Fracture Mechanics 2

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Applied Reliability

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Preface

This book is intended for technicians, engineers, designers, students, and teachers working in the fields of engineering and vocational education. Our main objective is to provide an assessment of indicators of quality and reliability to aid in decision-making. To this end, we recommend an *intuitive* and practical approach, based on mathematical rigor.

The first part of this book shows the fundamental basis of data analysis in both quality control and in studying the mechanical reliability of materials and structures. Laboratory and workshop results are discussed in accordance with the technological procedures inherent to the subject matter. We also discuss and interpret the standardization of manufacturing processes as a causal link with geometric and dimensional specifications (GPS, or *Geometrical Product Specification*). This is moreover the educational novelty of this work, in comparison here with consulted praiseworthy publications.

We discuss many laboratory examples, thereby covering a new, industrial organization of work. We also use mechanical components from our own real mechanisms, which we built and designed at our production labs. Finite element modification is thus relevant to real machined pieces, controlled and soldered in a dimensional metrology laboratory.

We also discuss mechanical component reliability. Since statistics are common to both this field and quality control, we will simply mention reliability indices in the context of using the structure for which we are performing the calculations.

Scientists from specialized schools and corporations often take an interest in the quality of measurement, and thus in the measurement of uncertainties. The so-called cutting-edge endeavors such as aeronautics, automotive, and nuclear industries, to

mention but a few, put an increasing emphasis on the *just measurement*. This text's educational content stands out due to the following:

1) The rigor of the probabilistic methods which support statistical–mathematical treatments of experimental or simulated data.

2) The presentation of varied lab models which are to come at the end of each of the following chapters: this should help the student to better understand how to:

- define and justify a quality and reliability control target;

- identify the appropriate tools to quantify reliability with respect to capabilities;

- interpret quality (capability) and reliability (reliability indices) indicators;

- choose the adequation test for the distribution (whether justified or used *a priori*);

- identify how trials can be accelerated and their limits;

- analyze the quality and reliability of materials and structures;

- size and *tolerance* (GPS) design structures and materials.

– What about uncertainty calculations in applied reliability?

The fracture behavior of structures is often characterized (in linear mechanics) by a local variation of the material's elastic properties. This inevitably leads to sizing calculations which seek to secure the structures derived from the materials. Much work has been, and still is, conducted in a wide range of disciplines from civil engineering to the different variants of mechanics. Here, we do not consider continuum mechanics, but rather probabilistic laws for cracking. Certain laws of statistical distributions are systematically repeated to better approach reliability.

Less severe adequation tests would confirm the fissure propagation hypothesis. In fields where safety is a priority, such as medicine (surgery and biomechanics), aviation and nuclear power plants, to mention but three, theorizing unverifiable concepts would be unacceptable. The relevant reliability calculations must therefore be as rigorous as possible.

Defining safety coefficients would be an important (or even major) element of structure sizing. This definition is costly, and does not really offer any real guarantee on safety previsions (unlike security previsions). Today, the interpretation and philosophy of these coefficients is reinforced by increasingly accurate probabilistic calculations. Well-developed computer tools largely contribute to the time and effort of calculations. Thus, we will use software commonly found in various schools

(Auto Desk Inventor Pro and ANSYS in modelization and design; MathCAD, GUM, and COSMOS in quality control, metrology, and uncertainty calculations).

Much work has been done to rationalize the concept of applied reliability; however, no "unified method" between the mechanical and statistical interpretations of rupture has yet been defined. Some of the many factors for this non-*consensus* are unpredictable events which randomly create the fault, its propagation, and the ensuing damage. Many researchers have worked on various random probabilistic and deterministic methods. This resulted in many simulation methods, the most common being the Monte Carlo simulation.

In this book, we present some documented applied cases to help teachers succinctly present probabilistic problems (reliability and/or degradation). The intuitive approach takes on an important part in our problem-solving methods, and among various points, the main goal of this book is to give this humble contribution. Many commendable works and books have talked about reliability, quality control, and uncertainty perfectly well, but as separate entities. However, our task here is to verify measurements and ensure that the measurand is well-taught. As Lord Kelvin said, "if you cannot measure it, you cannot improve it". Indeed, measuring identified quantities is an unavoidable part of laboratory life. Theoretical confirmation of physical phenomena must go through *measurement reliability* and its effects on the function are attributed to the material and/or structure, among other things.

Mechanical models (rupture criteria) of continuum mechanics discussed in Chapter 10 make up a reference pool of work used here and there in our case studies, such as the Paris–Erdogan law, the Manson–Coffin law, S-N curves (Wöhler curves), Weibull law (solid mechanics), etc. We could probably (and justly) wonder in what way is this chapter appropriate in works dedicated to reliability. The reason is that these criteria are deliberately targeted. We used them here to avoid the reader having to "digress" into specialized books.

Establishing *confidence* in our results is critical. Measuring a characteristic does not simply mean finding the value of the characteristic. We must also give it an *uncertainty* so as to show the measurement's *quality*. In this book, we will show educational laboratory examples of uncertainty (GUM: Guide to the Expression of Uncertainty in Measurement).

– Why then publish another book dedicated to quality control, uncertainties, and reliability?

Firstly, why publish a book which covers two seemingly distinct topics (quality control and reliability including uncertainties)? Because both these fields rely on probabilities, statistics, and a similar method in describing their hypothesis. In quality control, the process is often already known or appears to be under control

beforehand, hence the intervention of capability indices (SPC: statistical process control). Furthermore, the goal is sometimes the competitiveness between manufactured products. Security is shown in secondary terms. Indeed, it is in terms of maintainability and durability that quality control joins reliability as a means to guarantee the functions attributed to a mechanism, component, or even the entire system.

When considering the mechanical reliability of materials and structures, the reliability index is inherently a *safety* indicator. It is often very costly in terms of computation time and very serious in matters of consequence. The common aspect between both fields is still the probabilistic approach. Probabilities and statistical–mathematical tools are necessary to supply theoretical justifications for the computational methods. Again, this book intends to be pragmatic and leaves reasonable room for the intuitive approach of the hypotheses stated here and there.

Finally, we provide a succinct glossary to smooth the understanding of dimensional analysis (IMV: International Metrology Vocabulary) and structural mechanical reliability. This educational method allows us to "*agree*" on the international language used to define the mesurand, reliability index, or a succinct definition of the capability indicators largely used in quality control.

In terms of safety, component reliability (for both materials and structures) is absolutely essential in the field of safety and performance.

The field of reliability is used in many fields of engineering, from civil engineering to mechanical and electrical engineering: it is thus manifold. It often aims at the estimation of the functions at various phases of the lifecycle of components, subject to study. Reliability users increasingly depend on reproducible software, though they struggle to determine whether the component is active or passive, the size of the experience return and its imperative validation, the phenomena which tend to decrease the likelihood of failure or the reliability index, etc.

This book uses some methods provided here and there to estimate operational or target reliabilities. The apparent controversy between frequential and Bayesian probabilistic approaches are irrelevant in our humble opinion if we know how to set the problem *a priori*. Setting boundaries for the likelihood of rupture (failure or even degradation) is worth doing. As for us, we prefer calculating rupture through the damage indicator integral, made explicit by Madsen's work.

Just as estimating reliability can allow us to understand the history to better anticipate the future, we must show pragmatism in measuring the factors responsible for the likely rupture. Since the measurement is always inherently flawed and

uncertain, we must include uncertainty calculations in our reliability methods. Without such calculations, our results would lead to doubtfulness.

First, vocabulary: reliability has its own, specific terminology (see glossary) which, like for metrology, affects the decision's terms. Thus, we will abide by the EN 13306 standard (see Table A.45 of the Appendix of Volume 3). The definitions for reliability, durability, failure, and degradation can be found in the appendix and glossary.

Reliability data is necessary to:

– prioritize between components → occurrence rate of each mode (AMDEC);

– identify the degradation mechanism for a clear experience feedback;

– optimize preventative and corrective maintenance;

– gather data efficient for, and necessary to, the correct calculation of reliability parameters and especially their uncertainties.

Analysis and *validation* are done by analyzing the experience feedback with respect to critical failure criteria, such as failure modes, the mean time between failures (MTBF), probability of failure on demand (P_s) and its reliability index according to a "selected criterion", the repair and/or material unavailability time, the confidence intervals, or even the sample size.

We note that reliability is usually taken into account from design, based on the specifications. It is calculated and compared to the allocated reliability (reliability demand). It includes all phases of life (design, manufacture, and development trials).

During exploitation (operation), the planned reliability is calculated and compared to a threshold (e.g. rate of failure) such as physical calculations, with the intention to extend it beyond the lifespan (cycle) planned during design.

Reliability is mostly measured, therefore making its **metrology** a serious business; hence, the calculation of its uncertainties including instrument and measurement equipment calibration.

Among the diverse difficulties which are imperative upon the reliability function, we hold, among others, the type of component (repairable or not repairable, redundant active or passive) and even some controversial methods or model (*frequencial/Bayesian*).

The component can be active:

– various degradation mechanisms with different modes of failure,

– unknown or even complex physical modelization,

– classic reliability and Bayesian reliability (*if*) so sometimes adapted to the study case,

– incomplete or even truncated data,

– modelization using a Gaussian distribution, an exponential law, a two- or three-parameter Weibull law or especially a Birnbaum–Sanders law (as is often the case in cracking rupture: wear and tear in zone II of the Ritchie curve – see Figures 1.2 and 1.17).

The component can be active and passive:

– few degradation mechanisms;

– degradation type: slow or progressive;

– few failures (if any);

– physical modelization of degradation: fissure initiation and propagation;

– numerical methods (energetic methods, integral calculations, and finite elements);

– Bayesian approach for available failure data.

As we are a physicist, practical mathematician-statistician or engineer, sometimes, in many schools of thought, "controversies" appear in the method or model (ex. frequency/Bayesian) used. In our book, we will try to remain pragmatic and synthetize our opinions.

From a *physicist's* perspective, the experimental conditions of data-gathering are known, and their uncertainties well-bound. Its so-called frequential analysis is based only on objective data, because they were measured *correctly*. We know that measurements are costly and time-consuming. If the data from "our physicist's" experiments are insufficient and if the process turns out to be non-repetitive or the number of parameters to estimate is high, the frequential approach falsely introduces confirmation bias in the analysis. The paradox is that the calculations are correct, but they only answer to a logically mathematical demand. In other words, the mathematics are correct but are superficially stuck on an inappropriate case, hence a rejection of the solution and the birth of controversy…

The *engineering* approach is *attractive* due to its "applied arts and crafts" aspect (i.e. learning). Its analysis includes the knowledge that we must apply an "*a priori*" law, which must by definition be biased. Without rejecting the Bayesian approach, this is where we favor the *engineering* approach because it uses *decision-making*

tools for which preferences are clearly expressed. At the end of this approach, the uncertainty function greatly helps make the decision…

Finally, it is important to specify and frame the problem well: its context, hypotheses, available data, etc. Simulations (using software) are a helpful educational tool, but they should not be treated as *replacements* for real experiments. Relying on *real data* from the *experiment feedback* of the collection conditions is more suitable. Indeed, experiments and "real" data are a strategic necessity in case of preemptive validation.

In this book, we show (see Chapters 1 and 2) the qualitative analysis elements preceding quantitative, deterministic, and probabilistic analysis. The laws and tests shown in the first two chapters of Volume 1 are required reading for any probabilistic study of physical phenomena, and it falls to us to be pragmatic.

Regardless of the approach used, we must analyze the sensibility of factors and always apply common sense. Among many other methods of analysis, reliability is a tool for understanding the past. For example, many failures, degradations, and ruptures or ruin (damage) cannot be explained by deterministic models such as aging, mechanisms of degradation, models and laws (see Chapter 1, "Fracture mechanisms by fatigue"), etc. Studying reliability allows us to find the components and subcomponents to critique, the important variables (initial faults, factor of intensity of constraint *f.i.c.*, etc.) for which uncertainties should be reduced, and so on through a sound knowledge of physical phenomena.

Reliability anticipates and prepares for the future in order to improve performance and safety by optimizing exploitation strategies.

However, reliability alone cannot replace an experimental understanding of physical phenomena.

A. GROUS
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Glossary

Abrasion resistance

Hard materials also show good abrasion resistance; in other words, they are not easily worn down by friction. In practice they are harder to grind down.

Acceptable risk

"*Acceptable risk*" describes the structural and non-structural measures to be put in place to reduce probable damage to a reference level. A risk scale is often associated with dangers in order to classify them in order of seriousness.

Availability

Is a (dimensionless) attribute of **dependability**. It is the capacity of a system to properly deliver the service (quality) when the user has need of it. Availability is a unitless measurement; it corresponds to the ratio of uptime to total execution time of the system.

Chance

Imaginary cause of something that occurs for no apparent or explicable reason (dictionary definition).

Conditional probability

(Bayesian) probability of a consequence when the causal event will definitely occur. If we suppose that a fracture has reached the limit suggested

by a pre-established hypothesis, the probability of cracking is a conditional probability.

Corrective maintenance or restoration to a state of proper function

Maintenance performed when a breakdown is detected, aimed at *restoring* a product to a state where it can fulfill its required function.

Corrosion resistance

This denotes the ability of a material to withstand damage from the effects of the chemical reaction of oxygen with the metal. A ferrous metal which is resistant to corrosion does not rust.

Degradation

Irreversible evolution of one or more characteristics of a product related to time, to the duration of use or to an *external cause – alteration of function, constant phenomenon, physical aging*

Dependability

The property which enables users to justifiably place their faith in the service provided to them: *reliability, availability, safety, maintainability, security.*

Dilation and contraction

When a material is heated, it expands slightly; this is called dilation. Conversely, if it shrinks (clue to cold), this is a contraction. The level of dilation and contraction of a metal affects its weldability. The more the metal is expanded or contracted, the greater the risk of *cracks* or deformations appearing.

Distribution function

Integral function of the probability density (or cumulative probability function), calculated in order of ascending values of the random variable. It expresses the probability of the random variable assuming a value less than or equal to a given value.

Ductility

Represents the ability of a metal to be deformed without breaking. It can be stretched, elongated, or subjected to torsion forces. Ductile materials are difficult to break because the cracks or defects created by a deformation do not easily propagate.

Durability

The ability of a product to perform its required function, in given conditions of use and maintenance, until a critical state is reached.

Elasticity

Ability of a material to return to its original form after a deformation.

Failure

Alteration or suspension of the ability of a system to perform its required function(s) to the levels of performance defined in the technical specifications.

Fault (error) resistance

Fault resistance is implemented to detect and handle errors.

Fault tree

Logical diagram using a tree structure to represent the causes of failures and their combinations leading to a **feared state** (Bayes). Fault trees enable us to calculate the **unavailability** or the **reliability** of the system model.

FMECA – risk analysis

A method for systematic risk analysis of the causes and effects of failures that might affect the components of a system. FMECA analyzes the seriousness of each type of failure. It enables us to evaluate the impact of such failures on the reliability and safety of the system.

Frailty

Frailty describes the characteristic of a metal that breaks easily on impact or from a deformation. It deforms little or not at all, and is easily broken.

Hardness

The ability of a body to resist penetration by another body harder than it. It is also characterized by its scratch resistance.

Hazard

Describes any event, unpredictable phenomenon, or human activity which would result in the loss of human lives, or damages to commodities or the environment.

Heat-affected zone (HAZ)

The HAZ represents the heat affected region of the base metal that was not melted during welding process. Metallurgists define usually the HAZ as the area of base material which has had its microstructure and properties altered by welding or heat.

Maintainability

One of the aspects of **dependability**. The maintainability of a system expresses its capacity for repair and evolution, with maintenance supposedly completed under certain conditions with prescribed procedures and means.

Malleability

A characteristic which permits the metal to be molded. It is the relative resistance of a metal subjected to compression forces. The malleability of a material increases with increasing temperature.

Markov chains

Used to evaluate the *dependability* of systems in a quantitative manner, this technique is based on the hypothesis that failure and repair rates are constant and

that the stochastic process modeling its behavior is Markovian (a memoryless process). When the space of the potential states of the system is a discreet set, the Markovian process is called a Markov chain.

Materialized measure

The measuring instrument which replicates or permanently provides different types of values during use, each with an assigned value.

Metrology

The science of measurements and its different applications, which encompasses all theoretical and practical aspects of measuring, regardless of the uncertainty of the measurement or the domain to which it relates.

Measurand

A value to be measured.

Measuring accuracy

Proximity between a measured value and the true value of a measurand.

NOTE:

The measurement of accuracy does not produce a value and is not expressed numerically. A measurement is sometimes considered accurate if it offers a smaller uncertainty.

Although linked to the concepts of correctness and fidelity, it is better not to use the term “measuring accuracy” for measuring correctness or the term measuring fidelity for measuring accuracy.

Measuring accuracy is occasionally associated with the proximity between the measured values attributed to the measurand.

Measuring instrument, measuring apparatus

Usually a device used for making *measurements*, on its own or possibly in conjunction with other devices.

Measuring repeatability

This is the measuring fidelity according to a set of repeatable conditions.

Measuring reproducibility[1]

This is the measuring fidelity according to a set of *reproducibility* conditions.

Measuring uncertainty

Non-negative parameter which characterizes the dispersion of values attributed to a measurand, arising from information used according to the method (e.g. A or B of the GUM).

NOTE:

Measuring uncertainty includes elements caused by systematic effects (associated with the corrections and the assigned benchmark values) as well as definitional uncertainty. Estimated systematic effects are not always corrected. Elements associated with uncertainty would be added. The parameter is often the standard uncertainty (from the standard deviation, σ) or the half range (U/2) of an interval with a determined coverage probability (e.g. k = 2 for 95% confidence). Certain elements are evaluated with the type-A GUM from the statistical distribution of measured values characterized by standard deviations. The evaluation using the type-B GUM is characterized by standard deviations of probability density functions based on experience, among other things.

Preliminary hazard analysis (PHA)

Method for identifying and evaluating hazards, their causes, their consequences and the seriousness of these consequences. The aim of this analysis is to determine

1 For statistical terminology refer to ISO 5725-1:1994 and ISO 5725-2:1994.

the appropriate methods and corrective actions to eliminate or control dangerous situations or potential accidents.

Preventive maintenance

To *avoid loss of function; thus, it is a probabilistic notion, one of anticipation and prediction.* Such maintenance is performed at predetermined intervals, in accordance with prescribed criteria, intended to reduce the probability of failure or degradation of the function of a product.

Probability

Statistical concept which can either express a degree of confidence or a measurement of uncertainty (subjective probability) or be taken as the limit of a relative frequency in an infinite series (statistical probability).

Probability density (or distribution function)

This is a function describing the *relative likelihood* of a random variable assuming a particular value. It assigns a probability to each value of a random variable.

Random

Process in which the result varies even if the input data set remains identical (a protocol leads to different results).

Reliability

The reliability of a system (work) is its aptitude to meet its design objectives over a specified period of time, in the environmental conditions to which it is subject. It is based on the probabilities used to evaluate it.

Reliability is one of the aspects of *dependability*. It corresponds to the continuity of service that the system must provide to its users, with the system being considered as irreparable. Any accidental failure is taken into account, regardless of its severity. Reliability measures the rate of failure, the inverse of the MTTF (*mean time to failure*).

Risk

Risk is “a more or less predictable potential danger”, or in other words a drawback which is more or less probable to which we are exposed. The scientific definition of risk involves an aspect of hazard and an aspect of loss, both expressed as probabilities.

Risk analysis

System inevitably contains design errors, regardless of the amount of validation work done. The “zero error” criterion is not a realistic goal, in view of the development costs this would entail. Thus, it is important for so-called critical systems to evaluate the risks for the users of the following systems:

FMECA: failure mode, effects, and criticality analysis.

SEEA: software error effects analysis.

PHA: preliminary hazard analysis.

Risk assessment

Procedure to determine the probability of a hazard *occurring* and its possible consequences.

Safety case

This approach is mainly used in the oil, nuclear, and rail transport sectors. In practice, this procedure facilitates the monitoring of studies.

Security

We distinguish between safety and security. Thus:

Safety guards against catastrophic failures, for which the consequences are unacceptable in relation to the *risk*.

Security relates to the prevention of unauthorized access to information.

***Software* reliability**

Software inevitably contains *design errors*, no matter how strict the rules of design and validation. The ability of a software suite to provide acceptable service in spite of its residual errors defines its *reliability*.

Stress intensity factor (s.i.f., *ΔK*)

s.i.f (ΔK) is a function of the stress, crack size and crack shape. Stress intensity factors do not have variability. They have uncertainty and modeling errors. The crack shape may be unknown and be approximated by a semicircle.

In Mode 1 (during a fatigue cycle, in mechanics of rupture) (s.i.f = ΔK) is a measure of the stress-field intensity near the tip of an ideal crack in a linear-elastic solid when the crack surfaces are displaced in the opening mode. Stress-intensity factor range **(**ΔK**)** in fatigue crack growth rate is the variation in a cycle, that is, Kmax – Kmin.

Tenacity

The ability of materials to resist shock without breaking or chipping.

Trend test

Trend tests are used in *reliability* to obtain indicators of reliability, from data on failures, and determine fluctuations in reliability over time.

Undesirable event

An event that should not occur or which should be improbable in view of the objectives in terms of *dependability*.

Unit (measuring)

Real scalar value defined and adopted by convention, to which we can compare any other similar value to express the ratio between the two values as a number.

Value

Property of a phenomenon, (body, length, weight) expressed quantitatively by a number and a reference.

Abbreviations

CITAC	Cooperation on International Traceability in Analytical Chemistry
CSA	Canadian Standardisation Association
EA	European Cooperation for Accreditation
Eurachem	Focus for Analytical Chemistry in Europe
EUROLAB	European Federation of National Associations of Measurements, Testing and Analytical Laboratories
GUM	Guide to the expression of Uncertainty in Measurement (the reference document recognized by the CSA, EUROLAB, Eurachem, and EA)
IEC	International Electrotechnical Commission
ILAC	International Laboratory Accreditation Cooperation
ISO	International Organization for Standardization
S (or σ)	Standard deviation
SAS	*Le Service d'Accréditation Suisse* (Swiss Accreditation Service)
U	Uncertainty
VIM	International Vocabulary of basic and general terms in Metrology

Chapter 1

Fracture Mechanisms by Fatigue

1.1. Introduction

In the not so distant past, we often built *not from precise calculations*, but by intuition. Carpenters did not question the resistance of the wood they used to build their ships. However, there is no doubt that it is necessary to calculate beforehand, if we are to combine safety with economy in our works of art and professional projects. That is not to say that we can be one hundred percent sure about calculations, because they merely turn out to be the product of the transformation of figures that we put in. The figures themselves can also be marred by diverse mistakes, or not correspond with reality. Moreover, if we forget to calculate a particular part of the problem, there is no automatic mechanism which signals this omission. Therefore, we must go by calculations to obtain a satisfactory level of safety, bearing in mind the imprecision of figures, the irregularities of behavior in constructions, and even the defects of theoretical hypotheses.

The main problem, then, is to study how the construction of stability is modified by the random characteristics of the variables that govern it. We will first attempt to point out the importance of proper usage of materials and their propensity to crack. In this chapter, several important points concerning the analysis of the factors of cracking will be presented. Examples are based on a law renowned for being "simple", but which is representative of crack propagation in zone II (see Figure 1.18). For a greater understanding of behaviors during diverse fracture mechanisms, we will refer to works specialized in continuum mechanics. The reasons governing our choice of welded structures can be explained by their practical importance in metallic works and installations (offshore, building, cars, and other devices assembled by welding). Also, welded structures show a considerable amount

of sensitivity in terms of failure (damage), due to fatigue of the notch stemming from potential penetration lacunae (L), located at the root of the weld bead.

Fatigue, like a succession of mechanisms, constitutes a process (distortions, loading), which modifies the properties of a material. This causes cracks that, over time, tend toward the fracturing of a material and/or of the structure. Although the range of stress is smaller than the resistance of the traction, the fact remains that it has a considerable influence on the reliability of the structure. There are stages which occur over time, ranging from activation, slow propagation to final fracture, used to predict the behavior of the structure. These phases are taken into consideration by most of the models concerned with cracking. This is, in fact, the reason we thought this chapter would be useful in a work dedicated to reliability and quality control. In fatigue, damage occurs in zones where the alternating stress is at its most intense: diverse cavities, notches, blowholes once they have been welded, strong heterogeneity of the material, etc.

Moreover, from microscopic examination of facture, it is clear that typical facies are parallel to the crack propagation, followed by a *tear*, which is the final fracture. The most significant life expectancy (slow to moderate speed of crack propagation) corresponds to the activation of the crack. Life expectancy is relatively low. The problem lies with activation. In this phase, the material is subjected to damage, which cannot be detected by the naked eye. However, since the structure is not constantly under the microscope, it is beneficial to predict these phenomena using reliability calculations. It is this link between cause and effect which justifies reliability calculations. The *stress intensity factor* (*s.i.f.*, ΔK) starts at the foot of the weld bead [MAD 71, WAT 73, LAS 92]. The structure remains sensitive in terms of its resistance, presenting a high risk in fatigue.

1.2. Principal physical mechanisms of cracking by fatigue

1.2.1. *Fracture mechanics*

This section is intended to support the calculations of reliability indexes. Experimentally, it has been demonstrated that the presence of a crack in a part (structure) considerably modifies its resistance [GRO 98, LAS 92]. Additionally, we know that a crack could become unstable during loading. This crack could be propagated in increasing measures, before a brutal fracture occurs. To evaluate the residual strength of a cracked compound, fracture mechanics should be employed. The calculations of cracked solids are based on the crack (sometimes microscopic) in terms of a tributary surface discontinuity with *forced* decohesion between the neighboring atoms. Among the numerous studies on the topic, the work of Griffith [GRI 21], who pioneered the model on the crack resistance, is the most important. This original paper concerned fragile materials (e.g. glass). Irwin *et al.*, in 1948,

applied Griffith's work to solid components (structures). The following chart provides a short overview of the work on cracked components.

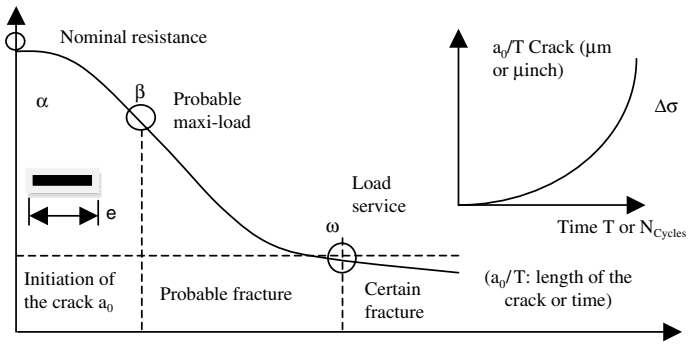


Figure 1.1. *Simplified illustration of fragile fracture in mechanics*

Estimates of the longevity of fatigue are based on rigorous calculations, hence the method employed for finite elements (Figure 1.4). Of course, there are other analytical methods for simple cases (boundary integral equations, photoelasticity, or extensometers), as experimental approaches. With the aim of determining the number of *da/dN* cycles per fracture, many laboratory tests were carried out on smooth test pieces during periodic loading. The literature confirms that during traction, at each one-fourth of a cycle, testing for traction gives a result which correlates to the maximum stress. Wöhler's curve can then be used to find the link between the alternating stress and the number of cycles per fracture, from which the relation of load (R) indicates the quotient between the minimum stress and the maximum stress.

Resistance to fatigue is often modified by a host of factors, such as concentration of stress, temperature, loading, the topography of the surface (rugosity), and random phenomena (wind, ice, waves, etc.). This inevitably leads us to additive considerations of conventional and classic calculations for the resistance of materials. It now becomes even more necessary to consider the statistical aspect of the test results for fatigue. For example:

i) During classic fatigue, dispersion is greater than during low-cycle fatigue.

ii) The minimal endurance limit equates to about half of the endurance limit.

iii) In the low-cycle domain, longevity is defined according to a realistic interval of $[\mu - 3\sigma, \mu + 3\sigma]$ in the *reputedly controlled* domains (aeronautics or "building").

At first glance, the problem of the depth of the initial crack (a_0) is simple. It becomes complicated when the variable representing the initial crack not only remains random but is also dependent on other parameters of the crack law:

$$\frac{da}{dN} = C \times (\Delta K)^m \qquad [1.1]$$

where:

da/dN is the geometric ratio of the number of cycles (mm/cycles);

C and m are intrinsic parameters of the material (adimensional);

ΔK is the *s.i.f.* (MPa$\times$ (m)$^{1/2}$ $\approx$ tenacity).

$$\Delta K = \Delta\sigma \times \sqrt{\pi \cdot a} \times \xi(a) \qquad [1.2]$$

where:

$\Delta\sigma$ is the stress amplitude in the normal direction of the crack (MPa);

a (or a_0/T) is the crack size (mm);

$\xi(a)$ is the indication of corrected geometry (form factor).

The questions we ask, the traditional objectives of fracture mechanics, can be summarized as follows:

– What will be the residual resistance of a cracked component (structure)?

– What critical crack (a_0/T) dimension would be tolerated (see Fig.1.19) by a given loading?

– How long does is take (see Fig.8.42, Chapter 8) for a microscopic crack to reach a critical length?

– Is there an effective experiment (recording gauges) to compare to the reliability model (*a priori* distribution law)?

– How frequently should a component and/or a structure be inspected?

– Which metrology is implemented for detection (non-destructive control, of course)?

1.2.2. *Criteria of fracture (plasticity) in mechanics*

It is easy to want to stay in the comfort zone of elastics (E), because the domain is so well-researched. Though, of course, this is not always the case. If a part is plastically deformed or remains unchanged during a given loading, there are always criteria to explain the nature of, what is called, plastic flow. The two most well-known criteria are TRESCA [TRE 81] and von Mises' (1913) criteria. It is customary

to represent the elastic limit as R_e, which appears after the limit of a plastic deformation, in the case of traction according to a single axis (xx). The stress is written as σ_{xx} and is inferior to R_e. Once modified by the safety factor (s), the stress becomes $\sigma_{xx \leq Re}/s$. This is the acceptable limit. Once applied to materials and to structures, this relation will take into account the diverse variations (forms, notches, and fillets) which are essentially the origin of concentrations of stress, hence the relation [1.2].

When the elastic limit is exceeded, plastic deformation occurs. It can therefore be shown by $\sigma_{xx \geq Re}$. These cases are encountered in the manufacturing process, during folding, stamping, laminating, or forging. The explanation for this resides in plastic deformation, which is shearing because the atoms slide and cause what is known as scission. The latter is maximal if the sliding angle (λ) is at 45° relative to the traction axis (xx).

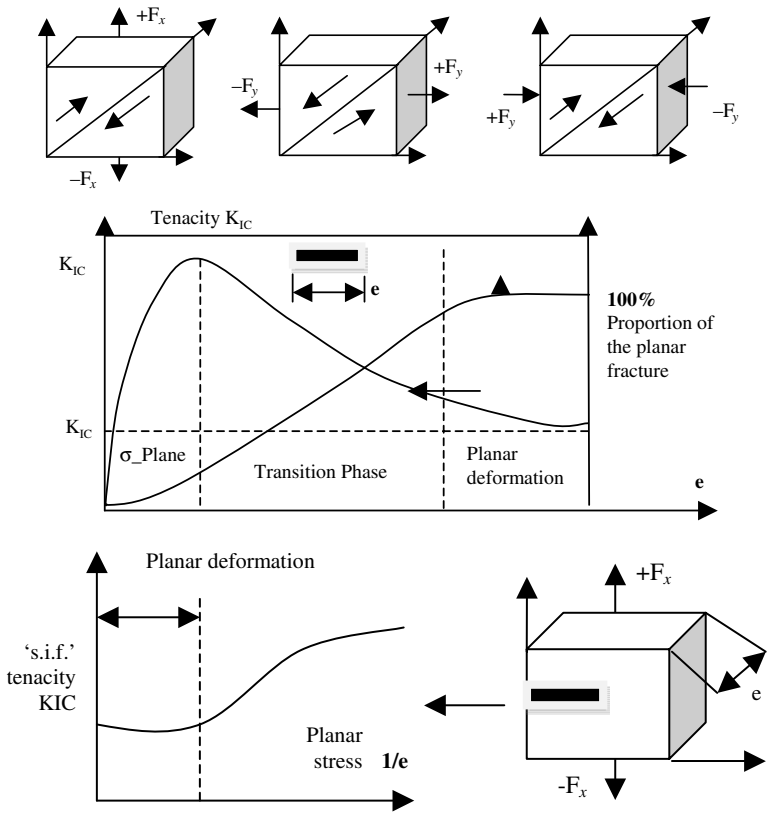


Figure 1.2. *Simplified illustration of fracture criteria for mechanical plasticity*

For a material that is stressed during loading, which has a known critical value (K_{IC} or G_{IC}), in mode I the graph takes into account external factors such as the rate of loading and temperature, to name but two examples. These factors are not tributary to the geometry of the solid component. In reality, for a crack component, the tenacity K_{IC} depends on the degree of biaxiality (see Figure 1.2 (top)), and even on the degree of triaxiality and of the stress of the *cfr* (crack front). In fact, this depends on the capacity of the solid component to endure plastic deformation in the *cfr*.

1.2.2.1. *Tresca's elastic limit criteria (1864)*

The elastic and plastic domains are separated by a hypersurface with five dimensions (isostatic system). The principal stresses are represented by $[\sigma_1, \sigma_2, \sigma_3]$. The hypersurface takes the form $f(\sigma_1, \sigma_2, \sigma_3) = 0$. Since plastic deformation occurs during shearing, Mohr's circle would normally be used to explain that for planar stress, the condition of elastic deformation being $R_e \geq |\sigma_1 - \sigma_2|$. In three dimensions, the stresses are shown as follows: $\{R_e \geq |\sigma_1 - \sigma_2|; R_e \geq |\sigma_1 - \sigma_3|; R_e \geq |\sigma_2 - \sigma_3|\}$.

1.2.2.2. *von Mises' elastic limit criteria (1913)*

von Mises' criteria translates the energy of elastic deformation $U = (\sigma \cdot \varepsilon)/2$ into traction–compression. During shearing, $U = (\tau \times \gamma)/2$ is used. Considering we often want to remain in the elastic domain, the energy must not exceed a maximum limit which can be formulated as follows: $R_e \geq \sqrt{\sigma_1^2 + \sigma_1^2 + \sigma_1\sigma_2}$.

This expresses the ellipse equation, hence the use of Mohr's circle. This topic could be developed further; however, it does not fall within the scope of this work. Therefore, the manuals concerned with the resistance of materials will be considered. To summarize, the literature proposes the following effective stresses:

– Tresca's stress: $\sigma_e = \max\left\{ |\sigma_1 - \sigma_2|; R_e \geq |\sigma_1 - \sigma_3|; R_e \geq |\sigma_2 - \sigma_3| \right\}$

– von Mises' stress: $\sigma_e = \frac{1}{2}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}$

For planar bidirectional stress:

– Tresca's stress: $\sigma_e = \sqrt{\sigma^2 + 4\tau^2}$

– von Mises' stress: $\sigma_e = \sqrt{\sigma^2 + 3\tau^2}$

– The surface that determines the plastic domain of the elastic: $\sigma_e = R_e$

– Tresca's stress: $\sigma_e \prec R_e$; von Mises' stress: $\sigma_e \succ R_e$

Finite element modeling (see Figure 1.3) represents the equivalent field of stress with a color chart. The metallic parts which are subjected to repeated or alternating efforts can break, even if the maximum effort is inferior to the elastic limit. The life span of these parts far exceeds that of the lowest effort (Wöhler or *S–N* curves).

Fatigue tests are carried out by subjecting a metallic test piece to traction/ compression or alternating bending efforts. For most steels there is a critical effort, below which the fracture appears only after a considerable amount of time. This effort is the fatigue limit of steel.

The *rupture* originates from a minuscule crack which progressively expands until a brutal fracture occurs. We calculate the metallic parts subjected to repeated efforts, so that at no point the effort, by a square millimeter, exceeds the fatigue limit. This requires the parts of different sections to be connected to a fillet with a large radius of curvature and the state of the surface to be cared for.

For each cycle of the law of cracking, it is possible to say whether the structure has broken or not. This leads to separating the space into two distinct regions, as is shown in the following figures of models by finite elements (software ANSYS). In conditions of speed, deformation, and temperature, materials show plastic deformation at the tip of the crack, which is sufficiently small to be handled with linear elastic theory.

Paris' law takes into account the stage of slow crack propagation by fatigue (activation stage of the crack in propagation phase). The crack is likely to propagate in three directions, which are linked to the applied efforts. Three modes of deformation can be distinguished as shown in Figure 1.3. According to the mode of crack propagation, three s.i.f. K can be defined. In the singular zone, the stress field shows a singularity in $(1/\sqrt{r})$ at the tip of the crack.

1.3. Modes of fracture

It is generally accepted that the crack propagates due to a combination of stresses, according to the three following modes:

Mode I or opening: The normal traction stress is applied to the plane of the crack. In mode *I*, K_I corresponds to the *s.i.f.* in the *mode of opening* of the crack edges (this fracture is extremely dangerous).

Mode II or straight slip: The shearing stress works in parallel to the plane of the crack and is perpendicular to the front of the crack. In mode *II*, K_{II} corresponds to the *s.i.f.* in the *mode of shearing* on the plane of the crack edges.

Mode III or screw slip: The shearing stress works in parallel to the plane of the crack and in parallel to the front of the crack. In mode *III*, K_{III} corresponds to the s.i.f. in the mode outside the plane of the crack edges.

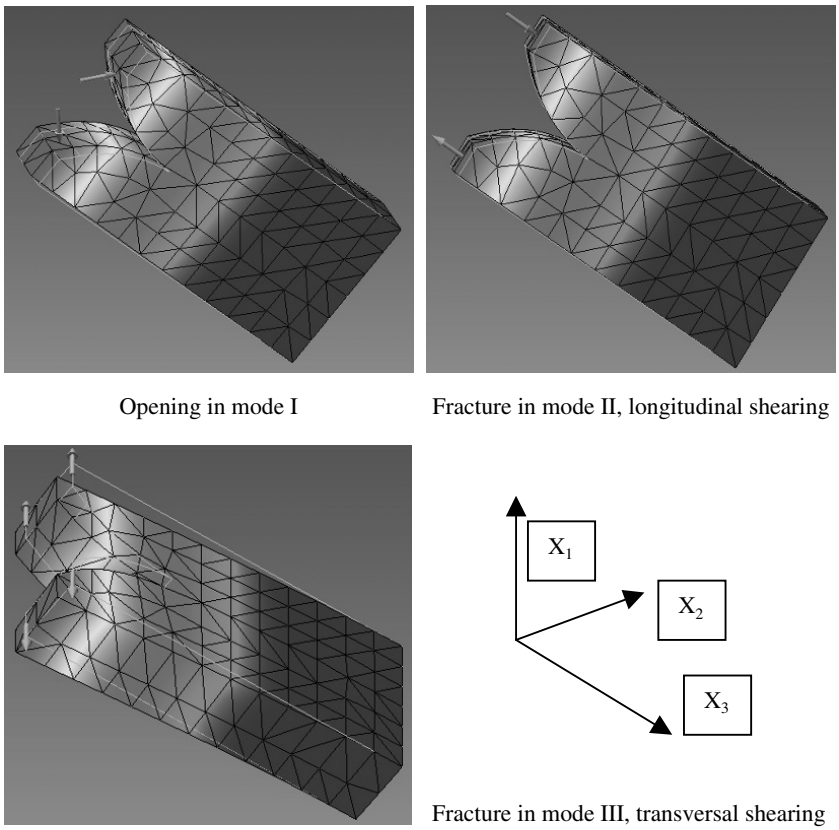


Figure 1.3. *Modes of deformation of a cracked body*

Factor K_I varies according to the nominal stress σ_n applied to a part which is half the length (a) of the crack. In the case of an infinite elastic medium, we use:

$$K_I = \sigma_N \times \sqrt{\pi a} \qquad [1.3]$$

For parts with finite dimensions, it has been demonstrated that:

$$K_I = \sigma_N \times \sqrt{\pi a} \times \xi(a) \qquad [1.4]$$

where $\xi(a)$ is a corrected coefficient of the geometry that allows K_I to take the following corrected K_{IC} values:

– ordinary steel: $K_{IC} \approx 10$ at 250 MPa $m^{1/2}$;

– steel with very high resistance: $K_{IC} \approx 30$ at 100 MPa $m^{1/2}$.

Using Irwin's theory of elasticity, we present, in deformation or in planar stress, displacements u_i and stresses σ_{ij}, in the singular zone, according to the mode considered.

Mode I is a mode of opening the crack, where the displacements are parallel to the direction of propagation. The following equations can be used:

$$\left\{\begin{array}{l} U_1(x,y) = U_1(x,-y) \\ V_1(x,y) = -V_1(x,-y) \end{array}\right\} \rightarrow \left\{\begin{array}{l} U_1 = \frac{1}{2}\left[U(x,y) + U(x,-y)\right] \\ V_1 = \frac{1}{2}\left[V(x,y) - V_1(x,-y)\right];\ W_1 = 0 \end{array}\right\} \quad [1.5]$$

Mode II is a mode of opening the shearing on the plane, where the displacements of the crack are parallel to the direction of propagation. We use:

$$\left\{\begin{array}{l} U_2(x,y) = -U_2(x,-y) \\ V_2(x,y) = V_2(x,-y) \end{array}\right\} \rightarrow \left\{\begin{array}{l} U_2 = \frac{1}{2}\left[U(x,y) - U(x,-y)\right] \\ V_1 = \frac{1}{2}\left[V(x,y) + V_1(x,-y)\right] \ \textit{and}\ W_2 = 0 \end{array}\right\} \quad [1.6]$$

Mode III is a mode of opening the anti-planar (outside-planar) shear, where the displacements of the crack are defined by U_i, V_i, W_i, where i is the index indicating the elementary mode of fracture, that is, $i = I$, II, or III, for example:

$$\left\{\ U_2 = 0;\ U_2 = 0;\ W_2 = 0\ \right\} \quad [1.7]$$

Fracture can be mixed. In this case, we proceed to the additivity of the displacements. The combination of modes I and II gives, for example:

$$\left\{U = U_1 + U_2 = U(x,y) \text{ and } V = V_1 + V_2 = V(x,y)\right\} \quad [1.8]$$

The mathematical equations of displacements U_i and the stresses σ_{ij}, in Irwin's sense, are written as follows:

$$\left\{\begin{array}{l} U_1 = \dfrac{K_I}{2\mu}\sqrt{\dfrac{r}{2\pi}}cos\left(\dfrac{\theta}{2}\right)\left(\kappa - \cos\theta\right) + \dfrac{K_{II}}{2\mu}\sqrt{\dfrac{r}{2\pi}}\sin\left(\dfrac{\theta}{2}\right)\left(\kappa + \cos\theta + 2\right) \\ U_2 = \dfrac{K_I}{2\mu}\sqrt{\dfrac{r}{2\pi}}\sin\left(\dfrac{\theta}{2}\right)\left(\kappa - \cos\theta\right) - \dfrac{K_{II}}{2\mu}\sqrt{\dfrac{r}{2\pi}}\sin\left(\dfrac{\theta}{2}\right)\left(\kappa + \cos\theta - 2\right) \end{array}\right\} \quad [1.9]$$

The stress equations, according to Irwin, are written as follows:

$$\left\{\begin{array}{l} \sigma_{11} = \dfrac{K_I}{\sqrt{2\pi r}}cos\left(\dfrac{\theta}{2}\right)\left[1 - \sin\left(\dfrac{\theta}{2}\right)\sin\left(\dfrac{30}{2}\right)\right] - \dfrac{K_{II}}{\sqrt{2\pi r}}\sqrt{\dfrac{r}{2\pi}}\sin\left(\dfrac{\theta}{2}\right)\left(2 + \cos\dfrac{\theta}{2}\cos\dfrac{30}{2}\right) \\ \sigma_{12} = \dfrac{K_I}{\sqrt{2\pi r}}cos\left(\dfrac{\theta}{2}\right)\sin\left(\dfrac{\theta}{2}\right)\cos\left(\dfrac{30}{2}\right) + \dfrac{K_{II}}{\sqrt{2\pi r}}\cos\left(\dfrac{\theta}{2}\right)\left(1 - \sin\left(\dfrac{\theta}{2}\right)\sin\left(\dfrac{30}{2}\right)\right) \\ \sigma_{22} = \dfrac{K_I}{\sqrt{2\pi r}}cos\left(\dfrac{\theta}{2}\right)\left(1 - \sin\left(\dfrac{\theta}{2}\right)\sin\left(\dfrac{30}{2}\right)\right) + \dfrac{K_{II}}{\sqrt{2\pi r}}\sin\left(\dfrac{\theta}{2}\right)cos\left(\dfrac{\theta}{2}\right)\cos\left(\dfrac{30}{2}\right) \end{array}\right\} \quad [1.10]$$

with $\kappa = \left(3 - \nu\right)/\left(3 + \nu\right)$ in planar deformation and $\kappa = (3-\nu)/(3+\nu)$ in planar stress. The shearing modulus is therefore

$$\mu = E/2\left(1 + \nu\right).$$

where:

r and θ are the radius and the angle, respectively, in polar coordinates;

ν and E are Poisson's coefficient and Young's modulus, respectively.

It is worth pointing out that in the case of anti-planar loading, the only displacement component remains U_3. The respective expressions of displacements and stresses are therefore the following:

$$\left\{U_3 = \frac{2K_{III}}{\mu}\cdot\sqrt{\frac{r}{2\pi}}Sin\left(\frac{\theta}{2}\right); \sigma_{13} = -\left(\frac{2K_{III}}{\sqrt{2\pi r}}\right)Sin\left(\frac{\theta}{2}\right); \sigma_{23} = -\left(\frac{2K_{III}}{\sqrt{2\pi r}}\right)Cos\left(\frac{\theta}{2}\right)\right\} \quad [1.11]$$

The *s.i.f.* (K_I, K_{II}, and K_{III}) remain independent of r and θ. They are distribution functions of external efforts and crack geometry. Griffith's theory was the first energetic approach on a cracked body. Moreover, there are other means of characterizing the singularity of the stress field in the neighborhood of the crack

front (*n.c.f.*) or of studying the contour integral, which is Rice's [RIC 68] integral. The aforementioned concepts are only useful for isotropic materials, which have an elastic behavior. Factors K_I, K_{II}, and K_{III} characterize both the detail of the geometry and of the crack, as well as the nature of the stress. Preventing fracture by fatigue means mastering parameters such as:

– conception of the part (materials and structure);

– some knowledge of loading;

– mastering residual stresses and the elaboration process of a material;

– accurately predicting life expectancy through inspections;

– appropriate dimensioning, periodic checks, and validating hypotheses.

By means of an example, for offshore structures, it is worth emphasizing fatigue inherent to waves, winds, and corrosion.

– For bridges and roads, it is advisable to keep a strict eye on Miner's damage when dimensioning the roads.

– Ensure that the calculations take into account the stressed materials.

– In aeronautics, cyclic stresses (vibrations and variations of temperature) are at the root of mechanic's particular interest in fracture.

1.3.1. *Directed works*

In this section we calculate the principal maximum and minimum stresses, as well as the normal stress and the shearing stress along the plane (δ = determined by e_f).

– normal stress on the *x*-axis: $\sigma_x = 100 \times 10^3$ Pa

– normal stress on the *x*-axis: $\sigma_x = 417 \times 10^3$ Pa

– shearing stress on *xy*: $\tau_{xy} = -47 \times 10^3$ Pa

– angle of planar stress: $\theta = 100 \times 10^3$ Pa

1.3.1.1. *Conditions and questions*

– The normal stress on x- and y-axes are σ_x and σ_y.

– The shearing stresses are τ_{xy} and τ_{xy}.

– What are the principal stresses, σ_{max} and σ_{min}, as well as the shearing stresses on the plane determined by AB?

Schematization of the problem:

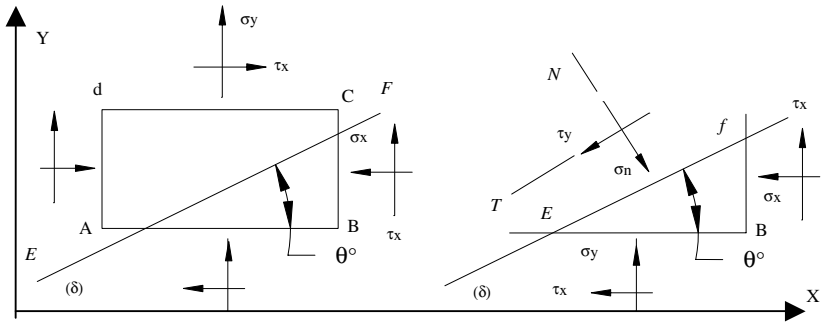


Figure 1.4. *Calculation of stresses*

Solution:

In accordance with the fundamental principles of mechanics of solid materials, we consider the following.

For planar stress, the normal stress is written as

$$\sigma_n(\theta) = \frac{\sigma_y + \sigma_x}{2} + \frac{\sigma_y - \sigma_x}{2} Cos(2\theta) + \tau_{xy} \times Sin(2\theta); therfore\ \sigma_n(\theta) = 3.171 \times 10^5\ Pa$$

i) Shearing stress on (xy)

$$\tau_n(\theta) = \frac{\sigma_y - \sigma_x}{2} Sin(2\theta) - \tau_{xy} \times Cos(2\theta); therfore\ \tau_n(\theta) = 1.490 \times 10^5\ Pa$$

ii) Maximum principal stress

$$\sigma_{major} = \frac{\sigma_y + \sigma_x}{2} + \sqrt{\left(\frac{\sigma_y - \sigma_x}{2}\right)^2 + \tau_{xy}^2};\ therfore\ \sigma_{major} = 4.242 \times 10^5\ Pa$$

iii) Maximum principal stress

$$\sigma_{\text{minor}} = \frac{\sigma_y + \sigma_x}{2} - \sqrt{\left(\frac{\sigma_y - \sigma_x}{2}\right)^2 + \tau_{xy}^2};\ therfore\ \sigma_{\text{minor}} = 1.098 \times 10^5\ Pa$$

iv) Maximum and minimum shearing stress

$$\tau_{\max} = \frac{\sigma_{\text{Major}} - \sigma_{\text{minor}}}{2} = 1.572 \times 10^5 \ Pa; \ and \ \tau_{\min} = -\tau_{\max} = -1.572 \times 10^5 \ Pa$$

The relation between the stresses and the tilt angle of the plane is demonstrated in the following graph, with an angle range of 0° to 360°. The range (R) of the angles $\theta = 0°, 1°, \ldots, 360°$.

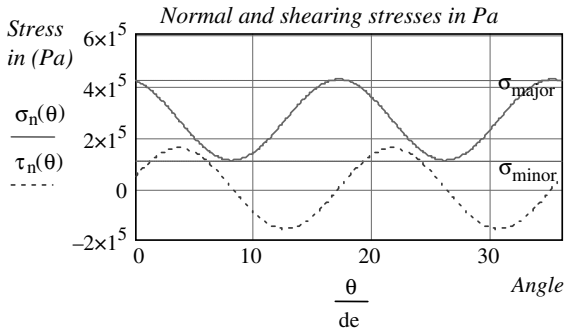


Figure 1.5. *Graph showing principal stresses*

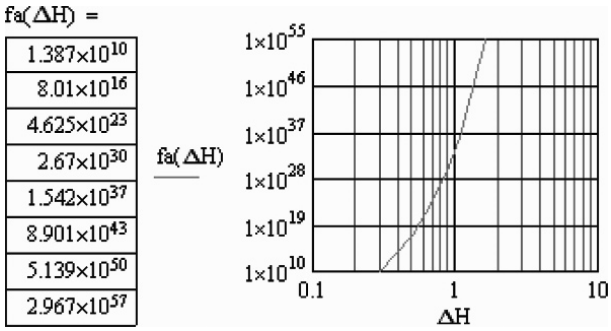


Figure 1.6. *Graph showing the maximum and minimum shearing stresses*

1.4. Fatigue of metals: analytical expressions used in reliability

What we call fatigue or damage by fatigue is the modification of materials due to the application of effort cycles, which, through repetition, lead to the fracture of component parts. As soon as there is an applied effort over *time*, what we call

fatigue occurs. Fractures can, therefore, appear for stresses which are often lower than the fracture limit of the material, and even the elasticity limit. Damage is accompanied with no apparent modification of form or the aspect of the part.

The origin of the fracture is due to a progressive crack which stretches until the remaining transversal section can no longer support the applied effort. When we subject test pieces to the cycles of periodic stress, to maximum amplitude (s) of constant frequency, for every fracture, we call N the number of cycles. (N, σ,)– *Stress number of cycles* (endurance curve). The three domains that have been previously detailed can be distinguished here.

1.4.1. *Wöhler's law*

It is believed by numerous authors that Wöhler's law (1860) is the oldest [WÖH 1860]. It allows a good representation that is completed by the middle part of the *S–N* curve. This can be explained by the fact that most curves have a low inflexion point in the neighborhood to which they are rectilinear. The general speed of the *S–N* diagram is described by the equiprobability curves of fracture, for which this is the expression:

$$LogN = a - \left(b \times \sigma\right) \qquad [1.12]$$

where:

N is the number of cycles;

a and b are two constants;

σ is the amplitude of the applied stress.

Wöhler's curves are established with experiments. They depend on numerous factors, such as the given maximum state, the loading, the environment, and the nature of the material. The *S–N* curve does not, in reality, represent a sharp "bend", but is a progressive curve connecting to the horizontal branch. Calculating the structures gives each cycle an average stress and an alternating stress, which are often calculated in elasticity. The inherent *S–N* curves are taken into consideration for each value attributable to the load ratio (R) where α and β are the characteristics of the material under average stress in a controlled temperature. In the laboratory the expression [1.3], taken from the literature, is:

$$N = \kappa\left(R\right) \times \left(R\right) \times S_{\alpha}^{-\beta \times (R)} \;;\; when \;\; \mathrm{S}_{\alpha} \succ S_{\alpha D}\left(R\right) \qquad [1.13]$$

where:

$S_{\alpha D}$ is the endurance limit;

K is a load factor;

R is the load ratio, or $R = \left(\sigma_{\min} / \sigma_{\max}\right)$.

In Chapter 8, which wholly dedicated to the valuation of Monte Carlo's (MC) method, a case study will be presented. However, it is worth concentrating on Wöhler's curve (*S–N*), because in terms of probability, it is true that the model we are looking to make more reliable is the one that results in Wöhler's famous curve. The objective of the probabilization of this model is, therefore, to obtain a network of *pS–N* curves, digitally generated by MC simulation. This causes uncertainties, which in turn contribute toward the variability of the curve *S–N*. This essentially depends upon:

– the state of the local stress σ, in MPa;

– the applied loading $S(t)$;

– the intrinsic parameters of the materials;

– the criteria of fatigue;

– rugosity (state of the surface), etc.

In our experimental opinion, metallurgy plays an important role in the variability of the *S–N* curves. It is advisable to bear this in mind in the spreading of uncertainties (see Chapter 3, volume 1).

1.4.2. *Basquin's law (1910)*

Basquin's equation represents a hyperbolic form from which a branch asymptotically links to the axis *N* (loading), as follows:

$$LogN = a - b \times L\log(c) \qquad [1.14]$$

This type of curve, like Wöhler's, cannot provide a fatigue limit. With a, b, and δ being constant, it can also be written as:

$$c = \left(\frac{a}{N}\right)^{\delta} \text{ when } A = e^{a} \text{ and } \delta = \left(\frac{1}{b}\right) \qquad [1.15]$$

1.4.3. *Stromayer's law (1914)*

Stromayer proposes another relation for greater precision. His law represents the logarithm of the number of cycles in a function of logarithms of applied stress, which however does not have a value to render endurance. It is presented as follows:

$$Log\left(N\right) = a - b \times L\log\left(c - \rho_e\right) \qquad [1.16]$$

where:

ρ_e represents the endurance limit;

N is the number of cycles;

a and b are constants;

c is the amplitude of the applied stress.

The curve that is obtained is identical to Basquin's curve. It shows, in addition, a horizontal asymptote of ordinates ρ_e, often not confused with the life expectancy axis. Specialists often believe that Stromayer's law is more realistic than Wöhler or Basquin's law. Nevertheless, it causes several difficulties of adjustment.

1.4.4. *Palmgren's law*

To avoid adjustment problems, Palmgren (1924) introduced the following:

$$C = \rho_e + \left(\frac{A}{N+B}\right) \text{ when } A = e^a \qquad [1.17]$$

where:

ρ_e represents the endurance limit;

N is the number of cycles;

B is the constant parameter;

C is the amplitude of applied stress.

For some authors, this law is realistic because it allows us to adjust the quality for a given data. The law can be written in two forms. The first is similar to Palmgren's law:

$$Log(N+B) = a - b \times L\log\left(c - \rho_e\right) \qquad [1.18]$$

The classic way of tackling the problem of variable amplitude loading, by Palmgren and Miner, consists of using damage accumulation laws. Palmgren assumes that the total damage of a structure is the sum of damages sustained during each of the loading levels of aptitude.

1.4.5. *Corson's law (1949)*

Corson [COR 49] tackled the damage accumulation problem. His relation is based on experiments, summarized in the following equation:

$$N = \frac{A}{(S-E)\times d^{(S-E)}} \qquad [1.19]$$

where:

S is the stress representing the endurance limit;

N is the number of cycles;

E, A, and d are constants.

By using $c = \mathrm{Log}(d)$, equation [10.19] becomes:

$$N = \frac{A\times \exp^{-C(S-E)}}{(S-E)} \qquad [1.20]$$

This law, though more recent than Palmgren's, deals with the problem by linking the number of cycles to stresses. The coefficients it integrates means that the unknown factors are added to the calculations, resulting in imprecisions. It is therefore rarely used.

1.4.6. *Bastenaire's law*

In the 1960s, Bastenaire [BAS 60] proposed a formula, which seems the most precise:

$$N = \left(\frac{A}{C-\rho_e}\right)\times Exp^{-B(C-\rho_e)} \qquad [1.21]$$

where:

ρ_e represents the endurance limit;

N is the number of cycles;

A and B are constant parameters;

C is the amplitude of the applied stress.

In terms of a critique, this law does not differ much in its formulation from the other laws that have been described here. Its credit lies in its being a relatively recent law; however, it brings very little new to the table. In fact, it integrates the same parameters (C and ρ_e).

1.4.7. *Weibull's law*

Weibull's law [WEI 55] does not provide any more explanation than Palmgren's. For reference, it is written as follows:

$$Log\left(N+B\right)=\frac{a-b\left(S-\rho_e\right)}{\left(R-\rho_e\right)} \qquad [1.22]$$

where:

ρ_e represents the endurance limit (constant);

N is the number of cycles;

R is the resistance to the traction of the material.

a, b, and B are constant parameters, which take into account the form, the position, and the scale of the probability density functions curve.

Weibull's law is often used in statistics to represent the life expectancy of structures whose rate of fracture depends on the parameter β. It will be studied in greater detail in Chapter 1, statistics and reliability.

1.4.8. *Henry's law*

This law expresses the damage for a level of stress and is written as follows:

$$D=\left(\frac{n}{N}\right)\times\frac{1}{1+\alpha\left(1-\frac{n}{N}\right)} \quad \text{and} \quad \alpha=\left(\frac{\rho_e}{\sigma-\rho_e}\right) \qquad [1.23]$$

where:

ρ_e represents the fatigue limit (endurance);

n is the constant parameter.

α takes into account the fact that the speed of damage is faster for greater efforts and due to increasing the number of imposed cycles N to the level of stress (σ).

1.4.9. *Corten and Dolen's law*

In this law, damage is seen to result from the germination of pores, which have developed from cracking. It is written as follows:

$$N_g = \frac{N_1}{\alpha_1 + \alpha_2 \left(\frac{C_2}{C_1} \right) \times d + \alpha_3 \left(\frac{C_3}{C_1} \right) \times d + \ldots + \alpha_n \left(\frac{C_n}{C_1} \right) \times d} \quad [1.24]$$

where:

N_g represents the number of stress cycles leading to fracture, for the chronological increase in stress amplitude;

N_1 is the number of cycles at the highest level of stress before fracture;

α_1, α_2, and α_n are the ratios of the number of cycles applied to the level of stress for the total number of applied cycles;

$C_1 > C_2 > C_n$ are different levels of alternating stress (applied amplitude);

d is the reverse of Wöhler's linear sloping partition.

Analytical expressions from the domain of low-cycle fatigue and other random aspects of the phenomena of fatigue consider that metal can become harder, soften, or remain stable under cyclic stress (plastic domain). From the evolution in mechanics of hysteresis loops, a strain-hardening curve has been determined [LIE 73, LIE 82], which is given by the following expression:

$$C_a = K' \times \left(\frac{\Delta \varepsilon_\rho}{2} \right) \times n' \quad [1.25]$$

where:

C_a represents the rational alternating stress (C_r) and K' is the coefficient;

$\Delta \varepsilon_\rho$ is the amplitude of rational plastic deformation (ε_r);

(n') is the coefficient for cyclic strain-hardening.

1.4.10. *Manson–Coffin's law*

Characterized by considerable deformation and a relatively low life expectancy, that is less than 10,000 cycles, Manson–Coffin's law is used in low-cycle fatigue. The facies of the fracture surfaces are drawn more closely together from those obtained during a static fracture, than from the facies of a fracture surface by fatigue. This means that this law is rarely used. The mechanisms which operate the phenomenon of facies differ to those governing ordinary fatigue. In fact, the amplitude of the deformation is considered, rather than the stress amplitude. Manson–Coffin's law assumes that the alternating number to fracture N_f is linked to the amplitude of the plastic deformation $\Delta\varepsilon_\rho$.

$$\left(\Delta\varepsilon_\rho\right)\times\left(N_f^\alpha\right)=\beta \qquad [1.26]$$

where:

α represents the coefficient between 0.5 and 0.7.

β is the constant linked to real deformation while the fracture is in traction, and it can also be expressed as follows:

$$\beta = 2^{(1-\alpha)}\times \mathrm{Log}\left(\frac{S_0}{S}\right) \qquad [1.27]$$

where S and S_0 are the initial damage sections for a traction test.

Manson–Coffin's law is generally proven when the amplitude of plastic deformation is greater than 1/100. This sensitivity limits its use. The life expectancy expressed in number of cycles leading to fracture is therefore provided by the Manson–Coffin relation in plastic deformation. For total deformation, another relation called Morrow's relation is employed:

$$\frac{\Delta\varepsilon_\rho}{2}=\Delta\varepsilon_\rho^{'}\times\left(2N_f\right)^{-c} \quad \textit{in plastic deformation} \qquad [1.28]$$

Morrow's relation, for total deformation, is expressed as follows:

$$\frac{\Delta\varepsilon_t}{2}=\Delta\varepsilon_\rho^{'}\times\left(2N_f\right)^{-c}+\left(\frac{C_f^{'}}{E}\right)\times\left(2N_f\right)^{b} \qquad [1.29]$$

where:

E is Young's modulus;

N_f is the number of cycles leading to fracture;

$\Delta\varepsilon_t$ is the amplitude of the total deformation;

ε'_t is the coefficient of ductility in fatigue;

C'_f is the coefficient of resistance to fatigue.

Morrow's relations demonstrate that ductile materials which have high coefficients show a good resistance to large deformations. The two essential factors for resistance to fatigue will lead us to introduce the probabilistic concepts for both fatigue and resistance. The main reasons for this being:

– stress fluctuations of a part during service;

– the dispersion of resistance characteristics in fatigue.

The causes of fatigue can originate from external factors and are due to the implementation of the material (thermal treatments, strain-hardening, etc.). The parameters affecting the resistance to fatigue of a material amount to the endurance limit. According to Shigley, the resistance of any part can be expressed as follows:

$$\rho_e = K_a \times K_b \times K_c \times K_d \times K_e \times K_f \times \left(\rho_e^{'}\right) \qquad [1.30]$$

where:

ρ'_e is the endurance limit on a smooth test piece in rotary bending;

K_a represents the scale effect;

K_b represents the surface effect;

K_c represents the temperature effect;

K_d represents the reliability effect;

K_e represents the notch effect;

K_f represents other effects.

The scale effect K_a allows the endurance of machine parts and the endurance of test pieces to be compared. The majority of fractures by fatigue K_b begin at the surface, hence the result machines fracture. According to Shigley, the temperature effect is written as:

$$K_c = 620/\left(460 + T\right), \textit{ when } T \succ 160^{\circ} \textit{ Fahrenheit} \qquad [1.31]$$

When the temperature rises, the elasticity limit and the resistance to traction are reduced. The same is true of ρ_e. For example, on one of Wöhler's classic curves, the endurance limit corresponds to a fracture probability of ½. When the endurance limit corresponds to a survival probability greater than ½, the number of standard deviations

must be subtracted from the endurance limit. Working from the hypothesis that standard deviation on $\rho_e \approx 80\%$, we consider:

$$K_d = \{1 - 0.08 \times D\}, \text{ with } D \textit{ diameter in mm or inches} \quad [1.32]$$

Factor K_d is expressed by means of stress concentrations, caused by changes in the section (holes, porosities, heterogeneities, notches, etc.). For welded structures, it is advisable to take the weld bead and the superficial or dense faults situated at the foot of the weld bead (micro-geometry) into consideration. According to the authors I.F.C. Smith and R.A. Smith [SMI 83], the sensitivity of the notch is calculated by the following relation:

$$q = (K_u - 1)/(K_t + 1) \quad [1.33]$$

To calculate K_t, Peterson proposes the following relation:

$$K_\tau = K_t \times \sqrt{1 - \nu \times \left(\frac{K_t - 1}{K_t}\right) + \nu \times \left(\frac{K_t - 1}{K_t}\right)^2} \quad [1.34]$$

Where K_t is the stress concentration factor.

The mathematical theory of elasticity provides many valuable solutions involving the stress distributions in bodies of simple geometries and loadings. A common use of these solutions is the determination of stress concentration factors (K_t) resulting from discontinuities or other localized disturbances in the stress field of the solid body.

v is Poisson's coefficient;

K_u is the ratio of endurance limits on smooth test pieces and notched test pieces, in which;

$K_u < K_t$ on account of the possibility of the material adapting;

$K_u \rightarrow K_t$ for materials with very high elastic limits.

