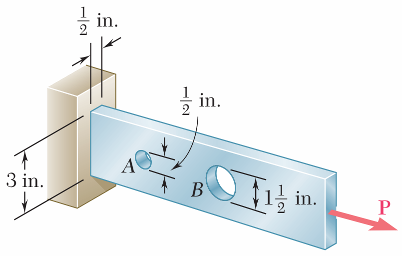


Fig. P9.65 and *P9.66*

9.65 Two holes have been drilled through a long steel bar that is subjected to a centric axial load as shown. For $P = 6.5$ kips, determine the maximum value of the stress (*a*) at *A*, (*b*) at *B*.

9.66 Knowing that $\sigma_{\text{all}} = 16$ ksi, determine the maximum allowable value of the centric axial load **P**.

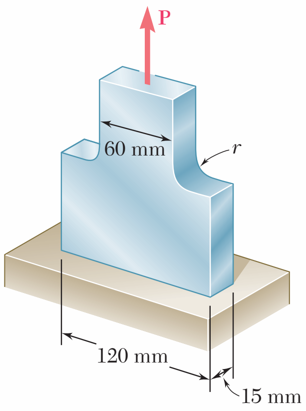


Fig. P9.67 and P9.68

9.67 Knowing that, for the plate shown, the allowable stress is 125 MPa, determine the maximum allowable value of *P* when (*a*) $r = 12$ mm, (*b*) $r = 18$ mm.

9.68 Knowing that $P = 38$ kN, determine the maximum stress when (*a*) $r = 10$ mm, (*b*) $r = 16$ mm, (*c*) $r = 18$ mm.

9.69 (*a*) Knowing that the allowable stress is 140 MPa, determine the maximum allowable magnitude of the centric load **P**. (*b*) Determine the percent change in the maximum allowable magnitude of **P** if the raised portions are removed at the ends of the specimen.

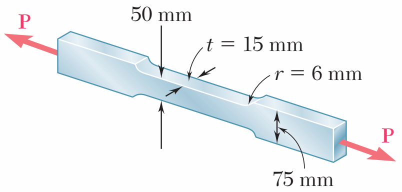


Fig. *P9.69*

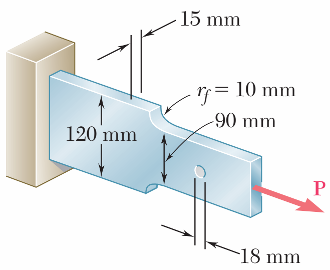


Fig. P9.70

9.70 A centric axial force is applied to the steel bar shown. Knowing that σ_{all} is 135 MPa, determine the maximum allowable load **P**.

9.71 Knowing that the hole has a diameter of $\frac{3}{8}$ in., determine (*a*) the radius r_f of the fillets for which the same maximum stress occurs at the hole *A* and at the fillets, (*b*) the corresponding maximum allowable load **P** if the allowable stress is 15 ksi.

9.72 For $P = 8.5$ kips, determine the minimum plate thickness *t* required if the allowable stress is 18 ksi.

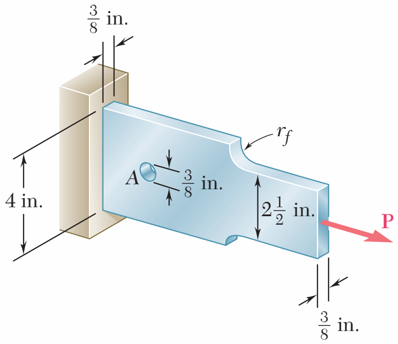


Fig. P9.71

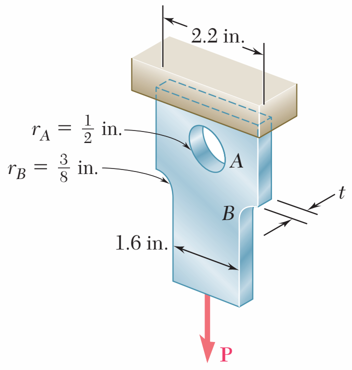


Fig. P9.72

REVIEW AND SUMMARY

This chapter was devoted to the introduction of the concept of *strain*, to the discussion of the relationship between stress and strain in various types of materials, and to the determination of the deformations of structural components under axial loading.

Normal strain

Considering a rod of length L and uniform cross section and denoting by δ its deformation under an axial load **P** (Fig. 9.56), we defined the *normal strain* ϵ in the rod as the *deformation per unit length* [Sec. 9.2]:

$$\epsilon = \frac{\delta}{L} \tag{9.1}$$

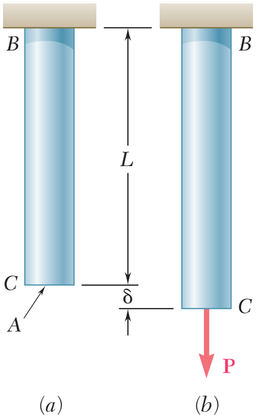


Fig. 9.56

In the case of a rod of variable cross section, the normal strain was defined at any given point Q by considering a small element of rod at Q. Denoting by Δx the length of the element and by $\Delta\delta$ its deformation under the given load, we wrote

$$\epsilon = \lim_{\Delta x \to 0} \frac{\Delta\delta}{\Delta x} = \frac{d\delta}{dx} \tag{9.2}$$

Stress-strain diagram

Plotting the stress σ versus the strain ϵ as the load increased, we obtained a *stress-strain diagram* for the material used [Sec. 9.3]. From such a diagram, we were able to distinguish between *brittle* and *ductile* materials: A specimen made of a brittle material ruptures without any noticeable prior change in the rate of elongation (Fig. 9.58), while a specimen made of a ductile material *yields* after a critical stress σ_Y, called the *yield strength*, has been reached, i.e., the specimen undergoes a large deformation before rupturing, with a relatively small increase in the applied load (Fig. 9.57). An example of brittle material with different properties in tension and in compression was provided by *concrete*.

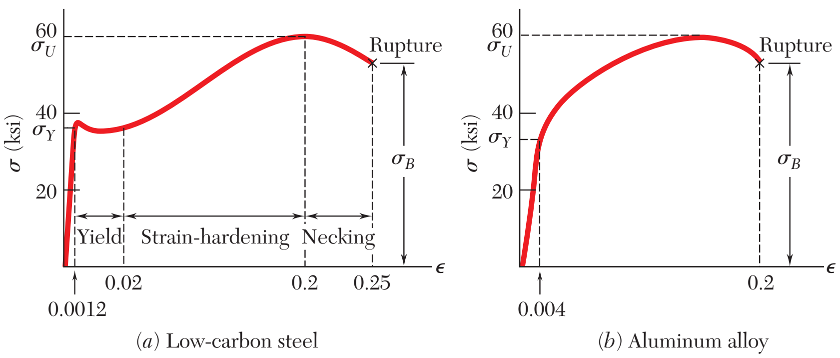


Fig. 9.57

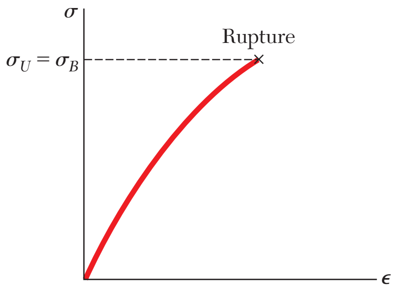


Fig. 9.58

Hooke's law
Modulus of elasticity

We noted in Sec. 9.4 that the initial portion of the stress-strain diagram is a straight line. This means that for small deformations, the stress is directly proportional to the strain:

$$\sigma = E\epsilon \tag{9.3}$$

This relation is known as *Hooke's law* and the coefficient E as the *modulus of elasticity* of the material. The largest stress for which Eq. (9.3) applies is the *proportional limit* of the material.

Materials considered up to this point were *isotropic,* i.e., their properties were independent of direction. In Sec. 9.4 we also considered a class of *anisotropic* materials, i.e., materials whose properties depend upon direction. They were *fiber-reinforced composite materials,* made of fibers of a strong, stiff material embedded in layers of a weaker, softer material (Fig. 9.59). We saw that different moduli of elasticity had to be used, depending upon the direction of loading.

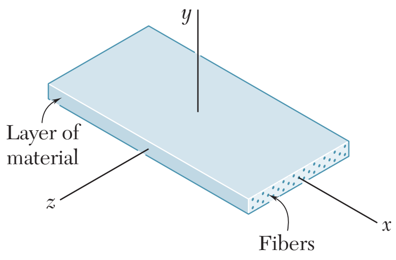


Fig. 9.59

Elastic limit. Plastic deformation

If the strains caused in a test specimen by the application of a given load disappear when the load is removed, the material is said to behave *elastically,* and the largest stress for which this occurs is called the *elastic limit* of the material [Sec. 9.5]. If the elastic limit is exceeded, the stress and strain decrease in a linear fashion when the load is removed and the strain does not return to zero (Fig. 9.60), indicating that a *permanent set* or *plastic deformation* of the material has taken place.

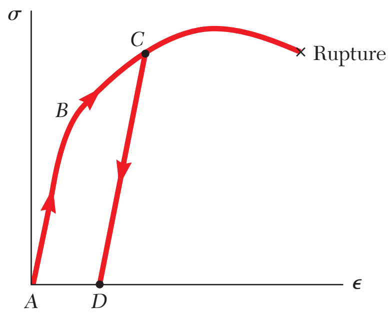


Fig. 9.60

Fatigue. Endurance limit

In Sec. 9.6, we discussed the phenomenon of *fatigue,* which causes the failure of structural or machine components after a very large number of repeated loadings, even though the stresses remain in the elastic range. A standard fatigue test consists in determining the number n of successive loading-and-unloading cycles required to cause the failure of a specimen for any given maximum stress level σ, and plotting the resulting σ-n curve. The value of σ for which failure does not occur, even for an indefinitely large number of cycles, is known as the *endurance limit* of the material used in the test.

Elastic deformation under axial loading

Section 9.7 was devoted to the determination of the elastic deformations of various types of machine and structural components under various conditions of axial loading. We saw that if a rod of length L and uniform cross section of area A is subjected at its end to a centric axial load **P** (Fig. 9.61), the corresponding deformation is

$$\delta = \frac{PL}{AE} \tag{9.6}$$

If the rod is loaded at several points or consists of several parts of various cross sections and possibly of different materials, the deformation δ of the rod must be expressed as the sum of the deformations of its component parts [Example 9.1]:

$$\delta = \sum_i \frac{P_i L_i}{A_i E_i} \tag{9.7}$$

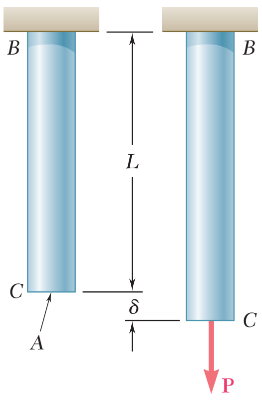


Fig. 9.61

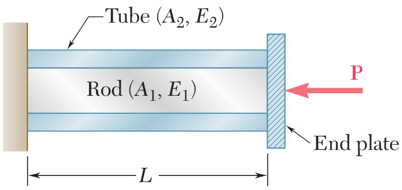


Fig. 9.62

Statically indeterminate problems

Section 9.8 was devoted to the solution of *statically indeterminate problems*, i.e., problems in which the reactions and the internal forces *cannot* be determined from statics alone. The equilibrium equations derived from the free-body diagram of the member under consideration were complemented by relations involving deformations and obtained from the geometry of the problem. The forces in the rod and in the tube of Fig. 9.62, for instance, were determined by observing, on the one hand, that their sum is equal to P, and on the other, that they cause equal deformations in the rod and in the tube [Example 9.2]. Similarly, the reactions at the supports of the bar of Fig. 9.63 could not be obtained from the free-body diagram of the bar alone [Example 9.3]; but they could be determined by expressing that the total elongation of the bar must be equal to zero.

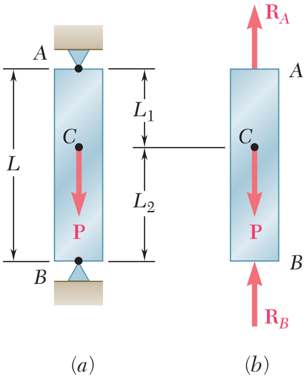


Fig. 9.63

Problems with temperature changes

In Sec. 9.9, we considered problems involving *temperature changes*. We first observed that if the temperature of an *unrestrained rod AB* of length L is increased by ΔT, its elongation is

$$\delta_T = \alpha(\Delta T)L \tag{9.20}$$

where α is the *coefficient of thermal expansion* of the material. We noted that the corresponding strain, called *thermal strain,* is

$$\epsilon_T = \alpha \Delta T \tag{9.21}$$

and that *no stress* is associated with this strain. However, if the rod AB is *restrained* by fixed supports (Fig. 9.64), stresses develop in the rod as the temperature increases because of the reactions at the supports. To determine the magnitude P of the reactions, we detached the rod from its support at B (Fig. 9.65) and considered separately the deformation δ_T of the rod as it expands freely because of the temperature change and the deformation δ_P caused by the force **P** required to bring it back to its original length, so that it may be reattached to the support at B. Writing that the total deformation $\delta = \delta_T + \delta_P$ is equal to zero, we obtained an equation that could be solved for P. While the final strain in rod AB is clearly zero, this will generally not be the case for rods and bars consisting of elements of different cross sections or materials, since the deformations of the various elements will usually *not* be zero [Example 9.6].

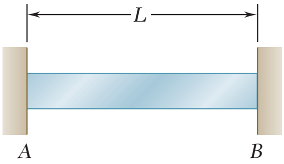


Fig. 9.64

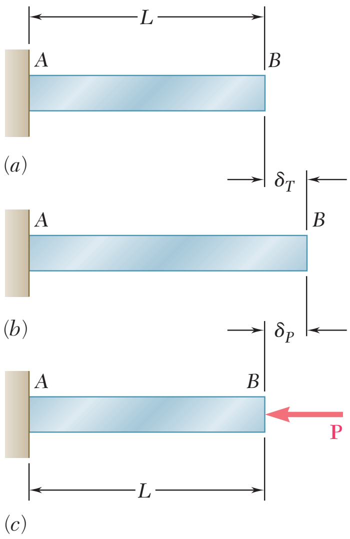


Fig. 9.65

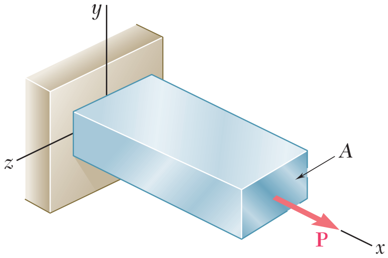


Fig. 9.66

Lateral strain. Poisson's ratio

When an axial load **P** is applied to a homogeneous, slender bar (Fig. 9.66), it causes a strain, not only along the axis of the bar but in any transverse direction as well [Sec. 9.10]. This strain is referred to as the *lateral strain*, and the ratio of the lateral strain over the axial strain is called *Poisson's ratio* and is denoted by ν (Greek letter nu). We wrote

$$\nu = -\frac{\text{lateral strain}}{\text{axial strain}} \tag{9.24}$$

Recalling that the axial strain in the bar is $\epsilon_x = \sigma_x/E$, we expressed as follows the condition of strain under an axial loading in the x direction:

$$\epsilon_x = \frac{\sigma_x}{E} \qquad \epsilon_y = \epsilon_z = -\frac{\nu\sigma_x}{E} \tag{9.26}$$

Multiaxial loading

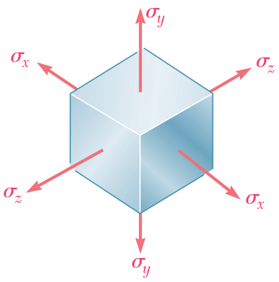


Fig. 9.67

This result was extended in Sec. 9.11 to the case of a *multiaxial loading* causing the state of stress shown in Fig. 9.67. The resulting strain condition was described by the following relations, referred to as the *generalized Hooke's law* for a multiaxial loading.

$$\begin{aligned}
\epsilon_x &= +\frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E} \\
\epsilon_y &= -\frac{\nu\sigma_x}{E} + \frac{\sigma_y}{E} - \frac{\nu\sigma_z}{E} \\
\epsilon_z &= -\frac{\nu\sigma_x}{E} - \frac{\nu\sigma_y}{E} + \frac{\sigma_z}{E}
\end{aligned} \tag{9.27}$$

Shearing strain. Modulus of rigidity

As we saw in Chap. 8, the state of stress in a material under the most general loading condition involves shearing stresses, as well as normal stresses (Fig. 9.68). The shearing stresses tend to deform a cubic element of material into an oblique parallelepiped [Sec. 9.12]. Considering, for instance, the stresses τ_{xy} and τ_{yx} shown in Fig. 9.69 (which, we recall, are equal in magnitude), we noted that they cause the angles formed by the faces on which they act to either increase or decrease by a small angle γ_{xy}; this angle, expressed in radians, defines the *shearing strain* corresponding to the x and y directions. Defining in a similar way the shearing strains γ_{yz} and γ_{zx}, we wrote the relations

$$\tau_{xy} = G\gamma_{xy} \qquad \tau_{yz} = G\gamma_{yz} \qquad \tau_{zx} = G\gamma_{zx} \tag{9.28, 9.29}$$

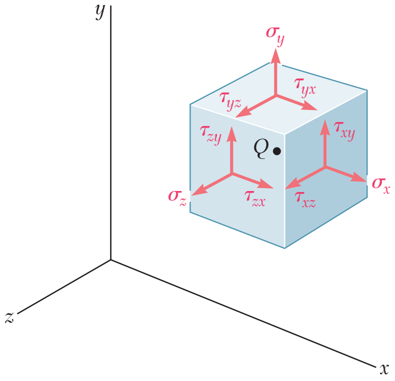


Fig. 9.68

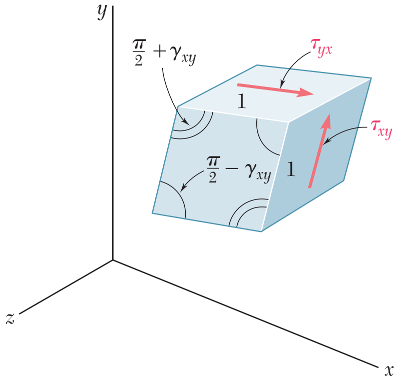


Fig. 9.69

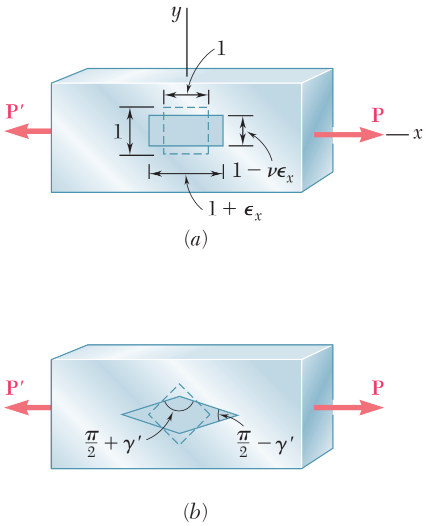


Fig. 9.70

which are valid for any homogeneous isotropic material within its proportional limit in shear. The constant G is called the *modulus of rigidity* of the material and the relations obtained express *Hooke's law for shearing stress and strain*. Together with Eqs. (9.27), they form a group of equations representing the generalized Hooke's law for a homogeneous isotropic material under the most general stress condition.

We observed in Sec. 9.13 that while an axial load exerted on a slender bar produces only normal strains—both axial and transverse—on an element of material oriented along the axis of the bar, it will produce both normal and shearing strains on an element rotated through 45° (Fig. 9.70). We also noted that the three constants E, ν, and G are not independent; they satisfy the relation

$$\frac{E}{2G} = 1 + \nu \tag{9.35}$$

which may be used to determine any of the three constants in terms of the other two.

Saint-Venant's principle

In Sec. 9.14, we discussed *Saint-Venant's principle,* which states that except in the immediate vicinity of the points of application of the loads, the distribution of stresses in a given member is independent of the actual mode of application of the loads. This principle makes it possible to assume a uniform distribution of stresses in a member subjected to concentrated axial loads, except close to the points of application of the loads, where stress concentrations will occur.

Stress concentrations

Stress concentrations will also occur in structural members near a discontinuity, such as a hole or a sudden change in cross section [Sec. 9.15]. The ratio of the maximum value of the stress occurring near the discontinuity over the average stress computed in the critical section is referred to as the *stress-concentration factor* of the discontinuity and is denoted by K:

$$K = \frac{\sigma_{\text{max}}}{\sigma_{\text{ave}}} \tag{9.36}$$

Values of K for circular holes and fillets in flat bars were given in Fig. 9.55 on p. 395.

REVIEW PROBLEMS

9.73 The aluminum rod ABC ($E = 10.1 \times 10^6$ psi), which consists of two cylindrical portions AB and BC, is to be replaced with a cylindrical steel rod DE ($E = 29 \times 10^6$ psi) of the same overall length. Determine the minimum required diameter d of the steel rod if its vertical deformation is not to exceed the deformation of the aluminum rod under the same load and if the allowable stress in the steel rod is not to exceed 24 ksi.

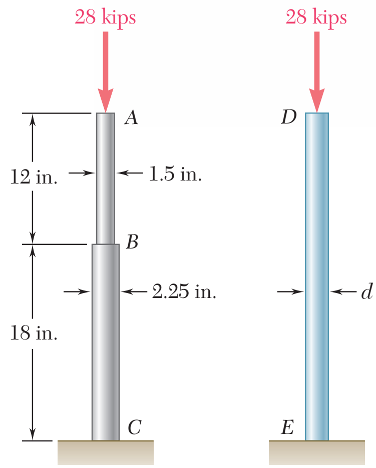


Fig. P9.73

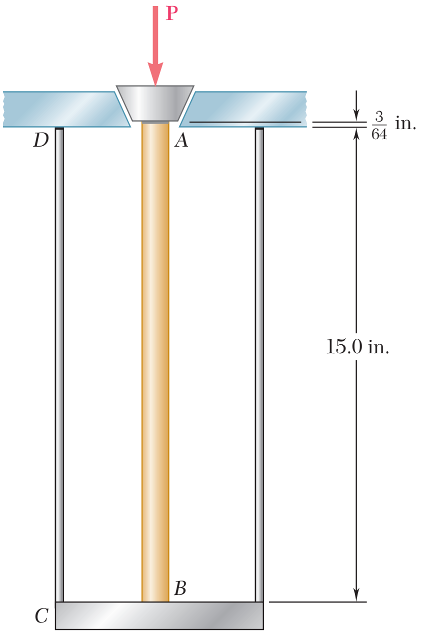


Fig. *P9.74*

9.74 The brass tube AB ($E = 15 \times 10^6$ psi) has a cross-sectional area of 0.22 in^2 and is fitted with a plug at A. The tube is attached at B to a rigid plate that is itself attached at C to the bottom of an aluminum cylinder ($E = 10.4 \times 10^6$ psi) with a cross-sectional area of 0.40 in^2. The cylinder is then hung from a support at D. In order to close the cylinder, the plug must move down through $\frac{3}{64}$ in. Determine the force $\mathbf{P}$ that must be applied to the cylinder.

9.75 The length of the 2-mm-diameter steel wire CD has been adjusted so that with no load applied, a gap of 1.5 mm exists between the end B of the rigid beam ACB and a contact point E. Knowing that $E = 200$ GPa, determine where a 20-kg block should be placed on the beam in order to cause contact between B and E.

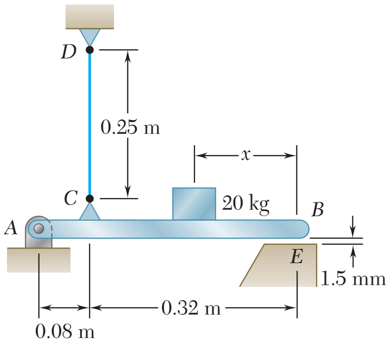


Fig. P9.75

9.76 The uniform rods AB and BC are made of steel and are loaded as shown. Knowing that $E = 29 \times 10^6$ psi, determine the magnitude and direction of the deflection of point B when $\theta = 22°$.

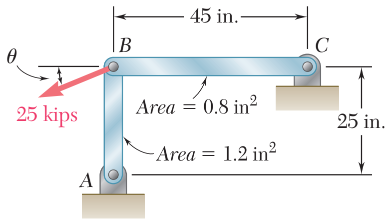


Fig. P9.76

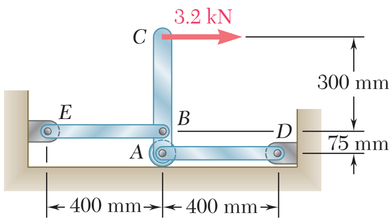


Fig. P9.77

9.77 The steel bars BE and AD each have a 6×18-mm cross section. Knowing that $E = 200$ GPa, determine the deflections of points A, B, and C of the rigid bar ABC.

9.78 In Prob. 9.77, the 3.2-kN force caused point C to deflect to the right. Using $\alpha = 11.7 \times 10^{-6}/°C$, determine ($a$) the overall change in temperature that causes point C to return to its original position, (b) the corresponding total deflection of points A and B.

9.79 An axial centric force P is applied to the composite block shown by means of a rigid end plate. Determine (a) the value of h if the portion of the load carried by the aluminum plates is half the portion of the load carried by the brass core, (b) the total load if the stress in the brass is 80 MPa.

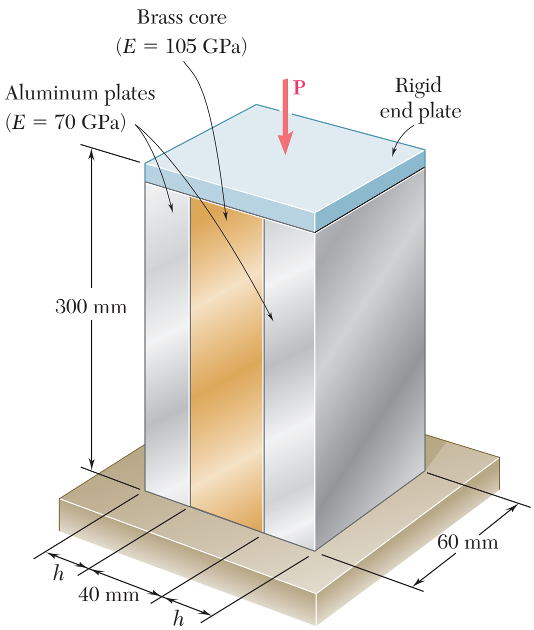


Fig. *P9.79*

9.80 A steel tube ($E = 29 \times 10^6$ psi) with a $1\frac{1}{4}$-in. outer diameter and a $\frac{1}{8}$-in. thickness is placed in a vise that is adjusted so that its jaws just touch the ends of the tube without exerting any pressure on them. The two forces shown are then applied to the tube. After these forces are applied, the vise is adjusted to decrease the distance between its jaws by 0.008 in. Determine (*a*) the forces exerted by the vise on the tube at *A* and *D*, (*b*) the change in length of the portion *BC* of the tube.

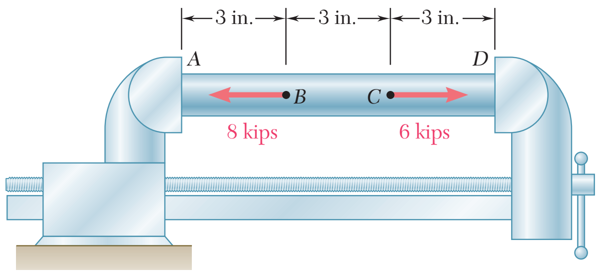


Fig. P9.80

9.81 The block shown is made of a magnesium alloy for which $E = 6.5 \times 10^6$ psi and $\nu = 0.35$. Knowing that $\sigma_x = -20$ ksi, determine (*a*) the magnitude of σ_y for which the change in the height of the block will be zero, (*b*) the corresponding change in the area of the face *ABCD*, (*c*) the corresponding change in the volume of the block.

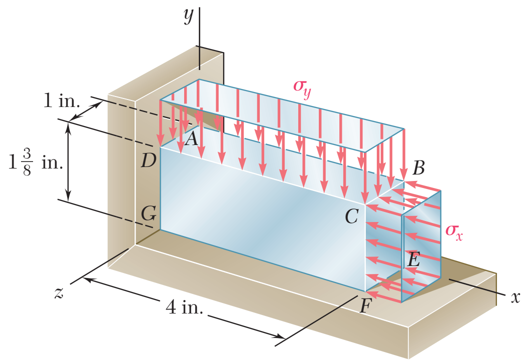


Fig. *P9.81*

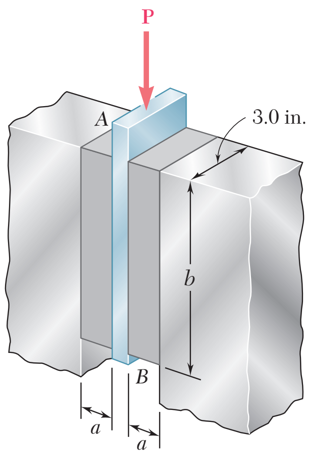


Fig. P9.82

9.82 A vibration isolation unit consists of two blocks of hard rubber bonded to plate *AB* and to rigid supports as shown. For the type and grade of rubber used, $\tau_{all} = 220$ psi and $G = 1800$ psi. Knowing that a centric vertical force of magnitude $P = 3.2$ kips must cause a 0.1-in. vertical deflection of the plate *AB*, determine the smallest allowable dimensions *a* and *b* of the block.

9.83 A hole is to be drilled in the plate at A. The diameters of the bits available to drill the hole range from 9 to 27 mm in 6-mm increments. If the allowable stress in the plate is 145 MPa, determine (a) the diameter d of the largest bit that can be used if the allowable load **P** at the hole is to exceed that at the fillets, (b) the corresponding allowable load **P**.

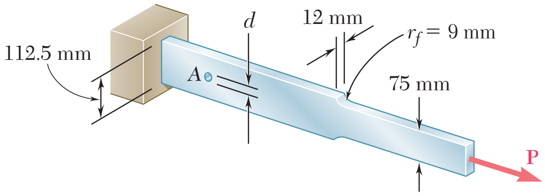
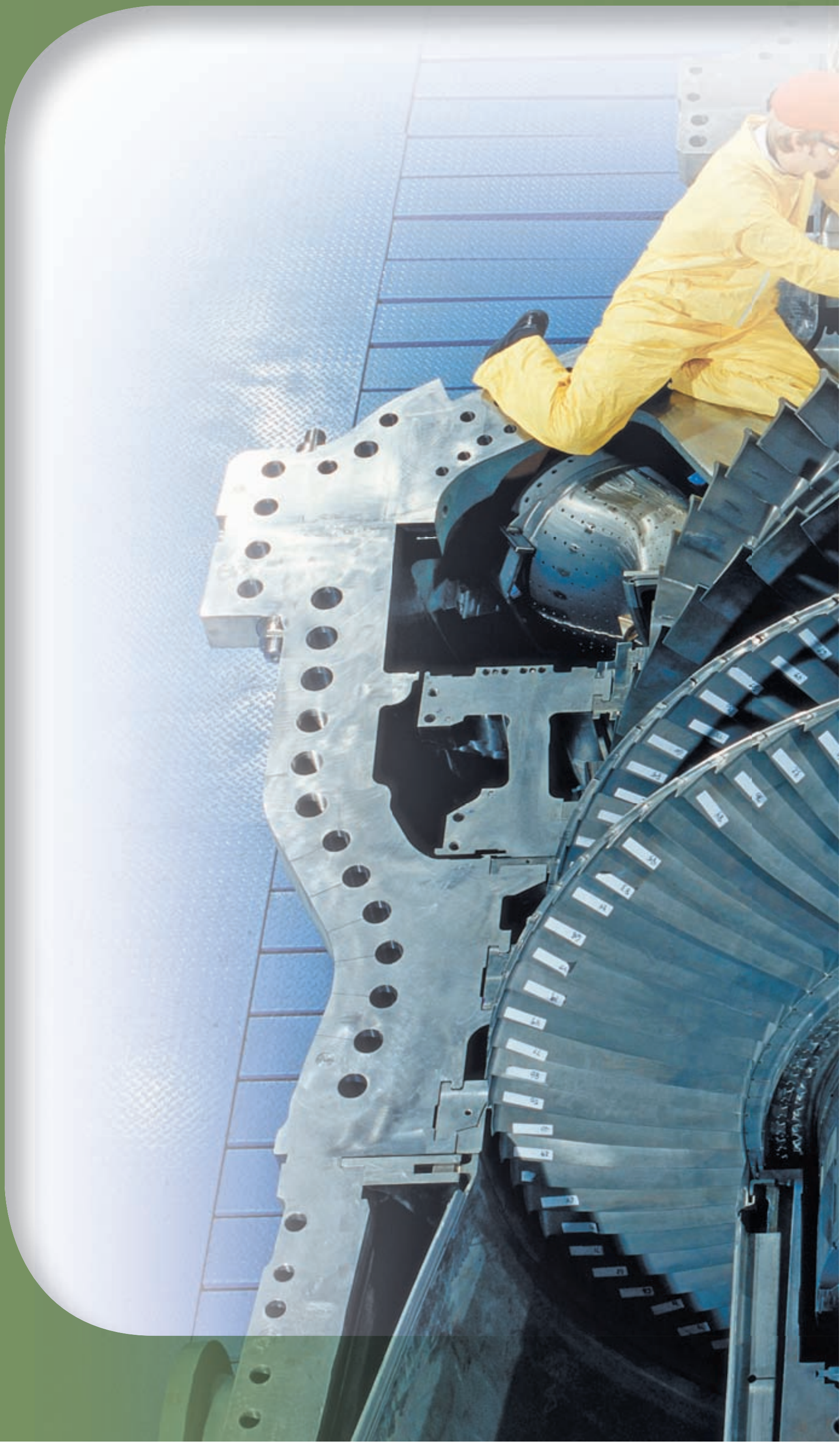


Fig. P9.83 and P9.84

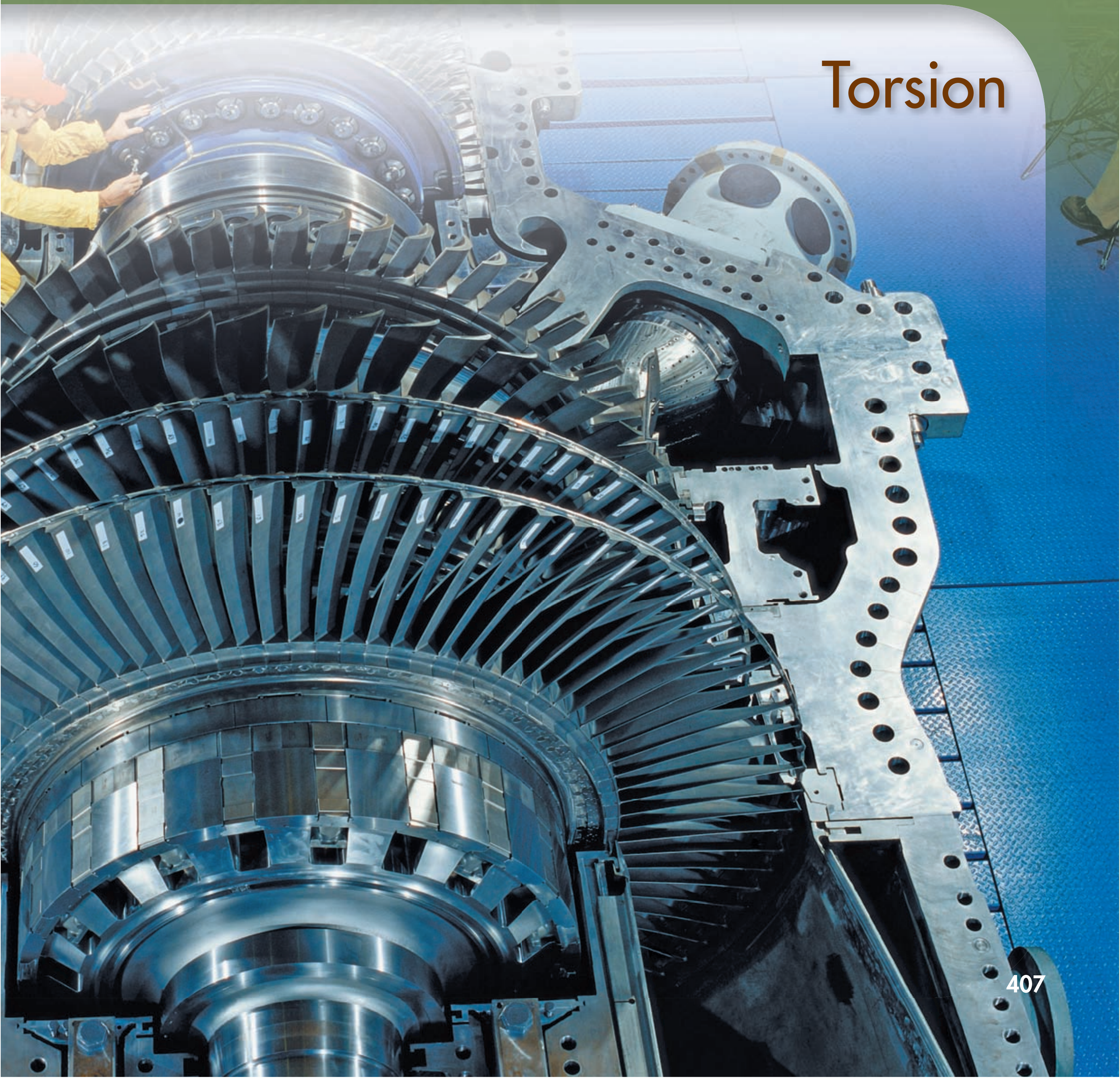
9.84 (a) For $P = 58$ kN and $d = 12$ mm, determine the maximum stress in the plate shown. (b) Solve part a assuming that the hole at A is not drilled.

This chapter is devoted to the study of torsion and of the stresses and deformations it causes. In the jet engine shown here, the central shaft links the components of the engine to develop the thrust that propels the plane.



CHAPTER 10

Torsion



Chapter 10 Torsion

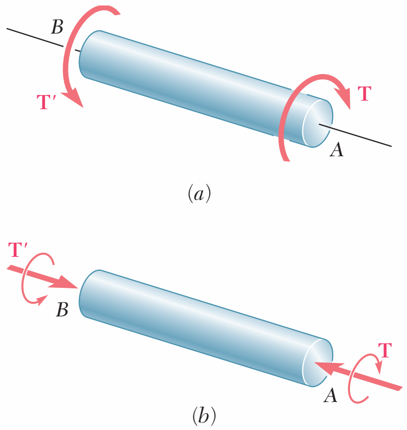


Fig. 10.1

10.1 INTRODUCTION

In Chaps. 8 and 9 you studied how to calculate the stresses and strains in structural members subjected to axial loads, that is, to forces directed along the axis of the member. In this chapter structural members and machine parts that are in *torsion* will be considered. More specifically, you will analyze the stresses and strains in members of circular cross section subjected to twisting couples, or *torques*, **T** and **T**′ (Fig. 10.1). These couples have a common magnitude T, and opposite senses. They are vector quantities and can be represented either by curved arrows as in Fig. 10.1*a*, or by couple vectors as in Fig. 10.1*b*.

Members in torsion are encountered in many engineering applications. The most common application is provided by *transmission shafts*, which are used to transmit power from one point to another. For example, the shaft shown in Photo 10.1 is used to transmit power from the engine to the rear wheels of an automobile. These shafts can be either solid, as shown in Fig. 10.1, or hollow.

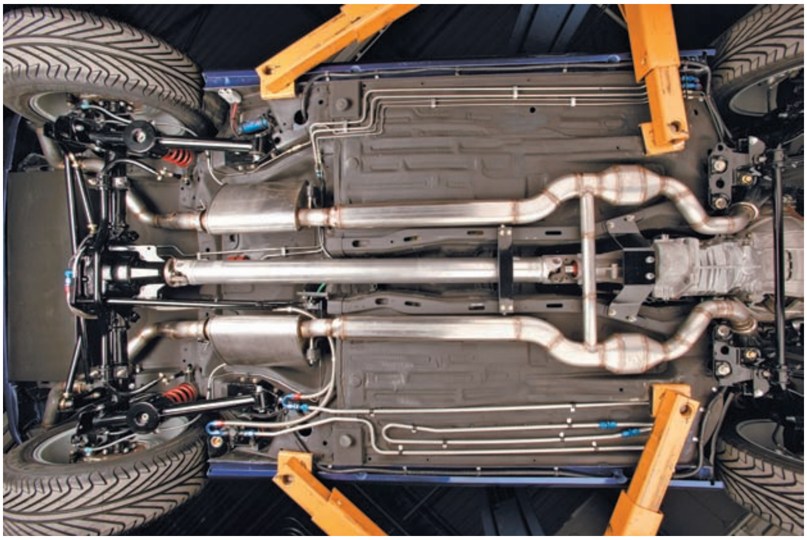


Photo 10.1 In the automotive power train shown, the shaft transmits power from the engine to the rear wheels.

Consider the system shown in Fig. 10.2*a*, which consists of a steam turbine A and an electric generator B connected by a transmission shaft AB. By breaking the system into its three component parts (Fig. 10.2*b*), you can see that the turbine exerts a twisting couple or torque **T** on the shaft and that the shaft exerts an equal torque on the generator. The generator reacts by exerting the equal and opposite torque **T**′ on the shaft, and the shaft by exerting the torque **T**′ on the turbine.

You will first analyze the stresses and deformations that take place in circular shafts. In Sec. 10.3, an important property of circular shafts is demonstrated: *When a circular shaft is subjected to torsion,*

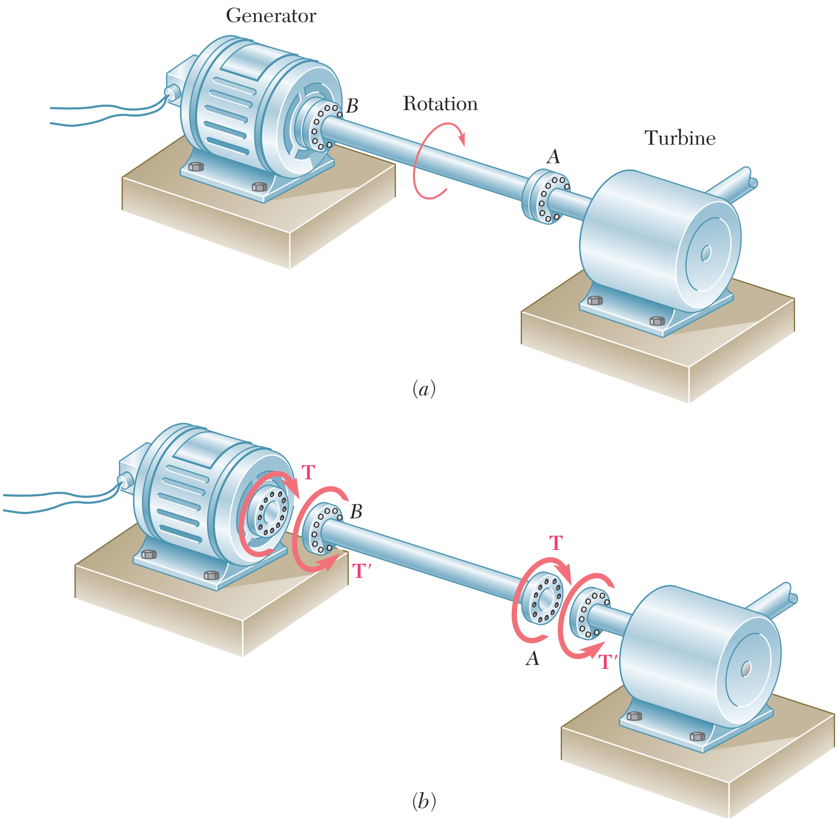


Fig. 10.2

every cross section remains plane and undistorted. In other words, while the various cross sections along the shaft rotate through different angles, each cross section rotates as a solid rigid slab. This property will enable you to determine the *distribution of shearing strains in a circular shaft and to conclude that the shearing strain varies linearly with the distance from the axis of the shaft.*

Considering deformations in the *elastic range* and using Hooke's law for shearing stress and strain, you will determine the *distribution of shearing stresses* in a circular shaft and derive the *elastic torsion formulas* (Sec. 10.4).

In Sec. 10.5, you will learn how to find the *angle of twist* of a circular shaft subjected to a given torque, assuming again elastic deformations. The solution of problems involving *statically indeterminate shafts* is considered in Sec. 10.6.

10.2 PRELIMINARY DISCUSSION OF THE STRESSES IN A SHAFT

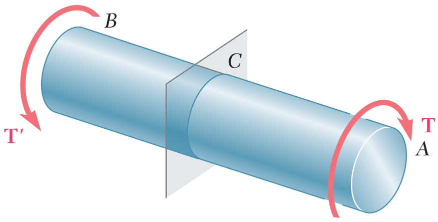


Fig. 10.3

Considering a shaft AB subjected at A and B to equal and opposite torques **T** and **T**′, we pass a section perpendicular to the axis of the shaft through some arbitrary point C (Fig. 10.3). The free-body diagram of the portion BC of the shaft must include the elementary

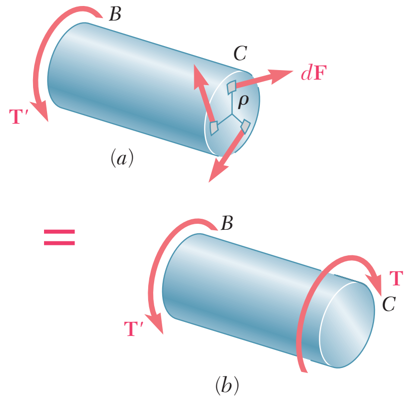


Fig. 10.4

shearing forces $d\mathbf{F}$, perpendicular to the radius of the shaft, that portion AC exerts on BC as the shaft is twisted (Fig. 10.4*a*). But the conditions of equilibrium for BC require that the system of these elementary forces be equivalent to an internal torque $\mathbf{T}$, equal and opposite to $\mathbf{T}'$ (Fig. 10.4*b*). Denoting by ρ the perpendicular distance from the force $d\mathbf{F}$ to the axis of the shaft, and expressing that the sum of the moments of the shearing forces $d\mathbf{F}$ about the axis of the shaft is equal in magnitude to the torque $\mathbf{T}$, we write

$$\int \rho dF = T$$

or, since $dF = \tau\, dA$, where τ is the shearing stress on the element of area dA,

$$\int \rho(\tau\, dA) = T \tag{10.1}$$

While the relation obtained expresses an important condition that must be satisfied by the shearing stresses in any given cross section of the shaft, it does *not* tell us how these stresses are distributed in the cross section. We thus observe, as we already did in Sec. 8.3, that the actual distribution of stresses under a given load is *statically indeterminate,* i.e., this distribution *cannot be determined by the methods of statics*. However, having assumed in Sec. 8.3 that the normal stresses produced by an axial centric load were uniformly distributed, we found later (Sec. 9.14) that this assumption was justified, except in the neighborhood of concentrated loads. A similar assumption with respect to the distribution of shearing stresses in an elastic shaft *would be wrong*. We must withhold any judgment regarding the distribution of stresses in a shaft until we have analyzed the *deformations* that are produced in the shaft. This will be done in the next section.

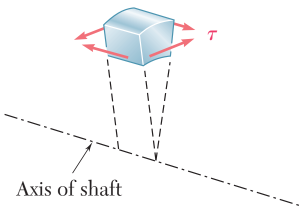


Fig. 10.5

One more observation should be made at this point. As was indicated in Sec. 8.9, shear cannot take place in one plane only. Consider the very small element of shaft shown in Fig. 10.5. We know that the torque applied to the shaft produces shearing stresses τ on the faces perpendicular to the axis of the shaft. But the conditions of equilibrium discussed in Sec. 8.9 require the existence of equal stresses on the faces formed by the two planes containing the axis of the shaft. That such shearing stresses actually occur in torsion can be demonstrated, by considering a "shaft" made of separate slats pinned at both ends to disks as shown in Fig. 10.6*a*. If markings have been painted on two adjoining slats, it is observed that the slats slide with respect to each other when equal and opposite torques are applied to the ends of the "shaft" (Fig. 10.6*b*). While sliding will not actually take place in a shaft made of a homogeneous and cohesive material, the tendency for sliding will exist, showing that stresses occur on longitudinal planes as well as on planes perpendicular to the axis of the shaft.†

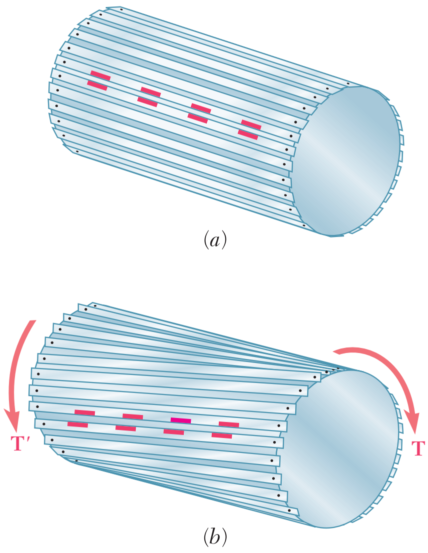


Fig. 10.6

†The twisting of a cardboard tube that has been slit lengthwise provides another demonstration of the existence of shearing stresses on longitudinal planes.

10.3 DEFORMATIONS IN A CIRCULAR SHAFT

Consider a circular shaft that is attached to a fixed support at one end (Fig. 10.7*a*). If a torque **T** is applied to the other end, the shaft will twist, with its free end rotating through an angle ϕ called *the angle of twist* (Fig. 10.7*b*). Observation shows that, within a certain range of values of T, the angle of twist ϕ is proportional to T. It also shows that ϕ is proportional to the length L of the shaft. In other words, the angle of twist for a shaft of the same material and same cross section, but twice as long, will be twice as large under the same torque **T**. One purpose of our analysis will be to find the specific relation existing among ϕ, L, and T; another purpose will be to determine the distribution of shearing stresses in the shaft, which we were unable to obtain in the preceding section on the basis of statics alone.

At this point, an important property of circular shafts should be noted: When a circular shaft is subjected to torsion, *every cross section remains plane and undistorted*. In other words, while the various cross sections along the shaft rotate through different amounts, each cross section rotates as a solid rigid slab. This is illustrated in Fig. 10.8*a*, which shows the deformations in a rubber model subjected to torsion. The property we are discussing is characteristic of circular shafts, whether solid or hollow; it is not enjoyed by members of noncircular cross section. For example, when a bar of square cross section is subjected to torsion, its various cross sections warp and do not remain plane (Fig. 10.8*b*).

The cross sections of a circular shaft remain plane and undistorted because a circular shaft is *axisymmetric,* i.e., its appearance remains the same when it is viewed from a fixed position and rotated about its axis through an arbitrary angle. (Square bars, on the other hand, retain the same appearance only if they are rotated through 90° or 180°.) As we will see presently, the axisymmetry of circular shafts may be used to prove theoretically that their cross sections remain plane and undistorted.

Consider the points C and D located on the circumference of a given cross section of the shaft, and let C' and D' be the positions they will occupy after the shaft has been twisted (Fig. 10.9*a*). The axisymmetry of the shaft and of the loading requires that the rotation which would have brought D into C should now bring D' into C'. Thus C' and D' must lie on the circumference of a circle, and the arc $C'D'$ must be equal to the arc CD (Fig. 10.9*b*). We will now examine whether the circle on which C' and D' lie is different from the original circle. Let us assume that C' and D' do lie on a different circle and that the new circle is located to the left of the original circle, as shown in Fig. 10.9*b*. The same situation will prevail for any other cross section, since all the cross sections of the shaft are subjected to the same internal torque T, and an observer looking at the shaft from its end A will conclude that the loading causes any given circle drawn on the shaft to move *away*. But an observer located at B, to whom the given loading looks the same (a clockwise couple in the foreground and a counterclockwise couple in the background) will reach the opposite conclusion, i.e., that the circle moves *toward*

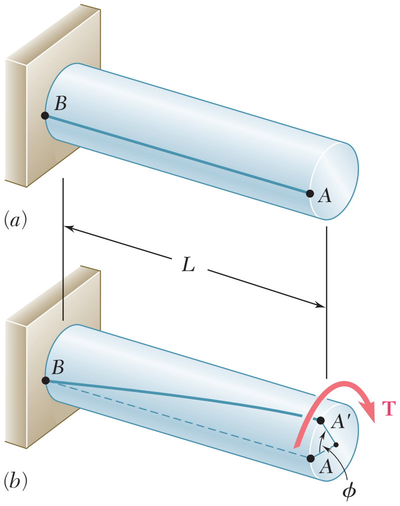


Fig. 10.7

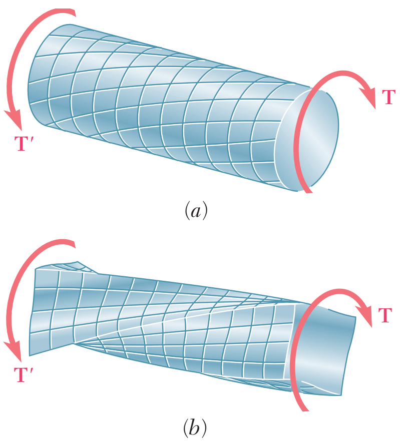


Fig. 10.8

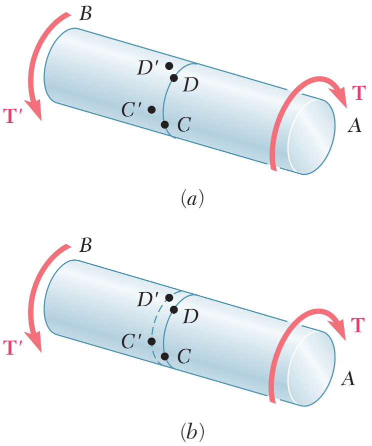


Fig. 10.9

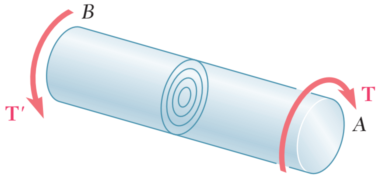


Fig. 10.10

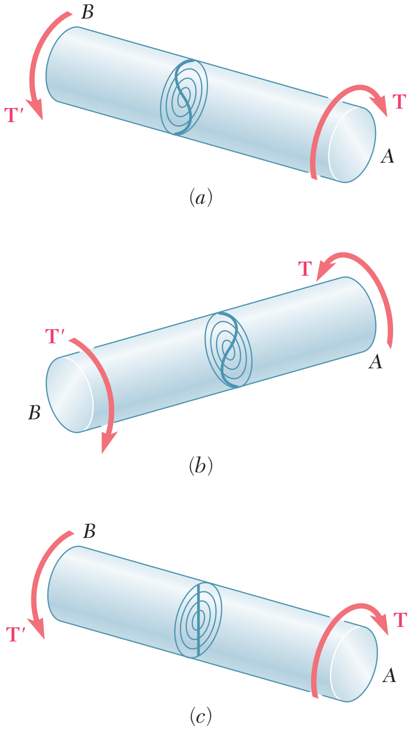


Fig. 10.11

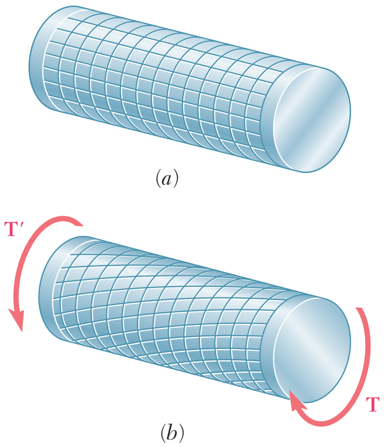


Fig. 10.12

him. This contradiction proves that our assumption is wrong and that C' and D' lie on the same circle as C and D. Thus, as the shaft is twisted, the original circle just rotates in its own plane. Since the same reasoning may be applied to any smaller, concentric circle located in the cross section under consideration, we conclude that the entire cross section remains plane (Fig. 10.10).

The above argument does not preclude the possibility for the various concentric circles of Fig. 10.10 to rotate by different amounts when the shaft is twisted. But if that were so, a given diameter of the cross section would be distorted into a curve which might look as shown in Fig. 10.11*a*. An observer looking at this curve from A would conclude that the outer layers of the shaft get more twisted than the inner ones, while an observer looking from B would reach the opposite conclusion (Fig. 10.11*b*). This inconsistency leads us to conclude that any diameter of a given cross section remains straight (Fig. 10.11*c*) and, therefore, that any given cross section of a circular shaft remains plane and undistorted.

Our discussion so far has ignored the mode of application of the twisting couples $\mathbf{T}$ and $\mathbf{T}'$. If *all* sections of the shaft, from one end to the other, are to remain plane and undistorted, we must make sure that the couples are applied in such a way that the ends of the shaft themselves remain plane and undistorted. This may be accomplished by applying the couples $\mathbf{T}$ and $\mathbf{T}'$ to rigid plates, which are solidly attached to the ends of the shaft (Fig. 10.12*a*). We can then be sure that all sections will remain plane and undistorted when the loading is applied and that the resulting deformations will occur in a uniform fashion throughout the entire length of the shaft. All of the equally spaced circles shown in Fig. 10.12*a* will rotate by the same amount relative to their neighbors, and each of the straight lines will be transformed into a curve (helix) intersecting the various circles at the same angle (Fig. 10.12*b*).

The derivations given in this and the following sections will be based on the assumption of rigid end plates. Loading conditions encountered in practice may differ appreciably from those corresponding to the model of Fig. 10.12. The chief merit of this model is that it helps us define a torsion problem for which we can obtain an exact solution, just as the rigid-end-plates model of Sec. 9.14 made it possible for us to define an axial-load problem which could be easily and accurately solved. By virtue of Saint-Venant's principle, the results obtained for our idealized model may be extended to most engineering applications. However, we should keep these results associated in our mind with the specific model shown in Fig. 10.12.

We will now determine the distribution of *shearing strains* in a circular shaft of length L and radius c which has been twisted through an angle ϕ (Fig. 10.13*a*). Detaching from the shaft a cylinder of radius ρ, we consider the small square element formed by two adjacent circles and two adjacent straight lines traced on the surface of the cylinder before any load is applied (Fig. 10.13*b*). As the shaft is subjected to a torsional load, the element deforms into a rhombus (Fig. 10.13*c*). We now recall from Sec. 9.12 that the shearing strain γ in a given element is measured by the change in the angles formed by the sides of that element. Since the circles defining two of the sides of the element considered here remain unchanged, the shearing strain γ must be equal to the angle between lines AB and $A'B$. (We recall that γ should be expressed in radians.)

We observe from Fig. 10.13*c* that, for small values of γ, we can express the arc length AA' as $AA' = L\gamma$. But, on the other hand, we have $AA' = \rho\phi$. It follows that $L\gamma = \rho\phi$, or

$$\gamma = \frac{\rho\phi}{L} \tag{10.2}$$

where γ and ϕ are both expressed in radians. The equation obtained shows, as we could have anticipated, that the shearing strain γ at a given point of a shaft in torsion is proportional to the angle of twist ϕ. It also shows that γ is proportional to the distance ρ from the axis of the shaft to the point under consideration. Thus, *the shearing strain in a circular shaft varies linearly with the distance from the axis of the shaft*.

It follows from Eq. (10.2) that the shearing strain is maximum on the surface of the shaft, where $\rho = c$. We have

$$\gamma_{\max} = \frac{c\phi}{L} \tag{10.3}$$

Eliminating ϕ from Eqs. (10.2) and (10.3), we can express the shearing strain γ at a distance ρ from the axis of the shaft as

$$\gamma = \frac{\rho}{c}\gamma_{\max} \tag{10.4}$$

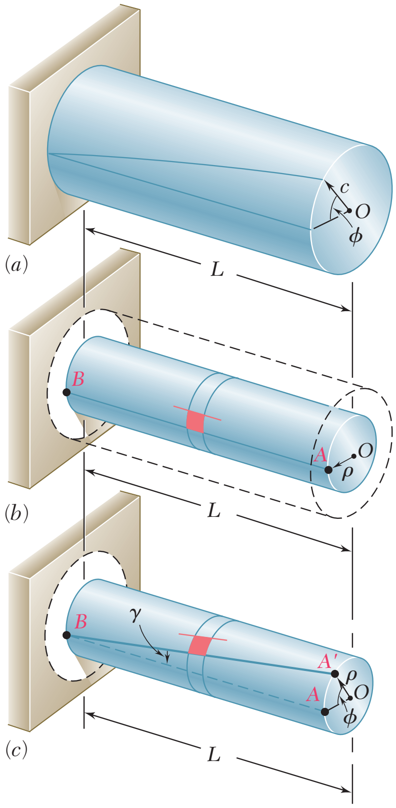


Fig. 10.13

10.4 STRESSES

No particular stress-strain relationship has been assumed so far in our discussion of circular shafts in torsion. Let us now consider the case when the torque **T** is such that all shearing stresses in the shaft remain below the yield strength τ_Y. We know from Chap. 9 that, for all practical purposes, this means that the stresses in the shaft will remain below the proportional limit and below the elastic limit as well. Thus, Hooke's law will apply, and there will be no permanent deformation.

Recalling Hooke's law for shearing stress and strain from Sec. 9.12, we write

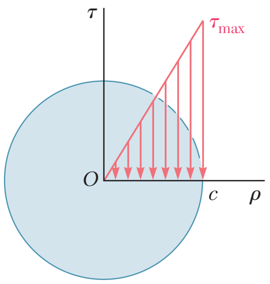
$$\tau = G\gamma \tag{10.5}$$

where G is the modulus of rigidity or shear modulus of the material. Multiplying both members of Eq. (10.5) by G, we write

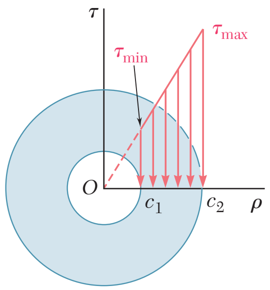
$$G\gamma = \frac{\rho}{c}G\gamma_{\max}$$

or, making use of Eq. (10.5),

$$\boxed{\tau = \frac{\rho}{c}\tau_{\max}} \tag{10.6}$$



(a)



(b)

Fig. 10.14

The equation obtained shows that, as long as the yield strength (or proportional limit) is not exceeded in any part of a circular shaft, *the shearing stress in the shaft varies linearly with the distance* ρ *from the axis of the shaft*. Figure 10.14*a* shows the stress distribution in a solid circular shaft of radius c, and Fig. 10.14*b* in a hollow circular shaft of inner radius c_1 and outer radius c_2. From Eq. (10.6), we find that, in the latter case,

$$\tau_{\min} = \frac{c_1}{c_2}\tau_{\max} \tag{10.7}$$

We now recall from Sec. 10.2 that the sum of the moments of the elementary forces exerted on any cross section of the shaft must be equal to the magnitude T of the torque exerted on the shaft:

$$\int \rho(\tau\, dA) = T \tag{10.1}$$

Substituting for τ from (10.6) into (10.1), we write

$$T = \int \rho\tau\, dA = \frac{\tau_{\max}}{c}\int \rho^2\, dA$$

But the integral in the last member represents the polar moment of inertia J of the cross section with respect to its center O. We have therefore

$$T = \frac{\tau_{\max} J}{c} \tag{10.8}$$

or, solving for $\tau_{\max}$,

$$\boxed{\tau_{\max} = \frac{Tc}{J}} \tag{10.9}$$

Substituting for $\tau_{\max}$ from (10.9) into (10.6), we express the shearing stress at any distance ρ from the axis of the shaft as

$$\boxed{\tau = \frac{T\rho}{J}} \tag{10.10}$$

Equations (10.9) and (10.10) are known as the *elastic torsion formulas*. We recall from statics that the polar moment of inertia of a circle of radius c is $J = \frac{1}{2}\pi c^4$. In the case of a hollow circular shaft of inner radius c_1 and outer radius c_2, the polar moment of inertia is

$$J = \tfrac{1}{2}\pi c_2^4 - \tfrac{1}{2}\pi c_1^4 = \tfrac{1}{2}\pi(c_2^4 - c_1^4) \tag{10.11}$$

We note that, if SI metric units are used in Eq. (10.9) or (10.10), T will be expressed in N · m, c or ρ in meters, and J in m^4; we check that the resulting shearing stress will be expressed in N/m^2, that is, pascals (Pa). If U.S. customary units are used, T should be expressed in lb · in., c or ρ in inches, and J in in^4, with the resulting shearing stress expressed in psi.

EXAMPLE 10.1 A hollow cylindrical steel shaft is 1.5 m long and has inner and outer diameters respectively equal to 40 and 60 mm (Fig. 10.15). (*a*) What is the largest torque that can be applied to the shaft if the shearing stress is not to exceed 120 MPa? (*b*) What is the corresponding minimum value of the shearing stress in the shaft?

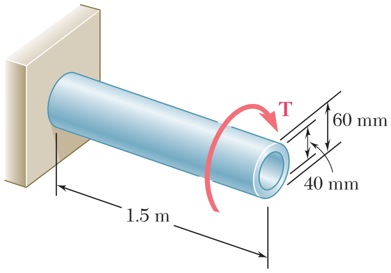


Fig. 10.15

(a) Largest Permissible Torque. The largest torque **T** that can be applied to the shaft is the torque for which $\tau_{\max} = 120$ MPa. Since this value is less than the yield strength for steel, we can use Eq. (10.9). Solving this equation for T, we have

$$T = \frac{J\tau_{\max}}{c} \tag{10.12}$$

Recalling that the polar moment of inertia J of the cross section is given by Eq. (10.11), where $c_1 = \frac{1}{2}(40 \text{ mm}) = 0.02$ m and $c_2 = \frac{1}{2}(60 \text{ mm}) = 0.03$ m, we write

$$J = \tfrac{1}{2}\pi\,(c_2^4 - c_1^4) = \tfrac{1}{2}\pi(0.03^4 - 0.02^4) = 1.021 \times 10^{-6} \text{ m}^4$$

Substituting for J and $\tau_{\max}$ into (10.12), and letting $c = c_2 = 0.03$ m, we have

$$T = \frac{J\tau_{\max}}{c} = \frac{(1.021 \times 10^{-6} \text{ m}^4)(120 \times 10^6 \text{ Pa})}{0.03 \text{ m}} = 4.08 \text{ kN} \cdot \text{m}$$

(b) Minimum Shearing Stress. The minimum value of the shearing stress occurs on the inner surface of the shaft. It is obtained from Eq. (10.7), which expresses that $\tau_{\min}$ and $\tau_{\max}$ are respectively proportional to c_1 and c_2:

$$\tau_{\min} = \frac{c_1}{c_2}\tau_{\max} = \frac{0.02 \text{ m}}{0.03 \text{ m}}(120 \text{ MPa}) = 80 \text{ MPa} \quad ■$$

The torsion formulas (10.9) and (10.10) were derived for a shaft of uniform circular cross section subjected to torques at its ends. However, they can also be used for a shaft of variable cross section or for a shaft subjected to torques at locations other than its ends (Fig. 10.16*a*). The distribution of shearing stresses in a given cross section S of the shaft is obtained from Eq. (10.9), where J denotes the polar moment of inertia of that section, and where T represents the *internal torque* in that section. The value of T is obtained by drawing the free-body diagram of the portion of shaft located on one side of the section (Fig. 10.16*b*) and writing that the sum of the torques applied to that portion, including the internal torque **T**, is zero (see Sample Prob. 10.1).

Up to this point, our analysis of stresses in a shaft has been limited to shearing stresses. This is due to the fact that the element we had selected was oriented in such a way that its faces were either parallel or perpendicular to the axis of the shaft (Fig. 10.5). We know from earlier discussions (Secs. 8.8 and 8.9) that normal stresses, shearing stresses, or a combination of both may be found under the same loading condition, depending upon the orientation of the element which has been chosen. Consider the two elements a and b

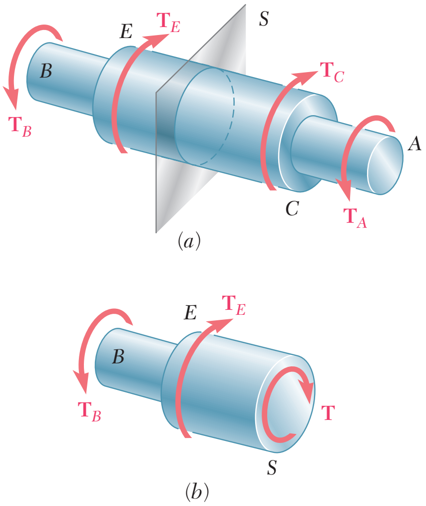


Fig. 10.16

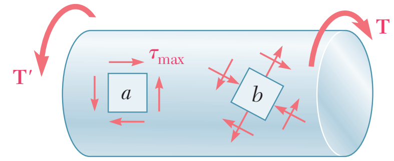


Fig. 10.17

located on the surface of a circular shaft subjected to torsion (Fig. 10.17). Since the faces of element a are respectively parallel and perpendicular to the axis of the shaft, the only stresses on the element will be the shearing stresses defined by formula (10.9), namely $\tau_{max} = Tc/J$. On the other hand, the faces of element b, which form arbitrary angles with the axis of the shaft, will be subjected to a combination of normal and shearing stresses.

Let us consider the particular case of an element c (not shown) at 45° to the axis of the shaft. In order to determine the stresses on the faces of this element, we consider the two triangular elements shown in Fig. 10.18 and draw their free-body diagrams. In the case of the element of Fig. 10.18a, we know that the stresses exerted on the faces BC and BD are the shearing stresses $\tau_{max} = Tc/J$. The magnitude of the corresponding shearing forces is thus $\tau_{max}A_0$, where A_0 denotes the area of the face. Observing that the components along DC of the two shearing forces are equal and opposite, we conclude that the force **F** exerted on DC must be perpendicular to that face. It is a tensile force, and its magnitude is

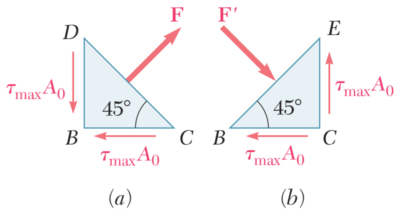


Fig. 10.18

$$F = 2(\tau_{max}A_0)\cos 45° = \tau_{max}A_0\sqrt{2} \qquad (10.13)$$

The corresponding stress is obtained by dividing the force F by the area A of face DC. Observing that $A = A_0\sqrt{2}$, we write

$$\sigma = \frac{F}{A} = \frac{\tau_{max}A_0\sqrt{2}}{A_0\sqrt{2}} = \tau_{max} \qquad (10.14)$$

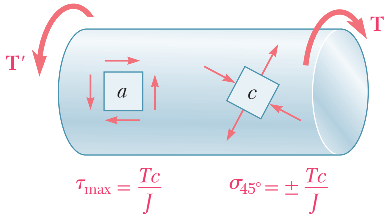


Fig. 10.19

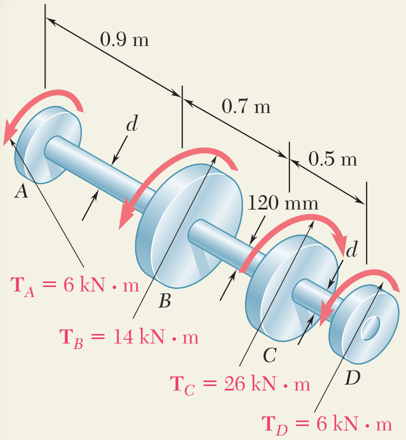
A similar analysis of the element of Fig. 10.18b shows that the stress on the face BE is $\sigma = -\tau_{max}$. We conclude that the stresses exerted on the faces of an element c at 45° to the axis of the shaft (Fig. 10.19) are normal stresses equal to $\pm\tau_{max}$. Thus, while the element a in Fig. 10.19 is in pure shear, the element c in the same figure is subjected to a tensile stress on two of its faces and to a compressive stress on the other two. We also note that all the stresses involved have the same magnitude, Tc/J.†

As you learned in Sec. 9.3, ductile materials generally fail in shear. Therefore, when subjected to torsion, a specimen J made of a ductile material breaks along a plane perpendicular to its longitudinal axis (Photo 10.2a). On the other hand, brittle materials are weaker in tension than in shear. Thus, when subjected to torsion, a specimen made of a brittle material tends to break along surfaces which are perpendicular to the direction in which tension is maximum, i.e., along surfaces forming a 45° angle with the longitudinal axis of the specimen (Photo 10.2b).

Photo 10.2

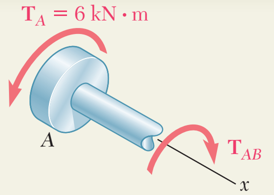
†Stresses on elements of arbitrary orientation, such as element b of Fig. 10.17, will be discussed in Chap. 14.

SAMPLE PROBLEM 10.1



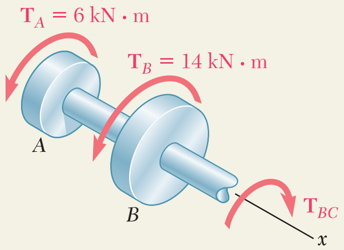
Shaft BC is hollow with inner and outer diameters of 90 mm and 120 mm, respectively. Shafts AB and CD are solid and of diameter d. For the loading shown, determine (*a*) the maximum and minimum shearing stress in shaft BC, (*b*) the required diameter d of shafts AB and CD if the allowable shearing stress in these shafts is 65 MPa.

SOLUTION



Equations of Statics. Denoting by $\mathbf{T}_{AB}$ the torque in shaft AB, we pass a section through shaft AB and, for the free body shown, we write

$$\Sigma M_x = 0: \qquad (6\text{ kN}\cdot\text{m}) - T_{AB} = 0 \qquad T_{AB} = 6\text{ kN}\cdot\text{m}$$

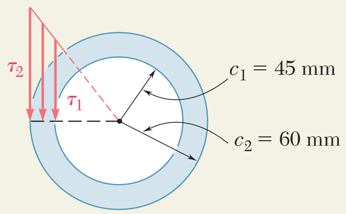


We now pass a section through shaft BC and, for the free body shown, we have

$$\Sigma M_x = 0: \quad (6\text{ kN}\cdot\text{m}) + (14\text{ kN}\cdot\text{m}) - T_{BC} = 0 \qquad T_{BC} = 20\text{ kN}\cdot\text{m}$$

a. Shaft *BC*. For this hollow shaft we have

$$J = \frac{\pi}{2}(c_2^4 - c_1^4) = \frac{\pi}{2}[(0.060)^4 - (0.045)^4] = 13.92 \times 10^{-6}\text{ m}^4$$

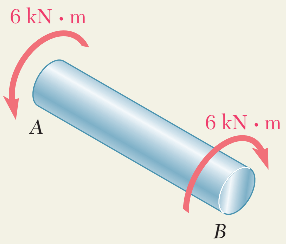


Maximum Shearing Stress. On the outer surface, we have

$$\tau_{\max} = \tau_2 = \frac{T_{BC}c_2}{J} = \frac{(20\text{ kN}\cdot\text{m})(0.060\text{ m})}{13.92\times 10^{-6}\text{ m}^4} \qquad \tau_{\max} = 86.2\text{ MPa} \blacktriangleleft$$

Minimum Shearing Stress. We write that the stresses are proportional to the distance from the axis of the shaft.

$$\frac{\tau_{\min}}{\tau_{\max}} = \frac{c_1}{c_2} \qquad \frac{\tau_{\min}}{86.2\text{ MPa}} = \frac{45\text{ mm}}{60\text{ mm}} \qquad \tau_{\min} = 64.7\text{ MPa} \blacktriangleleft$$



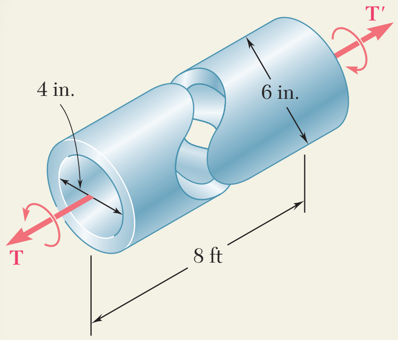
b. Shafts *AB* and *CD*. We note that in both of these shafts the magnitude of the torque is $T = 6$ kN · m and $\tau_{\text{all}} = 65$ MPa. Denoting by c the radius of the shafts, we write

$$\tau = \frac{Tc}{J} \qquad 65\text{ MPa} = \frac{(6\text{ kN}\cdot\text{m})c}{\frac{\pi}{2}c^4}$$

$$c^3 = 58.8\times 10^{-6}\text{ m}^3 \qquad c = 38.9\times 10^{-3}\text{ m}$$

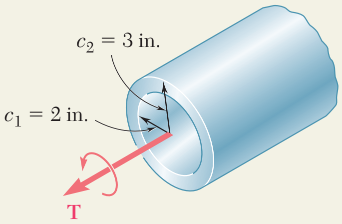
$$d = 2c = 2(38.9\text{ mm}) \qquad d = 77.8\text{ mm} \blacktriangleleft$$

SAMPLE PROBLEM 10.2



The preliminary design of a large shaft connecting a motor to a generator calls for the use of a hollow shaft with inner and outer diameters of 4 in. and 6 in., respectively. Knowing that the allowable shearing stress is 12 ksi, determine the maximum torque that can be transmitted (*a*) by the shaft as designed, (*b*) by a solid shaft of the same weight, (*c*) by a hollow shaft of the same weight and of 8-in. outer diameter.

SOLUTION



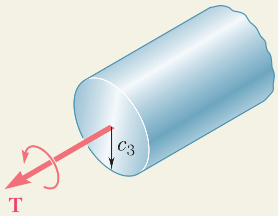
a. Hollow Shaft as Designed. For the hollow shaft we have

$$J = \frac{\pi}{2}(c_2^4 - c_1^4) = \frac{\pi}{2}[(3 \text{ in.})^4 - (2 \text{ in.})^4] = 102.1 \text{ in}^4$$

Using Eq. (10.9), we write

$$\tau_{max} = \frac{Tc_2}{J} \qquad 12 \text{ ksi} = \frac{T(3 \text{ in.})}{102.1 \text{ in}^4} \qquad T = 408 \text{ kip} \cdot \text{in.} \blacktriangleleft$$

b. Solid Shaft of Equal Weight. For the shaft as designed and this solid shaft to have the same weight and length, their cross-sectional areas must be equal.

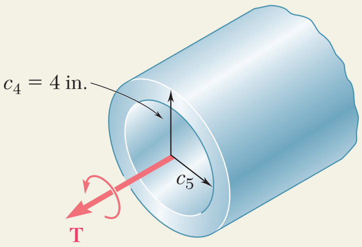


$$A_{(a)} = A_{(b)}$$
$$\pi[(3 \text{ in.})^2 - (2 \text{ in.})^2] = \pi c_3^2 \qquad c_3 = 2.24 \text{ in.}$$

Since $\tau_{all} = 12$ ksi, we write

$$\tau_{max} = \frac{Tc_3}{J} \qquad 12 \text{ ksi} = \frac{T(2.24 \text{ in.})}{\frac{\pi}{2}(2.24 \text{ in.})^4} \qquad T = 211 \text{ kip} \cdot \text{in.} \blacktriangleleft$$

c. Hollow Shaft of 8-in. Diameter. For equal weight, the cross-sectional areas again must be equal. We determine the inside diameter of the shaft by writing



$$A_{(a)} = A_{(c)}$$
$$\pi[(3 \text{ in.})^2 - (2 \text{ in.})^2] = \pi[(4 \text{ in.})^2 - c_5^2] \qquad c_5 = 3.317 \text{ in.}$$

For $c_5 = 3.317$ in. and $c_4 = 4$ in.,

$$J = \frac{\pi}{2}[(4 \text{ in.})^4 - (3.317 \text{ in.})^4] = 212 \text{ in}^4$$

With $\tau_{all} = 12$ ksi and $c_4 = 4$ in.,

$$\tau_{max} = \frac{Tc_4}{J} \qquad 12 \text{ ksi} = \frac{T(4 \text{ in.})}{212 \text{ in}^4} \qquad T = 636 \text{ kip} \cdot \text{in.} \blacktriangleleft$$

PROBLEMS

10.1 Determine the torque **T** that causes a maximum shearing stress of 70 MPa in the steel cylindrical shaft shown.

10.2 Determine the maximum shearing stress caused by a torque of magnitude $T = 800\ \text{N} \cdot \text{m}$.

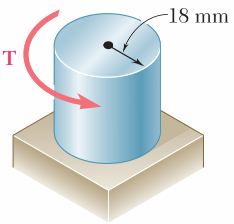


Fig. P10.1 and P10.2

10.3 (*a*) For the hollow shaft and loading shown, determine the maximum shearing stress. (*b*) Determine the diameter of a solid shaft for which the maximum shearing stress in the loading shown is the same as in part *a*.

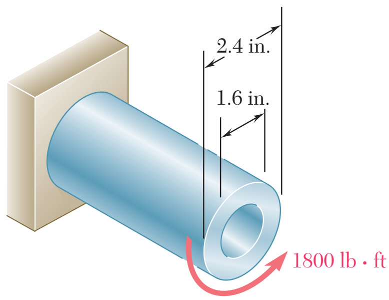


Fig. P10.3

10.4 (*a*) Determine the torque that can be applied to a solid shaft of 3.6-in. outer diameter without exceeding an allowable shearing stress of 10 ksi. (*b*) Solve part *a*, assuming that the solid shaft is replaced by a hollow shaft of the same mass and of 3.6-in. inner diameter.

10.5 (*a*) For the 3-in.-diameter solid cylinder and loading shown, determine the maximum shearing stress. (*b*) Determine the inner diameter of the hollow cylinder, of 4-in. outer diameter, for which the maximum stress is the same as in part *a*.

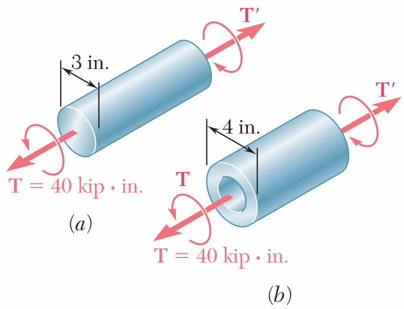


Fig. *P10.5*

10.6 (*a*) Determine the torque that can be applied to a solid shaft of 0.75-in. diameter without exceeding an allowable shearing stress of 10 ksi. (*b*) Solve part *a* assuming that the solid shaft has been replaced by a hollow shaft of the same cross-sectional area and with an inner diameter equal to half of its outer diameter.

10.7 The torques shown are exerted on pulleys *A*, *B*, and *C*. Knowing that both shafts are solid, determine the maximum shearing stress in (*a*) shaft *AB*, (*b*) shaft *BC*.

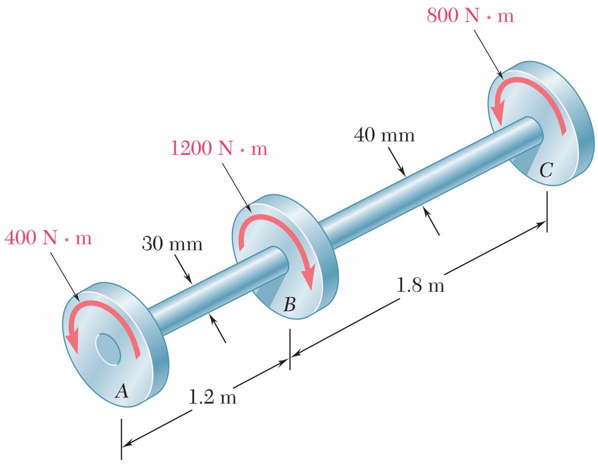


Fig. P10.7 and *P10.8*

10.8 The shafts of the pulley assembly shown are to be redesigned. Knowing that the allowable shearing stress in each shaft is 60 MPa, determine the smallest allowable diameter of (*a*) shaft *AB*, (*b*) shaft *BC*.

10.9 Knowing that each of the shafts AB, BC, and CD consist of solid circular rods, determine (*a*) the shaft in which the maximum shearing stress occurs, (*b*) the magnitude of that stress.

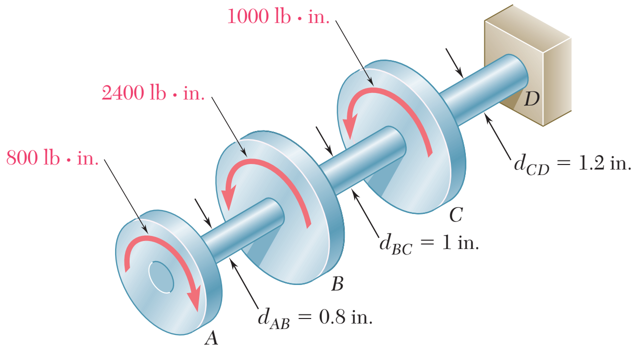


Fig. P10.9 and P10.10

10.10 Knowing that a 0.40-in.-diameter hole has been drilled through each of the shafts AB, BC, and CD, determine (*a*) the shaft in which the maximum shearing stress occurs, (*b*) the magnitude of that stress.

10.11 Under normal operating conditions, the electric motor exerts a torque of 2.4 kN · m on shaft AB. Knowing that each shaft is solid, determine the maximum shearing stress (*a*) in shaft AB, (*b*) in shaft BC, (*c*) in shaft CD.

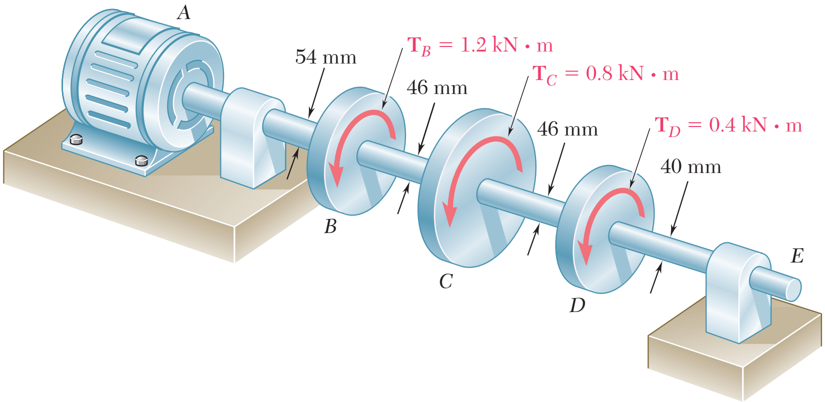


Fig. *P10.11*

10.12 In order to reduce the total mass of the assembly of Prob. 10.11, a new design is being considered in which the diameter of shaft BC will be smaller. Determine the smallest diameter of shaft BC for which the maximum value of the shearing stress in the assembly will not be increased.

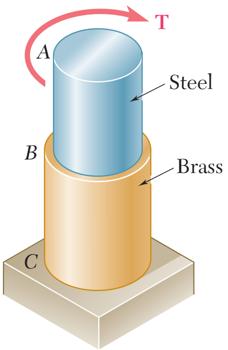


Fig. P10.13 and *P10.14*

10.13 The allowable shearing stress is 15 ksi in the 1.5-in.-diameter steel rod AB and 8 ksi in the 1.8-in.-diameter rod BC. Neglecting the effect of stress concentrations, determine the largest torque that can be applied at A.

10.14 The allowable shearing stress is 15 ksi in the steel rod AB and 8 ksi in the brass rod BC. Knowing that a torque of magnitude T = 10 kip · in. is applied at A and neglecting the effect of stress concentrations, determine the required diameter of (*a*) rod AB, (*b*) rod BC.

10.15 The solid rod AB has a diameter $d_{AB} = 60$ mm and is made of a steel for which the allowable shearing stress is 85 MPa. The pipe CD, which has an outer diameter of 90 mm and a wall thickness of 6 mm, is made of an aluminum for which the allowable shearing stress is 54 MPa. Determine the largest torque **T** that can be applied at A.

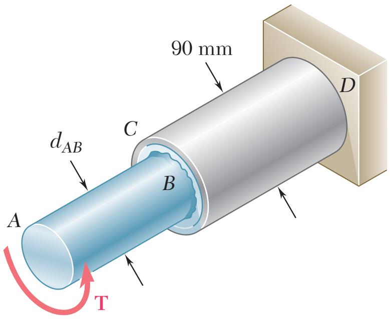


Fig. P10.15

10.16 The allowable shearing stress is 50 MPa in the brass rod AB and 25 MPa in the aluminum rod BC. Knowing that a torque of magnitude $T = 1250$ N · m is applied at A, determine the required diameter of (*a*) rod AB, (*b*) rod BC.

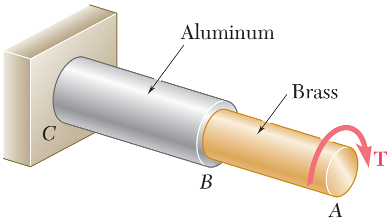


Fig. P10.16

10.17 The solid shaft shown is formed of a brass for which the allowable shearing stress is 55 MPa. Neglecting the effect of stress concentrations, determine the smallest diameters d_{AB} and d_{BC} for which the allowable shearing stress is not exceeded.

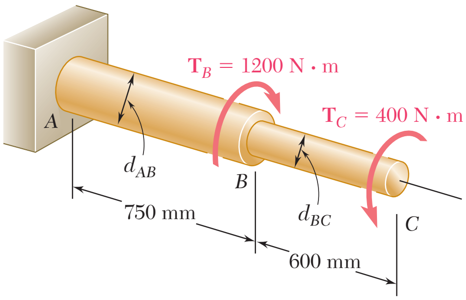


Fig. P10.17 and P10.18

10.18 Solve Prob. 10.17 assuming that the direction of $\mathbf{T}_C$ is reversed.

10.19 and 10.20 Under normal operating conditions a motor exerts a torque of magnitude $T_F = 1200$ lb · in. at F. Knowing that the allowable shearing stress is 10.5 ksi in each shaft, determine for the given data the required diameter of (*a*) shaft *CDE*, (*b*) shaft *FGH*.

10.19 $r_D = 8$ in., $r_G = 3$ in.

10.20 $r_D = 3$ in., $r_G = 8$ in.

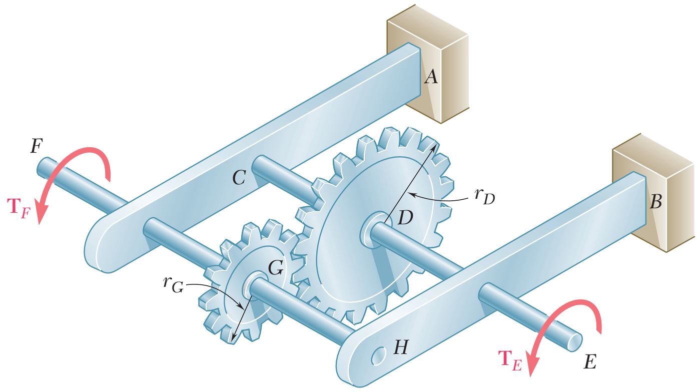


Fig. *P10.19* and P10.20

10.21 A torque of magnitude $T = 1000$ N · m is applied at D as shown. Knowing that the diameter of shaft *AB* is 56 mm and that the diameter of shaft *CD* is 42 mm, determine the maximum shearing stress in (*a*) shaft *AB*, (*b*) shaft *CD*.

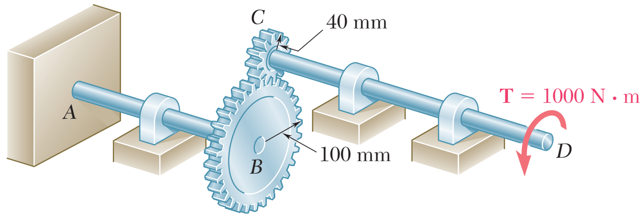


Fig. P10.21 and *P10.22*

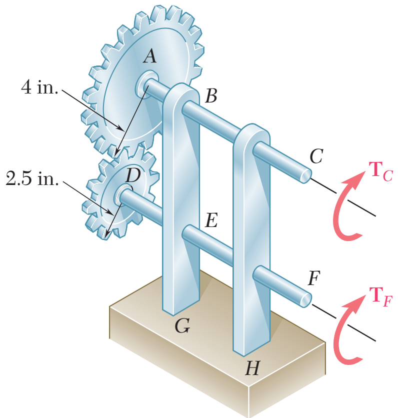


Fig. P10.23 and P10.24

10.22 A torque of magnitude $T = 1000$ N · m is applied at D as shown. Knowing that the allowable shearing stress is 60 MPa in each shaft, determine the required diameter of (*a*) shaft *AB*, (*b*) shaft *CD*.

10.23 Two solid shafts are connected by gears as shown and are made of a steel for which the allowable shearing stress is 8500 psi. Knowing that a torque of magnitude $T_C = 5$ kip · in. is applied at C and that the assembly is in equilibrium, determine the required diameter of (*a*) shaft *BC*, (*b*) shaft *EF*.

10.24 Two solid shafts are connected by gears as shown and are made of a steel for which the allowable shearing stress is 7000 psi. Knowing that the diameters of the two shafts are, respectively, $d_{BC} = 1.6$ in. and $d_{EF} = 1.25$ in., determine the largest torque $\mathbf{T}_C$ that can be applied at C.

10.5 ANGLE OF TWIST

In this section, a relation will be derived between the angle of twist ϕ of a circular shaft and the torque **T** exerted on the shaft. The entire shaft will be assumed to remain elastic. Considering first the case of a shaft of length L and of uniform cross section of radius c subjected to a torque **T** at its free end (Fig. 10.20), we recall from Sec. 10.3 that the angle of twist ϕ and the maximum shearing strain $\gamma_{\max}$ are related as follows:

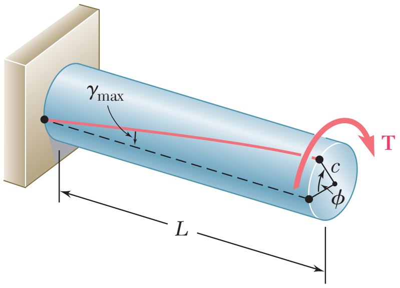


Fig. 10.20

$$\gamma_{\max} = \frac{c\phi}{L} \tag{10.3}$$

But, in the elastic range, the yield stress is not exceeded anywhere in the shaft, Hooke's law applies, and we have $\gamma_{\max} = \tau_{\max}/G$ or, recalling Eq. (10.9),

$$\gamma_{\max} = \frac{\tau_{\max}}{G} = \frac{Tc}{JG} \tag{10.15}$$

Equating the right-hand members of Eqs. (10.3) and (10.15), and solving for ϕ, we write

$$\boxed{\phi = \frac{TL}{JG}} \tag{10.16}$$

where ϕ is expressed in radians. The relation obtained shows that, within the elastic range, *the angle of twist ϕ is proportional to the torque T applied to the shaft*. This is in accordance with the experimental evidence cited at the beginning of Sec. 10.3.

Equation (10.16) provides us with a convenient method for determining the modulus of rigidity of a given material. A specimen of the material, in the form of a cylindrical rod of known diameter and length, is placed in a *torsion testing machine* (Photo 10.3). Torques

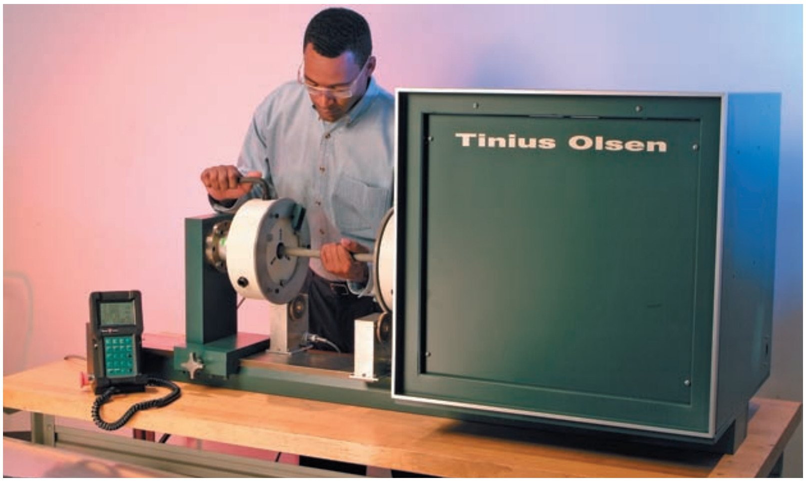


Photo 10.3 Torsion testing machine.

of increasing magnitude T are applied to the specimen, and the corresponding values of the angle of twist ϕ in a length L of the specimen are recorded. As long as the yield stress of the material is not exceeded, the points obtained by plotting ϕ against T will fall on a straight line. The slope of this line represents the quantity JG/L, from which the modulus of rigidity G may be computed.

EXAMPLE 10.2 What torque should be applied to the end of the shaft of Example 10.1 to produce a twist of 2°? Use the value $G = 77$ GPa for the modulus of rigidity of steel.

Solving Eq. (10.16) for T, we write

$$T = \frac{JG}{L}\phi$$

Substituting the given values

$$G = 77 \times 10^9 \text{ Pa} \qquad L = 1.5 \text{ m}$$
$$\phi = 2°\left(\frac{2\pi \text{ rad}}{360°}\right) = 34.9 \times 10^{-3} \text{ rad}$$

and recalling from Example 10.1 that, for the given cross section,

$$J = 1.021 \times 10^{-6} \text{ m}^4$$

we have

$$T = \frac{JG}{L}\phi = \frac{(1.021 \times 10^{-6} \text{ m}^4)(77 \times 10^9 \text{ Pa})}{1.5 \text{ m}}(34.9 \times 10^{-3} \text{ rad})$$
$$T = 1.829 \times 10^3 \text{ N} \cdot \text{m} = 1.829 \text{ kN} \cdot \text{m} \quad ■$$

EXAMPLE 10.3 What angle of twist will create a shearing stress of 70 MPa on the inner surface of the hollow steel shaft of Examples 10.1 and 10.2?

The method of attack for solving this problem that first comes to mind is to use Eq. (10.10) to find the torque T corresponding to the given value of τ, and Eq. (10.16) to determine the angle of twist ϕ corresponding to the value of T just found.

A more direct solution, however, may be used. From Hooke's law, we first compute the shearing strain on the inner surface of the shaft:

$$\gamma_{\min} = \frac{\tau_{\min}}{G} = \frac{70 \times 10^6 \text{ Pa}}{77 \times 10^9 \text{ Pa}} = 909 \times 10^{-6}$$

Recalling Eq. (10.2), which was obtained by expressing the length of arc AA′ in Fig. 10.13c in terms of both γ and ϕ, we have

$$\phi = \frac{L\gamma_{\min}}{c_1} = \frac{1500 \text{ mm}}{20 \text{ mm}}(909 \times 10^{-6}) = 68.2 \times 10^{-3} \text{ rad}$$

To obtain the angle of twist in degrees, we write

$$\phi = (68.2 \times 10^{-3} \text{ rad})\left(\frac{360°}{2\pi \text{ rad}}\right) = 3.91° \quad ■$$

Formula (10.16) for the angle of twist can be used only if the shaft is homogeneous (constant G), has a uniform cross section, and is

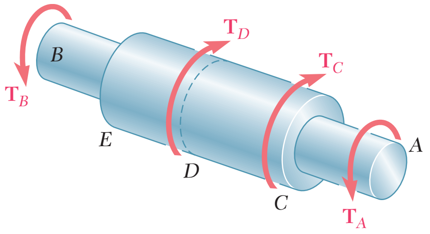


Fig. 10.21

loaded only at its ends. If the shaft is subjected to torques at locations other than its ends, or if it consists of several portions with various cross sections and possibly of different materials, we must divide it into component parts that satisfy individually the required conditions for the application of formula (10.16). In the case of the shaft AB shown in Fig. 10.21, for example, four different parts should be considered: AC, CD, DE, and EB. The total angle of twist of the shaft, i.e., the angle through which end A rotates with respect to end B, is obtained by adding *algebraically* the angles of twist of each component part. Denoting, respectively, by T_i, L_i, J_i, and G_i the internal torque, length, cross-sectional polar moment of inertia, and modulus of rigidity corresponding to part i, the total angle of twist of the shaft is expressed as

$$\phi = \sum_i \frac{T_i L_i}{J_i G_i} \tag{10.17}$$

The internal torque T_i in any given part of the shaft is obtained by passing a section through that part and drawing the free-body diagram of the portion of shaft located on one side of the section. This procedure, which has already been explained in Sec. 10.4 and illustrated in Fig. 10.16, is applied in Sample Prob. 10.3.

In the case of a shaft with a variable circular cross section, as shown in Fig. 10.22, formula (10.16) may be applied to a disk of thickness dx. The angle by which one face of the disk rotates with respect to the other is thus

$$d\phi = \frac{T\,dx}{JG}$$

where J is a function of x which may be determined. Integrating in x from 0 to L, we obtain the total angle of twist of the shaft:

$$\phi = \int_0^L \frac{T\,dx}{JG} \tag{10.18}$$

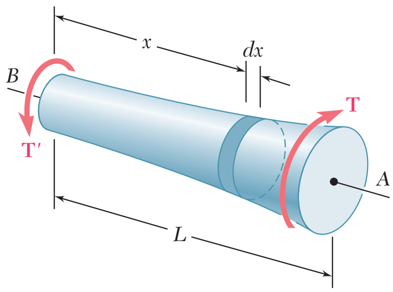


Fig. 10.22

The shaft shown in Fig. 10.20, which was used to derive formula (10.16), and the shaft of Fig. 10.15, which was discussed in Examples 10.2 and 10.3, both had one end attached to a fixed support. In each case, therefore, the angle of twist ϕ of the shaft was equal to the angle of rotation of its free end. When both ends of a shaft rotate,

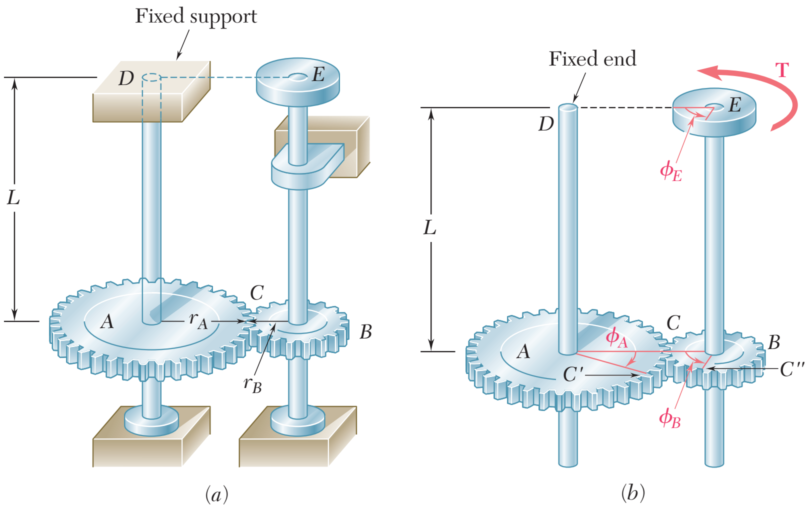


Fig. 10.23

however, the angle of twist of the shaft is equal to the angle through which one end of the shaft rotates *with respect to the other*. Consider, for instance, the assembly shown in Fig. 10.23*a*, consisting of two elastic shafts AD and BE, each of length L, radius c, and modulus of rigidity G, which are attached to gears meshed at C. If a torque **T** is applied at E (Fig. 10.23*b*), both shafts will be twisted. Since the end D of shaft AD is fixed, the angle of twist of AD is measured by the angle of rotation ϕ_A of end A. On the other hand, since both ends of shaft BE rotate, the angle of twist of BE is equal to the difference between the angles of rotation ϕ_B and ϕ_E, i.e., the angle of twist is equal to the angle through which end E rotates with respect to end B. Denoting this relative angle of rotation by $\phi_{E/B}$, we write

$$\phi_{E/B} = \phi_E - \phi_B = \frac{TL}{JG}$$

EXAMPLE 10.4 For the assembly of Fig. 10.23, knowing that $r_A = 2r_B$, determine the angle of rotation of end E of shaft BE when the torque **T** is applied at E.

We first determine the torque $\mathbf{T}_{AD}$ exerted on shaft AD. Observing that equal and opposite forces **F** and **F′** are applied on the two gears at C (Fig. 10.24), and recalling that $r_A = 2r_B$, we conclude that the torque exerted

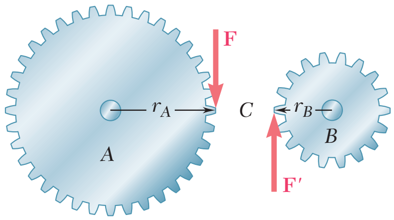


Fig. 10.24

on shaft AD is twice as large as the torque exerted on shaft BE; thus, $T_{AD} = 2T$.

Since the end D of shaft AD is fixed, the angle of rotation ϕ_A of gear A is equal to the angle of twist of the shaft and is obtained by writing

$$\phi_A = \frac{T_{AD}L}{JG} = \frac{2TL}{JG}$$

Observing that the arcs CC' and CC'' in Fig. 10.23*b* must be equal, we write $r_A\,\phi_A = r_B\,\phi_B$ and obtain

$$\phi_B = (r_A/r_B)\phi_A = 2\phi_A$$

We have, therefore,

$$\phi_B = 2\phi_A = \frac{4TL}{JG}$$

Considering now shaft BE, we recall that the angle of twist of the shaft is equal to the angle $\phi_{E/B}$ through which end E rotates with respect to end B. We have

$$\phi_{E/B} = \frac{T_{BE}L}{JG} = \frac{TL}{JG}$$

The angle of rotation of end E is obtained by writing

$$\begin{aligned}\phi_E &= \phi_B + \phi_{E/B} \\ &= \frac{4TL}{JG} + \frac{TL}{JG} = \frac{5TL}{JG} \quad \blacksquare\end{aligned}$$

10.6 STATICALLY INDETERMINATE SHAFTS

You saw in Sec. 10.4 that, in order to determine the stresses in a shaft, it was necessary to first calculate the internal torques in the various parts of the shaft. These torques were obtained from statics by drawing the free-body diagram of the portion of shaft located on one side of a given section and writing that the sum of the torques exerted on that portion was zero.

There are situations, however, where the internal torques cannot be determined from statics alone. In fact, in such cases the external torques themselves, i.e., the torques exerted on the shaft by the supports and connections, cannot be determined from the free-body diagram of the entire shaft. The equilibrium equations must be complemented by relations involving the deformations of the shaft and obtained by considering the geometry of the problem. Because statics is not sufficient to determine the external and internal torques, the shafts are said to be *statically indeterminate*. The following example, as well as Sample Prob. 10.5, will show how to analyze statically indeterminate shafts.

EXAMPLE 10.5 A circular shaft AB consists of a 10-in.-long, $\frac{7}{8}$-in.-diameter steel cylinder, in which a 5-in.-long, $\frac{5}{8}$-in.-diameter cavity has been drilled from end B. The shaft is attached to fixed supports at both ends, and a 90 lb · ft torque is applied at its midsection (Fig. 10.25). Determine the torque exerted on the shaft by each of the supports.

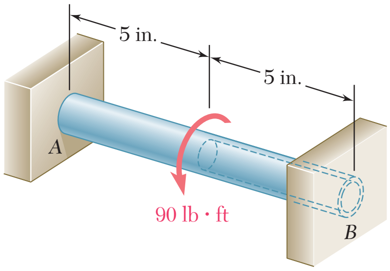


Fig. 10.25

Drawing the free-body diagram of the shaft and denoting by $\mathbf{T}_A$ and $\mathbf{T}_B$ the torques exerted by the supports (Fig. 10.26*a*), we obtain the equilibrium equation

$$T_A + T_B = 90 \text{ lb} \cdot \text{ft}$$

Since this equation is not sufficient to determine the two unknown torques $\mathbf{T}_A$ and $\mathbf{T}_B$, the shaft is statically indeterminate.

However, $\mathbf{T}_A$ and $\mathbf{T}_B$ can be determined if we observe that the total angle of twist of shaft AB must be zero, since both of its ends are restrained. Denoting by ϕ_1 and ϕ_2, respectively, the angles of twist of portions AC and CB, we write

$$\phi = \phi_1 + \phi_2 = 0$$

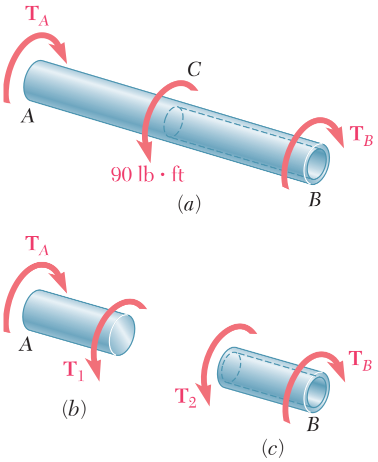


Fig. 10.26

From the free-body diagram of a small portion of shaft including end A (Fig. 10.26*b*), we note that the internal torque T_1 in AC is equal to T_A; from the free-body diagram of a small portion of shaft including end B (Fig. 10.26*c*), we note that the internal torque T_2 in CB is equal to T_B. Recalling Eq. (10.16) and observing that portions AC and CB of the shaft are twisted in opposite senses, we write

$$\phi = \phi_1 + \phi_2 = \frac{T_A L_1}{J_1 G} - \frac{T_B L_2}{J_2 G} = 0$$

Solving for T_B, we have

$$T_B = \frac{L_1 J_2}{L_2 J_1} T_A$$

Substituting the numerical data

$$L_1 = L_2 = 5 \text{ in.}$$
$$J_1 = \tfrac{1}{2}\pi(\tfrac{7}{16} \text{ in.})^4 = 57.6 \times 10^{-3} \text{ in}^4$$
$$J_2 = \tfrac{1}{2}\pi[(\tfrac{7}{16} \text{ in.})^4 - (\tfrac{5}{16} \text{ in.})^4] = 42.6 \times 10^{-3} \text{ in}^4$$

we obtain

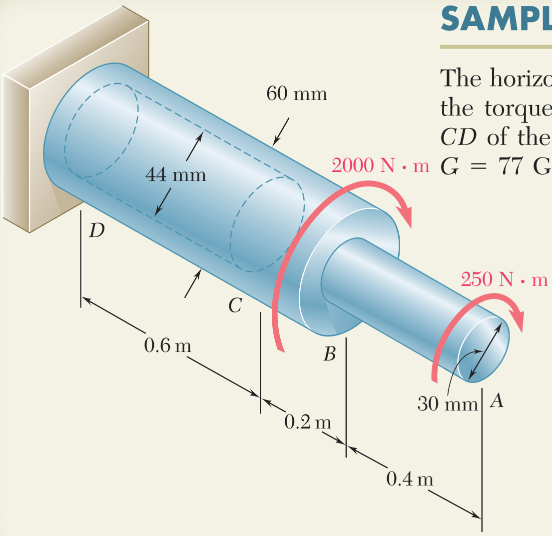
$$T_B = 0.740\ T_A$$

Substituting this expression into the original equilibrium equation, we write

$$1.740\ T_A = 90 \text{ lb} \cdot \text{ft}$$

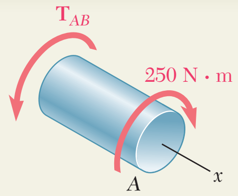
$$T_A = 51.7 \text{ lb} \cdot \text{ft} \qquad T_B = 38.3 \text{ lb} \cdot \text{ft} \ \blacksquare$$

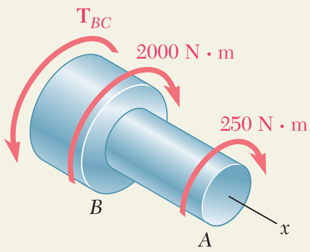
SAMPLE PROBLEM 10.3

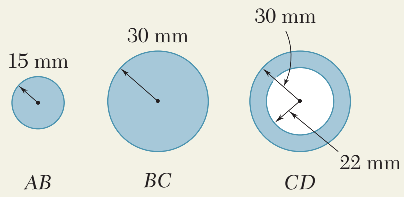


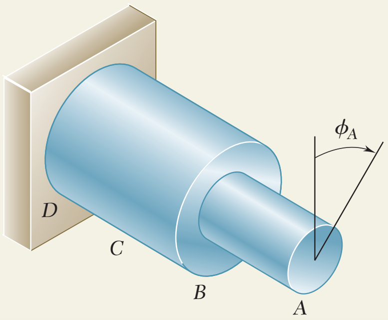
The horizontal shaft AD is attached to a fixed base at D and is subjected to the torques shown. A 44-mm-diameter hole has been drilled into portion CD of the shaft. Knowing that the entire shaft is made of steel for which $G = 77$ GPa, determine the angle of twist at end A.

SOLUTION









Since the shaft consists of three portions AB, BC, and CD, each of uniform cross section and each with a constant internal torque, Eq. (10.17) may be used.

Statics. Passing a section through the shaft between A and B and using the free body shown, we find

$$\Sigma M_x = 0: \qquad (250 \text{ N} \cdot \text{m}) - T_{AB} = 0 \qquad T_{AB} = 250 \text{ N} \cdot \text{m}$$

Passing now a section between B and C, we have

$$\Sigma M_x = 0: (250 \text{ N} \cdot \text{m}) + (2000 \text{ N} \cdot \text{m}) - T_{BC} = 0 \qquad T_{BC} = 2250 \text{ N} \cdot \text{m}$$

Since no torque is applied at C,

$$T_{CD} = T_{BC} = 2250 \text{ N} \cdot \text{m}$$

Polar Moments of Inertia

$$J_{AB} = \frac{\pi}{2}c^4 = \frac{\pi}{2}(0.015 \text{ m})^4 = 0.0795 \times 10^{-6} \text{ m}^4$$

$$J_{BC} = \frac{\pi}{2}c^4 = \frac{\pi}{2}(0.030 \text{ m})^4 = 1.272 \times 10^{-6} \text{ m}^4$$

$$J_{CD} = \frac{\pi}{2}(c_2^4 - c_1^4) = \frac{\pi}{2}[(0.030 \text{ m})^4 - (0.022 \text{ m})^4] = 0.904 \times 10^{-6} \text{ m}^4$$

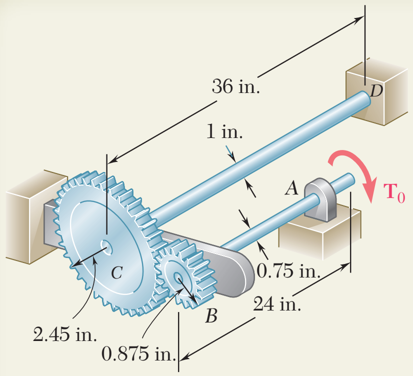
Angle of Twist. Using Eq. (10.17) and recalling that $G = 77$ GPa for the entire shaft, we have

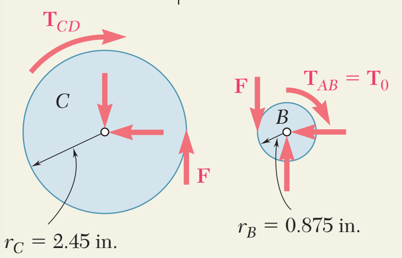
$$\phi_A = \sum_i \frac{T_i L_i}{J_i G} = \frac{1}{G}\left(\frac{T_{AB}L_{AB}}{J_{AB}} + \frac{T_{BC}L_{BC}}{J_{BC}} + \frac{T_{CD}L_{CD}}{J_{CD}}\right)$$

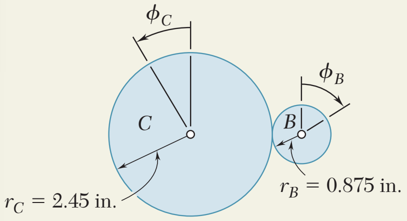
$$\phi_A = \frac{1}{77 \text{ GPa}}\left[\frac{(250 \text{ N} \cdot \text{m})(0.4 \text{ m})}{0.0795 \times 10^{-6} \text{ m}^4} + \frac{(2250)(0.2)}{1.272 \times 10^{-6}} + \frac{(2250)(0.6)}{0.904 \times 10^{-6}}\right]$$

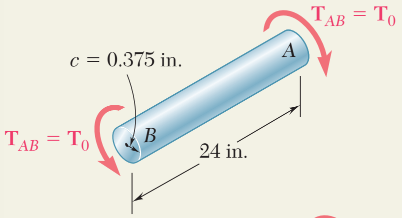
$$= 0.01634 + 0.00459 + 0.01939 = 0.0403 \text{ rad}$$

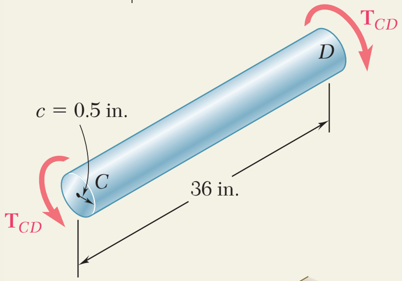
$$\phi_A = (0.0403 \text{ rad})\frac{360°}{2\pi \text{ rad}} \qquad \phi_A = 2.31° \blacktriangleleft$$

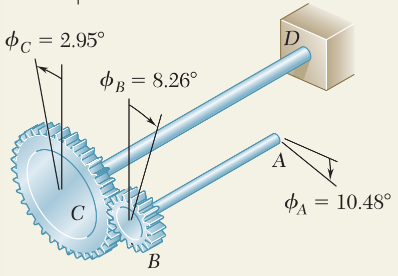












SAMPLE PROBLEM 10.4

Two solid steel shafts are connected by the gears shown. Knowing that for each shaft $G = 11.2 \times 10^6$ psi and that the allowable shearing stress is 8 ksi, determine (*a*) the largest torque $\mathbf{T}_0$ that may be applied to end *A* of shaft *AB*, (*b*) the corresponding angle through which end *A* of shaft *AB* rotates.

SOLUTION

Statics. Denoting by F the magnitude of the tangential force between gear teeth, we have

Gear B. $\Sigma M_B = 0$: $\quad F(0.875 \text{ in.}) - T_0 = 0$

Gear C. $\Sigma M_C = 0$: $\quad F(2.45 \text{ in.}) - T_{CD} = 0$

$$T_{CD} = 2.8T_0 \qquad (1)$$

Kinematics. Noting that the peripheral motions of the gears are equal, we write

$$r_B\phi_B = r_C\phi_C \qquad \phi_B = \phi_C\frac{r_C}{r_B} = \phi_C\frac{2.45 \text{ in.}}{0.875 \text{ in.}} = 2.8\phi_C \qquad (2)$$

a. Torque T_0

Shaft AB. With $T_{AB} = T_0$ and $c = 0.375$ in., together with a maximum permissible shearing stress of 8000 psi, we write

$$\tau = \frac{T_{AB}c}{J} \qquad 8000 \text{ psi} = \frac{T_0(0.375 \text{ in.})}{\frac{1}{2}\pi\,(0.375 \text{ in.})^4} \qquad T_0 = 663 \text{ lb} \cdot \text{in.} \blacktriangleleft$$

Shaft CD. From (1) we have $T_{CD} = 2.8T_0$. With $c = 0.5$ in. and $\tau_{\text{all}} = 8000$ psi, we write

$$\tau = \frac{T_{CD}c}{J} \qquad 8000 \text{ psi} = \frac{2.8T_0(0.5 \text{ in.})}{\frac{1}{2}\pi\,(0.5 \text{ in.})^4} \qquad T_0 = 561 \text{ lb} \cdot \text{in.} \blacktriangleleft$$

Maximum Permissible Torque. We choose the smaller value obtained for T_0

$$T_0 = 561 \text{ lb} \cdot \text{in.} \blacktriangleleft$$

b. Angle of Rotation at End *A*. We first compute the angle of twist for each shaft.

Shaft AB. For $T_{AB} = T_0 = 561$ lb · in., we have

$$\phi_{A/B} = \frac{T_{AB}L}{JG} = \frac{(561 \text{ lb} \cdot \text{in.})(24 \text{ in.})}{\frac{1}{2}\pi(0.375 \text{ in.})^4(11.2 \times 10^6 \text{ psi})} = 0.0387 \text{ rad} = 2.22°$$

Shaft CD. $T_{CD} = 2.8T_0 = 2.8(561 \text{ lb} \cdot \text{in.})$

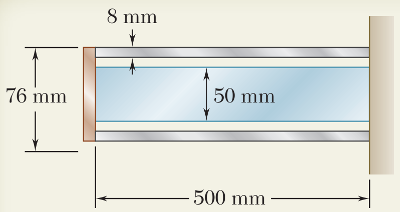
$$\phi_{C/D} = \frac{T_{CD}L}{JG} = \frac{2.8(561 \text{ lb} \cdot \text{in.})(36 \text{ in.})}{\frac{1}{2}\pi(0.5 \text{ in.})^4(11.2 \times 10^6 \text{ psi})} = 0.514 \text{ rad} = 2.95°$$

Since end *D* of shaft *CD* is fixed, we have $\phi_C = \phi_{C/D} = 2.95°$. Using (2), we find the angle of rotation of gear *B* to be

$$\phi_B = 2.8\phi_C = 2.8(2.95°) = 8.26°$$

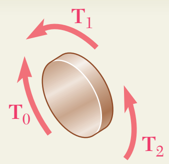
For end *A* of shaft *AB*, we have

$$\phi_A = \phi_B + \phi_{A/B} = 8.26° + 2.22° \qquad \phi_A = 10.48° \blacktriangleleft$$



SAMPLE PROBLEM 10.5

A steel shaft and an aluminum tube are connected to a fixed support and to a rigid disk as shown in the cross section. Knowing that the initial stresses are zero, determine the maximum torque $\mathbf{T}_0$ that can be applied to the disk if the allowable stresses are 120 MPa in the steel shaft and 70 MPa in the aluminum tube. Use G = 77 GPa for steel and G = 27 GPa for aluminum.



SOLUTION

Statics. ***Free Body of Disk.*** Denoting by $\mathbf{T}_1$ the torque exerted by the tube on the disk and by $\mathbf{T}_2$ the torque exerted by the shaft, we find

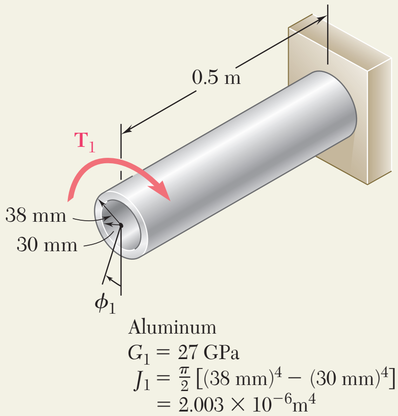
$$T_0 = T_1 + T_2 \tag{1}$$

Deformations. Since both the tube and the shaft are connected to the rigid disk, we have

$$\phi_1 = \phi_2: \qquad \frac{T_1 L_1}{J_1 G_1} = \frac{T_2 L_2}{J_2 G_2}$$

$$\frac{T_1(0.5 \text{ m})}{(2.003 \times 10^{-6} \text{ m}^4)(27 \text{ GPa})} = \frac{T_2(0.5 \text{ m})}{(0.614 \times 10^{-6} \text{ m}^4)(77 \text{ GPa})}$$

$$T_2 = 0.874 T_1 \tag{2}$$



Shearing Stresses. We *assume* that the requirement $\tau_{\text{alum}} \leqslant 70$ MPa is critical. For the aluminum tube, we have

$$T_1 = \frac{\tau_{\text{alum}} J_1}{c_1} = \frac{(70 \text{ MPa})(2.003 \times 10^{-6} \text{ m}^4)}{0.038 \text{ m}} = 3690 \text{ N} \cdot \text{m}$$

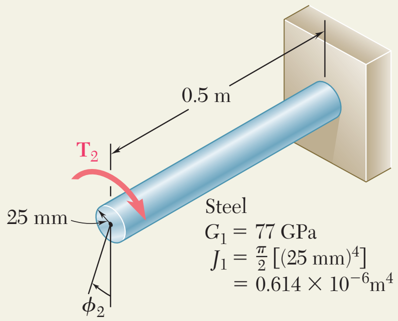
Using Eq. (2), we compute the corresponding value T_2 and then find the maximum shearing stress in the steel shaft.

$$T_2 = 0.874 T_1 = 0.874(3690) = 3225 \text{ N} \cdot \text{m}$$

$$\tau_{\text{steel}} = \frac{T_2 c_2}{J_2} = \frac{(3225 \text{ N} \cdot \text{m})(0.025 \text{ m})}{0.614 \times 10^{-6} \text{ m}^4} = 131.3 \text{ MPa}$$

We note that the allowable steel stress of 120 MPa is exceeded; our assumption was *wrong*. Thus, the maximum torque $\mathbf{T}_0$ will be obtained by making $\tau_{\text{steel}} = 120$ MPa. We first determine the torque $\mathbf{T}_2$.

$$T_2 = \frac{\tau_{\text{steel}} J_2}{c_2} = \frac{(120 \text{ MPa})(0.614 \times 10^{-6} \text{ m}^4)}{0.025 \text{ m}} = 2950 \text{ N} \cdot \text{m}$$



From Eq. (2), we have

$$2950 \text{ N} \cdot \text{m} = 0.874 T_1 \qquad T_1 = 3375 \text{ N} \cdot \text{m}$$

Using Eq. (1), we obtain the maximum permissible torque

$$T_0 = T_1 + T_2 = 3375 \text{ N} \cdot \text{m} + 2950 \text{ N} \cdot \text{m}$$

$$T_0 = 6.325 \text{ kN} \cdot \text{m} \quad ◀$$

PROBLEMS

10.25 For the aluminum shaft shown ($G = 3.9 \times 10^6$ psi), determine (*a*) the torque **T** that causes an angle of twist of 5°, (*b*) the angle of twist caused by the same torque **T** in a solid cylindrical shaft of the same length and cross-sectional area.

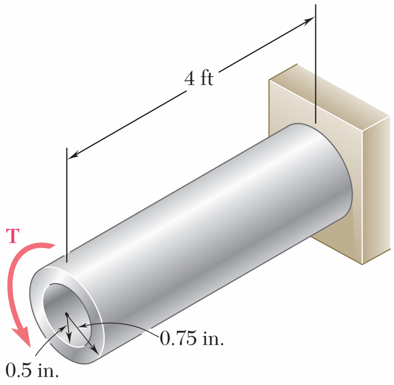


Fig. P10.25

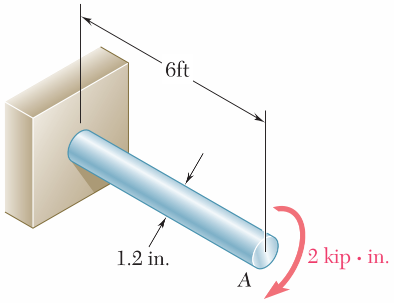


Fig. P10.26

10.26 (*a*) For the solid steel shaft shown ($G = 11.2 \times 10^6$ psi), determine the angle of twist at *A*. (*b*) Solve part (*a*), assuming that the steel shaft is hollow with a 1.2-in. outer diameter and a 0.8-in. inner diameter.