1.5. Reliability models commonly used in fracture mechanics by fatigue

Eyring's models of *stress* acceleration are simple. They tend to be used in the domains of applied chemistry and quantum mechanics, because stresses are heavily involved in the process. His model has several characteristics:

– a theoretical basis of chemistry and of quantum mechanics;

– a chemical process (diffusion, corrosion, migrations, etc.) is the cause of degradation which leads to damage due to varying rates of degradation with stress;

– a temperature that influences relevant stresses: similar case to Arrhenius' relation which is an empirical model (activation energy necessary to cross an energy barrier and to start a reaction).

$$\tau_f = AT^{\alpha} Exp\left\{\frac{\Delta H}{\kappa T} + \left(B + \frac{C}{T}\right) \times S_1\right\} \quad [1.35]$$

S_1 is a function of current–voltage, and the parameters α, H, B, and C serve to determine the acceleration between the combination of stresses. As in Arrhenius' model, κ is Boltzmann's constant and ΔT is Kelvin's variation of degrees of temperature. Arrhenius' model predicts failure by increasing (acceleration) the temperature. One of the early models of acceleration predicts the variation of temperature as in the following formula:

$$\tau_f = \gamma \times Exp\left\{\frac{\Delta H}{\kappa T}\right\} \quad [1.36]$$

With temperature T in Kelvin (+273.16°C) at the moment, the fracture occurs and κ as Boltzmann's constant (8.617×10^{-5} in eV/K). The constant γ is a scale factor which takes into account the calculation of acceleration factors, and ΔH designating activation energy, which is the critical parameter in this model. Indeed in this model, Arrhenius' activation energy, ΔH, is the factor that must be known to calculate the acceleration of temperature. ΔH generally varies between 0.3 and 1.5 and depends upon the fracture mechanism and the materials involved. The acceleration factors between the two temperatures exponentially increase with the rate of increase of ΔH. The acceleration factor between a higher temperature T_2 and a lower temperature T_1 is given by:

$$f_{acc} = Exp\left\{\frac{\Delta H}{\kappa}\left(\frac{1}{T_1} + \frac{1}{T_2}\right)\right\} \quad [1.37]$$

By substitution the value for Boltzmann's constant κ, the expression of AF is written as a function of T in °C as follows:

$$f_{acc} = Exp\left\{\Delta H \times 11605 \times \left(\frac{1}{T_1 + 273.16} + \frac{1}{T_2 + 273.16}\right)\right\} \quad [1.38]$$

Arrhenius' model is normally used to model the life expectancies of cases when the failing mechanisms depend upon chemical reactions, diffusion processes,

or a migration process (metallurgy). By adding a term to the stress (non-thermal), Eyring's model becomes:

$$\tau_f = AT^{\alpha} Exp\left\{\frac{\Delta H}{\kappa T} + \left(B + \frac{C}{T}\right) \times S_1 + \left(B + \frac{E}{T}\right) \times S_2\right\} \quad [1.39]$$

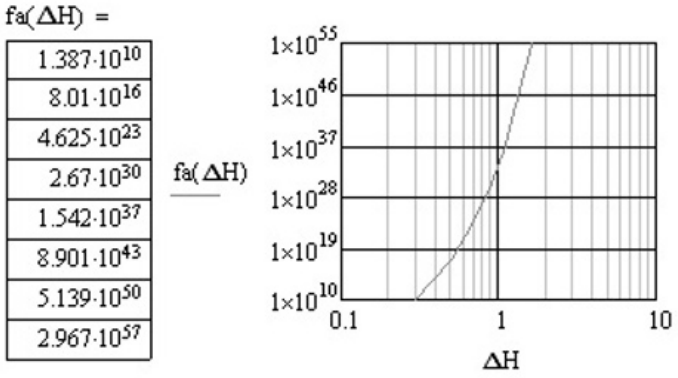


Figure 1.7. *Acceleration factor of Eyring's model in function of ΔH*

Most of the models used in mechanics do not comprise interaction terms. As well as temperature interactions, Eyring's model takes the *stress* into account. In the models without interaction, the acceleration factors for each stress can be calculated. Eyring's model is often complicated to use in its most general form, and must be simplified for every mechanism of a particular failure. It turns out to be disadvantageous on account of the numerous parameters it includes. Even with just two stresses, there are five parameters to estimate. Each additional stress adds another two unknown parameters. Certain parameters have a secondary effect. For example, when $\alpha = 0$, the model works quite well since the temperature term brings us closer to Arrhenius's model. Moreover, the constants C and E are only necessary if there is an interaction effect of temperature, with respect to other stress factors.

1.5.1. *Coffin–Manson's model for the analysis of crack propagation*

Coffin–Manson's model is suitable for the evaluation of crack propagation in fatigue of materials. A model of this type, known as Coffin–Manson's model, has been successfully used for crack propagation in metals assembled by welding. The model takes the following form:

$$N_f = {f_{acc}}^{-\alpha} \times \Delta T^{-\beta} \times G\left(\tau_{\max}\right) \quad [1.40]$$

where:

N_f is the number of cycles to fracture;

f is the cycle frequency;

ΔT is the temperature during a cycle;

$G(\tau_{max})$ is a factor valued at the maximum temperature reached each cycle.

The typical values for the exponent of cycle frequency (α) and the exponent of temperature range (β) vary between −1/3 and 2.

ΔH is the term of activation energy when $G(\tau_{max})$ is about 1.25.

Arrhenius's model defines damage as a result of acceleration due to an increase in temperature. The early model of acceleration is more successful in predicting the time of fracture (*time-to-fail*) which varies according to temperature.

EXAMPLE.– $G(\tau_{max}) = 1.25$; f = 50 Hz, ΔT = 20 to 50 °C, $\alpha = -1/3$ and $\beta = 2$, $f_{acc} = 1$(acceleration factor), the curve of the number of cycles in function of temperature is found and traced with a unit acceleration factor (=1).

Nf(ΔT) =

	0
0	0.012
1	0.011
2	0.01
3	9.962×10^{-3}
4	9.515×10^{-3}
5	9.096×10^{-3}
6	8.705×10^{-3}
7	...

continue

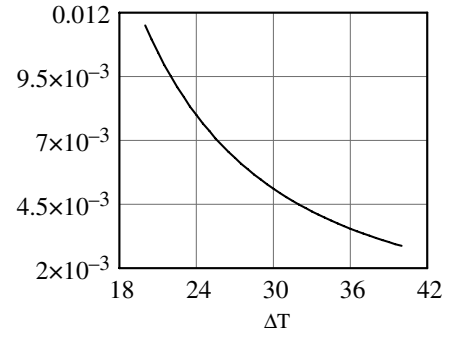


Figure 1.8. *Number of cycles to fracture in function of the variation of temperature and the acceleration factor in Arrhenius' model*

1.5.2. *Neuber's relation (1958)*

The classic methods of analysis are based either on Neuber's [NEU 58] coefficient, or on the local amplitude of deformation. The relations [1.41], [1.42],

and [1.3] are applied for sharp notches, where r is the radius of the notches and ρ′ is a characteristic of the material.

$$K_u = 1 + K_t \times 1 \Big/ \left(1 + \sqrt{\frac{\rho'}{r}} \right) \qquad [1.41]$$

The number of cycles at initiation N_i is expressed by the following relation:

$$Log\left(K_u\right) = 1.2969 + 0.1602 \times Log\left(N_i\right) \qquad [1.42]$$

where K_u is Neuber's coefficient as a result of relation [1.41].

Neuber found that ρ′ = 0.48 mm as agreed in bending tests. That the relation agrees with its values is also confirmed by Markovin and Moore [SMI 82, SMI 85] on fatigue tests for steel: SAE 1035, SAE 1010 and SAE 1038 (Canada and USA).

$$K_u = 1 + \frac{K_t - 1}{\left(1 + \left(\frac{\pi}{\pi - \omega} \right) \times \sqrt{\frac{\rho'}{r}} \right)} \qquad [1.43]$$

where K_u is the Neuber's coefficient which takes into account the opening angle of the notch (ω). The formulas [1.40] and [1.43] therefore allow Ky to be calculated by the notch effect which is partially due to the state of the combined stress, existing at the bottom of the notch. By combining von Mises' criteria and Neuber's theory in relation to the distribution of notch stress, we obtain a theoretical coefficient of notch effect K_u. K_t can therefore be deduced from Heydoo's relation, which is written as follows:

$$\left(\frac{K_t}{K_u} - 1 \right) = \left(\frac{M}{\sqrt{n \times r}} \right) \qquad [1.44]$$

where

n is the function coefficient of the notch type;

M is the constant of the material.

To find *M* and *n*, special manuals on the relevant topic will be used. Finally, Neuber and Heydoo explain the relation between K_t and K_u by a single parameter (*r*)

for a given material and a given notch type. The notch effect is given by the following relation:

$$K_e = 1/K_u \qquad [1.45]$$

It essentially depends upon:

cavitation, and/or aging;

corrosion and contact corrosion;

residual stresses;

radiations and other aggressions.

Neuber also presented a method founded on the s.i.f. (= ΔK) for a sharp notch ($\rho < 2/10$ mm) on a soft steel plate, more than 20 mm in length and 5 mm in thickness. The number of cycles N_i necessary for initiation of a crack measuring 1/10 mm is as follows:

$$N_i = \left(2.90 \times 10^8\right) \Big/ \Delta K^4 \,;\ \Delta K \ in\ MPa\sqrt{m} \qquad [1.46]$$

In general, caution must be paid to the formulations linking N_i and ΔK (intrinsic parameters of material C and m). It would seem that the initiation of a fatigue crack would make considerably more parameters intervene than those shown here and in technical literature. In our opinion, intensively using the electronic microscope should allow us to be more specific about these parameters and their actions on the safety of structures. However, mathematical simulations, as sophisticated as they may be, do not replace material observation of resistance phenomena in fracture mechanics by fatigue.

Coffin–Manson's model is not suitable for the use of Eyring's models for crack propagation by fatigue. The explanation of this model is as follows.

Coffin–Manson's model is one which takes into account crack propagation in stage 1 of Ritchie's representation (see Figure 1.18). It therefore takes into account the phenomena of deformation of materials by fatigue where Irving's theory (see [1.47]) is not applicable. Therefore, Coffin–Manson's model is extremely useful to study the stresses of cycles and the frequency of use with variable temperatures. This is a crucial point, since other models sometimes do not integrate the temperature in a significant way, which is the most important parameter in welded structures. The expression is written as follows:

$$N_r = A \times f^{-\alpha} \times \Delta T^{-\beta} \times G\left(T_{\max}\right) \qquad [1.47]$$

where:

N_r is the number of cycles to fracture;

f is the cycle frequency;

ΔT is the variation of the temperature during the cycle;

$G(T_{max})$ is one of Arrhenius' terms, valued at the maximum T °C reached during each cycle;

The typical values for each cycle frequency (α) and the temperature range (β) are $\alpha = -1/3$ and $\beta = 2$, respectively.

By reducing the cycle frequency, the number of cycles to fracture also reduces. Literature states that activation energy ΔH represented by $G(T_{max})$ is around 1¼.

1.5.3. *Arrhenius' model*

Arrhenius' model is an important model which is used particularly in its capacity to predict the acceleration of failures (damage) due to an increase in temperature. This early model of acceleration predicts the time to fracture (*time-to-fail*) as a function of the temperature. Arrhenius' empirical equation is written as:

$$T_r = \alpha \times Exp\left(\frac{\Delta H}{K \times T}\right) \qquad [1.48]$$

where:

T is the temperature measured in Kelvin (+273.16 °C) at the point of fracture.

α is the constant which represents the scale factor, initiated with acceleration factors ΔH expressing the activation energy (the latter is a critical parameter of Arrhenius' model).

ΔH incorporates the range of values (0.3 or 0.4 reaching 1.5 or more). It depends upon the damage mechanism and on the materials, and the acceleration factors between two temperatures exponentially increase as ΔH increases.).

K is the Stephan–Boltzmann's constant (= 8.617×10^{-5} in ev/K).

The acceleration factor (F_a), for T_2 (high temperature) and T_1 (low temperature) is expressed as:

$$F_a = Exp\left(\frac{\Delta H}{K} \times \left(\frac{1}{T_1} - \frac{1}{T_2}\right)\right) \qquad [1.49]$$

Activation energy ΔH is the only factor needed to calculate the acceleration of temperature. Using value K (in Kelvin) in degree Celsius, we obtain:

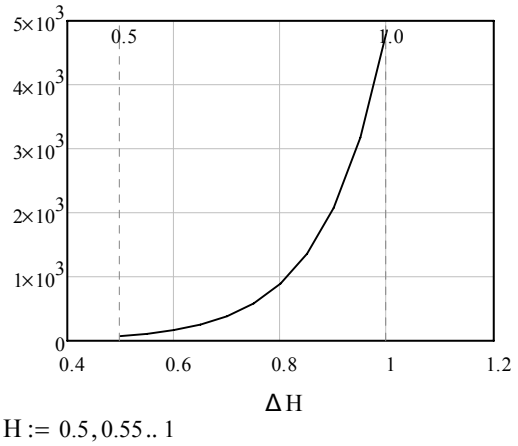
$$F_a = Exp\left(\Delta H \times 11605 \times \left(\frac{1}{T_1 + 273.16} - \frac{1}{T_2 + 273.16}\right)\right) \quad [1.50]$$

As has been previously stated, ΔH is unknown. Arrhenius' model has been successfully used to study the damage mechanisms which depend on corrosion (offshore structures assembled by welding and/or bolts), diffusion processes, or migration processes, to name but a few examples. This model is also widely used in the domain of electronic equipment. This work focuses principally on continuum mechanics.

Digital application: Let us calculate the acceleration factor F_a for T_1 = 20 °C and T_2 = 100 °C if ΔH varies between [½ and 1] with an increment of 0.05. Therefore, F_a.

$$Fa(\Delta H) = \quad Fa(\Delta H) = Exp\left[\Delta H \cdot 11605 \cdot \left(\frac{1}{T1 + 273.16} - \frac{1}{T2 + 273.16}\right)\right]$$

	0
0	69.64
1	106.45
2	162.71
3	248.72
4	380.18
5	581.13
6	888.30
7	$1.36 \cdot 10^3$
8	$2.08 \cdot 10^3$
9	$3.17 \cdot 10^3$
10	$4.85 \cdot 10^3$



with $\Delta H := 0.5, 0.55 .. 1$

Figure 1.9. *Evolution of the acceleration factor as a function of activation energy ΔH*

1.5.4. *Miner's law (1954)*

With respect to the physical description of damage, the reliability approach means the following problems can be resolved: oversizing, resistance to different stresses, appropriate geometry, and metrology of parts or of metallic assembly [LIN 65]. Statistical analysis takes into account the reliability treatments of dispersion causes, which imply that life expectancy on Wöhler's curve cannot be

represented by a point, but by the distribution of N_{cycles}. N_p and N_c indicate that the stressed part by σ_p has a probability p to break (see Figure 1.7) after $N_{p\text{-}cycles}$. The notion of damage assumes two distinct aspects, one aspect being physical and the other descriptive. It corresponds with variations of the physical properties of materials subjected to different stresses. It also corresponds to a quantitative description of the endurance of materials subjected to different stresses. Miner [MIN 54] proposed a simple damage law. The hypothesis is as follows.

n_i the number of cycles at the level of stresses C_1 (or σ_p) for which the average number of cycles to fracture is N_i which drives an increase in damage equal to (n_i/N_i). The fracture occurs when:

$$D = \left(\sum_{i=1}^{k} n_i \Big/ N_i \right) = 1 \quad (D \text{ for damage}) \qquad [1.51]$$

If the fraction (n_i/N_i) of life expectancy is carried out at a certain level of stress C_i, the remaining endurance at another level C_z will be $(n_Z/N_z = 1 - Z)$. Miner's law is not very precise, but it is very simple. It takes into account what is called understressing and overstressing.

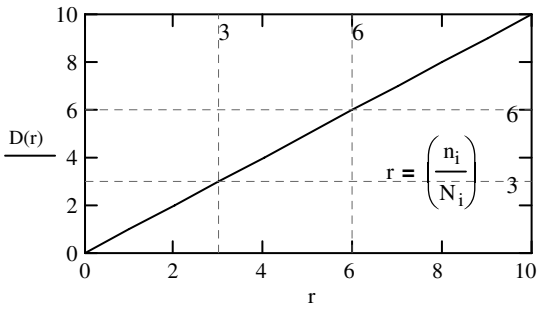


Figure 1.10. *Simplified diagram of damage in function of the ratio (n_i/N_i)*

For example, Wöhler's law cannot be applied to the study on reservoirs under pressure. We must, therefore, study the phenomenon of cracking causing leakages (in the case of spacecrafts). The main criticism of Miner's law is that it ignores the order in which the levels of loading occur. Indeed, experiments demonstrate that the cracking rate depends not only on the amplitude of loading at the moment it is considered, but also on the amplitudes during the precedent cycles. The phenomenon of *anterior memory* must therefore be considered.

In the uniaxial case, the load ratio R, at the bottom of the notch, tends toward −1 for an average of $N = 1000$ cycles. In the case of deformation imposed on smooth

test pieces under a lower number of cycles (N), there is a relaxation of the average stress σ_{av} and R therefore tends toward (−1). This is called structural damage in low-cycle fatigue. It is in conventional fatigue that the imposed tests are most suitable for the stability of R (not of the variation observed in tests). Miner provided a conventional form of damage created by N_{cycles}, as represented in [1.51]. This can be written in detail as follows:

$$D = \sum_{i=1}^{k} n_i \Big/ N_i = \frac{n_i}{\kappa(R)} \times S_{\alpha}^{\beta(R)} = |1| \qquad [1.52]$$

The effective stress for cycle (N) of fatigue is taken into consideration by the Smith–Watson–Topper (SWT) criteria. The SWT criteria take into account three important factors: alternating deformation ($\varepsilon_{alternating,}$ MPa), maximum stress (σ_{max}, MPa), and the elasticity modulus (E, MPa):

$$\sigma_{effective} = \sqrt{\varepsilon_{alterating} \times \sigma_{\max} \times E} \quad in\ MPa \qquad [1.53]$$

Without considering the fracture ratio (R), the number of cycles will be shown as follows:

$$N = \kappa \left(\sqrt{2\left(\frac{1}{1-R}\right) \times S_{\alpha}} \right)^{-\kappa} ;\ when\ S_{\alpha} = S_{\alpha D} \times 2\left(\frac{1}{1-R}\right);\ S_{\alpha D} \neq 0 \qquad [1.54]$$

1.6. Main common laws retained by fracture mechanics

Fracture mechanics helps us to quantitatively study the stages of slow propagation and brutal crack propagation. It helps us to calculate the characteristic parameters of local distribution of cracks and the deformations in the neighborhood of a crack, and also the critical lengths (a_c) of cracks, leading to a fracture for a given load.

For very acute cracks, G.R. Irwin [IRW 64] links the cracking rate (deformation) to the *s.i.f. K* as defined in Figure 1.18 representing Ritchie [RIT 79]. If K reaches K_{cr}, the crack brutally propagates. By knowing K_{cr}, we can find the critical length of the initial crack which leads to brutal fracture. Taking into consideration these effects, Henry established a formula which expresses the damage for level C of the previous relation. Of course, there are several formulas that take various levels of damage into account, Monson, Corten, and Dolen's theories are a few examples. Although less precise, Miner and Henry's formulas are often used because of their

simplicity. They allow us to predict the residual life expectancy of a part after a program of varied loads, and sometimes provide a sufficient approximation.

Welding techniques have not only developed, but improved over the years. They have certain advantages; however, they can also be the cause of specific problems. These problems can be very serious. The factors influencing the failures of welded assemblies are as follows:

– factors associated with the execution of welding;

– factors associated with metallurgical alteration due to welding;

– factors associated with the dimensioning of weld beads (geometry);

These three main factors are of a metallurgical, mechanical, and technological order respectively.

Faults can appear according to the way in which welding is carried out. These faults affect the mechanical behavior of the joint. The International Institute of Welding (IIW) has proposed a classification of these faults.

The alterations due to the thermal cycle of the welding operation cause local modifications of the mechanical properties that, in turn, constitute the causes of failure. The main factors are *quenching*, *aging*, *overheating*, and *the softening of the solidification structure.*

With respect to mechanical factors, the errors are of a conceptual order and not concerned with the execution. The severity of these factors is linked to the stress concentration they create. Methods of finite element analysis have stressed the importance of the shape of the weld bead.

In the domain of fracture by fatigue, good results have been obtained with the help of cracking laws based on fracture mechanics and in particular on Paris' law. Other laws are also commonly used, such as Fost and Digdale's law and MacEvily's law. There are other laws which are used in a different ways; however, this chapter will limit itself to Paris' law. Experience shows that the life expectancy of a welded structure strongly depends upon the size of the initial crack (a_0). For instance, according to Lawrence's calculations [LAW 73], the life expectancy quadruples when a_0 changes from 5 to 0.5 mm. When the weld shows no sign of a penetration fault, a_0 must be estimated, that is the half-small axis of the half-ellipse acting as a model, which is generally less than 0.5 mm. The fatigue of metals is tackled in four different ways:

– the reading of Wöhler's curves;

– Coffin's methods;

– the study of cyclic strain-hardening;

– measuring crack speed (*da/dN*).

The first method is the most used because it determines the endurance limit, which if maintained, low, protects from brutal fractures. The determination of Wöhler's curve allows us to make qualitative judgments with respect to the choice of materials for solid parts. Modeling the cracking phenomenon by fatigue integrates the following main notations:

– a, the crack length;

– N, the number of applied cycles;

– N_r, the number of cycles to fracture;

– σ, the nominal stress;

– K, the *s.i.f.*;

– C and m, the intrinsic parameters of the material.

From the crack length measurements on test pieces subjected to fatigue, different formulas have been proposed to calculate the propagation rate. The following are the most frequently used expressions:

1.6.1. *Fost and Dugdale's law*

Fost and Dugdale showed in 1960 that the diameter of the plastic zone (a) is proportional to one (diameter) obtained from ductile fracture in plasticity by integration of:

$$\frac{da}{dN} = \left\{ \alpha \times A \times \sigma^3 \right\} \qquad [1.55]$$

where:

α is the multiplier coefficient which depends on the state of the stress;

A is the constant, which depends on the material and on the stress;

σ is the amplitude of the stress.

To be precise, the plastic zone depends on the strain-hardening coefficient and on the mode of stress. For plastic deformation, there is an increase, in the peripheral zone, in the density of dislocations and of dislocation residues. So, when the crack opens during each cycle, there is a final rearrangement of the dislocations to accommodate the slip at the bottom of the crack. When the crack is closed, the rearrangement becomes definitive. We will therefore consider McEvily's law.

1.6.2. *McEvily's law (1979)*

This law is a function of the *s.i.f.* and the stress, which is written as follows:

$$da/dN = f\left(K_t, \sigma\right) \qquad [1.56]$$

where K_t is the stress concentration coefficient and σ is the nominal stress.

Both Fost–Dugdale and McEvily's laws take inter-inclusionary fractures into consideration, which occur according to a process of low-cycle fatigue. They are associated with Manson–Coffin's damage law at the bottom of the crack, which is presented as follows:

$$\left(4 \times \Delta N\right) \times \left(\frac{\varepsilon_p}{\varepsilon_f}\right)^{\frac{1}{C}} = 1 \qquad [1.57]$$

where:

C is the exponent from Manson–Coffin's law;

ε_p is deformation at plastic limit;

ε_p is deformation at fracture;

ΔN is the range of the number of cycles.

For the study of cracking, we consider the detailed expression of McEvily's law:

$$\frac{da}{dN} = \left(\frac{C}{E}\right) \times \left(\Delta K^2 - \Delta K_s^2\right) \times \left(1 + \frac{\Delta K}{K_c - \left(\frac{\Delta K}{1-R}\right)}\right) \qquad [1.58]$$

In 1979, McEvily proposed a relation applicable to alloys with low resilience, which is written as follows:

$$\frac{da}{dN} = \left(\frac{C'}{E}\right) \times \left(\Delta K - \Delta K_s\right)^2 \times \left(1 + \frac{\Delta K}{K_c - \left(\frac{\Delta K}{1-R}\right)}\right) \qquad [1.59]$$

where:

C and C' are the intrinsic factors of the material;

R is the load ratio (or the *s.i.f.*);

K_C is the critical *s.i.f.* which corresponds to brutal fracture;

ΔK_s is the *s.i.f.* threshold (see Figure 1.18, Ritchie later);

ΔK is the variation of *s.i.f.*

1.6.3. *Paris's law*

The study of the behavior of structures, in linear elasticity, often uses this law (1960). Fracture calculations gives the correlation between the intrinsic factors of the material (C and m), because tenacity is dependent upon it, through the means of the *s.i.f.* The crack propagation law gives the following equation:

$$da/dN = C \times (\Delta K)^m \quad \text{when} \quad (\Delta K) \succ 0 \qquad [1.60]$$

where C and m are intrinsic parameters of the material and da/dN expresses the crack propagation ratio. Linearizing expression [1.59] allows:

$$Log(da/dN) = Log(C) + m \times Log(\Delta K) \qquad [1.61]$$

Vicker's tests have shown that the zone affected by heat is localized at the foot of the weld bead for the four methods of welding[1]. The expression of the *s.i.f.* (or tenacity) is written as follows:

$$\Delta K = \Delta\sigma \times \xi(a) \times \sqrt{\pi \times a} \qquad [1.62]$$

The tenacity represents the resistance to deformations and to fracture. It is characterized by the elasticity limit R_e, resistance to fracture R_m, and the hardness HB, HRC, or HV for resistance to deformations. Work on welded cross-joints [GRO 94, GRO 95, LAS 92] for four different methods of welding has shown that fracture occurs at the foot of the weld bead. By replacing ΔK (ISO 12737: 1996) with its expression in [1.62], the following is obtained:

1 SAW, submerged-cored arc welding; FCAW, flux cored arc welding; SMAW 57 and SMAW 75, submerged metal arc welding.

$$\frac{da}{dN} = g\left(a/T\right)^m C \times \Delta\sigma^m \left(\sqrt{\pi a}\right)^m ;\ g\left(a/T\right) = \xi(a) ;\ \int_{N_i}^{N_f} dN = \int_{N_i}^{N_f} \frac{da}{C\left(\Delta K\right)^m} \rightarrow$$

$$when\ \Delta K = \Delta\sigma \times \xi(a) \times \sqrt{\pi a} \rightarrow dN = \frac{1}{\left(C\pi^{m/2}\right) \times \Delta\sigma^m} \times \int_{a_i}^{a_f} \frac{da}{a^{m/2} \times \left(\xi\left(a\right)\right)^m} \qquad [1.63]$$

$$dN = \frac{1}{C\left(\xi\left(a\right)\right)^m \pi^{m/2} \Delta\sigma^m} \int_{a_i}^{a_f} a^{-m/2} da = \frac{1}{C\left(\xi\left(a\right)\right)^m \pi^{m/2} \Delta\sigma^m} \cdot \frac{a_i^{1-\frac{m}{2}} - a_f^{1-\frac{m}{2}}}{\left(\frac{m}{2} - 1\right)}$$

where:

N is the number of cycles (by load);

a is the length of the crack in mm (or in μm);

T is the thickness of the stressed metal plate in mm.

The average of the intrinsic parameters of the material (C, m) for the four methods of welding is:

$$C_{average} = \frac{6,069 \times 10^{-8}}{24,64 \times m}\ \ MPa \times \sqrt{m}\ \ when\ \ R^2 = 0,963 \qquad [1.64]$$

The following photo shows resilience test piece of a welded cross-joint [LAS 92].

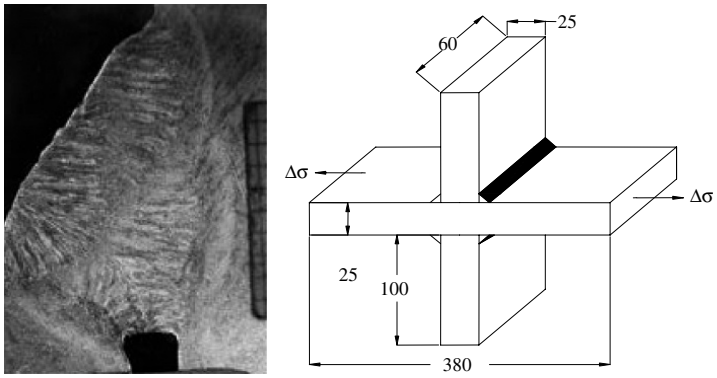


Figure 1.11. *Resilience test piece of a welded cross-joint*

In other instances, we can only go by previous experience or the correlations between the characteristics of materials and fracture behavior [GRO 94] of components (valuation from ISO 2553).

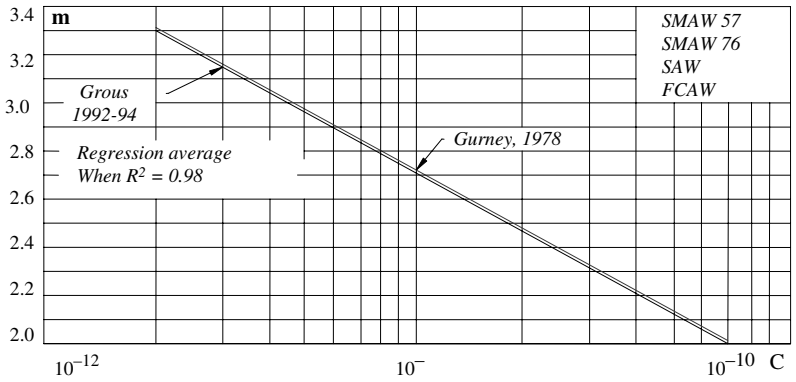


Figure 1.12. *Average relation of the intrinsic parameters of the material*

The procedure for measuring the dimensions of the weld bead (figure 1.11) is based on a procedure used in dentistry. Figure 1.12 illustrates the results from the four methods of welding. To measure the geometry of the weld bead, it was rolled in a dental paste. The imprint left by the bead was then measured. Thereby, the evolution of the geometry $g(a/T)$ used to calculate fracture parameters was deduced [GUR 78].

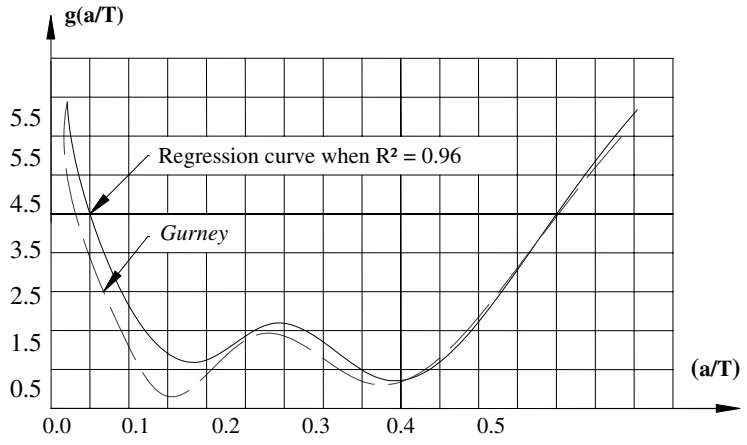


Figure 1.13. *Evolution of the correction factor of geometry by welding methods*

1.6.3.1. *Stress concentration factors*

The elements of stress concentration play a predominant role in RDM on machine elements, materials, and structures. Solids under stress load will present stress fields of low- to medium-levels with high gradient zones. At this level, shearing stresses reach dangerous levels. This is called the seat of crack by plasticizing. It leads to fracture by fatigue. Three cases illustrating a stepped shaft are shown in the following:

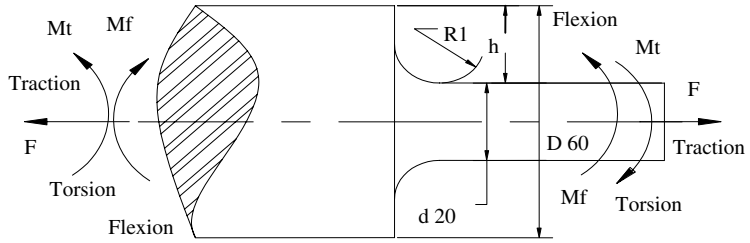


Figure 1.14. *Stress concentration factors for three figures: simple step shaft*

The respective relations to calculate stress concentration factors, for a stepped shaft, are shown in the following. In traction–flexion–torsion (TFT), the criteria are as follows:

$$\left\{\begin{array}{l} A=\left(\sqrt{\left(\dfrac{x.y}{1-\delta}\right)+1}\right)-1 \text{ and } B=\sqrt{x} \text{ when } x=\left(\dfrac{h}{r}\right) \text{ and } \delta=\left(\dfrac{d}{D}\right) \\ K_{Traction}=\dfrac{1}{\sqrt{\left(\dfrac{1}{0.88A}\right)^2+\left(\dfrac{1}{0.843B}\right)^2}}+1\ ; \quad \sigma_{\text{nominal}}=\dfrac{4F}{\pi d^2} \\ K_{Flexion}=\dfrac{1}{\sqrt{\left(\dfrac{1}{0.541A}\right)^2+\left(\dfrac{1}{0.843B}\right)^2}}+1\ ; \quad \sigma_{\text{nominal}}=\dfrac{32\text{M}}{\pi d^3} \\ K_{Torsion}=\dfrac{1}{\sqrt{\left(\dfrac{1}{0.263A}\right)^2+\left(\dfrac{1}{0.843B}\right)^2}}+1\ ; \quad \sigma_{\text{nominal}}=\dfrac{16F}{\pi d^3} \end{array}\right\} \quad [1.65]$$

The respective relations to calculate stress concentration factors, for one *with semi-circular bottom groove*, are shown in Figure 1.15.

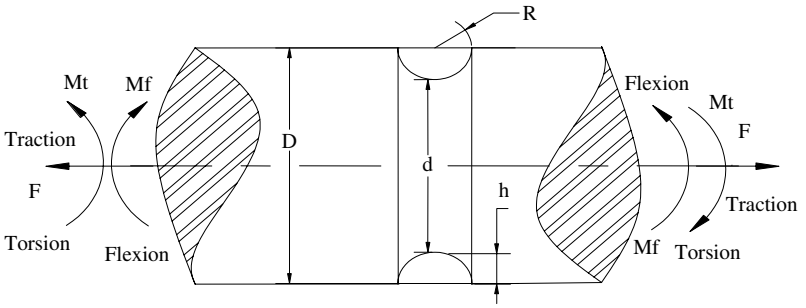


Figure 1.15. *Stress concentration factors for three figures: shaft with semi-circular bottom groove*

Observation: *It is worth pointing out that for the remaining cases, such as plates with symmetrical steps, reference should be made to manuals concerned with the resistance of materials for more detail.*

In TFT the criteria are as follows:

$$\left\{\begin{array}{l} A=\left(\sqrt{\left(\dfrac{x.y}{1-\delta}\right)+1}\right)-1 \text{ and } B=\sqrt{x} \text{ when } x=\left(\dfrac{h}{r}\right) \text{ and } \delta=\left(\dfrac{d}{D}\right) \\ K_{traction}=\dfrac{1}{\sqrt{\left(\dfrac{1}{1.197A}\right)^2+\left(\dfrac{1}{1.871B}\right)^2}}+1; \quad \sigma_{\text{nominal}}=\dfrac{4F}{\pi d^2} \\ K_{Flexion}=\dfrac{1}{\sqrt{\left(\dfrac{1}{0.715A}\right)^2+\left(\dfrac{1}{2B}\right)^2}}+1; \quad \sigma_{\text{nominal}}=\dfrac{32\text{M}}{\pi d^3} \\ K_{Torsion}=\dfrac{1}{\sqrt{\left(\dfrac{1}{0.365A}\right)^2+\left(\dfrac{1}{B}\right)^2}}+1; \quad \sigma_{\text{nominal}}=\dfrac{16F}{\pi d^3} \end{array}\right\} \quad [1.66]$$

1.6.4. *G.R. Sih's law*

Paris' law is useful only for simple mode crack propagation. Only one load parameter is involved (stress amplitude) and is only useful in mode I; simple propagation. G.R. Sih [SIH 79] proposes replacing the *s.i.f.* (ΔK) amplitude of Paris'

law with the amplitude of minimum deformation energy density $\Delta S_{\min}$. Sih's law is based on the concept of deformation energy density.

$$\frac{da}{dN} = C \times \left(\Delta S_{\min}^{m} \right) \qquad [1.67]$$

where:

C and m are the intrinsic factors of the material;

$\Delta S_{\min}$ is the minimum energy density;

da/dN is the crack propagation rate.

An approach to fatigue by linear fracture mechanics will now be illustrated. It also involves a host of mathematical expressions. Yamada and Albrecht's [YAM 77] model will be used to calculate the variation of the *s.i.f.* and Gurney's [CUR 78] model for the geometric correction $g(a/T)$.

1.7. Stress intensity factors in fracture mechanics

Dimensioning structures which are stressed in different ways leads to an even greater control of fatigue behavior in welded joints. To calculate the crack propagation, numerous works have recognized the link between the logarithm of the variation of the *s.i.f.* and the logarithm of propagation rate. Certainly, this is relative to stage II of Ritchie's [RIT 79] chart (see Figure 1.18), for which Paris proposed a law where the *s.i.f.* is expressed by relation [1.62], where $\Delta\sigma$ is the stress amplitude and $\xi(a)$ is the geometry of the crack.

1.7.1. *Maddox's model*

This model was originally intended for welded cross-joints without penetration fault. Maddox [MAD 75] assumes that the surface fault is a semi-elliptic of half-axes a and c. Therefore, he suggests calculating the *s.i.f.* by the following relation:

$$K = \sigma \times \sqrt{\pi \times a} \times \left(\frac{M_s \times M_t \times M_k}{\Phi_0} \right) \qquad [1.68]$$

where:

σ is the nominal stress (in MPa);

a is the crack length (in mm or inch);

M_s is the correction of the free surface (crack surface);

M_t is the correction of finite thickness;

M_k is the correction due to the stress concentration introduced by the geometry of the weld bead;

Φ_0 is the form factor = 1 for (a/c) = 0 crack length.

For welded cross-joints, Maddox proposes a connection angle $\theta = 0$ and for $\Phi_0 = 1$:

$$K = \sigma\sqrt{\pi a}\left\{1.122 - 0.23\left(\frac{a}{T}\right) + 10.55\left(\frac{a}{T}\right)^2 - 21.7\left(\frac{a}{T}\right)^3 + 33.19\left(\frac{a}{T}\right)^4\right\} \quad [1.69]$$

where T is the thickness of the welded sheet metal (in mm or inches). This expression has been applied several times. As early as 1965, B. Gross and J. E. Srawley [GRO 65] proposed a model which draws conclusions about the *s.i.f.*

1.7.2. *Gross and Srawley's model*

The proposed equation [1.70] is empirical and shows a model for a ratio of (a/W) ranging from 0.1 to 0.35.

$$Y^2 = \frac{K^2 \times B^2 \times W^2}{M^2} = 139\left(\frac{a}{W}\right) - 221\left(\frac{a}{W}\right)^2 + 783\left(\frac{a}{W}\right)^3 \quad [1.70]$$

where:

Y^2 is a dimensional quantity which is exclusively the function of a/W;

M^2 is the square of the bending moment ($M = P \cdot l/2$);

K is the s.i.f. in mode I crack opening;

B and W are the width and the thickness of the stressed test piece by M, respectively.

1.7.3. *Lawrence's model*

For butt joints, Lawrence [LAW 73] proposed a model from linear elastic mechanics. It comprises three distinct steps:

Step 1: Finite elements are used to determine, the stress field existing in a joint, along the crack. For triangular meshing, in planar deformation, we deduce the stress ratio (σ/S) along the crack.

Step 2: The curves are adjusted (σ/S) by the following polynomial function:

$$\frac{\sigma}{S} = b_0 + b_1\left(\frac{a}{t}\right) + b_2\left(\frac{a}{t}\right)^2 + b_3\left(\frac{a}{t}\right)^3 + b_4\left(\frac{a}{t}\right)^4 \quad [1.71]$$

where:

t is the thickness of the sheet metal;

b_i are constants dependent on the geometry of the weld bead;

a is the crack length;

σ/S is the stress ratio.

Step 3: Consists of calculating the *s.i.f.* by the following relation:

$$K = \sqrt{\pi a} \times \left\{1.1\sigma - \int_0^t f\left(\frac{a}{t}\right)\frac{d\sigma}{dt}dt\right\} \quad [1.72]$$

The function of a semi-finite crack subjected to a non-uniform load σ(a/t) is written as:

$$f\left(\frac{a}{t}\right) = 0.8\left(\frac{a}{t}\right) + 0.04\left(\frac{a}{t}\right)^2 + 3.62\times10^{-6} \times Exp^{11.18\left(\frac{a}{t}\right)} \quad [1.73]$$

The advantage of this model is that it replaces calculation using finite elements with calculation by the polynomial function which is dependent on the geometry of the weld bead. Lawrence's method, for martensitic steel, is used to study faults that are smaller than 0.025 mm. However, his method is hardly convincing with respect to estimating life expectancy. For this, it is advisable to refer to T.R. Gurney's [GUR 78] approach.

1.7.4. *Martin and Bousseau's model*

Martin and Bousseau's [MAR 76] method is linked to other methods in that it employs finite elements. It suggests determining the stress variation along the crack plane and, for a given type of joint, deducing the *s.i.f.* (K_t) at the foot of the weld bead, beside the crack activation. This method is expressed as follows:

$$da/dN = C \times \left(1.1 \times \sigma_N \times \sqrt{\pi a} \times K_\tau\right)^m \quad [1.74]$$

where:

C, m are intrinsic parameters of the material;

da/dN is the crack rate (life expectancy);

a is the crack length;

σ_N is the nominal stress;

K_τ is the nominal stress.

Employing this function leads to a markedly lower number of propagation cycles than those obtained by Lawrence's model.

1.7.5. *Gurney's model*

Gurney's model, in relation to the figure, proposes the following:

$$\Delta K = \Delta\sigma \times \sqrt{(\pi a)\sec\left(\frac{\pi a}{2b}\right)} \qquad [1.75]$$

For Figure 1.13 in zone I of Figure 1.18 we have:

$$\Delta K = \frac{\sqrt{\Delta P}}{BW}\left\{29.6\sqrt{\frac{a}{W}} - 185.53\sqrt{\frac{a}{W}} + 655.75\sqrt{\frac{a}{W}} - 1017\sqrt{\frac{a}{W}} + 638.99\sqrt{\frac{a}{W}}\right\} \qquad [1.76]$$

where:

ΔK is the variation of the *s.i.f.*;

$\Delta\sigma$ is the variation of the nominal stress;

a/W is the relative crack length ($2B = W$);

B is the thickness of the sheet metal.

1.7.6. *Engesvik's model*

Knut Engesvik's [ENG 82] model is based on Yamada and Albrecht's approach, from which the following can be written:

$$\Delta K = \Delta\sigma \times \sqrt{\pi a} \times f(a) \quad \text{when} \quad f(a) = f_s \times f_T \times f_E \times f_G \qquad [1.77]$$

where:

f_s is the correction factor of the free surface;

F_T is the correction factor of the finite thickness;

F_E is the correction factor of the shape at fracture;

F_G is the correction factor of geometry;

$\Delta\sigma$ is the variation of nominal stress.

$f_{(a)}$ coincides with $g(a/T)$. From Yamada and Albrecht's model we use:

$$f_G = \frac{2}{1.12 \times \pi \times \Delta a} \times \int_0^a \left(\frac{\sigma(x)}{\sqrt{a^2 - x^2}} \right) \times g\left(\frac{x}{a}\right) dx \qquad [1.78]$$

Using Lawrence's model as a base, we use the correction of geometry expression as follows:

$$g\left(\frac{x}{a}\right) = 0.8\left(\frac{x}{a}\right) + 0.04\left(\frac{x}{a}\right)^2 + 3.62 \times 10^{-6} \times Exp\left(11.18\left(\frac{x}{a}\right)\right) \qquad [1.79]$$

1.7.7. *Yamada and Albrecht's model*

This famous model [YAM 78] allows us to calculate ΔK in the case of a semi-elliptical fault of half-axis (a), given as follows:

$$\Delta K = \frac{1.12}{E_K} \times f_G \times \Delta\sigma \times \sqrt{\pi a} \times \sqrt{\frac{W}{\pi a} \tan\left(\frac{\pi a}{W}\right)} \qquad [1.80]$$

where:

$\Delta\sigma$ is the variation of nominal stress;

E_K is the elasticity modulus;

$W = 2B$ is the relative crack length $W=2B = a/T$ (See Figure 1.13);

a is the crack size;

f_G is the correction factor of the stress concentration effect (s.i.f.) due to the geometry of the weld bead.

Finally, to take into account the influence of short cracks subjected to a field of plastic deformation, El Haddad *et al.*'s [HAD 78, 80] model suggests using an elastoplastic solution. The relation is given:

$$\Delta K = E \times \Delta\varepsilon \times \sqrt{\pi(l + l_0)} \qquad [1.81]$$

where:

$\Delta\varepsilon$ is a local deformation in plastic regime that is obtained by using the finite element method;

E is the elasticity modulus;

l = is the crack length;

l_0 = is the constant of the material.

Neuber's [NEU 58] rule means that the solution can be approached using the following relation:

$$K_{equivalent} \times \Delta\sigma = \sqrt{\Delta\sigma \times \Delta\varepsilon \times E} \qquad [1.82]$$

where:

$\Delta\varepsilon$ and $\Delta\sigma$ are the deformation and the local stress at the tip of the crack, respectively.

$K_{\text{equivalent}}$ is an equivalent stress concentration coefficient, which is written as follows:

$$K_{equivalent} = \frac{M_k \times M_S \times M_t}{\Phi_0} \qquad [1.83]$$

where M_k, M_S, and M_t are correction factors explained in Maddox's model (a) (see also relation [1.66]).

1.7.8. *Tomkins and Scott's model*

Tomkins *et al.*'s [TOM 78] model is applied to analyze surface cracking around the intersection of a knot, using the following integral:

$$K = 2\sqrt{\frac{C}{\pi}} \times \int_0^C \frac{\sigma}{\sqrt{C^2 - x^2}}\, dx = Y_\sigma \times Y_S \times \sigma_R \times \sqrt{\pi a} \qquad [1.84]$$

where:

σ is the stress which considers the evolution of the initial *s.i.f.*

Y_σ is a factor which considers the evolution of the stress field;

Y_S is a factor which considers geometric effects;

σ_R is the radial stress at the hot spot;

x and C are the large and small axes of an ellipse (geometry), respectively;

a is the crack length.

1.7.9. *Harrison's model*

Harrison's [HAR 78] model is applied in the case of cracking due to welding faults (lack of penetration). It is written as follows:

$$\Delta K = \Delta\sigma \times \sqrt{\pi a} \times \sqrt{\left(\frac{W}{\pi a}\right) \times \tan\left(\frac{\pi a}{W}\right)} \qquad [1.85]$$

For the same application, Lawrence and Munse present the following relation for butt joints:

$$\Delta K = \Delta\sigma \times \sqrt{\pi a} \Big/ \sqrt{Cos\left(\frac{\pi a}{W}\right)} \qquad [1.86]$$

where:

$\Delta\sigma$ is the variation of nominal stress;

ΔK is the variation of the s.i.f.

W is the thickness of the sheet metal;

a is the crack length.

There are many models related to calculating *s.i.f.* according to the design, materials, the environment, and geometry. Some models have been shown here which are necessary for reliability calculations; but reference can also be made to literature on the topic for a deeper understanding of the subject.

1.8. Intrinsic parameters of the material (*C* and *m*)

There are a number of "relations between C and m" used to calculate the fatigue of welded structures (Paris' law –See Figure 1.12). For example, Japanese law WES

2805 [JWES]. Tanaka [TAN 81] and Kanazawa *et al.* [KAN 79] describe this standards using concepts from fracture mechanics of welded assemblies, where R_m = 100 Mpa. The norm proposes an average value equal to 2 for m. C is expressed by the linear regression of da/dN (mm/cycles) and ΔK by kP/mm$^{3/2}$. Their relation is given as follows:

$$C = \frac{5.53\times10^{-5}}{59.2^{m}} \ (\text{JWES}) \qquad [1.87]$$

There is also a Nordic regulation DnV for the construction of offshore steel platforms. In mode II (see Figure 1.18) for stable crack propagation in fatigue, the increase in m and the simultaneous decrease in C is often observed, when the elasticity limit increases. Therefore, several authors have proposed relations in m and Log(C). In our works [GRO 92, GRO 94], based on a large experimental project lead by Professor Tom Lassen [LAS 82, 92], we proposed four types of relations between C and m, stemming from mathematical regression. The results accompanied with their correlation coefficients are as follows:

$$\left\{\begin{array}{l} C_{SAW} = \dfrac{8.855\times10^{-8}}{29.39^{m}} \quad R^2 = 0.982; \quad C_{FCAW} = \dfrac{2.14\times10^{-8}}{18.13^{m}} \quad R^2 = 0.997 \\ C_{SMAW57} = \dfrac{4.092\times10^{-8}}{20.08^{m}} \ R^2 = 0.975; \ C_{SMAW76} = \dfrac{2.263\times10^{-8}}{16.23^{m}} \ R^2 = 0.97 \end{array}\right\} \qquad [1.88]$$

The average relation of the four methods of welding is written as follows:

$$C = 6.069\times10^{-5}/24.64^{m}; \ R^2 = 0.963 \ \left\{\text{units MPa}; \sqrt{\text{m}}\right\} \qquad [1.89]$$

We used a simple regression on the semi-Log scale for the experimental values represented by Paris' relation.

COMMENTS.– In stress concentration, zones efforts can cause cracks. Nevertheless, in this domain, a fault (lacuna) or a crack, sometimes extremely small, can be the cause of a fracture. On reflection, it seems that normal calculations from continuum mechanics cannot predict fractures since they do not assume the presence of a fault. It is therefore necessary, before adopting a *reliability approach*, to accurately calculate the:

– distinct *s.i.f.* (ΔK) for each method of welding used;

– initial crack lengths (a_0);

– correction factors of geometry $g(a/T)$ or $\xi(a)$;

– intrinsic parameters of the material, graded correctly (c and m).

1.9. Fracture mechanics elements used in reliability

This summary explains the behavior of existing cracks within the material, by estimating their evolution of the limiting conditions and the qualities required by the base material and the filler material. A fault is perceived at a given moment and we observe its slow propagation under a load effect, until the size of the fault becomes critical. Defined by fracture mechanics, the characteristics of stress distribution and deformations in the neighborhood of the crack front (*n.c.f.*) allow us to quantify the phenomenon of cracking. Fatigue itself is characterized by a large range of stress variation, which is inferior to resistance to traction of the stressed material. The main steps of fatigue are crack activation until final fracture (a_0 to a_f).

To predict the behavior of fatigue we use, in addition to the number of cycles, the amplitude of the stress, loading or imposed deformation $\Delta\sigma$ in MPa, the surface state (its finish), as well as the medium where the material (or the structure) evolves. The behavior of structures is influenced by different parameters, including geometric characteristics of the structure and weld beads, loading, and the limiting conditions.

In the case of a crack with sharp initiated notches, Irwin [IRW 58, 64] decided that Neuber's classic calculations were outdated. He therefore proposed a calculus that allows us to obtain the stress state at the crack at a point *n.c.f. (crack front)*.

$$\sigma_{xy} = K \times \left(\frac{f_{xy}}{\sqrt{2\pi \cdot r}} \right) \qquad [1.90]$$

where:

$f_{xy} = g(a/T)$ is a function of correction of geometry;

σ_y is the applied nominal stress;

r and θ are the radius and the angle in polar coordinates, respectively.

The *s.i.f.* (K) defines the stress state of the component, by taking into account the global and local geometry of the crack. When K reaches a value of K_{cr}, there is a brutal fracture of the part, as illustrated in Figure 1.14. Usually, the notion of brutal fracture is associated with the presence of an initial fault existing in the weld or in the structure. It is therefore worth reiterating the existence of a potential crack propagation from an initial fault (a_0).

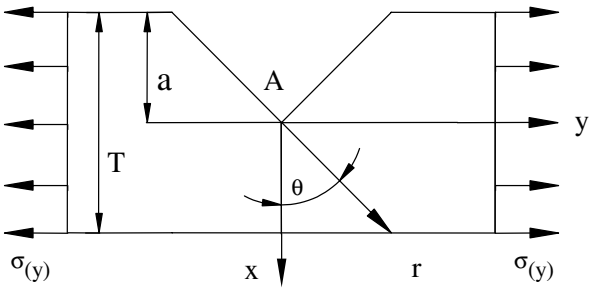


Figure 1.16. *Schematization of the fracture zone and the mechanical field*

Fracture can occur during a static load or after the propagation of a fault, under different stresses and until it reaches a critical dimension a_{cr}. Since Griffith [GRI 20] proposed his fracture theory, Rice [RIC 68] carried out more polished analyses than Griffith's, though his fundamental principles remained the same. In our case of singular structures, for example, we have applied Paris' crack law. Theories which are applicable to numerous materials and which authorize the use of critical fracture stress expressions can be expressed in the following relation:

$$\sigma = \sqrt{\left(E \cdot G_c\right)/\pi a} \qquad [1.91]$$

where G_c is the rate of releasing energy so that the crack propagates in a unitary length. To characterize the *s.i.f. K*, we consider the following relation:

$$K = \sigma \times \sqrt{\pi a} \times g\left(a/T\right) \qquad [1.92]$$

This equation is applied when the test piece is thin with very large lateral dimensions, in other words, when the crack length (2a) is very small in relation to its dimensions. As simple as it may be, the load reached in Paris' law allows us to have a significant influence on the crack length. In 1870, A. Wöhler [WÖH 70] focused on the study of premature fractures of rail car axles. In light of his conclusions, the behavior of materials subjected to fatigue can be determined. To do so, laboratory test pieces which are subjected to simple efforts of rotary flexion are used. There are different amplitude levels of stresses σ_a. Then, the number of cycles required to fracture the test piece is measured.

Bearing in mind the statistical character of fracture theory in linear elasticity, to study the reaction of the material subjected to fatigue the test should be repeated several times with different stress amplitudes. On a semi-log scale, a curve $\sigma_a = f(N_{cycles})$ is then plotted as shown in Figure 1.17.

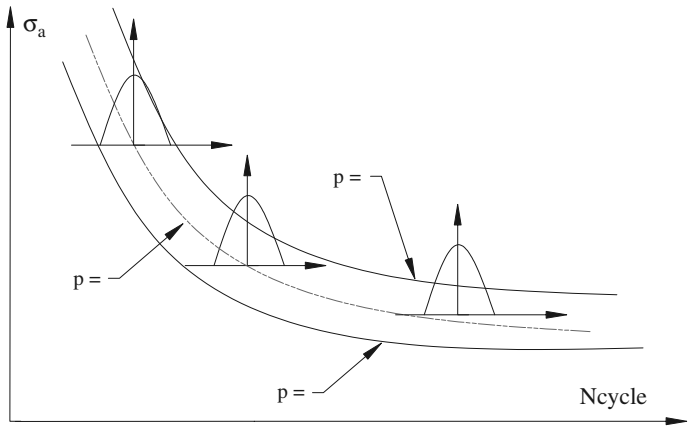


Figure 1.17. *Theoretical curves of equiprobability (p) of fracture (Wöhler)*

When a structure is subjected to a load of varying intensity, it is subjected to a phenomenon of fatigue which leads to its fracture, even though the charge remains constantly below its static resistance. This phenomenon is linked to crack propagation from a fault within the part itself, that is, a set of conditions favorable to a local decohesion of the material, because of a high stress concentration. Generally three steps in the life of a structure can be distinguished: crack initiation, slow crack propagation by fatigue, and finally fracture, as demonstrated in Figure 1.18 [RIT 79].

The three stages previously schematized are defined as follows:

Zone I: low-cycle fatigue zone under strong stress amplitudes where life expectancy is short ($N_{cycles} < 10^4$ cycles). Before fracture there is a plastic deformation.

Zone II: fatigue zone or limited endurance zone where fracture occurs after a number of cycles, which increases when the stress decreases ($10^4 < N_{cycles} < 10^6$ cycles).

Zone III: unlimited endurance zone, also known as safety zone, under weak stress amplitudes. Fracture does not occur, even after a high number of cycles ($N_{cycles} < 10^7$).

Fatigue cracks originate from the surface of parts, where there are numerous faults. Moreover, the surface is subjected to aggression of the environment. Dislocations are also as mobile at the surface as they are at the core. The number of cycles necessary for fatigue crack initiation essentially depends on the sharpness of

the fault where it originates. In the case of welded cross-structures, it is often said that the crack originates at the foot of the weld bead.

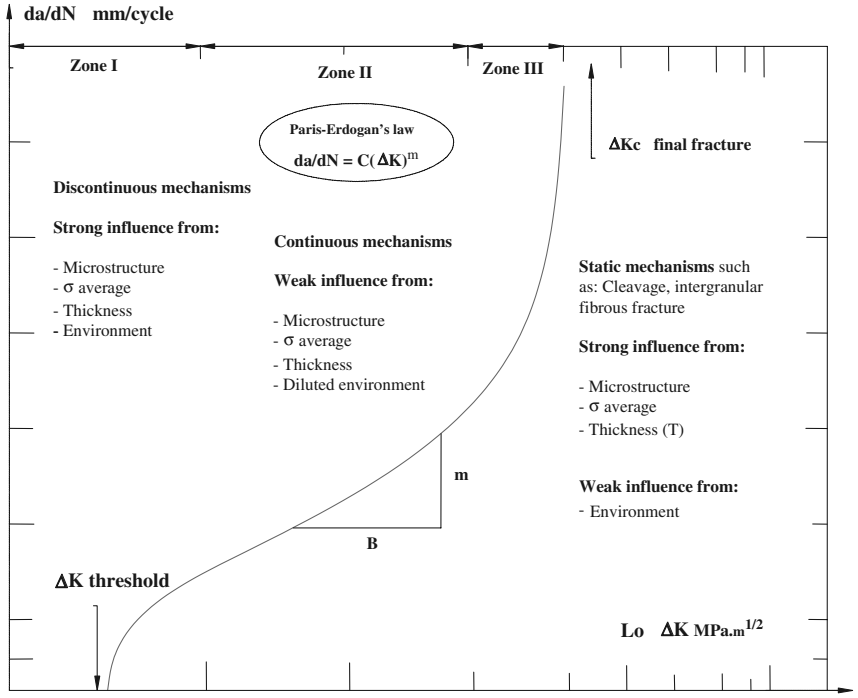


Figure 1.18. *Illustration of the propagation rate da/dN in function of the s.i.f. (Ritchie)*

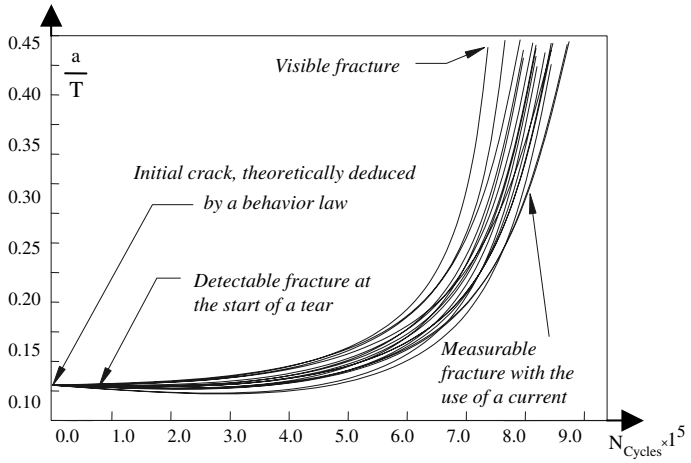
1.10. Crack rate (life expectancy) and s.i.f. (K_σ)

Materials contain inclusions, fabrication faults, heterogeneity, etc. Components show signs of section changes or surface states, which are more or less perfect. Insofar as these conditions favor the apparition of stress concentrations and consequently fatigue cracks under cyclic-load, not only the possibility of crack activation but also their propagation must be borne in mind. This explains why engineers spend a long time calculating when they need to design structures with cyclic loads. The structures must not only anticipate the possibility of cracks forming, but also evaluate their propagation rates, to be sure that these cracks will not reach a critical length and result in brutal fracture.

In the 1960s, P. Paris and F. Erdogan [PAR 63] proposed a relation for the evolution of a fatigue crack, through the measure of propagation, represented by *da/dN* and the *s.i.f.* (parameter that characterizes the fault) which is determined

analytically or by finite elements (see Figure 1.1) When a crack is formed, its length increases with increase in the number of load cycles.

Knowing that $\Delta K = \left(K_{\max} - K_{\min}\right)$ in a simple case when the stress amplitude is constant, the growth of the crack length (a) drives a rise in the value of the *s.i.f.* Indeed if the opposite varies, so does the *s.i.f.* In fact, when we represent the variation of crack rate (*da/dN*) as a function of *s.i.f.* (*ΔK*) in a logarithmic scale, the curve (see Figure 1.19) represents a linear part in accordance with Paris' equation, used for its simplicity and success in corroborating many experiment results. The test conditions, for metallic materials tested in an ambient atmosphere, are such that the value of the exponent (*m*) of Paris' law is generally between 2 and 4. We will attempt to demonstrate this in our case study later in this chapter. Many works have already proposed a number of convincing relations between *C* and *m*, the results published by [KAN 71, ENG 82, GRO 98].



Experimental curve: crack depth as a function of the number of applied cycles.

C has often been linked to the characteristics of the material (elasticity limit, resistance to traction, strain-hardening coefficient, lengthening to fracture). These correlations are not satisfactory. In 1971, for ΔK in *N*. $mm^{3/2}$ and *a* in mm, Kitagawa and Missumi [KIT] proposed the following:

$$C = \frac{10^{-4}}{2 \times \left(55^{m}\right)} \qquad [1.93]$$

a_Crack =

	0
0	0.450
1	0.870
2	1.280
3	1.600
4	2.000
5	2.350
6	2.850
7	3.125
8	3.500
9	3.850
10	4.150
11	4.700
12	5.000
13	5.450
14	5.720
15	6.200
16	6.500
17	...

N_Cycles =

	0
0	0.000
1	9.340×10^{6}
2	1.580×10^{7}
3	2.000×10^{7}
4	2.400×10^{7}
5	2.650×10^{7}
6	2.850×10^{7}
7	3.000×10^{7}
8	3.120×10^{7}
9	3.200×10^{7}
10	3.270×10^{7}
11	3.330×10^{7}
12	3.370×10^{7}
13	3.400×10^{7}
14	3.430×10^{7}
15	3.450×10^{7}
16	3.460×10^{7}
17	...

continuous

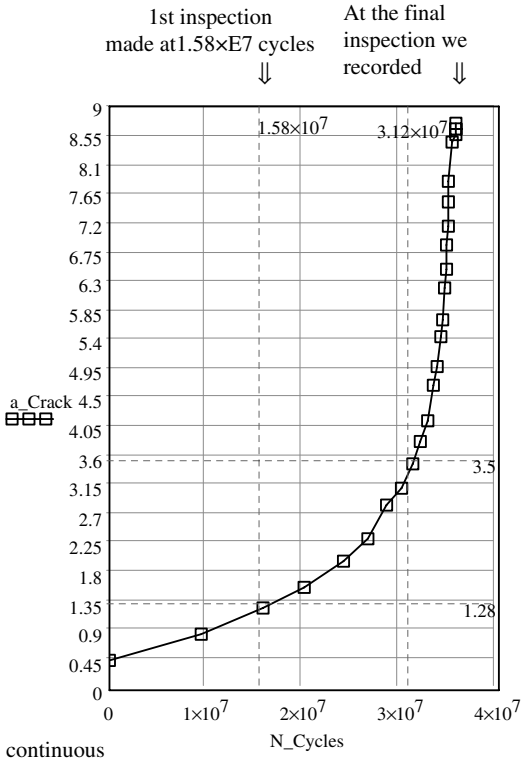


Figure 1.19. *Variation of the crack length (a/T) as a function of the number of cycles*

This relation is useful for a large number of steels, in particular for metals with central cubic structures. For austenitic steels and non-ferrous metals, however, the relation is less useful. In many cases, the data obtained from experiments are too dispersed to be able to determine in the exact values of C and m, which are included among the influencing factors for the behavior of solid structures in fatigue. Several typical values are presented in the following:

Grade of the material	M	C (SI)
Martensitic steels with 500 MPa < S_y < 2000 Mpa	1.35×10^{-10}	2.25
Steels with a ferritic-pearlitic structure	6.90×10^{-12}	3.00
Austenitic steels (stainless)	5.60×10^{-12}	3.25

Table 1.1. *Ideal values of the intrinsic parameters of the materials (without corrosion)*

Many factors influence the behavior of solids in fatigue, such as:

– the intrinsic parameters of the material, C and m;

– the variation of the *s.i.f.* (ΔK) and the stress σ;

– the function of the correction of geometry $g(a/T)$.

In a bid to correct the imperfections involved in calculating *s.i.f.*, factor $g(a/T)$ has often been introduced, which in turn is dependent on several factors [GUR 78, GRO 98]. It is also important to remember that every modification of the microstructure of the base material or of the weld leads to variations in the principal mechanical properties, such as the elasticity limit, the resistance to traction, ductility, and tenacity.

1.10.1. *Simplified version of Taylor's law for machining*

Taylor has established a relation which gives the life expectancy of a tool in function of the machining parameters. For example, the intrinsic parameters of the material appear in a simplified version of Taylor's law for machining.

$$\Delta T = C_v \times \left(V_{cutting} \right)^n \qquad [1.94]$$

where:

ΔT is the life expectancy of a cutting tool in min;

C_v is a Taylor constant, intrinsic value of the material;

V_c is the cutting rate (m/min or in ft/min);

n is a (negative) value, depending on the type of operation.

Example of application: A carbide tool GC 4015 for which the manufacturer recommends V_c = 315 m/min and f = 0.4 mm/min for a life expectancy of 15 min. The machining type, carbide turning of a steel gives n = −8.

If we plot, on bi-log paper, the life expectancy of a tool as a function of the cutting speed, we obtain the slope of the straight line n. For instance, calculating the life expectancy of a tool under the following cutting conditions, n and C are the intrinsic parameters of the material, e.g. coefficients provided by the manufacturer of cutting tools.

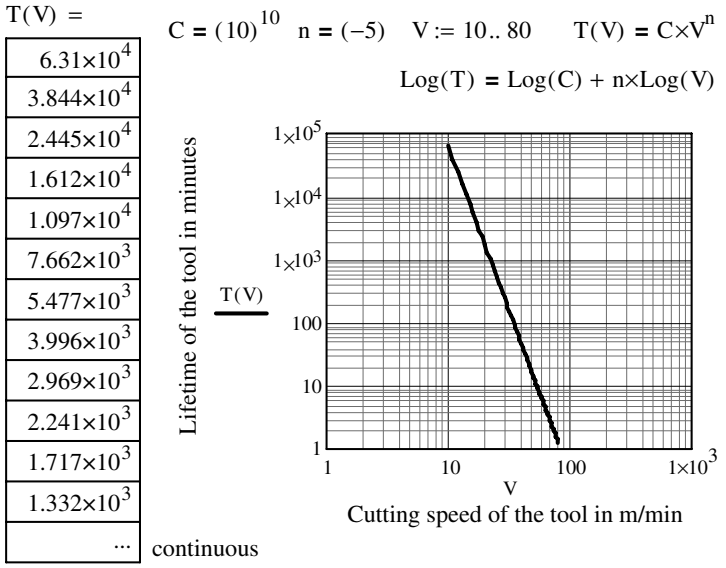


Figure 1.20. *Life expectancy of a tool (pellet) in function of the cutting speed V_c*

1.11. Elements of stress (*S*) and resistance theory (*R*)

In the following section, we are going to present another method known as stress/resistance (*S*/*R*). This will be accompanied by our own research results [GRO 94, GRO 95]. When fissures appear at the foot of the weld bead of a welded cross-assembly [GRO 98] and are random, they can be described by a probability distribution of initial cracks (variable or non-variable in function of the longevity of the assembly) that the parameters of Paris's law are themselves random, owing to the variability of the base material (as well as the filler metal for welding), of the dimensional variability, of that of treatments, etc. These can be described by a distribution of the probabilities of the intrinsic parameters of the material (*m*, *C*). The reliability of the assembly can be calculated in function of the life expectancy.

1.11.1. *Case study, part 2 – suspension bridge (Cirta)*

NOTE.– This case study is undertaken with a mechanical (static) approach. It will be referred to later for reliability calculations (Cornell and de Hosofer–Lind's reliability index).

PROBLEM.– We are looking for the "correct" environmental factors of the cable to support the tension and the deflection, supporting a uniform load. Secondary effects (wind) will be disregarded. The length of cable L and the deflection (f) must support a uniform load Q. Let us find, then, the expression and the value of this deflection, environmental factors H, and tension T.