10.27 Determine the largest allowable diameter of a 3-m-long steel rod ($G = 77$ GPa) if the rod is to be twisted through 30° without exceeding a shearing stress of 80 MPa.

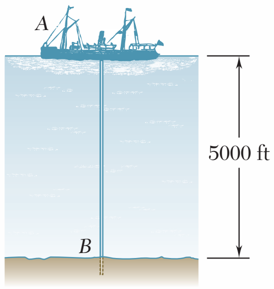


Fig. P10.28

10.28 The ship at *A* has just started to drill for oil on the ocean floor at a depth of 5000 ft. Knowing that the top of the 8-in.-diameter steel drill pipe ($G = 11.2 \times 10^6$ psi) rotates through two complete revolutions before the drill bit at *B* starts to operate, determine the maximum shearing stress caused in the pipe by torsion.

10.29 The torques shown are exerted on pulleys *A* and *B*. Knowing that the shafts are solid and made of steel ($G = 77$ GPa), determine the angle of twist between (*a*) *A* and *B*, (*b*) *A* and *C*.

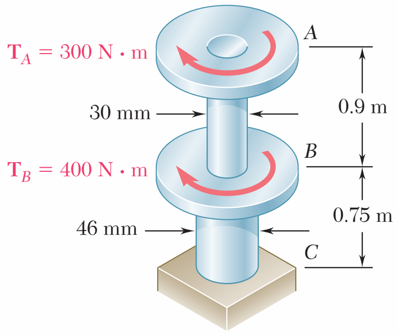


Fig. *P10.29*

10.30 The torques shown are exerted on pulleys B, C, and D. Knowing that the entire shaft is made of aluminum ($G = 27$ GPa), determine the angle of twist between (*a*) C and B, (*b*) D and B.

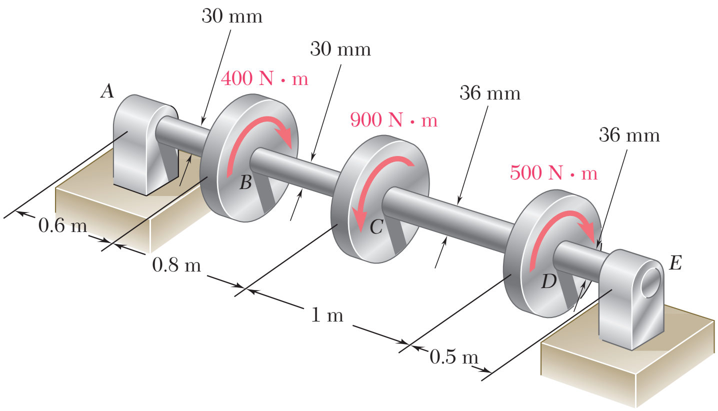


Fig. P10.30

10.31 The aluminum rod BC ($G = 3.9 \times 10^6$ psi) is bonded to the brass rod AB ($G = 5.6 \times 10^6$ psi). Knowing that each rod is solid and has a diameter of 0.5 in., determine the angle of twist (*a*) at B, (*b*) at C.

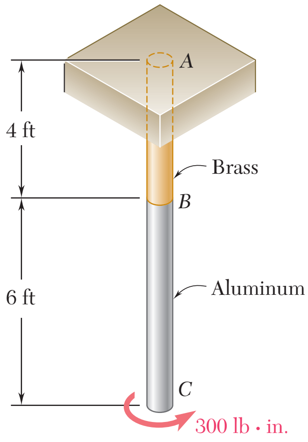


Fig. P10.31

10.32 The solid brass rod AB ($G = 39$ GPa) is bonded to the solid aluminum rod BC ($G = 27$ GPa). Determine the angle of twist (*a*) at B, (*b*) at A.

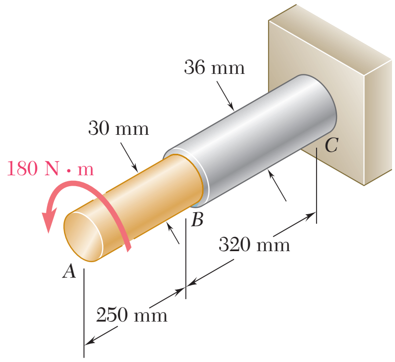


Fig. P10.32

10.33 Two solid steel shafts ($G = 77$ GPa) are connected by the gears shown. Knowing that the radius of gear B is $r_B = 20$ mm, determine the angle through which end A rotates when $T_A = 75$ N · m.

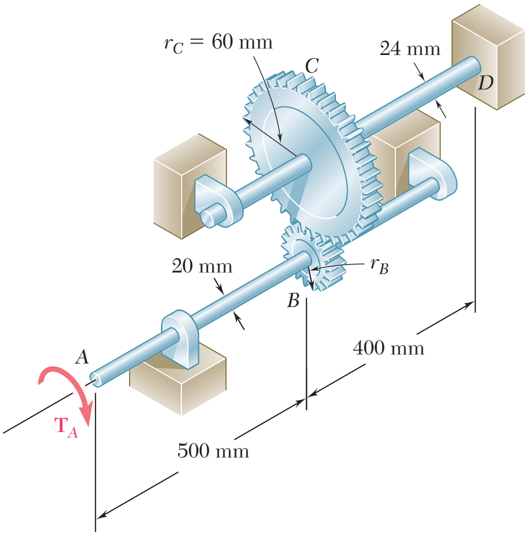


Fig. P10.33

10.34 Solve Prob. 10.33 assuming that a change in design of the assembly resulted in the radius of gear B being increased to 30 mm.

10.35 Two shafts, each of $\frac{3}{4}$-in. diameter, are connected by the gears shown. Knowing that $G = 11.2 \times 10^6$ psi and that the shaft at F is fixed, determine the angle through which end A rotates when a 750 lb · in. torque is applied at A.

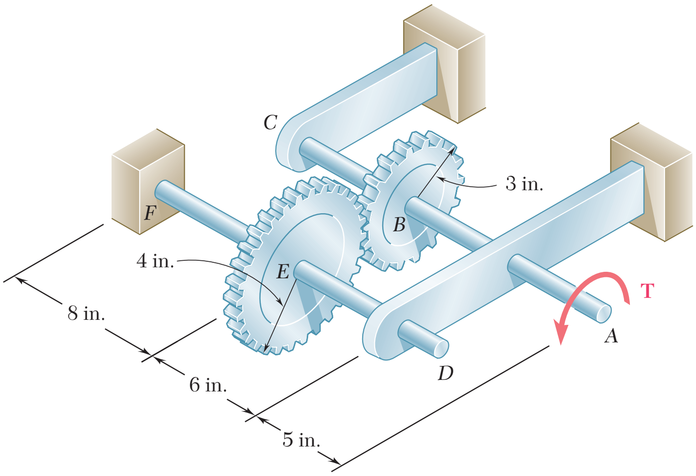


Fig. *P10.35*

10.36 Solve Prob. 10.35 assuming that after a design change the radius of gear B is 4 in. and the radius of gear E is 3 in.

10.37 The design specifications of a 1.2-m-long solid transmission shaft require that the angle of twist of the shaft not exceed 4° when a torque of 750 N · m is applied. Determine the required diameter of the shaft, knowing that the shaft is made of a steel with an allowable shearing stress of 90 MPa and a modulus of rigidity of 77.2 GPa.

10.38 The design specifications of a 2-m-long solid circular transmission shaft require that the angle of twist of the shaft not exceed 3° when a torque of 9 kN · m is applied. Determine the required diameter of the shaft, knowing that the shaft is made of (*a*) a steel with an allowable shearing stress of 90 MPa and a modulus of rigidity of 77 GPa, (*b*) a bronze with an allowable shearing stress of 35 MPa and a modulus of rigidity of 42 GPa.

10.39 The design of the gear-and-shaft system shown requires that steel shafts of the same diameter be used for both *AB* and *CD*. It is further required that $\tau_{max} \leq 9$ ksi and that the angle ϕ_D through which end *D* of shaft *CD* rotates not exceed 2°. Knowing that $G = 11.2 \times 10^6$ psi, determine the required diameter of the shafts.

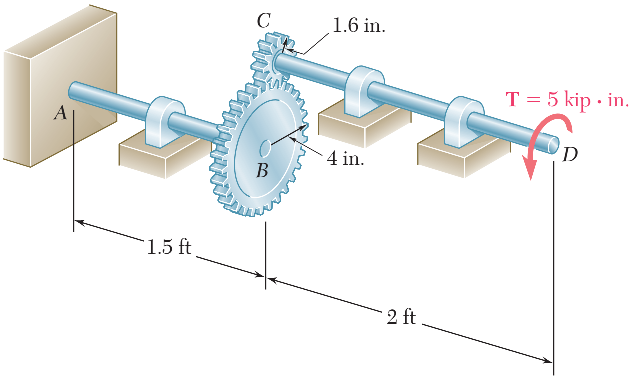


Fig. P10.39 and P10.40

10.40 In the gear-and-shaft system shown, the diameters of the shafts are $d_{AB} = 2$ in. and $d_{CD} = 1.5$ in. Knowing that $G = 11.2 \times 10^6$ psi, determine the angle through which end *D* of shaft *CD* rotates.

10.41 A torque of magnitude $T = 35$ kip · in. is applied at end *A* of the composite shaft shown. Knowing that the modulus of rigidity is 11.2×10^6 psi for the steel and 3.9×10^6 psi for the aluminum, determine (*a*) the maximum shearing stress in the steel core, (*b*) the maximum shearing stress in the aluminum jacket, (*c*) the angle of twist at *A*.

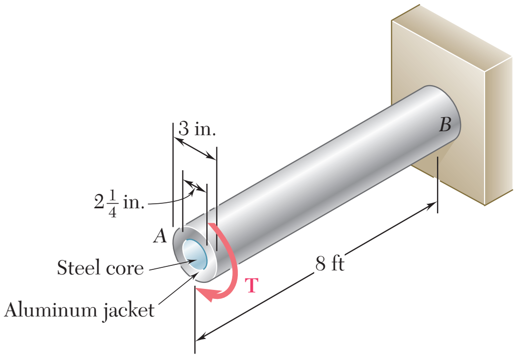


Fig. P10.41 and P10.42

10.42 The composite shaft shown is to be twisted by applying a torque **T** at end *A*. Knowing that the modulus of rigidity is 11.2×10^6 psi for the steel and 3.9×10^6 psi for the aluminum, determine the largest angle through which end *A* can be rotated if the following allowable stresses are not be exceeded: $\tau_{steel} = 8500$ psi and $\tau_{aluminum} = 6500$ psi.

10.43 The composite shaft shown consists of a 0.2-in.-thick brass jacket ($G_{brass} = 5.6 \times 10^6$ psi) bonded to a 1.2-in.-diameter steel core ($G_{steel} = 11.2 \times 10^6$ psi). Knowing that the shaft is subjected to 5 kip · in. torques, determine (*a*) the maximum shearing stress in the steel core, (*b*) the angle of twist of *B* relative to end *A*.

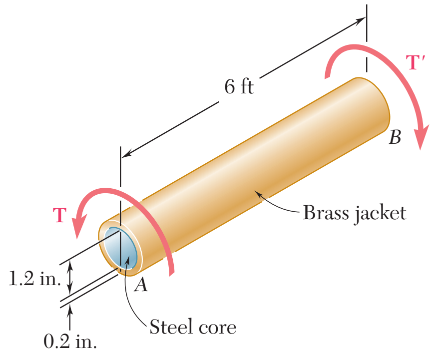


Fig. *P10.43* and P10.44

10.44 The composite shaft shown consists of a 0.2-in.-thick brass jacket ($G_{brass} = 5.6 \times 10^6$ psi) bonded to a 1.2-in.-diameter steel core ($G_{steel} = 11.2 \times 10^6$ psi). Knowing that the shaft is being subjected to the torques shown, determine the largest angle through which it can be twisted if the following allowable stresses are not to be exceeded: $\tau_{steel} = 15$ ksi and $\tau_{brass} = 8$ ksi.

10.45 Two solid steel shafts ($G = 77.2$ GPa) are connected to a coupling disk *B* and to fixed supports at *A* and *C*. For the loading shown, determine (*a*) the reaction at each support, (*b*) the maximum shearing stress in shaft *AB*, (*c*) the maximum shearing stress in shaft *BC*.

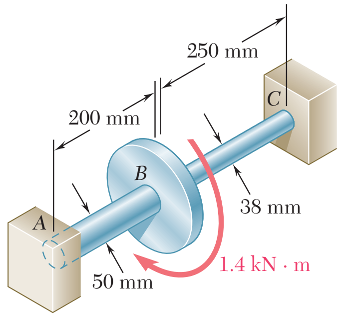


Fig. P10.45

10.46 Solve Prob. 10.45 assuming that shaft *AB* is replaced by a hollow shaft of the same outer diameter and of 25-mm inner diameter.

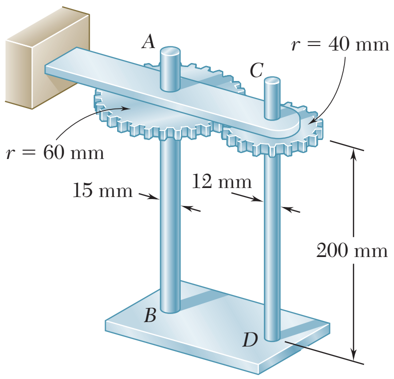


Fig. P10.47

10.47 At a time when rotation is prevented at the lower end of each shaft, a 50-N · m torque is applied to end *A* of shaft *AB*. Knowing that $G = 77$ GPa for both shafts, determine (*a*) the maximum shearing stress in shaft *CD*, (*b*) the angle of rotation at *A*.

10.48 Solve Prob. 10.47 assuming that the 50-N · m torque is applied to end *C* of shaft *CD*.

REVIEW AND SUMMARY

This chapter was devoted to the analysis and design of *shafts* subjected to twisting couples, or *torques*. Our discussion was limited to *circular shafts*.

Deformations in circular shafts

In a preliminary discussion [Sec. 10.2], it was pointed out that the distribution of stresses in the cross section of a circular shaft is *statically indeterminate*. The determination of these stresses, therefore, requires a prior analysis of the *deformations* occurring in the shaft [Sec. 10.3]. Having demonstrated that in a circular shaft subjected to torsion, *every cross section remains plane and undistorted*, we derived the following expression for the *shearing strain* in a small element with sides parallel and perpendicular to the axis of the shaft and at a distance ρ from that axis:

$$\gamma = \frac{\rho\phi}{L} \qquad (10.2)$$

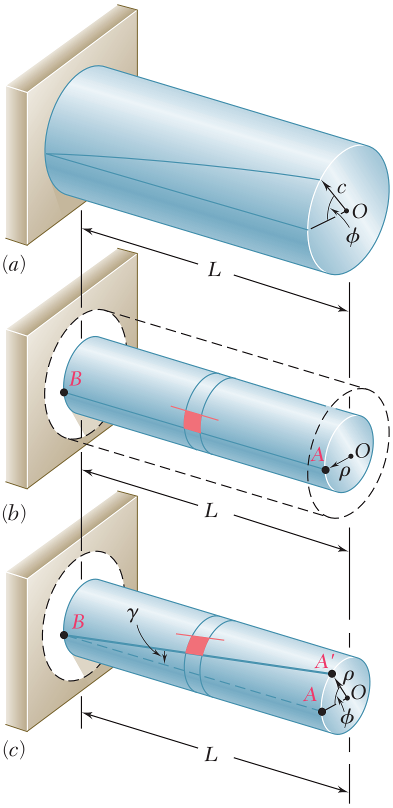


Fig.10.27

where ϕ is the angle of twist for a length L of the shaft (Fig. 10.27). Equation (10.2) shows that the *shearing strain in a circular shaft varies linearly with the distance from the axis of the shaft.* It follows that the strain is maximum at the surface of the shaft, where ρ is equal to the radius c of the shaft. We wrote

$$\gamma_{\max} = \frac{c\phi}{L} \qquad \gamma = \frac{\rho}{c}\gamma_{\max} \qquad (10.3,\ 10.4)$$

Considering *shearing stresses* in a circular shaft within the elastic range [Sec. 10.4] and recalling Hooke's law for shearing stress and strain, $\tau = G\gamma$, we derived the relation

$$\tau = \frac{\rho}{c}\tau_{\max} \qquad (10.6)$$

Shearing stresses in elastic range

which shows that within the elastic range, the *shearing stress* τ *in a circular shaft also varies linearly with the distance from the axis of the shaft.* Equating the sum of the moments of the elementary forces exerted on any section of the shaft to the magnitude T of the torque applied to the shaft, we derived the *elastic torsion formulas*

$$\tau_{\max} = \frac{Tc}{J} \qquad \tau = \frac{T\rho}{J} \qquad (10.9,\ 10.10)$$

where c is the radius of the cross section and J its centroidal polar moment of inertia. We noted that $J = \frac{1}{2}\pi c^4$ for a solid shaft and $J = \frac{1}{2}\pi(c_2^4 - c_1^4)$ for a hollow shaft of inner radius c_1 and outer radius c_2.

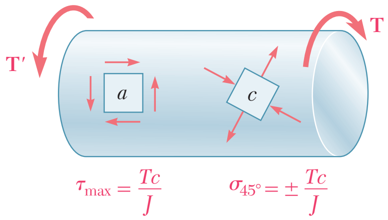


Fig. 10.28

We noted that while the element *a* in Fig. 10.28 is in pure shear, the element *c* in the same figure is subjected to normal stresses of the same magnitude, Tc/J, two of the normal stresses being tensile and two compressive. This explains why in a torsion test ductile materials, which generally fail in shear, will break along a plane perpendicular to the axis of the specimen, while brittle materials, which are weaker in tension than in shear, will break along surfaces forming a 45° angle with that axis.

Angle of twist

In Sec. 10.5, we found that within the elastic range, the angle of twist ϕ of a circular shaft is proportional to the torque T applied to it (Fig. 10.29). Expressing ϕ in *radians,* we wrote

$$\phi = \frac{TL}{JG} \tag{10.16}$$

where L = length of shaft
J = polar moment of inertia of cross section
G = modulus of rigidity of material

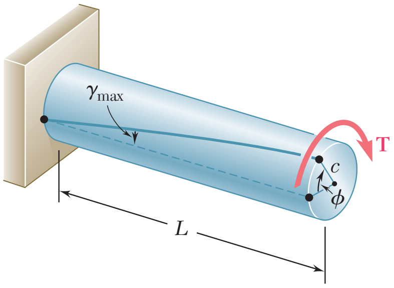


Fig. 10.29

If the shaft is subjected to torques at locations other than its ends or consists of several parts of various cross sections and possibly of different materials, the angle of twist of the shaft must be expressed as the *algebraic sum* of the angles of twist of its component parts [Sample Prob. 10.3]:

$$\phi = \sum_i \frac{T_i L_i}{J_i G_i} \tag{10.17}$$

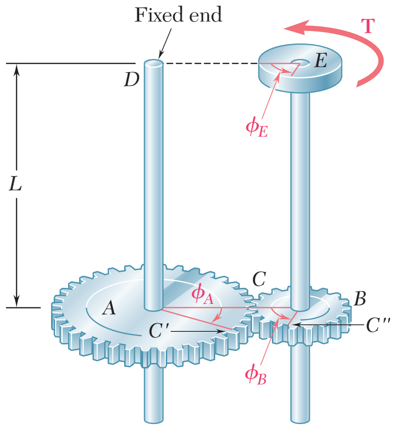


Fig. 10.30

We observed that when both ends of a shaft BE rotate (Fig. 10.30), the angle of twist of the shaft is equal to the *difference* between the angles of rotation ϕ_B and ϕ_E of its ends. We also noted that when two shafts AD and BE are connected by gears A and B, the torques applied, respectively, by gear A on shaft AD and by gear B on shaft BE are *directly proportional* to the radii r_A and r_B of the two gears—since the forces applied on each other by the gear teeth at C are equal and opposite. On the other hand, the angles ϕ_A and ϕ_B through which the two gears rotate are *inversely proportional* to r_A and r_B—since the arcs CC' and CC'' described by the gear teeth are equal [Example 10.4 and Sample Prob. 10.4].

Statically inderterminate shafts

If the reactions at the supports of a shaft or the internal torques cannot be determined from statics alone, the shaft is said to be *statically indeterminate* [Sec. 10.6]. The equilibrium equations obtained from free-body diagrams must then be complemented by relations involving the deformations of the shaft and obtained from the geometry of the problem [Example 10.5 and Sample Prob. 10.5].

REVIEW PROBLEMS

10.49 Knowing that the internal diameter of the hollow shaft shown is $d = 0.9$ in., determine the maximum shearing stress caused by a torque of magnitude $T = 9$ kip · in.

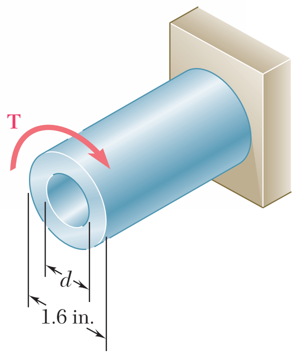


Fig. P10.49 and P10.50

10.50 Knowing that $d = 1.2$ in., determine the torque **T** that causes a maximum shearing stress of 7.5 ksi in the hollow shaft shown.

10.51 The solid spindle AB has a diameter $d_s = 1.5$ in. and is made of a steel with an allowable shearing stress of 12 ksi, while the sleeve CD is made of a brass with an allowable shearing stress of 7 ksi. Determine the largest torque **T** that can be applied at A.

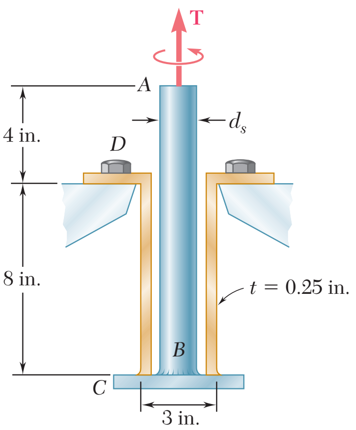


Fig. *P10.51* and P10.52

10.52 The solid spindle AB is made of a steel with an allowable shearing stress of 12 ksi, while sleeve CD is made of a brass with an allowable shearing stress of 7 ksi. Determine (a) the largest torque **T** that can be applied at A if the allowable shearing stress is not to be exceeded in sleeve CD, (b) the corresponding required value of the diameter d_s of spindle AB.

10.53 (*a*) Determine the torque that can be applied to a solid shaft of 90-mm outer diameter without exceeding an allowable shearing stress of 75 MPa. (*b*) Solve part *a* assuming that the solid shaft is replaced by a hollow shaft of the same mass and of 90-mm inner diameter.

10.54 Two solid brass rods *AB* and *CD* are brazed to a brass sleeve *EF*. Determine the ratio d_2/d_1 for which the same maximum shearing stress occurs in the rods and in the sleeve.

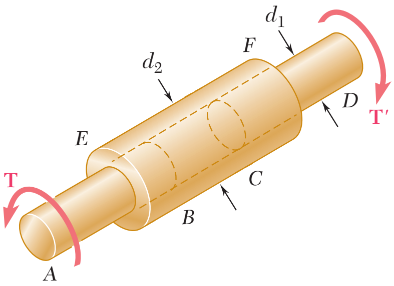


Fig. ***P10.54***

10.55 The aluminum rod *AB* (G = 27 GPa) is bonded to the brass rod *BD* (G = 39 GPa). Knowing that portion *CD* of the brass rod is hollow and has an inner diameter of 40 mm, determine the angle of twist at *A*.

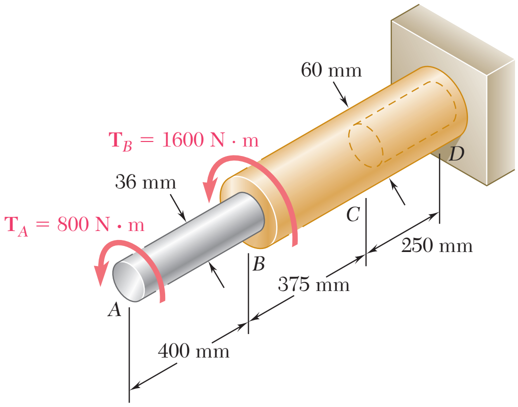


Fig. P10.55

10.56 In the bevel-gear system shown, α = 18.43°. Knowing that the allowable shearing stress is 8 ksi in each shaft and that the system is in equilibrium, determine the largest torque $\mathbf{T}_A$ that can be applied at *A*.

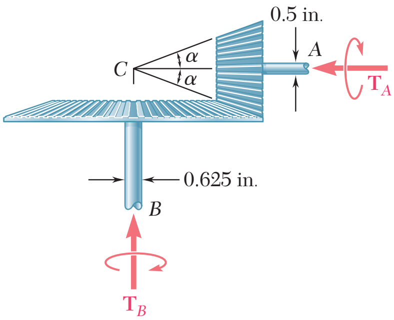


Fig. P10.56

10.57 The solid cylindrical steel rod *BC* of length L = 24 in. is attached to the rigid lever *AB* of length a = 15 in. and to the support at *C*. Design specifications require that the displacement of *A* not exceed 1 in. when a 100-lb force **P** is applied at *A*. Determine the required diameter of the rod. $G = 11.2 \times 10^6$ psi and τ_{all} = 15 ksi.

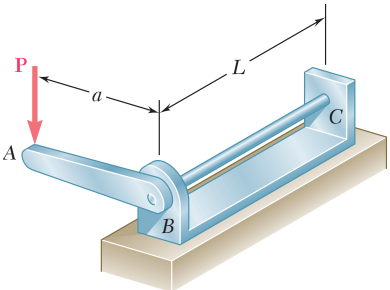


Fig. ***P10.57***

10.58 Two solid steel shafts, each of 30-mm diameter, are connected by the gears shown. Knowing that $G = 77$ GPa, determine the angle through which end A rotates when a torque of magnitude $T = 200$ N · m is applied at A.

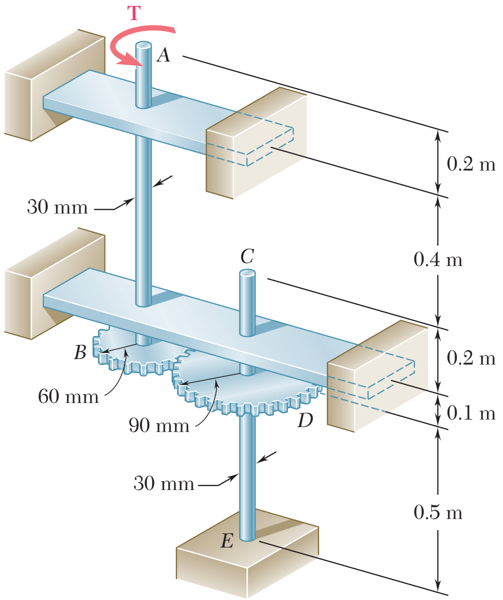


Fig. P10.58

10.59 Two solid steel shafts are fitted with flanges that are then connected by fitted bolts so that there is no relative rotation between the flanges. Knowing that $G = 77$ GPa, determine the maximum shearing stress in each shaft when a torque of magnitude $T = 500$ N · m is applied to flange B.

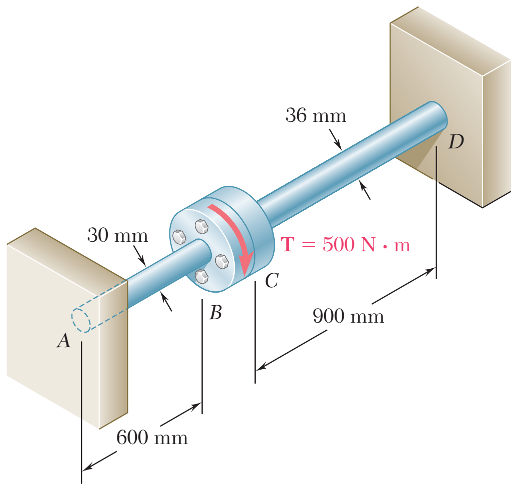


Fig. *P10.59*

10.60 The steel jacket CD has been attached to the 40-mm-diameter steel shaft AE by means of rigid flanges welded to the jacket and to the rod. The outer diameter of the jacket is 80 mm and its wall thickness is 4 mm. If 500 N · m torques are applied as shown, determine the maximum shearing stress in the jacket.

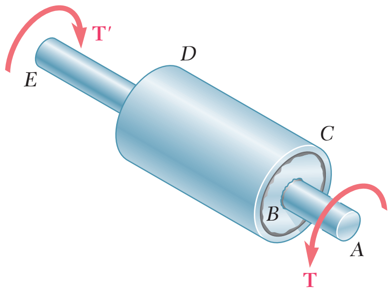


Fig. P10.60

The athlete shown holds the barbell with his hands placed at equal distances from the weights. This results in pure bending in the center portion of the bar. The normal stresses and the curvature resulting from pure bending will be determined in this chapter.

CHAPTER 11

Pure Bending

Chapter 11 Pure Bending

11.1 INTRODUCTION

In the preceding three chapters you studied how to determine the stresses in prismatic members subjected to axial loads or to twisting couples. In this chapter and in the following two you will analyze the stresses and strains in prismatic members subjected to *bending*. Bending is a major concept used in the design of many machine and structural components, such as beams and girders.

This chapter will be devoted to the analysis of prismatic members subjected to equal and opposite couples **M** and **M′** acting in the same longitudinal plane. Such members are said to be in *pure bending*. The members will be assumed to possess a plane of symmetry and the couples **M** and **M′** to be acting in that plane (Fig. 11.1).

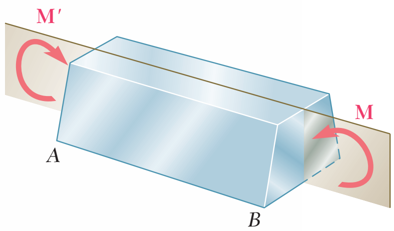


Fig. 11.1

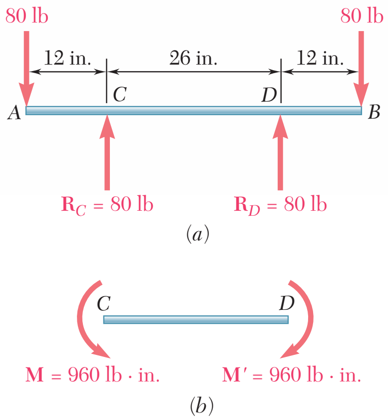


Fig. 11.2

An example of pure bending is provided by the bar of a typical barbell as it is held overhead by a weight lifter. The bar carries equal weights at equal distances from the hands of the weight lifter. Because of the symmetry of the free-body diagram of the bar (Fig. 11.2*a*), the reactions at the hands must be equal and opposite to the weights. Therefore, as far as the middle portion *CD* of the bar is concerned, the weights and the reactions can be replaced by two equal and opposite 960-lb · in. couples (Fig. 11.2*b*), showing that the middle portion of the bar is in pure bending. A similar analysis of the axle of a small sport baggy (Photo 11.1) would show that, between the two points where it is attached to the trailer, the axle is in pure bending.

Photo 11.1 For the sport buggy shown, the center portion of the rear axle is in pure bending.

As interesting as the direct applications of pure bending may be, devoting an entire chapter to its study would not be justified if it were not for the fact that the results obtained will be used in the analysis of other types of loadings as well, such as *eccentric axial loadings* and *transverse loadings.*

Photo 11.2 shows a 12-in. steel bar clamp used to exert 150-lb forces on two pieces of lumber as they are being glued together. Figure 11.3*a* shows the equal and opposite forces exerted by the lumber on the clamp. These forces result in an *eccentric loading* of the straight portion of the clamp. In Fig. 11.3*b* a section CC' has been passed through the clamp and a free-body diagram has been drawn of the upper half of the clamp, from which we conclude that the internal forces in the section are equivalent to a 150-lb axial tensile force **P** and a 750-lb · in. couple **M**. We can thus combine our knowledge of the stresses under a *centric* load and the results of our forthcoming analysis of stresses in pure bending to obtain the distribution of stresses under an *eccentric* load. This will be further discussed in Sec. 11.6.

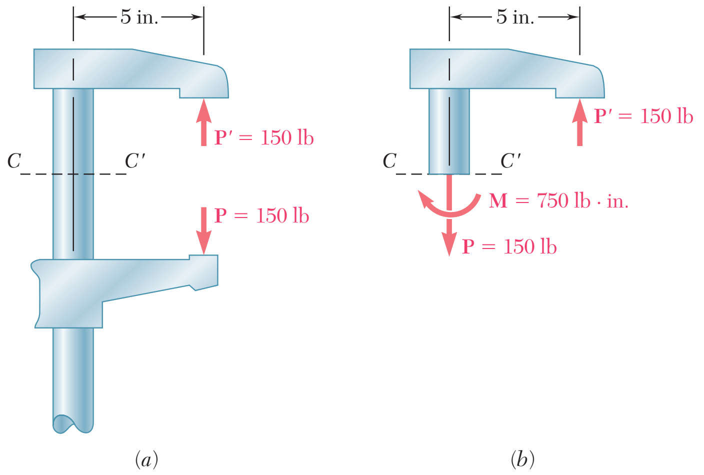


Fig. 11.3

Photo 11.2

The study of pure bending will also play an essential role in the study of beams, i.e., the study of prismatic members subjected to various types of *transverse loads*. Consider, for instance, a cantilever beam AB supporting a concentrated load **P** at its free end (Fig. 11.4*a*). If we pass a section through C at a distance x from A, we observe from the free-body diagram of AC (Fig. 11.4*b*) that the internal forces in the section consist of a force **P′** equal and opposite to **P** and a couple **M** of magnitude $M = Px$. The distribution of normal stresses in the section can be obtained from the couple **M** as if the beam were in pure bending. On the other hand, the shearing stresses in the section depend on the force **P′**, and you will learn in Chap. 13 how to determine their distribution over a given section.

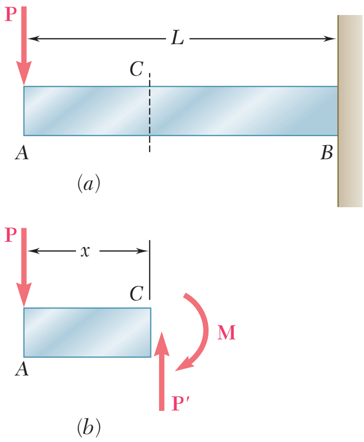


Fig. 11.4

The first part of the chapter is devoted to the analysis of the stresses and deformations caused by pure bending in a homogeneous member possessing a plane of symmetry and made of a material following Hooke's law. In a preliminary discussion of the stresses due to bending (Sec. 11.2), the methods of statics will be used to derive three fundamental equations which must be satisfied by the normal stresses in any given cross section of the member. In Sec. 11.3, it will be proved that *transverse sections remain plane* in a member subjected to pure bending, while in Sec. 11.4, formulas will be developed that can be used to determine the *normal stresses,* as well as the *radius of curvature* for that member within the elastic range.

In Sec. 11.5, you will study the stresses and deformations in *composite members* made of more than one material, such as reinforced-concrete beams, which utilize the best features of steel and concrete and are extensively used in the construction of buildings and bridges. You will learn to draw a *transformed section* representing the section of a member made of a homogeneous material that undergoes the same deformations as the composite member under the same loading. The transformed section will be used to find the stresses and deformations in the original composite member.

In Sec. 11.6, you will learn to analyze an *eccentric axial loading* in a plane of symmetry, such as the one shown in Photo 11.2, by superposing the stresses due to pure bending and the stresses due to a centric axial loading.

Your study of the bending of prismatic members will conclude with the analysis of *unsymmetric bending* (Sec. 11.7), and the study of the general case of *eccentric axial loading* (Sec. 11.8).

11.2 SYMMETRIC MEMBER IN PURE BENDING

Consider a prismatic member AB possessing a plane of symmetry and subjected to equal and opposite couples $\mathbf{M}$ and $\mathbf{M}'$ acting in that plane (Fig. 11.5*a*). We observe that if a section is passed through the member AB at some arbitrary point C, the conditions of equilibrium of the portion AC of the member require that the internal forces in the section be equivalent to the couple $\mathbf{M}$ (Fig. 11.5*b*). Thus, the internal forces in any cross section of a symmetric member in pure bending are equivalent to a couple. The moment M of that couple is referred to as the *bending moment* in the section. Following the usual convention, a positive sign will be assigned to M when the member is bent as shown in Fig. 11.5*a*, i.e., when the concavity of the beam faces upward, and a negative sign otherwise.

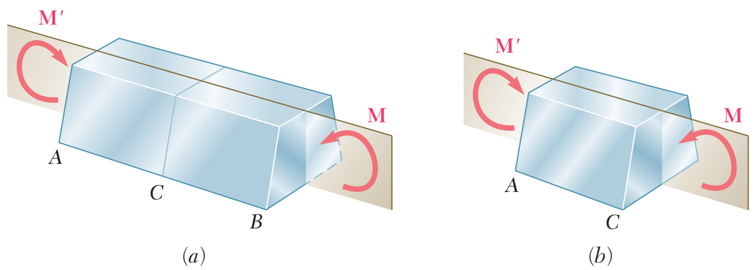


Fig. 11.5

Denoting by σ_x the normal stress at a given point of the cross section and by τ_{xy} and τ_{xz} the components of the shearing stress, we express that the system of the elementary internal forces exerted on the section is equivalent to the couple **M** (Fig. 11.6).

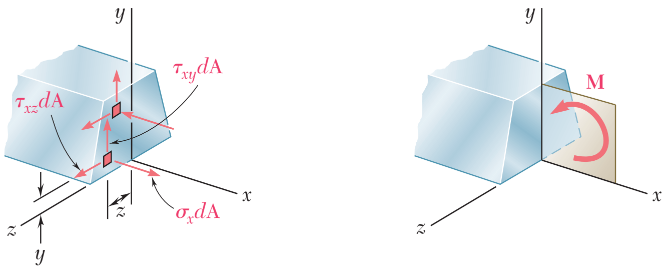


Fig. 11.6

We recall from statics that a couple **M** actually consists of two equal and opposite forces. The sum of the components of these forces in any direction is therefore equal to zero. Moreover, the moment of the couple is the same about *any* axis perpendicular to its plane, and is zero about any axis contained in that plane. Selecting arbitrarily the z axis as shown in Fig. 11.6, we express the equivalence of the elementary internal forces and of the couple **M** by writing that the sums of the components and of the moments of the elementary forces are equal to the corresponding components and moments of the couple **M**:

$$x \text{ components:} \qquad \int \sigma_x dA = 0 \tag{11.1}$$

$$\text{moments about } y \text{ axis:} \qquad \int z\sigma_x dA = 0 \tag{11.2}$$

$$\text{moments about } z \text{ axis:} \qquad \int (-y\sigma_x dA) = M \tag{11.3}$$

Three additional equations could be obtained by setting equal to zero the sums of the y components, z components, and moments about the x axis, but these equations would involve only the components of the shearing stress and, as you will see in the next section, the components of the shearing stress are both equal to zero.

Two remarks should be made at this point: (1) The minus sign in Eq. (11.3) is due to the fact that a tensile stress ($\sigma_x > 0$) leads to a negative moment (clockwise) of the normal force $\sigma_x\, dA$ about the z axis. (2) Equation (11.2) could have been anticipated, since the application of couples in the plane of symmetry of member AB will result in a distribution of normal stresses that is symmetric about the y axis.

Once more, we note that the actual distribution of stresses in a given cross section cannot be determined from statics alone. It is *statically indeterminate* and may be obtained only by analyzing the *deformations* produced in the member.

11.3 DEFORMATIONS IN A SYMMETRIC MEMBER IN PURE BENDING

Let us now analyze the deformations of a prismatic member possessing a plane of symmetry and subjected at its ends to equal and opposite couples **M** and **M′** acting in the plane of symmetry. The member will bend under the action of the couples, but will remain symmetric with respect to that plane (Fig. 11.7). Moreover, since the

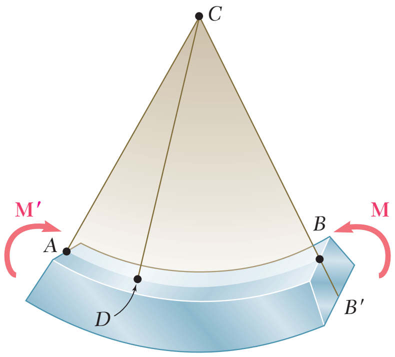


Fig. 11.7

bending moment M is the same in any cross section, the member will bend uniformly. Thus, the line AB along which the upper face of the member intersects the plane of the couples will have a constant curvature. In other words, the line AB, which was originally a straight line, will be transformed into a circle of center C, and so will the line $A'B'$ (not shown in the figure) along which the lower face of the member intersects the plane of symmetry. We also note that the line AB will decrease in length when the member is bent as shown in the figure, i.e., when $M > 0$, while $A'B'$ will become longer.

Next we will prove that any cross section perpendicular to the axis of the member remains plane and that the plane of the section passes through C. If this were not the case, we could find a point E of the original section through D (Fig. 11.8*a*) which, after the member has been bent, would *not* lie in the plane perpendicular to the plane of symmetry that contains line CD (Fig. 11.8*b*). But, because of the symmetry of the member, there would be another point E' that would be transformed exactly in the same way. Let us assume that, after the beam has been bent, both points would be located to the left of the plane defined by CD, as shown in Fig. 11.8*b*. Since the bending moment M is the same throughout the member, a similar situation would prevail in any other cross section, and the points corresponding to E and E' would also move to the left. Thus, an observer at A would conclude that the loading causes the points E and E' in the various cross sections to move forward (toward the observer). But an observer at B, to whom the loading looks the same, and who observes the points E and E' in the same positions (except that they are now inverted) would reach the opposite conclusion. This inconsistency leads us to conclude that E and E' will lie in the plane defined by CD and, therefore, that the section remains plane and passes through C. We should note, however, that this discussion does not rule out the possibility of deformations *within* the plane of the section.

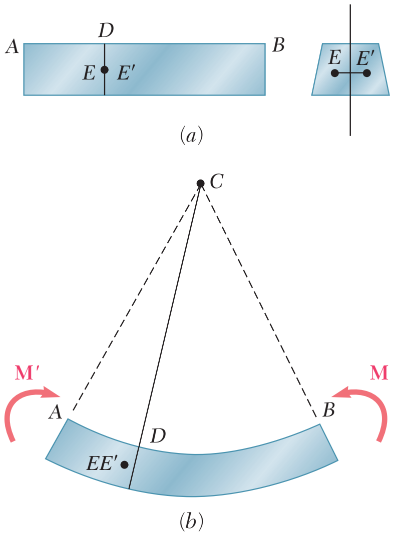
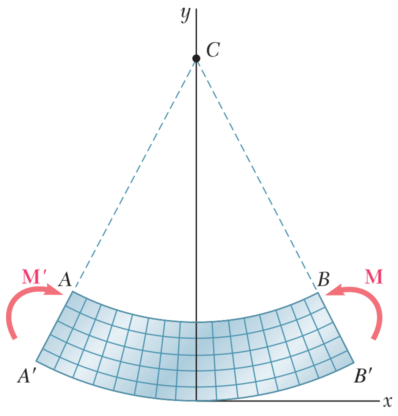


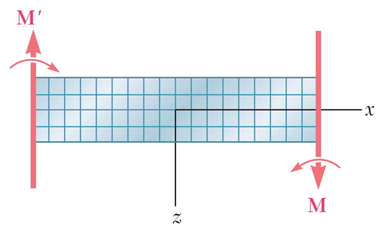
Fig. 11.8

Suppose that the member is divided into a large number of small cubic elements with faces respectively parallel to the three coordinate planes. The property we have established requires that these elements be transformed as shown in Fig. 11.9 when the member is subjected to the couples **M** and **M′**. Since all the faces represented in the two projections of Fig. 11.9 are at 90° to each other, we conclude that $\gamma_{xy} = \gamma_{zx} = 0$ and, thus, that $\tau_{xy} = \tau_{xz} = 0$. Regarding the three stress components that we have not yet discussed, namely, σ_y, σ_z, and τ_{yz}, we note that they must be zero on the surface of the member. Since, on the other hand, the deformations involved do not require any interaction between the elements of a given transverse cross section, we can assume that these three stress components are equal to zero throughout the member. This assumption is verified, both from experimental evidence and from the theory of elasticity, for slender members undergoing small deformations. We conclude that the only nonzero stress component exerted on any of the small cubic elements considered here is the normal component σ_x. Thus, at any point of a slender member in pure bending, we have a state of *uniaxial stress*. Recalling that, for $M > 0$, lines AB and $A'B'$ are observed, respectively, to decrease and increase in length, we note that the strain ϵ_x and the stress σ_x are negative in the upper portion of the member (*compression*) and positive in the lower portion (*tension*).

It follows from the above that there must exist a surface parallel to the upper and lower faces of the member, where ϵ_x and σ_x are zero. This surface is called the *neutral surface*. The neutral surface intersects the plane of symmetry along an arc of circle DE (Fig. 11.10a), and it intersects a transverse section along a straight line called the *neutral axis* of the section (Fig. 11.10b). The origin of coordinates will now be selected on the neutral surface, rather than on the lower face of the member as done earlier, so that the distance from any point to the neutral surface will be measured by its coordinate y.

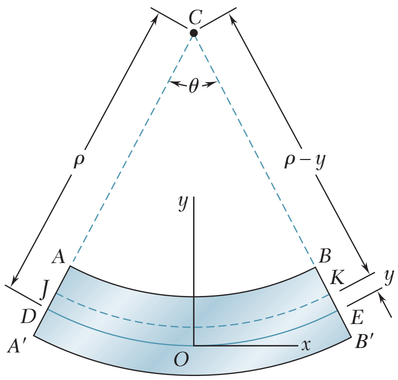


(a) Longitudinal, vertical section (plane of symmetry)

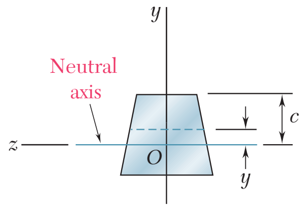


(b) Longitudinal, horizontal section

Fig. 11.9

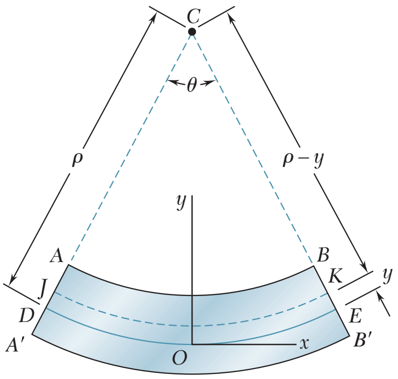


(a) Longitudinal, vertical section (plane of symmetry)



(b) Transverse section

Fig. 11.10



(a) Longitudinal, vertical section (plane of symmetry)

Fig. 11.10a *(repeated)*

Denoting by ρ the radius of arc DE (Fig. 11.10a), by θ the central angle corresponding to DE, and observing that the length of DE is equal to the length L of the undeformed member, we write

$$L = \rho\theta \tag{11.4}$$

Considering now the arc JK located at a distance y above the neutral surface, we note that its length L' is

$$L' = (\rho - y)\theta \tag{11.5}$$

Since the original length of arc JK was equal to L, the deformation of JK is

$$\delta = L' - L \tag{11.6}$$

or, if we substitute from (11.4) and (11.5) into (11.6),

$$\delta = (\rho - y)\theta - \rho\theta = -y\theta \tag{11.7}$$

The longitudinal strain ϵ_x in the elements of JK is obtained by dividing δ by the original length L of JK. We write

$$\epsilon_x = \frac{\delta}{L} = \frac{-y\theta}{\rho\theta}$$

or

$$\boxed{\epsilon_x = -\frac{y}{\rho}} \tag{11.8}$$

The minus sign is due to the fact that we have assumed the bending moment to be positive and, thus, the beam to be concave upward.

Because of the requirement that transverse sections remain plane, identical deformations will occur in all planes parallel to the plane of symmetry. Thus, the value of the strain given by Eq. (11.8) is valid anywhere, and we conclude that the *longitudinal normal strain* ϵ_x *varies linearly with the distance* y *from the neutral surface*.

The strain ϵ_x reaches its maximum absolute value when y itself is largest. Denoting by c the largest distance from the neutral surface (which corresponds to either the upper or the lower surface of the member), and by ϵ_m the *maximum absolute value* of the strain, we have

$$\epsilon_m = \frac{c}{\rho} \tag{11.9}$$

Solving (11.9) for ρ and substituting the value obtained into (11.8), we can also write

$$\boxed{\epsilon_x = -\frac{y}{c}\epsilon_m} \tag{11.10}$$

We conclude our analysis of the deformations of a member in pure bending by observing that we are still unable to compute the strain or stress at a given point of the member, since we have not yet located the neutral surface in the member. In order to locate this surface, we must first specify the stress-strain relation of the material used.†

†Let us note, however, that if the member possesses both a vertical and a horizontal plane of symmetry (e.g., a member with a rectangular cross section), and if the stress-strain curve is the same in tension and compression, the neutral surface will coincide with the plane of symmetry.

11.4 STRESSES AND DEFORMATIONS

We now consider the case when the bending moment M is such that the normal stresses in the member remain below the yield strength σ_Y. This means that, for all practical purposes, the stresses in the member will remain below the proportional limit and the elastic limit as well. There will be no permanent deformation, and Hooke's law for uniaxial stress applies. Assuming the material to be homogeneous, and denoting by E its modulus of elasticity, we have in the longitudinal x direction

$$\sigma_x = E\epsilon_x \tag{11.11}$$

Recalling Eq. (11.10), and multiplying both members of that equation by E, we write

$$E\epsilon_x = -\frac{y}{c}(E\epsilon_m)$$

or, using (11.11),

$$\boxed{\sigma_x = -\frac{y}{c}\sigma_m} \tag{11.12}$$

where σ_m denotes the *maximum absolute value* of the stress. This result shows that, *in the elastic range, the normal stress varies linearly with the distance from the neutral surface* (Fig. 11.11).

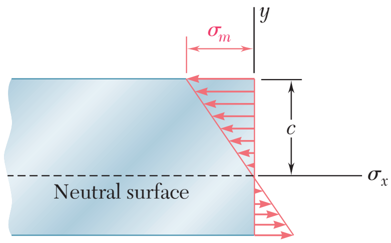


Fig. 11.11

It should be noted that, at this point, we do not know the location of the neutral surface, nor the maximum value σ_m of the stress. Both can be found if we recall the relations (11.1) and (11.3) which were obtained earlier from statics. Substituting first for σ_x from (11.12) into (11.1), we write

$$\int \sigma_x \, dA = \int \left(-\frac{y}{c}\sigma_m\right) dA = -\frac{\sigma_m}{c}\int y \, dA = 0$$

from which it follows that

$$\int y \, dA = 0 \tag{11.13}$$

This equation shows that the first moment of the cross section about its neutral axis must be zero. In other words, for a member subjected to pure bending, and *as long as the stresses remain in the elastic range, the neutral axis passes through the centroid of the section*.

We now recall Eq. (11.3), which was derived in Sec. 11.2 with respect to an *arbitrary* horizontal z axis,

$$\int (-y\sigma_x \, dA) = M \tag{11.3}$$

Specifying that the z axis should coincide with the neutral axis of the cross section, we substitute for σ_x from (11.12) into (11.3) and write

$$\int (-y)\left(-\frac{y}{c}\sigma_m\right) dA = M$$

or

$$\frac{\sigma_m}{c}\int y^2\,dA = M \tag{11.14}$$

Recalling that in the case of pure bending the neutral axis passes through the centroid of the cross section, we note that I is the moment of inertia, or second moment, of the cross section with respect to a centroidal axis perpendicular to the plane of the couple $\mathbf{M}$. Solving (11.14) for σ_m, we write therefore†

$$\boxed{\sigma_m = \frac{Mc}{I}} \tag{11.15}$$

Substituting for σ_m from (11.15) into (11.12), we obtain the normal stress σ_x at any distance y from the neutral axis:

$$\boxed{\sigma_x = -\frac{My}{I}} \tag{11.16}$$

Equations (11.15) and (11.16) are called the *elastic flexure formulas*, and the normal stress σ_x caused by the bending or "flexing" of the member is often referred to as the *flexural stress*. We verify that the stress is compressive ($\sigma_x < 0$) above the neutral axis ($y > 0$) when the bending moment M is positive, and tensile ($\sigma_x > 0$) when M is negative.

Returning to Eq. (11.15), we note that the ratio I/c depends only upon the geometry of the cross section. This ratio is called the *elastic section modulus* and is denoted by S. We have

$$\text{Elastic section modulus} = S = \frac{I}{c} \tag{11.17}$$

Substituting S for I/c into Eq. (11.15), we write this equation in the alternative form

$$\boxed{\sigma_m = \frac{M}{S}} \tag{11.18}$$

Since the maximum stress σ_m is inversely proportional to the elastic section modulus S, it is clear that beams should be designed with as large a value of S as practicable. For example, in the case of a wooden beam with a rectangular cross section of width b and depth h, we have

$$S = \frac{I}{c} = \frac{\frac{1}{12}bh^3}{h/2} = \tfrac{1}{6}bh^2 = \tfrac{1}{6}Ah \tag{11.19}$$

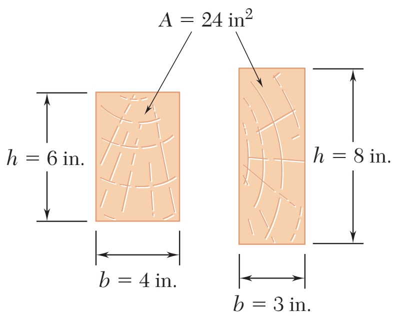


Fig. 11.12

where A is the cross-sectional area of the beam. This shows that, of two beams with the same cross-sectional area A (Fig. 11.12), the beam with the larger depth h will have the larger section modulus and, thus, will be the more effective in resisting bending.‡

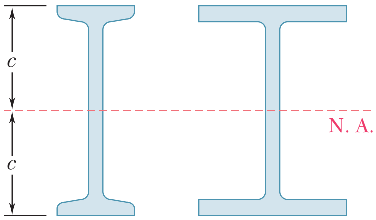
†We recall that the bending moment was assumed to be positive. If the bending moment is negative, M should be replaced in Eq. (11.15) by its absolute value $|M|$.

‡However, large values of the ratio h/b could result in lateral instability of the beam.

In the case of structural steel, American standard beams (S-beams) and wide-flange beams (W-beams), Photo 11.3, are preferred

Photo 11.3 Wide-flange steel beams form the frame of many buildings.

to other shapes because a large portion of their cross section is located far from the neutral axis (Fig. 11.13). Thus, for a given cross-sectional area and a given depth, their design provides large values of I and, consequently, of S. Values of the elastic section modulus of commonly manufactured beams can be obtained from tables listing the various geometric properties of such beams. To determine the maximum stress σ_m in a given section of a standard beam, the engineer needs only to read the value of the elastic section modulus S in a table and divide the bending moment M in the section by S.



(a) S-beam (b) W-beam

Fig. 11.13

The deformation of the member caused by the bending moment M is measured by the *curvature* of the neutral surface. The curvature is defined as the reciprocal of the radius of curvature ρ, and can be obtained by solving Eq. (11.9) for $1/\rho$:

$$\frac{1}{\rho} = \frac{\epsilon_m}{c} \tag{11.20}$$

But, in the elastic range, we have $\epsilon_m = \sigma_m/E$. Substituting for ϵ_m into (11.20), and recalling (11.15), we write

$$\frac{1}{\rho} = \frac{\sigma_m}{Ec} = \frac{1}{Ec}\frac{Mc}{I}$$

or

$$\boxed{\frac{1}{\rho} = \frac{M}{EI}} \tag{11.21}$$

EXAMPLE 11.1 A steel bar of 0.8×2.5-in. rectangular cross section is subjected to two equal and opposite couples acting in the vertical plane of symmetry of the bar (Fig. 11.14). Determine the value of the bending moment M that causes the bar to yield. Assume $\sigma_Y = 36$ ksi.

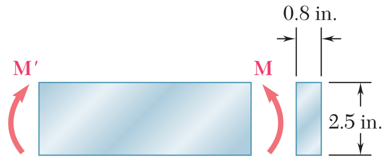


Fig. 11.14

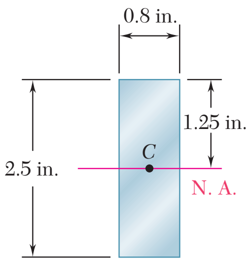


Fig. 11.15

Since the neutral axis must pass through the centroid C of the cross section, we have $c = 1.25$ in. (Fig. 11.15). On the other hand, the centroidal moment of inertia of the rectangular cross section is

$$I = \tfrac{1}{12}bh^3 = \tfrac{1}{12}(0.8 \text{ in.})(2.5 \text{ in.})^3 = 1.042 \text{ in}^4$$

Solving Eq. (11.15) for M, and substituting the above data, we have

$$M = \frac{I}{c}\sigma_m = \frac{1.042 \text{ in}^4}{1.25 \text{ in.}}(36 \text{ ksi})$$

$$M = 30 \text{ kip} \cdot \text{in.} \;\blacksquare$$

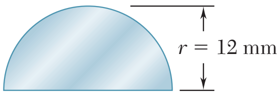


Fig. 11.16

EXAMPLE 11.2 An aluminum rod with a semicircular cross section of radius $r = 12$ mm (Fig. 11.16) is bent into the shape of a circular arc of mean radius $\rho = 2.5$ m. Knowing that the flat face of the rod is turned toward the center of curvature of the arc, determine the maximum tensile and compressive stress in the rod. Use $E = 70$ GPa.

We could use Eq. (11.21) to determine the bending moment M corresponding to the given radius of curvature ρ, and then Eq. (11.15) to determine σ_m. However, it is simpler to use Eq. (11.9) to determine ϵ_m and Hooke's law to obtain σ_m.

The ordinate $\bar{y}$ of the centroid C of the semicircular cross section is

$$\bar{y} = \frac{4r}{3\pi} = \frac{4(12 \text{ mm})}{3\pi} = 5.093 \text{ mm}$$

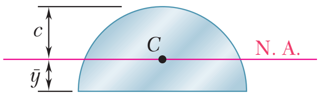


Fig. 11.17

The neutral axis passes through C (Fig. 11.17) and the distance c to the point of the cross section farthest away from the neutral axis is

$$c = r - \bar{y} = 12 \text{ mm} - 5.093 \text{ mm} = 6.907 \text{ mm}$$

Using Eq. (11.9), we write

$$\epsilon_m = \frac{c}{\rho} = \frac{6.907 \times 10^{-3} \text{ m}}{2.5 \text{ m}} = 2.763 \times 10^{-3}$$

and, applying Hooke's law,

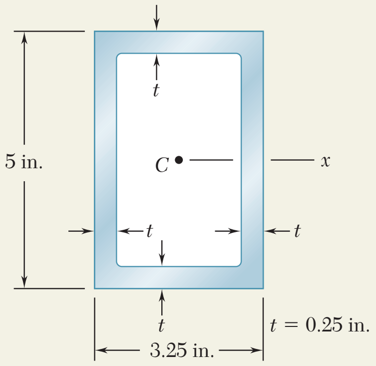
$$\sigma_m = E\epsilon_m = (70 \times 10^9 \text{ Pa})(2.763 \times 10^{-3}) = 193.4 \text{ MPa}$$

Since this side of the rod faces away from the center of curvature, the stress obtained is a tensile stress. The maximum compressive stress occurs on the flat side of the rod. Using the fact that the stress is proportional to the distance from the neutral axis, we write

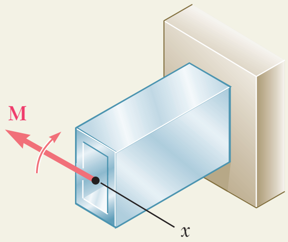
$$\sigma_{\text{comp}} = -\frac{\bar{y}}{c}\sigma_m = -\frac{5.093 \text{ mm}}{6.907 \text{ mm}}(193.4 \text{ MPa})$$

$$= -142.6 \text{ MPa} \;\blacksquare$$

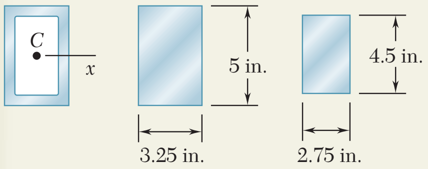
SAMPLE PROBLEM 11.1



The rectangular tube shown is extruded from an aluminum alloy for which $\sigma_Y = 40$ ksi, $\sigma_U = 60$ ksi, and $E = 10.6 \times 10^6$ psi. Neglecting the effect of fillets, determine (*a*) the bending moment M for which the factor of safety will be 3.00, (*b*) the corresponding radius of curvature of the tube.



SOLUTION



Moment of Inertia. Considering the cross-sectional area of the tube as the difference between the two rectangles shown and recalling the formula for the centroidal moment of inertia of a rectangle, we write

$$I = \tfrac{1}{12}(3.25)(5)^3 - \tfrac{1}{12}(2.75)(4.5)^3 \qquad I = 12.97 \text{ in}^4$$

Allowable Stress. For a factor of safety of 3.00 and an ultimate stress of 60 ksi, we have

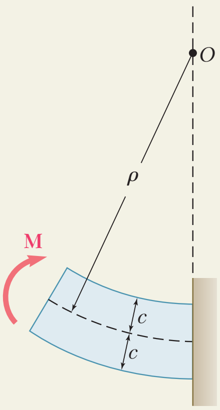
$$\sigma_{\text{all}} = \frac{\sigma_U}{F.S.} = \frac{60 \text{ ksi}}{3.00} = 20 \text{ ksi}$$

Since $\sigma_{\text{all}} < \sigma_Y$, the tube remains in the elastic range and we can apply the results of Sec. 11.4.

a. Bending Moment. With $c = \frac{1}{2}(5 \text{ in.}) = 2.5$ in., we write

$$\sigma_{\text{all}} = \frac{Mc}{I} \qquad M = \frac{I}{c}\sigma_{\text{all}} = \frac{12.97 \text{ in}^4}{2.5 \text{ in.}}(20 \text{ ksi}) \qquad M = 103.8 \text{ kip} \cdot \text{in.} \quad ◀$$

b. Radius of Curvature. Recalling that $E = 10.6 \times 10^6$ psi, we substitute this value and the values obtained for I and M into Eq. (11.21) and find



$$\frac{1}{\rho} = \frac{M}{EI} = \frac{103.8 \times 10^3 \text{ lb} \cdot \text{in.}}{(10.6 \times 10^6 \text{ psi})(12.97 \text{ in}^4)} = 0.755 \times 10^{-3} \text{ in}^{-1}$$

$$\rho = 1325 \text{ in.} \qquad \rho = 110.4 \text{ ft} \quad ◀$$

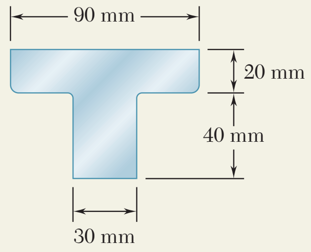
Alternative Solution. Since we know that the maximum stress is $\sigma_{\text{all}} =$ 20 ksi, we can determine the maximum strain ϵ_m and then use Eq. (11.9),

$$\epsilon_m = \frac{\sigma_{\text{all}}}{E} = \frac{20 \text{ ksi}}{10.6 \times 10^6 \text{ psi}} = 1.887 \times 10^{-3} \text{ in./in.}$$

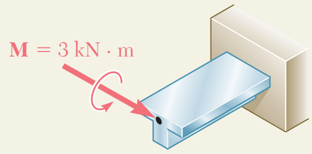
$$\epsilon_m = \frac{c}{\rho} \qquad \rho = \frac{c}{\epsilon_m} = \frac{2.5 \text{ in.}}{1.887 \times 10^{-3} \text{ in./in.}}$$

$$\rho = 1325 \text{ in.} \qquad \rho = 110.4 \text{ ft} \quad ◀$$

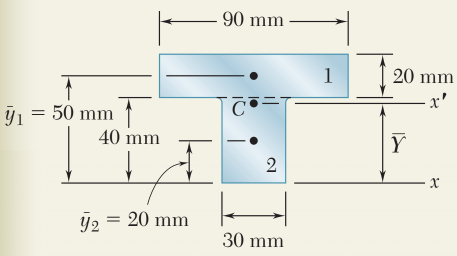
SAMPLE PROBLEM 11.2



A cast-iron machine part is acted upon by the 3 kN · m couple shown. Knowing that $E = 165$ GPa and neglecting the effect of fillets, determine (*a*) the maximum tensile and compressive stresses in the casting, (*b*) the radius of curvature of the casting.



SOLUTION

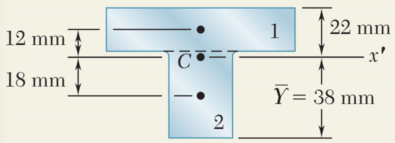


Centroid. We divide the T-shaped cross section into the two rectangles shown and write

	Area, mm^2	$\bar{y}$, mm	$\bar{y}A$, mm^3
1	$(20)(90) = 1800$	50	90×10^3
2	$(40)(30) = 1200$	20	24×10^3
	$\Sigma A = 3000$		$\Sigma\bar{y}A = 114 \times 10^3$

$$\bar{Y}\Sigma A = \Sigma\bar{y}A$$
$$\bar{Y}(3000) = 114 \times 10^6$$
$$\bar{Y} = 38 \text{ mm}$$

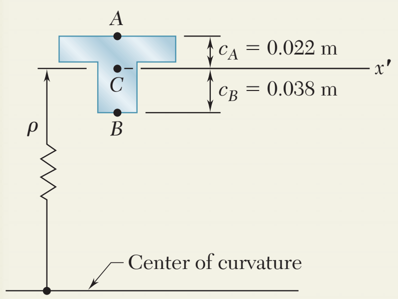
Centroidal Moment of Inertia. The parallel-axis theorem is used to determine the moment of inertia of each rectangle with respect to the axis x' that passes through the centroid of the composite section. Adding the moments of inertia of the rectangles, we write



$$\begin{aligned} I_{x'} &= \Sigma(\bar{I} + Ad^2) = \Sigma(\tfrac{1}{12}bh^3 + Ad^2) \\ &= \tfrac{1}{12}(90)(20)^3 + (90 \times 20)(12)^2 + \tfrac{1}{12}(30)(40)^3 + (30 \times 40)(18)^2 \\ &= 868 \times 10^3 \text{ mm}^4 \\ I &= 868 \times 10^{-9} \text{ m}^4 \end{aligned}$$

a. Maximum Tensile Stress. Since the applied couple bends the casting downward, the center of curvature is located below the cross section. The maximum tensile stress occurs at point A, which is farthest from the center of curvature.

$$\sigma_A = \frac{Mc_A}{I} = \frac{(3 \text{ kN} \cdot \text{m})(0.022 \text{ m})}{868 \times 10^{-9} \text{ m}^4} \qquad \sigma_A = +76.0 \text{ MPa} \blacktriangleleft$$



Maximum Compressive Stress. This occurs at point B; we have

$$\sigma_B = -\frac{Mc_B}{I} = -\frac{(3 \text{ kN} \cdot \text{m})(0.038 \text{ m})}{868 \times 10^{-9} \text{ m}^4} \qquad \sigma_B = -131.3 \text{ MPa} \blacktriangleleft$$

b. Radius of Curvature. From Eq. (11.21), we have

$$\begin{aligned} \frac{1}{\rho} &= \frac{M}{EI} = \frac{3 \text{ kN} \cdot \text{m}}{(165 \text{ GPa})(868 \times 10^{-9} \text{ m}^4)} \\ &= 20.95 \times 10^{-3} \text{ m}^{-1} \end{aligned} \qquad \rho = 47.7 \text{ m} \blacktriangleleft$$

PROBLEMS

11.1 and 11.2 Knowing that the couple shown acts in a vertical plane, determine the stress at (*a*) point *A*, (*b*) point *B*.

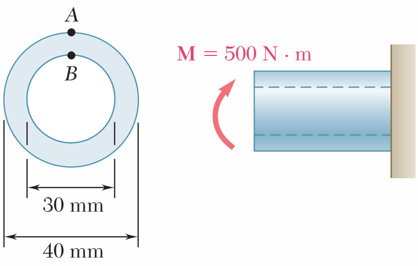


Fig. P11.1

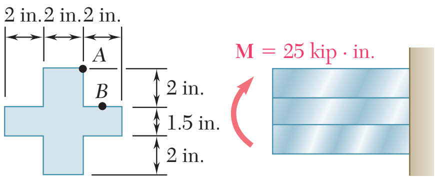


Fig. P11.2

11.3 Using an allowable stress of 155 MPa, determine the largest bending moment M_x that can be applied to the wide-flange beam shown. Neglect the effect of the fillets.

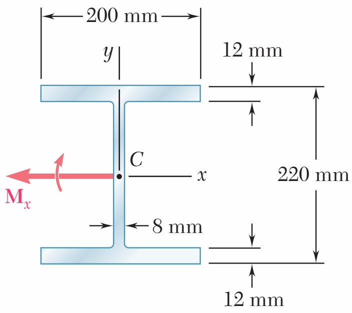


Fig. P11.3

11.4 Solve Prob. 11.3, assuming that the wide-flange beam is bent about the y axis by a couple of moment M_y.

11.5 A nylon spacing bar has the cross section shown. Knowing that the allowable stress for the grade of nylon used is 24 MPa, determine the largest couple $\mathbf{M}_z$ that can be applied to the bar.

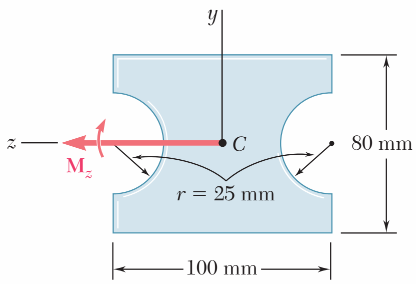


Fig. *P11.5*

11.6 Using an allowable stress of 16 ksi, determine the largest couple that can be applied to each pipe.

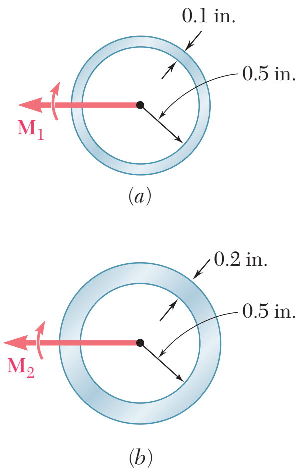


Fig. P11.6

11.7 and *11.8* Two W4 × 13 rolled sections are welded together as shown. Knowing that for the steel alloy used $\sigma_Y = 36$ ksi and $\sigma_U = 58$ ksi and using a factor of safety of 3.0, determine the largest couple that can be applied when the assembly is bent about the z axis.

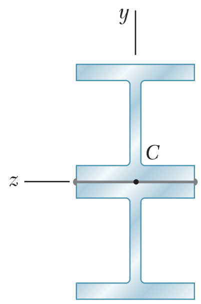


Fig. P11.7

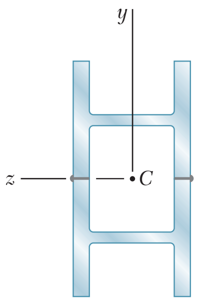


Fig. *P11.8*

11.9 through *11.11* Two vertical forces are applied to the beam of the cross section shown. Determine the maximum tensile and compressive stresses in portion BC of the beam.

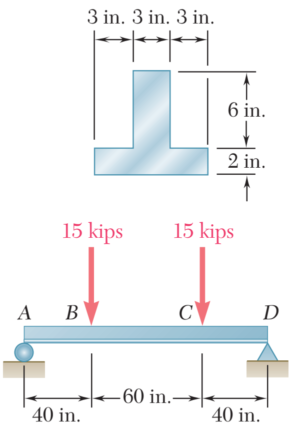


Fig. P11.9

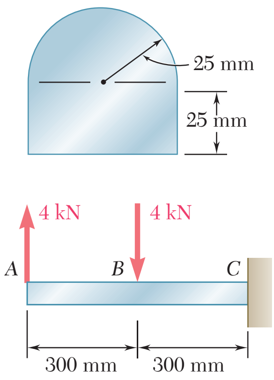


Fig. P11.10

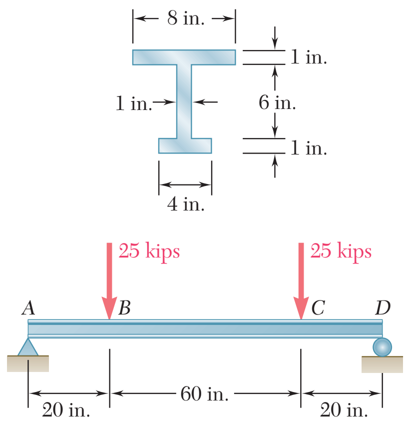


Fig. *P11.11*

11.12 Two equal and opposite couples of magnitude $M = 15\text{ kN}\cdot\text{m}$ are applied to the channel-shaped beam AB. Observing that the couples cause the beam to bend in a horizontal plane, determine the stress (*a*) at point C, (*b*) at point D, (*c*) at point E.

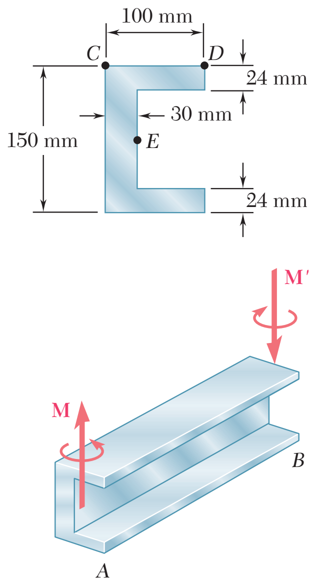


Fig. P11.12

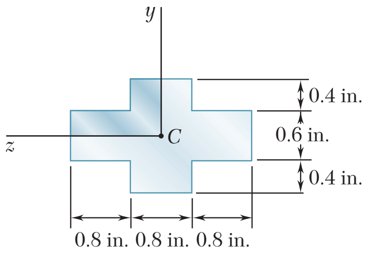


Fig. P11.13

11.13 Knowing that a beam of the cross section shown is bent about a horizontal axis and that the bending moment is $3.5\text{ kip}\cdot\text{in.}$, determine the total force acting on the shaded portion of the beam.

11.14 Solve Prob. 11.13 assuming that the beam is bent about a vertical axis and that the bending moment is $6\text{ kip}\cdot\text{in.}$

11.15 Knowing that a beam of the cross section shown is bent about a horizontal axis and that the bending moment is $8\text{ kN}\cdot\text{m}$, determine the total force acting on the top flange.

11.16 Knowing that a beam of the cross section shown is bent about a vertical axis and that the bending moment is $4\text{ kN}\cdot\text{m}$, determine the total force acting on the shaded portion of the lower flange.

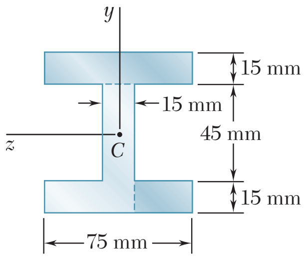


Fig. *P11.15* and P11.16

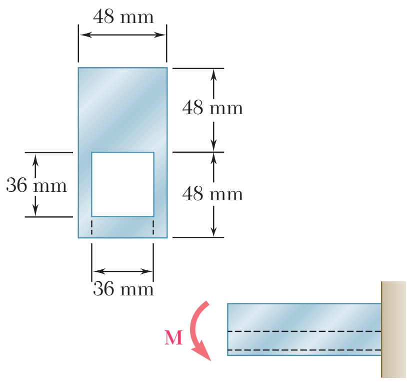


Fig. P11.17

11.17 Knowing that for the extruded beam shown the allowable stress is 120 MPa in tension and 150 MPa in compression, determine the largest couple **M** that can be applied.

11.18 Knowing that for the extruded beam shown the allowable stress is 12 ksi in tension and 16 ksi in compression, determine the largest couple **M** that can be applied.

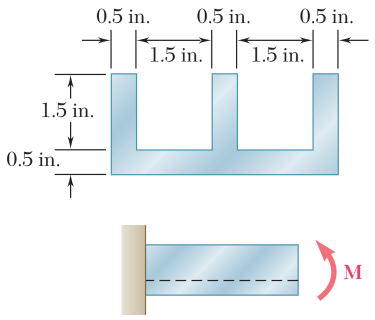


Fig. P11.18

11.19 For the casting shown, determine the largest couple **M** that can be applied without exceeding either of the following allowable stresses: $\sigma_{all} = +6$ ksi and $\sigma_{all} = -15$ ksi.

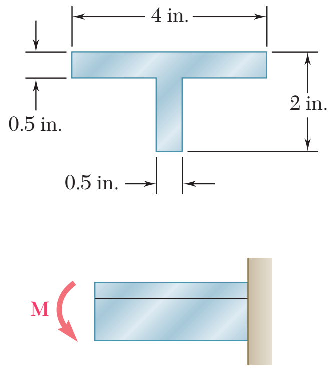


Fig. P11.19

11.20 The beam shown is made of a nylon for which the allowable stress is 24 MPa in tension and 30 MPa in compression. Determine the largest couple **M** that can be applied to the beam.

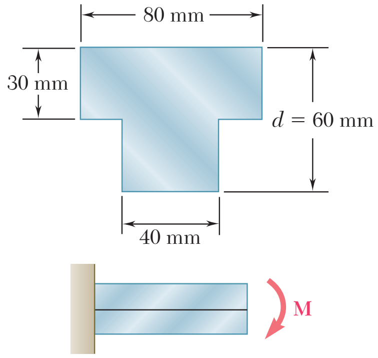


Fig. P11.20

11.21 Solve Prob. 11.20 assuming that $d = 80$ mm.

11.22 Knowing that for the beam shown the allowable stress is 12 ksi in tension and 16 ksi in compression, determine the largest couple **M** that can be applied.

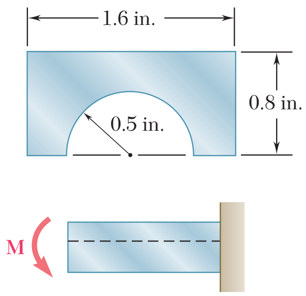


Fig. P11.22

11.23 Straight rods of 0.30-in. diameter and 200-ft length are sometimes used to clear underground conduits of obstructions or to thread wires through a new conduit. The rods are made of high-strength steel and, for storage and transportation, are wrapped on spools of 5-ft diameter. Assuming that the yield strength is not exceeded, determine (*a*) the maximum stress in a rod, when the rod, which was initially straight, is wrapped on the spool, (*b*) the corresponding bending moment in the rod. Use $E = 29 \times 10^6$ psi.

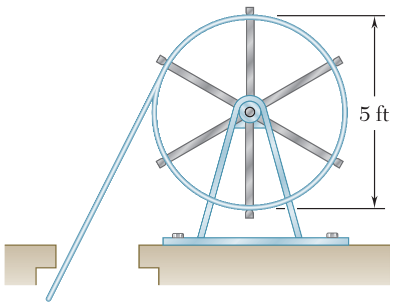


Fig. *P11.23*

11.24 A 24 kN · m couple is applied to the W200 × 46.1 rolled-steel beam shown. (*a*) Assuming that the couple is applied about the *z* axis as shown, determine the maximum stress and the radius of curvature of the beam. (*b*) Solve part *a* assuming that the couple is applied about the *y* axis. Use $E = 200$ GPa.

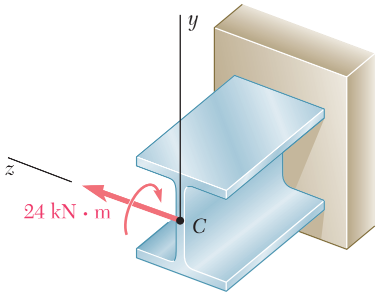


Fig. P11.24

11.5 BENDING OF MEMBERS MADE OF SEVERAL MATERIALS

The derivations given in Sec. 11.4 were based on the assumption of a homogeneous material with a given modulus of elasticity *E*. If the member subjected to pure bending is made of two or more materials with different moduli of elasticity, our approach to the determination of the stresses in the member must be modified.

Consider, for instance, a bar consisting of two portions of different materials bonded together as shown in cross section in Fig. 11.18. This composite bar will deform as described in Sec. 11.3, since its cross section remains the same throughout its entire length and since no assumption was made in Sec. 11.3 regarding the stress-strain relationship of the material or materials involved. Thus, the normal strain ϵ_x still varies linearly with the distance *y* from the neutral axis of the section (Fig. 11.19*a* and *b*), and formula (11.8) holds:

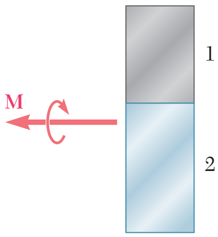


Fig. 11.18

$$\epsilon_x = -\frac{y}{\rho} \tag{11.8}$$

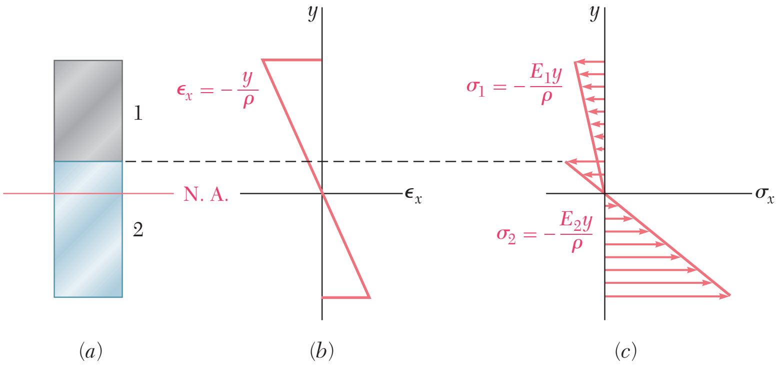


Fig. 11.19 Strain and stress distribution in bar made of two materials.

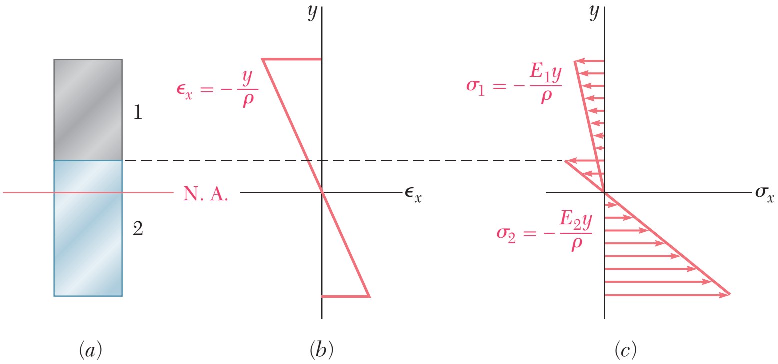


Fig. 11.19 (*repeated*) Strain and stress distribution in bar made of two materials.

However, we cannot assume that the neutral axis passes through the centroid of the composite section, and one of the goals of the present analysis will be to determine the location of this axis.

Since the moduli of elasticity E_1 and E_2 of the two materials are different, the expressions obtained for the normal stress in each material will also be different. We write

$$\begin{aligned} \sigma_1 &= E_1\epsilon_x = -\frac{E_1 y}{\rho} \\ \sigma_2 &= E_2\epsilon_x = -\frac{E_2 y}{\rho} \end{aligned} \tag{11.22}$$

and obtain a stress-distribution curve consisting of two segments of straight line (Fig. 11.19*c*). It follows from Eqs. (11.22) that the force dF_1 exerted on an element of area dA of the upper portion of the cross section is

$$dF_1 = \sigma_1\, dA = -\frac{E_1 y}{\rho}\, dA \tag{11.23}$$

while the force dF_2 exerted on an element of the same area dA of the lower portion is

$$dF_2 = \sigma_2\, dA = -\frac{E_2 y}{\rho}\, dA \tag{11.24}$$

But, denoting by n the ratio E_2/E_1 of the two moduli of elasticity, we can express dF_2 as

$$dF_2 = -\frac{(nE_1)y}{\rho}\, dA = -\frac{E_1 y}{\rho}(n\, dA) \tag{11.25}$$

Comparing Eqs. (11.23) and (11.25), we note that the same force dF_2 would be exerted on an element of area $n\, dA$ of the first material. In other words, the resistance to bending of the bar would remain the same if both portions were made of the first material, provided that the width of each element of the lower portion were multiplied by the factor n. Note that this widening (if $n > 1$), or narrowing (if $n < 1$), must be effected *in a direction parallel to the neutral axis of the section,* since it is essential that the distance y of each element from the neutral axis remain the same. The new cross section obtained in this way is called the *transformed section* of the member (Fig. 11.20).

Since the transformed section represents the cross section of a member made of a *homogeneous material* with a modulus of elasticity E_1, the method described in Sec. 11.4 can be used to determine the neutral axis of the section and the normal stress at various points of the section. The neutral axis will be drawn *through the centroid of the transformed section* (Fig. 11.21), and the stress σ_x at any point

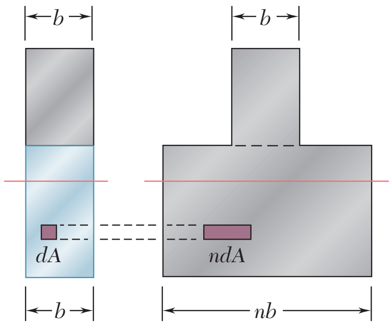


Fig. 11.20 Transformed section for composite bar.

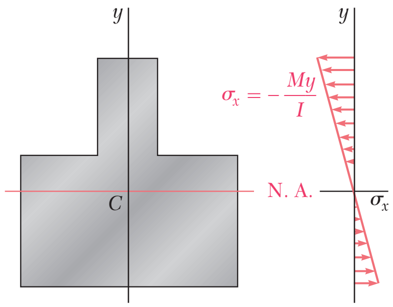


Fig. 11.21 Distribution of stresses in transformed section.

of the corresponding fictitious homogeneous member will be obtained from Eq. (11.16)

$$\sigma_x = -\frac{My}{I} \tag{11.16}$$

where y is the distance from the neutral surface, and I *the moment of inertia of the transformed section* with respect to its centroidal axis.

To obtain the stress σ_1 at a point located in the upper portion of the cross section of the original composite bar, we simply compute the stress σ_x at the corresponding point of the transformed section. However, to obtain the stress σ_2 at a point in the lower portion of the cross section, we must *multiply by n* the stress σ_x computed at the corresponding point of the transformed section. Indeed, as we saw earlier, the same elementary force dF_2 is applied to an element of area $n\ dA$ of the transformed section and to an element of area dA of the original section. Thus, the stress σ_2 at a point of the original section must be n times larger than the stress at the corresponding point of the transformed section.

The deformations of a composite member can also be determined by using the transformed section. We recall that the transformed section represents the cross section of a member, made of a homogeneous material of modulus E_1, which deforms in the same manner as the composite member. Therefore, using Eq. (11.21), we write that the curvature of the composite member is

$$\frac{1}{\rho} = \frac{M}{E_1 I}$$

where I is the moment of inertia of the transformed section with respect to its neutral axis.

EXAMPLE 11.3 A bar obtained by bonding together pieces of steel ($E_s = 29 \times 10^6$ psi) and brass ($E_b = 15 \times 10^6$ psi) has the cross section shown (Fig. 11.22). Determine the maximum stress in the steel and in the brass when the bar is in pure bending with a bending moment $M = 40$ kip · in.

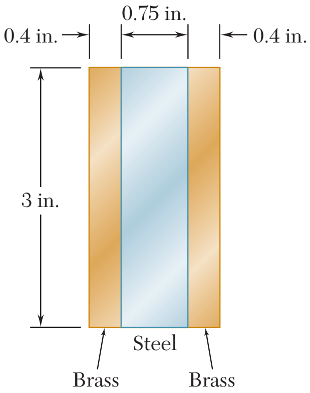


Fig. 11.22

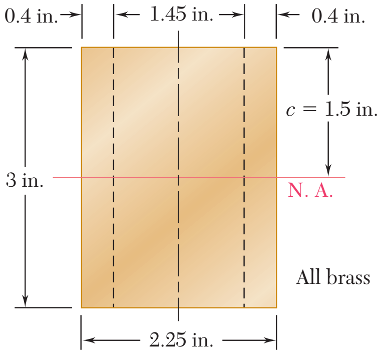


Fig. 11.23

The transformed section corresponding to an equivalent bar made entirely of brass is shown in Fig. 11.23. Since

$$n = \frac{E_s}{E_b} = \frac{29 \times 10^6 \text{ psi}}{15 \times 10^6 \text{ psi}} = 1.933$$

the width of the central portion of brass, which replaces the original steel portion, is obtained by multiplying the original width by 1.933, we have

$$(0.75 \text{ in.})(1.933) = 1.45 \text{ in.}$$

Note that this change in dimension occurs in a direction parallel to the neutral axis. The moment of inertia of the transformed section about its centroidal axis is

$$I = \tfrac{1}{12}bh^3 = \tfrac{1}{12}(2.25 \text{ in.})(3 \text{ in.})^3 = 5.063 \text{ in}^4$$

and the maximum distance from the neutral axis is $c = 1.5$ in. Using Eq. (11.15), we find the maximum stress in the transformed section:

$$\sigma_m = \frac{Mc}{I} = \frac{(40 \text{ kip} \cdot \text{in.})(1.5 \text{ in.})}{5.063 \text{ in}^4} = 11.85 \text{ ksi}$$

The value obtained also represents the maximum stress in the brass portion of the original composite bar. The maximum stress in the steel portion, however, will be larger than the value obtained for the transformed section, since the area of the central portion must be reduced by the factor $n = 1.933$ when we return from the transformed section to the original one. We thus conclude that

$$(\sigma_{\text{brass}})_{\max} = 11.85 \text{ ksi}$$
$$(\sigma_{\text{steel}})_{\max} = (1.933)(11.85 \text{ ksi}) = 22.9 \text{ ksi} \blacksquare$$

Photo 11.4

An important example of structural members made of two different materials is furnished by *reinforced concrete beams* (Photo 11.4). These beams, when subjected to positive bending moments, are reinforced by steel rods placed a short distance above their lower face (Fig. 11.24*a*). Since concrete is very weak in tension, it will crack below the neutral surface and the steel rods will carry the entire tensile load, while the upper part of the concrete beam will carry the compressive load.

To obtain the transformed section of a reinforced concrete beam, we replace the total cross-sectional area A_s of the steel bars by an equivalent area nA_s, where n is the ratio E_s/E_c of the moduli of elasticity of steel and concrete (Fig. 11.24*b*). On the other hand, since the concrete in the beam acts effectively only in compression, only the portion of the cross section located above the neutral axis should be used in the transformed section.

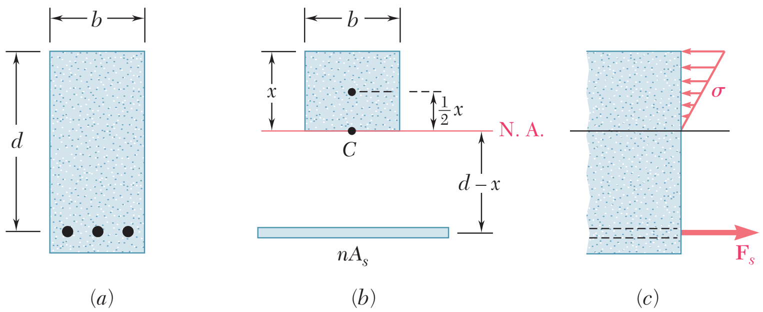


Fig. 11.24

The position of the neutral axis is obtained by determining the distance x from the upper face of the beam to the centroid C of the transformed section. Denoting by b the width of the beam, and by d the distance from the upper face to the center line of the steel rods, we write that the first moment of the transformed section with respect to the neutral axis must be zero. Since the first moment of each of the two portions of the transformed section is obtained by multiplying its area by the distance of its own centroid from the neutral axis, we have

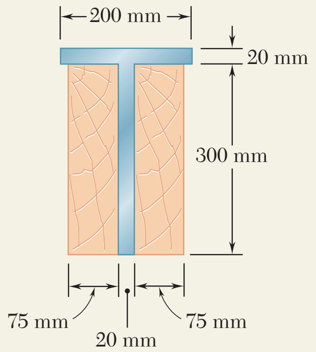
$$(bx)\frac{x}{2} - nA_s(d - x) = 0$$

or

$$\frac{1}{2}bx^2 + nA_sx - nA_sd = 0 \qquad (11.26)$$

Solving this quadratic equation for x, we obtain both the position of the neutral axis in the beam, and the portion of the cross section of the concrete beam which is effectively used.

The determination of the stresses in the transformed section is carried out as explained earlier in this section (see Sample Prob. 11.4). The distribution of the compressive stresses in the concrete and the resultant $\mathbf{F}_s$ of the tensile forces in the steel rods are shown in Fig. 11.24*c*.



SAMPLE PROBLEM 11.3

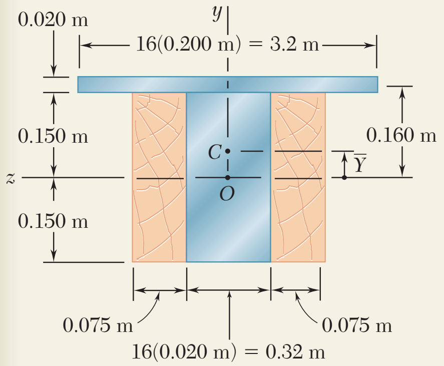
Two steel plates have been welded together to form a beam in the shape of a T that has been strengthened by securely bolting to it the two oak timbers shown. The modulus of elasticity is 12.5 GPa for the wood and 200 GPa for the steel. Knowing that a bending moment $M = 50$ kN · m is applied to the composite beam, determine (*a*) the maximum stress in the wood, (*b*) the stress in the steel along the top edge.

SOLUTION

Transformed Section. We first compute the ratio

$$n = \frac{E_s}{E_w} = \frac{200 \text{ GPa}}{12.5 \text{ GPa}} = 16$$

Multiplying the horizontal dimensions of the steel portion of the section by $n = 16$, we obtain a transformed section made entirely of wood.

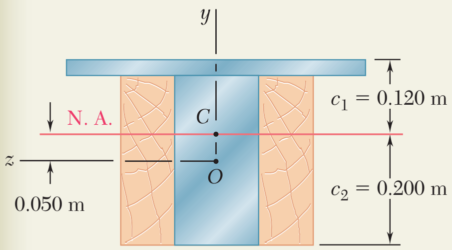


Neutral Axis. The neutral axis passes through the centroid of the transformed section. Since the section consists of two rectangles, we have

$$\overline{Y} = \frac{\Sigma \bar{y} A}{\Sigma A} = \frac{(0.160 \text{ m})(3.2 \text{ m} \times 0.020 \text{ m}) + 0}{3.2 \text{ m} \times 0.020 \text{ m} + 0.470 \text{ m} \times 0.300 \text{ m}} = 0.050 \text{ m}$$

Centroidal Moment of Inertia. Using the parallel-axis theorem:

$$\begin{aligned} I &= \tfrac{1}{12}(0.470)(0.300)^3 + (0.470 \times 0.300)(0.050)^2 \\ &\quad + \tfrac{1}{12}(3.2)(0.020)^3 + (3.2 \times 0.020)(0.160 - 0.050)^2 \\ I &= 2.19 \times 10^{-3} \text{ m}^4 \end{aligned}$$



a. Maximum Stress in Wood. The wood farthest from the neutral axis is located along the bottom edge, where $c_2 = 0.200$ m.

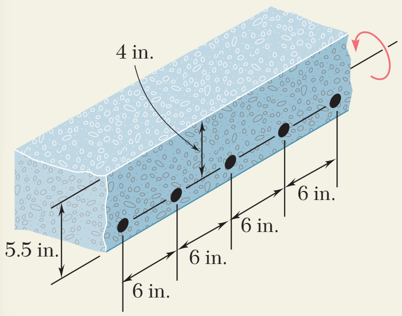
$$\sigma_w = \frac{Mc_2}{I} = \frac{(50 \times 10^3 \text{ N} \cdot \text{m})(0.200 \text{ m})}{2.19 \times 10^{-3} \text{ m}^4}$$

$$\sigma_w = 4.57 \text{ MPa} \quad \blacktriangleleft$$

b. Stress in Steel. Along the top edge $c_1 = 0.120$ m. From the transformed section we obtain an equivalent stress in wood, which must be multiplied by n to obtain the stress in steel.

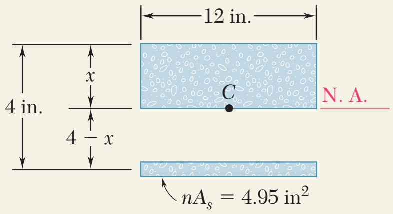
$$\sigma_s = n\frac{Mc_1}{I} = (16)\frac{(50 \times 10^3 \text{ N} \cdot \text{m})(0.120 \text{ m})}{2.19 \times 10^{-3} \text{ m}^4}$$

$$\sigma_s = 43.8 \text{ MPa} \quad \blacktriangleleft$$



SAMPLE PROBLEM 11.4

A concrete floor slab is reinforced by $\frac{5}{8}$-in.-diameter steel rods placed 1.5 in. above the lower face of the slab and spaced 6 in. on centers. The modulus of elasticity is 3.6×10^6 psi for the concrete used and 29×10^6 psi for the steel. Knowing that a bending moment of 40 kip · in. is applied to each 1-ft width of the slab, determine (*a*) the maximum stress in the concrete, (*b*) the stress in the steel.

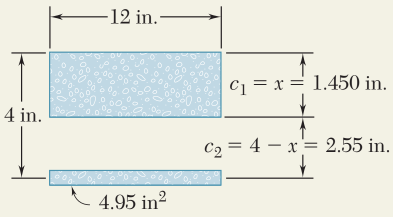


SOLUTION

Transformed Section. We consider a portion of the slab 12 in. wide, in which there are two $\frac{5}{8}$-in.-diameter rods having a total cross-sectional area

$$A_s = 2\left[\frac{\pi}{4}\left(\frac{5}{8}\text{ in.}\right)^2\right] = 0.614 \text{ in}^2$$

Since concrete acts only in compression, all the tensile forces are carried by the steel rods, and the transformed section consists of the two areas shown. One is the portion of concrete in compression (located above the neutral axis), and the other is the transformed steel area nA_s. We have



$$n = \frac{E_s}{E_c} = \frac{29 \times 10^6 \text{ psi}}{3.6 \times 10^6 \text{ psi}} = 8.06$$

$$nA_s = 8.06(0.614 \text{ in}^2) = 4.95 \text{ in}^2$$

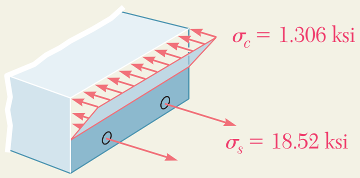
Neutral Axis. The neutral axis of the slab passes through the centroid of the transformed section. Summing moments of the transformed area about the neutral axis, we write

$$12x\left(\frac{x}{2}\right) - 4.95(4 - x) = 0 \qquad x = 1.450 \text{ in.}$$

Moment of Inertia. The centroidal moment of inertia of the transformed area is

$$I = \tfrac{1}{3}(12)(1.450)^3 + 4.95(4 - 1.450)^2 = 44.4 \text{ in}^4$$

a. Maximum Stress in Concrete. At the top of the slab, we have $c_1 = 1.450$ in. and



$$\sigma_c = \frac{Mc_1}{I} = \frac{(40 \text{ kip} \cdot \text{in.})(1.450 \text{ in.})}{44.4 \text{ in}^4} \qquad \sigma_c = 1.306 \text{ ksi} \blacktriangleleft$$

b. Stress in Steel. For the steel, we have $c_2 = 2.55$ in., $n = 8.06$ and

$$\sigma_s = n\frac{Mc_2}{I} = 8.06\frac{(40 \text{ kip} \cdot \text{in.})(2.55 \text{ in.})}{44.4 \text{ in}^4} \qquad \sigma_s = 18.52 \text{ ksi} \blacktriangleleft$$

PROBLEMS

11.25 and 11.26 A bar having the cross section shown has been formed by securely bonding brass and aluminum stock. Using the data given below, determine the largest permissible bending moment when the composite bar is bent about a horizontal axis.

	Aluminum	Brass
Modulus of elasticity	70 GPa	105 GPa
Allowable stress	100 MPa	160 MPa

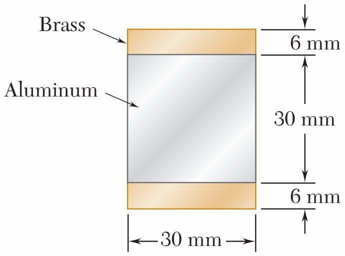


Fig. P11.25

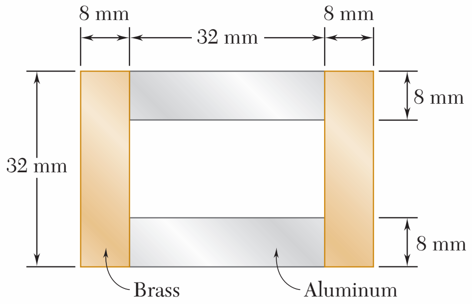


Fig. P11.26

11.27 and *11.28* For the composite bar indicated, determine the largest permissible bending moment when the bar is bent about a vertical axis.

11.27 Bar of Prob. 11.25.

11.28 Bar of Prob. 11.26.

11.29 through 11.31 Wooden beams and steel plates are securely bolted together to form the composite member shown. Using the data given below, determine the largest permissible bending moment when the composite member is bent about a horizontal axis.

	Wood	Steel
Modulus of elasticity	2×10^6 psi	30×10^6 psi
Allowable stress	2000 psi	22 ksi

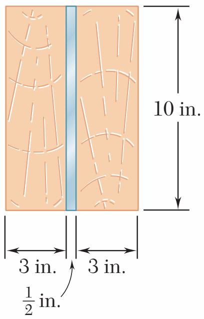


Fig. P11.29

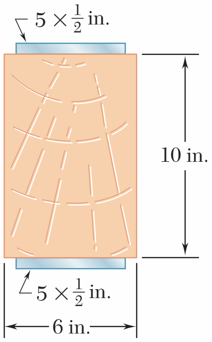


Fig. P11.30

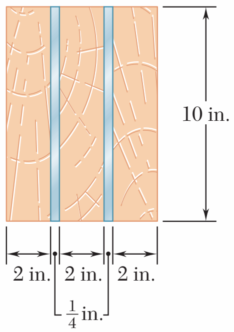


Fig. P11.31

11.32 For the composite member of Prob. 11.31, determine the largest permissible bending moment when the member is bent about a vertical axis.

11.33 and 11.34 A copper strip (E_c = 105 GPa) and an aluminum strip (E_a = 75 GPa) are bonded together to form the composite bar shown. Knowing that the bar is bent about a horizontal axis by a couple of moment 35 N · m, determine the maximum stress in (*a*) the aluminum strip, (*b*) the copper strip.

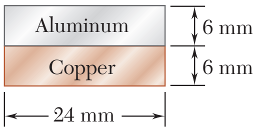


Fig. P11.33

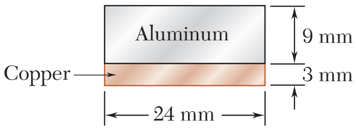


Fig. P11.34

11.35 and 11.36 The 6 × 12-in. timber beam has been strengthened by bolting to it the steel reinforcement shown. The modulus of elasticity for wood is 1.8 × 10^6 psi and for steel 29 × 10^6 psi. Knowing that the beam is bent about a horizontal axis by a couple of moment 450 kip · in., determine the maximum stress in (*a*) the wood, (*b*) the steel.

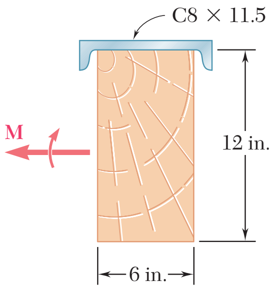


Fig. P11.35

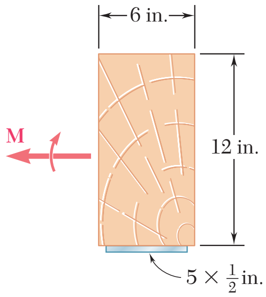


Fig. P11.36

***11.37* and 11.38** For the composite bar indicated, determine the radius of curvature caused by the couple of moment 35 N · m.

11.37 Bar of Prob. 11.33.
11.38 Bar of Prob. 11.34.

***11.39* and 11.40** For the composite bar indicated, determine the radius of curvature caused by the couple of moment 450 kip · in.

11.39 Bar of Prob. 11.35.
11.40 Bar of Prob. 11.36.

11.41 The reinforced concrete beam shown is subjected to a positive bending moment of 175 kN · m. Knowing that the modulus of elasticity is 25 GPa for the concrete and 200 GPa for the steel, determine (*a*) the stress in the steel, (*b*) the maximum stress in the concrete.

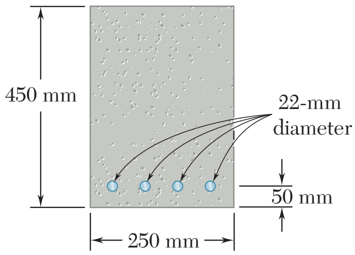


Fig. P11.41

11.42 Solve Prob. 11.41 assuming that the 450-mm depth of the beam is increased to 500 mm.

11.43 A concrete slab is reinforced by 16-mm-diameter steel rods placed on 180-mm centers as shown. The modulus of elasticity is 20 GPa for the concrete and 200 GPa for the steel. Using an allowable stress of 9 MPa for the concrete and 120 MPa for the steel, determine the largest allowable positive bending moment in a portion of the slab 1 m wide.

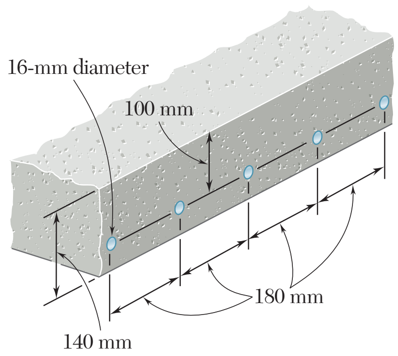


Fig. P11.43

11.44 Solve Prob. 11.43 assuming that the spacing of the 16-mm-diameter rods is increased to 225 mm on centers.

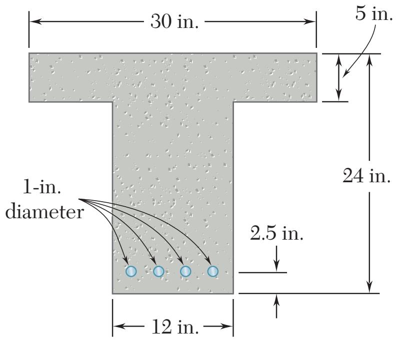


Fig. P11.45

11.45 Knowing that the bending moment in the reinforced concrete beam is $+150$ kip · ft and that the modulus of elasticity is 3.75×10^6 psi for the concrete and 30×10^6 psi for the steel, determine (*a*) the stress in the steel, (*b*) the maximum stress in the concrete.

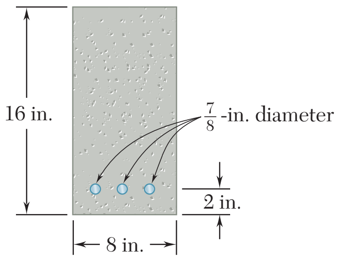


Fig. P11.46

11.46 A concrete beam is reinforced by three steel rods placed as shown. The modulus of elasticity is 3×10^6 psi for the concrete and 30×10^6 psi for the steel. Using an allowable stress of 1350 psi for the concrete and 20 ksi for the steel, determine the largest permissible positive bending moment in the beam.

11.47 and 11.48 Five metal strips, each of 0.5×1.5-in. cross section, are bonded together to form the composite beam shown. The modulus of elasticity is 30×10^6 psi for the steel, 15×10^6 psi for the brass, and 10×10^6 psi for the aluminum. Knowing that the beam is bent about a horizontal axis by a couple of moment 12 kip · in., determine (*a*) the maximum stress in each of the three metals, (*b*) the radius of curvature of the composite beam.

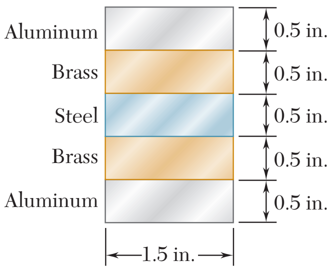


Fig. P11.47

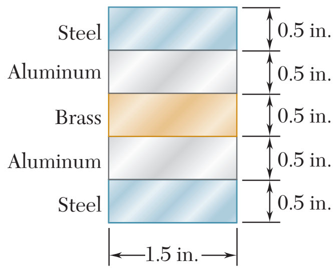


Fig. P11.48

11.6 ECCENTRIC AXIAL LOADING IN A PLANE OF SYMMETRY

We saw in Sec. 8.3 that the distribution of stresses in the cross section of a member under axial loading can be assumed to be uniform only if the line of action of the loads **P** and **P′** passes through the centroid of the cross section. Such a loading is said to be *centric*. Let us now analyze the distribution of stresses when the line of action of the loads does *not* pass through the centroid of the cross section, i.e., when the loading is *eccentric*.

Two examples of an eccentric loading are shown in Photos 11.5 and 11.6. In the case of the highway light, the weight of the lamp causes an eccentric loading on the post. Likewise, the vertical forces exerted on the press cause an eccentric loading on the back column of the press.

Photo 11.5

Photo 11.6

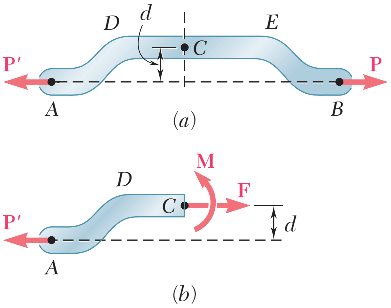


Fig. 11.25

In this section, our analysis will be limited to members which possess a plane of symmetry, and it will be assumed that the loads are applied in the plane of symmetry of the member (Fig. 11.25*a*). The internal forces acting on a given cross section may then be represented by a force **F** applied at the centroid C of the section and a couple **M** acting in the plane of symmetry of the member (Fig. 11.25*b*). The conditions of equilibrium of the free body AC require that the force **F** be equal and opposite to **P′** and that the moment of the couple **M** be equal and opposite to the moment of **P′** about C. Denoting by d the distance from the centroid C to the line of action AB of the forces **P** and **P′**, we have

$$F = P \qquad \text{and} \qquad M = Pd \tag{11.27}$$

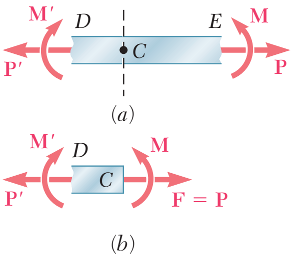


Fig. 11.26

We now observe that the internal forces in the section would have been represented by the same force and couple if the straight portion DE of member AB had been detached from AB and subjected simultaneously to the centric loads **P** and **P′** and to the bending couples **M** and **M′** (Fig. 11.26). Thus, the stress distribution due to the original eccentric loading can be obtained by superposing the uniform stress distribution corresponding to the centric loads **P** and **P′** and the linear distribution corresponding to the bending couples **M** and **M′** (Fig. 11.27). We write

$$\sigma_x = (\sigma_x)_{\text{centric}} + (\sigma_x)_{\text{bending}}$$

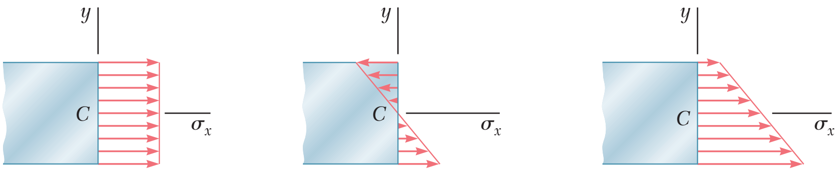


Fig. 11.27

or, recalling Eqs. (8.1) and (11.16):

$$\boxed{\sigma_x = \frac{P}{A} - \frac{My}{I}} \tag{11.28}$$

where A is the area of the cross section and I its centroidal moment of inertia, and where y is measured from the centroidal axis of the cross section. The relation obtained shows that the distribution of stresses across the section is *linear but not uniform*. Depending upon the geometry of the cross section and the eccentricity of the load, the combined stresses may all have the same sign, as shown in Fig. 11.27, or some may be positive and others negative, as shown in Fig. 11.28. In the latter case, there will be a line in the section, along which $\sigma_x = 0$. This line represents the *neutral axis* of the section. We note that the neutral axis does *not* coincide with the centroidal axis of the section, since $\sigma_x \neq 0$ for $y = 0$.

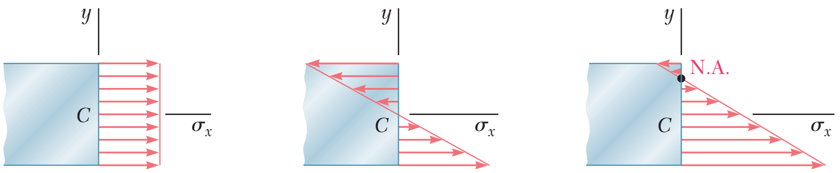


Fig. 11.28

The results obtained are valid only to the extent that the conditions of applicability of the superposition principle (Sec. 9.11) and of Saint-Venant's principle (Sec. 9.14) are met. This means that the stresses involved must not exceed the proportional limit of the material, that the deformations due to bending must not appreciably affect the distance d in Fig. 11.25, and that the cross section where the stresses are computed must not be too close to points D or E in the same figure.

EXAMPLE 11.4 An open-link chain is obtained by bending low-carbon steel rods of 0.5-in. diameter into the shape shown (Fig. 11.29). Knowing that the chain carries a load of 160 lb, determine (*a*) the largest tensile and compressive stresses in the straight portion of a link, (*b*) the distance between the centroidal and the neutral axis of a cross section.

(a) Largest Tensile and Compressive Stresses. The internal forces in the cross section are equivalent to a centric force **P** and a bending couple **M** (Fig. 11.30) of magnitudes

$$P = 160 \text{ lb}$$
$$M = Pd = (160 \text{ lb})(0.65 \text{ in.}) = 104 \text{ lb} \cdot \text{in.}$$

The corresponding stress distributions are shown in parts *a* and *b* of Fig. 11.31. The distribution due to the centric force **P** is uniform and equal to $\sigma_0 = P/A$. We have

$$A = \pi c^2 = \pi(0.25 \text{ in.})^2 = 0.1963 \text{ in}^2$$
$$\sigma_0 = \frac{P}{A} = \frac{160 \text{ lb}}{0.1963 \text{ in}^2} = 815 \text{ psi}$$

The distribution due to the bending couple **M** is linear with a maximum stress $\sigma_m = Mc/I$. We write

$$I = \tfrac{1}{4}\pi c^4 = \tfrac{1}{4}\pi(0.25 \text{ in.})^4 = 3.068 \times 10^{-3} \text{ in}^4$$
$$\sigma_m = \frac{Mc}{I} = \frac{(104 \text{ lb} \cdot \text{in.})(0.25 \text{ in.})}{3.068 \times 10^{-3} \text{ in}^4} = 8475 \text{ psi}$$

Superposing the two distributions, we obtain the stress distribution corresponding to the given eccentric loading (Fig. 11.31*c*). The largest tensile and compressive stresses in the section are found to be, respectively,

$$\sigma_t = \sigma_0 + \sigma_m = 815 + 8475 = 9290 \text{ psi}$$
$$\sigma_c = \sigma_0 - \sigma_m = 815 - 8475 = -7660 \text{ psi}$$

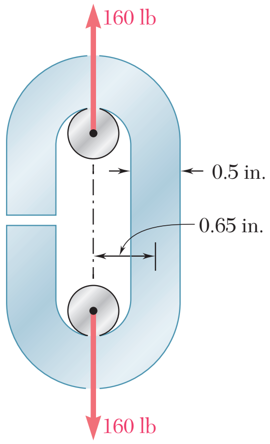


Fig. 11.29

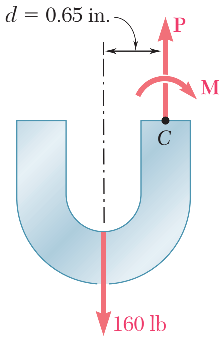


Fig. 11.30

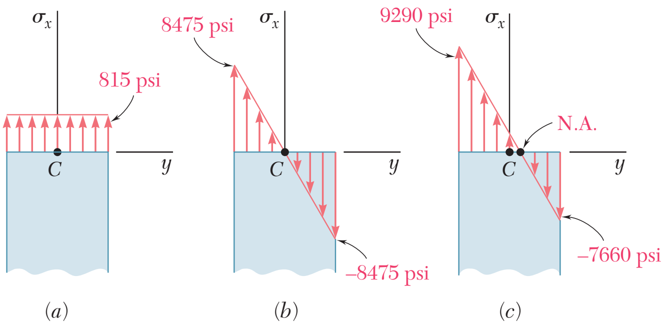
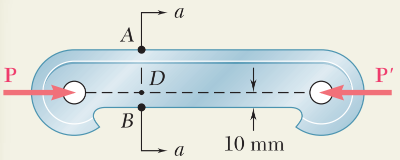


Fig. 11.31

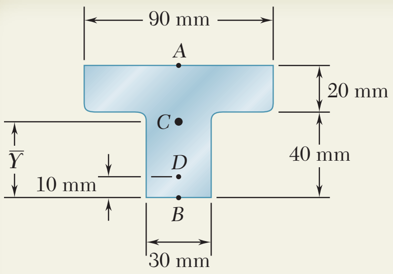
(b) Distance Between Centroidal and Neutral Axes. The distance y_0 from the centroidal to the neutral axis of the section is obtained by setting $\sigma_x = 0$ in Eq. (11.28) and solving for y_0:

$$0 = \frac{P}{A} - \frac{My_0}{I}$$
$$y_0 = \left(\frac{P}{A}\right)\left(\frac{I}{M}\right) = (815 \text{ psi})\frac{3.068 \times 10^{-3} \text{ in}^4}{104 \text{ lb} \cdot \text{in.}}$$
$$y_0 = 0.0240 \text{ in.} \quad ■$$

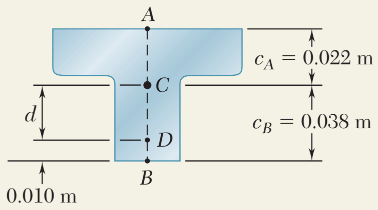


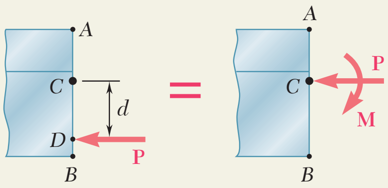
SAMPLE PROBLEM 11.5

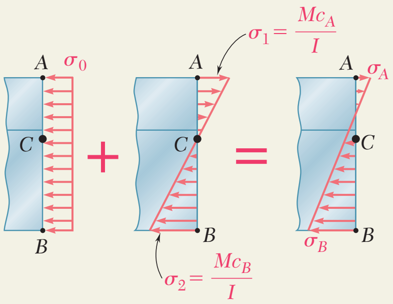
Knowing that for the cast iron link shown the allowable stresses are 30 MPa in tension and 120 MPa in compression, determine the largest force **P** which can be applied to the link. (*Note*: The T-shaped cross section of the link has previously been considered in Sample Prob. 11.2.)



Section *a–a*







(1) (2) (3)

SOLUTION

Properties of Cross Section. From Sample Prob. 11.2, we have

$$A = 3000 \text{ mm}^2 = 3 \times 10^{-3} \text{ m}^2 \qquad \bar{Y} = 38 \text{ mm} = 0.038 \text{ m}$$
$$I = 868 \times 10^{-9} \text{ m}^4$$

We now write: $\quad d = (0.038 \text{ m}) - (0.010 \text{ m}) = 0.028 \text{ m}$

Force and Couple at C. We replace **P** by an equivalent force-couple system at the centroid C.

$$P = P \qquad M = P(d) = P(0.028 \text{ m}) = 0.028P$$

The force **P** acting at the centroid causes a uniform stress distribution (Fig. 1). The bending couple **M** causes a linear stress distribution (Fig. 2).

$$\sigma_0 = \frac{P}{A} = \frac{P}{3 \times 10^{-3}} = 333P \qquad \text{(Compression)}$$
$$\sigma_1 = \frac{Mc_A}{I} = \frac{(0.028P)(0.022)}{868 \times 10^{-9}} = 710P \qquad \text{(Tension)}$$
$$\sigma_2 = \frac{Mc_B}{I} = \frac{(0.028P)(0.038)}{868 \times 10^{-9}} = 1226P \qquad \text{(Compression)}$$

Superposition. The total stress distribution (Fig. 3) is found by superposing the stress distributions caused by the centric force **P** and by the couple **M**. Since tension is positive, and compression negative, we have

$$\sigma_A = -\frac{P}{A} + \frac{Mc_A}{I} = -333P + 710P = +377P \qquad \text{(Tension)}$$
$$\sigma_B = -\frac{P}{A} - \frac{Mc_B}{I} = -333P - 1226P = -1559P \qquad \text{(Compression)}$$

Largest Allowable Force. The magnitude of **P** for which the tensile stress at point A is equal to the allowable tensile stress of 30 MPa is found by writing

$$\sigma_A = 377P = 30 \text{ MPa} \qquad P = 79.6 \text{ kN} \quad \blacktriangleleft$$

We also determine the magnitude of **P** for which the stress at B is equal to the allowable compressive stress of 120 MPa.

$$\sigma_B = -1559P = -120 \text{ MPa} \qquad P = 77.0 \text{ kN} \quad \blacktriangleleft$$

The magnitude of the largest force **P** that can be applied without exceeding either of the allowable stresses is the smaller of the two values we have found.

$$P = 77.0 \text{ kN} \quad \blacktriangleleft$$

PROBLEMS

11.49 Two forces **P** can be applied separately or at the same time to a plate that is welded to a solid circular bar of radius r. Determine the largest compressive stress in the circular bar (*a*) when both forces are applied, (*b*) when only one of the forces is applied.

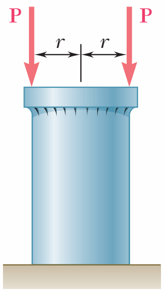


Fig. P11.49

11.50 As many as three axial loads each of magnitude $P = 10$ kips can be applied to the end of a W8 × 21 rolled-steel shape. Determine the stress at point A (*a*) for the loading shown, (*b*) if loads are applied at points 1 and 2 only.

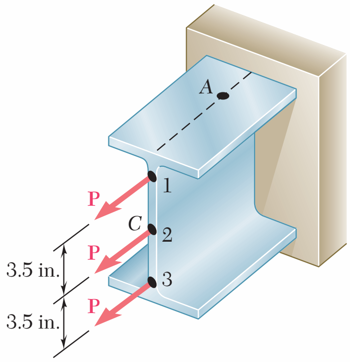


Fig. P11.50 and P11.51

11.51 As many as three axial loads each of magnitude $P = 10$ kips can be applied to the end of a W8 × 21 rolled-steel shape. Determine the stress at point A (*a*) for the loading shown, (*b*) if loads are applied at points 2 and 3 only.

11.52 Knowing that the magnitude of the horizontal force **P** is 8 kN, determine the stress at (*a*) point A, (*b*) point B.

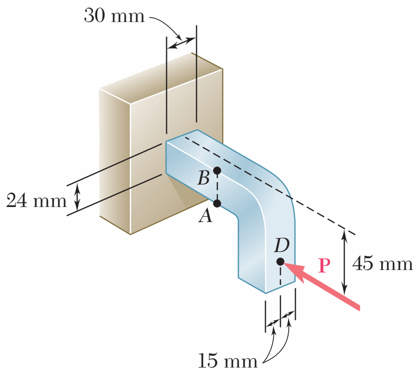


Fig. P11.52

11.53 The vertical portion of the press shown consists of a rectangular tube having a wall thickness $t = \frac{1}{2}$ in. Knowing that the press has been tightened on wooden planks being glued together until $P = 6$ kips, determine the stress (*a*) at point A, (*b*) point B.

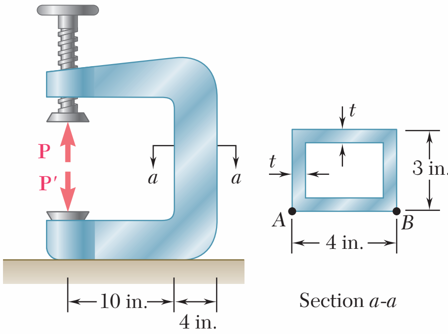


Fig. *P11.53*

11.54 Solve Prob. 11.53 assuming that $t = \frac{3}{8}$ in.

11.55 Determine the stress at points A and B (*a*) for the loading shown, (*b*) if the 60-kN loads are applied at points 1 and 2 only.

11.56 Determine the stress at points A and B (*a*) for the loading shown, (*b*) if the 60-kN loads applied at points 2 and 3 are removed.

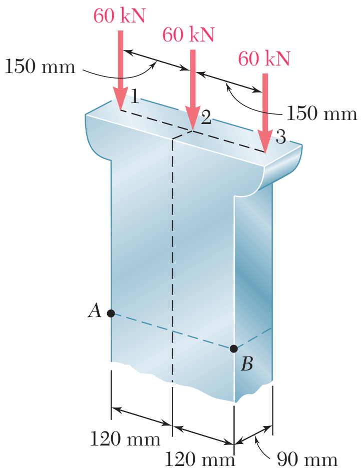


Fig. *P11.55* and P11.56

11.57 An offset h must be introduced into a solid circular rod of diameter d. Knowing that the maximum stress after the offset is introduced must not exceed four times the stress in the rod when it was straight, determine the largest offset that can be used.

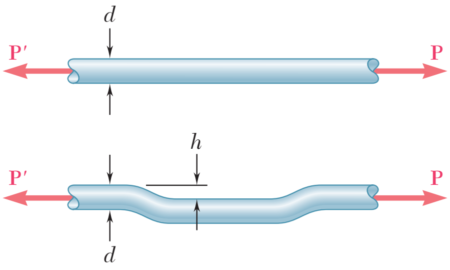


Fig. P11.57 and P11.58

11.58 An offset h must be introduced into a metal tube of 18-mm outer diameter and 2-mm wall thickness. Knowing that the maximum stress after the offset is introduced must not exceed four times the stress in the tube when it was straight, determine the largest offset that can be used.