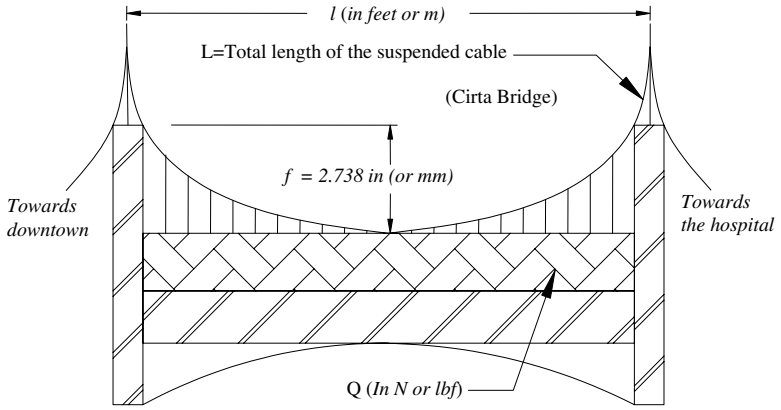


Figure 1.21. *Example of calculations for a suspension bridge (Cirta Bridge)*

– Uniform load: $Q = 0.01$ (lbf/ft).

– Stretch of the length is the total length of the cable $L = 100$ ft.

– f: deflection $f = 2.738$ in. (mm).

– l is the length in mm (or *inch*).

– τ is the tension in N (or in lbf).

SOLUTION.– The following equations come from the literature (RDM) dedicated to calculating cable design (in the case of suspension bridges with a parabolic shape) in static. The total length of the suspension cable is calculated by:

$$L = \ell \times \left[1 + \frac{8}{3}\left(\frac{f}{\ell}\right)^2 - \frac{32}{5}\left(\frac{f}{\ell}\right)^4 + \frac{256}{7}\left(\frac{f}{\ell}\right)^6 \right] \quad (in\ ft\ or\ m) \qquad [1.95]$$

The deflection (f) of the suspension cable is calculated by:

$$f = \frac{1}{8} \times \left(\frac{Q \times \ell^2}{\tau} \right) \quad (in\ ft\ or\ m) \qquad [1.96]$$

The cable tension is expressed as follows:

$$T = \frac{1}{2} \times Q \times \ell \times \sqrt{1 + \frac{\ell^2}{16 \times f^2}} \ \left(in\ N \text{ or } lb \right) \qquad [1.97]$$

In the case of $l = 90\% \times L$ (design of the horizontal range). Length l is written as follows:

$$L = \ell \times \left[1 + \frac{8}{3}\left(\frac{f}{\ell} \right)^2 - \frac{32}{5}\left(\frac{f}{\ell} \right)^4 + \frac{256}{7}\left(\frac{f}{\ell} \right)^6 \right] \rightarrow \ell = 119.854\ ft$$

From [1.96], the tension of the environment (τ) is deduced as follows:

$$\tau = \frac{1}{8} \times \left(\frac{Q \times \ell^2}{f} \right) = 14.028\ lbf \qquad [1.98]$$

The cable tension (T) is therefore calculated by [1.97]:

$$T = \frac{1}{2} \ell \times Q \times \sqrt{1 + \frac{\ell^2}{16 f^2}} = 14.079\ \left[lbf \right]$$

The range of (x) values is calculated as follows. The shape of y(x) parabola is given by the following equation:

$$y(x) = \left(f \times x^2 \right) \Big/ \left(\frac{\ell}{2} \right)^2 \quad when \ \ x = -\frac{\ell}{2}; -\left(\frac{\ell}{2} - \frac{1}{10} \times \frac{\ell}{2} \right) \ldots \left(\frac{\ell}{2} \right) \qquad [1.99]$$

1.11.2. *Case study: failure surface of geotechnical materials*

PROBLEM.– This case study establishes the factor of safety (*FS*) so that the fracture of the knot (*A*) of the slope circle can be managed or even avoided. The classic method, called the layer method, is used to this end (see Figure 1.23):

$y(x) =$		$x =$	
0.780	m	–18.266	m
0.632		–16.439	
0.499		–14.613	
0.382		–12.786	
0.281		–10.959	
0.195		–9.133	
0.125		–7.306	
0.070		–5.480	
0.031		–3.653	
7.803×10^{-3}		–1.827	
0.000		0.000	
7.803×10^{-3}		1.827	
...		...	

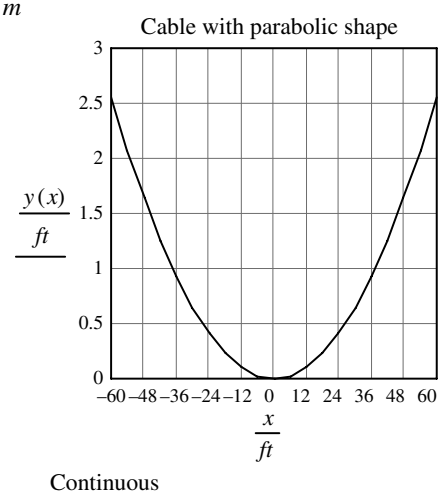


Figure 1.22. *Suspension cable with a parabolic shape for a bridge*

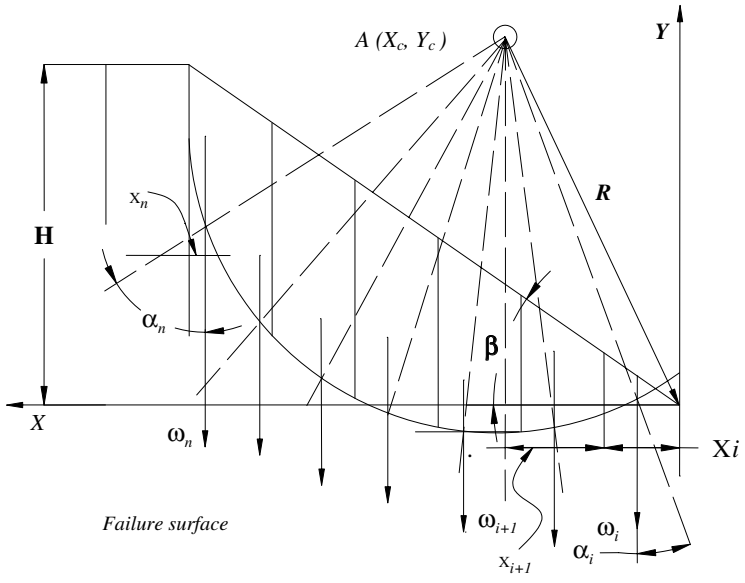


Figure 1.23. *Safety factor for a failure surface in geotechniques*

Initial data: What is the factor of safety (*FS*) to counter the fracture of the tilt slope, shown in Figure 1.23? The tilt angle is $\beta = 45°$ and the tilted height is $H = 18$ m. The circle has a radius of $\boldsymbol{R} = 25$ m and the center has the coordinates $A\ (X_c, Y_c)$. The ground has a density of $\rho = 17.56 \times 10^3$ N/m^3. The ground grip represented by the cohesion stress is *Cohesion* = 18×10^3 Pa = N/m^2 and a friction angle of $\phi = 18°$.

– Density of the ground: $\rho = 17.56 \times 10^3$ N/m^3.

– Cohesion: Cohesion = 18×10^3 Pa (N/m^2).

– Friction angle: $\phi = 18°$.

– Tilt angle of the slope: $\beta = 45°$.

– Height of the slope: $H = 18$ m.

– Radius of the circle: $R = 25$ m.

– Coordinates of the center: A [$X_c = 7.56$ m and $Y_c = 25$ m].

– Number of slices: $n = 117$.

Statements

– To express with a matrix the expression of the slope.

– To find, with linear interpolation, the relation which renders $y_{\text{slope}}(x)$ and $y_{\text{fracture}}(x)$ in function of (x).

– To find the maximum distance $X_{\text{Maxi}}(x, x_c; y_c,$ and $R)$ which is expressed in function of the circle equation.

– To plot the fracture circle, the slope, and the central point of the circle.

– To find the factor of safety *FS* for the 77 layers, that is $n = 1$ to 77.

– To find the weighted slice W_i, the (lowest) angle α_i of the tilted slice, and the friction length by slice ΔL_i.

– For layers 1 to 77, to explain with a graph the behavior of the structure, by emphasizing the *FS*.

– According to results from calculations, at what number layer can a singularity, that is a change of sign from (+) to (−), be observed?

Solutions

We will use the 2D function to develop an image of the failure surface and the slope. The 3 × 2 matrix is used to define the coordinates of key points along the slope. The defined points form the knot at the highest point on the slope (*A*).

$$slope(H,\beta)=\begin{bmatrix} 0\times H & 0\times H \\ \dfrac{H}{\tan(\beta)} & H \\ \dfrac{3\times H}{2\times\tan(\beta)} & H \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 18 & 18 \\ 27 & 18 \end{bmatrix} m \qquad [1.100]$$

The function of linear interpolation, which renders the expression of the slope, is given as follows:

$$y_{slope}(x)=\text{linterp}\left(slope^{\langle 0\rangle}, slope^{\langle 1\rangle}, x\right) \qquad [1.101]$$

We use a circle equation to express the function of fracture:

$$y_{fracture}(x)=y_c-\sqrt{R^2-\left(x_c-x\right)^2} \qquad [1.102]$$

We must define a series of *x* coordinates to plot the graph (see Figure 1.23). The maximum distance, X_{max}, required to plot the graph is the point where the fracture circle crosses the highest point of the slope, which is determined trignometrically as follows:

$$x_{max}(H,R,x_c,y_c)=x_c+\sqrt{R^2-\left(x_c-H\right)^2}=31.56\ m \qquad [1.103]$$

Once the variables of the first series are known, the *x* values can be calculated by iteration. The range is known by the hypothesis (from 1 to 77) or:

Range of variables of: i = [1, 2, …, n]

$$\text{Distance}(x)=\frac{x_i}{m}=x_i=\left(\frac{i}{n}\right)\times x_{\max} \qquad [1.104]$$

Digital calculations

$\frac{y_{slope}(x_i)}{m}$ =	$\frac{y_{Fracture}(x_i)}{m}$ =		$\frac{x_i}{m}$ =
0.27	1.087	$\frac{y_c}{m} = 25$	0.27
0.539	1.006		0.539
0.809	0.929		0.809
1.079	0.855		1.079
1.349	0.784	$\frac{x_c}{m} = 7.56$	1.349
1.618	0.716		1.618
1.888	0.652		1.888
2.158	0.591		2.158
2.428	0.532		2.428
2.697	0.477		2.697
2.967	0.426		2.967
...	...	Continuous	...

In this final analysis, the slope and the fracture circle are presented in the following graph:

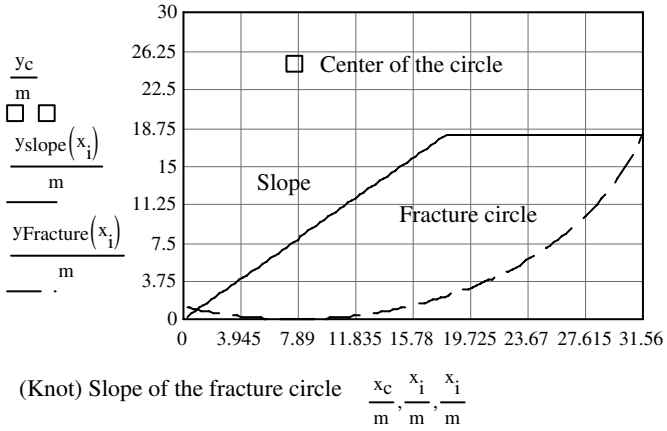


Figure 1.24. *Graph showing the slope and the fracture circle (and its precise center)*

Calculating the factor of safety *FS* by the classic method (slices) leads us to using the following equation. The derivative of the equation complies with geotechnical references.

$$\mathrm{FS} = \frac{\sum_{i=1}^{n}\left(Cohesion \times \Delta L_i + \Delta\omega_i \times Cos\left(\alpha_i\right) \times \tan\left(\phi\right)\right)}{\sum_{i=1}^{n}\left(\omega_i \times Sin\left(\alpha_i\right)\right)} \quad [1.105]$$

where the weighted slice ω_i is given by the following expression:

$$\omega_i = \frac{x_{\max}}{n} \times \left[y_{slope}\left(x_i\right) - y_{fracture}\left(x_i\right)\right] \times \rho \quad [1.106]$$

The (lowest) angle of the tilted slice is expressed with:

$$\alpha_i = \arctan\left(\frac{\left(x_i\right) - \left(x_c\right)}{y_c - y_{fracture}\left(x_i\right)}\right) \quad [1.107]$$

The length of friction by slice is written as follows:

$$\Delta L_i = \left(\frac{x_{\max}}{n}\right) \Big/ Cos\left(\alpha_i\right) \quad [1.108]$$

Digital calculations

Digital calculation of the factor of safety: from [1.105], we consider the following equation:

$$\mathrm{FS} = \sum_{i=1}^{n} Cohesion \times \Delta L_i + \Delta\omega_i \times Cos\left(\alpha_i\right) \times \tan\left(\phi\right) \Big/ \sum_{i=1}^{n} \omega_i \times Sin\left(\alpha_i\right) = 1.085$$

The calculation takes into account changes in safety, such as change in the number of layers (slices) used for the calculation which is changed. This can easily be visualized by redefining the *FS* in terms of *n* and plotting (determinant) *FS* (*n*) for a series of values from 1 to 77, and results are as follows:

$\omega_i =$

	0
0	$-3.869 \cdot 10^3$
1	$-2.21 \cdot 10^3$
2	-565.926
3	$1.062 \cdot 10^3$
4	$2.675 \cdot 10^3$
5	$4.273 \cdot 10^3$
6	$5.856 \cdot 10^3$
7	$7.424 \cdot 10^3$
8	$8.977 \cdot 10^3$
9	$1.052 \cdot 10^4$
10	$1.204 \cdot 10^4$
11	$1.355 \cdot 10^4$
12	...

$kg \cdot s^{-2}$

$\alpha_i =$

	0
0	-0.296
1	-0.285
2	-0.273
3	-0.262
4	-0.251
5	-0.240
6	-0.229
7	-0.218
8	-0.207
9	-0.196
10	...

$\Delta L_i =$

	0
0	0.282
1	0.281
2	0.28
3	0.279
4	0.278
5	0.278
6	0.277
7	0.276
8	0.276
9	0.275
10	0.274
11	...

m

etc..

FS(n) =

3.4332
4.7138
6.8937
11.3577
25.2990
–566.7485
–27.1282
–15.0117
–10.9019
...

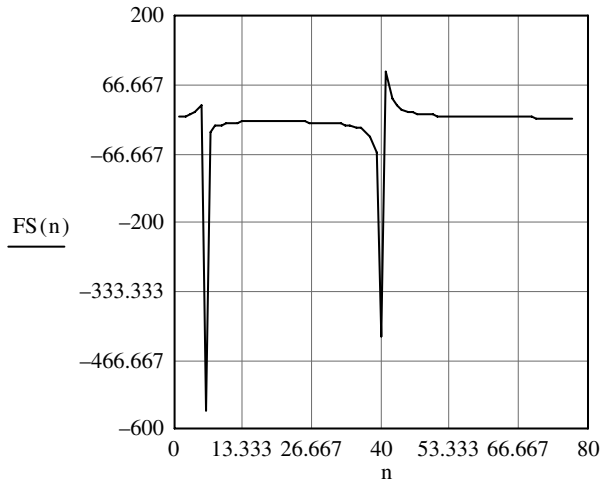


Figure 1.25. *Factor of safety in function of number of slices*

Behavior can be explained by plotting the numerator and the denominator on a graph:

$$\left\{\begin{array}{l}\text{Numerator}(n)=\sum_{i=1}^{n}\left(cohesion\times\Delta L_i\right)\times\omega_i\times Cos\left(\alpha_i\right)\times\tan\left(\phi\right)\\ \text{and denominator}(n)=\sum_{i=1}^{n}\omega_i\times Sin\left(\alpha_i\right)\end{array}\right\} \quad [1.109]$$

At n = 40, there is a singularity of the FS (n) owing to the denominator which passes below zero.

After calculations, the following result is obtained:

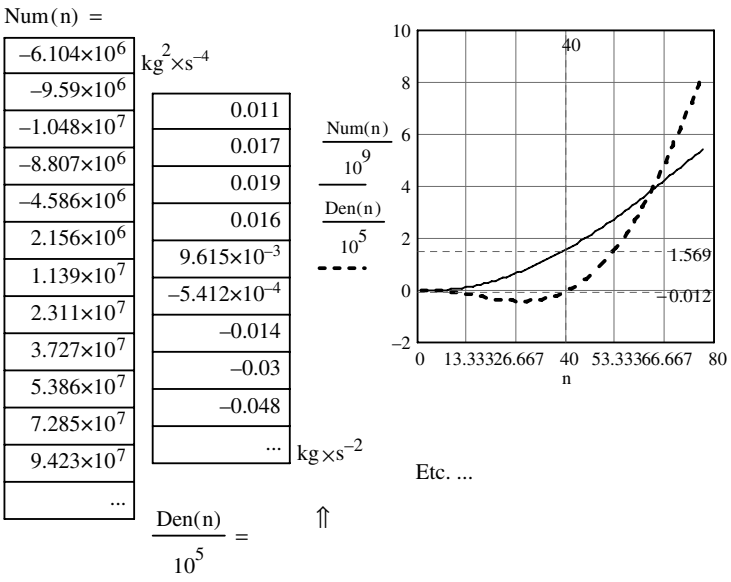


Figure 1.26. *Expressions of the denominator and the numerator in function of (n)*

COMMENTS.– Grapho-analytic proof: at the 40th layer, that is n = 40, there is a singularity = (−0.012).

$$\left\{\frac{Num(n)}{10^9}=1.569\frac{Kg^2}{s^{-4}}\ \ and\ \ \frac{Den(n)}{10^5}=-0.012\frac{Kg}{s^{-2}}\right\}$$

1.12. Conclusion

It should be borne in mind that this chapter has been purposefully developed here, even though the main focus of this work is reliability and quality control (volume 3). The reasons are as follows. First the case studies that follow are based on models of fracture mechanics, among others. Second, a high number of mechanical models exist, which are applied differently to materials and to structures. We therefore thought it useful to add this additional chapter, without which the readers would have had to do their own supplementary research. The present chapter has also included practical examples of reliability and uncertainties.

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Chapter 2

Analysis Elements for Determining the Probability of Rupture by Simple Bounds

2.1. Introduction

This chapter is dedicated to determining reliability by bounds. The failure probability P_f of a system presents approximations developed to quickly obtain estimations. From the technical literature, we know that approximations involve lengthy calculations and that their implementation offers results well below the simulation methods. By means of an example, for the two modes of failure (i) and (j) of occurrence system (case) [$\varepsilon_i \cap \varepsilon_j$], bounds are derived from Poincaré's methods. His well-known formula is written as follows:

$$P_f = \sum_{i=1}^{m} P\{M_i\} - \sum_{j=2}^{m} \sum_{i=1}^{j=1} P\{M_i \cap M_j\} + \ldots + (-1)^m \times P\{M_1 \cap M_2 \ldots \cap M_n\} \qquad [2.1]$$

We distinguish *first-order bounds*, known as *simple* bounds, for assembly systems *in series* or *in parallel*. These *bounds* only include failure mode probabilities:

$$\left\{ \left\{ \max_{i=1\ldots m} \{P_{f,i}\} \leq [P_{f,s}] \right\} \right\}_{\text{Good correlation between events}} \leq \left\{ \left\{ 1 - \prod_{i=1}^{m} [1 - P_{f,i}] \right\} \right\}_{\text{Independent failures mode}} \simeq \sum_{i=1}^{m} P_{f,i} \qquad [2.2]$$

2.1.1. *First-order bounds or simple bounds: systems in series*

The expression on the left-hand side corresponds to the case of a perfect correlation between events (fracture modes). The term on the right-hand side is an inherent approximation to statistically independent failure modes. It demonstrates that the failure probability of a system in series increases with the number of failure modes. This probability is strongly conditioned by the weakest elements. This is the principle of the weakest link.

2.1.2. *First-order bounds or simple bounds: systems in parallel*

The failure probability of a system in parallel is therefore outlined by:

$$\left\{\prod_{i=1}^{m}\left\{P_{f,i}\right\}\right\}_{\text{Good correlation between events}} \le \left\{P_{f,s}\right\} \le \left\{\min_{i=1\ldots m}\left\{P_{f,i}\right\}\right\}_{\text{Independent failures mode}} \quad [2.3]$$

As stated earlier, the term on the right-hand side expresses a good correlation between the failure modes of a system of components. The term on the left-hand side corresponds to statistically independent failure modes. In practice, these bounds are rather too far apart to be useful, as demonstrated by equations [2.2] and [2.3]. Ditlevsen's [DIT 79] bounds will now be introduced.

2.2. Second-order bounds or Ditlevsen's bounds

These bounds are defined by retaining the probability of intersections of two events $P\{G_i[X] < 0 \cap G_j[X] < 0\}$. A simple schematization is as follows:

By developing equation [2.1], it can be seen that alternating signs are used in the terms. Ditlevsen retained this characteristic to propose the following lower bound:

$$P_{(f,\mathrm{s})} \ge P_{(f,\mathrm{i})} + \sum_{i=2}^{m} \max\left\{0, \left[P_{(f,\mathrm{i})} - \sum_{j=1}^{i-1} P\left\{G_i[X] \prec 0 \cap G_j[X] \prec 0\right\}\right]\right\} \quad [2.4]$$

He also proposed the following upper bound:

$$P_{(f,\mathrm{s})} \le \sum_{i=1}^{m} P_{(f,\mathrm{i})} - \sum_{i=2}^{m} \max_{j \prec i}\left\{\sum_{j=1}^{i-1} P\left\{G_i[X] \prec 0 \cap G_j[X] \prec 0\right\}\right\} \quad [2.5]$$

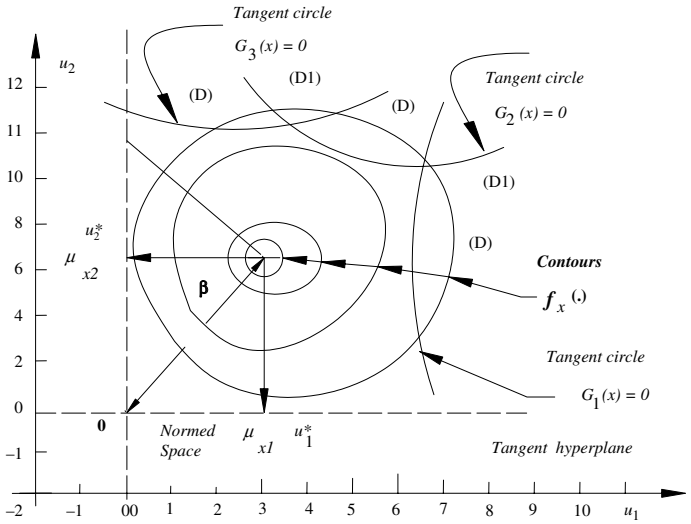

Figure 2.1. *Schematization of simple bounds: base system with two RV*

2.2.1. *Evaluating the probability of the intersection of two events*

The disadvantage of using Ditlevsen's bounds resides in evaluating joint probabilities $P\{G_i\,[X] \cap G_j[X]\}$. It is therefore possible to use approximation methods, Form and Sorm, as previously described. Figure 2.2 illustrates the *Gaussian* terms of intersection [MEL 99], such as domain D_1, obtained by limit state functions $G_i[X] = 0$ with $i = 1, 2, 3$.

Evaluating joint probabilities is generally carried out by linearizing, first, the limit states around their design points expressed according to the equation of failure probability P$f \approx \Phi(-\beta)$. Second, joint probability of the mean of binomial distribution $\varphi_2\,[y_1, y_2, \rho_{12}]$ is calculated, where y_1 and y_2 are two standard normal random variables with a correlation coefficient ρ_{12}. It should be noted that the hyperplanes from the Form calculations are represented as follows:

$$g_i(y) = \left\{ \beta_i + \sum_{j=1}^{n} \left(\alpha_{ij} \times y_j \right) \right\} \qquad [2.6]$$

In such a case, the joint failure probability can be deduced as follows:

$$P\left\{ G_1[X] \prec 0 \cap G_2[X] \prec 0 \right\} \simeq \varphi_2\left\{ -\beta_1; -\beta_2; \rho_{12} \right\} \qquad [2.7]$$

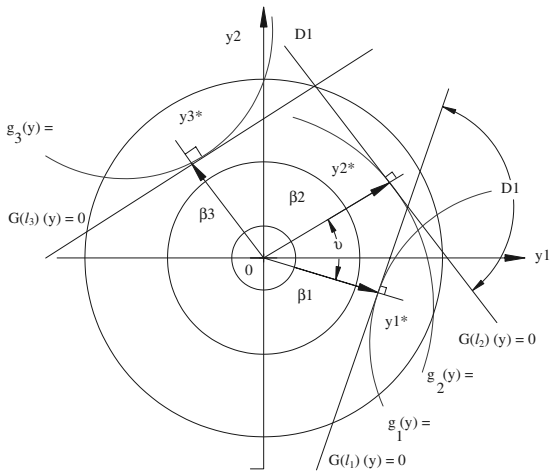

Figure 2.2. *Linearization of limit state function in space*

2.2.1.1. *Estimation of the first order of a system in series or in parallel*

To construct approximations of this order, the limit state surfaces (*M*) of structure system elements are linearized in Gaussian space at their respective design points. For an assembly first in series [DIT 81a] and then in parallel, the first-order approximation is presented as follows:

$$\left\{\begin{array}{l} P_{(f,serie)} = 1-\varphi_M\left\{(\beta);[\rho]\right\} \\ P_{(f,parallel)} = \varphi_M\left\{(-\beta);[\rho]\right\} \end{array}\right\} \qquad [2.8]$$

where:

$\varphi_{M(\cdot)}$ is the probability density function of the multinomial law.

β is the vector composed on *M* reliability index (β_{HD}). It is Hasofer–Lind's reliability index.

$[\rho]$ is the size correlation matrix [m × m] between the different linearized performance functions. This matrix comes from the coefficients of the tangent hyperplanes.

To evaluate the probability of failure for implicit limit states, structures can present complexities, that is hyperstaticity or even pronounced nonlinearity. Writing the limit state function (*M*) becomes problematic because Form and Sorm moments are no longer applicable. The MC simulation method, however, remains applicable,

but requires more time for calculation and is therefore expensive. This bears testament to the development of other techniques. Estimating multinomial distribution with a relation such as [2.9] poses numerous problems. This distribution can be presented as follows:

$$\Phi_{\left(\vec{x}_i[\rho]\right)} = \int_{-\infty}^{x_1} \int_{-\infty}^{x_m} \frac{1}{(2\pi)^{\frac{x_m}{2}}} \times \frac{1}{(\det)\cdot[\rho]} Exp\left(\frac{-\frac{1}{2}\vec{x}^T \vec{x}}{[\rho]} \right) dx_1 ... dx_1 \qquad [2.9]$$

For systems in series, we often resort to determining the probability of damage by upper and lower bounds. This is the case, for example, in relation [2.9]. Ditlevsen's simple bounds are expressed as follows:

$$\underset{l\varepsilon\{1,m\}}{MaxP_i} \le P_{Fs} \le \sum_{i=1}^{m} P_i \qquad [2.10]$$

This schematization is a simple and immediate demonstration to aid comprehension:

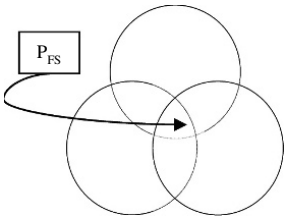

Figure 2.3. *Schematization of simple bounds, known as Ditlevsen's bounds*

P_{Fs} corresponds to the space where the three surfaces overlap. It is clearly smaller than the sum of the three elementary surfaces (which are equal if the overlapping ↔ correlation = 1). Let us remember that *Ditlevsen's bounds only call on binomial distribution*, that is the following expression:

$$\left\{ \begin{array}{l} P_{Fs} \le \sum_{i=1}^{m} P_i \; - \sum_{i=1}^{m} \underset{j\prec 1}{Max}\left[P_i \cap P_j \right] \\ \\ P_{Fs} \ge P_1 - \sum_{i=2}^{m} P_i \times \underset{j\prec 1}{Max}\left[P_i - \sum_{j=1}^{i-1} P_i \cap P_j ; 0 \right] \end{array} \right\} \qquad [2.11]$$

2.2.2. *Estimating multinomial distribution–normal distribution*

The previous expression demonstrates that estimating the reliability of structures relies, in a large part, on calculating multinomial distribution $\phi(m)$. The analysis of complex structures requires fast and effective digital tools, because the consumption sources in machine time are complicated. For example:

– the standardization of base variables, U;

– the iterative calculation of the security index (β_{HL}) for a large number of elements;

– reactualizing the rigidity matrix [ρ];

– calculating the equivalent safety margins (M), etc.

From this point on, we should not resort to digital integration techniques to calculate the values of $\phi(m)$. The following section will set out the most well-known and easy-to-use approximation techniques. They are concerned with the distribution of a standard normal random variable which is presented, *a priori*, in this simple classic form:

$$\Phi(m) = P\{X \le x\} = \frac{1}{2\pi}\int_{-\infty}^{x} Exp^{\left(-\frac{U^2}{2}\right)}du \qquad [2.12]$$

2.2.3. *Binomial distribution*

The expression of binomial distribution is given by relation ϕ_2.

$$\left\{\begin{array}{l} \Phi_2 = R\{R,S,\rho\} = \displaystyle\int_{-\infty}^{R}\int_{-\infty}^{S} \varphi_2(U_1,U_2,\rho)\, dU_1 dU_2 \ \textit{and}\ \varphi_2\ \textit{written} \\ \textit{as follows}\ \varphi_2 = U\{R,S,\rho\} = \dfrac{1}{2\pi\times\sqrt{1-\rho^2}}\, Exp^{-\frac{1}{2}\left(-\frac{U_1^2+U_2^2-2\rho\cdot U_1U_2}{1-\rho^2}\right)} \end{array}\right\} \qquad [2.13]$$

It is worth noting that $\varphi_2(R, S, \rho)$ takes the following forms:

$$\left\{\begin{array}{c} \varphi_2(R,S,\rho) = \dfrac{\partial^2\Phi_2(R,S,\rho)}{\partial R\partial S} = \dfrac{\partial\Phi_2(R,S,\rho)}{\partial\rho} = \varphi_2(R,S,0) + \displaystyle\int_0^{\rho} \frac{\partial\Phi_2(R,S,0)}{2\lambda_{\downarrow\lambda=U}}dU \\ \text{where}\ \varphi_2(R,S,\rho) = \varphi_2(R)\times\varphi_2(S) + \displaystyle\int_0^{\rho}\Phi_2(R,S,U)dU \end{array}\right\} \qquad [2.14]$$

The integral can be calculated by using a development of $\varphi_2(R, S, \rho)$, based on Hermite polynomials. For example:

$$\varphi_2(R,S,\rho) = \frac{1}{2\pi} Exp\left(-\frac{R^2}{2} - \frac{S^2}{2}\right) \times \sum_{i=1}^{\infty} \frac{U^i}{i!} \times H_i(R) \times H_i(S) \qquad [2.15]$$

The first Hermite polynomials are as follows:

$$\left\{\begin{array}{l} H_0(x) = 1 \\ H_1(x) = X \\ H_2(x) = X^2 - 1 \\ H_3(x) = X^3 - 3X \\ H_4(x) = X^4 - 6X^2 + 3 \\ H_5(x) = X^5 - 10X^3 + 15X \\ H_6(x) = X^6 - 15X^4 + 45X^2 - 15 \\ etc.\ \ldots \\ H_n(x) = \ldots\ \ldots\ \ldots \end{array}\right\}, \left\{\begin{array}{l} \hat{H}_0(x) = 1 \\ \hat{H}_1(x) = 2X \\ \hat{H}_2(x) = 4X^2 - 2 \\ \hat{H}_3(x) = 8X^3 - 12X \\ \hat{H}_4(x) = 16X^4 - 48X^2 + 12 \\ \hat{H}_5(x) = 32X^5 - 160X^3 + 120X \\ \hat{H}_6(x) = 64X^6 - 480X^4 + 720X^2 - 120 \\ etc.\ldots.\ \text{in practice, until } n = 20 \\ \hat{H}_n(x) = x \times H_{n-1}(x) - (n-1) \times H_{n-2}(x) \end{array}\right\} \qquad [2.16]$$

In probability, Hermite's polynomial (more precise that Lagrange's) constitutes a sequence of polynomials defined as follows:

$$H_n(x) = (-1)^n Exp^{x^2} \frac{d^n}{dx^n} Exp^{x^2} \qquad [2.17]$$

The two definitions are linked by the following scale property:

$$\hat{H}_n(x) = 2^{\left(\frac{n}{2}\right)} H_n\left\{\sqrt{2} \times x\right\} \qquad [2.18]$$

It can be demonstrated that in H_P, coefficients with the same parity as $(p - 1)$ are zero and that coefficients of order (P) and $(P - 2)$ are worth 1 and $[-(P\ (p - 1)/2]$,

respectively. In relation [2.18], x takes the values of R or S and we can deduce the following:

$$\int_0^{\rho} \varphi_2(R,S,\rho)dU = \frac{1}{2\pi} Exp\left(-\frac{R^2}{2}-\frac{S^2}{2}\right)\times\sum_{i=1}^{\infty}\frac{U^i}{i!}\times H_{i-1}(R)\times H_{i-1}(S) \quad [2.19]$$

In practice, the development of order 20 will be used. We have plotted the results of Hermite's polynomial in a graph:

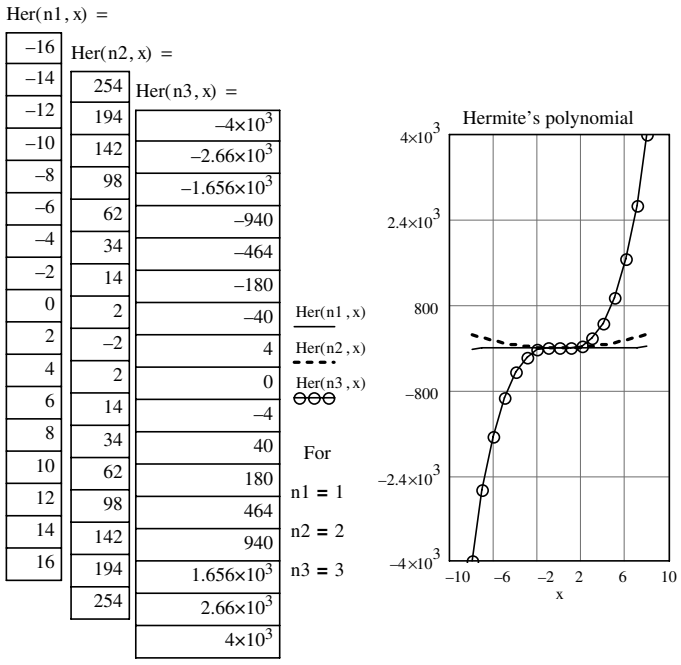

Figure 2.4. *Graph of Hermite's polynomial*

2.2.4. *Approximation of ϕ_2 (for $m \geq 3$)*

The expression of the normal law is given by the following relation:

$$\Phi_m\left(\vec{x},[\rho]\right) = \int_{-\infty}^{x_1}\int_{-\infty}^{x_m}\frac{1}{2\pi}\times\frac{1}{\sqrt{\det[\rho]}}Exp\left(-\frac{1}{2[\rho]}\times\left(\vec{x}^{T}\cdot\vec{x}\right)\right)dx.dx_m \quad [2.20]$$

where $\vec{x} = [x_1, x_2, x_3, ..., x_m]$ and $[\rho]$ is a correlation matrix of x_i.

The equicorrelated variables are represented in the case of the following matrix:

$$\left\{ \begin{array}{l} When\ [\rho] = \begin{bmatrix} 1 & \rho & ...\rho \\ \rho & 1 & ...\rho \\ \rho & \rho & ...1 \end{bmatrix}\ it\ can\ be\ shown\ that \\ \\ \Phi_m\left(\vec{x},[\rho]\right) = \int_{-\infty}^{+\infty} \varphi(U) \times \prod_{i=1}^{m} \Phi\left(\frac{x_i - \sqrt{\rho}\cdot U}{\sqrt{1-\rho}} \right) dU \end{array} \right\} \quad [2.21]$$

Relation [2.21] can be integrated without difficulty. The result will be used to get closer to the general case by replacing the correlation matrix [ρ] with the matrix [φ] as follows:

$$\left[\overline{\rho}\right] = \begin{bmatrix} 1 & \overline{\rho} & ...\overline{\rho} \\ \overline{\rho} & 1 & ...\overline{\rho} \\ \overline{\rho} & \overline{\rho} & ...1 \end{bmatrix}\ where\quad \left[\overline{\rho}\right] = \frac{1}{n(n-1)} \times \sum_{\substack{i,j=1 \\ i \neq j}}^{m} \rho_{ij} \quad [2.22]$$

$[\overline{\rho}]$ is the average correlation coefficient. The approximation is obtained by relation [2.20]. The majority of the cases amount to the following expression:

$$\Phi_m\left\{\vec{x},[\rho]\right\} \leq \Phi_m\left\{\vec{x}\left[\overline{\rho}\right]\right\} \quad [2.23]$$

As has been previously presented [2.8], by the determining of reliability by Ditlevsen's upper and lower bounds, relation [2.23] can be generalized in the case of the following matrix:

$$[\rho] = \begin{bmatrix} 1 & \lambda_1\lambda_2 & ...\lambda_1\lambda_n \\ \lambda_1\lambda_2 & 1 & ...\lambda_2\lambda_n \\ \lambda_n\lambda_1 & \lambda_n\lambda_2 & ...1 \end{bmatrix} \quad [2.24]$$

In this case, we will have the following:

$$\Phi_m\left(\vec{x},[\rho]\right) = \int_{-\infty}^{+\infty} \varphi(u) \times \prod_{i=1}^{m} \Phi\left(\frac{x_i - \lambda_i}{\sqrt{1-\lambda_i^2}} \right) du \quad [2.25]$$

On the other hand, it can be demonstrated that:

$$\Phi_n\left(\bar{x},[\rho']\right) \le \Phi_n\left(\bar{x},[\rho]\right) \quad [2.26]$$

$$\text{For each matrix } [\rho'] \quad \text{as } \rho'_{ij} = \rho_{ij};\ \forall i,j \in \{1,\ m\} \quad [2.27]$$

From this result and by using equation [2.9], we can obtain an *upper bound* of $\Phi_n\left(\bar{x},[\rho']\right)$ by choosing λ_i as:

$$\lambda_i = \sqrt{\underset{i\neq j}{Maxi}\left[\rho_{ij}\right]} \quad [2.28]$$

A lower bound of $\Phi_n\left(\bar{x},[\rho']\right)$ can be obtained by choosing (λ_i) as:

$$\lambda_i = \sqrt{\underset{i\neq j}{Mini}\left[\rho_{ij}\right]} \quad [2.29]$$

2.3. Hohenbichler's method

The Hohenbichler's method [HOH 84, BRE 84, HOH 87a, HOH 87b], though relatively *difficult* to implement, does give excellent results. A brief overview of the method is given in the following.

By definition:

$$\Phi_m\left(\bar{x},[\rho]\right) = P\left\{\bigcap_{i=1}^{m} X_i \le x_i / X_1 \le x_1\right\} \quad [2.30]$$

where X_i are the standard normal random variables of correlation matrices [ρ]. This identity can also be written in the following form:

$$\Phi_m\left(\bar{x},[\rho]\right) = P\left(X_i \le x_i\right) \times P\left\{\bigcap_{i=1}^{m} X_i \le x_i / X_1 \le x_1\right\} \quad [2.31]$$

By using Cholesky's factorization algorithm, a lower triangular matrix such as the following can be found:

$$[\alpha] = [\alpha] \times [\alpha]^T \quad [2.32]$$

If we consider:

$$\left\{\begin{array}{l} X_1 = \alpha_{11} U_i \\ X_1 = \alpha_{21} U_i + \alpha_{22} U_2 \\ X_1 = \alpha_{m1} U_i + \alpha_{m2} U_2 ... \alpha_{mm} U_m \end{array}\right\} \quad [2.33]$$

we can easily verify that the $[U_i]$ are *standard and independent normal random variables.* Moreover, $\boldsymbol{\alpha}_{11} = 1$. Equation [2.31] can be rendered in the $[U_i]$ space to obtain the following:

$$\Phi_m\left(\bar{x}, [\rho]\right) = P\left(X_i \le x_i\right) \times P\left\{\bigcap_{i=2}^{m}\left[\sum_{j=1}^{1} \frac{\alpha_{ij} U_j \le x_i}{U_1 \le x_i}\right]\right\} \quad [2.34]$$

In each linear combination $\left(\sum_{j=1}^{1} \alpha_{ij} U_j \le x_i\right)$, only the first term $[\alpha_{i1} x U_1]$ is affected by the condition $1 \le x_1$ because the U_i are independent. The conditional distribution functions of the U_i are therefore written as follows:

$$F_1(u) = P\left(U_1 \le x_1\right) = \left(\frac{\Phi(u)}{\Phi(x_1)}\right) \quad [2.35]$$

The condition can therefore be eliminated by replacing the variable $\left(\frac{U_1}{U_1} \le x_1\right)$ with the following variable:

$$\tilde{U} = \Phi^{-1}\left(\Phi\left(x_1\right)\right) \times \Phi\left(U_1\right) \ \left\{ \textit{or even } F_{\tilde{u}} = F_1 \right\} \quad [2.36]$$

Equation [2.26] can therefore be written as follows:

$$\begin{aligned} \Phi_m\left(\bar{x}, [\rho]\right) &= \Phi\left(x_1\right) \times P\left\{\bigcap_{i=2}^{m} \alpha_{i1} \Phi^{-1}\left[\Phi\left(x_1\right) \times P\left(U_1\right)\right] + \sum_{j=1}^{1} x_{ij} U_j \le x_i\right\} \\ &= \Phi\left(x_1\right) \times P\left[\bigcap_{i=2}^{m} G_i\left(\bar{u}\right) \le x_i\right] \end{aligned} \quad [2.37]$$

By linearization, as understood by Hasofer–Lind's [HAS 74] reliability index, we obtain the following:

$$\Phi_m\left(\bar{x},[\rho]\right)=\Phi\left(x_1\right)\times P\left\{M_i\le 0\right\} \qquad [2.38]$$

With $\left|\gamma_i\right|=1$, we obtain the following:

$$\Phi_m\left(\bar{x},[\rho]\right)=\Phi\left(x_1\right)\times P\left\{\bigcap_{i=2}^{m}\times\left(\sum_{j=1}^{1}\gamma_{ij}U_j\le x_1^{(2)}\right)\right\}\ with\ \left|\gamma_i\right|=1 \qquad [2.39]$$

Therefore, let

$$\Phi_m\left(\bar{x},[\rho]\right)=\Phi\left(x_1\right)\times\Phi_{m-1}\times\left\{\left(\bar{x}^{(2)}\right)\left[\rho^{(2)}\right]\right\} \qquad [2.40]$$

$$With\ \left[\rho^{(2)}\right]=\left[\left(\bar{\gamma}_i^T\right)\times\left(\bar{\gamma}_i\right)\right] \qquad [2.41]$$

This algorithm can lower the rank, by one degree, of the multinomial distribution to be estimated by recurrence. It therefore becomes easy to calculate $\Phi_n\left(x,[\rho]\right)$.

$$P_F=\Phi_n\left(\vec{\beta},[\rho]\right)\ when\ \beta=\left\{\beta,\beta,\ \ldots\ \beta\ \right\}\ and\ [\rho]=\begin{bmatrix}1 & \rho & \ldots\rho\\ \bar{\rho} & 1 & \ldots\rho\\ \rho & \rho & \ldots 1\end{bmatrix} \qquad [2.42]$$

Schematizing M. Hohenbichler's method is also precise for cases when the analytical expression of ϕ_{m} is known. The components for formulations can be read in literature by H. Hohenbichler and R. Rackwitz [HOH 87].

COMMENT.– We are going to present a hypothesis test of a Gaussian with two extremes (of unknown variance). What follows is a continuity of the analysis elements of reliability and of quality control, already developed in Chapters 1 and 2 of Volume 1.

2.4. Hypothesis test, through the example of a normal average with unknown variance

By forging, during the manufacturing process, we designed connecting rods by hot swaging. We then manufactured them by machining on an aluminum alloy 6061

(see the following definition drawing). According to the functional specifications document, 50 connecting rods must weigh 2 lb each.

In metrology, during measuring (weighing), we recorded the weights indicated in Table 2.1. It is not enough to determine the reliability by simple bounds, rather it is required to find the mean and the standard deviation, σ, at the threshold of significance a = 0.1 around the mean $\mu 0$ = 2 lb to analyze the distribution of the connecting rods within these 2 lb.

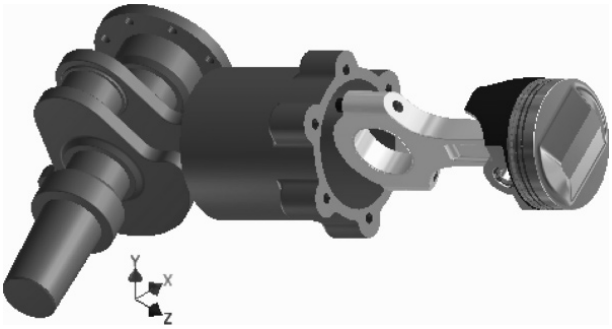

Figure 2.5. *Real mechanism designed in mechanical design laboratory (connecting rod)*

Connecting rod =

	0	1	2	3	4
0	2.35	2.12	2.12	2.56	2.00
1	2.00	1.95	1.89	3.25	2.00
2	3.45	2.33	1.45	1.53	3.00
3	2.21	2.54	2.54	2.00	1.91
4	3.35	1.51	2.54	2.15	3.00
5	2.00	1.58	2.54	2.95	2.52
6	1.75	2.35	2.16	2.50	3.58
7	1.25	2.15	2.33	3.00	3.35
8	3.15	1.56	3.00	2.65	1.69
9	2.52	2.00	2.58	3.71	1.88

Pb = Weight of each connecting rod

Table 2.1. *Data vector (weight of connecting rod in lb)*

Classifying the connecting rods into categories is not easy. For example, in which category should a connecting rod weighing 1.95 lb be classified and another weighing 3.25 lb?

First, this gap, which has been deliberately exaggerated to demonstrate the uncertainty, is intended for the limits of each weight category. We are going to study the characteristics of Gauss' curve, by following the central part of the curve quite closely: we are interested in the diagram of the dispersion of weights.

– Total number of weighed connecting rods, n = length (Pb) = 50 connecting rods.

– Let the value of the average to be tested be: $\mu_0 = 2$ lb.

– At significance level, $\alpha = 0.15$.

2.4.1. *Development and calculations*

– *Standard deviation:* $\sigma = \sqrt{\left(\frac{n}{n-1}\right)} \times \mathrm{VAR}\left(Pb\right) = 0.667;$

– *Test statistic:* $Weight = \frac{mean\left(Pb\right) - \mu_0}{\sigma/\sqrt{n}} = 3.645$;

– *Calculation for the degrees of freedom: dof* = d_f = [n − 1] = [50 − 1] = 49 connecting rods.

Three distinct case studies are presented in terms of the hypothesis tests:

CASE 1: Hypothesis test with two extremes: H_0 and $H_{1\rightarrow}$

– H_0: for $\mu = \mu_0$ and H_1 when $\mu \neq \mu_0$

– *Calculation for the critical value*: $t_t = qt\left\{1 - \frac{\alpha}{2}, d_f\right\} = 1.462$

– *If 1 does not exclude* $H_{0\rightarrow}$ *Test* H_0 *in function of p values*:

$$\left(\frac{\alpha}{2}\right) \prec pt\left\{Weight, df\right\} \prec \left(1 - \frac{\alpha}{2}\right) = 0$$

– *If 1 does not exclude* $H_{0\rightarrow}$ *Test* H_0 *in function of q values*: $\left|Weight\right| \prec t_t = 0$

Graphically the test for q = [−5, −4.9, … +5], and k = [0, … 1] is presented as follows:

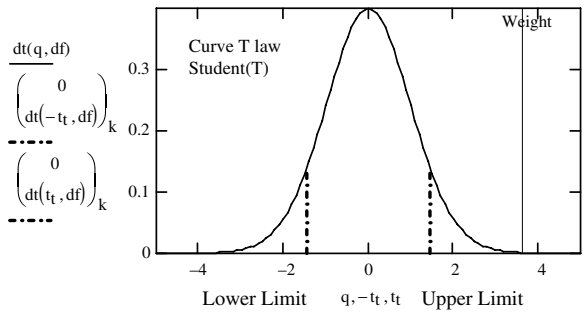

Figure 2.6. *Graph illustrating Case 1 test (49 connecting rods)*

CASE 2: Hypothesis test for a *left extreme*: H_0 and $H_{1\rightarrow}$

– H_0: for $\mu \geq \mu_0$ and H_1

– *Calculation for the critical value*: $t_L = qt\left\{\alpha, d_f\right\} = -1.048$

– *If 1 does not exclude* $H_{0\rightarrow}$ *Test* H_0 *in function of p values:*

$$pt\left\{Weight, df\right\} \succ \left(\alpha = 1\right)$$

– *If 1 does not exclude* $H_{0\rightarrow}$ *Test* H_0 *in function of q values*: $Weight \prec t_L = 1$

Graphically the test for q = [−5, −4.9, …, +5], and k = [0, …, 1] for $\mu < \mu_0$, is presented as follows:

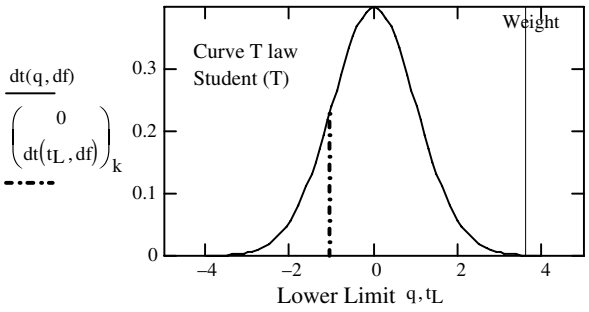

Figure 2.7. *Graph illustrating Case 2 test (49 connecting rods)*

CASE 3: Hypothesis test at a *right extreme*: H_0 and $H_{1\rightarrow}$

– H_0: for $\mu \leq \mu_0$ and H_1 for $\mu > \mu_0$

– *Calculation for the critical value*: $t_R = qt\left\{1-\alpha, d_f\right\} = +1.048$

– *If 1 does not exclude $H_{0\rightarrow}$Test H_0 in function of p values:*

$$pt\left\{Weight, df\right\} \prec \left(1-\alpha = 0\right)$$

– *If 1 does not exclude $H_{0\rightarrow}$Test H_0 in function of q values*: $Weight \prec t_R = 0$

Graphically the test, for q = [−5, −4.9, …, +5] and k = [0, …, 1], is presented as follows:

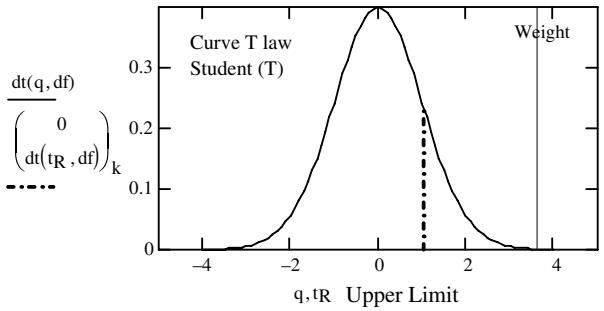

Figure 2.8. *Graph illustrating Case 3 test (49 connecting rods)*

We have just seen that two hypotheses H_0 and H_1 oppose one another. The decision on the behavior of the 50 connecting rods is based on real experimental data. The statistical hypothesis remains a quantitative statement relative to a characteristic (weights of the connecting rods). We sought to determine whether the process is *still* centered at μ_0 or whether there is a sliding (displacement) of the central tendency in (±). The decision depends on whether we accept or reject the hypothesis at the confidence threshold, at the risk of the error (α: 1st type error).

2.5. Confidence interval for estimating a normal mean: unknown variance

A case study for calculating the upper and lower bounds by confidence interval is presented as follows. The same experimental data have been used, as shown in Table 2.1.

SOLUTION.– To simply determine the probability of the confidence interval at significance level ($\alpha = 0.15$), let: $1 - \alpha = 85\%$.

Standard deviation (S) of the sample: $S = \sqrt{\left(\frac{n}{n-1}\right) \times VAR(Pb)} = 0.567$

Degrees of freedom (dof) $\rightarrow$ $d_f = n - 1 = 50 - 1 = 49$

Calculation of the critical value of the bounds (limit): $t1_0 = qt\left\{1 - \frac{\alpha}{2}, df\right\} = 1.462$

$$\text{Bounds} \left\{ \begin{array}{l} \text{Upper limit } \Rightarrow U = LS = mean(Pb) + t1_0 \times \frac{S}{\sqrt{n}} = 2.409 \\ \text{Lower limit } \Rightarrow LI = mean(Pb) - t1_0 \times \frac{S}{\sqrt{n}} = 2.175 \end{array} \right\}$$

2.6. Conclusion

This chapter having been dedicated to determining the probabilities by simple bounds, the importance of the method for systems of components assembled in series is clear. Specialized literature on the topic [DIT 73, DIT 79, DIT 81a, DIT 81b, DIT 86a, DIT 86b, HOH 84, BRE 84, HOH 87a, HOH 87b] offers many case studies. Presenting more studies here would be of little use.

The final two examples constitute the classical statistical approach to calculating confidence intervals for estimating the mean of a Gaussian which has an unknown variance. By this very fact, applied statistics remain a key tool for both reliability and quality control.

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Chapter 3

Analysis of the Reliability of Materials and Structures by the Bayesian Approach

3.1. Introduction to the Bayesian method used to evaluate reliability

In reliability, Bayesian analysis considers the parameters of the population of random variables (RV) to be non-fixed. Subjective judgment is sometimes used to determine the *a priori distribution* of the parameters of the population being studied. Numerous personal decisions are taken intuitively, that is, based on our experience and our judgments which are tainted with *subjectivity*. Mainstream statistical analysis aims at objectivity by generally restricting the information used for analysis, which is taken from a body of relevant data. Prior knowledge is not used, except to suggest the particular choice of a population model, which is then *adapted* to the data. An adequacy test later verifies that the characteristics of the data are reasonable. For the reliability of materials and structures, the life expectancy and the distribution of the components have one or several unknown parameters. The classic statistical approach considers these parameters to be fixed. These are unknown constants to be estimated, with the help of random samples taken from the population of the observed data.

In practice, few distribution models for life expectancy are successful. In literature, the population models of failure (ruin) times, stemming from fracture mechanisms, are treated under strict conditions based on probabilistic arguments on the mode of fracture (physical). The latter tend to justify the choice of a probability model. The model is used, in these cases, because of its empirical success in

adjusting the real data from the ruin of materials and/or of structures. Usually, around seven models are used. For example, one or two models are listed as follows:

1) Weibull (with two and/or three parameters).

2) Gamma (Erlang).

3) Birbaum–Saunders.

4) Exponential.

5) Extreme value (Gumbel).

6) Log-normal (Galton).

7) Proportional risks.

The Bayesian approach treats the parameters of the population model as random, non-fixed quantities. We base this on the history of the data, or on an *a priori subjective* judgment to construct an *a priori* distribution model. The model in question expresses the start evaluation, according to the probability of the unknown parameters. Then with the help of the Bayes formula, the previous evaluation is revised to arrive at the *posterior distribution model* for the population of the parameters of the model. The estimation of the average confidence interval's parameters is calculated from the posterior distribution.

Confidence intervals determine the probability of ruin from unknown parameters, since they are considered random and are not fixed (e.g. crack propagation). It is improbable, for most applications, to validate a distribution model, which has been previously selected. Bayesian parametric models are chosen for their adaptability and mathematical convenience. In particular, *a priori conjugate* models are a supposedly natural choice of a Bayesian distribution model, *a priori*.

3.2. Posterior distribution and conjugate models

Bayes formula combines prior knowledge of the physical phenomenon with actual (observed) data to produce a posterior distribution. Bayes method expresses the conditional probability of an event (A) which has occurred, given that event (B) has taken place. The following is noted: P(A|B). In terms of unconditional probabilities, it can be understood that the probability that event (B) would have occurred because (A) also occurred. The formula is therefore written as:

$$P(A/B) = \frac{P(A,B)}{P(B)} = \frac{P(A)\cdot P(B/A)}{P(B)} \qquad [3.1]$$

$P(B)$ is developed with the help of total probability:

$$P(B) = \sum_{i=1}^{n} P(B/A_i) \cdot P(A_i) \qquad [3.2]$$

Events A_j being mutually exclusive have exhausted all possibilities, notably event (a) of A_i. The same formula, written in terms of probability density function models, takes the following form:

$$g(\lambda/\tau) = \frac{f(\tau/\lambda) \cdot g(\tau)}{\int_0^{\infty} f(\tau/\lambda) \cdot g(\lambda) d\lambda} \qquad [3.3]$$

$f(\tau/\lambda)$ is the probability function of the observed data (x), from the unknown parameter (λ). g(λ) is the *a priori* distribution model for (λ). $g(\lambda|\tau)$ is therefore the posterior *distribution* model for (λ) for the observed data (x). When $g(\lambda|x)$ and $g(\lambda)$ belong to the same distribution family, $g(\lambda)$ and $f(\tau|\lambda)$ are therefore called *distribution conjugates* and $g(\lambda)$ is the *a priori conjugate* for $f(\tau|\lambda)$. *Beta* distribution (see Chapter 1 of this volume) is a preliminary condition for the proportion of success (p) of the conjugate when the samples have a binomial distribution.

The Gamma model is an *a priori* conjugate for the rate of ruin (λ) when the time to fail or repair comes from an exponential population. These conjugate laws, that is gamma law and exponent law, are largely used in reliability with the help of a Bayesian system.

This is how Bayes analysis is used in the evaluation of reliability:

a) Classic paradigm to evaluate system reliability:

- Mean time before failures (MTBF) is a fixed and unknown value. No "probability" is associated with it.

- The data for failure during a test or observation period allow us to draw conclusions, with respect to the value of the true unknown MTBF.

- With the exception of what has previously been said, no other data are used, nor "judgment" made: the procedure is an objective based solely on the test data, and the homogeneous Poisson process (HPP) model is presumed to be valid.

b) Bayesian paradigm to evaluate system reliability:

- MTBF is a random quantity with a probability distribution.

- The *test* component "chooses" an MTBF from the distribution of failure data which follow an HPP model with this MTBF.

- MTBF is based on prior test data or a consensus stemming from engineering judgment (design).

Using Bayes' analysis methods in reliability is justified by the cost of time and of the material of the test. Adding to these grounds are the following:

Advantages of the Bayes' approach	Disadvantages of the Bayes' approach
Uses "logical" preliminary information	Preliminary information is not accurate
If the preliminary information is encouraging, fewer new tests are needed to confirm an MTBF with a given confidence	The mode of collecting prior information cannot be *correct*, because chosen *a priori*
The confidence intervals are intervals for the MTBF (random) – sometimes called "*credible intervals*"	The validity of previous data or judgments of the designers (engineering) are *not*, *de facto*, *accepted*
There are few applications coming from literature on continuum mechanics	A "correct way" of collecting preliminary information does not exist. Different approaches give different results
It is worth making sure that the results are not strictly confined to theoretical mathematical manipulations (many experiences must be made)	Results cannot always be guaranteed

Table 3.1. *Advantages and disadvantages of the Bayes' method*

Sometimes we want to know the probability of a given event (E_1), not as a singular event but relative to another event (E_2) (or events), which has already occurred. Event (E_1) is conditioned by the preliminary performance of event (E_2). Definition space is a noted sample space (Ω). Therefore, the definition of the conditional probability of E_1 in relation to E_2 in this space (Ω) can be shown as follows:

$$P(E_1/E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)} \quad just\ as \quad P(E_2/E_1) = \frac{P(E_1 \cap E_2)}{P(E_1)} \qquad [3.4]$$

By analogy if an event E_3 was added, we will say that the latter will be carried out when we know that E_1 and E_2 will have been already carried out. Let:

$$P(E_3/E_1 \cap E_2) = \frac{P\left[E_3 \cap (E_1 \cap E_2)\right]}{P(E_1 \cap E_2)} = \frac{P[E_3 \cap E_1 \cap E_2]}{P(E_1 \cap E_2)} \quad with\ P(E_1 \cap E_2) \neq 0 \qquad [3.5]$$

Multiplication is a direct consequence of the definition of conditional probability. The following rule allows us to find the intersection:

$$\left\{\begin{array}{ll} P(E_1 \cap E_2) & = P(E_1) \times P(E_1/E_2) \\ P[E_1 \cap E_2 \cap E_3] & = P(E_1 \cap E_2) \times P[E_3/E_1 \cap E_2] \\ & = P(E_1) \times P(E_2/E_1) \times P[E_3/E_1 \cap E_2] \end{array}\right\} \quad [3.6]$$

The generalization of relation [3.6] takes the following form:

$$\left\{\begin{array}{l} P\{E_1 \cap E_2 \cap E_3 \cap ... \cap E_i\} = P(E_1) \times P(E_2/E_1) \times P[E_3/E_1 \cap E_2] \times \\ \qquad \times \ ... \times P\{E_i/[E_1 \cap E_2 \cap E_3 \cap ... \cap E_{i-1}]\} \end{array}\right\} \quad [3.7]$$

3.2.1. *Independent events*

Sometimes events are independent, meaning that the failure of a structure is not influenced by the failure of the other. It can therefore be understood that events E_1 and E_2 are not dependent and that their respective probabilities are not conditioned in relation to the other. The *sine qua non* condition reads: event E_1 is independent of event E_2 if and only if:

$$P(E_1/E_2) = P(E_1) \text{ just as } P(E_2/E_1) = P(E_2) \quad [3.8]$$

If we replace $P(E_1/E_2)$ with $P(E_1 \cap E_2)/P(E_2)$ we will have $P(E_1)$ or $P(E_1 \cap E_2) = P(E_1) \times P(E_2)$.

In the reliability of components and structures, this formulation provides us with the independent verification criteria, where our intuition fails. A generalization of the independence definition is as follows:

$$P\{E_1 \cap E_2 \cap E_3 \cap ... \cap E_i\} = \{P(E_1) \times P(E_2) \times P(E_3) \times ... \times P(E_i)\} = \prod_{k=1}^{i} P(E_k) \quad [3.9]$$

Practical application 1 in reliability of structures

Let $P(S_1) = 1/10$, the probability that a singular structure is failing, and another similar structure $P(S_2) = 1/15$. Let us suppose that the two structures are independently stressed. We need to determine:

1) the probability that the two structures are *failing simultaneously*;

2) the probability that no structure will ruin;

3) the probability that any of the structure will ruin.

Solution

1. The expression that considers the simultaneity of ruin is written as follows:

$$Let:\ P\{S_1 \cap S_2\} = \{P(S_1) \times P(S_2)\} = \left(\frac{1}{40}\right) \times \left(\frac{1}{50}\right) = 0.0005 = \frac{1}{2000}$$

2. The expression for the probability that no structure will ruin is written as follows:

$$Let:\ P\{S_1' \cap S_2'\} = \{P(S_1') \times P(S_2')\} = [1 - P(S_1)] \times [1 - P(S_2)]$$
$$= \left(\frac{39}{40}\right) \times \left(\frac{49}{50}\right) = 0.995 = \frac{1911}{2000}$$

3. The expression probability that any of the structure will ruin is written as follows:

$$P\{S_1 \cup S_2\} = [P(S_1) + P(S_2)] - P[S_1 \cap S_2] =$$
$$= [P(S_1) + P(S_2)] - [P(S_1) \times P(S_2)] = 0.045 = \frac{89}{2000}$$

Practical application 2 in reliability of structures

Another typical case in the reliability of materials and structures is the following. Let the structure system of four (04) welded cross-structures with an arc be schematized as follows:

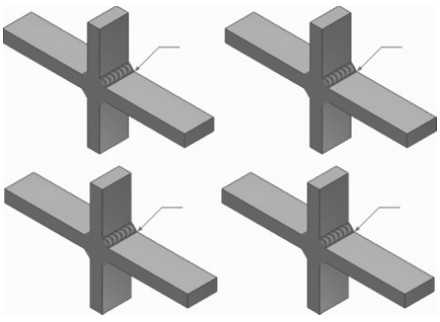


Figure 3.1. *Welded cross-structure*

To have a *reputed normal* operation, the probability of acceptable operation conforms to the hypothesis that $p = [0.52 \rightarrow 0.85]$.

$$\Phi = \left\{\left[S_1 \cap S_2\right] \cup \left[S_3 \cap S_4\right]\right\} \text{ therefore } P(\Phi) = P\left\{\left[S_1 \cap S_2\right] \cup \left[S_3 \cap S_4\right]\right\}$$

$$\text{Let } \varepsilon_1 = \left[S_1 \cap S_2\right];\ \varepsilon_2 = \left[S_3 \cap S_4\right] \text{ etc... } \varepsilon_{i,k} = \left[S_i \cap S_k\right].$$

Let us apply the union rule: $P(\Phi) = P\left\{\varepsilon_1 \cup \varepsilon_2\right\}$

$$P(\Phi) = P\left(S_1\right) \times P\left(S_2\right) + P\left(S_3\right) \times P\left(S_4\right) - P\left(S_1\right) \times P\left(S_2\right) \times P\left(S_3\right) \times P\left(S_4\right)$$

Given that $(S_k) = p$ with $k = 1, 2, 3, \ldots, n$ (here $n = 4$), the following can be deduced:

$$P(\Phi) = p \times p + (p \times p) - (p \times p \times p \times p) = \left(p^2\right) \times \left(p^2\right) \times \left(p^4\right) = p^2 \times \left(2 - p^2\right)$$

The system will operate well, in accordance with the hypothesis that between $p = [0.52 \rightarrow 0.85]$, each singular structure will not fail.

Using the probability approach to study the reliability of structures is simple when the hypothesis is well chosen and clearly demonstrated. Among many cases, certain cases can be summarized in an explanatory calculation, as previously shown. It would be tedious to devote ourselves to theories which are as complex as they are vague to intuitively resolve a problem with a Bayesian approach.

Events can be independent as well as incompatible. Since it is often easy in mathematics to represent incompatible events in a graph, we are often inclined to adopt the same approach for independent events. *A priori*, we believe that these two notions (independence and incompatibility) oppose each another. We have rarely come across simultaneity. For two *independent* and *incompatible* events, it would be advisable that one of the two events have a probability of zero (= 0). The mathematical formula for our hypothesis is as follows:

$$P(E_1/E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)} = P(E_1) \qquad [3.10]$$

Moreover, E_1 and E_2 are incompatible. Therefore, let $E_1 \cap E_2 = \emptyset$ and $P(E_1 \cap E_2)$, hence the independent condition will be:

$$P(E_1/E_2) = \left\{0/P(E_2)\right\} = P(E_1), \text{ logically } P(E_1) = 0$$

When would we have independent events in mechanical reliability?