11.59 A short column is made by nailing two 1 × 4-in. planks to a 2 × 4-in. timber. Determine the largest compressive stress created in the column by a 12-kip load applied as shown at the center of the top section of the timber if (*a*) the column is as described, (*b*) plank 1 is removed, (*c*) both planks are removed.

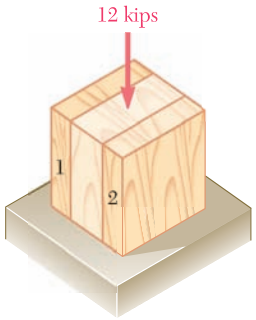


Fig. P11.59

11.60 Knowing that the allowable stress in section ABD is 10 ksi, determine the largest force **P** that can be applied to the bracket shown.

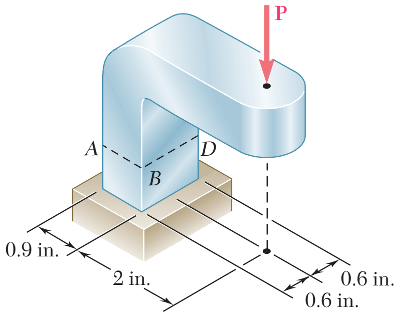


Fig. P11.60

11.61 A milling operation was used to remove a portion of a solid bar of square cross section. Knowing that $a = 1.2$ in., $d = 0.8$ in., and $\sigma_{all} = 8$ ksi, determine the magnitude P of the largest forces that can be safely applied at the centers of the ends of the bar.

11.62 A milling operation was used to remove a portion of a solid bar of square cross section. Forces of magnitude $P = 4$ kips are applied at the center of the ends of the bar. Knowing that $a = 1.2$ in. and $\sigma_{all} = 8$ ksi, determine the smallest allowable depth d of the milled portion of the rod.

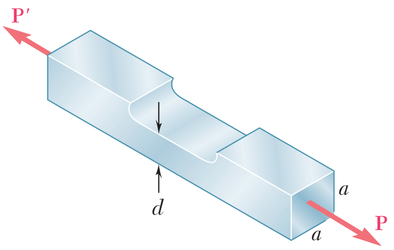


Fig. *P11.61* and P11.62

11.63 The two forces shown are applied to a rigid plate supported by a steel pipe of 140-mm outer diameter and 120-mm inner diameter. Knowing that the allowable compressive stress is 100 MPa, determine the range of allowable values of P.

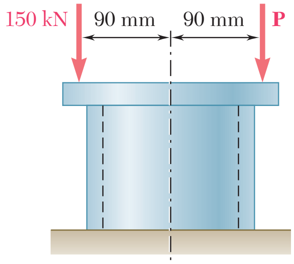


Fig. *P11.63* and P11.64

11.64 The two forces shown are applied to a rigid plate supported by a steel pipe of 140-mm outer diameter and 120-mm inner diameter. Determine the range of allowable values of P for which all stresses in the pipe are compressive and less than 100 MPa.

11.65 The shape shown was formed by bending a thin steel plate. Assuming that the thickness t is small compared to the length a of a side of the shape, determine the stress (a) at A, (b) at B, (c) at C.

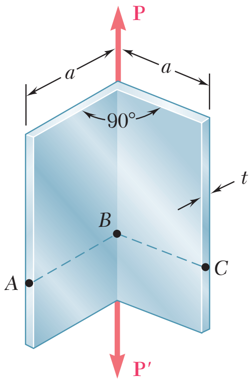


Fig. P11.65

11.66 Knowing that the allowable stress in section *a-a* of the hydraulic press shown is 40 MPa in tension and 80 MPa in compression, determine the largest force **P** that can be exerted by the press.

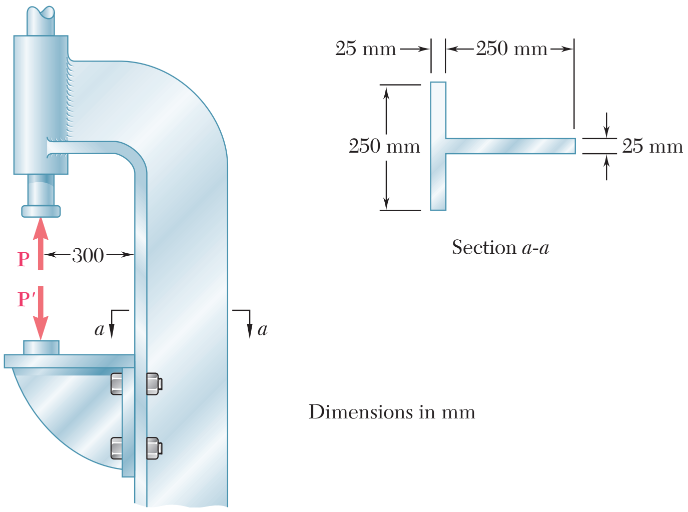


Fig. P11.66

11.67 A vertical force **P** of magnitude 20 kips is applied at a point *C* located on the axis of symmetry of the cross section of a short column. Knowing that $y = 5$ in., determine (*a*) the stress at point *A*, (*b*) the stress at point *B*, (*c*) the location of the neutral axis.

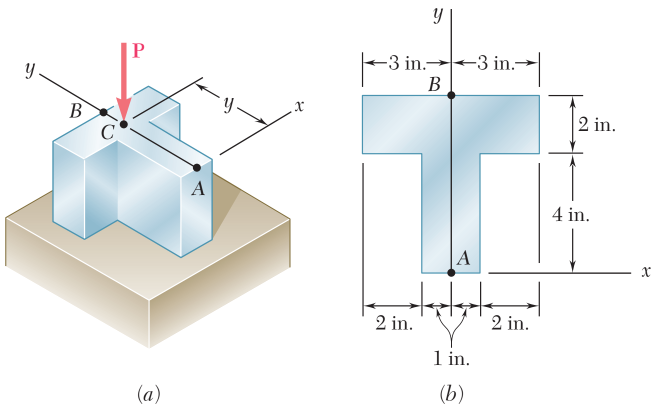


Fig. *P11.67* and P11.68

11.68 A vertical force **P** is applied at a point *C* located on the axis of symmetry of the cross section of a short column. Determine the range of values of *y* for which tensile stresses do not occur in the column.

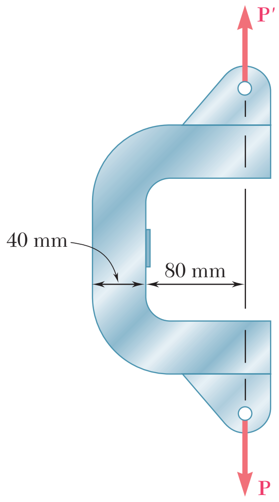


Fig. *P11.69*

11.69 The C-shaped steel bar is used as a dynamometer to determine the magnitude *P* of the forces shown. Knowing that the cross section of the bar is a square of side 40 mm and that the strain on the inner edge was measured and found to be 450 μ, determine the magnitude *P* of the forces. Use $E = 200$ GPa.

11.70 A short length of a rolled-steel column supports a rigid plate on which two loads **P** and **Q** are applied as shown. The strains at two points A and B on the center lines of the outer faces of the flanges have been measured and found to be $\epsilon_A = -400 \times 10^{-6}$ in./in. and $\epsilon_B = -300 \times 10^{-6}$ in./in. Knowing that $E = 29 \times 10^6$ psi, determine the magnitude of each load.

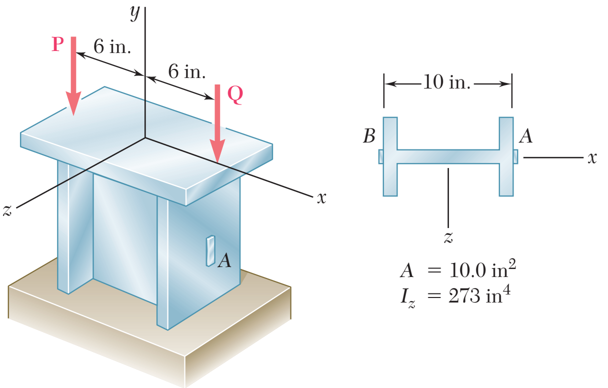


Fig. P11.70

11.71 Solve Prob. 11.70 assuming that the measured strains are $\epsilon_A = -350 \times 10^{-6}$ in./in. and $\epsilon_B = -50 \times 10^{-6}$ in./in.

11.72 An eccentric force **P** is applied as shown to a steel bar of 25 × 90-mm cross section. The strains at A and B have been measured and found to be $\epsilon_A = +350\ \mu$ and $\epsilon_B = -70\ \mu$. Knowing that $E = 200$ GPa, determine (*a*) the distance d, (*b*) the magnitude of the force **P**.

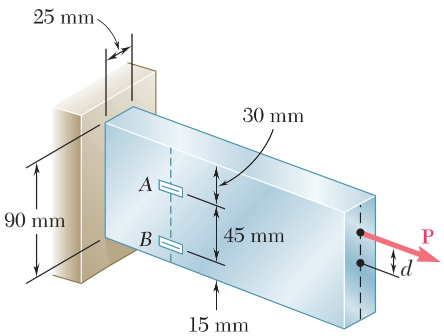


Fig. P11.72

11.7 UNSYMMETRIC BENDING

Our analysis of pure bending has been limited so far to members possessing at least one plane of symmetry and subjected to couples acting in that plane. Because of the symmetry of such members and of their loadings, we concluded that the members would remain symmetric with respect to the plane of the couples and thus bend in that plane

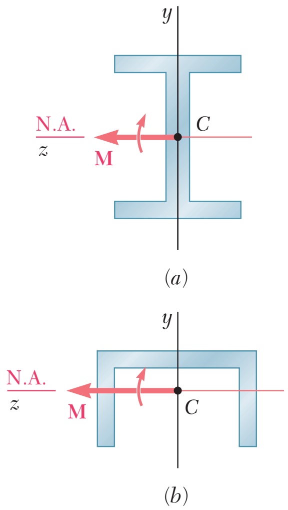


Fig. 11.32

(Sec. 11.3). This is illustrated in Fig. 11.32; part *a* shows the cross section of a member possessing two planes of symmetry, one vertical and one horizontal, and part *b* the cross section of a member with a single, vertical plane of symmetry. In both cases the couple exerted on the section acts in the vertical plane of symmetry of the member and is represented by the horizontal couple vector **M**, and in both cases the neutral axis of the cross section is found to coincide with the axis of the couple.

Let us now consider situations where the bending couples do *not* act in a plane of symmetry of the member, either because they act in a different plane, or because the member does not possess any plane of symmetry. In such situations, we cannot assume that the member will bend in the plane of the couples. This is illustrated in Fig. 11.33. In each part of the figure, the couple exerted on the section has again been assumed to act in a vertical plane and has been represented by a horizontal couple vector **M**. However, since the vertical plane is not a plane of symmetry, *we cannot expect the member to bend in that plane or the neutral axis of the section to coincide with the axis of the couple*.

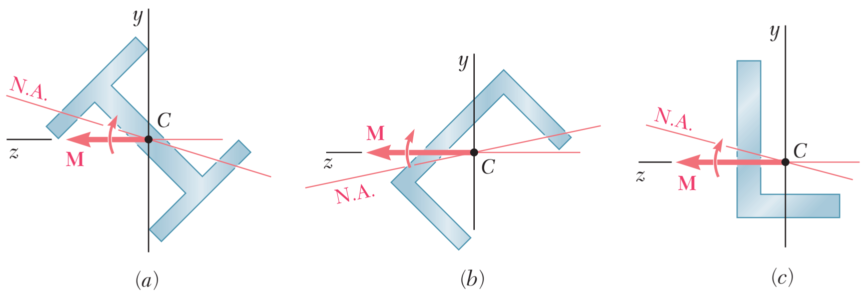


Fig. 11.33

We propose to determine the precise conditions under which the neutral axis of a cross section of arbitrary shape coincides with the axis of the couple **M** representing the forces acting on that section. Such a section is shown in Fig. 11.34, and both the couple vector **M** and the neutral axis have been assumed to be directed along the z axis. We recall from Sec. 11.2 that, if we then express

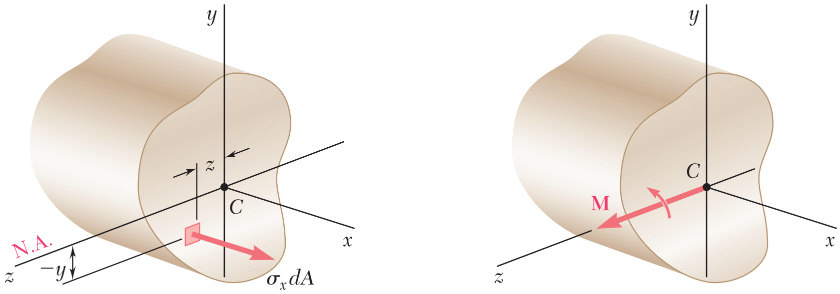


Fig. 11.34

that the elementary internal forces $\sigma_x dA$ form a system equivalent to the couple $\mathbf{M}$, we obtain

$$x \text{ components:} \qquad \int \sigma_x dA = 0 \qquad (11.1)$$

$$\text{moments about } y \text{ axis:} \qquad \int z\sigma_x dA = 0 \qquad (11.2)$$

$$\text{moments about } z \text{ axis:} \qquad \int (-y\sigma_x dA) = M \qquad (11.3)$$

As we saw earlier, when all the stresses are within the proportional limit, the first of these equations leads to the requirement that the neutral axis be a centroidal axis, and the last to the fundamental relation $\sigma_x = -My/I$. Since we had assumed in Sec. 11.2 that the cross section was symmetric with respect to the y axis, Eq. (11.2) was dismissed as trivial at that time. Now that we are considering a cross section of arbitrary shape, Eq. (11.2) becomes highly significant. Assuming the stresses to remain within the proportional limit of the material, we can substitute $\sigma_x = -\sigma_m y/c$ into Eq. (11.2) and write

$$\int z\left(-\frac{\sigma_m y}{c}\right) dA = 0 \qquad \text{or} \qquad \int yz\, dA = 0 \qquad (11.29)$$

The integral $\int yz dA$ represents the product of inertia I_{yz} of the cross section with respect to the y and z axes, and will be zero if these axes are the *principal centroidal axes of the cross section*.† We thus conclude that the neutral axis of the cross section will coincide with the axis of the couple $\mathbf{M}$ representing the forces acting on that section *if, and only if, the couple vector* $\mathbf{M}$ *is directed along one of the principal centroidal axes of the cross section*.

We note that the cross sections shown in Fig. 11.32 are symmetric with respect to at least one of the coordinate axes. It follows that, in each case, the y and z axes are the principal centroidal axes of the section. Since the couple vector $\mathbf{M}$ is directed along one of the principal centroidal axes, we verify that the neutral axis will coincide with the axis of the couple. We also note that, if the cross sections are rotated through 90° (Fig. 11.35), the couple vector $\mathbf{M}$ will still be directed along a principal centroidal axis, and the neutral axis will again coincide with the axis of the couple, even though in case b the couple does *not* act in a plane of symmetry of the member.

In Fig. 11.33, on the other hand, neither of the coordinate axes is an axis of symmetry for the sections shown, and the coordinate axes are not principal axes. Thus, the couple vector $\mathbf{M}$ is not directed along a principal centroidal axis, and the neutral axis does not coincide with the axis of the couple. However, any given section possesses principal centroidal axes, even if it is unsymmetric, as in the section shown in Fig. 11.33c. If the couple vector $\mathbf{M}$ is directed along one of the principal centroidal axes of the section, the neutral axis will coincide with the axis of the couple (Fig. 11.36) and the equations derived in Secs. 11.3 and 11.4 for symmetric members can be used to determine the stresses in this case as well.

†See Ferdinand P. Beer, E. Russell Johnston, Jr., David F. Mazurek, and Elliot R. Eisenberg, *Vector Mechanics for Engineers,* 9th ed., McGraw-Hill, New York, 2010, secs. 9.8–9.10.

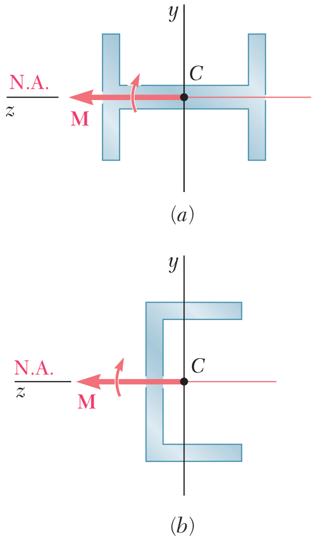


Fig. 11.35

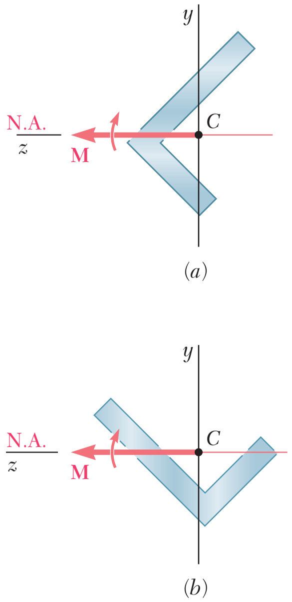


Fig. 11.36

As you will see presently, the principle of superposition can be used to determine stresses in the most general case of unsymmetric bending. Consider first a member with a vertical plane of symmetry, which is subjected to bending couples **M** and **M′** acting in a plane forming an angle θ with the vertical plane (Fig. 11.37). The couple vector **M** representing the forces acting on a given cross section will

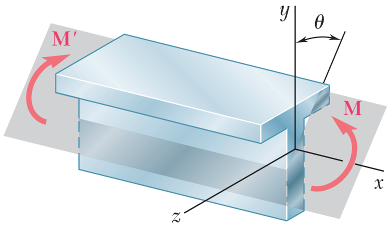


Fig. 11.37

form the same angle θ with the horizontal z axis (Fig. 11.38). Resolving the vector **M** into component vectors $\mathbf{M}_z$ and $\mathbf{M}_y$ along the z and y axes, respectively, we write

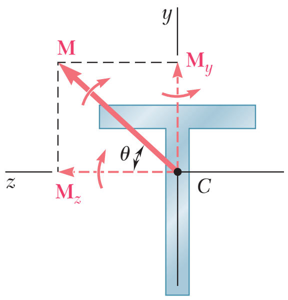


Fig. 11.38

$$M_z = M \cos \theta \qquad M_y = M \sin \theta \tag{11.30}$$

Since the y and z axes are the principal centroidal axes of the cross section, we can use Eq. (11.16) to determine the stresses resulting from the application of either of the couples represented by $\mathbf{M}_z$ and $\mathbf{M}_y$. The couple $\mathbf{M}_z$ acts in a vertical plane and bends the member in that plane (Fig. 11.39). The resulting stresses are

$$\sigma_x = -\frac{M_z y}{I_z} \tag{11.31}$$

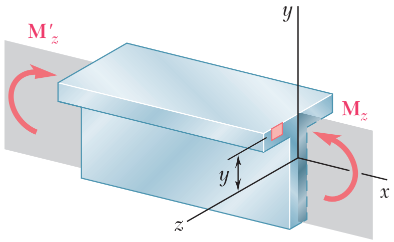


Fig. 11.39

where I_z is the moment of inertia of the section about the principal centroidal z axis. The negative sign is due to the fact that we have compression above the xz plane ($y > 0$) and tension below ($y < 0$). On the other hand, the couple $\mathbf{M}_y$ acts in a horizontal plane and bends the member in that plane (Fig. 11.40). The resulting stresses are found to be

$$\sigma_x = +\frac{M_y z}{I_y} \tag{11.32}$$

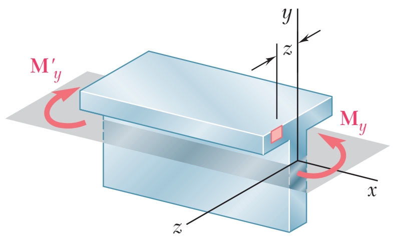


Fig. 11.40

where I_y is the moment of inertia of the section about the principal centroidal y axis, and where the positive sign is due to the fact that we have tension to the left of the vertical xy plane ($z > 0$) and compression to its right ($z < 0$). The distribution of the stresses caused by the original couple **M** is obtained by superposing the stress distributions defined by Eqs. (11.31) and (11.32), respectively. We have

$$\sigma_x = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y} \tag{11.33}$$

We note that the expression obtained can also be used to compute the stresses in an unsymmetric section, such as the one shown in Fig. 11.41, once the principal centroidal y and z axes have been determined. On the other hand, Eq. (11.33) is valid only if the conditions of applicability of the principle of superposition are met. In other words, it should not be used if the combined stresses exceed the proportional limit of the material, or if the deformations caused by one of the component couples appreciably affect the distribution of the stresses due to the other.

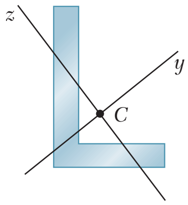


Fig. 11.41

Equation (11.33) shows that the distribution of stresses caused by unsymmetric bending is linear. However, as we have indicated earlier in this section, the neutral axis of the cross section will not, in general, coincide with the axis of the bending couple. Since the normal stress is zero at any point of the neutral axis, the equation defining that axis can be obtained by setting $\sigma_x = 0$ in Eq. (11.33). We write

$$-\frac{M_z y}{I_z} + \frac{M_y z}{I_y} = 0$$

or, solving for y and substituting for M_z and M_y from Eqs. (11.30),

$$y = \left(\frac{I_z}{I_y}\tan\theta\right)z \qquad (11.34)$$

The equation obtained is that of a straight line of slope $m = (I_z/I_y)\tan\theta$. Thus, the angle ϕ that the neutral axis forms with the z axis (Fig. 11.42) is defined by the relation

$$\boxed{\tan\phi = \frac{I_z}{I_y}\tan\theta} \qquad (11.35)$$

where θ is the angle that the couple vector **M** forms with the same axis. Since I_z and I_y are both positive, ϕ and θ have the same sign. Furthermore, we note that $\phi > \theta$ when $I_z > I_y$, and $\phi < \theta$ when $I_z < I_y$. Thus, the neutral axis is always located between the couple vector **M** and the principal axis corresponding to the minimum moment of inertia.

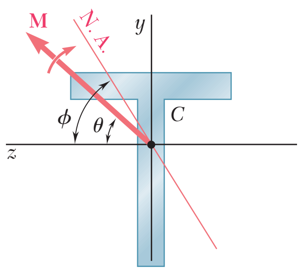


Fig. 11.42

EXAMPLE 11.5 A 1600-lb · in. couple is applied to a wooden beam, of rectangular cross section 1.5 by 3.5 in., in a plane forming an angle of 30° with the vertical (Fig. 11.43). Determine (*a*) the maximum stress in the beam, (*b*) the angle that the neutral surface forms with the horizontal plane.

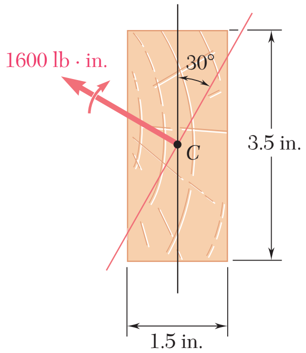


Fig. 11.43

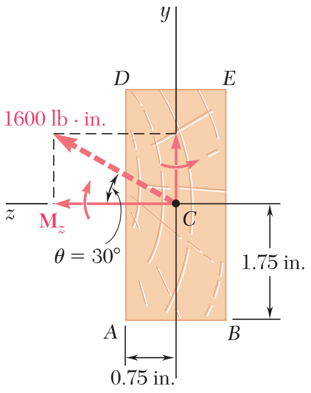


Fig. 11.44

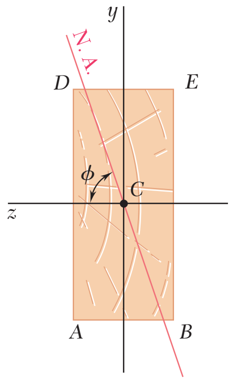


Fig. 11.45

(a) Maximum Stress. The components $\mathbf{M}_z$ and $\mathbf{M}_y$ of the couple vector are first determined (Fig. 11.44):

$$M_z = (1600 \text{ lb} \cdot \text{in.}) \cos 30° = 1386 \text{ lb} \cdot \text{in.}$$
$$M_y = (1600 \text{ lb} \cdot \text{in.}) \sin 30° = 800 \text{ lb} \cdot \text{in.}$$

We also compute the moments of inertia of the cross section with respect to the z and y axes:

$$I_z = \tfrac{1}{12}(1.5 \text{ in.})(3.5 \text{ in.})^3 = 5.359 \text{ in}^4$$
$$I_y = \tfrac{1}{12}(3.5 \text{ in.})(1.5 \text{ in.})^3 = 0.9844 \text{ in}^4$$

The largest tensile stress due to $\mathbf{M}_z$ occurs along AB and is

$$\sigma_1 = \frac{M_z y}{I_z} = \frac{(1386 \text{ lb} \cdot \text{in.})(1.75 \text{ in.})}{5.359 \text{ in}^4} = 452.6 \text{ psi}$$

The largest tensile stress due to $\mathbf{M}_y$ occurs along AD and is

$$\sigma_2 = \frac{M_y z}{I_y} = \frac{(800 \text{ lb} \cdot \text{in.})(0.75 \text{ in.})}{0.9844 \text{ in}^4} = 609.5 \text{ psi}$$

The largest tensile stress due to the combined loading, therefore, occurs at A and is

$$\sigma_{\max} = \sigma_1 + \sigma_2 = 452.6 + 609.5 = 1062 \text{ psi}$$

The largest compressive stress has the same magnitude and occurs at E.

(b) Angle of Neutral Surface with Horizontal Plane. The angle ϕ that the neutral surface forms with the horizontal plane (Fig. 11.45) is obtained from Eq. (11.35):

$$\tan \phi = \frac{I_z}{I_y} \tan \theta = \frac{5.359 \text{ in}^4}{0.9844 \text{ in}^4} \tan 30° = 3.143$$
$$\phi = 72.4°$$

The distribution of the stresses across the section is shown in Fig. 11.46. ■

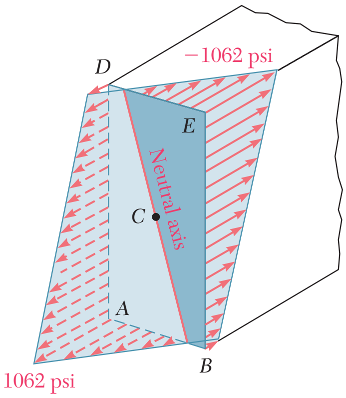


Fig. 11.46

11.8 GENERAL CASE OF ECCENTRIC AXIAL LOADING

In Sec. 11.6 you analyzed the stresses produced in a member by an eccentric axial load applied in a plane of symmetry of the member. You will now study the more general case when the axial load is not applied in a plane of symmetry.

Consider a straight member AB subjected to equal and opposite eccentric axial forces **P** and **P′** (Fig. 11.47*a*), and let a and b denote the distances from the line of action of the forces to the principal centroidal axes of the cross section of the member. The eccentric force **P** is statically equivalent to the system consisting of a centric force **P** and of the two couples $\mathbf{M}_y$ and $\mathbf{M}_z$ of moments $M_y = Pa$ and $M_z = Pb$ represented in Fig. 11.47*b*. Similarly, the eccentric force **P′** is equivalent to the centric force **P′** and the couples $\mathbf{M}'_y$ and $\mathbf{M}'_z$.

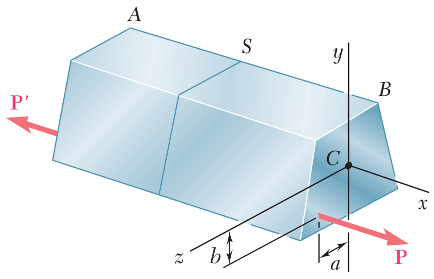
By virtue of Saint-Venant's principle (Sec. 9.14), we can replace the original loading of Fig. 11.47*a* by the statically equivalent loading of Fig. 11.47*b* in order to determine the distribution of stresses in a section S of the member, as long as that section is not too close to either end of the member. Furthermore, the stresses due to the loading of Fig. 11.47*b* can be obtained by superposing the stresses corresponding to the centric axial load **P** and to the bending couples $\mathbf{M}_y$ and $\mathbf{M}_z$, as long as the conditions of applicability of the principle of superposition are satisfied (Sec. 9.11). The stresses due to the centric load **P** are given by Eq. (8.1), and the stresses due to the bending couples by Eq. (11.33), since the corresponding couple vectors are directed along the principal centroidal axes of the section. We write, therefore,

$$\sigma_x = \frac{P}{A} - \frac{M_z y}{I_z} + \frac{M_y z}{I_y} \tag{11.36}$$

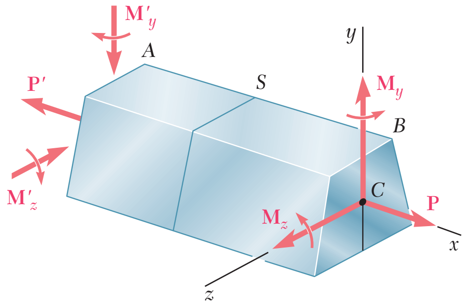
where y and z are measured from the principal centroidal axes of the section. The relation obtained shows that the distribution of stresses across the section is *linear*.

In computing the combined stress σ_x from Eq. (11.36), care should be taken to correctly determine the sign of each of the three terms in the right-hand member, since each of these terms can be positive or negative, depending upon the sense of the loads **P** and **P′** and the location of their line of action with respect to the principal centroidal axes of the cross section. Depending upon the geometry of the cross section and the location of the line of action of **P** and **P′**, the combined stresses σ_x obtained from Eq. (11.36) at various points of the section may all have the same sign, or some may be positive and others negative. In the latter case, there will be a line in the section, along which the stresses are zero. Setting $\sigma_x = 0$ in Eq. (11.36), we obtain the equation of a straight line, which represents the *neutral axis* of the section:

$$\frac{M_z}{I_z}y - \frac{M_y}{I_y}z = \frac{P}{A}$$



(*a*)



(*b*)

Fig. 11.47

EXAMPLE 11.6 A vertical 4.80-kN load is applied as shown on a wooden post of rectangular cross section, 80 by 120 mm (Fig. 11.48). (*a*) Determine the stress at points A, B, C, and D. (*b*) Locate the neutral axis of the cross section.

(a) Stresses. The given eccentric load is replaced by an equivalent system consisting of a centric load **P** and two couples $\mathbf{M}_x$ and $\mathbf{M}_z$ represented by

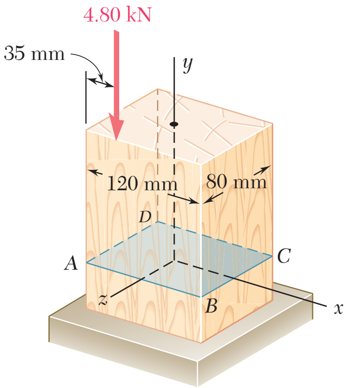


Fig. 11.48

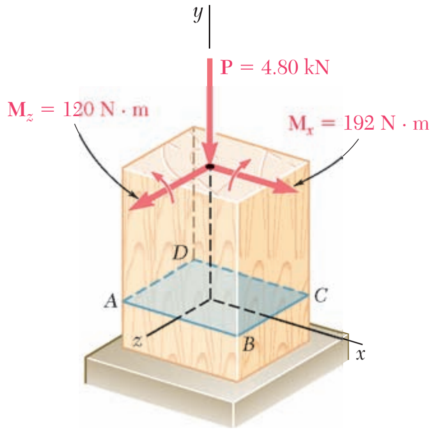


Fig. 11.49

vectors directed along the principal centroidal axes of the section (Fig. 11.49). We have

$$M_x = (4.80 \text{ kN})(40 \text{ mm}) = 192 \text{ N} \cdot \text{m}$$
$$M_z = (4.80 \text{ kN})(60 \text{ mm} - 35 \text{ mm}) = 120 \text{ N} \cdot \text{m}$$

We also compute the area and the centroidal moments of inertia of the cross section:

$$A = (0.080 \text{ m})(0.120 \text{ m}) = 9.60 \times 10^{-3} \text{ m}^2$$
$$I_x = \tfrac{1}{12}(0.120 \text{ m})(0.080 \text{ m})^3 = 5.12 \times 10^{-6} \text{ m}^4$$
$$I_z = \tfrac{1}{12}(0.080 \text{ m})(0.120 \text{ m})^3 = 11.52 \times 10^{-6} \text{ m}^4$$

The stress σ_0 due to the centric load **P** is negative and uniform across the section. We have

$$\sigma_0 = \frac{P}{A} = \frac{-4.80 \text{ kN}}{9.60 \times 10^{-3} \text{ m}^2} = -0.5 \text{ MPa}$$

The stresses due to the bending couples $\mathbf{M}_x$ and $\mathbf{M}_z$ are linearly distributed across the section, with maximum values equal, respectively, to

$$\sigma_1 = \frac{M_x z_{\max}}{I_x} = \frac{(192 \text{ N} \cdot \text{m})(40 \text{ mm})}{5.12 \times 10^{-6} \text{ m}^4} = 1.5 \text{ MPa}$$
$$\sigma_2 = \frac{M_z x_{\max}}{I_z} = \frac{(120 \text{ N} \cdot \text{m})(60 \text{ mm})}{11.52 \times 10^{-6} \text{ m}^4} = 0.625 \text{ MPa}$$

The stresses at the corners of the section are

$$\sigma_y = \sigma_0 \pm \sigma_1 \pm \sigma_2$$

where the signs must be determined from Fig. 11.49. Noting that the stresses due to $\mathbf{M}_x$ are positive at C and D, and negative at A and B, and that the stresses due to $\mathbf{M}_z$ are positive at B and C, and negative at A and D, we obtain

$$\sigma_A = -0.5 - 1.5 - 0.625 = -2.625 \text{ MPa}$$
$$\sigma_B = -0.5 - 1.5 + 0.625 = -1.375 \text{ MPa}$$
$$\sigma_C = -0.5 + 1.5 + 0.625 = +1.625 \text{ MPa}$$
$$\sigma_D = -0.5 + 1.5 - 0.625 = +0.375 \text{ MPa}$$

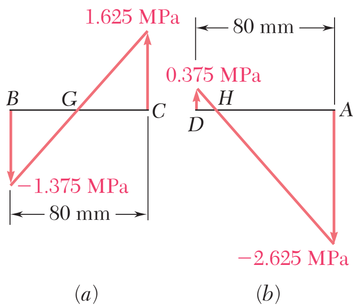


Fig. 11.50

(b) Neutral Axis. We note that the stress will be zero at a point G between B and C, and at a point H between D and A (Fig. 11.50). Since the stress distribution is linear, we write

$$\frac{BG}{80 \text{ mm}} = \frac{1.375}{1.625 + 1.375} \qquad BG = 36.7 \text{ mm}$$
$$\frac{HA}{80 \text{ mm}} = \frac{2.625}{2.625 + 0.375} \qquad HA = 70 \text{ mm}$$

The neutral axis can be drawn through points G and H (Fig. 11.51).

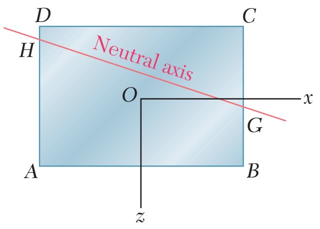


Fig. 11.51

The distribution of the stresses across the section is shown in Fig. 11.52. ■

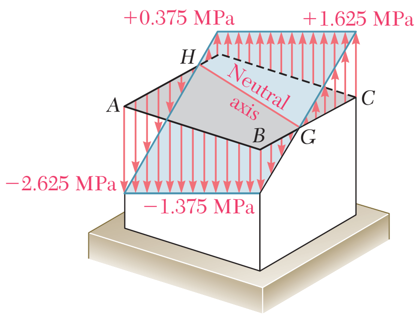
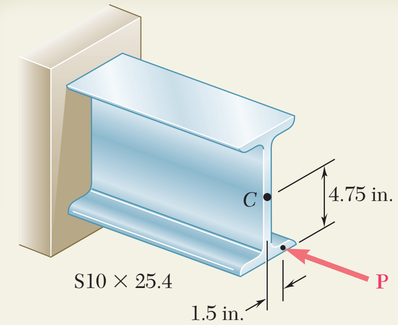
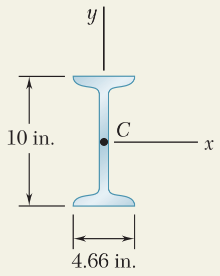


Fig. 11.52



SAMPLE PROBLEM 11.6

A horizontal load **P** is applied as shown to a short section of an S10 × 25.4 rolled-steel member. Knowing that the compressive stress in the member is not to exceed 12 ksi, determine the largest permissible load **P**.



SOLUTION

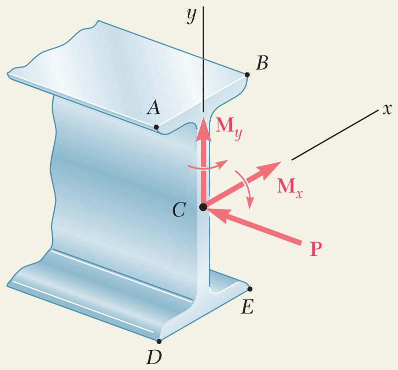
Properties of Cross Section. The following data are taken from Appendix B.

$$\textit{Area: } A = 7.45 \text{ in}^2$$
$$\textit{Section moduli: } S_x = 24.6 \text{ in}^3 \qquad S_y = 2.89 \text{ in}^3$$

Force and Couple at C. We replace **P** by an equivalent force-couple system at the centroid C of the cross section.

$$M_x = (4.75 \text{ in.})P \qquad M_y = (1.5 \text{ in.})P$$

Note that the couple vectors $\mathbf{M}_x$ and $\mathbf{M}_y$ are directed along the principal axes of the cross section.



Normal Stresses. The absolute values of the stresses at points A, B, D, and E due, respectively, to the centric load **P** and to the couples $\mathbf{M}_x$ and $\mathbf{M}_y$ are

$$\sigma_1 = \frac{P}{A} = \frac{P}{7.45 \text{ in}^2} = 0.1342P$$
$$\sigma_2 = \frac{M_x}{S_x} = \frac{4.75P}{24.6 \text{ in}^3} = 0.1931P$$
$$\sigma_3 = \frac{M_y}{S_y} = \frac{1.5P}{2.89 \text{ in}^3} = 0.5190P$$

Superposition. The total stress at each point is found by superposing the stresses due to **P**, $\mathbf{M}_x$, and $\mathbf{M}_y$. We determine the sign of each stress by carefully examining the sketch of the force-couple system.

$$\sigma_A = -\sigma_1 + \sigma_2 + \sigma_3 = -0.1342P + 0.1931P + 0.5190P = +0.578P$$
$$\sigma_B = -\sigma_1 + \sigma_2 - \sigma_3 = -0.1342P + 0.1931P - 0.5190P = -0.460P$$
$$\sigma_D = -\sigma_1 - \sigma_2 + \sigma_3 = -0.1342P - 0.1931P + 0.5190P = +0.192P$$
$$\sigma_E = -\sigma_1 - \sigma_2 - \sigma_3 = -0.1342P - 0.1931P - 0.5190P = -0.846P$$

Largest Permissible Load. The maximum compressive stress occurs at point E. Recalling that $\sigma_{\text{all}} = -12$ ksi, we write

$$\sigma_{\text{all}} = \sigma_E \qquad -12 \text{ ksi} = -0.846P \qquad P = 14.18 \text{ kips} \quad ◀$$

PROBLEMS

11.73 through 11.78 The couple **M** is applied to a beam of the cross section shown in a plane forming an angle β with the vertical. Determine the stress at (*a*) point *A*, (*b*) point *B*, (*c*) point *D*.

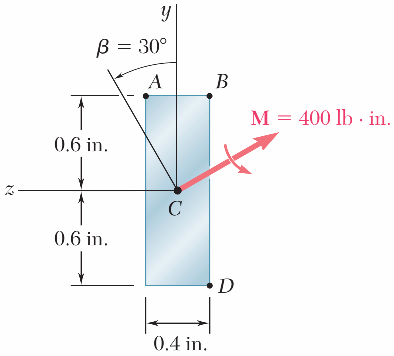


Fig. P11.73

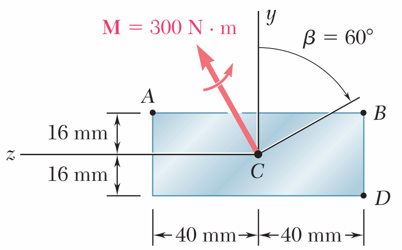


Fig. P11.74

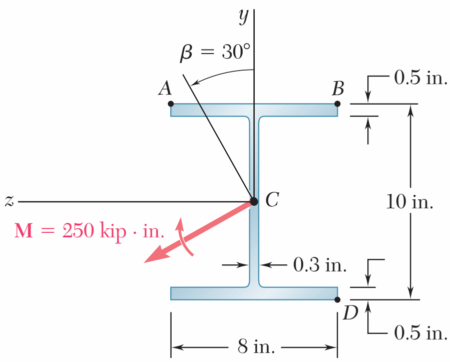


Fig. P11.75

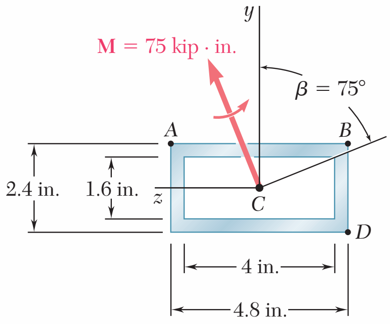


Fig. P11.76

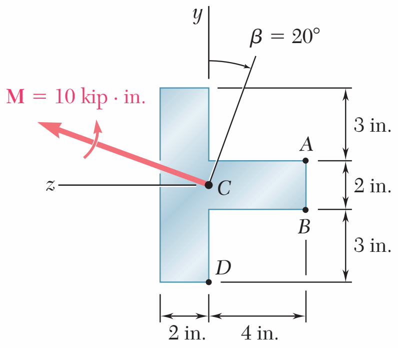


Fig. *P11.77*

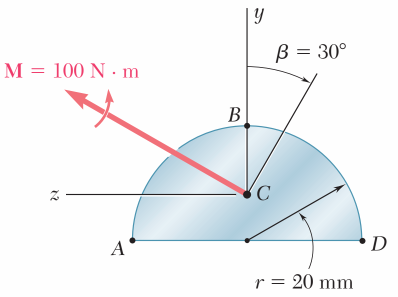


Fig. P11.78

11.79 through 11.84 The couple **M** acts in a vertical plane and is applied to a beam oriented as shown. Determine (*a*) the angle that the neutral axis forms with the horizontal plane, (*b*) the maximum tensile stress in the beam.

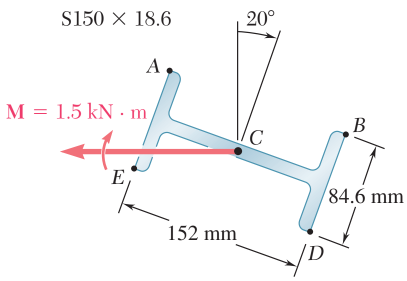


Fig. *P11.79*

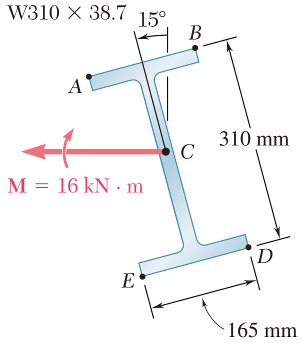
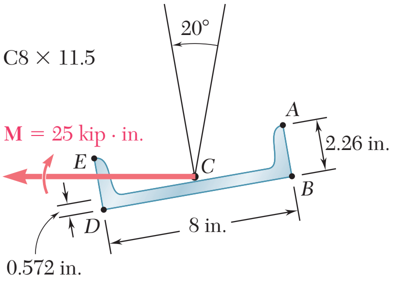
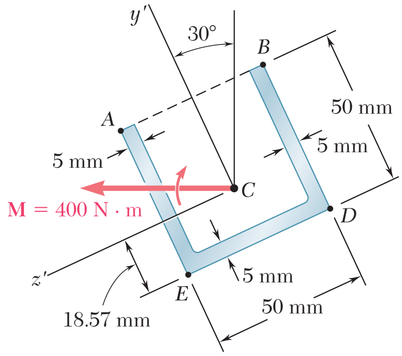


Fig. P11.80

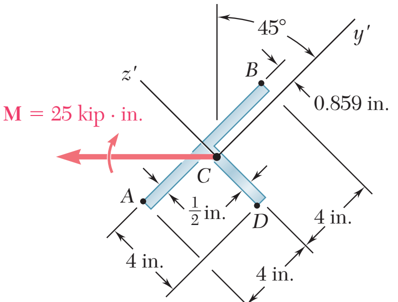




$I_{y'} = 281 \times 10^3 \text{ mm}^4$
$I_{z'} = 176.9 \times 10^3 \text{ mm}^4$

Fig. P11.81

Fig. P11.82



$I_{y'} = 6.74 \text{ in}^4$
$I_{z'} = 21.4 \text{ in}^4$

Fig. P11.83

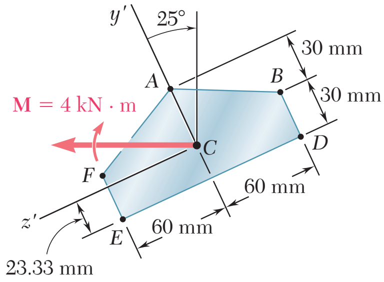


Fig. P11.84

11.85 For the loading shown, determine (*a*) the stress at points *A* and *B*, (*b*) the point where the neutral axis intersects line *ABD*.

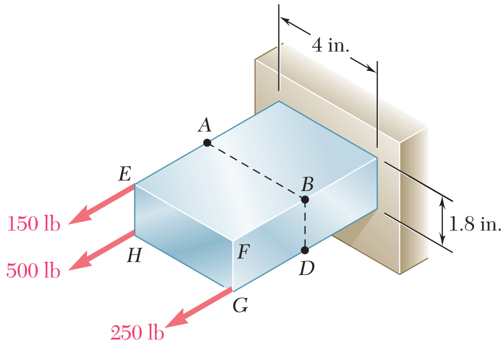


Fig. P11.85

11.86 Solve Prob. 11.85 assuming that the magnitude of the force applied at *G* is increased from 250 lb to 400 lb.

11.87 The tube shown has a uniform wall thickness of 0.5 in. For the given loading, determine (*a*) the stress at points *A* and *B*, (*b*) the point where the neutral axis intersects line *ABD*.

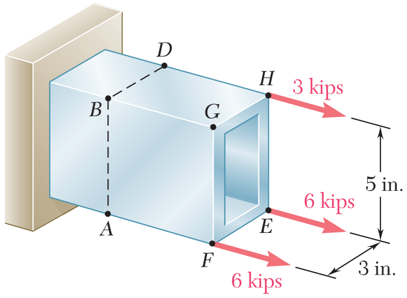


Fig. P11.87

11.88 Solve Prob. 11.87 assuming that the 6-kip force at point *E* is removed.

11.89 An axial load **P** of magnitude 50 kN is applied as shown to a short section of a W150 × 24 rolled-steel member. Determine the largest distance *a* for which the maximum compressive stress does not exceed 90 MPa.

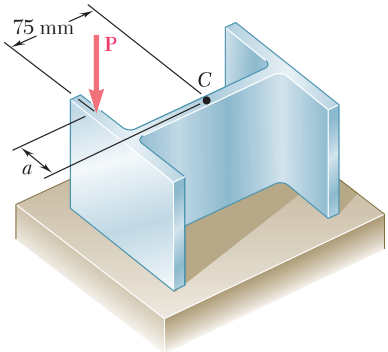


Fig. P11.89

11.90 An axial load **P** of magnitude 30 kN is applied as shown to a short section of a C150 × 12.2 rolled-steel channel. Determine the largest distance a for which the maximum compressive stress does not exceed 60 MPa.

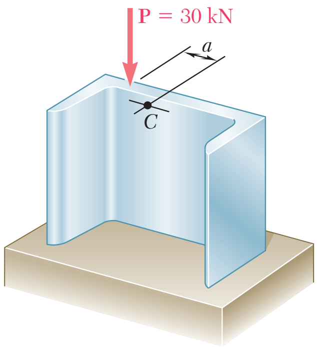


Fig. *P11.90*

11.91 A horizontal load **P** is applied to the beam shown. Knowing that $a = 20$ mm and that the tensile stress in the beam is not to exceed 75 MPa, determine the largest permissible load **P**.

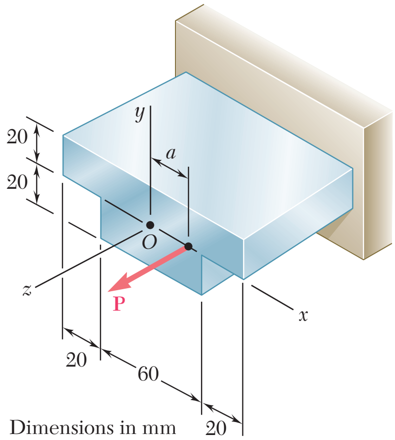


Fig. P11.91 and *P11.92*

11.92 A horizontal load **P** of magnitude 100 kN is applied to the beam shown. Determine the largest distance a for which the maximum tensile stress in the beam does not exceed 75 MPa.

REVIEW AND SUMMARY

This chapter was devoted to the analysis of members in *pure bending*. That is, we considered the stresses and deformation in members subjected to equal and opposite couples **M** and **M**′ acting in the same longitudinal plane (Fig. 11.53).

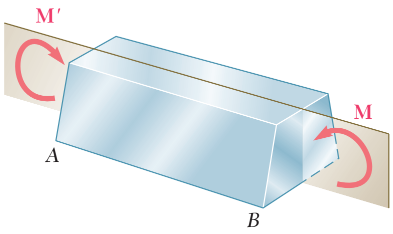


Fig. 11.53

We first studied members possessing a plane of symmetry and subjected to couples acting in that plane. Considering possible deformations of the member, we proved that *transverse sections remain plane* as a member is deformed [Sec. 11.3]. We then noted that a member in pure bending has a *neutral surface* along which normal strains and stresses are zero and that the longitudinal *normal strain* ϵ_x *varies linearly* with the distance y from the neutral surface:

Normal strain in bending

$$\epsilon_x = -\frac{y}{\rho} \tag{11.8}$$

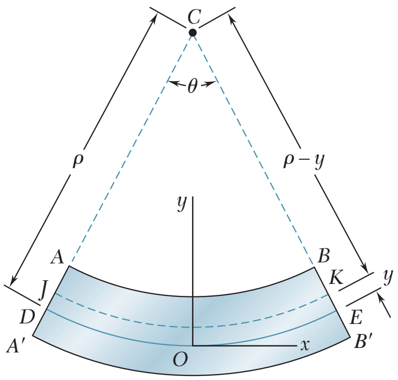


Fig. 11.54

where ρ is the *radius of curvature* of the neutral surface (Fig. 11.54). The intersection of the neutral surface with a transverse section is known as the *neutral axis* of the section.

Normal stress in elastic range

For members made of a material that follows Hooke's law [Sec. 11.4], we found that the *normal stress* σ_x *varies linearly* with the distance from the neutral axis (Fig. 11.55). Denoting by σ_m the maximum stress, we wrote

$$\sigma_x = -\frac{y}{c}\sigma_m \tag{11.12}$$

where c is the largest distance from the neutral axis to a point in the section.

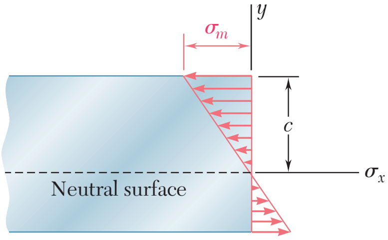


Fig. 11.55

Elastic flexure formula

By setting the sum of the elementary forces, $\sigma_x dA$, equal to zero, we proved that the *neutral axis passes through the centroid* of the cross section of a member in pure bending. Then by setting the sum of the moments of the elementary forces equal to the bending moment, we derived the *elastic flexure formula* for the maximum normal stress

$$\sigma_m = \frac{Mc}{I} \tag{11.15}$$

where I is the moment of inertia of the cross section with respect to the neutral axis. We also obtained the normal stress at any distance y from the neutral axis:

$$\sigma_x = -\frac{My}{I} \tag{11.16}$$

Noting that I and c depend only on the geometry of the cross section, we introduced the *elastic section modulus*

Elastic section modulus

$$S = \frac{I}{c} \tag{11.17}$$

and then used the section modulus to write an alternative expression for the maximum normal stress:

$$\sigma_m = \frac{M}{S} \tag{11.18}$$

Curvature of member

Recalling that the curvature of a member is the reciprocal of its radius of curvature, we expressed the *curvature* of the member as

$$\frac{1}{\rho} = \frac{M}{EI} \tag{11.21}$$

Members made of several materials

Next we considered the bending of members made of several materials with *different moduli of elasticity* [Sec. 11.5]. While transverse sections remain plane, we found that, in general, the *neutral axis does not pass through the centroid* of the composite cross section (Fig. 11.56). Using the ratio of the moduli of elasticity of the materials,

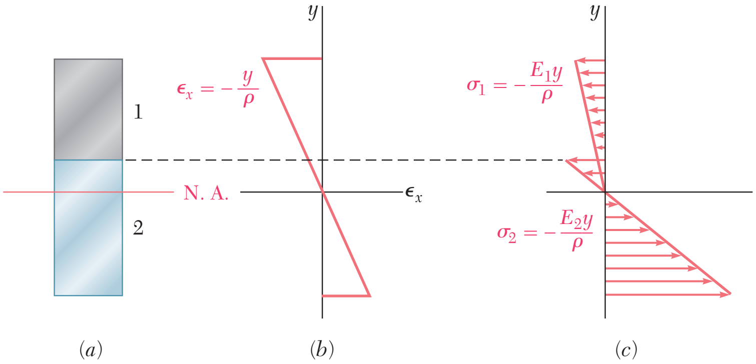


Fig. 11.56

we obtained a *transformed section* corresponding to an equivalent member made entirely of one material. We then used the methods previously developed to determine the stresses in this equivalent homogeneous member (Fig. 11.57) and then again used the ratio of the moduli of elasticity to determine the stresses in the composite beam [Sample Probs. 11.3 and 11.4].

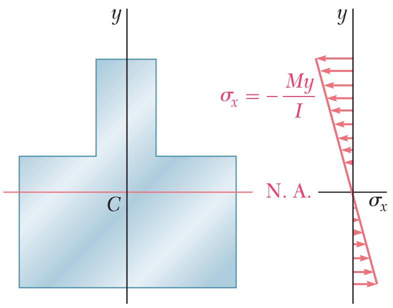


Fig. 11.57

Eccentric axial loading

In Sec. 11.6, we studied the stresses in members loaded *eccentrically in a plane of symmetry*. Our analysis made use of methods developed earlier. We replaced the *eccentric load* by a force-couple system located at the centroid of the cross section (Fig. 11.58) and then superposed stresses due to the centric load and the bending couple (Fig. 11.59):

$$\sigma_x = \frac{P}{A} - \frac{My}{I} \tag{11.28}$$

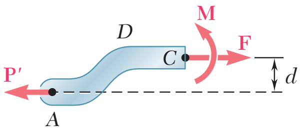


Fig. 11.58

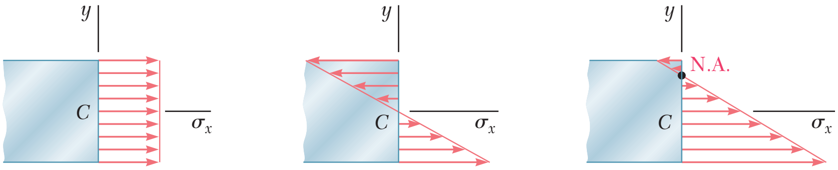


Fig. 11.59

Unsymmetric bending

The bending of members of *unsymmetric cross section* was considered next [Sec. 11.7]. We found that the flexure formula may be used, provided that the couple vector **M** is directed along one of the principal centroidal axes of the cross section. When necessary we resolved **M** into components along the principal axes and superposed the stresses due to the component couples (Figs. 11.60 and 11.61).

$$\sigma_x = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y} \tag{11.33}$$

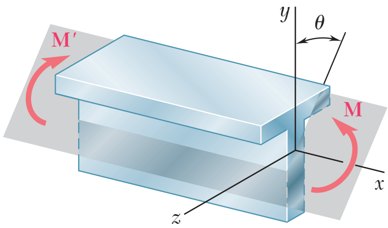


Fig. 11.60

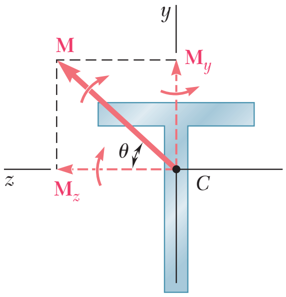


Fig. 11.61

For the couple **M** shown in Fig. 11.62, we determined the orientation of the neutral axis by writing

$$\tan \phi = \frac{I_z}{I_y} \tan \theta \tag{11.35}$$

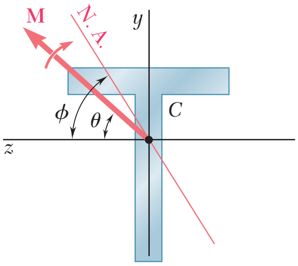


Fig. 11.62

General eccentric axial loading

The general case of *eccentric axial loading* was considered in Sec. 11.8, where we again replaced the load by a force-couple system located at the centroid. We then superposed the stresses due to the centric load and two component couples directed along the principal axes:

$$\sigma_x = \frac{P}{A} - \frac{M_z y}{I_z} + \frac{M_y z}{I_y} \tag{11.36}$$

REVIEW PROBLEMS

11.93 Knowing that the hollow beam shown has a uniform wall thickness of 0.25 in., determine (*a*) the largest couple that can be applied without exceeding the allowable stress of 20 ksi, (*b*) the corresponding radius of curvature of the beam. Use $E = 10.6 \times 10^6$ psi.

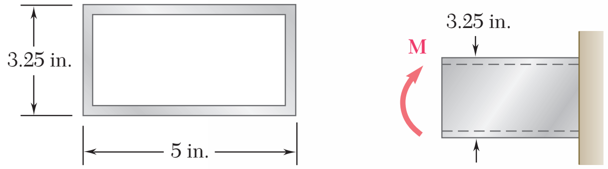


Fig. P11.93

11.94 (*a*) Using an allowable stress of 120 MPa, determine the largest couple **M** that can be applied to a beam of the cross section shown. (*b*) Solve part *a* assuming that the cross section of the beam is an 80-mm square.

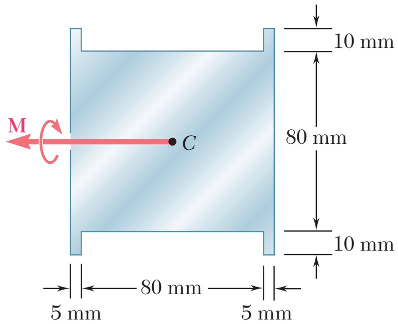


Fig. P11.94

11.95 A steel bar ($E_s = 210$ GPa) and an aluminum bar ($E_a = 70$ GPa) are bonded together to form the composite bar shown. Determine the maximum stress in (*a*) the aluminum and (*b*) the steel when the bar is bent about a horizontal axis with $M = 60$ N · m.

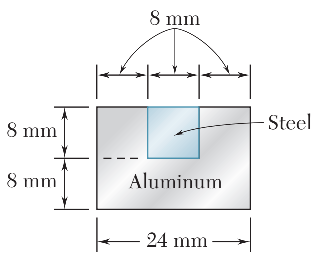


Fig. *P11.95*

11.96 A single vertical force **P** is applied to a short steel post as shown. Gages located at A, B, and C indicate the following strains: $\epsilon_A = -500\ \mu$, $\epsilon_B = -1000\ \mu$, and $\epsilon_C = -200\ \mu$. Knowing that $E = 29 \times 10^6$ psi, determine (*a*) the magnitude of **P**, (*b*) the line of action of **P**, (*c*) the corresponding strain at the hidden edge of the post, where $x = -2.5$ in. and $z = -1.5$ in.

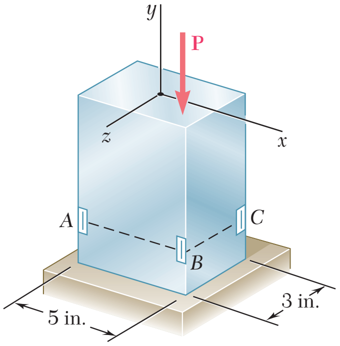


Fig. P11.96

11.97 Two vertical forces are applied to a beam of the cross section shown. Determine the maximum tensile and compressive stresses in portion BC of the beam.

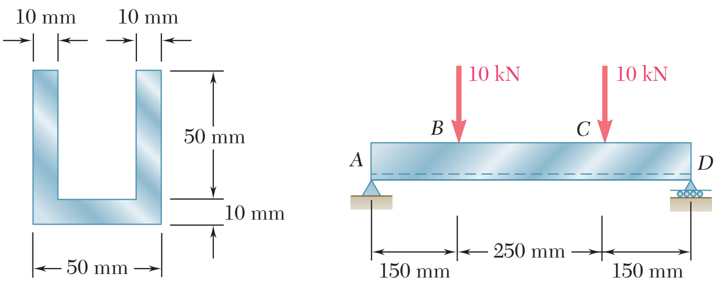


Fig. P11.97

11.98 In order to increase corrosion resistance, a 0.08-in.-thick cladding of aluminum has been added to a steel bar as shown. The modulus of elasticity is $E = 29 \times 10^6$ psi for steel and $E = 10.4 \times 10^6$ psi for aluminum. For a bending moment of 12 kip · in., determine (*a*) the maximum stress in the steel, (*b*) the maximum stress in the aluminum, (*c*) the radius of curvature of the bar.

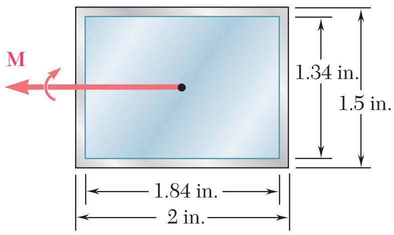


Fig. *P11.98*

11.99 A 6 × 10-in. timber beam has been strengthened by bolting to it the steel straps shown. The modulus of elasticity is $E = 1.5 \times 10^6$ psi for the wood and $E = 30 \times 10^6$ psi for the steel. Knowing that the beam is bent about a horizontal axis by a couple of moment 200 kip · in., determine the maximum stress in (*a*) the wood, (*b*) the steel.

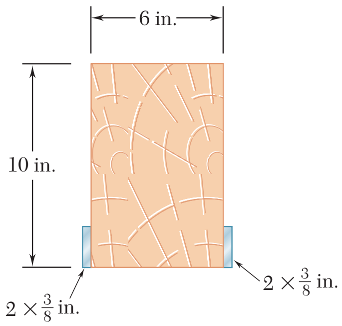


Fig. P11.99

11.100 The four forces shown are applied to a rigid plate supported by a solid steel post of radius a. Determine the maximum stress in the post when (*a*) all four forces are applied, (*b*) the force at D is removed, (*c*) the forces at C and D are removed.

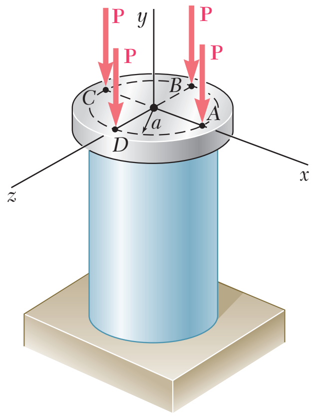


Fig. *P11.100*

11.101 A couple **M** of moment 8 kN · m acting in a vertical plane is applied to a W200 × 19.3 rolled-steel beam as shown. Determine (*a*) the angle that the neutral axis forms with the horizontal plane, (*b*) the maximum stress in the beam.

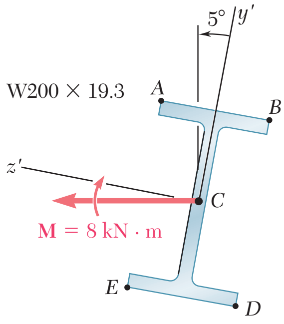


Fig. P11.101

11.102 The couple **M**, which acts in a vertical plane ($\beta = 0$), is applied to an aluminum beam of the cross section shown. Determine (*a*) the stress at point *A*, (*b*) the stress at point *B*, (*c*) the radius of curvature of the beam. Use $E = 72$ GPa.

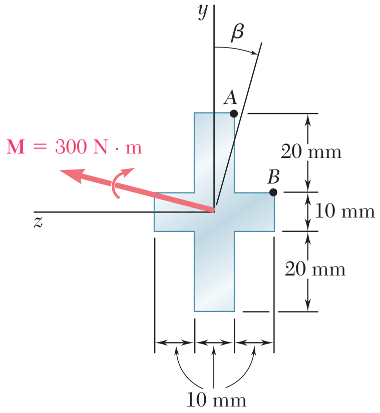


Fig. P11.102 and *P11.103*

11.103 The couple **M** is applied to a beam of the cross section shown in a plane forming an angle $\beta = 15°$ with the vertical. Determine (*a*) the stress at point *A*, (*b*) the stress at point *B*, (*c*) the angle that the neutral axis forms with the horizontal.

11.104 A couple **M** will be applied to a beam of rectangular cross section that is to be sawed from a log of circular cross section. Determine the ratio d/b for which (*a*) the maximum stress σ_m will be as small as possible, (*b*) the radius of curvature of the beam will be maximum.

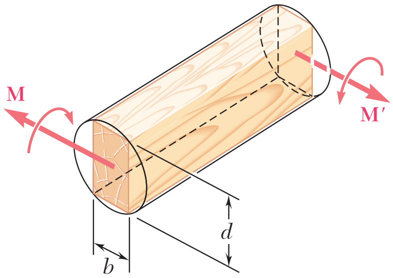
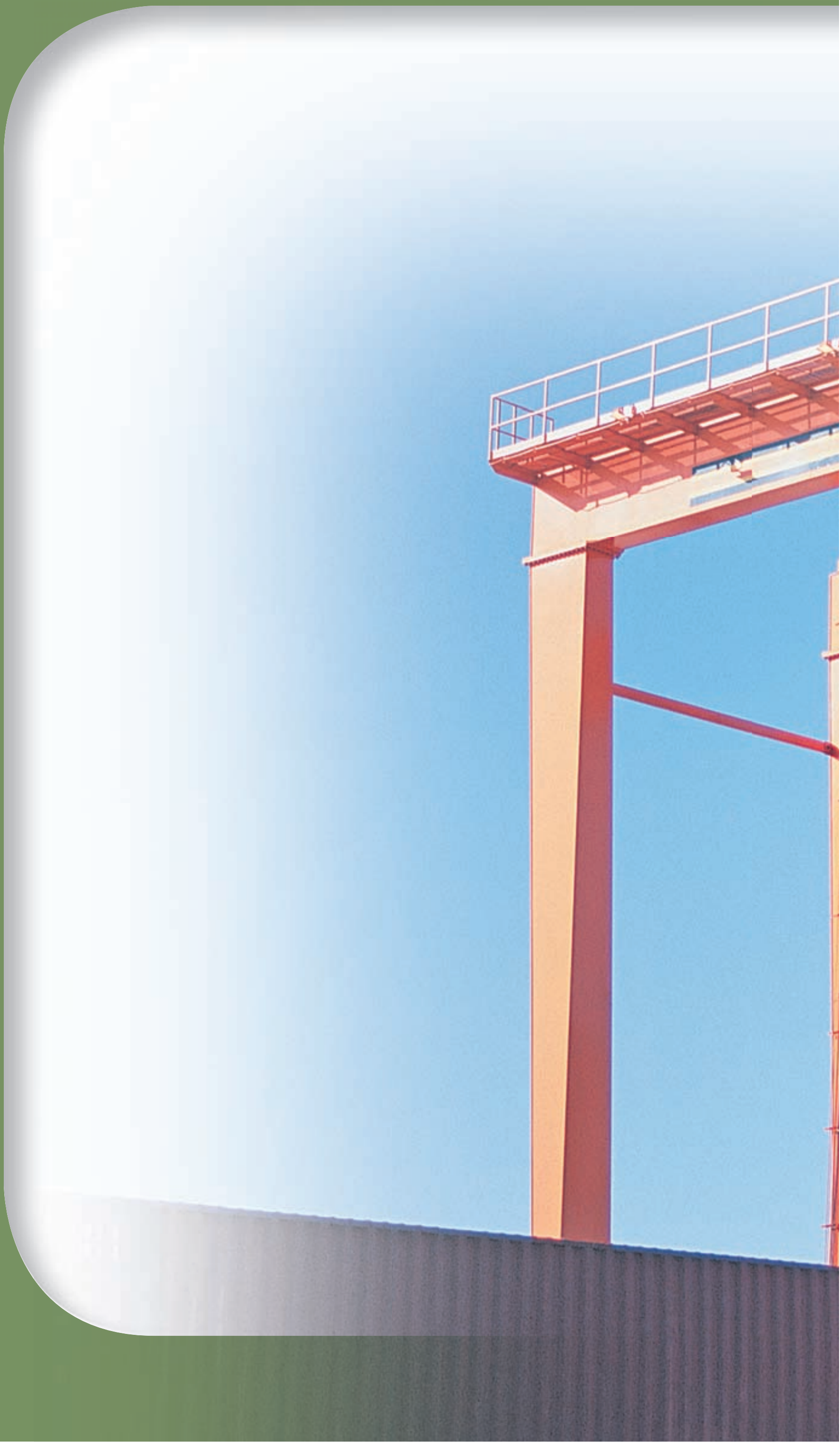


Fig. P11.104

The beams supporting the multiple overhead cranes system shown in this picture are subjected to transverse loads causing the beams to bend. The normal stresses resulting from such loadings will be determined in this chapter.



CHAPTER 12

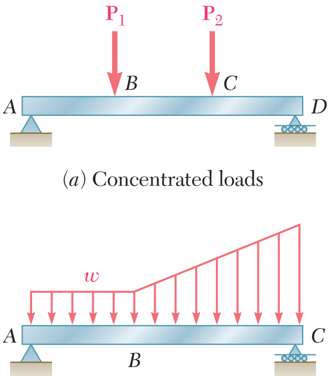
Analysis and Design of Beams for Bending

Chapter 12 Analysis and Design of Beams for Bending

12.1 INTRODUCTION

This chapter and most of the next one will be devoted to the analysis and the design of *beams,* i.e., structural members supporting loads applied at various points along the member. Beams are usually long, straight prismatic members, as shown in the photo on the previous page. Steel and aluminum beams play an important part in both structural and mechanical engineering. Timber beams are widely used in home construction (Photo 12.1). In most cases, the loads are perpendicular to the axis of the beam. Such a *transverse loading* causes only bending and shear in the beam. When the loads are not at a right angle to the beam, they also produce axial forces in the beam.

Photo 12.1



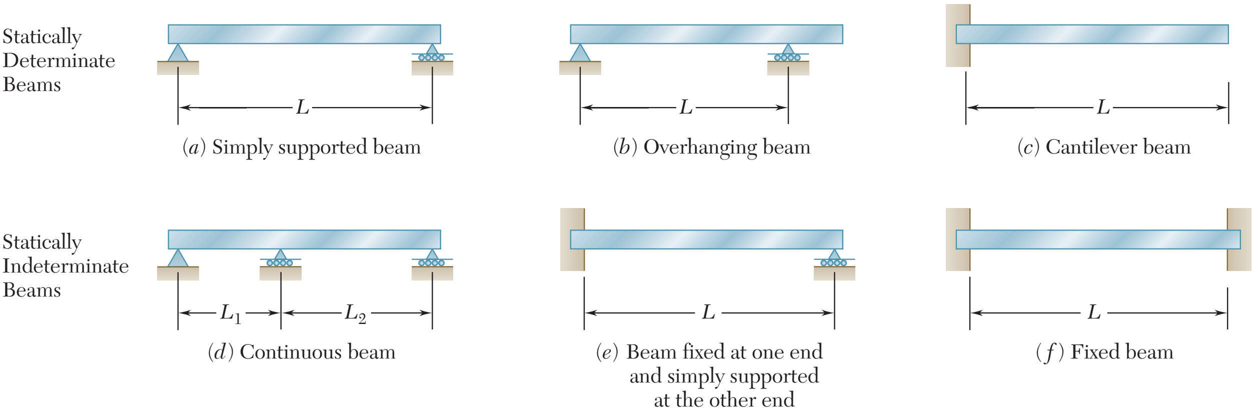
(*a*) Concentrated loads

(*b*) Distributed load

Fig. 12.1

The transverse loading of a beam may consist of *concentrated loads* $\mathbf{P}_1$, $\mathbf{P}_2$, . . . , expressed in newtons, pounds, or their multiples, kilonewtons and kips (Fig. 12.1*a*), of a *distributed load* w, expressed in N/m, kN/m, lb/ft, or kips/ft (Fig. 12.1*b*), or of a combination of both. When the load w per unit length has a constant value over part of the beam (as between A and B in Fig. 12.1*b*), the load is said to be *uniformly distributed* over that part of the beam.

Beams are classified according to the way in which they are supported. Several types of beams frequently used are shown in Fig. 12.2. The distance L shown in the various parts of the figure is called the *span*. Note that the reactions at the supports of the beams in parts a, b, and c of the



(*a*) Simply supported beam

(*b*) Overhanging beam

(*c*) Cantilever beam

(*d*) Continuous beam

(*e*) Beam fixed at one end and simply supported at the other end

(*f*) Fixed beam

Fig. 12.2

figure involve a total of only three unknowns and, therefore, can be determined by the methods of statics. Such beams are said to be *statically determinate* and will be discussed in this chapter and the next. On the other hand, the reactions at the supports of the beams in parts *d*, *e*, and *f* of Fig. 12.2 involve more than three unknowns and cannot be determined by the methods of statics alone. The properties of the beams with regard to their resistance to deformations must be taken into consideration. Such beams are said to be *statically indeterminate* and their analysis will be postponed until Chap. 15, where deformations of beams will be discussed.

Sometimes two or more beams are connected by hinges to form a single continuous structure. Two examples of beams hinged at a point H are shown in Fig. 12.3. It will be noted that the reactions at the supports involve four unknowns and cannot be determined from the free-body diagram of the two-beam system. They can be determined, however, by considering the free-body diagram of each beam separately; six unknowns are involved (including two force components at the hinge), and six equations are available.

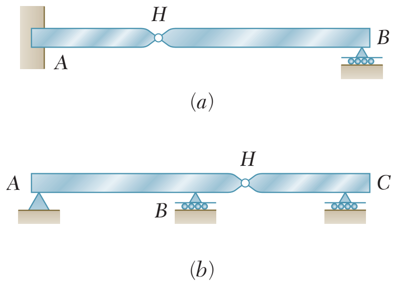


Fig. 12.3

It was shown in Sec. 11.1 that if we pass a section through a point C of a cantilever beam supporting a concentrated load $\mathbf{P}$ at its end (Fig. 11.4), the internal forces in the section are found to consist of a shear force $\mathbf{P}'$ equal and opposite to the load $\mathbf{P}$ and a bending couple $\mathbf{M}$ of moment equal to the moment of $\mathbf{P}$ about C. A similar situation prevails for other types of supports and loadings. Consider, for example, a simply supported beam AB carrying two concentrated loads and a uniformly distributed load (Fig. 12.4*a*). To determine the internal forces in a section through point C, we first draw the free-body diagram of the entire beam to obtain the reactions at the supports (Fig. 12.4*b*). Passing a section through C, we then draw the free-body diagram of AC (Fig. 12.4*c*), from which we determine the shear force $\mathbf{V}$ and the bending couple $\mathbf{M}$.

The bending couple $\mathbf{M}$ creates *normal stresses* in the cross section, while the shear force $\mathbf{V}$ creates *shearing stresses* in that section. In most cases the dominant criterion in the design of a beam for strength is the maximum value of the normal stress in the beam. The determination of the normal stresses in a beam will be the subject of this chapter, while shearing stresses will be discussed in Chap. 13.

Since the distribution of the normal stresses in a given section depends only upon the value of the bending moment M in that section

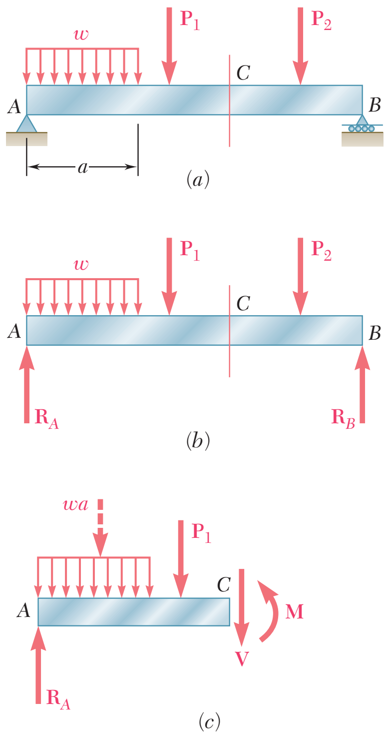


Fig. 12.4

and the geometry of the section,† the elastic flexure formulas derived in Sec. 11.4 can be used to determine the maximum stress, as well as the stress at any given point, in the section. We write‡

$$\sigma_m = \frac{|M|c}{I} \qquad \sigma_x = -\frac{My}{I} \tag{12.1, 12.2}$$

where I is the moment of inertia of the cross section with respect to a centroidal axis perpendicular to the plane of the couple, y is the distance from the neutral surface, and c is the maximum value of that distance (Fig. 11.11). We also recall from Sec. 11.4 that, introducing the elastic section modulus $S = I/c$ of the beam, the maximum value σ_m of the normal stress in the section can be expressed as

$$\sigma_m = \frac{|M|}{S} \tag{12.3}$$

The fact that σ_m is inversely proportional to S underlines the importance of selecting beams with a large section modulus. Section moduli of various rolled-steel shapes are given in App. B, while the section modulus of a rectangular shape can be expressed, as shown in Sec. 11.4, as

$$S = \tfrac{1}{6}bh^2 \tag{12.4}$$

where b and h are, respectively, the width and the depth of the cross section.

Equation (12.3) also shows that, for a beam of uniform cross section, σ_m is proportional to $|M|$: Thus, the maximum value of the normal stress in the beam occurs in the section where $|M|$ is largest. It follows that one of the most important parts of the design of a beam for a given loading condition is the determination of the location and magnitude of the largest bending moment.

This task is made easier if a *bending-moment diagram* is drawn, i.e., if the value of the bending moment M is determined at various points of the beam and plotted against the distance x measured from one end of the beam. It is further facilitated if a *shear diagram* is drawn at the same time by plotting the shear V against x.

The sign convention to be used to record the values of the shear and bending moment will be discussed in Sec. 12.2. The values of V and M will then be obtained at various points of the beam by drawing free-body diagrams of successive portions of the beam. In Sec. 12.3 relations among load, shear, and bending moment will be derived and used to obtain the shear and bending-moment diagrams. This approach facilitates the determination of the largest absolute value of the bending moment and, thus, the determination of the maximum normal stress in the beam.

In Sec. 12.4 you will learn to design a beam for bending, i.e., so that the maximum normal stress in the beam will not exceed its allowable value. As indicated earlier, this is the dominant criterion in the design of a beam.

†It is assumed that the distribution of the normal stresses in a given cross section is not affected by the deformations caused by the shearing stresses.

‡We recall from Sec. 11.2 that M can be positive or negative, depending upon whether the concavity of the beam at the point considered faces upward or downward. Thus, in the case considered here of a transverse loading, the sign of M can vary along the beam. On the other hand, since σ_m is a positive quantity, the absolute value of M is used in Eq. (12.1).

12.2 SHEAR AND BENDING-MOMENT DIAGRAMS

As indicated in Sec. 12.1, the determination of the maximum absolute values of the shear and of the bending moment in a beam are greatly facilitated if V and M are plotted against the distance x measured from one end of the beam. Besides, as you will see in Chap. 15, the knowledge of M as a function of x is essential to the determination of the deflection of a beam.

In the examples and sample problems of this section, the shear and bending-moment diagrams will be obtained by determining the values of V and M at selected points of the beam. These values will be found in the usual way, i.e., by passing a section through the point where they are to be determined (Fig. 12.5*a*) and considering the equilibrium of the portion of beam located on either side of the section (Fig. 12.5*b*). Since the shear forces $\mathbf{V}$ and $\mathbf{V}'$ have opposite senses, recording the shear at point C with an up or down arrow would be meaningless, unless we indicated at the same time which of the free bodies AC and CB we are considering. For this reason, the shear V will be recorded with a sign: a *plus sign* if the shearing forces are directed as shown in Fig. 12.5*b*, and a *minus sign* otherwise. A similar convention will apply for the bending moment M. It will be considered as positive if the bending couples are directed as shown in that figure, and negative otherwise.† Summarizing the sign conventions we have presented, we state:

The shear V and the bending moment M at a given point of a beam are said to be positive when the internal forces and couples acting on each portion of the beam are directed as shown in Fig. 12.6a.

These conventions can be more easily remembered if we note that

1. *The shear at any given point of a beam is positive when the* **external** *forces (loads and reactions) acting on the beam tend to shear off the beam at that point as indicated in Fig. 12.6b.*
2. *The bending moment at any given point of a beam is positive when the* **external** *forces acting on the beam tend to bend the beam at that point as indicated in Fig. 12.6c.*

It is also of help to note that the situation described in Fig. 12.6, in which the values of the shear and of the bending moment are positive, is precisely the situation that occurs in the left half of a simply supported beam carrying a single concentrated load at its midpoint. This particular case is fully discussed in the next example.

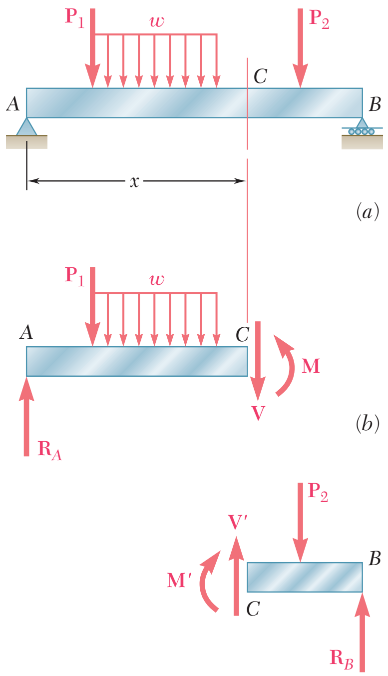
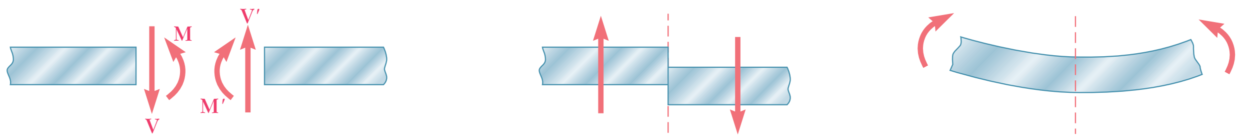


Fig. 12.5



(*a*) Internal forces (positive shear and positive bending moment)

(*b*) Effect of external forces (positive shear)

(*c*) Effect of external forces (positive bending moment)

Fig. 12.6

†Note that this convention is the same that we used earlier in Sec. 11.2.

EXAMPLE 12.1 Draw the shear and bending-moment diagrams for a simply supported beam AB of span L subjected to a single concentrated load **P** at it midpoint C (Fig. 12.7).

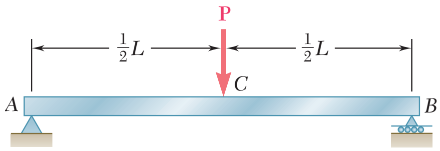


Fig. 12.7

We first determine the reactions at the supports from the free-body diagram of the entire beam (Fig. 12.8a); we find that the magnitude of each reaction is equal to $P/2$.

Next we cut the beam at a point D between A and C and draw the free-body diagrams of AD and DB (Fig. 12.8b). *Assuming that shear and bending moment are positive,* we direct the internal forces **V** and **V′** and the internal couples **M** and **M′** as indicated in Fig. 12.6a. Considering the free body AD and writing that the sum of the vertical components and the sum of the moments about D of the forces acting on the free body are zero, we find $V = +P/2$ and $M = +Px/2$. Both the shear and the bending moment are therefore positive; this may be checked by observing that the reaction at A tends to shear off and to bend the beam at D as indicated in Figs. 12.6b and c. We now plot V and M between A and C (Figs. 12.8d and e); the shear has a constant value $V = P/2$, while the bending moment increases linearly from $M = 0$ at $x = 0$ to $M = PL/4$ at $x = L/2$.

Cutting, now, the beam at a point E between C and B and considering the free body EB (Fig. 12.8c), we write that the sum of the vertical components and the sum of the moments about E of the forces acting on the free body are zero. We obtain $V = -P/2$ and $M = P(L - x)/2$. The shear is therefore negative and the bending moment positive; this can be checked by observing that the reaction at B bends the beam at E as indicated in Fig. 12.6c but tends to shear it off in a manner opposite to that shown in Fig. 12.6b. We can complete, now, the shear and bending-moment diagrams of Figs. 12.8d and e; the shear has a constant value $V = -P/2$ between C and B, while the bending moment decreases linearly from $M = PL/4$ at $x = L/2$ to $M = 0$ at $x = L$. ■

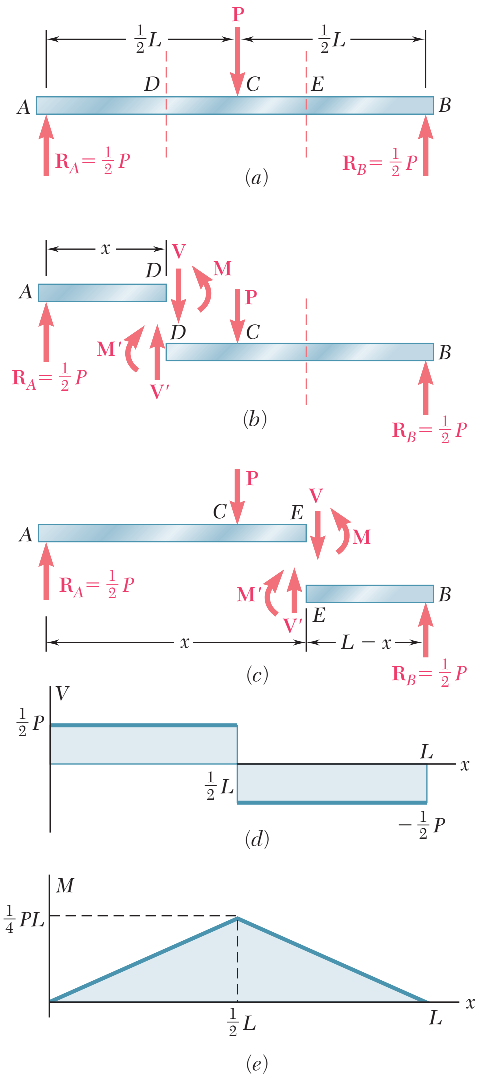


Fig. 12.8

We note from the foregoing example that, when a beam is subjected only to concentrated loads, the shear is constant between loads and the bending moment varies linearly between loads. In such situations, therefore, the shear and bending-moment diagrams can easily be drawn, once the values of V and M have been obtained at sections selected just to the left and just to the right of the points where the loads and reactions are applied (see Sample Prob. 12.1).

EXAMPLE 12.2 Draw the shear and bending-moment diagrams for a cantilever beam AB of span L supporting a uniformly distributed load w (Fig. 12.9).

We cut the beam at a point C between A and B and draw the free-body diagram of AC (Fig. 12.10a), directing **V** and **M** as indicated in Fig. 12.6a. Denoting by x the distance from A to C and replacing the distributed load over AC by its resultant wx applied at the midpoint of AC, we write

$$+\uparrow \Sigma F_y = 0: \qquad -wx - V = 0 \qquad V = -wx$$

$$+\circlearrowleft \Sigma M_C = 0: \qquad wx\left(\frac{x}{2}\right) + M = 0 \qquad M = -\frac{1}{2}wx^2$$

We note that the shear diagram is represented by an oblique straight line (Fig. 12.10b) and the bending-moment diagram by a parabola (Fig. 12.10c). The maximum values of V and M both occur at B, where we have

$$V_B = -wL \qquad M_B = -\tfrac{1}{2}wL^2 \;\blacksquare$$

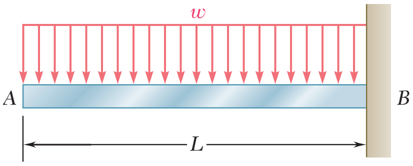


Fig. 12.9

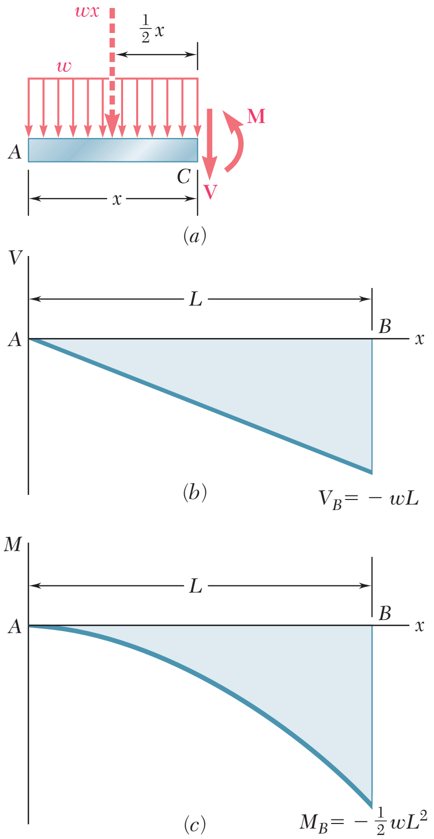
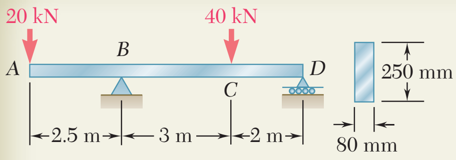
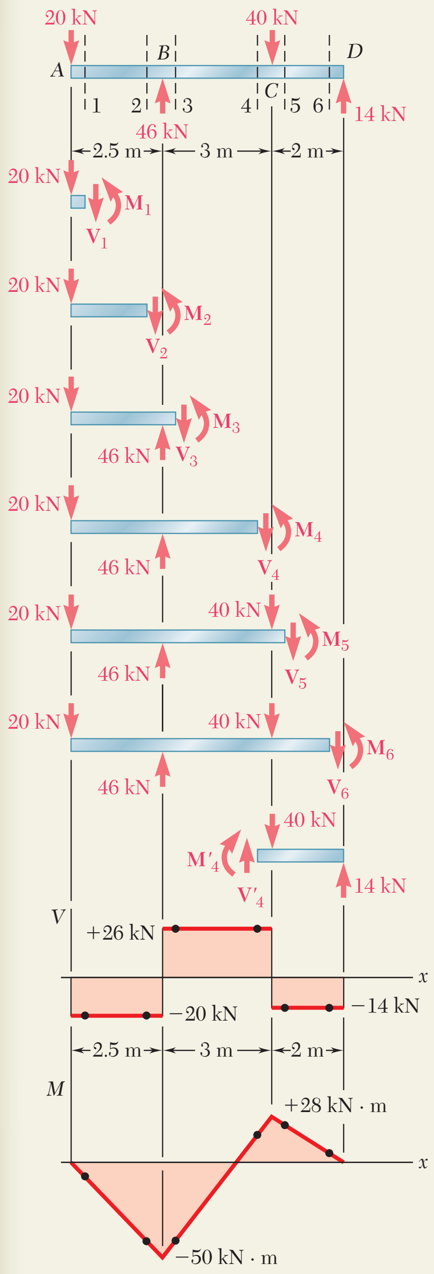


Fig. 12.10



SAMPLE PROBLEM 12.1

For the timber beam and loading shown, draw the shear and bending-moment diagrams and determine the maximum normal stress due to bending.



SOLUTION

Reactions. Considering the entire beam as a free body, we find

$$\mathbf{R}_B = 46 \text{ kN} \uparrow \qquad \mathbf{R}_D = 14 \text{ kN} \uparrow$$

Shear and Bending-Moment Diagrams. We first determine the internal forces just to the right of the 20-kN load at A. Considering the stub of beam to the left of section *1* as a free body and assuming V and M to be positive (according to the standard convention), we write

$$+\uparrow \Sigma F_y = 0: \qquad -20 \text{ kN} - V_1 = 0 \qquad V_1 = -20 \text{ kN}$$
$$+\circlearrowleft \Sigma M_1 = 0: \qquad (20 \text{ kN})(0 \text{ m}) + M_1 = 0 \qquad M_1 = 0$$

We next consider as a free body the portion of beam to the left of section *2* and write

$$+\uparrow \Sigma F_y = 0: \qquad -20 \text{ kN} - V_2 = 0 \qquad V_2 = -20 \text{ kN}$$
$$+\circlearrowleft \Sigma M_2 = 0: \qquad (20 \text{ kN})(2.5 \text{ m}) + M_2 = 0 \qquad M_2 = -50 \text{ kN} \cdot \text{m}$$

The shear and bending moment at sections *3*, *4*, *5*, and *6* are determined in a similar way from the free-body diagrams shown. We obtain

$$V_3 = +26 \text{ kN} \qquad M_3 = -50 \text{ kN} \cdot \text{m}$$
$$V_4 = +26 \text{ kN} \qquad M_4 = +28 \text{ kN} \cdot \text{m}$$
$$V_5 = -14 \text{ kN} \qquad M_5 = +28 \text{ kN} \cdot \text{m}$$
$$V_6 = -14 \text{ kN} \qquad M_6 = 0$$

For several of the latter sections, the results may be more easily obtained by considering as a free body the portion of the beam to the right of the section. For example, for the portion of the beam to the right of section *4*, we have

$$+\uparrow \Sigma F_y = 0: \qquad V_4 - 40 \text{ kN} + 14 \text{ kN} = 0 \qquad V_4 = +26 \text{ kN}$$
$$+\circlearrowleft \Sigma M_4 = 0: \qquad -M_4 + (14 \text{ kN})(2 \text{ m}) = 0 \qquad M_4 = +28 \text{ kN} \cdot \text{m}$$

We can now plot the six points shown on the shear and bending-moment diagrams. As indicated earlier in this section, the shear is of constant value between concentrated loads, and the bending moment varies linearly; we obtain therefore the shear and bending-moment diagrams shown.

Maximum Normal Stress. It occurs at B, where $|M|$ is largest. We use Eq. (12.4) to determine the section modulus of the beam:

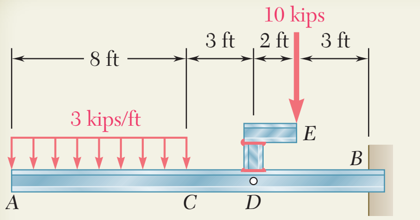
$$S = \tfrac{1}{6}bh^2 = \tfrac{1}{6}(0.080 \text{ m})(0.250 \text{ m})^2 = 833.33 \times 10^{-6} \text{ m}^3$$

Substituting this value and $|M| = |M_B| = 50 \times 10^3 \text{ N} \cdot \text{m}$ into Eq. (12.3):

$$\sigma_m = \frac{|M_B|}{S} = \frac{(50 \times 10^3 \text{ N} \cdot \text{m})}{833.33 \times 10^{-6}} = 60.00 \times 10^6 \text{ Pa}$$

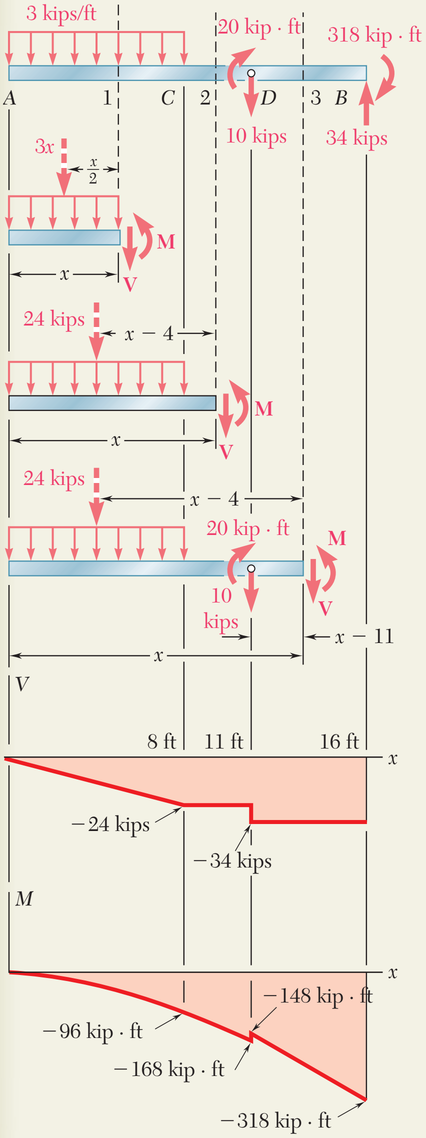
Maximum normal stress in the beam $= 60.0$ MPa ◀

SAMPLE PROBLEM 12.2



The structure shown consists of a W10 × 112 rolled-steel beam *AB* and of two short members welded together and to the beam. (*a*) Draw the shear and bending-moment diagrams for the beam and the given loading. (*b*) Determine the maximum normal stress in sections just to the left and just to the right of point *D*.

SOLUTION



Equivalent Loading of Beam. The 10-kip load is replaced by an equivalent force-couple system at *D*. The reaction at *B* is determined by considering the beam as a free body.

a. Shear and Bending-Moment Diagrams

From A to C. We determine the internal forces at a distance x from point *A* by considering the portion of beam to the left of section *1*. That part of the distributed load acting on the free body is replaced by its resultant, and we write

$$+\uparrow \Sigma F_y = 0: \qquad -3x - V = 0 \qquad V = -3x \text{ kips}$$

$$+\circlearrowleft \Sigma M_1 = 0: \qquad 3x(\tfrac{1}{2}x) + M = 0 \qquad M = -1.5x^2 \text{ kip} \cdot \text{ft}$$

Since the free-body diagram shown can be used for all values of x smaller than 8 ft, the expressions obtained for V and M are valid in the region $0 < x < 8$ ft.

From C to D. Considering the portion of beam to the left of section 2 and again replacing the distributed load by its resultant, we obtain

$$+\uparrow \Sigma F_y = 0: \qquad -24 - V = 0 \qquad V = -24 \text{ kips}$$

$$+\circlearrowleft \Sigma M_2 = 0: \qquad 24(x - 4) + M = 0 \qquad M = 96 - 24x \qquad \text{kip} \cdot \text{ft}$$

These expressions are valid in the region 8 ft $< x <$ 11 ft.

From D to B. Using the position of beam to the left of section 3, we obtain for the region 11 ft $< x <$ 16 ft

$$V = -34 \text{ kips} \qquad M = 226 - 34x \qquad \text{kip} \cdot \text{ft}$$

The shear and bending-moment diagrams for the entire beam can now be plotted. We note that the couple of moment 20 kip · ft applied at point *D* introduces a discontinuity into the bending-moment diagram.

b. Maximum Normal Stress to the Left and Right of Point *D*. From App. B we find that for the W10 × 112 rolled-steel shape, $S = 126$ in^3 about the *X-X* axis.

To the left of D: We have $|M| = 168$ kip · ft = 2016 kip · in. Substituting for $|M|$ and S into Eq. (12.3), we write

$$\sigma_m = \frac{|M|}{S} = \frac{2016 \text{ kip} \cdot \text{in.}}{126 \text{ in}^3} = 16.00 \text{ ksi} \qquad \sigma_m = 16.00 \text{ ksi} \blacktriangleleft$$

To the right of D: We have $|M| = 148$ kip · ft = 1776 kip · in. Substituting for $|M|$ and S into Eq. (12.3), we write

$$\sigma_m = \frac{|M|}{S} = \frac{1776 \text{ kip} \cdot \text{in.}}{126 \text{ in}^3} = 14.10 \text{ ksi} \qquad \sigma_m = 14.10 \text{ ksi} \blacktriangleleft$$

PROBLEMS

12.1 through 12.4 For the beam and loading shown, (*a*) draw the shear and bending-moment diagrams, (*b*) determine the equations of the shear and bending-moment curves.

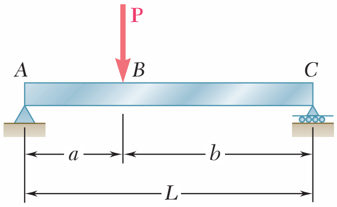


Fig. P12.1

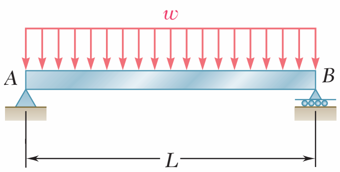


Fig. P12.2

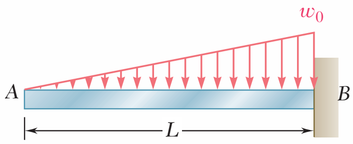


Fig. P12.3

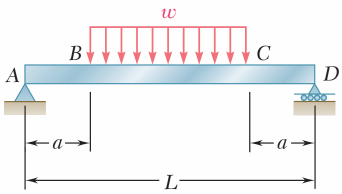


Fig. P12.4

12.5 and *12.6* Draw the shear and bending-moment diagrams for the beam and loading shown, and determine the maximum absolute value (*a*) of the shear, (*b*) of the bending moment.

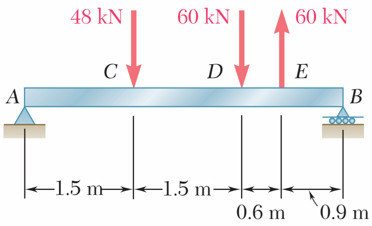


Fig. P12.5

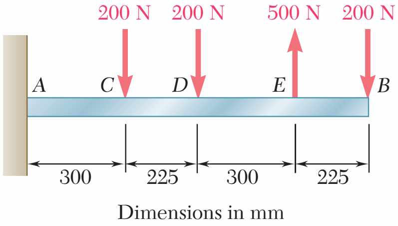


Fig. *P12.6*

12.7 and *12.8* Draw the shear and bending-moment diagrams for the beam and loading shown, and determine the maximum absolute value (*a*) of the shear, (*b*) of the bending moment.

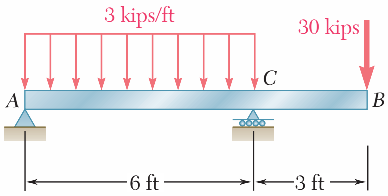


Fig. P12.7

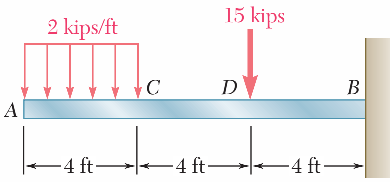


Fig. *P12.8*

12.9 and 12.10 Draw the shear and bending-moment diagrams for the beam and loading shown, and determine the maximum absolute value (*a*) of the shear, (*b*) of the bending moment.

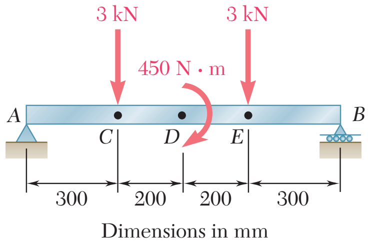


Fig. P12.9

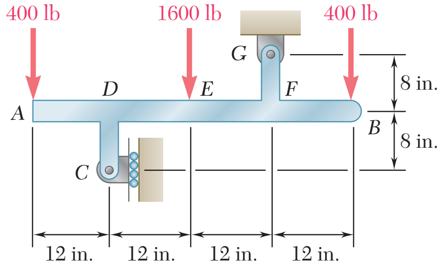


Fig. P12.10

12.11 and 12.12 Assuming the upward reaction of the ground to be uniformly distributed, draw the shear and bending-moment diagrams for the beam *AB* and determine the maximum absolute value (*a*) of the shear, (*b*) of the bending moment.

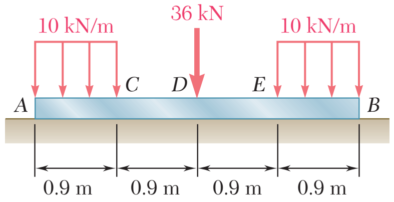


Fig. P12.11

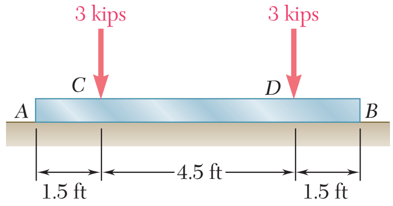


Fig. P12.12

12.13 and 12.14 For the beam and loading shown, determine the maximum normal stress due to bending on a transverse section at *C*.

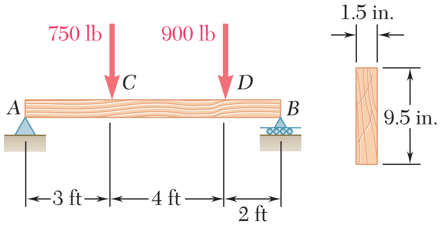


Fig. P12.13

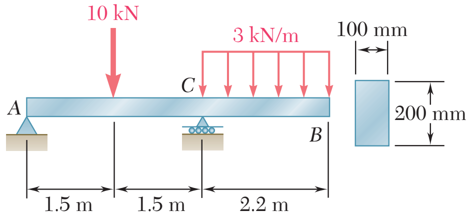


Fig. P12.14

12.15 For the beam and loading shown, determine the maximum normal stress due to bending on section *a-a*.

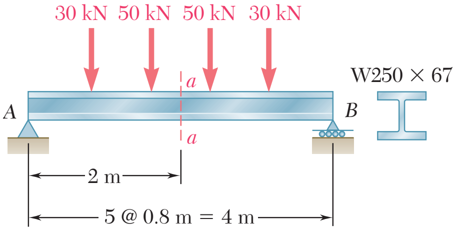


Fig. P12.15

12.16 For the beam and loading shown, determine the maximum normal stress due to bending on a transverse section at C.

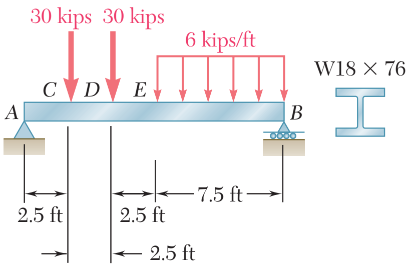


Fig. P12.16

***12.17* and 12.18** For the beam and loading shown, determine the maximum normal stress due to bending on a transverse section at C.

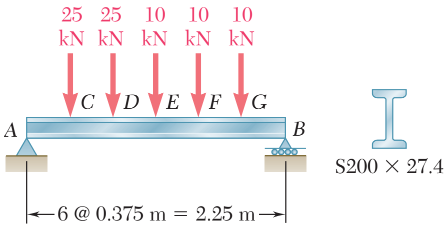


Fig. *P12.17*

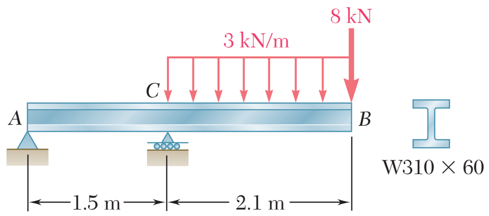


Fig. P12.18

12.19 and 12.20 Draw the shear and bending-moment diagrams for the beam and loading shown, and determine the maximum normal stress due to bending.

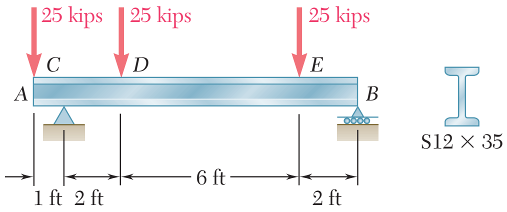


Fig. P12.19

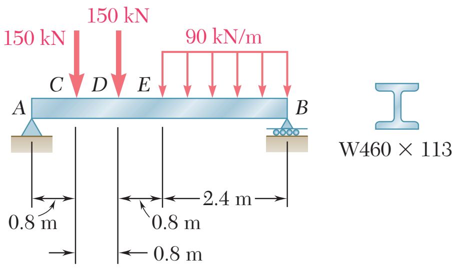


Fig. P12.20

12.21* and *12.22 Draw the shear and bending-moment diagrams for the beam and loading shown, and determine the maximum normal stress due to bending.

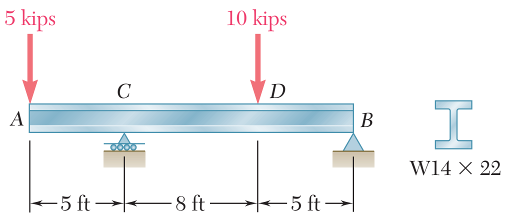


Fig. *P12.21*

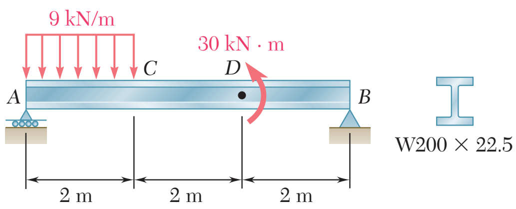


Fig. *P12.22*

12.23 Draw the shear and bending-moment diagrams for the beam and loading shown, and determine the maximum normal stress due to bending.

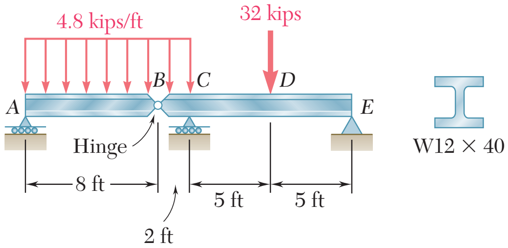


Fig. P12.23

12.24 Knowing that $W = 3$ kips, draw the shear and bending-moment diagrams for beam AB and determine the maximum normal stress due to bending.

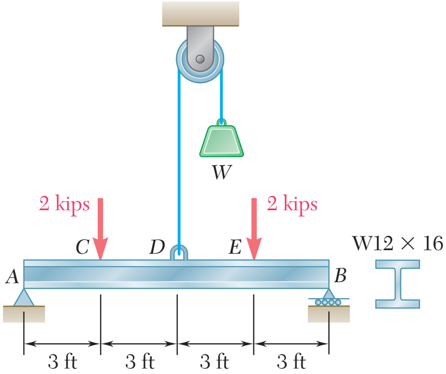


Fig. P12.24

12.25 Determine (*a*) the distance a for which the maximum absolute value of the bending moment in the beam is as small as possible, (*b*) the corresponding maximum normal stress due to bending. (*Hint:* Draw the bending-moment diagram, and equate the absolute values of the largest positive and negative bending moments obtained.)

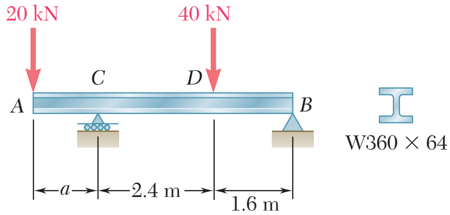


Fig. P12.25

12.26 Determine (*a*) the distance a for which the maximum absolute value of the bending moment in the beam is as small as possible, (*b*) the corresponding maximum normal stress due to bending. (See hint of Prob. 12.25.)

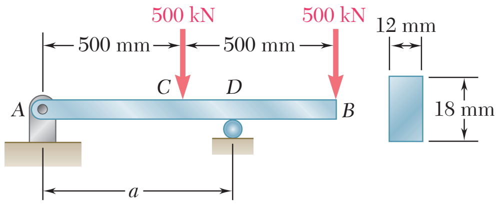


Fig. P12.26

12.27 Determine (*a*) the distance a for which the maximum absolute value of the bending moment in the beam is as small as possible, (*b*) the corresponding maximum normal stress due to bending. (See hint of Prob. 12.25.)

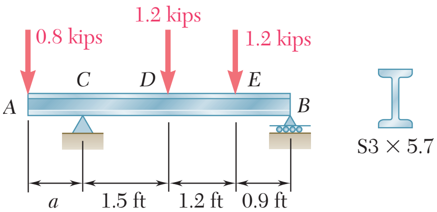


Fig. P12.27

12.28 A solid steel rod of diameter d is supported as shown. Knowing that for steel $\gamma = 490$ lb/ft^3, determine the smallest diameter d that can be used if the normal stress due to bending is not to exceed 4 ksi.

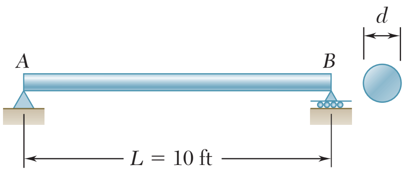


Fig. *P12.28*

12.3 RELATIONS AMONG LOAD, SHEAR, AND BENDING MOMENT

When a beam carries more than two or three concentrated loads, or when it carries distributed loads, the method outlined in Sec. 12.2 for plotting shear and bending moment can prove quite cumbersome. The construction of the shear diagram and, especially, of the bending-moment diagram will be greatly facilitated if certain relations existing among load, shear, and bending moment are taken into consideration.

Let us consider a simply supported beam AB carrying a distributed load w per unit length (Fig. 12.11*a*), and let C and C' be two points of the beam at a distance Δx from each other. The shear and bending moment at C will be denoted by V and M, respectively, and will be assumed positive; the shear and bending moment at C' will be denoted by $V + \Delta V$ and $M + \Delta M$.

We now detach the portion of beam CC' and draw its free-body diagram (Fig. 12.11*b*). The forces exerted on the free body include a load of magnitude $w\,\Delta x$ and internal forces and couples at C and C'. Since shear and bending moment have been assumed positive, the forces and couples will be directed as shown in the figure.

Relations between Load and Shear. Writing that the sum of the vertical components of the forces acting on the free body CC' is zero, we have

$$+\uparrow \Sigma F_y = 0: \qquad V - (V + \Delta V) - w\,\Delta x = 0$$
$$\Delta V = -w\,\Delta x$$

Dividing both members of the equation by Δx and then letting Δx approach zero, we obtain

$$\boxed{\frac{dV}{dx} = -w} \qquad (12.5)$$

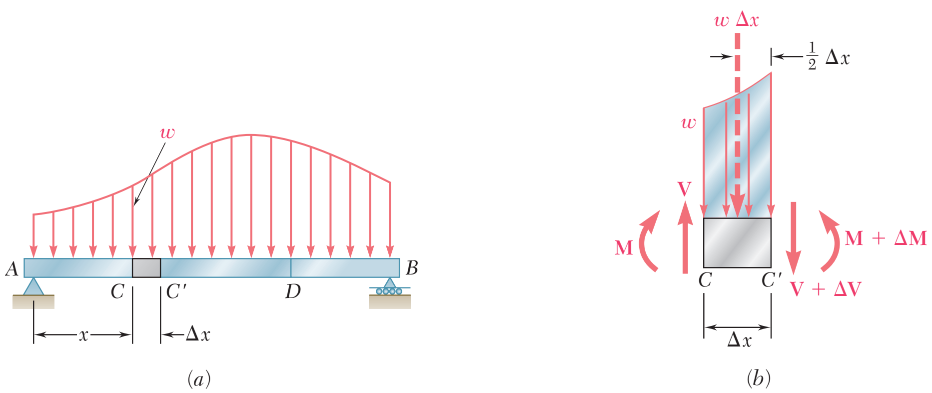


Fig. 12.11

Equation (12.5) indicates that, for a beam loaded as shown in Fig. 12.11*a*, the slope dV/dx of the shear curve is negative; the numerical value of the slope at any point is equal to the load per unit length at that point.

Integrating (12.5) between points C and D, we write

$$V_D - V_C = -\int_{x_C}^{x_D} w\, dx \tag{12.6}$$

$$V_D - V_C = -(\text{area under load curve between } C \text{ and } D) \tag{12.6$'$}$$

Note that this result could also have been obtained by considering the equilibrium of the portion of beam CD, since the area under the load curve represents the total load applied between C and D.

It should be observed that Eq. (12.5) is not valid at a point where a concentrated load is applied; the shear curve is discontinuous at such a point, as seen in Sec. 12.2. Similarly, Eqs. (12.6) and (12.6′) cease to be valid when concentrated loads are applied between C and D, since they do not take into account the sudden change in shear caused by a concentrated load. Equations (12.6) and (12.6′), therefore, should be applied only between successive concentrated loads.

Relations between Shear and Bending Moment. Returning to the free-body diagram of Fig. 12.11*b*, and writing now that the sum of the moments about C' is zero, we have

$$+\circlearrowleft \Sigma M_{C'} = 0: \qquad (M + \Delta M) - M - V\,\Delta x + w\,\Delta x \frac{\Delta x}{2} = 0$$

$$\Delta M = V\,\Delta x - \frac{1}{2} w\,(\Delta x)^2$$

Dividing both members of the equation by Δx and then letting Δx approach zero, we obtain

$$\frac{dM}{dx} = V \tag{12.7}$$

Equation (12.7) indicates that the slope dM/dx of the bending-moment curve is equal to the value of the shear. This is true at any point where the shear has a well-defined value, i.e., at any point where no concentrated load is applied. Equation (12.7) also shows that $V = 0$ at points where M is maximum. This property facilitates the determination of the points where the beam is likely to fail under bending.

Integrating (12.7) between points C and D, we write

$$M_D - M_C = \int_{x_C}^{x_D} V\, dx \tag{12.8}$$

$$M_D - M_C = \text{area under shear curve between } C \text{ and } D \tag{12.8$'$}$$

Note that the area under the shear curve should be considered positive where the shear is positive and negative where the shear is negative. Equations (12.8) and (12.8′) are valid even when concentrated loads are applied between C and D, as long as the shear curve has been correctly drawn. The equations cease to be valid, however, if a couple is applied at a point between C and D, since they do not take into account the sudden change in bending moment caused by a couple (see Sample Prob. 12.6).

EXAMPLE 12.3 Draw the shear and bending-moment diagrams for the simply supported beam shown in Fig. 12.12 and determine the maximum value of the bending moment.

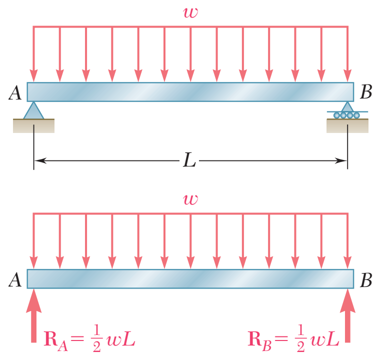


Fig. 12.12

From the free-body diagram of the entire beam, we determine the magnitude of the reactions at the supports.

$$R_A = R_B = \tfrac{1}{2}wL$$

Next, we draw the shear diagram. Close to the end A of the beam, the shear is equal to R_A, that is, to $\frac{1}{2}wL$, as we can check by considering as a free body a very small portion of the beam. Using Eq. (12.6), we then determine the shear V at any distance x from A; we write

$$V - V_A = -\int_0^x w\,dx = -wx$$

$$V = V_A - wx = \tfrac{1}{2}wL - wx = w(\tfrac{1}{2}L - x)$$

The shear curve is thus an oblique straight line which crosses the x axis at $x = L/2$ (Fig. 12.13*a*). Considering, now, the bending moment, we first observe that $M_A = 0$. The value M of the bending moment at any distance x from A may then be obtained from Eq. (12.8); we have

$$M - M_A = \int_0^x V\,dx$$

$$M = \int_0^x w(\tfrac{1}{2}L - x)\,dx = \tfrac{1}{2}w(Lx - x^2)$$

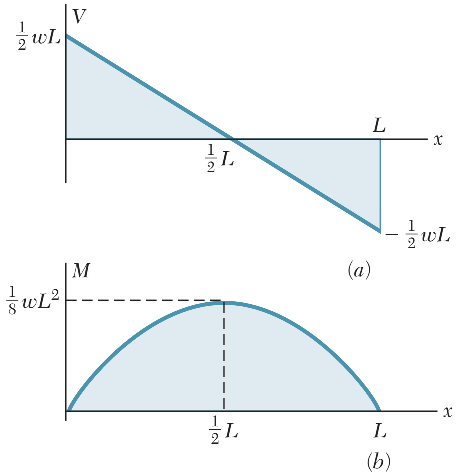


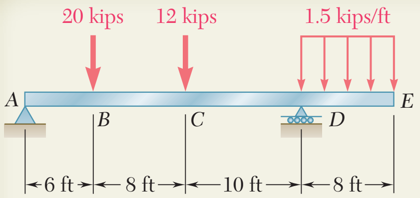
Fig. 12.13

The bending-moment curve is a parabola. The maximum value of the bending moment occurs when $x = L/2$, since V (and thus dM/dx) is zero for that value of x. Substituting $x = L/2$ in the last equation, we obtain $M_{\max} = wL^2/8$ (Fig. 12.13*b*). ■

In most engineering applications, one needs to know the value of the bending moment only at a few specific points. Once the shear diagram has been drawn, and after M has been determined at one of the ends of the beam, the value of the bending moment can then be obtained at any given point by computing the area under the shear curve and using Eq. (12.8′). For instance, since $M_A = 0$ for the beam of Example 12.3, the maximum value of the bending moment for that beam can be obtained simply by measuring the area of the shaded triangle in the shear diagram of Fig. 12.13*a*. We have

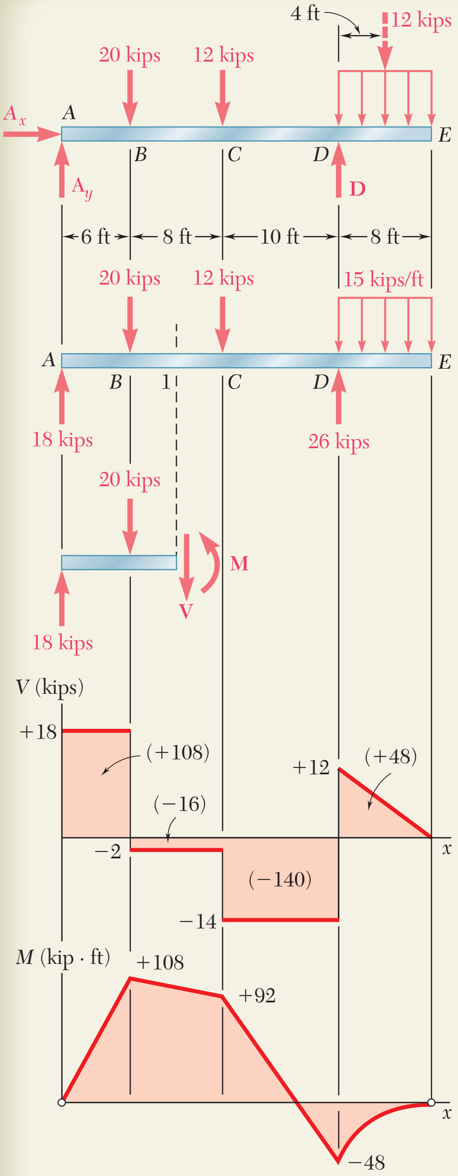
$$M_{\max} = \frac{1}{2}\frac{L}{2}\frac{wL}{2} = \frac{wL^2}{8}$$

We note that, in this example, the load curve is a horizontal straight line, the shear curve an oblique straight line, and the bending-moment curve a parabola. If the load curve had been an oblique straight line (first degree), the shear curve would have been a parabola (second degree) and the bending-moment curve a cubic (third degree). The shear and bending-moment curves will always be, respectively, one and two degrees higher than the load curve. With this in mind, we should be able to sketch the shear and bending-moment diagrams without actually determining the functions $V(x)$ and $M(x)$, once a few values of the shear and bending moment have been computed. The sketches obtained will be more accurate if we make use of the fact that, at any point where the curves are continuous, the slope of the shear curve is equal to $-w$ and the slope of the bending-moment curve is equal to V.



SAMPLE PROBLEM 12.3

Draw the shear and bending-moment diagrams for the beam and loading shown.



SOLUTION

Reactions. Considering the entire beam as a free body, we write

$+\circlearrowleft \Sigma M_A = 0$:

$$D(24 \text{ ft}) - (20 \text{ kips})(6 \text{ ft}) - (12 \text{ kips})(14 \text{ ft}) - (12 \text{ kips})(28 \text{ ft}) = 0$$

$$D = +26 \text{ kips} \qquad \mathbf{D} = 26 \text{ kips} \uparrow$$

$$+\uparrow \Sigma F_y = 0: \qquad A_y - 20 \text{ kips} - 12 \text{ kips} + 26 \text{ kips} - 12 \text{ kips} = 0$$

$$A_y = +18 \text{ kips} \qquad \mathbf{A}_y = 18 \text{ kips} \uparrow$$

$$\xrightarrow{+} \Sigma F_x = 0: \qquad A_x = 0 \qquad \mathbf{A}_x = 0$$

We also note that at both A and E the bending moment is zero; thus, two points (indicated by dots) are obtained on the bending-moment diagram.

Shear Diagram. Since $dV/dx = -w$, we find that between concentrated loads and reactions the slope of the shear diagram is zero (i.e., the shear is constant). The shear at any point is determined by dividing the beam into two parts and considering either part as a free body. For example, using the portion of beam to the left of section *1*, we obtain the shear between B and C:

$$+\uparrow \Sigma F_y = 0: \qquad +18 \text{ kips} - 20 \text{ kips} - V = 0 \qquad V = -2 \text{ kips}$$

We also find that the shear is +12 kips just to the right of D and zero at end E. Since the slope $dV/dx = -w$ is constant between D and E, the shear diagram between these two points is a straight line.

Bending-Moment Diagram. We recall that the area under the shear curve between two points is equal to the change in bending moment between the same two points. For convenience, the area of each portion of the shear diagram is computed and is indicated in parentheses on the diagram. Since the bending moment M_A at the left end is known to be zero, we write

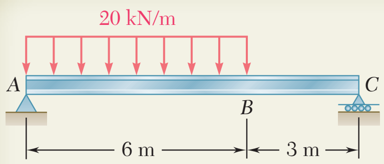
$$M_B - M_A = +108 \qquad M_B = +108 \text{ kip} \cdot \text{ft}$$
$$M_C - M_B = -16 \qquad M_C = +92 \text{ kip} \cdot \text{ft}$$
$$M_D - M_C = -140 \qquad M_D = -48 \text{ kip} \cdot \text{ft}$$
$$M_E - M_D = +48 \qquad M_E = 0$$

Since M_E is known to be zero, a check of the computations is obtained.

Between the concentrated loads and reactions, the shear is constant; thus, the slope dM/dx is constant, and the bending-moment diagram is drawn by connecting the known points with straight lines. Between D and E where the shear diagram is an oblique straight line, the bending-moment diagram is a parabola.

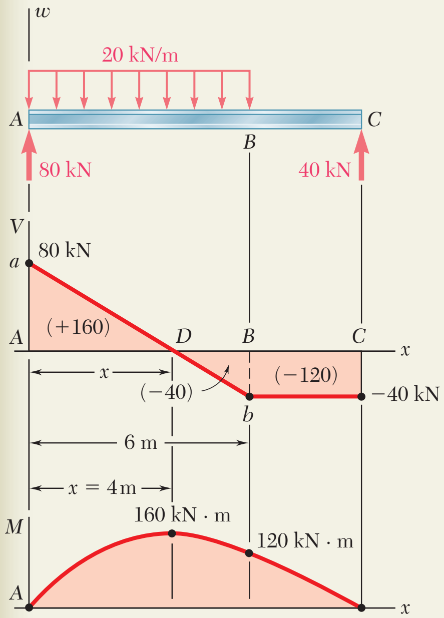
From the V and M diagrams we note that $V_{max} = 18$ kips and $M_{max} = 108$ kip · ft.

SAMPLE PROBLEM 12.4



The W360 × 79 rolled-steel beam AC is simply supported and carries the uniformly distributed load shown. Draw the shear and bending-moment diagrams for the beam and determine the location and magnitude of the maximum normal stress due to bending.

SOLUTION



Reactions. Considering the entire beam as a free body, we find

$$\mathbf{R}_A = 80 \text{ kN} \uparrow \qquad \mathbf{R}_C = 40 \text{ kN} \uparrow$$

Shear Diagram. The shear just to the right of A is $V_A = +80$ kN. Since the change in shear between two points is equal to *minus* the area under the load curve between the same two points, we obtain V_B by writing

$$V_B - V_A = -(20 \text{ kN/m})(6 \text{ m}) = -120 \text{ kN}$$
$$V_B = -120 + V_A = -120 + 80 = -40 \text{ kN}$$

The slope $dV/dx = -w$ being constant between A and B, the shear diagram between these two points is represented by a straight line. Between B and C, the area under the load curve is zero; therefore,

$$V_C - V_B = 0 \qquad V_C = V_B = -40 \text{ kN}$$

and the shear is constant between B and C.

Bending-Moment Diagram. We note that the bending moment at each end of the beam is zero. In order to determine the maximum bending moment, we locate the section D of the beam where $V = 0$. We write

$$V_D - V_A = -wx$$
$$0 - 80 \text{ kN} = -(20 \text{ kN/m})x$$

and, solving for x: $x = 4$ m ◀

The maximum bending moment occurs at point D, where we have $dM/dx = V = 0$. The areas of the various portions of the shear diagram are computed and are given (in parentheses) on the diagram. Since the area of the shear diagram between two points is equal to the change in bending moment between the same two points, we write

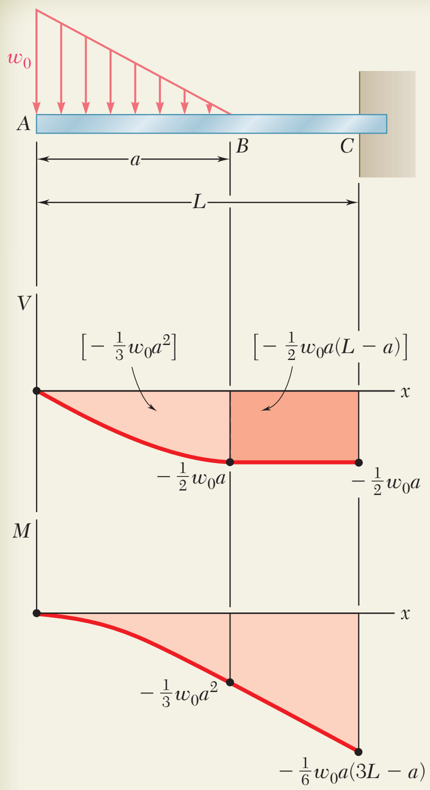
$$\begin{aligned} M_D - M_A &= +160 \text{ kN} \cdot \text{m} & M_D &= +160 \text{ kN} \cdot \text{m} \\ M_B - M_D &= -\ \ 40 \text{ kN} \cdot \text{m} & M_B &= +120 \text{ kN} \cdot \text{m} \\ M_C - M_B &= -120 \text{ kN} \cdot \text{m} & M_C &= 0 \end{aligned}$$

The bending-moment diagram consists of an arc of parabola followed by a segment of straight line; the slope of the parabola at A is equal to the value of V at that point.

Maximum Normal Stress. It occurs at D, where $|M|$ is largest. From App. B we find that for a W360 × 79 rolled-steel shape, $S = 1270 \text{ mm}^3$ about a horizontal axis. Substituting this value and $|M| = |M_D| = 160 \times 10^3$ N · m into Eq. (12.3), we write

$$\sigma_m = \frac{|M_D|}{S} = \frac{160 \times 10^3 \text{ N} \cdot \text{m}}{1270 \times 10^{-6} \text{ m}^3} = 126.0 \times 10^6 \text{ Pa}$$

Maximum normal stress in the beam = 126.0 MPa ◀



SAMPLE PROBLEM 12.5

Sketch the shear and bending-moment diagrams for the cantilever beam shown.

SOLUTION

Shear Diagram. At the free end of the beam, we find $V_A = 0$. Between A and B, the area under the load curve is $\frac{1}{2}w_0a$; we find V_B by writing

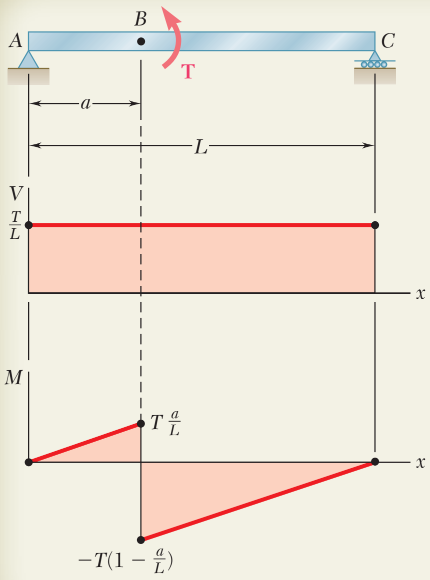
$$V_B - V_A = -\tfrac{1}{2}w_0a \qquad V_B = -\tfrac{1}{2}w_0a$$

Between B and C, the beam is not loaded; thus $V_C = V_B$. At A, we have $w = w_0$ and, according to Eq. (12.5), the slope of the shear curve is $dV/dx = -w_0$, while at B the slope is $dV/dx = 0$. Between A and B, the loading decreases linearly, and the shear diagram is parabolic. Between B and C, $w = 0$, and the shear diagram is a horizontal line.

Bending-Moment Diagram. The bending moment M_A at the free end of the beam is zero. We compute the area under the shear curve and write

$$M_B - M_A = -\tfrac{1}{3}w_0a^2 \qquad M_B = -\tfrac{1}{3}w_0a^2$$
$$M_C - M_B = -\tfrac{1}{2}w_0a(L-a)$$
$$M_C = -\tfrac{1}{6}w_0a(3L-a)$$

The sketch of the bending-moment diagram is completed by recalling that $dM/dx = V$. We find that between A and B the diagram is represented by a cubic curve with zero slope at A, and between B and C by a straight line.



SAMPLE PROBLEM 12.6

The simple beam AC is loaded by a couple of moment T applied at point B. Draw the shear and bending-moment diagrams of the beam.

SOLUTION

The entire beam is taken as a free body, and we obtain

$$\mathbf{R}_A = \frac{T}{L}\uparrow \qquad \mathbf{R}_C = \frac{T}{L}\downarrow$$

The shear at any section is constant and equal to T/L. Since a couple is applied at B, the bending-moment diagram is discontinuous at B; it is represented by two oblique straight lines and decreases suddenly at B by an amount equal to T.

PROBLEMS

12.29 Using the method of Sec. 12.3, solve Prob. 12.1*a*.

12.30 Using the method of Sec. 12.3, solve Prob. 12.2*a*.

12.31 Using the method of Sec. 12.3, solve Prob. 12.3*a*.

12.32 Using the method of Sec. 12.3, solve Prob. 12.4*a*.

12.33 Using the method of Sec. 12.3, solve Prob. 12.5.

12.34 Using the method of Sec. 12.3, solve Prob. 12.6.

12.35 Using the method of Sec. 12.3, solve Prob. 12.7.

12.36 Using the method of Sec. 12.3, solve Prob. 12.8.

12.37 through 12.40 Draw the shear and bending-moment diagrams for the beam and loading shown, and determine the maximum absolute value (*a*) of the shear, (*b*) of the bending moment.

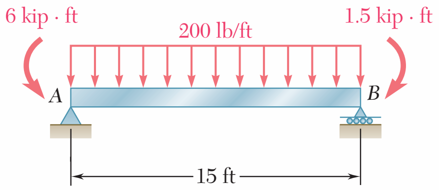


Fig. P12.37

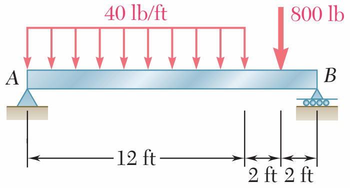


Fig. P12.38

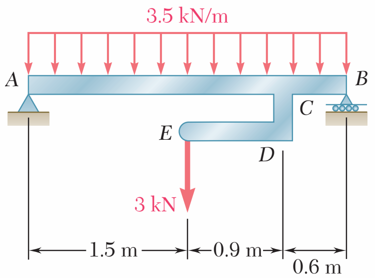


Fig. P12.39

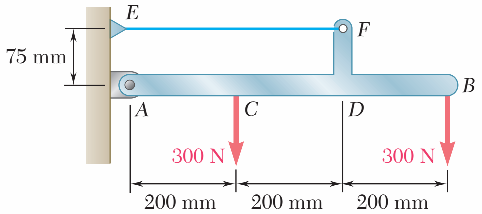


Fig. P12.40

12.41 Using the method of Sec. 12.3, solve Prob. 12.13.

12.42 Using the method of Sec. 12.3, solve Prob. 12.14.

12.43 Using the method of Sec. 12.3, solve Prob. 12.15.

12.44 Using the method of Sec. 12.3, solve Prob. 12.16.

12.45 and *12.46* Determine (*a*) the equations of the shear and bending-moment curves for the beam and loading shown, (*b*) the maximum absolute value of the bending moment in the beam.

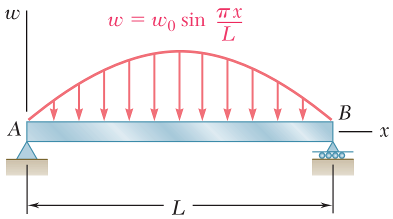


Fig. P12.45

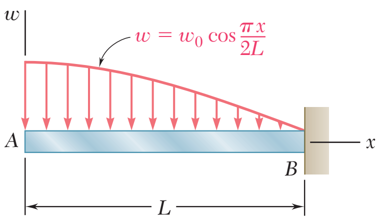


Fig. *P12.46*

12.47 Determine (*a*) the equations of the shear and bending-moment curves for the beam and loading shown, (*b*) the maximum absolute value of the bending moment in the beam.

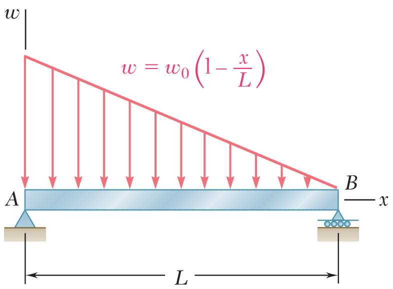


Fig. P12.47

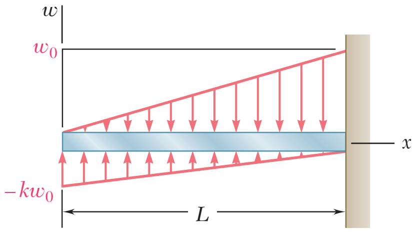


Fig. *P12.48*

12.48 For the beam and loading shown, determine the equations of the shear and bending-moment curves and the maximum absolute value of the bending moment in the beam, knowing that (*a*) $k = 1$, (*b*) $k = 0.5$.

12.49 and 12.50 Draw the shear and bending-moment diagrams for the beam and loading shown, and determine the maximum normal stress due to bending.

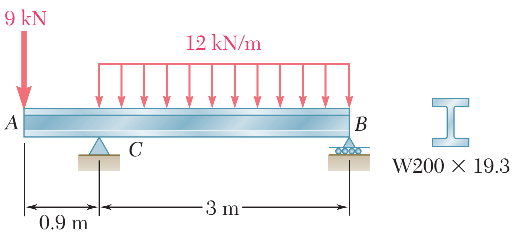


Fig. P12.49

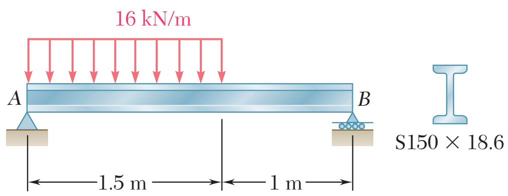


Fig. P12.50

12.51 and 12.52 Draw the shear and bending-moment diagrams for the beam and loading shown, and determine the maximum normal stress due to bending.

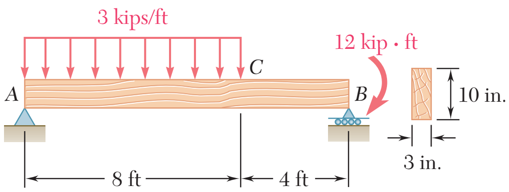


Fig. P12.51

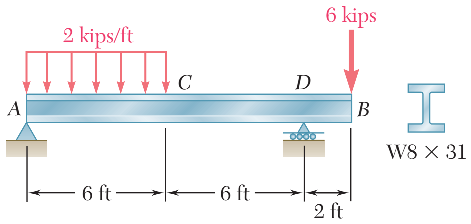


Fig. P12.52

12.53 **and 12.54** Draw the shear and bending-moment diagrams for the beam and loading shown, and determine the maximum normal stress due to bending.

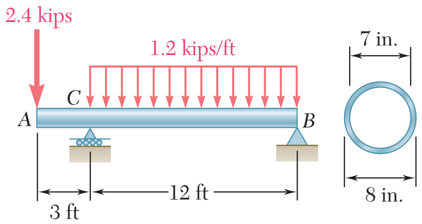


Fig. *P12.53*

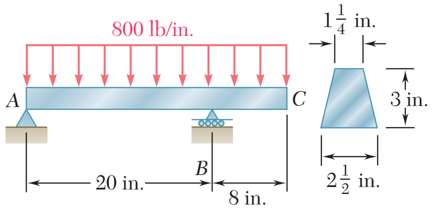


Fig. P12.54

12.55 and ***12.56*** Draw the shear and bending-moment diagrams for the beam and loading shown, and determine the maximum normal stress due to bending.

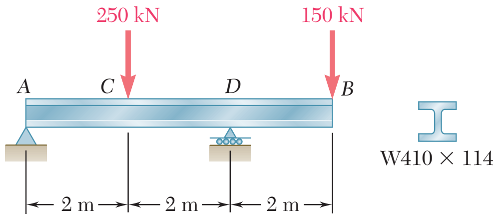


Fig. P12.55

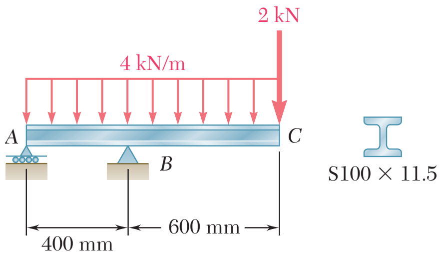


Fig. *P12.56*

12.4 DESIGN OF PRISMATIC BEAMS FOR BENDING

As indicated in Sec. 12.1, the design of a beam is usually controlled by the maximum absolute value $|M|_{max}$ of the bending moment that will occur in the beam. The largest normal stress σ_m in the beam is found at the surface of the beam in the critical section where $|M|_{max}$ occurs and can be obtained by substituting $|M|_{max}$ for $|M|$ in Eq. (12.1) or Eq. (12.3).† We write

$$\sigma_m = \frac{|M|_{max}c}{I} \qquad \sigma_m = \frac{|M|_{max}}{S} \qquad (12.1', 12.3')$$

A safe design requires that $\sigma_m \leq \sigma_{all}$, where σ_{all} is the allowable stress for the material used. Substituting σ_{all} for σ_m in (12.3′) and solving for S yields the minimum allowable value of the section modulus for the beam being designed:

$$S_{min} = \frac{|M|_{max}}{\sigma_{all}} \qquad (12.9)$$

The design of common types of beams, such as timber beams of rectangular cross section and rolled-steel beams of various cross-sectional shapes, will be considered in this section. A proper procedure should lead to the most economical design. This means that, among beams of the same type and the same material, and other things being equal, the beam with the smallest weight per unit length—and, thus, the smallest cross-sectional area—should be selected, since this beam will be the least expensive.

The design procedure will include the following steps‡:

1. First determine the value of σ_{all} for the material selected from a table of properties of materials or from design specifications. You can also compute this value by dividing the ultimate strength σ_U of the material by an appropriate factor of safety (Sec. 8.10). Assuming for the time being that the value of σ_{all} is the same in tension and in compression, proceed as follows.
2. Draw the shear and bending-moment diagrams corresponding to the specified loading conditions, and determine the maximum absolute value $|M|_{max}$ of the bending moment in the beam.
3. Determine from Eq. (12.9) the minimum allowable value S_{min} of the section modulus of the beam.
4. For a timber beam, the depth h of the beam, its width b, or the ratio h/b characterizing the shape of its cross section will probably have been specified. The unknown dimensions may then be selected by recalling from Eq. (11.19) of Sec. 11.4 that b and h must satisfy the relation $\frac{1}{6}bh^2 = S \geq S_{min}$.
5. For a rolled-steel beam, consult the appropriate table in App. B. Of the available beam sections, consider only those with a section

†For beams that are not symmetrical with respect to their neutral surface, the largest of the distances from the neutral surface to the surfaces of the beam should be used for c in Eq. (12.1) and in the computation of the section modulus $S = I/c$.

‡We assume that all beams considered in this chapter are adequately braced to prevent lateral buckling and that bearing plates are provided under concentrated loads applied to rolled-steel beams to prevent local buckling (crippling) of the web.

modulus $S \geq S_{\min}$ and select from this group the section with the smallest weight per unit length. This is the most economical of the sections for which $S \geq S_{\min}$. Note that this is not necessarily the section with the smallest value of S (see Example 12.4). In some cases, the selection of a section may be limited by other considerations, such as the allowable depth of the cross section, or the allowable deflection of the beam (cf. Chap. 15).

The foregoing discussion was limited to materials for which σ_{all} is the same in tension and in compression. If σ_{all} is different in tension and in compression, you should make sure to select the beam section in such a way that $\sigma_m \leq \sigma_{\text{all}}$ for both tensile and compressive stresses. If the cross section is not symmetric about its neutral axis, the largest tensile and the largest compressive stresses will not necessarily occur in the section where $|M|$ is maximum. One may occur where M is maximum and the other where M is minimum. Thus, step 2 should include the determination of both $M_{\max}$ and $M_{\min}$, and step 3 should be modified to take into account both tensile and compressive stresses.

Finally, keep in mind that the design procedure described in this section takes into account only the normal stresses occurring on the surface of the beam. Short beams, especially those made of timber, may fail in shear under a transverse loading. The determination of shearing stresses in beams will be discussed in Chap. 13.

EXAMPLE 12.4 Select a wide-flange beam to support the 15-kip load as shown in Fig. 12.14. The allowable normal stress for the steel used is 24 ksi.

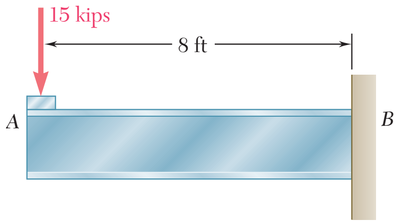


Fig. 12.14

1. The allowable normal stress is given: $\sigma_{\text{all}} = 24$ ksi.
2. The shear is constant and equal to 15 kips. The bending moment is maximum at B. We have

$$|M|_{\max} = (15 \text{ kips})(8 \text{ ft}) = 120 \text{ kip} \cdot \text{ft} = 1440 \text{ kip} \cdot \text{in.}$$

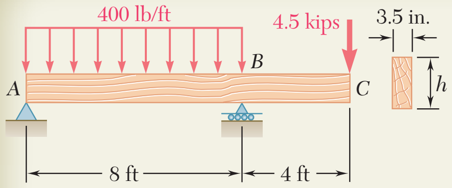
3. The minimum allowable section modulus is

$$S_{\min} = \frac{|M|_{\max}}{\sigma_{\text{all}}} = \frac{1440 \text{ kip} \cdot \text{in.}}{24 \text{ ksi}} = 60.0 \text{ in}^3$$

4. Referring to the table of *Properties of Rolled-Steel Shapes* in App. B, we note that the shapes are arranged in groups of the same depth and that in each group they are listed in order of decreasing weight. We choose in each group the lightest beam having a section modulus $S = I/c$ at least as large as $S_{\min}$ and record the results in the following table.

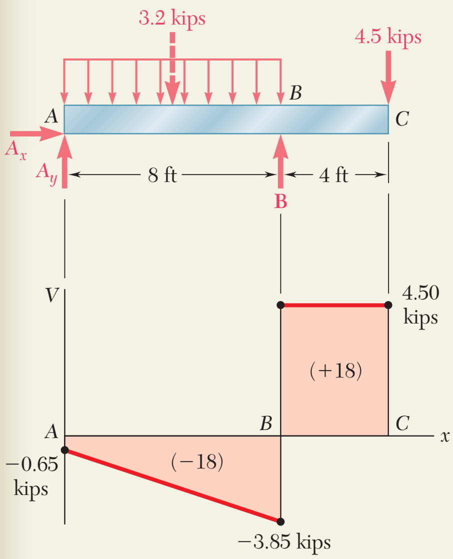
Shape	S, in^3
W21 × 44	81.6
W18 × 50	88.9
W16 × 40	64.7
W14 × 43	62.6
W12 × 50	64.2
W10 × 54	60.0

The most economical is the W16 × 40 shape since it weighs only 40 lb/ft, even though it has a larger section modulus than two of the other shapes. We also note that the total weight of the beam will be (8 ft) × (40 lb) = 320 lb. This weight is small compared to the 15,000-1b load and can be neglected in our analysis. ■



SAMPLE PROBLEM 12.7

A 12-ft-long overhanging timber beam AC with an 8-ft span AB is to be designed to support the distributed and concentrated loads shown. Knowing that timber of 4-in. nominal width (3.5-in. actual width) with a 1.75-ksi allowable stress is to be used, determine the minimum required depth h of the beam.



SOLUTION

Reactions. Considering the entire beam as a free body, we write

$$+\circlearrowleft \Sigma M_A = 0\colon B(8 \text{ ft}) - (3.2 \text{ kips})(4 \text{ ft}) - (4.5 \text{ kips})(12 \text{ ft}) = 0$$
$$B = 8.35 \text{ kips} \qquad \mathbf{B} = 8.35 \text{ kips} \uparrow$$
$$\xrightarrow{+} \Sigma F_x = 0\colon \qquad A_x = 0$$
$$+\uparrow \Sigma F_y = 0\colon A_y + 8.35 \text{ kips} - 3.2 \text{ kips} - 4.5 \text{ kips} = 0$$
$$A_y = -0.65 \text{ kips} \qquad \mathbf{A} = 0.65 \text{ kips} \downarrow$$

Shear Diagram. The shear just to the right of A is $V_A = A_y = -0.65$ kips. Since the change in shear between A and B is equal to *minus* the area under the load curve between these two points, we obtain V_B by writing

$$V_B - V_A = -(400 \text{ lb/ft})(8 \text{ ft}) = -3200 \text{ lb} = -3.20 \text{ kips}$$
$$V_B = V_A - 3.20 \text{ kips} = -0.65 \text{ kips} - 3.20 \text{ kips} = -3.85 \text{ kips.}$$

The reaction at B produces a sudden increase of 8.35 kips in V, resulting in a value of the shear equal to 4.50 kips to the right of B. Since no load is applied between B and C, the shear remains constant between these two points.

Determination of $|M|_{\max}$. We first observe that the bending moment is equal to zero at both ends of the beam: $M_A = M_C = 0$. Between A and B the bending moment decreases by an amount equal to the area under the shear curve, and between B and C it increases by a corresponding amount. Thus, the maximum absolute value of the bending moment is $|M|_{\max} = 18.00$ kip · ft.

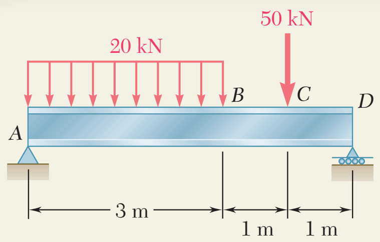
Minimum Allowable Section Modulus. Substituting into Eq. (12.9) the given value of σ_{all} and the value of $|M|_{\max}$ that we have found, we write

$$S_{\min} = \frac{|M|_{\max}}{\sigma_{\text{all}}} = \frac{(18 \text{ kip} \cdot \text{ft})(12 \text{ in./ft})}{1.75 \text{ ksi}} = 123.43 \text{ in}^3$$

Minimum Required Depth of Beam. Recalling the formula developed in part 4 of the design procedure described in Sec. 12.4 and substituting the values of b and $S_{\min}$, we have

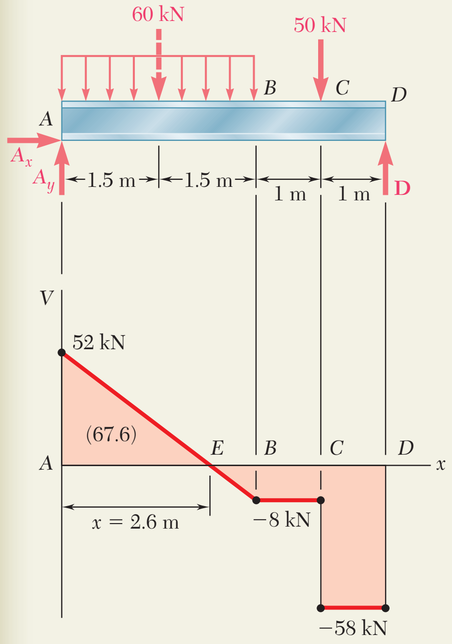
$$\tfrac{1}{6}bh^2 \geq S_{\min} \qquad \tfrac{1}{6}(3.5 \text{ in.})h^2 \geq 123.43 \text{ in}^3 \qquad h \geq 14.546 \text{ in.}$$

The minimum required depth of the beam is $\qquad h = 14.55$ in. ◀



SAMPLE PROBLEM 12.8

A 5-m-long, simply supported steel beam *AD* is to carry the distributed and concentrated loads shown. Knowing that the allowable normal stress for the grade of steel to be used is 160 MPa, select the wide-flange shape that should be used.



SOLUTION

Reactions. Considering the entire beam as a free body, we write

$$+\circlearrowleft \Sigma M_A = 0: D(5\text{ m}) - (60\text{ kN})(1.5\text{ m}) - (50\text{ kN})(4\text{ m}) = 0$$

$$D = 58.0\text{ kN} \qquad \mathbf{D} = 58.0\text{ kN} \uparrow$$

$$\xrightarrow{+} \Sigma F_x = 0: \qquad A_x = 0$$

$$+\uparrow \Sigma F_y = 0: A_y + 58.0\text{ kN} - 60\text{ kN} - 50\text{ kN} = 0$$

$$A_y = 52.0\text{ kN} \qquad \mathbf{A} = 52.0\text{ kN} \uparrow$$

Shear Diagram. The shear just to the right of *A* is $V_A = A_y = +52.0$ kN. Since the change in shear between *A* and *B* is equal to *minus* the area under the load curve between these two points, we have

$$V_B = 52.0\text{ kN} - 60\text{ kN} = -8\text{ kN}$$

The shear remains constant between *B* and *C*, where it drops to −58 kN, and keeps this value between *C* and *D*. We locate the section *E* of the beam where $V = 0$ by writing

$$V_E - V_A = -wx$$

$$0 - 52.0\text{ kN} = -(20\text{ kN/m})x$$

Solving for x, we find $x = 2.60$ m.

Determination of $|M|_{max}$. The bending moment is maximum at *E*, where $V = 0$. Since *M* is zero at the support *A*, its maximum value at *E* is equal to the area under the shear curve between *A* and *E*. We have, therefore, $|M|_{max} = M_E = 67.6$ kN · m.

Minimum Allowable Section Modulus. Substituting into Eq. (12.9) the given value of σ_{all} and the value of $|M|_{max}$ that we have found, we write

$$S_{min} = \frac{|M|_{max}}{\sigma_{all}} = \frac{67.6\text{ kN} \cdot \text{m}}{160\text{ MPa}} = 422.5 \times 10^{-6}\text{ m}^3 = 422.5 \times 10^3\text{ mm}^3$$

Selection of Wide-Flange Shape. From App. B we compile a list of shapes that have a section modulus larger than S_{min} and are also the lightest shape in a given depth group.

Shape	S, mm^3
W410 × 38.8	629
W360 × 32.9	475
W310 × 38.7	547
W250 × 44.8	531
W200 × 46.1	451

We select the lightest shape available, namely W360 × 32.9 ◀

PROBLEMS

12.57 and 12.58 For the beam and loading shown, design the cross section of the beam knowing that the grade of timber used has an allowable normal stress of 12 MPa.

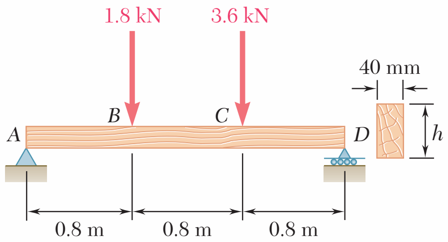


Fig. P12.57

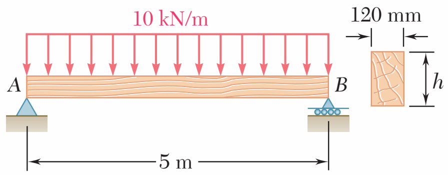


Fig. P12.58

***12.59* and 12.60** For the beam and loading shown, design the cross section of the beam knowing that the grade of timber used has an allowable normal stress of 1750 psi.

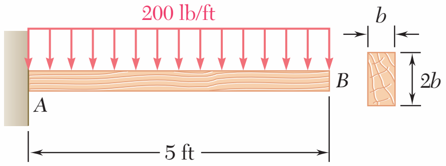


Fig. *P12.59*

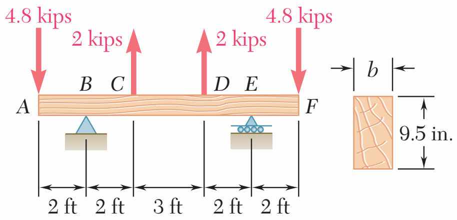


Fig. P12.60

12.61 For the beam and loading shown, design the cross section of the beam knowing that the grade of timber used has an allowable normal stress of 12 MPa.

12.62 For the beam and loading shown, design the cross section of the beam knowing that the grade of timber used has an allowable normal stress of 1750 psi.

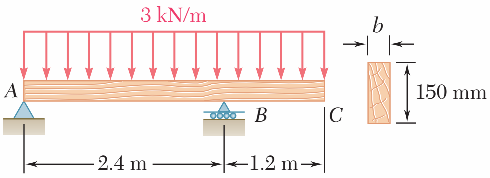


Fig. *P12.61*

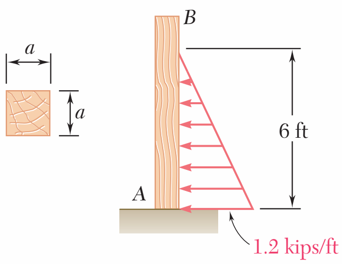


Fig. P12.62

12.63 and 12.64 Knowing that the allowable normal stress for the steel used is 24 ksi, select the most economical wide-flange beam to support the loading shown.

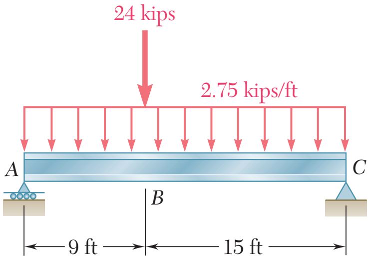


Fig. P12.63

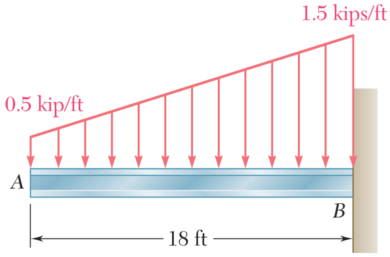


Fig. P12.64

12.65 and 12.66 Knowing that the allowable normal stress for the steel used is 160 MPa, select the most economical wide-flange beam to support the loading shown.

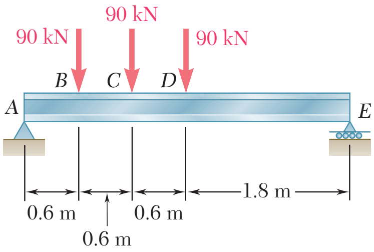


Fig. P12.65

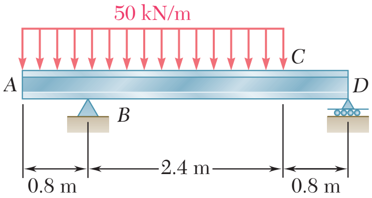


Fig. P12.66

12.67 and *12.68* Knowing that the allowable normal stress for the steel used is 160 MPa, select the most economical S-shape beam to support the loading shown.

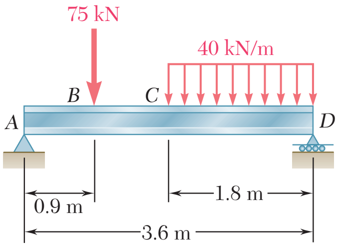


Fig. P12.67

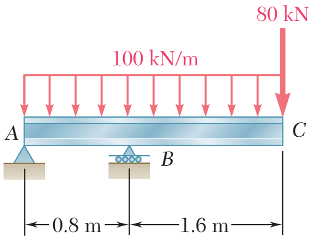


Fig. *P12.68*

12.69 and *12.70* Knowing that the allowable normal stress for the steel used is 24 ksi, select the most economical S-shape beam to support the loading shown.

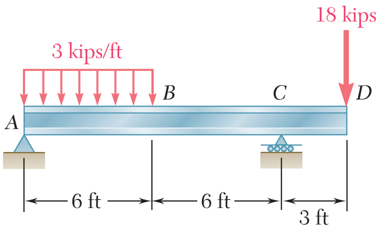


Fig. P12.69

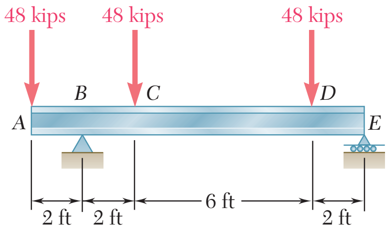


Fig. *P12.70*

12.71 Two metric rolled-steel channels are to be welded back to back and used to support the loading shown. Knowing that the allowable normal stress for the steel used is 200 MPa, determine the most economical channels that can be used.

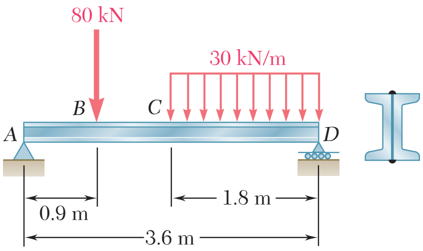


Fig. P12.71

12.72 Two metric rolled-steel channels are to be welded along their edges and used to support the loading shown. Knowing that the allowable normal stress for the steel used is 150 MPa, determine the most economical channels that can be used.

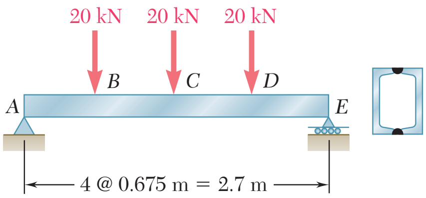


Fig. P12.72

12.73 Two L4 × 3 rolled-steel angles are bolted together and used to support the loading shown. Knowing that the allowable normal stress for the steel used is 24 ksi, determine the minimum angle thickness that can be used.

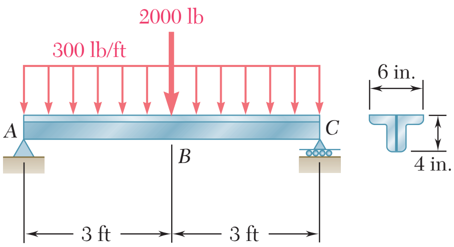


Fig. P12.73

12.74 A steel pipe of 4-in. diameter is to support the loading shown. Knowing that the stock of pipes available has thicknesses varying from $\frac{1}{4}$ in. to 1 in. in $\frac{1}{8}$-in. increments and that the allowable normal stress for the steel used is 24 ksi, determine the minimum wall thickness t that can be used.

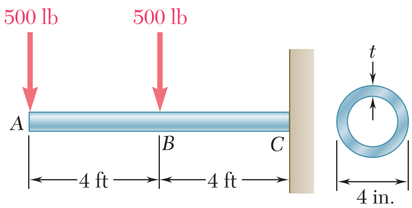


Fig. P12.74

12.75 Assuming the upward reaction of the ground to be uniformly distributed and knowing that the allowable normal stress for the steel used is 170 MPa, select the most economical wide-flange beam to support the loading shown.

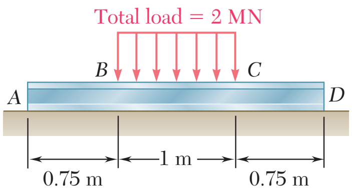


Fig. P12.75

12.76 Assuming the upward reaction of the ground to be uniformly distributed and knowing that the allowable normal stress for the steel used is 24 ksi, select the most economical S-shape beam to support the loading shown.

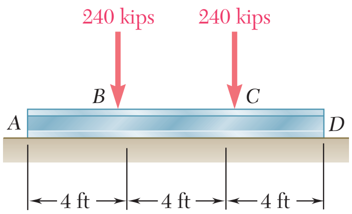


Fig. P12.76

REVIEW AND SUMMARY

Considerations for the design of prismatic beams

This chapter was devoted to the analysis and design of beams under transverse loadings. Such loadings can consist of concentrated loads or distributed loads and the beams themselves are classified according to the way they are supported (Fig. 12.15). Only *statically determinate* beams were considered in this chapter, the analysis of statically indeterminate beams being postponed until Chap. 15.

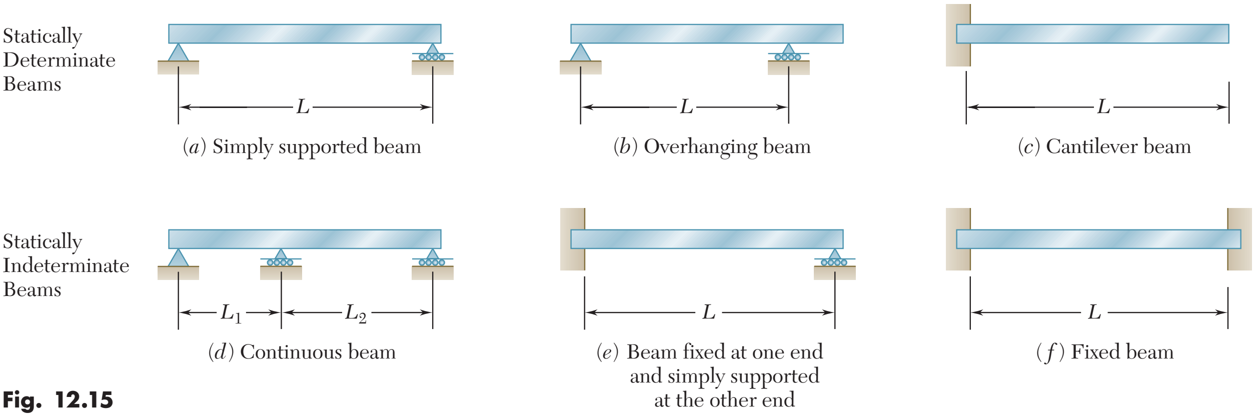


Fig. 12.15

Normal stresses due to bending

While transverse loadings cause both bending and shear in a beam, the normal stresses caused by bending are the dominant criterion in the design of a beam for strength [Sec. 12.1]. Therefore, this chapter dealt only with the determination of the normal stresses in a beam, the effect of shearing stresses being examined in the next one.

We recalled from Sec. 11.4 the flexure formula for the determination of the maximum value σ_m of the normal stress in a given section of the beam,

$$\sigma_m = \frac{|M|c}{I} \tag{12.1}$$

where I is the moment of inertia of the cross section with respect to a centroidal axis perpendicular to the plane of the bending couple $\mathbf{M}$ and c is the maximum distance from the neutral surface (Fig. 12.16). We also recalled from Sec. 11.4 that, introducing the elastic section modulus $S = I/c$ of the beam, the maximum value σ_m of the normal stress in the section can be expressed as

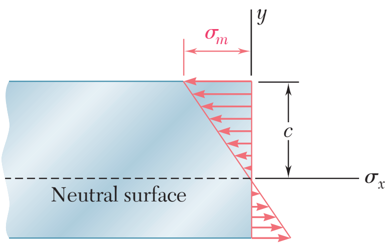
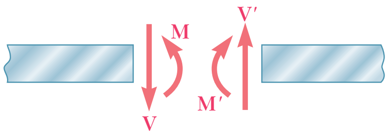


Fig. 12.16

$$\sigma_m = \frac{|M|}{S} \tag{12.3}$$

Shear and bending-moment diagrams

It follows from Eq. (12.1) that the maximum normal stress occurs in the section where $|M|$ is largest, at the point farthest from the neutral axis. The determination of the maximum value of $|M|$ and of the critical section of the beam in which it occurs is greatly simplified if we draw



(*a*) Internal forces
(positive shear and positive bending moment)

Fig. 12.17

a *shear diagram* and a *bending-moment diagram*. These diagrams represent, respectively, the variation of the shear and of the bending moment along the beam and were obtained by determining the values of V and M at selected points of the beam [Sec. 12.2]. These values were found by passing a section through the point where they were to be determined and drawing the free-body diagram of either of the portions of beam obtained in this fashion. To avoid any confusion regarding the sense of the shearing force **V** and of the bending couple **M** (which act in opposite sense on the two portions of the beam), we followed the sign convention adopted earlier in the text and illustrated in Fig. 12.17 [Examples 12.1 and 12.2, and Sample Probs. 12.1 and 12.2].

Relations among load, shear, and bending moment

The construction of the shear and bending-moment diagrams is facilitated if the following relations are taken into account [Sec. 12.3]. Denoting by w the distributed load per unit length (assumed positive if directed downward), we wrote

$$\frac{dV}{dx} = -w \qquad \frac{dM}{dx} = V \qquad (12.5,\ 12.7)$$

or, in integrated form,

$$V_D - V_C = -(\text{area under load curve between } C \text{ and } D) \qquad (12.6')$$

$$M_D - M_C = \text{area under shear curve between } C \text{ and } D \qquad (12.8')$$

Equation (12.6′) makes it possible to draw the shear diagram of a beam from the curve representing the distributed load on that beam and the value of V at one end of the beam. Similarly, Eq. (12.8′) makes it possible to draw the bending-moment diagram from the shear diagram and the value of M at one end of the beam. However, concentrated loads introduce discontinuities in the shear diagram and concentrated couples in the bending-moment diagram, none of which is accounted for in these equations [Sample Probs. 12.3 and 12.6]. Finally, we noted from Eq. (12.7) that the points of the beam where the bending moment is maximum or minimum are also the points where the shear is zero [Sample Prob. 12.4].

Design of prismatic beams

A proper procedure for the design of a prismatic beam was described in Sec. 12.4 and is summarized here:

Having determined σ_{all} for the material used and assuming that the design of the beam is controlled by the maximum normal stress in the beam, compute the minimum allowable value of the section modulus:

$$S_{min} = \frac{|M|_{max}}{\sigma_{all}} \qquad (12.9)$$

For a timber beam of rectangular cross section, $S = \frac{1}{6}bh^2$, where b is the width of the beam and h its depth. The dimensions of the section, therefore, must be selected so that $\frac{1}{6}bh^2 \geq S_{min}$.

For a rolled-steel beam, consult the appropriate table in App. B. Of the available beam sections, consider only those with a section modulus $S \geq S_{min}$ and select from this group the section with the smallest weight per unit length. This is the most economical of the sections for which $S \geq S_{min}$.

REVIEW PROBLEMS

12.77 and 12.78 Draw the shear and bending-moment diagrams for the beam and loading shown, and determine the maximum absolute value (*a*) of the shear, (*b*) of the bending moment.

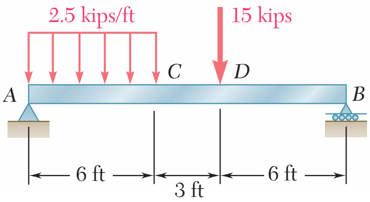


Fig. P12.77

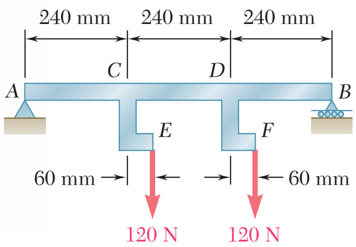


Fig. P12.78

12.79 Determine (*a*) the equations of the shear and bending-moment curves for the beam and loading shown, (*b*) the maximum absolute value of the bending moment in the beam.

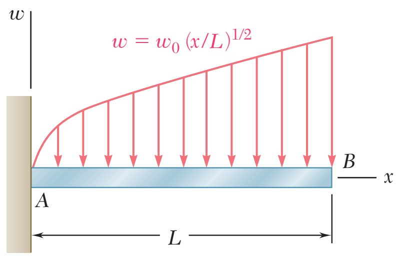


Fig. P12.79

12.80 For the beam and loading shown, determine the maximum normal stress due to bending on a transverse section at the center of the beam.

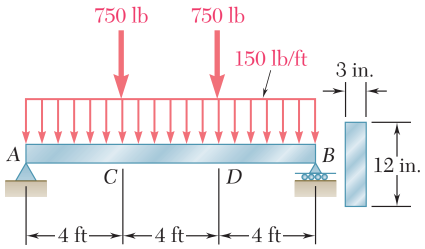


Fig. P12.80

12.81 Draw the shear and bending-moment diagrams for the beam and loading shown, and determine the maximum normal stress due to bending.

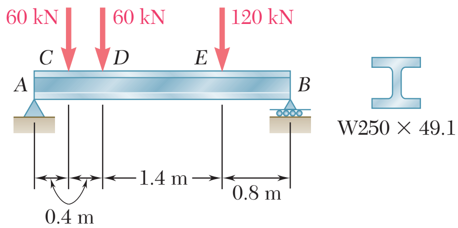


Fig. P12.81

12.82 Determine (*a*) the distance *a* for which the maximum absolute value of the bending moment in the beam is as small as possible, (*b*) the corresponding maximum normal stress due to bending. (*Hint:* Draw the bending-moment diagram, and equate the absolute values of the largest positive and negative bending moments obtained.)

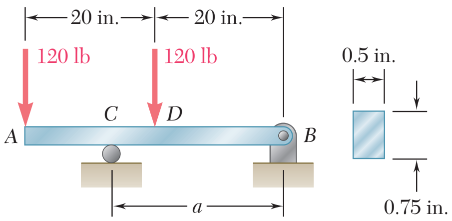


Fig. P12.82

12.83 Beam AB, of length L and square cross section of side a, is supported by a pivot at C and loaded as shown. (*a*) Check that the beam is in equilibrium. (*b*) Show that the maximum stress due to bending occurs at C and is equal to $w_0L^2/(1.5a)^3$.

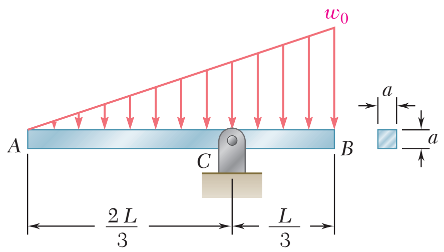


Fig. P12.83

12.84 Knowing that rod AB is in equilibrium under the loading shown, draw the shear and bending-moment diagrams and determine the maximum normal stress due to bending.

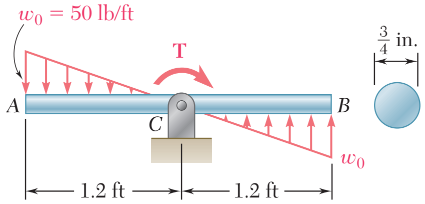


Fig. P12.84

12.85 For the beam and loading shown, design the cross section of the beam knowing that the grade of timber used has an allowable normal stress of 1750 psi.

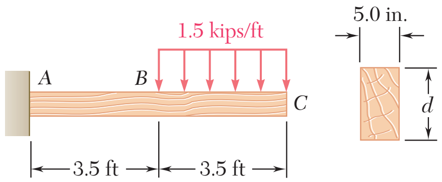


Fig. P12.85

12.86 For the beam and loading shown, design the cross section of the beam knowing that the grade of timber used has an allowable normal stress of 12 MPa.

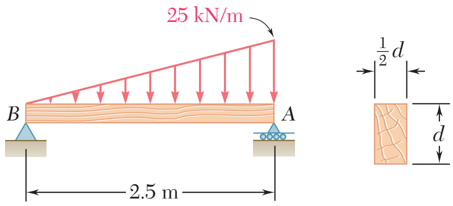


Fig. P12.86

12.87 Knowing that the allowable normal stress for the steel used is 160 MPa, select the most economical wide-flange beam to support the loading shown.

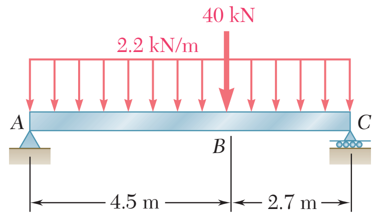


Fig. P12.87

12.88 Knowing that the allowable normal stress for the steel used is 24 ksi, select the most economical wide-flange beam to support the loading shown.

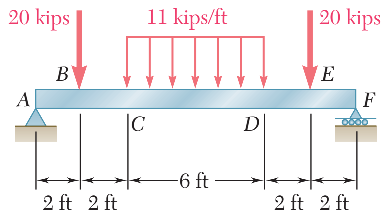
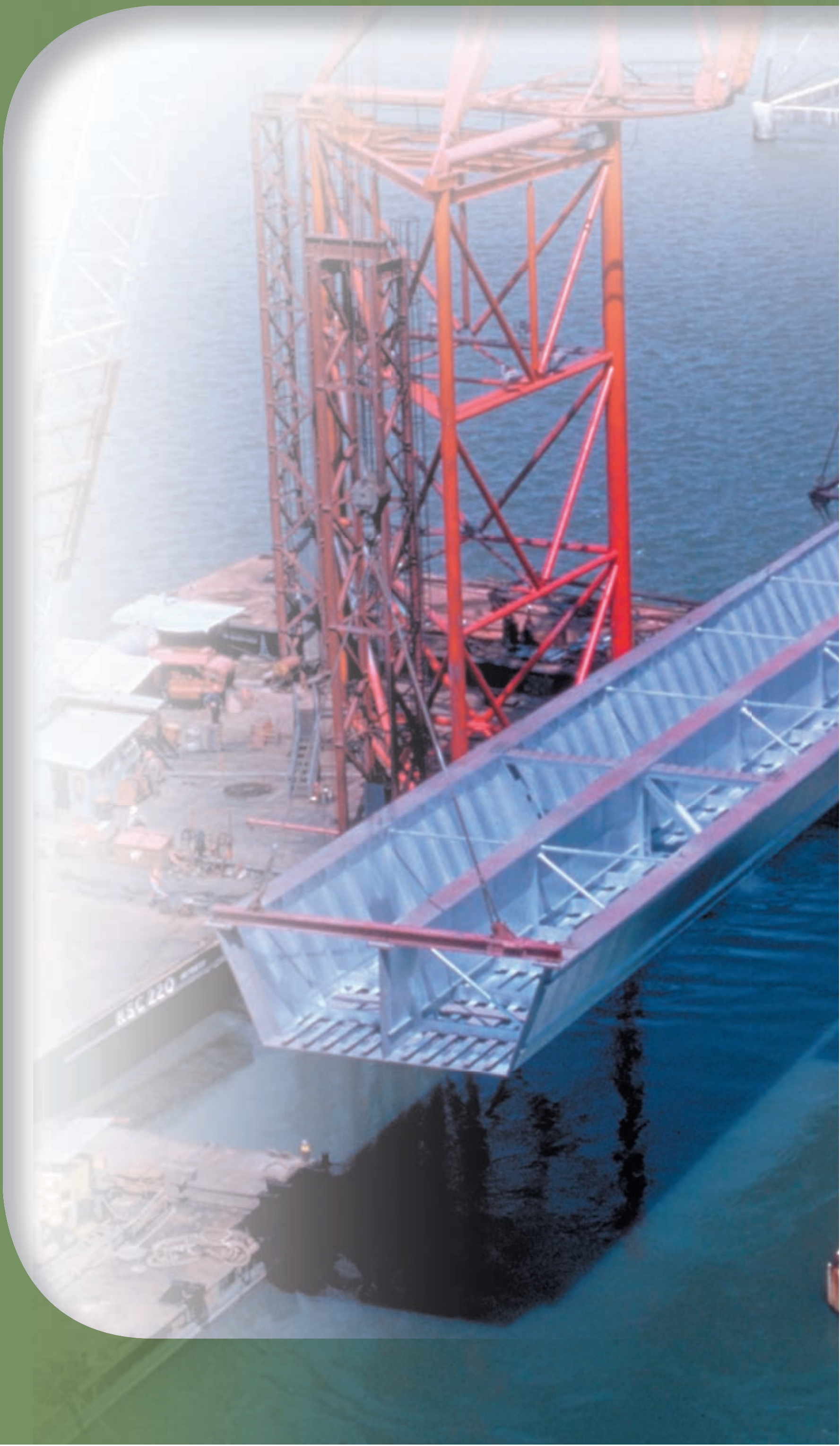


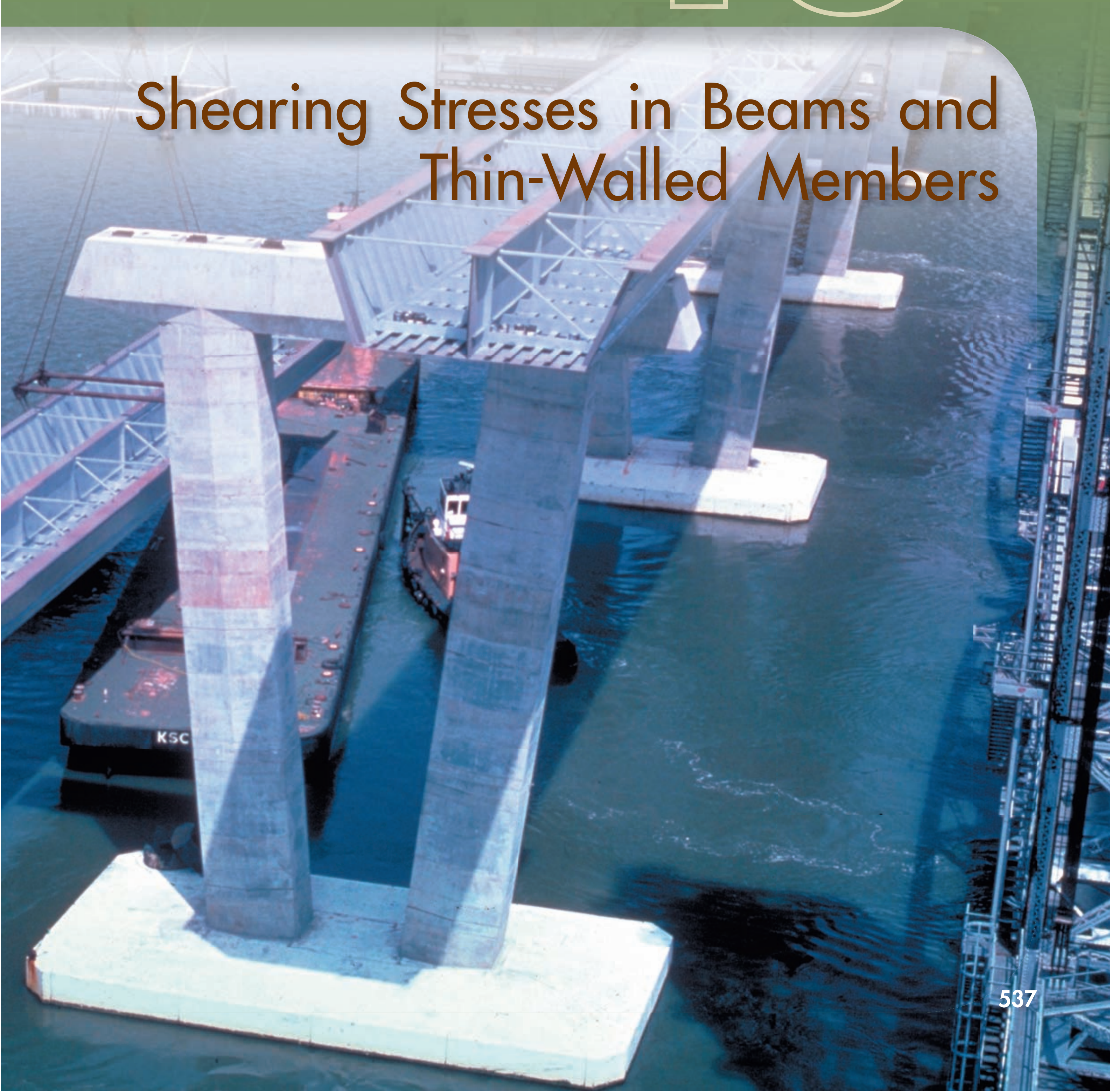
Fig. *P12.88*



A reinforced concrete deck will be attached to each of the steel sections shown to form a composite box girder bridge. In this chapter the shearing stresses will be determined in various types of beams and girders.

CHAPTER 13

Shearing Stresses in Beams and Thin-Walled Members



13.1 INTRODUCTION

You saw in Sec. 12.1 that a transverse loading applied to a beam will result in normal and shearing stresses in any given transverse section of the beam. The normal stresses are created by the bending couple **M** in that section and the shearing stresses by the shear **V**. Since the dominant criterion in the design of a beam for strength is the maximum value of the normal stress in the beam, our analysis was limited in Chap. 12 to the determination of the normal stresses. Shearing stresses, however, can be important, particularly in the design of short, stubby beams, and their analysis will be the subject of the first part of this chapter.

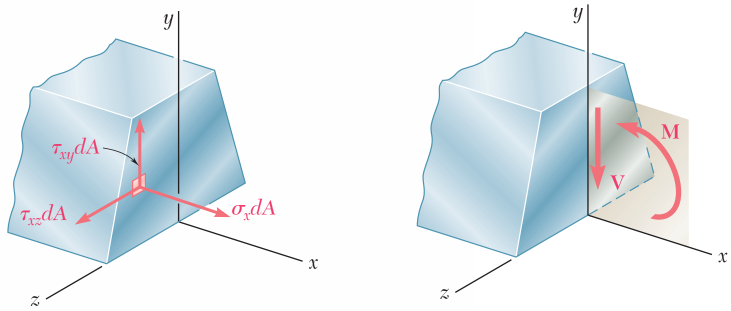


Fig. 13.1

Figure 13.1 expresses graphically that the elementary normal and shearing forces exerted on a given transverse section of a prismatic beam with a vertical plane of symmetry are equivalent to the bending couple **M** and the shearing force **V**. Six equations can be written to express that fact. Three of these equations involve only the normal forces $\sigma_x\, dA$ and have already been discussed in Sec. 11.2; they are Eqs. (11.1), (11.2), and (11.3), which express that the sum of the normal forces is zero and that the sums of their moments about the y and z axes are equal to zero and M, respectively. Three more equations involving the shearing forces $\tau_{xy}\, dA$ and $\tau_{xz}\, dA$ can now be written. One of them expresses that the sum of the moments of the shearing forces about the x axis is zero and can be dismissed as trivial in view of the symmetry of the beam with respect to the xy plane. The other two involve the y and z components of the elementary forces and are

$$y \text{ components:} \qquad \int \tau_{xy}\, dA = -V \tag{13.1}$$

$$z \text{ components:} \qquad \int \tau_{xz}\, dA = 0 \tag{13.2}$$

The first of these equations shows that vertical shearing stresses must exist in a transverse section of a beam under transverse loading. The second equation indicates that the average horizontal shearing stress in any section is zero. However, this does not mean that the shearing stress τ_{xz} is zero everywhere.

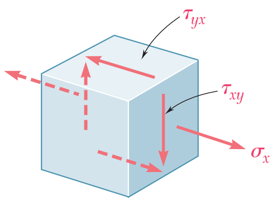


Fig. 13.2

Let us now consider a small cubic element located in the vertical plane of symmetry of the beam (where we know that τ_{xz} must be zero) and examine the stresses exerted on its faces (Fig. 13.2). As we have just seen, a normal stress σ_x and a shearing stress τ_{xy} are exerted on each of the two faces perpendicular to the x axis. But we know from Chap. 8 that, when shearing stresses τ_{xy} are exerted on the vertical faces of an element, equal stresses must be exerted on the horizontal faces of the same element. We thus conclude that longitudinal shearing stresses must exist in any member subjected to a transverse loading. This can be verified by considering a cantilever beam made of separate planks clamped together at one end (Fig. 13.3*a*). When a transverse load **P** is applied to the free end of this composite beam, the planks are observed to slide with respect to each other (Fig. 13.3*b*). In contrast, if a couple **M** is applied to the free end of the same composite beam (Fig. 13.3*c*), the various planks will bend into concentric arcs of circle and will not slide with respect to each other, thus verifying the fact that shear does not occur in a beam subjected to pure bending (cf. Sec. 11.3).

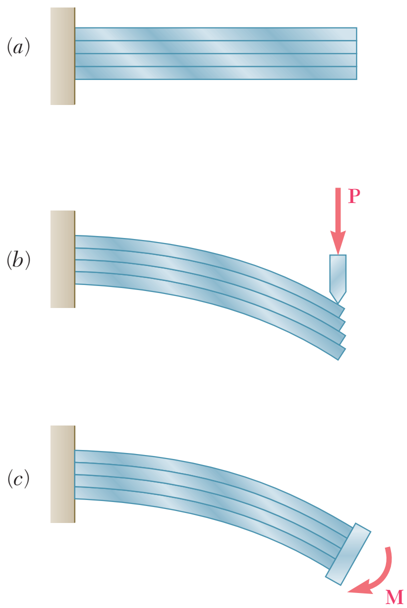


Fig. 13.3

While sliding does not actually take place when a transverse load **P** is applied to a beam made of a homogeneous and cohesive material such as steel, the tendency to slide does exist, showing that stresses occur on horizontal longitudinal planes as well as on vertical transverse planes. In the case of timber beams, whose resistance to shear is weaker between fibers, failure due to shear will occur along a longitudinal plane rather than a transverse plane (Photo 13.1).

In Sec. 13.2, a beam element of length Δx bounded by two transverse planes and a horizontal one will be considered and the shearing force $\Delta \mathbf{H}$ exerted on its horizontal face will be determined, as well as the shear per unit length, q, also known as *shear flow*. A formula for the shearing stress in a beam with a vertical plane of symmetry will be derived in Sec. 13.3 and used in Sec. 13.4 to determine the shearing stresses in common types of beams.

The derivation given in Sec. 13.2 will be extended in Sec. 13.5 to cover the case of a beam element bounded by two transverse planes and a curved surface. This will allow us in Sec. 13.6 to determine the shearing stresses at any point of a symmetric thin-walled member, such as the flanges of wide-flange beams and box beams.

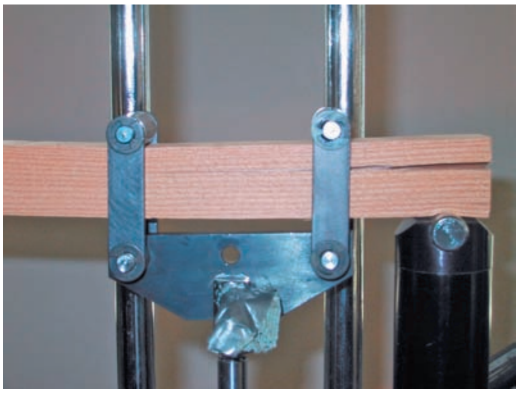


Photo 13.1

13.2 SHEAR ON THE HORIZONTAL FACE OF A BEAM ELEMENT

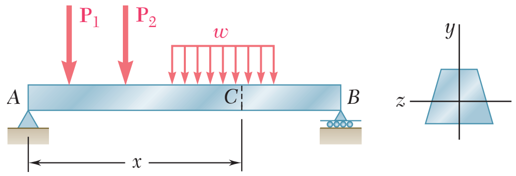


Fig. 13.4

Consider a prismatic beam AB with a vertical plane of symmetry that supports various concentrated and distributed loads (Fig. 13.4). At a distance x from end A we detach from the beam an element $CDD'C'$ of length Δx extending across the width of the beam from the upper surface of the beam to a horizontal plane located at a distance y_1 from the neutral axis (Fig. 13.5). The forces exerted on this element consist

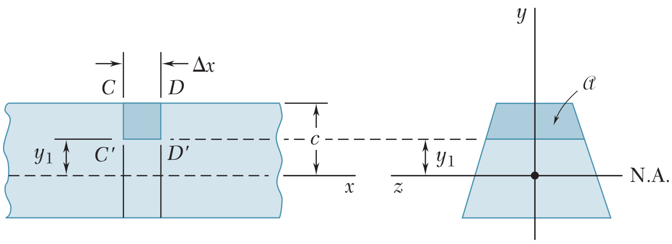


Fig. 13.5

of vertical shearing forces $\mathbf{V}'_C$ and $\mathbf{V}'_D$, a horizontal shearing force $\Delta\mathbf{H}$ exerted on the lower face of the element, elementary horizontal normal forces $\sigma_C\, dA$ and $\sigma_D\, dA$, and possibly a load $w\,\Delta x$ (Fig. 13.6). We write the equilibrium equation

$$\stackrel{+}{\rightarrow}\Sigma F_x = 0: \qquad \Delta H + \int_{\mathcal{A}} (\sigma_D - \sigma_C)\, dA = 0$$

where the integral extends over the shaded area $\mathcal{A}$ of the section located above the line $y = y_1$. Solving this equation for ΔH and using Eq. (12.2) of Sec. 12.1, $\sigma = My/I$, to express the normal stresses in

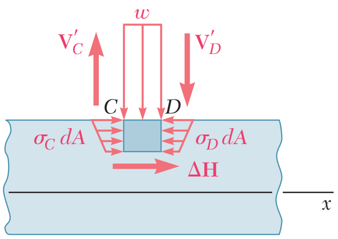


Fig. 13.6

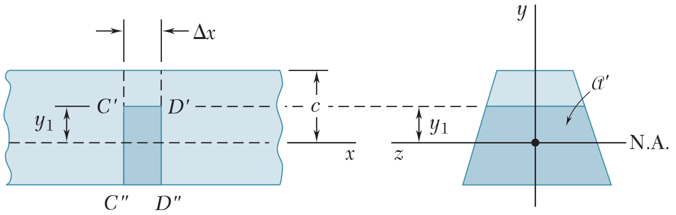


Fig. 13.7

terms of the bending moments at C and D, we have

$$\Delta H = \frac{M_D - M_C}{I} \int_{\mathcal{A}} y \, dA \tag{13.3}$$

The integral in (13.3) represents the *first moment* with respect to the neutral axis of the portion $\mathcal{A}$ of the cross section of the beam that is located above the line $y = y_1$ and will be denoted by Q. On the other hand, recalling Eq. (12.7) of Sec. 12.3, we can express the increment $M_D - M_C$ of the bending moment as

$$M_D - M_C = \Delta M = (dM/dx)\,\Delta x = V\,\Delta x$$

Substituting into (13.3), we obtain the following expression for the horizontal shear exerted on the beam element

$$\Delta H = \frac{VQ}{I} \Delta x \tag{13.4}$$

The same result would have been obtained if we had used as a free body the lower element $C'D'D''C''$, rather than the upper element $CDD'C'$ (Fig. 13.7), since the shearing forces $\mathbf{\Delta H}$ and $\mathbf{\Delta H'}$ exerted by the two elements on each other are equal and opposite. This leads us to observe that the first moment Q of the portion $\mathcal{A}$ of the cross section located below the line $y = y_1$ (Fig. 13.7) is equal in magnitude and opposite in sign to the first moment of the portion $\mathcal{A}$ located above that line (Fig. 13.5). Indeed, the sum of these two moments is equal to the moment of the area of the entire cross section with respect to its centroidal axis and, thus, must be zero. This property can sometimes be used to simplify the computation of Q. We also note that Q is maximum for $y_1 = 0$, since the elements of the cross section located above the neutral axis contribute positively to the integral in (13.3) that defines Q, while the elements located below that axis contribute negatively.

The *horizontal shear per unit length,* which will be denoted by the letter q, is obtained by dividing both members of Eq. (13.4) by Δx:

$$q = \frac{\Delta H}{\Delta x} = \frac{VQ}{I} \tag{13.5}$$

We recall that Q is the first moment with respect to the neutral axis of the portion of the cross section located either above or below the point at which q is being computed, and that I is the centroidal moment of inertia of the *entire* cross-sectional area. For a reason that will become apparent later (Sec. 13.6), the horizontal shear per unit length q is also referred to as the *shear flow*.

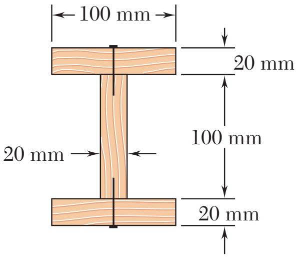


Fig. 13.8

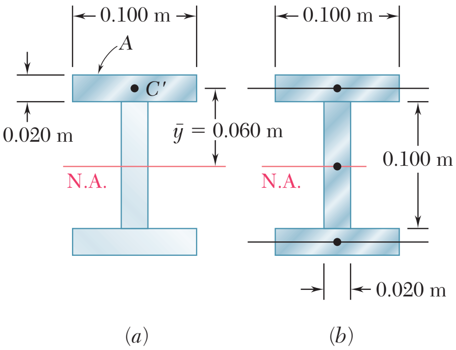


Fig. 13.9

EXAMPLE 13.1 A beam is made of three planks, 20 by 100 mm in cross section, nailed together (Fig. 13.8). Knowing that the spacing between nails is 25 mm and that the vertical shear in the beam is $V = 500$ N, determine the shearing force in each nail.

We first determine the horizontal force per unit length, q, exerted on the lower face of the upper plank. We use Eq. (13.5), where Q represents the first moment with respect to the neutral axis of the shaded area A shown in Fig. 13.9*a*, and where I is the moment of inertia about the same axis of the entire cross-sectional area (Fig. 13.9*b*). Recalling that the first moment of an area with respect to a given axis is equal to the product of the area and of the distance from its centroid to the axis,† we have

$$\begin{aligned} Q &= A\bar{y} = (0.020 \text{ m} \times 0.100 \text{ m})(0.060 \text{ m}) \\ &= 120 \times 10^{-6} \text{ m}^3 \\ I &= \tfrac{1}{12}(0.020 \text{ m})(0.100 \text{ m})^3 \\ &\quad + 2[\tfrac{1}{12}(0.100 \text{ m})(0.020 \text{ m})^3 \\ &\quad + (0.020 \text{ m} \times 0.100 \text{ m})(0.060 \text{ m})^2] \\ &= 1.667 \times 10^{-6} + 2(0.0667 + 7.2)10^{-6} \\ &= 16.20 \times 10^{-6} \text{ m}^4 \end{aligned}$$

Substituting into Eq. (13.5), we write

$$q = \frac{VQ}{I} = \frac{(500 \text{ N})(120 \times 10^{-6} \text{ m}^3)}{16.20 \times 10^{-6} \text{ m}^4} = 3704 \text{ N/m}$$

Since the spacing between the nails is 25 mm, the shearing force in each nail is

$$F = (0.025 \text{ m})q = (0.025 \text{ m})(3704 \text{ N/m}) = 92.6 \text{ N} \quad ■$$

13.3 DETERMINATION OF THE SHEARING STRESSES IN A BEAM

Consider again a beam with a vertical plane of symmetry, subjected to various concentrated or distributed loads applied in that plane. We saw in the preceding section that if, through two vertical cuts and one horizontal cut, we detach from the beam an element of length Δx (Fig. 13.10), the magnitude ΔH of the shearing force exerted on the horizontal face of the element can be obtained from Eq. (13.4). The *average shearing stress* τ_{ave} on that face of the element is obtained by dividing ΔH by the area ΔA of the face. Observing that $\Delta A = t\,\Delta x$, where t is the width of the element at the cut, we write

$$\tau_{\text{ave}} = \frac{\Delta H}{\Delta A} = \frac{VQ}{I}\,\frac{\Delta x}{t\,\Delta x}$$

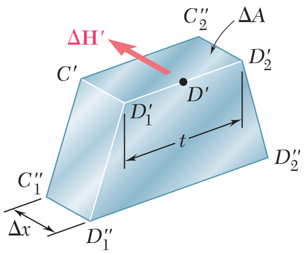


Fig. 13.10

†See Sec. 5.4.

or

$$\tau_{\text{ave}} = \frac{VQ}{It} \tag{13.6}$$

We note that, since the shearing stresses τ_{xy} and τ_{yx} exerted respectively on a transverse and a horizontal plane through D' are equal, the expression obtained also represents the average value of τ_{xy} along the line $D'_1 D'_2$ (Fig. 13.11).

We observe that $\tau_{yx} = 0$ on the upper and lower faces of the beam, since no forces are exerted on these faces. It follows that $\tau_{xy} = 0$ along the upper and lower edges of the transverse section (Fig. 13.12). We also note that, while Q is maximum for $y = 0$ (see Sec. 13.2), we cannot conclude that τ_{ave} will be maximum along the neutral axis, since τ_{ave} depends upon the width t of the section as well as upon Q.

As long as the width of the beam cross section remains small compared to its depth, the shearing stress varies only slightly along the line $D'_1 D'_2$ (Fig. 13.11) and Eq. (13.6) can be used to compute τ_{xy} at any point along $D'_1 D'_2$. Actually, τ_{xy} is larger at points D'_1 and D'_2 than at D', but the theory of elasticity shows† that, for a beam of rectangular section of width b and depth h, and as long as $b \le h/4$, the value of the shearing stress at points C_1 and C_2 (Fig. 13.13) does not exceed by more than 0.8% the average value of the stress computed along the neutral axis.‡

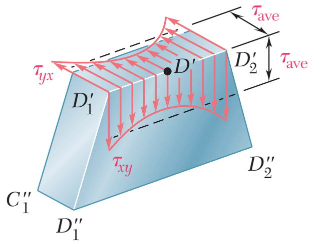


Fig. 13.11

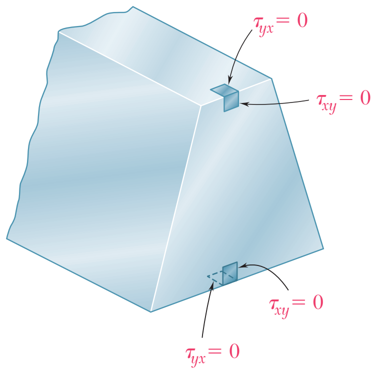


Fig. 13.12

13.4 SHEARING STRESSES τ_{xy} IN COMMON TYPES OF BEAMS

We saw in the preceding section that, for a *narrow rectangular beam*, i.e., for a beam of rectangular section of width b and depth h with $b \le \frac{1}{4}h$, the variation of the shearing stress τ_{xy} across the width of the beam is less than 0.8% of τ_{ave}. We can, therefore, use Eq. (13.6) in practical applications to determine the shearing stress at any point of the cross section of a narrow rectangular beam and write

$$\tau_{xy} = \frac{VQ}{It} \tag{13.7}$$

where t is equal to the width b of the beam, and where Q is the first moment with respect to the neutral axis of the shaded area A (Fig. 13.14).

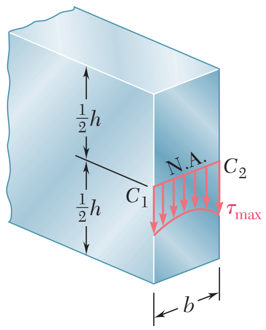


Fig. 13.13

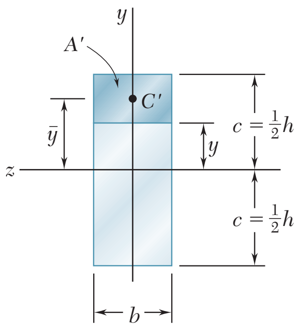


Fig. 13.14

†See S. P. Timoshenko and J. N. Goodier, *Theory of Elasticity*, McGraw-Hill, New York, 3d ed., 1970, sec. 124.

‡On the other hand, for large values of b/h, the value $\tau_{\max}$ of the stress at C_1 and C_2 may be many times larger then the average value τ_{ave} computed along the neutral axis, as we may see from the following table:

b/h	0.25	0.5	1	2	4	6	10	20	50
$\tau_{\max}/\tau_{\text{ave}}$	1.008	1.033	1.126	1.396	1.988	2.582	3.770	6.740	15.65
$\tau_{\min}/\tau_{\text{ave}}$	0.996	0.983	0.940	0.856	0.805	0.800	0.800	0.800	0.800

Observing that the distance from the neutral axis to the centroid C' of A is $\bar{y} = \frac{1}{2}(c + y)$ and recalling that $Q = A\bar{y}$, we write

$$Q = A\bar{y} = b(c - y)\tfrac{1}{2}(c + y) = \tfrac{1}{2}b(c^2 - y^2) \qquad (13.8)$$

Recalling, on the other hand, that $I = bh^3/12 = \frac{2}{3}bc^3$, we have

$$\tau_{xy} = \frac{VQ}{Ib} = \frac{3}{4}\frac{c^2 - y^2}{bc^3}V$$

or, noting that the cross-sectional area of the beam is $A = 2bc$,

$$\tau_{xy} = \frac{3}{2}\frac{V}{A}\left(1 - \frac{y^2}{c^2}\right) \qquad (13.9)$$

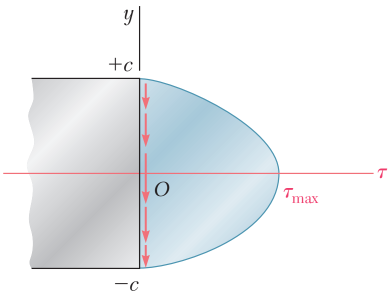


Fig. 13.15

Equation (13.9) shows that the distribution of shearing stresses in a transverse section of a rectangular beam is *parabolic* (Fig. 13.15). As we have already observed in the preceding section, the shearing stresses are zero at the top and bottom of the cross section ($y = \pm c$). Making $y = 0$ in Eq. (13.9), we obtain the value of the maximum shearing stress in a given section of a *narrow rectangular beam:*

$$\tau_{max} = \frac{3}{2}\frac{V}{A} \qquad (13.10)$$

The relation obtained shows that the maximum value of the shearing stress in a beam of rectangular cross section is 50% larger than the value V/A that would be obtained by wrongly assuming a uniform stress distribution across the entire cross section.

In the case of an *American standard beam* (S-beam) or a *wide-flange beam* (W-beam), Eq. (13.6) can be used to determine the average value of the shearing stress τ_{xy} over a section aa'or bb' of the transverse cross section of the beam (Figs. 13.16*a* and *b*). We write

$$\tau_{ave} = \frac{VQ}{It} \qquad (13.6)$$

where V is the vertical shear, t the width of the section at the elevation considered, Q the first moment of the shaded area with respect to the neutral axis cc', and I the moment of inertia of the entire cross-sectional area about cc'. Plotting τ_{ave} against the vertical distance y, we obtain the curve shown in Fig. 13.16*c*. We note the discontinuities existing in this curve, which reflect the difference

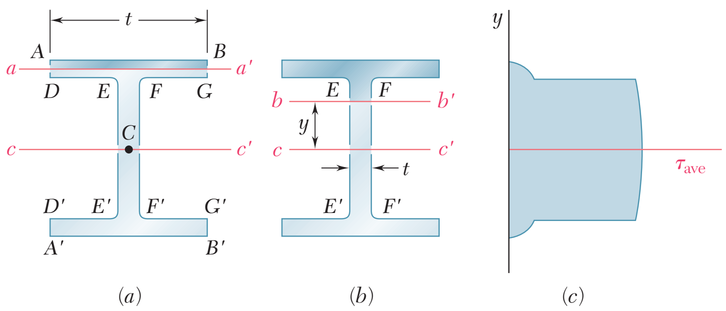


Fig. 13.16

between the values of t corresponding respectively to the flanges $ABGD$ and $A'B'G'D'$ and to the web $EFF'E'$.

In the case of the web, the shearing stress τ_{xy} varies only very slightly across the section bb' and can be assumed equal to its average value τ_{ave}. This is not true, however, for the flanges. For example, considering the horizontal line $DEFG$, we note that τ_{xy} is zero between D and E and between F and G, since these two segments are part of the free surface of the beam. On the other hand the value of τ_{xy} between E and F can be obtained by making $t = EF$ in Eq. (13.6). In practice, one usually assumes that the entire shear load is carried by the web, and that a good approximation of the maximum value of the shearing stress in the cross section can be obtained by dividing V by the cross-sectional area of the web.

$$\tau_{max} = \frac{V}{A_{web}} \tag{13.11}$$

We should note, however, that while the vertical component τ_{xy} of the shearing stress in the flanges can be neglected, its horizontal component τ_{xz} has a significant value that will be determined in Sec. 13.6.

EXAMPLE 13.2 Knowing that the allowable shearing stress for the timber beam of Sample Prob. 12.7 is $\tau_{all} = 0.250$ ksi, check that the design obtained in that sample problem is acceptable from the point of view of the shearing stresses.

We recall from the shear diagram of Sample Prob. 12.7 that $V_{max} = 4.50$ kips. The actual width of the beam was given as $b = 3.5$ in. and the value obtained for its depth was $h = 14.55$ in. Using Eq. (13.10) for the maximum shearing stress in a narrow rectangular beam, we write

$$\tau_{max} = \frac{3}{2}\frac{V}{A} = \frac{3}{2}\frac{V}{bh} = \frac{3(4.50 \text{ kips})}{2(3.5 \text{ in.})(14.55 \text{ in.})} = 0.1325 \text{ ksi}$$

Since $\tau_{max} < \tau_{all}$, the design obtained in Sample Prob. 12.7 is acceptable. ■

EXAMPLE 13.3 Knowing that the allowable shearing stress for the steel beam of Sample Prob. 12.8 is $\tau_{all} = 90$ MPa, check that the W360 × 32.9 shape obtained in that sample problem is acceptable from the point of view of the shearing stresses.

We recall from the shear diagram of Sample Prob. 12.8 that the maximum absolute value of the shear in the beam is $|V|_{max} = 58$ kN. As we saw in this section it may be assumed in practice that the entire shear load is carried by the web and that the maximum value of the shearing stress in the beam can be obtained from Eq. (13.11). From App. B we find that for a W360 × 32.9 shape the depth of the beam and the thickness of its web are, respectively, $d = 348$ mm and $t_w = 5.84$ mm. We thus have

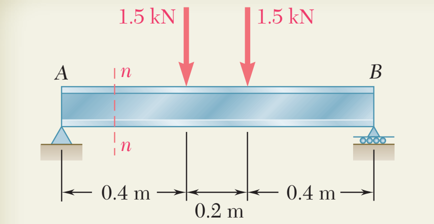
$$A_{web} = d\,t_w = (348 \text{ mm})(5.84 \text{ mm}) = 2032 \text{ mm}^2$$

Substituting the values of $|V|_{max}$ and A_{web} into Eq. (13.11), we obtain

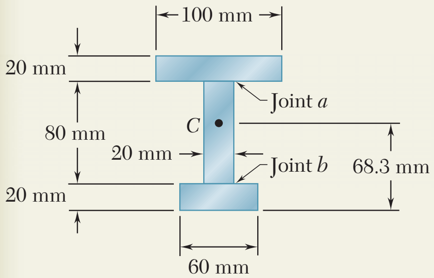
$$\tau_{max} = \frac{|V|_{max}}{A_{web}} = \frac{58 \text{ kN}}{2032 \text{ mm}^2} = 28.5 \text{ MPa}$$

Since $\tau_{max} < \tau_{all}$, the design obtained in Sample Prob. 12.8 is acceptable. ■

SAMPLE PROBLEM 13.1

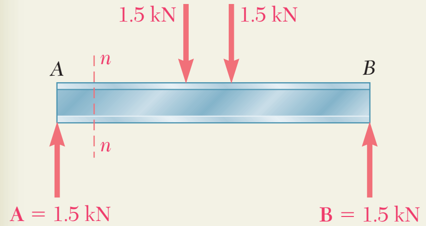


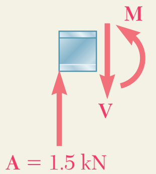
Beam AB is made of three planks glued together and is subjected, in its plane of symmetry, to the loading shown. Knowing that the width of each glued joint is 20 mm, determine the average shearing stress in each joint at section n-n of the beam. The location of the centroid of the section is given in the sketch and the centroidal moment of inertia is known to be $I = 8.63 \times 10^{-6}\ \text{m}^4$.



SOLUTION

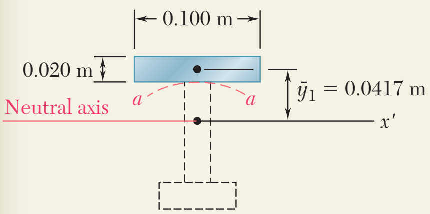
Vertical Shear at Section *n-n*. Since the beam and loading are both symmetric with respect to the center of the beam, we have $\mathbf{A} = \mathbf{B} = 1.5\ \text{kN}\uparrow$.





Considering the portion of the beam to the left of section n-n as a free body, we write

$$+\uparrow \sum F_y = 0: \qquad 1.5\ \text{kN} - V = 0 \qquad V = 1.5\ \text{kN}$$

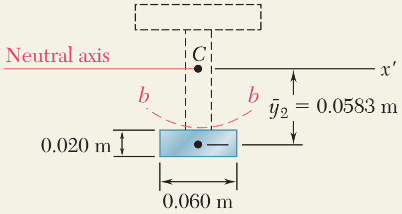


Shearing Stress in Joint *a*. We pass the section a-a through the glued joint and separate the cross-sectional area into two parts. We choose to determine Q by computing the first moment with respect to the neutral axis of the area above section a-a.

$$Q = A\bar{y}_1 = [(0.100\ \text{m})(0.020\ \text{m})](0.0417\ \text{m}) = 83.4 \times 10^{-6}\ \text{m}^3$$

Recalling that the width of the glued joint is $t = 0.020$ m, we use Eq. (13.7) to determine the average shearing stress in the joint.

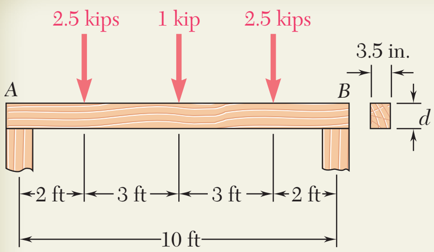
$$\tau_{\text{ave}} = \frac{VQ}{It} = \frac{(1500\ \text{N})(83.4 \times 10^{-6}\ \text{m}^3)}{(8.63 \times 10^{-6}\ \text{m}^4)(0.020\ \text{m})} \qquad \tau_{\text{ave}} = 725\ \text{kPa} \blacktriangleleft$$



Shearing Stress in Joint *b*. We now pass section b-b and compute Q by using the area below the section.

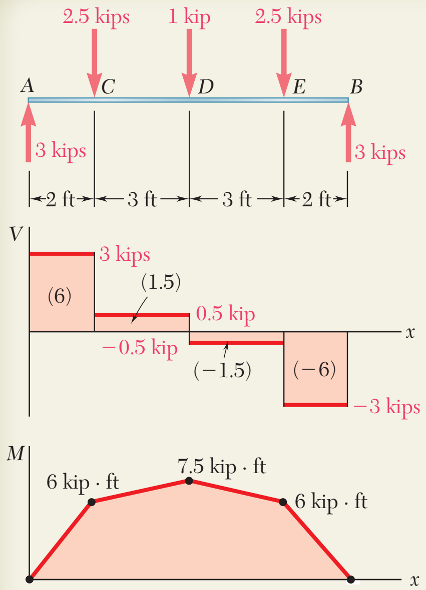
$$Q = A\bar{y}_2 = [(0.060\ \text{m})(0.020\ \text{m})](0.0583\ \text{m}) = 70.0 \times 10^{-6}\ \text{m}^3$$

$$\tau_{\text{ave}} = \frac{VQ}{It} = \frac{(1500\ \text{N})(70.0 \times 10^{-6}\ \text{m}^3)}{(8.63 \times 10^{-6}\ \text{m}^4)(0.020\ \text{m})} \qquad \tau_{\text{ave}} = 608\ \text{kPa} \blacktriangleleft$$



SAMPLE PROBLEM 13.2

A timber beam AB of span 10 ft and nominal width 4 in. (actual width = 3.5 in.) is to support the three concentrated loads shown. Knowing that for the grade of timber used σ_{all} = 1800 psi and τ_{all} = 120 psi, determine the minimum required depth d of the beam.



SOLUTION

Maximum Shear and Bending Moment. After drawing the shear and bending-moment diagrams, we note that

$$M_{max} = 7.5 \text{ kip} \cdot \text{ft} = 90 \text{ kip} \cdot \text{in.}$$
$$V_{max} = 3 \text{ kips}$$

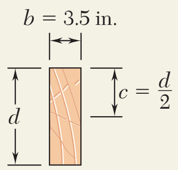
Design Based on Allowable Normal Stress. We first express the elastic section modulus S in terms of the depth d. We have

$$I = \frac{1}{12}bd^3 \qquad S = \frac{I}{c} = \frac{1}{6}bd^2 = \frac{1}{6}(3.5)d^2 = 0.5833d^2$$

For M_{max} = 90 kip · in. and σ_{all} = 1800 psi, we write

$$S = \frac{M_{max}}{\sigma_{all}} \qquad 0.5833d^2 = \frac{90 \times 10^3 \text{ lb} \cdot \text{in.}}{1800 \text{ psi}}$$
$$d^2 = 85.7 \qquad\qquad d = 9.26 \text{ in.}$$

We have satisfied the requirement that $\sigma_m \leq 1800$ psi.



Check Shearing Stress. For V_{max} = 3 kips and d = 9.26 in., we find

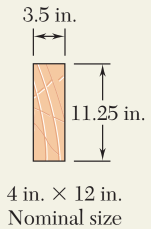
$$\tau_m = \frac{3}{2}\frac{V_{max}}{A} = \frac{3}{2}\frac{3000 \text{ lb}}{(3.5 \text{ in.})(9.26 \text{ in.})} \qquad \tau_m = 138.8 \text{ psi}$$

Since τ_{all} = 120 psi, the depth d = 9.26 in. is *not* acceptable and we must redesign the beam on the basis of the requirement that $\tau_m \leq 120$ psi.

Design Based on Allowable Shearing Stress. Since we now know that the allowable shearing stress controls the design, we write

$$\tau_m = \tau_{all} = \frac{3}{2}\frac{V_{max}}{A} \qquad 120 \text{ psi} = \frac{3}{2}\frac{3000 \text{ lb}}{(3.5 \text{ in.})d}$$

$$d = 10.71 \text{ in.} \quad ◀$$



The normal stress is, of course, less than σ_{all} = 1800 psi, and the depth of 10.71 in. is fully acceptable.

Comment. Since timber is normally available in depth increments of 2 in., a 4 × 12-in. nominal size timber should be used. The actual cross section would then be 3.5 × 11.25 in.

PROBLEMS

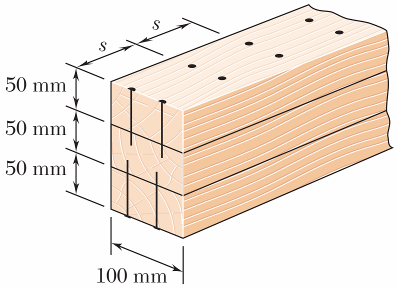


Fig. P13.1

13.1 Three full-size 50 × 100-mm boards are nailed together to form a beam that is subjected to a vertical shear of 1500 N. Knowing that the allowable shearing force in each nail is 400 N, determine the largest longitudinal spacing s that can be used between each pair of nails.

13.2 For the built-up beam of Prob. 13.1, determine the allowable shear if the spacing between each pair of nails is $s = 45$ mm.

13.3 A square box beam is made of two $\frac{3}{4} \times 3.5$-in. planks and two $\frac{3}{4} \times 5$-in. planks nailed together as shown. Knowing that the spacing between nails is $s = 1.25$ in. and that the vertical shear in the beam is $V = 250$ lb, determine (*a*) the shearing force in each nail, (*b*) the maximum shearing stress in the beam.

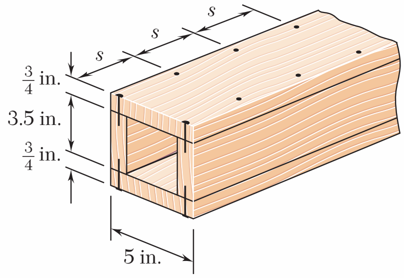


Fig. P13.3 and P13.4

13.4 A square box beam is made of two $\frac{3}{4} \times 3.5$-in. planks and two $\frac{3}{4} \times 5$-in. planks nailed together as shown. Knowing that the spacing between nails is s = 2 in. and that the allowable shearing force in each nail is 75 lb, determine (*a*) the largest allowable vertical shear in the beam, (*b*) the corresponding maximum shearing stress in the beam.

13.5 The American Standard rolled-steel beam shown has been reinforced by attaching to it two 16 × 200-mm plates using 18-mm-diameter bolts spaced longitudinally every 120 mm. Knowing that the average allowable shearing stress in the bolts is 90 MPa, determine the largest permissible vertical shearing force.

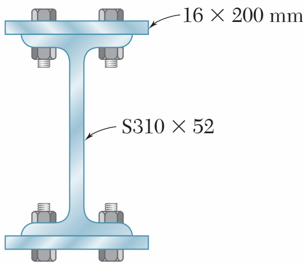


Fig. P13.5

13.6 Solve Prob. 13.5 assuming that the reinforcing plates are only 12 mm thick.

13.7 and 13.8 A column is fabricated by connecting the rolled-steel members shown by bolts of $\frac{3}{4}$-in. diameter spaced longitudinally every 5 in. Determine the average shearing stress in the bolts caused by a shearing force of 30 kips parallel to the y axis.

13.9 through 13.12 For the beam and loading shown, consider section n-n and determine (a) the largest shearing stress in that section, (b) the shearing stress at point a.

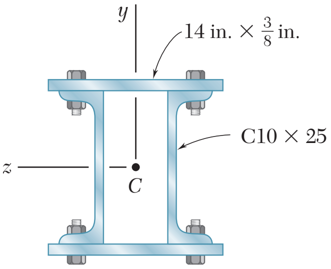


Fig. P13.7

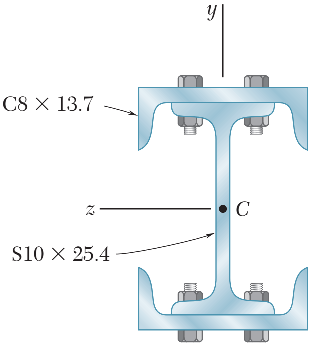


Fig. P13.8

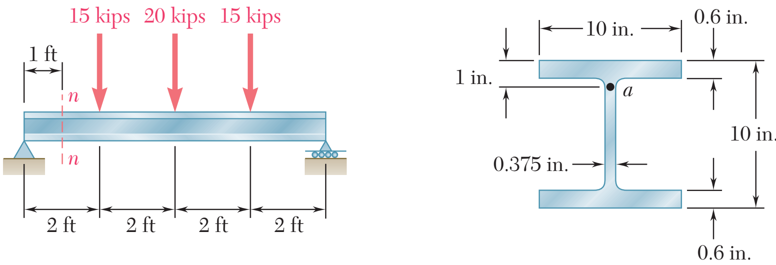


Fig. P13.9

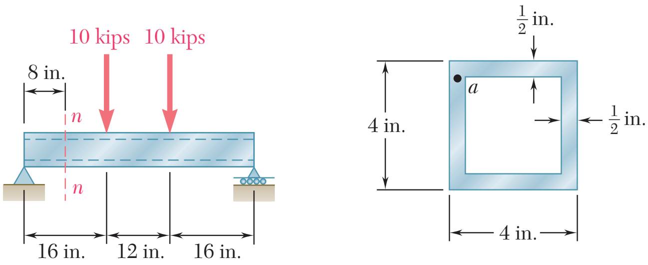


Fig. P13.10

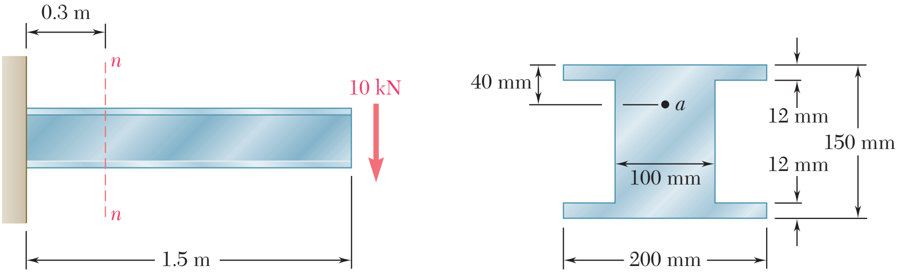


Fig. P13.11

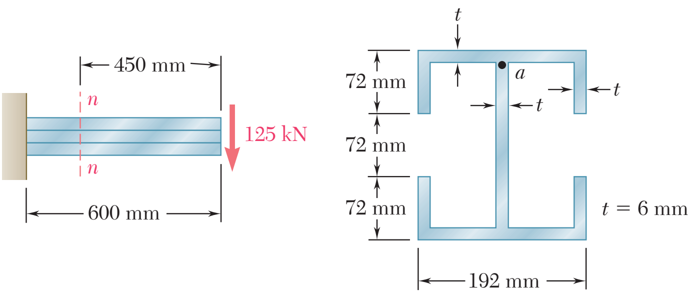


Fig. P13.12

13.13 For the beam and loading shown, determine the minimum required depth h knowing that for the grade of timber used $\sigma_{all} = 1750$ psi and $\tau_{all} = 130$ psi.

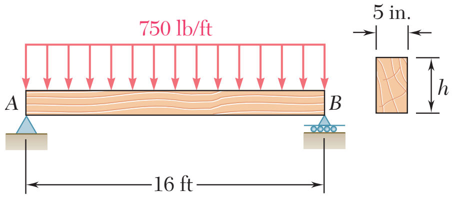


Fig. P13.13

13.14 For the beam and loading shown, determine the minimum required width b knowing that for the grade of timber used $\sigma_{all} = 12$ MPa and $\tau_{all} = 825$ kPa.

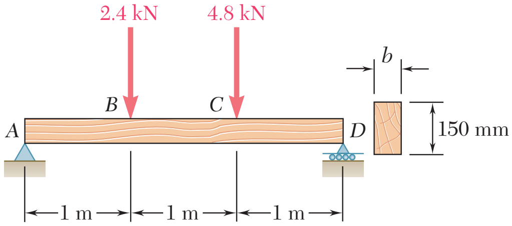


Fig. P13.14

13.15 For the wide-flange beam with the loading shown, determine the largest load **P** that can be applied knowing that the maximum normal stress is 24 ksi and the largest shearing stress using the approximation $\tau_m = V/A_{web}$ is 14.5 ksi.

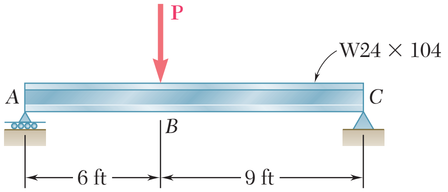


Fig. *P13.15*

13.16 For the wide-flange beam with the loading shown, determine the largest load **P** that can be applied knowing that the maximum normal stress is 160 MPa and the largest shearing stress using the approximation $\tau_m = V/A_{web}$ is 100 MPa.

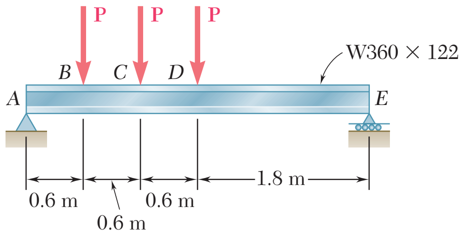


Fig. *P13.16*

13.17 and 13.18 For the beam and loading shown, consider section n-n and determine the shearing stress at (a) point a, (b) point b.

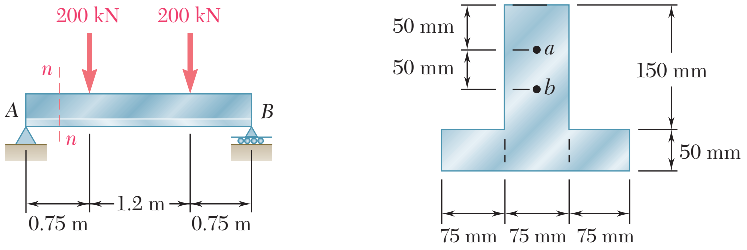


Fig. P13.17 and P13.19

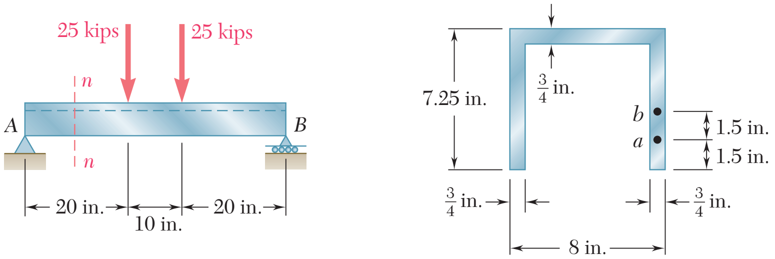


Fig. P13.18 and P13.20

13.19 and 13.20 For the beam and loading shown, determine the largest shearing stress in section n-n.

***13.21* through 13.24** A beam having the cross section shown is subjected to a vertical shear **V**. Determine (a) the horizontal line along which the shearing stress is maximum, (b) the constant k in the following expression for the maximum shearing stress

$$\tau_{\max} = k\frac{V}{A}$$

where A is the cross-sectional area of the beam.

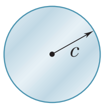


Fig. *P13.21*

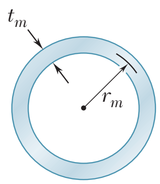


Fig. P13.22

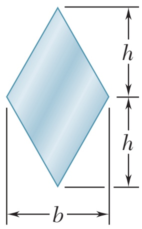


Fig. *P13.23*

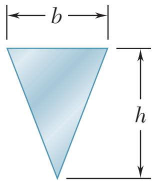


Fig. P13.24

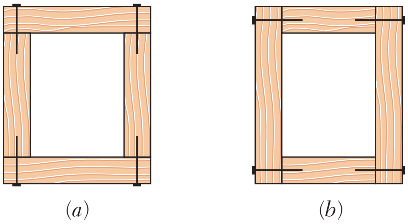


Fig. 13.17

13.5 LONGITUDINAL SHEAR ON A BEAM ELEMENT OF ARBITRARY SHAPE

Consider a box beam obtained by nailing together four planks, as shown in Fig. 13.17*a*. You learned in Sec. 13.2 how to determine the shear per unit length, q, on the horizontal surfaces along which the planks are joined. But could you determine q if the planks had been joined along *vertical* surfaces, as shown in Fig. 13.17*b*? We examined in Sec. 13.4 the distribution of the vertical components τ_{xy} of the stresses on a transverse section of a W-beam or an S-beam and found that these stresses had a fairly constant value in the web of the beam and were negligible in its flanges. But what about the *horizontal* components τ_{xz} of the stresses in the flanges?

To answer these questions we must extend the procedure developed in Sec. 13.2 for the determination of the shear per unit length, q, so that it will apply to the cases just described.

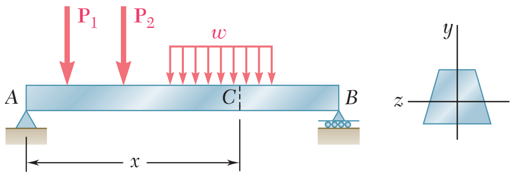


Fig. 13.4 (*repeated*)

Consider the prismatic beam AB of Fig. 13.4, which has a vertical plane of symmetry and supports the loads shown. At a distance x from end A we detach again an element $CDD'C'$ of length Δx. This element, however, will now extend from two sides of the beam to an arbitrary curved surface (Fig. 13.18). The forces exerted on the

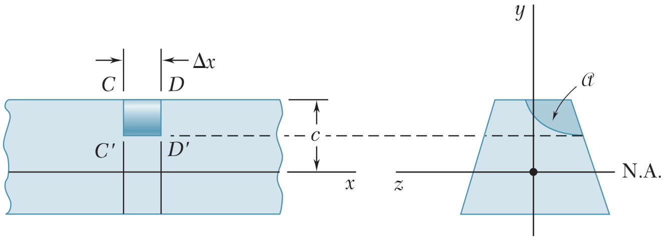


Fig. 13.18

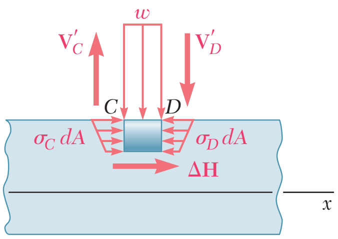


Fig. 13.19

element include vertical shearing forces $\mathbf{V}'_C$ and $\mathbf{V}'_D$, elementary horizontal normal forces $\sigma_C\, dA$ and $\sigma_D\, dA$, possibly a load $w\,\Delta x$, and a longitudinal shearing force $\mathbf{\Delta H}$ representing the resultant of the elementary longitudinal shearing forces exerted on the curved surface (Fig. 13.19). We write the equation of equilibrium

$$\xrightarrow{+} \Sigma F_x = 0: \qquad \Delta H + \int_{\mathcal{a}} (\sigma_D - \sigma_C)\, dA = 0$$

where the integral is to be computed over the shaded area $\mathcal{A}$ of the section. We observe that the equation obtained is the same as the one we obtained in Sec. 13.2, but that the shaded area $\mathcal{A}$ over which the integral is to be computed now extends to the curved surface.

The remainder of the derivation is the same as in Sec. 13.2. We find that the longitudinal shear exerted on the beam element is

$$\Delta H = \frac{VQ}{I}\Delta x \tag{13.4}$$

where I is the centroidal moment of inertia of the entire section, Q the first moment of the shaded area $\mathcal{A}$ with respect to the neutral axis, and V the vertical shear in the section. Dividing both members of Eq. (13.4) by Δx, we obtain the horizontal shear per unit length, or shear flow:

$$q = \frac{\Delta H}{\Delta x} = \frac{VQ}{I} \tag{13.5}$$

EXAMPLE 13.4 A square box beam is made of two 0.75×3-in. planks and two 0.75×4.5-in. planks, nailed together as shown (Fig. 13.20). Knowing that the spacing between nails is 1.75 in. and that the beam is subjected to a vertical shear of magnitude $V = 600$ lb, determine the shearing force in each nail.

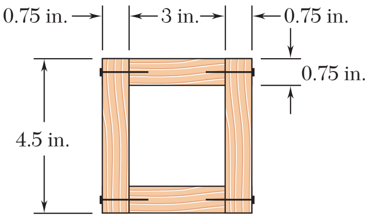


Fig. 13.20

We isolate the upper plank and consider the total force per unit length, q, exerted on its two edges. We use Eq. (13.5), where Q represents the first moment with respect to the neutral axis of the shaded area A' shown in Fig. 13.21a, and where I is the moment of inertia about the same axis of the entire cross-sectional area of the box beam (Fig. 13.21b). We have

$$Q = A'\bar{y} = (0.75 \text{ in.})(3 \text{ in.})(1.875 \text{ in.}) = 4.22 \text{ in}^3$$

Recalling that the moment of inertia of a square of side a about a centroidal axis is $I = \frac{1}{12}a^4$, we write

$$I = \tfrac{1}{12}(4.5 \text{ in.})^4 - \tfrac{1}{12}(3 \text{ in.})^4 = 27.42 \text{ in}^4$$

Substituting into Eq. (13.5), we obtain

$$q = \frac{VQ}{I} = \frac{(600 \text{ lb})(4.22 \text{ in}^3)}{27.42 \text{ in}^4} = 92.3 \text{ lb/in.}$$

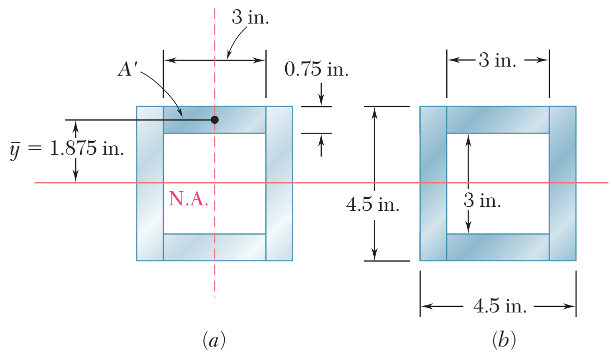


Fig. 13.21

Because both the beam and the upper plank are symmetric with respect to the vertical plane of loading, equal forces are exerted on both edges of the plank. The force per unit length on each of these edges is thus $\frac{1}{2}q = \frac{1}{2}(92.3) = 46.15$ lb/in. Since the spacing between nails is 1.75 in., the shearing force in each nail is

$$F = (1.75 \text{ in.})(46.15 \text{ lb/in.}) = 80.8 \text{ lb} \blacksquare$$

13.6 SHEARING STRESSES IN THIN-WALLED MEMBERS

We saw in the preceding section that Eq. (13.4) may be used to determine the longitudinal shear $\Delta\mathbf{H}$ exerted on the walls of a beam element of arbitrary shape and Eq. (13.5) to determine the corresponding shear flow q. These equations will be used in this section to calculate both the shear flow and the average shearing stress in thin-walled members such as the flanges of wide-flange beams (Photo 13.2) and box beams, or the walls of structural tubes (Photo 13.3).

Photo 13.2

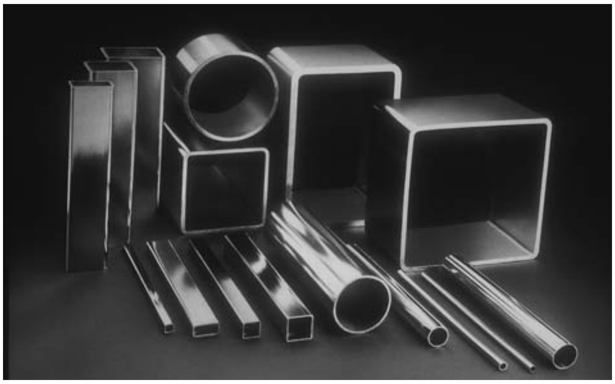


Photo 13.3

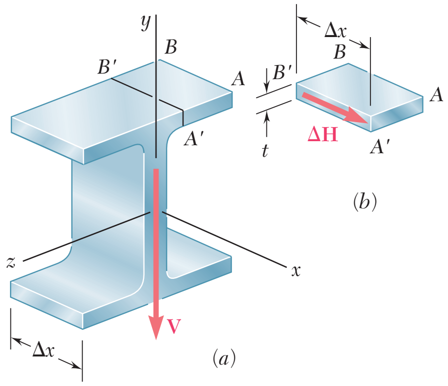


Fig. 13.22

Consider, for instance, a segment of length Δx of a wide-flange beam (Fig. 13.22*a*) and let $\mathbf{V}$ be the vertical shear in the transverse section shown. Let us detach an element $ABB'A'$ of the upper flange (Fig. 13.22*b*). The longitudinal shear $\Delta\mathbf{H}$ exerted on that element can be obtained from Eq. (13.4):

$$\Delta H = \frac{VQ}{I}\Delta x \tag{13.4}$$

Dividing ΔH by the area $\Delta A = t\,\Delta x$ of the cut, we obtain for the average shearing stress exerted on the element the same expression that we had obtained in Sec. 13.3 in the case of a horizontal cut:

$$\tau_{\text{ave}} = \frac{VQ}{It} \tag{13.6}$$

Note that τ_{ave} now represents the average value of the shearing stress τ_{zx} over a vertical cut, but since the thickness t of the flange

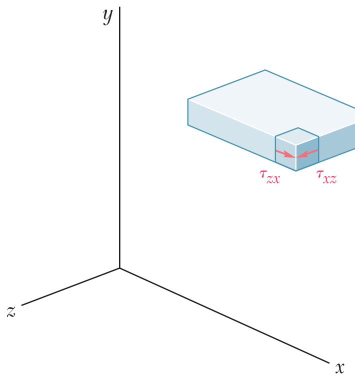


Fig. 13.23

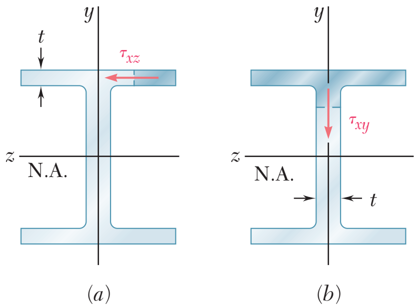


Fig. 13.24

is small, there is very little variation of τ_{zx} across the cut. Recalling that $\tau_{xz} = \tau_{zx}$ (Fig. 13.23), we conclude that the horizontal component τ_{xz} of the shearing stress at any point of a transverse section of the flange can be obtained from Eq. (13.6), where Q is the first moment of the shaded area about the neutral axis (Fig. 13.24*a*). We recall that a similar result was obtained in Sec. 13.4 for the vertical component τ_{xy} of the shearing stress in the web (Fig. 13.24*b*). Equation (13.6) can be used to determine shearing stresses in box beams (Fig. 13.25), half pipes (Fig. 13.26), and other thin-walled members, as long as the loads are applied in a plane of symmetry of the member. In each case, the cut must be perpendicular to the surface of the member, and Eq. (13.6) will yield the component of the shearing stress in the direction of the tangent to that surface. (The other component may be assumed equal to zero, in view of the proximity of the two free surfaces.)

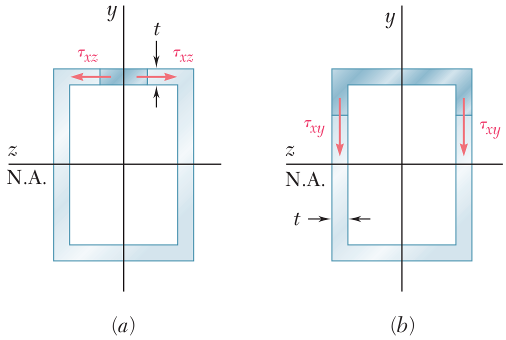


Fig. 13.25

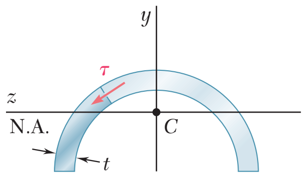


Fig. 13.26

Comparing Eqs. (13.5) and (13.6), we note that the product of the shearing stress τ at a given point of the section and of the thickness t of the section at that point is equal to q. Since V and I are constant in any given section, q depends only upon the first moment Q and, thus, can easily be sketched on the section. In the case of a

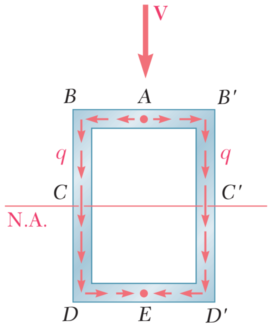


Fig. 13.27 Variation of q in box-beam section.

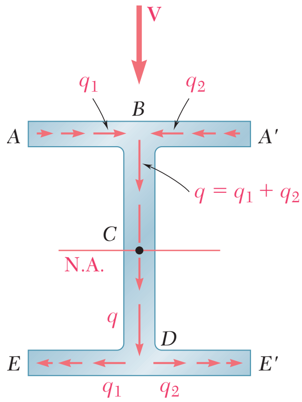


Fig. 13.28 Variation of q in wide-flange beam section.

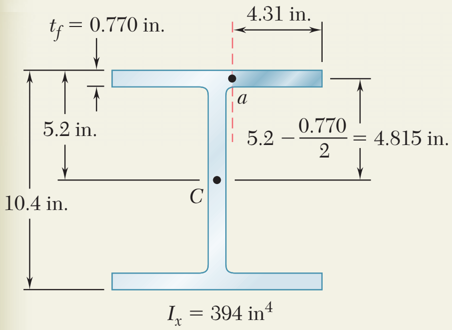
box beam, for example (Fig. 13.27), we note that q grows smoothly from zero at A to a maximum value at C and C' on the neutral axis, and then decreases back to zero as E is reached. We also note that there is no sudden variation in the magnitude of q as we pass a corner at B, D, B', or D', and that the sense of q in the horizontal portions of the section may be easily obtained from its sense in the vertical portions (which is the same as the sense of the shear **V**). In the case of a wide-flange section (Fig. 13.28), the values of q in portions AB and $A'B$ of the upper flange are distributed symmetrically. As we turn at B into the web, the values of q corresponding to the two halves of the flange must be combined to obtain the value of q at the top of the web. After reaching a maximum value at C on the neutral axis, q decreases, and at D splits into two equal parts corresponding to the two halves of the lower flange. The term *shear flow* commonly used to refer to the shear per unit length, q, reflects the similarity between the properties of q that we have just described and some of the characteristics of a fluid flow through an open channel or pipe.

So far we have assumed that all the loads were applied in a plane of symmetry of the member. In the case of members possessing two planes of symmetry, such as the wide-flange beam of Fig. 13.24 or the box beam of Fig. 13.25, any load applied through the centroid of a given cross section can be resolved into components along the two axes of symmetry of the section. Each component will cause the member to bend in a plane of symmetry, and the corresponding shearing stresses can be obtained from Eq. (13.6). The principle of superposition can then be used to determine the resulting stresses.

However, if the member considered possesses no plane of symmetry, or if it possesses a single plane of symmetry and is subjected to a load that is not contained in that plane, the member is observed to *bend and twist* at the same time, except when the load is applied at a specific point, called the *shear center*.† Note that the shear center generally does *not* coincide with the centroid of the cross section.

†See Ferdinand P. Beer, E. Russell Johnston, Jr., John T. DeWolf, and David F. Mazurek, *Mechanics of Materials,* 5th ed., McGraw-Hill, New York, 2009, sec. 6.9.

SAMPLE PROBLEM 13.3



Knowing that the vertical shear is 50 kips in a W10 × 68 rolled-steel beam, determine the horizontal shearing stress in the top flange at a point a located 4.31 in. from the edge of the beam. The dimensions and other geometric data of the rolled-steel section are given in App. B.

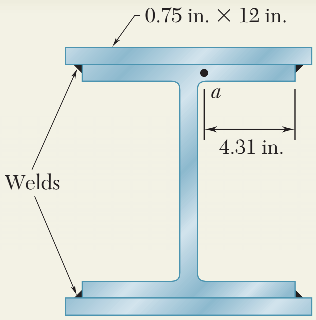
SOLUTION

We isolate the shaded portion of the flange by cutting along the dashed line that passes through point a.

$$Q = (4.31 \text{ in.})(0.770 \text{ in.})(4.815 \text{ in.}) = 15.98 \text{ in}^3$$

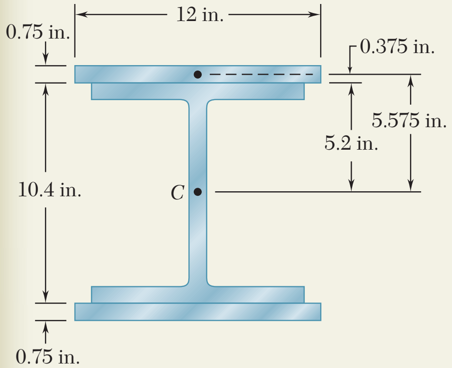
$$\tau = \frac{VQ}{It} = \frac{(50 \text{ kips})(15.98 \text{ in}^3)}{(394 \text{ in}^4)(0.770 \text{ in})} \qquad \tau = 2.63 \text{ ksi} \blacktriangleleft$$

SAMPLE PROBLEM 13.4



Solve Sample Prob. 13.3, assuming that 0.75 × 12-in. plates have been attached to the flanges of the W10 × 68 beam by continuous fillet welds as shown.

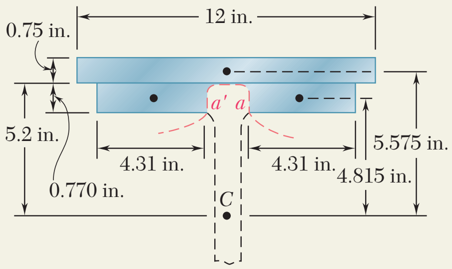
SOLUTION



For the composite beam the centroidal moment of inertia is

$$I = 394 \text{ in}^4 + 2[\tfrac{1}{12}(12 \text{ in.})(0.75 \text{ in.})^3 + (12 \text{ in.})(0.75 \text{ in.})(5.575 \text{ in.})^2]$$
$$I = 954 \text{ in}^4$$

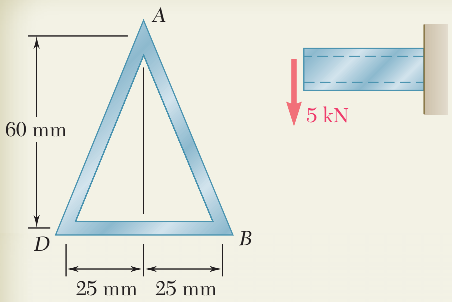
Since the top plate and the flange are connected only at the welds, we find the shearing stress at a by passing a section through the flange at a, *between* the plate and the flange, and again through the flange at the symmetric point a'.



For the shaded area that we have isolated, we have

$$t = 2t_f = 2(0.770 \text{ in.}) = 1.540 \text{ in.}$$
$$Q = 2[(4.31 \text{ in.})(0.770 \text{ in.})(4.815 \text{ in.})] + (12 \text{ in.})(0.75 \text{ in.})(5.575 \text{ in.})$$
$$Q = 82.1 \text{ in}^3$$

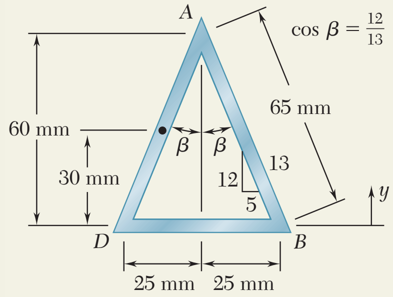
$$\tau = \frac{VQ}{It} = \frac{(50 \text{ kips})(82.1 \text{ in}^3)}{(954 \text{ in}^4)(1.540 \text{ in.})} \qquad \tau = 2.79 \text{ ksi} \blacktriangleleft$$



SAMPLE PROBLEM 13.5

The thin-walled extruded beam shown is made of aluminum and has a uniform 3-mm wall thickness. Knowing that the shear in the beam is 5 kN, determine (*a*) the shearing stress at point *A*, (*b*) the maximum shearing stress in the beam. *Note:* The dimensions given are to lines midway between the outer and inner surfaces of the beam.

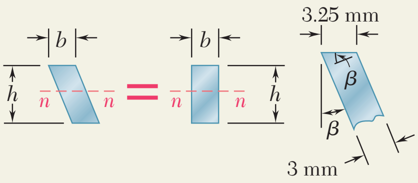
SOLUTION

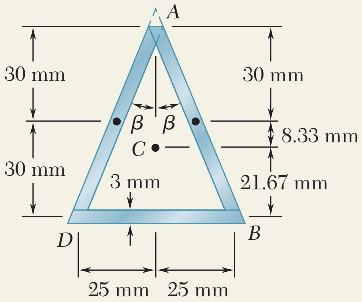


Centroid. We note that $AB = AD = 65$ mm.

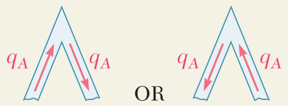
$$\bar{Y} = \frac{\Sigma\ \bar{y}A}{\Sigma\ A} = \frac{2[(65\text{ mm})(3\text{ mm})(30\text{ mm})]}{2[(65\text{ mm})(3\text{ mm})] + (50\text{ mm})(3\text{ mm})}$$
$$\bar{Y} = 21.67\text{ mm}$$

Centroidal Moment of Inertia. Each side of the thin-walled beam can be considered as a parallelogram, and we recall that for the case shown $I_{nn} = bh^3/12$ where b is measured parallel to the axis nn.