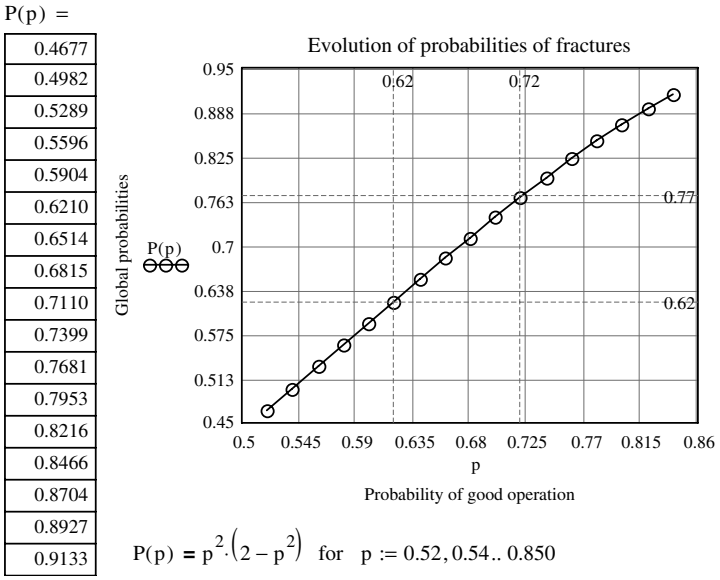


Figure 3.2. *Probability of good operation for four independent structures*

Let us express relation [3.10] using the empirical definition of probability of frequencies by this definition:

We know that:

$$P(E_1) = \left\{ \frac{\text{Number of structure elements for } E_1}{\text{Number of structure elements for } N_{total}} \right\}$$

Therefore:

$$P(E_1/E_2) = \left\{ \frac{\text{Number of structure elements for } E_1 \cap E_2}{\text{Number of structure elements for } /N_{total}} \right\} \qquad [3.11]$$

Expression [3.11] is now written as:

$$P(E_1/E_2) = P(E_1) \text{ hence } \frac{\text{Number of elements for } E_1 \cap E_2}{\text{Number of elements for } E_2/N_{total}} = \frac{\text{Number of elements for } E_1}{N_{total}} \qquad [3.12]$$

In this way, the proportion of elements represented by event (E_1) in event (E_2) relative to the number of elements of structures represented by (E_2) is identical to that of all the elements of E1 relative to the total of the entire structure represented

by the reference (Ω). In this case, it would be difficult to prove *a priori* that two events would be independent by intuition. Do we really need to do all this reasoning to be able to state that the independence of the structure's elements is physically indefensible on account of the bonds of the entire structure (e.g. offshore)? There are, however, confusions that must be cleared up. Here are three main definitions:

$$\left\{\begin{array}{l} \textit{Incompatible events}, \rightarrow E_1 \textit{ and } E_2 \textit{ where } E_1 \cap E_2 = \varnothing \textit{ et } P(\varnothing) = 0 \\ \textit{Incompatible events}, \rightarrow E_1 \textit{ and } E_2 \textit{ where } P\left[E_1 \cap E_2\right] = P(E_1) \times P(E_1) \\ \textit{Incompatible events}, \rightarrow E_1 \textit{ and } E_2 \textit{ where } P\left[E_1 / E_2\right] \neq P(E_2) \end{array}\right\} \quad [3.13]$$

3.2.2. *Counting diagram*

There are cases when counting is desirable. Let us take an example. Consider a bolted structure for transporting electrical energy produced by Hydro-Quebec that is composed of several levels. Welded cross-assemblies maintain the joints between components (Figure 3.1).

Problem of structures assembled by bolting

We noted that we could estimate the chances of ensuring a reliability represented by event E_1 to be *75%*. If this can be verified and ensured then the chances are two (*2 chances*) against one (1 chance), for the reliability of the entire structure. This event is therefore represented by E_2. However, if the first hypothesis is not ensured, then the reliability of the whole structure cannot be guaranteed (contract not ensured from functional specifications document). In the end, we would estimate the chances of the bolted structure remaining reliable to be 56%. What is the probability that the pylon completely breaks down and that there will be many harmful consequences?

Solution

According to the following diagram, there are four possible incompatibilities.

$$\left\{\begin{array}{l} 1.\ \textit{The two cases ensure reliability} \rightarrow \textit{we note } E_1 \cap E_2 \\ 2.\ \textit{Case } E_1 \textit{ ensures reliability but not } E_2 \rightarrow \textit{we note } E_1 \cap E_2^{'} \\ 3.\textit{Case } E_1 \textit{ does not ensure reliability but } E_2 \textit{ does} \rightarrow \textit{we note } E_1^{'} \cap E_2 \\ 4.\textit{None of case } E_1 \textit{ ensures reliability} \rightarrow \textit{we therefore note } E_1^{'} \cap E_2^{'} \end{array}\right.$$

The probability that does not ensure reliability is written as follows:

$$P\left[E_1^{'} \cap E_2^{'}\right] = P\left(E_1^{'}\right) \times P\left(E_2^{'} / E_1^{'}\right) = (0.25) \times (0.44) = 0.11 = (11/100)$$

Let us first plot the counting diagram as follows:

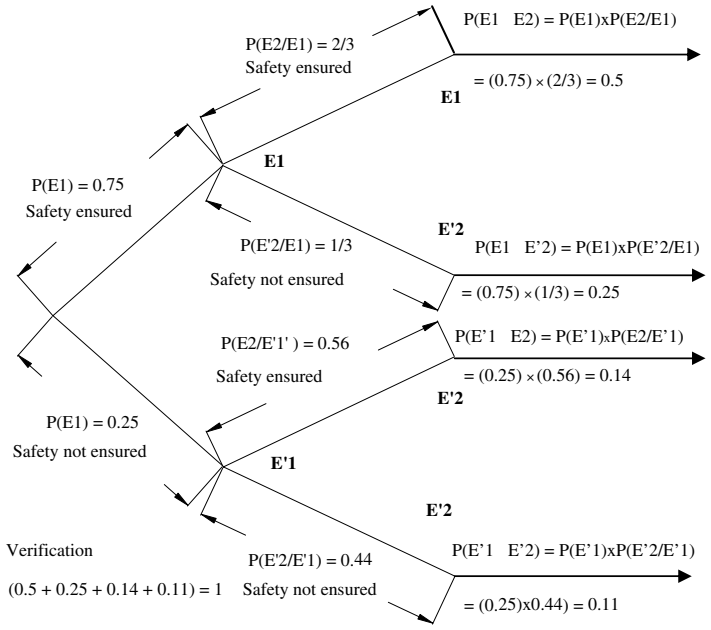


Figure 3.3. *Tree diagram of counting (a)*

The logic behind reasoning through counting is as follows:

In order for the reliability of two cases to be ensured, the probability is (1 – 0.11) = 89%. According to this counting diagram, there is an 89% chance that Hydro-Quebec's pylon will remain safe.

– Probabilities $P(E_1)$ and $P(E'_1)$ are *marginal*. The performance of (E_1) is not conditioned by other events.

– Probabilities $P(E_2/E_1)$, $P(E'_2/E_1)$, and $P(E'_2/E'_1)$ are *conditional probabilities.*

– Probabilities $P(E_1 \cap E_2)$; $P(E_1 \cap E'_2)$, $P(E'_1 \cap E_2)$, $P(E'_1 \cap E'_2)$, are *joint probabilities*.

We could also proceed with rules of elimination:

The performance of an event is often conditioned by the performance of another event. If a section is ruined (unbolted pylon) during an event, it is only logical that what follows will be conditioned by this weakness (welded cracking seat and dangerous section due to the ruined link by sudden unbolting). In the opposite case, the structure continues to ensure that energy is transported to the *Cheyennes*.

Problem

During design, we always assemble mechanical components. For example, of six (06) assembled parts, two (02) components turn out not to conform to mechanical bonding. Two components are therefore faulty. After the *hypothetical* assembly has taken place, the mechanism is verified component-by-component, until the two faulty components have been identified in the mechanism. Let us calculate the probability that verifications stops at the second test or at the third test? If the first component is reliable (not faulty), what is the probability that the following verifications stop at the third test?

Solution

Let us logically plot a tree diagram, shown in the following, which relates to what has been stated earlier:

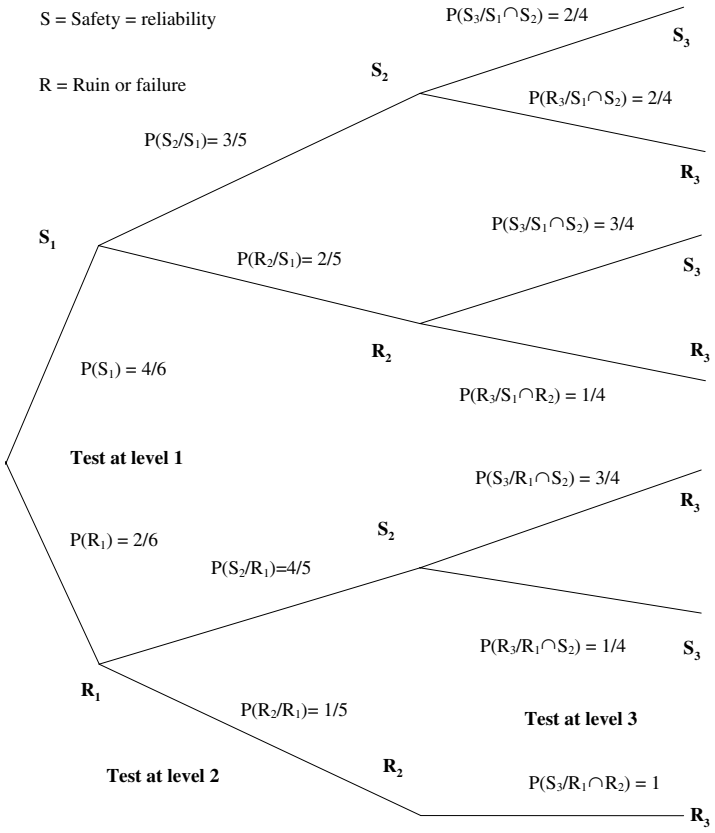


Figure 3.4. *Tree diagram of counting (b)*

1) The verification principle would be stopped at the second test if the two ruined components were *pulled*. Let us therefore identify this event by Ω when:

$$\Omega = R_1 \cap R_2 \text{ and } \mathrm{P}(\Omega) = P(R_1) \times P(R_2 | R_2) = \left(\frac{2}{6}\right) \times \left(\frac{1}{5}\right) = 0.067$$

2) The verification principle is stopped at the third test and the event is shown according to the following duality:

$S_1 \cap R_2 \cap R_3$ *et* $R_1 \cap R_2 \cap R_3 \rightarrow$, therefore Ω will be written as follows:

$\Omega = [S_1 \cap R_2 \cap R_3] \cup [R_1 \bigcap S_2 \cap R_3]$, which permits

$$P(\Omega) = P[S_1 \cap R_2 \cap R_3] + P[R_1 \cap S_2 \cap R_3]$$

Multiplication leads to the following relation:

$$P[S_1 \cap R_2 \cap R_3] = P(S_1) \times P(R_2 | R_1) \times P(S_3) \times P(S_1 | R_2)$$
$$= \left(\frac{4}{6}\right) \times \left(\frac{2}{5}\right) \times \left(\frac{1}{4}\right) = 0.067$$

Similarly:

$$P[S_1 \cap R_2 \cap R_3] = P(S_1) \times P(R_2 | R_1) \times P(S_3) \times P(S_1 | R_2)$$
$$= \left(\frac{2}{6}\right) \times \left(\frac{4}{5}\right) \times \left(\frac{1}{4}\right) = 0.067$$

It can be concluded as follows: $P(\Omega) = (0.067) + (0.067) = 0.133$.

3) The principle of verification is stopped at the third test, when the first component is reliable. This fact corresponds to Ω/S_1 when Ω has been previously defined in response 1). The corresponding probability is therefore written as follows:

$$P(\Omega | S_1) = \frac{P(\Omega \cap S_1)}{P(S_1)} = \frac{P[S_1 \cap R_2 \cap R_3]}{P(S_1)} = \left(\frac{0.067}{4/6}\right) = 0.101 \cong \left(\frac{1}{10}\right)$$

The above problem is a simple example of probability in mechanical design. It is worth pointing out that many case studies on probability have been carried out.

With respect to pedagogy, we would like users (technical schools) to make use of probability approaches alongside concrete cases during their studies. Unfortunately

some classes on probability and statistics are often confined to abstract theories which are off-putting, since they are not followed up with an application by means of explanation. And then the time comes for concrete projects and initiation to design and production.

3.3. Conditional probability or Bayes' law

In the 19th Century (in Scotland), Reverend Thomas Bayes gave a definition of conditional probability which allows us to express the rules of multiplication and elimination. This theorem now has different applications in engineering sciences. Several case studies will be presented here in an attempt to popularize, in the most concise way possible, Bayes' law. Formulating the behavior of the welded cross-structure by classic laws of fracture mechanics has allowed us to link the essential dependence parameters of the Paris–Erdogan law, specifically c and m. If we designate the number of cycles to fracture by N and the relation between nominal stress S ($\Delta\sigma$) and the number of cycles to fracture (N) by a law on fatigue [BAS 60], this relation is naturally expressed as follows:

$$N = \Delta K \times S^m \qquad [3.14]$$

where:

$m = 3$ is an intrinsic coefficient of the material, adimensional;

ΔK is the stress intensity factor (*s.i.f.*) in MPa $m^{3/2}$;

S(N) corresponds to the nominal stress in Mpa.

$\Delta\sigma_i =$ $N_i =$ $i := 1..6$

	0
0	200
1	98
2	58
3	35
4	25
5	25

	0
0	$1 \cdot 10^4$
1	$1 \cdot 10^5$
2	$1 \cdot 10^6$
3	$1 \cdot 10^7$
4	$1 \cdot 10^8$
5	$1 \cdot 10^9$

Wöhler's (S–N) curve

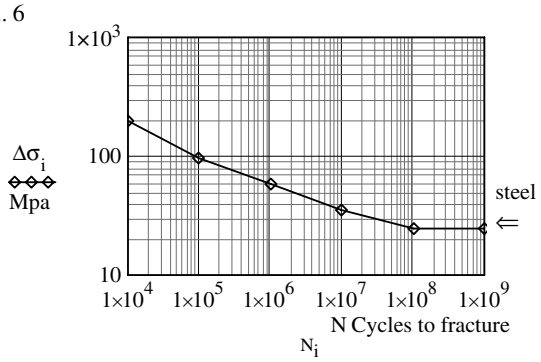


Figure 3.5. *S-N curve (or Wöhler's curve)*

Taking into account the dispersion of experimental points, we will make reference to statistical methods, including the following. Through experience we have deduced that for an insufficient number of experimental data, we end up with a Galton distribution (log-normal, see Chapter 1, Volume 1), naturally in the domain of limited endurance of fractures at level σ_a. The law becomes Gaussian (normal) when N varies. This is a normal phenomenon of cumulative wear.

In 1972, Jack and Price [JAC 72] showed that for a sheet of width ≥20 mm and of thickness >5 mm, the number of activation cycles is a simple dependent function of the *s.i.f.* For example, for mild-steel these authors parameterized the relation as follows:

$$N_a = \frac{2.9\times10^8}{(\Delta K)^4} \qquad [3.15]$$

We retrieved this relation from literature to check the way our simulated plot looked and then to be able to compare it to these authors' results. At our last observation, we found an excellent similarity. This allows us to continue the development as follows:

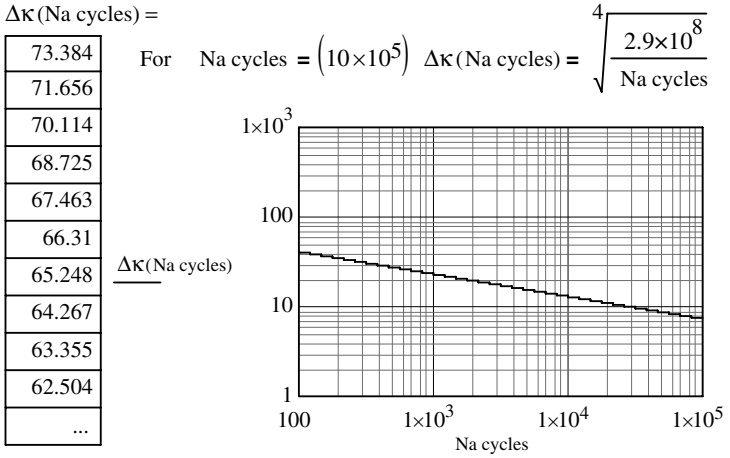


Figure 3.6. *Evolution of the s.i.f. in function of N_{cycles} for activation N_a*

Under a given stress (S = σ_{max} = 200 MPa), we subject a series of test pieces to stress cycles. The number N of cycles required to provoke the fracture of the test piece ($N = N_f$) is then measured. The test is repeated at different stress amplitudes $\sigma_{applied}$. The plot on the semi-log scale, from the curve S = σ_a = f(N), is as shown in

the following figure (mild-steel). Wöhler's curve shows the resistance of a material in accordance with the mode of stress. It is an experimental relation between the maximum stress S = σ_{max} and the number of cycles N. Wöhler's curve corresponds to the median value of N (50% survival at a given stress).

The method of obtaining results is as follows:

Figure 3.5 shows the results (S*i*, N*i*), i = 1, 2 … *r*, for each typical experiment. The results are often represented by transforming (x,y) in (*Log*S, *Log*N). Let $x = Ln(S)$ and $y = Ln(N)$. The model of normal regression and linear regression with a constant variance is formulated for the independent variable x and the dependent variable y. The value of *Y*, which can be read as $X = x_i$, is written as follows:

$$\left(Y_i / X\right) = x = \alpha - \beta\left[x_i - \mu\right] + \sigma \times U_i \qquad [3.16]$$

U_i is a standard normal variable, written as follows: $U_i \in$ N [0, 1]. The average values and the variance of $Y_i/X_i = x_i$ are written as follows:

$$\left\{ \begin{array}{l} Mean = E\left(\dfrac{Y_i}{X} = x_i\right) = \alpha - \beta\left[x_i - \mu\right] \\ Mathematical\ variance = VAR\left(\dfrac{Y_i}{X} = x_i\right) = \sigma^2 \\ Average\ of\ x\_values = \left(\dfrac{1}{\kappa}\right) \times \sum_{i=1}^{\kappa} x_i \end{array} \right\} \qquad [3.17]$$

For the limited number of statistical tests, parameters α, β, and σ cannot be determined with any degree of certainty. Moreover, they can be considered as random variables of parameters A, B, and Σ. The respective laws of A, B, and Σ are obtained once the statistical tests are known. The respective distributions are obtained by a Bayesian theorem developed by D.V. Lindley [LIN 65] in 1965. It can be summarized as follows:

$$_{AB\Sigma}\frac{\left(\alpha,\beta,\sigma\right)}{\left(x_1,...,x_\kappa;y_1,...y_\kappa\right)} = C \times f_{(Y_1,...,Y_\kappa)}\frac{\left(y_1,...y_\kappa\right)}{\left(x_1,...,x_\kappa,\alpha,\beta,\sigma\right)} \times f_{AB\Sigma}\left(\alpha,\beta,\sigma\right) \qquad [3.18]$$

The left-hand side represents the probability density function, according to the Lindley–Bayes' law. It is proportional to the product of the most probable function and the first probability density function. The constant C is proportional to a normalized factor. Random variables U_i are considered to be mutually independent and each corresponds to their respective statistical test. The probable function is therefore written by the following relations:

$$f_{(Y_1,\ldots Y_\kappa)}\frac{(y_1,\ldots y_\kappa)}{(x_1,\ldots,x_\kappa,\alpha,\beta,\sigma)}=\frac{f_{Y_1}(Y_1)}{(x_1,\alpha,\beta,\sigma)}\times\frac{f_{Y_2}(Y_2)}{(x_2,\alpha,\beta,\sigma)}\times\ldots \times\frac{f_{Y_\kappa}(Y_\kappa)}{(x_\kappa,\alpha,\beta,\sigma)}=\prod_{i=1}^{\kappa}\frac{1}{\sigma}\times\varphi\left(\frac{y_i-\alpha+\beta(x_1-\mu)}{\sigma}\right) \quad [3.19]$$

Let us calculate this similarity and let:

$$f_{(Y_1,\ldots Y_\kappa)}\frac{(y_1,\ldots y_\kappa)}{(x_1,\ldots,x_\kappa,\alpha,\beta,\sigma)}\mu\frac{1}{\sigma^2}Exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{\kappa}\left[y_i-\alpha+\beta(x_1-\mu)\right]^2\right\} =\frac{1}{\sigma^2}Exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{\kappa}\left[D^2+\kappa(a-\alpha)^2+S_{xx}(b-\beta)^2\right]^2\right\} \quad [3.20]$$

When $\varphi(\cdot)$ is the standard normal random variable of the probability density function, which is proportional to the averages, then $\propto$ is similar. The constants a, b, and S_{xx}, as well as D are determined by their respective expressions which are written as follows:

$$\left\{\begin{array}{l} a=(\bar{y})=\dfrac{1}{\kappa}\displaystyle\sum_{i=1}^{\kappa}y_i;\quad b=\frac{S_{xy}}{S_{yy}}=\frac{\displaystyle\sum_{i=1}^{\kappa}\left(x_iy_i-\frac{r}{\bar{x}_i\bar{y}_i}\right)}{\displaystyle\sum_{i=1}^{\kappa}\left(y_i^2-\frac{r}{y^2}\right)} \\ S_{xx}=\displaystyle\sum_{i=1}^{\kappa}(x_i-\mu_i)^2=\sum_{i=1}^{\kappa}\left(x_i^2-\frac{r}{\mu^2}\right) \\ S_{yy}=\displaystyle\sum_{i=1}^{\kappa}(y_i-\bar{y}_i)^2=\sum_{i=1}^{\kappa}\left(y_i^2-\frac{r}{y^2}\right) \\ S_{xy}=\displaystyle\sum_{i=1}^{\kappa}(x_i-\mu_i)\times(y_i-\bar{y}_i)=\sum_{i=1}^{\kappa}\left(x_iy_i-\frac{r}{\bar{x}_i\bar{y}_i}\right) \end{array}\right\} \quad [3.21]$$

$$D^2=S_{yy}-\frac{S_{xy}^2}{S_{xx}}=\sum_{i=1}^{\kappa}\left(y_i^2-\frac{r}{y^2}\right)-\frac{\displaystyle\sum_{i=1}^{\kappa}\left(x_iy_i-\frac{r}{\bar{x}_i\bar{y}_i}\right)^2}{\displaystyle\sum_{i=1}^{\kappa}\left(x_i^2-\frac{r}{\mu^2}\right)} \quad [3.22]$$

Many propositions have been made to represent the iso-probability curve of fracture. The plot of Wöhler's curve (Europe), which is the equivalent to S-N curves

in Canada and the USA, requires an experimental program, which is sometimes very costly. Some S–N curves can be found, however, that are dedicated to certain materials.

Expression	Suggested equation	Parameters[a]
Wöhler (1870)	$Log(N) = a - (b \times S)$	a and b
Basquin (1910)	$Log(N) = a - \{b \times Log(S)\}$	a and b
Strohmeyer (1914)	$Log(N) = a - \{b \times Log(S - S_e)\}$	a and b
Palmgren (1924)	$Log(N + B) = a - \{b \times Log(S - S_e)\}$	a, b, and B
Corson (1955)	$N = \frac{A}{(S - S_e) \times d^{(S - S_e)}}$	a and d
Weibull (≈1955)	$Log(N + B) = a - \left\{\frac{b \times Log(S - S_e)}{(S_u - S_e)}\right\}$	a, b, and B
Etc.	*See Chapter 1, Volume 2*	

[a]Constants adjusted to each material considered.

Table 3.2. *Some important equations for the iso-probability of fracture*

3.4. Anterior and posterior distributions

The probability density function, stated earlier by H.O. Madsen's [MAD 84], corresponds for A, B, Σ to the mutually independent parameters in the following proportions:

$$f_{AB\Sigma}(\alpha, \beta, \sigma) = f_A(\alpha) \times f_B(\beta) \times f_\Sigma(\sigma) \qquad [3.23]$$

According to H.O. Madsen, selecting the anterior density function is bound by a representation of the knowledge of it and the desire to have a simple density function, said in turn to be posterior, which is in fact a joint probability density function. The latter completes the conjugate distribution function. Knowing anterior parameters (A, B, Σ) concerns, in particular, offshore structures with cross welding, assembled in *series*.

Most information is still related to parameter A. Consequently, it is sufficient to limit ourselves to it. The value of parameter B is not completely known in advance, but can be generally determined by Ditlevsen's [DIT 79, 81] bound theory, among others, if the choice of anterior value A remains inconclusive. At the end of statistical calculations, it is worth carefully checking the chosen hypothesis by a conformity statistical test, using for example Pearson's confidence interval $\{m_1, m_2\}$ or (KS) Kolmogorov–Smirnov's confidence interval (see Chapter 2, Volume 1).

Standard deviation (Σ) is always positive. When numerous tests in *series* can be polarized around the average of S, we can sometimes consider them known, for which a theoretical demonstration is as follows.

f(A, B, Σ), the anterior probability density function can be known as follows:

$$f_{AB\Sigma}(\alpha,\beta,\sigma) \propto \left(\frac{1}{\sigma}\right) \text{ with } \sigma \succ 0 \text{ and } m_1 \le \beta \le m_1 \quad [3.24]$$

When $(1/\sigma)$ represents the asymptotic limit of the density form of the anterior sequences of (A, Σ), which are approached by uniform probability densities (α, Lnσ) planar. The posterior distribution function is therefore written as follows:

$$\left\{\begin{array}{l} \text{With } \sigma \succ 0 \text{ and } m_1 \le \beta \le m_1 \rightarrow f_{AB\Sigma} \dfrac{(\alpha,\beta,\sigma)}{(x_1,...,x_\kappa,y_1,...,y_\kappa)} \propto \dfrac{1}{\sigma^{\kappa+1}} \times \\ \qquad \times Exp\left\{-\dfrac{1}{2\sigma^{\kappa+1}}\left[D^2+\kappa(a-\alpha)^2+S_{xx}(b-\beta)\right]^2\right\} \end{array}\right\} \quad [3.25]$$

The joint posterior distribution is often complicated to find. The simplified version of this problem will be done by introducing the random variables (T_1, T_2, and T_3) which in turn will be calculated as follows:

$$\left\{T_1 = \left(\frac{\sqrt{\kappa}}{\Sigma}\right)\times(A-a);\ T_2 = \left(\frac{\sqrt{S_{xx}}}{\Sigma}\right)\times(B+b) \text{ and } T_3 = \left(\frac{D}{\Sigma}\right)^2\right\} \quad [3.26]$$

By introducing variables (T_1, T_2, and T_3), the posterior distribution function will take the form:

$$\left\{\begin{array}{l} f_{T_1T_2T_3}(t_1t_2t_3) \ \propto\ Exp\left(-\dfrac{1}{2}t_1^2\right)\times Exp\left(-\dfrac{1}{2}t_2^2\right)\times t_3^{\left(\frac{\kappa-2}{2}\right)}\times Exp\left(-\dfrac{1}{2}t_3\right) \text{ with} \\ -\infty \prec t_1 \prec +\infty\ ;\ 0 \le t_3 \le +\infty\ ;\ \dfrac{\sqrt{S_{xx}\cdot t_3}\times(m_1+b)}{D} \prec t_2 \prec \dfrac{\sqrt{S_{xx}\cdot t_3}\times(m_2+b)}{D} \end{array}\right\} \quad [3.27]$$

From equation [3.26] it appears that T_1 is independent of T_2 and T_3 with:

$$T_1 \in N[0,1] \text{ and } T_1 \in \chi^2_{(\kappa-1)} \rightarrow \text{This is Pearson's adequacy test } \chi^2.$$

This equation expresses Pearson's test at (κ − 2) *degrees of freedom* (*dof*). The conditional probability density function of T_2 on T_3 is a normal truncated distribution, presented as follows:

$$\left\{ \begin{array}{l} f_{T_2}\left(t_2 \left| t_3 \right.\right) = \dfrac{\Phi\left(t_2\right)}{\Phi\left(\dfrac{\sqrt{S_{xx}t_3}\times(m_2+b)}{D}\right) - \Phi\left(\dfrac{\sqrt{S_{xx}t_3}\times(m_1+b)}{D}\right)} \\ \quad with \left(\dfrac{\sqrt{S_{xx}t_3}\times(m_1+b)}{D}\right) \prec t_2 \prec \left(\dfrac{\sqrt{S_{xx}t_3}\times(m_2+b)}{D}\right) \end{array} \right\} \quad [3.28]$$

If the "anterior" distribution of (B) is widespread, that is to say that $m_1 = -\infty$ and $m_2 = +\infty$, knowing that T_2 and T_3 are independent and T_2 has a standard normal distribution (tabulated reading), the value of (B) is fixed for $m_1 = m_2 = m$ and T_1 has a standard normal distribution which is independent of T_3, which is defined as follows:

$$T_3 = \frac{D_1^2}{\Sigma^2} \in \chi^2_{(\kappa-1)} \quad [3.29]$$

where the constant D_1 will be defined by the following relation:

$$D_1^2 = D^2 + S_{xx}\left(b+m\right)^2 \quad [3.30]$$

The dependent parameters of the materials, for example the m coefficients of Paris–Erdogan's behavior law of fatigue and the stress intensity factor (ΔK), can be expressed through Bayesian theory [MAD 84] using the relations:

$$m = B = -b + \left(\frac{D\times T_2}{\sqrt{S_{xx}\cdot T_3}}\right) \quad [3.31]$$

$$\Delta K = Exp\left(A + B\bar{x} + \Sigma U\right) = Exp\left(a + \frac{DT_2}{\sqrt{\kappa\cdot T_3}} + \left(-b + \frac{DT_2}{\sqrt{S_{xx}\cdot T_3}}\right)\right)\bar{x} + \frac{DU}{\sqrt{T_3}} \quad [3.32]$$

$T_2 = 0$ and D will therefore be replaced by D_1 if the value of B is fixed. U is initially considered a standard normal distribution and therefore a tabulated value. The number of cycles to fracture is expressed by the following relation:

$$N_{cycles} = Exp\left(a + \frac{DT_1}{\sqrt{\kappa\cdot T_3}} + \left(-b + \frac{DT_2}{\sqrt{S_{xx}\cdot T_3}}\right)\right)\bar{x} + \frac{DU}{\sqrt{T_3}} \quad [3.33]$$

T_1 and U are standard normal variable functions, independent of T_2 and T_3. T_3 follows one of the Pearson's laws χ^2 defined by relation [3.29] with $(\kappa - 2) = 6 - 2 = 4$ *dof*, because we have hypothesized two parameters, specifically T_2 and T_3. T_2 follows a normal truncated law.

3.5. Reliability analysis by moments methods, FORM/SORM

We have, until now, designated reliability by the probability of ruin P_F or sometimes by its reliability index β_F, which is expressed by $\beta_F = \phi^{-1}(P_F)$ and which remains random. Reliability is therefore expressed by the distribution function of the N_{cycles} as follows:

$$P_F = P\{N \succ n\} = 1 - F_N(n) \qquad [3.34]$$

when n is the number of cycles applied to the welded cross-structure. In terms of S, U, T_1, T_2, and T_3, fracture is expressed by the following relation:

$$g(S,U,T_1,T_2,T_3) = \left(a + \frac{DT_1}{\sqrt{\kappa T_3}}\right)\left(-b + \frac{DT_2}{\sqrt{S_{xx}T_3}}\right)\left(\bar{x} - LnS\right) + \frac{DU}{\sqrt{T_3}} - Ln \le 0 \qquad [3.35]$$

when the logarithm of the N_{cycles} is taken into account by relation [3.38], function g is a state limit function and (S, U, T_1, T_2, and T_3) are base variables which express the behavior law. Reliability is taken into account by the first- or second-order moment, traditionally known as FORM and SORM, respectively. The first order consists of transforming, step-by-step, the base variables S, U, T_1, T_2, and T_3 across five important stages in standard normal variables U_1, U_2, U_3, U_4, and U_5. Rosenblatt's [ROS 52] transformation was introduced by Hohenbichler and Rackwitz [HOH 82, 87a, 87b] with the following mode:

$$\left\{\begin{array}{l} U_1 = \Phi^1\left\{F_s(S)\right\};\ U_2 = \Phi^{-1}\left\{F_u\left(\dfrac{U}{S}\right)\right\} = \mathrm{U};\ U_3 = \Phi^{-1}\left\{F_{T_1}\left(\dfrac{T_1}{S,U}\right)\right\} = T_1 \\ U_4 = \Phi^{-1}\left\{F_{T_3}\left(\dfrac{T_3}{S,U,T_1}\right)\right\} = \Phi^{-1}F_{T_3}(T_3); \\ U_5 = \Phi^{-1}\left\{F_{T_2}\left(\dfrac{T_2}{S,U,T_1,T_3}\right)\right\} = \Phi^{-1}F_{T_2}\left(\dfrac{T_2}{T_3}\right) \end{array}\right\} \qquad [3.36]$$

3.6. Control margins from the results of fracture mechanics

There are two types of fracture control to consider:

$$\left\{a\left(N_i\right) \leq A_{di},\ i=1,\ 2,\ldots,\ r\ \ et\ \ a\left(N_j\right)=A_j,\ i=1,\ 2,\ldots,\ s\right\} \qquad [3.37]$$

In the first case, there is no fracture at control after N_i number of cycles to fracture, which results in a crack size so small and detectable with A_{di}. The term A_{di} is generally random, because of the fact that the fracture is detectable with a certain dependent probability of the crack size by the method of control used. The results [KIR 88, MAD 87] and models of direct control are only effective if we calculate "as we should" the statistical distributions of variables and parameters of the behavior law and also if we respect the conditions of the physical and chemical environment. Their papers [KIR 88, MAD 87] emphasize the diagrams at sea (in the North sea), the transfer functions, and the relating *s.i.f.*

Weibull's distribution of two parameters, represented by β and η, is used to model specific events. Therefore, let:

$$P\left(F|U\right)=\frac{P\left(F\cap U\right)}{P\left(U\right)} \qquad [3.38]$$

where U is an event of any case and $P(F/U)$ a conditional probability when case F is known. For example, if F is a reliability component of the structure and U the control event, we will write the fracture probability by the sum of the control cases P(F/U). If events *F and U are independent*, P = F ∩ U, which is the product of the probability of events F and U.

In this case, $P(F/U) = P(F)$. Furthermore if events *F and U are dependent*, estimating the probability of fractures will be given by the control event U and $P(F/U)$ remain different to probability P(F) outside of inspection. The result of control is therefore given by relations [3.47], in accordance with the previously stated hypotheses.

The distribution of the A_{ji} is a distribution where we could detect effective cracks, that is visible cracks which can be measured immediately by metrology. It is known by its statistical distribution law, during the inspection method, according to Madsen's theory [MAD 87].

In the second case of relation [3.47], the crack size A_j (or a_0) is visible after N_j number of cycles to fracture. It is generally random owing to the errors or unknown

factors due to uncertainties in measuring the size, or even of the signal emitted by the crack (mark in the form of spectrum if it is an ultrasound recording). In the same vein of analysis, it is also worth considering the event Mi as a similar margin to a safety margin which is written as follows:

$$M_i = C \times N_i \times A^m \times \Gamma\left(1+\frac{m}{B}\right) - \int_{a_0}^{A_{di}} \frac{dx}{Y^m \times \left(\sqrt{\pi \cdot x}\right)^m} \leq 0 \quad i = 1,2,...,r \qquad [3.39]$$

According to the first relation of [3.38], M_i is considered negative and for measuring the event as defined by the second relation of [3.38], M_j is also compared to a safety margin which takes the following form:

$$M_j = \int_{a_0}^{A_j} \frac{dx}{Y^m \times \left(\sqrt{\pi \cdot x}\right)^m} - C \times N_j \times A^m \times \Gamma\left(1+\frac{m}{B}\right) = 0 \quad j = 1,2,...,s \qquad [3.40]$$

These safety margins are zero, as the second relation of [3.40] indicates. In this case, the crack is still not detected in R inspections. Probability P_F^U is therefore calculated by the following relation:

$$P_F^U = P\left\{M \leq 0 \middle| M_1 \leq 0 \cap M_2 \leq 0 \cap ... \cap M_2 \leq 0\right\} \qquad [3.41]$$

This expression can be resolved for a reliability problem where the structure is assembled in a parallel system with other structures. It is discussed theoretically in this section. Expression [3.41] is used by a reliability system in the form of [3.42] where the numerator and the denominator are two terms for the indicative ratio for the probability of fracture (ruin).

$$P_F^U = \frac{P\left\{M \leq 0 \cap M_1 \leq 0 \cap ... \cap M_r \leq 0\right\}}{P\left\{M_1 \leq 0 \cap ... \cap M_r \leq 0\right\}} \qquad [3.42]$$

This situation is considered when the crack is not detected at first inspection. At the moment when the crack is detected at the (r + 1)th inspection and its size can be measured by conventional metrology, and so on until (s − 1) inspections. The probability of a new fracture is therefore shown as follows:

$$P_F^U = P\left\{M \leq 0 \middle| M_1 \leq 0 \cap M_r \leq 0 \cap M_{r+1} ... \cap M_{r+s} = 0\right\} \qquad [3.43]$$

These reliability techniques [MAD 87] are easy and “neat” in application; however, they are still useful in the case of systems in parallel and in series, as

developed by Ditlevsen [DIT 79, 81]. In addition to the inspection methods explained here, it is important to remember that new probabilities of ruin can occur P_F^U. When repair is carried out at time N_r (number of cycles to fracture) the length of the crack a_r is measured. The event margin M_r is therefore defined by the following expression:

$$M_r = \int_{a_0}^{a_r} \frac{dx}{Y^m \times \left(\sqrt{\pi \cdot x}\right)^m} - C \times N_r \times A^m \times \Gamma\left(1 + \frac{m}{B}\right) = 0 \qquad [3.44]$$

The size of the crack after repair and final inspection is expressed with a new random variable a_{0nv}. The intrinsic properties of the materials from the Paris–Erdogan law after repair will be represented by m_{nv} and C_{nv}. These variables can be considered original if physical influence and, therefore, mathematical statistics are not very consistent. They can also change, as for example in the case of continuum mechanics (fatigue) in offshore. Consequently, it is worth reconsidering the original hypothesis of calculations by fracture mechanics by fatigue. In any case, the new safety margin will be calculated by the new expression:

$$M_{nv} = \int_{a_{0,nv}}^{a_{critical}} \frac{dx}{Y^{m_{nv}} \times \left(\sqrt{\pi \cdot x}\right)^{m_{nv}}} - C_{nv} \times \left(N - N_{fracture}\right) \times A^{m_{nv}} \times \Gamma\left(1 + \frac{m_{nv}}{B}\right) = 0 \qquad [3.45]$$

The updated failure probability after repair will be calculated by the following relation:

$$P_F^U = P\left\{M_{new} \le 0 \middle| M_{repaired} = 0\right\} \qquad [3.46]$$

This expression is exactly identical to the fracture probability calculated by the Bayesian theory of conditional probabilities (see also [3.43]), presented at the beginning of this chapter. The reliability index (β) could therefore be calculated by the expression we have already seen. Here is a reminder:

$$\beta_R = -\Phi^{-1}\left(P_F\right) \text{ and } P_F = P\left(M \le 0\right);\ \text{also } \beta_R \approx \beta \text{ with } \beta = \left|U^*\right| \to \infty$$

The generalization of this solution to parallel systems is written as follows:

$$\beta_R = -\Phi^{-1} \cdot \Phi\left[-\beta; \rho\right] \text{ et } \left|U^*\right| \to \infty \qquad [3.47]$$

3.7. Bayesian model by exponential gamma distribution

On account of its constant failure rate, exponential distribution is an excellent "intrinsic failure" model, relative to the long and flat part of the bath curve (see Figure 3.6). Since most components and systems pass most of their lifetime in this part of the bath curve, using exponential distribution is justified (in a case of early failure).

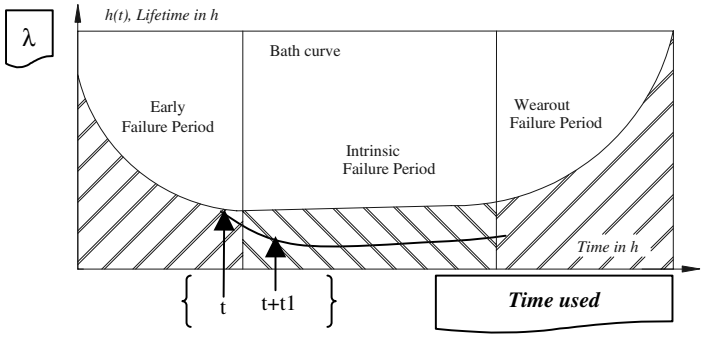


Figure 3.7. *Bath curve (classic reliability)*

To approach a curve in segments of straight lines, it is often useful to estimate a curve by its failure rate. Period after period, the failure rate remains constant. In this way, we can estimate any exponential distribution model by segments (pieces) that are stuck together. The exponential mode, with a single unknown parameter, is the most simple of all lifetime distribution models. The equations on exponential distributions have been presented at length in Chapter 1. As a reminder, the following are the main points:

– $Average: \mu = \frac{1}{\lambda}$; $Median: \mu = \frac{\ln 2}{\lambda} \simeq \left(\frac{0.693}{\lambda}\right)$; $Variance: VAR = \frac{1}{\lambda^2}$.

Standard deviation $= \sigma = \sqrt{VAR} = \frac{1}{\lambda}$, and $failure\ rate: h(\tau) = \lambda = Cste$:

– Probability density function $F(\tau) \rightarrow F(\tau) = 1 - \exp^{-\lambda\tau}$;

– Distribution function $f(\tau) \rightarrow f(\tau) = \lambda \times \exp^{-\lambda\tau}$;

– Reliability $R(\tau) \rightarrow R(\tau) = \exp^{-\lambda\tau}$.

Exponential distribution is the only distribution to have a constant failure rate $h(\tau) = (\lambda)$. It is also worth pointing out that the average of this law represents the mean time to fail $MTTF = 1/\lambda$.

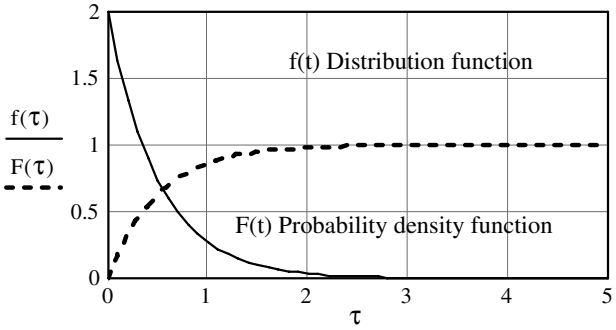


Figure 3.8. *Distribution functions f(τ) and probability density F(τ)*

Exponential models (flat part of the bath curve) represent a large part of the lifetime of systems. The reliability functions $R(\tau)$ and the failure rate $h(\tau)$ are represented below. The risk function represents the integral of the failure rate $H(t) = (\lambda)\tau$. The relative graphs are as follows:

h(τ) =	R(τ) =
2	1
2	0.819
2	0.67
2	0.549
2	0.449
2	0.368
2	0.301
2	0.247
2	0.202
2	0.165
2	0.135
2	0.111
...	...

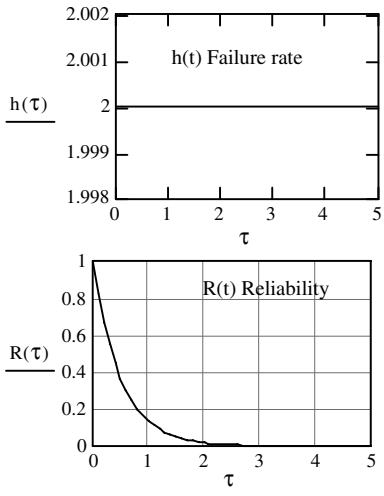


Figure 3.9. *Reliability functions R(τ) and failure rate h(τ)*

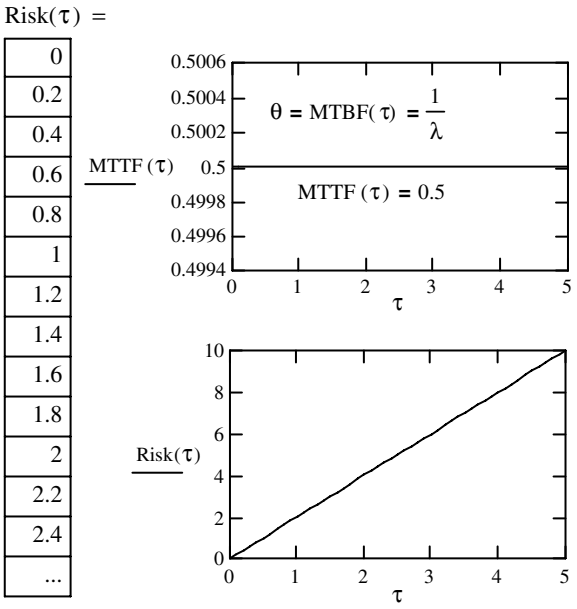


Figure 3.10. *Reliability functions: MTTF(τ) and risk Risk(τ)*

3.8. Homogeneous Poisson process and rate of occurrence of failure

This model is used for independent breakdowns with a failure rate λ. Function F(t) = 1 − Exp [$-\lambda\tau$] is the probability function of inter-arrival time between failures. $N(T)$ are the accumulated failures between 0 and the time T. The probability is written as follows:

$$P\left\{N\left(\tau\right)=\kappa\right\}=\frac{\left(\lambda\times T\right)^{\kappa}}{\kappa!}\times Exp\left\{-\lambda.T\right\} \qquad [3.48]$$

M(T) = (λT) is the number of expected failures after the time (T).

M′(T) = m(τ) = λ is therefore the repair rate by units of time or by rate of occurrence of failure (τ) (ROCOF) at time (τ). The MTBF (mean time before failures) or the average time of good operation is MTBF = 1/λ. Estimating the MTBF can be determined by using upper and lower confidence limits. Most of the systems spend their useful lifetimes at the flat part of the bath curve. It is quite easy to plan tests and to calculate the confidence intervals of an exponential model.

3.9. Estimating the maximum likelihood

The method of maximum likelihood is a powerful method used for large samples. Estimation of the maximum likelihood is based on the *likelihood function* of the sample data. The probability of a group of data is the probability of obtaining this particular group of data. This expression contains the unknown parameters of the model. The values of these parameters, which maximize the likelihood of the sample, are known as the *maximum likelihood estimates* (*MLE*). The estimation of the maximum likelihood is an completely analytical maximization procedure. It is applied to all data, censored or multicensored.

Examples of the likelihood function for reliability data: Let $f(\tau)$ be the distribution function and $F(\tau)$ the probability density from the distribution model chosen to represent lifetime. They are functions of τ and the unknown parameters of the model. The likelihood function is as follows:

$$L = C \times \prod_{i=1}^{r} f\left(\tau_i\right) \times \left[1 - F\left(T\right)\right]^{n-r} \qquad [3.49]$$

where C is a constant. Without censor, probability is reduced to the product of densities, each evaluated at a failure time. For type II censored data, replacing T by τ_r is sufficient. The likelihood function is written as follows:

$$L = C\left(\prod_{i=1}^{\kappa} F\left(T_i - F\left(T_{i-1}\right)\right)^{r_i}\right) \times \left(1 - F\left(T\right)\right)^{n-\sum_{i=1}^{\kappa} r_i}, \text{ when } F(T_0) \text{ is defined at 0.}$$

3.9.1. *Type I censored exponential model*

$$L = \left(C \times \lambda^r\right) \times Exp\left(-\lambda \sum_{i=1}^{r} \tau_i\right) \times Exp\left[-\lambda\left(n-r\right)T\right] \quad \ln L = \ln C + r \ln \lambda - \lambda\left(n-r\right)T$$

$$\frac{\partial \ln L}{\partial \lambda} = \frac{r}{\lambda} - \sum_{i=1}^{r} \tau_i + \left(n-r\right)T = 0 \text{ hence} \rightarrow \hat{\lambda} = \frac{r}{\sum_{i=1}^{r} \tau_i + \left(n-r\right)T} \qquad [3.50]$$

3.9.2. *Estimating the MTBF (or rate of repair/rate of failure)*

The best estimation of the MTBF is the "total time", divided by total "failures".

$$MTBF = \left(\frac{Total\ time\ of\ operations}{Total\ number\ of\ failures}\right) \rightarrow \hat{\lambda} = \left(\frac{Total\ number\ of\ failures}{Total\ time\ of\ operations}\right)$$

3.9.3. *MTBF and confidence interval*

Reading Tables 3.3 through to 3.5 from literature [POR 98], [TOB] is useful to estimate the MTBF with the help of confidence intervals. The multiplying factors of the confidence interval for the estimation of the MTBF are presented as follows:

Failure number, *r*	60%		80%	
	LI_ MTBF	LS_ MTBF	LI_ MTBF	LS_ MTBF
0	0.6213	–	0.4343	–
1	0.3340	4.4814	0.2571	9.4912
2	0.4674	2.4260	0.3758	3.7607
3	0.5440	1.9543	0.4490	2.7222
4	0.5952	1.7416	0.5004	2.2926
5	0.6324	1.6184	0.5391	2.0554
6	0.6611	1.5370	0.5697	1.9036
7	0.6841	1.4788	0.5947	1.7974
8	0.7030	1.4347	0.6156	1.7182
9	0.7189	1.4000	0.6335	1.6567
10	0.7326	1.3719	0.6491	1.6074
11	0.7444	1.3485	0.6627	1.5668
12	0.7548	1.3288	0.6749	1.5327
13	0.7641	1.3118	0.6857	1.5036
14	0.7724	1.2970	0.6955	1.4784
15	0.7799	1.2840	0.7045	1.4564
20	0.8088	1.2367	0.7395	1.3769
25	0.8288	1.2063	0.7643	1.3267
30	0.8436	1.1848	0.7830	1.2915
35	0.8552	1.1687	0.7978	1.2652
40	0.8645	1.1560	0.8099	1.2446
45	0.8722	1.1456	0.8200	1.2280
50	0.8788	1.1371	0.8286	1.2142
75	0.9012	1.1090	0.8585	1.1694
100	0.9145	1.0929	0.8766	1.1439
500	0.9614	1.0401	0.9436	1.0603

Source: http://www.itl.nist.gov/div898/handbook

Table 3.3. *Multiplying factors of the confidence interval (estimation MTBF)*

Multiplying factors of the confidence interval for the estimation of the MTBF:

Failure number, r	90%		95%	
	LI_ MTBF	LS_ MTBF	LI_ MTBF	LS_ MTBF
0	0.3338	–	0.2711	–
1	0.2108	19.4958	0.1795	39.4978
2	0.3177	5.6281	0.2768	8.2573
3	0.3869	3.6689	0.3422	4.8491
4	0.4370	2.9276	0.3906	3.6702
5	0.4756	2.5379	0.4285	3.0798
6	0.5067	2.2962	0.4594	2.7249
7	0.5324	2.1307	0.4853	2.4872
8	0.5542	2.0096	0.5075	2.3163
9	0.5731	1.9168	0.5268	2.1869
10	0.5895	1.8432	0.5438	2.0853
11	0.6041	1.7831	0.5589	2.0032
12	0.6172	1.7330	0.5725	1.9353
13	0.6290	1.6906	0.5848	1.8781
14	0.6397	1.6541	0.5960	1.8291
15	0.6494	1.6223	0.6063	1.7867
20	0.6882	1.5089	0.6475	1.6371
25	0.7160	1.4383	0.6774	1.5452
30	0.7373	1.3893	0.7005	1.4822
35	0.7542	1.3529	0.7190	1.4357
40	0.7682	1.3247	0.7344	1.3997
45	0.7800	1.3020	0.7473	1.3710
50	0.7901	1.2832	0.7585	1.3473
75	0.8252	1.2226	0.7978	1.2714
100	0.8469	1.1885	0.8222	1.2290
500	0.9287	1.0781	0.9161	1.0938

Source: http://www.itl.nist.gov/div898/handbook

Table 3.4. *Multiplying factors of the confidence interval (estimation MTBF)*

The formula used to calculate the confidence intervals is the following:

$$P\left\{\frac{MTBF \times 2r}{\chi^2_{\alpha/2;2(r+1)}} \leq True\ MTBF \leq \frac{MTBF \times 2r}{\chi^2_{(1-\alpha/2);2r}}\right\} \geq \{1-\alpha\} \qquad [3.51]$$

$\chi^2_{\left(1-\alpha/2\right);2r}$ is a value of χ^2 when $(2r)$ *dll* is larger than with a probability of $[1 - \alpha/2]$. The right tail of the distribution has a probability of $[1 - \alpha/2]$. A simpler version of this formula is written by using the total time T of the test:

$$P\left\{\frac{2\times MTBF}{\chi^2_{\alpha/2;2(r+1)}} \le True\ MTBF \le \frac{2\times MTBF}{\chi^2_{\left(1-\alpha/2\right);2r}}\right\} \ge \{1-\alpha\} \qquad [3.52]$$

These bounds are accurate in the case of one or several reparable systems or for a fixed duration. When a new component is installed in a mechanism, it is strictly monitored for a considerable period of time. If the new part does not present notable failures during this period, the equipment passes the reliability *acceptance test*, called a qualification test or production reliability acceptance test (*PRAT*). This is linked to satisfying the costumer's MTBF requirement, at a specified level of confidence.

For how long should a part of equipment or a system be tested to ensure a specified MTBF at a given confidence? We start with a given MTBF objective, at a confidence level, for example $100 \times (1 - \alpha)$.

The recommended procedure consists of making iterations on r, which is the acceptable failure number. For example, to confirm an objective of 200 h of MTBF at a confidence of 90%, which permits up to four failures on the test, the test duration must be of $200 \times 7.99 = 1598$ h. If this does not suffice, only three failures will be authorized for a test duration of $(200 \times 6.68) = 1336$ h. The shortest test that would not permit the failure (200×2.3) = last 460 h. All these tests guarantee an MTBF of 200 h at the confidence threshold of 90%, when the equipment passes the test. The following table allows us to read the length of the test, to determine the useful time when testing the component (equipment). The following is the guide for the length test:

EXAMPLE.– A part must satisfy an MTBF requirement of 400 h at the confidence threshold of 80%. We are giving ourselves up to two months to decide whether the part satisfies the mission and if the test is acceptable. Two months of operation, with several days rest for maintenance and repairs, reaches a maximum of ≈1300 h. The confidence threshold of 80% for $r = 1$ is 2.99 (see Table 3.5) therefore a test of $(400 \times 2.99) = \approx 1200$ h is the best that can be done by *accepting* one failure.

Acceptable failure number	Level factor of a given confidence					
R	50%	60%	75%	80%	90%	95%
0	0.693	0.916	1.39	1.61	2.30	3.00
1	1.68	2.02	2.69	**2.99**	3.89	4.74
2	2.67	3.11	3.92	4.28	5.32	6.30
3	3.67	4.18	5.11	5.52	6.68	7.75
4	4.67	5.24	6.27	6.72	7.99	9.15
5	5.67	6.29	7.42	7.90	9.28	10.51
6	6.67	7.35	8.56	9.07	10.53	11.84
7	7.67	8.38	9.68	10.23	11.77	13.15
8	8.67	9.43	10.80	11.38	13.00	14.43
9	9.67	1048	11.91	12.52	14.21	15.70
10	10.67	11.52	13.02	13.65	15.40	16.96
15	15.67	16.69	18.48	19.23	21.29	23.10
20	20.68	21.84	23.88	24.73	27.05	29.06

Source: http://www.itl.nist.gov/div898/handbook

Table 3.5. *Length test*

3.10. Repair rate or ROCOF

Repair rate models are based on counting the number of accumulated failures over time. This approach is used to model the accident rate that occurs during failures of a repairable system. These rates are called repair rates, which should not be confused with the length of time for repair. The failures occur at different time spaces (ages) in the lifetime of a system, components, or structures. The frequency of repair (weld joints, riveting, bolting, etc.) varies at a relatively constant rate.

Let $N(t)$ be a counting function which allows us to follow the accumulated number of failures of a given system from zero time at (τ). $N(t)$ is a step function which skips a count each time a failure occurs and then returns to normal until the next level, and so on until the next failure. $M(\tau)$ is the expected number (average number) of accumulated failures by the time (τ) of the system. The derivative of $M(\tau)$, written as $m(\tau)$, is defined as being the *repair rate* or the *rate of occurrence of failure* at time (τ) or *ROCOF*.

$$m(\tau) = \exp\{\alpha + \beta\tau\} \qquad [3.53]$$

It is said that the model follows an exponential law or that the model is log-linear. This is called the *Cox–Lewis model*. A system whose repair rate follows this

flexible model is to be improved, if $\beta < 0$ and the deterioration is $\beta > 0$. When $\beta = 0$, the exponential law is reduced to a homogeneous poisson process (HPP) for constant speed repair.

3.10.1. *Power law*: *non-homogeneous Poisson process*

For a power law, the repair rate (non-homogeneous poisson process, NHPP) is approached as follows:

$$M(\tau) = a \times \tau^b; \; for \; a \succ 0 \; and \; b \succ 0 \qquad [3.54]$$

It is a flexible model which has proved itself in many industrial environments to study the expected failures at the first time (τ):

$$m(\tau) = a.b.\tau^{b-1} = \alpha.\tau^{-\beta}; \; for \; \alpha \succ 0 \; and \; \beta \prec 1 \qquad [3.55]$$

The HPP model has a constant repair rate (λ). By substituting an arbitrary function $\lambda(\tau)$ by (λ), an NHPP is obtained with an intensity function (λ). If, say, $\lambda(\tau) = m(\tau) = \alpha\tau^{-\beta}$, there will be an NHPP with a power function. The probability of failure for the NHPP is calculated by one of Poisson's laws, expressed as follows:

$$P\{N(T) = \kappa\} = \frac{M(T)^{\kappa} \times Exp\{-M(T)\}}{\kappa!} \qquad [3.56]$$

For the model of a power law, let

$$P\{N(T) = \kappa\} = \frac{\left(aT^b\right)^{\kappa} \times \left(\exp^{-aT^b}\right)}{\kappa!} = \frac{\left(a^{\kappa}T^{b\kappa}\right) \times \left(\exp^{-aT^b}\right)}{\kappa!} \qquad [3.57]$$

This power law is also called the Duane model or the ASMAA model (in the 1970s in the USA this acronym stood for the United States *Army Materials System Analysis Activity*). This process is sometimes called the Weibull process, because for a power law the form parameter $a = \alpha$ characterizes the lifetime (see Chapter 1, Volume 1). For every NHPP with an intensity function m(τ), of distribution $F_T(\tau)$ to express the rate of occurrence of failure at time T, is written as follows:

$$F_T(\tau) = 1 - \exp\left\{-\int_0^{\tau} m(T+\tau)\, d\tau\right\} \qquad [3.58]$$

To calculate the expected time before the next failure the following relation is used:

$$F_T(\tau) = 1 - \exp\left\{-a\left((T+\tau)^b - T^b\right)\right\} \quad [3.59]$$

3.10.2. *Distribution law – gamma (reminder)*

There are two ways of writing (parameterizing) gamma distribution. Below, we present the gamma law with $a = (\alpha) = (\gamma)$, is the "form" parameter and the scale parameter is $b = 1/\beta$. The first choice of parameters (a, b) will be the most convenient for the later applications of the gamma law (below is a reminder of the formulas).

Average: $\mu = \left(\frac{a}{b}\right)$ *or* $(\alpha \times \beta)$ *or* $(\gamma \times \beta)$

Variance: $VAR = a/b^2$ *or* $\alpha.\beta^2$ *or* $\gamma.\beta^2$ and **standard deviation** $\sigma = \beta\sqrt{\alpha}$

Distribution function: f(τ)

$$f(\tau, a, b) = \left(\frac{b^a}{\Gamma(a)}\right) \times \left(\tau^{a-1}\right) \times \left(\exp^{-b\tau}\right) = \left(\frac{1}{\beta^\alpha \Gamma(\alpha)}\right) \times \left(\tau^{a-1}\right) \times Exp\left\{-\frac{\tau}{\beta}\right\}$$
$$or\ even\ f(\tau, \gamma, \beta) = \left(\frac{1}{\beta^\gamma \Gamma(\gamma)}\right) \times \left(\tau^{\gamma-1}\right) \times Exp\left\{-\frac{\tau}{\beta}\right\} \quad [3.60]$$

Probability density F(τ): $F(\tau) = \int_0^\tau f(\tau) d\tau$

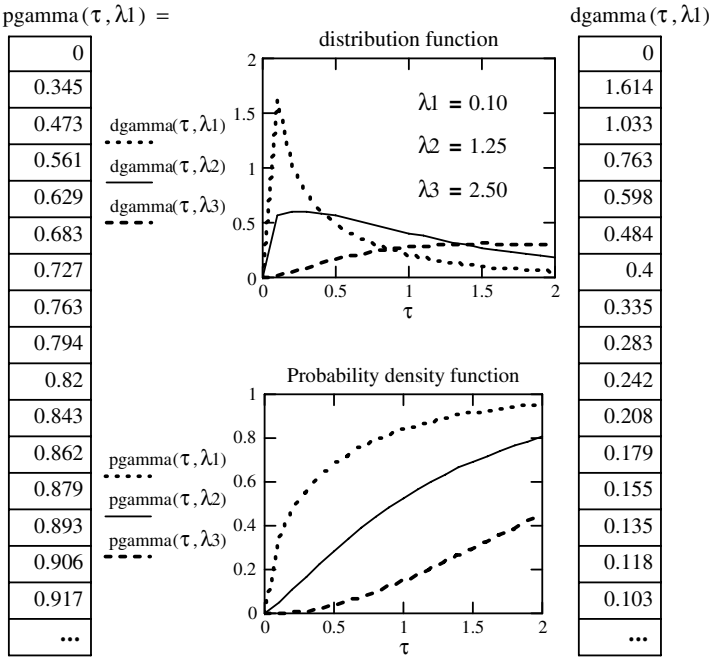
Reliability R(τ): $R(\tau) = 1 - F(\tau)$

Failure rate: $h(\tau) = \frac{f(\tau)}{R(\tau)}$

These formulas have already been discussed at length in Chapter 1 of volume 1.

Exponential distribution is a specific case of gamma law. For $a = 1$, gamma law is reduced to an exponential distribution when $b = (\lambda)$. Another well-known statistic distribution is the chi-square law (see Chapter 1, volume 1), which is also a

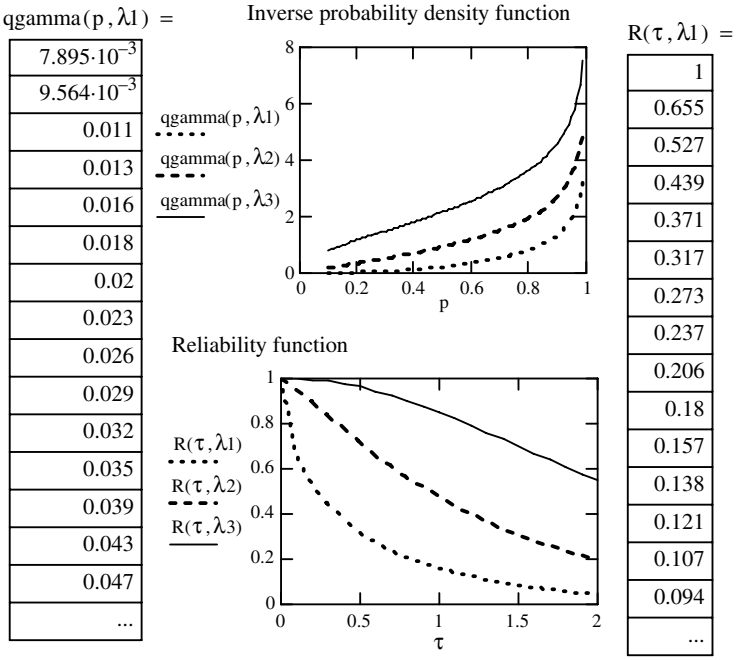
specific case of gamma law (see Chapter 1). Distribution χ^2 with n degrees of freedom is the same for a gamma law with $a = n/2$ and $b = 0.5$ (or $\beta = 2$). We will present the way the main functions of a gamma law look. We calculated the latter with the help of MathCAD for different (λ).



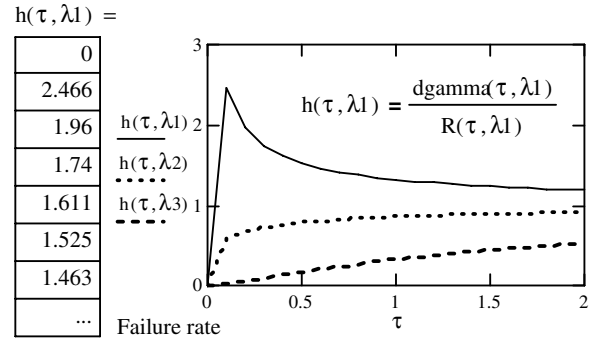
Figures 3.11. *Given gamma forms for $\lambda_1 = 0.1$; $\lambda_2 = 0.1$; $\lambda_3 = 0.1$*

Values of random numbers (picked at random), distributed according to a gamma law.

$$m = 7 \qquad \lambda 1 = 0.5 \qquad \text{rgamma}(m, \lambda 1) = \begin{pmatrix} 0.054 \\ 0.643 \\ 0.185 \\ 0.187 \\ 0.594 \\ 0.139 \\ 0.047 \end{pmatrix} \qquad \lambda 2 = 1.25 \qquad \text{rgamma}(m, \lambda 2) = \begin{pmatrix} 1.87 \\ 1.172 \\ 1.919 \\ 2.795 \\ 4.543 \\ 0.153 \\ 1.711 \end{pmatrix} \qquad \lambda 3 = 2.5 \qquad \text{rgamma}(m, \lambda 3) = \begin{pmatrix} 2.537 \\ 5.998 \\ 0.991 \\ 2.984 \\ 1.001 \\ 0.967 \\ 1.201 \end{pmatrix}$$



Failure rate curve for the gamma law



Figures 3.12. *Forms of the failure rate curve for the gamma law, for $\lambda_1 = 0.1$; $\lambda_2 = 0.1$; $\lambda_3 = 0.1$*

The statistical law of distribution, gamma, is used to model standby systems, as well as for a Bayesian analysis of reliability. The gamma law is a flexible model for the distribution of lifetimes. The data on the failures can be well-adjusted. However, it is not well adapted to distribution models whose failure mechanism's lifetime is cumulative (accumulated wear of the components).

If there are ($n - 1$) backup devices and all backup systems have a lifetime that is exponential to parameter λ, then the total lifetime follows a gamma distribution with $a = n$ and b. λ as has been already demonstrated in Chapter 1. For a, a positive integer, gamma law is said to be of Erlang distribution (see Chapter 1). The latter distribution (Erlang) is frequently used for queuing theory (by Bell, Canada and USA).

In the mechanical reliability of materials and structures, the current use of the model from gamma law is concerned with Bayesian cases. When a system follows an HPP with a constant repair rate (λ), it is preferable, beforehand, to make *a priori* a practical choice, relative to the failure rate (λ). This has already been covered in Chapter 2, with the help of a Poisson distribution (binomial).

3.10.3. *Bayesian model of a priori gamma distribution*

How can a Bayesian test be planned to confirm that a system responds to its objective, that is, the averages of good operation (MTBF)? We will start at the moment when gamma law parameters (a and b) have already been determined. Let us suppose that we have a given MTBF objective, for example M, and a desired confidence threshold, for example $100 \times (1 - \alpha)$.