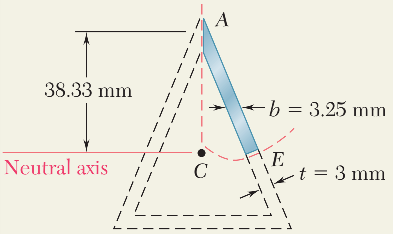


$$b = (3\text{ mm})/\cos\beta = (3\text{ mm})/(12/13) = 3.25\text{ mm}$$
$$I = \Sigma(\bar{I} + Ad^2) = 2[\tfrac{1}{12}(3.25\text{ mm})(60\text{ mm})^3 + (3.25\text{ mm})(60\text{ mm})(8.33\text{ mm})^2] + [\tfrac{1}{12}(50\text{ mm})(3\text{ mm})^3 + (50\text{ mm})(3\text{ mm})(21.67\text{ mm})^2]$$
$$I = 214.6 \times 10^3\text{ mm}^4 \qquad I = 0.2146 \times 10^{-6}\text{ m}^4$$



a. Shearing Stress at *A*. If a shearing stress τ_A occurs at *A*, the shear flow will be $q_A = \tau_A t$ and must be directed in one of the two ways shown. But the cross section and the loading are symmetric about a vertical line through *A*, and thus the shear flow must also be symmetric. Since neither of the possible shear flows is symmetric, we conclude that $\tau_A = 0$ ◀

b. Maximum Shearing Stress. Since the wall thickness is constant, the maximum shearing stress occurs at the neutral axis, where Q is maximum. Since we know that the shearing stress at *A* is zero, we cut the section along the dashed line shown and isolate the shaded portion of the beam. In order to obtain the largest shearing stress, the cut at the neutral axis is made perpendicular to the sides and is of length $t = 3$ mm.



$$Q = [(3.25\text{ mm})(38.33\text{ mm})]\left(\frac{38.33\text{ mm}}{2}\right) = 2387\text{ mm}^3$$
$$Q = 2.387 \times 10^{-6}\text{ m}^3$$
$$\tau_E = \frac{VQ}{It} = \frac{(5\text{ kN})(2.387 \times 10^{-6}\text{ m}^3)}{(0.2146 \times 10^{-6}\text{ m}^4)(0.003\text{ m})} \qquad \tau_{max} = \tau_E = 18.54\text{ MPa} ◀$$

PROBLEMS

13.25 The built-up timber beam is subjected to a 6-kN vertical shear. Knowing that the longitudinal spacing of the nails is $s = 60$ mm and that each nail is 90 mm long, determine the shearing force in each nail.

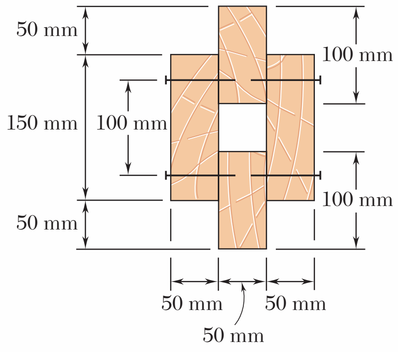


Fig. P13.25

13.26 The built-up timber beam is subjected to a vertical shear of 1200 lb. Knowing that the allowable shearing force in the nails is 75 lb, determine the largest permissible spacing s of the nails.

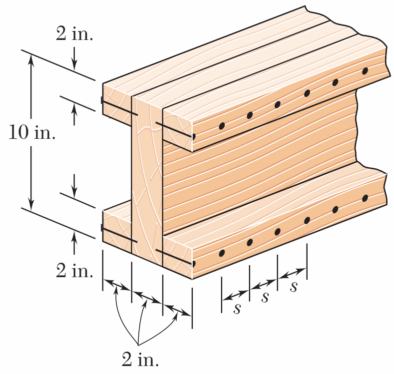
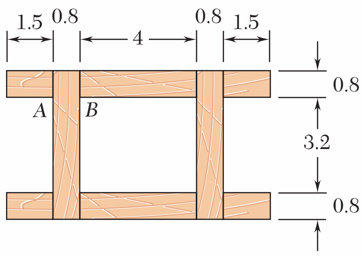


Fig. P13.26



Dimensions in inches

Fig. P13.27

13.27 The built-up beam was made by gluing together several wooden planks. Knowing that the beam is subjected to a 1200-lb vertical shear, determine the average shearing stress in the glued joint (*a*) at *A*, (*b*) at *B*.

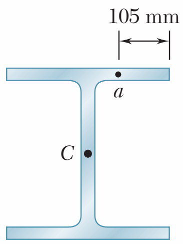


Fig. P13.28

13.28 Knowing that a W360 × 122 rolled-steel beam is subjected to a 250-kN vertical shear, determine the shearing stress (*a*) at point *a*, (*b*) at the centroid *C* of the section.

13.29 and *13.30* An extruded aluminum beam has the cross section shown. Knowing that the vertical shear in the beam is 150 kN, determine the shearing stress at (*a*) point *a*, (*b*) point *b*.

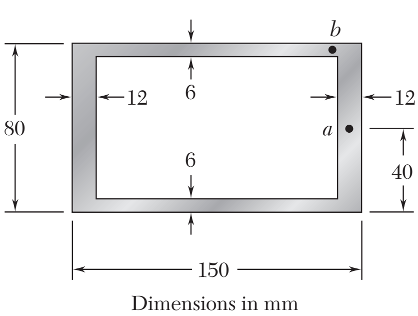


Fig. P13.29

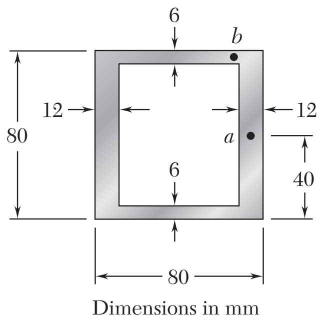


Fig. *P13.30*

13.31 and *13.32* The extruded beam shown has a uniform wall thickness of $\frac{1}{8}$ in. Knowing that the vertical shear in the beam is 2 kips, determine the shearing stress at each of the five points indicated.

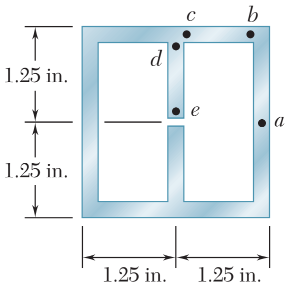


Fig. P13.31

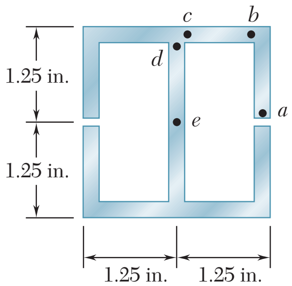


Fig. *P13.32*

13.33 Knowing that a given vertical shear **V** causes a maximum shearing stress of 75 MPa in the hat-shaped extrusion shown, determine the corresponding shearing stress at (*a*) point *a*, (*b*) point *b*.

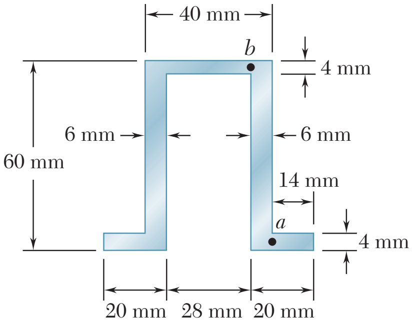


Fig. P13.33

13.34 Knowing that a given vertical shear **V** causes a maximum shearing stress of 50 MPa in a thin-walled member having the cross section shown, determine the corresponding shearing stress at (*a*) point *a*, (*b*) point *b*, (*c*) point *c*.

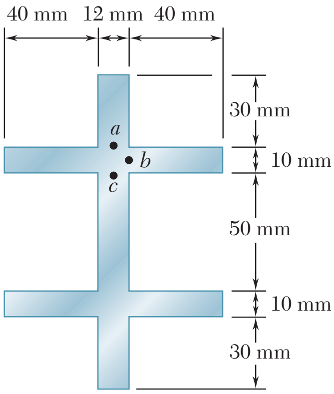


Fig. P13.34

13.35 The vertical shear is 1200 lb in a beam having the cross section shown. Knowing that $d = 4$ in., determine the shearing stress at (a) point a, (b) point b.

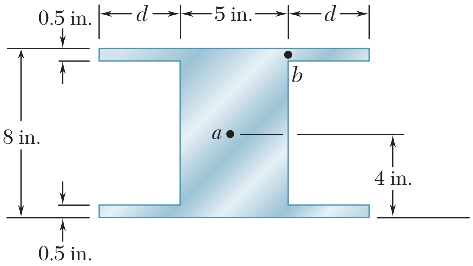


Fig. P13.35 and P13.36

13.36 The vertical shear is 1200 lb in a beam having the cross section shown. Determine (a) the distance d for which $\tau_a = \tau_b$, (b) the corresponding shearing stress at points a and b.

13.37 A beam consists of three planks connected by steel bolts with a longitudinal spacing of 225 mm. Knowing that the shear in the beam is vertical and equal to 6 kN and that the allowable average shearing stress in each bolt is 60 MPa, determine the smallest permissible bolt diameter that can be used.

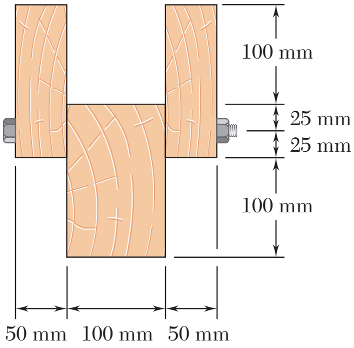


Fig. P13.37

13.38 Four L102 × 102 × 9.5 steel angle shapes and a 12 × 400-mm steel plate are bolted together to form a beam with the cross section shown. The bolts are of 22-mm diameter and are spaced longitudinally every 120 mm. Knowing that the beam is subjected to a vertical shear of 240 kN, determine the average shearing stress in each bolt.

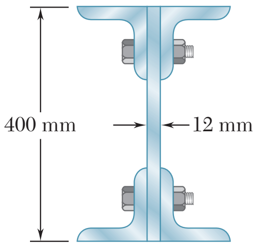


Fig. *P13.38*

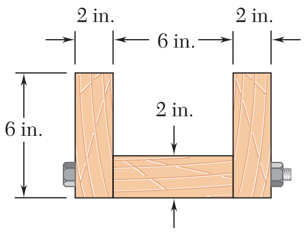


Fig. P13.39

13.39 A beam consists of three planks connected as shown by $\frac{3}{8}$-in.-diameter bolts spaced every 12 in. along the longitudinal axis of the beam. Knowing that the beam is subjected to a 2500-lb vertical shear, determine the average shearing stress in the bolts.

13.40 A beam consists of five planks of 1.5 × 6-in. cross section connected by steel bolts with a longitudinal spacing of 9 in. Knowing that the shear in the beam is vertical and equal to 2000 lb and that the allowable average shearing stress in each bolt is 7500 psi, determine the smallest permissible bolt diameter that can be used.

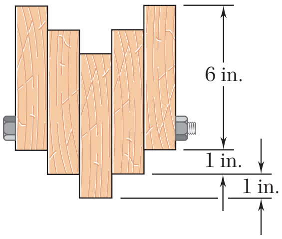


Fig. *P13.40*

13.41 Three plates, each 12 mm thick, are welded together to form the section shown. For a vertical shear of 100 kN, determine the shear flow through the welded surfaces, and sketch the shear flow in the cross section.

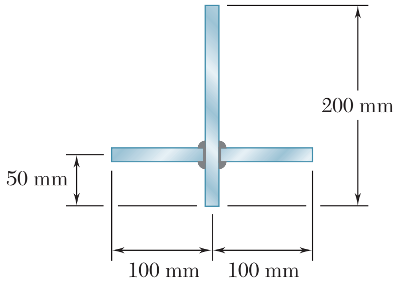


Fig. P13.41

13.42 A plate of 2-mm thickness is bent as shown and then used as a beam. For a vertical shear of 5 kN, determine the shearing stress at the five points indicated, and sketch the shear flow in the cross section.

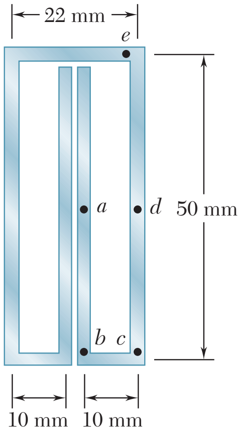


Fig. P13.42

13.43 A plate of $\frac{1}{4}$-in. thickness is corrugated as shown and then used as a beam. For a vertical shear of 1.2 kips, determine (*a*) the maximum shearing stress in the section, (*b*) the shearing stress at point *B*. Also sketch the shear flow in the cross section.

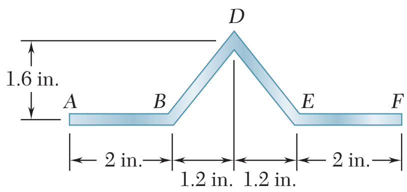


Fig. P13.43

13.44 A plate of thickness t is bent as shown and then used as a beam. For a vertical shear of 600 lb, determine (*a*) the thickness t for which the maximum shearing stress is 300 psi, (*b*) the corresponding shearing stress at point *E*. Also sketch the shear flow in the cross section.

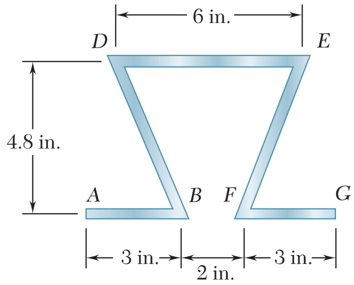


Fig. *P13.44*

13.45 For a beam made of two or more materials with different moduli of elasticity, show that Eq. (13.6)

$$\tau_{\text{ave}} = \frac{VQ}{It}$$

remains valid provided that both Q and I are computed by using the transformed section of the beam (see Sec. 11.5) and provided further that t is the actual width of the beam where τ_{ave} is computed.

13.46 A composite beam is made by attaching the timber and steel portions shown with bolts of 12-mm diameter spaced longitudinally every 200 mm. The modulus of elasticity is 10 GPa for the wood and 200 GPa for the steel. For a vertical shear of 4 kN, determine (*a*) the average shearing stress in the bolts, (b) the shearing stress at the center of the cross section. (*Hint:* Use the method indicated in Prob. 13.45.)

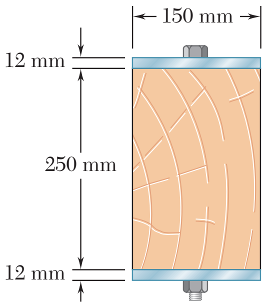


Fig. P13.46

13.47 A composite beam is made by attaching the timber and steel portions shown with bolts of $\frac{5}{8}$-in. diameter spaced longitudinally every 8 in. The modulus of elasticity is 1.9×10^6 psi for the wood and 29×10^6 psi for the steel. For a vertical shear of 4000 lb, determine (*a*) the average shearing stress in the bolts, (*b*) the shearing stress at the center of the cross section. (*Hint:* Use the method indicated in Prob. 13.45.)

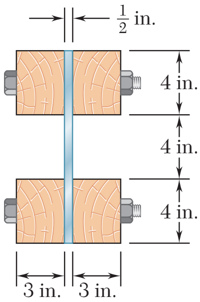


Fig. P13.47

13.48 A steel bar and an aluminum bar are bonded together as shown to form a composite beam. Knowing that the vertical shear in the beam is 6 kN and that the modulus of elasticity is 210 GPa for the steel and 70 GPa for the aluminum, determine (*a*) the average stress at the bonded surface, (*b*) the maximum shearing stress in the beam. (*Hint:* Use the method indicated in Prob. 13.45.)

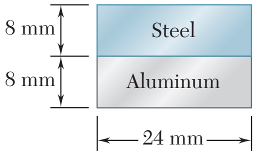


Fig. *P13.48*

13.49 A steel bar and an aluminum bar are bonded together as shown to form a composite beam. Knowing that the vertical shear in the beam is 4 kips and that the modulus of elasticity is 29×10^6 psi for the steel and 10.6×10^6 psi for the aluminum, determine (*a*) the average stress at the bonded surface, (*b*) the maximum shearing stress in the beam. (*Hint:* Use the method indicated in Prob. 13.45.)

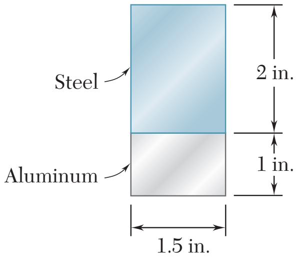


Fig. P13.49

REVIEW AND SUMMARY

This chapter was devoted to the analysis of beams and thin-walled members under transverse loadings.

Stresses on a beam element

In Sec. 13.1 we considered a small element located in the vertical plane of symmetry of a beam under a transverse loading (Fig. 13.29) and found that normal stresses σ_x and shearing stresses τ_{xy} were exerted on the transverse faces of that element, while shearing stresses τ_{yx}, equal in magnitude to τ_{xy}, were exerted on its horizontal faces.

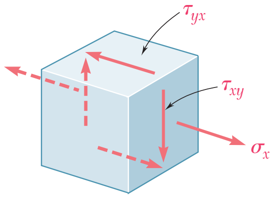


Fig. 13.29

In Sec. 13.2 we considered a prismatic beam AB with a vertical plane of symmetry supporting various concentrated and distributed loads (Fig. 13.30). At a distance x from end A we detached from the

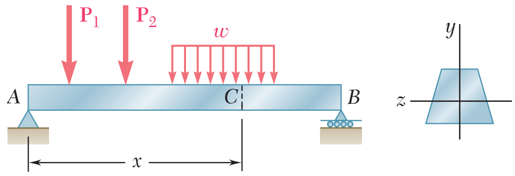


Fig. 13.30

beam an element $CDD'C'$ of length Δx extending across the width of the beam from the upper surface of the beam to a horizontal plane located at a distance y_1 from the neutral axis (Fig. 13.31). We found

Horizontal shear in a beam

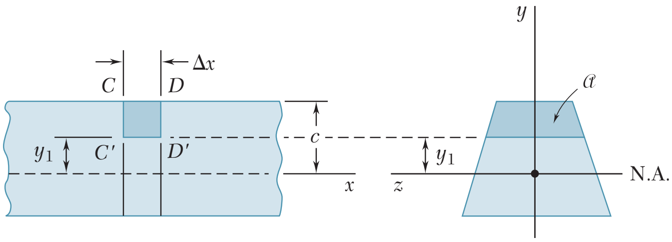


Fig. 13.31

that the magnitude of the shearing force $\Delta\mathbf{H}$ exerted on the lower face of the beam element was

$$\Delta H = \frac{VQ}{I}\Delta x \tag{13.4}$$

where V = vertical shear in the given transverse section
Q = first moment with respect to the neutral axis of the shaded portion $\mathcal{a}$ of the section
I = centroidal moment of inertia of the entire cross-sectional area

Shear flow

The *horizontal shear per unit length,* or *shear flow,* which was denoted by the letter q, was obtained by dividing both members of Eq. (13.4) by Δx:

$$q = \frac{\Delta H}{\Delta x} = \frac{VQ}{I} \tag{13.5}$$

Shearing stresses in a beam

Dividing both members of Eq. (13.4) by the area ΔA of the horizontal face of the element and observing that $\Delta A = t\,\Delta x$, where t is the width of the element at the cut, we obtained in Sec. 13.3 the following expression for the *average shearing stress* on the horizontal face of the element

$$\tau_{\text{ave}} = \frac{VQ}{It} \tag{13.6}$$

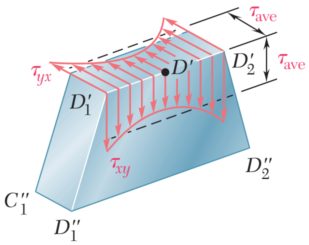


Fig. 13.32

We further noted that, since the shearing stresses τ_{xy} and τ_{yx} exerted, respectively, on a transverse and a horizontal plane through D' are equal, the expression in (13.6) also represents the average value of τ_{xy} along the line $D'_1 D'_2$ (Fig. 13.32).

Shearing stresses in a beam of rectangular cross section

In Sec. 13.4 we analyzed the shearing stresses in a beam of rectangular cross section. We found that the distribution of stresses is parabolic and that the maximum stress, which occurs at the center of the section, is

$$\tau_{\max} = \frac{3}{2}\frac{V}{A} \tag{13.10}$$

where A is the area of the rectangular section. For wide-flange beams, we found that a good approximation of the maximum shearing stress can be obtained by dividing the shear V by the cross-sectional area of the web.

Longitudinal shear on curved surface

In Sec. 13.5 we showed that Eqs. (13.4) and (13.5) could still be used to determine, respectively, the longitudinal shearing force $\Delta \mathbf{H}$ and the shear flow q exerted on a beam element if the element was bounded by an arbitrary curved surface instead of a horizontal plane (Fig. 13.33). This made it possible for us in Sec. 13.6 to extend the use of Eq. (13.6) to the determination of the average shearing stress in thin-walled members such as wide-flange beams and box beams, in the flanges of such members, and in their webs (Fig. 13.34).

Shearing stresses in thin-walled members

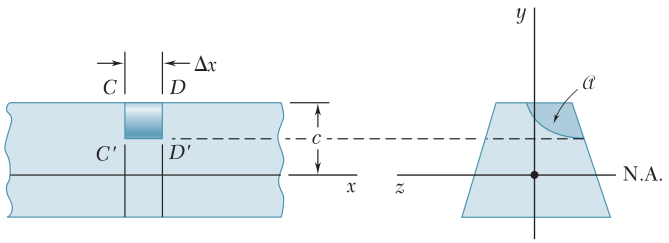


Fig. 13.33

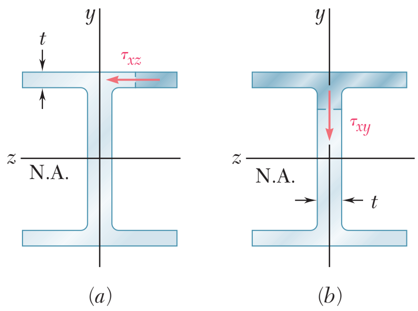


Fig. 13.34

REVIEW PROBLEMS

13.50 Three boards are nailed together to form the beam shown, which is subjected to a vertical shear. Knowing that the spacing between the nails is $s = 75$ mm and that the allowable shearing force in each nail is 400 N, determine the allowable shear.

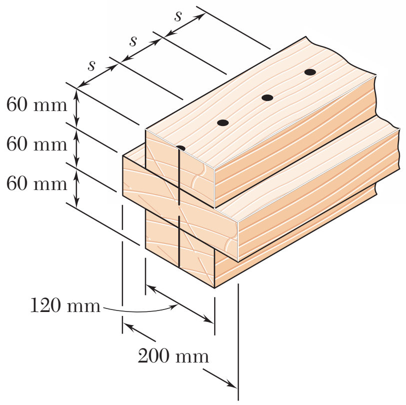


Fig. P13.50

13.51 For the beam and loading shown, consider section *n-n* and determine (*a*) the largest shearing stress in that section, (*b*) the shearing stress at point *a*.

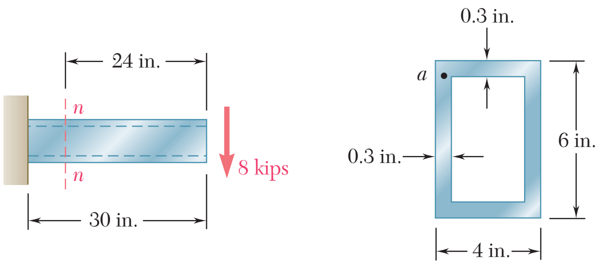


Fig. P13.51

13.52 For the beam and loading shown, consider section *n-n* and determine (*a*) the largest shearing stress in that section, (*b*) the shearing stress at point *a*.

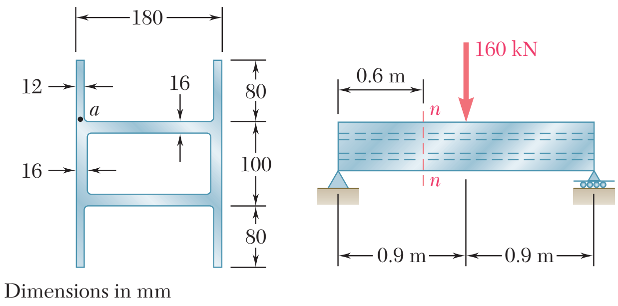


Fig. P13.52

13.53 A timber beam AB of length L and rectangular cross section carries a uniformly distributed load w and is supported as shown. (*a*) Show that the ratio τ_m/σ_m of the maximum values of the shearing and normal stresses in the beam is equal to $2h/L$, where h and L are, respectively, the depth and the length of the beam. (*b*) Determine the depth h and the width b of the beam, knowing that $L = 5$ m, $w = 8$ kN/m, $\tau_m = 1.08$ MPa, and $\sigma_m = 12$ MPa.

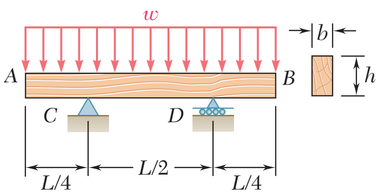


Fig. P13.53

13.54 For the beam and loading shown, consider section n-n and determine the shearing stress at (*a*) point a, (*b*) point b.

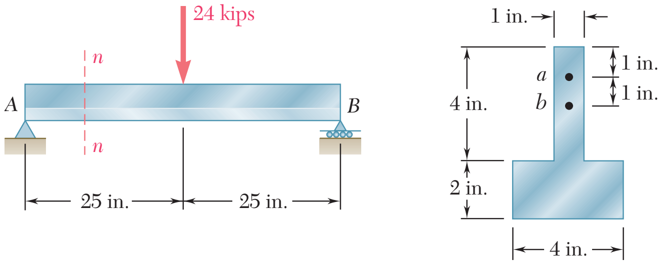


Fig. P13.54

13.55 Two W8 × 31 rolled sections can be welded at A and B in either of the two ways shown in order to form a composite beam. Knowing that for each weld the allowable horizontal shearing force is 3000 lb per inch of weld, determine the maximum allowable vertical shear in the composite beam for each of the two arrangements shown.

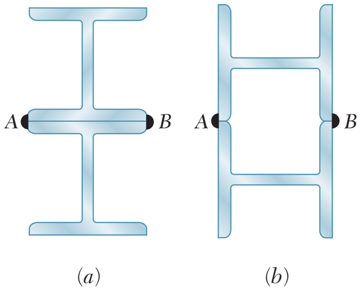


Fig. P13.55

13.56 The built-up wooden beam shown is subjected to a vertical shear of 8 kN. Knowing that the nails are spaced longitudinally every 60 mm at A and every 25 mm at B, determine the shearing force in the nails (*a*) at A, (*b*) at B. (*Given:* $I_x = 1.504 \times 10^9$ mm^4.)

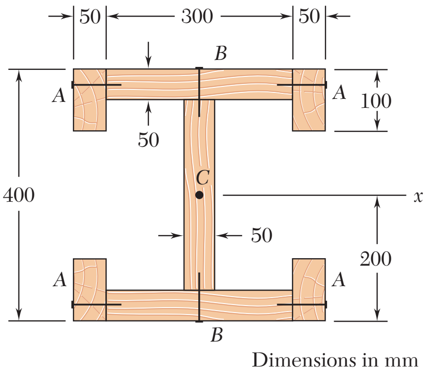


Fig. P13.56

13.57 The built-up beam shown is made up by gluing together five planks. Knowing that the allowable average shearing stress in the glued joints is 60 psi, determine the largest permissible vertical shear in the beam.

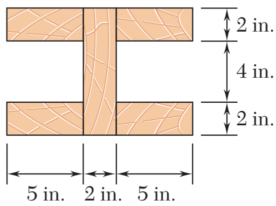


Fig. P13.57

13.58 An extruded beam has the cross section shown and a uniform wall thickness of 0.20 in. Knowing that a given vertical shear **V** causes a maximum shearing stress $\tau = 9$ ksi, determine the shearing stress at the four points indicated.

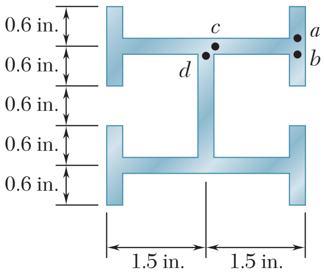


Fig. P13.58

13.59 Solve Prob. 13.58 assuming that the beam is subjected to a horizontal shear **V**.

13.60 Three 20 × 450-mm steel plates are bolted to four L152 × 152 × 19.0 angles to form a beam with the cross section shown. The bolts have a 22-mm diameter and are spaced longitudinally every 125 mm. Knowing that the allowable average shearing stress in the bolts is 90 MPa, determine the largest permissible vertical shear in the beam. (*Given:* $I_x = 1901 \times 10^6\ \text{mm}^4$.)

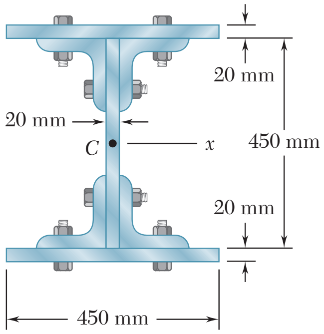


Fig. P13.60

13.61 An extruded beam has the cross section shown and a uniform wall thickness of 3 mm. For a vertical shear of 10 kN, determine (*a*) the shearing stress at point *A*, (*b*) the maximum shearing stress in the beam. Also sketch the shear flow in the cross section.

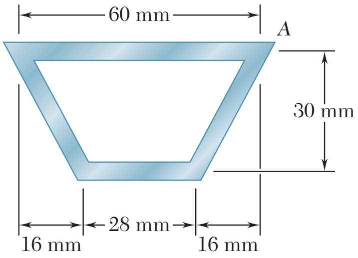


Fig. P13.61

The plane shown is being tested to determine how the forces due to lift would be distributed over the wing. This chapter deals with stresses and strains in structures and machine components.

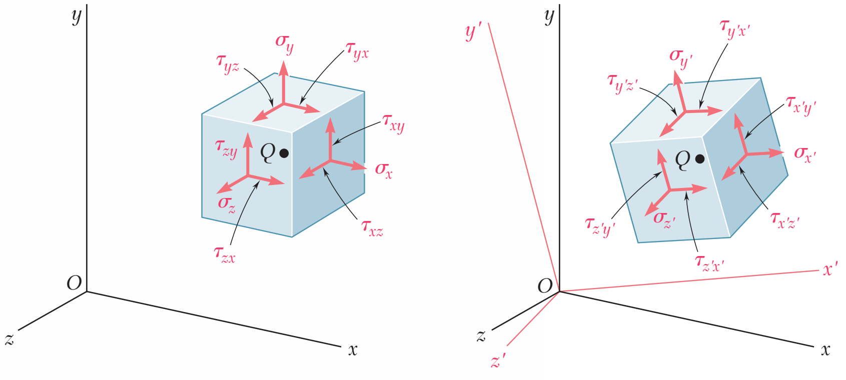
CHAPTER 14

Transformation of Stress

Chapter 14 Transformation of Stress

14.1 INTRODUCTION

We saw in Sec. 8.9 that the most general state of stress at a given point Q may be represented by six components. Three of these components, σ_x, σ_y, and σ_z, define the normal stresses exerted on the faces of a small cubic element centered at Q and of the same orientation as the coordinate axes (Fig. 14.1*a*), and the other three, τ_{xy}, τ_{yz}, and τ_{zx},† the components of the shearing stresses on the same element. As we remarked at the time, the same state of stress will be represented by a different set of components if the coordinate axes are rotated (Fig. 14.1*b*). We propose in the first part of this chapter to determine how the components of stress are transformed under a rotation of the coordinate axes.



(*a*) (*b*)

Fig. 14.1

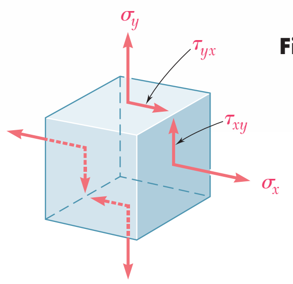


Fig. 14.2

Our discussion of the transformation of stress will deal mainly with *plane stress*, i.e., with a situation in which two of the faces of the cubic element are free of any stress. If the z axis is chosen perpendicular to these faces, we have $\sigma_z = \tau_{zx} = \tau_{zy} = 0$, and the only remaining stress components are σ_x, σ_y, and τ_{xy} (Fig. 14.2). Such a situation occurs in a thin plate subjected to forces acting in the midplane of the plate (Fig. 14.3). It also occurs on the free surface of a structural element or machine component, i.e., at any point of the surface of that element or component that is not subjected to an external force (Fig. 14.4).

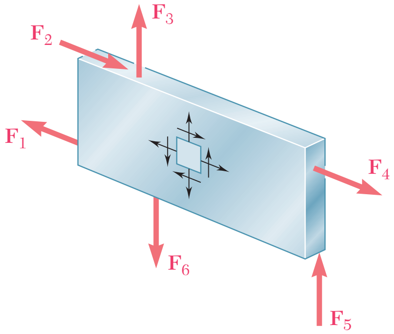


Fig. 14.3

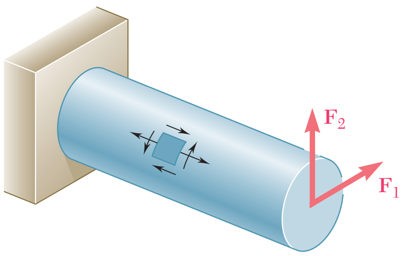


Fig. 14.4

†We recall that $\tau_{yx} = \tau_{xy}$, $\tau_{zy} = \tau_{yz}$, and $\tau_{xz} = \tau_{zx}$.

Considering in Sec. 14.2 a state of plane stress at a given point Q characterized by the stress components σ_x, σ_y, and τ_{xy} associated with the element shown in Fig. 14.5*a*, you will learn to determine the components $\sigma_{x'}$, $\sigma_{y'}$, and $\tau_{x'y'}$ associated with that element after it has been rotated through an angle θ about the z axis (Fig. 14.5*b*). In Sec. 14.3, you will determine the value θ_p of θ for which the stresses $\sigma_{x'}$ and $\sigma_{y'}$ are, respectively, maximum and minimum; these values of the normal stress are the *principal stresses* at point Q, and the faces of the corresponding element define the *principal planes of stress* at that point. You will also determine the value θ_s of the angle of rotation for which the shearing stress is maximum, as well as the value of that stress.

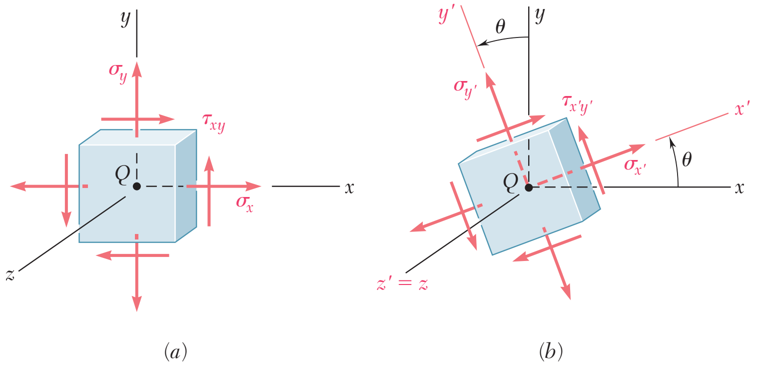


Fig. 14.5

In Sec. 14.4, an alternative method for the solution of problems involving the transformation of plane stress, based on the use of *Mohr's circle,* will be presented.

Thin-walled pressure vessels provide an important application of the analysis of plane stress. In Sec. 14.5, we will discuss stresses in both cylindrical and spherical pressure vessels (Photos 14.1 and 14.2).

Photo 14.1

Photo 14.2

14.2 TRANSFORMATION OF PLANE STRESS

Let us assume that a state of plane stress exists at point Q (with $\sigma_z = \tau_{zx} = \tau_{zy} = 0$), and that it is defined by the stress components σ_x, σ_y, and τ_{xy} associated with the element shown in Fig. 14.5*a*. We propose to determine the stress components $\sigma_{x'}$, $\sigma_{y'}$, and $\tau_{x'y'}$ associated with the element after it has been rotated through an angle θ about the z axis (Fig. 14.5*b*) and to express these components in terms of σ_x, σ_y, τ_{xy}, and θ.

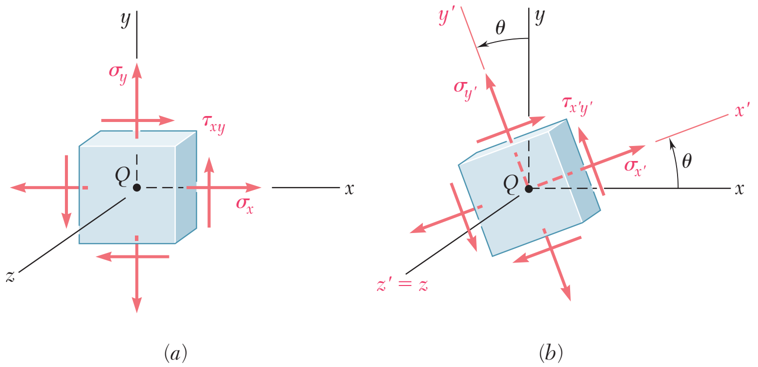


Fig. 14.5 (*repeated*)

In order to determine the normal stress $\sigma_{x'}$ and the shearing stress $\tau_{x'y'}$ exerted on the face perpendicular to the x' axis, we consider a prismatic element with faces respectively perpendicular to the x, y, and x' axes (Fig. 14.6*a*). We observe that, if the area of the oblique face is denoted by ΔA, the areas of the vertical and horizontal faces are respectively equal to $\Delta A \cos\theta$ and $\Delta A \sin\theta$. It follows that the *forces* exerted on the three faces are as shown in Fig. 14.6*b*.

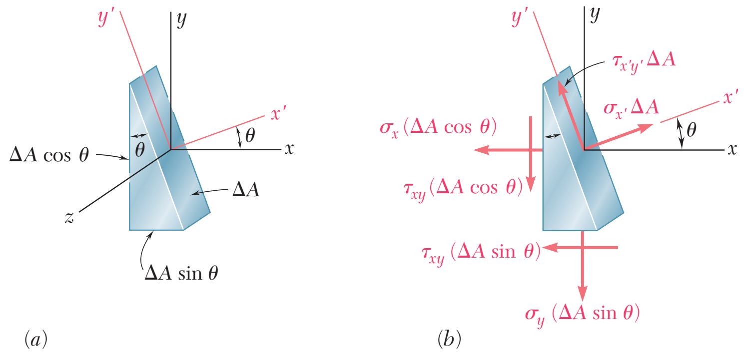


Fig. 14.6

(No forces are exerted on the triangular faces of the element, since the corresponding normal and shearing stresses have all been assumed equal to zero.) Using components along the x' and y' axes, we write the following equilibrium equations:

$$\sum F_{x'} = 0: \quad \sigma_{x'}\,\Delta A - \sigma_x(\Delta A\cos\theta)\cos\theta - \tau_{xy}(\Delta A\cos\theta)\sin\theta$$
$$-\sigma_y(\Delta A\sin\theta)\sin\theta - \tau_{xy}(\Delta A\sin\theta)\cos\theta = 0$$

$$\Sigma F_{y'} = 0: \quad \tau_{x'y'}\,\Delta A + \sigma_x(\Delta A \cos\theta)\sin\theta - \tau_{xy}(\Delta A\cos\theta)\cos\theta$$
$$-\sigma_y(\Delta A\sin\theta)\cos\theta + \tau_{xy}(\Delta A\sin\theta)\sin\theta = 0$$

Solving the first equation for $\sigma_{x'}$ and the second for $\tau_{x'y'}$, we have

$$\sigma_{x'} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta \qquad (14.1)$$

$$\tau_{x'y'} = -(\sigma_x - \sigma_y)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta) \quad (14.2)$$

Recalling the trigonometric relations

$$\sin 2\theta = 2\sin\theta\cos\theta \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta \quad (14.3)$$

and

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \qquad \sin^2\theta = \frac{1-\cos 2\theta}{2} \qquad (14.4)$$

we write Eq. (14.1) as follows:

$$\sigma_{x'} = \sigma_x \frac{1+\cos 2\theta}{2} + \sigma_y \frac{1-\cos 2\theta}{2} + \tau_{xy}\sin 2\theta$$

or

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \qquad (14.5)$$

Using the relations (14.3), we write Eq. (14.2) as

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \qquad (14.6)$$

The expression for the normal stress $\sigma_{y'}$ is obtained by replacing θ in Eq. (14.5) by the angle $\theta + 90°$ that the y' axis forms with the x axis. Since $\cos(2\theta + 180°) = -\cos 2\theta$ and $\sin(2\theta + 180°) = -\sin 2\theta$, we have

$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta \qquad (14.7)$$

Adding Eqs. (14.5) and (14.7) member to member, we obtain

$$\sigma_{x'} + \sigma_{y'} = \sigma_x + \sigma_y \qquad (14.8)$$

Since $\sigma_z = \sigma_{z'} = 0$, we thus verify in the case of plane stress that the sum of the normal stresses exerted on a cubic element of material is independent of the orientation of that element.

14.3 PRINCIPAL STRESSES. MAXIMUM SHEARING STRESS

The equations (14.5) and (14.6) obtained in the preceding section are the parametric equations of a circle. This means that, if we choose a set of rectangular axes and plot a point M of abscissa $\sigma_{x'}$ and ordinate $\tau_{x'y'}$ for any given value of the parameter θ, all the points thus obtained will lie on a circle. To establish this property we eliminate θ from Eqs. (14.5) and (14.6); this is done by first transposing $(\sigma_x + \sigma_y)/2$ in Eq. (14.5) and squaring both members of the equation, then squaring both members of Eq. (14.6), and finally adding member to member the two equations obtained in this fashion. We have

$$\left(\sigma_{x'} - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{x'y'}^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2 \qquad (14.9)$$

Setting

$$\sigma_{\text{ave}} = \frac{\sigma_x + \sigma_y}{2} \quad \text{and} \quad R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \qquad (14.10)$$

we write the identity (14.7) in the form

$$(\sigma_{x'} - \sigma_{\text{ave}})^2 + \tau_{x'y'}^2 = R^2 \qquad (14.11)$$

which is the equation of a circle of radius R centered at the point C of abscissa σ_{ave} and ordinate 0 (Fig. 14.7). It can be observed that, due to the symmetry of the circle about the horizontal axis, the same result would have been obtained if, instead of plotting M, we had plotted a point N of abscissa $\sigma_{x'}$ and ordinate $-\tau_{x'y'}$ (Fig. 14.8). This property will be used in Sec. 14.4.

The two points A and B where the circle of Fig. 14.7 intersects the horizontal axis are of special interest: Point A corresponds to the maximum value of the normal stress $\sigma_{x'}$, while point B corresponds to its minimum value. Besides, both points correspond to a zero value of the shearing stress $\tau_{x'y'}$. Thus, the values θ_p of the parameter θ which correspond to points A and B can be obtained by setting $\tau_{x'y'} = 0$ in Eq. (14.6). We write†

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \qquad (14.12)$$

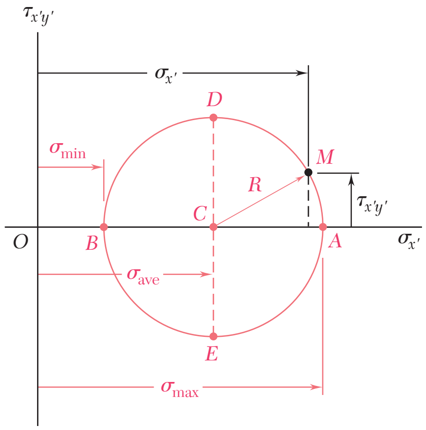


Fig. 14.7

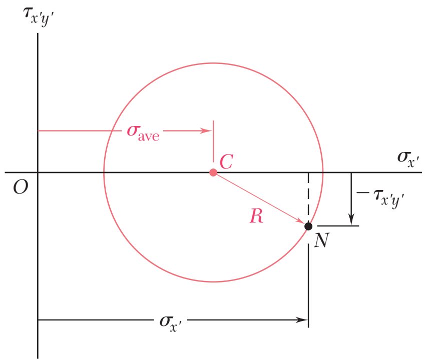


Fig. 14.8

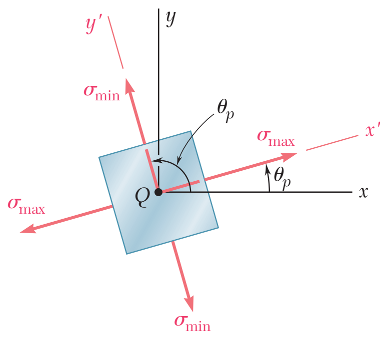


Fig. 14.9

This equation defines two values $2\theta_p$ that are 180° apart, and thus two values θ_p that are 90° apart. Either of these values can be used to determine the orientation of the corresponding element (Fig. 14.9). The planes containing the faces of the element obtained in this way are called the *principal planes of stress* at point Q, and the corresponding values $\sigma_{\max}$ and $\sigma_{\min}$ of the normal stress exerted on these planes are called the *principal stresses* at Q. Since the two values θ_p defined by Eq. (14.12) were obtained by setting $\tau_{x'y'} = 0$ in Eq. (14.6), it is clear that no shearing stress is exerted on the principal planes.

We observe from Fig. 14.7 that

$$\sigma_{\max} = \sigma_{\text{ave}} + R \quad \text{and} \quad \sigma_{\min} = \sigma_{\text{ave}} - R \qquad (14.13)$$

†This relation can also be obtained by differentiating $\sigma_{x'}$ in Eq. (14.5) and setting the derivative equal to zero: $d\sigma_{x'}/d\theta = 0$.

Substituting for σ_{ave} and R from Eq. (14.10), we write

$$\sigma_{max,\,min} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \tag{14.14}$$

Unless it is possible to tell by inspection which of the two principal planes is subjected to σ_{max} and which is subjected to σ_{min}, it is necessary to substitute one of the values θ_p into Eq. (14.5) in order to determine which of the two corresponds to the maximum value of the normal stress.

Referring again to the circle of Fig. 14.7, we note that the points D and E located on the vertical diameter of the circle correspond to the largest numerical value of the shearing stress $\tau_{x'y'}$. Since the abscissa of points D and E is $\sigma_{ave} = (\sigma_x + \sigma_y)/2$, the values θ_s of the parameter θ corresponding to these points are obtained by setting $\sigma_{x'} = (\sigma_x + \sigma_y)/2$ in Eq. (14.5). It follows that the sum of the last two terms in that equation must be zero. Thus, for $\theta = \theta_s$, we write†

$$\frac{\sigma_x - \sigma_y}{2} \cos 2\theta_s + \tau_{xy} \sin 2\theta_s = 0$$

or

$$\tan 2\theta_s = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}} \tag{14.15}$$

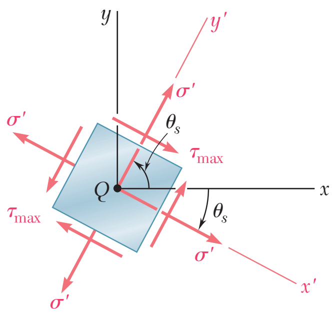


Fig. 14.10

This equation defines two values $2\theta_s$ that are 180° apart, and thus two values θ_s that are 90° apart. Either of these values can be used to determine the orientation of the element corresponding to the maximum shearing stress (Fig. 14.10). Observing from Fig. 14.7 that the maximum value of the shearing stress is equal to the radius R of the circle and recalling the second of Eqs. (14.10), we write

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \tag{14.16}$$

As observed earlier, the normal stress corresponding to the condition of maximum shearing stress is

$$\sigma' = \sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} \tag{14.17}$$

Comparing Eqs. (14.12) and (14.15), we note that $\tan 2\theta_s$ is the negative reciprocal of $\tan 2\theta_p$. This means that the angles $2\theta_s$ and $2\theta_p$ are 90° apart and, therefore, that the angles θ_s and θ_p are 45° apart. We thus conclude that *the planes of maximum shearing stress are at 45° to the principal planes*. This confirms the results obtained earlier in Sec. 8.9 in the case of a centric axial loading (Fig. 8.37) and in Sec. 10.4 in the case of a torsional loading (Fig. 10.19.)

†This relation may also be obtained by differentiating $\tau_{x'y'}$ in Eq. (14.6) and setting the derivative equal to zero: $d\tau_{x'y'}/d\theta = 0$.

We should be aware that our analysis of the transformation of plane stress has been limited to rotations *in the plane of stress*. If the cubic element of Fig. 14.5 is rotated about an axis other than the z axis, its faces may be subjected to shearing stresses larger than the stress defined by Eq. (14.16). In such cases, the value given by Eq. (14.16) is referred to as the maximum *in-plane* shearing stress.

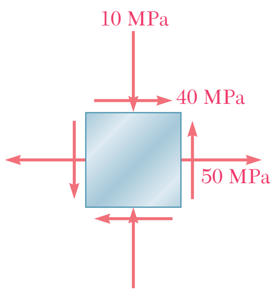


Fig. 14.11

EXAMPLE 14.1 For the state of plane stress shown in Fig. 14.11, determine (a) the principal planes, (b) the principal stresses, (c) the maximum shearing stress and the corresponding normal stress.

(a) Principal Planes. Following the usual sign convention, we write the stress components as

$$\sigma_x = +50\text{ MPa} \qquad \sigma_y = -10\text{ MPa} \qquad \tau_{xy} = +40\text{ MPa}$$

Substituting into Eq. (14.12), we have

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2(+40)}{50 - (-10)} = \frac{80}{60}$$

$$2\theta_p = 53.1^\circ \quad \text{and} \quad 180^\circ + 53.1^\circ = 233.1^\circ$$

$$\theta_p = 26.6^\circ \quad \text{and} \quad 116.6^\circ$$

(b) Principal Stresses. Formula (14.14) yields

$$\sigma_{\max,\min} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= 20 \pm \sqrt{(30)^2 + (40)^2}$$

$$\sigma_{\max} = 20 + 50 = 70\text{ MPa}$$

$$\sigma_{\min} = 20 - 50 = -30\text{ MPa}$$

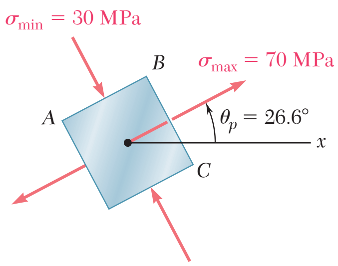


Fig. 14.12

The principal planes and principal stresses are sketched in Fig. 14.12. Making $\theta = 26.6^\circ$ in Eq. (14.5), we check that the normal stress exerted on face BC of the element is the maximum stress:

$$\sigma_{x'} = \frac{50 - 10}{2} + \frac{50 + 10}{2}\cos 53.1^\circ + 40\sin 53.1^\circ$$

$$= 20 + 30\cos 53.1^\circ + 40\sin 53.1^\circ = 70\text{ MPa} = \sigma_{\max}$$

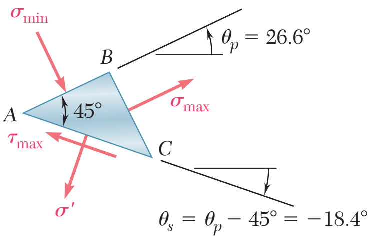


Fig. 14.13

(c) Maximum Shearing Stress. Formula (14.16) yields

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sqrt{(30)^2 + (40)^2} = 50\text{ MPa}$$

Since $\sigma_{\max}$ and $\sigma_{\min}$ have opposite signs, the value obtained for $\tau_{\max}$ actually represents the maximum value of the shearing stress at the point considered. The orientation of the planes of maximum shearing stress and the sense of the shearing stresses are best determined by passing a section along the diagonal plane AC of the element of Fig. 14.12. Since the faces AB and BC of the element are contained in the principal planes, the diagonal plane AC must be one of the planes of maximum shearing stress (Fig. 14.13). Furthermore, the equilibrium conditions for the prismatic element ABC require that the shearing stress exerted on AC be directed as shown. The cubic element corresponding to the maximum shearing stress is shown in Fig. 14.14. The normal stress on each of the four faces of the element is given by Eq. (14.17):

$$\sigma' = \sigma_{\text{ave}} = \frac{\sigma_x + \sigma_y}{2} = \frac{50 - 10}{2} = 20\text{ MPa} \ ■$$

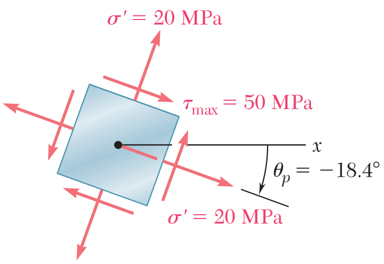
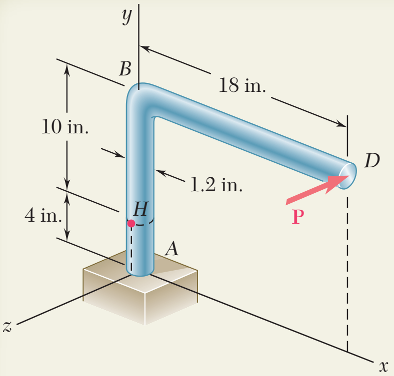
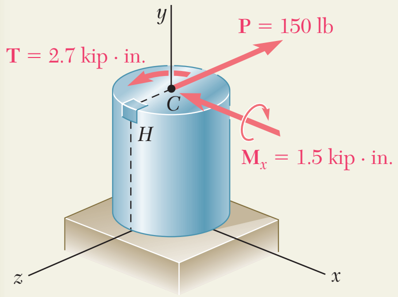


Fig. 14.14

SAMPLE PROBLEM 14.1



A single horizontal force **P** of magnitude 150 lb is applied to end D of lever ABD. Knowing that portion AB of the lever has a diameter of 1.2 in., determine (*a*) the normal and shearing stresses on an element located at point H and having sides parallel to the x and y axes, (*b*) the principal planes and the principal stresses at point H.



SOLUTION

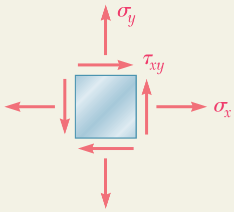
Force-Couple System. We replace the force **P** by an equivalent force-couple system at the center C of the transverse section containing point H:

$$P = 150 \text{ lb} \qquad T = (150 \text{ lb})(18 \text{ in.}) = 2.7 \text{ kip} \cdot \text{in.}$$
$$M_x = (150 \text{ lb})(10 \text{ in.}) = 1.5 \text{ kip} \cdot \text{in.}$$

a. Stresses σ_x, σ_y, τ_{xy} at Point H. Using the sign convention shown in Fig. 14.2, we determine the sense and the sign of each stress component by carefully examining the sketch of the force-couple system at point C:

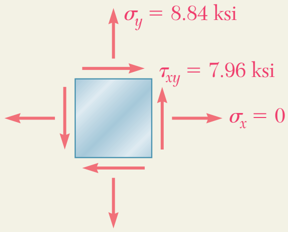
$$\sigma_x = 0 \qquad \sigma_y = +\frac{Mc}{I} = +\frac{(1.5 \text{ kip} \cdot \text{in.})(0.6 \text{ in.})}{\frac{1}{4}\pi\,(0.6 \text{ in.})^4} \qquad \sigma_y = +8.84 \text{ ksi} \blacktriangleleft$$

$$\tau_{xy} = +\frac{Tc}{J} = +\frac{(2.7 \text{ kip} \cdot \text{in.})(0.6 \text{ in.})}{\frac{1}{2}\pi\,(0.6 \text{ in.})^4} \qquad \tau_{xy} = +7.96 \text{ ksi} \blacktriangleleft$$



We note that the shearing force **P** does not cause any shearing stress at point H.

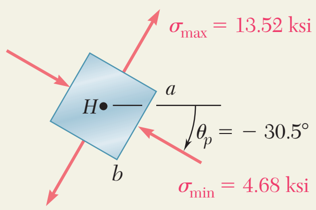
b. Principal Planes and Principal Stresses. Substituting the values of the stress components into Eq. (14.12), we determine the orientation of the principal planes:



$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2(7.96)}{0 - 8.84} = -1.80$$
$$2\theta_p = -61.0° \qquad \text{and} \qquad 180° - 61.0° = +119°$$
$$\theta_p = -30.5° \qquad \text{and} \qquad +59.5° \blacktriangleleft$$

Substituting into Eq. (14.14), we determine the magnitudes of the principal stresses:

$$\sigma_{\max,\,\min} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$
$$= \frac{0 + 8.84}{2} \pm \sqrt{\left(\frac{0 - 8.84}{2}\right)^2 + (7.96)^2} = +4.42 \pm 9.10$$
$$\sigma_{\max} = +13.52 \text{ ksi} \blacktriangleleft$$
$$\sigma_{\min} = -4.68 \text{ ksi} \blacktriangleleft$$



Considering face ab of the element shown, we make $\theta_p = -30.5°$ in Eq. (14.5) and find $\sigma_{x'} = -4.68$ ksi. We conclude that the principal stresses are as shown.

PROBLEMS

14.1 through 14.4 For the given state of stress, determine the normal and shearing stresses exerted on the oblique face of the shaded triangular element shown. Use a method of analysis based on the equilibrium of that element as was done in the derivations of Sec. 14.2.

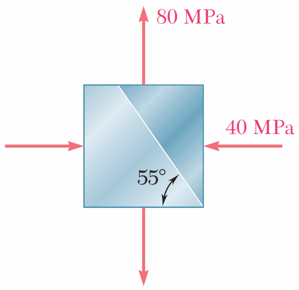


Fig. P14.1

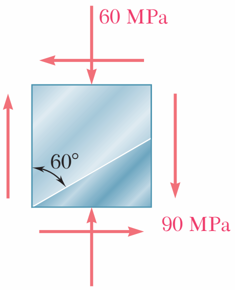


Fig. P14.2

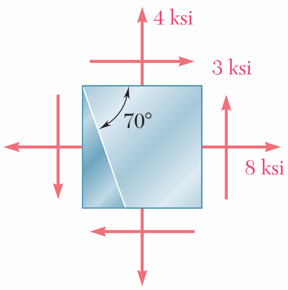


Fig. P14.3

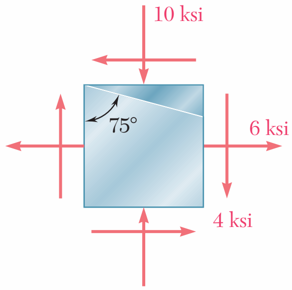


Fig. P14.4

14.5 through *14.8* For the given state of stress, determine (*a*) the principal planes, (*b*) the principal stresses.

14.9 through 14.12 For the given state of stress, determine (*a*) the orientation of the planes of maximum in-plane shearing stress, (*b*) the maximum in-plane shearing stress, (*c*) the corresponding normal stress.

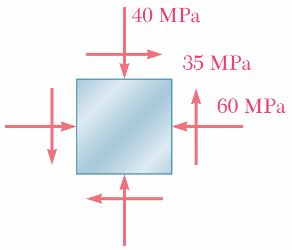


Fig. P14.5 and P14.9

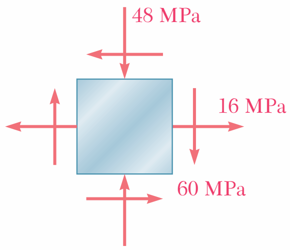


Fig. *P14.6* and P14.10

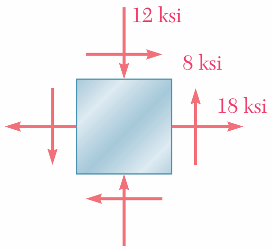


Fig. *P14.7* and *P14.11*

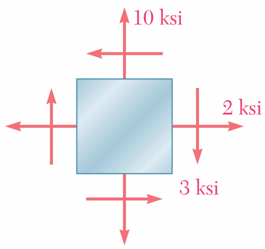


Fig. *P14.8* and P14.12

14.13 through 14.16 For the given state of stress, determine the normal and shearing stresses after the element shown has been rotated through (*a*) 25° clockwise, (*b*) 10° counterclockwise.

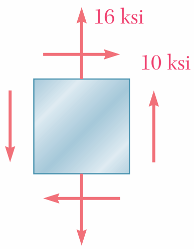


Fig. P14.13

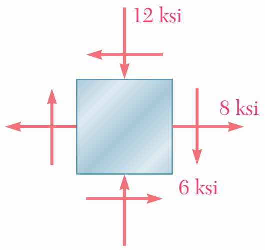


Fig. P14.14

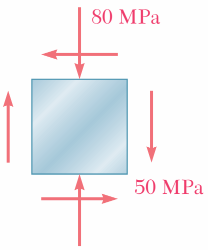


Fig. *P14.15*

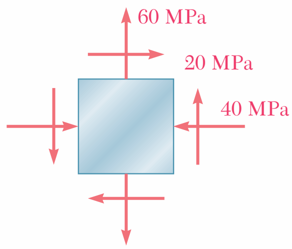


Fig. P14.16

14.17 and 14.18 The grain of a wooden member forms an angle of 15° with the vertical. For the state of stress shown, determine (*a*) the in-plane shearing stress parallel to the grain, (*b*) the normal stress perpendicular to the grain.

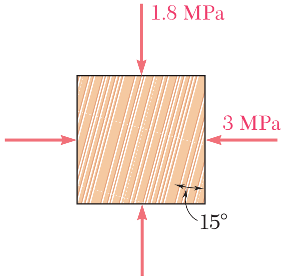


Fig. P14.17

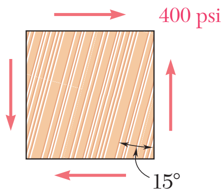


Fig. P14.18

14.19 Two plates of uniform cross section 10 × 80 mm are welded together as shown. Knowing that centric 100-kN forces are applied to the welded plates and that the in-plane shearing stress parallel to the weld is 30 MPa, determine (*a*) the angle β, (*b*) the corresponding normal stress perpendicular to the weld.

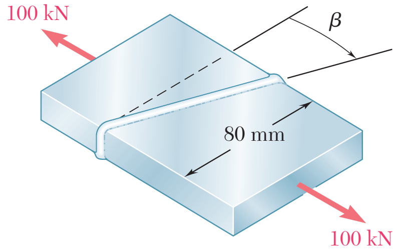


Fig. P14.19

14.20 The centric force **P** is applied to a short post as shown. Knowing that the stresses on plane *a-a* are $\sigma = -15$ ksi and $\tau = 5$ ksi, determine (*a*) the angle β that plane *a-a* forms with the horizontal, (*b*) the maximum compressive stress in the post.

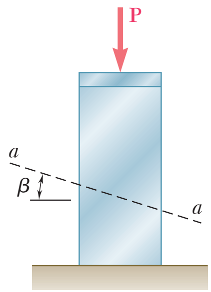


Fig. P14.20

14.21 A 400-lb vertical force is applied at *D* to a gear attached to the solid 1-in. diameter shaft *AB*. Determine the principal stresses and the maximum shearing stress at point *H* located as shown on top of the shaft.

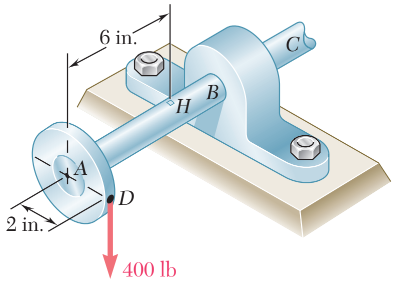


Fig. *P14.21*

14.22 A mechanic uses a crowfoot wrench to loosen a bolt at E. Knowing that the mechanic applies a vertical 24-lb force at A, determine the principal stresses and the maximum shearing stress at point H located as shown on top of the $\frac{3}{4}$-in. diameter shaft.

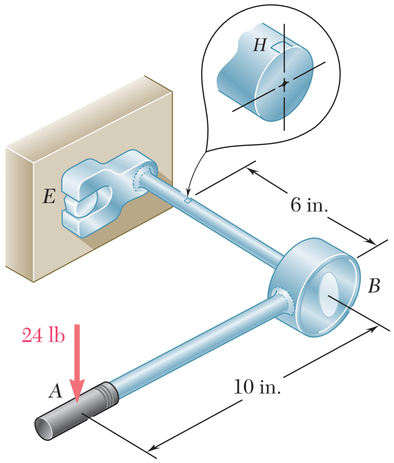


Fig. P14.22

14.23 The steel pipe AB has a 102-mm outer diameter and a 6-mm wall thickness. Knowing that arm CD is rigidly attached to the pipe, determine the principal stresses and the maximum shearing stress at point H.

14.24 The steel pipe AB has a 102-mm outer diameter and a 6-mm wall thickness. Knowing that arm CD is rigidly attached to the pipe, determine the principal stresses and the maximum shearing stress at point K.

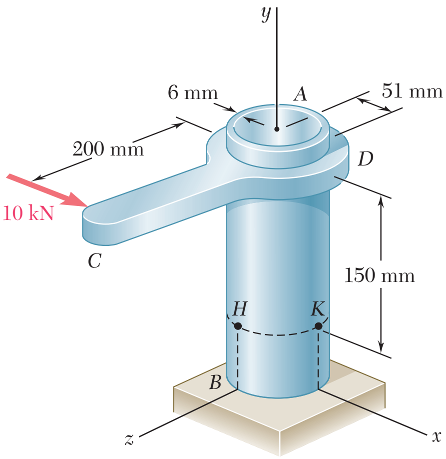


Fig. *P14.23* and P14.24

14.4 MOHR'S CIRCLE FOR PLANE STRESS

The circle used in the preceding section to derive some of the basic formulas relating to the transformation of plane stress was first introduced by the German engineer Otto Mohr (1835–1918) and is known as *Mohr's circle* for plane stress. As you will see presently, this circle can be used to obtain an alternative method for the solution of the various problems considered in Secs. 14.2 and 14.3. This method is based on simple geometric considerations and does not require the use of specialized formulas. While originally designed for graphical solutions, it lends itself well to the use of a calculator.

Consider a square element of a material subjected to plane stress (Fig. 14.15a), and let σ_x, σ_y, and τ_{xy} be the components of the stress exerted on the element. We plot a point X of coordinates σ_x and $-\tau_{xy}$, and a point Y of coordinates σ_y and $+\tau_{xy}$ (Fig. 14.15*b*). If τ_{xy} is positive, as assumed in Fig. 14.15*a*, point X is located below the σ axis and point Y is located above, as shown in Fig. 14.15*b*. If τ_{xy} is negative, X is located above the σ axis and Y is located below. Joining X and Y by a straight line, we define the point C of intersection of line XY with the σ axis and draw the circle of center C and diameter XY. Noting that the abscissa of C and the radius of the circle are respectively equal to the quantities σ_{ave} and R defined by Eqs. (14.10), we conclude that the circle obtained is Mohr's circle for plane stress. Thus the abscissas of points A and B where the circle intersects the σ axis represent respectively the principal stresses σ_{max} and σ_{min} at the point considered.

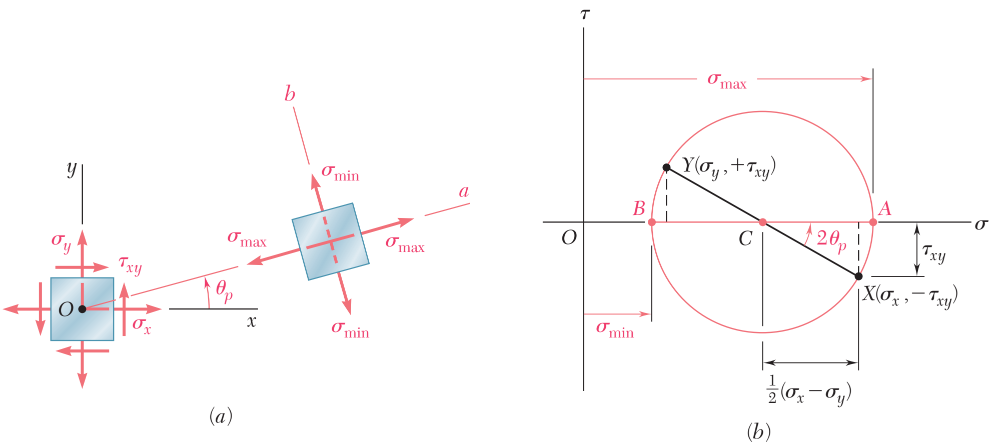


Fig. 14.15

We also note that, since $\tan(XCA) = 2\tau_{xy}/(\sigma_x - \sigma_y)$, the angle XCA is equal in magnitude to one of the angles $2\theta_p$ that satisfy Eq. (14.12). Thus, the angle θ_p that defines in Fig. 14.15*a* the orientation of the principal plane corresponding to point A in Fig. 14.15*b* can be obtained by dividing in half the angle XCA measured on Mohr's circle. We further observe that if $\sigma_x > \sigma_y$ and $\tau_{xy} > 0$, as in the case considered here, the rotation that brings CX into CA is counterclockwise. But, in that case, the angle θ_p obtained from Eq. (14.12) and defining the direction of the normal Oa to the principal plane is positive; thus, the rotation bringing Ox into Oa is also counterclockwise. We conclude that the senses of rotation in both parts of Fig. 14.15 are the same; if a counterclockwise rotation through $2\theta_p$ is required to bring CX into CA on Mohr's circle, a counterclockwise rotation through θ_p will bring Ox into Oa in Fig. 14.15*a*.†

Since Mohr's circle is uniquely defined, the same circle can be obtained by considering the stress components $\sigma_{x'}$, $\sigma_{y'}$, and $\tau_{x'y'}$,

†This is due to the fact that we are using the circle of Fig 14.8 rather than the circle of Fig. 14.7 as Mohr's circle.

corresponding to the x' and y' axes shown in Fig. 14.16*a*. The point X' of coordinates $\sigma_{x'}$ and $-\tau_{x'y'}$, and the point Y' of coordinates $\sigma_{y'}$ and $+\tau_{x'y'}$, are therefore located on Mohr's circle, and the angle $X'CA$ in Fig. 14.16*b* must be equal to twice the angle $x'Oa$ in Fig. 14.16*a*. Since, as noted before, the angle XCA is twice the angle xOa, it follows that the angle XCX' in Fig. 14.16*b* is twice the angle xOx' in Fig. 14.16*a*. Thus the diameter $X'Y'$ defining the normal and shearing stresses $\sigma_{x'}$, $\sigma_{y'}$, and $\tau_{x'y'}$ can be obtained by rotating the diameter XY through an angle equal to twice the angle θ formed by the x' and x axes in Fig. 14.16*a*. We note that the rotation that brings the diameter XY into the diameter $X'Y'$ in Fig. 14.16*b* has the same sense as the rotation that brings the xy axes into the $x'y'$ axes in Fig. 14.16*a*.

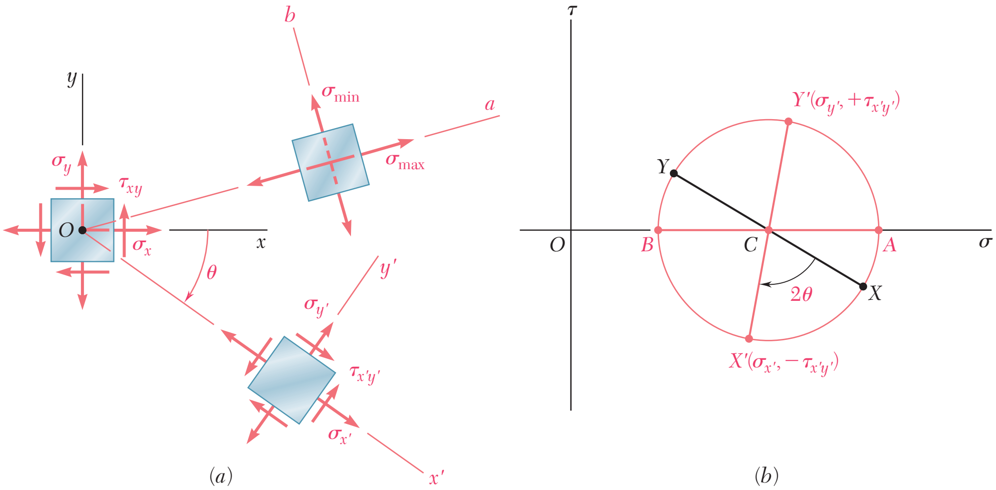


Fig. 14.16

The property we have just indicated can be used to verify the fact that the planes of maximum shearing stress are at 45° to the principal planes. Indeed, we recall that points D and E on Mohr's circle correspond to the planes of maximum shearing stress, while A and B correspond to the principal planes (Fig. 14.17*b*). Since the diameters AB and DE of Mohr's circle are at 90° to each other, it follows that the faces of the corresponding elements are at 45° to each other (Fig. 14.17*a*).

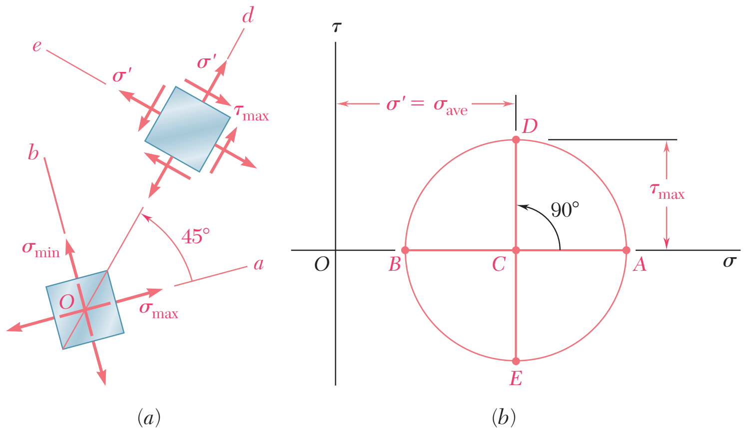


Fig. 14.17

The construction of Mohr's circle for plane stress is greatly simplified if we consider separately each face of the element used to define the stress components. From Figs. 14.15 and 14.16 we observe that, when the shearing stress exerted *on a given face* tends to rotate the element *clockwise*, the point on Mohr's circle corresponding to that face is located *above* the σ axis. When the shearing stress on a given face tends to rotate the element *counterclockwise*, the point corresponding to that face is located *below* the σ axis (Fig. 14.18).† As far as the normal stresses are concerned, the usual convention holds, i.e., a tensile stress is considered as positive and is plotted to the right, while a compressive stress is considered as negative and is plotted to the left.

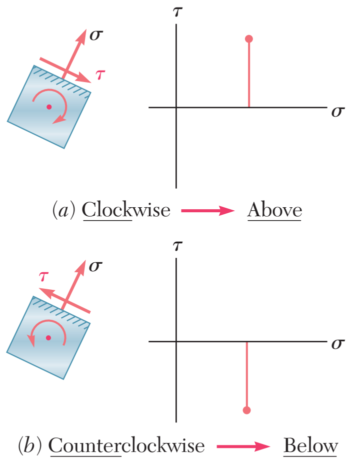


Fig. 14.18

EXAMPLE 14.2 For the state of plane stress already considered in Example 14.1, (*a*) construct Mohr's circle, (*b*) determine the principal stresses, (*c*) determine the maximum shearing stress and the corresponding normal stress.

(a) Construction of Mohr's Circle. We note from Fig. 14.19*a* that the normal stress exerted on the face oriented toward the *x* axis is tensile (positive)

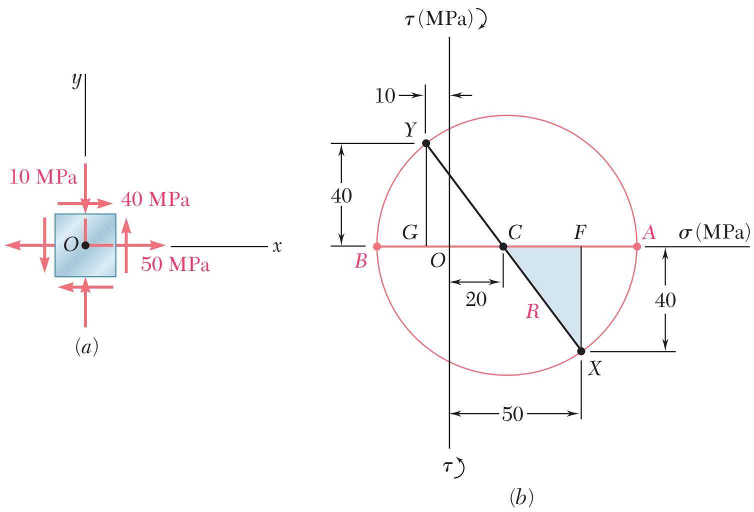
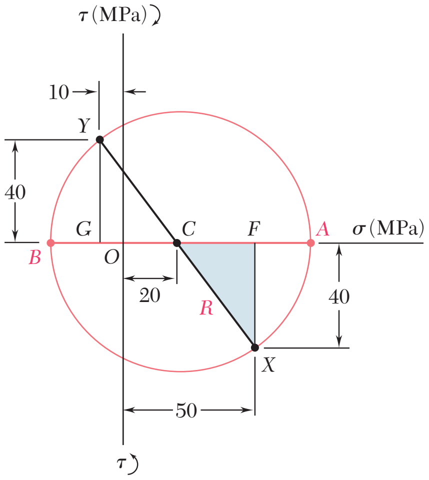


Fig. 14.19

and that the shearing stress exerted on that face tends to rotate the element counterclockwise. Point X of Mohr's circle, therefore, will be plotted to the right of the vertical axis and below the horizontal axis (Fig. 14.19*b*). A similar inspection of the normal stress and shearing stress exerted on the upper face of the element shows that point Y should be plotted to the left of the vertical axis and above the horizontal axis. Drawing the line XY, we obtain the center C of Mohr's circle; its abscissa is

$$\sigma_{\text{ave}} = \frac{\sigma_x + \sigma_y}{2} = \frac{50 + (-10)}{2} = 20 \text{ MPa}$$

†The following jingle is helpful in remembering this convention. "In the kitchen, the *clock* is above, and the *counter* is below."



(b)

Fig. 14.19b *(repeated)*

Since the sides of the shaded triangle are

$$CF = 50 - 20 = 30 \text{ MPa} \qquad \text{and} \qquad FX = 40 \text{ MPa}$$

the radius of the circle is

$$R = CX = \sqrt{(30)^2 + (40)^2} = 50 \text{ MPa}$$

(b) Principal Planes and Principal Stresses. The principal stresses are

$$\sigma_{max} = OA = OC + CA = 20 + 50 = 70 \text{ MPa}$$

$$\sigma_{min} = OB = OC - BC = 20 - 50 = -30 \text{ MPa}$$

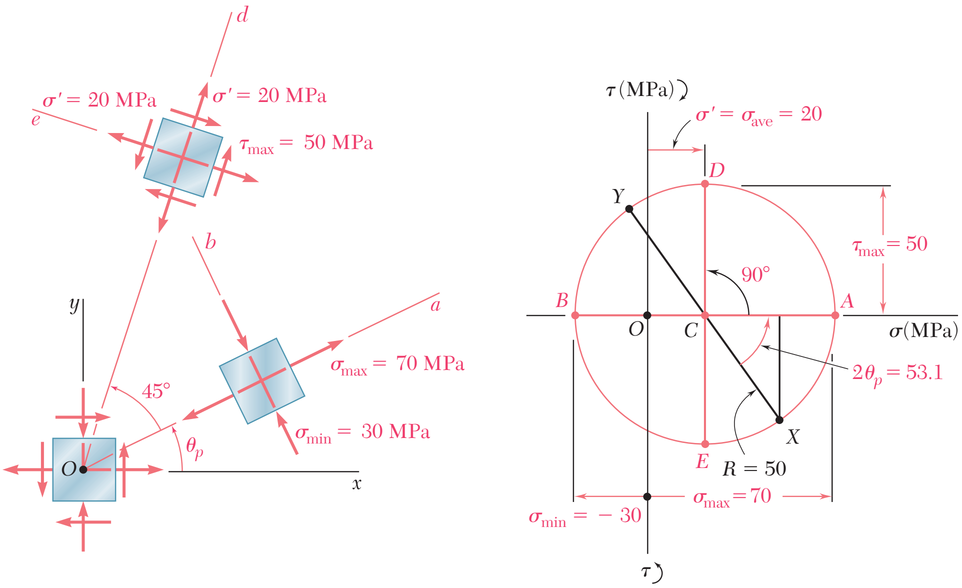
Recalling that the angle ACX represents $2\theta_p$ (Fig. 14.19b), we write

$$\tan 2\theta_p = \frac{FX}{CF} = \frac{40}{30}$$

$$2\theta_p = 53.1° \qquad \theta_p = 26.6°$$

Since the rotation which brings CX into CA in Fig. 14.20*b* is counterclockwise, the rotation that brings Ox into the axis Oa corresponding to $\sigma_{\max}$ in Fig. 14.20*a* is also counterclockwise.

(c) Maximum Shearing Stress. Since a further rotation of 90° counterclockwise brings CA into CD in Fig. 14.20*b*, a further rotation of 45° counterclockwise



(a) (b)

Fig. 14.20

will bring the axis Oa into the axis Od corresponding to the maximum shearing stress in Fig. 14.20*a*. We note from Fig. 14.20*b* that $\tau_{\max} = R = 50$ MPa and that the corresponding normal stress is $\sigma' = \sigma_{\text{ave}} = 20$ MPa. Since point D is located above the σ axis in Fig. 14.20*b*, the shearing stresses exerted on the faces perpendicular to Od in Fig. 14.20*a* must be directed so that they will tend to rotate the element clockwise. ■

Mohr's circle provides a convenient way of checking the results obtained earlier for stresses under a centric axial loading (Sec. 8.9) and under a torsional loading (Sec. 10.4). In the first case (Fig. 14.21*a*), we have $\sigma_x = P/A$, $\sigma_y = 0$, and $\tau_{xy} = 0$. The corresponding points X and Y define a circle of radius $R = P/2A$ that passes through the origin of coordinates (Fig. 14.21*b*). Points D and E yield the orientation of the planes of maximum shearing stress (Fig. 14.21*c*), as well as the values of $\tau_{\max}$ and of the corresponding normal stresses σ':

$$\tau_{\max} = \sigma' = R = \frac{P}{2A} \tag{14.18}$$

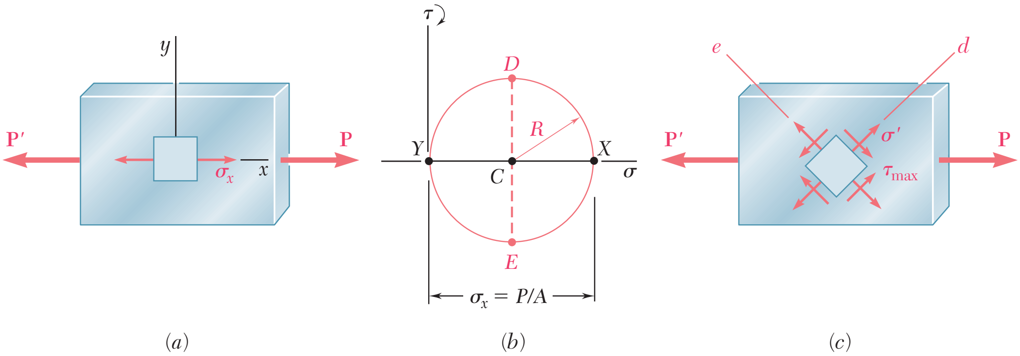


Fig. 14.21 Mohr's circle for centric axial loading.

In the case of torsion (Fig. 14.22*a*), we have $\sigma_x = \sigma_y = 0$ and $\tau_{xy} = \tau_{\max} = Tc/J$. Points X and Y, therefore, are located on the τ axis, and Mohr's circle is a circle of radius $R = Tc/J$ centered at the origin (Fig. 14.22*b*). Points A and B define the principal planes (Fig. 14.22*c*) and the principal stresses:

$$\sigma_{\max,\,\min} = \pm R = \pm \frac{Tc}{J} \tag{14.19}$$

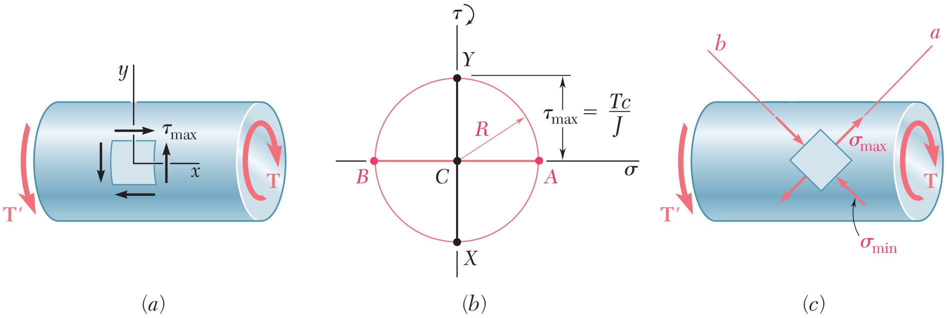
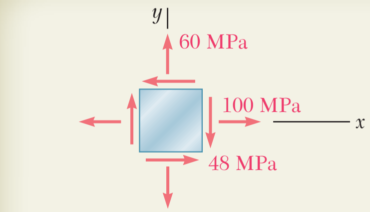


Fig. 14.22 Mohr's circle for torsional loading.

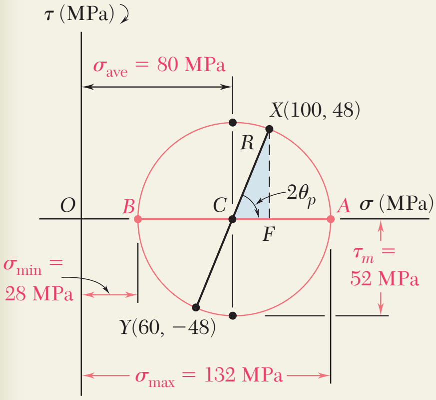
SAMPLE PROBLEM 14.2

For the state of plane stress shown, determine (*a*) the principal planes and the principal stresses, (*b*) the stress components exerted on the element obtained by rotating the given element counterclockwise through 30°.



SOLUTION

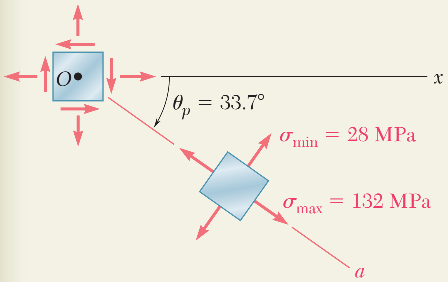
Construction of Mohr's Circle. We note that on a face perpendicular to the x axis, the normal stress is tensile and the shearing stress tends to rotate the element clockwise; thus, we plot X at a point 100 units to the right of the vertical axis and 48 units above the horizontal axis. In a similar fashion, we examine the stress components on the upper face and plot point Y (60, −48). Joining points X and Y by a straight line, we define the center C of Mohr's circle. The abscissa of C, which represents σ_{ave}, and the radius R of the circle can be measured directly or calculated as follows:



$$\sigma_{ave} = OC = \tfrac{1}{2}(\sigma_x + \sigma_y) = \tfrac{1}{2}(100 + 60) = 80 \text{ MPa}$$
$$R = \sqrt{(CF)^2 + (FX)^2} = \sqrt{(20)^2 + (48)^2} = 52 \text{ MPa}$$

a. Principal Planes and Principal Stresses. We rotate the diameter XY clockwise through $2\theta_p$ until it coincides with the diameter AB. We have

$$\tan 2\theta_p = \frac{XF}{CF} = \frac{48}{20} = 2.4 \quad 2\theta_p = 67.4° \circlearrowright \quad \theta_p = 33.7° \circlearrowright \quad ◀$$

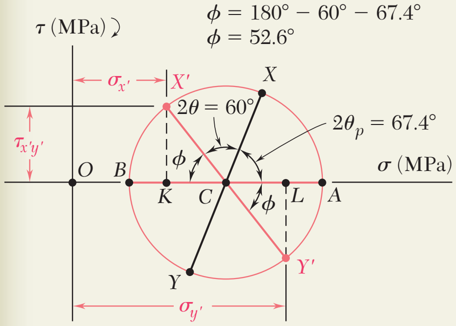


The principal stresses are represented by the abscissas of points A and B:

$$\sigma_{max} = OA = OC + CA = 80 + 52 \qquad \sigma_{max} = +132 \text{ MPa} \quad ◀$$
$$\sigma_{min} = OB = OC - BC = 80 - 52 \qquad \sigma_{min} = +\ 28 \text{ MPa} \quad ◀$$

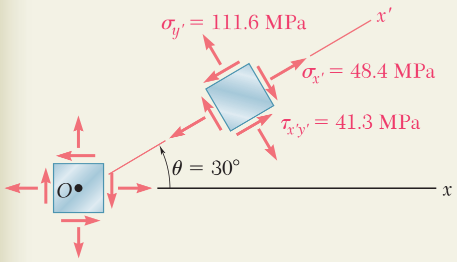
Since the rotation that brings XY into AB is clockwise, the rotation that brings Ox into the axis Oa corresponding to σ_{max} is also clockwise; we obtain the orientation shown for the principal planes.

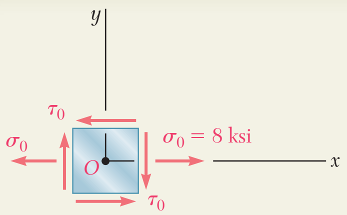
b. Stress Components on Element Rotated 30°↺. Points X' and Y' on Mohr's circle that correspond to the stress components on the rotated element are obtained by rotating XY counterclockwise through $2\theta = 60°$. We find



$$\phi = 180° - 60° - 67.4° \qquad \phi = 52.6° \quad ◀$$
$$\sigma_{x'} = OK = OC - KC = 80 - 52 \cos 52.6° \qquad \sigma_{x'} = +\ 48.4 \text{ MPa} \quad ◀$$
$$\sigma_{y'} = OL = OC + CL = 80 + 52 \cos 52.6° \qquad \sigma_{y'} = +111.6 \text{ MPa} \quad ◀$$
$$\tau_{x'y'} = KX' = 52 \sin 52.6° \qquad \tau_{x'y'} = \ 41.3 \text{ MPa} \quad ◀$$

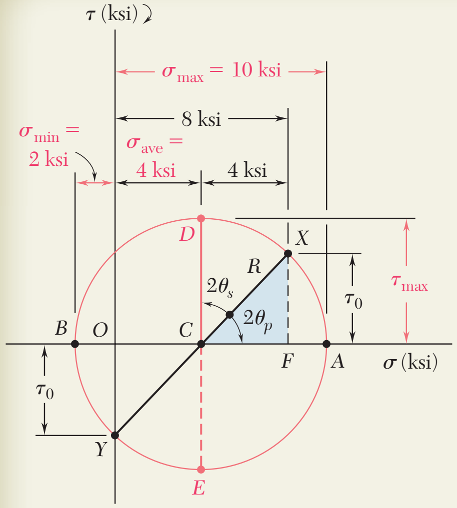
Since X' is located above the horizontal axis, the shearing stress on the face perpendicular to Ox' tends to rotate the element clockwise.





SAMPLE PROBLEM 14.3

A state of plane stress consists of a tensile stress $\sigma_0 = 8$ ksi exerted on vertical surfaces and of unknown shearing stresses. Determine (*a*) the magnitude of the shearing stress τ_0 for which the largest normal stress is 10 ksi, (*b*) the corresponding maximum shearing stress.



SOLUTION

Construction of Mohr's Circle. We assume that the shearing stresses act in the senses shown. Thus, the shearing stress τ_0 on a face perpendicular to the x axis tends to rotate the element clockwise, and we plot the point X of coordinates 8 ksi and τ_0 above the horizontal axis. Considering a horizontal face of the element, we observe that $\sigma_y = 0$ and that τ_0 tends to rotate the element counterclockwise; thus, we plot point Y at a distance τ_0 below O.

We note that the abscissa of the center C of Mohr's circle is

$$\sigma_{ave} = \tfrac{1}{2}(\sigma_x + \sigma_y) = \tfrac{1}{2}(8 + 0) = 4 \text{ ksi}$$

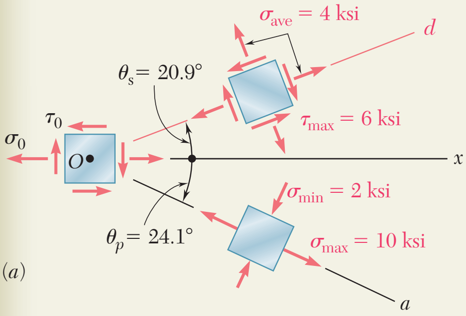
The radius R of the circle is determined by observing that the maximum normal stress, $\sigma_{max} = 10$ ksi, is represented by the abscissa of point A and writing

$$\sigma_{max} = \sigma_{ave} + R$$
$$10 \text{ ksi} = 4 \text{ ksi} + R \qquad R = 6 \text{ ksi}$$

a. Shearing Stress τ_0. Considering the right triangle CFX, we find

$$\cos 2\theta_p = \frac{CF}{CX} = \frac{CF}{R} = \frac{4 \text{ ksi}}{6 \text{ ksi}} \qquad 2\theta_p = 48.2^\circ \downarrow \qquad \theta_p = 24.1^\circ \downarrow$$

$$\tau_0 = FX = R \sin 2\theta_p = (6 \text{ ksi}) \sin 48.2^\circ \qquad \tau_0 = 4.47 \text{ ksi} \blacktriangleleft$$



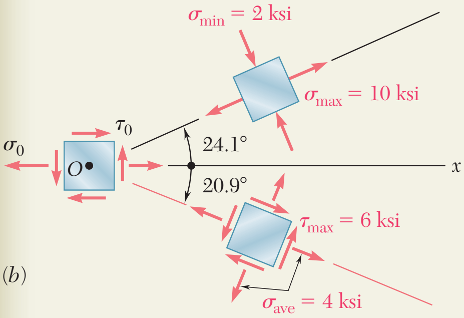
(*a*)

b. Maximum Shearing Stress. The coordinates of point D of Mohr's circle represent the maximum shearing stress and the corresponding normal stress.

$$\tau_{max} = R = 6 \text{ ksi} \qquad \tau_{max} = 6 \text{ ksi} \blacktriangleleft$$
$$2\theta_s = 90^\circ - 2\theta_p = 90^\circ - 48.2^\circ = 41.8^\circ \uparrow \qquad \theta_s = 20.9^\circ \uparrow$$

The maximum shearing stress is exerted on an element that is oriented as shown in Fig. *a*. (The element upon which the principal stresses are exerted is also shown.)

Note. If our original assumption regarding the sense of τ_0 was reversed, we would obtain the same circle and the same answers, but the orientation of the elements would be as shown in Fig. *b*.



(*b*)

PROBLEMS

14.25 Solve Probs. 14.5 and 14.9, using Mohr's circle.

14.26 Solve Probs. 14.6 and 14.10, using Mohr's circle.

14.27 Solve Prob. 14.11, using Mohr's circle.

14.28 Solve Prob. 14.12, using Mohr's circle.

14.29 Solve Prob. 14.13, using Mohr's circle.

14.30 Solve Prob. 14.14, using Mohr's circle

14.31 Solve Prob. 14.15, using Mohr's circle.

14.32 Solve Prob. 14.16, using Mohr's circle.

14.33 Solve Prob. 14.17, using Mohr's circle.

14.34 Solve Prob. 14.18, using Mohr's circle.

14.35 Solve Prob. 14.19, using Mohr's circle.

14.36 Solve Prob. 14.20, using Mohr's circle.

14.37 Solve Prob. 14.21, using Mohr's circle.

14.38 Solve Prob. 14.22, using Mohr's circle.

14.39 Solve Prob. 14.23, using Mohr's circle.

14.40 Solve Prob. 14.24, using Mohr's circle.

14.41 For the state of plane stress shown, use Mohr's circle to determine (*a*) the largest value of τ_{xy} for which the maximum in-plane shearing stress is equal to or less than 12 ksi, (*b*) the corresponding principal stresses.

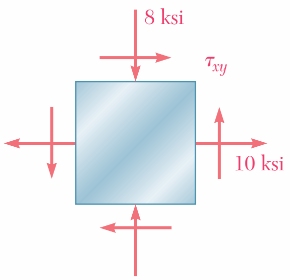


Fig. P14.41

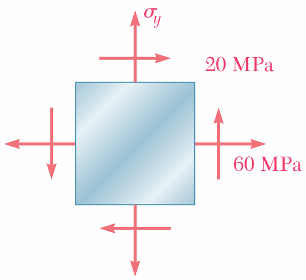


Fig. *P14.42*

14.42 For the state of plane stress shown, use Mohr's circle to determine the largest value of σ_y for which the maximum in-plane shearing stress is equal to or less than 75 MPa.

14.43 For the state of plane stress shown, use Mohr's circle to determine (*a*) the value of τ_{xy} for which the in-plane shearing stress parallel to the weld is zero, (*b*) the corresponding principal stresses.

14.44 Solve Prob. 14.43 assuming that the weld forms an angle of 60° with the horizontal.

***14.45* through 14.48** Determine the principal planes and the principal stresses for the state of plane stress resulting from the superposition of the two states of stress shown.

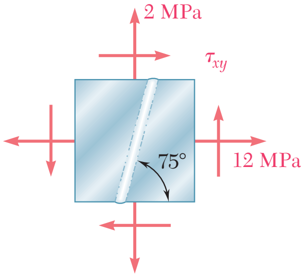


Fig. P14.43

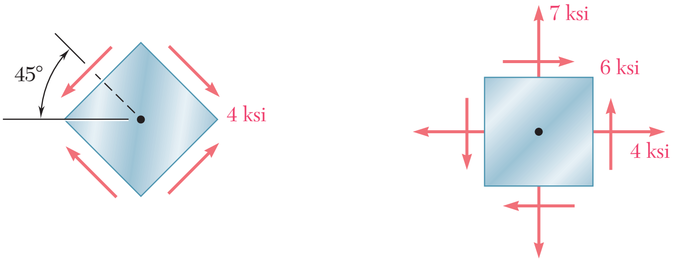


Fig. *P14.45*

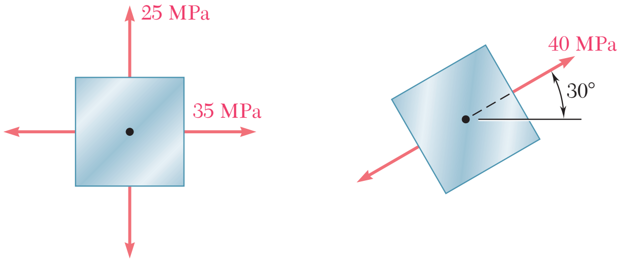


Fig. P14.46

Fig. P14.47

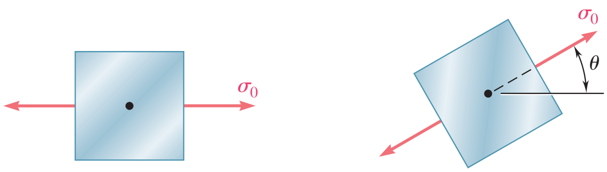


Fig. P14.48

14.5 STRESSES IN THIN-WALLED PRESSURE VESSELS

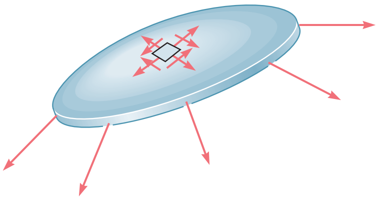


Fig. 14.23

Thin-walled pressure vessels provide an important application of the analysis of plane stress. Since their walls offer little resistance to bending, it can be assumed that the internal forces exerted on a given portion of wall are tangent to the surface of the vessel (Fig. 14.23). The resulting stresses on an element of wall will thus be contained in a plane tangent to the surface of the vessel.

Our analysis of stresses in thin-walled pressure vessels will be limited to the two types of vessels most frequently encountered: cylindrical pressure vessels and spherical pressure vessels (Photos 14.3 and 14.4).

Photo 14.3

Photo 14.4

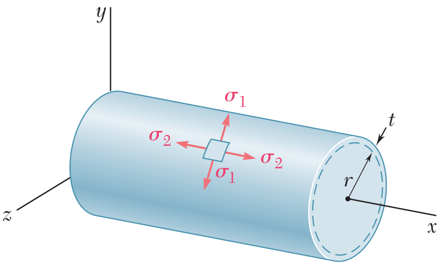


Fig. 14.24

Consider a cylindrical vessel of inner radius r and wall thickness t containing a fluid under pressure (Fig. 14.24). We propose to determine the stresses exerted on a small element of wall with sides respectively parallel and perpendicular to the axis of the cylinder. Because of the axisymmetry of the vessel and its contents, it is clear that no shearing stress is exerted on the element. The normal stresses σ_1 and σ_2 shown in Fig. 14.24 are therefore principal stresses. The stress σ_1 is known as the *hoop stress*, because it is the type of stress found in hoops used to hold together the various slats of a wooden barrel, and the stress σ_2 is called the *longitudinal stress*.

In order to determine the hoop stress σ_1, we detach a portion of the vessel and its contents bounded by the xy plane and by two planes parallel to the yz plane at a distance Δx from each other (Fig. 14.25). The forces parallel to the z axis acting on the free body defined in this fashion consist of the elementary internal forces $\sigma_1\, dA$ on the wall sections, and of the elementary pressure forces $p\, dA$ exerted on the portion of fluid included in the free body. Note that p denotes the *gage pressure* of the fluid, i.e., the excess of the inside pressure over the outside atmospheric pressure. The resultant of the internal forces $\sigma_1\, dA$ is equal to the product of σ_1 and of the cross-sectional area $2t\,\Delta x$ of the wall, while the resultant of the pressure forces $p\, dA$ is equal to the product of p and of the area $2r\,\Delta x$. Writing the equilibrium equation $\Sigma F_z = 0$, we have

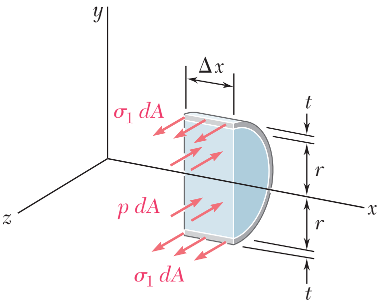


Fig. 14.25

$$\Sigma F_z = 0: \qquad \sigma_1(2t\,\Delta x) - p(2r\,\Delta x) = 0$$

and, solving for the hoop stress σ_1,

$$\boxed{\sigma_1 = \frac{pr}{t}} \tag{14.20}$$

To determine the longitudinal stress σ_2, we now pass a section perpendicular to the x axis and consider the free body consisting of the portion of the vessel and its contents located to the left of the section (Fig. 14.26). The forces acting on this free body are the elementary internal forces $\sigma_2\, dA$ on the wall section and the elementary pressure forces $p\, dA$ exerted on the portion of fluid included in the free body. Noting that the area of the fluid section is πr^2 and that the area of the wall section can be obtained by multiplying the circumference $2\pi r$ of the cylinder by its wall thickness t, we write the equilibrium equation:†

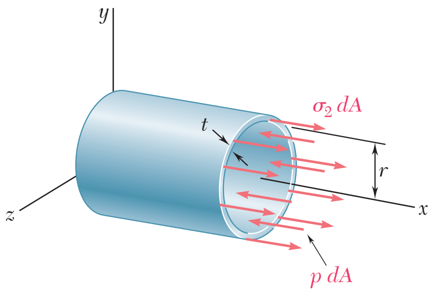


Fig. 14.26

$\Sigma F_x = 0$: $\qquad \sigma_2(2\pi r t) - p(\pi r^2) = 0$

and, solving for the longitudinal stress σ_2,

$$\boxed{\sigma_2 = \frac{pr}{2t}} \tag{14.21}$$

We note from Eqs. (14.20) and (14.21) that the hoop stress σ_1 is twice as large as the longitudinal stress σ_2:

$$\sigma_1 = 2\sigma_2 \tag{14.22}$$

We now consider a spherical vessel of inner radius r and wall thickness t containing a fluid under a gage pressure p. For reasons of symmetry, the stresses exerted on the four faces of a small element of wall must be equal (Fig. 14.27). We have

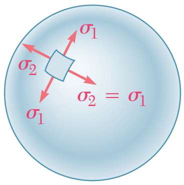


Fig. 14.27

$$\sigma_1 = \sigma_2 \tag{14.23}$$

To determine the value of the stress, we pass a section through the center C of the vessel and consider the free body consisting of the portion of the vessel and its contents located to the left of the section (Fig. 14.28). The equation of equilibrium for this free body is the same as for the free body of Fig. 14.26. We thus conclude that, for a spherical vessel,

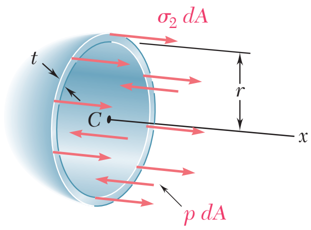


Fig. 14.28

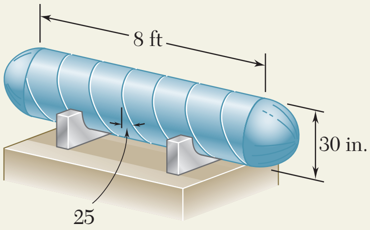
$$\boxed{\sigma_1 = \sigma_2 = \frac{pr}{2t}} \tag{14.24}$$

†Using the mean radius of the wall section, $r_m = r + \frac{1}{2}t$, in computing the resultant of the forces on that section, we would obtain a more accurate value of the longitudinal stress, namely,

$$\sigma_2 = \frac{pr}{2t}\frac{1}{1 + \dfrac{t}{2r}} \tag{14.21$'$}$$

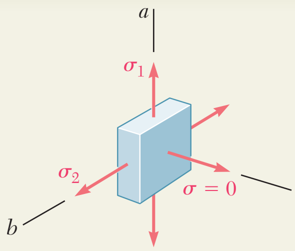
However, for a thin-walled pressure vessel, the term $t/2r$ is sufficiently small to allow the use of Eq. (14.21) for engineering design and analysis. If a pressure vessel is not thin-walled (i.e., if $t/2r$ is not small), the stresses σ_1 and σ_2 vary across the wall and must be determined by the methods of the theory of elasticity.

SAMPLE PROBLEM 14.4



A compressed-air tank is supported by two cradles as shown; one of the cradles is designed so that it does not exert any longitudinal force on the tank. The cylindrical body of the tank has a 30-in. outer diameter and is fabricated from a $\frac{3}{8}$-in. steel plate by butt welding along a helix that forms an angle of 25° with a transverse plane. The end caps are spherical and have a uniform wall thickness of $\frac{5}{16}$ in. For an internal gage pressure of 180 psi, determine (*a*) the normal stress in the spherical caps, (*b*) the stresses in directions perpendicular and parallel to the helical weld.

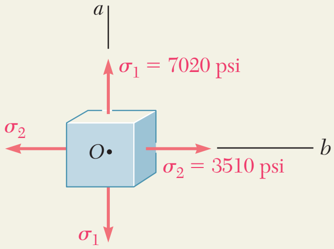
SOLUTION



a. Spherical Cap. Using Eq. (14.24), we write

$$p = 180 \text{ psi}, t = \tfrac{5}{16} \text{ in.} = 0.3125 \text{ in.}, r = 15 - 0.3125 = 14.688 \text{ in.}$$

$$\sigma_1 = \sigma_2 = \frac{pr}{2t} = \frac{(180 \text{ psi})(14.688 \text{ in.})}{2(0.3125 \text{ in.})} \qquad \sigma = 4230 \text{ psi} \blacktriangleleft$$



b. Cylindrical Body of the Tank. We first determine the hoop stress σ_1 and the longitudinal stress σ_2. Using Eqs. (14.20) and (14.22), we write

$$p = 180 \text{ psi}, t = \tfrac{3}{8} \text{ in.} = 0.375 \text{ in.}, r = 15 - 0.375 = 14.625 \text{ in.}$$

$$\sigma_1 = \frac{pr}{t} = \frac{(180 \text{ psi})(14.625 \text{ in.})}{0.375 \text{ in.}} = 7020 \text{ psi} \qquad \sigma_2 = \tfrac{1}{2}\sigma_1 = 3510 \text{ psi}$$

$$\sigma_{\text{ave}} = \tfrac{1}{2}(\sigma_1 + \sigma_2) = 5265 \text{ psi} \qquad R = \tfrac{1}{2}(\sigma_1 - \sigma_2) = 1755 \text{ psi}$$

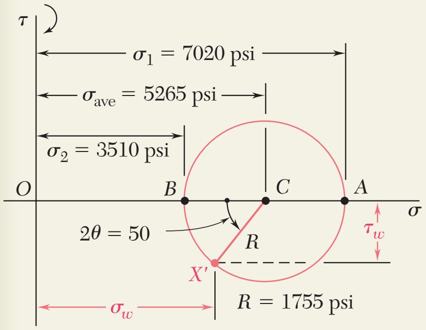
Stresses at the Weld. Noting that both the hoop stress and the longitudinal stress are principal stresses, we draw Mohr's circle as shown.

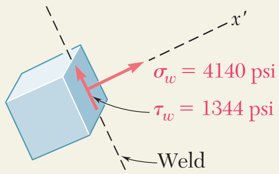
An element having a face parallel to the weld is obtained by rotating the face perpendicular to the axis *Ob* counterclockwise through 25°. Therefore, on Mohr's circle we locate the point X' corresponding to the stress components on the weld by rotating radius *CB* counterclockwise through $2\theta = 50°$.

$$\sigma_w = \sigma_{\text{ave}} - R \cos 50° = 5265 - 1755 \cos 50° \qquad \sigma_w = +4140 \text{ psi} \blacktriangleleft$$

$$\tau_w = R \sin 50° = 1755 \sin 50° \qquad \tau_w = 1344 \text{ psi} \blacktriangleleft$$

Since X' is below the horizontal axis, τ_w tends to rotate the element counterclockwise.





PROBLEMS

14.49 Determine the normal stress in a basketball of 9.5-in. outer diameter and 0.125-in. wall thickness that is inflated to a gage pressure of 9 psi.

14.50 A spherical gas container made of steel has an 18-ft outer diameter and a wall thickness of $\frac{3}{8}$ in. Knowing that the internal pressure is 60 psi, determine the maximum normal stress in the container.

14.51 The maximum gage pressure is known to be 8 MPa in a spherical steel pressure vessel having a 250-mm outer diameter and a 6-mm wall thickness. Knowing that the ultimate stress in the steel used is $\sigma_U = 400$ MPa, determine the factor of safety with respect to tensile failure.

14.52 A spherical gas container having an outer diameter of 15 ft and a wall thickness of 0.90 in. is made of a steel for which $E = 29 \times 10^6$ psi and $\nu = 0.29$. Knowing that the gage pressure in the container is increased from zero to 250 psi, determine (*a*) the maximum normal stress in the container, (*b*) the increase in the diameter of the container.

14.53 A spherical pressure vessel has an outer diameter of 3 m and a wall thickness of 12 mm. Knowing that for the steel used $\sigma_{all} = 80$ MPa, $E = 200$ GPa, and $\nu = 0.29$, determine (*a*) the allowable gage pressure, (*b*) the corresponding increase in the diameter of the vessel.

14.54 A spherical pressure vessel of 750-mm outer diameter is to be fabricated from a steel having an ultimate stress $\sigma_U = 400$ MPa. Knowing that a factor of safety of 4.0 is desired and that the gage pressure can reach 4.2 MPa, determine the smallest wall thickness that should be used.

14.55 When filled to capacity, the unpressurized storage tank shown contains water to a height of 15.5 m above its base. Knowing that the lower portion of the tank has a wall thickness of 16 mm, determine the maximum normal stress in the tank. (Density of water = 1000 kg/m^3.)

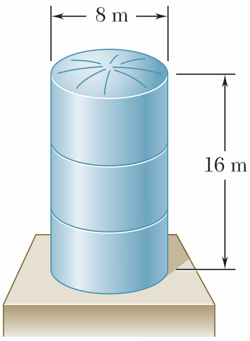


Fig. *P14.55*

14.56 Determine the largest internal pressure that can be applied to a cylindrical tank of 1.75-m outer diameter and 16-mm wall thickness if the ultimate normal stress of the steel used is 450 MPa and a factor of safety of 5.0 is desired.

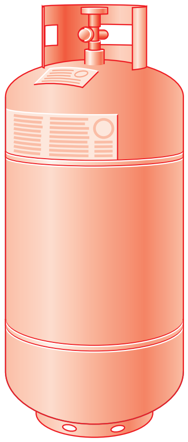


Fig. *P14.57*

14.57 The storage tank shown contains liquefied propane under a pressure of 210 psi at a temperature of 100°F. Knowing that the tank has an outer diameter of 12.6 in. and a wall thickness of 0.11 in., determine the maximum normal stress in the tank.

14.58 The bulk storage tank shown in Photo 14.3 has an outer diameter of 3.3 m and a wall thickness of 18 mm. At a time when the internal pressure of the tank is 1.5 MPa, determine the maximum normal stress in the tank.

14.59 A steel penstock has a 36-in. outer diameter and a 0.5-in. wall thickness, and connects a reservoir at A with a generating station at B. Knowing that the specific weight of water is 62.4 lb/ft^3, determine the maximum normal stress in the penstock under static conditions.

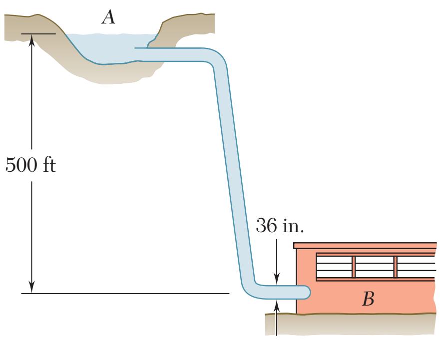


Fig. P14.59 and P14.60

14.60 A steel penstock has a 36-in. outer diameter and connects a reservoir at A with a generating station at B. Knowing that the specific weight of water is 62.4 lb/ft^3 and that the allowable normal stress in the steel is 12.5 ksi, determine the smallest wall thickness that can be used for the penstock.

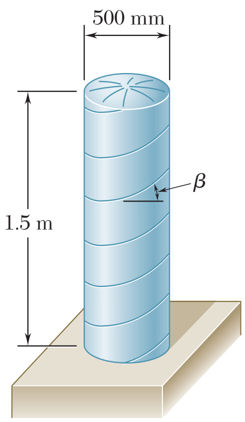


Fig. *P14.61* and P14.62

14.61 The cylindrical portion of the compressed air tank shown is fabricated of 6-mm-thick plate welded along a helix forming an angle $\beta = 30°$ with the horizontal. Knowing that the allowable stress normal to the weld is 75 MPa, determine the largest gage pressure that can be used in the tank.

14.62 The cylindrical portion of the compressed air tank shown is fabricated of 6-mm-thick plate welded along a helix forming an angle $\beta = 30°$ with the horizontal. Determine the gage pressure that will cause a shearing stress parallel to the weld of 30 MPa.

14.63 The pressure tank shown has a $\frac{3}{8}$-in. wall thickness and butt welded seams forming an angle $\beta = 20°$ with a transverse plane. For a gage pressure of 85 psi, determine (*a*) the normal stress perpendicular to the weld, (*b*) the shearing stress parallel to the weld.

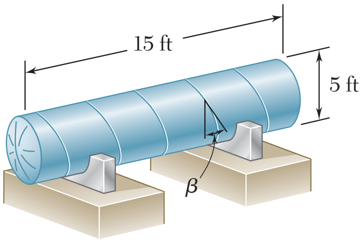


Fig. *P14.63* and P14.64

14.64 The pressure tank shown has a $\frac{3}{8}$-in. wall thickness and butt welded seams forming an angle $\beta = 25°$ with a transverse plane. Determine the largest allowable gage pressure knowing that the allowable normal stress perpendicular to the weld is 18 ksi and the allowable shearing stress parallel to the weld is 10 ksi.

14.65 The pipe shown was fabricated by welding strips of plate along a helix forming an angle β with a transverse plane. Determine the largest value of β that can be used if the normal stress perpendicular to the weld is not to be larger than 85 percent of the maximum stress in the pipe.

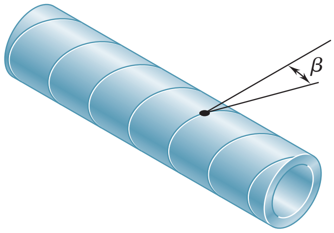


Fig. P14.65 and P14.66

14.66 The pipe shown has an outer diameter of 600 mm and was fabricated by welding strips of 10-mm-thick plate along a helix forming an angle $\beta = 25°$ with a transverse plane. Knowing that the ultimate normal stress perpendicular to the weld is 450 MPa and that a factor of safety of 6.0 is desired, determine the largest allowable gage pressure that can be used.

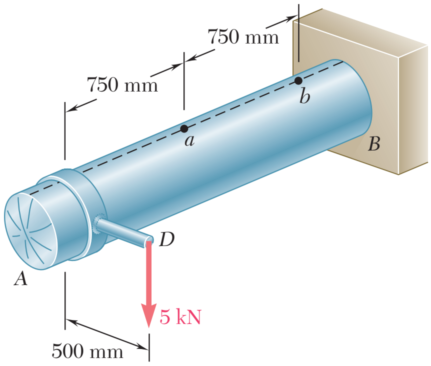


Fig. *P14.67*

14.67 The compressed-air tank AB has an inner diameter of 450 mm and a uniform wall thickness of 6 mm. Knowing that the gage pressure inside the tank is 1.2 MPa, determine the maximum normal stress and the maximum in-plane shearing stress at point a on the top of the tank.

14.68 For the compressed-air tank and loading of Prob. 14.67, determine the maximum normal stress and the maximum in-plane shearing stress at point b on the top of the tank.

14.69 A pressure vessel of 10-in. inner diameter and 0.25-in. wall thickness is fabricated from a 4-ft section of spirally welded pipe *AB* and is equipped with two rigid end plates. The gage pressure inside the vessel is 300 psi, and 10-kip centric axial forces **P** and **P′** are applied to the end plates. Determine (*a*) the normal stress perpendicular to the weld, (*b*) the shearing stress parallel to the weld.

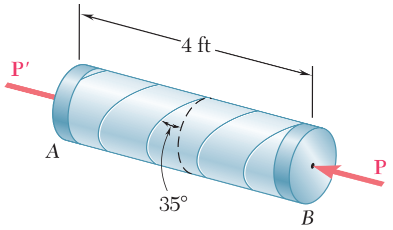


Fig. P14.69

14.70 Solve Prob. 14.69 assuming that the magnitude of *P* of the two forces is increased to 30 kips.

14.71 The cylindrical tank *AB* has an 8-in. inner diameter and a 0.32-in. wall thickness. Knowing that the pressure inside the tank is 600 psi, determine the maximum normal stress and the maximum in-plane shearing stress at point *K*.

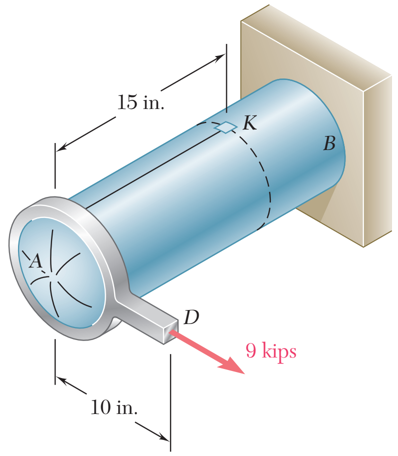


Fig. P14.71

14.72 Solve Prob. 14.71 assuming that the 9-kip force applied at point *D* is directed vertically downward.

REVIEW AND SUMMARY

The first part of this chapter was devoted to a study of the *transformation of stress* under a rotation of axes and to its application to the solution of engineering problems.

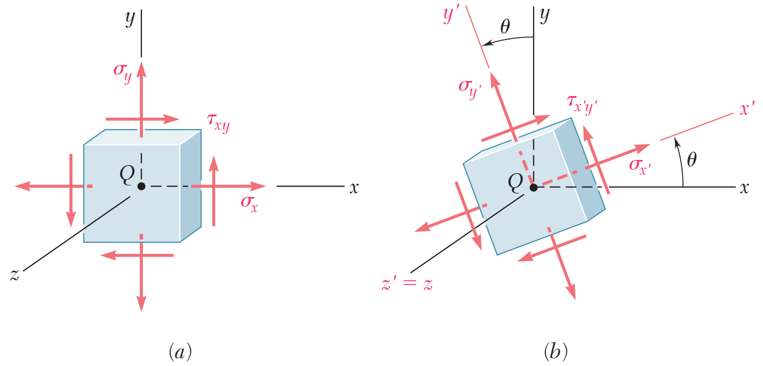


Fig. 14.29

Transformation of plane stress

Considering first a state of *plane stress* at a given point Q [Sec. 14.2] and denoting by σ_x, σ_y, and τ_{xy} the stress components associated with the element shown in Fig. 14.29*a*, we derived the following formulas defining the components $\sigma_{x'}$, $\sigma_{y'}$, and $\tau_{x'y'}$ associated with that element after it had been rotated through an angle θ about the z axis (Fig. 14.29*b*):

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \qquad (14.5)$$

$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta \qquad (14.7)$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \qquad (14.6)$$

In Sec. 14.3, we determined the values θ_p of the angle of rotation which correspond to the maximum and minimum values of the normal stress at point Q. We wrote

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \qquad (14.12)$$

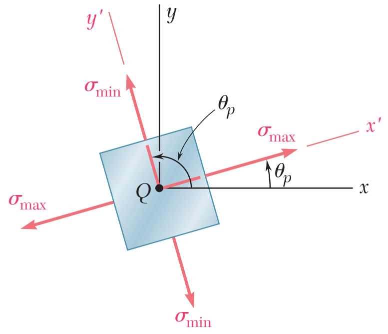


Fig. 14.30

Principal planes. Principal stresses

The two values obtained for θ_p are 90° apart (Fig. 14.30) and define the *principal planes of stress* at point Q. The corresponding values of the normal stress are called the *principal stresses* at Q; we obtained

$$\sigma_{\max,\min} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \qquad (14.14)$$

We also noted that the corresponding value of the shearing stress is zero. Next, we determined the values θ_s of the angle θ for which the largest value of the shearing stress occurs. We wrote

$$\tan 2\theta_s = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}} \tag{14.15}$$

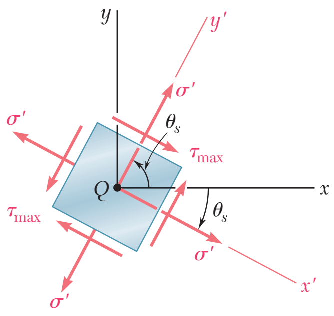


Fig. 14.31

The two values obtained for θ_s are 90° apart (Fig. 14.31). We also noted that the planes of maximum shearing stress are at 45° to the principal planes. The maximum value of the shearing stress for a rotation *in the plane of stress is*

Maximum in-plane shearing stress

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \tag{14.16}$$

and the corresponding value of the normal stresses is

$$\sigma' = \sigma_{\text{ave}} = \frac{\sigma_x + \sigma_y}{2} \tag{14.17}$$

Mohr's circle for stress

We saw in Sec. 14.4 that *Mohr's circle* provides an alternative method, based on simple geometric considerations, for the analysis of the transformation of plane stress. Given the state of stress shown in black in Fig. 14.32*a*, we plot point X of coordinates σ_x, $-\tau_{xy}$ and point Y of coordinates σ_y, $+\tau_{xy}$ (Fig. 14.32*b*). Drawing the circle of

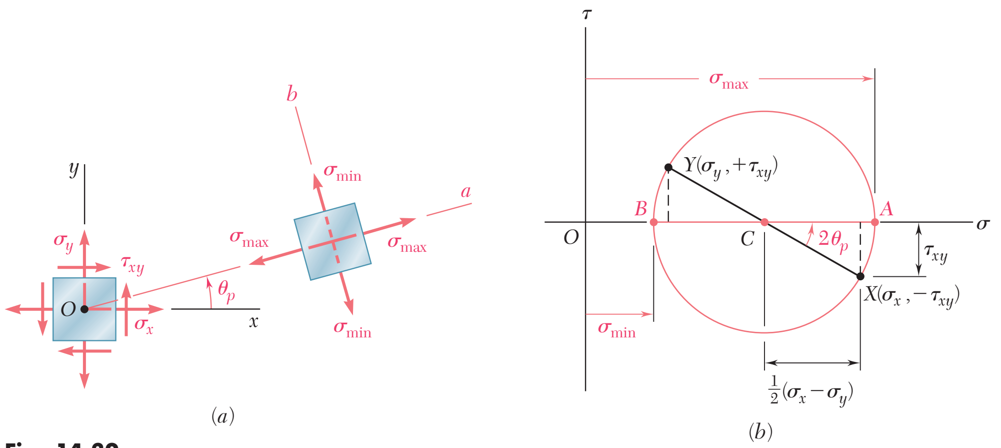


Fig. 14.32

diameter XY, we obtain Mohr's circle. The abscissas of the points of intersection A and B of the circle with the horizontal axis represent the principal stresses, and the angle of rotation bringing the diameter XY into AB is twice the angle θ_p defining the principal planes in Fig. 14.32*a*, with both angles having the same sense. We also noted that diameter DE defines the maximum shearing stress and the orientation of the corresponding plane (Fig. 14.33) [Example 14.2, Sample Probs. 14.2 and 14.3].

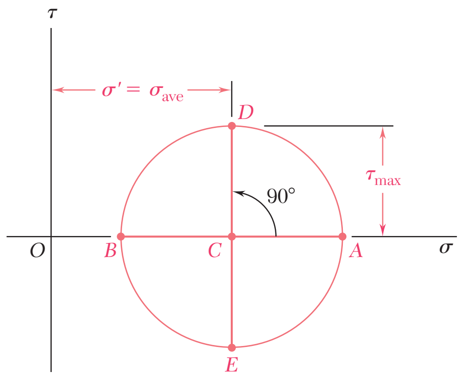


Fig. 14.33

Cylindrical pressure vessels

In Sec. 14.5, we discussed the stresses in *thin-walled pressure vessels* and derived formulas relating the stresses in the walls of the vessels and the *gage pressure* p in the fluid they contain. In the case of a *cylindrical vessel* of inside radius r and thickness t (Fig. 14.34), we obtained the following expressions for the *hoop stress* σ_1 and the *longitudinal stress* σ_2:

$$\sigma_1 = \frac{pr}{t} \qquad \sigma_2 = \frac{pr}{2t} \qquad (14.20,\ 14.21)$$

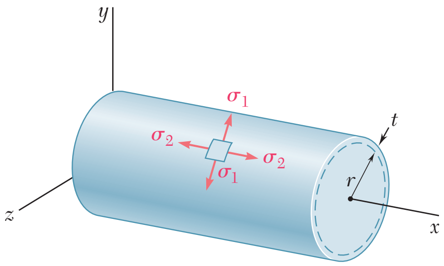


Fig. 14.34

Spherical pressure vessels

In the case of a *spherical vessel* of inside radius r and thickness t (Fig. 14.35), we found that the two principal stresses are equal:

$$\sigma_1 = \sigma_2 = \frac{pr}{2t} \qquad (14.24)$$

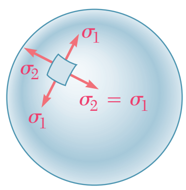


Fig. 14.35

REVIEW PROBLEMS

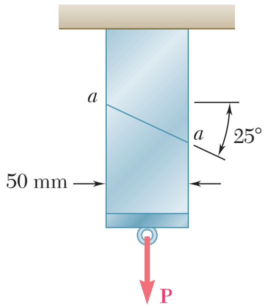


Fig. P14.73

14.73 Two members of uniform cross section 50 × 80 mm are glued together along plane a-a that forms an angle of 25° with the horizontal. Knowing that the allowable stresses for the glued joint are $\sigma = 800$ kPa and $\tau = 600$ kPa, determine the largest axial load **P** that can be applied.