We are looking to confirm that the system will have a minimum MTBF M of confidence threshold $100 \times (1 - \alpha)$. We accept that there will be a certain number of failures, r, which will be allowed during the test. We need a specific test time T so that we can observe until r fails but still "passes" the test. If the duration of the test is too long (or too short), we can cover the process with a different choice of r. When the test ends, the posterior gamma distribution will have (in the worst case – supposing exactly r failures) new parameters of:

$$a' = \{a + r\} \text{ and } b' = \{b + T\} \qquad [3.61]$$

The test signifies that the failure rate $\lambda_{1-\alpha}$ of the upper part $100 \times (1 - \alpha)$ percentile for the posterior gamma law must be equal to the target failure rate of $1/M$. But this centile is, by definition, $G^{-1}(1 - \alpha;\ a',\ b')$, when G^{-1} designates inverse gamma distribution of parameters (a', b'). We can find the value of T which satisfies $G^{-1}(1-\alpha;\ a',\ b') = 1/M$ by trial and error, or by using a *target value* in Excel. However, on the basis of gamma distribution properties, it turns out that we can calculate T directly with:

$$T = \left\{0.5M \times \left(G^{-1}\right)(1-\alpha;\ 2a';\ 0.5) - b\right\};\left(G^{-1}\right) \text{ is inverse gamma law}$$

At the time (T) of the Bayesian test, using the Excel expression for the duration of the Bayesian test, which is required to confirm an objective of M at $100 \times (1 - \alpha)\%$ confidence, which permits r failures and by supposing that the gamma parameters before a and b is written as:

$$= 0.5 \times M \times INVERSE.GAMMA.LAW\left(\left(1-\alpha\right),\left(\left(a+r\right)\right),2\right)-b$$

there is a very simple way to calculate the required Bayesian test time, when $a = 1$. Table 3.6 can be used to calculate the duration of the classic test. Let T_c. The Bayesian test is carried out by $(T = T_c - b)$. If parameter b has been fixed at $(ln2) \times MTBF_{50}$ (when $MTBF_{50}$ is the accepted choice), let:

$$T = T_c - \left(\ln 2\right) \times MTBF_{50} \qquad [3.62]$$

This shows that when α is used, the duration of the Bayesian test is still lower than the duration of the corresponding classic test. This is the reason for this technique being so simple. In general, the time of the Bayesian test can be shorter, or longer, than the corresponding times of a classic test, according to the choice of anterior parameters. However, the Bayesian time will always be shorter when the parameter is lower or equal to 1.

Calculation example of a Bayesian test time, with the help of Excel: An equipment component must respond to the requirement of an MTBF of 500 h at a confidence level of 80%. We decided to use a Bayesian gamma law *a priori*. We believe that 600 h correspond to a probable MTBF by remaining "confident" that the MTBF will exceed the 250 h. Gamma law parameters, *a priori*, are $a = 2.863$ and $b = 1522.46$.

We want to determine a suitable test time so that we can confirm an MTBF of 500 h at a confidence threshold of at least 80%, provided that there are no more than two failures. With the help of an Excel spreadsheet, let → = 0.5*500* INVERSE.GAMMA.LAW (0.8, ((a + r)), 2) − 1522.46. The result is quick and quite amusing → 1756 h.

Excel:= 0.5*500* INVERSE.GAMMA.LAW (0.8;4.863;2)-1522.46 = 1756.1172 hrs

To compare this result to the classic test for time needed, we use the table of the length test (see Table 3.6). The factor from the table is 4.28, meaning that the necessary test time is $500 \times 4.28 = 2140$ h for a non-Bayesian test. The Bayesian test allows around ≈384 h to be saved, or a gain in efficiency of 18%. If the test is executed for 1756 h, with no more than two failures, then an MTBF of at least 500 h

will have been confirmed at a confidence level of 80%. If, conversely, we had decided to use one *a priori* with an $MTBF_{50}$ of 600 h, the duration of the required test would have been 2140 – (600 − 2) = *ln* 1724 h.

3.10.4. *Distribution tests for exponential life (or HPP model)*

Tests using exponential law are common in industry to check that components or systems respond to their reliability requirement. The principal indicator of this is the MTBF. According to the hypothesis, a system that has a constant failure rate (or repair rate) is the inverse of the MTBF. The expected time between two breakdowns follows the exponential distribution model. A typical test situation consists of monitoring a component subjected to the real stresses of normal operation. After a period of several months, if the component does not break (is without failure) during this period, the equipment "passes" the reliability acceptance test. This type of reliability test is often called a qualification test, or a product reliability acceptation test (PRAT).

How much time should the test take for a part or a system to ensure a specified MTBF at a given confidence threshold?

Let M represent a given MTBF objective at a confidence level of [100 × (1 − α)]. Let us consider a mechanical part (a mechanical component) system, needed to determine the *length of the test*.

The recommended procedure consists of making iterations on *r* and observing the acceptable failure number up to a larger *r*. An unacceptable length of test would be needed. For every case of *r*, the length of the corresponding test is calculated by multiplying M (the objective) by the factor found in Table 3.6 which corresponds to the *r*th row and to the column corresponding to the desired confidence level. For example, to confirm an objective of 200-h MTBF at a confidence threshold of 90%, which permits up to four failures in the test, the test length must be (200 × 7.99) = 1598 h. If this is not long enough, just three failures will be authorized for a test duration of (200 × 6.68) = 1336 h. The shortest test does not permit failure and is (200 × 2.3) = 460 h. All these tests guarantee an MTBF of 200 h at a confidence threshold of 90%, when the equipment "passes" the test. However, the shortest test length turns out to be far less "*fair*" for the supplier in the sense that there is a good chance that the test will be classed marginally acceptable to the system as a whole.

Guide for the length of test table

Accepted failure number	Level factor of a given confidence					
r	50%	60%	75%	80%	90%	95%
0	0.693	0.916	1.39	1.61	**2.30**	3.00
1	1.68	2.02	2.69	**2.99**	3.89	4.74
2	2.67	3.11	3.92	4.28	5.32	6.30
3	3.67	4.18	5.11	5.52	6.68	7.75
4	4.67	5.24	6.27	6.72	7.99	9.15
5	5.67	6.29	7.42	7.90	9.28	10.51
6	6.67	7.35	8.56	9.07	10.53	11.84
7	7.67	8.38	9.68	10.23	11.77	13.15
8	8.67	9.43	10.80	11.38	13.00	14.43
9	9.67	10.48	11.91	12.52	14.21	15.70
10	10.67	11.52	13.02	13.65	15.40	16.96
15	15.67	16.69	18.48	19.23	21.29	23.10
20	20.68	21.84	23.88	24.73	27.05	29.06

Source: http://www.itl.nist.gov/div898/handbook

Table 3.6. *Guide for the test length*

The formula to calculate the factors of the table is the following: The comparison of $\left(\chi^2_{\alpha;2(\gamma+1)}/2\right)$ *with* $\left(\chi^2_{\alpha;2(\gamma+1)}\right)$ demonstrates that the upper limit 100 × (1 − α) percentile of Pearson's chi-two distribution test with 2(r + 1) degrees of freedom to calculate the test length factors.

Application example: A part (connecting rod, pump, or other) must satisfy an MTBF requirement of 400 h at a confidence level of 80%. We give ourselves up to two months (or 52 working days) with an interval of three operations to decide whether the part is acceptable. Two months of operation, with several days for rest, maintenance, and repairs, will amount to a maximum of ≈1300 h. The confidence factor of 80% for $r = 1$ is 2.99 (Table 3.6) therefore a test of (400 × 2.99) = ≈1200 h (with a maximum of one failure test is not authorized).

How to shorten the time of the required test, by testing more than one system: The test times of the exponential law can be considerably shortened if several similar tools can be tested at the same time. The duration of the test signifies that the

"test hours" for a part operating for 1000 h is equivalent (for an exponential mode) to two parts being used for 500 h each, or 10 parts being used for 100 h each. It is sufficient to count all the failures and to count all the parts and the sum of the test hours. The Bayesian paradigm was the focus of a previous discussion, where the advantages and disadvantages of this approach in reliability of materials and structures have been outlined. The underlying hypotheses of the gamma law will be described here, as well as the exponential model, known as the Bayesian model. We have also seen how to use the model from the Bayesian system to calculate the required test time to confirm (or invalidate) an MTBF of a system at a given confidence level.

3.10.5. *Bayesian procedure for the exponential system model*

The objective of Bayesian reliability procedures is to obtain the most precise *a posteriori* distribution as possible and then use this distribution to calculate the failure rate (or MTBF) of confidence intervals estimates (called Bayesian credibile intervals). For a graph, it is worth referring to Figure 3.8.

3.10.5.1. *Gamma law (Erlang), probability density function F(τ, λ)*

Z6(τ) =	Z7(τ) =	Z8(τ) =
4.525×10^{-3}	0.090	0.905
9.688×10^{-3}	0.129	0.861
0.016	0.164	0.82
0.024	0.195	0.78
0.033	0.223	0.743
0.043	0.248	0.709
0.054	0.270	0.676
0.065	0.290	0.645
0.077	0.308	0.615
...	...	...

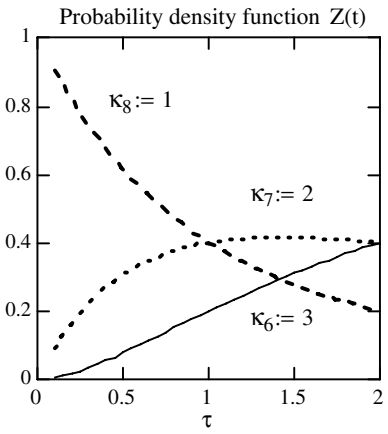


Figure 3.13. *Model of the system from Erlang's law*

$$Z_8(\tau) = \lambda \times (\lambda \times \tau)^{\kappa_8 - 1} \Big/ (\kappa_8 - 1)! \times \sum_{i=1}^{\kappa_6 - 1} \frac{(\lambda \times \tau)^i}{i!} \qquad [3.63]$$

3.10.5.2. *Probability density from Erlang's law (specific gamma)*

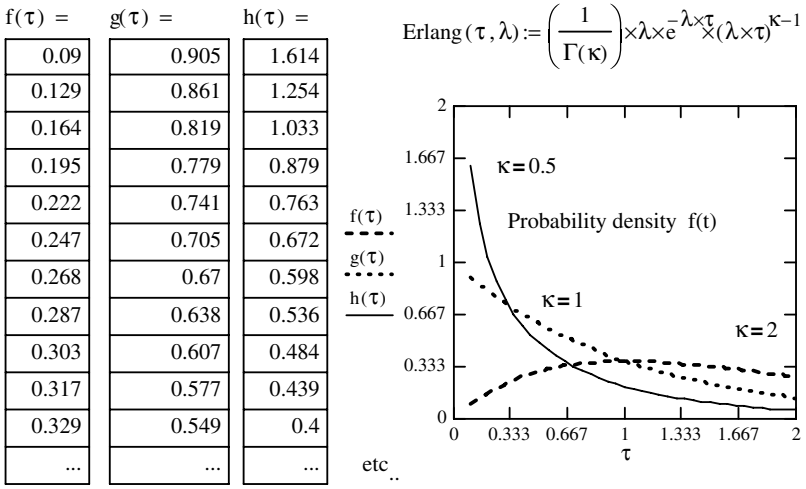


Figure 3.14. *Probability density from Erlang's law (specificity of gamma)*

3.10.5.3. *Reliability of Erlang's law (gamma)*

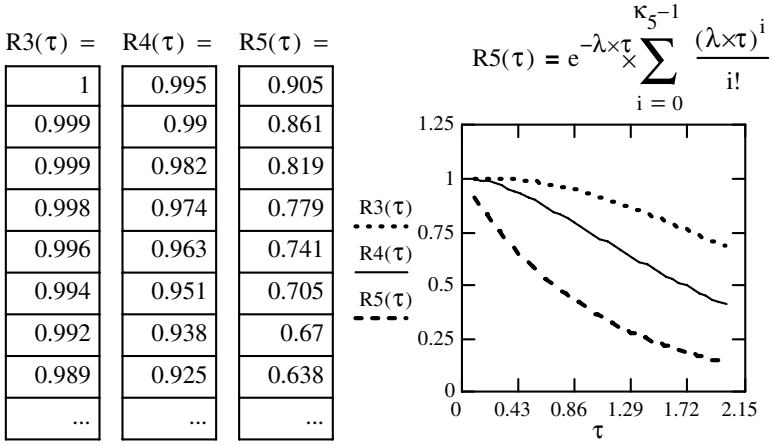


Figure 3.15. *Reliability of Erlang's law (gamma)*

3.10.5.4. *Probability density of an exponential law applied in reliability*

f(τ) =	g(τ) =	h(τ) =
2	1	0.5
1.637	0.905	0.476
1.341	0.819	0.452
1.098	0.741	0.43
0.899	0.67	0.409
0.736	0.607	0.389
0.602	0.549	0.37
0.493	0.497	0.352
0.404	0.449	0.335
0.331	0.407	0.319
...	...	...

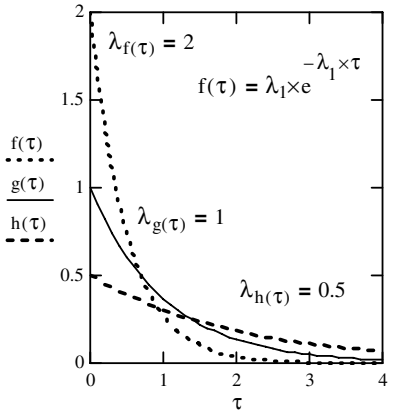


Figure 3.16. *Probability density of an exponential law applied in reliability*

3.10.5.5. *Estimation of the MTBF (determining bounds) on the basis of gamma distribution, a posteriori*

Once the test has been performed and *r* failures have been recorded, the posterior parameters of gamma law (see [3.61]) are the following:

$$a' = \{a + r\} \; et \; b' = \{b + T\}$$

By using an Excel spreadsheet, an estimation of the median value of the MTBF is calculated quite simply as follows: $=1/\text{GAMMAINV}(0.5, a', 1/b)$.

The inverse of the average *a posteriori* distribution can also be used to estimate the MTBF. Here the average represents the mean square error (MSE), which is an estimator of (λ). It is, however, recommended to use the inverse of the average to estimate the MTBF. At a confidence level of 80%, which is considered "weak", the MTBF is obtained from the Excel spreadsheet by: = 1/GAMMAINV (0,8, *a*', (1/*b*')) and, in general, lower than 100 × (1 − α)% like the lower limit given by this equation: = 1/GAMMAINV ((1 - α), *a*', (1/*b*')).

Also, let a confidence interval (credible) in %, that is 100 × (1 − α), for the MTBF, be written as follows:

[{=1/GAMMAINV((1,α/2), *a* ',(1/*b* '))},{=1/GAMMAINV ((α/2), *a* ',(1/*b* '))}]

Finally, the equation = GAMMA.LAW ((1/M), a', (1/b'), VRAI) calculates the probability of the MTBF which is higher than M.

Here is a Bayesian example using Excel to estimate the MTBF to calculate probability by upper and lower limits.

Consider a system which has come to the end of the reliability test, which aims to confirm an MTBF of 600 h at a confidence level of 80%. Before the test, there is a gamma distribution. *A priori*, when $a = 2$ and $b = 1400$, as agreed from tests. The model is considered to be Bayesian to plan the tests, permiting up to two new failures, with the need of another test of a duration of 1909 h. When the latter test has been carried out, two failures were indeed recorded.

What can be said about the Bayes test system?

A posterior gamma law is plotted with parameters $a' = 4$ and $b' = 3309$. The curve below shows the values on the y-axis, plotted as a function of $(1/\lambda)$ and the MTBF on the x-axis. Probability, on the y-axis, according to the curve represents the MTBF. Any of the percentile points of the MTBF can be read here. A simple Excel spreadsheet (see below) will give the percentile MTBF values, as seen in this graph:

Probabilities	(Excel) formula of MTBF in hours	MTBF (h)
1	= 1/GAMMAINV (1.0, 4, (1/3309))	113.5014735
0.9	= 1/GAMMAINV (0.9, 4, (1/3309))	495.3012193
0.8	**= 1/GAMMAINV (0.8, 4, (1/3309))**	**599.9950265**
0.7	= 1/GAMMAINV (0.7, 4, (1/3309))	694.8426740
0.6	= 1/GAMMAINV (0.6, 4, (1/3309))	792.5249764
0.5	= 1/GAMMAINV (0.5, 4, (1/3309))	901.1288667
0.4	= 1/GAMMAINV (0.4, 4, (1/3309))	1030.416492
0.3	= 1/GAMMAINV (0.3, 4, (1/3309))	1197.303160
0.2	= 1/GAMMAINV (0.2, 4, (1/3309))	1440.708370
0.1	= 1/GAMMAINV (0.1, 4, (1/3309))	1896.525514

Table 3.7. *Table of the Results calculations from an Excel spreadsheet*

The values of the MTBF are presented below, with the help of an Excel spreadsheet.

The test confirmed an MTBF = 600 h at a confidence level of 80%, an MTBF = 495 h at a confidence level of 90% ,and 495.1897) is a confidence interval of 90% for the MTBF. A single number (point) of the estimation of the MTBF of the system is equal to 901 h. Alternatively, the inverse of the average of the *a posteriori*

distribution b'/a'' = 3309/4 = 827 h can be used as an estimation. The reciprocal average is in this case 57% lower, as shown here: = GAMMADIST ((4/3309), 4, (1/3309), True).

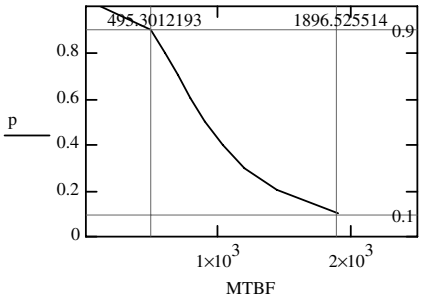
p =

	0
0	1
1	0.9
2	0.8
3	0.7
4	0.6
5	0.5
6	0.4
7	0.3
8	0.2
9	0.1

MTBF =

	0
0	113.501
1	495.301
2	599.995
3	694.844
4	792.525
5	901.129
6	1.03×10^3
7	1.197×10^3
8	1.441×10^3
9	1.897×10^3

Probabilities exceeding the objectives of gamma distribution law, *a priority*



MTBF, the objectives

Figure 3.17. *Graph of the probabilities exceeding the fixed objectives a priori, of the MTBF, of a Bayesian model with the help of Gamma distribution, a priori*

DISCUSSION.– The classic reliability approach does not contain enough details about the failure rate of components which constitute a batch. It therefore presents "lacunae" during the testing of models. As has previously been said, the Bayesian approach proposes that each component of the batch is characterized by its own capacity to resist stress. It is, moreover, this modeling that postulates the existence of a singular failure rate (λ) which is attached to each component of the mechanism and therefore to the statistical distribution known as the λ of the batch.

In this section, we considered an "*a priori*" distribution law; a Bayesian treatment therefore serves reliability modeling. A formalism of the unconditional failure rate (breakdown or ruin) follows, which reflects the failures of the batch as a whole, if the veracity of the parameters can be demonstrated. By choosing an "*a priori*" distribution law (gamma, exponential, Erlang, Poisson, or even binomial), a Bayesian treatment is implemented, from which we get the formalization of an unconditional failure rate (mechanical ruin of the fracture). Since the previously plotted laws justify a gamma type law, using both practical and theoretical considerations, it has been concisely demonstrated that the unconditional breakdown rate hyperbolically decreases as a function of time. This has a considerable impact on the famous "bath curve" which no longer presents a horizontal bearing but rather a slight tilt.

3.11. Bayesian case study applied in fracture mechanics

General argument: Bayesian modeling of the failure rate in the reliability of materials and of structures. Usually in classic reliability, the link between the failure rate (λ) of all components that constitute the total structure (batch), which is considered homogeneous, and the HPP (homogeneous Poisson process) is rarely referenced. From the point of view of the reliability of the materials and the structures, the failure rate (ruin) $\lambda = \lambda(\tau)$ is strictly linked to the survival of the mechanism. Furthermore, by designating R(τ) by its reliability (otherwise called its survival probability) at the moment (τ) and by (dR) elementary probability of the damages of the mechanism (component) to count (τ) at (τ + dτ), the failure rate $\lambda(\tau)$ is presented as follows:

$$\lambda(\tau) = -\left(\frac{1}{R(\tau)}\right) \times \left(\frac{dR}{d\tau}\right) \qquad [3.64]$$

In literature, the famous bath curve is often used as a point of reference to explain the three life-stages of a component or of a mechanism. From the intuitive point of view this remains true; however, it is worth nuancing approaches as before.

In the domain confined to fractures by cracking (fatigue) we noted the same results as Ritchie's diagram (Chapter 7), in mode II when the Paris–Erdogan law is largely justified. The strengthened meaning of $\lambda(\tau)$ is not as simple as certain texts claim it to be. Must each component be attributed an identical failure rate even when the batch presents a physical heterogeneity? In the slight tilt zone of the bath curve, the "useful life" considerations of components are brought together.

If we start from the fact (not established) that the bearing is not tilted, it can be intuitively presumed that there is no accumulated wear (to justify a constant of the failure rate). Indeed, it is inconceivable to disregard the phenomenon of wear even when it would complicate calculating lifetime probabilities. Certainly the rate is likely to be very weak, explaining its supposedly "true" constancy.

Mathematically, the failure rate $\lambda(\tau)$ is considered to represent the relatively important capacity of a component or of a structure to successfully resist the stresses from the environment which are imposed on it. From a Bayesian perspective, the failure rate considers that the analyzed component (structure) comes from the batch from the same chain of design; for example, the same structures assembled by welding with the same filler metal, in the same assembly conditions [GRO 94].

Let i be each component from the batch of welded crosses of a metal structure (see Chapter 7). Let us allocate a resistance capacity $r_{i,(\kappa)}$ to the stress spectrum, represented by (κ). This leads us to declare the existence of a statistical distribution $g\{r_i, (\kappa)\}$ of all quantities $\{r_i, (\kappa)\}$. The ruin rate λ_i (for cracking we will use ruin

rather than failure) of the first component of the structure represents a certain mathematical quantity, directly correlated to the resistance to stress $\{r_i, (\kappa)\}$ relative to this component. In other words, it means that every function $\{r_i, (\kappa)\}$, has a univocal corresponding value $g(\lambda_i)$.

$$g\{ r_i, (\kappa)\} \rightarrow univocal\ g(\lambda_i) \qquad [3.65]$$

From a reliability perspective of the classic Bayes approach, the quantity (λ_i) is an "objective" value (see Figure 3.7). In what follows, we will attempt to simplify the writing of $g(\lambda_i)$ in $g(\lambda)$. The latter writing is employed in fracture mechanics. It represents the physical distribution of quantities to be determined (failure rate of ruin of the components of the structure – batches).

Let $f(\tau/\lambda)$ be the instantaneous probability density of the "breakdown" of a component of the structure, represented by its failure rate (λ). Also, let $f(\tau)$ be the unconditional distribution at the moment of breakdown, expressing the whole of the structure under analysis (calculations). The relation of unconditional probability $f(\tau)$ is established in accordance with Bayes method to represent a marginal probability at the moment of breakdown which affects the entirety of the (λ). This is where the preferred term "*a priori*" stems from. Therefore, $f(\tau)$ is written as follows:

$$f(\tau) = \int_0^{\infty} f\left(\frac{\tau}{\lambda}\right) \times g(\lambda)\, d\lambda \qquad [3.66]$$

The representation of unconditional probability is therefore plotted [GRO 11, GRO 94, GRO 98] at the moment of breakdown for the entirety of structures assembled by welding.

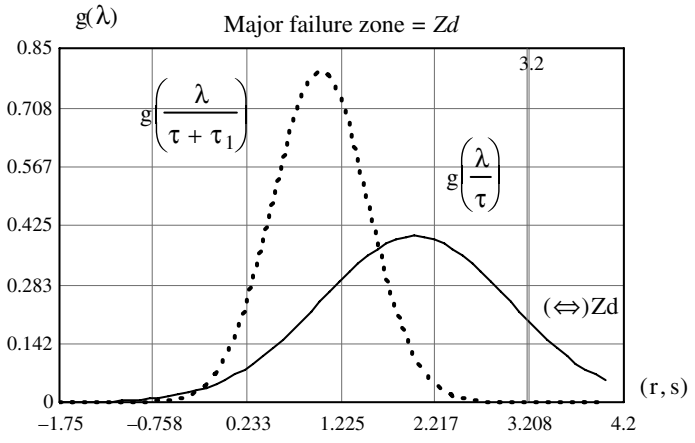


Figure 3.18. *Graph showing unconditional distribution at the moment of breakdown*

In the Bayesian approach, the "thorny" issue is the choice of *a priori* distribution, which dictates the rest of the model. On this subject, at the moment "0" has $g_0(\lambda)$, which must express at best the dispersion of the (λ) of the components of the structure at the start of active life.

$$g_0(\lambda) = \frac{Exp\left\{-\frac{\lambda}{\beta}\right\} \times \lambda^{\alpha}}{\beta^{\alpha+1} \times \Gamma(\alpha+1)} \qquad [3.67]$$

If a maximum uncertainty occurs [GRO 11], it would be preferable to opt for a uniform law, such as an *a priori* law. If the dispersion is not badly known, the reputed laws in cracking mechanics, such as Weibull, Galton, or particularly Birnbaum–Saunder (see Chapter 1), are largely representative. In the case of our present analysis, we will favor a gamma law (Erlang) with two parameters α and β. The reasons guiding our choice of *a priori* law are as follows:

i) Gamma law (Erlang is a specific case of gamma) $f(\tau/\lambda)$ constitutes the "natural" Bayesian conjugate of exponential law. This is the reason we deliberately presented this law as exponential before.

ii) Also, the capacity of the gamma law with two parameters to adapt (flexibly) to the Bayesian natural process.

Bearing this in mind, let:

$$g_0(\lambda) = \frac{Exp\left(-\frac{\lambda}{\beta}\right) \times \lambda^{\alpha}}{\beta^{\alpha+1} \times \Gamma(\alpha+1)} \qquad [3.68]$$

where $\Gamma(\alpha+1)$ is the incomplete gamma function (see Chapter 1) for (α + 1) of the variable under analysis (e.g. the crack size, etc.). Our ruin process is inherent to a power law (exponential). Indeed, in fracture mechanics by fatigue (cracking) the crack size is born (microscopic) well before the moment "0". It progressively propagates according to a Paris–Erdogan-type law. In relation to this hypothesis of exponential law, let:

$$f\left(\frac{\tau}{\lambda}\right) = \lambda \times \exp^{-\lambda\tau} \qquad [3.69]$$

It can be clearly seen that the failure rate (λ) is intrinsic to the component under analysis (calculation). Unconditional distribution at the moment of breakdown, relative to the total structure, is therefore written as:

$$f(\tau)=\int_0^{\infty}\frac{\lambda\times Exp\{-\lambda\tau\}\times Exp\left\{-\frac{\lambda}{\beta}\right\}\times\lambda^{\alpha}}{\beta^{\alpha+1}\times\Gamma(\alpha+1)}d\lambda \qquad [3.70]$$

Integration by part leads to the following:

$$f(\tau)=\left(\frac{(\alpha+1)\times\beta}{(1+\beta.\tau)^{\alpha+2}}\right) \qquad [3.71]$$

We have already represented instantaneous reliability by R(τ) which is written as follows:

$$R(\tau)=1-\int_0^{\tau}f(u)du=\int_{\tau}^{\infty}\frac{(\alpha+1)\times\beta}{(1+\beta.u)^{\alpha+2}}du=\left(\frac{1}{(1+\beta.\tau)}\right)^{\alpha+1} \qquad [3.72]$$

By $\Psi(\tau)$ we designate an unconditional failure rate at time (τ):

$$\Psi(\tau)=\frac{f(\tau)}{R(\tau)}$$

and the combination of [3.71] and [3.72] obtains:

$$\Psi(\tau)=\left(\frac{\beta\times(\alpha+1)}{\beta.\tau+1}\right) \qquad [3.73]$$

Here is the proof of the hyperbolic plot, based on our calculations:

$\Psi(\tau)$ =

	0
0	1.1
1	0.55
2	0.367
3	0.275
4	0.22
5	0.183
6	0.157
7	0.138
8	0.122
9	0.11
10	0.1

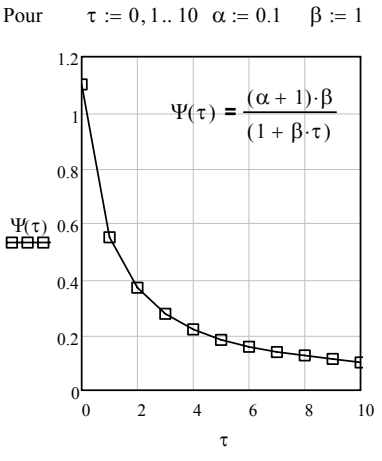


Figure 3.19. *Unconditional failure rate at the moment (τ)*

The previous expression shows that from a Bayesian point of view, on an *a priori* law of a gamma type, the unconditional failure rate decreases with the time the structure serves. This expression is valid in the limit of the useful lifetime of the component. It is worth underscoring that the plotted curve, in Figure 3.19, on the phenomenon of aging (not the subject of our present study) must be distinguished.

The resistance to stress (stress of 150 MPa = constant in our case) [GRO 94, GRO 98], is represented by $r_{i(j)}$. Considering aging would impose a corrected and general model for each t of the unconditional failure rate, which is written according to literature [RIN 81] as follows:

$$\psi(\tau) = \left(\frac{\beta \times (\alpha + 1)}{\beta.\tau + 1}\right) \times \exp^{\gamma.\tau} \qquad [3.74]$$

It is clear that the distribution of the failure rates of components still in service at time (τ) takes the following form:

$$g\left(\frac{\lambda}{\tau}\right) = \frac{exp^{-\frac{\lambda}{\beta}} \times g_0(\lambda)}{\int_{\tau}^{\infty} g_0(\lambda) \times exp^{-\lambda.\tau} d\lambda} = \frac{exp^{-\left(\frac{1}{\beta}+\tau\right)^{\lambda}} \times \left(\lambda^{\alpha}\right)}{\int_{0}^{\infty} exp^{-\lambda\left(\frac{1}{\beta}+\tau\right)} \times \left(\lambda^{\alpha}\right) d\lambda} \qquad [3.75]$$

It is not difficult to see that the law is a gamma type of scale parameter (α') of position and (β') (see Chapter 1, Volume 1).

$$\left\{ \alpha = \alpha^{'} \; and \; \left(\frac{1}{\beta} + \tau\right) = \frac{1}{\beta^{'}} \right\} \qquad [3.76]$$

Therefore, the average of the new distribution represents the failure rate at the moment (τ) and is therefore written as follows:

$$m^{'}(\lambda) = \left\{\beta^{'} \times \left(\alpha^{'} + 1\right)\right\} = \left(\frac{\beta \times (\alpha + 1)}{\beta.\tau + 1}\right) = \left(\frac{m_0(\alpha)}{\beta.\tau + 1}\right) \qquad [3.77]$$

$m_0(\lambda)$ is nothing but the average of initial distribution $g_0(\lambda)$, which has already been explained. To the left of the distribution $g_0(\lambda/\tau)$, related to time (τ) is "valued" – even quantified – by the quantity ($\beta \cdot t + 1$). The latter represents the relationship between $g_0(\lambda/\tau)$ and $g_0(\lambda)$. We will now try to understand where a certain standard deviation will be situated. Gradually, the service of the structure (component) moves away from a cetain quantity, represented by the standard deviation $\sigma'(\lambda)$ of the

distribution at time (τ). We will therefore write the expression of this deviation quantity (standard deviation) as:

$$\sigma'(\lambda) = \left\{\beta' \times \sqrt{\alpha + 1}\right\} = \sigma(\lambda)\frac{\beta'}{\beta} = \left(\frac{\sigma(\lambda)}{\beta.\tau + 1}\right) \quad [3.78]$$

Therefore, $\sigma(\lambda)$ is the standard deviation of the distribution at time $g(\lambda/\tau)$ characterized by this quantity ($\beta t + 1$) as has been previously explained.

DISCUSSION.– It has been shown that the Bayesian approach relative to the failure rate of components influences the shape of the famous "bath curve", which is hardly an unchanging point. The term "bath curve" should not, in our opinion, be guaranteed owing to the horizontality (light slope) of the zone in the middle where the majority of components and structures remain.

Digital application: We simulated an application with the data from table $\tau_{i,}$ which were made to correspond to the instantaneous failure rates (this was a simple simulation and not an experimental result). We then plotted the straight line evolution of $\left(1/\hat{\lambda}\right) \times 10^5 \, hours$.

The medium time is calculated by $\tau_m = \tau_1 + \tau_2/2$ and the inverse of the estimator of the instantaneous failure rate as follows:

$$\frac{1}{\hat{\lambda}} = \frac{n \times \left(\tau_2 - \tau_1\right)}{C} \rightarrow hence \; \hat{\lambda} = \frac{C}{n \times \left(\tau_2 - \tau_1\right)}$$

where:

C is the number of failing components;

N is the number of surviving components.

From expression [3.73] the inverse of the unconditional breakdown rate is as follows:

$$\frac{1}{\psi(\tau)} = \left(\frac{1}{(\alpha + 1) \times \beta}\right) + \left(\frac{\tau}{(\alpha + 1)}\right) \rightarrow \psi(\tau) = ...$$

$$At \; time \; \tau = xhours \rightarrow \psi(xh) = \cdots/_h; \; At \; time \; \tau = yhours \rightarrow \psi(yh) = \cdots/_h, \text{ etc.}$$

This brings us back to reading the expression of an increasing linear straight line as a function of time with an ordinate at the origin (a) and as a slope (b).

$$\left\{a = \frac{1}{\beta \times (\alpha + 1)}; b = \frac{1}{(\alpha + 1)}\right\} \quad [3.79]$$

Experimentally, the coordinate (a) and the slope (b), in an aging test for components and structures, would allow us to write the respective estimators a and b as follows:

$$\left\{\hat{\alpha} = \frac{1-b}{b}; \hat{\beta} = \frac{b}{a}\right\} \quad [3.80]$$

This allows us to plot the straight line of the following figure:

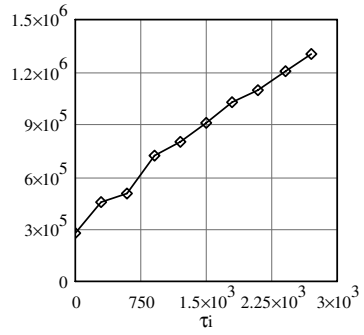
$\tau i =$

	0
0	0
1	300
2	600
3	900
4	1.2×10^{3}
5	1.5×10^{3}
6	1.8×10^{3}
7	2.1×10^{3}
8	2.4×10^{3}
9	2.7×10^{3}

$\lambda(\tau i) =$

	0
0	2.75×10^{5}
1	4.5×10^{5}
2	5×10^{5}
3	7.25×10^{5}
4	8.01×10^{5}
5	9.05×10^{5}
6	1.025×10^{6}
7	1.1×10^{6}
8	1.2×10^{6}
9	1.3×10^{6}

$\lambda(\tau i)$



$\frac{1}{\lambda} = \lambda(\tau i)$

Figure 3.20. *Evolution of the inverse failure rate as a function of time*

The obtained straight line must be adjusted, at best with experimental points, with the help of the regression method (least squares, see Chapter 2, Volume 1).

COMMENT.– We have not included an experiment in this section. This allowed us to comment in more depth on the straight line from the simulation. The latter seemed amply sufficient to support this chapter.

3.12. Conclusion

Throughout this chapter it has been pointed out that the Bayes method is an excellent approach for evaluating the risk probabilities of materials and structures. It

is a method "without temporal memory" and can be applied well to the moment the event (failure) occurs, without necessarily referring to what happened earleir.

Of course, it is worth being vigilant with repect to the application of the Bayes method in all continuum mechanics cases, including "memory" which has an unavoidable effect on the fracture process.

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Chapter 4

Elements of Analysis for the Reliability of Components by Markov Chains

4.1. Introduction

The methods of studying random processes examined in the previous chapters of fracture mechanics by fatigue and the *probabilization* of mechanical assemblies [GRO 94] requires knowledge of at least the first moment of the random function *first order reliability method* (FORM) and in some cases even the second moment *second order reliability method* (SORM). Random function theory only considers the two first moments of values with a random function, specifically the depth of initial cracks (a_0), known as correlation theory. Unfortunately, practical problems are far from agreeing on a solution by simple correlation theory.

For example, in the analysis of crack propagation, the problem of determining probability so that the value of the random function oversteps the given limits often arises. To this end, we used a statistical approach through Weibull's law with two parameters (β and η). It can, therefore, be understood that correlation theory allows us to obtain the probability characteristics of the differential equation solution with separable variables, that is to say, Paris–Erdogan's behavior law. On the other hand, correlation theory would not allow us to resolve the problem caused by crack modeling, because the Paris–Erdogan behavior law is a *nonlinear* differential equation. The solution to this equation can be obtained through Markov processes, which are characterized completely and solely by the knowledge of two-dimensional distribution laws (see Chapter 5).

In the theory of Markov processes, it is assumed that the distribution law of the ordinate of the process at any *future instant* T_i (or even at the number of cycles N_i cyles) depends solely on the value of the ordinate at the instant T_{i-1} (or even N_{i-1} cyles) and *not* on function values *from the past*. In other words, the additional knowledge of random function values for $T < T_{i-1}$ (or even ($N < N_{i-1}$ cyles) does not modify the distribution characters of the ordinates of the function for $T \geq T_i$ (or even $N \geq N_{i-1}$ cycles). Physically, this specificity of Markovian random processes is equivalent to *processes without consequences*. By this, we mean consequences which do not depend on previous history. For this reason, they are known as "memoryless processes". For Markov processes, any multidimensional distribution law can be expressed from a bi-dimensional law (see Chapter 5, Monte-Carlo (MC) double integral) as used by some authors [MAD 86, 87].

Ultimately, initial crack modeling (a_0) by the theory of Markov chains was proposed and applied by certain authors, such as Kozin [KOZ 83] and Bogdanov and Kozin [BOG 85].

4.2. Applying Markov chains to a fatigue model

In fracture mechanics by fatigue, modeling the crack size is sometimes analyzed by the Markov chain process on discrete values. The discretization of the random parameters of the process (a_0) or even $X(\tau)$ cannot occur in a Markovian space, unless there was a relation [4.1] between the conditional probability of formalized functions as follows:

$$\left(\frac{a}{T}\right) = f\left\{a_0, m\right\} \text{ is analogical to } \left(\frac{a}{T}\right) = \mathrm{T}\left\{X_n, \tau_n\right\} \qquad [4.1]$$

Expression [4.1] is in fact presented as follows:

$$\mathrm{T}\left\{X_n, \left(\frac{\tau_n}{X_{n-1}}\right), \tau_{n-1}, ..., X_1; X_0, \tau_0\right\} = \mathrm{T}\left\{X_n, \left(\frac{\tau_n}{X_{n-1}}\right), \tau_{n-1}\right\} \qquad [4.2]$$

with $\tau_n \succ \tau_{n-1} \succ ... \succ \tau_1 \succ \tau_0$ **analogical to** $a_n \succ a_{n-1} \succ ... \succ a_1 \succ a_{0,\,final\ crack}$

Expression [4.2] is an equation of probability states for a random process where $X(\tau)$ (or even X(a)) takes value X_n at time τ_n, provided that the sum of the values at time τ_n is known and depends solely on recent values $X(\tau) = X_{n-1}$ at time $\tau = \tau_{n-1}$. This relation is therefore written because of the term OSM, or the "One Step

Memory" of the random process, where it is clear that the Markov chain process does not have a memory.

The conditional probability function appears on the right-hand side terms of relation [4.2]. It is known by the transition of the probability function P_τ in the Markov process $X(\tau)$. The latter is discrete and entirely defined in a probability space by the first probability function $T(X, \tau)$ and the probability transition function T_τ. Consequently, the discrete vector of the Markov process is $X(\tau)$. It takes the following form:

$$X(\tau) = \left[X_1(\tau), X_2(\tau), X_3(\tau), ..., X_n(\tau) \right]^T \qquad [4.3]$$

This relation is defined by the initial probability vector $T(X, \tau_0)$ and by its matrix of transition probabilities T_τ.

The stochastic Markovian process for the depth of cracks (a_0) due to fatigue will only be used if the *independence condition* of the amplitude of the applied load ($\Delta\sigma$) is ensured with no effects on *the history or the memory of the mechanical phenomenon of fracture mechanics by fatigue.*

COMMENT.– *In this work, the method of Markov processes will be developed on teaching grounds. It has not been proven that load amplitude is exempt from the dependence of the initial crack (see Chapter 7). All we can do is suppose this exemption from independence. In any case, it seems plausible because the load starts its physical effect as soon as the stresses start to act, that is, at time τ_1 where history starts (load recording). If and only if this assumption became physically true, would it also become true in the stochastic sense. Ultimately, we could proceed to probability modeling to estimate the probability of ruin, at the safety index or at a margin of this "reputed" safety*.

If history, in the Markovian meaning of the term (load memory), suggests some influence and if we were to apply Markov theory to these events, it could be assumed that the first condition, as expressed above, takes on *independent probabilities*.

Given the "veracity" of the applicability of Markovian theory for initial cracks a_0 and the various supporting statements from Hohenbichler *et al.* [HOH 81], as well as Schuëller *et al*. [SCH 85, SCH 87], we can consider an approach where the crack length vector is discretized in (b + 1) intervals a_i, where the *states* of the crack lengths are taken into consideration. The a_0 express the crack lengths of the states of fracture (ruin). Beyond these load limits, the average initial crack is equal to or

larger than a crack supposed to be initial. For example, let us suppose [GRO 94, GRO 98] an initial crack a_0, when initiated smaller than ~0.1 mm. By using these experimental data [LAS 92], we could demonstrate that they started at ~0.0019 mm, on a welded cross-structure by four distinct methods of welding, under a load $\Delta\sigma$ = 150 MPa. The transition probabilities are expressed by the following matrix T_τ:

$$T_\tau = \begin{bmatrix} T_{0,0} & T_{0,1} & T_{0,\beta} \\ 0 & T_{1,1} & T_{1,\beta} \\ \dots & \dots & \dots \\ 0 & 0 & T_{\beta-1,\beta} \\ 0 & 0 & 1 \end{bmatrix} \quad [4.4]$$

Here the transition probabilities $T_{i,\kappa}$ of the state κ are defined by the following:

$$T_{i,\kappa}(\tau) = T\left[a(\tau_i + 1)\right] = \frac{a_\kappa}{a(\tau_j)} = a_i \text{ with } \sum_{i=1}^{\kappa} T_{i,\kappa}(\tau) = 1 \quad [4.5]$$

which is also written in the following analogical form:

$$T_{i,\kappa}(\tau) = \text{Probability}\left\{\left(\kappa - i - \frac{1}{2}\right)\Delta a \prec \zeta_i \prec \left(\kappa - i + \frac{1}{2}\right)\Delta a\right\} \quad [4.6]$$

where ζ_i is an increment in the crack due to the load event induced by the *i*th level. To simplify the index ζ_i, expression [4.7] has deliberately been omitted:

$$T_{i,\kappa}(\tau) = \int_{\left(\kappa - i - \frac{1}{2}\right)\Delta a}^{\left(\kappa - i + \frac{1}{2}\right)\Delta a} f(\zeta)\, d\zeta \quad [4.7]$$

The increment (ζ_i) is random. It is framed by the limits in relation [4.6]. It is determined by using the Paris–Erdogan type behavior law which is, in turn, based on random parameters. The discrete distribution of crack lengths after (*j*) *i*th load event ($\Delta\sigma$) = constant, for which the probability in state κ = 1, 2, 3, …, β) and at the *j*th event, we propose the following expression which uses vectors:

$$T_{i,\kappa}(\tau) = \left[T_1(j), T_2(j), T_3(j) \dots, T_\beta(j)\right]^T \quad [4.8]$$

This corresponds, in Markov theory, to homogeneous chains which are expressed by the following relation:

$$T_\tau(j) = \left\{T(0), T_\tau(j)\right\} \qquad [4.9]$$

where T(0) is a discrete distribution of the crack length of level 0 and and P_j will be the jth row of the matrix of transition probabilities. For a more detailed understanding of the theoretical expressions of statistical characteristics (average, variance, standard deviation, etc.), resulting from a Markovian analysis of the welded cross-structure, we will refer to the results presented by Professor Lassen in his scientific report [LAS 92]. Moreover, for comparison we can also use the direct calculations of the crack lengths by the following simple relation:

$$a(j) = a_0 + \sum_{i=1}^{j} \zeta(i) \qquad [4.10]$$

where the length of the initial crack, the random increment $\zeta(i)$, and the results of the (j) crack lengths after the j load event are random. Expression [4.10] will follow a normal distribution (Gaussian) – or we assume so, at least, the explanation given by MC simulation – because the analytical solution versus the Markov chains process turns out to be quite difficult and even, we would say, unreliable. For this reason prioritizing MC simulation is advised.

Finally, let us mention the case of a corrosive environment [API 87, CSA 11, DNV 99] where it becomes necessary to correct the intrinsic factor of the material, specifically C. According to calculation codes, we propose the following expression which takes into account corrosion (offshore) in the Paris-type behavior law.

$$\left\{ \begin{array}{l} \dfrac{da}{dN} = C_\xi \times \Delta K^{m_{a0}} \quad With \;\; C_\xi \;\; and \;\; m_{a_0} \\ C_\xi = \mathrm{C} \times C_{corrosion} \;\; and \;\; m_{a_0} = \left(\dfrac{1}{\lambda}\right) + a^*; m_{a_0} = 6.5 \end{array} \right\} \qquad [4.11]$$

where a* is invariable, but the average m_{a0} will be reduced from 12.5 to 6.5 mm according to current codes [API 87, CSA 11, DNV 99].

4.3. Case study with the help of Markov chains for a fatigue model

Let there be a physical system of *n* possible states. At any moment, the system would find itself in one and only one of its *n* states. Also, during the κth period of a

period of observations, the probability that the system is in a particular state depends solely on its *status at the period* ($\kappa - 1$). It is the typical system of a Markov chain, sometimes called a Markovian process. We will present an example to aid understanding of this important process in random cracking.

4.3.1. *Position of the problem*

Let there be a mechanical structure with three ramifications, as schematized in Figure 4.1, stressed, respectively, at 30%, 30%, and 40%, where α, β, and η are the central, right, and left elements, respectively

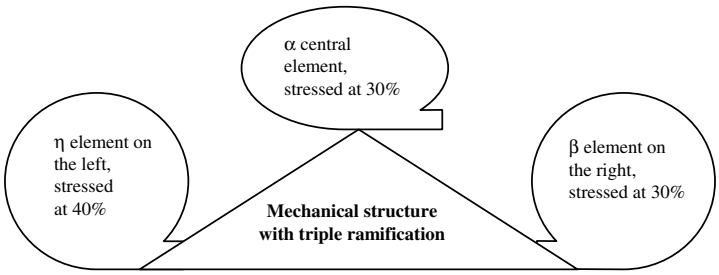

Figure 4.1. *Triply loaded structure (α, β, η)*

– The central element (α) is stressed at 30%.

– The element on the right (β) is stressed at 30%.

– The element on the left (η) is stressed at 40%.

In turn, according to the purely random direction of the *wind* and/or the *waves* (on an *offshore structure*), the following was observed:

a) recordings of loads at the center of the structure, that is α, $\rightarrow$ 30% propagates to the right (β) and 40% to the left, that is η.

b) recordings of loads to the right of the structure, that is β, $\rightarrow$ 40% propagates to the center (α) and 40% to the right (the same branch), that is β, and 20% end up under the load of the left branch, that is (η).

c) recordings of loads to the left of the structure, that is η, $\rightarrow$ 50% propagates to the center (α) and 20% to the left (the same branch), that is η, and 30% end up under the load of the right branch, that is β.

We installed catchment gauges (recordings) of the stresses to be able to finally state the following: The load "slides" toward the branch next to it (taking into account the resistance of the material at all levels). Therefore, the stress at moment (τ) is *solely* dependent on its previous location. On this specific basis of observation, we write the transition matrix of the system, represented by M = M(τ). This matrix models the instantaneous position of the Markovian process of our case structure, which is prone to cracking.

$$M := \begin{pmatrix} 0.3 & 0.4 & 0.5 \\ 0.3 & 0.4 & 0.3 \\ 0.4 & 0.2 & 0.2 \end{pmatrix} \quad \text{therefore} \quad M^2 = \begin{pmatrix} 0.41 & 0.38 & 0.37 \\ 0.33 & 0.34 & 0.33 \\ 0.26 & 0.28 & 0.30 \end{pmatrix} \begin{matrix} \leftarrow \alpha \\ \leftarrow \beta \\ \leftarrow \eta \end{matrix}$$

$$\begin{matrix} \uparrow & \uparrow & \uparrow \\ \alpha & \beta & \eta \end{matrix}$$

Previously, a state was represented by a specific random load in the structure system at time (τ). The arrival ε_{ji} of matrix M below represents the transition probability of the state corresponding to i at the state corresponding to j. For example, the state corresponding to $\beta = 2$ and at $\eta = 3$.

For simplicity, we will start from the plausible "fact" that the time taken to reach a critical stress (acceptable stress) for each branch is 15 min from one branch to another. According to the accumulated observations in the "year 101 of solitude", the stresses were initiated at branch α. Therefore, 30% of stresses are still at α because it is the starting point (inking) of recordings by physical gauges. Then 30% will be at β and 40% at η.

It is worth remembering that the structure being considered is trivial. The singular probability of (even extremely transient) stress is one-to-one in the three branches p = 1. This justifies our use of value (1) in each column of the transition matrix as presented earlier. In the field of probability, each entry is 0 and 1 inclusively.

Question: What justifies the Markovian process in cracking?

Response: The known fact that the stress chain depends *exclusively* on the current load, that is at moment (τ) and no other loads.

It is only in this case that the Markov chain is justified for use with cracking, without which it is but pure "probability contortion" deprived of effective use. It is

certain that the probability matrix will not change in the *observation time* of the stress. For fracture mechanics this is true; however, it is worth emphasizing in particular the extremely transient time of the observation of crack propagation. This is also how we justify the use of recording gauges for the stresses in the branches.

a) An example of a sudden case is as follows: What would happen to the probability of being at branch β if the stress was initiated at branch η after two consecutive recordings?

b) How would we reach β? We can imagine, for example, a propagation from $\eta \rightarrow \eta$ and then from η to β. Just as we can imagine a propagation from η to β and then from β to β.

c) Can we also consider a propagation from η to α and then from α to β?

Let $P(\eta\beta) = P(\eta \rightarrow \beta)$ be the probability of the propagation of η toward β after a duly recorded stress.

From the probability theory, we already know that if two (or more) *independent events* occur simultaneously, probability multiples. The case of *additions* to probabilities is reserved for distinct events occurring at the same time. In accordance with what has previously been said, the probability of going from $P(\eta \rightarrow \beta)$ $\eta \rightarrow \beta$ after two stresses is formulated as follows:

$$P\{\eta \times \alpha\} \times P\{\alpha \times \beta\} + P\{\eta \times \beta\} \times P\{\eta \times \eta\} \times P\{\eta \times \beta\} \qquad [4.12]$$

Digital application of probability:

$$\{0.5\} \times \{0.3\} + \{0.3\} \times \{0.4\} \times \{0.2\} \times \{0.3\} = 0.333$$

The result shows that if the stress (and hence the crack) is initiated at sector η there is a 33% chance of ending at sector β after two recorded stresses. Let us see what would give us another combination (pair): if the stress is initiated at sector β, and what is the probability of being at β after two stresses? The probability of going from sector β to sector β in two stresses is:

$$P\{\beta \times \alpha\} \times P\{\alpha \times \beta\} + P\{\beta \times \beta\} \times P\{\beta \times \beta\} + P\{\beta \times \eta\} \times P\{\eta \times \beta\} \qquad [4.13]$$

Digital application of probability:

$$\{4\} \times \{0.3\} + \{0.4\} \times \{0.4\} \times \{2\} \times \{0.3\} = 34.00$$

We now know exactly where "the stresses momentarily end". What will happen after ($\sigma = 5$) five stresses? This is exactly where the Markovian process is extremely useful.

4.3.2. *Discussion*

– The oriented stress [$\eta \to \beta$] in accordance with two stresses is equivalent to the product of two scalars (line 2 and column 3) of matrix M.

– Going from [$\beta \to \beta$] in two stresses is the equivalent to considering the scalar product of line 2 and of column 2 of matrix M.

– The product of matrix M by itself (M × M) = M^2 allows us to obtain the same responses as before after the two stresses.

It is demonstrated as follows:

$$M^2 = \begin{pmatrix} 0.410 & 0.380 & 0.370 \\ 0.330 & 0.340 & 0.330 \\ 0.260 & 0.280 & 0.300 \end{pmatrix} \begin{matrix} \leftarrow \alpha \\ \leftarrow \beta \\ \leftarrow \eta \end{matrix}$$
$$\begin{matrix} \uparrow & \uparrow & \uparrow \\ \alpha & \beta & \eta \end{matrix}$$

The sum of the elements in each column is continually equal to 1. Each element is therefore located between 0 and 1, inclusively. Moreover, it is this validation condition of the process of a Markov chain. This matrix indicates the probabilities of going from place *i* to place *j* in two stresses. Once you know the constitution of the base matrix, it is simple to know the value of crack propagation (a_0) after three stresses. Let $p(\alpha\beta)$ be the probability of going from [$\alpha \to \beta$] in two stresses.

We will therefore find the probability of going from [$\eta \to \beta$] in three stresses:

$$P\{\eta \times \alpha\} \times P\{\alpha \times \beta\} + P\{\eta \times \beta\} \times P\{\beta \times \beta\} + P\{\eta \times \eta\} \times P\{\eta \times \beta\} \qquad [4.14]$$

Digital application: $\{37\} \times \{0.3\} + \{33\} \times \{0.4\} \times \{3\} \times \{0.3\} = 333.0$

4.3.3. *Explanatory information*

This probability is the scalar product of line 2 of M^2 and of column 3 of M. If we multiply the matrix vector M^2 by M, we will obtain the probability matrix for the three stresses as follows:

$$M^3 = \begin{pmatrix} 0.385 & 0.39 & 0.393 \\ 0.333 & 0.334 & 0.333 \\ 0.282 & 0.276 & 0.274 \end{pmatrix} \begin{matrix} \leftarrow \alpha \\ \leftarrow \beta \\ \leftarrow \eta \end{matrix}$$
$$\begin{matrix} \uparrow & \uparrow & \uparrow \\ \alpha & \beta & \eta \end{matrix}$$

Subsequently, we know exactly how to proceed with the matrix of probabilities for four or more parts (according to the mechanical structure under stress). Let us remember that the elements on each column are always added at 1. It is, however, important not to round off numbers. Maintaining accuracy and using as many decimal places as possible is strongly recommended. Here is the rest of the Markov process approach, applied to structures subjected to random stress.

First M^4 and M^5

$$M^4 = \begin{pmatrix} 0.3897 & 0.3886 & 0.3881 \\ 0.3333 & 0.3334 & 0.3333 \\ 0.2770 & 0.2780 & 0.2786 \end{pmatrix} \begin{matrix} \leftarrow \alpha \\ \leftarrow \beta \\ \leftarrow \eta \end{matrix} \qquad M^5 = \begin{pmatrix} 0.38873 & 0.38894 & 0.38905 \\ 0.33333 & 0.33334 & 0.33333 \\ 0.27794 & 0.27772 & 0.27762 \end{pmatrix} \begin{matrix} \leftarrow \alpha \\ \leftarrow \beta \\ \leftarrow \eta \end{matrix}$$
$$\begin{matrix} \uparrow & \uparrow & \uparrow \\ \alpha & \beta & \eta \end{matrix} \qquad \begin{matrix} \uparrow & \uparrow & \uparrow \\ \alpha & \beta & \eta \end{matrix}$$

Then M^6 and M^7

$$M^6 = \begin{pmatrix} 0.388921 & 0.388878 & 0.388857 \\ 0.333333 & 0.333334 & 0.333333 \\ 0.277746 & 0.277788 & 0.277810 \end{pmatrix} \begin{matrix} \leftarrow \alpha \\ \leftarrow \beta \\ \leftarrow \eta \end{matrix} \qquad M^7 = \begin{pmatrix} 0.3888825 & 0.3888910 & 0.3888953 \\ 0.3333333 & 0.3333334 & 0.3333333 \\ 0.2777842 & 0.2777756 & 0.2777714 \end{pmatrix} \begin{matrix} \leftarrow \alpha \\ \leftarrow \beta \\ \leftarrow \eta \end{matrix}$$
$$\begin{matrix} \uparrow & \uparrow & \uparrow \\ \alpha & \beta & \eta \end{matrix} \qquad \begin{matrix} \uparrow & \uparrow & \uparrow \\ \alpha & \beta & \eta \end{matrix}$$

$$M^{15} = \begin{pmatrix} 0.3888889 & 0.3888889 & 0.3888889 \\ 0.3333333 & 0.3333333 & 0.3333333 \\ 0.2777778 & 0.2777778 & 0.2777778 \end{pmatrix} \begin{matrix} \leftarrow \alpha \\ \leftarrow \beta \\ \leftarrow \eta \end{matrix}$$
$$\begin{matrix} \uparrow & \uparrow & \uparrow \\ \alpha & \beta & \eta \end{matrix}$$

etc.

The more the branches are stressed, the more the numbers in each line seem to converge toward a particular number. In the long term, this means that after a high

number of stresses, no importance is directed at where the structure was initiated. In terms of recordings (period to be defined by reliability maintenance) we have a 38.88% chance of being at sector α, a 33.33% chance of being at sector β and a 27.7% chance of being at sector η. Often this convergence occurs in the majority of transition matrices considered. If all the arrivals of the transition matrix are between 0 and 1, *exclusively*, then convergence is guaranteed. Convergence can occur when 0 and 1 are in the transition matrix.

Let us now consider matrix [α] as:

$$\alpha = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Matrix [α] can be written as A^{κ}. Let us look closely at why κ in the matrix [α] oscillates with the increase of κ. The vector represented below by [τ_0] of the initial distributions (stresses) indicates the fraction (ratio) of stresses that are initiated in each sector. If we start with a uniform distribution, one-third of our stresses in each sector will be represented by [τ_0], such as:

$$\tau_0 = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}$$

[τ_0] is the vector of initial distributions. After one stress, the distribution will be 40% of the total stresses in region α, 33.33% in β, and 26.6% in region η. By multiplying the initial distributions matrix by the transition matrix, we find vector [τ_1]:

$$\tau_1 = \begin{pmatrix} 0.3 \\ 0.3 \\ 0.3 \end{pmatrix} \times \begin{pmatrix} 0.3 & 0.4 & 0.5 \\ 0.3 & 0.4 & 0.3 \\ 0.4 & 0.2 & 0.2 \end{pmatrix} = \begin{pmatrix} 0.360 \\ 0.300 \\ 0.240 \end{pmatrix} \qquad [4.15]$$

During the middle of the month, the fractions converge into some specific numbers, meaning that the starting sector is no longer important. After numerous stresses, the same right-hand side will be the same regardless of the initial distribution, from which the first stress is initiated. For example:

$$\begin{pmatrix} 0.38 & 0.38 & 0.38 \\ 0.33 & 0.33 & 0.33 \\ 0.27 & 0.27 & 0.27 \end{pmatrix} \times \begin{pmatrix} 0.3 \\ 0.3 \\ 0.3 \end{pmatrix} = \begin{pmatrix} 0.34 \\ 0.30 \\ 0.24 \end{pmatrix} \textit{ and } \begin{pmatrix} 0.38 & 0.38 & 0.38 \\ 0.33 & 0.33 & 0.33 \\ 0.27 & 0.27 & 0.27 \end{pmatrix} \times \begin{pmatrix} 0.2 \\ 0.2 \\ 0.6 \end{pmatrix} = \begin{pmatrix} 0.38 \\ 0.33 \\ 0.27 \end{pmatrix}$$

After observing these results, it can be seen that the right-hand side is identical to one of the lines of our transition matrix after numerous stresses. This is exactly what we predicted in our hypotheses, that is to say that 38.8% of stresses will be in sector α after numerous stresses, without consideration for the percentage of stresses which were in sector α according to initial distribution.

We show in the following how this fact can be verified with numerous initial distributions of N_{cyles} or even a_0. If initial distribution indicates the real number of items (branches) in the system, the following equation represents our system after a recording (with onsite gauges):

$$\begin{pmatrix} 0.3 & 0.4 & 0.5 \\ 0.3 & 0.4 & 0.3 \\ 0.4 & 0.2 & 0.2 \end{pmatrix} \times \begin{pmatrix} 18 \\ 18 \\ 18 \end{pmatrix} = \begin{pmatrix} 21.60 \\ 18.00 \\ 14.40 \end{pmatrix}.$$

We now have a rational number of components in sectors α and η after a stress (*we rounded off the responses*). We know that this cannot occur, but it gives us a good idea of approximately how many cases of stresses will happen in each sector (branches of the welded cross-structure). After numerous stresses, the right-hand side of this identity will also be very close to a specific vector, as proven by the following:

$$\begin{pmatrix} 0.38 & 0.38 & 0.38 \\ 0.33 & 0.33 & 0.33 \\ 0.27 & 0.27 & 0.27 \end{pmatrix} \times \begin{pmatrix} 18 \\ 18 \\ 18 \end{pmatrix} = \begin{pmatrix} 20.52 \\ 17.82 \\ 14.58 \end{pmatrix} = \begin{pmatrix} 21 \\ 18 \\ 15 \end{pmatrix}.$$

The specific vector toward which the product converges is the total number of items in the system (54 in this case) multiplied by any column of matrix A^{κ} toward which the matrix converges as κ increases.

$$54 \times \begin{pmatrix} 0.38 \\ 0.33 \\ 0.27 \end{pmatrix} = \begin{pmatrix} 20.52 \\ 17.82 \\ 14.58 \end{pmatrix} = \begin{pmatrix} 21 \\ 18 \\ 15 \end{pmatrix}.$$

This was an explicit case of the Markov chain process. However, it is always worth explaining in detail the application of Markov chains to the phenomenon of cracking. Referring simply to Markov chains is not enough and the applicability of this theory to fracture mechanics by fatigue must be monitored. The general concepts of Markov chains are as follows.

For a Markov chain with n states, the *state vector* is a vector, the column of which the ith component represents the probability of the system in this state i at time τ. The sum of all inputs to a state vector is 1. For example, vectors T_0 and T_1 in the following example are state vectors. If p_{ij} is the transition probability from state j to state i, then matrix $T = [p_{ij}]$ and is called the *transition matrix* of the Markov chain.

The following theorem gives the relation between two consecutive state vectors:

If X_{n+1} and X_n are two consecutive state vectors of a Markov chain with a transition matrix T, then $X_{n+1} = T.X_n$.

For a Markov chain, we are usually perplexed by the long-term behavior of a general state vector X_n. In other words, we would like to find the limit of X_n when $n \to \infty$. It is possible that this limit does not exist. For example:

$$\text{Let the matrix } a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and the matrix vector } X_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$\text{therefore } \left\{ \begin{array}{l} X_1 = a \times X_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}; X_2 = a \times X_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; X_3 = a \times X_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ X_4 = a \times X_3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; X_5 = a \times X_4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}; X_6 = a \times X_5 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{array} \right\}, \text{ etc.}$$

Vector X_n oscillates between vectors [0.1] and [1.0] and therefore does not approach a fixed vector. The question therefore is: what makes a vector X_n approach a limited vector when $n \to \infty$. The following theorem will answer this question. A transition matrix T is regular if the inputs of the matrix T^r are positive for a given whole number (0 is not considered positive). For example, matrix:

$$\rho = \begin{pmatrix} 0.5 & 0.5 & 0.0 \\ 0.5 & 0.0 & 1.0 \\ 0.0 & 0.5 & 0.0 \end{pmatrix} \text{ is regular because of the fact that } \rho^5 = \begin{pmatrix} 0.4375 & 0.3125 & 0.5000 \\ 0.3125 & 0.6250 & 0.1250 \\ 0.2500 & 0.0625 & 0.3750 \end{pmatrix}.$$

From the literature, we know that a Markov chain process is called regular if its transition matrix is regular. The main theorem of the Markov chains theory is as follows:

– If T is a regular transition matrix, then if n tends toward infinity, $T^n \to \rho$, where ρ is a matrix of the form [v, v,...,v] with v as a constant vector.

– If T is a regular transition matrix of a Markov chain process and if X is any state vector, then if n tends toward infinity, $T^n X \rightarrow p$, where p is a fixed probability vector (the sum of its arrivals is 1) with all inputs being positive.

Let us consider a Markov chain with a regular transition matrix T, and let ρ be the limit of T^n when n approaches infinity, therefore $T^n X \rightarrow \rho \cdot X = p$ and the system tends toward a fixed state vector p, known as the *balanced vector* of the system. Since $T^{n+1} = T.T^n$ and T^{n+} and T^n approach ρ we have $(\rho) = T \cdot (\rho)$. Any column of this equation of matrixes gives $T \cdot p = p$. Therefore, the balanced vector of a regular Markov chain with a transition matrix T is the only probability vector p satisfying $Tp = p$.

Is there a way of calculating the balanced vector of a regular Markov chain without using the notion of limit? Here is a popularized approach?

Let there be a squared matrix (ξ). Let λ be an *eigenvalue* of ξ if there is a vector $\neq [0, X]$ satisfying $[\xi_x X] = [\lambda_x X]$.

Therefore, we say that X is an *eigenvalue* of ξ corresponding to the eigenvalue λ. It is now clear that a balanced vector of a regular Markov chain is an eigenvalue for the transition matrix corresponding to the eigenvalue 1. The eigenvalues of an ξ matrix are the solutions to the equation $\det\{A - \lambda \times I\} = 0$, where I is the identity matrix which is the same size as ξ. If λ is an eigenvalue of C, then an eigenvalue corresponding to λ is a solution of the homogeneous system, $\{\xi - \lambda \times I\} \times X = 0$. Consequently, there is an infinite number of eigenvalues corresponding to a particular eigenvalue.

4.3.4. *Directed Works*

Let us suppose that there is an offshore platform whose structure is constituted of a welded cross-assembly. A series of observations (support contact gauges) in open sea, demonstrated that it is practically impossible:

1) To have two consecutive weeks when the waves (or the winds) do not move in the same direction.

2) If the sea is calm, there is the same probability of having a rough sea the following week.

3) If the sea is rough or the winds strong, there is an equal chance of having the same probability the week after the storm.

4) There is a change of storm or icy winds only half of the time where we witness a lull at open sea.

4.3.5. *Approach for solving the problem*

1) Writing the transition matrix which models this system.

2) If it is not windy (calm, good weather), what is the probability that 3) the sea will be calm in a week?

3) To understand the behavior of the weather in the long term in open sea.

Solution 1: Since tomorrow's *weather* depends *solely* on today's, it is a Markov process. The transition matrix of this system is written as follows:

$$Y := \begin{pmatrix} 0 & 0.25 & 0.25 \\ 0.5 & 0.5 & 0.25 \\ 0.5 & 0.25 & 0.5 \end{pmatrix} \begin{matrix} \leftarrow \text{C} \\ \leftarrow \text{V} \\ \leftarrow \text{T} \end{matrix}$$

$$\begin{matrix} \uparrow & \uparrow & \uparrow \\ \text{C} & \text{V} & \text{T} \end{matrix}$$

C, *W*, *S* indicates that in open sea it is $C = calm \rightarrow W = windy \rightarrow S = stormy$.

Solution 2: If the weather is good today, then the initial state vector ξ is written as follows:

$$\xi = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{matrix} \leftarrow C = calm \\ \leftarrow V = Windy \\ \leftarrow T = Stormy \end{matrix}$$

After seven days (a week) for $n = 7$, the state vector Y will be:

$$Y := \begin{pmatrix} 0 & 0.25 & 0.25 \\ 0.5 & 0.5 & 0.25 \\ 0.5 & 0.25 & 0.5 \end{pmatrix} \begin{matrix} \leftarrow \text{C} \\ \leftarrow \text{V} \\ \leftarrow \text{T} \end{matrix} \quad \xi := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{matrix} \uparrow & \uparrow & \uparrow \\ \text{C} & \text{V} & \text{T} \end{matrix}$$

$$n := 7 \qquad \Pi := Y^{n} \cdot \xi \quad \text{therefore} \quad \Pi = \begin{pmatrix} 0.1999512 \\ 0.4000244 \\ 0.4000244 \end{pmatrix}$$

$$\text{For } U := \begin{pmatrix} 0.1999511 & 0.2000122 & 0.2000122 \\ 0.4000244 & 0.4000244 & 0.3999633 \\ 0.4000244 & 0.3999633 & 0.4000244 \end{pmatrix} \begin{matrix} \leftarrow \text{C} \\ \leftarrow \text{V} \\ \leftarrow \text{T} \end{matrix}$$

$$\begin{matrix} \uparrow & \uparrow & \uparrow \\ \text{C} & \text{V} & \text{T} \end{matrix}$$

We will have the following:

$$\text{Knowing } \xi := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ and } X := U^{n} \cdot \xi \text{ therefore } X = \begin{pmatrix} 0.1999999 \\ 0.3999997 \\ 0.3999997 \end{pmatrix}$$

Therefore, there is a 20% chance of having one calm day a week.

Solution 3: We are dealing with a regular Markov chain since the transition matrix is regular. We are therefore sure that the balanced vector exists. To find it, we will resolve the homogeneous system $[T - I]_x X = 0$ of coefficients:

$$\begin{pmatrix} -1 & 0.25 & 0.25 \\ 0.5 & -0.5 & 0.5 \\ 0.5 & 0.5 & -0.5 \end{pmatrix} \text{ of a reduced row echelon form: } \begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\text{The general solution of system } \{T - I\} \times X = 0 \text{ is } \begin{pmatrix} 0.5\tau \\ \tau \\ \tau \end{pmatrix} \textit{ with } \tau \in \Re$$

4.3.6. *Which solution should we choose?*

A balanced vector is specifically a probability vector that is to say that the sum of its components is 1. We can therefore understand that $[0.5\tau + \tau + \tau] = 1$ which gives $\tau = 4/10$. Therefore, the balanced vector is written as follows:

$$p = \begin{pmatrix} 0.2 \\ 0.4 \\ 0.4 \end{pmatrix}$$

In the long term, there is a 20% probability of the day being calm, 40% of it being windy, and 40% of there being a storm.

The above example does not resolve the problem of crack propagation of the welded cross-structure. Rathers it provides information about the natural climate out in open sea. Moreover, it does not provide information about the direction of the waves or the winds. No theory, however imaginative, proposes "stopped" solutions over time. The safety of the structures shows the effects of it. In our opinion, the theory of Markov chains is a good tool, but it is unsatisfactory for obtaining credible

statistics in continuum mechanics. That said, these statistics are, without a shadow of a doubt, useful for compiling a database.

4.4. Conclusion

There is reason to state that the theory of Markov chains is *questionable* in terms of studying the reliability of a singular structure on memoryless parameters. Nevertheless, the probability approach by the Markov chains process is recommended for structures assembled in series and/or in parallel. Certainly, this approach has relatively few applications in comparison to the method of determining using Ditlevsen's bounds, classic MC simulation, or other integral methods. It is worth remembering that correlation theory does not allow us to obtain solutions to the problem, which were posed by the Paris–Erdogan-type cracking law. Being non-linear, the latter's solution can be obtained by, among others, Markov chain processes which are solely characterized by the knowledge of the two-dimensional distribution laws. Therefore, we advocate the use of the classic MC simulation method even though it is costly in terms of time spent calculating.

4.5. Bibliography

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Chapter 5

Reliability Indices

5.1. Introduction

In recent years, models for the theory of probability and applied statistics have been characterized by a sharp development. The scope for studying the predictions or replacements of traditional safety coefficients is detrimental toward both calculation laws and the economics of the project. In terms of the approach to fatigue by fracture mechanics, laboratory and construction practices create a whole realm of problems for which the use of probability and statistical methods is proving successful and even indispensable. The developments in the theory of probability and of applied statistics, the abundance of new data, and some unforeseen catastrophes have caused safety issues to be examined more closely, but at times with an acute economical and technical concern. This has led to the abundant introduction of reliability analysis methods.

Numerous research studies focus on *probability mechanics* with the aim of predicting the behavior of components to establish a decision support system (expert systems). This work exists as part of the criteria of fracture mechanics used to predict, assess, analyze, and simulate the reliability of mechanical components. The reliability of structural components is dependent on mathematics, which estimate the probability of a component (structure) reaching a certain state of *predictive failure.* This is carried out using the *probability of resistant elements* feature of a structure, along with a load that is applied to it.

As opposed to the classical *regulatory* approach, a *risk* is estimated at the same time as being aware that any rule*,* no matter how *conservative*, can never guarantee perfect safety. Users of reliability techniques have to define the *failure rate* of a

structure or a component. If in some cases this effectively corresponds to the formation of a failure mechanism, for many components and structures, a failure criterion will be adopted to a certain level of the so-called *damage.*

The important parameters, either the resistance (R) of the structure or the stresses (S) that are applied to it, can no longer be uniquely defined in terms of nominal or weighted values but instead in terms of unpredictable variables characterized by their mathematical expectations, variances/co-variances, and laws of distribution. The true estimation of the reliability of a structure (components) can only generally be tackled with the help of a relatively simplified model. The analysis of this model will be carried out with the help of mathematical algorithms, which often present themselves as approximation techniques where rigorous calculation can sometimes lead to a time-consuming process.

To evaluate weak failure probabilities of components, a standard technique is to proceed with a random integration of probability density function on the area of failure, i.e. via simulation techniques. The advantage of these techniques is that they necessitate the location of an explicit form in the area of failure. The downside is the weakness of the convergence speed, added to an increased cost in calculation time. Numerous distribution laws have been used to simulate the reliability of components and structures. These laws did not appear with reliability but instead with fundamental research from various fields. We will now introduce a few of them.

Choosing a probability model in which the experimental data is appropriate to the theoretical justifications imposed by the distribution law is not always easy. Evidence for this is the exaggeration surrounding a Gauss law, which is incorporated into nearly all theories. Distribution models for lifecycles are chosen according to:

– the physical factor of a failure mechanism which corresponds to a model of lifecycle distribution;

– the model which has already successfully been used by a similar failure mechanism;

– the model which ensures a theory/practice adequacy applicable to all failure data.

Whatever the recommended method for choosing a significant probability model, it is a good idea to justify it properly. For example, it would not be appropriate to choose a model with a consistent failure rate by an exponential law, as this law is more suited to breakdowns where the failure mechanism is accidental. Some Galton (log-normal) and Weibull models are very flexible and accommodate breakdown modes even for weak experimental data, particularly when they are projected by means of acceleration models once used in conditions that are far removed from the test data. These two models are incredibly useful in terms of

preventing breakdown rates of differing magnitudes. This fact is unusual in directed university research, which prefers to find some theoretical justification for using a particular distribution.

Introducing reliability as an essential tool for the quality of a component starts at the design stage. It is undeniable that reliability imposes itself as a discipline that rigorously analyses weaknesses because it is based on experimental data. Because reliability is tightly linked to quality, the laws of distribution that are used are often the same or related. Intuition tells us that components are numerous and complex in a mechanism (structures). Consequently, reliability calculations become less evident because of, among other things, restricting hypotheses.

5.2. Design of material and structure reliability

Reliability does not just mean "quality". These two distinct disciplines have statistical–mathematical tools in common but lead to different conclusions. Reliability serves to provide a safety (or reliability) index and quality serves to provide capabilities (C_p, C_{pk}, C_m, and C_{mk}) which are in fact indicators of quality. Both disciplines aid decision making. Reliability aims for the probability of smooth functioning during a well-controlled given time (τ). In other words, it is an attribution of agreed *confidence* in a component, a piece of equipment or a mechanism to render the service succinctly composed in the specifications from the onset. The word "confidence" is used because a failure or a weakness in this trust could prove to be enormously substantial (airplanes, nuclear stations, etc.). This is the principal reason that the reliability function is carried out from the beginnings of the project's conception, to:

– ensure feasibility in terms of reliability "contracts";

– guide reliability objectives through various sub-systems;

– predict means of control and maintenance at the time of use;

– define (physical, economical) stresses so that the system is well adapted to its clearly intended task, according to precise criteria.

In industrial automation, it is strongly advised to ensure an *intrinsic* reliability of the designed mechanism. To do this, it is highly recommended to proceed during or at the end of the manufacturing process with some reliability trials. Conceptual studies cannot in general reproduce real functioning conditions. Simulation remains the "weak laboratory" experience. It is also for this reason that simulation techniques are often proposed in the literature as being technical and scientific. The following diagram summarizes the principal tasks of components' reliability.

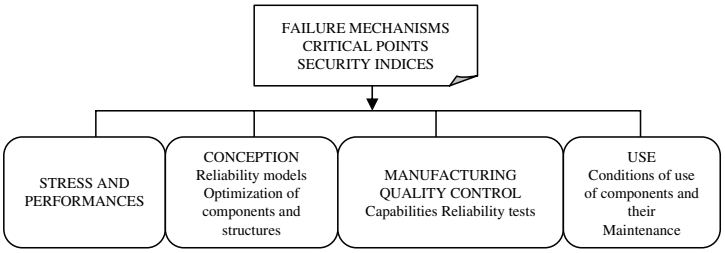


Figure 5.1. *Diagram showing principal tasks of components' reliability*

5.2.1. *Reliability of materials and structures*

The sizing of structures is normally based on deterministic analyses. Loading, and variability in physical properties of materials, uncertainties, and imperfections of analysis models, to cite just a few factors, contribute differently, along with making materials put up a resistance that is difficult to predict and makes it hard to calculate with certainty. We have to reflect here on a defined probability as an indicator of weakness in the case of a system or an operation of failure. In the past, safety coefficients constituted a more or less "reassuring" response, but did not completely erase the risk of failure. The introduction of safety coefficients is expensive but does not always carry the best response. Sometimes it is even the contrary that would occur if the load is strengthened even more. The duality of load/geometry is an extremely particular compromise. Reliability is thus defined by various calculations and is considered as the additional probability of fracture. It constitutes a means of measuring the degree of safety at work.

As calculation models result from probability analyses, the reliability that follows is therefore based on *sensitivity* analyses and uncertainties that affect variables. Proposing a quantified measure by reliability stemming from different calculation methods makes reliability a decisive factor of safety. An equally important demand can also come from optimization. There are no methods exempt from uncertainties and therefore not tainted with errors. Faced with the serious problem of safety, numerous authors have proposed solutions for controlling probability, through limits and acceptable thresholds. This led to a more or less specific notion of *threshold probability*, to mark out certainty and thus to avoid failure from subsistence probability.

In our modest opinion, it seems convincing to speak of *threshold probability* and to generalize it to the whole structure. The structure is often complex and difficult, not to say costly, to put into action if simplistic hypotheses for the distribution of random variables are not carried out. It is in fact a constituent element from the overlapping design, having a maximal admissible value of failure probability P_f. From this assessment, we can pose the following questions:

– In what reliability index, no matter how well calculated, is the representative of an overlapping design ever singular?

– Is the issue of heterogeneity never managed by the variability of choice of materials which obey other specification requirements?

– Is the systematic generalization of a model somewhat penalizing?

The probability approach remains significant for calibrating semi-probability codes to determine partial safety coefficients so that the code ensures a target reliability level with the maximum possible homogeneity. In Europe, the level of threshold probability is akin to [10^{E-9} to 10^{E-6}] in a deemed period. In Canada, the code is controlled by principal rules, codes, and technical standards (see Table A.12 in the appendix). The necessity for a failure criteria being *limit state criteria* is apparent. In the Eurocodes, probability is expressed by a reliability index of 3.8 for the so-called ultimate limit states, valid for a service duration of 50 years. To do this, it is agreed to take action on the following analysis process:

1) To succinctly establish a reliability objective.

2) To clearly identify the modes of failure.

3) To formulate a criterion of failure (limit state function).

4) To identify dominant random variables and parameters.

5) To specify the probability distributions (and their appropriate criteria).

6) To calculate the reliability (according to the failure mode).

7) To evaluate the sufficiency of the estimated reliability.

8) To put together necessary modifications (if necessary).

Reliability methods can be distinguished by the following distinct levels:

– **Level I:** corresponds to specific values (resistance/charge) where each random variable is represented by its average value $\mu(\tau)$.

– **Level II:** random variables are characterized by their average value $\mu(\tau)$ and their variance RV (τ).

– **Level III:** the distribution law is formulated to be linked to all random variables. The aim is to obtain a reliability index and a breaking probability, and thus the quantified measure of security.

– **Level IV:** a level of reliability is attributed in function with various economic parameters from the revised specifications (costs of upkeep and repair, etc.).

From experience, level I methods are dedicated to studying the reliability of simple structures. Level II is valid for all reliability methods. The evaluation of the reliability of offshore platforms is traditionally dedicated to level III. The methods in level IV are normally employed for the analysis of very high design (*Reliability Assessment*) where we can mention the *Advanced Level II Method.* In Chapter 8, we have attached a study relevant to failure criteria from the point of view of fracture mechanics. The modes of failure are mentioned from the perspective of experimental mechanics. We have distinctly set a criterion which allows the separation of the following two zones:

– security zone, *S*;

– *failure* zone, *F*.

These two zones define the analyzed *element failure mode* of the structure. This could act as an applied stress, exceeding the so-called acceptable stress with reference to the material: shift of a spire, etc. This is the same definition as a limit state, as the failure mode expresses itself in terms of distance between a realization of a mechanical state of the component and a realization of an ultimate state of resistance (see mechanical liaisons, Resistance of Material, design, etc.).

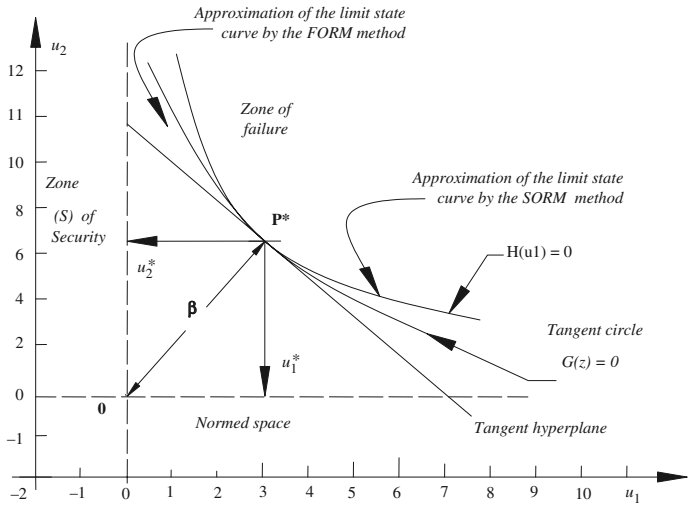


Figure 5.2. *Graph to show boundaries of failure and security*

Usually the choice of failure mode allows us to establish the equation of the *limit state surface* (or of failure) which is located in the *boundary* between the zone of security (*S*) and the zone of failure (*F*) (see Figure 5.2 where the boundary limits can be observed). At the same time, we express our modest opinion again on the

FORM–SORM that we do not favor with regard to their limiting restrictions, which can themselves be subject to caution.

5.3. First-order reliability method

This hypothesis is created so that the curves are negligible and the area of failure can be linearized. The probability of failure is thus written as follows:

$$\wp\{Y \succ S\} = \Phi(-\beta) \qquad [5.1]$$

The cost of this method, in the algorithmic sense, is entirely carried by the research from design point $\underline{u}^*$ (see Figure 5.2). Usually, other calculations do not necessitate the additional evaluation of P_f.

5.4. Second-order reliability method

In design, the zone of failure is represented by a paraboloid. The calculation for failure probability is thus more complex, and is limited to the result proposed by K.W. Breitung's method [BRE 84], whose algorithmic cost is relatively higher than in the case of FORM approximation. It is therefore necessary to calculate the curves κ_i, and thus the hessian of P_f into $\underline{u}^*$.We select a margin of security (M) for designing the breaking criteria and thus propose:

$$M = r - s = R\{z_1, z_2, ..., z_n\} - S\{z_1, z_2, ..., z_n\} \qquad [5.2]$$

$S(z)$ represents the stress (load) and $R(z)$ the resistance opposed to this load. $z = \left[z_1,\ z_2,\ z_3,\ ..., z_n \right]^T$ represents the vector of random variables, thus describing the element, its load, and its failure model. The margin of security (M) is considered a random variable as well as resistance (R) and solicitation $S(z)$. Conventionally, $M > 0$ represents the structure in the zone of security. When $M < 0$, it is placed in the zone of failure, and $M = 0$ corresponds to the set of realizations of z_1 (for a null margin) where the named surface is defined the *limit state surface.* For a formulation of the failure surface, we can define a function $g(z)$ as representing values taken by the margin of security. This function is called the limit state function. Conventionally, it verifies the following criteria:

1) **g(z) > 0**, which corresponds to the so-called *zone of security* (ZS).

2) **g(z) < 0**, which corresponds to the so-called *zone of failure* (ZF).

3) **g(z) = 0**, which corresponds to the *zone of failure (limited)* (ZL).

The limit state function usually comes from a deterministic analysis. This leads us to consider the reliability methods from level III, which provide an estimation of the probability P_f (*probability of failure*). This shows that the structure is located in the zone of failure and that it is time to act.

$$P_f = P\left(g(z) \le 0\right) = \int_{(F)} p_z\left\{z_1, z_2, ..., z_n\right\} dz_1 ... dz_n \qquad [5.3]$$

where p_z (z) is the *linked distribution* law of random variables z_1, z_2, z_3, …, z_n.

As integration presents highly complex mathematics, we have to make do with integration through Monte Carlo (MC) simulations. This moves the problem of convergence toward a stable result (a little dispersed with the smallest confidence interval). The reliability index is proposed in the literature to be a measure of reliability. Once the reliability index is known, and in particular the *most probable* point (ultra precise zone) of failure, the estimation of P_f by MC simulations becomes more simple. The measure of reliability by a reliability index is largely held in the structures' relevant Eurocodes.

During the last two decades, "*index mania*" in the field of reliability has spread widely in engineering schools: the Cornell index, the Hasofer–Lind index, the Madsen integral indicator of damage, etc. The function $g(z)$ plays a decisive role in terms of risk. It is thus notably important that it possesses regularity properties in terms of continuity and derivability. In fact, we are taking a rigorous mathematical step, which makes us cautious when using algorithms. We present here two indices, for educational purposes. We will not spend too much time on this subject, which has already been dealt with in depth in works such as [MAD 86, DIT 86].

5.5. Cornell's reliability index

Cornell's reliability index is defined as the ratio between the average value $E(M)$ and the standard deviation $\sqrt{RV\left(M\right)}$, by the following equation:

$$\beta_{cornell} = \left\{ \frac{E\left(M\right)}{\sqrt{RV\left(M\right)}} \right\} \qquad [5.4]$$

In the one-dimensional case, the failure surface is represented by the point $M = 0$. The idea expressed by this definition of the reliability index is that the distance from the average value $E(M)$ to the failure surface (expressed in number of intervals) provides an acceptable measure of reliability. If the limit state function

$g(z)$ is linear, then $E(M)$ and $\sqrt{RV(M)}$ will be calculated in function with base variable characteristics. In the case where $g(z)$ is not linear, Cornell's method suggests approaching it by its Taylor series from 1 to the average point. This evaluation of $\beta_{cornell}$ only demands the awareness of the first moments of the random variables. Also, we can resort to the MC simulations for non-explicit forms of $g(z)$.

To calculate $E(M)$ and $\sqrt{RV(M)}$, an example of statistical application is as follows.

Let a matrix of data (X, Y) be analyzed where (x-coordinates are the first column, y-coordinates the second column):

$$data = \begin{pmatrix} 1 & 1.25 \\ 2 & 1.65 \\ 3 & 1.55 \\ 4 & 1.25 \\ 5 & 1.26 \\ 6 & 3.00 \\ 7 & 3.25 \\ 8 & 3.89 \\ 9 & 2.54 \\ 10 & 3.11 \\ 11 & 2.00 \\ 12 & 2.11 \\ 13 & 2.33 \\ 14 & 2.00 \end{pmatrix}$$

Conventional statistics

$X = data^{\langle 0 \rangle}; Y = data^{\langle 1 \rangle}$

$n = rows(data); n = 14$

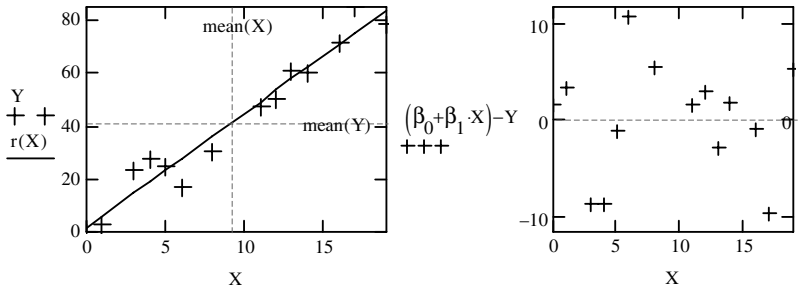
SD = Standard deviation

$$SD(X) = stdev(x) \times \sqrt{\frac{n}{n-1}}$$

Statistics	X-coordinates	Y-coordinates
Mean	Mean (X) = 9.357	Mean (Y) = 2.228
Median	Median (X) = 9.000	Median (Y) = 2.055
Standard deviation (SD)	Standard deviation (X) = 5.930	Standard deviation (Y) = 0.834
Variance RV	Variance (X) = 35.17	Variance (Y) = 0.696
Regression statistics		
Interception	b_0 = intercept (X,Y) = 1.72	
Slope	b_1 = slope (X,Y) = 0.054	
Correlation coefficient	Con(X,Y) = 0.386	
R^2	Con(X,Y)^2 = 0.149	
Covariance	CVAR(X,Y) = 0.774	
Standard error	Stdent(X,Y) = 0.801	
Scale plotting	$r(x) = b_0 + b_1 x$ Scale = $\max\left\|\overrightarrow{r(x-Y)}\right\| \times 1.1$	

Cornell's reliability index ($\beta_{Cornell}$) is calculated, in this case, as follows:

$$Let\ SD(Y)^2 = \sqrt{RV(M)}\ and\ E(M) = mean(Y) \rightarrow \beta_{cornell} = \left\{\frac{mean(Y)}{SD(Y)^2}\right\} = 3.2$$



Linear Regression of Y on X *Residual Plot*

_____ *The least squares method and* +++ *data : (X, Y)*

Figure 5.3. *Regression method (least squares)*

5.6. Hasofer–Lind's reliability index

The reliability index is based on a geometrical interpretation of $\beta_{cornell}$. The Hasofer–Lind index β_{HL} represents the smallest distance (in the Euclidian sense) from the origin to the failure surface in the space of normal, centered, reduced, and independent variables (standard space). The corresponding point to β_{HL} in the standard space is the *most probable point of failure*, the so-called functioning point, written as P^* (see Figure 5.1). This is the point on the failure surface for which the linked variables density is the highest.

$$\beta_{HL} = \left\|OP^*\right\| = \min\left\|u\right\|\ with\ g(u) = 0 \qquad [5.5]$$

where u is the transformed vector of z in the change in space (passage of variables from physical space to standard space). Let:

$$z = \{z_1, z_2, ..., z_n,\} \xrightarrow{T} u = \{u_1, u_2, ..., u_n\} \qquad [5.6]$$

This transformation, according to Rosenblatt, is only linear for physically Gaussian variables. In general, if Z_i is the random base variable taking the value z_i,

and the distribution law P_{zi}, the random normal, centered, and reduced correspondent U_i, taking the value u_i, results in the following:

$$\Phi\left(u_i\right) = P_{zi}\left(z_i\right) \; let \;\; u_i = T\left(z_i\right) = \Phi^{-1} P_{zi}\left(z_i\right) \qquad [5.7]$$

where $\Phi(\cdot)$ is the distribution function of a Gaussian centered reduced variable. The previous relation is thus only applicable when the variables Z_i are independent. In this case, the conditional distribution laws must be established, and the result of the Rosenblatt transformation depends on the order in which the variables are transformed. If, for example, X follows a normal law $N(\mu_z, \sigma_z)$ or even for a log-normal distribution $LN(\mu_z, \sigma_z)$, the transformation is explicit and presents itself in the following form:

$$\left\{ u = \left(\frac{z - \mu_z}{\sigma_z} \right) (Normal) \; or \; u = \left(\frac{\ln(x) - \mu_{Lx}}{\sigma_{Lx}} \right) (Galton) \right\} \qquad [5.8]$$

In fact, the Rosenblatt transformation carries a vector $\underline{X}$ of any law to a vector $\underline{U}$ of the same dimension but with independent, Gaussian, centered, and reduced components. This transformation takes place in the following two stages:

$$\left\{ \begin{array}{l} Stage\ 1 \; : z_i = F_{i|1,\ldots,i-1}\left\{ X_i \left| X_1, \ldots, X_{i-1} \right. \right\} \\ Stage\ 2 \; : U_i = \Phi^{-1}\left(Z_i\right) \end{array} \right\} \qquad [5.9]$$

– The first stage of this transformation is similar to the general sampling method, where Z_i are variables *ii*d of law U ([0; 1]) and the function $F_{i|1,\ldots,i-1}$ represents the distribution function of variable $X_i \left| X_{1,\ldots,X_{i-1}} \right.$.

– The second stage consists of a choice of an obligatory Gaussian variable as it is the only spherical law for independent components.

The reliability indices β_{cornell} and β_{HL} meet when the failure surface is linear. In the case of the failure surface not being linear, it will be (as already noted) approached by Taylor's theorem from 1 to the point P^*.

It has been succinctly observed that the difficult part of reliability methods consists of finding the functioning point, which is, in fact, the result of a minimization procedure under stress. The Rackwitz–Fiessler [RAC 79] algorithm, based on the projected gradient method, provides a simple response to this problem. Leaving any point in the standard space, and being part of the failure surface, we can

determine the following point $u_{(k+1)}$ by projecting the former at the end of the origin, parallel to the gradient of the limit state function $\nabla g(u)$ and directed toward the zone of failure (see Figure 5.1). This projection is displaced with the aim of taking into consideration the initial point situated outside the failure surface. By correctly applying this algorithm, we can obtain a succession of points, defined in the literature as follows:

$$u_{\{\kappa+1\}} = \left(u_{\{\kappa\}}^{T} \alpha_{\{\kappa\}} + \frac{g\left(u_{\{\kappa\}}\right)}{\left\| \nabla g\left(u_{\{\kappa\}}\right) \right\|} \right) \alpha_{\{\kappa\}} \; with \; \alpha_{\{\kappa\}} = -\left(\frac{\nabla g\left(u_{\{\kappa\}}\right)}{\left\| \nabla g\left(u_{\{\kappa\}}\right) \right\|} \right) \qquad [5.10]$$

The Rackwitz–Fiessler algorithm [RAC 79] takes into account the numerical precision of the physical model for choosing a calculation increment used at the time of estimating the partial derivatives of the limit state function (finite differences). The procedure used in this algorithm allows us to correctly estimate gradient vectors. It also allows us to economize limit state function calculations. In this book, we do not see the worth in adding to an already expendable literature [RAC 79]. The educational aim of this work is not to go into depth on these algorithms but instead to explain their simple and appropriate use.

In addition, the *advanced level II* reliability methods can be applied when they are known at the time, the distributions of base variables, and the limit state surface $g(z)$. The measure of reliability is given by probability on subsistence, thus the expression:

$$P_s = P\left\{ g(z) \succ 0 \right\} = 1 - P_f = \int_{(S)} P_z\left(z_1, z_2, ..., z_n\right) dz_n \qquad [5.11]$$

This integral cannot be calculated analytically. We find ourselves confronted by the same methods as for level III. Hence, we favor the option of reliability indices to be able to better approximate P_f. When the base variables follow a normal linked distribution law, and the fracture surface is a hyperplane [$g(z)$ linear], the breaking probability can be calculated by the following simple equation:

$$P_f = \Phi\left(-\beta_{Cornell}\right) \qquad [5.12]$$

In the case where the failure surface is not a hyperplane as shown in Figure 5.1, but the base variables are of normal law, we can often obtain a good approximation of the failure probability with the help of the Hasofer–Lind reliability index β_{HL}:

$$P_f = \Phi\left(-\beta_{HL}\right) \quad [5.13]$$

This approximation brings us back to the previous case in approaching the fracture surface, near the functioning point (or the most probable breaking point), by a hyperplane. In practice, nothing can prove that the variables are normal or that the failure surface is linear. However, the two previous cases have shown that we can easily calculate the breaking probability from the respective reliability index. Having determined the index β_{HL}, we can linearize the failure surface to the functioning point P^* to obtain $g(u)$ as follows:

$$g\left(u\right) \approx L\left(u\right) = \beta_{HL} + \sum_{i=1}^{n} \alpha_i u_i = 0 \quad [5.14]$$

The failure probability is thus given by the following FORM:

$$P_f = P\left\{g\left(u\right) \leq 0\right\} = \Phi\left(-\beta_{HL}\right) \quad [5.15]$$

5.7. Reliability of material and structure components

At a conceptual level, i.e. "calculations and design", mathematics of reliability is measured by the evaluation of probability so that the *entity* realizes its task, in the given conditions, in a specified time. Hence, the reliability of a material at the time τ is the probability that the non-negative random variable, X, representing the lifecycle of the component, is greater than τ. We therefore suppose the expression of reliability as follows:

$$\Re(\tau) = P\left\{X \succ \tau\right\} = 1 - P\left\{x \leq \tau\right\} = 1 - F\left(\tau\right) \quad [5.16]$$

where $F(\cdot)$ is the cumulative distribution function known by the usual statistical name of the probability distribution function. The conditional probability of *subsistence* of a component or a system, beyond time τ, allows the definition of the *failure rate* of this entity:

$$\lambda(\tau)d\tau = \frac{f(\tau)dt}{1 - F\left(\tau\right)} \quad [5.17]$$

where $f(\cdot)$ is the probability density function of $F(\cdot)$. Reliability mathematically stems from an integration technique, which links the two previous principal functions of probability. Hence, let:

$$Log\left\{1-F\left(\tau\right)\right\}=-\int_{0}^{\tau}\lambda(x)dx \qquad [5.18]$$

From [5.16], we can deduce the reliability function by:

$$\Re(\tau)=P\left\{X\succ\tau\right\}=1-P\left\{x\leq\tau\right\}=1-F\left(\tau\right)=Exp\left\{\int_{0}^{\tau}\lambda(x)dx\right\} \qquad [5.19]$$

In electrical components, the failure rate $\lambda(\tau)$ is consistent with time τ. Expression [5.18] expresses an exponential function, hence the overuse of this law in electronics, in automatism, and even in the mechanics of offshore structures assembled in a series. The reliability thus reduces its expression to the following:

$$\Re(\tau)=Exp\left\{\lambda.\tau\right\} \qquad [5.20]$$

From [5.19] and the preceding equations, we can understand that the probability of failure is proved random. Consequently, we cannot "*establish*" a precise maintenance frequency. It is for this reason that we instead speak of the *instantaneous failure rate* $\lambda(\tau)$ of a component. This rate can be understood between τ and $\tau + \Delta\tau$. As at the time $\tau = 0$ failure would not be an issue, we propose:

$$\lambda(\tau)=\lim_{\Delta\tau\to0}\left(\frac{\Re(\tau)-\Re(\tau+\Delta\tau)}{\Delta\tau\times\Re(\tau)}\right)=\frac{dF(\tau)}{d\tau}\frac{1}{d\tau}=-\frac{dLog\Re(\tau)}{d\tau}=\frac{f(\tau)}{\Re(\tau)} \qquad [5.21]$$

5.8. Reliability of systems in parallels and series

Assembly systems of irreparable materials are often grouped in parallels and sometimes in series. When they are failing, their reliability laws in function with the elementary material reliability are evaluated differently, depending on whether they are redundant or sequential.

5.8.1. *Parallel system*

The probability of this failure system of components (C_n) at the time τ is proportional to the probability of all the elements in the system breaking down at τ. The reliability of the global system is written as follows:

$$\Re_{system}\left(\tau\right)=1-P\left\{C_1\leq\tau,C_2\leq\tau,C_3\leq\tau,...,C_n\leq\tau\right\} \qquad [5.22]$$

For independent events, we propose the following:

$$\Re_{system}(\tau) = 1 - \prod_{i=1}^{n} \{1 - \Re_i(\tau)\} \quad [5.23]$$

The failures expressed by exponential distributions appear as follows:

$$\Re_{system}(\tau) = 1 - \prod_{i=1}^{n} (1 - r_i) = 1 - \prod_{i=1}^{m} \left[1 - Exp\{-\lambda.\tau\}\right] \quad [5.24]$$

5.8.2. *Parallel system (m/n)*

Sometimes, an assembly allows the system to function even if at least m materials among the *n components* are still functioning. The number of materials in use thus follows a binomial law of parameters r and n, and the system's reliability is written as follows:

$$\Re(\tau) = \sum_{m} \left(\prod_{i \in A_m} \Re_i \right) \times \left(\prod_{i \in A_m} \{1 - \Re_i\} \right) \quad [5.25]$$

A_m represents the minimum boundary (from 1 to m). This arrangement consists of at least κ components in use. If all the components are identical, with a basic reliability r, we can suppose:

$$\Re(\tau) = \sum_{\kappa=m}^{n} C_n^{\kappa} r^{\kappa} \times (1 - r)^{n-\kappa} \quad [5.26]$$

5.8.3. *Serial assembly system*

At the time τ, a system (a mechanism, parts, etc.) Ω must have a probability of r_i for it to function. Each material component Ω_i of the mechanism will be functional at the time τ. If Ω_i corresponds to the component event M_i, functioning at the time τ, its reliability $\Re(\tau)$ takes the following form:

$$\Re(\tau) = P\{\Omega \succ \tau\} = P\{\Omega_1 \succ \tau; \Omega_2 \succ \tau; \Omega_3 \succ \tau; ...; \Omega_n \succ \tau\} \quad [5.27]$$

When the events are independent, we can suppose:

$$\Re(\tau) = \prod_{i=1}^{n} \Re_i(\tau) = \prod_{i=1}^{n} r_i \qquad [5.28]$$

Subsequently, the failure rate of the serial system is written as:

$$\lambda_{system}(\tau) = -\frac{dLog\Re_{sys}(\tau)}{d\tau} = -\frac{dLog\prod_{i=1}^{n}\Re_i(\tau)}{d\tau} = \sum_{i=1}^{n}\left(-\frac{dLog\Re_i(\tau)}{d\tau}\right) \qquad [5.29]$$

As previously noted, the exponential law is almost omnipresent and is therefore responsible for the distributions of probabilities:

$$\text{As } \sum_{i=1}^{n}\lambda_i(\tau) = \lambda_{system}(\tau) \text{ then } \Re_{system}(\tau) = \exp\left\{-\tau \times \sum_{i=1}^{n}\lambda_i\right\} \qquad [5.30]$$

In the design stage of mechanical engineering techniques, it is clear that components are assembled with respect to their functional valuation. *Geometrical Products Specifications* (GPS) standards help to determine sizing (valuation). Assemblies are only dependent on the functional layout of the mechanism's parts, defined in a finished product outline. It is fairly simple to calculate and assess these components' reliability. In civil engineering structures and constructions in general, the issue is completely different and new in the mathematical and physical world.

Until recently (toward the end of the 19th Century), constructions were designed and executed in part intuitively (empirically), as in ancient castles, bridges, pyramids, etc. Metallurgy, informatics (calculating with finite elements), new materials, and, above all, the concern for *safety* have all led to principles of calculation based on the *resistance* of materials. Since then, the need for guaranteed safety has been understood and adopted. The notion of acceptable stress [σ, in MPa] in resistance of material is fully quantifiable and justified. Design, while remaining an "art", is becoming more and more a geometrically controlled "art".

The classic safety ratio $S \leq [R_{Max\_Effort}/R_{Material}]$ served to reassure designers that the maximum effect in the so-called *critical* zone (an intuitive concept) remained smaller than an acceptable stress, which experience has proved correct. Through time, safety coefficients have, therefore, provided a so-called *clumsily quantitative* measure (often costly) that we can qualify as the *probability of fracture*. This term is largely justified, as the resistant effects of materials constitute random variables

(effects of power, wind, water, load, fatigue, etc.). Their influences justify the probabilistic approach to what is conventionally called *failure* or *fracture.*

The engineering approach has evolved, thanks to probabilities. The very definition of a construction's safety, *deemed safe*, linked to its probability of breaking, is now less important than an anticipated fixed value which is dependent on numerous factors, including the *lifecycle.* It would be too simplistic to assume that the breaking factors have already been assessed. The economics of keeping the structure in use is absolutely dependent on the quality of materials, the ingenuity of the construction, and hazards of the nature of use (fatigue, everyday use, etc.).

It is generally agreed that imposing a *single,* costly, safety coefficient on the resistance of materials is not sufficient. The reliability index β as a quantitative measure for the probability of failure (fracture) has been accepted by scientists from across the world. This current work, dedicated to reliability, attempts to justify the use of this reliability index to make constructions safe. We will first introduce the theoretical approach, which has already proved its worth in terms of reliability. Then we will comment on the assumptions which favor or do not favor the calculation results.

PROBLEM.– Imagine any structure whose safety we will check by scalar dimensions. The effects of *stress* (represented by *S*) meet resistance *R*, acceptable by two random independent variables, presumed Gaussian, with the respective averages and standard deviations of μ_S, μ_R and σ_S, σ_R. The relative distance is written as follows:

$$Z = \{R - S\} \qquad [5.31]$$

As a solution to be guaranteed, Z is strictly ≥0.

$Z = \{R–S\}$ is a RV which follows an average Gaussian $\mu_Z = \{\mu_R - \mu_S\}$ and standard deviation $\sigma_Z = \sqrt{\sigma_R^2 - \sigma_S^2}$. With this in mind, the probability of failure P_f of the structure is the result of the integral representing Gaussian law as follows:

$$P_f = P(Z \le 0) = \frac{1}{\sqrt{2\pi} \times \sigma_Z} \int_{-\infty}^{0} \exp^{\frac{x-\mu_Z}{2\sigma_Z^2}} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\mu_Z}{\sigma_Z}} \exp^{\frac{u^2}{2}} du = \Phi\left(-\frac{\mu_Z}{\sigma_Z}\right) \qquad [5.32]$$

where Φ(·) is a tabulated function from the normal law. Recall that this hypothesis is only valid if we assume that the distribution law is Gaussian. Therefore, the reliability (or safety) index β becomes the result of a simple tabulated reading from a

changing variable of the classical Gaussian law. We thus suppose the following expression of the β index:

$$\beta = \frac{\mu_Z}{\sigma_Z} = \frac{\mu_R - \mu_S}{\sqrt{\sigma_R^2 - \sigma_S^2}} \qquad [5.33]$$

Figure 5.4 shows the relationship between the probability of failure and the reliability index β. This *reliability index* β, by carrying [5.33] in the last expression of [5.32], provides the famous expression of the construction's failure probability:

$$P_f = \Phi\left(-\frac{\mu_Z}{\sigma_Z}\right) = \Phi(-\beta) \qquad [5.34]$$

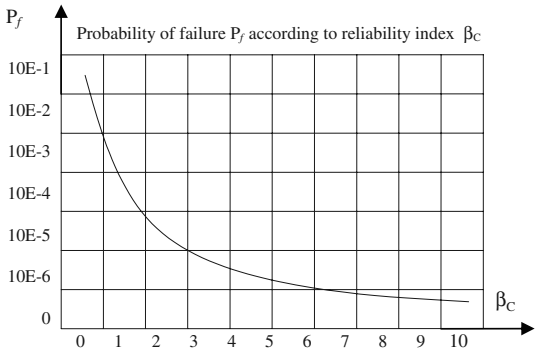


Figure 5.4. *Relationship between the reliability index β and the probability of failure P_f*

Equation [5.34] is true for multiple RV functions, which are independent of one another. Imagine R to be the resistance and S the stress of a real structure. The expression of this generalization appears as follows:

$$R = g_R\{X_1, X_2, X_3, ..., X_m\} \text{ and } S = g_S\{X_{m+1}, X_{m+2}, X_{m+3}, ..., X_n\} \qquad [5.35]$$

The purpose of this operation is to find what is conventionally known as a *limit state function,* which accounts for all RV. Hence, the limit state function will present itself in terms of the representativeness of all the RV:

$$Z\{R,\ S\} = g\{X_1, X_2, X_3, ..., X_n\} \qquad [5.36]$$

Going back to the expression of probability of failure according to the safety index, the structure's probability will be:

$$P_f = P\{Z \le 0\} = \Phi\left(-\frac{\mu_Z}{\sigma_Z}\right) = \Phi(-\beta) = P\{g(Z) \le 0\} \quad [5.37]$$

This is in fact the limit state function. If this function does not display linearity, it is advisable to linearize it in proximity of the most probable point of coordinates (μ_1, μ_2, …, μ_n), taking into account only the first derivatives. If the limit state function is non-linear, it should be linearized in proximity of the point which represents the *probable state*, ignoring secondary derivatives. This gives the expression of Z as follows:

$$Z \cong g(\mu_Z) + \sum_{i=1}^{n} \frac{\partial g}{\partial X_i}\{X_i - \mu_{X_i}\} \quad [5.38]$$

To simplify integral calculations with at times "cumbersome" and even "irritating" remnants for precision purposes, we can employ the standard space. This corresponds to the RV(X^*) from the X variables by the following transformation:

$$X_i^* = \left(\frac{X_i - \mu_i}{\sigma_i}\right) \quad [5.39]$$

By the simple transformation into RV from centered and reduced Gaussian variables around the ("real") average, and based on the fact that these RV are ordinary and more importantly *always independent*, we can thus suppose:

$$X_i^* = \left(\frac{X_i - \mu_i}{\sigma_i}\right) = \Phi^{-1}\{F(X_i)\} \quad [5.40]$$

We can clearly see that $F_i(\cdot)$ is the *distribution function* of X_i. This space is none other than the system of reduced coordinates from a common Gaussian law.

Imagine a corresponding point C [$X_i = \mu_i$] which represents the probable state of the structure. $Z = g(X^*)$ constitutes the space (surface) of the normal functional structure, i.e. without weakness or failure. Through the orthogonal projection (P) from C on the surface, we can call the functioning point of the structure a probable point of weakness which would be the closest possible point of failure. By analogy, this is the point that represents a point of stress before reaching the most acceptable stress, as in classic resistance of material. The reliability index β is, therefore,

defined as the distance between the origin and the functioning point of the structure. In other words, it is the minimum distance between the probable state and the limit state surface, which is the Euclidian distance. The case of two independent Gaussian RVs confirms the following relation:

$$X_p^* = \left\{ X_C^* - \beta \times \alpha \right\} \quad [5.41]$$

where X_p^* and X_C^* are respective vectors of the coordinates from point *P* and point *C*, and α represents a common vector called the *vector of influence*. Limit State Functions, the coordinates $Z = g(X^*)$ are provided by linearization which is expressed as follows:

$$\alpha_i = \frac{\left(\left. \frac{\partial g}{\partial X_i^*} \right|_P \right)}{\sqrt{\sum_{i=1}^{n} \left(\left. \frac{\partial g}{\partial X_i^*} \right|_P \right)^2}} \quad [5.42]$$

During our various design projects, we have never come across the functioning point of a structure at the time (τ). Through simulation, we are getting closer, but it is impossible to predict this point with great precision. There are some specialized route algorithms that can calculate the reliability index. In the subsequent case studies, we will provide an example to fully back up our assertions. The following points summarize the reliability approach in the mechanics of materials and structures:

1) It defines the equation of limit state $Z = g(X)$.

2) It transforms random base variables into reduced centered RV.

3) It postulates the initial position of the functioning point (*P*). Typically, this initial point may be identical to the point of possible state $X_p^* = X_C^*$.

4) It evaluates partial derivatives and the vector of influence (α) to point *P*.

5) It resolves β by means of [5.33] and [5.34] in the limit state equation $g(x) = g\left\{X_1, X_2, X_3, ..., X_n\right\} = Z\left\{R,\ S\right\}$.

6) It re-evaluates the functioning point with the acquired reliability index β.

7) It repeats steps 4 to 6 until the convergence of β.

5.9. Conclusion

Indicators of safety are based on reliability indices, which are often detrimental in terms of calculation costs, and are not completely "reassuring" in terms of the methods, stability, or robustness. The omnipresence of statistical distribution laws reinforces the importance of the probabilistic approach. It is important to not choose a distribution law *a priori*, but instead to properly *determine* its appropriateness for the predicted case.

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Chapter 6

Fracture Criteria Reliability Methods through an Integral Damage Indicator

6.1. Introduction

In this chapter, we will be using random variables from the cracking law, in this case {a_0, a_c, (C, m), and N}, to calculate the failure probability of a structure joined in a cruciform by four distinct joining processes: SAW, FCAW, SMAW 56, and SMAW 76. The latter were the object of a commendable experimental campaign [MEL 83, COR 67a. COR 67b, LIN 73, FRE 47, JOH 53, BAS 60, LAS 92, GRO 94, GUM 08, WAT 70, KOZ 83, WÖH 60, YAM 77]. The development of reliability theory in structures dates back to the 1920s. Back then, people relied on material choice and assembly process choice. At that time, there were dazzling developments in physical metallurgy, which can account for this approach. Nowadays, however, we rely on numerous method applications based on probability and statistics. With safety considerations in mind, these essential methods have largely developed and established themselves with their apparently random data, which until now was obligingly considered deterministic. For example, how can we consider the intrinsic parameters of material (C and m) in Paris's law to be deterministic, when we don't know the degree to which the material is consistent? And how can we suppose that the stress intensity factor (*s.i.f,* ΔK) compared to tenacity is deterministic, when we don't know in advance that the load is in itself variable. In several cases, it is unpredictable (wind, ice, aging, saline erosion on offshore structures, etc.).

In 1926, Mayer and subsequently in 1960 E. Basler *et al*. [BAS 60] proposed an estimation for a safety margin, which was then continued in 1967 by C.A. Cornell

[COR 67a, COR 67b], who formulated the reliability index βc. The Cornell formulation was not greatly implemented because of the *unknown factors* which it implicated. Already numerous authors, including C. Forssel, had worked on the same idea, not just confined to this period, but it was later on (1967) that the idea was developed by Cornell with a qualified success. Optimization of constructions became difficult toward the 1950s. For economic reasons, in 1953 A.I. Johnson [JOH 53] proposed a *comprehensive* formulation of the safety margin theory, by introducing the statistical theory developed by W. Weibull in 1939. This statistical approach is so universal that we felt it necessary to include it in the present work, to statistically simulate initial cracking (a_0) in a cross-joined structure. In 1947, the author A.M. Freudenthal [FRE 47] presented the fundamental problems of structural safety measures. It was the first time in the history of structural safety that unknown factors were treated as defects. This is one of the reasons that we felt it would be useful in this work to tackle the metrology approach of errors and uncertainties in measuring.

In the 1970s, there was an initial transition period where probability theory became almost inescapable in the analysis of crack spreading in components and other structures. It is worth adding that informatics has played a key role in terms of the powerful programs which support complex calculation models. The same reflection can be made for the role which informatics has played in the simulation of structures using finite elements. In fact, toward 1970 many research papers on material and structural reliability began to appear again, among which was one by Ferry-Borges and Castanheta [FER 71]. There are many conflicting opinions on the use of safety margin methods. The titles of these papers are at times eye-catching, but not necessarily conclusive. This state of affairs is essentially due to the systematic and unjustified application of statistical distribution laws or uncertainty. The latter has often been put aside, taking for granted that complex calculations from powerful computers would not be questioned. And yet, the error is sometimes at this level. It is not the statistical laws which are erroneous.

Among the notable works on approximation methods for safety from the 1960s, it is worth referring again to N.C. Lind *et al.* [LIN 64]. There were also some commendable works from 1973 to 1983 by authors such as M.K. Ravindra and N.C. Lind [RAV 83] on the optimization of calculation methods for structural safety, which allowed a reliable calculation code to be proposed. Through the 1970s and 1980s, other works emerged, aiming for a better simulation of reliability indices in terms of random multi-variables. Independently from one another, O. Ditlevsen [DIT 73] and N.C. Lind [LIN 73] proposed a method to resolve the problem of invariance in the reliability index. They argue that the Cornell index (β_c) [COR 67] is not constant. This poses yet more problems linked to structural mechanics as well as to the statistical methods inherent in model processing.

To resolve the problem of index invariance, A.M. Hasofer and N.C. Lind [HAS 74] attempted to clarify the notion of invariance limitation in the reliability index. This index was made popular by the so-called *first-order reliability method* **(**FORM). To do this, they suggested a new calculation method which would allow the calculation of the safety index for a law of correlated and uncorrelated random variables. Whilst on the subject**,** we can refer to the critical idea formulated in the same year (1974) by D. Veneziano [VEN 74]. She presented a scientific report (R74-33) to the Department of Civil Engineering at Massachusetts Institute of Technology (USA). This report rested on its "Contributions to Second Moment Reliability Theory". Numerous papers have been revised on the basis of a Canadian calculation code CSA [CSA 81] and also the English CIRIA [CIR 81], where the limit state function (M) is based on the probabilistic approach.

We know that human error can be the cause of insecurity in structures and other components, which at times causes failure. With this in mind, some serious works have been carried out. We can refer to the research papers of R.E. Melchers and M.V. Harrington [MEL 83], not forgetting the commendable scientific contributions resulting from the works of H.O. Madsen, S. Krenk, and N.C. Lind [MAD 86] *et al.* The work that particularly springs to mind is that of Professor Knut M. Engesvik [ENG 82].

This current work has already entered the domain of uncertainties in fracture mechanics through fatigue. In our experience, this book is one of the rare works, along with Danielle Veneziano [VEN 75], which emphasizes the uncertainties in the subject. The works of Professor K.M. Engesvik have not been used enough outside of the Scandinavia. However, we can pay homage here to their contributions, whose educational benefits no longer need to be proved. Our inspiration to include a chapter dedicated to errors and uncertainties comes from what we have just mentioned. We can also add the recent techniques [GUM 08] from the *Guide to the Expression of Uncertainty in Measurement*.

Within the domain of applied science, there is no such notion called *absolutely exact* measures, because *measures* are always tainted by errors which can be attributed to various causes, both human and material. Qualifying an error to then quantify an uncertainty proves that we *doubt* the validity of a measuring result. Therefore, evaluating uncertainties on measures stemming from errors can sometimes be quite a complex task. To *mark* the influencing factors on which the kind of measure depends on, we will first develop some mathematical principles.

The development of measuring uncertainty from the so-called *classic or true value* approach toward an uncertainty approach has led to the reconsideration of certain concepts. This comes from the fact that instruments and measurements do not provide this *true value*. We can therefore realize that it is possible to

distinguish two categories of errors. It is advisable to treat them differently in the *spreading of errors*. However, no justified rule is given to combine systematic and random errors adding to a total error thus characterizing the measuring result. It is, however, possible to estimate an upper limit of the total error, *clumsily* named uncertainty.

The components of measurement uncertainty are conventionally grouped into two categories. The first, type A, is estimated by statistical methods. Type B is estimated by other methods based on laws founded *a priori*. It is in fact the user who must evaluate the sources of error. The manufacturer provides some data such as the *class of device*, the *standard*, the *resolution,* etc. In addition, a well-founded knowledge based on experience is needed. By combining the two categories A and B, we can obtain composite uncertainties, marked $U_c(y)$. In the case of the type B approach, the *Guide to the Expression of Uncertainty in Measurement* of 1993 (revised in 1995) provides a description of the uncertainty approach. The emphasis is placed in the mathematical treatment of uncertainty, with the help of an explicit measuring model where the measurand is characterized by a value of a unique nature. The aim of measuring in the uncertainty approach is not to determine a true value. It is necessary to draw up an *assessment of errors*. In 2008, the VIM 3 increased the accuracy of the terminology used in metrology. In other areas of engineering science, the focus was on reliability indices.

On the basis of what has just been said, the aim of measuring is to establish the probability that the measured values are compatible with the definition of *measurand, measure, measuring, true value,* etc. These are not terms to be tampered with. The terms given in the 3rd edition of the VIM, along with their formats, are in keeping with ISO 704, ISO 1087-1, and ISO 10241. For example, the terms “measure” and “measuring” have several meanings. This is the reason that the word “measuring” was introduced to qualify the action to measure. The word “measure” occurs in many instances in the VIM. Measuring instrument, measuring device, measuring unit, and measuring method are also noted. The source factors of uncertainties can generally be of three types as follows:

– human: manipulation, maintenance of trial installation, etc.;

– technical: trial method, properties of tested material, calibration, etc.;

– environmental: trial environment and random components.

In metrology, measure is an experimental operation which aims to determine the value of a physical size. To realize this, a *measuring method* is used. This necessitates the use of devices and measuring instruments which in many cases can reveal the source of errors. This allows us to easily see that metrology heavily rests on concepts of uncertainty. Uncertainty takes account of the way in which size has

been measured. It therefore expresses the confidence that we can give to a result. It is well known that measuring requires the use of instruments. This implicates *calibrations* and manipulations, thus creating appropriate procedures and calculations. It is for these motives that systems of calculation and *measuring* have been introduced.

6.2. Literature review of the integral damage indicator method

Before presenting the results from probabilistic processes of random variables a_0, m, and N_{cycles} from the cracking law on a cross-joined structure [LAS 92, GRO 94, GRO 95, GRO 98] using an integral damage indicator method, it is highly useful to first present the hypothesis which supports metrological measuring of experimental data. We had calculated, through the Monte Carlo (MC) simulation of random variables from the cracking law, F{a_0, m, and N_{cycles}}, a limit state function for four joining processes [LAS 92]. This task has allowed us to find the failure probability and the safety index by joining processes. We will revisit these results in their entirety in [GRO 94]. In the following, we will present another approach, introduced by Madsen *et al*. [MAD 86] to calculate:

a) a limit state function M = g(z);

b) a failure probability P_f;

c) a safety index β_c.

This method is succinctly laid out in the work of Madsen *et al*. [MAD 86]. Other authors have applied it in commendable works, such as the paper by F. Kirkemo [KIR 88]. The general problem in the reliability of joined structures is the unavoidable unknown factors that the probabilistic approach takes into account. The issue of uncertainties [VEN 75, ENG 82] remains prominent. The methods which are used in structural reliability problems are not scientifically agreed upon. These methods are subject to caution when the hypothesis and the parameters which make up the probabilistic formulation of the *true value* are not clearly presented.

In 1987, H.O. Madsen argued that reliability was still vital. Of course, Madsen, like Lassen, came from Northern Europe where offshore platforms had spread. The lifecycle predictions for solid structures (if laboratory trials are carried out) are generally analyzed by a probabilistic-statistical approach, which takes the following functions into account:

– probability density f(x) and the distribution function F(x);

– failure rate λ (MTBF: time before failure, cf. Chapter 1);

– auto-correlation function;

– reliability index β;

– spectrum of probability density, the transfer function, etc.

In addition, *material dynamics and geometry* pose a serious problem which causes important dependents to be put into a differential equation by the Paris law. Predictions of lifecycle remain randomly dependent on the physical parameters that the cracking law permits. The general problem realized by D. Veneziano [VEN 75], M.K. Engeskik [ENG 82], Madsen [MAD 85], and Ditleysen [DIT 86], to name just a few, comes from the statistical uncertainties in mechanical structure reliability. It helps to understand that a singular structure would not forcibly impose a calculation by the first-order reliability method/second-order reliability method (FORM/SORM).

The Cornell reliability index [COR 67b] from the FORM has been proven sufficient to provide a concise idea of the safety index (β). The works of Madsen in our opinion are still incredibly relevant. His method of an integral damage indicator is favored for its simplicity of execution.

6.2.1. *Brief recap of the FORM/SORM method*

Failure probability is represented by P_F. The reliability index β_c can then be calculated, as previously shown:

$$\beta_c = \left\{\Phi^{-1} \times P_R\right\} = \left\{\Phi^{-1} \times P_F\right\} \ \textit{where} \ \ P_R = \left\{1 - P_F\right\} \qquad [6.1]$$

where:

P_R is the subsistence probability known under the general term of reliability (R).

P_F is the fracture probability known under the general term of failure probability.

The limit state function g(z) = M, in the space U**,** is represented in Figure 5.1. M divides the U-space into two distinct zones: the zone of failure, represented by F or R, and the zone of security, represented by S. In the FORM approach, the analysis of the limit state function is estimated by a hyperplane tangent to the Euclidian plane U*. The latter is the point where the limit state function surface is stopped at origin, i.e. the point where the zone of failure is located in a large area of probability density.

In the SORM approach (see Chapter 5), the analysis of the limit state function is estimated by a hyperboloid rotation surface, tangent to the same hyperplane and

curves to the Euclidian point U*. This corresponds to reliability by the analytical expression.

6.2.1.1. *FORM*

$$P_{Failure} = P_F = \Phi(-\beta_c) \qquad [6.2]$$

where:

P_f is the failure probability;

β_c is the reliability index.

6.2.1.2. *SORM*

$$\left\{ \begin{array}{l} 1 - P_R = \Phi(-\beta) \times \prod_{j=1}^{\kappa-1} \sqrt{(1-\beta_{\kappa j})} + \left[\beta \times \Phi(-\beta) - \varphi(\beta)\right] \times \\ \qquad \times \left\{ \prod_{j=1}^{\kappa-1} \sqrt{(1-\beta_{\kappa j})} - \prod_{j=1}^{\kappa-1} \sqrt{1-(\beta+1)_{\kappa j}} \right\} + \\ \qquad + (\beta+1) \times \left[\beta \times \Phi(-\beta) - \varphi(\beta)\right] \times \\ \qquad \times \left\{ \prod_{j=1}^{\kappa-1} \sqrt{(1-\beta_{\kappa j})} - \mathrm{RE}\left\{ \prod_{j=1}^{\kappa-1} \sqrt{1-(\beta+i)_{\kappa j}} \right\} \right\} \end{array} \right. \qquad [6.3]$$

where:

B is the length between the U* point and κ_j to the curve points in the Euclidian space [U*], where κ is the number of base variables;

i is the imaginary part (imaginary unit of a complex);

RE{·} is the true part of a complex.

This method has experienced numerous applications, in this case the applicationone used by Tvedt [TVE 83]. He is cautious to differentiate between the FORM and the index β_c. Relation [6.2] just allows us to see that it is possible to calculate the index β_c by the FORM.

6.2.2. *Recap of the Hasofer–Lind index method*

The Hasofer–Lind index [HAS 74], β_{HL}, allows us to see the distribution laws for random variables and to be invariant for the formulation of a same limit state. It

causes the transformation of base variables into Gaussian variables Z_i, and then into reduced centered Gaussian variables U_i, by the simple classical relationship:

$$U_i = \left\{ \frac{Z_i - \mu(Z_i)}{\sigma(Z_i)} \right\} \quad [6.4]$$

where $\mu(Z_i)$ and $\sigma(Z_i)$ are the average and standard deviation of the considered RV, respectively.

In the 1970s, this formulation had a considerable success. The area of failure is constructed in a system of coordinates linked to reduced centered variables. The curves of equal probability density are therefore centered hyper-spheres at origin (see Chapter 5). The representation of the Hasofer–Lind index is thus expressed as follows:

$$\beta_{HL} = \min \sqrt{\left(\sum_{i=1}^{n} U_i^2 \right)} \text{ with } D \prec U_i \quad [6.5]$$

where $D(U_i)$ is the area of failure in the system of reduced random variables U_i. The calculation of β_{HL} requires the use of repeated methods when the zone of failure is not linear.

6.3. Literature review of the probabilistic approach of cracking law parameters in region II of the Paris law

In the following, we will present the results in terms of probabilistic calculations of initial cracking (a_0). The aim is that the initial crack is an indicator of the evolution of the distribution laws. The initial crack a_0 is the focus of numerous theoretical disagreements. A reliability calculation with the help of the PROBAN software has been found by Professor Lassen *et al.* [LAS 92], on experimental data from a cross-joined structure, whose results are shown in Table 6.1.

Our calculations [GRO 94, 95, 98], on a deducted sample from the same experimental data, have allowed us to determine an average crack of 0.0019 mm. The distribution of average cracks follows a Weibull law of two parameters, with a tendency toward an exponential law. The distribution model presented in the report by T. Lassen considered a development of initial cracks between 0.35 mm and 0.5 mm. Whatever the method used to obtain the safety margin of a structure, its reliability index β, or inherent failure probabilities, it is advisable to note the following:

– simulation through fracture mechanics must strictly observe the appropriate material choice for the structure, to aid assembly;

– precise calculation of intrinsic material parameters (C and m);

– the strict correction of global and local geometry g(a/T) [GUR 78];

– an adequate statistical simulation of the parameters (a_0, C, m, and N_{cyles}).

P {N(x) ≤ n}	Number of cycles, n	FORM	SORM	SIM	Truncated exponential function (1)
	For a_0 = 0.35 mm				
	140.000	8.84×10^{-2}	6.64×10^{-2}	$(664 \pm 2) \times 10^{-4}$	8.07×10^{-2}
	180.000	0.379	0.351	$(23504 \pm 2) \times 10^{-4}$	0.422
	220.000	0.636	0.626	$(6186 \pm 2) \times 10^{-4}$	0.637
	260.000	0.798	0.796	$(7876 \pm 2) \times 10^{-4}$	0.771
	For a_0 = 5.00 mm				
	200.000	4.88×10^{-2}	2.77×10^{-4}	$(2.75 \pm 0.13) \times 10^{-4}$	(*) Rem.
	300.000	$\mathbf{6.84 \times 10^{-2}}$	$\mathbf{5.18 \times 10^{-2}}$	$\mathbf{(5.20 \pm 0.04) \times 10^{-2}}$	

SAW, submerged-arc welding; FCAW, flux-cored arc welding; SMAW 57, SMAW 76, *shielded metal-arc welding.*
Source: T. Lassen [LAS 92].

Table 6.1. *Cross-joined structure, by four different joining processes*

On the basis of the experimental results from Professor T. Lassen, we have presented the results [GRO 94, GRO 98] for a cross-joined structure of 25 mm in thickness, where the initial average crack is a_0 = 0.0019 mm, from a dichotomy calculation. In 1982, J.H. Rogerson and W.K. Wong [ROG 82] presented some very precise results on tubular nodes mounted in the North Sea, with the help of magnetic recording methods (MPI) under permanent inspection by ultrasound on a welded joint 1000-m long. In 1981, the authors Bokalrud and Karlsen measured cracks appearing on plaques of 10 to 25 mm in thickness. On the tubular nodes mounted in the North Sea, these authors presented results on 827 selected measuring points, and 502 measurements were without screw threads. The leads of the screw threads were reproduced in *scrubbed* silicone.

In this study, 325 positions follow an exponential distribution with an average measurement of 0.11 mm taken on a histogram and measured with screw threads. The majority of screw threads are 0.5 mm and the results are represented by the summarized relationship: $P(a_0) = 9_x\text{Exp}(9_x a_0)$.

Other authors such as Burdekin and Townend [BUR 81] have presented data conducted under real conditions (without melting) on defects due to the lack of

penetration in tubes at 90° and 45° on manually produced joints welded in an arch in which the grade was BS 4360: steel grade 50D. Defects were found by ultrasound on 440 welded joints [BUR 81]. They present an average crack as [$0.03 \leq a_0 \leq 0.25$] with an average of 0.095 and a standard deviation of 0.046 for joints welded in a cross form.

In 1976, O.I. Eide [EID 76] presented his thesis on cracks on screw threads from 0.025 to 0.25 mm with an average of 0.095 mm and a standard deviation of 0.046. Authors such as F. Watkinson *et al.* [WAT 70] found that in a manual metal arc welding with screw threads, the average initial crack did not exceed 0.4 mm. In 1967, authors such as E.G. Signes *et al.* [SIG 67] proposed that initial effective cracks varied between 0.1 and 0.5 mm with an average of 0.15 mm.

6.4. Crack spreading by a classical fatigue model

In fracture mechanics through fatigue, the statistical model which explains the behavior of the structure or mechanical component is often marked with multiple uncertainties. This assessment is carried out for many reasons, including:

– the empiricism of the behavior law of the structure, e.g. Paris–Erdogan law;

– the depth of the initial calculated crack a_0 (dichotomy in our case);

– uncertainties due to calculations of relationships between intrinsic material parameters (C, m) in the Paris–Erdogan law;

– the uncertainties from geometrical correction factors (a_0/T);

– the calculation methods for the variation in the stress intensity factor (ΔK);

– the potential variation of the nominal load $\Delta\sigma$;

– physical–chemical aggressions (offshore structures, for example), etc.

The physical cracking mechanisms through fatigue (see Chapter 7), in region II of the Ritchie diagram, are taken into consideration by a Paris law:

$$\frac{da}{dN} = C \times (\Delta K)^m \qquad [6.6]$$

where:

ΔK the *f.i.c.* is the stress intensity factor, in MPa $\cdot$ $m^{-3/2}$;

da/dN expresses the lifecycle in mm/cycle, (da/dN) > 10^{-6};

C and *m* are intrinsic material parameters.

The linearization of [6.6] allows us to easily express C and m:

$$Log\left(\frac{da}{dN}\right) = Log\left(C\right) + m \times Log\left(\Delta K\right) \qquad [6.7]$$

The expression [YAM 77] from the *f.i.c.* is written as:

$$\Delta K = \Delta\sigma \times g\left(\frac{a}{T}\right) \times \sqrt{\pi \times a} \qquad [6.8]$$

By replacing ΔK (ISO 12737: 1996) by its expression in [6.7] we get:

$$\frac{da}{dN} = g\left(\frac{a}{T}\right)^m \times C \times \Delta\sigma^m \times \left(\sqrt{\pi \times a}\right)^m \qquad [6.9]$$

where:

N is the number of cycles (per load) ;

a is the length of the crack in mm (or in inches) ;

T is the thickness of the metal sheet in mm.

The final result of the average between C and m, the intrinsic material parameters for the four joining processes [GRO 94], is as follows:

$$C_{average} = \frac{6,069 \times 10^{-8}}{24,64 \times m} \ \ MPa \times \sqrt{m} \ \ with\ a\ correlation\ \ R^2 = 0,963 \qquad [6.10]$$

The number of cycles is expressed by the integral of cracking as follows:

$$N = \frac{1}{C} \times T^{\left(1-\frac{m}{2}\right)} \times \Delta\sigma^m \int_{\frac{a_0}{T}}^{\frac{a_f}{T}} \frac{d\left(\frac{a}{T}\right)}{\left[\sqrt{\pi a} \times g\left(\frac{a}{T}\right)\right]^m} \qquad [6.11]$$

where g(a/T) is a factor of geometrical correction [GUR 78, ENG 82]. It is a vector of random parameters. Extending expression [6.8] we can suppose:

$$C \times \Delta\sigma^m \times dN = \frac{da}{\left(\sqrt{\pi a}\right)^m \times Y\left(a\right)^m} \qquad [6.12]$$

D.A. Virkler *et al*. [VIR 79] carried out a vast experimental campaign to plot a = *f* (N). The following graphic representation shows the spread of cracking with respect to the number of cycles.