14.74 Determine the largest internal pressure that can be applied to a cylindrical tank of 5.5-ft outer diameter and $\frac{5}{8}$-in. wall thickness if the ultimate normal stress of the steel used is 65 ksi and a factor of safety of 5.0 is desired.

14.75 A spherical pressure tank has a 1.2-m outer diameter and a uniform wall thickness of 10 mm. Knowing that the gage pressure is 1.25 MPa in the tank, determine the maximum normal stress. (Use E = 200 GPa and $\nu = 0.30$.)

14.76 For a state of plane stress it is known that the normal and shearing stresses are directed as shown and that $\sigma_x = 5$ ksi, $\sigma_y = 12$ ksi, and $\sigma_{\max} = 18$ ksi. Determine (*a*) the orientation of the principal planes, (*b*) the maximum in-plane shearing stress.

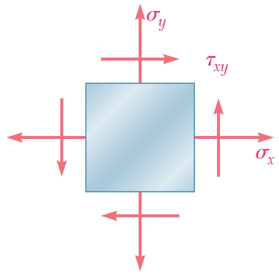


Fig. P14.76

14.77 For the state of plane stress shown, determine (*a*) the principal planes, (*b*) the principal stresses, (*c*) the maximum shearing stress.

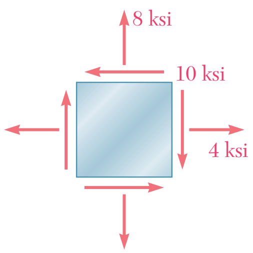


Fig. *P14.77*

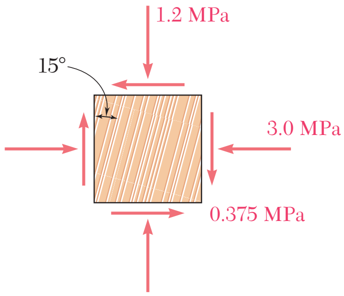


Fig. P14.78

14.78 The grain of a wooden member forms an angle of 15° with the vertical. For the state of plane stress shown, determine (*a*) the in-plane shearing stress parallel to the grain, (*b*) the normal stress perpendicular to the grain.

14.79 A cylindrical steel pressure tank has a 26-in. inner diameter and a uniform $\frac{1}{4}$-in. wall thickness. Knowing that the ultimate stress of the steel used is 65 ksi, determine the maximum allowable gage pressure if a factor of safety of 5.0 must be maintained.

14.80 Two wooden members of 80 × 120-mm uniform rectangular cross section are joined by the simple glued scarf splice shown. Knowing that $\beta = 22°$ and that the maximum allowable stresses in the joint are, respectively, 400 kPa in tension (perpendicular to the splice) and 600 kPa in shear (parallel to the splice), determine the largest centric load **P** that can be applied.

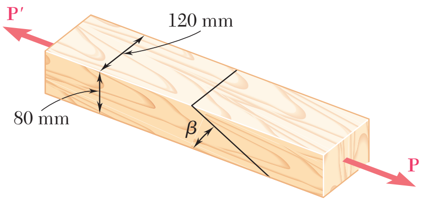


Fig. *P14.80* and P14.81

14.81 Two wooden members of 80 × 120-mm uniform rectangular cross section are joined by the simple glued scarf splice shown. Knowing that $\beta = 25°$ and that the centric loads of magnitude $P = 10$ kN are applied to the member as shown, determine (*a*) the in-plane shearing stress parallel to the splice, (*b*) the normal stress perpendicular to the splice.

14.82 The axle of an automobile is acted upon by the forces and couple shown. Knowing that the diameter of the solid axle is 1.25 in., determine (*a*) the principal planes and principal stresses at point H located on top of the axle, (*b*) the maximum shearing stress at the same point.

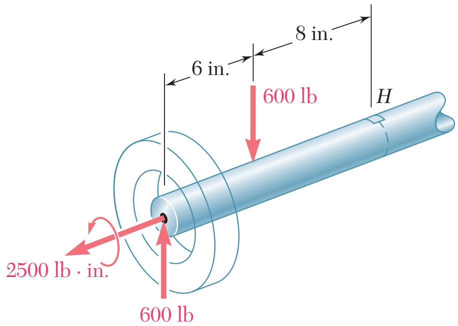


Fig. P14.82

14.83 Square plates, each of 0.5-in. thickness, can be bent and welded together in either of the two ways shown to form the cylindrical portion of the compressed air tank. Knowing that the allowable normal stress perpendicular to the weld is 12 ksi, determine the largest allowable gage pressure in each case.

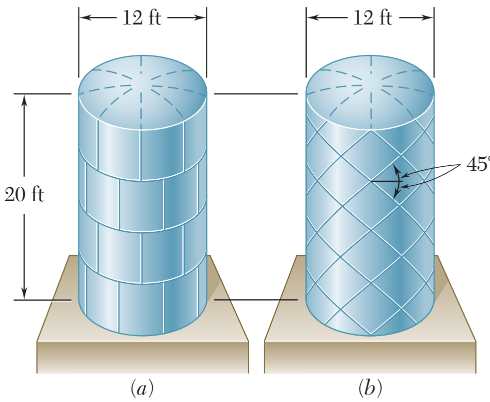


Fig. *P14.83*

14.84 A torque of magnitude $T = 12$ kN · m is applied to the end of a tank containing compressed air under a pressure of 8 MPa. Knowing that the tank has a 180-mm inner diameter and a 12-mm wall thickness, determine the maximum normal stress and the maximum in-plane shearing stress in the tank.

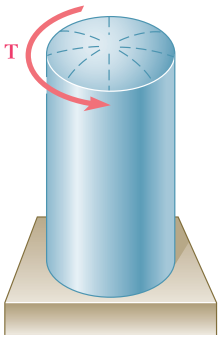


Fig. P14.84

The photo shows a multiple-girder bridge during construction. The design of the steel girders is based on both strength considerations and deflection evaluations.

CHAPTER 15

Deflection of Beams

Chapter 15 Deflection of Beams

15.1 INTRODUCTION

In the preceding chapter we learned to design beams for strength. In this chapter we will be concerned with another aspect in the design of beams, namely, the determination of the *deflection*. Of particular interest is the determination of the *maximum deflection* of a beam under a given loading, since the design specifications of a beam will generally include a maximum allowable value for its deflection. Also of interest is that a knowledge of the deflections is required to analyze *indeterminate beams*. These are beams in which the number of reactions at the supports exceeds the number of equilibrium equations available to determine these unknowns.

We saw in Sec. 11.4 that a prismatic beam subjected to pure bending is bent into an arc of circle and that, within the elastic range, the curvature of the neutral surface can be expressed as

$$\frac{1}{\rho} = \frac{M}{EI} \tag{11.21}$$

where M is the bending moment, E the modulus of elasticity, and I the moment of inertia of the cross section about its neutral axis.

When a beam is subjected to a transverse loading, Eq. (11.21) remains valid for any given transverse section, provided that Saint-Venant's principle applies. However, both the bending moment and the curvature of the neutral surface will vary from section to section. Denoting by x the distance of the section from the left end of the beam, we write

$$\frac{1}{\rho} = \frac{M(x)}{EI} \tag{15.1}$$

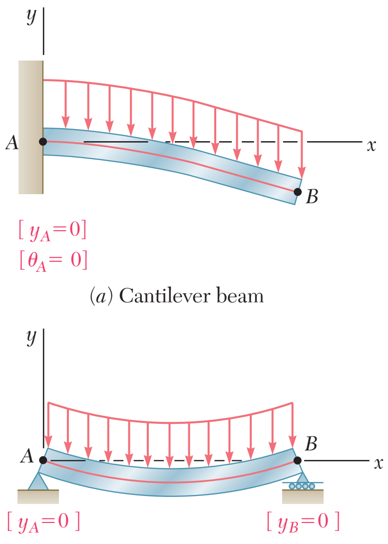
The knowledge of the curvature at various points of the beam will enable us to draw some general conclusions regarding the deformation of the beam under loading (Sec. 15.2).

To determine the slope and deflection of the beam at any given point, we first derive the following second-order linear differential equation, which governs the *elastic curve* characterizing the shape of the deformed beam (Sec. 15.3):

$$\frac{d^2y}{dx^2} = \frac{M(x)}{EI}$$

If the bending moment can be represented for all values of x by a single function $M(x)$, as in the case of the beams and loadings shown in Fig. 15.1, the slope $\theta = dy/dx$ and the deflection y at any point of the beam may be obtained through two successive integrations. The two constants of integration introduced in the process will be determined from the boundary conditions indicated in the figure.

However, if different analytical functions are required to represent the bending moment in various portions of the beam, different differential equations will also be required, leading to different



(*a*) Cantilever beam

(*b*) Simply supported beam

Fig. 15.1

functions defining the elastic curve in the various portions of the beam. In the case of the beam and loading of Fig. 15.2, for example, two differential equations are required, one for the portion of beam AD and the other for the portion DB. The first equation yields the functions θ_1 and y_1, and the second the functions θ_2 and y_2. Altogether, four constants of integration must be determined; two will be obtained by writing that the deflection is zero at A and B, and the other two by expressing that the portions of beam AD and DB have the same slope and the same deflection at D.

You will observe in Sec. 15.4 that in the case of a beam supporting a distributed load $w(x)$, the elastic curve can be obtained directly from $w(x)$ through four successive integrations. The constants introduced in this process will be determined from the boundary values of V, M, θ, and y.

In Sec. 15.5, we will discuss *statically indeterminate beams* where the reactions at the supports involve four or more unknowns. The three equilibrium equations must be supplemented with equations obtained from the boundary conditions imposed by the supports.

The next part of the chapter (Secs. 15.6 and 15.7) is devoted to the *method of superposition,* which consists of determining separately, and then adding, the slope and deflection caused by the various loads applied to a beam. This procedure can be facilitated by the use of the table in App. C, which gives the slopes and deflections of beams for various loadings and types of support.

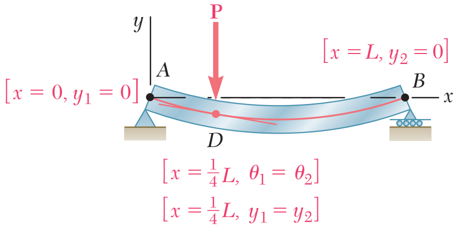


Fig. 15.2

15.2 DEFORMATION OF A BEAM UNDER TRANSVERSE LOADING

At the beginning of this chapter, we recalled Eq. (11.21) of Sec. 11.4, which relates the curvature of the neutral surface and the bending moment in a beam in pure bending. We pointed out that this equation remains valid for any given transverse section of a beam subjected to a transverse loading provided that Saint-Venant's principle applies. However, both the bending moment and the curvature of the neutral surface will vary from section to section. Denoting by x the distance of the section from the left end of the beam, we write

$$\boxed{\frac{1}{\rho} = \frac{M(x)}{EI}} \qquad (15.1)$$

Consider, for example, a cantilever beam AB of length L subjected to a concentrated load $\mathbf{P}$ at its free end A (Fig. 15.3a). We have $M(x) = -Px$ and, substituting into (15.1),

$$\frac{1}{\rho} = -\frac{Px}{EI}$$

which shows that the curvature of the neutral surface varies linearly with x, from zero at A, where ρ_A itself is infinite, to $-PL/EI$ at B, where $|\rho_B| = EI/PL$ (Fig. 15.3b).

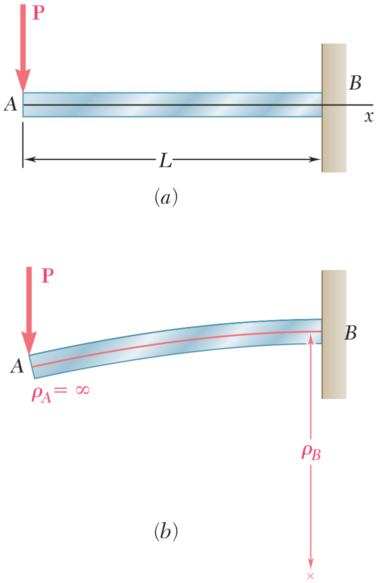


Fig. 15.3

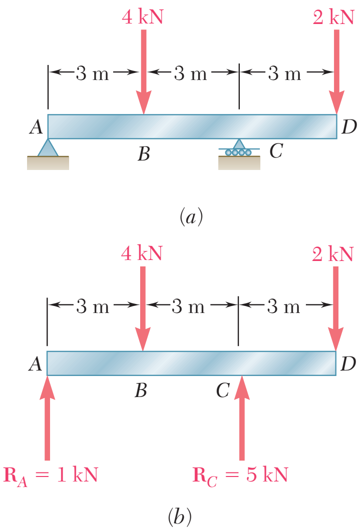


Fig. 15.4

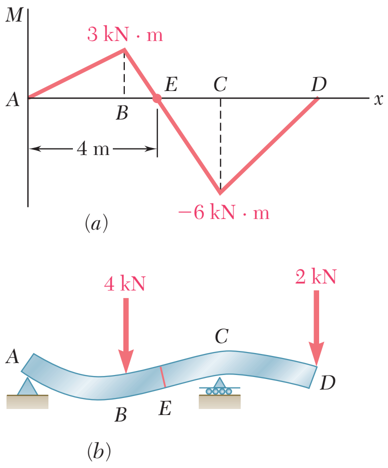


Fig. 15.5

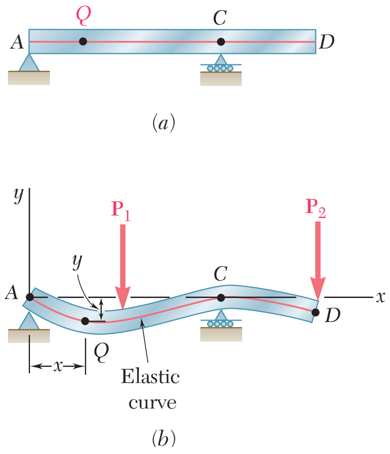


Fig. 15.6

Consider now the overhanging beam AD of Fig. 15.4*a* that supports two concentrated loads as shown. From the free-body diagram of the beam (Fig. 15.4*b*), we find that the reactions at the supports are $R_A = 1$ kN and $R_C = 5$ kN, respectively, and draw the corresponding bending-moment diagram (Fig. 15.5*a*). We note from the diagram that M, and thus the curvature of the beam, are both zero at each end of the beam, and also at a point E located at $x = 4$ m. Between A and E the bending moment is positive and the beam is concave upward; between E and D the bending moment is negative and the beam is concave downward (Fig. 15.5*b*). We also note that the largest value of the curvature (i.e., the smallest value of the radius of curvature) occurs at the support C, where $|M|$ is maximum.

From the information obtained on its curvature, we get a fairly good idea of the shape of the deformed beam. However, the analysis and design of a beam usually require more precise information on the *deflection* and the *slope* of the beam at various points. Of particular importance is the knowledge of the *maximum deflection* of the beam. In the next section Eq. (15.1) will be used to obtain a relation between the deflection y measured at a given point Q on the axis of the beam and the distance x of that point from some fixed origin (Fig. 15.6). The relation obtained is the equation of the *elastic curve,* i.e., the equation of the curve into which the axis of the beam is transformed under the given loading (Fig. 15.6*b*).†

15.3 EQUATION OF THE ELASTIC CURVE

We first recall from elementary calculus that the curvature of a plane curve at a point $Q(x,y)$ of the curve can be expressed as

$$\frac{1}{\rho} = \frac{\dfrac{d^2y}{dx^2}}{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2}} \tag{15.2}$$

where dy/dx and d^2y/dx^2 are the first and second derivatives of the function $y(x)$ represented by that curve. But, in the case of the elastic curve of a beam, the slope dy/dx is very small, and its square is negligible compared to unity. We write, therefore,

$$\frac{1}{\rho} = \frac{d^2y}{dx^2} \tag{15.3}$$

Substituting for $1/\rho$ from (15.3) into (15.1), we have

$$\boxed{\frac{d^2y}{dx^2} = \frac{M(x)}{EI}} \tag{15.4}$$

†It should be noted that, in this chapter, y represents a vertical displacement, while it was used in previous chapters to represent the distance of a given point in a transverse section from the neutral axis of that section.

The equation obtained is a second-order linear differential equation; it is the governing differential equation for the elastic curve.

The product EI is known as the *flexural rigidity* and, if it varies along the beam, as in the case of a beam of varying depth, we must express it as a function of x before proceeding to integrate Eq. (15.4). However, in the case of a prismatic beam, which is the case considered here, the flexural rigidity is constant. We may thus multiply both members of Eq. (15.4) by EI and integrate in x. We write

$$EI\frac{dy}{dx} = \int_0^x M(x)\,dx + C_1 \tag{15.5}$$

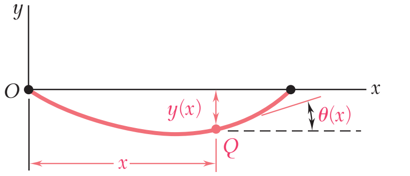


Fig. 15.7

where C_1 is a constant of integration. Denoting by $\theta(x)$ the angle, measured in radians, that the tangent to the elastic curve at Q forms with the horizontal (Fig. 15.7), and recalling that this angle is very small, we have

$$\frac{dy}{dx} = \tan\theta \simeq \theta(x)$$

Thus, we write Eq. (15.5) in the alternative form

$$EI\,\theta(x) = \int_0^x M(x)\,dx + C_1 \tag{15.5$'$}$$

Integrating both members of Eq. (15.5) in x, we have

$$EI\,y = \int_0^x \left[\int_0^x M(x)\,dx + C_1\right]dx + C_2$$

$$EI\,y = \int_0^x dx \int_0^x M(x)\,dx + C_1x + C_2 \tag{15.6}$$

where C_2 is a second constant, and where the first term in the right-hand member represents the function of x obtained by integrating twice in x the bending moment $M(x)$. If it were not for the fact that the constants C_1 and C_2 are as yet undetermined, Eq. (15.6) would define the deflection of the beam at any given point Q, and Eq. (15.5) or (15.5′) would similarly define the slope of the beam at Q.

The constants C_1 and C_2 are determined from the *boundary conditions* or, more precisely, from the conditions imposed on the beam by its supports. Limiting our analysis in this section to *statically determinate beams,* i.e., to beams supported in such a way that the reactions at the supports can be obtained by the methods of statics, we note that only three types of beams need to be considered here (Fig. 15.8): (*a*) the *simply supported beam,* (*b*) the *overhanging beam,* and (*c*) the *cantilever beam.*

In the first two cases, the supports consist of a pin and bracket at A and of a roller at B, and require that the deflection be zero at each of these points. Letting first $x = x_A$, $y = y_A = 0$ in Eq. (15.6), and then $x = x_B$, $y = y_B = 0$ in the same equation, we obtain two equations that can be solved for C_1 and C_2. In the case of the cantilever beam (Fig. 15.8*c*), we note that both the deflection and the slope at A must be zero. Letting $x = x_A$, $y = y_A = 0$ in Eq. (15.6), and $x = x_A$, $\theta = \theta_A = 0$ in Eq. (15.5′), we obtain again two equations which can be solved for C_1 and C_2.

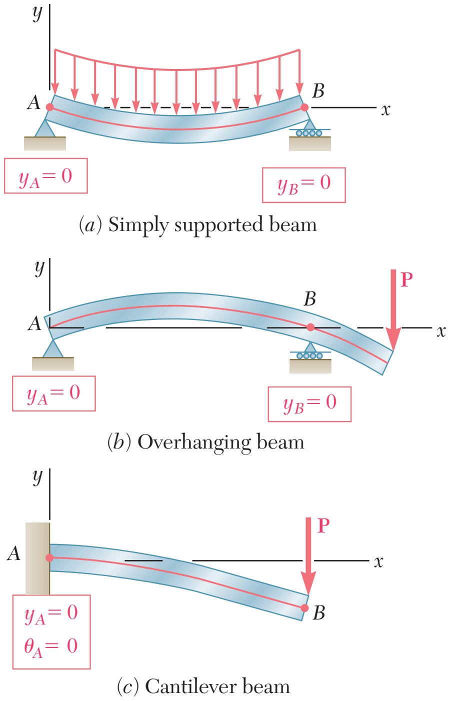


Fig. 15.8 Boundary conditions for statically determinate beams.

EXAMPLE 15.1 The cantilever beam AB is of uniform cross section and carries a load $\mathbf{P}$ at its free end A (Fig. 15.9). Determine the equation of the elastic curve and the deflection and slope at A.

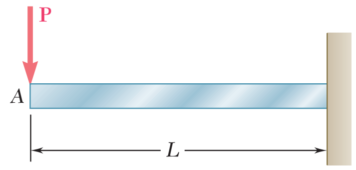


Fig. 15.9

Using the free-body diagram of the portion AC of the beam (Fig. 15.10), where C is located at a distance x from end A, we find

$$M = -Px \tag{15.7}$$

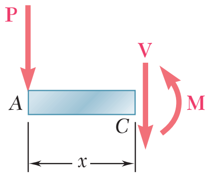


Fig. 15.10

Substituting for M into Eq. (15.4) and multiplying both members by the constant EI, we write

$$EI\frac{d^2y}{dx^2} = -Px$$

Integrating in x, we obtain

$$EI\frac{dy}{dx} = -\tfrac{1}{2}Px^2 + C_1 \tag{15.8}$$

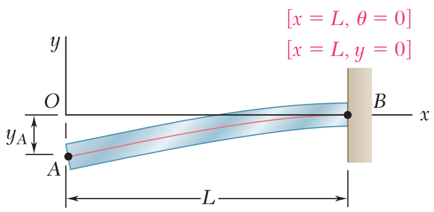


Fig. 15.11

We now observe that at the fixed end B we have $x = L$ and $\theta = dy/dx = 0$ (Fig. 15.11). Substituting these values into (15.8) and solving for C_1, we have

$$C_1 = \tfrac{1}{2}PL^2$$

which we carry back into (15.8):

$$EI\frac{dy}{dx} = -\tfrac{1}{2}Px^2 + \tfrac{1}{2}PL^2 \tag{15.9}$$

Integrating both members of Eq. (15.9), we write

$$EI\,y = -\tfrac{1}{6}Px^3 + \tfrac{1}{2}PL^2x + C_2 \tag{15.10}$$

But, at B we have $x = L$, $y = 0$. Substituting into (15.10), we have

$$0 = -\tfrac{1}{6}PL^3 + \tfrac{1}{2}PL^3 + C_2$$
$$C_2 = -\tfrac{1}{3}PL^3$$

Carrying the value of C_2 back into Eq. (15.10), we obtain the equation of the elastic curve:

$$EI\,y = -\tfrac{1}{6}Px^3 + \tfrac{1}{2}PL^2x - \tfrac{1}{3}PL^3$$

or

$$y = \frac{P}{6EI}(-x^3 + 3L^2x - 2L^3) \tag{15.11}$$

The deflection and slope at A are obtained by letting $x = 0$ in Eqs. (15.11) and (15.9). We find

$$y_A = -\frac{PL^3}{3EI} \qquad \text{and} \qquad \theta_A = \left(\frac{dy}{dx}\right)_A = \frac{PL^2}{2EI} \blacksquare$$

EXAMPLE 15.2 The simply supported prismatic beam AB carries a uniformly distributed load w per unit length (Fig. 15.12). Determine the equation of the elastic curve and the maximum deflection of the beam.

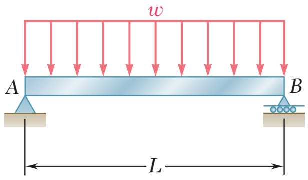


Fig. 15.12

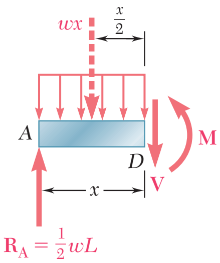


Fig. 15.13

Drawing the free-body diagram of the portion AD of the beam (Fig. 15.13) and taking moments about D, we find that

$$M = \tfrac{1}{2}wLx - \tfrac{1}{2}wx^2 \qquad (15.12)$$

Substituting for M into Eq. (15.4) and multiplying both members of this equation by the constant EI, we write

$$EI\frac{d^2y}{dx^2} = -\frac{1}{2}wx^2 + \frac{1}{2}wLx \qquad (15.13)$$

Integrating twice in x, we have

$$EI\frac{dy}{dx} = -\frac{1}{6}wx^3 + \frac{1}{4}wLx^2 + C_1 \qquad (15.14)$$

$$EI\,y = -\frac{1}{24}wx^4 + \frac{1}{12}wLx^3 + C_1x + C_2 \qquad (15.15)$$

Observing that $y = 0$ at both ends of the beam (Fig. 15.14), we first let $x = 0$ and $y = 0$ in Eq. (15.15) and obtain $C_2 = 0$. We then make $x = L$ and $y = 0$ in the same equation and write

$$0 = -\tfrac{1}{24}wL^4 + \tfrac{1}{12}wL^4 + C_1L$$
$$C_1 = -\tfrac{1}{24}wL^3$$

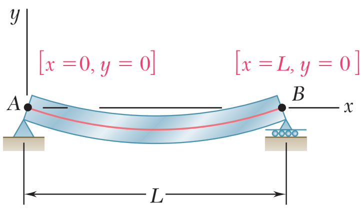


Fig. 15.14

Carrying the values of C_1 and C_2 back into Eq. (15.15), we obtain the equation of the elastic curve:

$$EI\,y = -\tfrac{1}{24}wx^4 + \tfrac{1}{12}wLx^3 - \tfrac{1}{24}wL^3x$$

or

$$y = \frac{w}{24EI}(-x^4 + 2Lx^3 - L^3x) \qquad (15.16)$$

Substituting into Eq. (15.14) the value obtained for C_1, we check that the slope of the beam is zero for $x = L/2$ and that the elastic curve has a minimum at the midpoint C of the beam (Fig. 15.15). Letting $x = L/2$ in Eq. (15.16), we have

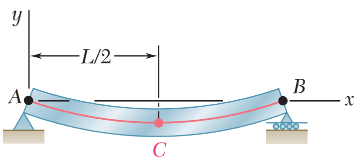


Fig. 15.15

$$y_C = \frac{w}{24EI}\left(-\frac{L^4}{16} + 2L\frac{L^3}{8} - L^3\frac{L}{2}\right) = -\frac{5wL^4}{384EI}$$

The maximum deflection or, more precisely, the maximum absolute value of the deflection, is thus

$$|y|_{max} = \frac{5wL^4}{384EI}$$

Photo 15.1 A different function $M(x)$ is required in each portion of the cantilever arms.

In each of the two examples considered so far, only one free-body diagram was required to determine the bending moment in the beam. As a result, a single function of x was used to represent M throughout the beam. This, however, is not generally the case. Concentrated loads, reactions at supports, or discontinuities in a distributed load will make it necessary to divide the beam into several portions and to represent the bending moment by a different function $M(x)$ in each of these portions of beam (Photo 15.1). Each of the functions $M(x)$ will then lead to a different expression for the slope $\theta(x)$ and for the deflection $y(x)$. Since each of the expressions obtained for the deflection must contain two constants of integration, a large number of constants will have to be determined. As you will see in the next example, the required additional boundary conditions can be obtained by observing that, while the shear and bending moment can be discontinuous at several points in a beam, the *deflection* and the *slope* of the beam *cannot be discontinuous* at any point.

EXAMPLE 15.3 For the prismatic beam and the loading shown (Fig. 15.16), determine the slope and deflection at point D.

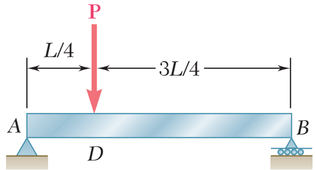


Fig. 15.16

We must divide the beam into two portions, AD and DB, and determine the function $y(x)$ which defines the elastic curve for each of these portions.

1. From A to D ($x < L/4$). We draw the free-body diagram of a portion of beam AE of length $x < L/4$ (Fig. 15.17). Taking moments about E, we have

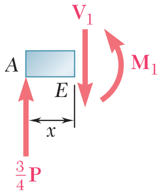


Fig. 15.17

$$M_1 = \frac{3P}{4}x \tag{15.17}$$

or, recalling Eq. (15.4),

$$EI\frac{d^2y_1}{dx^2} = \frac{3}{4}Px \tag{15.18}$$

where $y_1(x)$ is the function which defines the elastic curve *for portion AD of the beam*. Integrating in x, we write

$$EI\,\theta_1 = EI\frac{dy_1}{dx} = \frac{3}{8}Px^2 + C_1 \tag{15.19}$$

$$EI\,y_1 = \frac{1}{8}Px^3 + C_1x + C_2 \tag{15.20}$$

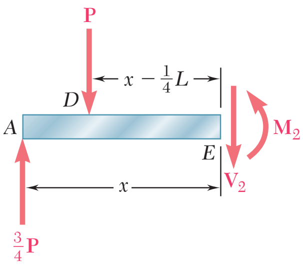


Fig. 15.18

2. From D to B ($x > L/4$). We now draw the free-body diagram of a portion of beam AE of length $x > L/4$ (Fig. 15.18) and write

$$M_2 = \frac{3P}{4}x - P\left(x - \frac{L}{4}\right) \tag{15.21}$$

or, recalling Eq. (15.4) and rearranging terms,

$$EI\frac{d^2y_2}{dx^2} = -\frac{1}{4}Px + \frac{1}{4}PL \tag{15.22}$$

where $y_2(x)$ is the function which defines the elastic curve *for portion DB of the beam*. Integrating in x, we write

$$EI\,\theta_2 = EI\frac{dy_2}{dx} = -\frac{1}{8}Px^2 + \frac{1}{4}PLx + C_3 \tag{15.23}$$

$$EI\,y_2 = -\frac{1}{24}Px^3 + \frac{1}{8}PLx^2 + C_3x + C_4 \tag{15.24}$$

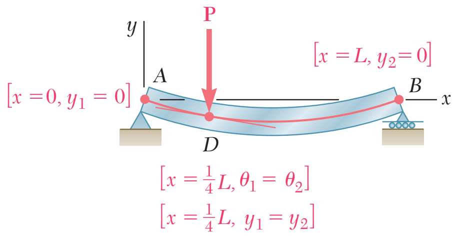


Fig. 15.19

Determination of the Constants of Integration. The conditions that must be satisfied by the constants of integration have been summarized in Fig. 15.19. At the support A, where the deflection is defined by Eq. (15.20), we must have $x = 0$ and $y_1 = 0$. At the support B, where the deflection is defined by Eq. (15.24), we must have $x = L$ and $y_2 = 0$. Also, the fact that there can be no sudden change in deflection or in slope at point D requires that $y_1 = y_2$ and $\theta_1 = \theta_2$ when $x = L/4$. We have therefore:

$[x = 0, y_1 = 0]$, Eq. (15.20): $\quad 0 = C_2 \qquad (15.25)$

$[x = L, y_2 = 0]$, Eq. (15.24): $\quad 0 = \frac{1}{12}PL^3 + C_3L + C_4 \qquad (15.26)$

$[x = L/4, \theta_1 = \theta_2]$, Eqs. (15.19) and (15.23):

$$\frac{3}{128}PL^2 + C_1 = \frac{7}{128}PL^2 + C_3 \tag{15.27}$$

$[x = L/4, y_1 = y_2]$, Eqs. (15.20) and (15.24):

$$\frac{PL^3}{512} + C_1\frac{L}{4} = \frac{11PL^3}{1536} + C_3\frac{L}{4} + C_4 \tag{15.28}$$

Solving these equations simultaneously, we find

$$C_1 = -\frac{7PL^2}{128},\ C_2 = 0,\ C_3 = -\frac{11PL^2}{128},\ C_4 = \frac{PL^3}{384}$$

Substituting for C_1 and C_2 into Eqs. (15.19) and (15.20), we write that for $x \le L/4$,

$$EI\,\theta_1 = \frac{3}{8}Px^2 - \frac{7PL^2}{128} \tag{15.29}$$

$$EI\,y_1 = \frac{1}{8}Px^3 - \frac{7PL^2}{128}x \tag{15.30}$$

Letting $x = L/4$ in each of these equations, we find that the slope and deflection at point D are, respectively,

$$\theta_D = -\frac{PL^2}{32EI} \qquad \text{and} \qquad y_D = -\frac{3PL^3}{256EI}$$

We note that, since $\theta_D \neq 0$, the deflection at D is *not* the maximum deflection of the beam. ■

15.4 DIRECT DETERMINATION OF THE ELASTIC CURVE FROM THE LOAD DISTRIBUTION

We saw in Sec. 15.3 that the equation of the elastic curve can be obtained by integrating twice the differential equation

$$\frac{d^2y}{dx^2} = \frac{M(x)}{EI} \tag{15.4}$$

where $M(x)$ is the bending moment in the beam. We now recall from Sec. 12.3 that, when a beam supports a distributed load $w(x)$, we have $dM/dx = V$ and $dV/dx = -w$ at any point of the beam. Differentiating both members of Eq. (15.4) with respect to x and assuming EI to be constant, we have therefore

$$\frac{d^3y}{dx^3} = \frac{1}{EI}\frac{dM}{dx} = \frac{V(x)}{EI} \tag{15.31}$$

and, differentiating again,

$$\frac{d^4y}{dx^4} = \frac{1}{EI}\frac{dV}{dx} = -\frac{w(x)}{EI}$$

We conclude that, when a prismatic beam supports a distributed load $w(x)$, its elastic curve is governed by the fourth-order linear differential equation

$$\frac{d^4y}{dx^4} = -\frac{w(x)}{EI} \tag{15.32}$$

Multiplying both members of Eq. (15.32) by the constant EI and integrating four times, we write

$$\begin{aligned}
EI\frac{d^4y}{dx^4} &= -w(x) \\
EI\frac{d^3y}{dx^3} &= V(x) = -\int w(x)\,dx + C_1 \\
EI\frac{d^2y}{dx^2} &= M(x) = -\int dx\int w(x)\,dx + C_1x + C_2 \\
EI\frac{dy}{dx} &= EI\,\theta(x) = -\int dx\int dx\int w(x)\,dx + \frac{1}{2}C_1x^2 + C_2x + C_3 \\
EI\,y(x) &= -\int dx\int dx\int dx\int w(x)\,dx + \frac{1}{6}C_1x^3 + \frac{1}{2}C_2x^2 + C_3x + C_4
\end{aligned} \tag{15.33}$$

The four constants of integration can be determined from the boundary conditions. These conditions include (*a*) the conditions imposed on the deflection or slope of the beam by its supports (cf. Sec. 15.3), and (*b*) the condition that V and M be zero at the free end of a cantilever beam or that M be zero at both ends of a simply supported beam (cf. Sec. 12.3). This has been illustrated in Fig. 15.20.

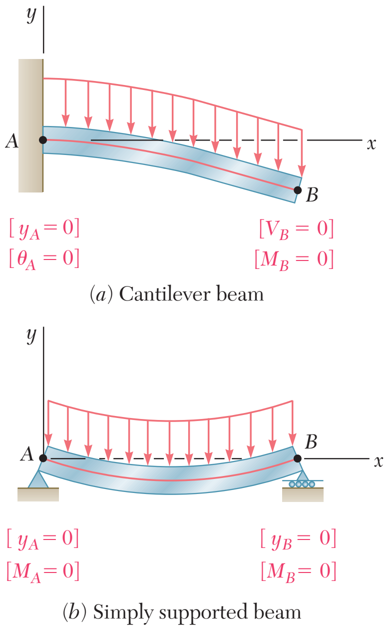


Fig. 15.20 Boundary conditions for beams carrying a distributed load.

The method presented here can be used effectively with cantilever or simply supported beams carrying a distributed load. In the case of overhanging beams, however, the reactions at the supports will cause discontinuities in the shear, i.e., in the third derivative of y, and different functions would be required to define the elastic curve over the entire beam.

EXAMPLE 15.4 The simply supported prismatic beam AB carries a uniformly distributed load w per unit length (Fig. 15.21). Determine the equation of the elastic curve and the maximum deflection of the beam. (This is the same beam and loading as in Example 15.2.)

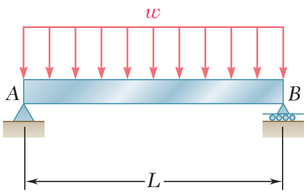


Fig. 15.21

Since w = constant, the first three of Eqs. (15.33) yield

$$EI\frac{d^4y}{dx^4} = -w$$

$$EI\frac{d^3y}{dx^3} = V(x) = -wx + C_1$$

$$EI\frac{d^2y}{dx^2} = M(x) = -\frac{1}{2}wx^2 + C_1x + C_2 \tag{15.34}$$

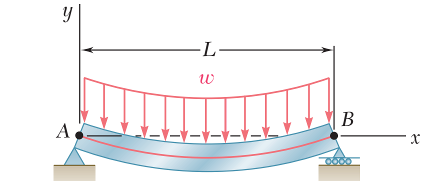


Fig. 15.22

Noting that the boundary conditions require that $M = 0$ at both ends of the beam (Fig. 15.22), we first let $x = 0$ and $M = 0$ in Eq. (15.34) and obtain $C_2 = 0$. We then make $x = L$ and $M = 0$ in the same equation and obtain $C_1 = \frac{1}{2}wL$.

Carrying the values of C_1 and C_2 back into Eq. (15.34), and integrating twice, we write

$$EI\frac{d^2y}{dx^2} = -\frac{1}{2}wx^2 + \frac{1}{2}wLx$$

$$EI\frac{dy}{dx} = -\frac{1}{6}wx^3 + \frac{1}{4}wLx^2 + C_3$$

$$EI\,y = -\frac{1}{24}wx^4 + \frac{1}{12}wLx^3 + C_3x + C_4 \tag{15.35}$$

But the boundary conditions also require that $y = 0$ at both ends of the beam. Letting $x = 0$ and $y = 0$ in Eq. (15.35), we obtain $C_4 = 0$; letting $x = L$ and $y = 0$ in the same equation, we write

$$0 = -\tfrac{1}{24}wL^4 + \tfrac{1}{12}wL^4 + C_3L$$
$$C_3 = -\tfrac{1}{24}wL^3$$

Carrying the values of C_3 and C_4 back into Eq. (15.35) and dividing both members by EI, we obtain the equation of the elastic curve:

$$y = \frac{w}{24EI}(-x^4 + 2Lx^3 - L^3x) \tag{15.36}$$

The value of the maximum deflection is obtained by making $x = L/2$ in Eq. (15.36). We have

$$|y|_{\max} = \frac{5wL^4}{384EI}$$

15.5 STATICALLY INDETERMINATE BEAMS

In the preceding sections, our analysis was limited to statically determinate beams. Consider now the prismatic beam AB (Fig. 15.23*a*), which has a fixed end at A and is supported by a roller at B. Drawing the free-body diagram of the beam (Fig. 15.23*b*), we note that the reactions involve four unknowns, while only three equilibrium equations are available, namely

$$\Sigma F_x = 0 \qquad \Sigma F_y = 0 \qquad \Sigma M_A = 0 \tag{15.37}$$

Since only A_x can be determined from these equations, we conclude that the beam is *statically indeterminate*.

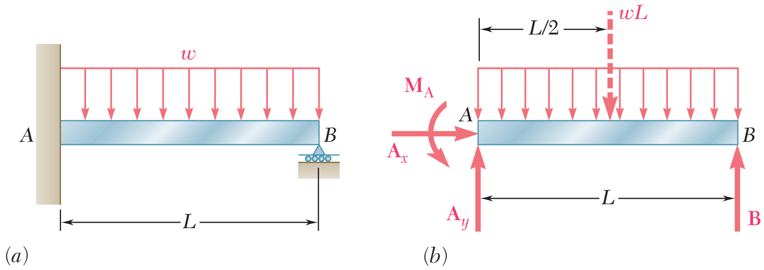


Fig. 15.23

However, we recall from Chaps. 9 and 10 that, in a statically indeterminate problem, the reactions can be obtained by considering the *deformations* of the structure involved. We should, therefore, proceed with the computation of the slope and deformation along the beam. Following the method used in Sec. 15.3, we first express the bending moment $M(x)$ at any given point of AB in terms of the distance x from A, the given load, and the unknown reactions. Integrating in x, we obtain expressions for θ and y which contain two additional unknowns, namely the constants of integration C_1 and C_2. But altogether six equations are available to determine the reactions and the constants C_1 and C_2; they are the three equilibrium equations (15.37) and the three equations expressing that the boundary conditions are satisfied, i.e., that the slope and deflection at A are zero and that the deflection at B is zero (Fig. 15.24). Thus, the reactions at the supports can be determined, and the equation of the elastic curve can be obtained.

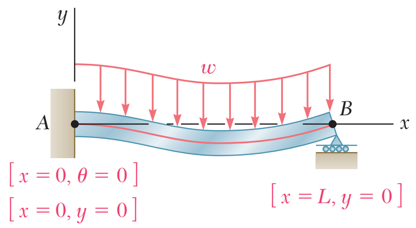


Fig. 15.24

EXAMPLE 15.5 Determine the reactions at the supports for the prismatic beam of Fig. 15.23*a*.

Equilibrium Equations. From the free-body diagram of Fig. 15.23*b* we write

$$\xrightarrow{+} \Sigma F_x = 0: \qquad A_x = 0$$
$$+\uparrow \Sigma F_y = 0: \qquad A_y + B - wL = 0 \qquad (15.38)$$
$$+\circlearrowleft \Sigma M_A = 0: \qquad M_A + BL - \tfrac{1}{2}wL^2 = 0$$

Equation of Elastic Curve. Drawing the free-body diagram of a portion of beam AC (Fig. 15.25), we write

$$+\circlearrowleft \Sigma M_C = 0: \qquad M + \tfrac{1}{2}wx^2 + M_A - A_y x = 0 \qquad (15.39)$$

Solving Eq. (15.39) for M and carrying into Eq. (15.4), we write

$$EI\frac{d^2y}{dx^2} = -\frac{1}{2}wx^2 + A_y x - M_A$$

Integrating in x, we have

$$EI\,\theta = EI\frac{dy}{dx} = -\frac{1}{6}wx^3 + \frac{1}{2}A_y x^2 - M_A x + C_1 \qquad (15.40)$$

$$EI\,y = -\frac{1}{24}wx^4 + \frac{1}{6}A_y x^3 - \frac{1}{2}M_A x^2 + C_1 x + C_2 \qquad (15.41)$$

Referring to the boundary conditions indicated in Fig. 15.24, we make $x = 0$, $\theta = 0$ in Eq. (15.40), $x = 0$, $y = 0$ in Eq. (15.41), and conclude that $C_1 = C_2 = 0$. Thus, we rewrite Eq. (15.41) as follows:

$$EI\,y = -\tfrac{1}{24}wx^4 + \tfrac{1}{6}A_y x^3 - \tfrac{1}{2}M_A x^2 \qquad (15.42)$$

But the third boundary condition requires that $y = 0$ for $x = L$. Carrying these values into (15.42), we write

$$0 = -\tfrac{1}{24}wL^4 + \tfrac{1}{6}A_y L^3 - \tfrac{1}{2}M_A L^2$$

or

$$3M_A - A_y L + \tfrac{1}{4}wL^2 = 0 \qquad (15.43)$$

Solving this equation simultaneously with the three equilibrium equations (15.38), we obtain the reactions at the supports:

$$A_x = 0 \qquad A_y = \tfrac{5}{8}wL \qquad M_A = \tfrac{1}{8}wL^2 \qquad B = \tfrac{3}{8}wL \quad ■$$

In the example we have just considered, there was one redundant reaction, i.e., there was one more reaction than could be determined from the equilibrium equations alone. The corresponding beam is said to be *statically indeterminate to the first degree*. Another example of a beam indeterminate to the first degree is provided in Sample Prob. 15.3. If the beam supports are such that two reactions are redundant (Fig. 15.26*a*), the beam is said to be *indeterminate to the second degree*. While there are now five unknown reactions (Fig. 15.26*b*), we find that four equations may be obtained from the boundary conditions (Fig. 15.26*c*). Thus, altogether seven equations are available to determine the five reactions and the two constants of integration.

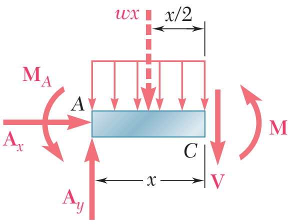


Fig. 15.25

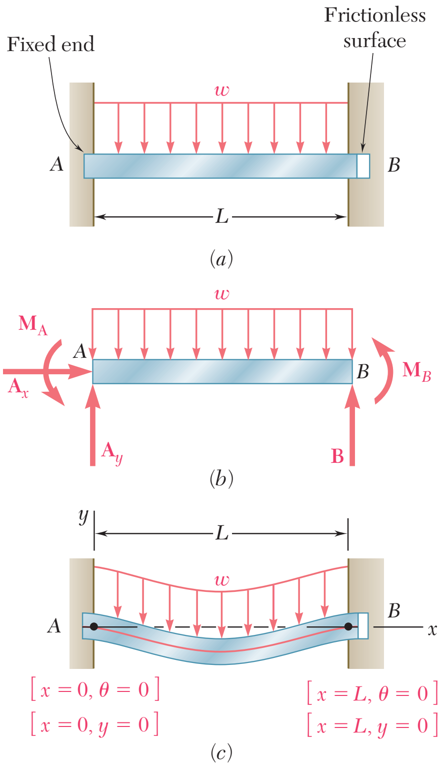
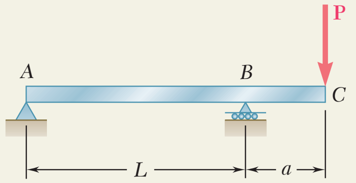


Fig. 15.26

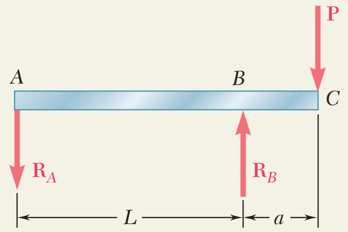
SAMPLE PROBLEM 15.1

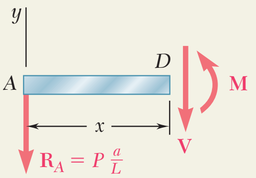


The overhanging steel beam ABC carries a concentrated load $\mathbf{P}$ at end C. For portion AB of the beam, (*a*) derive the equation of the elastic curve, (*b*) determine the maximum deflection, (*c*) evaluate $y_{\max}$ for the following data:

W14 × 68	$I = 723 \text{ in}^4$	$E = 29 \times 10^6$ psi
$P = 50$ kips	$L = 15$ ft $= 180$ in.	$a = 4$ ft $= 48$ in.

SOLUTION





Free-Body Diagrams. Reactions: $\mathbf{R}_A = Pa/L \downarrow \quad \mathbf{R}_B = P(1 + a/L)\uparrow$
Using the free-body diagram of the portion of beam AD of length x, we find

$$M = -P\frac{a}{L}x \qquad (0 < x < L)$$

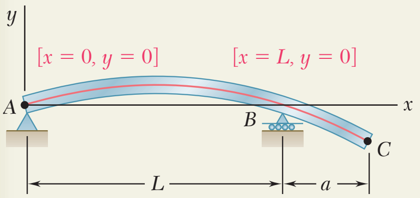
Differential Equation of the Elastic Curve. We use Eq. (15.4) and write

$$EI\frac{d^2y}{dx^2} = -P\frac{a}{L}x$$

Noting that the flexural rigidity EI is constant, we integrate twice and find

$$EI\frac{dy}{dx} = -\frac{1}{2}P\frac{a}{L}x^2 + C_1 \qquad (1)$$

$$EI\,y = -\frac{1}{6}P\frac{a}{L}x^3 + C_1x + C_2 \qquad (2)$$



Determination of Constants. For the boundary conditions shown, we have

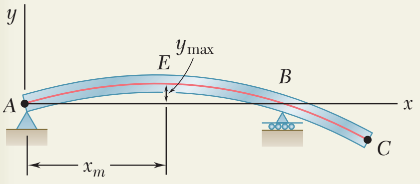
$[x = 0, y = 0]$: From Eq. (2), we find $C_2 = 0$
$[x = L, y = 0]$: Again using Eq. (2), we write

$$EI(0) = -\frac{1}{6}P\frac{a}{L}L^3 + C_1L \qquad C_1 = +\frac{1}{6}PaL$$

a. Equation of the Elastic Curve. Substituting for C_1 and C_2 into Eqs. (1) and (2), we have

$$EI\frac{dy}{dx} = -\frac{1}{2}P\frac{a}{L}x^2 + \frac{1}{6}PaL \qquad \frac{dy}{dx} = \frac{PaL}{6EI}\left[1 - 3\left(\frac{x}{L}\right)^2\right] \qquad (3)$$

$$EI\,y = -\frac{1}{6}P\frac{a}{L}x^3 + \frac{1}{6}PaLx \qquad y = \frac{PaL^2}{6EI}\left[\frac{x}{L} - \left(\frac{x}{L}\right)^3\right] \qquad (4) \blacktriangleleft$$



b. Maximum Deflection in Portion *AB*. The maximum deflection $y_{\max}$ occurs at point E where the slope of the elastic curve is zero. Setting $dy/dx = 0$ in Eq. (3), we determine the abscissa x_m of point E:

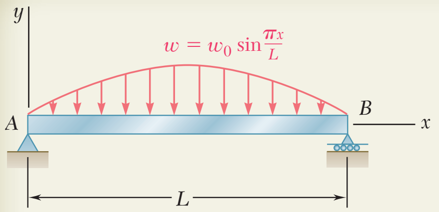
$$0 = \frac{PaL}{6EI}\left[1 - 3\left(\frac{x_m}{L}\right)^2\right] \qquad x_m = \frac{L}{\sqrt{3}} = 0.577L$$

We substitute $x_m/L = 0.577$ into Eq. (4) and have

$$y_{\max} = \frac{PaL^2}{6EI}[(0.577) - (0.577)^3] \qquad y_{\max} = 0.0642\frac{PaL^2}{EI} \blacktriangleleft$$

c. Evaluation of $y_{\max}$. For the data given, the value of $y_{\max}$ is

$$y_{\max} = 0.0642\frac{(50 \text{ kips})(48 \text{ in.})(180 \text{ in.})^2}{(29 \times 10^6 \text{ psi})(723 \text{ in}^4)} \qquad y_{\max} = 0.238 \text{ in.} \blacktriangleleft$$



SAMPLE PROBLEM 15.2

For the beam and loading shown, determine (*a*) the equation of the elastic curve, (*b*) the slope at end *A*, (*c*) the maximum deflection.

SOLUTION

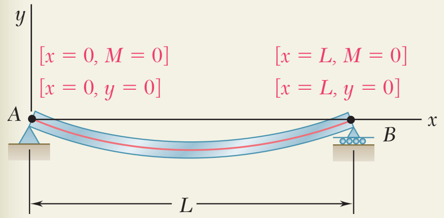
Differential Equation of the Elastic Curve. From Eq. (15.32),

$$EI\frac{d^4y}{dx^4} = -w(x) = -w_0 \sin\frac{\pi x}{L} \tag{1}$$

Integrate Eq. (1) twice:

$$EI\frac{d^3y}{dx^3} = V = +w_0\frac{L}{\pi}\cos\frac{\pi x}{L} + C_1 \tag{2}$$

$$EI\frac{d^2y}{dx^2} = M = +w_0\frac{L^2}{\pi^2}\sin\frac{\pi x}{L} + C_1x + C_2 \tag{3}$$



Boundary Conditions:

$[x = 0, M = 0]$: From Eq. (3), we find $C_2 = 0$

$[x = L, M = 0]$: Again using Eq. (3), we write

$$0 = w_0\frac{L^2}{\pi^2}\sin\pi + C_1L \qquad C_1 = 0$$

Thus:

$$EI\frac{d^2y}{dx^2} = +w_0\frac{L^2}{\pi^2}\sin\frac{\pi x}{L} \tag{4}$$

Integrate Eq. (4) twice:

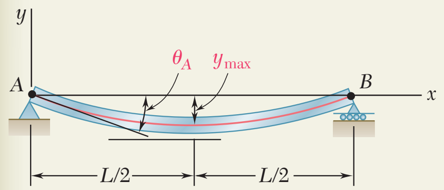
$$EI\frac{dy}{dx} = EI\,\theta = -w_0\frac{L^3}{\pi^3}\cos\frac{\pi x}{L} + C_3 \tag{5}$$

$$EI\,y = -w_0\frac{L^4}{\pi^4}\sin\frac{\pi x}{L} + C_3x + C_4 \tag{6}$$

Boundary Conditions:

$[x = 0, y = 0]$: Using Eq. (6), we find $C_4 = 0$

$[x = L, y = 0]$: Again using Eq. (6), we find $C_3 = 0$



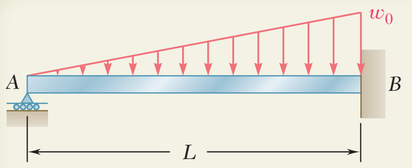
a. Equation of Elastic Curve $\quad EIy = -w_0\dfrac{L^4}{\pi^4}\sin\dfrac{\pi x}{L}$ ◀

b. Slope at End A. For $x = 0$, we have

$$EI\,\theta_A = -w_0\frac{L^3}{\pi^3}\cos 0 \qquad \theta_A = \frac{w_0L^3}{\pi^3EI} \; ◀$$

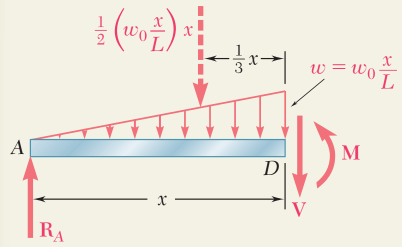
c. Maximum Deflection. For $x = \frac{1}{2}L$

$$ELy_{\max} = -w_0\frac{L^4}{\pi^4}\sin\frac{\pi}{2} \qquad y_{\max} = \frac{w_0L^4}{\pi^4EI} \downarrow \; ◀$$



SAMPLE PROBLEM 15.3

For the uniform beam AB, (a) determine the reaction at A, (b) derive the equation of the elastic curve, (c) determine the slope at A. (Note that the beam is statically indeterminate to the first degree.)



SOLUTION

Bending Moment. Using the free body shown, we write

$$+\circlearrowright\Sigma M_D = 0: \qquad R_A x - \frac{1}{2}\left(\frac{w_0 x^2}{L}\right)\frac{x}{3} - M = 0 \qquad M = R_A x - \frac{w_0 x^3}{6L}$$

Differential Equation of the Elastic Curve. We use Eq. (15.4) and write

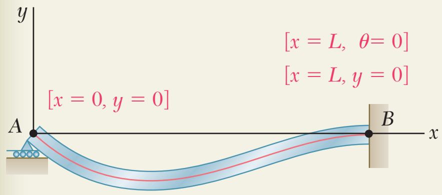
$$EI\frac{d^2y}{dx^2} = R_A x - \frac{w_0 x^3}{6L}$$

Noting that the flexural rigidity EI is constant, we integrate twice and find

$$EI\frac{dy}{dx} = EI\,\theta = \frac{1}{2}R_A x^2 - \frac{w_0 x^4}{24L} + C_1 \qquad (1)$$

$$EI\,y = \frac{1}{6}R_A x^3 - \frac{w_0 x^5}{120L} + C_1 x + C_2 \qquad (2)$$

Boundary Conditions. The three boundary conditions that must be satisfied are shown on the sketch



$$[x = 0, y = 0]: \quad C_2 = 0 \qquad (3)$$

$$[x = L, \theta = 0]: \quad \frac{1}{2}R_A L^2 - \frac{w_0 L^3}{24} + C_1 = 0 \qquad (4)$$

$$[x = L, y = 0]: \quad \frac{1}{6}R_A L^3 - \frac{w_0 L^4}{120} + C_1 L + C_2 = 0 \qquad (5)$$

a. Reaction at *A*. Multiplying Eq. (4) by L, subtracting Eq. (5) member by member from the equation obtained, and noting that $C_2 = 0$, we have

$$\tfrac{1}{3}R_A L^3 - \tfrac{1}{30}w_0 L^4 = 0 \qquad \mathbf{R}_A = \tfrac{1}{10}w_0 L \uparrow \quad ◀$$

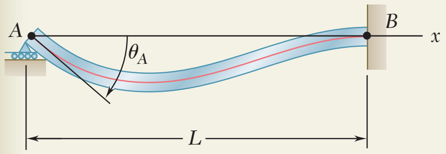
We note that the reaction is independent of E and I. Substituting $R_A = \frac{1}{10}w_0 L$ into Eq. (4), we have

$$\tfrac{1}{2}(\tfrac{1}{10}w_0 L)L^2 - \tfrac{1}{24}w_0 L^3 + C_1 = 0 \qquad C_1 = -\tfrac{1}{120}w_0 L^3$$

b. Equation of the Elastic Curve. Substituting for R_A, C_1, and C_2 into Eq. (2), we have

$$EI\,y = \frac{1}{6}\left(\frac{1}{10}w_0 L\right)x^3 - \frac{w_0 x^5}{120L} - \left(\frac{1}{120}w_0 L^3\right)x$$

$$y = \frac{w_0}{120EIL}(-x^5 + 2L^2x^3 - L^4x) \quad ◀$$



c. Slope at *A*. We differentiate the above equation with respect to x:

$$\theta = \frac{dy}{dx} = \frac{w_0}{120EIL}(-5x^4 + 6L^2x^2 - L^4)$$

Making $x = 0$, we have $\qquad \theta_A = -\dfrac{w_0 L^3}{120EI} \qquad \theta_A = \dfrac{w_0 L^3}{120EI} \measuredangle \quad ◀$

PROBLEMS

In the following problems assume that the flexural rigidity *EI* of each beam is constant.

15.1 through 15.4 For the loading shown, determine (*a*) the equation of the elastic curve for the cantilever beam *AB*, (*b*) the deflection at the free end, (*c*) the slope at the free end.

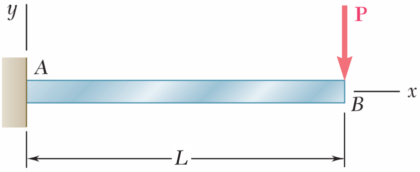


Fig. P15.1

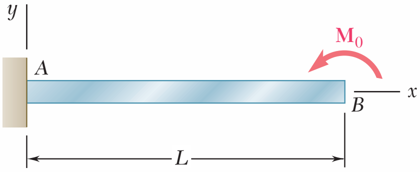


Fig. P15.2

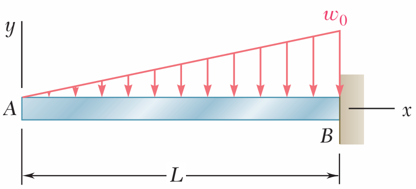


Fig. P15.3

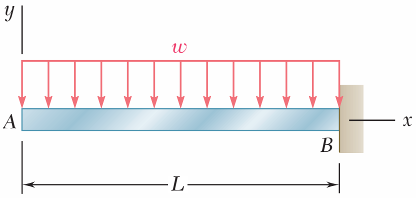


Fig. P15.4

***15.5* and 15.6** For the cantilever beam and loading shown, determine (*a*) the equation of the elastic curve for portion *AB* of the beam, (*b*) the deflection at *B*, (*c*) the slope at *B*.

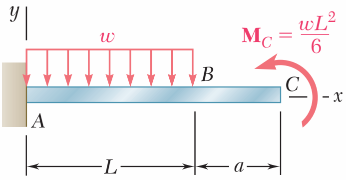


Fig. *P15.5*

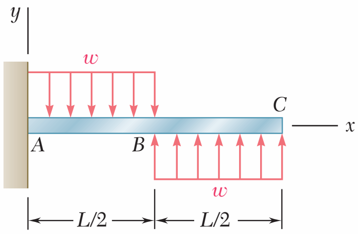


Fig. P15.6

15.7 For the beam and loading shown, determine (*a*) the equation of the elastic curve for portion *AB* of the beam, (*b*) the slope at *A*, (*c*) the slope at *B*.

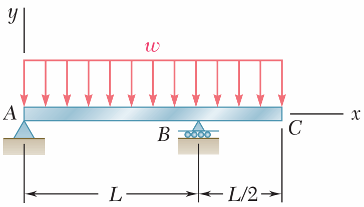


Fig. P15.7

15.8 For the beam and loading shown, determine (*a*) the equation of the elastic curve for portion *BC* of the beam, (*b*) the deflection at midspan, (*c*) the slope at *B*.

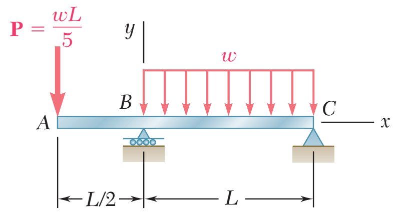


Fig. *P15.8*

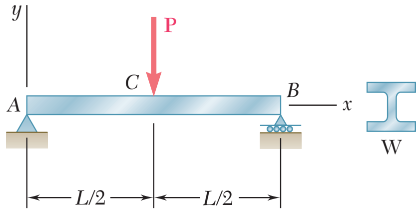


Fig. P15.9

15.9 Knowing that beam *AB* is a W130 × 23.8 rolled shape and that $P = 50$ kN, $L = 1.25$ m, and $E = 200$ GPa, determine (*a*) the slope at *A*, (*b*) the deflection at *C*.

15.10 Knowing that beam *AB* is an S8 × 18.4 rolled shape and that $w_0 = 4$ kips/ft, $L = 9$ ft, and $E = 29 \times 10^6$ psi, determine (*a*) the slope at *A*, (*b*) the deflection at *C*.

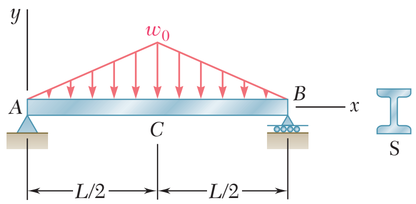


Fig. P15.10

15.11 (*a*) Determine the location and magnitude of the maximum deflection of beam *AB*. (*b*) Assuming that beam *AB* is a W360 × 64, $L = 3.5$ m, and $E = 200$ GPa, calculate the maximum allowable value of the applied moment $\mathbf{M}_0$ if the maximum deflection is not to exceed 1 mm.

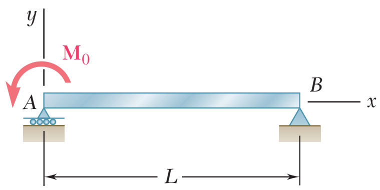


Fig. P15.11

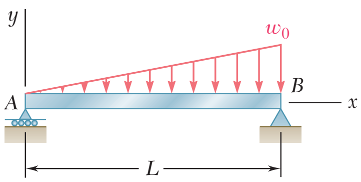


Fig. P15.12

15.12 For the beam and loading shown, (*a*) express the magnitude and location of the maximum deflection in terms of w_0, *L*, *E*, and *I*. (*b*) Calculate the value of the maximum deflection, assuming that beam *AB* is a W18 × 50 rolled shape and that $w_0 = 4.5$ kips/ft, $L = 18$ ft, and $E = 29 \times 10^6$ psi.

15.13 and *15.14* For the beam and loading shown, determine the deflection at point C. Use $E = 200$ GPa.

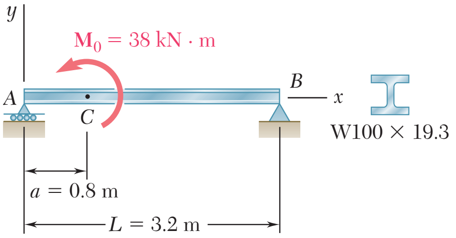


Fig. P15.13

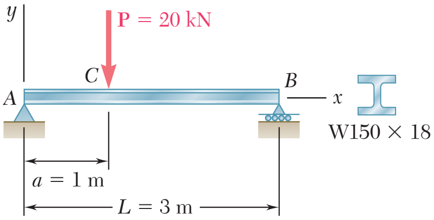


Fig. *P15.14*

15.15 For the beam and loading shown, determine (*a*) the equation of the elastic curve, (*b*) the slope at end A, (*c*) the deflection at the midpoint of the span.

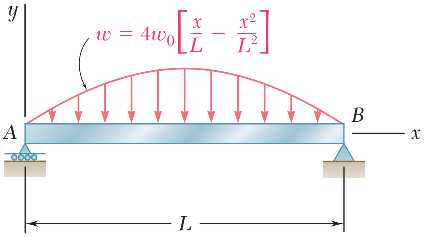


Fig. P15.15

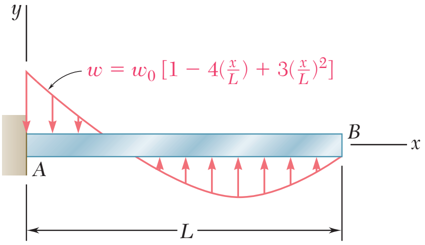


Fig. *P15.16*

15.16 For the beam and loading shown, determine (*a*) the equation of the elastic curve, (*b*) the deflection at the free end.

15.17 through 15.20 For the beam and loading shown, determine the reaction at the roller support.

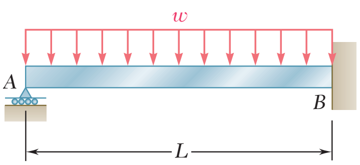


Fig. P15.17

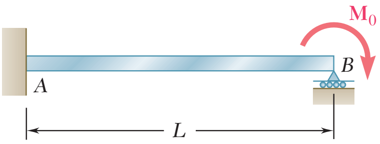


Fig. P15.18

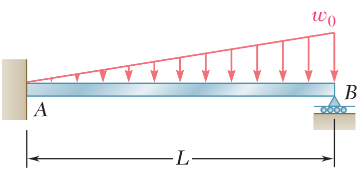


Fig. *P15.19*

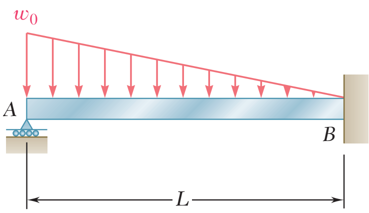


Fig. P15.20

15.21 and 15.22 Determine the reaction at the roller support, and draw the bending moment diagram for the beam and loading shown.

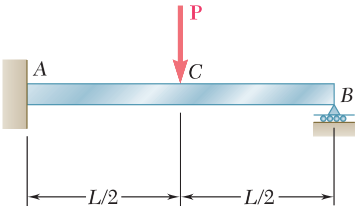


Fig. P15.21

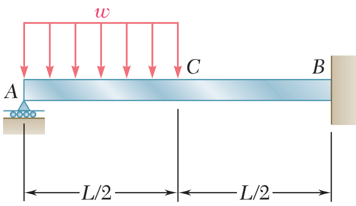


Fig. P15.22

15.23 and *15.24* Determine the reaction at the roller support and the deflection at point D if a is equal to $L/3$.

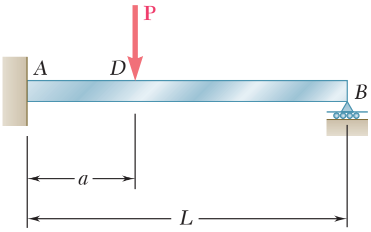


Fig. P15.23

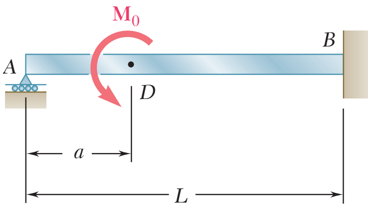


Fig. *P15.24*

15.25 and 15.26 Determine the reaction at A, and draw the bending moment diagram for the beam and loading shown.

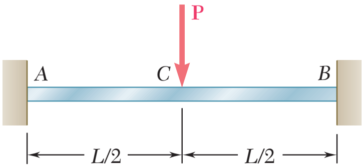


Fig. P15.25

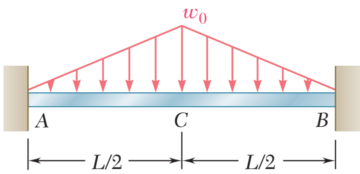


Fig. P15.26

15.6 METHOD OF SUPERPOSITION

When a beam is subjected to several concentrated or distributed loads, it is often found convenient to compute separately the slope and deflection caused by each of the given loads. The slope and deflection due to the combined loads are then obtained by applying the principle of superposition (Sec. 9.11) and adding the values of the slope or deflection corresponding to the various loads.

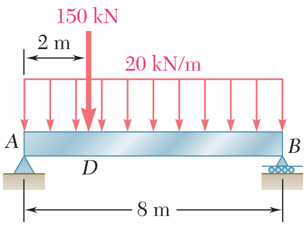


Fig. 15.27

EXAMPLE 15.6 Determine the slope and deflection at D for the beam and loading shown (Fig. 15.27) knowing that the flexural rigidity of the beam is $EI = 100 \text{ MN} \cdot \text{m}^2$.

The slope and deflection at any point of the beam can be obtained by superposing the slopes and deflections caused respectively by the concentrated load and by the distributed load (Fig. 15.28).

Since the concentrated load in Fig. 15.28*b* is applied at quarter span, we can use the results obtained for the beam and loading of Example 15.3 and write

$$(\theta_D)_P = -\frac{PL^2}{32EI} = -\frac{(150 \times 10^3)(8)^2}{32(100 \times 10^6)} = -3 \times 10^{-3} \text{ rad}$$

$$(y_D)_P = -\frac{3PL^3}{256EI} = -\frac{3(150 \times 10^3)(8)^3}{256(100 \times 10^6)} = -9 \times 10^{-3} \text{ m}$$
$$= -9 \text{ mm}$$

On the other hand, recalling the equation of the elastic curve obtained for a uniformly distributed load in Example 15.2, we express the deflection in Fig. 15.28*c* as

$$y = \frac{w}{24EI}(-x^4 + 2Lx^3 - L^3x) \qquad (15.44)$$

and, differentiating with respect to x,

$$\theta = \frac{dy}{dx} = \frac{w}{24EI}(-4x^3 + 6Lx^2 - L^3) \qquad (15.45)$$

Making $w = 20$ kN/m, $x = 2$ m, and $L = 8$ m in Eqs. (15.45) and (15.44), we obtain

$$(\theta_D)_w = \frac{20 \times 10^3}{24(100 \times 10^6)}(-352) = -2.93 \times 10^{-3} \text{ rad}$$

$$(y_D)_w = \frac{20 \times 10^3}{24(100 \times 10^6)}(-912) = -7.60 \times 10^{-3} \text{ m}$$
$$= -7.60 \text{ mm}$$

Combining the slopes and deflections produced by the concentrated and the distributed loads, we have

$$\theta_D = (\theta_D)_P + (\theta_D)_w = -3 \times 10^{-3} - 2.93 \times 10^{-3}$$
$$= -5.93 \times 10^{-3} \text{ rad}$$
$$y_D = (y_D)_P + (y_D)_w = -9 \text{ mm} - 7.60 \text{ mm} = -16.60 \text{ mm}$$

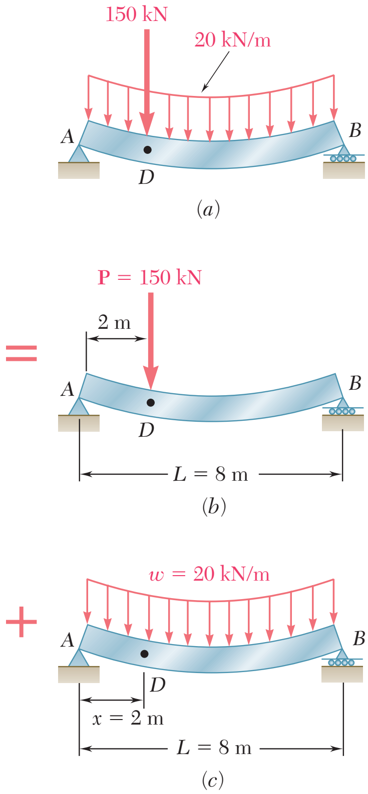


Fig. 15.28

To facilitate the task of practicing engineers, most structural and mechanical engineering handbooks include tables giving the deflections and slopes of beams for various loadings and types of support. Such a table will be found in App. C. We note that the slope and deflection of the beam of Fig. 15.27 could have been determined from that table. Indeed, using the information given under cases 5 and 6, we could have expressed the deflection of the beam for any value $x \leq L/4$. Taking the derivative of the expression obtained in this way would have yielded the slope of the beam over the same interval. We also note that the slope at both ends of the beam can be obtained by simply adding the corresponding values given in the table. However, the maximum deflection of the beam of Fig. 15.27 *cannot* be obtained by adding the maximum deflections of cases 5 and 6, since these deflections occur at different points of the beam.†

†An approximate value of the maximum deflection of the beam can be obtained by plotting the values of y corresponding to various values of x. The determination of the exact location and magnitude of the maximum deflection would require setting equal to zero the expression obtained for the slope of the beam and solving this equation for x.

15.7 APPLICATION OF SUPERPOSITION TO STATICALLY INDETERMINATE BEAMS

Photo 15.2 The continuous beams supporting this highway overpass have three supports and are thus indeterminate.

We often find it convenient to use the method of superposition to determine the reactions at the supports of a statically indeterminate beam. Considering first the case of a beam indeterminate to the first degree (cf. Sec. 15.5), such as the beam shown in Photo 15.2, we follow the approach described in Sec. 9.8. We designate one of the reactions as redundant and eliminate or modify accordingly the corresponding support. The redundant reaction is then treated as an unknown load that, together with the other loads, must produce deformations that are compatible with the original supports. The slope or deflection at the point where the support has been modified or eliminated is obtained by computing separately the deformations caused by the given loads and by the redundant reaction, and by superposing the results obtained. Once the reactions at the supports have been found, the slope and deflection can be determined in the usual way at any other point of the beam.

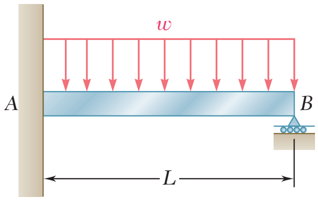


Fig. 15.29

EXAMPLE 15.7 Determine the reactions at the supports for the prismatic beam and loading shown in Fig. 15.29. (This is the same beam and loading as in Example 15.5 of Sec. 15.5.)

We consider the reaction at B as redundant and release the beam from the support. The reaction $\mathbf{R}_B$ is now considered as an unknown load (Fig. 15.30*a*) and will be determined from the condition that the deflection of the beam at B must be zero. The solution is carried out by considering separately the deflection $(y_B)_w$ caused at B by the uniformly distributed load w (Fig. 15.30b) and the deflection $(y_B)_R$ produced at the same point by the redundant reaction $\mathbf{R}_B$ (Fig. 15.30*c*).

From the table of App. C (cases 2 and 1), we find that

$$(y_B)_w = -\frac{wL^4}{8EI} \qquad (y_B)_R = +\frac{R_BL^3}{3EI}$$

Writing that the deflection at B is the sum of these two quantities and that it must be zero, we have

$$y_B = (y_B)_w + (y_B)_R = 0$$

$$y_B = -\frac{wL^4}{8EI} + \frac{R_BL^3}{3EI} = 0$$

and, solving for R_B, $\quad R_B = \frac{3}{8}wL \qquad \mathbf{R}_B = \frac{3}{8}wL \uparrow$

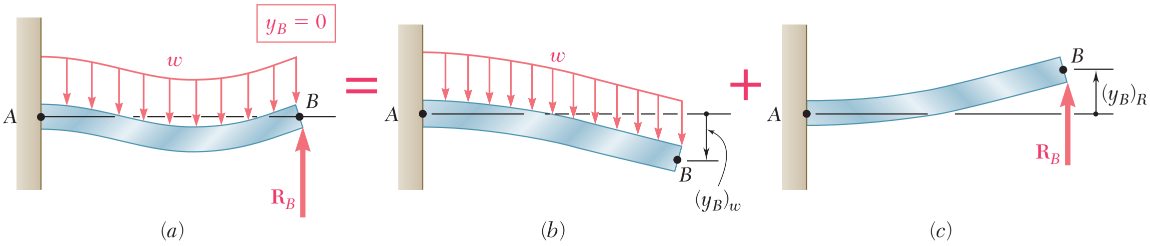


Fig. 15.30

Drawing the free-body diagram of the beam (Fig. 15.31) and writing the corresponding equilibrium equations, we have

$$+\uparrow \sum F_y = 0: \qquad R_A + R_B - wL = 0 \tag{15.46}$$

$$R_A = wL - R_B = wL - \tfrac{3}{8}wL = \tfrac{5}{8}wL$$

$$\mathbf{R}_A = \tfrac{5}{8}wL \uparrow$$

$$+\circlearrowleft \sum M_A = 0: \qquad M_A + R_B L - (wL)(\tfrac{1}{2}L) = 0 \tag{15.47}$$

$$M_A = \tfrac{1}{2}wL^2 - R_B L = \tfrac{1}{2}wL^2 - \tfrac{3}{8}wL^2 = \tfrac{1}{8}wL^2$$

$$\mathbf{M}_A = \tfrac{1}{8}wL^2 \circlearrowleft$$

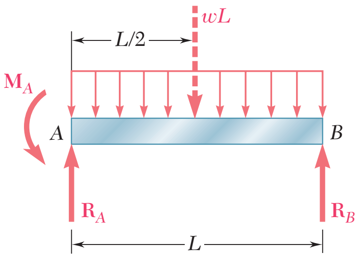


Fig. 15.31

Alternative Solution. We may consider the couple exerted at the fixed end A as redundant and replace the fixed end by a pin-and-bracket support. The couple $\mathbf{M}_A$ is now considered as an unknown load (Fig. 15.32*a*) and will be determined from the condition that the slope of the beam at A must be zero. The solution is carried out by considering separately the slope $(\theta_A)_w$ caused at A by the uniformity distributed load w (Fig. 15.32*b*) and the slope $(\theta_A)_M$ produced at the same point by the unknown couple $\mathbf{M}_A$ (Fig. 15.32*c*).

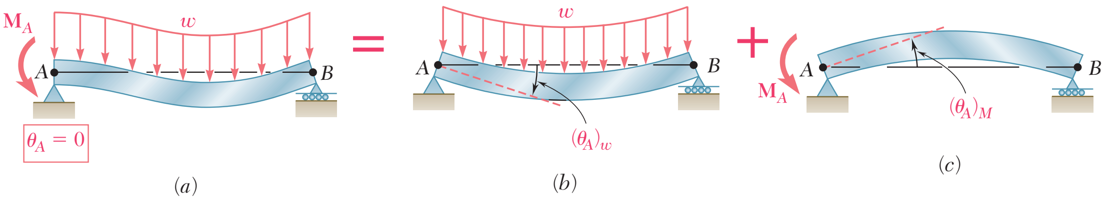


Fig. 15.32

Using the table of App. C (cases 6 and 7), and noting that in case 7, A and B must be interchanged, we find that

$$(\theta_A)_w = -\frac{wL^3}{24EI} \qquad (\theta_A)_M = \frac{M_A L}{3EI}$$

Writing that the slope at A is the sum of these two quantities and that it must be zero, we have

$$\theta_A = (\theta_A)_w + (\theta_A)_M = 0$$

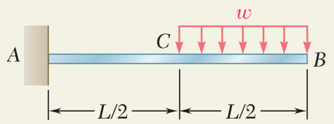
$$\theta_A = -\frac{wL^3}{25EI} + \frac{M_A L}{3EI} = 0$$

and, solving for M_A,

$$M_A = \tfrac{1}{8}wL^2 \qquad \mathbf{M}_A = \tfrac{1}{8}wL^2 \circlearrowleft$$

The values of R_A and R_B may then be found from the equilibrium equations (15.46) and (15.47). ■

The beam considered in the preceding example was indeterminate to the first degree. In the case of a beam indeterminate to the second degree (cf. Sec. 15.5), two reactions must be designated as redundant, and the corresponding supports must be eliminated or modified accordingly. The redundant reactions are then treated as unknown loads which, simultaneously and together with the other loads, must produce deformations which are compatible with the original supports. (See Sample Prob. 15.6.)

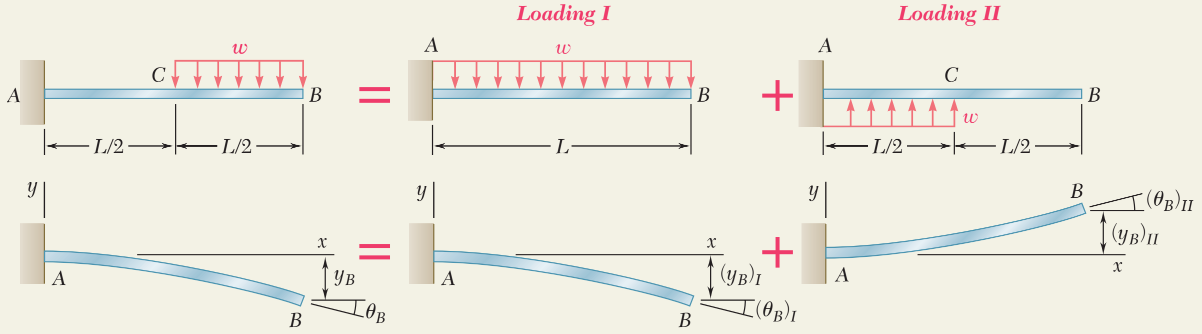


SAMPLE PROBLEM 15.4

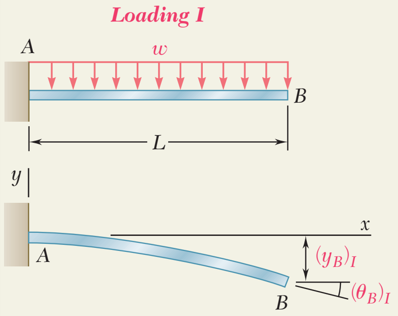
For the beam and loading shown, determine the slope and deflection at point B.

SOLUTION

Principle of Superposition. The given loading can be obtained by superposing the loadings shown in the following "picture equation." The beam AB is, of course, the same in each part of the figure.

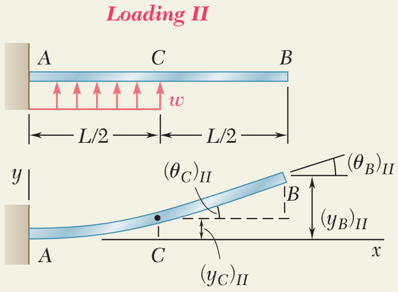


For each of the loadings I and II, we now determine the slope and deflection at B by using the table of *Beam Deflections and Slopes* in App. C.



Loading I

$$(\theta_B)_{\mathrm{I}} = -\frac{wL^3}{6EI} \qquad (y_B)_{\mathrm{I}} = -\frac{wL^4}{8EI}$$



Loading II

$$(\theta_C)_{\mathrm{II}} = +\frac{w(L/2)^3}{6EI} = +\frac{wL^3}{48EI} \qquad (y_C)_{\mathrm{II}} = +\frac{w(L/2)^4}{8EI} = +\frac{wL^4}{128EI}$$

In portion CB, the bending moment for loading II is zero and thus the elastic curve is a straight line.

$$(\theta_B)_{\mathrm{II}} = (\theta_C)_{\mathrm{II}} = +\frac{wL^3}{48EI} \qquad (y_B)_{\mathrm{II}} = (y_C)_{\mathrm{II}} + (\theta_C)_{\mathrm{II}}\left(\frac{L}{2}\right)$$

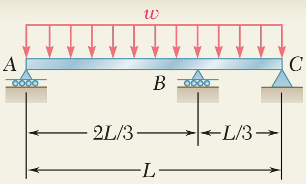
$$= \frac{wL^4}{128EI} + \frac{wL^3}{48EI}\left(\frac{L}{2}\right) = +\frac{7wL^4}{384EI}$$

Slope at Point B

$$\theta_B = (\theta_B)_{\mathrm{I}} + (\theta_B)_{\mathrm{II}} = -\frac{wL^3}{6EI} + \frac{wL^3}{48EI} = -\frac{7wL^3}{48EI} \qquad \theta_B = \frac{7wL^3}{48EI} \measuredangle \blacktriangleleft$$

Deflection at B

$$y_B = (y_B)_{\mathrm{I}} + (y_B)_{\mathrm{II}} = -\frac{wL^4}{8EI} + \frac{7wL^4}{384EI} = -\frac{41wL^4}{384EI} \qquad y_B = \frac{41wL^4}{384EI} \downarrow \blacktriangleleft$$

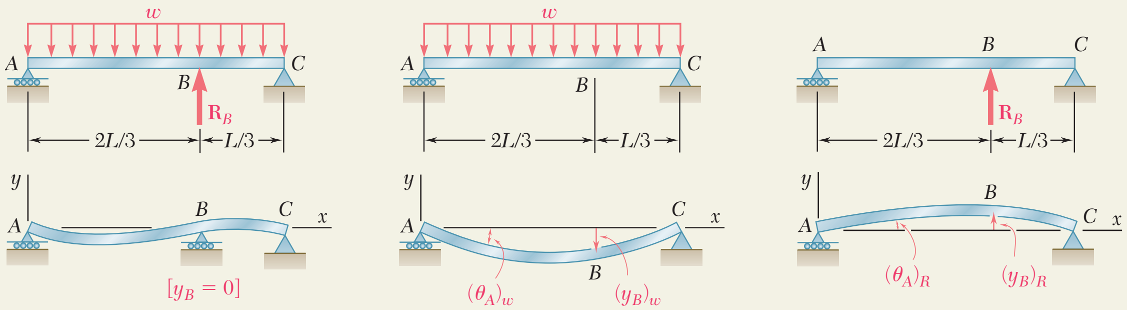


SAMPLE PROBLEM 15.5

For the uniform beam and loading shown, determine (*a*) the reaction at each support, (*b*) the slope at end A.

SOLUTION

Principle of Superposition. The reaction $\mathbf{R}_B$ is designated as redundant and considered as an unknown load. The deflections due to the distributed load and to the reaction $\mathbf{R}_B$ are considered separately as shown below.



For each loading the deflection at point B is found by using the table of *Beam Deflections and Slopes* in App. C.

Distributed Loading. We use case 6, App. C.

$$y = -\frac{w}{24EI}(x^4 - 2Lx^3 + L^3x)$$

At point B, $x = \frac{2}{3}L$:

$$(y_B)_w = -\frac{w}{24EI}\left[\left(\frac{2}{3}L\right)^4 - 2L\left(\frac{2}{3}L\right)^3 + L^3\left(\frac{2}{3}L\right)\right] = -0.01132\frac{wL^4}{EI}$$

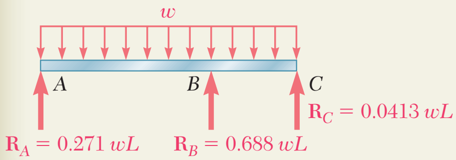
Redundant Reaction Loading. From case 5, App. C, with $a = \frac{2}{3}L$ and $b = \frac{1}{3}L$, we have

$$(y_B)_R = -\frac{Pa^2b^2}{3EIL} = +\frac{R_B}{3EIL}\left(\frac{2}{3}L\right)^2\left(\frac{L}{3}\right)^2 = 0.01646\frac{R_BL^3}{EI}$$

a. Reactions at Supports. Recalling that $y_B = 0$, we write

$$y_B = (y_B)_w + (y_B)_R$$

$$0 = -0.01132\frac{wL^4}{EI} + 0.01646\frac{R_BL^3}{EI} \qquad \mathbf{R}_B = 0.688wL\uparrow \blacktriangleleft$$



Since the reaction R_B is now known, we may use the methods of statics to determine the other reactions: $\mathbf{R}_A = 0.271wL\uparrow \quad \mathbf{R}_C = 0.0413wL\uparrow \blacktriangleleft$

b. Slope at End *A*. Referring again to App. C, we have

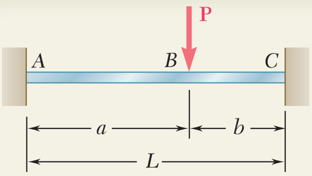
Distributed Loading. $(\theta_A)_w = -\dfrac{wL^3}{24EI} = -0.04167\dfrac{wL^3}{EI}$

Redundant Reaction Loading. For $P = -R_B = -0.688wL$ and $b = \frac{1}{3}L$

$$(\theta_A)_R = -\frac{Pb(L^2 - b^2)}{6EIL} = +\frac{0.688wL}{6EIL}\left(\frac{L}{3}\right)\left[L^2 - \left(\frac{L}{3}\right)^2\right] \qquad (\theta_A)_R = 0.03398\frac{wL^3}{EI}$$

Finally, $\theta_A = (\theta_A)_w + (\theta_A)_R$

$$\theta_A = -0.04167\frac{wL^3}{EI} + 0.03398\frac{wL^3}{EI} = -0.00769\frac{wL^3}{EI} \qquad \theta_A = 0.00769\frac{wL^3}{EI} \blacktriangleleft$$

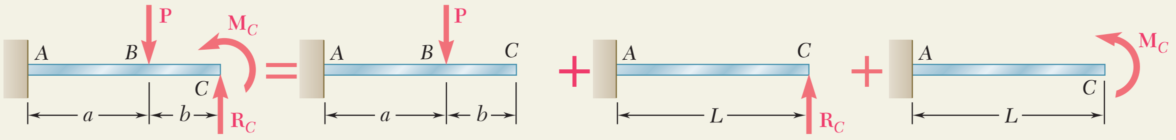


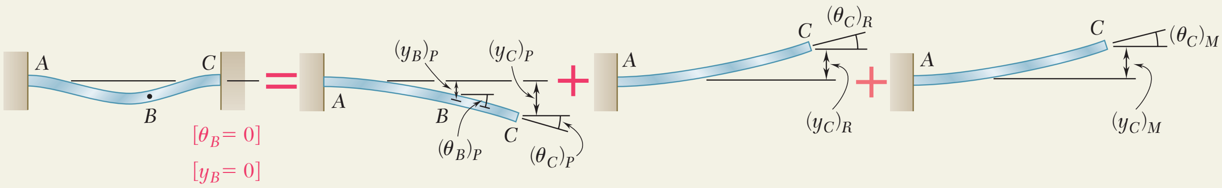
SAMPLE PROBLEM 15.6

For the beam and loading shown, determine the reaction at the fixed support C.

SOLUTION

Principle of Superposition. Assuming the axial force in the beam to be zero, the beam ABC is indeterminate to the second degree and we choose two reaction components as redundant, namely, the vertical force $\mathbf{R}_C$ and the couple $\mathbf{M}_C$. The deformations caused by the given load $\mathbf{P}$, the force $\mathbf{R}_C$, and the couple $\mathbf{M}_C$ will be considered separately as shown.





For each load, the slope and deflection at point C will be found by using the table of *Beam Deflections and Slopes* in App. C.

Load P. We note that, for this loading, portion BC of the beam is straight.

$$(\theta_C)_P = (\theta_B)_P = -\frac{Pa^2}{2EI} \qquad (y_C)_P = (y_B)_P + (\theta_B)_P b$$

$$= -\frac{Pa^3}{3EI} - \frac{Pa^2}{2EI}b = -\frac{Pa^2}{6EI}(2a + 3b)$$

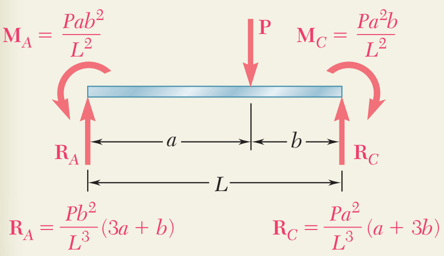
Force R$_C$
$$(\theta_C)_R = +\frac{R_C L^2}{2EI} \qquad (y_C)_R = +\frac{R_C L^3}{3EI}$$

Couple M$_C$
$$(\theta_C)_M = +\frac{M_C L}{EI} \qquad (y_C)_M = +\frac{M_C L^2}{2EI}$$

Boundary Conditions. At end C the slope and deflection must be zero:

$[x = L, \theta_C = 0]$:
$$\theta_C = (\theta_C)_P + (\theta_C)_R + (\theta_C)_M$$
$$0 = -\frac{Pa^2}{2EI} + \frac{R_C L^2}{2EI} + \frac{M_C L}{EI} \tag{1}$$

$[x = L, y_C = 0]$:
$$y_C = (y_C)_P + (y_C)_R + (y_C)_M$$
$$0 = -\frac{Pa^2}{6EI}(2a + 3b) + \frac{R_C L^3}{3EI} + \frac{M_C L^2}{2EI} \tag{2}$$



Reaction Components at C. Solving simultaneously Eqs. (1) and (2), we find after reductions

$$R_C = +\frac{Pa^2}{L^3}(a + 3b) \qquad \mathbf{R}_C = \frac{Pa^2}{L^3}(a + 3b)\uparrow \blacktriangleleft$$

$$M_C = -\frac{Pa^2b}{L^2} \qquad \mathbf{M}_C = \frac{Pa^2b}{L^2} \circlearrowright \blacktriangleleft$$

Using the methods of statics, we can now determine the reaction at A.

PROBLEMS

Use the method of superposition to solve the following problems and assume that the flexural rigidity *EI* of each beam is constant.

15.27 through 15.30 For the beam and loading shown, determine (a) the deflection at point C, (b) the slope at end A.

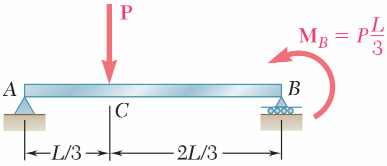


Fig. P15.27

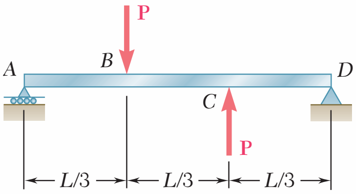


Fig. P15.28

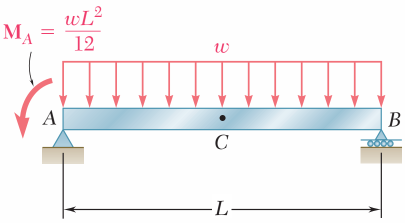


Fig. P15.29

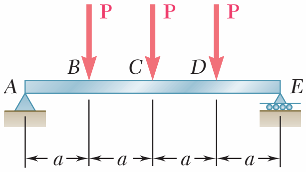


Fig. P15.30

15.31 and 15.32 For the cantilever beam and loading shown, determine the slope and deflection at the free end.

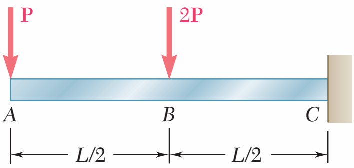


Fig. P15.31

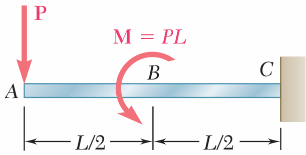


Fig. P15.32

15.33* and *15.34 For the cantilever beam and loading shown, determine the slope and deflection at point C.

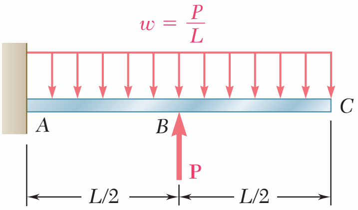


Fig. *P15.33*

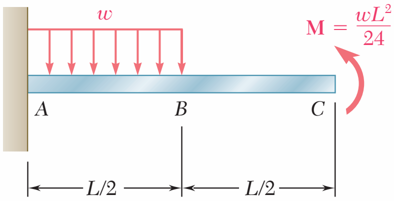


Fig. *P15.34*

15.35 For the cantilever beam and loading shown, determine the slope and deflection at end C. Use $E = 29 \times 10^6$ psi.

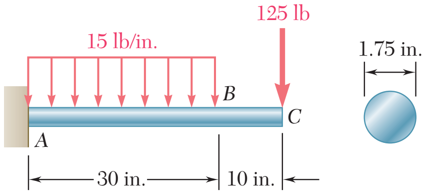


Fig. P15.35 and P15.36

15.36 For the cantilever beam and loading shown, determine the slope and deflection at point B. Use $E = 29 \times 10^6$ psi.

15.37 and *15.38* For the beam and loading shown, determine (*a*) the slope at end A, (*b*) the deflection at point C. Use $E = 200$ GPa.

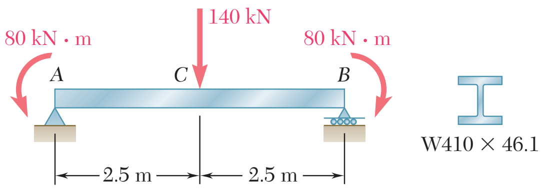


Fig. P15.37

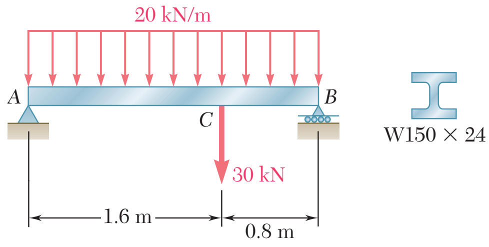


Fig. *P15.38*

15.39 and 15.40 For the uniform beam shown, determine (*a*) the reaction at A, (*b*) the reaction at B.

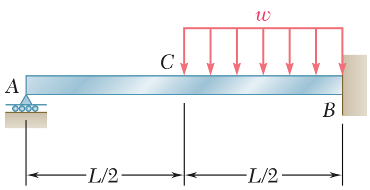


Fig. P15.39

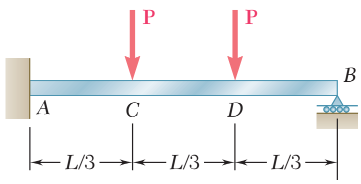


Fig. P15.40

***15.41* and 15.42** For the uniform beam shown, determine the reaction at each of the three supports.

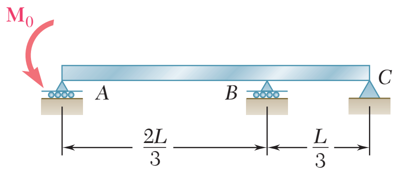


Fig. *P15.41*

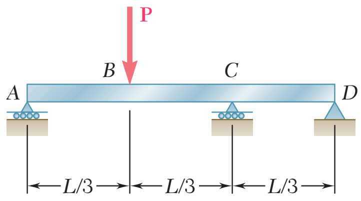


Fig. P15.42

15.43 and *15.44* For the beam shown, determine the reaction at B.

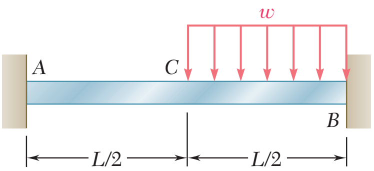


Fig. P15.43

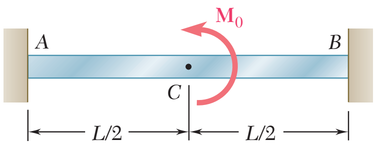


Fig. *P15.44*

15.45 The two beams shown have the same cross section and are joined by a hinge at C. For the loading shown, determine (*a*) the slope at point A, (*b*) the deflection at point B. Use $E = 29 \times 10^6$ psi.

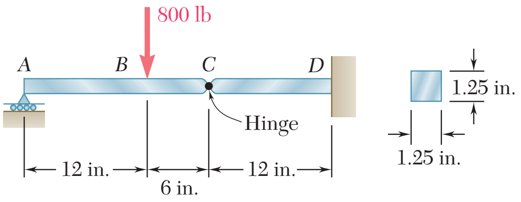


Fig. P15.45

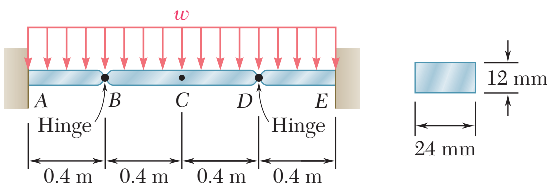


Fig. P15.46

15.46 A central beam BD is joined by hinges to two cantilever beams AB and DE. All beams have the cross section shown. For the loading shown, determine the largest w so that the deflection at C does not exceed 3 mm. Use $E = 200$ GPa.

15.47 For the loading shown, and knowing that beams AB and DE have the same flexural rigidity, determine the reaction (*a*) at B, (*b*) at E.

15.48 Knowing that the rod ABC and the cable BD are both made of steel, determine (*a*) the deflection at B, (*b*) the reaction at A. Use $E = 200$ GPa.

15.49 A $\frac{5}{8}$-in.-diameter rod ABC has been bent into the shape shown. Determine the deflection of end C after the 30-lb force is applied. Use $E = 29 \times 10^6$ psi and $G = 11.2 \times 10^6$ psi.

15.50 Two 24-mm-diameter aluminum rods are welded together to form the T-shaped hanger shown. Knowing that $E = 70$ GPa and $G = 26$ GPa, determine the deflection at (*a*) end A, (*b*) end B.

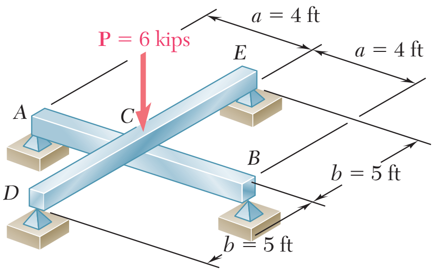


Fig. *P15.47*

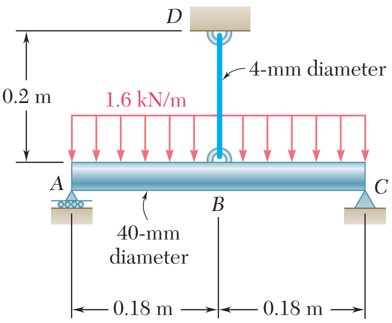


Fig. P15.48

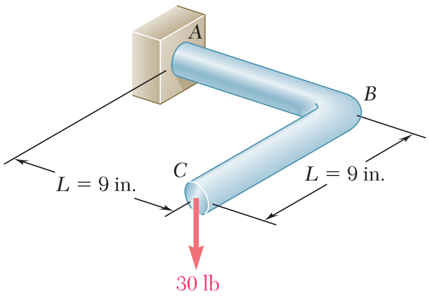


Fig. P15.49

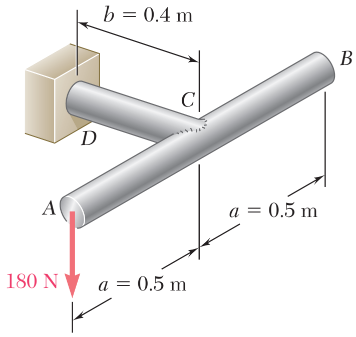


Fig. P15.50

REVIEW AND SUMMARY

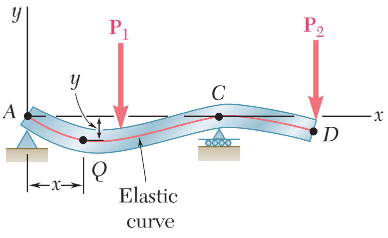


Fig. 15.33

Deformation of a beam under transverse loading

This chapter was devoted to the determination of slopes and deflections of beams under transverse loadings. We used a mathematical method based on the method of integration of a differential equation to get the slopes and deflections at any point along the beam. We also applied this method for determining deflections to the analysis of *indeterminate beams*, those in which the number of reactions at the supports exceeds the number of equilibrium equations available to determine these unknowns.

We noted in Sec. 15.2 that Eq. (11.21) of Sec. 11.4, which relates the curvature $1/\rho$ of the neutral surface and the bending moment M in a prismatic beam in pure bending, can be applied to a beam under a transverse loading, but that both M and $1/\rho$ will vary from section to section. Denoting by x the distance from the left end of the beam, we wrote

$$\frac{1}{\rho} = \frac{M(x)}{EI} \tag{15.1}$$

This equation enabled us to determine the radius of curvature of the neutral surface for any value of x and to draw some general conclusions regarding the shape of the deformed beam.

In Sec. 15.3, we discussed how to obtain a relation between the deflection y of a beam, measured at a given point Q, and the distance x of that point from some fixed origin (Fig. 15.33). Such a relation defines the *elastic curve* of a beam. Expressing the curvature $1/\rho$ in terms of the derivatives of the function $y(x)$ and substituting into (15.1), we obtained the following second-order linear differential equation:

$$\frac{d^2y}{dx^2} = \frac{M(x)}{EI} \tag{15.4}$$

Integrating this equation twice, we obtained the following expressions defining the slope $\theta(x) = dy/dx$ and the deflection $y(x)$, respectively:

$$EI\frac{dy}{dx} = \int_0^x M(x)\,dx + C_1 \tag{15.5}$$

$$EI\,y = \int_0^x dx \int_0^x M(x)\,dx + C_1x + C_2 \tag{15.6}$$

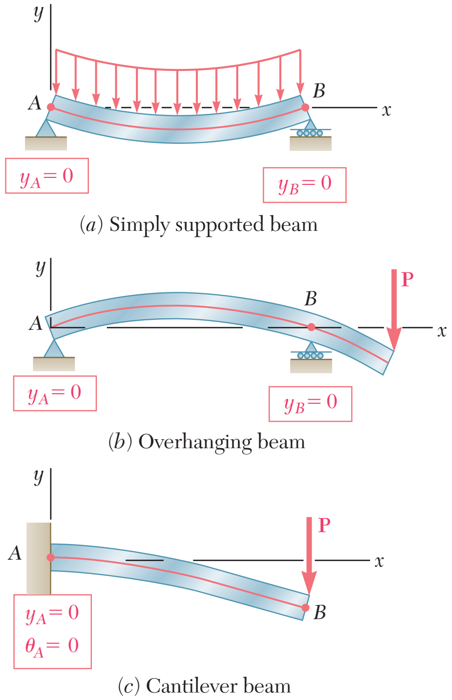


Fig. 15.34 Boundary conditions for statically determinate beams.

Boundary conditions

The product EI is known as the *flexural rigidity* of the beam; C_1 and C_2 are two constants of integration that can be determined from the *boundary conditions* imposed on the beam by its supports (Fig. 15.34) [Example 15.1]. The maximum deflection can then be obtained by determining the value of x for which the slope is zero and the corresponding value of y [Example 15.2 and Sample Prob. 15.1].

When the loading is such that different analytical functions are required to represent the bending moment in various portions of the beam, then different differential equations are also required, leading to different functions representing the slope $\theta(x)$ and the deflection $y(x)$ in the various portions of the beam. In the case of the beam and loading considered in Example 15.3 (Fig. 15.35), two differential equations were required, one for the portion of beam AD and the other for the portion DB. The first equation yielded the functions θ_1 and y_1, and the second the functions θ_2 and y_2. Altogether, four constants of integration had to be determined; two were obtained by writing that the deflections at A and B were zero, and the other two by expressing that the portions of beam AD and DB had the same slope and the same deflection at D.

Elastic curve defined by different function

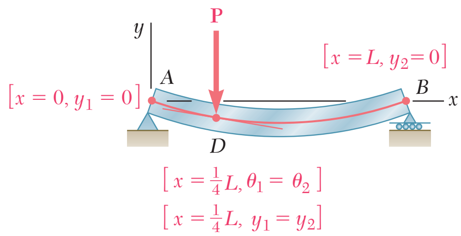
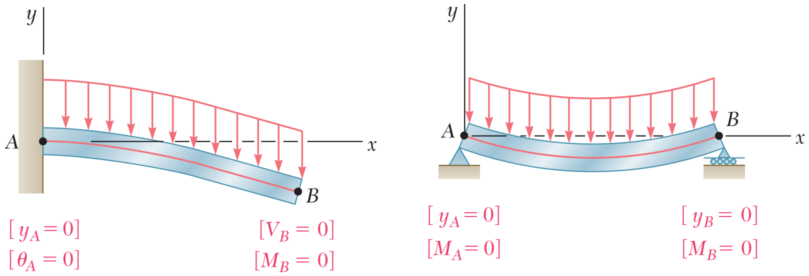


Fig. 15.35

We observed in Sec. 15.4 that in the case of a beam supporting a distributed load $w(x)$, the elastic curve can be determined directly from $w(x)$ through four successive integrations yielding V, M, θ, and y in that order. For the cantilever beam of Fig. 15.36a and the simply supported beam of Fig. 15.36b, the resulting four constants of integration can be determined from the four boundary conditions indicated in each part of the figure [Example 15.4 and Sample Prob. 15.2].

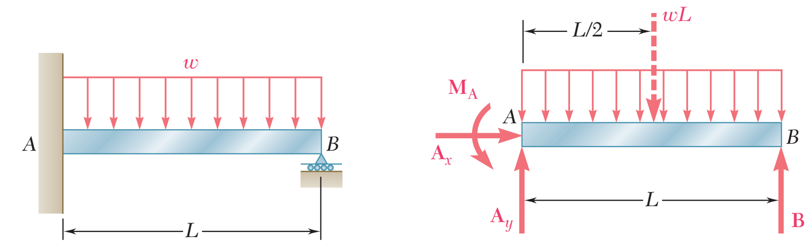


(a) Cantilever beam (b) Simply supported beam

Fig. 15.36 Boundary conditions for beams carrying a distributed load.

Statically indeterminate beams

In Sec. 15.5, we discussed *statically indeterminate beams,* i.e., beams supported in such a way that the reactions at the supports involved four or more unknowns. Since only three equilibrium equations are available to determine these unknowns, the equilibrium equations had to be supplemented by equations obtained from the boundary conditions imposed by the supports. In the case of the beam of Fig. 15.37, we noted that the reactions at the supports involved four



(a) (b)

Fig. 15.37

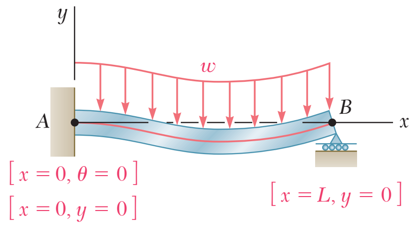


Fig. 15.38

unknowns, namely, M_A, A_x, A_y, and B. Such a beam is said to be *indeterminate to the first degree*. (If five unknowns were involved, the beam would be indeterminate to the *second degree.*) Expressing the bending moment $M(x)$ in terms of the four unknowns and integrating twice [Example 15.5], we determined the slope $\theta(x)$ and the deflection $y(x)$ in terms of the same unknowns and the constants of integration C_1 and C_2. The six unknowns involved in this computation were obtained by solving simultaneously the three equilibrium equations for the free body of Fig. 15.37*b* and the three equations expressing that $\theta = 0, y = 0$ for $x = 0$, and that $y = 0$ for $x = L$ (Fig. 15.38) [see also Sample Prob. 15.3].

Method of superposition

The next section was devoted to the *method of superposition*, which consists of determining separately, and then adding, the slope and deflection caused by the various loads applied to a beam [Sec. 15.6]. This procedure was facilitated by the use of the table of App. C, which gives the slopes and deflections of beams for various loadings and types of support [Example 15.6 and Sample Prob. 15.4].

Statically indeterminate beams by superposition

The method of superposition can be used effectively with *statically indeterminate beams* [Sec. 15.7]. In the case of the beam of Example 15.7 (Fig. 15.39), which involves four unknown reactions and is thus indeterminate to the *first degree*, the reaction at B was considered as *redundant* and the beam was released from that support. Treating the reaction $\mathbf{R}_B$ as an unknown load and considering separately the deflections caused at B by the given distributed load and by $\mathbf{R}_B$, we wrote that the sum of these deflections was zero (Fig. 15.40). The equation obtained was solved for R_B [see also Sample Prob. 15.5]. In the case of a beam indeterminate to the *second degree*, i.e., with reactions at the supports involving five unknowns, two reactions must be designated as redundant, and the corresponding supports must be eliminated or modified accordingly [Sample Prob. 15.6].

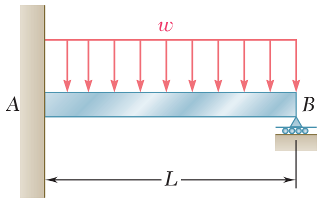


Fig. 15.39

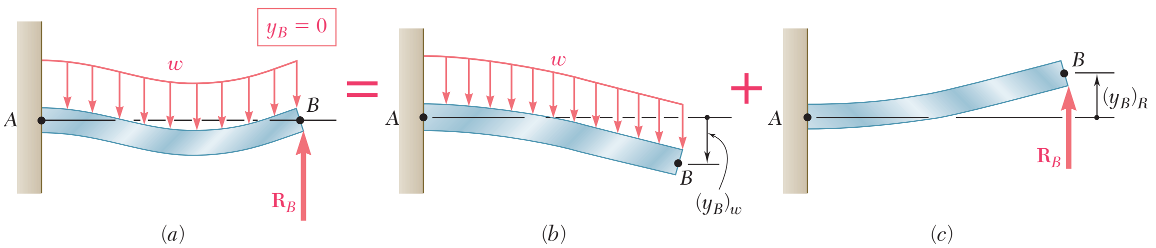


Fig. 15.40

REVIEW PROBLEMS

15.51 For the beam and loading shown, determine (*a*) the equation of the elastic curve for portion *AB* of the beam, (*b*) the slope at *A*, (*c*) the slope at *B*.

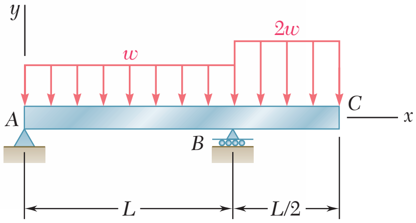


Fig. P15.51

15.52 (*a*) Determine the location and magnitude of the maximum absolute deflection in *AB* between *A* and the center of the beam. (*b*) Assuming that beam *AB* is a W460 × 113, $M_0 = 224$ kN · m, and $E = 200$ GPa, determine the maximum allowable length *L* so that the maximum deflection does not exceed 1.2 mm.

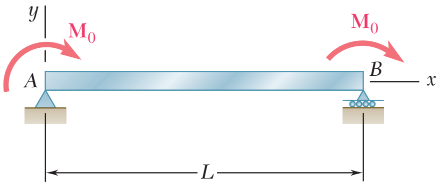


Fig. P15.52

15.53 Knowing that beam *AE* is an S200 × 27.4 rolled shape and that $P = 17.5$ kN, $L = 2.5$ m, $a = 0.8$ m, and $E = 200$ GPa, determine (*a*) the equation of the elastic curve for portion *BD*, (*b*) the deflection at the center *C* of the beam.

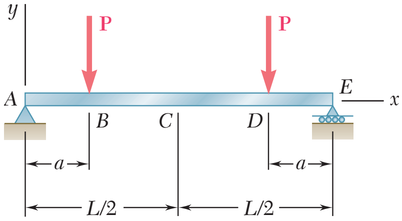


Fig. P15.53

15.54 For the beam and loading shown, determine (*a*) the equation of the elastic curve, (*b*) the slope at the free end, (*c*) the deflection at the free end.

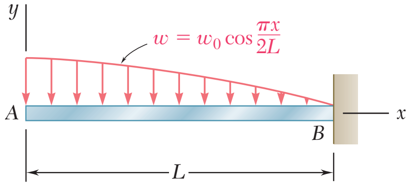


Fig. P15.54

15.55 For the beam shown, determine the reaction at the roller support when $w_0 = 6$ kips/ft.

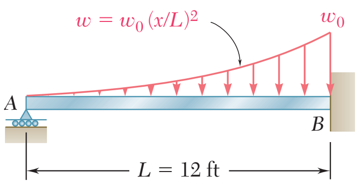


Fig. P15.55

15.56 Determine the reaction at the roller support and draw the bending moment diagram for the beam and loading shown.

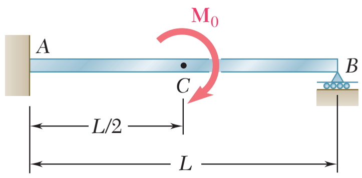


Fig. P15.56

15.57 For the cantilever beam and loading shown, determine the slope and deflection at point B.

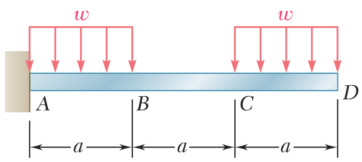


Fig. P15.57

15.58 For the beam and loading shown, determine (a) the deflection at point C, (b) the slope at end A.

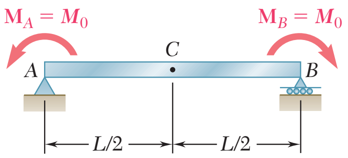


Fig. *P15.58*

15.59 For the cantilever beam and loading shown, determine the slope and deflection at point B. Use $E = 200$ GPa.

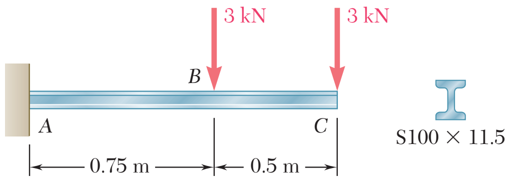


Fig. P15.59

15.60 For the uniform beam shown, determine the reaction at each of the three supports.

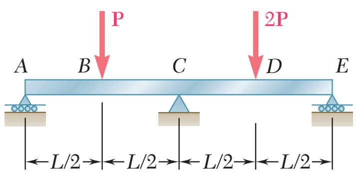


Fig. P15.60

15.61 The cantilever beam BC is attached to the steel cable AB as shown. Knowing that the cable is initially taut, determine the tension in the cable caused by the distributed load shown. Use $E = 200$ GPa.

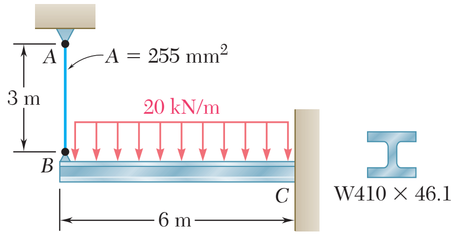


Fig. P15.61

15.62 A $\frac{7}{8}$-in.-diameter rod BC is attached to the lever AB and to the fixed support at C. Lever AB has a uniform cross section $\frac{3}{8}$ in. thick and 1 in. deep. For the loading shown, determine the deflection of point A. Use $E = 29 \times 10^6$ psi and $G = 11.2 \times 10^6$ psi.

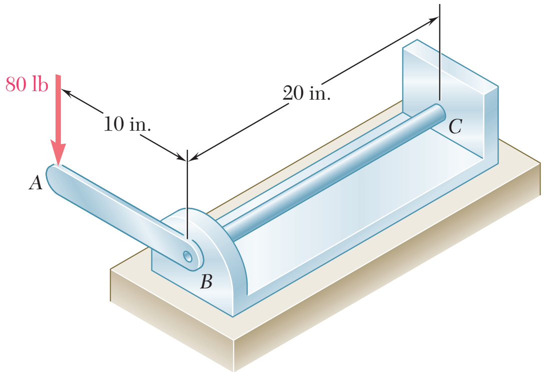
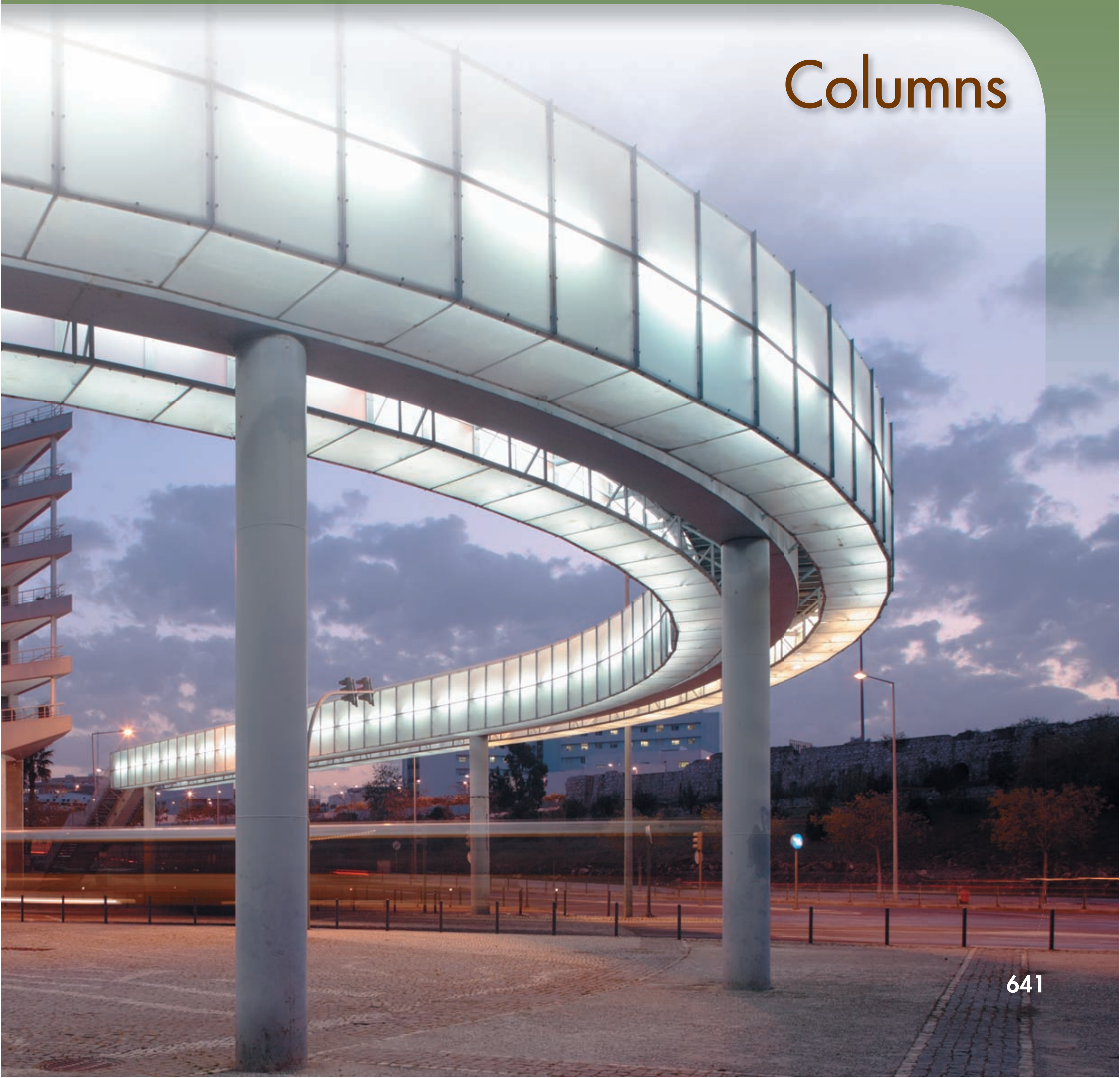


Fig. *P15.62*

The curved pedestrian bridge is supported by a series of columns. The analysis and design of members supporting axial compressive loads will be discussed in this chapter.

CHAPTER 16

Columns



Chapter 16 Columns

16.1 INTRODUCTION

In the preceding chapters, we had two primary concerns: (1) the strength of the structure, i.e., its ability to support a specified load without experiencing excessive stress; (2) the ability of the structure to support a specified load without undergoing unacceptable deformations. In this chapter, our concern will be with the stability of the structure, i.e., with its ability to support a given load without experiencing a sudden change in its configuration. Our discussion will relate chiefly to columns, i.e., to the analysis and design of vertical prismatic members supporting axial loads.

In Sec. 16.2, the stability of a simplified model of a column, consisting of two rigid rods connected by a pin and a spring and supporting a load **P**, will first be considered. You will observe that if its equilibrium is disturbed, this system will return to its original equilibrium position as long as P does not exceed a certain value P_{cr}, called the *critical load.* However, if $P > P_{cr}$, the system will move away from its original position and settle in a new position of equilibrium. In the first case, the system is said to be *stable,* and in the second case, it is said to be *unstable.*

In Sec. 16.3, you will begin the study of the *stability of elastic columns* by considering a pin-ended column subjected to a centric axial load. *Euler's formula* for the critical load of the column will be derived and from that formula the corresponding critical normal stress in the column will be determined. By applying a factor of safety to the critical load, you will be able to determine the allowable load that can be applied to a pin-ended column.

In Sec. 16.4, the analysis of the stability of columns with different end conditions will be considered. You will simplify these analyses by learning how to determine the *effective length* of a column, i.e., the length of a pin-ended column having the same critical load.

In the first sections of the chapter, each column is initially assumed to be a straight homogeneous prism. In the last part of the chapter, you will consider real columns which are designed and analyzed using empirical formulas set forth by professional organizations. In Sec. 16.5, formulas will be presented for the allowable stress in columns made of steel, aluminum, or wood and subjected to a centric axial load.

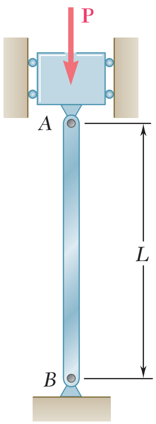


Fig. 16.1

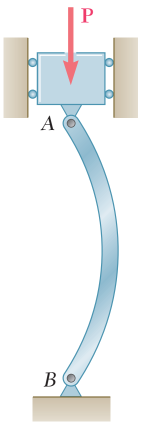


Fig. 16.2

16.2 STABILITY OF STRUCTURES

Suppose we are to design a column AB of length L to support a given load **P** (Fig. 16.1). The column will be pin-connected at both ends and we assume that **P** is a centric axial load. If the cross-sectional area A of the column is selected so that the value $\sigma = P/A$ of the stress on a transverse section is less than the allowable stress σ_{all} for the material used, and if the deformation $\delta = PL/AE$ falls within the given specifications, we might conclude that the column has been properly designed. However, it may happen that, as the load is applied, the column will *buckle;* instead of remaining straight, it will suddenly become sharply curved (Fig. 16.2). Photo 16.1 shows a column that has been loaded so that it is no longer straight; the

Photo 16.1 Test column that has buckled

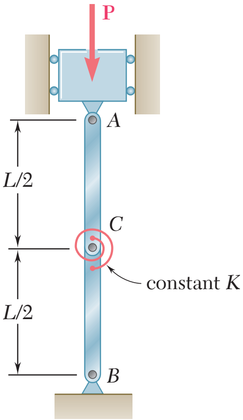
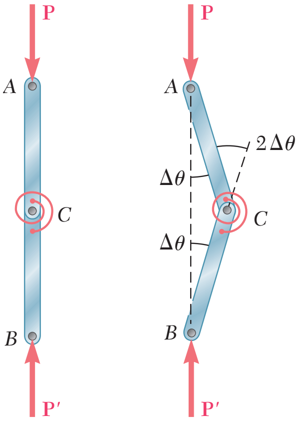


Fig. 16.3



(a) (b)

Fig. 16.4

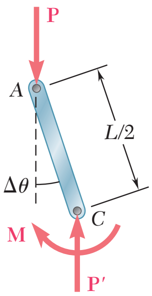


Fig. 16.5

column has buckled. Clearly, a column that buckles under the load it is to support is not properly designed.