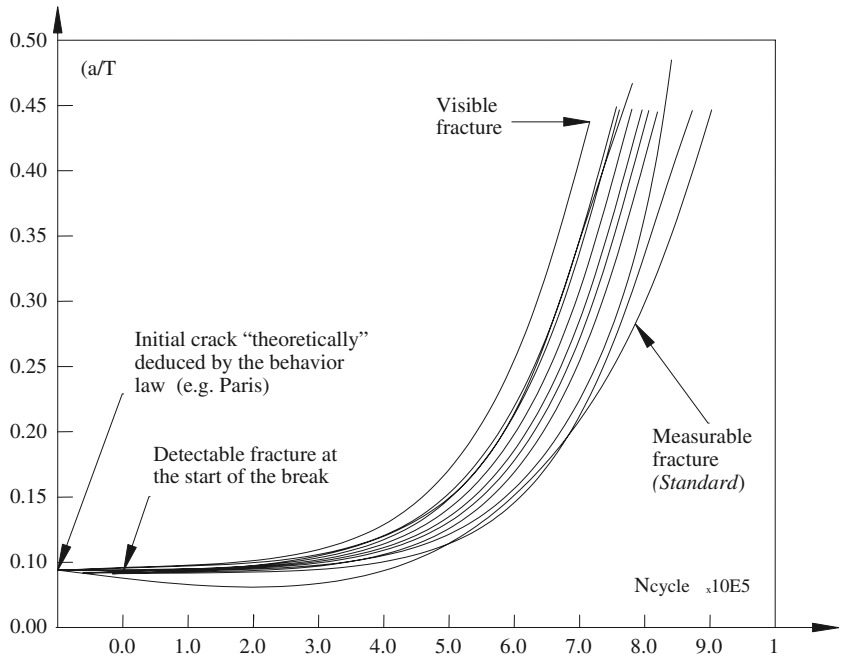


Figure 6.1. *The Crack Growth with respect to the number of cycles (N)*

Among the more conclusive stochastic models on the subject of the simulation of initial crack spreading, we can refer to the Markov's theory, which has been used by Ortiz *et al.* [ORT 84, 88, KOZ 83] and Lin-Yang [LIN 83]. The average intrinsic material parameter C is expressed as:

$$C = C(a) = \frac{C_1}{C_2(a)} \qquad [6.13]$$

where:

C_1 is a random variable which describes the probable variations between average values for each joining process.

$C_2(a)$ is an average positive value from the random process along the spreading crack for each joining process.

Hence for a constant loading amplitude (150 MPa in our case), the lifecycle, by means of the Paris–Erdogan law, takes the following form:

$$\frac{da}{dN} = Y^m \times C \times S^m \times \left(\sqrt{\pi \cdot a}\right)^m \quad [6.14]$$

where S^m ($\Delta\sigma$) is a nominal stress. The variables can thus be separated as follows:

$$\frac{da}{Y^m \times \left(\sqrt{\pi \cdot a}\right)^m} = C \times S^m \times dN \quad [6.15]$$

The integration of relationship [6.15] then allows us to write:

$$\Psi\left(a_c\right) = C \times S^m \times \left(N - N_0\right) \quad [6.16]$$

where the cumulated integral damage indicator $\Psi(a_c)$ is written as:

$$\Psi\left(a_c\right) = \int_{a_0}^{a_N} \frac{da}{Y^m \times \left(\sqrt{\pi \cdot a}\right)^m} \quad [6.17]$$

Expression [6.16] shows the integral damage indicator accumulated by crack spreading, displayed on the *marked history* of initial cracks which is represented by the so-called final cracks $\Psi(a_0)$ (stopped physically during experiments) $\Psi(a_c)$. The sequential order of loading has an unavoidable influence on crack spreading. For example, we cited the problems of delayed spread or potential leaps in acceleration.

These phenomena are purely physical and well mastered by material science. Treating the physical morphology of materials with disdain to concentrate on mathematical developments, no matter how commendable, is in our opinion an infringement on the *truth*. To pragmatically *establish* our demonstrations, deductions, and reliability results, it would be wise to control them using a complementary uncertainty calculation of the *measurands* around the *true value*. Numerous models have been proposed on the phenomenon of slowing or acceleration of crack spreading, which is more physical than theoretical. Here we can mention the works of Willenborg *et al*., Elber, and de Wheeler. In 1984, Fleck and Smith had already studied the same phenomenon under variable loads. This led them to propose the following relation:

$$\Delta a_i = Y^m \times \left(\sqrt{\pi \cdot a}\right)^m \times C \times S_i^m \quad [6.18]$$

The increment Ψ(a) takes the following form:

$$\Delta\Psi_i = C \times S_i^m \qquad [6.19]$$

The value of Ψ(a) after N cycles is written as follows:

$$\Psi\left(a_N\right) = C \times \sum_{i=1}^{N} S_i^m \qquad [6.20]$$

Based on the fact that stress intensity is a random process, its variables ΣS_m^i are equally random. When the number of cycles is high, the uncertainty in the sum dN may be neglected. This sum can be replaced by the expression ΣS_m^i. Some authors [MAD 87] find that the distribution of the S-stresses variable follows a Weibull law with two parameters, formalized as follows:

$$F_s\left(S\right) = 1 - Exp\left(-\frac{s}{\alpha}\right)^{\beta} \quad \left(with\ \alpha\text{=A and } \beta\text{=B}\right) \qquad [6.21]$$

We can now present a tabulated function Γ(a) and reformulate the relationship [6.20] of $\psi(a_N)$ under the following form, which is in fact the expression of damage:

$$\Psi\left(a_N\right) = C \times \alpha^m \times \Gamma\left(1 + \frac{m}{\beta}\right) - N_0 \qquad [6.22]$$

where Γ(a) is a reduction factor, defined as $0 \leq \Gamma(a) \leq 1$.

Γ(a), in our case, can be calculated using equation [6.23]. This expression was introduced in 1987 by Wirshing *et al.* [WIR 87], where the usefulness of the theoretical f.i.c. (ΔK_{th}) was particularly emphasized. Authors such as Madsen *et al.* [MAD 87] completed this expression by adding the function Γ(·) or sometimes Γ(· ; ·). We find this idea ingeniously well utilized to the benefit of applied mathematical simulation in fracture mechanics.

$$\Gamma\left(a\right) = \Gamma\left\{\left(1 + \frac{m}{\beta}\right); \left(\frac{\Delta K_{th}}{Y \times \alpha \times \left(\sqrt{\pi \cdot a}\right)}\right)\right\}^{\beta} \Big/ \Gamma\left(1 + \frac{m}{\beta}\right) \qquad [6.23]$$

If the observed process is truly *Gaussian and stationary*, a good estimation of accumulated damage is analyzed by replacing this process with an equivalent process. The latter is just a spectral density function whose spectrums are

represented by υ_0 and υ_i. The average values of S_i^m are therefore expressed as follows:

$$\varepsilon\left(S_i^m\right)=\left(2\sqrt{2}\right)^m\times\left(\upsilon_0^{m/2}\right)\times\Gamma\left(1+\frac{m}{\beta}\right) \qquad [6.24]$$

The average number of cycles (N_T) in a period [0, T] provides the following form:

$$N_T=\nu_0\times T=\frac{1}{2\pi}\times\left(\sqrt{\frac{\upsilon_i}{\upsilon_0}}\right)\times \mathrm{T} \qquad [6.25]$$

At the time (T), the result expressed by [6.22], $\psi(a_T)$, will be:

$$\Psi\left(a_T\right)=C\times T=\frac{1}{2\pi}\times\left(\sqrt{\frac{\upsilon_i}{\upsilon_0}}\right)\times\left(2\sqrt{2}\right)^m\times\left(\upsilon_0^{m/2}\right)\times\Gamma\left(1+\frac{m}{i}\right) \qquad [6.26]$$

By introducing the initial crack to the ith spectral frequency υ_i and to the period T_0, expression [6.26] will become:

$$\Psi\left(a_T\right)=C\times\left(T-T_0\right)=\frac{1}{2\pi}\times\left(\sqrt{\frac{\upsilon_i}{\upsilon_0}}\right)\times\left(2\sqrt{2}\right)^m\times\left(\upsilon_0^{m/2}\right)\times\Gamma\left(1+\frac{m}{i}\right) \qquad [6.27]$$

A number of uncertainties already existed in mechanical reliability. These still persist mainly in offshore structures. One could mention, for example, other commendable scientific contributions in the spirit of the works of J.D. Sørensen *et al.* [SØR 85], P. Thoft Christensen and J. Baker [CHR 82]. In reliability, as already mentioned, the problems of crack spreading through fatigue could be limited to the margins of safety. It is enough that spreading is restricted when the problem occurs.

$$\left\{\ \left(a_c-a_c\right)\le 0\ \textit{and}\ \left(K_{1c}-K_{applied}\left(a_N\right)\right)\le 0\ \right\} \qquad [6.28]$$

These two types of fracture criteria separation, among others, have been considered by ASCe (*Committee on Fatigue and Fracture Reliability*, 1982). The traditional approach, such as that developed in the previous chapter, means that the limit state function (M) is written as:

$$M=\left(R-S\right)\prec 0 \qquad [6.29]$$

In this precise case, the limit state function is expressed, according to Madsen by:

$$M = \left\{ \Psi\left(a_c\right) - \Psi\left(a_N\right) \right\} \prec 0 \qquad [6.30]$$

In the first case, the critical crack represented by a_c is chosen beforehand. The size of the crack could be based on service considerations. Within the context of this work, this first factor is restricted to approximately $a_c \leq 15$ mm. It often appears when the spreading of the crack becomes unstable with other reliability occurrences. Starting from the fact that the damage function $\psi(a_0)$ is a monotonous and increasing function of a_0, the occurrence of a fracture from damage therefore requires the spreading of a crack of critical size $\psi(a_c)$, which exceeds through the accumulation of the loading effect, thus:

$$C \times S_N^m = C \times \sum_{i=1}^{N} S_i^N \qquad [6.31]$$

The result of the fracture criterion, as much for a constant nominal stress as for this variable, is written as follows:

$$M = \left\{ \begin{array}{l} \left(\int_{a_0}^{a_c} \frac{da}{Y^m \times \left(\sqrt{\pi \cdot a}\right)^m} - C \times S^m \times N \leq 0 \right) for\ \Delta\sigma = \text{ constant} \\ \left(\int_{a_0}^{a_c} \frac{da}{Y^m \times \left(\sqrt{\pi \cdot a}\right)^m} - C \times \sum_{i=1}^{N} S^m \times N \leq 0 \right) for\ \Delta\sigma \text{ variable} \end{array} \right\} \qquad [6.32]$$

The final expression of safety M, for a variable a_0 and an amplitude of constant load, is written according to Madsen *et al.* [MAD 85, MAD 86] as shown in [6.32]. Fracture probability, i.e. the probability that the crack exceeds critical size, to the number of cycles N_i in the period (T) is written as:

$$P_F = P\left\{ M \leq 0 \right\} \qquad [6.33]$$

This approach represents a damage indictor which has already been formalized by some authors such as Virkler *et al.* [VIR 79].

$$D_{domage} = \left\{ \frac{\Psi\left(a_N\right)}{\Psi\left(a_c\right)} \right\} \qquad [6.34]$$

If we echo this in terms of *f.i.c*., the limit state function appears as follows:

$$g(z) = K_{IC} - Y\{a(N)\} \times \left(\sigma_a + \frac{1}{2} \cdot S \right) \times \frac{1}{\sqrt{\pi \cdot a(N)}} \qquad [6.35]$$

where σ_a is the nominal stress and a(N) is obtained through the solution in expression [6.31]. The first expression [6.32] will not be the object of calculations in the present work because our experimental data arises from a constant stress.

6.5. Reliability calculations using the integral damage indicator method

Before tackling failure probability calculations by the integral damage indicator method [MAD 76], it would be useful to summarize what we have already mentioned: let $\psi(a_c)$ be a damage indicator function caused by a critical crack which is known or will be known, and let $\psi(a_N)$ be a damage indicator function caused by a crack due to fatigue, i.e. after the application of N_{cycles} in accordance with A. Wöhler's theory [WÖH 60] (S-N curves). Consider a limit state function ($M < 0$) which integrates the difference of these two previous functions and we get equation [6.30]. The failure probability [6.33] will be known, once the limit state function M is calculated.

By virtue of what has just been said, we will now present our results on a singular cross-joined structure by four different welding processes. With traditional safety coefficients, there is normally a certain *amount of subjectivity* in the authors' specifications. It could be thought that reinforcing a structure would free it from failure or at least delay its effects. This intuition, which is incidentally unaesthetic, is *costly* because it leads to reinforcement through safeguarding. Time and circumstance have shown us the inverse of this absurd *intuition*. To break free from this subjectivity, we are basing our calculations on proven precise methods which are based on probabilistic calculations. Do the following for this:

1) Calculate the number of cycles by choosing appropriate parameters: $\Delta\sigma$, a_0 and $a_f = a_{critical}$. This first step allows us to calculate a damage indicator function $\psi(a_c)$ under constant $\Delta\sigma$, according to the critical size of the crack.

2) Establish the product [6.22], $\{C.\Delta\sigma.m.N\} = \psi(a_N)$ to calculate a crack indicator function after applying N_{cycles} under constant amplitude $\Delta\sigma$.

3) Establish the relationship of the functions previously found in 1and 2 allowing the two damage indicator functions to convey this safety margin [6.30] as follows: $M = \{\psi(a_c) - \psi(a_N)\}$.

4) Calculate the failure probability of the structure by the relationship $P_F = P\ (M < 0)$ and then plot the corresponding graph $P_F = f(\beta_c)$ knowing that $\beta_c = -\Phi^{-1}(P_F)$.

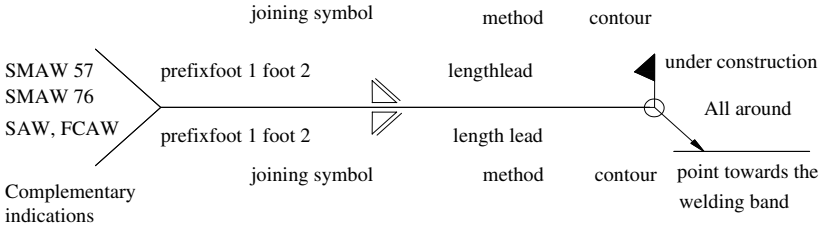
Application: To evaluate the failure probability P_F and to deduce the safety margin (M) of this structure by an appropriate method.

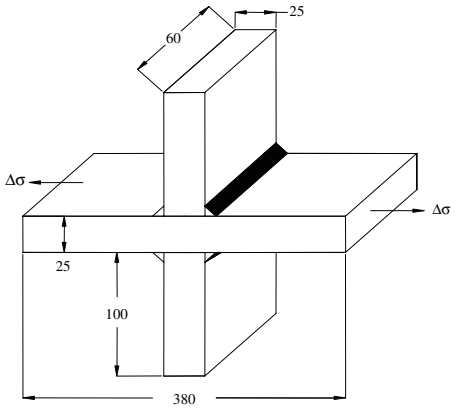
Hypothesis and definition outline of the cross-joined structure:

– joining processes: SAW, FCAW, SMAW 57, and SMAW 76;

– thickness of metal sheet to weld, T = 25 mm;

– constant stress, $\Delta\sigma = 150$ MPa.





Source of photograph: T. Lassen [LAS 92]

Figure 6.2. *Valuation according to ISO 2553 of the test on four joints welded into a cross form*

Solution: In accordance with [6.30], we can express the safety margin as follows:

$$M = \left\{\Psi\left(a_c\right) - \Psi\left(a_N\right)\right\} \prec 0 \text{ and } P_F = M \leq 0$$

The safety margin (M) is the difference of the two indicative functions of damage, expressed in the number of cumulative cycles (N) and by the critical size of the length of the crack (a_0). From relationship [6.32] we get:

$$M = \left\{\Psi\left(a_c\right) - \Psi\left(a_N\right)\right\} = \left\{\int_{a_0}^{a_c} \frac{da}{Y^m \times \left(\sqrt{\pi \cdot a}\right)^m} - C \times S^m \times N\right\} \leq 0 \text{ for } \Delta\sigma = \text{cst}$$

The failure probability is calculated from where the crack is greater than a critical size (a_c) to the Nth number of cycles, or even the period T_c. For a constant $\Delta\sigma$ (150 MPa) and a corrected geometry g(a/T) in relation to the thickness of the sheet metal assembled by welding, the safety margin can be expressed as:

$$M = \left\{\Psi\left(a_c\right) - \Psi\left(a_N\right)\right\} = \left\{\int_{\frac{a_0}{T}}^{\frac{a_c}{T}} \frac{d\left(\frac{a}{T}\right)}{g\left(\frac{a}{T}\right)^m \times \left(\sqrt{\pi \cdot a}\right)^m} - C \times \Delta\sigma^m \times N\right\} \leq 0$$

The expression of N_{cyles} takes the form:

$$N \geq \frac{\int_{\frac{a_0}{T}}^{\frac{a_c}{T}} \frac{d\left(\frac{a}{T}\right)}{g\left(\frac{a}{T}\right)^m \times \left(\sqrt{\pi \cdot a}\right)^m}}{C \times \Delta\sigma^m} \quad with\ N \geq \overline{N} \qquad [6.36]$$

6.6. Conclusion

The sustained belief, in tests on propagating fissures, is still important in several accounts, because probability theories and laws on behavior are empirical in their turn. The works by Virkler *et al.* [VIR 79] explained this succinctly and stressed the heterogeneousness of materials from which their *questionable intrinsic coefficients* are derived. This is also true for the choice of classic modules of elasticity. In welded joints, the propagation of cracks occurs, because of metallic inclusions and the influence of the size of the crack, in the heat-affected zone (HAZ) and in the metal. Based on these considerations and the probabilistic theories essentially

used by numerous authors, it can be observed that the fundamental problem lies in the appropriateness between the parameters of the behavior law of components, such as:

– m and C are deterministic.

– m is deterministic and C random.

– C is deterministic and m random.

– m and C are random *and* correlated.

– m and C are random and *not* correlated.

Without consisdering the various cases, the cracking law remains dependent on a chiefly random subordination:

$$N = f\{a_0, a, (c, m), \Delta\sigma, \Delta K \text{ and } \textit{implicitly } g(a/T)\}$$

Among the models proposed here to describe the relations between the random variables C and m, the most important model [GRO 92, GRO 95, GRO 98, ENG 82] will be used. Some authors, such as A.B. Lidiard [LID 79], considered C to be a random variable and m deterministic. What is deserving of sustained attention is the work of authors such as S. Tanaka, M. Ishikwaw, and S. Akita [TAN 81] who proposed an effective and correlated relationship between C and m even permitting for slight differences of identical materials. According to the works published by Madsen, the fundamental problem of relationships between C and m is not just physical. It is also inherent in the techniques of randomization of these coefficients.

Besides this, if we refer to such things as the DnV standards of northern Europe [DnV 82], the Canadian standards [CSA 81], or the CIRIA [CIR 81] in the UK, the necessary recommendations are there. Since the parameter a_0 is responsible for its longevity, it is strongly recommended we pay close attention to the properties of the base materials and to those of the welding electrodes. To emphasize the point, it is imperative to determine the size of the initial fault a_0 to the period (T_0) or to the cycle N_0. The factor a_0 is often deductible through a statistical approach, when it is not measurable.

The problem of the m and C relationship is also resolved either by the secant method, or by the least squared regression method [TAN 81]. The limitations, in terms of reliability models, could sometimes turn out to be questionable with repect to the precisions offered by other methods. For example, the SIM method (MC simulation), although costly in terms of calculation time, permits for an estimate of the safety margins for each of the four previously presented welding procedures (SAW, FCAW, SMAW 57, and SMAW 76). The calculation for the safety margins

varies depending on whether the structure is singular or constructed in parallel or in series with other similar structures. The following indications will guide our choice of the most effective method for a potentially accurate calculation:

– **Particular features of singular structures:** The Monte Carlo simulation method is applied to calculate the safety margin corresponding to the reliability index (β) as well as its failure probability $P_{F.}$ The MC simulation is reliable but *time-consuming* in terms of calculation.

– **Particular features of structures in parallel**: The so-called Ditlevsen's bounds' method [DIT 81] is applied, as it is simple and precise compared to that of M. Hohenbichler and M. Rackwitz [HOH 87] which requires a relatively sustained mathematical adaptation aptitude.

– **Particular features of structures in series:** The so-called Madsen *et al.*'s [MAD 86] method is applied, which, similar to the Monte Carlo simulation, proves to be simple, pragmatic, and elegant. Some statistical dispersions are in danger of being accentuated, and so it is imperative to correct the geometry g(a/T) by known methods [GUR 78]. The most appropriate method in this case of studying a singular structure was the subject of a study by K. Yamada and P. Albrecht [YAM 77].

We will find a wide range of references in the bibliography. We are inspired by them. The work which has retained our particular attention is first and foremost the wealth of experimental data by Professor Lassen, who generously allowed us to use them for our own calculations. Of course, the work of H.O. Madsen *et al*. guided us through the theoretical approach, from which the results exploited here are derived.

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Chapter 7

Monte Carlo Simulation

7.1. Introduction

During recent decades, probability theory models for mathematical simulations (particularly that of Monte Carlo (MC)) have developed intensely. The domains for studying safety coefficients have become disadvantageous. In the last two decades, more than 3,000 published articles [RUB 81] have dealt with variable simulation using the Monte Carlo (MC) method. With the help of informatics, the use of this often time consuming method, in terms of calculation time, was encouraged. The first applications of the MC method were carried out on the analysis results of Stefan–Boltzmann's equation. In 1908, the statistician student was already using this method to estimate the correlation of coefficients using his famous test of Student's law. We will now clarify some of the terminology that characterizes simulation methods.

7.1.1. *From the origin of the Monte Carlo method!*

The method, contrary to popular belief, does not come from a person named Monte Carlo, but instead from the name of the town Monte Carlo, in the principality of Monaco (mid-France). It was not von Neumann who invented it, but instead von Neumann and Ulam [NEU 51] who introduced the concept, during the Second World War. The motif was not linked to gambling games from the Monte Carlo casinos. It was instead a secret war code from an American scientific work which was preparing for the atomic bomb in *Los Alamos* (USA). The problem lay in using the method to simulate random variables relating to material behavior (neutrons), leading to diffusion with difficulty. During that period (without the current power

of computers), the complex and multidimensional evaluation of integrals led researchers to resolve integral equations by means of an analytical solution, using the so-called MC method. It would now seem restrictive to believe that this method only applies to purely random phenomena. It is also used in stochastic solutions to rational deterministic problems.

The simulation of variable analysis was defined as an efficient technique to analyze the experiences of singular or structural models. The true definition, according to R.Y. Rubenstein [RUB 81], is not often completely linked to the MC method. When we integrate random variables, apart from their statistical law distributions, we can precisely see MC simulation. Consider that random variables are independent (in our case a_0 and m), and uniformly distributed on the interval [0, 1]. The generation of random numbers between 0 and 9 is approximated with equal probability.

7.1.2. *The terminology*

We indicate here the *academic* (yet largely simple) confusion surrounding the method: the simulation of analyzed variables, whether *correlated or not*, is not always an MC simulation. It may be a simple combination of variables, certainly similar to the MC method, but not quite the method itself. For example, to simulate random variables which are differently distributed and not correlated, a random average run can be carried out or indeed the Kolmogorov–Smimov method can be conducted to qualify the number of runs. This is an MC simulation method of random variables.

Numerous authors [SCH 85, SCH 87, FOG 82] have worked on MC simulation. Of course, not all used the same procedures but the final interpretation was often the same. We believe that this is essentially due to the calculation time that each author attempted to reduce. The problem of *convergence of stability and failure probability* is omnipresent. This can seriously jeopardize a simulation of the MC type.

1) In MC simulation, observations are made on random independent variables.

2) MC simulation is a method which provides answers from simple stochastic model functions, by introducing random variables chosen at random in the respective distribution functions.

3) MC simulation is complicated for a multidimensional model because the calculation time is excessively costly, independent from software efficiency.

The essential problem lies in studying how the safety (stability) of constructions is affected by the alteration of unfamiliar conditions (which are likely to intervene) when these conditions depart from the *norm*. The variables which characterize these

various elements have a random character. We can therefore state that safety is a function of random variables and its study should be carried out by probabilities. If a structure is made with the purpose of being very unlikely to "fail", it is necessary to know the abnormal factors which it is exposed to, before we can decide on a degree of safety (or of failure).

Constructions, in general, and welded structures, in particular, are designed with the aim of responding to various needs with the implicit understanding of a certain level of safety or probable failure. Given that the probability which guarantees failure, i.e. the probability that a structure reaches failure level, is not a physically palpable entity, it is instead mathematically estimated based on physical and/or unknown data. We could say that this probability is "justifiably" calculated, as it rests on the large assurance of reputed safety [FRE 47, FRE 66].

In the case studies outlined in this work, we suppose that the parameters of resistance (R), stress (S), geometry g(a/T), coefficients (ρ, θ) respective to local geometrical correction (taking into account the radius and angle of the joint at the foot of the welding band) and intrinsic material coefficients (C, m) are all correctly estimated. Yet they are nearly all variable, added to the issue of random external factors such as wind, waves, corrosion, fatigue, etc.

There are many codes and rules [CSA 81, CSA 11, LIN 73, CIR 77, 81, AME 87, API 87, SIG 67, HOH 84, DIT 79, DIT 86a, DIT86b, GRO 98, AME 84, DNV 82, CIR 81] which suggest to the user the limits to which they are authorized to consider proportional hypotheses to ensure sufficient construction safety. Nowadays, modern rules use the best knowledge that the designer could possibly have in the dynamics of materials. Sometimes we can see the so-called safety coefficients used in a *purely academic* way. This is a great danger when we consider that it is not only materials which evolve and develop, but also their dynamics. How then can we trust a Young's coefficient or an admissible stress selected from a manual?

As serious as it is, the recommended manual [COR 69, COR 67, CIR 81] remains erroneous in the values that it proposes if it is not continually updated. As an example, the Canadian and European rules for metallic construction suppose that the probabilities of reaching a limit state are 10^{-5} in limit state, and 5×10^{-5} in service limit state.

As we have already mentioned, cross-joined structures have a limit state of fatigue that fracture mechanics takes into account, through the Paris–Erdogan law. These structures are often governed by a very rigorous design, especially at the foot of the welding band. The causes of uncertainty are often linked to physical phenomena: hydrodynamic loads, particularly aggressive environments, stress intensity factors, tenacity, empiricism of behavior laws, etc. Fatigue cracking in

metals is a multidisciplinary phenomenon which remains the object of many scientific debates [SIG 67]. Fatigue cracking often leads to unknown mechanical structures being analyzed using probabilistic theories with multiple correlations developed by Markov's chain theory, among others.

Faced with convergence problems, some authors such as Hohenbichler [HOH 84] have preferred to settle on a linear approximation of the limit state surface near the "*design point*". This calculation is carried out beforehand by optimizing the area of true failure or on a quadratic approximation of this area.

For example, a linear approximation of the limit state surface provides the huge advantage of allowing a numerical integration of probability densities in the area of failure. It is entirely sufficient with regard to the highly possible distortion on the true surface. This surface is of course strengthened by the hyperplane surface. The results from level II are sometimes similar to those arising from level III [SEL 93]. Long before the 1990s, an interesting study carried out by Schuëller *et al*. [SCH 79] dealt with this issue. It is well worth consulting.

With calculation systems, the inherent risks to engineering science essentially result from what we do not know with sufficient accuracy. Intuition plays a large part in the field of reliability [PRO 87]. When in doubt, "*you cannot doubt badly*". The data relevant to projects involve materials, their geometric dimensions, and their diverse faculties of "*aging*". Many physical parameters are marked with uncertainty, leading to their random and unknown character. We cannot calculate with certainty what has already existed. Starting new design projects can also lead toward mitigation, and even the opposite of the expected development of parameters which constitute the mathematical skeleton of models.

The project becomes less disadvantageous when we consider a hypothesis where sufficient information is collected to establish a quantitative estimation of risk. Statistical tools rationalize the measuring of uncertainties to try and make structures reliable. We will limit ourselves here to just present the necessary statistical tools for the probabilistic approach in the domain of dimensional metrology. Their use aims to estimate parameters of probability laws, which have been chosen *a priori* and/or on the basis of conformity tests.

To emphasize this point, the statistical approach, in many other areas (science and equipment in medicine), is very risky and even hazardous. One simple way to properly understand the effect of unknown uncertainties affecting material and structural behavior is to place them in conditions where they are subjected to *simulated* uncertainties. It is then possible to estimate mathematical expectations and the variance in variables necessary for the understanding of many phenomena (stresses, resistances, displacement, safety margins). We can also treat simulated

samples as a group of observations and apply an appropriate statistical treatment to them. The making of the atomic bomb (at Los Alamos, USA) is an example of the use of the MC simulation method. The principle of MC simulations is to generate a group of pseudo-random realizations from a random variable or random field. To obtain a realization x from the variable X obtained from the distribution function $P_X(\cdot)$, we simulate a number z uniformly distributed on [0,1], and then apply a formula of the following type:

$$x = P_x^{-1}(z) \qquad [7.1]$$

Many simulation algorithms on a uniformly distributed number [0,1] are based on linear congruence, as follows:

$$Z_{n+1} = \left\{ (\alpha \times Z_n + \beta) \bmod (m) / m \right\} \qquad [7.2]$$

In this congruence relationship, it is necessary to define an initial value z_0 of Z. The quality of generation is linked to the period of return for the same realizations of Z, as well as the dispersion of these realizations. The choice of α, β, and m depends on this quality. As an example, we can mention an effective choice arising from the literature such as the "Mersenne Twister" from Matsumoto and Nishimura:

– $\{\alpha, \beta, m\} = \{1664525, 1013904223, 2^{32}\}$ suggested by Knuth and Lewis;

– $\{\alpha, \beta, m\} = \{69069,0, 2^{32}\}$ suggested by Marsaglia;

– $\{\alpha, \beta, m\} = \{3116728,1, 2^{48}\}$ suggested by Lavaux and Jenssens;

In the present work, we have used the software MathCAD to generate a group of pseudo-random variables of a RV.

7.2. Simulation of a singular variable of a Gaussian

According to the Box and Muller method, if z_1 and z_2 are two numbers uniformly distributed on [0,1], the number u defined by $u = \sin(2\pi z_1)\sqrt{-2\ln(z_2)}$ is a reduced centered Gaussian variable.

– To simulate the Gaussian variable with mathematical expectations of μ_x and of variance σ_x^2 we apply: $x = u \times \sigma_X + \mu_X$.

– To simulate a log-normal variable with mathematical expectations of μ_x and of variance σ_x^2 we apply $x = Exp(u \times \sigma_{Lx} + \mu_{Lx})$ where μ_{Lx} and σ_{Lx}^2 are the average and the variance of $ln(x)$.

7.2.1. *Simulation of non-Gaussian variable*

In this case we can use the general relationship $x = P_x^{-1}(z)$. Note that for the majority of laws of exponential type (Weibull), the inversion of the distribution function does not create a problem. For other laws, a numeric inversion would often prove necessary (see beta law).

7.2.2. *Simulation of correlated variables*

The vector X of correlated variables is characterized by its expected mathematical vector μ_x, its variance–covariance matrix Σ_x, and the group of marginal distribution functions P_{X1}, …, P_{Xn}. The awareness of the latter is often more obvious than that of linked probability density.

7.2.3. *Simulation of correlated Gaussian variables*

Starting from the simulation of a vector U to independent reduced centered Gaussian components, we can build a construction of Gaussian X by:

$$x = C.u + \mu_x \qquad [7.3]$$

$\Sigma x = [C] \times [C]^T$, where [C] is an inferior triangular matrix, obtained by a Choleski transformation.

7.2.4. *Simulation of correlated non-Gaussian variables*

To do this, we must use the so-called Rosenblatt transformations. By simply knowing the marginal probability densities of X_1 and X_2, it is possible to express their linked density by the following relationship:

$$P_{x1,x2}(x_1, x_2) = \Phi_2(y_1, y_2, \rho_{0,12}) = \frac{p_{x1}(x_1)\, p_{x2}(x_2)}{\Phi_2(y_1)\Phi_2(y_2)} \qquad [7.4]$$

where:

Y_1 and Y_2 are two reduced centered Gaussian variables with the so-called fictitious correlation $\rho_{0,12}$;

$\Phi_2(\cdot)$ is the reduced centered binominal density.

With the correlation ρ_{12} already known between X_1 and X_2, the following relation is expressed as:

$$\rho_{12} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left(\frac{x_1-\mu_{x_1}}{\sigma_{x_1}}\right)\left(\frac{x_2-\mu_{x_2}}{\sigma_{x_2}}\right)\Phi_2\left(y_1,y_2,\rho_{0,12}\right)\frac{p_{x1}\left(x_1\right)p_{x2}\left(x_2\right)}{\Phi\left(y_1\right)\Phi\left(y_2\right)}dx_1dx_2 \qquad [7.5]$$

This is a relationship from which we can extract the fictitious correlation value $\rho_{0,12}$. In the literature, some approximate expressions of $\rho_{0,12}$ have been proposed by Professor Der Kiureghian and Liu [KIU 85, LIU 86, LIU 91] for different marginal probability laws. These suggestions are subject to restrictions on coefficients of variations of variables X_1 and X_2 and on the value range of their correlation, according to marginal laws.

For example, note that for two log-normal variables, $\rho_{0,12}$ is expressed exactly as follows:

$$\rho_{0,12} = \frac{\ln\left(1+\rho_{12}\dfrac{\sigma_{x_1}\sigma_{x_2}}{\mu_{x_1}\mu_{x_2}}\right)}{\rho_{12}\left\{\ln\left(1+\left(\dfrac{\sigma_{x_1}}{\mu_{x_1}}\right)^2\right)\ln\left(1+\left(\dfrac{\sigma_{x_2}}{\mu_{x2}}\right)^2\right)\right\}} \qquad [7.6]$$

The simulation procedure of an X construction therefore consists of determining the fictitious covariance matrix Σ_y and then simulating Y by $x=[c]\times u+\mu_x$ with $\mu_y = 0$ to finally end up at $x_1,\ldots,x_n$ by the following:

$$\left\{\begin{array}{l} x_1 = P_{x1}^{-1}\left(\Phi^{-1}\left(y_1\right)\right) \\ x_1 = P_{x1}^{-1}\left(\Phi^{-1}\left(y_1\right)\right) \\ \ldots = \ldots\ldots\ldots\ldots \\ x_n = P_{xn}^{-1}\left(\Phi^{-1}\left(y_n\right)\right) \end{array}\right\} \qquad [7.7]$$

In the present work, we limit ourselves to an MC simulation for basic applications arising from mechanical engineering techniques.

7.3. Determining safety indices using Monte Carlo simulation

7.3.1. *General tools and problem outline*

The considered variables in this chapter are random. They have been simulated [GRO 94] by their respective distribution laws, i.e. Weibull law for initial cracking (a_0), log-normal (Galton) law for *m*-coefficients and normal (Gaussian) law for the number of cycles N_{cycles}.

– S is the resistance that the cross-joined structure can oppose to the load (e.g. $\Delta\sigma$ = 150 MPa) to remain in (and ensure) service.

– R is the resistance that the structure puts up against the constant load $\Delta\sigma$.

Beforehand, it is considered that failure is possible. This eventuality is therefore more or less probable. The degree of failure will be characterized by the *probability* when the equality ($R - S$) < 0 is verified. Therefore, let this probability P_F be expressed as:

$$P_F = \{R - S\} \prec 0 \;\; or \; even \;\; \{R - S\} = M \prec 0 \qquad [7.8]$$

where:

M is the safety margin;

P_F is the probability of fracture (failure).

The model considered presents uncertainties which are essentially due to unknown factors brought about by initial cracking (a_0) by material properties. It is vital to pay attention to the choice of conditions which risk being considered as a normal distribution. The limit state function g(z) as a mode of reliability is formulated in finite (n) terms. In general, one or more non-random base variables correspond to the limit state function presented by a safety margin M. This is defined by the following expression:

$$g(Z) = M = g(z_1, z_2, ..., z_n) \qquad [7.9]$$

where Z represents (n) vectors of random base variables (a_0 and m).

These random base variables can be dependent (or independent). The failure probability which results when M < 0 is represented by P_F and is written as follows:

$$P_F = \int_{g(z) \le 0} f_z(Z) dz \qquad [7.10]$$

F(z) is the probability density function of (z_1, z_2, …, z_n). This allows the reliability index to be calculated, with the help of the expression:

$$\beta_c = \Phi^{-1}\left(P_F\right) \text{ and } P_F = \left(1 - P_R\right) \quad [7.11]$$

where $\Phi(\cdot)$ is the function of the reduced centered random variable from normal law.

The relationship between the reliability index β_c and the failure probability P_F takes the graphical form as presented in Chapter 5. De Moivre's law, reported by Laplace and then by Gauss (lending to the habitual name of Gaussian law or normal law), is largely utilized, which at times erroneously causes certain adequacy tests to be used in manipulations. The use of Gaussian law is often justified, if not indispensable, but in many problems fundamentally incorrect, if not absurd. It does not seem inappropriate here for us to comment on these observations of the abusive use of the normal law.

In our case study, we will be considering three distinct laws to simulate our parameters of cracking. The increment of ($100 \times a_0$) is made on the probability density function of Weibull law with two parameters. The m coefficient is also random and is chosen in the same way as the a_0 but on a probability density function of Galton (log-normal) law.

After having read and calculated the parameters which will allow us to find the numbers of corresponding cycles and will integrate the simulated random variables a_0 and m, we can then proceed to a mathematical statistical simulation of the results of Ni found by this classic variable simulation. Note that the random variables of the distribution function $F_N(n)$ are calculated as being close to 7.5×10^{-8}.

We will also calculate the parameters a_0 and m using the direct safety margin method with the help of the integral damage indicator. The safety margin calculation often arises when double integrals need to be calculated. To simplify this problem, we can employ the MC simulation method.

Designed by O. Ditlevsen and P. Bjerager [DIT 86] as the method of Gaussian safety margins, it is often used to resolve problems of mechanical reliability in structures and components. It is also used to design the geometrical configuration of Gaussian distributions. According to O. Ditlevsen [DIT 79, DIT 81], all structures can use a reliability application through *boundary* theory, especially when the system is redundant (and/or in a series). Sometimes, boundary theory can be hugely complex in terms of application.

The reliability index β can also be calculated using the Hasofer–Lind theory (β_{HL}) [HAS 74], which is the smallest Euclidian distance from the origin of the

failure surface (see Figure 5.2) in a reduced space of *non-correlated*, *reduced*, *centered*, *normal variables*. This is very relevant in our scenario of treating a_0 and m *a priori* as two parameters from which we cannot presume a correlation. In the last instance, the MC simulation method does not require any particular remarks. It is, as a matter of course, the most appropriate reliability model for a singular structure [DIT 86]. Also, it is simple in terms of application. The advantages and disadvantages of the MC method are as follows.

Advantages:

– very precise results;

– simplicity of graphical reading;

– merit of its direct application on singular structures.

Disadvantages:

– long (but efficient) calculation time;

– inelegant method with regard to generating random data;

– often disadvantageous *convergence* issues.

In reality, the problem with this method rests with the "*calculation time*" factor and the complexity of the procedure generalization concern, for all types of structural problems, i.e. the capacity to treat multiple cases of reliability (aeronautic, maritime structures, buildings, and other industrial equipment). Numerous authors have analyzed and studied MC simulation techniques to try and free it from its inability to resolve problems with double integrals to attain failure probability. We class these methods into the following three types:

– Simulation, in a small number, coupled with a simulation of random variable distributions or of the random variable of limit state.

– Direct simulation of all implicated variables in the behavior model and counting the simulations which lead to a failed structure. Professor K.M. Engesvik [ENG 82] (Trondheim, Norway, 1982) conducted a commendable study on a cross-joined structure. The results of his laborious work merit particular academic attention.

– Simulation by separating variables or by conditioning techniques.

7.3.2. *Presentation and discussion of our experimental results*

The results from the probabilistic treatment of a number of cycles from previous simulations of random variables allows us to see the importance of the engineering

risk of fracture in the singular cross-joined structure. After having listed what we have just mentioned, we will attempt to show the efficiency of MC simulation in calculating the failure probability of a structure. Remember that the constant parameters of the behavior law are T, $\Delta\sigma$, a_{cr}, with the metal sheet thickness (25 mm), the nominal stress (150 MPa), and the critical crack (mm, arbitrarily fixed).

– Consider the relationships between C and m where the correlation R^2 is deemed perfect.

– Choose at random 100 values of initial cracks a_0.

– Choose at random 100 values of m-coefficients intrinsic to the material.

We can also remark that the random choice of these parameters is carried out on probability density functions which have already been plotted. We are choosing an increment of 10^{-5} which will allow us to sort the 10 randomly chosen values among the a_0 and m of the respective probability density functions. The 100 chosen values among the a_0 and m are grouped into the following table:

Welding process[a]	a_0_1st to a_0_100th	m_1st to m_100th
SAW	0.0027 to 0.0119	2.1821 to 3.3335
FCAW	0.0027 to 0.0180	2.1563 to 3.0648
SMAW 57	0.0013 to 0.0134	2.5590 to 3.0580
SMAW 76	0.0027 to 0.0196	2.4285 to 3.0392

[a]The experiments were carried out by Professor T. Lassen (Norway). With his kind and generous permission we have recovered the data to conduct a thesis study [GRO 94, GRO 98].

Table 7.1. *Results of MC simulations on four welding processes*

7.3.3. *Use of the randomly selected numbers table*

To select an n number from the group {0, 1, 2, 3, 4, 5, 6, 7, 8, 8, 9} we set a rule for ourselves from the table of random numbers. For example, we choose the number in the left corner and then the number below each of the N first blocks, which gives N = 5, 5, 8, 1, 5. Other techniques permit a direct reading from the table of random numbers. In our case, the result is simple. For $n = 5 \times 5$, an N at random will be 68645.

$$N\left\{N \geq \left(\frac{K_u}{D}\right)^2\right\} \qquad [7.12]$$

The problem studied here is simple and based on a necessary drawing from the Kolmogorov–Smirnov (KS) test. For example, by trying to limit the uncertainty to ± 1%, i.e. the uncertainty $D = 0.01$ on the simulation of probability density function, we choose a level of confidence *99.9%*, i.e. $\alpha = 0.01$ and we obtain $N \geq 2.657 \times 10^4$ draws. K_u is a coefficient read in the following table:

Determining the number of necessary draws $N\left\{N \geq \left(\frac{K_u}{D}\right)^2\right\}$					
Level of confidence $(1 - \alpha)$	0.80	0.85	0.90	0.95	***0.99***
Risk ***u***	0.20	0.15	0.10	0.05	***0.01***
Coefficient K_u	1.07	1.14	1.22	1.36	***1.63***

Source: J. Cadiou [CAD 84] Techniques of the Engineer, pp. T-4 301–4.

Table 7.2. *Determination of the number of necessary draws in MC SIM*

$$N\left\{N \geq \left(\frac{K_u}{D}\right)^2\right\} = \left(\frac{1.63}{0.01}\right)^2 = 2.657 \times 10^4 \approx 27000 \text{ draws}$$

The problem of drawing at random is variously realized. We have already mentioned the drawing technique based on the KS test. Having judged this test as being well adapted to our simulation variables, we will now apply it in this study. Now we will use the MC simulation of random variables from a Weibull law with two parameters. The random drawing process is carried out in the following stages:

First stage: Obtain five random draws from the a_{0i} in Weibull's distribution of parameter β with scale η. Call U_i a random uniform number and we get the Weibull's distribution of

$$U_i = F\left(X_i\right) = 1 - Exp\left(-\frac{X_i}{\eta}\right)^{\beta} \qquad [7.13]$$

In the Eularian law Γ (see table A.1. in the Appendix). We can recover the values of β and η, and thus for each joining process we will have X_i, a random *steady* variable:

$$X_i = \eta\left[-\ln\left(1-U_i\right)\right]^{\frac{1}{\beta}} \rightarrow \left\{ \begin{array}{l} X_i\left\{SAW\right\} = 10.240\times10^{-3}\left[-\ln\left(1-U_i\right)\right]^{\frac{1}{1.9}} \\ X_i\left\{FCAW\right\} = 10.640\times10^{-3}\left[-\ln\left(1-U_i\right)\right]^{\frac{1}{2}} \\ X_i\left\{SMAW57\right\} = 8.191\times10^{-3}\left[-\ln\left(1-U_i\right)\right]^{\frac{1}{2.5}} \\ X_i\left\{SMAW76\right\} = 15.440\times10^{-3}\left[-\ln\left(1-U_i\right)\right]^{\frac{1}{2.5}} \end{array} \right\} \quad [7.14]$$

We have used the software MathCAD for $U_i = 100$. The results are as follows:

Calculation results of distribution functions for welding processes		
X_i	$F(X_i)$	$F_p(X_i) = 1 - F(X_i)$
5.38624×10^{-3}	5.554×10^{-3}	**0.7446**
1.04272×10^{-3}	9.554×10^{-3}	**0.9904**
3.27640×10^{-3}	9.6064×10^{-3}	**0.9039**
6.17600×10^{-3}	9.606×10^{-3}	**0.9039**
Etc.	*Etc.*	*Etc.*

Table 7.3. *Results of statistical distribution functions for $U_i = 100$*

In this way we can *steady* the random variables X_i for the Weibull law with two parameters and thus calculate the distribution function and probability densities for each random draw, with the help of the Kolmogorov–Smirnov approach.

Second stage: The work set out in the first stage is resumed to calculate the number of cycles, relative to each of the four welding processes. We will thus carry out a statistical study to imitate the number of cycles found through the simulation of random variables a_0 and m. Recall that the distribution laws for each random variable which intervene in probability failure density calculations all obey the following laws:

Distribution laws by welding processes	
Random variables X_i	Statistical law $F_p(X_i) = 1 - F(X_i)$
Initial cracks, a_0	Weibull law with two parameters
Paris coefficients, m	Log-normal law (Galton)
Number of cycles N	Normal law (Laplace–Gauss)

Table 7.4. *Statistical distribution laws applied to the four welding processes [LAS 92, GRO 94]*

We will now calculate the reduced, centered, normal law with an accuracy of (7.5×10^{-8}) for a reduced, centered, random variable u by the following relationship:

$$u = \left\{ \frac{1}{1 + 0.2316419 \times N} \right\} \; with \;\; N \succ 0 \qquad [7.15]$$

The results from the distribution function program permit a "*setting*" of random variables from the function $F(X_i)$. As the latter is defined in R^2 on the interval [0,1], we are freed from 10^{-5}, i.e. instead of considering the true value of N_{cycles}, we have suggested ($N \times 10^{+5}$) because ($10^{-5}/10^{+5}$) = 1 allows us to program the distribution function in the interval [0,1], as follows:

```
********** NORMAL LAW PROGRAM WITH RV SETTING ************
IMPLICIT NONE
      CHARACTER*24 DOCNAME
            REAL*8 FX(100), X91000, X1(100)
      + U(100)
      + A1, A2, A3, A4, A5
      + PI, AVG, STDV
INTEGER N, I, CHOICE, CHOICE1
PI=4DO*ATAN91.D0)
      A1 = +0.319815300
      A2 = −0.356563782
      A3 = +1.781477937
      A4 = −1.821255978
      A5 = +1.330274429
C ******************** INTRODUCTION OF DATA ********************
WRITE (6,*) 'WOULD YOU LIKE TO EXPORT DATA INTO A DOCUMENT?'
WRITE (6,*) '(1) → YES (0) → NO'
READ (5,*) CHOICE
IF (CHOICE.EQ.1) THEN
      WRITE (6,*) 'INSERT NAME OF DOCUMENT'
      READ (5, '(A)' DOCNAME
      OPEN (86, FILE = DOCNAME, STATUS = 'UNKNOWN')
ENDIF
15    WRITE (6, *) 'INSERT AVERAGE AND STANDARD DEVIATION'
      READ (5, *) AVG, STDV
      WRITE (6, *) 'INSERT NUMBER OF VARIABLES'
      READ (5, *) N
      DO I = 1, N
```

```
        WRITE (6, *) 'XREAL ('I, ') = '
        READ (6, *) X1 (I)'
        END DO
    C ****************** CALCULATION OF F(X) AND U(X) ******************
    IF (CHOICE.EQ.1) THEN
        WRITE (86, 5)
    ELSE
    WRITE (6, 5)
    ENDIF
    05    FORMAT (6X, ' X ', 12X' U(X) ', 12X, ' F(X) ')
DO I      = 1, N
X(1)      = X1 (I) – AVG)/STDV
U(I)      = 1/(1+0.2316419*X(I)
F(X)      = 1−1/(SQRT92*PI)*EXP(− X(I)**2/2)*(A1*U(I) + A2*U(I) **2 + A3*U(I)**3 +
        A4*U(I)**4+ A5*U(i)**5 +
    IF (CHOICE.EQ.1) THEN
                WRITE (86, 10) X(I), U(I), FX(I)
    ELSE
        WRITE (6, 10) X I), U(I), FX(I)
    ENDIF
    ENDDO
    10    FORMAT (2X, 3 (E14, 6, 2X))
    WRITE (6,*) ' (1) → YES (0) → NO'
                READ (5, *) CHOICE1
        IF (CHOICE1EQ.1) GO TO 15
    IF (CHOICE1EQ.1) CLOSE (86)
    STOP
    END
```

X	u(X)	f(X)
0.116327 E + 02	0.270665 E + 00	0.100000 E + 01
−0.121076 E + 01	0.138978 E + 01	0.110584 E + 01
−0.146302 E + 01	0.151262 E + 01	0.652430 E − 01
−0.576923 E + 00	0.115425 E + 01	0.281909 E + 00

Table 7.5. *Results: X, u(X) and f(X): statistical distribution laws*

We have conducted ten simulations (n = 10, 20, 30, 40, 50, 60, 70, 80, 90, 100) by considering random variables a_0 and m by their respective distributions.

N =	0
0	10
1	20
2	30
3	40
4	50
5	60
6	70
7	80
8	90
9	100

PF(N) =	0
0	1.018
1	1.025
2	1.015
3	1.020
4	1.020
5	1.020
6	1.020
7	1.020
8	1.020
9	1.020

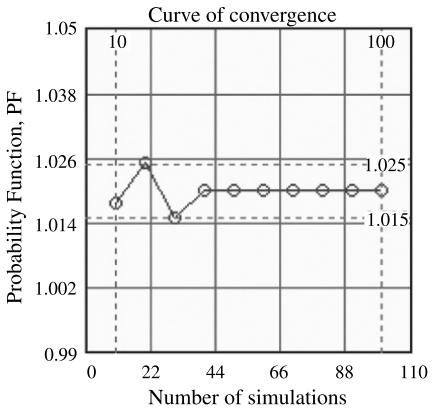

Figure 7.1. *Convergence of MC simulation*

We can clearly see the convergence of MC simulation. We feared beforehand that there would be convergence difficulties, because we knew from the literature that one of the problems with SM was the convergence of simulated variable probability results on randomly drawn N (100 in our case).

We will now present another mathematical technique used to generate random numbers and succinctly show how we are able to approach them with our four statistical distribution laws: uniform, exponential, Gauss, and Weibull. To do this, this time we require a model programmed with the help of the MathCAD software.

7.4. Applied mathematical techniques to generate random numbers by MC simulation on four principle statistical laws

This mathematical approach is used to create a vector of random numbers from three laws on a given interval: a uniform law, a normal law, and an exponential law. This mathematical technique includes two programs which allow us to control the *initial value* used to generate random numbers.

7.4.1. *Uniform law*

First, it is advisable to enter the number of random centered variables and to limit (*set*) them by their highest and lowest extremities, as follows:

Enter highest and lowest extremities: $l0 = 2hi = 2$

Enter the number of groups for the histogram: $ens\_hist = 20$

Vector of random centered variables: N = runif(*n, lo, hi*)

Frequency distribution: Lower = floor(min(N))

upper = ceil(max(N))

$n = 1x10^3$

$$\left\{ hr = \left(\frac{upper - lower}{ens\_hist} \right), \quad j = 0..ens\_hist \ and \ \text{int}_j = lower + hr.j \right\}$$
$$\left\{ f = hist(\text{int}, N) \ and \ \text{int} = \text{int} + hr.0.5 \right\}$$

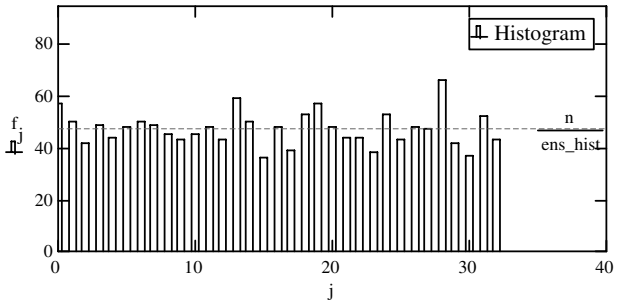

Figure 7.2. *Frequency distribution of uniform law using MC simulation*

7.4.2. *Laplace–Gauss (normal) law*

Enter the number of random centered variables: $n = 1 \times 10^3$

Enter the average and standard deviation: $\mu = 0$ and $\sigma = 2$

Vector of random centered variables: N = norm(n, μ, σ)

Frequency distribution: Lower = floor(min(N))

upper = ceil(max(N))/2

$$\left\{ hr = \left(\frac{upper - lower}{ens\_hist} \right), \quad j = 0.. \ ens\_hist \right\}$$
$$\left\{ \text{int}_j = lower + hr.j \quad f = hist(\text{int}, N) \ and \ \text{int} = \text{int} + hr.\frac{1}{2} \right\}$$

Normal adjustment function: F(x) = *n.hr.dnorm*(n, μ, σ)

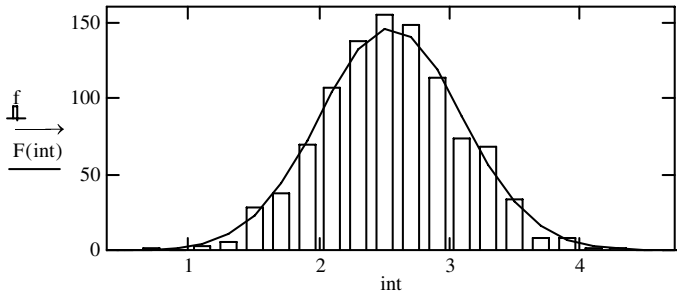

Figure 7.3. *Frequency distribution of Gaussian law using MC simulation*

7.4.3. *Exponential law*

Enter the number of random centered variables: $n = 1 \times 10^3$

Enter the rate a: $\alpha = 0.5$

Enter the number of groups for the histogram: ens_hist = 20

Vector of random centered variables: N = rexp (n, α)

Frequency distribution: Lower = floor (min (N))

upper = ceil (max (N))/2

$$\left\{ \begin{array}{l} hr = \left(\dfrac{upper - lower}{ens\_hist} \right), \quad j = 0..\, ens\_hist \\ \text{int}_j = lower + hr.j \quad f = hist\left(\text{int}, N\right) \; and \quad \text{int} = \text{int} + hr.\dfrac{1}{2} \end{array} \right\}$$

Exponential adjustment function: F(x) = *n*. hr. dexp(x,α)

7.4.4. *Initial value control*

The following program allows us to generate the same set of random numbers following a normal law at each iteration, by resetting the initial value. Let the number of iterations $n = 10^3$ with an average $\mu = 0$ and standard deviation $\sigma = 2$.

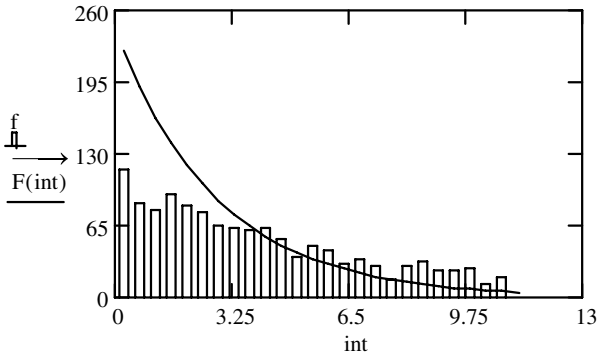

Figure 7.4. *Frequency distribution of exponential law using MC simulation*

$n = 1 \times 10^3$

$\mu = 2.5462$

$\sigma = 0.5462$

SameNormal$(n, \mu, \sigma i) :=$ for $i \in 0..5$

  Seed(1)

  nums $\leftarrow$ rnorm(n, μ, σ)

  $M^{\langle i \rangle} \leftarrow$ nums

M

DiffNormal$(n, \mu, \sigma) =$

	0	1	2	3	4
0	2.3064	2.4892	2.7219	2.7856	2.8017
1	2.1751	1.5986	2.1684	1.5953	2.1618
2	2.2877	2.2405	2.2362	2.3375	2.2146
3	2.0265	2.7612	1.9111	2.2290	2.1115
4	1.6255	1.8906	2.8920	2.3726	2.5833
5	2.5700	2.2959	1.8797	2.4988	2.3458
6	2.4803	2.6512	1.4234	2.4392	2.9030
7	2.8501	2.7230	2.8549	2.7788	1.7354
8	3.7434	2.9679	2.1433	2.2413	3.0181
9	2.9879	2.2165	2.7377	3.3617	2.3326
10	3.0843	2.5140	1.9468	2.4471	...

Table 7.6. *Generation of random numbers following a Gaussian at each iteration (n) by resetting and controlling the initial value*

Note that the exact same set of random numbers is generated each time; each column includes the same numbers in the same order. In the same vein, this program allows us to control and to repeat the generation process with a new set of random numbers from a given initial value.

$$M = \text{DiffNormal}(n, \mu, \sigma) \rightarrow DiffNormal(n, \mu, \sigma) = \left| \begin{array}{l} \text{for } i \in 0..7 \\ \quad \left| \begin{array}{l} Seed(i+1) \\ nums \leftarrow nnorm(n, \mu, \sigma) \\ M^{\langle i \rangle} \leftarrow nums \end{array} \right. \end{array} \right.$$

DiffNormal(n, μ, σ) =

	0	1	2	3	4	5	6	7
0	2.306	2.489	2.722	2.786	2.802	1.592	3.377	1.589
1	2.175	1.599	2.168	1.595	2.162	2.172	2.155	1.576
2	2.288	2.241	2.236	2.338	2.215	2.431	2.384	2.590
3	2.027	2.761	1.911	2.229	2.112	3.665	2.455	2.166
4	1.625	1.891	2.892	2.373	2.583	3.465	3.015	2.937
5	2.570	2.296	1.880	2.499	2.346	2.973	1.715	3.303
6	2.480	2.651	1.423	2.439	2.903	2.457	2.577	2.538
7	2.850	2.723	2.855	2.779	1.735	3.470	1.911	2.705
8	3.743	2.968	2.143	2.241	3.018	3.310	2.951	1.863
9	2.988	2.217	2.738	3.362	2.333	2.788	2.475	2.907
10	3.084	2.514	1.947	2.447	2.426	3.099	3.045	...

Table 7.7. *Generation of random numbers following a Gaussian at each iteration (n) by resetting and controlling the initial value*

We can also represent the columns graphically to clearly observe the effects. With the help of the software MathCAD, the selection technique is as follows:

$$\text{lower} = \text{floor}\left(\min\left(M^{\langle sel-1 \rangle}\right)\right) \qquad \text{upper} = \text{ceil}\left(\max\left(M^{\langle sel-1 \rangle}\right)\right)$$

$$\text{int}_j = (\text{lower} + \text{hr} \cdot j) \qquad \text{int} = (\text{int} + 0.5 \cdot \text{hr})$$

$$\text{hr} = \left(\frac{\text{upper} - \text{lower}}{\text{ens\_hist}}\right) \qquad j := 0..\text{ens\_hist}$$

$$\text{Adjust}(x) = n \cdot \text{hr} \cdot \text{dnorm}(x, \mu, \sigma) \qquad Q = \text{hist}\left(\text{int}, M^{\langle sel-1 \rangle}\right)$$

Using MathCAD software, it is possible to graphically represent these columns to observe the effects of the Gaussian curves plotted in the following for four cases of simulation:

sel :=
- Initiale 1
- Initiale 2
- Initiale 3
- Initiale 4
- Initiale 5

sel :=
- Initiale 1
- Initiale 2
- Initiale 3
- Initiale 4
- Initiale 5

sel :=
- Initiale 1
- Initiale 2
- Initiale 3
- Initiale 4
- Initiale 5

sel :=
- Initiale 1
- Initiale 2
- Initiale 3
- Initiale 4
- Initiale 5

The graphic results are presented as follows.

On this basis, we can now plot the histograms and inherent laws:

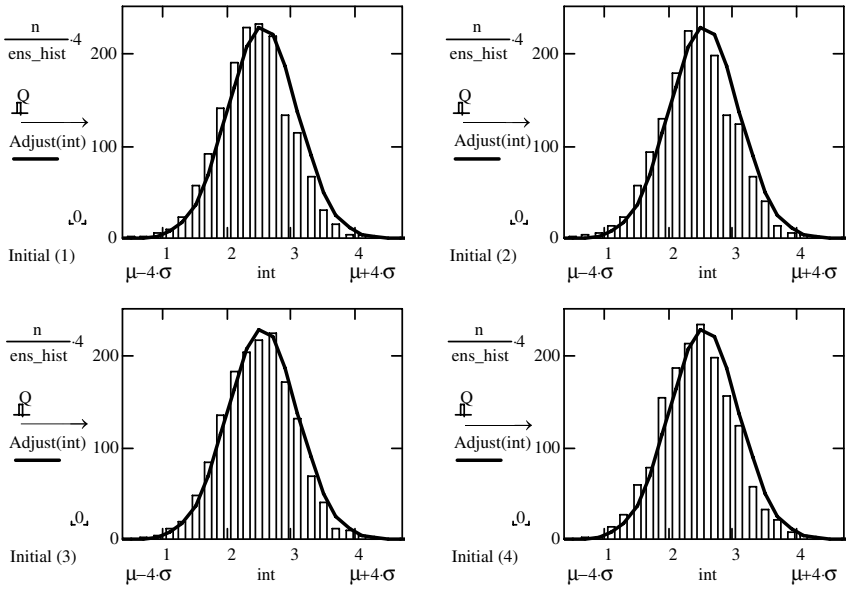

Figure 7.5. *Adjustment of normal law frequencies using MC simulation*

The following uses random number generators to explain how important sampling methods can estimate probabilities of quantities with unknown distributions. The estimation of an average in a logistical distribution sample is used for educational purposes.

PROBLEM.– We will demonstrate in an *empirical* manner that the average in the sample (based on a sample size, SSize = 100) of a logistical distribution with the parameters L and S, is located in the interval [a, b].

– Define the parameters of the problem: L = 1; S = 0.5; a = 0.9; and b = 1.1.

– Define the sample parameters of MC D; let

- **NSamples = 1 × 10^4**, the number of samples to collect, and then;

- **SSize = 100**, the number of points of data per sample.

– Carry out the sample and collect statistics on this sample:

I = 0. NSamples – 1 and $Means_i$ = mean (rlogis (SSize, L, S))

– Estimate the desired probability by the following expression:

$$Success = \sum_{i=1}^{NSample} \left(a \le Means_{i-1} \le b\right) \text{ therefore } \text{Prob} = \frac{Success}{NSamplees} = 0.731$$

We will shortly present the results of the MC simulation through drawing random variables a_0 and m at chance. We will calculate the failure probabilities of the structure, through welding processes, on the results from 100 drawn values. We can establish a good stability of convergence toward the 40th drawn values (see Figure 7.2).

The next stage will provide a calculated interpretation of averages and standard deviations, which we can calculate the reliability index by using a limit state function, as clarified: M = (R − S) < 0.

Third stage: The results from the MC simulation calculations, which have already been used to plot distribution functions, can be grouped by average and standard deviation.

Welding processes	μ Average N_{cycles}	σ Standard deviation N_{cycles}	Law for N_{cycles}	Law for a_0	Law for m
SAW	7.15E + 5	2.45E + 4	Gauss	Weibull	Galton
FCAW	1.54E + 6	4.46E + 5	Gauss	Weibull	Galton
SMAW 57	1.91E + 6	6.62E + 5	Gauss	Weibull	Galton
SMAW 76	1.18E + 6	3.12E + 5	Gauss	Weibull	Galton

Table 7.8. *Average and standard deviations from MC simulation of variables on the number of theoretical cycles to imitate parameters a_0 and m*

After having carried out a "*setting*" of functions from random variables (a_0, m, and N), we can apply the method of load-stress to calculate the failure probability P_F and the safety index. Our results must be similar to the following theoretical graph:

We will now attempt to answer the following question: With what probability of P_F would there be certain failure, if we know its capacity *R* to resist load *S*, and what would its probable safety index β_c be?

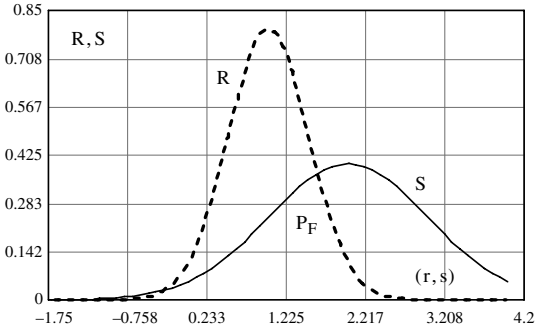

Figure 7.6. *Simplification of failure probability P_F according to a Gaussian distribution*

To objectively answer this question we follow the following stages:

a) First stage

According to Figure 7.6, we can clearly see that it is necessary to first calculate and statistically simulate the number of cycles, by welding processes. This is what we have done until now through the simulation of random variables. Then, we can consider that the simulated N_{cycles} correspond to *R* and the "added" simulated N_{cycles} correspond to *S*.

b) Second stage

This stage involves choosing, based on an industrial demand (functional specifications), a load S which allows us to verify whether the structure will resist or break, by adding a supplementary load to the one designed at hypothesis stage. In our precise case, there is good reason to calculate a limit state function $M < 0$, a failure probability P_F, and a safety index β_c. In accordance with what has been said in this chapter so far, it would seem wise to consider a hypothesis with $S > 1.5R$, according to which we could calculate the failure probability P_F and its safety index β_c.