Before getting into the actual discussion of the stability of elastic columns, some insight will be gained on the problem by considering a simplified model consisting of two rigid rods AC and BC connected at C by a pin and a torsional spring of constant K (Fig. 16.3).

If the two rods and the two forces $\mathbf{P}$ and $\mathbf{P}'$ are perfectly aligned, the system will remain in the position of equilibrium shown in Fig. 16.4*a* as long as it is not disturbed. But suppose that we move C slightly to the right, so that each rod now forms a small angle $\Delta\theta$ with the vertical (Fig. 16.4*b*). Will the system return to its original equilibrium position, or will it move further away from that position? In the first case, the system is said to be *stable*, and in the second case, it is said to be *unstable*.

To determine whether the two-rod system is stable or unstable, we consider the forces acting on rod AC (Fig. 16.5). These forces consist of two couples, namely the couple formed by $\mathbf{P}$ and $\mathbf{P}'$, of moment $P(L/2)\sin\Delta\theta$, which tends to move the rod away from the vertical, and the couple $\mathbf{M}$ exerted by the spring, which tends to bring the rod back into its original vertical position. Since the angle of deflection of the spring is $2\,\Delta\theta$, the moment of the couple $\mathbf{M}$ is $M = K(2\,\Delta\theta)$.

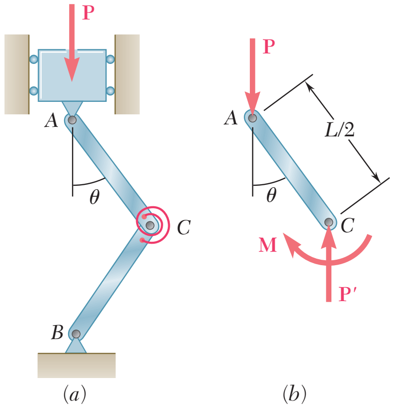


Fig. 16.6

If the moment of the second couple is larger than the moment of the first couple, the system tends to return to its original equilibrium position; the system is stable. If the moment of the first couple is larger than the moment of the second couple, the system tends to move away from its original equilibrium position; the system is unstable. The value of the load for which the two couples balance each other is called the *critical load* and is denoted by P_{cr}. We have

$$P_{cr}(L/2) \sin \Delta\theta = K(2\,\Delta\theta) \tag{16.1}$$

or, since $\sin\,\Delta\theta \approx \Delta\theta$,

$$P_{cr} = 4K/L \tag{16.2}$$

Clearly, the system is stable for $P < P_{cr}$, that is, for values of the load smaller than the critical value, and unstable for $P > P_{cr}$.

Let us assume that a load $P > P_{cr}$ has been applied to the two rods of Fig. 16.3 and that the system has been disturbed. Since $P > P_{cr}$, the system will move further away from the vertical and, after some oscillations, will settle into a new equilibrium position (Fig. 16.6*a*). Considering the equilibrium of the free body AC (Fig. 16.6*b*), we obtain an equation similar to Eq. (16.1), but involving the finite angle θ, namely

$$P(L/2) \sin \theta = K(2\theta)$$

or

$$\frac{PL}{4K} = \frac{\theta}{\sin\theta} \tag{16.3}$$

The value of θ corresponding to the equilibrium position represented in Fig. 16.6 is obtained by solving Eq. (16.3) by trial and error. But we observe that, for any positive value of θ, we have $\sin\theta < \theta$. Thus, Eq. (16.3) yields a value of θ different from zero only when the left-hand member of the equation is larger than one. Recalling Eq. (16.2), we note that this is indeed the case here, since we have assumed $P > P_{cr}$. But, if we had assumed $P < P_{cr}$, the second equilibrium position shown in Fig. 16.6 would not exist and the only possible equilibrium position would be the position corresponding to $\theta = 0$. We thus check that, for $P < P_{cr}$, the position $\theta = 0$ must be stable.

This observation applies to structures and mechanical systems in general, and will be used in the next section where the stability of elastic columns will be discussed.

16.3 EULER'S FORMULA FOR PIN-ENDED COLUMNS

Returning to the column AB considered in the preceding section (Fig. 16.1), we propose to determine the critical value of the load **P**, i.e., the value P_{cr} of the load for which the position shown in Fig. 16.1 ceases to be stable. If $P > P_{cr}$, the slightest misalignment or disturbance will cause the column to buckle, i.e., to assume a curved shape as shown in Fig. 16.2.

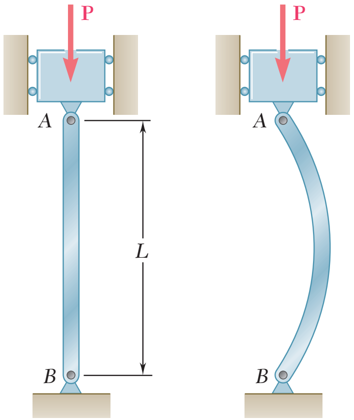


Fig. 16.1 (*repeated*)

Fig. 16.2 (*repeated*)

Our approach will be to determine the conditions under which the configuration of Fig. 16.2 is possible. Since a column can be considered as a beam placed in a vertical position and subjected to an axial load, we proceed as in Chap. 15 and denote by x the distance from end A of the column to a given point Q of its elastic curve, and by y the deflection of that point (Fig. 16.7*a*). It follows that the x axis will be vertical and directed downward, and the y axis horizontal and directed to the right. Considering the equilibrium of the free body AQ (Fig. 16.7*b*), we find that the bending moment at Q is $M = -Py$. Substituting this value for M in Eq. (15.4) of Sec. 15.3, we write

$$\frac{d^2y}{dx^2} = \frac{M}{EI} = -\frac{P}{EI}y \tag{16.4}$$

or, transposing the last term,

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = 0 \tag{16.5}$$

This equation is a linear, homogeneous differential equation of the second order with constant coefficients. Setting

$$p^2 = \frac{P}{EI} \tag{16.6}$$

we write Eq. (16.5) in the form

$$\frac{d^2y}{dx^2} + p^2y = 0 \tag{16.7}$$

which is the same as that of the differential equation for simple harmonic motion except, of course, that the independent variable is now the distance x instead of the time t. The general solution of Eq. (16.7) is

$$y = A \sin px + B \cos px \tag{16.8}$$

as we easily check by computing d^2y/dx^2 and substituting for y and d^2y/dx^2 into Eq. (16.7).

Recalling the boundary conditions that must be satisfied at ends A and B of the column (Fig. 16.7*a*), we first make $x = 0, y = 0$ in Eq. (16.8) and find that $B = 0$. Substituting next $x = L, y = 0$, we obtain

$$A \sin pL = 0 \tag{16.9}$$

This equation is satisfied either if $A = 0$, or if $\sin pL = 0$. If the first of these conditions is satisfied, Eq. (16.8) reduces to $y = 0$ and the column is straight (Fig. 16.1). For the second condition to be satisfied, we must have $pL = n\pi$ or, substituting for p from Eq. (16.6) and solving for P,

$$P = \frac{n^2\pi^2EI}{L^2} \tag{16.10}$$

The smallest of the values of P defined by Eq. (16.10) is that corresponding to $n = 1$. We thus have

$$P_{cr} = \frac{\pi^2EI}{L^2} \tag{16.11}$$

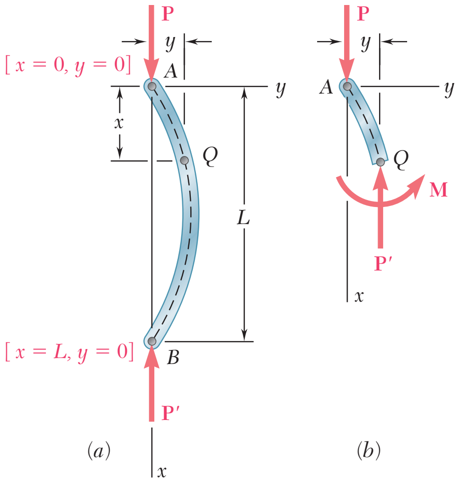


Fig. 16.7

The expression obtained is known as *Euler's formula,* after the Swiss mathematician Leonhard Euler (1707–1783). Substituting this expression for P into Eq. (16.6) and the value obtained for p into Eq. (16.8), and recalling that $B = 0$, we write

$$y = A \sin \frac{\pi x}{L} \tag{16.12}$$

which is the equation of the elastic curve after the column has buckled (Fig. 16.2). We note that the value of the maximum deflection, $y_m = A$, is indeterminate. This is due to the fact that the differential equation (16.5) is a linearized approximation of the actual governing differential equation for the elastic curve.†

If $P < P_{cr}$, the condition $\sin pL = 0$ cannot be satisfied, and the solution given by Eq. (16.12) does not exist. We must then have $A = 0$, and the only possible configuration for the column is a straight one. Thus, for $P < P_{cr}$ the straight configuration of Fig. 16.1 is stable.

In the case of a column with a circular or square cross section, the moment of inertia I of the cross section is the same about any centroidal axis, and the column is as likely to buckle in one plane as another, except for the restraints that can be imposed by the end connections. For other shapes of cross section, the critical load should be computed by making $I = I_{min}$ in Eq. (16.11); if buckling occurs, it will take place in a plane perpendicular to the corresponding principal axis of inertia.

The value of the stress corresponding to the critical load is called the *critical stress* and is denoted by σ_{cr}. Recalling Eq. (16.11) and setting $I = Ar^2$, where A is the cross-sectional area and r its radius of gyration, we have

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 E A r^2}{A L^2}$$

or

$$\sigma_{cr} = \frac{\pi^2 E}{(L/r)^2} \tag{16.13}$$

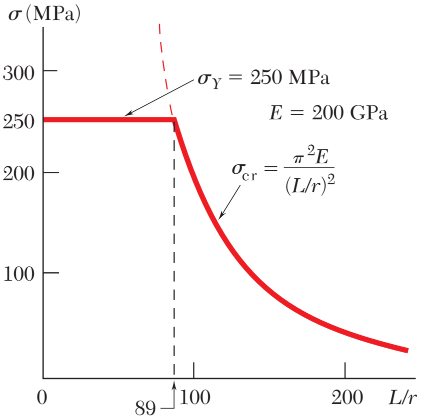


Fig. 16.8

The quantity L/r is called the *slenderness ratio* of the column. It is clear, in view of the remark of the preceding paragraph, that the minimum value of the radius of gyration r should be used in computing the slenderness ratio and the critical stress in a column.

Equation (16.13) shows that the critical stress is proportional to the modulus of elasticity of the material, and inversely proportional to the square of the slenderness ratio of the column. The plot of σ_{cr} versus L/r is shown in Fig. 16.8 for structural steel, assuming $E = 200$ GPa and $\sigma_Y = 250$ MPa. We should keep in mind that no factor of safety has been used in plotting σ_{cr}. We also note that, if the

†We recall that the equation $d^2y/dx^2 = M/EI$ was obtained in Sec. 15.3 by assuming that the slope dy/dx of the beam could be neglected and that the exact expression given in Eq. (15.3) for the curvature of the beam could be replaced by $1/\rho = d^2y/dx^2$.

value obtained for σ_{cr} from Eq. (16.13) or from the curve of Fig. 16.8 is larger than the yield strength σ_Y, this value is of no interest to us, since the column will yield in compression and cease to be elastic before it has a chance to buckle.

EXAMPLE 16.1 A 2-m-long pin-ended column of square cross section is to be made of wood. Assuming $E = 13$ GPa, $\sigma_{all} = 12$ MPa, and using a factor of safety of 2.5 in computing Euler's critical load for buckling, determine the size of the cross section if the column is to safely support (*a*) a 100-kN load, (*b*) a 200-kN load.

(a) For the 100-kN Load. Using the given factor of safety, we make

$$P_{cr} = 2.5(100\text{ kN}) = 250\text{ kN} \qquad L = 2\text{ m} \qquad E = 13\text{ GPa}$$

in Euler's formula (16.11) and solve for I. We have

$$I = \frac{P_{cr}L^2}{\pi^2 E} = \frac{(250 \times 10^3\text{ N})(2\text{ m})^2}{\pi^2(13 \times 10^9\text{ Pa})} = 7.794 \times 10^{-6}\text{ m}^4$$

Recalling that, for a square of side a, we have $I = a^4/12$, we write

$$\frac{a^4}{12} = 7.794 \times 10^{-6}\text{ m}^4 \qquad a = 98.3\text{ mm} \approx 100\text{ mm}$$

We check the value of the normal stress in the column:

$$\sigma = \frac{P}{A} = \frac{100\text{ kN}}{(0.100\text{ m})^2} = 10\text{ MPa}$$

Since σ is smaller than the allowable stress, a 100 × 100-mm cross section is acceptable.

(b) For the 200-kN Load. Solving again Eq. (16.11) for I, but making now $P_{cr} = 2.5(200) = 500$ kN, we have

$$I = 15.588 \times 10^{-6}\text{ m}^4$$

$$\frac{a^4}{12} = 15.588 \times 10^{-6} \qquad a = 116.95\text{ mm}$$

The value of the normal stress is

$$\sigma = \frac{P}{A} = \frac{200\text{ kN}}{(0.11695\text{ m})^2} = 14.62\text{ MPa}$$

Since this value is larger than the allowable stress, the dimension obtained is not acceptable, and we must select the cross section on the basis of its resistance to compression. We write

$$\text{A} = \frac{P}{\sigma_{all}} = \frac{200\text{ kN}}{12\text{ MPa}} = 16.67 \times 10^{-3}\text{ m}^2$$

$$a^2 = 16.67 \times 10^{-3}\text{ m}^2 \qquad a = 129.1\text{ mm}$$

A 130 × 130-mm cross section is acceptable. ■

16.4 EXTENSION OF EULER'S FORMULA TO COLUMNS WITH OTHER END CONDITIONS

Euler's formula (16.11) was derived in the preceding section for a column that was pin-connected at both ends. Now the critical load P_{cr} will be determined for columns with different end conditions.

In the case of a column with one free end A supporting a load **P** and one fixed end B (Fig. 16.9*a*), we observe that the column will behave as the upper half of a pin-connected column (Fig. 16.9*b*). The critical load for the column of Fig. 16.9*a* is thus the same as for the pin-ended column of Fig. 16.9*b* and can be obtained from Euler's

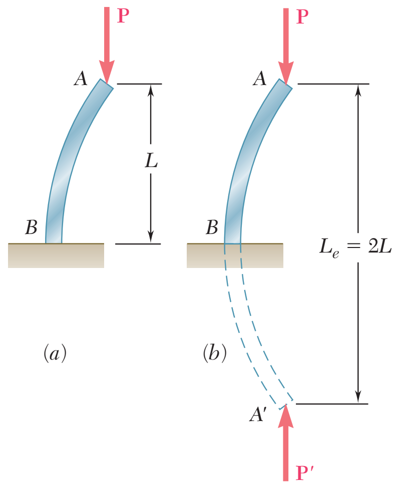


Fig. 16.9

formula (16.11) by using a column length equal to twice the actual length L of the given column. We say that the *effective length* L_e of the column of Fig. 16.9 is equal to $2L$ and substitute $L_e = 2L$ in Euler's formula:

$$P_{cr} = \frac{\pi^2 EI}{L_e^2} \tag{16.11′}$$

The critical stress is found in a similar way from the formula

$$\sigma_{cr} = \frac{\pi^2 E}{(L_e/r)^2} \tag{16.13′}$$

The quantity L_e/r is referred to as the *effective slenderness ratio* of the column and, in the case considered here, is equal to $2L/r$.

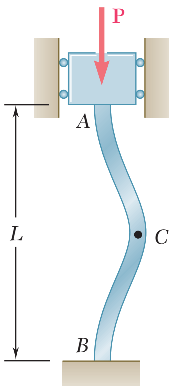


Fig. 16.10

Consider next a column with two fixed ends A and B supporting a load **P** (Fig. 16.10). The symmetry of the supports and of the loading about a horizontal axis through the midpoint C requires that the shear at C and the horizontal components of the reactions at A and B be zero (Fig. 16.11). It follows that the restraints imposed upon the upper half AC of the column by the support at A and by the lower half CB are identical (Fig. 16.12). Portion AC must thus be symmetric about its midpoint D, and this point must be a point of inflection, where the bending moment is zero. A similar reasoning shows that the bending moment at the midpoint E of the lower half of the column must also be zero (Fig. 16.13*a*). Since the bending moment at the ends of a pin-ended column is zero, it follows that the portion DE of the column of Fig. 16.13*a* must behave as a pin-ended column (Fig. 16.13*b*). We thus conclude that the effective length of a column with two fixed ends is $L_e = L/2$.

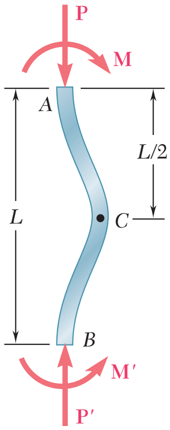


Fig. 16.11

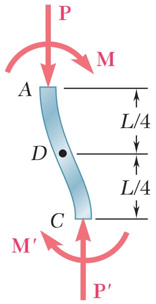


Fig. 16.12

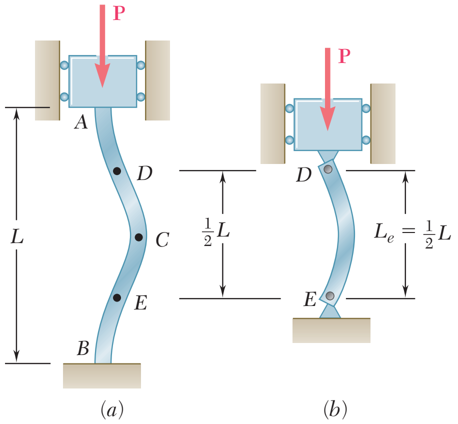


Fig. 16.13

In the case of a column with one fixed end B and one pin-connected end A supporting a load **P** (Fig. 16.14), we must write and solve the differential equation of the elastic curve to determine the effective length of the column. From the free-body diagram of the entire column (Fig. 16.15), we first note that a transverse force **V** is exerted at end A, in addition to the axial load **P**, and that **V** is statically indeterminate. Considering now the free-body diagram of a portion AQ of the column (Fig. 16.16), we find that the bending moment at Q is

$$M = -Py - Vx$$

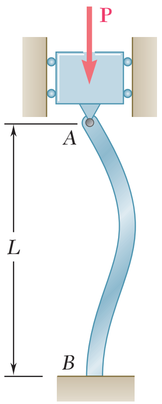


Fig. 16.14

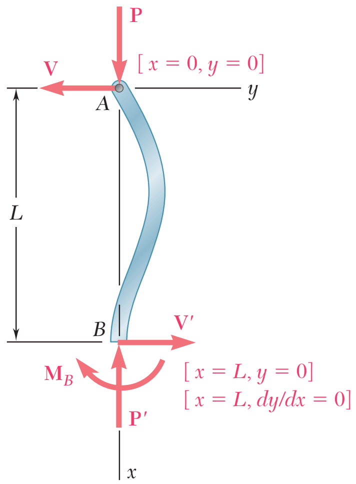


Fig. 16.15

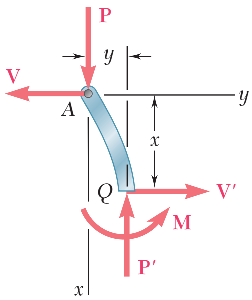


Fig. 16.16

Substituting this value into Eq. (15.4) of Sec. 15.3, we write

$$\frac{d^2y}{dx^2} = \frac{M}{EI} = -\frac{P}{EI}y - \frac{V}{EI}x$$

Transposing the term containing y and setting

$$p^2 = \frac{P}{EI} \tag{16.6}$$

as we did in Sec. 16.3, we write

$$\frac{d^2y}{dx^2} + p^2y = -\frac{V}{EI}x \tag{16.14}$$

This equation is a linear, nonhomogeneous differential equation of the second order with constant coefficients. Observing that the left-hand members of Eqs. (16.7) and (16.14) are identical, we conclude that the general solution of Eq. (16.14) can be obtained by adding a particular solution of Eq. (16.14) to the solution (16.8) obtained for Eq. (16.7). Such a particular solution is easily seen to be

$$y = -\frac{V}{p^2EI}x$$

or, recalling Eq. (16.6),

$$y = -\frac{V}{P}x \tag{16.15}$$

Adding the solutions to Eqs. (16.8) and (16.15), we write the general solution of Eq. (16.14) as

$$y = A \sin px + B \cos px - \frac{V}{P}x \tag{16.16}$$

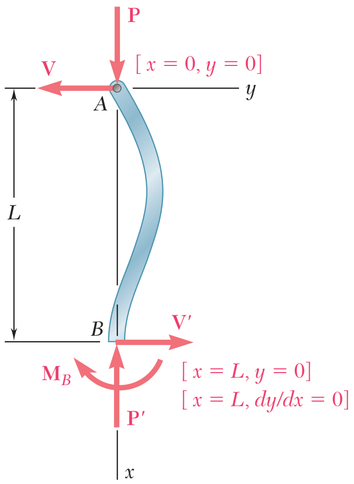


Fig. 16.15 *(repeated)*

The constants A and B, and the magnitude V of the unknown transverse force **V** are obtained from the boundary conditions indicated in Fig. (16.15). Making first $x = 0, y = 0$ in Eq. (16.16), we find that $B = 0$. Making next $x = L, y = 0$, we obtain

$$A \sin pL = \frac{V}{P}L \tag{16.17}$$

Finally, computing

$$\frac{dy}{dx} = Ap \cos px - \frac{V}{P}$$

and making $x = L, dy/dx = 0$, we have

$$Ap \cos pL = \frac{V}{P} \tag{16.18}$$

Dividing Eq. (16.17) by Eq. (16.18) member by member, we conclude that a solution of the form for Eq. (16.16) can exist only if

$$\tan pL = pL \tag{16.19}$$

Solving this equation by trial and error, we find that the smallest value of pL which satisfies Eq. (16.19) is

$$pL = 4.4934 \tag{16.20}$$

Carrying the value of p defined by Eq. (16.20) into Eq. (16.6) and solving for P, we obtain the critical load for the column of Fig. 16.14

$$P_{cr} = \frac{20.19EI}{L^2} \tag{16.21}$$

The effective length of the column is obtained by equating the right-hand members of Eqs. (16.11′) and (16.21):

$$\frac{\pi^2 EI}{L_e^2} = \frac{20.19EI}{L^2}$$

Solving for L_e, we find that the effective length of a column with one fixed end and one pin-connected end is $L_e = 0.699L \approx 0.7L$.

The effective lengths corresponding to the various end conditions considered in this section are shown in Fig. 16.17.

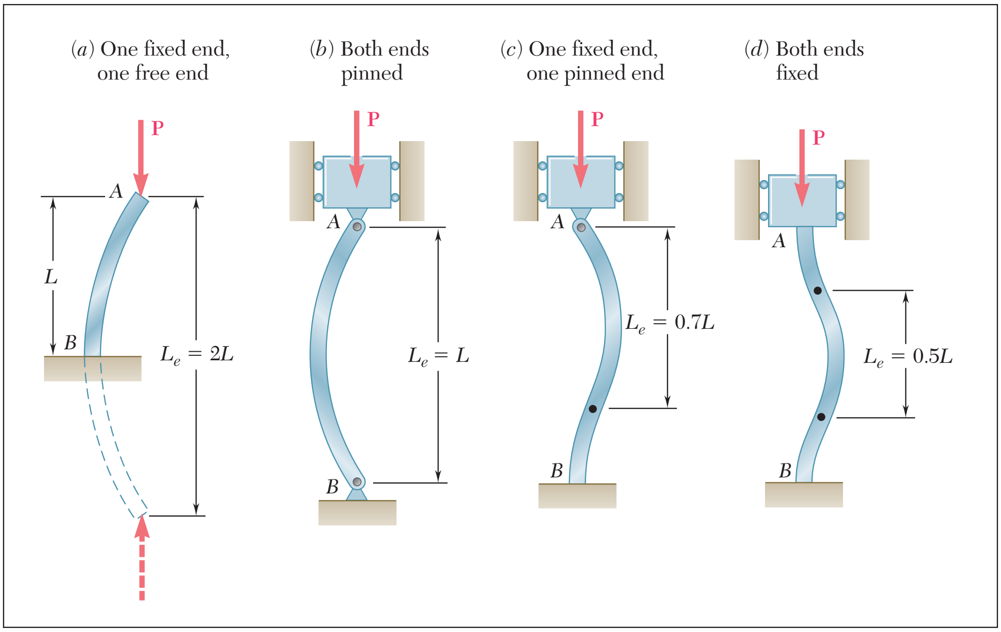
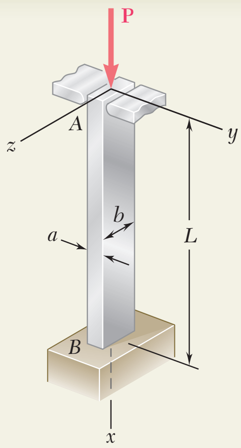


Fig. 16.17 Effective length of column for various end conditions.



SAMPLE PROBLEM 16.1

An aluminum column of length L and rectangular cross section has a fixed end B and supports a centric load at A. Two smooth and rounded fixed plates restrain end A from moving in one of the vertical planes of symmetry of the column but allow it to move in the other plane. (a) Determine the ratio a/b of the two sides of the cross section corresponding to the most efficient design against buckling. (b) Design the most efficient cross section for the column knowing that $L = 20$ in., $E = 10.1 \times 10^6$ psi, $P = 5$ kips, and that a factor of safety of 2.5 is required.

SOLUTION

Buckling in *xy* Plane. Referring to Fig. 16.17, we note that the effective length of the column with respect to buckling in this plane is $L_e = 0.7L$. The radius of gyration r_z of the cross section is obtained by writing

$$I_x = \tfrac{1}{12}ba^3 \qquad A = ab$$

and, since $I_z = Ar_z^2$,
$$r_z^2 = \frac{I_z}{A} = \frac{\frac{1}{12}ba^3}{ab} = \frac{a^2}{12} \qquad r_z = a/\sqrt{12}$$

The effective slenderness ratio of the column with respect to buckling in the xy plane is

$$\frac{L_e}{r_z} = \frac{0.7L}{a/\sqrt{12}} \tag{1}$$

Buckling in *xz* Plane. The effective length of the column with respect to buckling in this plane is $L_e = 2L$, and the corresponding radius of gyration is $r_y = b/\sqrt{12}$. Thus,

$$\frac{L_e}{r_y} = \frac{2L}{b/\sqrt{12}} \tag{2}$$

a. Most Efficient Design. The most efficient design is that for which the critical stresses corresponding to the two possible modes of buckling are equal. Referring to Eq. (16.13′), we note that this will be the case if the two values obtained above for the effective slenderness ratio are equal. We write

$$\frac{0.7L}{a/\sqrt{12}} = \frac{2L}{b/\sqrt{12}}$$

and, solving for the ratio a/b,
$$\frac{a}{b} = \frac{0.7}{2} \qquad \frac{a}{b} = 0.35 \quad ◀$$

b. Design for Given Data. Since $F.S. = 2.5$ is required,

$$P_{cr} = (F.S.)P = (2.5)(5 \text{ kips}) = 12.5 \text{ kips}$$

Using $a = 0.35b$, we have $A = ab = 0.35b^2$ and

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{12{,}500 \text{ lb}}{0.35b^2}$$

Making $L = 20$ in. in Eq. (2), we have $L_e/r_y = 138.6/b$. Substituting for E, L_e/r, and σ_{cr} into Eq. (16.13′), we write

$$\sigma_{cr} = \frac{\pi^2 E}{(L_e/r)^2} \qquad \frac{12{,}500 \text{ lb}}{0.35b^2} = \frac{\pi^2(10.1 \times 10^6 \text{ psi})}{(138.6/b)^2}$$

$$b = 1.620 \text{ in.} \qquad a = 0.35b = 0.567 \text{ in.} \quad ◀$$

16.1 Knowing that the spring at A is of constant k and that the bar AB is rigid, determine the critical load P_{cr}.

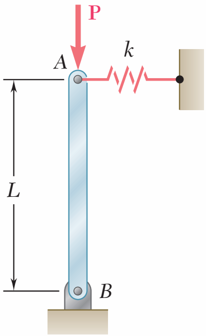


Fig. P16.1

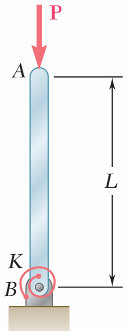


Fig. P16.2

16.2 Knowing that the torsional spring at B is of constant K and that the bar AB is rigid, determine the critical load P_{cr}.

16.3 Two rigid bars AC and BC are connected as shown to a spring of constant k. Knowing that the spring can act in either tension or compression, determine the critical load P_{cr} for the system.

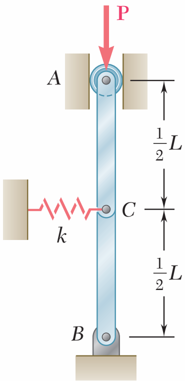


Fig. P16.3

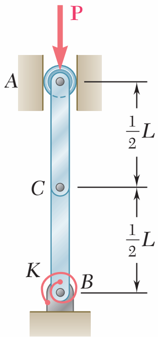


Fig. P16.4

16.4 Two rigid bars AC and BC are connected by a pin at C as shown. Knowing that the torsional spring at B is of constant K, determine the critical load P_{cr} for the system.

16.5 The rigid rod AB is attached to a hinge at A and to two springs, each of constant $k = 2$ kips/in., that can act in either tension or compression. Knowing that $h = 2$ ft, determine the critical load.

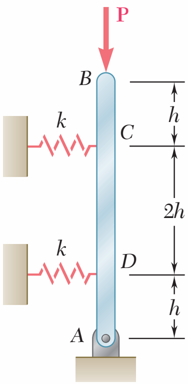


Fig. P16.5

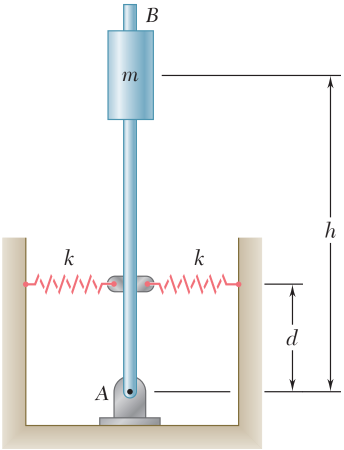


Fig. *P16.6*

16.6 If m = 125 kg, h = 700 mm, and the constant in each spring is $k = 2.8$ kN/m, determine the range of values of the distance d for which the equilibrium of rod AB is stable in the position shown. Each spring can act in either tension or compression.

16.7 Determine the critical load of a round wooden dowel that is 48 in. long and has a diameter of (a) 0.375 in., (b) 0.5 in. Use $E = 1.6 \times 10^6$ psi.

16.8 Determine the critical load of an aluminum tube that is 1.5 m long and has a 16-mm outer diameter and a 1.25-mm wall thickness. Use $E = 70$ GPa.

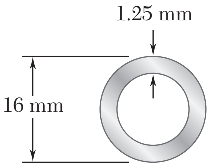


Fig. *P16.8*

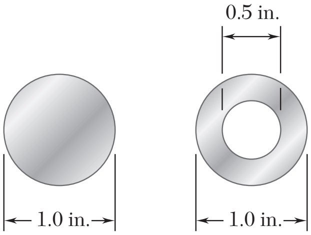


Fig. P16.9

16.9 A compression member of 20-in. effective length consists of a solid 1-in.-diameter aluminum rod. In order to reduce the weight of the member by 25%, the solid rod is replaced by a hollow rod of the cross section shown. Determine (a) the percent reduction in the critical load, (b) the value of the critical load for the hollow rod. Use $E = 10.6 \times 10^6$ psi.

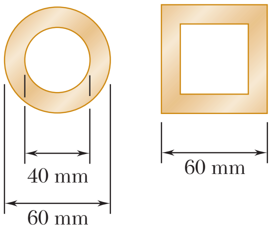


Fig. P16.10

16.10 Two brass rods used as compression members, each of 3-m effective length, have the cross sections shown. (a) Determine the wall thickness of the hollow square rod for which the rods have the same cross-sectional area. (b) Using $E = 105$ GPa, determine the critical load of each rod.

16.11 Determine the radius of the round strut so that the round and square struts have the same cross-sectional area and compute the critical load for each. Use $E = 200$ GPa.

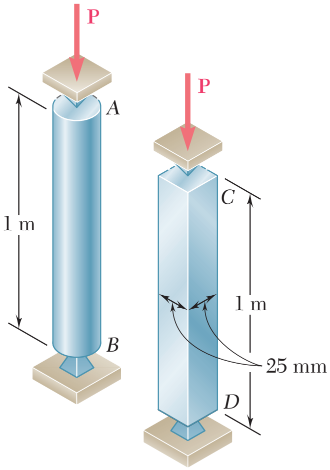


Fig. *P16.11*

16.12 A column of effective length L can be made by gluing together identical planks in either of the arrangements shown. Determine the ratio of the critical load using the arrangement a to the critical load using the arrangement b.

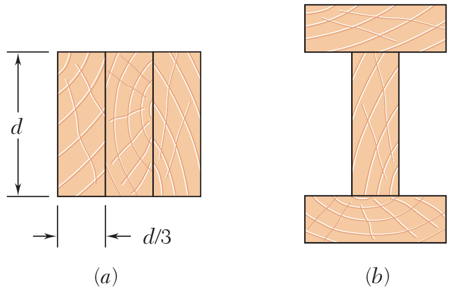


Fig. P16.12

16.13 A compression member of 7-m effective length is made by welding together two L152 × 102 × 12.7 angles as shown. Using $E = 200$ GPa, determine the allowable centric load for the member if a factor of safety of 2.2 is required.

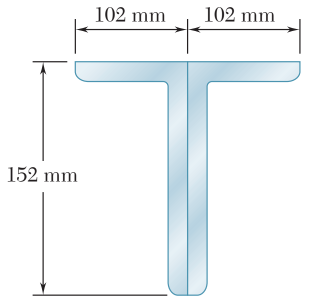


Fig. P16.13

16.14 A column of 26-ft effective length is made from half a W16 × 40 rolled-steel shape. Knowing that the centroid of the cross section is located as shown, determine the factor of safety if the allowable centric load is 20 kips. Use $E = 29 \times 10^6$ psi.

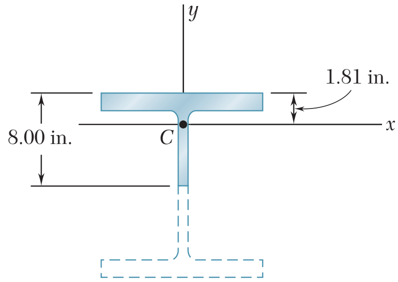


Fig. P16.14

16.15 A column of 22-ft effective length is to be made by welding two 9 × 0.5-in. plates to a W8 × 35 as shown. Determine the allowable centric load if a factor of safety 2.3 is required. Use $E = 29 \times 10^6$ psi.

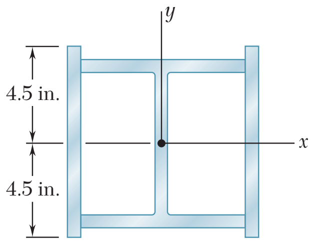


Fig. *P16.15*

16.16 A column of 3-m effective length is to be made by welding together two C130 × 13 rolled-steel channels. Using $E = 200$ GPa, determine for each arrangement shown the allowable centric load if a factor of safety of 2.4 is required.

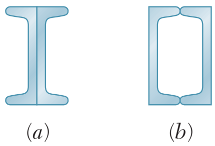


Fig. P16.16

16.17 Knowing that $P = 5.2$ kN, determine the factor of safety for the structure shown. Use $E = 200$ GPa and consider only buckling in the plane of the structure.

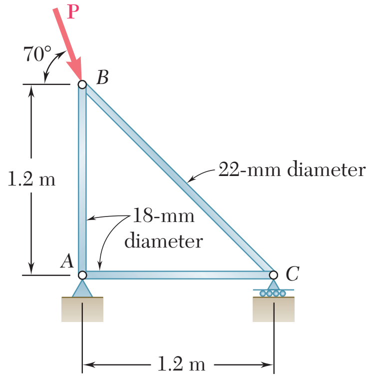


Fig. P16.17

16.18 Members AB and CD are 30-mm-diameter steel rods, and members BC and AD are 22-mm-diameter steel rods. When the turnbuckle is tightened, the diagonal member AC is put in tension. Knowing that a factor of safety with respect to buckling of 2.75 is required, determine the largest allowable tension in AC. Use $E = 200$ GPa and consider only buckling in the plane of the structure.

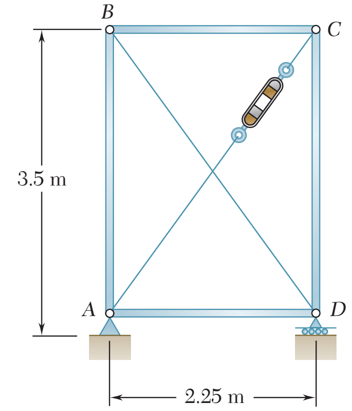


Fig. P16.18

16.19 A 25-mm-square aluminum strut is maintained in the position shown by a pin support at A and by sets of rollers at B and C that prevent rotation of the strut in the plane of the figure. Knowing that $L_{AB} = 1.0$ m, $L_{BC} = 1.25$ m, and $L_{CD} = 0.5$ m, determine the allowable load **P** using a factor of safety with respect to buckling of 2.8. Consider only buckling in the plane of the figure and use $E = 75$ GPa.

16.20 A 32-mm-square aluminum strut is maintained in the position shown by a pin support at A and by sets of rollers at B and C that prevent rotation of the strut in the plane of the figure. Knowing that $L_{AB} = 1.4$ m, determine (*a*) the largest values of L_{BC} and L_{CD} that can be used if the allowable load **P** is to be as large as possible, (*b*) the magnitude of the corresponding allowable load if the factor of safety is to be 2.8. Consider only buckling in the plane of the figure and use $E = 72$ GPa.

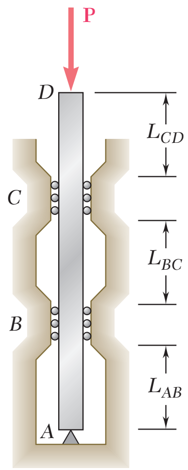


Fig. *P16.19* and P16.20

16.21 The aluminum column ABC has a uniform rectangular cross section and is braced in the xz plane at its midpoint C. (*a*) Determine the ratio b/d for which the factor of safety is the same with respect to buckling in the xz and yz planes. (*b*) Using the ratio found in part *a*, design the cross section of the column so that the factor of safety will be 2.7 when $P = 1.2$ kips, $L = 24$ in., and $E = 10.6 \times 10^6$ psi.

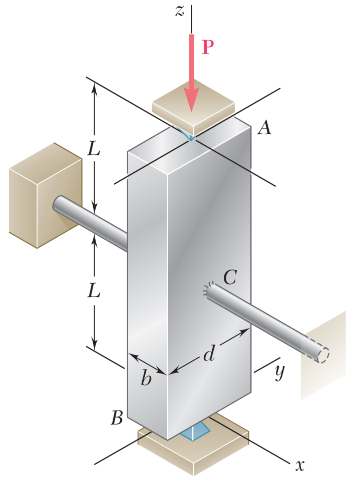


Fig. *P16.21* and P16.22

16.22 The aluminum column ABC has a uniform rectangular cross section with $b = \frac{1}{2}$ in. and $d = \frac{7}{8}$ in. The column is braced in the xz plane at its midpoint C and carries a centric load **P** of magnitude 1.1 kips. Knowing that a factor of safety of 2.5 is required, determine the largest allowable length L. Use $E = 10.6 \times 10^6$ psi.

16.23 A W8 × 21 rolled-steel shape is used with the support and cable arrangement shown. Cables BC and BD are taut and prevent motion of point B in the xz plane. Knowing that $L = 24$ ft, determine the allowable centric load **P** if a factor of safety of 2.2 is required. Use $E = 29 \times 10^6$ psi.

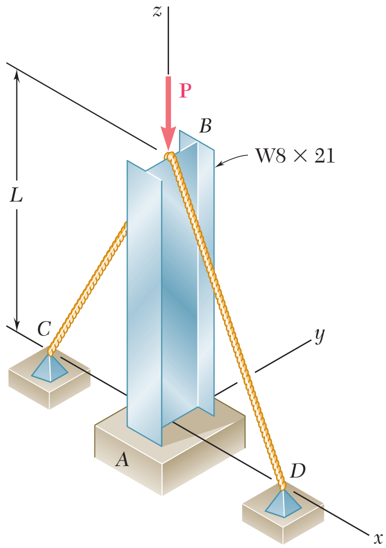


Fig. P16.23

16.24 Two columns are used to support a block weighing 3.25 kips in each of the four ways shown. (*a*) Knowing that the column of Fig. (1) is made of steel with a 1.25-in. diameter, determine the factor of safety with respect to buckling for the loading shown. (*b*) Determine the diameter of each of the other columns for which the factor of safety is the same as the factor of safety obtained in part *a*. Use $E = 29 \times 10^6$ psi.

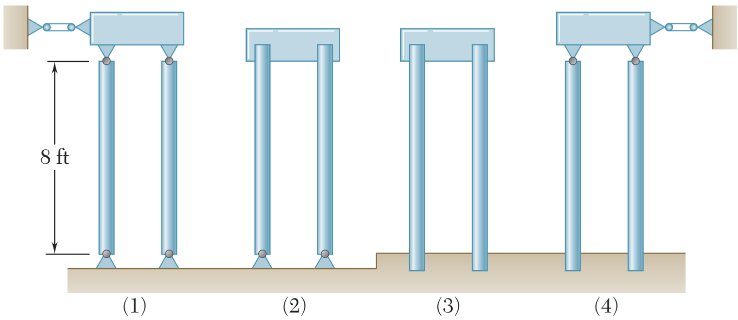


Fig. P16.24

*16.5 DESIGN OF COLUMNS UNDER A CENTRIC LOAD

In the preceding sections, we have determined the critical load of a column by using Euler's formula. We assumed that all stresses remained below the proportional limit and that the column was initially a straight homogeneous prism. Real columns fall short of such an idealization, and in practice the design of columns is based on empirical formulas that reflect the results of numerous laboratory tests.

Over the last century, many steel columns have been tested by applying to them a centric axial load and increasing the load until failure occurred. The results of such tests are represented in Fig. 16.18 where, for each of many tests, a point has been plotted with its ordinate equal to the normal stress σ_{cr} at failure, and its abscissa equal to the corresponding value of the effective slenderness ratio, L_e/r. Although there is considerable scatter in the test results, regions corresponding to three types of failure can be observed. For long columns, where L_e/r is large, failure is closely predicted by Euler's formula, and the value of σ_{cr} is observed to depend on the modulus of elasticity E of the steel used, but not on its yield strength σ_Y. For very short columns and compression blocks, failure occurs essentially as a result of yield, and we have $\sigma_{cr} \approx \sigma_Y$. Columns of intermediate length comprise those cases where failure is dependent on both σ_Y and E. In this range, column failure is an extremely complex phenomenon, and test data have been used extensively to guide the development of specifications and design formulas.

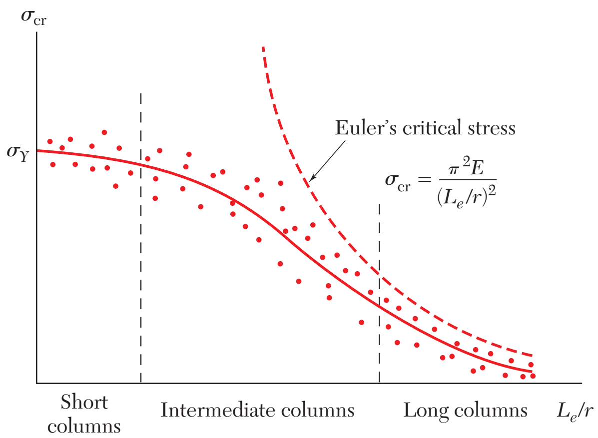


Fig. 16.18

Empirical formulas that express an allowable stress or critical stress in terms of the effective slenderness ratio were first introduced over a century ago and since then have undergone a continuous process of refinement and improvement. Typical empirical formulas

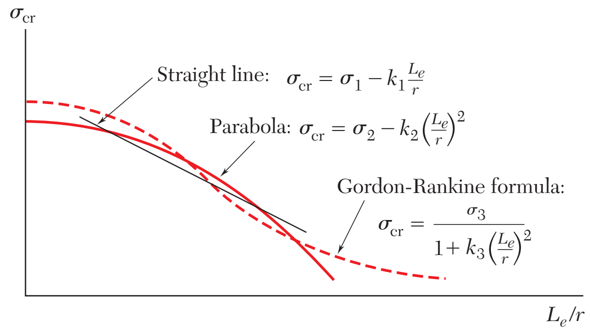


Fig. 16.19

previously used to approximate test data are shown in Fig. 16.19. It is not always feasible to use a single formula for all values of L_e/r. Most design specifications use different formulas, each with a definite range of applicability. In each case we must check that the formula we propose to use is applicable for the value of L_e/r for the column involved. Furthermore, we must determine whether the formula provides the value of the critical stress for the column, in which case we must apply the appropriate factor of safety, or whether it provides directly an allowable stress.

Specific formulas for the design of steel, aluminum, and wood columns under centric loading will now be considered. Photo 16.2 shows examples of columns that would be designed using these formulas. The design for the three different materials using *Allowable Stress Design* is shown in this section.†

(*a*) (*b*)

Photo 16.2 The water tank in (*a*) is supported by steel columns and the building in construction in (*b*) is framed with wood columns.

†In specific design formulas, the letter L will always refer to the effective length of the column.

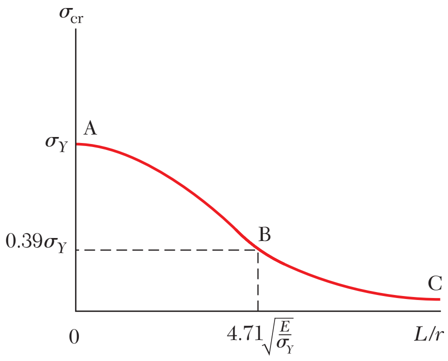


Fig. 16.20

Structural Steel. The formulas most widely used for the allowable stress design of steel columns under a centric load are found in the Specification for Structural Steel Buildings of the American Institute of Steel Construction (AISC).† As we shall see, an exponential expression is used to predict σ_{all} for columns of short and intermediate lengths, and an Euler-based relation is used for long columns. The design relations are developed in two steps:

1. First a curve representing the variation of σ_{cr} with L/r is obtained (Fig. 16.20). It is important to note that this curve does not incorporate any factor of safety.‡ The portion AB of this curve is defined by the equation

$$\sigma_{cr} = [0.658^{(\sigma_Y/\sigma_e)}]\sigma_Y \tag{16.22}$$

where

$$\sigma_e = \frac{\pi^2 E}{(L/r)^2} \tag{16.23}$$

The portion BC is defined by the equation

$$\sigma_{cr} = 0.877\sigma_e \tag{16.24}$$

We note that when $L/r = 0$, $\sigma_{cr} = \sigma_Y$ in Eq. (16.22). At point B, Eq. (16.22) joins Eq. (16.24). The value of slenderness L/r at the junction between the two equations is

$$\frac{L}{r} = 4.71\sqrt{\frac{E}{\sigma_Y}} \tag{16.25}$$

If L/r is smaller than the value in Eq. (16.25), σ_{cr} is determined from Eq. (16.22), and if L/r is greater, σ_{cr} is determined from Eq. (16.24). At the value of the slenderness L/r specified in Eq. (16.25), the stress $\sigma_e = 0.44\ \sigma_Y$. Using Eq. (16.24), $\sigma_{cr} = 0.877\ (0.44\ \sigma_Y) = 0.39\ \sigma_Y$.

2. A factor of safety must be introduced to obtain the final AISC design formulas. The factor of safety specified by the specification is 1.67. Thus,

$$\sigma_{all} = \frac{\sigma_{cr}}{1.67} \tag{16.26}$$

The formulas obtained can be used with SI or U.S. customary units.

We observe that, by using Eqs. (16.22), (16.24), (16.25), and (16.26), we can determine the allowable axial stress for a given grade of steel and any given value of L/r. The procedure is to first compute the value of L/r at the intersection between the two equations from Eq. (16.25). For given values of L/r smaller than that in Eq. (16.25), we use Eqs. (16.22) and (16.26) to calculate σ_{all}, and for values greater than that in Eq. (16.25), we use Eqs. (16.24) and (16.26) to calculate σ_{all}. Figure 16.21 provides a general illustration of how σ_e varies as a function of L/r for different grades of structural steel.

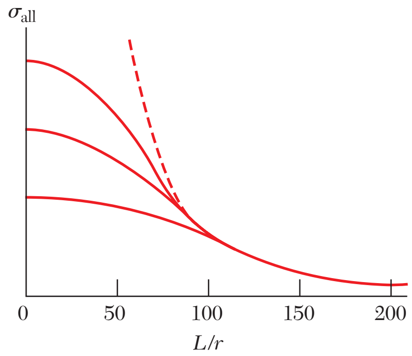


Fig. 16.21

†*Manual of Steel Construction*, 13th ed., American Institute of Steel Construction, Chicago, 2005.

‡In the *Specification for Structural Steel for Buildings*, the symbol F is used for stresses.

EXAMPLE 16.2 Determine the longest unsupported length L for which the S100 × 11.5 rolled-steel compression member AB can safely carry the centric load shown (Fig. 16.22). Assume $\sigma_Y = 250$ MPa and $E = 200$ GPa.

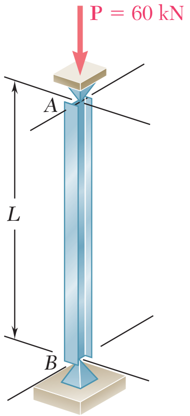


Fig. 16.22

From App. C we find that, for an S100 × 11.5 shape,

$$A = 1460 \text{ mm}^2 \qquad r_x = 41.7 \text{ mm} \qquad r_y = 14.6 \text{ mm}$$

If the 60-kN load is to be safely supported, we must have

$$\sigma_{\text{all}} = \frac{P}{A} = \frac{60 \times 10^3 \text{ N}}{1460 \times 10^{-6} \text{ m}^2} = 41.1 \times 10^6 \text{ Pa}$$

We must compute the critical stress σ_{cr}. Assuming L/r is larger than the slenderness specified by Eq. (16.25), we use Eq. (16.24) with (16.23) and write

$$\sigma_{\text{cr}} = 0.877\, \sigma_e = 0.877 \frac{\pi^2 E}{(L/r)^2}$$

$$= 0.877 \frac{\pi^2 (200 \times 10^9 \text{ Pa})}{(L/r)^2} = \frac{1.731 \times 10^{12} \text{ Pa}}{(L/r)^2}$$

Using this expression in Eq. (16.26) for σ_{all}, we write

$$\sigma_{\text{all}} = \frac{\sigma_{\text{cr}}}{1.67} = \frac{1.037 \times 10^{12} \text{ Pa}}{(L/r)^2}$$

Equating this expression to the required value of σ_{all}, we write

$$\frac{1.037 \times 10^{12} \text{ Pa}}{(L/r)^2} = 1.41 \times 10^6 \text{ Pa} \qquad L/r = 158.8$$

The slenderness ratio from Eq. (16.25) is

$$\frac{L}{r} = 4.71 \sqrt{\frac{200 \times 10^9}{250 \times 10^6}} = 133.2$$

Our assumption that L/r is greater than this slenderness ratio was correct. Choosing the smaller of the two radii of gyration, we have

$$\frac{L}{r_y} = \frac{L}{14.6 \times 10^{-3} \text{ m}} = 158.8 \qquad L = 2.32 \text{ m} \;\blacksquare$$

Aluminum. Many aluminum alloys are available for use in structural and machine construction. For most columns the specifications of the Aluminum Association† provide two formulas for the allowable stress in columns under centric loading. The variation of σ_{all} with L/r defined by these formulas is shown in Fig. 16.23. We note that for short columns a linear relation between σ_{all} with L/r is used and for long columns an Euler-type formula is used. Specific formulas for use in the design of buildings and similar structures are given below in both SI and U.S. customary units for two commonly used alloys.

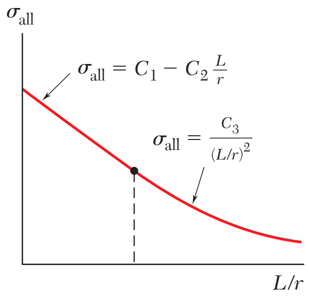


Fig. 16.23

†*Specifications and Guidelines for Aluminum Structures,* Aluminum Association, Inc., Washington D.C., 2005.

Alloy 6061-T6:

$$L/r < 66: \qquad \sigma_{\text{all}} = [20.2 - 0.126(L/r)] \text{ ksi} \tag{16.27}$$
$$= [139 - 0.868(L/r)] \text{ MPa} \tag{16.27$'$}$$
$$L/r \geq 66: \qquad \sigma_{\text{all}} = \frac{51{,}000 \text{ ksi}}{(L/r)^2} = \frac{351 \times 10^3 \text{ MPa}}{(L/r)^2} \tag{16.28}$$

Alloy 2014-T6:

$$L/r < 55: \qquad \sigma_{\text{all}} = [30.7 - 0.23(L/r)] \text{ ksi} \tag{16.29}$$
$$= [212 - 1.585(L/r)] \text{ MPa} \tag{16.29$'$}$$
$$L/r \geq 55: \qquad \sigma_{\text{all}} = \frac{54{,}000 \text{ ksi}}{(L/r)^2} = \frac{372 \times 10^3 \text{ MPa}}{(L/r)^2} \tag{16.30}$$

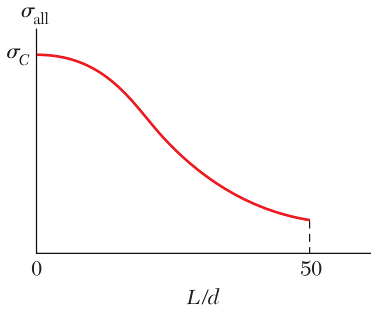


Fig. 16.24

Wood. For the design of wood columns the specifications of the American Forest & Paper Association† provides a single equation that can be used to obtain the allowable stress for short, intermediate, and long columns under centric loading. For a column with a *rectangular* cross section of sides b and d, where $d < b$, the variation of σ_{all} with L/d is shown in Fig. 16.24.

For solid columns made from a single piece of wood or made by gluing laminations together, the allowable stress σ_{all} is

$$\sigma_{\text{all}} = \sigma_C C_P \tag{16.31}$$

where σ_C is the adjusted allowable stress for compression parallel to the grain.‡ Adjustments used to obtain σ_C are included in the specifications to account for different variations, such as in the load duration. The column stability factor C_P accounts for the column length and is defined by the following equation:

$$C_P = \frac{1 + (\sigma_{CE}/\sigma_C)}{2c} - \sqrt{\left[\frac{1 + (\sigma_{CE}/\sigma_C)}{2c}\right]^2 - \frac{\sigma_{CE}/\sigma_C}{c}} \tag{16.32}$$

The parameter c accounts for the type of column, and it is equal to 0.8 for sawn lumber columns and 0.90 for glued laminated wood columns. The value of σ_{CE} is defined as

$$\sigma_{CE} = \frac{0.822E}{(L/d)^2} \tag{16.33}$$

Where E is an adjusted modulus of elasticity for column buckling. Columns in which L/d exceeds 50 are not permitted by the *National Design Specification for Wood Construction*.

†*National Design Specification for Wood Construction*, American Forest & Paper Association, American Wood Council, Washington, D.C., 2005.

‡In the *National Design Specification for Wood Construction*, the symbol F is used for stresses.

EXAMPLE 16.3 Knowing that column AB (Fig. 16.25) has an effective length of 14 ft, and that it must safely carry a 32-kip load, design the column using a square glued laminated cross section. The adjusted modulus of elasticity for the wood is $E = 800 \times 10^3$ psi, and the adjusted allowable stress for compression parallel to the grain is $\sigma_C = 1060$ psi.

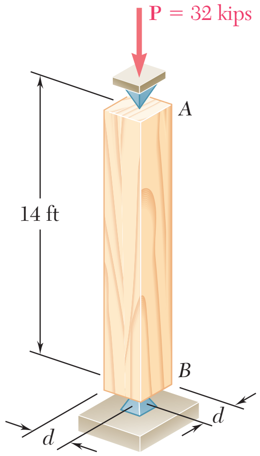


Fig. 16.25

We note that $c = 0.90$ for glued laminated wood columns. We must compute the value of σ_{CE}. Using Eq. (16.33) we write

$$\sigma_{CE} = \frac{0.822E}{(L/d)^2} = \frac{0.822(800 \times 10^3 \text{ psi})}{(168 \text{ in.}/d)^2} = 23.299d^2 \text{ psi}$$

We then use Eq. (16.32) to express the column stability factor in terms of d, with $(\sigma_{CE}/\sigma_C) = (23.299d^2/1.060 \times 10^3) = 21.98 \times 10^{-3}\, d^2$,

$$C_P = \frac{1 + (\sigma_{CE}/\sigma_C)}{2c} - \sqrt{\left[\frac{1 + (\sigma_{CE}/\sigma_C)}{2c}\right]^2 - \frac{\sigma_{CE}/\sigma_C}{c}}$$

$$= \frac{1 + 21.98 \times 10^{-3}\, d^2}{2(0.90)} - \sqrt{\left[\frac{1 + 21.98 \times 10^{-3}\, d^2}{2(0.90)}\right]^2 - \frac{21.98 \times 10^{-3}\, d^2}{0.90}}$$

Since the column must carry 32 kips, which is equal to $\sigma_C d^2$, we use Eq. (16.31) to write

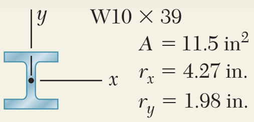
$$\sigma_{\text{all}} = \frac{32 \text{ kips}}{d^2} = \sigma_C C_P = 1.060 C_P$$

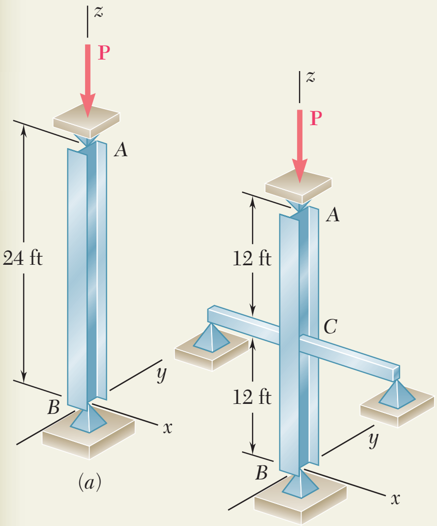
Solving this equation for C_P and substituting the value obtained into the previous equation, we write

$$\frac{30.19}{d^2} = \frac{1 + 21.98 \times 10^{-3}\, d^2}{2(0.90)} - \sqrt{\left[\frac{1 + 21.98 \times 10^{-3}\, d^2}{2(0.90)}\right]^2 - \frac{21.98 \times 10^{-3}\, d^2}{0.90}}$$

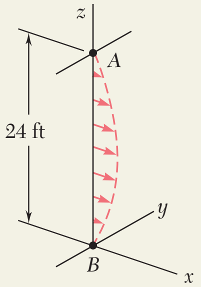
Solving for d by trial and error yields $d = 6.45$ in. ■

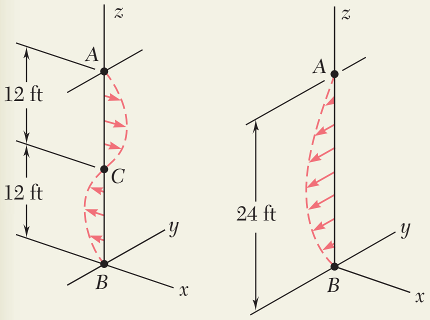
Note: The design formulas presented in this section are intended to provide examples of different design approaches. These formulas do not provide all the requirements that are needed for many designs, and the student should refer to the appropriate design specifications before attempting actual designs.





(a) (b)





Buckling in xz plane Buckling in yz plane

SAMPLE PROBLEM 16.2

Column AB consists of a W10 × 39 rolled-steel shape made of a grade of steel for which $\sigma_Y = 36$ ksi and $E = 29 \times 10^6$ psi. Determine the allowable centric load $\mathbf{P}$ (a) if the effective length of the column is 24 ft in all directions, (b) if bracing is provided to prevent the movement of the midpoint C in the xz plane. (Assume that the movement of point C in the yz plane is not affected by the bracing.)

SOLUTION

We first compute the value of the slenderness ratio from Eq. 16.25 corresponding to the given yield strength $\sigma_Y = 36$ ksi.

$$\frac{L}{r} = 4.71\sqrt{\frac{29 \times 10^6}{36 \times 10^3}} = 133.7$$

a. Effective Length = 24 ft. Since $r_y < r_x$, buckling will take place in the xz plane. For $L = 24$ ft and $r = r_y = 1.98$ in., the slenderness ratio is

$$\frac{L}{r_y} = \frac{(24 \times 12)\text{ in.}}{1.98\text{ in.}} = \frac{288\text{ in.}}{1.98\text{ in.}} = 145.5$$

Since $L/r > 133.7$, we use Eq. (16.23) in Eq. (16.24) to determine σ_{cr}

$$\sigma_{cr} = 0.877\,\sigma_e = 0.877\,\frac{\pi^2 E}{(L/r)^2} = 0.877\,\frac{\pi^2(29 \times 10^3\text{ ksi})}{(145.5)^2} = 11.86\text{ ksi}$$

The allowable stress, determined using Eq. (16.26), and P_{all} are

$$\sigma_{\text{all}} = \frac{\sigma_{cr}}{1.67} = \frac{11.86\text{ ksi}}{1.67} = 7.10\text{ ksi}$$

$$P_{\text{all}} = \sigma_{\text{all}}A = (7.10\text{ ksi})(11.5\text{ in}^2) = 81.7\text{ kips} \quad ◀$$

b. Bracing at Midpoint C. Since bracing prevents movement of point C in the xz plane but not in the yz plane, we must compute the slenderness ratio corresponding to buckling in each plane and determine which is larger.

***xz* Plane:** Effective length = 12 ft = 144 in., $r = r_y = 1.98$ in.
$L/r = (144\text{ in.})/(1.98\text{ in.}) = 72.7$

***yz* Plane:** Effective length = 24 ft = 288 in., $r = r_x = 4.27$ in.
$L/r = (288\text{ in.})/(4.27\text{ in.}) = 67.4$

Since the larger slenderness ratio corresponds to a smaller allowable load, we choose $L/r = 72.7$. Since this is smaller than $L/r = 145.5$, we use Eqs. (16.23) and (16.22) to determine σ_{cr}

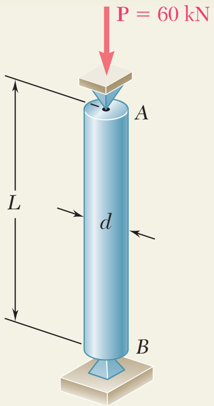
$$\sigma_e = \frac{\pi^2 E}{(L/r)^2} = \frac{\pi^2(29 \times 10^3\text{ ksi})}{(72.7)^2} = 54.1\text{ ksi}$$

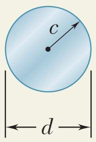
$$\sigma_{\text{cr}} = \left[0.658^{(\sigma_Y/\sigma_e)}\right] F_Y = \left[0.658^{(36\text{ ksi}/54.1\text{ ksi})}\right] 36\text{ ksi} = 27.3\text{ ksi}$$

We now calculate the allowable stress using Eq. (16.26) and the allowable load.

$$\sigma_{\text{all}} = \frac{\sigma_{cr}}{1.67} = \frac{27.3\text{ ksi}}{1.67} = 16.32\text{ ksi}$$

$$P_{\text{all}} = \sigma_{\text{all}}A = (16.32\text{ ksi})(11.5\text{ in}^2) \qquad P_{\text{all}} = 187.7\text{ ksi} \quad ◀$$





SAMPLE PROBLEM 16.3

Using the aluminum alloy 2014-T6, determine the smallest diameter rod which can be used to support the centric load $P = 60$ kN if (*a*) $L = 750$ mm, (*b*) $L = 300$ mm.

SOLUTION

For the cross section of a solid circular rod, we have

$$I = \frac{\pi}{4}c^4 \qquad A = \pi c^2 \qquad r = \sqrt{\frac{I}{A}} = \sqrt{\frac{\pi c^4/4}{\pi c^2}} = \frac{c}{2}$$

a. Length of 750 mm. Since the diameter of the rod is not known, a value of L/r must be assumed; we *assume* that $L/r > 55$ and use Eq. (16.30). For the centric load **P**, we have $\sigma = P/A$ and write

$$\frac{P}{A} = \sigma_{\text{all}} = \frac{372 \times 10^3 \text{ MPa}}{(L/r)^2}$$

$$\frac{60 \times 10^3 \text{ N}}{\pi c^2} = \frac{372 \times 10^9 \text{ Pa}}{\left(\dfrac{0.750 \text{ m}}{c/2}\right)^2}$$

$$c^4 = 115.5 \times 10^{-9} \text{ m}^4 \qquad c = 18.44 \text{ mm}$$

For $c = 18.44$ mm, the slenderness ratio is

$$\frac{L}{r} = \frac{L}{c/2} = \frac{750 \text{ mm}}{(18.44 \text{ mm})/2} = 81.3 > 55$$

Our assumption is correct, and for $L = 750$ mm, the required diameter is

$$d = 2c = 2(18.44 \text{ mm}) \qquad d = 36.9 \text{ mm} \quad ◀$$

b. Length of 300 mm. We again *assume* that $L/r > 55$. Using Eq. (16.30), and following the procedure used in part *a*, we find that $c = 11.66$ mm and $L/r = 51.5$. Since L/r is less than 55, our assumption is wrong; we now assume that $L/r < 55$ and use Eq. (16.29′) for the design of this rod.

$$\frac{P}{A} = \sigma_{\text{all}} = \left[212 - 1.585\left(\frac{L}{r}\right)\right] \text{MPa}$$

$$\frac{60 \times 10^3 \text{ N}}{\pi c^2} = \left[212 - 1.585\left(\frac{0.3 \text{ m}}{c/2}\right)\right] 10^6 \text{ Pa}$$

$$c = 12.00 \text{ mm}$$

For $c = 12.00$ mm, the slenderness ratio is

$$\frac{L}{r} = \frac{L}{c/2} = \frac{300 \text{ mm}}{(12.00 \text{ mm})/2} = 50$$

Our second assumption that $L/r < 55$ is correct. For $L = 300$ mm, the required diameter is

$$d = 2c = 2(12.00 \text{ mm}) \qquad d = 24.0 \text{ mm} \quad ◀$$

PROBLEMS

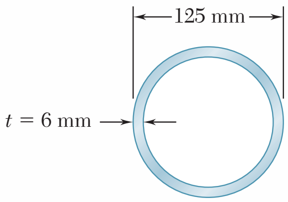


Fig. P16.25

16.25 A steel pipe having the cross section shown is used as a column. Using the AISC allowable stress design formulas, determine the allowable centric load if the effective length of the column is (*a*) 6 m, (*b*) 4 m. Use $\sigma_Y = 250$ MPa and $E = 200$ GPa.

16.26 A column with the cross section shown has a 13.5-ft effective length. Using AISC allowable stress design, determine the largest centric load that can be applied to the column. Use $\sigma_Y = 36$ ksi and $E = 29 \times 10^6$ psi.

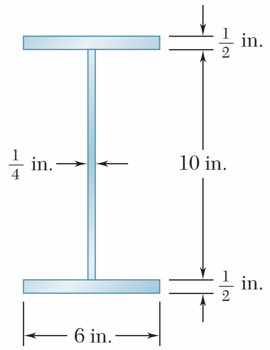


Fig. P16.26

16.27 Using allowable stress design, determine the allowable centric load for a column of 6-m effective length that is made from the following rolled-steel shape: (*a*) W200 × 35.9, (*b*) W200 × 86. Use $\sigma_Y = 250$ MPa and $E = 200$ GPa.

16.28 A W8 × 31 rolled-steel shape is used for a column of 21-ft effective length. Using allowable stress design, determine the allowable centric load if the yield strength of the grade of steel used is (*a*) $\sigma_Y = 36$ ksi, (*b*) $\sigma_Y = 50$ ksi. Use $E = 29 \times 10^6$ psi.

16.29 A column having a 3.5-m effective length is made of sawn lumber with a 114 × 140-mm cross section. Knowing that for the grade of wood used the adjusted allowable stress for compression parallel to the grain is $\sigma_C = 7.6$ MPa and the adjusted modulus $E = 2.8$ GPa, determine the maximum allowable centric load for the column.

16.30 A sawn lumber column with a 7.5 × 5.5-in. cross section has an 18-ft effective length. Knowing that for the grade of wood used the adjusted allowable stress for compression parallel to the grain is $\sigma_C = 1200$ psi and that the adjusted modulus $E = 470 \times 10^3$ psi, determine the maximum allowable centric load for the column.

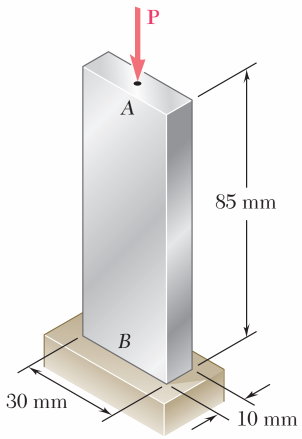


Fig. P16.31

16.31 Bar AB is free at its end A and fixed at its base B. Determine the allowable centric load **P** if the aluminum alloy is (*a*) 6061-T6, (*b*) 2014-T6.

16.32 A compression member has the cross section shown and an effective length of 5 ft. Knowing that the aluminum alloy used is 6061-T6, determine the allowable centric load.

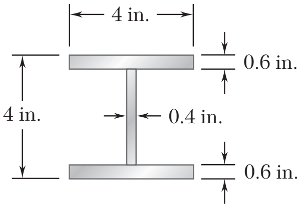


Fig. P16.32

16.33 and 16.34 A compression member of 9-m effective length is obtained by welding two 10-mm-thick steel plates to a W250 × 80 rolled-steel shape as shown. Knowing that σ_Y = 345 MPa and E = 200 GPa and using allowable stress design, determine the allowable centric load for the compression member.

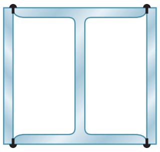


Fig. P16.33

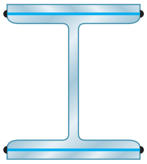


Fig. P16.34

16.35 A compression member of 2.3-m effective length is obtained by bolting together two L127 × 76 × 12.7-mm steel angles as shown. Using allowable stress design, determine the allowable centric load for the column. Use σ_Y = 250 MPa and E = 200 GPa.

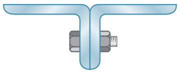


Fig. *P16.35*

16.36 A column of 21-ft effective length is obtained by connecting two C10 × 20 steel channels with lacing bars as shown. Using allowable stress design, determine the allowable centric load for the column. Use σ_Y = 36 ksi and $E = 29 \times 10^6$ psi.

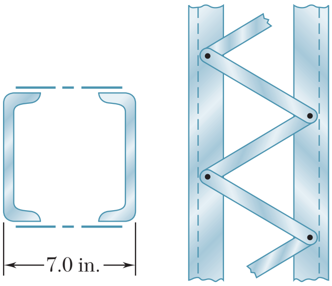


Fig. P16.36

16.37 A rectangular column with a 4.4-m effective length is made of glued laminated wood. Knowing that for the grade of wood used the adjusted allowable stress for compression parallel to the grain is σ_C = 8.3 MPa and the adjusted modulus E = 4.6 GPa, determine the maximum allowable centric load for the column.

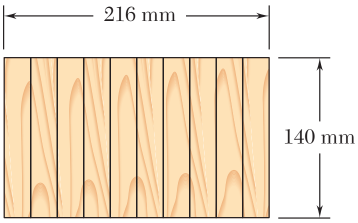


Fig. P16.37

16.38 An aluminum structural tube is reinforced by bolting two plates to it as shown for use as a column of 5.6-ft effective length. Knowing that all the material is aluminum alloy 2014-T6, determine the maximum allowable centric load.

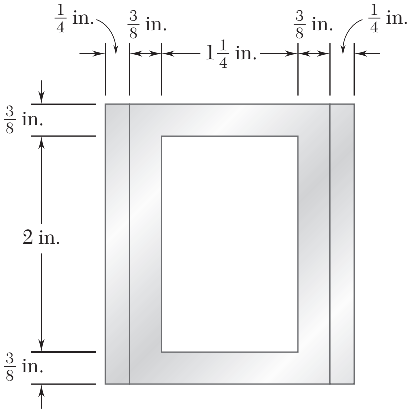


Fig. *P16.38*

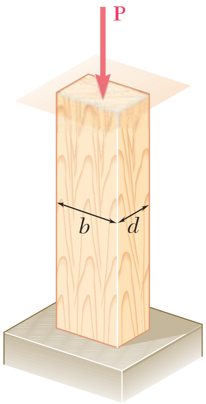


Fig. P16.39

16.39 An 18-kip centric load is applied to a rectangular sawn lumber column of 22-ft effective length. Using sawn lumber for which the adjusted allowable stress for compression parallel to the grain is $\sigma_C = 1050$ psi and the adjusted modulus is $E = 440 \times 10^3$ psi, determine the smallest cross section that can be used. Use $b = 2d$.

16.40 A column of 2.1-m effective length is to be made by gluing together laminated wood pieces of 25×150-mm cross section. Knowing that for the grade of wood used the adjusted allowable stress for compression parallel to the grain is $\sigma_C = 7.7$ MPa and the adjusted modulus is $E = 5.4$ GPa, determine the number of wood pieces that must be used to support the concentric load shown when (*a*) $P = 52$ kN, (*b*) $P = 108$ kN.

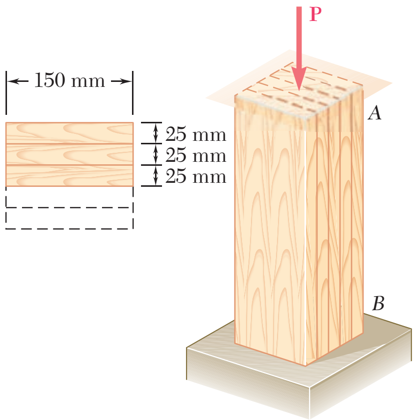


Fig. P16.40

16.41 A 16-kip centric load must be supported by an aluminum column as shown. Using the aluminum alloy 6061-T6, determine the minimum dimension b that can be used.

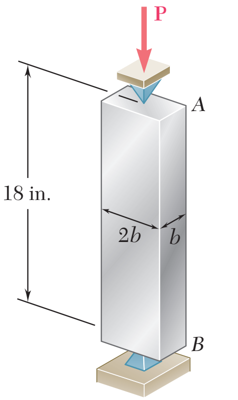


Fig. P16.41

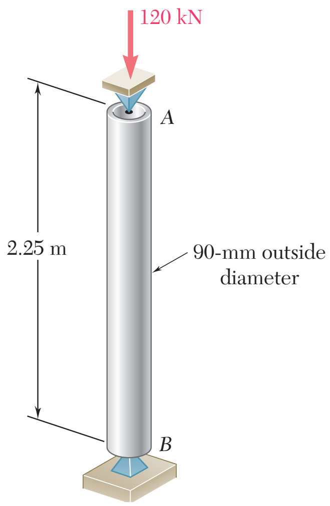


Fig. P16.42

16.42 An aluminum tube of 90-mm outer diameter is used to carry a centric load of 120 kN. Knowing that the stock of tubes available for use are made of alloy 2014-T6 and with wall thicknesses in increments of 3 mm from 6 mm to 15 mm, determine the lightest tube that can be used.

16.43 A centric load **P** must be supported by the steel bar AB. Using allowable stress design, determine the smallest dimension d of the cross section that can be used when (*a*) $P = 24$ kips, (*b*) $P = 36$ kips. Use $\sigma_Y = 36$ ksi and $E = 29 \times 10^6$ psi.

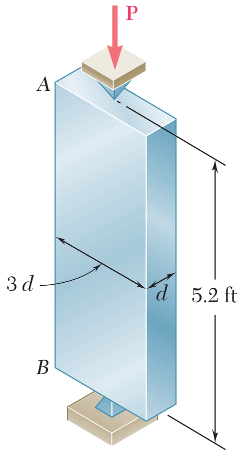


Fig. P16.43

16.44 A column of 4.5-m effective length must carry a centric load of 900 kN. Knowing that $\sigma_Y = 345$ MPa and $E = 200$ GPa, use allowable stress design to select the steel wide-flange shape of 250-mm nominal depth that should be used.

16.45 A column of 22.5-ft effective length must carry a centric load of 288 kips. Using allowable stress design, select the steel wide-flange shape of 14-in. nominal depth that should be used. Use $\sigma_Y = 50$ ksi and $E = 29 \times 10^6$ psi.

16.46 A column of 4.6-m effective length must carry a centric load of 525 kN. Knowing that $\sigma_Y = 345$ MPa and $E = 200$ GPa, use allowable stress design to select the steel wide-flange shape of 200-mm nominal depth that should be used.

16.47 A square steel tube having the cross section shown is used as a column of 26-ft effective length to carry a centric load of 65 kips. Knowing that the tubes available for use are made with wall thicknesses ranging from $\frac{1}{4}$ to $\frac{3}{4}$ in. in increments of $\frac{1}{16}$ in., use allowable stress design to determine the lightest tube that can be used. Use $\sigma_Y = 36$ ksi and $E = 29 \times 10^6$ psi.

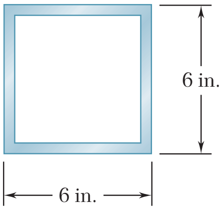


Fig. P16.47

16.48 Two $3\frac{1}{2} \times 2\frac{1}{2}$-in. steel angles are bolted together as shown for use as a column of 6-ft effective length to carry a centric load of 54 kips. Knowing that the angles available have thicknesses of $\frac{1}{4}$, $\frac{3}{8}$, and $\frac{1}{2}$ in., use allowable stress design to determine the lightest angles that can be used. Use $\sigma_Y = 36$ ksi and $E = 29 \times 10^6$ psi.

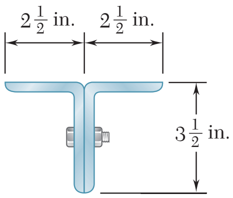


Fig. P16.48

REVIEW AND SUMMARY

This chapter was devoted to the design and analysis of columns, i.e., prismatic members supporting axial loads. In order to gain insight into the behavior of columns, we first considered in Sec. 16.2 the equilibrium of a simple model and found that for values of the load P exceeding a certain value P_{cr}, called the *critical load,* two equilibrium positions of the model were possible: the original position with zero transverse deflections and a second position involving deflections that could be quite large. This led us to conclude that the first equilibrium position was unstable for $P > P_{cr}$, and stable for $P < P_{cr}$, since in the latter case it was the only possible equilibrium position.

Critical load

In Sec. 16.3, we considered a pin-ended column of length L and of constant flexural rigidity EI subjected to an axial centric load P. Assuming that the column had buckled (Fig. 16.26), we noted that the bending moment at point Q was equal to $-Py$ and wrote

$$\frac{d^2y}{dx^2} = \frac{M}{EI} = -\frac{P}{EI}y \qquad (16.4)$$

Solving this differential equation, subject to the boundary conditions corresponding to a pin-ended column, we determined the smallest load P for which buckling can take place. This load, known as the *critical load* and denoted by P_{cr}, is given by *Euler's formula:*

Euler's formula

$$P_{cr} = \frac{\pi^2 EI}{L^2} \qquad (16.11)$$

where L is the length of the column. For this load or any larger load, the equilibrium of the column is unstable and transverse deflections will occur.

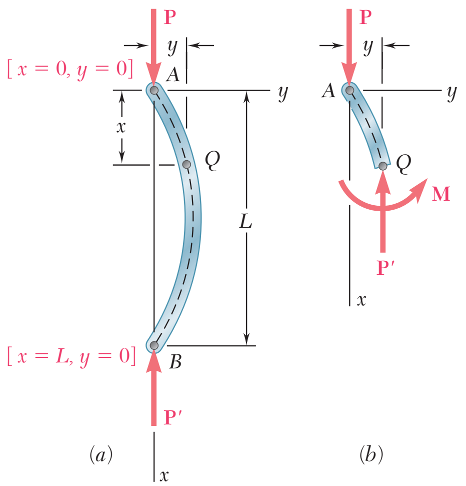


Fig. 16.26

Denoting the cross-sectional area of the column by A and its radius of gyration by r, we determined the critical stress σ_{cr} corresponding to the critical load P_{cr}:

$$\sigma_{cr} = \frac{\pi^2 E}{(L/r)^2} \tag{16.13}$$

Slenderness ratio

The quantity L/r is called the *slenderness ratio* and we plotted σ_{cr} as a function of L/r (Fig. 16.27). Since our analysis was based on stresses remaining below the yield strength of the material, we noted that the column would fail by yielding when $\sigma_{cr} > \sigma_Y$.

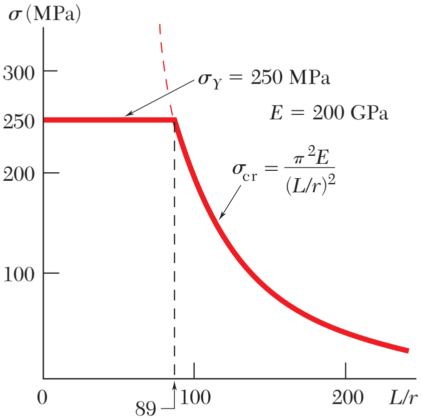


Fig. 16.27

In Sec. 16.4, we discussed the critical load of columns with various end conditions and wrote

$$P_{cr} = \frac{\pi^2 EI}{L_e^2} \tag{16.11$'$}$$

Effective length

where L_e is the *effective length* of the column, i.e., the length of an equivalent pin-ended column. The effective lengths of several columns with various end conditions were calculated and shown in Fig. 16.17 on page 651.

Design of real columns

Centrically loaded columns

In the first part of the chapter we considered each column as a straight homogeneous prism. Since imperfections exist in all real columns, the *design of real columns* is done by using empirical formulas based on laboratory tests and set forth in specifications and codes issued by professional organizations. In Sec. 16.5, we discussed the design of *centrically loaded columns* made of steel, aluminum, and wood. For each material, the design of the column was based on formulas expressing the allowable stress as a function of the slenderness ratio L/r of the column.

REVIEW PROBLEMS

16.49 A column of 3.5-m effective length is made by welding together two L89 × 64 × 6.4-mm angles as shown. Using Euler's formula with $E = 200$ GPa, determine the allowable centric load if a factor of safety of 2.8 is required.

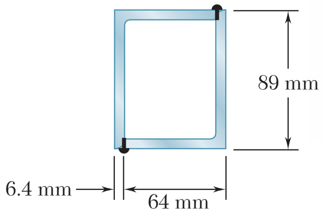


Fig. P16.49

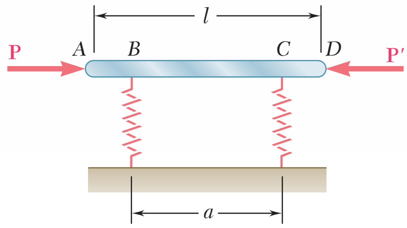


Fig. P16.50

16.50 A rigid bar *AD* is attached to two springs of constant k and is in equilibrium in the position shown. Knowing that the equal and opposite loads **P** and **P**′ *remain horizontal,* determine the magnitude P_{cr} of the critical load for the system.

16.51 The steel rod *BC* is attached to the rigid bar *AB* and to the fixed support at *C*. Knowing that $G = 11.2 \times 10^6$ psi, determine the critical load P_{cr} of the system when $d = \frac{1}{2}$ in.

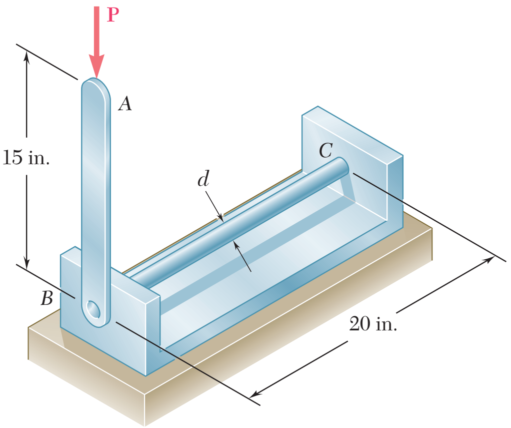


Fig. *P16.51* and P16.52

16.52 The steel rod *BC* is attached to the rigid bar *AB* and to the fixed support at *C*. Knowing that $G = 11.2 \times 10^6$ psi, determine the diameter of the rod *BC* for which the critical P_{cr} of the system is 80 lb.

16.53 Supports A and B of the pin-ended column shown are at a fixed distance L from each other. Knowing that at a temperature T_0 the force in the column is zero and that buckling occurs when the temperature is $T_1 = T_0 + \Delta T$, express ΔT in terms of b, L, and the coefficient of thermal expansion α.

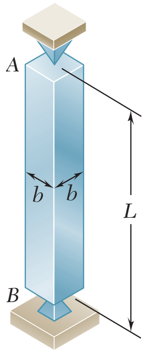


Fig. P16.53

16.54 Member AB consists of a single C130 × 10.4 steel channel of length 2.5 m. Knowing that the pins at A and B pass through the centroid of the cross section of the channel, determine the factor of safety for the load shown with respect to buckling in the plane of the figure when $\theta = 30°$. Use Euler's formula with $E = 200$ GPa.

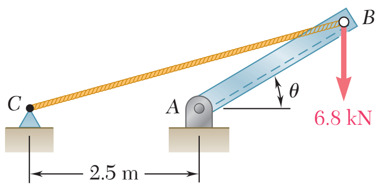


Fig. *P16.54*

16.55 (a) Considering only buckling in the plane of the structure shown and using Euler's formula, determine the value of θ between 0 and 90° for which the allowable magnitude of the load **P** is maximum. (b) Determine the corresponding maximum value of P knowing that a factor of safety of 3.2 is required. Use $E = 29 \times 10^6$ psi.

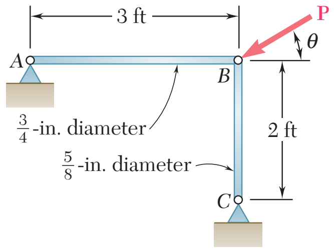


Fig. *P16.55*

16.56 Knowing that a factor of safety of 2.6 is required, determine the largest load **P** that can be applied to the structure shown using Euler's formula. Use $E = 200$ GPa and consider only buckling in the plane of the structure.

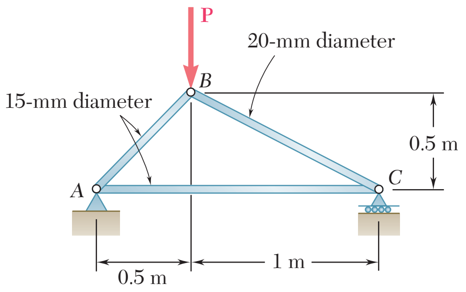


Fig. P16.56

16.57 Determine (*a*) the critical load for the brass strut, (*b*) the dimension d for which the aluminum strut will have the same critical load, (*c*) the weight of the aluminum strut as a percent of the weight of the brass strut.

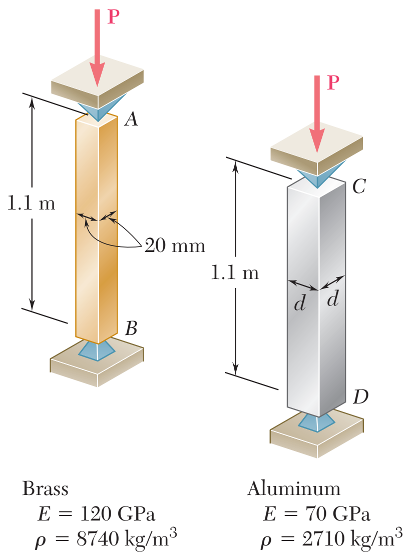


Fig. *P16.57*

16.58 A compression member has the cross section shown and an effective length of 5 ft. Knowing that the aluminum alloy used is 2014-T6, use Sec. 16.5 to determine the allowable centric load.

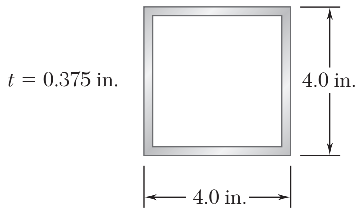


Fig. P16.58

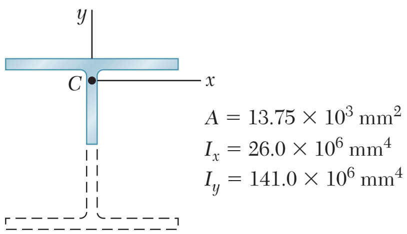


Fig. P16.59

16.59 A column is made from half of a W360 × 216 rolled-steel shape with the geometric properties as shown. Using allowable stress design in Sec. 16.5, determine the allowable centric load if the effective length of the column is (*a*) 4.0 m, (*b*) 6.5 m. Use σ_Y = 345 MPa and E = 200 GPa.

16.60 A column of 17-ft effective length must carry a centric load of 235 kips. Using allowable stress design in Sec. 16.5, select the steel wide-flange shape of 10-in. nominal depth that should be used. Use σ_Y = 36 ksi and $E = 29 \times 10^6$ psi.

Appendices

†Courtesy of the American Institute of Steel Construction, Chicago, Illinois.

Typical Properties of Selected Materials Used in Engineering[1,5] (U.S. Customary Units)

Material	Specific Weight, lb/in^3	Ultimate Strength			Yield Strength[3]		Modulus of Elasticity, 10^6 psi	Modulus of Rigidity, 10^6 psi	Coefficient of Thermal Expansion, 10^{-6}/°F	Ductility, Percent Elongation in 2 in.
		Tension, ksi	Compression,[2] ksi	Shear, ksi	Tension, ksi	Shear, ksi				
Steel										
Structural (ASTM-A36)	0.284	58			36	21	29	11.2	6.5	21
High-strength-low-alloy										
ASTM-A709 Grade 50	0.284	65			50		29	11.2	6.5	21
ASTM-A913 Grade 65	0.284	80			65		29	11.2	6.5	17
ASTM-A992 Grade 50	0.284	65			50		29	11.2	6.5	21
Quenched & tempered										
ASTM-A709 Grade 100	0.284	110			100		29	11.2	6.5	18
Stainless, AISI 302										
Cold-rolled	0.286	125			75		28	10.8	9.6	12
Annealed	0.286	95			38	22	28	10.8	9.6	50
Reinforcing Steel										
Medium strength	0.283	70			40		29	11	6.5	
High strength	0.283	90			60		29	11	6.5	
Cast Iron										
Gray Cast Iron										
4.5% C, ASTM A-48	0.260	25	95	35			10	4.1	6.7	0.5
Malleable Cast Iron										
2% C, 1% Si,										
ASTM A-47	0.264	50	90	48	33		24	9.3	6.7	10
Aluminum										
Alloy 1100-H14										
(99% Al)	0.098	16		10	14	8	10.1	3.7	13.1	9
Alloy 2014-T6	0.101	66		40	58	33	10.9	3.9	12.8	13
Alloy 2024-T4	0.101	68		41	47		10.6		12.9	19
Alloy 5456-H116	0.095	46		27	33	19	10.4		13.3	16
Alloy 6061-T6	0.098	38		24	35	20	10.1	3.7	13.1	17
Alloy 7075-T6	0.101	83		48	73		10.4	4	13.1	11
Copper										
Oxygen-free copper										
(99.9% Cu)										
Annealed	0.322	32		22	10		17	6.4	9.4	45
Hard-drawn	0.322	57		29	53		17	6.4	9.4	4
Yellow Brass										
(65% Cu, 35% Zn)										
Cold-rolled	0.306	74		43	60	36	15	5.6	11.6	8
Annealed	0.306	46		32	15	9	15	5.6	11.6	65
Red Brass										
(85% Cu, 15% Zn)										
Cold-rolled	0.316	85		46	63		17	6.4	10.4	3
Annealed	0.316	39		31	10		17	6.4	10.4	48
Tin bronze	0.318	45			21		14		10	30
(88 Cu, 8Sn, 4Zn)										
Manganese bronze	0.302	95			48		15		12	20
(63 Cu, 25 Zn, 6 Al, 3 Mn, 3 Fe)										
Aluminum bronze	0.301	90	130		40		16	6.1	9	6
(81 Cu, 4 Ni, 4 Fe, 11 Al)										

(Table continued on page 678)

Material	Density kg/m^3	Ultimate Strength: Tension, MPa	Ultimate Strength: Compression,[2] MPa	Ultimate Strength: Shear, MPa	Yield Strength[3]: Tension, MPa	Yield Strength[3]: Shear, MPa	Modulus of Elasticity, GPa	Modulus of Rigidity, GPa	Coefficient of Thermal Expansion, $10^{-6}/°C$	Ductility, Percent Elongation in 50 mm
Steel										
Structural (ASTM-A36)	7860	400			250	145	200	77.2	11.7	21
High-strength-low-alloy										
ASTM-A709 Grade 345	7860	450			345		200	77.2	11.7	21
ASTM-A913 Grade 450	7860	550			450		200	77.2	11.7	17
ASTM-A992 Grade 345	7860	450			345		200	77.2	11.7	21
Quenched & tempered										
ASTM-A709 Grade 690	7860	760			690		200	77.2	11.7	18
Stainless, AISI 302										
Cold-rolled	7920	860			520		190	75	17.3	12
Annealed	7920	655			260	150	190	75	17.3	50
Reinforcing Steel										
Medium strength	7860	480			275		200	77	11.7	
High strength	7860	620			415		200	77	11.7	
Cast Iron										
Gray Cast Iron										
4.5% C, ASTM A-48	7200	170	655	240			69	28	12.1	0.5
Malleable Cast Iron										
2% C, 1% Si,										
ASTM A-47	7300	345	620	330	230		165	65	12.1	10
Aluminum										
Alloy 1100-H14										
(99% Al)	2710	110		70	95	55	70	26	23.6	9
Alloy 2014-T6	2800	455		275	400	230	75	27	23.0	13
Alloy-2024-T4	2800	470		280	325		73		23.2	19
Alloy-5456-H116	2630	315		185	230	130	72		23.9	16
Alloy 6061-T6	2710	260		165	240	140	70	26	23.6	17
Alloy 7075-T6	2800	570		330	500		72	28	23.6	11
Copper										
Oxygen-free copper										
(99.9% Cu)										
Annealed	8910	220		150	70		120	44	16.9	45
Hard-drawn	8910	390		200	265		120	44	16.9	4
Yellow-Brass										
(65% Cu, 35% Zn)										
Cold-rolled	8470	510		300	410	250	105	39	20.9	8
Annealed	8470	320		220	100	60	105	39	20.9	65
Red Brass										
(85% Cu, 15% Zn)										
Cold-rolled	8740	585		320	435		120	44	18.7	3
Annealed	8740	270		210	70		120	44	18.7	48
Tin bronze	8800	310			145		95		18.0	30
(88 Cu, 8Sn, 4Zn)										
Manganese bronze	8360	655			330		105		21.6	20
(63 Cu, 25 Zn, 6 Al, 3 Mn, 3 Fe)										
Aluminum bronze	8330	620	900		275		110	42	16.2	6
(81 Cu, 4 Ni, 4 Fe, 11 Al)										

(Table continued on page 679)

APPENDIX A Typical Properties of Selected Materials Used in Engineering[1,5] (U.S. Customary Units)

Continued from page 676

Material	Specific Weight, lb/in^3	Ultimate Strength			Yield Strength[3]		Modulus of Elasticity, 10^6 psi	Modulus of Rigidity, 10^6 psi	Coefficient of Thermal Expansion, 10^{-6}/°F	Ductility, Percent Elongation in 2 in.
		Tension, ksi	Compression,[2] ksi	Shear, ksi	Tension, ksi	Shear, ksi				
Magnesium Alloys										
Alloy AZ80 (Forging)	0.065	50		23	36		6.5	2.4	14	6
Alloy AZ31 (Extrusion)	0.064	37		19	29		6.5	2.4	14	12
Titanium										
Alloy (6% Al, 4% V)	0.161	130			120		16.5		5.3	10
Monel Alloy 400(Ni-Cu)										
Cold-worked	0.319	98			85	50	26		7.7	22
Annealed	0.319	80			32	18	26		7.7	46
Cupronickel (90% Cu, 10% Ni)										
Annealed	0.323	53			16		20	7.5	9.5	35
Cold-worked	0.323	85			79		20	7.5	9.5	3
Timber, air dry										
Douglas fir	0.017	15	7.2	1.1			1.9	.1	Varies	
Spruce, Sitka	0.015	8.6	5.6	1.1			1.5	.07	1.7 to 2.5	
Shortleaf pine	0.018		7.3	1.4			1.7			
Western white pine	0.014		5.0	1.0			1.5			
Ponderosa pine	0.015	8.4	5.3	1.1			1.3			
White oak	0.025		7.4	2.0			1.8			
Red oak	0.024		6.8	1.8			1.8			
Western hemlock	0.016	13	7.2	1.3			1.6			
Shagbark hickory	0.026		9.2	2.4			2.2			
Redwood	0.015	9.4	6.1	0.9			1.3			
Concrete										
Medium strength	0.084		4.0				3.6		5.5	
High strength	0.084		6.0				4.5		5.5	
Plastics										
Nylon, type 6/6, (molding compound)	0.0412	11	14		6.5		0.4		80	50
Polycarbonate	0.0433	9.5	12.5		9		0.35		68	110
Polyester, PBT (thermoplastic)	0.0484	8	11		8		0.35		75	150
Polyester elastomer	0.0433	6.5		5.5			0.03			500
Polystyrene	0.0374	8	13		8		0.45		70	2
Vinyl, rigid PVC	0.0520	6	10		6.5		0.45		75	40
Rubber	0.033	2							90	600
Granite (Avg. values)	0.100	3	35	5			10	4	4	
Marble (Avg. values)	0.100	2	18	4			8	3	6	
Sandstone (Avg. values)	0.083	1	12	2			6	2	5	
Glass, 98% silica	0.079		7				9.6	4.1	44	

[1]Properties of metals vary widely as a result of variations in composition, heat treatment, and mechanical working.
[2]For ductile metals the compression strength is generally assumed to be equal to the tension strength.
[3]Offset of 0.2 percent.
[4]Timber properties are for loading parallel to the grain.
[5]See also *Marks' Mechanical Engineering Handbook*, 10th ed., McGraw-Hill, New York, 1996; *Annual Book of ASTM*, American Society for Testing Materials, Philadelphia, Pa.; *Metals Handbook*, American Society for Metals, Metals Park, Ohio; and *Aluminum Design Manual*, The Aluminum Association, Washington, DC.

Continued from page 677

Material	Density kg/m^3	Ultimate Strength: Tension, MPa	Ultimate Strength: Compression,[2] MPa	Ultimate Strength: Shear, MPa	Yield Strength[3]: Tension, MPa	Yield Strength[3]: Shear, MPa	Modulus of Elasticity, GPa	Modulus of Rigidity, GPa	Coefficient of Thermal Expansion, $10^{-6}/°C$	Ductility, Percent Elongation in 50 mm
Magnesium Alloys										
Alloy AZ80 (Forging)	1800	345		160	250		45	16	25.2	6
Alloy AZ31 (Extrusion)	1770	255		130	200		45	16	25.2	12
Titanium										
Alloy (6% Al, 4% V)	4730	900			830		115		9.5	10
Monel Alloy 400(Ni-Cu)										
Cold-worked	8830	675			585	345	180		13.9	22
Annealed	8830	550			220	125	180		13.9	46
Cupronickel (90% Cu, 10% Ni)										
Annealed	8940	365			110		140	52	17.1	35
Cold-worked	8940	585			545		140	52	17.1	3
Timber, air dry										
Douglas fir	470	100	50	7.6			13	0.7	Varies	
Spruce, Sitka	415	60	39	7.6			10	0.5	3.0 to 4.5	
Shortleaf pine	500		50	9.7			12			
Western white pine	390		34	7.0			10			
Ponderosa pine	415	55	36	7.6			9			
White oak	690		51	13.8			12			
Red oak	660		47	12.4			12			
Western hemlock	440	90	50	10.0			11			
Shagbark hickory	720		63	16.5			15			
Redwood	415	65	42	6.2			9			
Concrete										
Medium strength	2320		28				25		9.9	
High strength	2320		40				30		9.9	
Plastics										
Nylon, type 6/6, (molding compound)	1140	75	95		45		2.8		144	50
Polycarbonate	1200	65	85		35		2.4		122	110
Polyester, PBT (thermoplastic)	1340	55	75		55		2.4		135	150
Polyester elastomer	1200	45		40			0.2			500
Polystyrene	1030	55	90		55		3.1		125	2
Vinyl, rigid PVC	1440	40	70		45		3.1		135	40
Rubber	910	15							162	600
Granite (Avg. values)	2770	20	240	35			70	4	7.2	
Marble (Avg. values)	2770	15	125	28			55	3	10.8	
Sandstone (Avg. values)	2300	7	85	14			40	2	9.0	
Glass, 98% silica	2190		50				65	4.1	80	

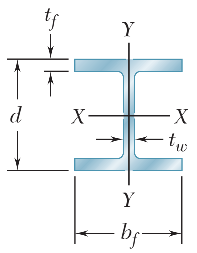
[1] Properties of metals very widely as a result of variations in composition, heat treatment, and mechanical working.
[2] For ductile metals the compression strength is generally assumed to be equal to the tension strength.
[3] Offset of 0.2 percent.
[4] Timber properties are for loading parallel to the grain.
[5] See also *Marks' Mechanical Engineering Handbook,* 10th ed., McGraw-Hill, New York, 1996; *Annual Book of ASTM,* American Society for Testing Materials, Philadelphia, Pa.; *Metals Handbook,* American Society of Metals, Metals Park, Ohio; and *Aluminum Design Manual,* The Aluminum Association, Washington, DC.

APPENDIX B Properties of Rolled-Steel Shapes

(U.S. Customary Units)

W Shapes

(Wide-Flange Shapes)



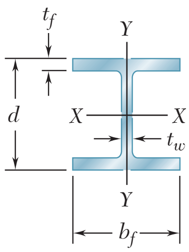
Designation†	Area A, in^2	Depth d, in.	Flange Width b_f, in.	Flange Thickness t_f, in.	Web Thickness t_w, in.	Axis X-X I_x, in^4	Axis X-X S_x, in^3	Axis X-X r_x, in.	Axis Y-Y I_y, in^4	Axis Y-Y S_y, in^3	Axis Y-Y r_y, in.
W36 × 302	88.8	37.3	16.7	1.68	0.945	21100	1130	15.4	1300	156	3.82
135	39.7	35.6	12.0	0.790	0.600	7800	439	14.0	225	37.7	2.38
W33 × 201	59.2	33.7	15.7	1.15	0.715	11600	686	14.0	749	95.2	3.56
118	34.7	32.9	11.5	0.740	0.550	5900	359	13.0	187	32.6	2.32
W30 × 173	51.0	30.4	15.0	1.07	0.655	8230	541	12.7	598	79.8	3.42
99	29.1	29.7	10.50	0.670	0.520	3990	269	11.7	128	24.5	2.10
W27 × 146	43.1	27.4	14.0	0.975	0.605	5660	414	11.5	443	63.5	3.20
84	24.8	26.70	10.0	0.640	0.460	2850	213	10.7	106	21.2	2.07
W24 × 104	30.6	24.1	12.8	0.750	0.500	3100	258	10.1	259	40.7	2.91
68	20.1	23.7	8.97	0.585	0.415	1830	154	9.55	70.4	15.7	1.87
W21 × 101	29.8	21.4	12.3	0.800	0.500	2420	227	9.02	248	40.3	2.89
62	18.3	21.0	8.24	0.615	0.400	1330	127	8.54	57.5	14.0	1.77
44	13.0	20.7	6.50	0.450	0.350	843	81.6	8.06	20.7	6.37	1.26
W18 × 106	31.1	18.7	11.2	0.940	0.590	1910	204	7.84	220	39.4	2.66
76	22.3	18.2	11.0	0.680	0.425	1330	146	7.73	152	27.6	2.61
50	14.7	18.0	7.50	0.570	0.355	800	88.9	7.38	40.1	10.7	1.65
35	10.3	17.7	6.00	0.425	0.300	510	57.6	7.04	15.3	5.12	1.22
W16 × 77	22.6	16.5	10.3	0.76	0.455	1110	134	7.00	138	26.9	2.47
57	16.8	16.4	7.12	0.715	0.430	758	92.2	6.72	43.1	12.1	1.60
40	11.8	16.0	7.00	0.505	0.305	518	64.7	6.63	28.9	8.25	1.57
31	9.13	15.9	5.53	0.440	0.275	375	47.2	6.41	12.4	4.49	1.17
26	7.68	15.7	5.50	0.345	0.250	301	38.4	6.26	9.59	3.49	1.12
W14 × 370	109	17.9	16.5	2.66	1.66	5440	607	7.07	1990	241	4.27
145	42.7	14.8	15.5	1.09	0.680	1710	232	6.33	677	87.3	3.98
82	24.0	14.3	10.1	0.855	0.510	881	123	6.05	148	29.3	2.48
68	20.0	14.0	10.0	0.720	0.415	722	103	6.01	121	24.2	2.46
53	15.6	13.9	8.06	0.660	0.370	541	77.8	5.89	57.7	14.3	1.92
43	12.6	13.7	8.00	0.530	0.305	428	62.6	5.82	45.2	11.3	1.89
38	11.2	14.1	6.77	0.515	0.310	385	54.6	5.87	26.7	7.88	1.55
30	8.85	13.8	6.73	0.385	0.270	291	42.0	5.73	19.6	5.82	1.49
26	7.69	13.9	5.03	0.420	0.255	245	35.3	5.65	8.91	3.55	1.08
22	6.49	13.7	5.00	0.335	0.230	199	29.0	5.54	7.00	2.80	1.04

†A wide-flange shape is designated by the letter W followed by the nominal depth in inches and the weight in pounds per foot.

(Table continued on page 682)

APPENDIX B Properties of Rolled-Steel Shapes
(SI Units)

W Shapes
(Wide-Flange Shapes)



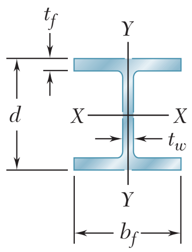
			Flange		Web	Axis X-X			Axis Y-Y		
Designation†	Area A, mm^2	Depth d, mm	Width b_f, mm	Thickness t_f, mm	Thickness t_w, mm	I_x 10^6 mm^4	S_x 10^3 mm^3	r_x mm	I_y 10^6 mm^4	S_y 10^3 mm^3	r_y mm
W920 × 449	57300	947	424	42.7	24.0	8780	18500	391	541	2560	97.0
201	25600	904	305	20.1	15.2	3250	7190	356	93.7	618	60.5
W840 × 299	38200	856	399	29.2	18.2	4830	11200	356	312	1560	90.4
176	22400	836	292	18.8	14.0	2460	5880	330	77.8	534	58.9
W760 × 257	32900	772	381	27.2	16.6	3430	8870	323	249	1310	86.9
147	18800	754	267	17.0	13.2	1660	4410	297	53.3	401	53.3
W690 × 217	27800	696	356	24.8	15.4	2360	6780	292	184	1040	81.3
125	16000	678	254	16.3	11.7	1190	3490	272	44.1	347	52.6
W610 × 155	19700	612	325	19.1	12.7	1290	4230	257	108	667	73.9
101	13000	602	228	14.9	10.5	762	2520	243	29.3	257	47.5
W530 × 150	19200	544	312	20.3	12.7	1010	3720	229	103	660	73.4
92	11800	533	209	15.6	10.2	554	2080	217	23.9	229	45.0
66	8390	526	165	11.4	8.89	351	1340	205	8.62	104	32.0
W460 × 158	20100	475	284	23.9	15.0	795	3340	199	91.6	646	67.6
113	14400	462	279	17.3	10.8	554	2390	196	63.3	452	66.3
74	9480	457	191	14.5	9.02	333	1460	187	16.7	175	41.9
52	6650	450	152	10.8	7.62	212	944	179	6.37	83.9	31.0
W410 × 114	14600	419	262	19.3	11.6	462	2200	178	57.4	441	62.7
85	10800	417	181	18.2	10.9	316	1510	171	17.9	198	40.6
60	7610	406	178	12.8	7.75	216	1060	168	12.0	135	39.9
46.1	5890	404	140	11.2	6.99	156	773	163	5.16	73.6	29.7
38.8	4950	399	140	8.76	6.35	125	629	159	3.99	57.2	28.4
W360 × 551	70300	455	419	67.6	42.2	2260	9950	180	828	3950	108
216	27500	376	394	27.7	17.3	712	3800	161	282	1430	101
122	15500	363	257	21.7	13.0	367	2020	154	61.6	480	63.0
101	12900	356	254	18.3	10.5	301	1690	153	50.4	397	62.5
79	10100	353	205	16.8	9.40	225	1270	150	24.0	234	48.8
64	8130	348	203	13.5	7.75	178	1030	148	18.8	185	48.0
57.8	7230	358	172	13.1	7.87	160	895	149	11.1	129	39.4
44	5710	351	171	9.78	6.86	121	688	146	8.16	95.4	37.8
39	4960	353	128	10.7	6.48	102	578	144	3.71	58.2	27.4
32.9	4190	348	127	8.51	5.84	82.8	475	141	2.91	45.9	26.4

†A wide-flange shape is designated by the letter W followed by the nominal depth in millimeters and the mass in kilograms per meter.

(Table continued on page 683)

APPENDIX B Properties of Rolled-Steel Shapes

(U.S. Customary Units)
Continued from page 680

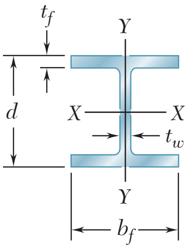


W Shapes

(Wide-Flange Shapes)

			Flange		Web	Axis X-X			Axis Y-Y		
Designation†	Area A, in²	Depth d, in.	Width b_f, in.	Thickness t_f, in.	Thickness t_w, in.	I_x, in⁴	S_x, in³	r_x, in.	I_y, in⁴	S_y, in³	r_y, in.
W12 × 96	28.2	12.7	12.2	0.900	0.550	833	131	5.44	270	44.4	3.09
72	21.1	12.3	12.0	0.670	0.430	597	97.4	5.31	195	32.4	3.04
50	14.6	12.2	8.08	0.640	0.370	391	64.2	5.18	56.3	13.9	1.96
40	11.7	11.9	8.01	0.515	0.295	307	51.5	5.13	44.1	11.0	1.94
35	10.3	12.5	6.56	0.520	0.300	285	45.6	5.25	24.5	7.47	1.54
30	8.79	12.3	6.52	0.440	0.260	238	38.6	5.21	20.3	6.24	1.52
26	7.65	12.2	6.49	0.380	0.230	204	33.4	5.17	17.3	5.34	1.51
22	6.48	12.3	4.03	0.425	0.260	156	25.4	4.91	4.66	2.31	0.848
16	4.71	12.0	3.99	0.265	0.220	103	17.1	4.67	2.82	1.41	0.773
W10 × 112	32.9	11.4	10.4	1.25	0.755	716	126	4.66	236	45.3	2.68
68	20.0	10.4	10.1	0.770	0.470	394	75.7	4.44	134	26.4	2.59
54	15.8	10.1	10.0	0.615	0.370	303	60.0	4.37	103	20.6	2.56
45	13.3	10.1	8.02	0.620	0.350	248	49.1	4.32	53.4	13.3	2.01
39	11.5	9.92	7.99	0.530	0.315	209	42.1	4.27	45.0	11.3	1.98
33	9.71	9.73	7.96	0.435	0.290	171	35.0	4.19	36.6	9.20	1.94
30	8.84	10.5	5.81	0.510	0.300	170	32.4	4.38	16.7	5.75	1.37
22	6.49	10.2	5.75	0.360	0.240	118	23.2	4.27	11.4	3.97	1.33
19	5.62	10.2	4.02	0.395	0.250	96.3	18.8	4.14	4.29	2.14	0.874
15	4.41	10.0	4.00	0.270	0.230	68.9	13.8	3.95	2.89	1.45	0.810
W8 × 58	17.1	8.75	8.22	0.810	0.510	228	52.0	3.65	75.1	18.3	2.10
48	14.1	8.50	8.11	0.685	0.400	184	43.2	3.61	60.9	15.0	2.08
40	11.7	8.25	8.07	0.560	0.360	146	35.5	3.53	49.1	12.2	2.04
35	10.3	8.12	8.02	0.495	0.310	127	31.2	3.51	42.6	10.6	2.03
31	9.12	8.00	8.00	0.435	0.285	110	27.5	3.47	37.1	9.27	2.02
28	8.24	8.06	6.54	0.465	0.285	98.0	24.3	3.45	21.7	6.63	1.62
24	7.08	7.93	6.50	0.400	0.245	82.7	20.9	3.42	18.3	5.63	1.61
21	6.16	8.28	5.27	0.400	0.250	75.3	18.2	3.49	9.77	3.71	1.26
18	5.26	8.14	5.25	0.330	0.230	61.9	15.2	3.43	7.97	3.04	1.23
15	4.44	8.11	4.01	0.315	0.245	48.0	11.8	3.29	3.41	1.70	0.876
13	3.84	7.99	4.00	0.255	0.230	39.6	9.91	3.21	2.73	1.37	0.843
W6 × 25	7.34	6.38	6.08	0.455	0.320	53.4	16.7	2.70	17.1	5.61	1.52
20	5.87	6.20	6.02	0.365	0.260	41.4	13.4	2.66	13.3	4.41	1.50
16	4.74	6.28	4.03	0.405	0.260	32.1	10.2	2.60	4.43	2.20	0.967
12	3.55	6.03	4.00	0.280	0.230	22.1	7.31	2.49	2.99	1.50	0.918
9	2.68	5.90	3.94	0.215	0.170	16.4	5.56	2.47	2.20	1.11	0.905
W5 × 19	5.56	5.15	5.03	0.430	0.270	26.3	10.2	2.17	9.13	3.63	1.28
16	4.71	5.01	5.00	0.360	0.240	21.4	8.55	2.13	7.51	3.00	1.26
W4 × 13	3.83	4.16	4.06	0.345	0.280	11.3	5.46	1.72	3.86	1.90	1.00

†A wide-flange shape is designated by the letter W followed by the nominal depth in inches and the weight in pounds per foot.



APPENDIX B Properties of Rolled-Steel Shapes

(SI Units)

Continued from page 681

W Shapes

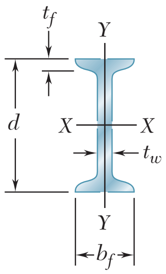
(Wide-Flange Shapes)

			Flange			Axis X-X			Axis Y-Y		
Designation†	Area A, mm^2	Depth d, mm	Width b_f, mm	Thickness t_f, mm	Web Thickness t_w, mm	I_x 10^6 mm^4	S_x 10^3 mm^3	r_x mm	I_y 10^6 mm^4	S_y 10^3 mm^3	r_y mm
W310 × 143	18200	323	310	22.9	14.0	347	2150	138	112	728	78.5
107	13600	312	305	17.0	10.9	248	1600	135	81.2	531	77.2
74	9420	310	205	16.3	9.40	163	1050	132	23.4	228	49.8
60	7550	302	203	13.1	7.49	128	844	130	18.4	180	49.3
52	6650	318	167	13.2	7.62	119	747	133	10.2	122	39.1
44.5	5670	312	166	11.2	6.60	99.1	633	132	8.45	102	38.6
38.7	4940	310	165	9.65	5.84	84.9	547	131	7.20	87.5	38.4
32.7	4180	312	102	10.8	6.60	64.9	416	125	1.94	37.9	21.5
23.8	3040	305	101	6.73	5.59	42.9	280	119	1.17	23.1	19.6
W250 × 167	21200	290	264	31.8	19.2	298	2060	118	98.2	742	68.1
101	12900	264	257	19.6	11.9	164	1240	113	55.8	433	65.8
80	10200	257	254	15.6	9.4	126	983	111	42.9	338	65.0
67	8580	257	204	15.7	8.89	103	805	110	22.2	218	51.1
58	7420	252	203	13.5	8.00	87.0	690	108	18.7	185	50.3
49.1	6260	247	202	11.0	7.37	71.2	574	106	15.2	151	49.3
44.8	5700	267	148	13.0	7.62	70.8	531	111	6.95	94.2	34.8
32.7	4190	259	146	9.14	6.10	49.1	380	108	4.75	65.1	33.8
28.4	3630	259	102	10.0	6.35	40.1	308	105	1.79	35.1	22.2
22.3	2850	254	102	6.86	5.84	28.7	226	100	1.20	23.8	20.6
W200 × 86	11000	222	209	20.6	13.0	94.9	852	92.7	31.3	300	53.3
71	9100	216	206	17.4	10.2	76.6	708	91.7	25.3	246	52.8
59	7550	210	205	14.2	9.14	60.8	582	89.7	20.4	200	51.8
52	6650	206	204	12.6	7.87	52.9	511	89.2	17.7	174	51.6
46.1	5880	203	203	11.0	7.24	45.8	451	88.1	15.4	152	51.3
41.7	5320	205	166	11.8	7.24	40.8	398	87.6	9.03	109	41.1
35.9	4570	201	165	10.2	6.22	34.4	342	86.9	7.62	92.3	40.9
31.3	3970	210	134	10.2	6.35	31.3	298	88.6	4.07	60.8	32.0
26.6	3390	207	133	8.38	5.84	25.8	249	87.1	3.32	49.8	31.2
22.5	2860	206	102	8.00	6.22	20.0	193	83.6	1.42	27.9	22.3
19.3	2480	203	102	6.48	5.84	16.5	162	81.5	1.14	22.5	21.4
W150 × 37.1	4740	162	154	11.6	8.13	22.2	274	68.6	7.12	91.9	38.6
29.8	3790	157	153	9.27	6.60	17.2	220	67.6	5.54	72.3	38.1
24	3060	160	102	10.3	6.60	13.4	167	66.0	1.84	36.1	24.6
18	2290	153	102	7.11	5.84	9.20	120	63.2	1.24	24.6	23.3
13.5	1730	150	100	5.46	4.32	6.83	91.1	62.7	0.916	18.2	23.0
W130 × 28.1	3590	131	128	10.9	6.86	10.9	167	55.1	3.80	59.5	32.5
23.8	3040	127	127	9.14	6.10	8.91	140	54.1	3.13	49.2	32.0
W100 × 19.3	2470	106	103	8.76	7.11	4.70	89.5	43.7	1.61	31.1	25.4

†A wide-flange shape is designated by the letter W followed by the nominal depth in millimeters and the mass in kilograms per meter.

APPENDIX B Properties of Rolled-Steel Shapes

(U.S. Customary Units)



S Shapes

(American Standard Shapes)

			Flange		Web	Axis X-X			Axis Y-Y		
Designation†	Area A, in^2	Depth d, in.	Width b_f, in.	Thickness t_f, in.	Thickness t_w, in.	I_x, in^4	S_x, in^3	r_x, in.	I_y, in^4	S_y, in^3	r_y, in.
S24 × 121	35.5	24.5	8.05	1.09	0.800	3160	258	9.43	83.0	20.6	1.53
106	31.1	24.5	7.87	1.09	0.620	2940	240	9.71	76.8	19.5	1.57
100	29.3	24.0	7.25	0.870	0.745	2380	199	9.01	47.4	13.1	1.27
90	26.5	24.0	7.13	0.870	0.625	2250	187	9.21	44.7	12.5	1.30
80	23.5	24.0	7.00	0.870	0.500	2100	175	9.47	42.0	12.0	1.34
S20 × 96	28.2	20.3	7.20	0.920	0.800	1670	165	7.71	49.9	13.9	1.33
86	25.3	20.3	7.06	0.920	0.660	1570	155	7.89	46.6	13.2	1.36
75	22.0	20.0	6.39	0.795	0.635	1280	128	7.62	29.5	9.25	1.16
66	19.4	20.0	6.26	0.795	0.505	1190	119	7.83	27.5	8.78	1.19
S18 × 70	20.5	18.0	6.25	0.691	0.711	923	103	6.70	24.0	7.69	1.08
54.7	16.0	18.0	6.00	0.691	0.461	801	89.0	7.07	20.7	6.91	1.14
S15 × 50	14.7	15.0	5.64	0.622	0.550	485	64.7	5.75	15.6	5.53	1.03
42.9	12.6	15.0	5.50	0.622	0.411	446	59.4	5.95	14.3	5.19	1.06
S12 × 50	14.6	12.0	5.48	0.659	0.687	303	50.6	4.55	15.6	5.69	1.03
40.8	11.9	12.0	5.25	0.659	0.462	270	45.1	4.76	13.5	5.13	1.06
35	10.2	12.0	5.08	0.544	0.428	228	38.1	4.72	9.84	3.88	0.980
31.8	9.31	12.0	5.00	0.544	0.350	217	36.2	4.83	9.33	3.73	1.00
S10 × 35	10.3	10.0	4.94	0.491	0.594	147	29.4	3.78	8.30	3.36	0.899
25.4	7.45	10.0	4.66	0.491	0.311	123	24.6	4.07	6.73	2.89	0.950
S8 × 23	6.76	8.00	4.17	0.425	0.441	64.7	16.2	3.09	4.27	2.05	0.795
18.4	5.40	8.00	4.00	0.425	0.271	57.5	14.4	3.26	3.69	1.84	0.827
S6 × 17.2	5.06	6.00	3.57	0.359	0.465	26.2	8.74	2.28	2.29	1.28	0.673
12.5	3.66	6.00	3.33	0.359	0.232	22.0	7.34	2.45	1.80	1.08	0.702
S5 × 10	2.93	5.00	3.00	0.326	0.214	12.3	4.90	2.05	1.19	0.795	0.638
S4 × 9.5	2.79	4.00	2.80	0.293	0.326	6.76	3.38	1.56	0.887	0.635	0.564
7.7	2.26	4.00	2.66	0.293	0.193	6.05	3.03	1.64	0.748	0.562	0.576
S3 × 7.5	2.20	3.00	2.51	0.260	0.349	2.91	1.94	1.15	0.578	0.461	0.513
5.7	1.66	3.00	2.33	0.260	0.170	2.50	1.67	1.23	0.447	0.383	0.518

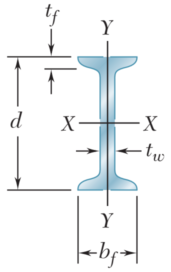
†An American Standard Beam is designated by the letter S followed by the nominal depth in inches and the weight in pounds per foot.

APPENDIX B Properties of Rolled-Steel Shapes

(SI Units)

S Shapes

(American Standard Shapes)

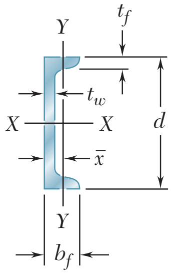


Designation†	Area A, mm^2	Depth d, mm	Flange: Width b_f, mm	Flange: Thickness t_f, mm	Web Thickness t_w, mm	Axis X-X: I_x 10^6 mm^4	Axis X-X: S_x 10^3 mm^3	Axis X-X: r_x mm	Axis Y-Y: I_y 10^6 mm^4	Axis Y-Y: S_y 10^3 mm^3	Axis Y-Y: r_y mm
S610 × 180	22900	622	204	27.7	20.3	1320	4230	240	34.5	338	38.9
158	20100	622	200	27.7	15.7	1220	3930	247	32.0	320	39.9
149	18900	610	184	22.1	18.9	991	3260	229	19.7	215	32.3
134	17100	610	181	22.1	15.9	937	3060	234	18.6	205	33.0
119	15200	610	178	22.1	12.7	874	2870	241	17.5	197	34.0
S510 × 143	18200	516	183	23.4	20.3	695	2700	196	20.8	228	33.8
128	16300	516	179	23.4	16.8	653	2540	200	19.4	216	34.5
112	14200	508	162	20.2	16.1	533	2100	194	12.3	152	29.5
98.2	12500	508	159	20.2	12.8	495	1950	199	11.4	144	30.2
S460 × 104	13200	457	159	17.6	18.1	384	1690	170	10.0	126	27.4
81.4	10300	457	152	17.6	11.7	333	1460	180	8.62	113	29.0
S380 × 74	9480	381	143	15.8	14.0	202	1060	146	6.49	90.6	26.2
64	8130	381	140	15.8	10.4	186	973	151	5.95	85.0	26.9
S310 × 74	9420	305	139	16.7	17.4	126	829	116	6.49	93.2	26.2
60.7	7680	305	133	16.7	11.7	112	739	121	5.62	84.1	26.9
52	6580	305	129	13.8	10.9	94.9	624	120	4.10	63.6	24.9
47.3	6010	305	127	13.8	8.89	90.3	593	123	3.88	61.1	25.4
S250 × 52	6650	254	125	12.5	15.1	61.2	482	96.0	3.45	55.1	22.8
37.8	4810	254	118	12.5	7.90	51.2	403	103	2.80	47.4	24.1
S200 × 34	4360	203	106	10.8	11.2	26.9	265	78.5	1.78	33.6	20.2
27.4	3480	203	102	10.8	6.88	23.9	236	82.8	1.54	30.2	21.0
S150 × 25.7	3260	152	90.7	9.12	11.8	10.9	143	57.9	0.953	21.0	17.1
18.6	2360	152	84.6	9.12	5.89	9.16	120	62.2	0.749	17.7	17.8
S130 × 15	1890	127	76.2	8.28	5.44	5.12	80.3	52.1	0.495	13.0	16.2
S100 × 14.1	1800	102	71.1	7.44	8.28	2.81	55.4	39.6	0.369	10.4	14.3
11.5	1460	102	67.6	7.44	4.90	2.52	49.7	41.7	0.311	9.21	14.6
S75 × 11.2	1420	76.2	63.8	6.60	8.86	1.21	31.8	29.2	0.241	7.55	13.0
8.5	1070	76.2	59.2	6.60	4.32	1.04	27.4	31.2	0.186	6.28	13.2

†An American Standard Beam is designated by the letter S followed by the nominal depth in millimeters and the mass in kilograms per meter.

APPENDIX B Properties of Rolled-Steel Shapes
(U.S. Customary Units)

C Shapes
(American Standard Channels)



Designation†	Area A, in^2	Depth d, in.	Flange Width b_f, in.	Flange Thickness t_f, in.	Web Thickness t_w, in.	Axis X-X I_x, in^4	Axis X-X S_x, in^3	Axis X-X r_x, in.	Axis Y-Y I_y, in^4	Axis Y-Y S_y, in^3	Axis Y-Y r_y, in.	Axis Y-Y x, in.
C15 × 50	14.7	15.0	3.72	0.650	0.716	404	53.8	5.24	11.0	3.77	0.865	0.799
40	11.8	15.0	3.52	0.650	0.520	348	46.5	5.45	9.17	3.34	0.883	0.778
33.9	10.0	15.0	3.40	0.650	0.400	315	42.0	5.62	8.07	3.09	0.901	0.788
C12 × 30	8.81	12.0	3.17	0.501	0.510	162	27.0	4.29	5.12	2.05	0.762	0.674
25	7.34	12.0	3.05	0.501	0.387	144	24.0	4.43	4.45	1.87	0.779	0.674
20.7	6.08	12.0	2.94	0.501	0.282	129	21.5	4.61	3.86	1.72	0.797	0.698
C10 × 30	8.81	10.0	3.03	0.436	0.673	103	20.7	3.42	3.93	1.65	0.668	0.649
25	7.34	10.0	2.89	0.436	0.526	91.1	18.2	3.52	3.34	1.47	0.675	0.617
20	5.87	10.0	2.74	0.436	0.379	78.9	15.8	3.66	2.80	1.31	0.690	0.606
15.3	4.48	10.0	2.60	0.436	0.240	67.3	13.5	3.87	2.27	1.15	0.711	0.634
C9 × 20	5.87	9.00	2.65	0.413	0.448	60.9	13.5	3.22	2.41	1.17	0.640	0.583
15	4.41	9.00	2.49	0.413	0.285	51.0	11.3	3.40	1.91	1.01	0.659	0.586
13.4	3.94	9.00	2.43	0.413	0.233	47.8	10.6	3.49	1.75	0.954	0.666	0.601
C8 × 18.7	5.51	8.00	2.53	0.390	0.487	43.9	11.0	2.82	1.97	1.01	0.598	0.565
13.7	4.04	8.00	2.34	0.390	0.303	36.1	9.02	2.99	1.52	0.848	0.613	0.554
11.5	3.37	8.00	2.26	0.390	0.220	32.5	8.14	3.11	1.31	0.775	0.623	0.572
C7 × 12.2	3.60	7.00	2.19	0.366	0.314	24.2	6.92	2.60	1.16	0.696	0.568	0.525
9.8	2.87	7.00	2.09	0.366	0.210	21.2	6.07	2.72	0.957	0.617	0.578	0.541
C6 × 13	3.81	6.00	2.16	0.343	0.437	17.3	5.78	2.13	1.05	0.638	0.524	0.514
10.5	3.08	6.00	2.03	0.343	0.314	15.1	5.04	2.22	0.860	0.561	0.529	0.500
8.2	2.39	6.00	1.92	0.343	0.200	13.1	4.35	2.34	0.687	0.488	0.536	0.512
C5 × 9	2.64	5.00	1.89	0.320	0.325	8.89	3.56	1.83	0.624	0.444	0.486	0.478
6.7	1.97	5.00	1.75	0.320	0.190	7.48	2.99	1.95	0.470	0.372	0.489	0.484
C4 × 7.2	2.13	4.00	1.72	0.296	0.321	4.58	2.29	1.47	0.425	0.337	0.447	0.459
5.4	1.58	4.00	1.58	0.296	0.184	3.85	1.92	1.56	0.312	0.277	0.444	0.457
C3 × 6	1.76	3.00	1.60	0.273	0.356	2.07	1.38	1.08	0.300	0.263	0.413	0.455
5	1.47	3.00	1.50	0.273	0.258	1.85	1.23	1.12	0.241	0.228	0.405	0.439
4.1	1.20	3.00	1.41	0.273	0.170	1.65	1.10	1.17	0.191	0.196	0.398	0.437

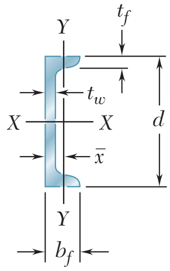
†An American Standard Channel is designated by the letter C followed by the nominal depth in inches and the weight in pounds per foot.

APPENDIX B Properties of Rolled-Steel Shapes

(SI Units)

C Shapes

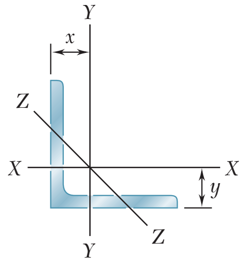
(American Standard Channels)



			Flange		Web	Axis X-X			Axis Y-Y			
Designation†	Area A, mm^2	Depth d, mm	Width b_f, mm	Thickness t_f, mm	Thickness t_w, mm	I_x 10^6 mm^4	S_x 10^3 mm^3	r_x mm	I_y 10^6 mm^4	S_y 10^3 mm^3	r_y mm	x mm
C380 × 74	9480	381	94.5	16.5	18.2	168	882	133	4.58	61.8	22.0	20.3
60	7610	381	89.4	16.5	13.2	145	762	138	3.82	54.7	22.4	19.8
50.4	6450	381	86.4	16.5	10.2	131	688	143	3.36	50.6	22.9	20.0
C310 × 45	5680	305	80.5	12.7	13.0	67.4	442	109	2.13	33.6	19.4	17.1
37	4740	305	77.5	12.7	9.83	59.9	393	113	1.85	30.6	19.8	17.1
30.8	3920	305	74.7	12.7	7.16	53.7	352	117	1.61	28.2	20.2	17.7
C250 × 45	5680	254	77.0	11.1	17.1	42.9	339	86.9	1.64	27.0	17.0	16.5
37	4740	254	73.4	11.1	13.4	37.9	298	89.4	1.39	24.1	17.1	15.7
30	3790	254	69.6	11.1	9.63	32.8	259	93.0	1.17	21.5	17.5	15.4
22.8	2890	254	66.0	11.1	6.10	28.0	221	98.3	0.945	18.8	18.1	16.1
C230 × 30	3790	229	67.3	10.5	11.4	25.3	221	81.8	1.00	19.2	16.3	14.8
22	2850	229	63.2	10.5	7.24	21.2	185	86.4	0.795	16.6	16.7	14.9
19.9	2540	229	61.7	10.5	5.92	19.9	174	88.6	0.728	15.6	16.9	15.3
C200 × 27.9	3550	203	64.3	9.91	12.4	18.3	180	71.6	0.820	16.6	15.2	14.4
20.5	2610	203	59.4	9.91	7.70	15.0	148	75.9	0.633	13.9	15.6	14.1
17.1	2170	203	57.4	9.91	5.59	13.5	133	79.0	0.545	12.7	15.8	14.5
C180 × 18.2	2320	178	55.6	9.30	7.98	10.1	113	66.0	0.483	11.4	14.4	13.3
14.6	1850	178	53.1	9.30	5.33	8.82	100	69.1	0.398	10.1	14.7	13.7
C150 × 19.3	2460	152	54.9	8.71	11.1	7.20	94.7	54.1	0.437	10.5	13.3	13.1
15.6	1990	152	51.6	8.71	7.98	6.29	82.6	56.4	0.358	9.19	13.4	12.7
12.2	1540	152	48.8	8.71	5.08	5.45	71.3	59.4	0.286	8.00	13.6	13.0
C130 × 13	1700	127	48.0	8.13	8.26	3.70	58.3	46.5	0.260	7.28	12.3	12.1
10.4	1270	127	44.5	8.13	4.83	3.11	49.0	49.5	0.196	6.10	12.4	12.3
C100 × 10.8	1370	102	43.7	7.52	8.15	1.91	37.5	37.3	0.177	5.52	11.4	11.7
8	1020	102	40.1	7.52	4.67	1.60	31.5	39.6	0.130	4.54	11.3	11.6
C75 × 8.9	1140	76.2	40.6	6.93	9.04	0.862	22.6	27.4	0.125	4.31	10.5	11.6
7.4	948	76.2	38.1	6.93	6.55	0.770	20.2	28.4	0.100	3.74	10.3	11.2
6.1	774	76.2	35.8	6.93	4.32	0.687	18.0	29.7	0.0795	3.21	10.1	11.1

†An American Standard Channel is designated by the letter C followed by the nominal depth in millimeters and the mass in kilograms per meter.

APPENDIX B Properties of Rolled-Steel Shapes
(U.S. Customary Units)



Angles
Equal Legs

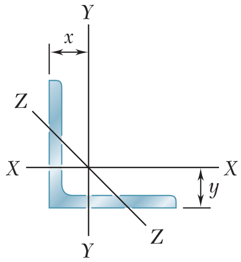
Size and Thickness, in.	Weight per Foot, lb/ft	Area, in^2	Axis X-X and Axis Y-Y				Axis Z-Z
			I, in^4	S, in^3	r, in.	x or y, in.	r_z, in.
L8 × 8 × 1	51.0	15.0	89.1	15.8	2.43	2.36	1.56
¾	38.9	11.4	69.9	12.2	2.46	2.26	1.57
½	26.4	7.75	48.8	8.36	2.49	2.17	1.59
L6 × 6 × 1	37.4	11.0	35.4	8.55	1.79	1.86	1.17
¾	28.7	8.46	28.1	6.64	1.82	1.77	1.17
⅝	24.2	7.13	24.1	5.64	1.84	1.72	1.17
½	19.6	5.77	19.9	4.59	1.86	1.67	1.18
⅜	14.9	4.38	15.4	3.51	1.87	1.62	1.19
L5 × 5 × ¾	23.6	6.94	15.7	4.52	1.50	1.52	0.972
⅝	20.0	5.86	13.6	3.85	1.52	1.47	0.975
½	16.2	4.75	11.3	3.15	1.53	1.42	0.980
⅜	12.3	3.61	8.76	2.41	1.55	1.37	0.986
L4 × 4 × ¾	18.5	5.44	7.62	2.79	1.18	1.27	0.774
⅝	15.7	4.61	6.62	2.38	1.20	1.22	0.774
½	12.8	3.75	5.52	1.96	1.21	1.18	0.776
⅜	9.80	2.86	4.32	1.50	1.23	1.13	0.779
¼	6.60	1.94	3.00	1.03	1.25	1.08	0.783
L3½ × 3½ × ½	11.1	3.25	3.63	1.48	1.05	1.05	0.679
⅜	8.50	2.48	2.86	1.15	1.07	1.00	0.683
¼	5.80	1.69	2.00	0.787	1.09	0.954	0.688
L3 × 3 × ½	9.40	2.75	2.20	1.06	0.895	0.929	0.580
⅜	7.20	2.11	1.75	0.825	0.910	0.884	0.581
¼	4.90	1.44	1.23	0.569	0.926	0.836	0.585
L2½ × 2½ × ½	7.70	2.25	1.22	0.716	0.735	0.803	0.481
⅜	5.90	1.73	0.972	0.558	0.749	0.758	0.481
¼	4.10	1.19	0.692	0.387	0.764	0.711	0.482
3/16	3.07	0.900	0.535	0.295	0.771	0.687	0.482
L2 × 2 × ⅜	4.70	1.36	0.476	0.348	0.591	0.632	0.386
¼	3.19	0.938	0.346	0.244	0.605	0.586	0.387
⅛	1.65	0.484	0.189	0.129	0.620	0.534	0.391

APPENDIX B Properties of Rolled-Steel Shapes

(SI Units)

Angles

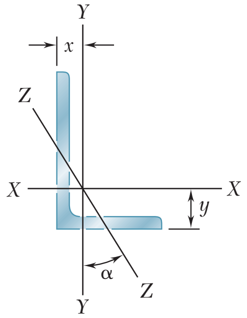
Equal Legs



Size and Thickness, mm	Mass per Meter, kg/m	Area, mm^2	Axis X-X: I 10^6 mm^4	Axis X-X: S 10^3 mm^3	Axis X-X: r mm	Axis X-X: x or y mm	Axis Z-Z: r_z mm
L203 × 203 × 25.4	75.9	9680	37.1	259	61.7	59.9	39.6
19	57.9	7350	29.1	200	62.5	57.4	39.9
12.7	39.3	5000	20.3	137	63.2	55.1	40.4
L152 × 152 × 25.4	55.7	7100	14.7	140	45.5	47.2	29.7
19	42.7	5460	11.7	109	46.2	45.0	29.7
15.9	36.0	4600	10.0	92.4	46.7	43.7	29.7
12.7	29.2	3720	8.28	75.2	47.2	42.4	30.0
9.5	22.2	2830	6.41	57.5	47.5	41.1	30.2
L127 × 127 × 19	35.1	4480	6.53	74.1	38.1	38.6	24.7
15.9	29.8	3780	5.66	63.1	38.6	37.3	24.8
12.7	24.1	3060	4.70	51.6	38.9	36.1	24.9
9.5	18.3	2330	3.65	39.5	39.4	34.8	25.0
L102 × 102 × 19	27.5	3510	3.17	45.7	30.0	32.3	19.7
15.9	23.4	2970	2.76	39.0	30.5	31.0	19.7
12.7	19.0	2420	2.30	32.1	30.7	30.0	19.7
9.5	14.6	1850	1.80	24.6	31.2	28.7	19.8
6.4	9.80	1250	1.25	16.9	31.8	27.4	19.9
L89 × 89 × 12.7	16.5	2100	1.51	24.3	26.7	26.7	17.2
9.5	12.6	1600	1.19	18.8	27.2	25.4	17.3
6.4	8.60	1090	0.832	12.9	27.7	24.2	17.5
L76 × 76 × 12.7	14.0	1770	0.916	17.4	22.7	23.6	14.7
9.5	10.7	1360	0.728	13.5	23.1	22.5	14.8
6.4	7.30	929	0.512	9.32	23.5	21.2	14.9
L64 × 64 × 12.7	11.4	1450	0.508	11.7	18.7	20.4	12.2
9.5	8.70	1120	0.405	9.14	19.0	19.3	12.2
6.4	6.10	768	0.288	6.34	19.4	18.1	12.2
4.8	4.60	581	0.223	4.83	19.6	17.4	12.2
L51 × 51 × 9.5	7.00	877	0.198	5.70	15.0	16.1	9.80
6.4	4.70	605	0.144	4.00	15.4	14.9	9.83
3.2	2.40	312	0.0787	2.11	15.7	13.6	9.93

APPENDIX B Properties of Rolled-Steel Shapes
(U.S. Customary Units)

Angles
Unequal Legs



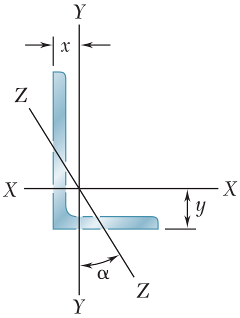
Size and Thickness, in.	Weight per Foot, lb/ft	Area, in^2	Axis X-X I_x, in^4	Axis X-X S_x, in^3	Axis X-X r_x, in.	Axis X-X y, in.	Axis Y-Y I_y, in^4	Axis Y-Y S_y, in^3	Axis Y-Y r_y, in.	Axis Y-Y x, in.	Axis Z-Z r_z, in.	Axis Z-Z tan α
L8 × 6 × 1	44.2	13.0	80.9	15.1	2.49	2.65	38.8	8.92	1.72	1.65	1.28	0.542
¾	33.8	9.94	63.5	11.7	2.52	2.55	30.8	6.92	1.75	1.56	1.29	0.550
½	23.0	6.75	44.4	8.01	2.55	2.46	21.7	4.79	1.79	1.46	1.30	0.557
L6 × 4 × ¾	23.6	6.94	24.5	6.23	1.88	2.07	8.63	2.95	1.12	1.07	0.856	0.428
½	16.2	4.75	17.3	4.31	1.91	1.98	6.22	2.06	1.14	0.981	0.864	0.440
⅜	12.3	3.61	13.4	3.30	1.93	1.93	4.86	1.58	1.16	0.933	0.870	0.446
L5 × 3 × ½	12.8	3.75	9.43	2.89	1.58	1.74	2.55	1.13	0.824	0.746	0.642	0.357
⅜	9.80	2.86	7.35	2.22	1.60	1.69	2.01	0.874	0.838	0.698	0.646	0.364
¼	6.60	1.94	5.09	1.51	1.62	1.64	1.41	0.600	0.853	0.648	0.652	0.371
L4 × 3 × ½	11.1	3.25	5.02	1.87	1.24	1.32	2.40	1.10	0.858	0.822	0.633	0.542
⅜	8.50	2.48	3.94	1.44	1.26	1.27	1.89	0.851	0.873	0.775	0.636	0.551
¼	5.80	1.69	2.75	0.988	1.27	1.22	1.33	0.585	0.887	0.725	0.639	0.558
L3$\frac{1}{2}$ × 2$\frac{1}{2}$ × ½	9.40	2.75	3.24	1.41	1.08	1.20	1.36	0.756	0.701	0.701	0.532	0.485
⅜	7.20	2.11	2.56	1.09	1.10	1.15	1.09	0.589	0.716	0.655	0.535	0.495
¼	4.90	1.44	1.81	0.753	1.12	1.10	0.775	0.410	0.731	0.607	0.541	0.504
L3 × 2 × ½	7.70	2.25	1.92	1.00	0.922	1.08	0.667	0.470	0.543	0.580	0.425	0.413
⅜	5.90	1.73	1.54	0.779	0.937	1.03	0.539	0.368	0.555	0.535	0.426	0.426
¼	4.10	1.19	1.09	0.541	0.953	0.980	0.390	0.258	0.569	0.487	0.431	0.437
L2$\frac{1}{2}$ × 2 × ⅜	5.30	1.55	0.914	0.546	0.766	0.826	0.513	0.361	0.574	0.578	0.419	0.612
¼	3.62	1.06	0.656	0.381	0.782	0.779	0.372	0.253	0.589	0.532	0.423	0.624

APPENDIX B Properties of Rolled-Steel Shapes

(SI Units)

Angles

Unequal Legs



			Axis X-X				Axis Y-Y				Axis Z-Z	
Size and Thickness, mm	Mass per Meter kg/m	Area mm^2	I_x 10^6 mm^4	S_x 10^3 mm^3	r_x mm	y mm	I_y 10^6 mm^4	S_y 10^3 mm^3	r_y mm	x mm	r_z mm	tan α
L203 × 152 × 25.4	65.5	8390	33.7	247	63.2	67.3	16.1	146	43.7	41.9	32.5	0.542
19	50.1	6410	26.4	192	64.0	64.8	12.8	113	44.5	39.6	32.8	0.550
12.7	34.1	4350	18.5	131	64.8	62.5	9.03	78.5	45.5	37.1	33.0	0.557
L152 × 102 × 19	35.0	4480	10.2	102	47.8	52.6	3.59	48.3	28.4	27.2	21.7	0.428
12.7	24.0	3060	7.20	70.6	48.5	50.3	2.59	33.8	29.0	24.9	21.9	0.440
9.5	18.2	2330	5.58	54.1	49.0	49.0	2.02	25.9	29.5	23.7	22.1	0.446
L127 × 76 × 12.7	19.0	2420	3.93	47.4	40.1	44.2	1.06	18.5	20.9	18.9	16.3	0.357
9.5	14.5	1850	3.06	36.4	40.6	42.9	0.837	14.3	21.3	17.7	16.4	0.364
6.4	9.80	1250	2.12	24.7	41.1	41.7	0.587	9.83	21.7	16.5	16.6	0.371
L102 × 76 × 12.7	16.4	2100	2.09	30.6	31.5	33.5	0.999	18.0	21.8	20.9	16.1	0.542
9.5	12.6	1600	1.64	23.6	32.0	32.3	0.787	13.9	22.2	19.7	16.2	0.551
6.4	8.60	1090	1.14	16.2	32.3	31.0	0.554	9.59	22.5	18.4	16.2	0.558
L89 × 64 × 12.7	13.9	1770	1.35	23.1	27.4	30.5	0.566	12.4	17.8	17.8	13.5	0.485
9.5	10.7	1360	1.07	17.9	27.9	29.2	0.454	9.65	18.2	16.6	13.6	0.495
6.4	7.30	929	0.753	12.3	28.4	27.9	0.323	6.72	18.6	15.4	13.7	0.504
L76 × 51 × 12.7	11.5	1450	0.799	16.4	23.4	27.4	0.278	7.70	13.8	14.7	10.8	0.413
9.5	8.80	1120	0.641	12.8	23.8	26.2	0.224	6.03	14.1	13.6	10.8	0.426
6.4	6.10	768	0.454	8.87	24.2	24.9	0.162	4.23	14.5	12.4	10.9	0.437
L64 × 51 × 9.5	7.90	1000	0.380	8.95	19.5	21.0	0.214	5.92	14.6	14.7	10.6	0.612
6.4	5.40	684	0.273	6.24	19.9	19.8	0.155	4.15	15.0	13.5	10.7	0.624

APPENDIX C Beam Deflections and Slopes

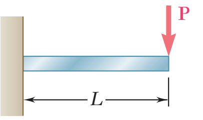
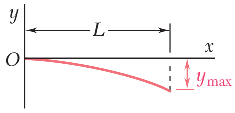
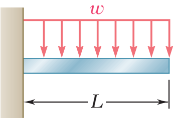
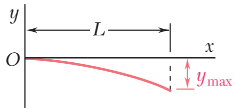
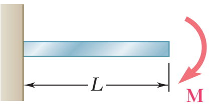
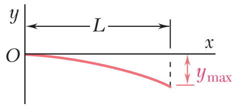
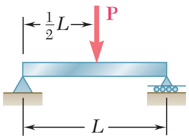
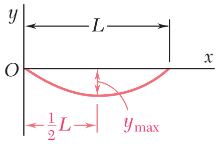
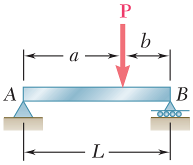
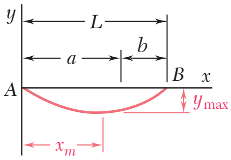
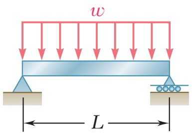
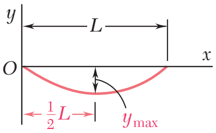
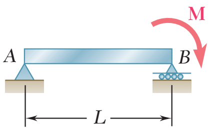
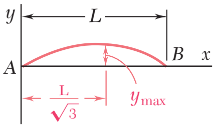
Beam and Loading	Elastic Curve	Maximum Deflection	Slope at End	Equation of Elastic Curve
1 		$-\dfrac{PL^3}{3EI}$	$-\dfrac{PL^2}{2EI}$	$y = \dfrac{P}{6EI}(x^3 - 3Lx^2)$
2 		$-\dfrac{wL^4}{8EI}$	$-\dfrac{wL^3}{6EI}$	$y = -\dfrac{w}{24EI}(x^4 - 4Lx^3 + 6L^2x^2)$
3 		$-\dfrac{ML^2}{2EI}$	$-\dfrac{ML}{EI}$	$y = -\dfrac{M}{2EI}x^2$
4 		$-\dfrac{PL^3}{48EI}$	$\pm\dfrac{PL^2}{16EI}$	For $x \le \frac{1}{2}L$: $y = \dfrac{P}{48EI}(4x^3 - 3L^2x)$
5 		For $a > b$: $-\dfrac{Pb(L^2 - b^2)^{3/2}}{9\sqrt{3}EIL}$ at $x_m = \sqrt{\dfrac{L^2 - b^2}{3}}$	$\theta_A = -\dfrac{Pb(L^2 - b^2)}{6EIL}$ $\theta_B = +\dfrac{Pa(L^2 - a^2)}{6EIL}$	For $x < a$: $y = \dfrac{Pb}{6EIL}[x^3 - (L^2 - b^2)x]$ For $x = a$: $y = -\dfrac{Pa^2b^2}{3EIL}$
6 		$-\dfrac{5wL^4}{384EI}$	$\pm\dfrac{\Sigma L^3}{24EI}$	$y = -\dfrac{w}{24EI}(x^4 - 2Lx^3 + L^3x)$
7 		$\dfrac{ML^2}{9\sqrt{3}EI}$	$\theta_A = +\dfrac{ML}{6EI}$ $\theta_B = -\dfrac{ML}{3EI}$	$y = -\dfrac{M}{6EIL}(x^3 - L^2x)$

Photo Credits

CHAPTER 1

Opener: Sr. Isaac Newton, Photo courtesy of Jeremy Whitaker; p. 4, 1.1: © PhotoLink/Getty Images RF.

CHAPTER 2

Opener: © Tom Paiva/Getty Images; p. 35, 2.1: © H. David Seawell/CORBIS; p. 52, 2.2: © WIN-Initiative/Getty Images.

CHAPTER 3

Opener: © Daniel Sheehan.

CHAPTER 4

Opener: © Alfredo Maiquez/Getty Images; p. 133, 4.1: © McGraw-Hill/Photo by Lucinda Dowell; 4.2: © McGraw-Hill/Photo by Lucinda Dowell; p. 134, 4.3: © McGraw-Hill/Photo by Lucinda Dowell; 4.4: Godden Collection, National Information Service for Earthquake Engineering, University of California, Berkeley; 4.5: Godden Collection, National Information Service for Earthquake Engineering, University of California, Berkeley; p. 156, 4.6: ©McGraw-Hill/Photo by Lucinda Dowell; 4.7: Courtesy of SKF Limited; p. 171, 4.8: © Chuck Savage/CORBIS.

CHAPTER 5

Opener: Photo courtesy of Massachusetts Turnpike Authority; p. 188, 5.1: © Christies Images/SuperStock; p. 203, 5.2: Value RF/© Kevin Burke/CORBIS; p. 210, 5.3: © Ghislain & Marie David de Lossy/Getty Images; p. 213, 5.4: NASA.

CHAPTER 6

Opener: © Alan Schein Photography/CORBIS; p. 229, 6.1: Godden Collection, National Information Service for Earthquake Engineering, University of California, Berkeley; p. 231, 6.2: © Ferdinand Beer; p. 232, 6.3: © McGraw-Hill/Photo by Sabina Dowell; p. 261, 6.4: Courtesy of Luxo Lamp Corporation.

CHAPTER 7

Opener: © Lester Lefkowitz/Getty Images; p. 288, 7.1: © Ed Eckstein/CORBIS.

CHAPTER 8

Opener: © Construction Photography/CORBIS; p. 303, 8.1: © Vince Streano/CORBIS; p. 306, 8.2: © John DeWolf.

CHAPTER 9

Opener: © Construction Photography/CORBIS; p. 346, 9.1: © John DeWolf; p. 347, 9.2: Courtesy of Tinius Olsen Testing Machine Co., Inc.; 9.3: © John DeWolf; p. 348, 9.4: © John DeWolf; p. 349, 9.5: © John DeWolf.

CHAPTER 10

Opener: © Brownie Harris/CORBIS; p. 408, 10.1: © 2008 Ford Motor Company and Wieck Media Services, Inc.; p. 416, 10.2: © John DeWolf; p. 423, 10.3: Courtesy of Tinius Olsen Testing Machine Co., Inc.

CHAPTER 11

Opener: © Lawrence Manning/CORBIS; p. 444, 11.1: Courtesy of Flexifoil; p. 445, 11.2: © Tony Freeman/Photo Edit; p. 453, 11.3: © Hisham Ibrahim/Getty Images RF; p. 464, 11.4: © Kevin R. Morris/CORBIS; p. 471, 11.5: © Tony Freeman/Photo Edit; 11.6: © John DeWolf.

CHAPTER 12

Opener: © Mark Segal/Digital Vision/Getty Images, RF; p. 502, 12.1: © David Papazian/CORBIS.

CHAPTER 13

Opener: Godden Collection, National Information Service for Earthquake Engineering, University of California, Berkeley; p. 539, 13.1: © John DeWolf; p. 554, 13.2: Courtesy of Nucor-Yamato Steel Company; 13.3: Courtesy of Leavitt Tube Company.

CHAPTER 14

Opener: NASA/Tony Landis; p. 573, 14.1: © Radlund & Associates/Getty Images, RF; p. 573, 14.2: © Spencer C. Grant/Photo Edit; p. 592, 14.3: © Clair Dunn/Alamy; p. 592, 14.4: © Spencer C. Grant/Photo Edit.

CHAPTER 15

Opener: © Construction Photography/CORBIS; p. 612, 15.1: © Royalty-Free/CORBIS; p. 626, 15.2: © John DeWolf.

CHAPTER 16

Opener: © Jose Manuel/Photographer's Choice RF/Getty Images; p. 643, 16.1: Courtesy of Fritz Engineering Laboratory, Lehigh University; p. 659, 16.2a: Godden Collection, National Information Service for Earthquake Engineering, University of California, Berkeley; 16.2b: © Peter Marlow/Magnum Photos.

This page intentionally left blank

Index

Appendices showing properties of materials, shapes and beam deflections begin on page 675.

D

E

F

G

H

I

J

K

L

Q

R

S

T

U

V

W

Y

Z

Answers to Problems

Answers to problems with a number set in straight type are given on this and the following pages. Answers to problems set in italic are not listed.

CHAPTER 2

2.1 1391 N $\measuredangle$ 47.8°.
2.2 906 lb $\measuredangle$ 26.6°.
2.3 14.3 kN $\measuredangle$ 19.9°.
2.4 4000 lb $\measuredangle$ 9.2°.
2.5 32.4°.
2.7 (*a*) 3660 N. (*b*) 3730 N.
2.9 $P = 14.73$ lb; $R = 30.2$ lb.
2.10 $T_{AC} = 666$ lb; $\alpha = 34.3°$.
2.12 43.6 lb $\measuredangle$ 78.4°.
2.13 1391 N $\measuredangle$ 47.8°.
2.14 4000 N $\measuredangle$ 9.2°.
2.16 350-N force: $F_x = 317$ N, $F_y = 147.9$ N;
800-N force: $F_x = 274$ N, $F_y = 752$ N;
600-N force: $F_x = -300$ N, $F_y = 520$ N.
2.17 80-lb force: $F_x = 69.3$ lb, $F_y = -40.0$ lb;
120-lb force: $F_x = 31.1$ lb, $F_y = -115.9$ lb;
150-lb force: $F_x = -114.9$ lb, $F_y = -96.4$ lb.
2.18 145-lb force: $F_x = 100$ lb, $F_y = 105$ lb;
200-lb force: $F_x = 192$ lb, $F_y = -56$ lb.
2.19 255-N force: $F_x = 225$ N, $F_y = 120$ N;
340-N force: $F_x = -160$ N, $F_y = 300$ N.
2.20 $\mathbf{A}_x = 25$ lb $\longrightarrow$, $\mathbf{A}_y = 60$ lb $\downarrow$.
2.23 (*a*) 2109 N. (*b*) 2060 N $\measuredangle$ 30°.
2.24 1391 N $\measuredangle$ 47.8°.
2.25 906 lb $\measuredangle$ 26.6°.
2.26 253 lb $\measuredangle$ 86.7°.
2.27 425 N $\measuredangle$ 81.2°.
2.29 (*a*) 177.9 lb. (*b*) 410 lb.
2.31 (*a*) 48.2°. (*b*) impossible.
2.32 $T_{AC} = 530$ N, $T_{BC} = 350$ N.
2.33 $T_{AC} = 326$ lb, $T_{BC} = 265$ lb.
2.34 $T_{AC} = 586$ N, $T_{BC} = 2190$ N.
2.35 $T_{AC} = 2860$ lb, $T_{BC} = 1460$ lb.
2.36 $T_{AC} = 305$ N, $T_{BC} = 514$ N.
2.38 $T_B = 16.73$ kips, $T_D = 14.00$ kips.
2.40 65.2 lb $< P <$ 150 lb.
2.41 (*a*) 784 N. (*b*) 71.0°.
2.42 $F = 2.87$ kN $\measuredangle$ 75°.
2.43 (*a*) 30°. (*b*) $T_{AC} = 300$ lb, $T_{BC} = 520$ lb.
2.45 (*a*) 35°; $T_{AC} = 4.91$ kN, $T_{BC} = 3.44$ kN.
(*b*) 55°; $T_{AC} = T_{BC} = 3.66$ kN.
2.46 36.0 in.
2.47 913 N $\measuredangle$ 82.5°.
2.48 41.9°.
2.49 (*a*) 2.45 kN. (*b*) 1.839 kN.
2.50 50.0 in.
2.51 (*a*) 1226 N. (*b*) 1226 N. (*c*) 817 N. (*d*) 817 N. (*e*) 613 N.
2.54 75.6 mm.
2.55 (*a*) 18.00 lb. (*b*) 24.0 lb.
2.56 (*a*) $F_x = 113.3$ N, $F_y = 217$ N, $F_z = -52.8$ N.
(*b*) $\theta_x = 63.1°$, $\theta_y = 30.0°$, $\theta_z = 102.2°$.
2.57 (*a*) $F_x = 65.9$ N, $F_y = 230$ N, $F_z = 181.2$ N.
(*b*) $\theta_x = 77.3°$, $\theta_y = 40.0°$, $\theta_z = 52.8°$.
2.58 (*a*) $F_x = -78.6$ lb, $F_y = 282$ lb, $F_z = -66.0$ lb.
(*b*) $\theta_x = 105.2°$, $\theta_y = 20.0°$, $\theta_z = 102.7°$.
2.59 (*a*) $F_x = 78.6$ lb, $F_y = 282$ lb, $F_z = -66.0$ N.
(*b*) $\theta_x = 74.8°$, $\theta_y = 20.0°$, $\theta_z = 102.7°$.
2.60 (*a*) $F_x = 224$ N, $F_y = -459$ N, $F_z = 615$ N.
(*b*) $\theta_x = 73.7°$, $\theta_y = 125.0°$, $\theta_z = 39.8°$.
2.62 $F = 721$ lb; $\theta_x = 109.4°$, $\theta_y = 116.3°$, $\theta_z = 33.7°$.
2.63 $F = 950$ lb; $\theta_x = 43.4°$, $\theta_y = 71.6°$, $\theta_z = 127.6°$.
2.64 $F = 48.4$ N; $\theta_x = 34.3°$.
2.65 $\theta_z = 61.0°$; $F_x = 105.7$ lb, $F_y = 191.5$ lb, $F_z = 121.0$ lb.
2.66 (*a*) $F_x = 199.6$ lb, $F_z = -395$ lb; $F = 584$ lb.
(*b*) $\theta_y = 46.7°$.
2.67 (*a*) $F_y = 654$ N, $F_z = 1186$ N; $F = 1549$ N.
(*b*) $\theta_x = 119.0°$.
2.69 $C_x = -300$ N, $C_y = 300$ N, $C_z = 150$ N.
2.71 $(T_{CA})_x = -270$ lb, $(T_{CA})_y = 180$ lb, $(T_{CA})_z = -276$ lb.
2.72 $R = 940$ lb; $\theta_x = 65.7°$, $\theta_y = 28.3°$, $\theta_z = 16.4°$.
2.73 $R = 623$ lb; $\theta_x = 37.4°$, $\theta_y = 122.0°$, $\theta_z = 72.6°$.
2.75 (*a*) 54.7° and 125.3°. (*b*) 60° and 120°.
2.76 $T_{AC} = 21.0$ kN, $T_{AD} = 64.3$ kN.
2.77 $T_{AB} = 52.0$ kN, $T_{AD} = 85.7$ kN.
2.79 548 N.
2.80 13.98 kN.
2.81 9.71 kN.
2.82 $T_{AB} = 4.00$ kN, $T_{AC} = 3.67$ kN, $T_{AD} = 4.13$ kN.
2.83 2775 lb.
2.84 888 lb.
2.86 $T_{DA} = 119.7$ lb, $T_{DB} = T_{DC} = 98.4$ lb.
2.87 $T_{DA} = 7.21$ lb, $T_{DB} = T_{DC} = 6.50$ lb.
2.89 (*a*) $\mathbf{P} = -(25.2 \text{ kN})\,\mathbf{i}$.
(*b*) $T_{AB} = 2.25$ kN, $T_{AC} = 16.65$ kN.
2.90 $T_{AB} = 30.8$ lb, $T_{AC} = 62.5$ lb.
2.91 $T_{AB} = 81.3$ lb, $T_{AC} = 22.2$ lb.
2.92 (*a*) $P = 120.0$ N.
(*b*) $T_{AB} = 234$ N, $T_{AC} = 174.0$ N.
2.93 (*a*) $P = 135.0$ N.
(*b*) $T_{AB} = 156.0$ N, $T_{AC} = 261$ N.
2.95 1372 N.
2.97 (*a*) $P = 305$ lb.
(*b*) $T_{AD} = 40.9$ lb, $T_{BAC} = 117.0$ lb.
2.98 $T_{DA} = 103.7$ N, $T_{DB} = 51.8$ N, $T_{DC} = 89.8$ N.
2.99 (*a*) 6.30 lb. (*b*) 7.20 lb.
2.101 $T_{CA} = 1192$ lb, $T_{CB} = 898$ lb.
2.103 320 mm.
2.104 (*a*) 2450 N. (*b*) 2220 N.
2.105 168.3 lb $\measuredangle$ 13.5°.

2.106 52.2 lb $\leq P \leq$ 176.3 lb.
2.108 T_{AC} = 134.6 lb, T_{BC} = 110.4 lb.
2.110 1210 N.
2.112 θ_z = 61.0°; F_x = 507 N, F_y = 919 N, F_z = 581 N.
2.113 R = 1171 N; θ_x = 89.5°, θ_y = 36.2°, θ_z = 126.2°.
2.115 T_{DA} = 14.33 lb, $T_{DB} = T_{DC}$ = 12.92 lb.

CHAPTER 3

3.1 115.6 lb · in ↻.
3.2 23.2°.
3.3 P = 400 N; α = 22.6°.
3.5 (*a*) 88.8 N · m ↻. (*b*) 237 N ∠ 53.1°.
3.6 (*a*) 88.8 N · m ↻. (*b*) 395 N ←. (*c*) 279 N ∠ 45°.
3.7 (*a*), (*b*), and (*c*) 167.0 lb · in ↺.
3.9 140.0 N · m ↺.
3.10 61.6 N · m ↺.
3.12 520 lb.
3.14 (*a*) −23**i** − 11**j** + 2**k**. (*b*) −30**j** + 18**k**. (*c*) 0.
3.15 (*a*) and (*b*) −(2160 lb · in)**i** + (4320 lb · in)**j** + (360 lb · in)**k**.
3.16 (36 N · m)**i** + (24 N · m)**j** + (32 N · m)**k**.
3.17 (*a*) −(7200 lb · ft)**i** − (1600 lb · ft)**j** + (3200 lb · ft)**k**.
(*b*) (5600 lb · ft)**j** + (3200 lb · ft)**k**.
3.18 (*a*) −(1200 lb · ft)**j** + (2400 lb · ft)**k**.
(*b*) (5400 lb · ft)**i** + (4200 lb · ft)**j** + (2400 lb · ft)**k**.
3.19 (7.50 N · m)**i** − (6.00 N · m)**j** − (10.39 N · m)**k**.
3.20 (492 lb · ft)**i** + (144 lb · ft)**j** − (372 lb · ft)**k**.
3.23 4.86 ft.
3.24 207 mm.
3.25 **P** · **Q** = 0; **P** · **S** = −11; **Q** · **S** = 2.
3.27 (*a*) 59.0°. (*b*) 720 N.
3.28 (*a*) 70.5°. (*b*) 300 N.
3.29 63.6°.
3.31 (*a*) and (*b*) 26.8°.
3.33 **P** · (**Q** × **S**) = −1; (**P** × **Q**) · **S** = −1; (**S** × **Q**) · P = 1.
3.34 −6.
3.35 M_x = 24.0 kN · m, M_y = −16.00 kN · m,
M_z = −38.4 kN · m.
3.39 M_x = −1598 N · m, M_y = 959 N · m, M_z = 0.
3.40 M_x = −1283 N · m, M_y = 770 N · m, M_z = 1824 N · m.
3.41 61.5 lb.
3.42 6.23 ft.
3.43 (*a*) −299 lb · in. (*b*) 212 lb · in.
3.44 (*a*) 144.0 lb · in. (*b*) 127.1 lb · in.
3.46 124.2 N · m.
3.48 −176.6 lb · ft.
3.49 (*a*) 271 N. (*b*) 390 N. (*c*) 250 N.
3.50 280 lb · in ↻.
3.51 (*a*) 7.33 N · m ↺. (*b*) 91.6 mm.
3.52 (*a*) 26.7 N. (*b*) 50.0 N. (*c*) 23.5 N.
3.53 (*a*) 1170 lb · in ↺.
(*b*) With pegs *A* and *D*: ∠ 53.1° at *A*, ∠ 53.1° at *D*;
with pegs *B* and *C*: ∠ 53.1° at *B*, ∠ 53.1° at *C*.
(*c*) 70.9 lb.
3.54 d = 1.125 in.
3.56 M = 13.00 lb · ft; θ_x = 67.4°, θ_y = 90.0°, θ_z = 22.6°.
3.57 M = 3.22 N · m; θ_x = 90.0°, θ_y = 53.1°, θ_z = 36.9°.
3.58 M = 2.72 N · m; θ_x = 134.9°, θ_y = 58.0°, θ_z = 61.9°.
3.59 M = 2150 lb · ft; θ_x = 113.0°, θ_y = 92.7°, θ_z = 23.2°.
3.61 (*a*) 60.0 lb ↓, 450 lb · in ↺.
(*b*) **B** = 100.0 lb ←; **C** = 100.0 lb →.
3.63 (*a*) 960 N ∠ 60°, 28.9 mm to the right of *O*.
(*b*) 960 N ∠ 60°, 50.0 mm below *O*.
3.65 (*a*) 300 N ∠ 30°, 75.0 N · m ↺.
(*b*) **B** = 800 N ∠ 30°, **C** = 500 N ∠ 30°.
3.66 (*a*) **P** = 60.0 lb ∠ 50°; 3.24 in. from *A*.
(*b*) **P** = 60.0 lb ∠ 50°; 3.87 in. below *A*.
3.67 −(250 kN)**j**; (15.00 kN · m)**i** + (7.50 kN · m)**k**.
3.68 (4.00 kips)**i**; −(3.18 kip · in)**j** − (16.00 kip · in)**k**.
3.71 **F** = −(2.40 kips)**j** + (1.000 kip)**k**, **M** = (15.00 kip · ft)**i** −
(10.00 kip · ft)**j** − (24.0 kip · ft)**k**.
3.72 **F** = −(173.2 N)**j** + (100.0 N)**k**, **M** = (7.50 N · m)**i** −
(6.00 N · m)**j** − (10.39 N · m)**k**.
3.73 Loadings c and f.
3.74 Loading e.
3.75 (*a*) 2.00 m from front axle.
(*b*) 50.0 kN located 2.80 m from front axle.
3.76 1300 lb ↓ at 8.69 ft to the right of *A*.
3.77 (*a*) 0.600 m. (*b*) 1.000 m. (*c*) 1.800 m.
3.80 (*a*) 1562 N ∠ 50.2°, 300 N · m ↺.
(*b*) 250 mm to the right of *C* and 300 mm above *C*.
3.81 (*a*) 29.9 lb ∠ 23.0°.
(*b*) 1.70 in. to the right of *A* and 3.64 in. above *C*.
3.82 (*a*) 100.0 lb ∠ 36.9°; at *A*.
(*b*) 100.0 lb ∠ 36.9°; 8.00 in. to the right of *B* on *BC*.
(*c*) 100.0 lb ∠ 36.9°; 3.00 in. below *C* on *CD*.
3.83 (*a*) 3.80 kN →; 22.8 kN · m ↺.
(*b*) 3.80 kN →; 6.00 m below *DE*.
3.84 329 kN ∠ 61.7°; 6.82 m to the right of *A*.
3.85 −(100 lb)**i** − (900 lb)**j** − (200 lb)**k**; −(1200 lb · ft)**i** −
(600 lb · ft)**k**.
3.87 R = 385 N; θ_x = 141.2°, θ_y = 128.6°, θ_z = 86.3°.
M = 16.50 N · m; θ_x = 100.5°, θ_y = 35.1°, θ_z = 56.9°.
3.89 (*a*) 60°. (*b*) (20.0 lb)**i** − (34.6 lb)**j**; (520 lb · in)**i**.
3.90 **R** = (20.0 lb)**i** − (34.6 lb)**j**; $\mathbf{M}_D^R$ = (520 lb · in)**i** −
(500 lb · in)**k**.
(*a*) neither loosen nor tighten. (*b*) tighten.
3.91 500 kN ↓; 2.56 m from *AD* and 2.00 m from *DC*.
3.92 70.0 kips ↓; at x = 2.50 ft, z = −0.619 ft.
3.95 200 N at y = 63.4 mm, z = 200 mm.
3.96 72.2 N.
3.97 (*a*) −(1200 lb · in)**i** + (4800 lb · in)**j** + (7200 lb · in)**k**.
(*b*) 3090 lb · in.
3.99 M_x = 78.9 kN · m, M_y = 13.15 kN · m,
M_z = −9.86 kN · m.
3.101 23.0 N · m.
3.102 (*a*) 20.0 lb. (*b*) 16.00 lb. (*c*) 12.00 lb.
3.103 (*a*) 500 N ∠ 60°; 86.2 N · m ↻.
(*b*) **A** = 689 N ↑, **B** = 1150 N ∠ 77.4°.
3.105 (*a*) −(75 lb)**j**. (*b*) x = −3.20 in., z = 0.640 in.
3.106 12.00 kips ↓ at 17.33 ft to the right of A.
3.108 P = 72.1 kN ↓ at x = 4.16 m, z = 2.77 m.

CHAPTER 4

4.1 **A** = 200 lb ↓, **B** = 200 lb ↑.
4.2 (*a*) 15.21 kN ↑, (*b*) 5.89 kN ↑.
4.3 8.40 lb ↑.
4.5 1.25 kN $\leq Q \leq$ 27.5 kN.
4.6 1.50 kN $\leq Q \leq$ 9.00 kN.
4.7 60 lb $\leq P \leq$ 560 lb.

4.9 T = 29.9 kips, **A** = 33.0 kips ∠ 31.5°.
4.10 (*a*) $W \cos \theta/\cos \frac{\theta}{2}$. (*b*) 11.74 lb.
4.12 (*a*) 400 N. (*b*) 458 N ∠ 49.1°.
4.14 (*a*) 125 lb ↓. (*b*) 325 lb ∠ 22.6°.
4.15 600 lb.
4.16 (*a*) **A** = 4.27 kN ∠ 20.6°; **B** = 4.50 kN ↑.
(*b*) **A** = 1.50 kN ↓; **B** = 6.02 kN ∠ 48.4°.
(*c*) **A** = 2.05 kN ∠ 47.0°; **B** = 5.20 kN ∠ 60°.
4.17 T_{BE} = 196.2 N, **A** = 73.6 N →, **D** = 73.6 N ←.
4.18 (*a*) **B** = 920 N ∠ 53.1°, **C** = 80 N ∠ 53.1°, **D** = 600 N ↑.
(*b*) rollers 1 and 3.
4.19 (*a*) 128.0 lb. (*b*) **A** = 80.0 lb ↑, **B** = 64.0 lb →.
4.20 11.06 in.
4.23 (*a*) 11.20 kips. (*b*) $|M_E|$ = 28.8 kip · ft.
4.24 (*a*) **A** = 5540 N ∠ 87.3°, **C** = 683 N ∠ 67.4°.
(*b*) **A** = 4900 N ↑, $\mathbf{M}_A$ = 1890 N · m ↺.
(*c*) **A** = 6740 N ∠ 83.6°, $\mathbf{M}_A$ = 3510 N · m ↻, **C** = 1950 N ∠ 67.4°.
4.25 (*a*) 1, 3, 4, 7, and 8 are completely constrained.
2, and 5 are improperly constrained.
6 is partially constrained.
(*b*) Reactions for 1, 3, 6, and 7 are statically determinate.
Reactions for 2, 4, 5, and 8 are statically indeterminate.
(*c*) Equilibrium maintained for any loading for 1, 3, 4, 7, 8.
Equilibrium maintained for given loading for 6.
No equilibrium for 2 and 5.
4.26 (*a*) 1, 2, 3, 5, and 9 are completely constrained.
4 and 6 are partially constrained.
7 and 8 are improperly constrained.
(*b*) Reactions for 1, 2, 4 and 5 are statically determinate.
Reactions for 6 are determined from dynamics.
Reactions for 3, 7, 8, and 9 are statically indeterminate.
(*c*) Equilibrium maintained for any loading for 1, 2, 3, 5, and 9. Equilibrium maintained for given loading for 4.
No equilibrium for 6, 7, and 8.
4.27 **B** = 501 N ∠ 56.3°; **C** = 324 N ∠ 31.0°.
4.28 **A** = 2230 N ∠ 7.7°; **B** = 2210 N →.
4.29 **A** = 124.8 lb ∠ 15.9°; T = 147.5 lb.
4.31 **A** = 185.3 N ∠ 62.4°; T = 92.8 N.
4.32 (*a*) 400 N. (*b*) 458 N ∠ 49.1°.
4.33 **A** = 346 N ∠ 60.6°; **B** = 196.2 N ∠ 30°.
4.34 (*a*) 125 lb ↓. (*b*) 325 lb ∠ 22.6°.
4.37 (*a*) 36.9°. (*b*) **A** = 400 N ↑, **E** = 300 N ←.
4.38 **A** = 97.6 lb ∠ 50.2°; **B** = 62.5 lb ←.
4.40 (*a*) 59.2°. (*b*) T_{AB} = 0.596 W, T_{CD} = 1.164 W.
4.42 **A** = 170.0 lb ∠ 28.1°; **B** = 150.0 lb ←.
4.43 10.00 in.
4.44 **A** = 170.0 N ∠ 33.9°; **C** = 160.0 N ∠ 28.1°.
4.45 **A** = 170.0 N ∠ 56.1°; **C** = 300 N ∠ 28.1°.
4.46 α = 73.9°; T_A = 4160 lb, T_B = 2310 lb.
4.47 **A** = 7.07 lb →; **B** = 40.6 lb ∠ 80.0°.
4.48 **A** = 225 N ∠ 30°; T = 225 N.
4.50 (*a*) $4P - 3Q$. (*b*) 30.0 lb.
4.51 **A** = (120.0 N)**j** + (133.3 N)**k**; **D** = (60.0 N)**j** + (166.7 N)**k**.
4.52 **A** = (125.3 N)**j** + (137.8 N)**k**; **D** = (62.7 N)**j** + (172.2 N)**k**.
4.53 **A** = (24.0 lb)**j** − (2.31 lb)**k**; **B** = (16.00 lb)**j** − (9.24 lb)**k**; **C** = (11.55 lb)**k**.
4.54 (*a*) 96.0 lb. (*b*) **A** = (2.4 lb)**j**; **B** = (214 lb)**j**.
4.56 (*a*) 1039 N. (*b*) **C** = (346 N)**i** + (1200 N)**j**; **D** = −(1386 N)**i** − (480 N)**j**.
4.57 T_A = 30.0 lb, T_B = 10.00 lb, T_C = 40.0 lb.
4.59 T_A = 24.5 N. T_B = 73.6 N, T_C = 98.1 N.
4.61 (*a*) T_{BC} = 975 lb, T_{BD} = 700 lb.
(*b*) **A** = (1500 lb)**i** + (425 lb)**j**.
4.62 (*a*) T_{BC} = 1950 lb, T_{BD} = 1400 lb.
(*b*) **A** = (3000 lb)**i**.
4.64 (*a*) $T_{DE} = T_{DF}$ = 1.284 kN.
(*b*) **A** = −(3.93 kN)**i** + (7.57 kN)**j**.
4.65 **A** = −(56.3 lb)**i**; **B** = −(56.3 lb)**i** + (150.0 lb)**j** − (75.0 lb)**k**
F_{CE} = 202 lb compression.
4.66 (*a*) 2.40 kN. (*b*) −0.600 kN.
4.67 T = 37.5 lb; **A** = (36.3 lb)**i** + (65.6 lb)**j**; **B** = (75.0 lb)**j**.
4.70 P = 118.9 N; **A** = (42.9 N)**i** − (69.9 N)**k**; **B** = (61.1 N)**i** + (196.2 N)**j** + (84.7 N)**k**.
4.71 F_{CE} = 202 lb compression;
B = −(112.5 lb)**i** + (150.0 lb)**j** − (75.0 lb)**k**;
$\mathbf{M}_B$ = −(225 lb · ft)**j**.
4.72 F_{CD} = 19.62 N compression; **B** = −(19.22 N)**i** + (94.2 N)**j**;
$\mathbf{M}_B$ = −(40.6 N · m)**i** − (17.30 N · m)**j**.
4.73 T_{BD} = 780 lb, T_{BE} = 650 lb, T_{CF} = 650 lb; **A** = (1920 lb)**i** − (300 lb)**k**.
4.74 **A** = (600 N)**j** − (750 N)**k**; **B** = (900 N)**i** + (750 N)**k**;
C = −(900 N)**i** + (600 N)**j**.
4.75 Equilibrium; 172.6 N ∠ 25.0°.
4.76 Moves down; 279 N ∠ 30.0°.
4.77 Moves up; 36.1 lb ∠ 30.0°.
4.78 Equilibrium; 36.3 lb ∠ 30.0°.
4.80 5.77 lb.
4.81 $P = W \sin(\alpha + \phi_s)$; $\theta = \alpha + \phi_s$.
4.82 (*a*) 116.2 N ∠ 36.3°. (*b*) 46.5 N ∠ 13.7°.
4.83 (*a*) 403 N. (*b*) 229 N.
4.85 (*a*) 206 N →. (*b*) 177.6 N →. (*c*) 72.5 N →.
4.87 (*a*) 58.1°. (*b*) 166.4 N.
4.88 (*a*) 138.6 N. (*b*) Slide.
4.90 40.0 lb →.
4.91 0.955 lb.
4.93 (*a*) 33.4°. (*b*) 0.287 W.
4.94 Equilibrium; 0.250 W →.
4.95 No equilibrium.
4.97 $Wr\,\mu_s\,(1 + \mu_s)/(1 + \mu_s^2)$
4.98 (*a*) 0.400 Wr. (*b*) 0.464 Wr.
4.99 30.0 kN ≤ P ≤ 210 kN.
4.101 (*a*) **A** = 60.0 lb ↑, **B** = 136.1 lb →, **C** = 32.2 lb ←.
(*b*) **A** = 0, **B** = 120 lb ←, **C** = 240 lb →.
4.102 (*a*) 600 N. (*b*) **A** = 4.00 kN ←; **B** = 4.00 kN →.
4.104 (*a*) 1500 N ∠ 30°. (*b*) 593 N ∠ 30°.
4.105 (*a*) T_B = 24.0 lb, T_B' = 12.00 lb.
(*b*) A_y = 55.2 lb, A_z = −12.49 lb; E_y = 33.4 lb, E_z = −2.50 lb.
A_x and E_x are indeterminate.
4.107 (*a*) **A** = 0.745 P ∠ 63.4°; **D** = 0.471 P ∠ 45°.
(*b*) **A** = P →; **D** = 1.414 P ∠ 45°.
(*c*) **A** = 0.471 P ∠ 45°; **D** = 0.745 P ∠ 63.4°.
(*d*) **A** = 0.707 P ∠ 45°; **C** = 0.707 P ∠ 45°; **D** = P ↑.
4.109 (*a*) 36.3 N T. (*b*) 29.7 N ←.
4.110 (*a*) 8.00 lb. (*b*) 12.00 lb.

CHAPTER 5

5.1 $\bar{x}$ = 55.4 mm, $\bar{y}$ = 93.8 mm.
5.2 $\bar{x}$ = 3.27 in., $\bar{y}$ = 2.82 in.
5.3 $\bar{x}$ = 1.045 in., $\bar{y}$ = 3.59 in.

5.5 $\bar{x} = \bar{y} = 8.09$ in.
5.6 $\bar{x} = \bar{y} = 16.75$ mm.
5.7 $\bar{x} = -62.4$ mm, $\bar{y} = 0$.
5.9 $\bar{x} = 120.0$ mm, $\bar{y} = 60.0$ mm.
5.10 $\bar{x} = 10.11$ in., $\bar{y} = 3.87$ in.
5.11 $\bar{x} = 0$, $\bar{y} = 4.57$ ft.
5.12 $\bar{x} = 386$ mm, $\bar{y} = 66.4$ mm.
5.13 42.25×10^3 mm^3 for A_1, -42.25×10^3 mm^3 for A_2.
5.14 0.2352 in^3 for A_1, -0.2352 in^3 for A_2.
5.17 $\bar{x} = 53.0$ mm, $\bar{y} = 91.5$ mm.
5.18 $\bar{x} = 3.38$ in., $\bar{y} = 2.93$ mm.
5.19 $\bar{x} = 172.5$ mm, $\bar{y} = 97.5$ mm.
5.20 $\bar{x} = 3.19$ in., $\bar{y} = 6.00$ in.
5.21 300 mm.
5.23 (*a*) 5.09 lb. (*b*) 9.48 lb $\measuredangle$ 57.5°.
5.25 $\bar{x} = 2b/3$, $y = h/3$.
5.26 $\bar{x} = 2a/5$, $\bar{y} = 3b/7$.
5.29 $\bar{x} = (n+1)a/(n+2)$, $\bar{y} = (n+1)h/(4n+2)$.
5.30 $\bar{x} = 4a/3\pi$, $\bar{y} = 4b/3\pi$.
5.31 $\bar{x} = 0$, $\bar{y} = 4r/3\pi$.
5.32 $\bar{x} = 3a/8$, $\bar{y} = 3h/5$.
5.33 $\bar{x} = 0.300\,a$.
5.34 $\bar{y} = 0.310\,h$.
5.35 $\bar{x} = \bar{y} = 1.027$ in.
5.36 $\bar{x} = \bar{y} = (2a^2 - 1)/2a(1 + 2\ln a)$.
5.37 (*a*) 584 in^3. (*b*) 679 in^3.
5.39 (*a*) $\pi a^2 h/2$. (*b*) $8\pi a h^2/15$.
5.41 (*a*) 0.226 m^3. (*b*) 131.2 kg.
5.42 1.508 m^2.
5.43 314 in^2.
5.44 $V = 655$ in^3; $W = 23.6$ lb.
5.45 $V = 3.96$ in^3; $W = 1.211$ lb.
5.48 300×10^3 mm^3.
5.49 $\mathbf{R} = 9.45$ kN $\downarrow$, $\bar{x} = 2.57$ m; $\mathbf{A} = 4.05$ kN $\uparrow$, $\mathbf{B} = 5.40$ kN $\uparrow$.
5.51 $\mathbf{A} = 1260$ lb $\uparrow$, $\mathbf{M}_A = 14040$ lb · in $\circlearrowleft$.
5.53 $\mathbf{B} = 1200$ N $\uparrow$, $\mathbf{M}_B = 800$ N · m $\circlearrowleft$.
5.54 $\mathbf{A} = 10800$ lb $\uparrow$, $\mathbf{B} = 3600$ lb $\uparrow$.
5.55 $\mathbf{A} = 2860$ lb $\uparrow$, $\mathbf{B} = 740$ lb $\uparrow$.
5.56 $\mathbf{A} = 105$ N $\uparrow$, $\mathbf{B} = 270$ N $\uparrow$.
5.57 $21\,h/16$ above the vertex of the cone.
5.58 (*a*) $0.448\,h$. (*b*) $0.425\,h$.
5.59 0.707.
5.60 $\bar{x} = 0$, $\bar{y} = -0.608\,h$, $\bar{z} = 0$.
5.61 0.610 in.
5.63 40.3 mm.
5.65 $\bar{x} = 105.2$ mm, $\bar{y} = 175.8$ mm, $\bar{z} = 105.2$ mm.
5.66 $\bar{x} = 0.0729$ in., $\bar{y} = -1.573$ in., $\bar{z} = 0$.
5.69 $\bar{x} = 205$ mm, $\bar{y} = 255$ mm, $\bar{z} = 75$ mm.
5.70 $\bar{x} = 0$, $\bar{y} = 10.05$ in., $\bar{z} = 5.15$ in.
5.71 $\bar{x} = 0$, $\bar{y} = 3.44$ in., $\bar{z} = 0$.
5.72 On center axis, 27.6 mm above base.
5.73 $\bar{x} = 105.0$ mm, $\bar{y} = 90.0$ mm.
5.75 $\bar{x} = 105.6$ mm, $\bar{y} = 97.6$ mm.
5.76 (*a*) $1.427\,r$. (*b*) $2.113\,r$.
5.77 $\bar{x} = 1.607\,a$, $\bar{y} = 0.332\,h$.
5.78 $0.611\,L$.
5.79 275×10^3 mm^3.
5.81 $\mathbf{B} = 5657$ lb $\uparrow$, $\mathbf{C} = 643$ lb $\uparrow$.
5.83 $\bar{x} = 3.79$ in., $\bar{y} = 0.923$ in., $\bar{z} = 3.00$ in.
5.84 On vertical symmetry axis 81.8 mm above the base.

CHAPTER 6

6.1 $F_{AB} = 1600$ lb C, $F_{AC} = 2000$ lb T, $F_{BC} = 1709$ lb T.
6.2 $F_{AB} = 52.0$ kN T, $F_{AC} = 64.0$ kN T, $F_{BC} = 80.0$ kN C.
6.3 $F_{AB} = 1080$ lb T, $F_{BC} = 1170$ lb C, $F_{AC} = 1800$ lb C.
6.4 $F_{AD} = 125.0$ kN T, $F_{CD} = 120.0$ kN C, $F_{AB} = 175.0$ kN T, $F_{AC} = 84.0$ kN C, $F_{BC} = 120.0$ kN C.
6.6 $F_{BA} = 3900$ N T, $F_{BC} = 3600$ N C, $F_{CA} = 4500$ N C.
6.8 $F_{AB} = 0$, $F_{AD} = 5.00$ kN C, $F_{BD} = 34.0$ kN C, $F_{DE} =$ 30.0 kN T, $F_{BE} = 12.00$ kN T.
6.9 $F_{BD} = 0$, $F_{AB} = 12.00$ kips C, $F_{AC} = 5.00$ kips C, $F_{AD} =$ 13.00 kips T, $F_{CD} = 30.0$ kips C, $F_{DF} = 5.00$ kips T, $F_{CF} =$ 32.5 kips T, $F_{CE} = 17.5$ kips C, $F_{EF} = 0$.
6.10 $F_{BE} = 5.00$ kN T, $F_{DE} = 4.00$ kN C, $F_{AB} = 4.00$ kN T, $F_{BD} = 9.00$ kN C, $F_{AD} = 15.00$ kN T, $F_{CD} = 16.00$ kN C.
6.11 $F_{AD} = 260$ lb C, $F_{DC} = 125.0$ lb T, $F_{BE} = 832$ lb C, $F_{CE} =$ 400 lb T, $F_{AC} = 400$ lb T, $F_{BC} = 125.0$ lb T, $F_{AB} = 420$ lb C.
6.12 $F_{DA} = 41.2$ kips T, $F_{DC} = 40.0$ kips C, $F_{CA} = 22.4$ kips T, $F_{CB} = 60.0$ kips C, $F_{BA} = 0$.
6.13 $F_{EC} = 360$ lb T, $F_{ED} = 390$ lb C, $F_{DB} = 360$ lb C, $F_{DC} =$ 150.0 lb T, $F_{CA} = 390$ lb T, $F_{CB} = 0$.
6.15 $F_{CD} = 24.0$ kips T, $F_{DH} = 26.0$ kips C, $F_{CH} = 0$, $F_{GH} =$ 26.0 kips C, $F_{CG} = 0$, $F_{BC} = 24.0$ kips T, $F_{BG} = 0$, $F_{FG} =$ 26.0 kips C, $F_{BF} = 0$, $F_{AB} = 24.0$ kips T, $F_{AF} = 30.0$ kips C, $F_{AE} = 38.4$ kips, $F_{EF} = 24.0$ kips C.
6.17 $F_{AB} = 15.00$ kN T, $F_{AD} = 17.00$ kN C, $F_{BC} = 15.00$ kN T, $F_{CE} = 8.00$ kN T, $F_{EF} = 8.00$ kN T, $F_{DF} = 17.00$ kN C, $F_{BE} = 0$, $F_{BD} = 0$, $F_{DE} = 0$.
6.18 $F_{AB} = F_{DE} = 8.00$ kN C, $F_{AF} = F_{HE} = 6.93$ kN T, $F_{FG} =$ $F_{GH} = 6.93$ kN T, $F_{BF} = F_{DH} = 4.00$ kN T, $F_{BC} = F_{CD} =$ 4.00 kN C, $F_{BG} = F_{DG} = 4.00$ kN C, $F_{CG} = 4.00$ kN T.
6.19 6.17 and 6.21 are simple trusses.
6.23 is not a simple truss.
6.20 6.12, 6.14, and 6.24 are simple trusses.
6.22 is not a simple truss.
6.21 *EI, BE, FG, GH, IJ, HI.*
6.24 *FJ, EJ, EB, BD, DH, AH, AG.*
6.25 $F_{BD} = 36.0$ kips C, $F_{CD} = 45.0$ kips C.
6.26 $F_{DF} = 60.0$ kips C, $F_{DG} = 15.00$ kips C.
6.27 $F_{FG} = 70.0$ kN C, $F_{FH} = 240$ kN T.
6.28 $F_{EF} = 69.5$ kN T, $F_{EG} = 250$ kN C.
6.29 $F_{DE} = 25.0$ kips T, $F_{DF} = 13.00$ kips C.
6.31 $F_{DF} = 91.4$ kN T, $F_{DE} = 38.6$ kN C.
6.33 $F_{BD} = 37.5$ kN T, $F_{DE} = 22.5$ kN T.
6.34 $F_{FH} = 12.50$ kN T, $F_{DH} = 90.0$ kN T.
6.35 $F_{FH} = 16.97$ kips T, $F_{GH} = 12.00$ kips C, $F_{GI} = 18.00$ kips C.
6.37 $F_{CE} = 40.0$ kN C, $F_{DE} = 16.00$ kN C, $F_{DF} = 40$ kN T.
6.39 $F_{AD} = 3.38$ kips C, $F_{CD} = 0$, $F_{CE} = 14.03$ kips T.
6.40 $F_{DG} = 18.75$ kips C, $F_{FG} = 14.03$ kips T, $F_{FH} =$ 17.43 kips T.
6.41 22.5 kN C.
6.42 $F_{AB} = 0.833$ P(T), $F_{KL} = 1.167$ P(T).
6.44 $F_{BE} = 10.00$ kips T, $F_{EF} = 5.00$ kips T, $F_{DE} = 0$.
6.45 $F_{BE} = 12.50$ kips T, $F_{EF} = 2.50$ kips T, $F_{DE} = 0$.
6.47 (*a*) Completely constrained and indeterminate.
(*b*) Completely constrained and determinate.
(*c*) Partially constrained.
6.48 (*a*) Partially constrained.
(*b*) Completely constrained and determinate.
(*c*) Completely constrained and indeterminate.

6.49 F_{BD} = 1750 N C; $\mathbf{C}_x$ = 1400 N ←, $\mathbf{C}_y$ = 700 N ↓.
6.50 F_{BD} = 300 lb T; $\mathbf{C}_x$ = 150.0 lb ←, $\mathbf{C}_y$ = 180.0 lb ↑.
6.51 F_{BD} = 375 N C; $\mathbf{C}_x$ = 205 N ←, $\mathbf{C}_y$ = 360 N ↓.
6.52 $\mathbf{A}_x$ = 120.0 lb →, $\mathbf{A}_y$ = 30.0 lb ↑; $\mathbf{B}_x$ = 120.0 lb ←, $\mathbf{B}_y$ = 80.0 lb ↓; **C** = 30.0 lb ↓, **D** = 80.0 lb ↑.
6.53 **A** = 150.0 lb →; $\mathbf{B}_x$ = 150.0 lb ←, $\mathbf{B}_y$ = 60.0 lb ↑; **C** = 20.0 lb ↑; **D** = 80.0 lb ↓.
6.55 (*a*) 2.44 kN ∠ 8.4°. (*b*) 1.930 kN ∠ 51.3° on each arm.
6.57 **B** = 152.0 lb ↓; $\mathbf{C}_x$ = 60.0 lb ←, $\mathbf{C}_y$ = 200 lb ↑; $\mathbf{D}_x$ = 60.0 lb →, $\mathbf{D}_y$ = 42.0 lb ↑.
6.58 (*a*) 1465 kN T. (*b*) 1105 kN C. (*c*) 1663 kN ∠ 62.0°.
6.59 (*a*) $\mathbf{D}_x$ = 750 N →, $\mathbf{D}_y$ = 250 N ↓; $\mathbf{E}_x$ = 750 N ←, $\mathbf{E}_y$ = 250 N ↑. (*b*) $\mathbf{D}_x$ = 375 N →, $\mathbf{D}_y$ = 250 N ↓; $\mathbf{E}_x$ = 375 N ←, $\mathbf{E}_y$ = 250 N ↑.
6.61 (*a*) **A** = 78.0 lb ∠ 22.6°, **C** = 144.0 lb →. **G** = 72.0 lb ←, **I** = 30.0 lb ↑.
(*b*) **A** = 78.0 lb ∠ 22.6°, **C** = 72.0 lb →, **G** = 0, **I** = 30.0 lb ↑.
6.62 (*a*) **A** = 78.0 lb ∠ 22.6°, **C** = 144.0 lb →, **G** = 72.0 lb ←, **I** = 30.0 lb ↑.
(*b*) **A** = 78.0 lb ∠ 22.6°, **C** = 120.0 lb →, **G** = 96.0 lb ←, **I** = 30.0 lb ↑.
6.64 (*a*) 828 N T. (*b*) 1197 N ∠ 86.2°.
6.65 $\mathbf{A}_x$ = 250 lb ←, $\mathbf{A}_y$ = 600 lb ↑; $\mathbf{C}_x$ = 250 lb →, $\mathbf{C}_y$ = 600 lb ↑; B_x = 790 lb ←, $\mathbf{B}_y$ = 0.
6.66 (*a*) $\mathbf{E}_x$ = 960 lb ←, $\mathbf{E}_y$ = 1280 lb ↑.
(*b*) $\mathbf{C}_x$ = 2640 lb ←, $\mathbf{C}_y$ = 3520 lb ↑.
6.67 $\mathbf{D}_x$ = 13.60 kN →, $\mathbf{D}_y$ = 7.50 kN ↑; $\mathbf{E}_x$ = 13.60 kN ←, $\mathbf{E}_y$ = 2.70 kN ↓.
6.69 (*a*) *A*: 15.76 kips ↑, *B*: 26.2 kips ↑ (each wheel)
(*b*) **C** = 34.6 kips ←; $\mathbf{D}_x$ = 34.6 kips →, $\mathbf{D}_y$ = 2.48 kips ↓.
6.71 (*a*) *A*: 117.5 kN ↑, *B*: 176.9 kN ↑ (each wheel)
(*b*) **C** = 8.28 kN →, $\mathbf{D}_x$ = 8.28 kN ←, $\mathbf{D}_y$ = 256 kN ↓.
6.72 (*a*) *A*: 3980 N ↑, *B*: 4170 N ↑, *C*: 2890 N ↑
(*b*) *B*: 1326 N, *C*: −398 N. (each wheel).
6.73 (*a*) 1200 N →. (*b*) 1230 N ∠ 12.7°.
6.74 (*a*) 103.6 lb ←. (*b*) 114.7 lb T.
6.75 (*a*) 2860 N ↓. (*b*) 2700 N ∠ 68.5°.
6.76 T_{DE} = 18.00 lb; **B** = 48.0 lb ↓.
6.78 **C** = 4.65 kips →; **E** = 6.14 kips ∠ 40.7°.
6.80 $\mathbf{A}_x$ = 210 N ←, $\mathbf{A}_y$ = 2400 N ↓; **B** = 2720 N ∠ 61.9°; **C** = 1070 N ←.
6.81 (*a*) 252 N · m ↻. (*b*) 108.0 N · m ↻.
6.82 (*a*) 3.00 kN ↓. (*b*) 7.00 kN ↓.
6.83 (*a*) 1261 lb · in. ↺. (*b*) $\mathbf{C}_x$ = 54.3 lb ←, $\mathbf{C}_y$ = 21.7 lb ↑.
6.85 (*a*) 2500 N. (*b*) 2760 N ∠ 63.1°.
6.86 14 800 lb.
6.88 720 lb.
6.89 18.75 lb.
6.91 140.0 N.
6.92 260 N.
6.94 *EF*: 9.61 kips C; *CD*: 4.27 kips T; *AB*: 18.97 kips C.
6.95 *AB*: 1.051 kN C; *DE*: 40.8 kN T; *FI*: 4.74 kN C.
6.96 (*a*) 3000 lb T. (*b*) $\mathbf{H}_x$ = 2400 lb ←, $\mathbf{H}_y$ = 4800 lb ↓.
6.97 F_{AC} = 80.0 kN T, F_{CE} = 45.0 kN T, F_{DE} = 51.0 kN C, F_{BD} = 51.0 kN C, F_{CD} = 48.0 kN T, F_{BC} = 19.00 kN C.
6.99 F_{EF} = 2400 lb T, F_{FG} = 1500 lb C, F_{GI} = 2600 lb C.
6.100 F_{CE} = 4690 lb T, F_{CD} = 3600 lb C, F_{CB} = 0.
6.101 7.36 kN C.
6.103 $\mathbf{A}_x$ = 3.32 kN ←, $\mathbf{A}_y$ = 14.26 kN ↓; $\mathbf{C}_x$ = 3.72 kN →, $\mathbf{C}_y$ = 14.26 kN ↑.
6.104 28.6 lb.
6.106 F_s = 1611 lb C; **A** = 500 lb ←; $\mathbf{D}_x$ = 500 lb →, $\mathbf{D}_y$ = 861 lb ↓.
6.108 Case (1) (*a*) $\mathbf{A}_x$ = 0, $\mathbf{A}_y$ = 7.85 kN ↑, $\mathbf{M}_A$ = 15.70 kN · m ↺.
(*b*) **D** = 22.2 kN ∠ 45°.
Case (2) (*a*) $\mathbf{A}_x$ = 0, $\mathbf{A}_y$ = 3.92 kN ↑, $\mathbf{M}_A$ = 8.34 kN · m ↺.
(*b*) **D** = 11.10 kN ∠ 45°.
Case (3) (*a*) $\mathbf{A}_x$ = 0, $\mathbf{A}_y$ = 3.92 kN ↑, $\mathbf{M}_A$ = 8.34 kN · m ↺.
(*b*) **D** = 18.95 kN ∠ 45°
Case (4) (*a*) $\mathbf{A}_x$ = 3.92 kN →, $\mathbf{A}_y$ = 3.92 kN ↑, $\mathbf{M}_A$ = 2.35 kN · m ↻.
(*b*) **D** = 11.10 kN ∠ 45°.

CHAPTER 7

7.1 $a^3(h_1 + 3h_2)/12$.
7.2 $2a^3b/7$.
7.3 $ha^3/5$.
7.4 $a^3b/20$.
7.5 $a(h_1^2 + h_2^2)(h_1 + h_2)/12$.
7.6 $2ab^3/15$.
7.9 $I_x = ab^3/30$; $r_x = 0.365\ b$.
7.10 $I_x = \pi ab^3/8$; $r_x = 0.500\ b$.
7.11 $I_x = ab^3/9$; $r_x = 0.430\ b$.
7.12 $I_x = 3ab^3/35$; $r_x = 0.507\ b$.
7.13 $I_y = a^3b/6$; $r_y = 0.816\ a$.
7.14 $I_y = \pi a^3b/8$; $r_y = 0.500\ a$.
7.17 (*a*) $J_O = 4a^4/3$; $r_O = 0.816\ a$.
(*b*) $J_O = 17a^4/6$; $r_O = 1.190\ a$.
7.18 $J_O = 10a^4/3$; $r_O = 1.291\ a$.
7.20 $J_O = \pi ab(a^2 + b^2)/8$; $r_O = 0.500\sqrt{a^2 + b^2}$.
7.21 (*a*) $J_O = \pi(R_2^4 - R_1^4)/2$; $I_x = \pi(R_2^4 - R_1^4)/4$.
7.23 $4a^3/9$.
7.24 $0.935\ a$.
7.25 $I_x = 614 \times 10^3$ mm^4; $r_x = 19.01$ mm.
7.26 $I_x = 28.0$ in^4; $r_x = 2.25$ in.
7.27 $I_x = 501 \times 10^6$ mm^4; $r_x = 149.4$ mm.
7.30 $I_y = 6.99$ in^4; $r_y = 1.127$ in^4.
7.31 $I_y = 150.3 \times 10^6$ mm^4; $r_y = 81.9$ mm.
7.32 $I_y = 185.4$ in^4; $r_y = 2.81$ in.
7.33 $A = 3000$ mm^2; $\bar{I} = 325 \times 10^3$ mm^4.
7.35 $\bar{I}_x = 204$ in^4; $\bar{I}_y = 135.0$ in^4.
7.36 $\bar{I}_x = 2.08 \times 10^6$ mm^4; $\bar{I}_y = 2.08 \times 10^6$mm^4.
7.38 $J_c = 379$ in^4.
7.39 (*a*) 11.57×10^6 mm^4. (*b*) 7.81×10^6 mm^4.
7.40 (*a*) 129.2 in^4. (*b*) 25.8 in^4.
7.41 (*a*) 512 in^4; (*b*) = 366 in^4.
7.43 $\bar{I}_x = 186.7 \times 10^6$ mm^4; $\bar{r}_x = 118.6$ mm;
$\bar{I}_y = 167.7 \times 10^6$ mm^4; $\bar{r}_y = 112.4$ mm.
7.44 227 mm.
7.45 $\bar{I}_x = 325$ in^4; $\bar{I}_y = 41.8$ in^4.
7.46 $\bar{I}_x = 9.54$ in^4; $\bar{I}_y = 104.5$ in^4.
7.47 $\bar{I}_x = 7.04 \times 10^6$ mm^4; $\bar{I}_y = 63.9 \times 10^6$ mm^4.
7.48 $\bar{I}_x = 7.32 \times 10^6$ mm^4; $\bar{I}_y = 101.3 \times 10^6$ mm^4.
7.49 $a^3b/28$.
7.51 $I_x = 0.0945\ ah^3$; $r_x = 0.402\ h$.
7.53 $J_O = 0.1804\ a^4$; $r_O = 0.645\ a$.
7.54 $\bar{I}_x = 1.268 \times 10^6$ mm^4; $\bar{I}_y = 339 \times 10^3$ mm^4.
7.55 (*a*) $I_x = 174.7$ in^4; $I_y = 1851$ in^4. (*b*) 22.4 in.
7.56 3.78×10^6 mm^4.

7.58 $J_c = 25.1\text{ in}^4$; $r_c = 1.606\ a$.
7.59 (*a*) 185.9 in^4. (*b*) 154.0 in^4.
7.60 $\bar{I}_x = 6120\text{ in}^4$; $\bar{r}_x = 7.90$ in.; $\bar{I}_y = 1360\text{ in}^4$; $\bar{r}_y = 3.73$ in.

CHAPTER 8

8.1 (*a*) 35.7 MPa. (*b*) 42.4 MPa.
8.2 $d_1 = 25.2$ mm, $d_2 = 16.52$ mm.
8.3 (*a*) 12.73 ksi. (*b*) −2.83 ksi.
8.4 18.46 kips.
8.6 62.7 kN.
8.7 1.084 ksi.
8.8 (*a*) 14.64 ksi. (*b*) −9.96 ksi.
8.9 8.52 ksi.
8.10 4.29 in^2.
8.11 (*a*) 17.86 kN. (*b*) −41.4 MPa.
8.12 (*a*) 12.73 MPa. (*b*) −4.77 MPa.
8.14 43.4 mm.
8.16 12.57 kips.
8.17 321 mm.
8.18 178.6 mm.
8.20 (*a*) 1.030 in. (*b*) 38.8 ksi.
8.21 (*a*) 7.28 ksi. (*b*) 18.30 ksi.
8.22 (*a*) 10.84 ksi. (*b*) 5.11 ksi.
8.24 8.31 kN.
8.25 $\sigma = 55.1$ psi, $\tau = 65.7$ psi.
8.26 (*a*) 3290 lb. (*b*) 75.5 psi.
8.27 $\sigma = 565$ kPa, $\tau = 206$ kPa.
8.28 (*a*) 5.31 kN. (*b*) 182.0 kPa.
8.30 (*a*) 180.0 kips. (*b*) 45°. (*c*) −2.5 ksi. (*d*) −5 ksi.
8.31 $\sigma = -37.1$ MPa, $\tau = 17.28$ MPa.
8.33 168.1 mm^2.
8.34 3.64.
8.35 4.55 kips.
8.36 (*a*) 13.47 mm. (*b*) 14.61 mm.
8.38 1.800.
8.39 4.49 kips.
8.41 (*a*) 1.550 in. (*b*) 8.05 in.
8.42 3.47.
8.44 3.97 kN.
8.46 283 lb.
8.47 2.42.
8.48 2.05.
8.49 (*a*) 3.33 MPa. (*b*) 525 mm.
8.51 0.408 in.
8.53 (*a*) −640 psi. (*b*) −320 psi.
8.54 9.22 kN.
8.55 (*a*) 9.94 ksi. (*b*) 6.25 ksi.
8.57 15.08 kN.
8.58 3.49.
8.60 $21.3° \le \theta \le 32.3°$.

CHAPTER 9

9.1 (*a*) 0.0303 in. (*b*) 15.28 ksi.
9.2 (*a*) 81.8 MPa. (*b*) 1.712.
9.3 (*a*) 0.01819 in. (*b*) 7.70 ksi.
9.4 (*a*) 11.31 kN. (*b*) 400 MPa.
9.6 (*a*) 0.1784 in. (*b*) 58.6 in.
9.8 (*a*) 17.25 MPa. (*b*) 2.82 mm.
9.9 48.4 kips.
9.11 1.988 kN.
9.12 0.429 in.
9.13 (*a*) 9.53 kips. (*b*) 1.254×10^{-3} in.
9.14 (*a*) 32.8 kN. (*b*) 0.0728 mm.
9.15 (*a*) 0.01819 mm. (*b*) −0.0909 mm.
9.17 (*a*) 5.62×10^{-3} in. (*b*) 8.52×10^{-3} in. ↓. (*c*) 16.30 ksi.
9.18 (*a*) 2.95 mm. (*b*) 5.29 mm.
9.19 50.4 kN.
9.20 $S_{AB} = -0.0753$ in., $S_{AD} = 0.0780$ in.
9.21 (*a*) 0.1727 in. (*b*) 0.1304 in.
9.23 0.1095 mm.
9.25 (*a*) 47.5 MPa. (*b*) 0.1132 mm.
9.26 (*a*) 75.9 kN. (*b*) 120 MPa.
9.27 steel: −8.34 ksi; concrete: −1.208 ksi.
9.28 695 kips.
9.30 (*a*) 62.8 kN ← at *A*; 37.2 kN ← at *E*. (*b*) 46.3 μm →.
9.32 (*a*) 11.92 kips ← at *A*; 20.08 kips ← at *D*. (*b*) 3.34×10^{-3} in.
9.33 177.4 lb.
9.35 *A*: 0.525 *P*; *B*: 0.200 *P*; *C*: 0.275 *P*.
9.36 *A*: 0.1 *P*; *B*: 0.2 *P*; *C*: 0.3 *P*; *D*: 0.4 *P*.
9.37 75.4 °C.
9.39 steel: −1883 psi; concrete: 53.6 psi.
9.40 (*a*) −17.91 ksi. (*b*) −2.42 ksi.
9.41 (*a*) *AB*: −44.4 MPa; *BC*: −100.0 MPa. (*b*) 0.500 mm ↓.
9.42 (*a*) *AB*: −21.1 ksi; *BC*: −6.50 ksi. (*b*) 0.00364 in. ↑.
9.44 (*a*) 217 kN. (*b*) 0.2425 mm.
9.46 (*a*) −22.1 ksi. (*b*) 0.01441 in.
9.47 (*a*) −7.55 ksi. (*b*) 10.00467 in.
9.48 (*a*) 21.4°C. (*b*) 3.68 MPa.
9.50 $E = 216$ MPa, $\nu = 0.451$, $G = 74.5$ MPa.
9.52 422 kN.
9.53 1.99551:1.
9.54 (*a*) 1.324×10^{-3} in. (*b*) -99.3×10^{-6} in. (*c*) -12.41×10^{-6} in. (*d*) $-12.41 \times 10^{-6}\text{ in}^2$.
9.55 (*a*) 5.13×10^{-3} in. (*b*) -0.570×10^{-3} in.
9.56 (*a*) 7630 lb compression. (*b*) 4580 lb compression.
9.57 −0.0518%.
9.58 (*a*) 0.0754 mm. (*b*) 0.1028 mm. (*c*) 0.1220 mm.
9.61 1.091 mm ↓.
9.63 105.6×10^3 lb/in.
9.64 (*a*) 262 mm. (*b*) 21.4 mm.
9.65 (*a*) 13.31 ksi. (*b*) 18.72 ksi.
9.67 (*a*) 58.3 kN. (*b*) 64.3 kN.
9.68 (*a*) 87.0 MPa. (*b*) 75.2 MPa. (*c*) 73.9 MPa.
9.70 58.1 kN.
9.71 (*a*) 0.475 in. (*b*) 7.50 kips.
9.72 0.866 in.
9.73 1.219 in.
9.75 $x = 92.6$ mm.
9.76 0.0455 in. at $\phi = 8.51°$.
9.77 *A*: 0.237 mm ←; *B*: 0.296 mm →; *C*: 2.43 mm →.
9.80 (*a*) 14.72 kips → at *A*; 12.72 kips ← at *D*. (*b*) -1.574×10^{-3} in.
9.82 $a = 0.818$ in., $b = 2.42$ in.
9.83 (*a*) 9 mm. (*b*) 62 kN.
9.84 (*a*) 134.7 MPa. (*b*) 135.3 MPa.

CHAPTER 10

10.1 641 N · m.
10.2 87.3 MPa.
10.3 (*a*) 9.92 ksi. (*b*) 2.23 in.
10.4 (*a*) 7.63 kip · ft. (*b*) 16.19 kip · ft.
10.6 (*a*) 828 lb · in. (*b*) 1196 lb · in.
10.7 (*a*) 75.5 MPa. (*b*) 63.7 MPa.
10.9 (*a*) *BC*. (*b*) 8.15 ksi.
10.10 (*a*) *AB*. (*b*) 8.49 ksi.
10.12 42.8 mm.
10.13 9.16 kip · in.
10.15 3.37 kN · m.
10.16 (*a*) 50.3 mm. (*b*) 63.4 mm.
10.17 *AB*: 42.0 mm; *BC*: 33.3 mm.
10.18 *AB*: 52.9 mm; *BC*: 33.3 mm.
10.20 (*a*) 0.602 in. (*b*) 0.835 in.
10.21 (*a*) 72.5 MPa. (*b*) 68.7 MPa.
10.23 (*a*) 1.442 in. (*b*) 1.233 in.
10.24 4.30 kip · in.
10.25 (*a*) 2.83 kip · in. (*b*) 13.00°.
10.26 (*a*) 3.62°. (*b*) 4.51°.
10.27 11.91 mm.
10.28 9.38 ksi.
10.30 (*a*) 8.54°. (*b*) 2.11°.
10.32 (*a*) 0.741°. (*b*) 1.573°.
10.33 7.94°.
10.34 4.52°.
10.36 1.914°.
10.37 36.1 mm.
10.39 2.05 in.
10.40 3.07°.
10.41 (*a*) 8.93 ksi. (*b*) 4.14 ksi. (*c*) 3.90°.
10.42 3.71°.
10.44 7.37°.
10.45 (*a*) *A*: 1105 N · m; *C*: 295 N · m. (*b*) 45.0 MPa. (*c*) 27.4 MPa.
10.47 (*a*) 47.1 MPa. (*b*) 0.779°.
10.48 (*a*) 70.7 MPa. (*b*) 1.169°.
10.49 12.44 ksi.
10.50 4.12 kip · in.
10.52 (*a*) 19.21 kip · in. (*b*) 2.01 in.
10.53 (*a*) 10.74 kN · m. (*b*) 22.8 kN · m.
10.55 6.02°.
10.56 127.8 kip · in.
10.58 3.79°.
10.60 12.24 MPa.

CHAPTER 11

11.1 (*a*) −116.4 MPa. (*b*) −87.3 MPa.
11.2 (*a*) −2.38 ksi. (*b*) −0.650 ksi.
11.3 80.2 kN · m.
11.4 24.8 kN · m.
11.6 (*a*) 1.405 kip · in. (*b*) 3.19 kip · in.
11.7 259 kip · in.
11.9 top: −14.71 ksi; bottom: 8.82 ksi.
11.10 top: −81.8 MPa; bottom: 67.8 MPa.
11.12 (*a*) 83.7 MPa. (*b*) −146.4 MPa. (*c*) 14.67 MPa.
11.13 2.22 kips.
11.14 2.05 kips.
11.16 37.9 kN.
11.17 7.67 kN · m.
11.18 20.4 kip · in.
11.19 7.39 kip · in.
11.20 849 N · m.
11.22 1.372 kip · in.
11.24 (*a*) 53.2 MPa; 382 m. (*b*) 157.9 MPa; 128.3 m.
11.25 1.240 kN · m.
11.26 887 N · m.
11.27 720 N · m.
11.29 330 kip · in.
11.30 685 kip · in.
11.31 330 kip · in.
11.33 (*a*) −56.0 MPa. (*b*) 66.4 MPa.
11.34 (*a*) −56.0 MPa. (*b*) 68.4 MPa.
11.35 (*a*) 2.03 ksi. (*b*) −14.68 ksi.
11.36 (*a*) −1.979 ksi. (*b*) 16.48 ksi.
11.38 8.59 m.
11.40 625 ft.
11.41 (*a*) 330 MPa. (*b*) −26.0 MPa.
11.42 (*a*) 292 MPa. (*b*) −21.3 MPa.
11.44 9.50 kN · m.
11.45 (*a*) 29.0 ksi. (*b*) −1.163 ksi.
11.46 32.4 kip · ft.
11.48 (*a*) steel: 8.96 ksi; aluminum: 1.792 ksi; brass: 0.896 ksi. (*b*) 349 ft.
11.49 (*a*) $-2P/\pi r^2$. (*b*) $-5P/\pi r^2$.
11.50 (*a*) 4.87 ksi. (*b*) 5.17 ksi.
11.51 (*a*) 4.87 ksi. (*b*) 1.322 ksi.
11.52 (*a*) −102.8 MPa. (*b*) 80.6 MPa.
11.54 (*a*) 16.34 ksi. (*b*) −13.78 ksi.
11.56 (*a*) −8.33 MPa. (*b*) *A*: −13.19 MPa; *B*: 7.64 MPa.
11.57 0.375 *d*.
11.58 10.83 mm.
11.59 (*a*) −0.750 ksi. (*b*) −2.00 ksi. (*c*) −1.500 ksi.
11.60 623 lb.
11.62 0.877 in.
11.64 94.8 kN $\leq P \leq$ 177.3 kN.
11.65 (*a*) $-P/2at$. (*b*) $2P/at$. (*c*) $-P/2at$.
11.66 96.0 kN.
11.68 2.485 in. $< y <$ 4.561 in.
11.70 P = 44.2 kips, Q = 57.3 kips.
11.71 P = 9.21 kips, Q = 48.8 kips.
11.72 (*a*) 30.0 mm. (*b*) 94.5 kN.
11.73 (*a*) 9.86 ksi. (*b*) −2.64 ksi. (*c*) −9.86 ksi.
11.74 (*a*) −3.37 MPa. (*b*) −18.60 MPa. (*c*) 3.37 MPa.
11.75 (*a*) −17.16 ksi. (*b*) 6.27 ksi. (*c*) 17.16 ksi.
11.76 (*a*) 7.20 ksi. (*b*) −18.39 ksi. (*c*) −7.20 ksi.
11.77 (*a*) 0.321 ksi. (*b*) −0.107 ksi. (*c*) 0.427 ksi.
11.78 (*a*) 57.8 MPa. (*b*) −56.8 MPa. (*c*) 25.9 MPa.
11.80 (*a*) 57.4°. (*b*) 75.7 MPa.
11.81 (*a*) 19.16°. (*b*) 11.31 ksi.
11.82 (*a*) 10.03°. (*b*) 54.2 MPa.
11.83 (*a*) 27.5°. (*b*) 8.44 ksi.
11.84 (*a*) 19.52°. (*b*) 95.0 MPa.
11.85 (*a*) 41.7 psi at *A*, 292 psi at *B*. (*b*) Intersects *AB* at 0.500 in. from *A*. Intersects *BD* at 0.750 in. from *D*.
11.87 (*a*) 4.09 ksi at *A*; −1.376 ksi at *B*. (*b*) Intersects AB at 3.741 in. above *A*.
11.89 37.0 mm.

11.91 91.3 kN.
11.93 71.8 ft.
11.94 (*a*) 9.17 kN · m. (*b*) 10.24 kN · m.
11.96 (*a*) 152.25 kips. (*b*) $x = 0.595$ in., $z = 0.571$ in. (*c*) 8.70 ksi.
11.97 73.2 MPa; −102.4 MPa.
11.99 (*a*) −1.526 ksi. (*b*) 17.67 ksi.
11.101 (*a*) 46.7°. (*b*) 80.2 MPa.
11.102 (*a*) −70.9 MPa. (*b*) −14.17 MPa. (*c*) 25.4 m.
11.104 (*a*) 1.414. (*b*) 1.732.

CHAPTER 12

12.1 (*a*) $V_{max} = PL/L$, $V_{min} = -Pa/L$; $M_{max} = Pab/L$, $M_{min} = 0$.
(*b*) $0 \leq x < a$: $V = Pb/L$; $M = Pbx/L$;
$a \leq x < L$: $V = -Pa/L$; $M = Pa(L - x)/L$.
12.2 (*a*) $V_{max} = wL/2$, $V_{min} = -wL/2$; $M_{max} = wL^2/8$.
(*b*) $V = w(L/2 - x)$; $M = wx(L - x)/2$.
12.3 (*a*) $|V|_{max} = w_0L/2$; $|M|_{max} = w_0L^2/6$.
(*b*) $V = -w_ox^2/2L$; $M = -w_ox^3/6L$.
12.4 (*a*) $|V|_{max} = w(L - 2a)/2$; $|M|_{max} = w(L^2/8 - a^2/2)$.
(*b*) $0 \leq x \leq a$: $V = w(L - 2a)/2$; $M = w\ (L - 2a)x/2$;
$a \leq x \leq L - a$: $V = w(L/2 - x)$; $M = w[x(L - x) - a^2]/2$.
$L - a \leq x \leq L$: $V = -w(L - 2a)/2$; $M = w(L - 2a)(L - x)/2$.
12.5 (*a*) 68.0 kN. (*b*) 60.0 kN · m.
12.7 (*a*) 30.0 kips. (*b*) 90.0 kip · ft.
12.9 (*a*) 3.45 kN. (*b*) 1125 N · m.
12.10 (*a*) 2000 lb. (*b*) 19200 lb · in.
12.11 (*a*) 18.00 kN. (*b*) 12.15 kN · m.
12.12 (*a*) 1.800 kips. (*b*) 1.125 kip · ft.
12.13 1.117 ksi.
12.14 10.89 MPa.
12.15 129.0 MPa.
12.16 11.56 ksi.
12.18 27.7 MPa.
12.19 $|V|_{max} = 27.5$ kips; $|M|_{max} = 45.0$ kip · ft; $\sigma = 14.17$ ksi.
12.20 $|V|_{max} = 279$ kN; $|M|_{max} = 326$ kN · m; $\sigma = 136.6$ MPa.
12.23 $|V|_{max} = 28.8$ kips; $|M|_{max} = 56.0$ kip · ft; $\sigma = 13.05$ ksi.
12.24 $|V|_{max} = 1.500$ kips; $|M|_{max} = 3.00$ kip · ft; $\sigma = 2.11$ ksi.
12.25 (*a*) 1.371 m. (*b*) 26.6 MPa.
12.26 (*a*) 866 mm. (*b*) 5.74 MPa.
12.27 (*a*) 1.260 ft. (*b*) 7.24 ksi.
12.29 See Prob. 12.1.
12.30 See Prob. 12.2.
12.31 See Prob. 12.3.
12.32 See Prob. 12.4.
12.33 See Prob. 12.5.
12.34 See Prob. 12.6.
12.35 See Prob. 12.7.
12.36 (*a*) 23.0 kips. (*b*) 140.0 kip · ft.
12.37 (*a*) 1.800 kips. (*b*) 6.00 kip · ft.
12.38 (*a*) 880 lb. (*b*) 2000 lb · ft.
12.39 (*a*) 6.75 kN. (*b*) 6.51 kN · m.
12.40 (*a*) 600 N. (*b*) 180.0 N · m.
12.41 1.117 ksi.
12.42 10.89 MPa.
12.43 129.2 MPa.
12.44 11.56 MPa.
12.45 (*a*) $V = (w_0L/\pi)\cos(\pi x/L)$; $M = (w_0L^2/\pi^2)\sin(\pi x/L)$.
(*b*) w_0L^2/π^2.
12.47 (*a*) $V = w_0\ (L/3 + x^2/2L - x)$; $M = w_0\ (Lx/3 + x^3/6L - x^2/2)$. (*b*) $0.06415\ w_0L^2$.
12.49 $|V|_{max} = 20.7$ kN; $|M|_{max} = 9.75$ kN · m; $\sigma = 60.2$ MPa.
12.50 $|V|_{max} = 16.80$ kN; $|M|_{max} = 8.82$ kN · m; $\sigma = 73.5$ MPa.
12.51 $|V|_{max} = 15.00$ kips; $|M|_{max} = 37.5$ kip · ft; $\sigma = 9.00$ ksi.
12.52 $|V|_{max} = 8.00$ kips; $|M|_{max} = 16.00$ kip · ft; $\sigma = 6.98$ ksi.
12.54 $|V|_{max} = 9.28$ kips; $|M|_{max} = 28.2$ kip · in; $\sigma = 11.58$ ksi.
12.55 $|V|_{max} = 150$ kN; $|M|_{max} = 300$ kN · m; $\sigma = 136.4$ MPa.
12.57 $h = 173.2$ mm.
12.58 $h = 361$ mm.
12.60 $b = 6.20$ in.
12.62 $a = 6.67$ in.
12.63 W27 × 84.
12.64 W18 × 50.
12.65 W410 × 60.
12.66 W250 × 28.4.
12.67 S310 × 47.3.
12.69 S12 × 31.8.
12.71 C230 × 19.9.
12.72 C180 × 14.4.
12.73 3/8 in.
12.74 3/8 in.
12.76 S24 × 80.
12.77 (*a*) 18.00 kips. (*b*) 72.0 kip · ft.
12.78 (*a*) 140 N. (*b*) 33.6 kN · m.
12.80 950 psi.
12.81 $|V|_{max} = 128$ kN; $|M|_{max} = 89.6$ kN · m; $\sigma = 156.1$ MPa.
12.84 $|V|_{max} = 30$ lb; $|M|_{max} = 24$ lb · ft; $\sigma = 6.95$ ksi.
12.85 $d = 15.06$ in.
12.87 W310 × 38.7.

CHAPTER 13

13.1 60.0 mm.
13.2 2.00 kN.
13.3 (*a*) 31.5 lb. (*b*) 43.2 psi.
13.4 (*a*) 372 lb. (*b*) 64.4 psi.
13.5 193.2 kN.
13.7 9.95 ksi.
13.9 (*a*) 7.40 ksi. (*b*) 6.70 ksi.
13.10 (*a*) 3.17 ksi. (*b*) 2.40 ksi.
13.11 (*a*) 920 kPa. (*b*) 765 kPa.
13.12 (*a*) 114.1 MPa. (*b*) 96.9 MPa.
13.13 14.05 in.
13.14 88.9 mm.
13.17 (*a*) 12.55 MPa. (*b*) 18.82 MPa.
13.18 (*a*) 1.745 ksi. (*b*) 2.82 ksi.
13.19 19.61 MPa.
13.20 3.21 ksi.
13.22 2.00.
13.23 1.125.
13.24 1.500.
13.25 728 N.
13.26 1.672 in.
13.27 (*a*) 59.9 psi. (*b*) 79.8 psi.
13.28 (*a*) 12.21 MPa. (*b*) 58.6 MPa.
13.29 (*a*) 95.2 MPa. (*b*) 112.9 MPa.
13.31 3.93 ksi at *a*, 2.67 ksi at *b*, 0.63 ksi at *c*, 1.02 ksi at *d*, 0 at *e*.
13.33 (*a*) 41.4 MPa. (*b*) 41.4 MPa.
13.34 (*a*) 18.23 MPa. (*b*) 14.59 MPa. (*c*) 46.2 MPa.
13.35 (*a*) 40.5 psi. (*b*) 55.2 psi.
13.36 (*a*) 2.67 in. (*b*) 41.6 psi.
13.37 9.05 mm.
13.39 20.1 ksi.
13.41 266 kN/m.

13.42 10.76 MPa at *a*, 0 at *b*, 11.21 MPa at *c*, 22.0 MPa at *d*, 9.35 MPa at *e*.
13.43 (*a*) 2.025 ksi. (*b*) 1.800 ksi.
13.46 (*a*) 23.3 MPa. (*b*) 109.7 kPa.
13.47 (*a*) 2.59 ksi. (*b*) 0.967 ksi.
13.49 (*a*) 0.888 ksi. (*b*) 1.453 ksi.
13.50 738 N.
13.51 (*a*) 2.73 ksi. (*b*) 1.665 ksi.
13.53 (*b*) $h = 225$ mm, $b = 61.7$ mm.
13.55 (*a*) 84.2 kips. (*b*) 60.2 kips.
13.56 (*a*) 239 N. (*b*) 549 N.
13.57 1835 lb.
13.58 1.167 ksi at *a*, 0.513 ksi at *b*, 4.03 ksi at *c*, 8.40 ksi at *d*.
13.59 2.50 ksi at *a*, 2.50 ksi at *b*, 9.00 ksi at *c*, 0 at *d*.
13.60 255 kN.
13.61 (*a*) 50.9 MPa. (*b*) 36.0 MPa.

CHAPTER 14

14.1 $\sigma = -0.521$ MPa, $\tau = 56.4$ MPa.
14.2 $\sigma = 32.9$ MPa, $\tau = 71.0$ MPa.
14.3 $\sigma = 9.46$ ksi, $\tau = 1.013$ ksi.
14.4 $\sigma = 10.93$ ksi, $\tau = 0.536$ ksi.
14.5 (*a*) $-37.0°$, $53.0°$. (*b*) -13.60 MPa, -86.4 MPa.
14.7 (*a*) $14.0°$, $104.0°$. (*b*) 20.0 ksi, -14.00 ksi.
14.9 (*a*) $8.0°$, $98.0°$. (*b*) 36.4 MPa. (*c*) -50.0 MPa.
14.10 (*a*) $14.0°$, $104.0°$. (*b*) 68.0 MPa. (*c*) -16.00 MPa.
14.12 (*a*) $-26.6°$, $63.4°$. (*b*) 5.00 ksi. (*c*) 6.00 ksi.
14.13 (*a*) $\sigma_{x'} = -4.80$ ksi, $\tau_{x'y'} = 0.30$ ksi, $\sigma_{y'} = 20.8$ ksi.
(*b*) $\sigma_{x'} = 3.90$ ksi, $\tau_{x'y'} = 12.13$ ksi, $\sigma_{y'} = 12.10$ ksi.
14.14 (*a*) $\sigma_{x'} = 9.02$ ksi, $\tau_{x'y'} = 3.80$ ksi, $\sigma_{y'} = -13.02$ ksi.
(*b*) $\sigma_{x'} = 5.34$ ksi, $\tau_{x'y'} = -9.06$ ksi, $\sigma_{y'} = -9.34$ ksi.
14.16 (*a*) $\sigma_{x'} = -37.5$ MPa, $\tau_{x'y'} = -25.4$ MPa, $\sigma_{y'} = 57.5$ MPa.
(*b*) $\sigma_{x'} = -30.1$ MPa, $\tau_{x'y'} = 35.9$ MPa, $\sigma_{y'} = 50.1$ MPa.
14.17 (*a*) -0.300 MPa. (*b*) -2.92 MPa.
14.18 (*a*) 346 psi. (*b*) -200 psi.
14.19 (*a*) $14.3°$. (*b*) 117.3 MPa.
14.20 (*a*) $18.4°$. (*b*) 16.67 ksi.
14.22 $\sigma_a = 5.12$ ksi, $\sigma_b = -1.64$ ksi, $\tau_{max} = 3.38$ ksi.
14.24 $\sigma_a = 12.18$ MPa, $\sigma_b = -48.7$ MPa, $\tau_{max} = 30.5$ MPa.
14.25 See 14.5 and 14.9.
14.26 See 14.6 and 14.10.
14.28 See 14.12.
14.29 See 14.13.
14.30 See 14.14.
14.32 See 14.16.
14.33 See 14.17.
14.34 See 14.18.
14.35 See 14.19.
14.36 See 14.20.
14.38 See 14.22.
14.40 See 14.24.
14.41 (*a*) 7.94 ksi. (*b*) 13.00 ksi, -11.00 ksi.
14.43 (*a*) -2.89 MPa. (*b*) 12.77 MPa, 1.23 MPa.
14.44 (*a*) -8.66 MPa. (*b*) 17.00 MPa, -3.00 MPa.
14.46 $24.6°$, $114.6°$; 72.9 MPa, 27.1 MPa.
14.47 $60°$, $-30°$; $1.732\ \tau_0$, $-1.732\ \tau_0$.
14.48 $\frac{1}{2}\theta$, $\frac{1}{2}\theta + 90°$; $\sigma_0\,(1 + \cos\theta)$, $\sigma_0\,(1 - \cos\theta)$.
14.49 166.5 psi.
14.50 8.61 ksi.
14.51 5.04.
14.52 (*a*) 12.38 ksi. (*b*) 0.0545 in.
14.53 (*a*) 1.290 MPa. (*b*) 0.0852 mm.
14.54 7.71 mm.
14.56 1.676 MPa.
14.58 136.0 MPa.
14.59 7.58 ksi.
14.60 0.307 in.
14.61 2.95 MPa.
14.62 3.41 MPa.
14.64 387 psi.
14.65 $56.8°$.
14.66 2.84 MPa.
14.68 $\sigma_{max} = 45.1$ MPa, $\tau_{max(in\text{-}plane)} = 7.49$ MPa.
14.69 (*a*) 3.15 ksi. (*b*) 1.993 ksi.
14.71 8.48 ksi, 2.85 ksi.
14.72 13.09 ksi, 3.44 ksi.
14.73 3.90 kN.
14.74 251 psi.
14.76 (*a*) $-34.2°$, $55.8°$. (*b*) 9.50 ksi.
14.78 (*a*) 0.775 MPa. (*b*) -2.69 MPa.
14.79 250 psi.
14.81 (*a*) 399 kPa. (*b*) 186.0 kPa.
14.82 (*a*) $27.1°$ ↻, $62.9°$ ↺. (*b*) -20.8 ksi, 2.04 ksi.
(c) 11.43 ksi.
14.84 $\sigma_{max} = 68.6$ MPa, $\tau_{max(in\text{-}plane)} = 23.6$ MPa.

CHAPTER 15

15.1 (*a*) $y = -Px^2(3L - x)/6EI$.
(*b*) $PL^3/3EI \downarrow$. (*c*) $PL^2/2EI$ ⦩.
15.2 (*a*) $y = M_0x^2/2EI$. (*b*) $M_0L^2/2EI \uparrow$. (*c*) M_0L/EI ⦨.
15.3 (*a*) $y = -w_0(x^5 - 5L^4x + 4L^5)/120\ EIL$.
(*b*) $W_0L^4/30\ EI \downarrow$. (*c*) $W_0L^3/24\ EI$ ⦨.
15.4 (*a*) $y = -w(x^4 - 4L^3x + 3L^4)/24\ EI$.
(*b*) $wL^4/8\ EI \downarrow$. (*c*) $wL^3/6\ EI$ ⦨.
15.6 (*a*) $y = w(L^2x^2/8 - x^4/24)/EI$.
(*b*) $11\ wL^4/384\ EI \uparrow$. (*c*) $5\ wL^3/48\ EI$ ⦨.
15.7 (*a*) $y = w(Lx^3/16 - x^4/24 - L^3x/48)/EI$.
(*b*) $wL^3/48\ EI$ ⦨. (*c*) 0.
15.9 (*a*) 2.74×10^{-3} rad ⦩. (*b*) 1.142 mm ↓.
15.10 (*a*) 6.56×10^{-3} rad ⦩. (*b*) 0.227 in ↓.
15.11 (*a*) $x_m = 0.423\ L$, $y_m = 0.06415\ M_0L^2/EI \uparrow$.
(*b*) 45.3 kN · m.
15.12 (*a*) $x_m = 0.519\ L$, $y_m = 0.00652\ w_0L^4/EI \downarrow$.
(*b*) 0.229 in.
15.13 12.94 mm ↑.
15.15 (*a*) $y = w_0(x^6/90 - Lx^5/30 + L^3x^3/18 - L^5x/30)/EIL^2$.
(*b*) $w_0L^3/30\ EI$ ⦩. (*c*) $61\ w_0L^4/5760\ EI \downarrow$.
15.17 $3\ wL/8 \uparrow$.
15.18 $3\ M_0/2L \uparrow$.
15.20 $11\ w_0L/40 \uparrow$.
15.21 $\mathbf{R}_A = 11P/16 \uparrow$, $\mathbf{M}_A = 3PL/16$ ↺, $\mathbf{R}_B = 5P/16 \uparrow$, $\mathbf{M}_B = 0$; $M = -3PL/16$ at *A*, $M = 5PL/32$ at *C*, $M = 0$ at *B*.
15.22 $\mathbf{R}_A = 41\ wL/128 \uparrow$, $\mathbf{M}_A = 0$, $\mathbf{R}_B = 23\ wL/128 \uparrow$, $\mathbf{M}_B = 7\ wL^2/128$ ↻; $M = 0$ at *A*, $M = 0.0513\ wL^2$ at $x = 0.320\ L$, $M = 0.01351\ wL^2$ at *C*, $M = -0.0547\ wL^2$ at *B*.
15.23 $R_B = 4P/27 \uparrow$, $y_D = 11\ PL^3/2187\ EI \downarrow$.
15.25 $\mathbf{R}_A = \frac{1}{2}\ P \uparrow$, $\mathbf{M}_A = PL/8$ ↺; $M = -PL/8$ at *A*, $M = PL/8$ at *C*, $M = -PL/8$ at *B*.
15.26 $R_A = w_0L/4 \uparrow$, $M_A = 5w_0L^2/96$ ↺; $M = -5\ w_0\ L^2/96$ at *A*, $M = w_0L^2/32$ at *C*, $M = -5\ w_0L^2/96$ at *B*.
15.27 (*a*) $8PL^3/243 \downarrow$. (*b*) $19\ PL^2/162\ EI$ ⦩.
15.28 (*a*) $PL^3/486\ EI \uparrow$. (*b*) $PL^2/81\ EI$ ⦩.

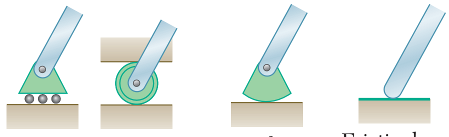
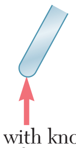
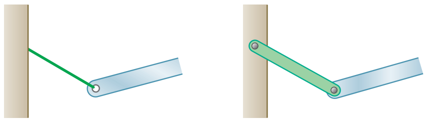
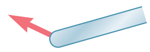
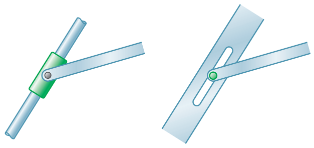
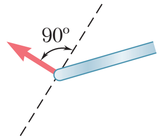
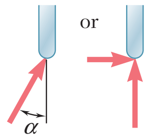
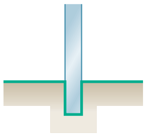
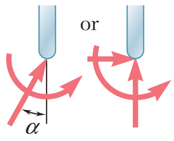
15.29 (*a*) $wL^4/128\,EI \downarrow$. (*b*) $wL^3/72\,EI$ ⦪.
15.30 (*a*) $19\,Pa^3/6\,EI \downarrow$. (*b*) $5\,Pa^2/2\,EI$ ⦪.
15.31 $3\,PL^2/4\,EI$ ⦩, $13\,PL^3/24\,EI \downarrow$.
15.32 PL^2/EI ⦩, $17PL^3/24\,EI \downarrow$.
15.35 12.55×10^{-3} rad ⦪, 0.364 in. $\downarrow$.
15.36 12.08×10^{-3} rad ⦪, 0.240 in. $\downarrow$.
15.37 (*a*) 0.601×10^{-3} rad ⦪, (*b*) 3.67 mm $\downarrow$.
15.39 (*a*) $7\,wL/128$. (*b*) $57\,wL/128 \uparrow$, $9\,wL^2/128$ ↻.
15.40 (*a*) $4P/3 \uparrow$, $PL/3$ ↺. (*b*) $2P/3 \uparrow$.
15.42 $3P/8 \uparrow$ at A, $7P/8 \uparrow$ at C, $P/4 \downarrow$ at D.
15.43 $13\,wL/32 \uparrow$, $11\,wL^2/192$ ↻.
15.45 (*a*) 5.06×10^{-3} rad ⦪. (*b*) 47.7×10^{-3} in. $\downarrow$.
15.46 121.5 N/m.
15.48 (*a*) 0.00937 mm $\downarrow$. (*b*) 229 N.
15.49 0.1975 in.
15.50 (*a*) 31.2 mm. (*b*) 17.89 mm $\uparrow$.
15.52 (*a*) 0.211 L, $0.1604\,M_0L^2/EI \downarrow$. (*b*) 6.08 m.
15.54 (*a*) $y = 2w_0L^4\,[-8\cos(\pi x/2L) - \pi^2x^2/L^2 + 2\pi(\pi - 2)x/L + \pi(4 - \pi)]/\pi^4EI$. (*b*) $0.1473\,w_0L^3/EI$ ⦩.
(*c*) $0.1089\,w_0L^4/EI \downarrow$.
15.55 3.00 kips.
15.56 $9M_0/8L \uparrow$; M_0/L at A, $-7M_0/16$ just to the left of C, $9M_0/16$ just to the right of C, 0 at B.
15.57 $13\,wa^3/6\,EI$ ⦪, $29\,wa^3/24\,EI \downarrow$.
15.59 5.58×10^{-3} rad ⦪, 2.51 mm $\downarrow$.
15.60 $7P/32 \uparrow$ at A, $23P/32 \uparrow$ at B, $33\,P/16 \uparrow$ at C.
15.61 43.9 kN.

CHAPTER 16

16.1 kL.
16.2 K/L.
16.3 $kL/4$.
16.4 K/L.
16.5 120 kips.
16.7 (*a*) 6.65 lb. (*b*) 21.0 lb.
16.9 (*a*) 6.25%. (*b*) 12.04 kips.
16.10 (*a*) 7.48 mm. (*b*) 58.8 kN for round, 84.8 kN for square.
16.12 1.421.
16.13 168.4 kN.
16.14 2.125.
16.16 (*a*) 93.0 kN. (*b*) 448 kN.
16.17 2.27.
16.18 2.77 kN.
16.20 (*a*) $L_{BC} = 1.960$ m, $L_{CD} = 0.490$ m. (*b*) 23.1 kN.
16.22 16.29 in.
16.23 29.5 kips.
16.24 (*a*) 2.29. (*b*) 1.768 in. for (2), 1.250 in. for (3), 1.046 in. for (4).
16.25 (*a*) 114.7 kN. (*b*) 208 kN.
16.26 95.5 kips.
16.27 (*a*) 220 kN. (*b*) 841 kN.
16.28 (*a*) 86.6 kips. (*b*) 88.1 kips.
16.31 (*a*) 26.4 kN. (*b*) 32.2 kN.
16.32 76.6 kips.
16.33 1598 kN.
16.34 903 kN.
16.36 173.8 kips.
16.37 107.7 kN.
16.39 6.53 in.
16.40 (*a*) 3. (*b*) 5.
16.41 0.884 in.
16.42 9 mm.
16.43 (*a*) 1.256 in. (*b*) 1.390 in.
16.44 W250 × 67.
16.47 3/8 in.
16.48 L .
16.49 79.0 kN.
16.50 $ka^2/2l$.
16.52 0.384 in.
16.53 $\pi^2b^2/12L^2\alpha$.
16.56 4.00 kN.
16.58 116.5 kips.
16.59 (*a*) 1531 kN. (*b*) 638 kN.
16.60 W10 × 54.

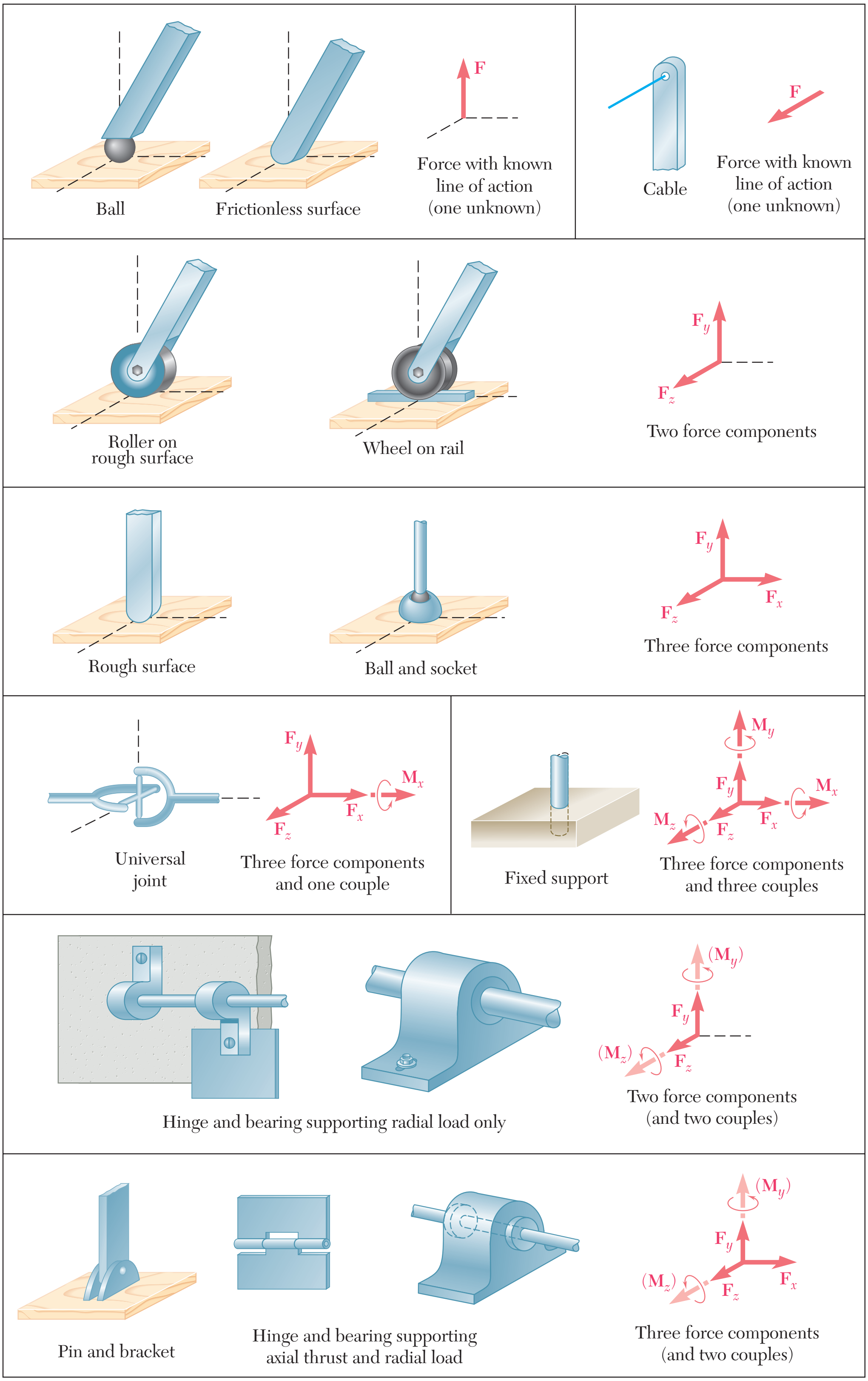
This page intentionally left blank

Reactions at Supports and Connections for a Two-Dimensional Structure

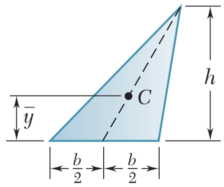
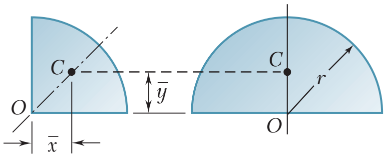
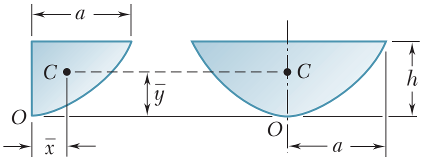
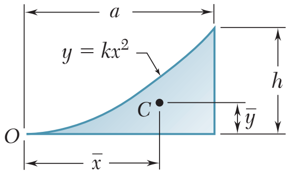
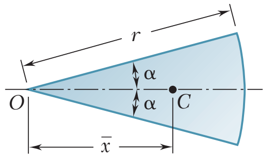
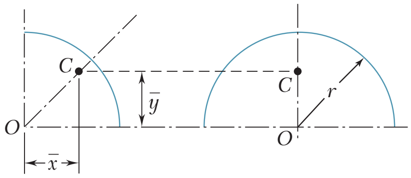
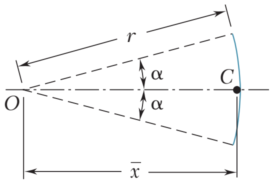
Support or Connection	Reaction	Number of Unknowns
 Rollers, Rocker, Frictionless surface	 Force with known line of action	1
 Short cable, Short link	 Force with known line of action	1
 Collar on frictionless rod, Frictionless pin in slot	 Force with known line of action	1
 Frictionless pin or hinge, Rough surface	 Force of unknown direction	2
 Fixed support	 Force and couple	3

The first step in the solution of any problem concerning the equilibrium of a rigid body is to construct an appropriate free-body diagram of the body. As part of that process, it is necessary to show on the diagram the reactions through which the ground and other bodies oppose a possible motion of the body. The figures on this and the facing page summarize the possible reactions exerted on two- and three-dimensional bodies.

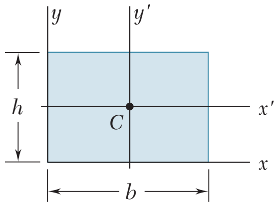
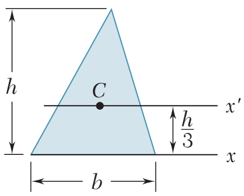
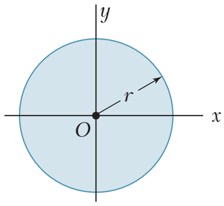
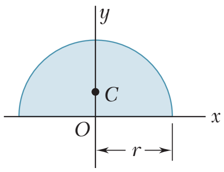
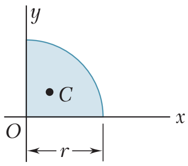
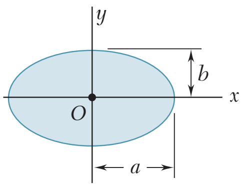
Reactions at Supports and Connections for a Three-Dimensional Structure



Centroids of Common Shapes of Areas and Lines

Shape		$\bar{x}$	$\bar{y}$	Area
Triangular area			$\dfrac{h}{3}$	$\dfrac{bh}{2}$
Quarter-circular area		$\dfrac{4r}{3\pi}$	$\dfrac{4r}{3\pi}$	$\dfrac{\pi r^2}{4}$
Semicircular area		0	$\dfrac{4r}{3\pi}$	$\dfrac{\pi r^2}{2}$
Semiparabolic area		$\dfrac{3a}{8}$	$\dfrac{3h}{5}$	$\dfrac{2ah}{3}$
Parabolic area		0	$\dfrac{3h}{5}$	$\dfrac{4ah}{3}$
Parabolic spandrel		$\dfrac{3a}{4}$	$\dfrac{3h}{10}$	$\dfrac{ah}{3}$
Circular sector		$\dfrac{2r\sin\alpha}{3\alpha}$	0	αr^2
Quarter-circular arc		$\dfrac{2r}{\pi}$	$\dfrac{2r}{\pi}$	$\dfrac{\pi r}{2}$
Semicircular arc		0	$\dfrac{2r}{\pi}$	πr
Arc of circle		$\dfrac{r\sin\alpha}{\alpha}$	0	$2\alpha r$

Moments of Inertia of Common Geometric Shapes

Rectangle		$\bar{I}_{x'} = \frac{1}{12}bh^3$ $\bar{I}_{y'} = \frac{1}{12}b^3h$ $I_x = \frac{1}{3}bh^3$ $I_y = \frac{1}{3}b^3h$ $J_C = \frac{1}{12}bh(b^2 + h^2)$
Triangle		$\bar{I}_{x'} = \frac{1}{36}bh^3$ $I_x = \frac{1}{12}bh^3$
Circle		$\bar{I}_x = \bar{I}_y = \frac{1}{4}\pi r^4$ $J_O = \frac{1}{2}\pi r^4$
Semicircle		$I_x = I_y = \frac{1}{8}\pi r^4$ $J_O = \frac{1}{4}\pi r^4$
Quarter circle		$I_x = I_y = \frac{1}{16}\pi r^4$ $J_O = \frac{1}{8}\pi r^4$
Ellipse		$\bar{I}_x = \frac{1}{4}\pi ab^3$ $\bar{I}_y = \frac{1}{4}\pi a^3b$ $J_O = \frac{1}{4}\pi ab(a^2 + b^2)$