To do this, we must calculate and plot the probability density functions of the N_{cycles}, i.e. R, and on the same graph add the results from the probability density

functions of load S. Graphically, the intersection of the two curves allows us to locate the failure probability as shown in Figure 7.6. Recall that analytically, the failure probability is evaluated across the limit state function M, as shown in the relationships [7.1] and [7.4]. After having defined the foundations of the random variable calculations (a_0, m, and N), we will present the results of these parameters from MC simulation as follows:

Statistical characteristics	SAW process	FCAW process	SMAW 57 process	SMAW 76 process
Average of μ_R	7.15E + 5	1.54E + 6	1.91E + 6	1.18E + 6
Average of μ_S	1.07E + 6	2.13E + 6	2.87E + 6	1.77E + 6
Standard deviation of σ_R	2.45E + 5	4.46E + 5	6.22E + 5	3.12E + 5
Standard deviation of σ_S	3.68E + 5	6.69E + 5	9.33E + 5	4.68E + 5
$\mu_R - \mu_S$	3.55E + 5	5.90E + 5	9.60E + 5	5.90E + 5
$\sqrt{\sigma_R - \sigma_S}$	4.42E + 5	8.04E + 5	1.12E + 6	5.62E + 5

Table 7.9. *Calculation results for the theory of resistance and stress (R − S) < 0*

The graphical results which follow indeed correspond to the theoretical plot in Figure 7.6.

i) Welding process: SAW

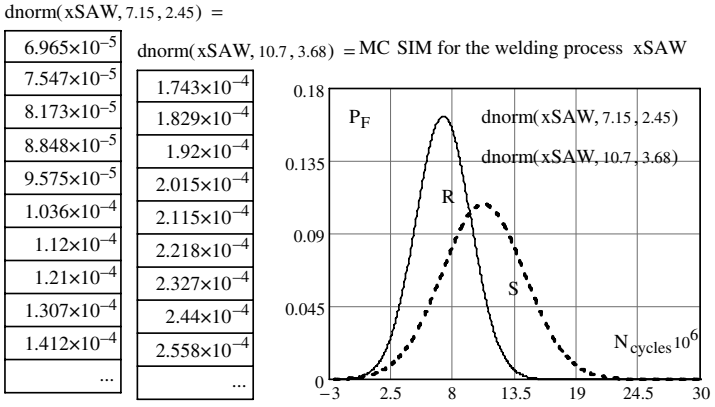

Figure 7.7. *Graphical results of failure probability P_F calculations, simulated according to a Gaussian distribution using MC simulation and an arbitrary choice by the engineer*

ii) Welding process: FCAW

dnorm(xFCAW, 15.4, 4.46) =

1.802×10^{-5}
1.976×10^{-5}
2.166×10^{-5}
2.373×10^{-5}
2.598×10^{-5}
2.843×10^{-5}
3.11×10^{-5}
3.4×10^{-5}
3.716×10^{-5}
4.059×10^{-5}
...

dnorm(xFCAW, 21.3, 6.69) =

8.139×10^{-5}
8.592×10^{-5}
9.068×10^{-5}
9.569×10^{-5}
1.009×10^{-4}
1.065×10^{-4}
1.123×10^{-4}
1.184×10^{-4}
1.248×10^{-4}
...

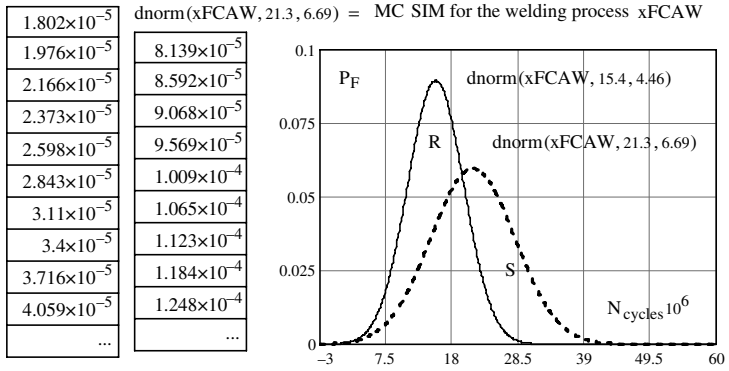

Figure 7.8. *Graphical results of failure probability P_F calculations, simulated according to a Gaussian distribution using MC simulation and an arbitrary choice by the engineer*

iii) Welding process: SMAW 57

dnorm(xSMAW57, 19.1, 6.22) =

1.164×10^{-4}
1.232×10^{-4}
1.304×10^{-4}
1.38×10^{-4}
1.459×10^{-4}
1.543×10^{-4}
1.632×10^{-4}
1.725×10^{-4}
1.823×10^{-4}
1.925×10^{-4}
...

dnorm(xSMAW57, 28.7, 9.33) =

1.331×10^{-4}
1.381×10^{-4}
1.432×10^{-4}
1.484×10^{-4}
1.539×10^{-4}
1.595×10^{-4}
1.653×10^{-4}
1.713×10^{-4}
1.775×10^{-4}
...

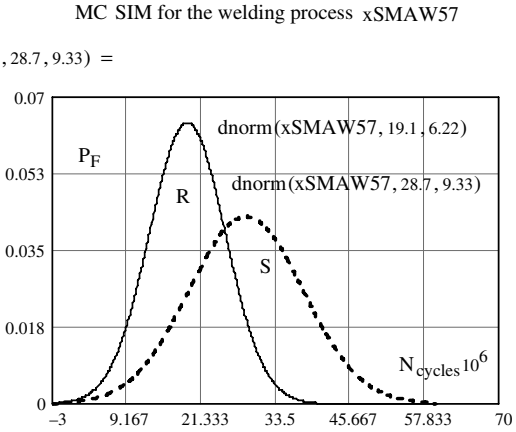

Figure 7.9. *Graphical results of failure probability P_F calculations, simulated according to a Gaussian distribution using MC simulation and an arbitrary choice by the engineer*

iv) Welding process: SMAW 76

dnorm(xSMAW76, 11.18, 3.12) =

4.182×10^{-6}
4.835×10^{-6}
5.585×10^{-6}
6.444×10^{-6}
7.428×10^{-6}
8.553×10^{-6}
9.839×10^{-6}
1.131×10^{-5}
1.298×10^{-5}
1.488×10^{-5}
1.705×10^{-5}
...

dnorm(xSMAW76, 17.7, 4.68) =

4.814×10^{-6}
5.29×10^{-6}
5.81×10^{-6}
6.379×10^{-6}
7×10^{-6}
7.678×10^{-6}
8.417×10^{-6}
9.224×10^{-6}
1.01×10^{-5}
1.106×10^{-5}
...

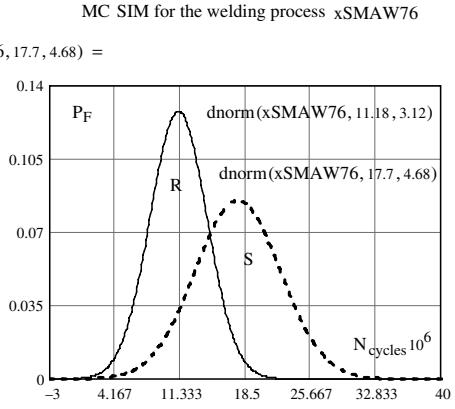

Figure 7.10. *Graphical results of failure probability P_F calculations, simulated according to a Gaussian distribution using MC simulation and an arbitrary choice by the engineer*

c) Third stage: graphical interpretation of probability densities arising from MC simulation (resistance/stress (R − S) < 0)

PF(β) =

	0
0	0.30622
1	0.27385
2	0.24491
3	0.21902
4	0.19587
5	0.17517
6	0.15666
7	0.14010
8	0.12529
9	0.11205
10	0.10021
11	0.08961

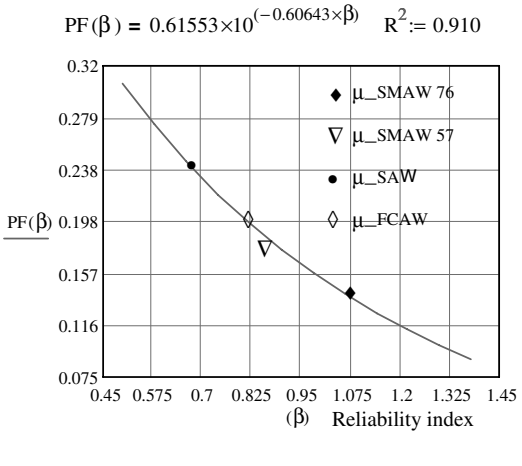

Figure 7.11. *Graphical results of the safety index (β) by joining processes*

We can establish on inspection of the two probability density functions that the structure will not be able to withstand the load that is imposed on it.

This process is more economic compared to traditional safety coefficient calculations, which take material physics into consideration in a direct method. This method does not have the pretence of being perfect, but it does have the merit of considering three random variables from the crack growth law [a_0, (C and/or m) and N]. In terms of these results, we have calculated, by exponential regression, an average relationship which allows us to read the failure probability according to the safety index (β_c) with a good correlation ($R^2 = 0.91$).

7.5. Conclusion

We have presented the MC simulation method, which has allowed us to obtain very weak probabilities, in the region of 10^{-6}. This method is relatively well adapted to singular structures because first it allows us to determine the safety index β_c, which is used to design structures and constructions. Also, with very few additional calculations and without a restrictive hypothesis, it completes the independence between resistance R and load S. By knowing the Hasofer–Lind or the Cornell index, this method allows us to attain a so-called level III precision, with a few supplementary calculations. On the other hand, this method remains dependent on the random drawing of random variables a_0 and m. This is what we have done throughout the examples shown in this chapter.

We are equally convinced that for 27,000 draws from the KS model, we have a convergence of failure probability. However, we note the complexity of the MC simulation method. Structural reliability analysis requires the control of structural and material behavior. It needs ample advanced knowledge in probabilistic simulation to reach a correct imitation of the random variables which intervene in the problem, in this case a_0, m, and N. It also requires highly evolved mathematical techniques to carry out multidimensional integrations from probability densities and zones of failure ($M < 0$).

The method of using pseudo-random simulations has clearly spread to the study of safety in numerous fields. The most commonly used technique in civil engineering is the MC simulations technique. This very general technique is simple to put into place: in this chapter we have provided its principal features. The issue that these techniques raise (which is dealt with in the following chapter) is their convergence rate toward a stable and faintly variant result, when this result is a failure probability, and therefore very weak. Technical literature (informatics programs) on numerous pseudo-random algorithms provides uniformly distributed

values. A very widespread class of generators uses a linear congruence. For our part, we have used programs from the software MathCAD rather than the Fibonacci suite.

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Chapter 8

Case Studies

8.1. Introduction

In this chapter, we present practical cases that are taken from our own laboratories and workshops. First, the aim of this research is to confirm good theoretical–practical compatibility, and second to present educational tools with the aim of drawing up purely formative supervised works.

8.2. Reliability indicators (λ) and MTBF

8.2.1. *Model of parallel assembly*

A system with four parallel components (K) can tolerate the failure of half of its elements. Reliability calculation is presented in the following, which supports our reasoning:

$$R = \sum_{i=m}^{n} \left\{ \left(\frac{n!}{i! \times (n-i)!} \right) \times R^i \left(1 - R\right)^{n-i} \right\} \qquad [8.1]$$

where:

R is the reliability of the components; for R = 0, 0.1, …, 1.

n is the number of components; n = 4 components.

m is the minimum number of components in use; $m = 2$.

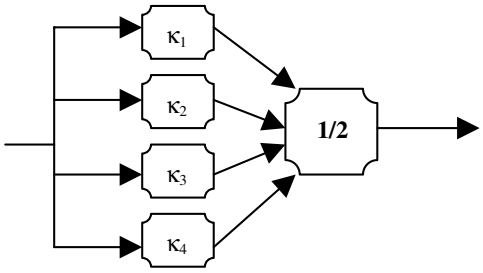


Figure 8.1. *Assembly (arrangement) of parallel components*

Reliab(R) =

0.0000
0.0523
0.1808
0.3483
0.5248
0.6875
0.8208
0.9163
0.9728
0.9963
1.0000

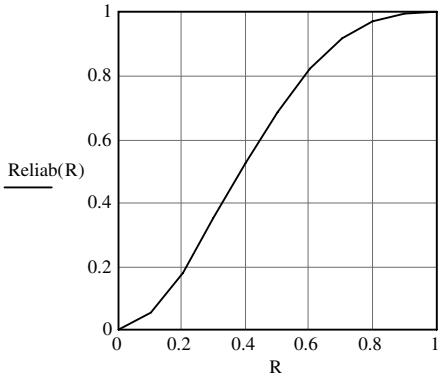


Figure 8.2. *Failure curve for a system of parallel components*

8.2.2. *Model of serial assembly*

The model of serial assembly supposes that the system does not reach the final stage of its task if the first component breaks down. This assembly system only applies to irreplaceable components. Let κ be the failure modes of (n) components in an assembly system, then

a) Each component functions or fails independently of all other component, at least until the failure of the first component.

b) The system fails when the first component fails.

c) For each n (possibly different) the system's components have a lifecycle known by the distribution model $F_i(\tau)$.

Reliability, in the case of serial assembly, is written as:

$$R_S(\tau) = \prod_{i=1}^{n} R_i(\tau);\ F_s(\tau) = 1 - \prod_{i=1}^{n} \{1 - F_i(\tau)\};\ h_s(\tau) = \sum_{i=1}^{n} h_i(\tau) \qquad [8.2]$$

where:

$R_s(\tau)$ is the reliability.

$F_s(\tau)$ is the distribution function.

$h_s(\tau)$ is the failure rate.

The index (s) refers to the whole system and the index (i) refers to the ith component. The whole system has (n) serial components as shown in Figure 8.3.

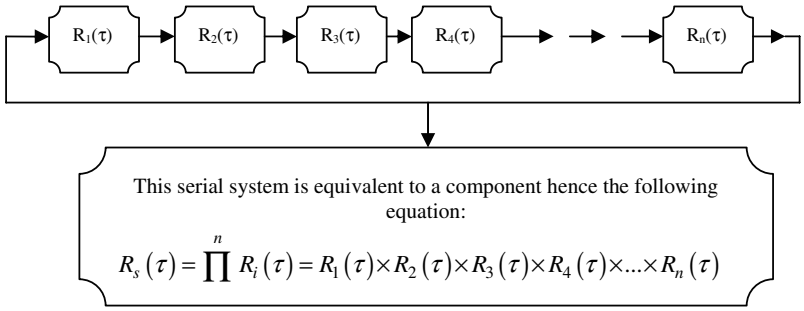


Figure 8.3. *Arrangement of serial components*

8.3. Parallel or redundant model

In the case of a parallel system, we suppose that all the (n) components that make up the system function in an independent way, and that the system continues to function as long as one component is still functioning. This is in fact the opposite of a serial system, in which the failure of the first component results in the failure of the system.

In a parallel model, all the components should have failed for the system to break down. If there are (n) components, all ($n - 1$) will be redundant to the remaining

component (if all components are different). Once the system is *started up*, all the components function until they fail. The system reaches failure point when the last component fails.

1) All components function independently of one another.

2) The system functions as long as at least one component is still functioning. The system fails when the last component fails.

3) The distribution function for each component is shown in equation [8.3].

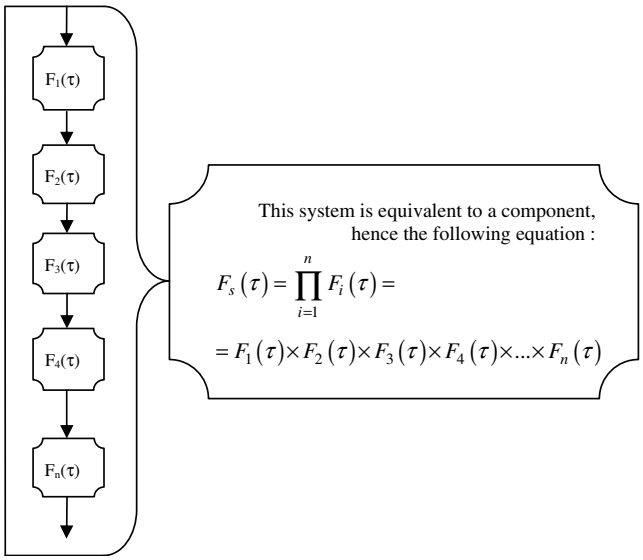


Figure 8.4a. *Parallel system*

$$F_s(\tau) = \prod_{i=1}^{n} F_i(\tau) \qquad [8.3]$$

EXAMPLE.– Imagine a model composed of (n) components. We would say that a composite system survives when at least (r) components continue to function, whatever the (r). We call this an "r on n" system, which is in fact a system combining a serial assembly and a parallel assembly. The system has (n) components, which function independently from the others. As long as at least (r) of these components ($\forall$ r) are functioning, so too will the system. The failure of the

system occurs when the $(n - 1)$th component fails r. Two scenarios are given as follows:

– When $r = n$, the model "r on n" is reduced to a serial model.

– When $r = 1$, the model "r on n" is a parallel model.

In the following, we present the case where all components are identical.

1) All components have an identical reliability function $R(\tau)$.

2) All components function independently from one another (*failure*).

The system's reliability is given by adding the survival probability of (r) components, at the time (τ). The total of these probabilities is in fact the probability of a binominal law with $p = R(\tau)$, so the system's reliability is written as follows:

$$R_S(\tau) = \sum_{i=r}^{n} \binom{n}{i} \times R(\tau)^i \times \left(1 - R(\tau)\right)^{n-1} \text{ and} \qquad [8.4]$$

$$R = \sum_{i=m}^{n} \left\{ \left(\frac{n!}{i! \times (n-i)!} \right) \times R^i (1-R)^{n-i} \right\} \qquad [8.5]$$

If not all the components are identical, then $R_s(\tau)$ is the sum of evaluated probabilities for all the possible terms, formed by choosing at least (r) surviving components and their corresponding failures. The probability for each term is evaluated as being a product of $R(\tau)$. For example, if $n = 4$ and $r = 2$, the system's reliability would be:

$$R_S(\tau) = \left\{ \begin{array}{l} \{R_1 \times R_2 \times F_3 \times F_4\} + \{R_1 \times R_3 \times F_2 \times F_4\} + \{R_1 \times R_4 \times F_2 \times F_3\} + \\ \{R_2 \times R_3 \times F_1 \times F_4\} + \{R_2 \times R_4 \times F_1 \times F_3\} + \{R_3 \times R_4 \times F_1 \times F_2\} + \\ \{R_1 \times R_2 \times R_3 \times F_4\} + \{R_1 \times R_3 \times R_4 \times F_2\} + \{R_1 \times R_2 \times R_4 \times F_3\} + \\ + \{R_2 \times R_3 \times R_4 \times F_1\} + \{R_1 \times R_2 \times R_3 \times R_4\} \end{array} \right\} \qquad [8.6]$$

8.4. Reliability and structural redundancy: systems without distribution

8.4.1. *Serial model*

It is clear that in such an assembly of components, just one failing component causes the failure of the whole system. For a system including components of

different types (i = 1, 2, …, κ), the respective reliability for each component is written as $R_1(\tau)$, $R_2(\tau)$, …, $R_\kappa(\tau)$. The reliability of the whole system can thus be written as:

$$R_{series} = R_1 \times R_2 \times ... \times R_\kappa = \prod_{i=1}^{\kappa} R_i \qquad [8.7]$$

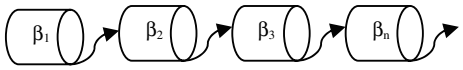


Figure 8.4b. *Tree diagram of serial components failure*

If the reliabilities are near unity (≈1), we can, instead, use expression $Q_i = 1 - R_i$, which represents the failure probability. Expression [8.8] is written as follows:

$$R_{series} \simeq 1 - \sum_{i=1}^{\kappa} Q_i \qquad [8.8]$$

We accept an error of:

$$Error \prec \left(\sum_{i=1}^{\kappa} Q_i \right)^2 \Big/ 2 \qquad [8.9]$$

It is worth mentioning that the reliability of a system composed of a serial assembly presents a smaller reliability than the weakest component presents. But, if all the components are identical and present a reliability of (r), we can suppose the following:

$$R_{series} = r^n \simeq \left\{ 1 - \left(n \times r \right) \right\} \qquad [8.10]$$

We accept an error of:

$$Error \prec \left(n \times r \right)^2 / 2 \qquad [8.11]$$

8.5. Rate of constant failure

If λ represents a constant failure rate and if the system is composed of pieces organized in a series with a constant λ, we can suppose that for an ith component:

$$R_i(\tau) = Exp\{-\lambda_i \tau\} \quad [8.12]$$

The system's reliability is written as:

$$R_i(\tau) = Exp\{-\lambda_s \times \tau\} \quad [8.13]$$

With $\lambda_s = \sum_i \lambda_i$ the system presents a mean time before failure (MTBF):

$$\theta_s = 1/\lambda_s \quad [8.14]$$

By analogy, the MTBF: $\theta_s \prec \theta^{smallest}_{component}$

The reliability is written as:

$$R_{series}(\tau) = Exp\{-n\lambda\tau\} \quad [8.15]$$

For $\lambda_s \cong \sum \lambda_i = (n \times \lambda)$ which is quite weak

We suppose:

$$R_{series}(\tau) \simeq \{1 - (\kappa_s \times \tau)\} \quad [8.16]$$

We accept an error of:

$$Error \prec (\lambda_s \times \tau)^2 / 2 \quad [8.17]$$

It is common for the system's components not to present a constant failure rate. As an example, consider a system where $Z_i(\tau)$ is the failure rate of a component (i):

$$R_{series}(\tau) = \prod_i R_i(\tau) = \prod_i Exp\left(-\int_0^\tau Z_i(u)\,du\right) = Exp\left(-\int_0^\tau \sum_i Z_i(u)\,du\right) \quad [8.18]$$

The system's failure rate, as a whole, is simply written as:

$$Z_{series}(\tau) = \sum_{i=1}^{\kappa} Z_i(\tau) \quad [8.19]$$

The cumulated risk therefore becomes:

$$\Psi_{series}(\tau) = \left(\int_0^{\tau} \sum_{i=1}^{\kappa} Z_i(u)\,du \right) = \sum_{i=1}^{\kappa} \int_0^{\tau} Z_i(u)\,du \qquad [8.20]$$

Drawing on the relationships we have just mentioned and those which we developed in Chapter 1, we will now present a case study applied to material and structural reliability. Here is the initial hypothesis. Imagine a system of five components assembled in a series:

– Component **1** has a constant failure rate, i.e. λ_1 = Constant.

– Component **2** presents a distributed lifecycle according to Weibull law with two parameters (η and β, see Chapter 1), its failure rate is λ_2.

– Component **3** presents a distributed lifecycle according to Gamma law of parameters (λ_3 and k = 1.75, see Chapter 1, Part 1).

We can add two other components, i.e.:

– Component **4** presents a distributed lifecycle according to Galton (Log-Normal) law of parameters (λ_4 and k = 1.75, see Chapter 1).

– Component **5** presents a distributed lifecycle according to Saunders (Birnbaum–Saunders) law of parameters (λ_4 and k = 1.75, see Chapter 1).

ISSUES.–

1) Writing the reliability expressions of this system, i.e. $Z_s(\tau)$, $H_s(\tau)$, and $R_s(\tau)$.

2) Writing the same respective functions after a task initiated after a period of time (T, age of components), i.e. $Z_s(\tau/T)$, $H_s(\tau/T)$, and $R_s(\tau/T)$.

3) Conducting a numerical calculation for the most appropriate law for fracture mechanics through fatigue, supported by inherent graph plotting.

SOLUTION.– Following the mathematical relationships already presented in Chapter 1, we suppose the following:

Failure rate $Z(\tau)$:

$$Z_{series}(\tau) = \lambda_1 + \left(\frac{\beta}{\eta}\right) \times \left(\frac{\tau}{\eta}\right)^{\beta-1} + \lambda_3 \times \left(\frac{\lambda_3 \times \tau}{1 + \lambda_3 \times \tau}\right) \qquad [8.21]$$

Failure rate $\psi(\tau)$:

$$\Psi_{series}\left(\tau\right)=\lambda_1\times\tau+\left(\frac{\tau}{\eta}\right)^{\beta}+\lambda_3\times\tau-\ln\times\left(1+\lambda_3\times\tau\right) \quad [8.22]$$

Reliability of system R(τ):

$$R_{series}\left(\tau\right)=Exp\left\{-H_{Series}\left(\tau\right)\right\}=\left(1+\lambda_3\times\tau\right)Exp\left\{-\left(\left(\lambda_1+\lambda_3\right)\tau+\left(\frac{\tau}{\eta}\right)^{\beta}\right)\right\} \quad [8.23]$$

After a certain age (T), we suppose the corresponding expressions as follows:

$$Z_{series}\left(\frac{\tau}{T}\right)=Z_{Series}\left(\tau+T\right)=\lambda_1+\left(\frac{\beta}{\eta}\right)\times\left(\frac{\tau+T}{\eta}\right)^{\beta-1}+\lambda_3\times\left(\frac{\lambda_3\times\left(\tau+T\right)}{1+\lambda_3\times\left(\tau+T\right)}\right) \quad [8.24]$$

$$\Psi_{series}\left(\frac{\tau}{T}\right)=\Psi_{series}\left(T+u\right)du=\left(\lambda_1+\lambda_3\right)\tau+\frac{\left(\tau+T\right)^{\beta}-T^{\beta}}{\eta^{\beta}}-\ln\left(\frac{1+\lambda_3\left(\tau+T\right)}{1+\lambda_3\times T}\right) \quad [8.25]$$

$$R_{series}\left(\frac{\tau}{T}\right)=\frac{1+\lambda_3\left(\tau+T\right)}{1+\lambda_3\times T}Exp\left\{-\left(\left(\lambda_1+\lambda_3\right)\tau+\frac{\left(\tau+T\right)^{\beta}-T^{\beta}}{\eta^{\beta}}\right)\right\} \quad [8.26]$$

All calculations from the previous relationships are carried out after the age (T).

8.5.1. *Reliability of systems without repairing: parallel model*

In this example, to reach total failure, all the components must break down.

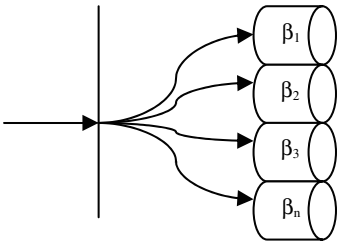


Figure 8.5. *Tree diagram of parallel components failure*

It is clear that in such an assembly, one sole failing component causes the failure of the whole system. In a system which includes (κ) components of different types (i = 1, 2, …, κ), the reliability of the whole system is:

$$R_{Parallel}(\tau) = 1 - \prod(1 - R_i) = 1 - \prod Q_i \ \textit{because}\ Q_i = (1 - R_i) \qquad [8.27]$$

$Q_{Parallel}$ represents the failure probability of the whole system, and is written as $Q_p(\tau) = \prod Q_i$. We can express the development of the reliability R_1, R_2, …, R_p for structures S_1, S_2, …, S_n when they are assembled in parallel as:

$$R_p(\tau) = S_1 - S_2 + ... + (-1)^{\kappa+1} S_\kappa + ... + (-1)^{n+1} S_n \qquad [8.28]$$

$$\left\{ \begin{array}{l} S_1 = \sum_i R_i(\tau);\ S_2 = \sum_{i,j} R_i(\tau) R_j(\tau);\ \textit{avec}\ i \neq j \\ S_3 = \sum_{i,j,\kappa} R_i(\tau) R_j(\tau) R_\kappa(\tau);\ i \neq j \neq \kappa,\ \textit{etc...}\ S_n = \prod_i R_i(\tau) \end{array} \right\} \qquad [8.29]$$

Note that the reliability of a parallel system is greater than the reliability of the component with the highest reliability. Also, it is important to remember that if the components' failures are not independent, it is highly advisable to use the Bayes approach (conditional probabilities, see Chapter 3). Components with constant failure rates are thus expressed as in the following:

Each component will have a reliability expressed as: $R_i(\tau) = Exp\{-\lambda_i \times \tau\}$ and the reliability of the whole system will be expressed as:

$$R_p(\tau) = 1 - \prod\left[1 - Exp\{-\lambda_i \times \tau\}\right] \qquad [8.30]$$

As previously developed, we also have the following expression for a variable failure rate:

$$R_p(\tau) = S_1 - S_2 + S_3 - S_4 + ... + (-1)^{\kappa+1} S_\kappa + ... + (-1)^{n+1} S_n \qquad [8.31]$$

$$\left\{ \begin{array}{l} S_1 = \sum_i \left(Exp^{-\lambda_i \times \tau}\right);\ S_2 = \sum_{i,j} Exp^{\left(-(\lambda_i + \lambda_j) \times \tau\right)};\ \textit{with}\ i \neq j \\ S_3 = \sum_{i,j,\kappa} Exp^{\left(-(\lambda_i + \lambda_j + \lambda_\kappa) \times \tau\right)};\ i \neq j \neq \kappa,\ \textit{etc...}\ S_n = Exp^{-\tau \sum \lambda_i} \end{array} \right\} \qquad [8.32]$$

Numerical application: Imagine *three* components which make up a parallel system whose failure rate is $\lambda = 0.5$. After a certain longevity T = 1 relative to a task of $\tau = (0.1, 0.2, \ldots, 10)$, checking the system shows that the *two* components are able (fine), and we will therefore get a reliability of $R(\tau) = ?$

$$R_{parallel}(\tau) = Exp^{-\lambda\times\tau} \times \left(2 - Exp^{-\lambda\times\tau}\right) \qquad [8.33]$$

Summarized solution: If $\lambda = 0.5$ and $\tau = (0.1, 0.2, \ldots, 10)$ we get:

For 2 components $Rp2(\tau) := e^{-\lambda\times\tau}\times\left(2 - e^{-\lambda\times\tau}\right)$ *then* $Rp2(\tau) =$

0.998
0.991
0.981
0.967
0.951
0.933
0.913
0.891
...

For 1 component $Rp1(\tau) := e^{-\lambda\times\tau}$ *then* $Rp1(\tau) =$

0.951
0.905
0.861
0.819
0.779
0.741
0.705
0.67
...

etc. ...

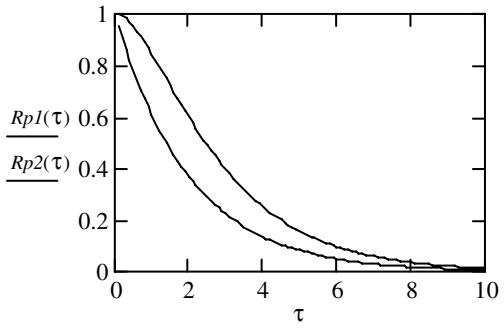


Figure 8.6. *Reliability, at the time (τ) of a parallel assembly system*

If the system is not checked after the age T = (0.75), we get $R(\tau/T)$:

$$R_{parallel}\left(\frac{\tau}{T}\right) = Exp^{-\lambda\times\tau} \times \left(\frac{2 - Exp^{-\lambda\times(\tau+T)}}{2 - Exp^{-\lambda\times(T)}}\right) \text{ if } T = 0.75 \text{ we get:}$$

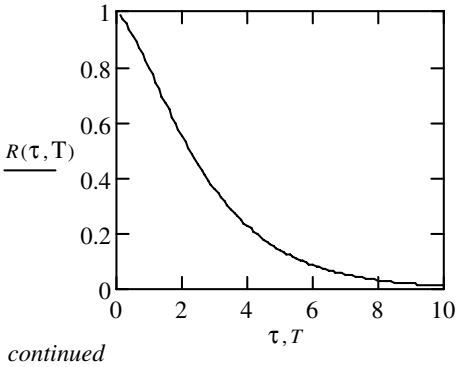
$$For\ \lambda_1 = \lambda_2 = 1 : \theta_P(\lambda) = \left(\frac{3}{2}\right)$$

The MTBF is thus found as:

$$\theta_{parallel(2)}(\lambda) = \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right) - \left(\frac{1}{\lambda_1 + \lambda_2}\right) \qquad [8.34]$$

$R(\tau, T) =$

0.9879
0.9728
0.9554
0.9359
0.9148
0.8924
0.8690
...



continued

Figure 8.7. *Reliability of a system of parallel components after the age (T)*

This is in fact the combination of the previous expression of $R_p(\tau)$, giving the following:

$$\left\{ \theta_{P(\text{instance of n components})} = \sum_i \frac{1}{\lambda_i} - \sum_{i \neq j} \left(\frac{1}{\lambda_i + \lambda_j}\right) + \right.$$
$$\left. + \sum_{i \neq j \neq \kappa} \left(\frac{1}{\lambda_i + \lambda_j + \lambda_O}\right) + \ldots + (-1)^{n+1} \sum_i \frac{1}{\lambda_i} \right\} \qquad [8.35]$$

Specific example: In the reliability expression R_i, at the specific times (τ), $R_i(\tau)$, we substitute the $R_i(\tau)$ with R_i to eventually lead to $R_p(\tau)$. The latter corresponds to the average time until failure from $R_p(\tau)$.

Attention: These relationships are valid for a first task, in the case of original (new) components. If the task started at a great age, i.e. after a certain lifecycle (of functioning), there is a good reason to use the following relationships:

$$R_p\left(\frac{\tau}{T}\right)^{Uncontrolled\ components}_{before\ the\ new\ task} = \frac{R_p(T+\tau)}{R_p(T)} \quad [8.36]$$

$$R_i\left(\frac{\tau}{T}\right)^{Controlled\ components\ all}_{functioning\ correctly...} = \frac{R_i(T+\tau)}{R_i(T)} \quad [8.37]$$

Here is an applied example for educational purposes: At the time $\tau = 0.2$ and T = 1, we present a case of two components whose statistics are simulated by Weibull law with two parameters, as follows:

Case 1 $\rightarrow$ ($\lambda_1 = 1$ and $\beta_1 = 3/2$) and **Case 2** $\rightarrow$ ($\lambda_2 = 2$ and $\beta_2 = 2$).

Parallel system. For two components, we will get the following:

a) First task according to time (τ):

$\tau = 0.1, 0.5, \ldots, 5$ for $\eta_1 = 0.5658$, $\beta_1 = 0.5658$ and for $\eta_2 = 2.5654$, $\beta_2 = 3.5654$

$$Rp(\tau) = \left[-\left(\frac{\tau}{\eta 1}\right)^{\beta 1} + e^{-\left(\frac{\tau}{\eta 2}\right)^{\beta 2}}\right] - e^{-\left[\left(\frac{\tau}{\eta 1}\right)^{\beta 1} + \left(\frac{\tau}{\eta 2}\right)^{\beta 2}\right]}$$

Rp(τ) =

0.999997
0.998222
0.982831
0.932309
0.826011
0.660150
0.460623
0.275917
0.146620
0.078960
0.051238
0.040039
0.033683

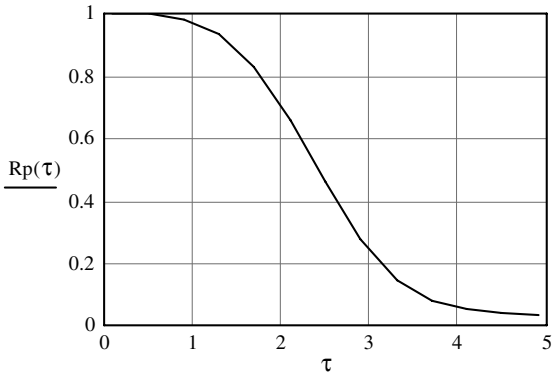


Figure 8.8. *Evolution of Weibull's reliability for a parallel system*

b) First task after the age (T, functioning lifecycle, Weibull):

$$Rp2(\tau, T) = \frac{e^{-\left(\frac{\tau+T}{\eta 1}\right)^{\beta 1}} + e^{-\left(\frac{\tau+T}{\eta 2}\right)^{\beta 2}} - e^{-\left[\left(\frac{\tau+T}{\eta 1}\right)^{\beta 1}+\left(\frac{\tau+T}{\eta 2}\right)^{\beta 2}\right]}}{e^{-\left(\frac{T}{\eta 1}\right)^{\beta 1}} + e^{-\left(\frac{T}{\eta 2}\right)^{\beta 2}} - e^{-\left[\left(\frac{T}{\eta 1}\right)^{\beta 1}+\left(\frac{T}{\eta 2}\right)^{\beta 2}\right]}}$$

$Rp2(\tau, T) =$

0.994697
0.949463
0.850007
0.689454
0.490017
0.299091
0.160546
0.085399
0.053884
0.041399
0.034665
0.029648
0.025526

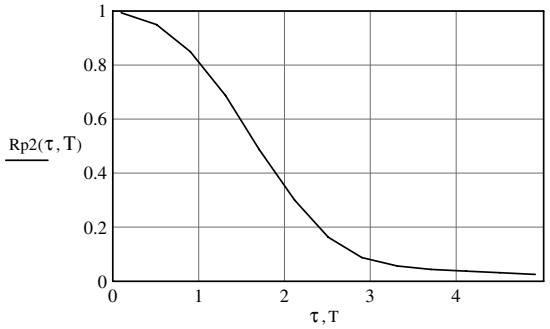


Figure 8.9. *Evolution of reliability for a parallel system after age (T)*

c) Second task at age T = 0.75:

If $\lambda_1 = 1.5654$ and $\lambda_2 = 1.5658$, the MTBF $\theta_{p2}(\lambda)$ will therefore be written as:

$$\theta_{p2}(\lambda) = \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right) - \left(\frac{1}{\lambda_1 + \lambda_2}\right) = 0.9581$$

8.6. Reliability applications in cases of redundant systems

There are various cases of redundancy. We present here the essential cases. When we make more components work than there is need, we say that there is redundancy. Redundancy is deemed *active* when the components are permanently in use. Redundancy is *sequential* or *backup* if only certain components are activated, according to need. In the case of *structural* redundancy, a structure is deemed redundant when a local damage is not *fatal* for the global safety. Such a behavior is often encountered in experiments and can be explained, at least partially, by the following reasons:

1) An overabundance of components' safety, because of sizing, is not strongly optimized with regard to *normal* conditions of service (e.g. availability and supply constraints). If the sizing codes do not sufficiently take account of uncertainties (global safety coefficients) this gives rise to heterogeneous levels of reliability in the global structure.

2) At the time of damage (failure of a rod), the whole structure is mobilized to redistribute its efforts and resist it (e.g. the geometric hyperstaticity of the system).

$$RP2(\tau, T) = \left[e^{-\left(\frac{\tau+T}{\eta 1}\right)^{\beta 1}} \right] \Big/ \left[e^{-\left(\frac{T}{\eta 1}\right)^{\beta 1}} \right] + \left[e^{-\left(\frac{\tau+T}{\eta 2}\right)^{\beta 2}} \right] \Big/ \left[e^{-\left(\frac{T}{\eta 2}\right)^{\beta 2}} \right] - e^{-\left[\left(\frac{\tau+T}{\eta 1}\right)^{\beta 1} + \left(\frac{\tau+T}{\eta 2}\right)^{\beta 2}\right]} \Big/ e^{-\left[\left(\frac{T}{\eta 1}\right)^{\beta 1} + \left(\frac{T}{\eta 2}\right)^{\beta 2}\right]}$$

$RP2(\tau, T) =$

0.999424
0.979676
0.914519
0.790062
0.618484
0.439615
0.296927
0.207556
0.158997
0.130841
0.110915
0.094976
0.081776

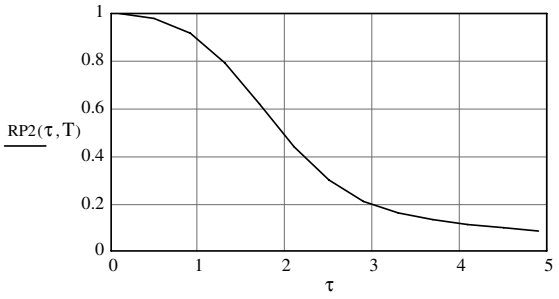


Figure 8.10. *Evolution of reliability for a parallel system after age (T)*

Conventional codes do not essentially apply to the local level. They do not take into account the *geometric hyperstaticity of the system*. However, reliability methods offer more precision and coherence in terms of structural safety, because they take account of structural redundancy. The reliability framework of failure modes such as plasticization, buckling, or fatigue allows us to standardize the design and maintenance of structures.

The probability that any first element on the platform breaks is P_{AFF} (*Probability of Any-First member-Failure event*). To calculate this probability, we combine all individual failure events. The system's failure probability is represented by P_{SF} (*Probability of System Failure*). The P_{SF} is calculated by exploring the fault tree. It is calculated by substituting a certain number of predominant mechanisms in a series. The probability of a mechanism is calculated by substituting a sequence of

individual successive failures in a parallel assembly. This sequence gives rise to a global failure. As a measure of the system's effect, the literature [BAR 65], [JOH 64], [KOV 97], [GRA 66] suggests using the following probabilities relation:

$$R_0 = P\left\{SF \middle| AFF\right\} = P_{SF} / P_{AFF} \qquad [8.38]$$

R_0 represents the conditional probability of the system failing, knowing that a first element has already broken. We suggest using the following index as a probabilistic measure of redundancy:

$$R_p = \left\{1 - R_0\right\} = 1 - P\left\{SF \middle| AFF\right\} \qquad [8.39]$$

R_p represents the conditional probability of the system surviving, knowing that at least one element has already broken. R_0 is its inverse measure. R_p is more natural, i.e. (R_p = 0) means that the structure is *not redundant* (no survival probability if a first element is broken). For this condition, $0 < R_p < 1$ means that the structure is *redundant*. A structure possesses a great number of failure mechanisms.

For example, a hyperstatic structure of S degrees, presenting (κ) plastic joints, will have [$\kappa(\kappa - 1) \times (\kappa - 2)$, …, $(\kappa - s)$] failure mechanisms, just taking into account the failure mode. As it is seemingly impossible to determine a list of all failure mechanisms, it would appear wise to adopt a method that allows us to determine dominant mechanisms.

To do this, we can resort to the so-called *Branch Bound* method, to identify failure mechanisms in descending order according to their probability of occurrence. Then we must construct two incomplete representations of the structure, which frame the smallest and highest boundaries of the system's failure probability P_{fs} or of the reliability index β_s. To construct the *fault tree*, we proceed in stages, as shown in Figure 8.11.

1) Class the failure elements in ascending order according to their reliability index. The strongest failure probability corresponds to the weakest reliability index, and it is therefore this index that will be kept for the future tree diagram.

2) Then consider the failure element of the reliability index (β_n), and calculate the new reliability indices of the elements remaining on the structure. Continue in this way, constructing this tree using the previously selected element of failure. Here is an outline of the schematization, shown in Figure 8.11:

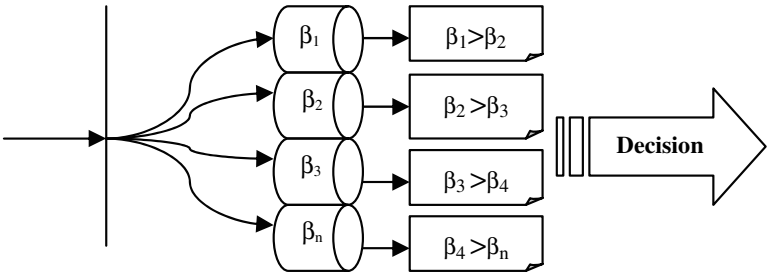


Figure 8.11. *Primary fault tree*

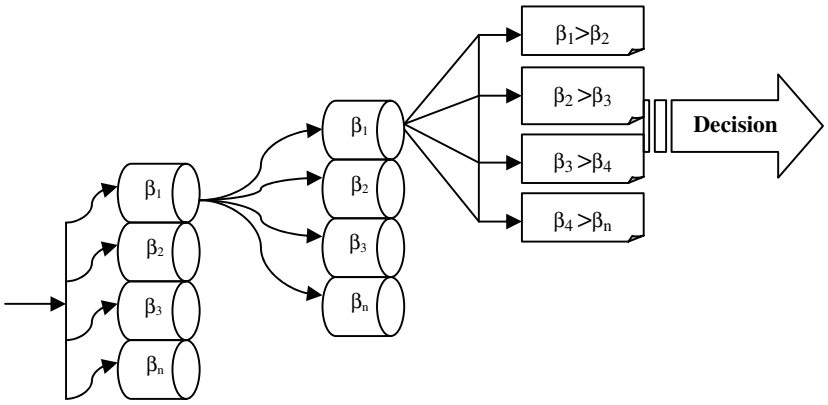


Figure 8.12. *Primary and secondary fault tree*

Prospecting the fault tree only involves components subject to the structure's failure. The trick (*or the cleverness*) consists in aiming at an upper limit of the prospecting index from which we could deliberately liberate ourselves from inspecting routes to failure (or ruin). An initial point corresponding to the first branch of the fault tree is chosen on the element whose reliability index (β_{weak}) is the weakest. In reality, this element presents a *post-failure* behavior. Then, we can redefine the new prospection indices for the remaining elements.

By classing the indices obtained in this way, we can select a new element that will represent the extremity of the previous branch and the departing branch point. By repeating these different stages each time and by considering the post-failure behavior of selected elements (calculation of new indices), we can then define a first branch of the fault tree with the following form:

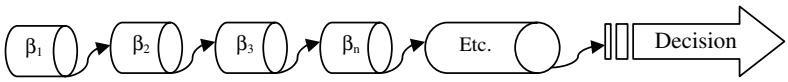


Figure 8.13. *First branch of the fault tree*

Faults (routes to failure) of superior elements only intervene if the inferior elements have followed a *post-failure* behavior. This is conditional failure. The difficulty lies in prospecting the other branches of the fault tree. In effect, the behavior of elements involving the first branch, in particular the failure criterion, must not restrict the prospection of the other branches.

8.6.1. *Total active redundancy*

The system has failed only when the last component breaks. R is the reliability of a component of the structure. For (n) components prone to redundancy, we suppose:

$$R_n = \left\{1-\left(1-R\right)^n\right\} = \sum_{i=1}^{n}\left(-1\right)^{i+1} \times C_n^i \times R^i \text{ and the } C_n^i = \frac{n\left(n-1\right)...\left(n-i+1\right)}{i!} \quad [8.40]$$

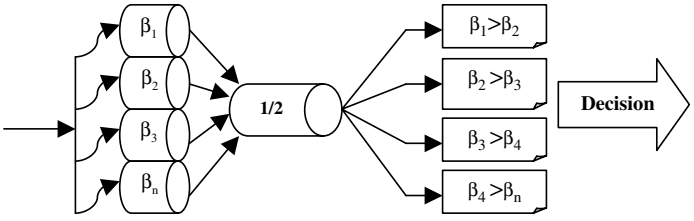


Figure 8.14. *Fault tree branch for a structure with four components*

Following the binominal law (see Chapters 1 and 2, volume 1), we suppose:

$$R_n = \sum_{i=1}^{n} C_n^i \times R^i \times \left(1-R\right)^{\left(n-i\right)} \quad [8.41]$$

Numerical application: Imagine a structure with four components (κ) in parallel, which allows the failure of half its elements:

– R is the reliability of components with R = 0, 0.1, …, 1.

– n is the number of components, n = (4, 2, and 3).

– κ is the number of minimum components, $\kappa = 2$.

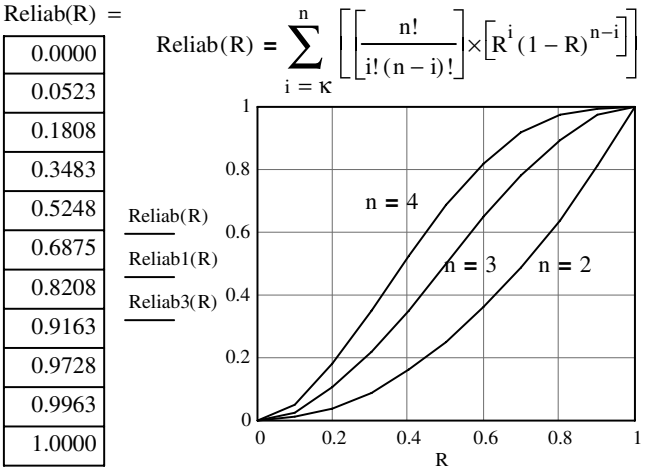


Figure 8.15. *Reliability of a structure with four components (total active redundancy)*

According to Figure 8.15: $R = \sum_{i=\kappa}^{n} \left(\left(\frac{n!}{i!(n-1)!} \right) \times \left(R^{i} (1-R)^{n-i} \right) \right)$

We can clearly see the evolution of (R) according to (R) if n = 2, 3, and 4. For the first task, with a constant failure rate (λ = Constant), we get:

$$\theta_n = \left(\frac{1}{\lambda} \right) \times \sum_{i=1}^{n} \left(\frac{1}{i} \right) \qquad [8.42]$$

8.6.2. *Partial active redundancy*

The null backup failure rate is considered, and then a non-null backup failure rate followed by practical applications on the components with constant failure rates. Imagine (n) unfailed components. The structural system functions and allows ($n \leq \kappa$)

breakages. The system as a whole allows ($n - \kappa$) breakages. Using the binominal distribution, we suppose:

$$R = R_{\kappa/n} = \sum_{i=\kappa}^{n} C_n^i \times \mathrm{R}^i \times (1-R)^{n-i} = 1 - \sum_{i=0}^{\kappa=1} C_n^i \times \mathrm{R}^i \times (1-R)^{n-i} \qquad [8.43]$$

We apply the recursive reasoning method to get $R_{\kappa/n}$:

$$R_{\kappa/n} = R_{(\kappa+1)/n} + C_n^{\kappa} \times R^{\kappa} \times (1-R)^{n-\kappa};\ R_{n/n} \equiv R^n \{n \text{ components in a series}\} \qquad [8.44]$$

F(R) =	F1(R) =	F2(R) =
0.0000	0.0000	0.0000
0.1000	0.0523	3.7000×10^{-3}
0.2000	0.1808	0.0272
0.3000	0.3483	0.0837
0.4000	0.5248	0.1792
0.5000	0.6875	0.3125
0.6000	0.8208	0.4752
0.7000	0.9163	0.6517
0.8000	0.9728	0.8192
0.9000	0.9963	0.9477
1.0000	1.0000	1.0000

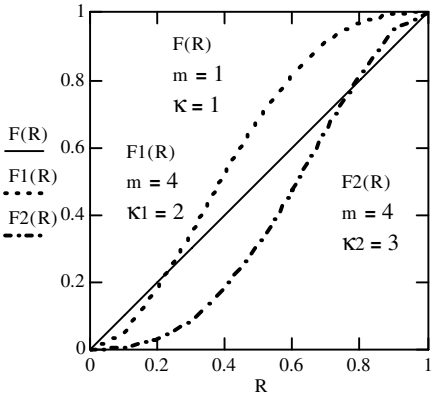


Figure 8.16. *Reliability of a structure with four components (partial active redundancy)*

We can see clearly the evolution of (R) according to (R) when n = m = 1, 2, and 3. If (m = 1 and κ = 1), we can establish a straight line. This corresponds to the so-called *total active redundancy*. For the first task, with a constant failure rate (λ = Constant) we get:

$$\theta_{\kappa/m} = \left(\frac{1}{\lambda}\right) \times \sum_{i=\kappa}^{m} \left(\frac{1}{i}\right) \qquad [8.45]$$

Imagine another case in which a structure has five components assembled in parallel.

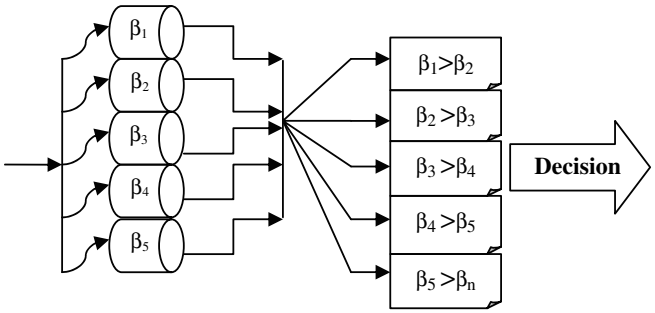


Figure 8.17. *Parallel arrangement (the so-called total active redundancy)*

A brief reading is taken where the signal from the entire structure represents the total signals. Let the reliability of a single component $R_c = 1$ and the reliability $R = ½$. Thus if $n = 3$, the reliability decision $R_S = R_{system}$ is expressed as:

$$R_{signal} = R_{system} = R_{component} \times \left[3R^2 - 2R^3 \right] \; with \; R_S \prec R_C \qquad [8.46]$$

$$R_S = R_C \times \left(3R^2 - 2R^3 \right) = 1 \times \left(3 \times \left(\frac{1}{2} \right)^2 - 2 \times 1^3 \right) = \left(\frac{1}{2} \right) \leftrightarrow QED$$

Reliability is limited by the following boundaries:

$$\left\{ 0.75 \times \left(1 - \sqrt{1 - \left(\frac{8}{9} \right) \times \left(\frac{1}{R_c} \right)} \right) \right\} \prec R \prec \left\{ 0.75 \times \left(1 + \sqrt{1 - \left(\frac{8}{9} \right) \times \left(\frac{1}{R_c} \right)} \right) \right\} \qquad [8.47]$$

$$\underset{\left\{ 0.75 \times \left(1 - \sqrt{1 - \frac{8}{9R_c}} \right) \right\}}{\overset{\text{Lower boundary}}{\uparrow}} = 0.5 \;\; and \;\; \underset{\left\{ 0.75 \times \left(1 + \sqrt{1 - \frac{8}{9R_c}} \right) \right\}}{\overset{\text{Upper boundary}}{\uparrow}} = 1$$

In summary, for the majority of redundancy, the reliability of a system corresponds to the reliability of the partially active redundant system ($R_{\kappa/n}$) by the reliability of the decision system. Here κ is a first integer greater than (n/2).

EXAMPLE.– Sequential redundancy practice: Imagine a component in use with a given failure rate (λ). We have already mentioned reliability and statistical laws

(see Chapter 1, volume 1). Suppose that the system is approached by a Gamma type law, with (n) number of components. The reliability of this gamma law is written as:

$$R_{Syst}(\tau) = Exp^{-\lambda\times\tau} \times \sum_{i=0}^{n-1} \frac{(\lambda\times\tau)^i}{i!} \text{ and the MTBF } = \theta_{Syst}(\tau) = \left(\frac{n}{\lambda}\right) \quad [8.48]$$

If it was a question of *partial sequential redundancy*, i.e. that the system necessitates κ components assembled in a series, with $n > \kappa$ components, we express the system's reliability as follows:

$$R_{Syst\,\kappa/n}(\tau) = Exp^{-\lambda\times\tau\times\kappa} \times \sum_{i=0}^{n-\kappa} \frac{(\lambda\times\tau\times\kappa)^i}{i!} \text{ and the MTBF } = \theta_{Syst\,\kappa/n}(\tau) = \left(\frac{n-\kappa+1}{\kappa\times\lambda}\right) \quad [8.49]$$

Numerical application: Imagine a mechanism that expresses a power law model (Weibull, exponential, etc.). As an example in practice consider the pin of a milling machine, which has a failure rate of $\lambda = 10^{-3}$ RPM. The belt of the motor involves a partial sequential redundancy of 4, i.e. $\kappa = 4$ and $n = 10$ components. Calculate and plot the evolution of the system's reliability R_{system} when the pin rotates between 300 and 3200 RPM, supposing that the milling machine does not break down for any component's failure.

Solution: We have programmed the function $R_{sys}(\tau)$ and the results are given as follows:

Now imagine components with a constant failure rate (i.e. a failure rate different to 0 backup). Let λ_0 be the backup failure rate and λ_1 the failure rate in use. If $n = 2$ components, we suppose, for example, the following reliability expression:

$$R_{Syst=2}(\tau) = Exp^{-\lambda_1\times\tau} + \int_0^{\tau} \lambda_1 \times Exp^{-\lambda_1\times\tau} \times Exp^{-\lambda_0\times\tau} \times Exp^{-\lambda_1\times(T-\tau)} d\tau \quad [8.50]$$

$$\text{Let: } R_{Syst=2}(\tau) = \left\{ \begin{array}{l} Exp^{-\lambda_1\times\tau} \times \left(\left(1 + \frac{\lambda_1}{\lambda_0}\right) - \left(\frac{\lambda_1}{\lambda_0}\right) \times Exp^{-\lambda_0\times\tau} \right) = \\ = \left(\frac{(\lambda_0 + \lambda_1) \times Exp^{-\lambda_1\times\tau} - \lambda_1 \times Exp^{-(\lambda_0+\lambda_1)\times\tau}}{\lambda_0} \right) \end{array} \right\} \quad [8.51]$$

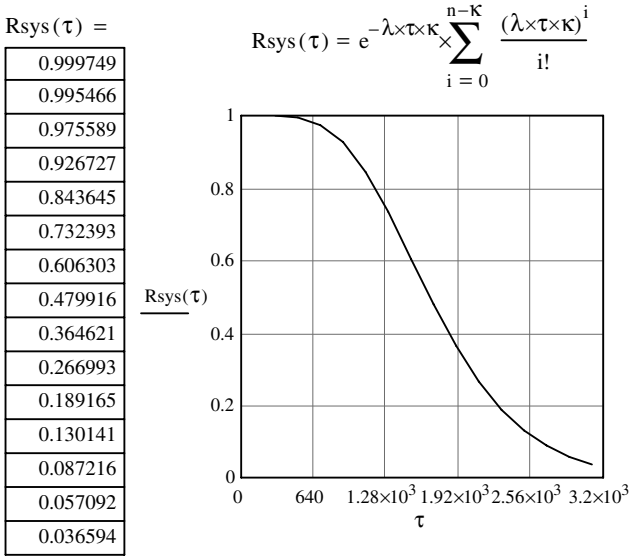


Figure 8.18. *Reliability of a structure with 10 components (partial active redundancy)*

The MTBF (θ) is written as:

$$\theta_{Syst=2}(\tau) = \frac{1}{\lambda_1} \times \left(\frac{\lambda_0 + 2\lambda_1}{\lambda_0 + \lambda_1} \right) \qquad [8.52]$$

If $n = 3$ components, the reliability is expressed as:

$$R_{Syst=3}(\tau) = Exp^{-\lambda_1 \times \tau} \times \left(1 + \frac{1}{2} \times \frac{\lambda_1}{\lambda_0} \right) \times \left(\begin{array}{l} \left(3 + \frac{\lambda_1}{\lambda_0} \right) - 2 \left(2 + \frac{\lambda_1}{\lambda_0} \right) \times Exp^{-\lambda_0 \times \tau} + \\ + \left(1 + \frac{\lambda_1}{\lambda_0} \right) \times Exp^{-2\lambda_0 \times \tau} \end{array} \right) \qquad [8.53]$$

and the MTBF (θ) is therefore written as:

$$\theta_{Syst=3}(\tau) = \frac{1}{2} \times \frac{1}{\lambda_0^2} \times \left(\frac{\lambda_1^2 + 3\lambda_0\lambda_1 + 2\lambda_0^2}{\lambda_1} - \frac{2\lambda_1(\lambda_1 + 2\lambda_0)}{\lambda_0 + \lambda_1} + \frac{\lambda_1(\lambda_1 + \lambda_0)}{2\lambda_0 + \lambda_1} \right) \qquad [8.54]$$

8.7. Reliability and availability of repairable systems

A redundant system with (n) components at the moment (τ) in one of the ith states ($m + 1$) ε_i is characterized by the number ith of components in use. We can represent the state where all components are in use without failure with ε_m. By contrast, ε_0 expresses the state where all components are out of service (total breakdown = failure). It is customary to present the evolution of a system using Figure 8.19:

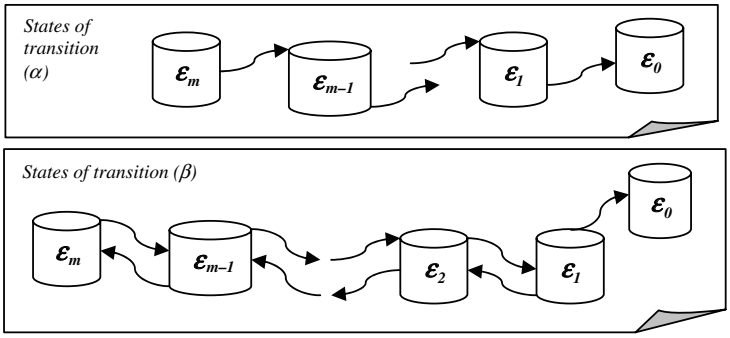


Figure 8.19. *Graphical illustrations showing states of transition*

The cylinders represent the state and the arrows express the transition from one state to another. We can now do the mathematical reading:

In probability terms, at the time (τ) in the state ε_i, the probability will be $P_i(\tau)$. Thus the differential of the probability will be $P_i(\tau)\mathrm{d}\tau$. There is a good reason to take from the probability (reliability) R_i that the system is passing from a nominative state ε_i to $\varepsilon_{(i-1)}$ between the allotted times of the process, i.e. τ and ($\tau + \mathrm{d}\tau$). It is on this level that we can see the failure of a given component. By analogy, there is reason to believe that between τ and ($\tau + \mathrm{d}\tau$), thus the state ε_i and the state $\varepsilon_{(i-1)}$, there is component repair. For the simplicity of probabilistic understanding, it is advisable not to consider the two terms to express the inherent probability in the state ε_i at the time ($\tau + \mathrm{d}\tau$), as follows:

$$\left\{\begin{array}{c} P_i(\tau+d\tau)=P_{i+1}(\tau)d_{i+1}(\tau)d\tau+P_i(\tau)\times\ldots \\ \times\left\{1-d_i(\tau)d(\tau)-R_i(\tau)d\tau\right\}+P_{i-1}(\tau)R_{i-1}(\tau)d\tau \end{array}\right\} \qquad [8.55]$$

This is schematized as shown in Figure 8.20:

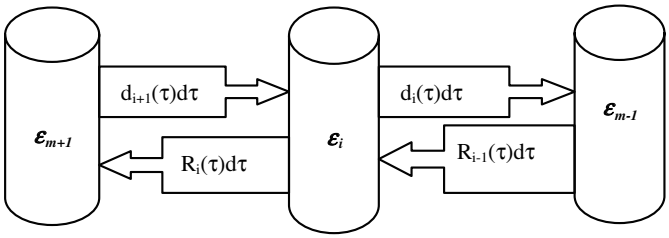


Figure 8.20. *Graphical illustration showing states (m) of transition*

For the probability equation $= \left(1 = \sum_{i=0}^{m} P_i(\tau)\right)$ with (m) components, i.e. an unfailed system with total reliability = 1, we can use the differential equation:

$$\frac{\partial P_i(\tau)}{d\tau} = P_{i+1}(\tau)\, d_{i+1}(\tau)\, d\tau - P_i(\tau)\left\{d_i(\tau) + R_i(\tau)\right\} + P_{i-1}(\tau)\, R_{i-1}(\tau) \qquad [8.56]$$

In limited conditions, (i.e. 0 and m) we suppose:

$$\left\{\begin{array}{l} \dfrac{\partial P_0(\tau)}{d\tau} = P_1(\tau)\, d_1(\tau)\, d\tau - P_0(\tau)\, R_0(\tau) \leftarrow \text{In state (0)} \\ \dfrac{\partial P_m(\tau)}{d\tau} = -P_m(\tau)\, d_m(\tau)\, d\tau + P_{m-1}(\tau)\, R_{m-1}(\tau) \leftarrow \text{In state } (m) \end{array}\right\} \qquad [8.57]$$

To properly apply these equations to material and structural reliability, it is worth laying down limited conditions (a starting hypothesis). At the start of a new design (the equation in case 1), we start from the principle that no component is faulty. But, if we start from the idea that just (κ) components are functional, we can consider the equation in (case 2).

$$\left\{\begin{array}{l} \left(case\ 1\right) \Rightarrow P_n(0) = 1\ ;\ P_i(0) = 0\ ; \forall i \neq m \leftarrow \text{In state (0)} \\ \left(case\ 2\right) \Rightarrow P_\kappa(0) = 1\ ;\ P_i(0) = 0\ ; \forall i \neq \kappa \leftarrow \text{In state } t(\kappa) \end{array}\right\} \qquad [8.58]$$

Normally, before calculating the reliability of the system, it is advisable to find and clearly note the state ($\varepsilon_{indice\ >\ 0}$) of the upper index, i.e. the probabilistic indicator that would cause system failure. We call this an "absorbent state," and we write its reliability as:

$$R(\tau)=\sum_{i=d+1}^{m}P_i(\tau)=1-\sum_{i=0}^{d}P_i(\tau);\ \textit{Associated with}\ \left\{R_i(\tau)=0\ ,\ \forall_j\le\kappa\right\} \qquad [8.59]$$

The general matrix is written as follows:

$$\begin{bmatrix}
1-m\lambda\Delta\tau & \mu\Delta\tau & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
m\lambda\Delta\tau & 1-\left[\mu+(m-1)\lambda\Delta\tau\right] & \mu\Delta\tau & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \ldots & \ldots & \ldots & 0 & 0 & 0 & 0\\
0 & 0 & \ldots & \ldots & \ldots & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & (i+1)\lambda\Delta\tau & 1-(\mu+i\lambda\Delta\tau) & \mu\lambda\Delta\tau & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & \ldots & \ldots & 0\\
0 & 0 & 0 & 0 & 0 & 0 & \ldots & \ldots & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 2\lambda\Delta\tau & \left[1-(\lambda+\mu)\Delta\tau\right] & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu\Delta\tau & 1
\end{bmatrix}\times$$

$$\times\begin{bmatrix}P_m(\tau)\\P_{m-1}(\tau)\\\ldots\\\ldots\\P_i(\tau)\\\ldots\\\ldots\\P_1(\tau)\\P_0(\tau)\end{bmatrix}=\begin{bmatrix}P_m(\tau+\Delta\tau)\\P_{m-1}(\tau+\Delta\tau)\\\ldots\\\ldots\\P_i(\tau+\Delta\tau)\\\ldots\\\ldots\\P_1(\tau+\Delta\tau)\\P_0(\tau+\Delta\tau)\end{bmatrix} \qquad [8.60]$$

In continuum mechanics through fatigue, we have stated a hypothesis where the system's components failure rate is variable. In our case study, we prefer to use the Laplace transformation that presents the matrix in equation [8.60] as follows:

$$\left\{\begin{array}{ll}
\dfrac{\partial P_m(\tau)}{\partial\tau} & =-m\times\lambda\times P_m(\tau)+\mu\times P_{m-1}(\tau)\\
\dfrac{\partial P_i(\tau)}{\partial\tau} & =P_{i+1}(\tau)\times(i+1)\lambda-(i\times\lambda+\mu)\times P_i(\tau)+\mu P_{i-1}(\tau)\\
\ldots & =\ \ldots.\ \ldots\ \ldots.\ \ldots\\
\dfrac{\partial P_0(\tau)}{\partial\tau} & =\lambda P_1(\tau)
\end{array}\right\} \qquad [8.61]$$

If $m = 2$ components, the equation system is presented as follows:

$$\left\{\begin{array}{l} \dfrac{\partial P_2(\tau)}{\partial \tau} = -2\times\lambda\times P_m(\tau)+\mu\times P_{m-1}(\tau) \\ \dfrac{\partial P_1(\tau)}{\partial \tau} = 2\times\lambda\times P_2(\tau)-(\lambda+\mu)\times P_1(\tau) \\ \dfrac{\partial P_0(\tau)}{\partial \tau} = \lambda P_1(\tau) \end{array}\right\}$$

Or even as the matrix form:

$$\begin{bmatrix} (1-2\times\lambda\times\Delta\tau) & (\mu\times\Delta\tau) & 0 \\ (2\times\lambda\times\Delta\tau) & (1-(\lambda+\mu)\Delta\tau) & 0 \\ 0 & (\lambda\times\Delta\tau) & 1 \end{bmatrix}\times\begin{bmatrix} P_2(\tau) \\ P_1(\tau) \\ P_0(\tau) \end{bmatrix}=\begin{bmatrix} P_2(\tau+\Delta\tau) \\ P_1(\tau+\Delta\tau) \\ P_0(\tau+\Delta\tau) \end{bmatrix} \qquad [8.62]$$

First of all, here is a simple example of the Laplace transformation:

$$(1-2\times\lambda\times\Delta\tau)\,Laplace\rightarrow -\frac{2\times\lambda\times\Delta\tau}{S},\ (\mu\Delta\tau)\,Laplace\rightarrow\frac{\lambda\times\Delta\tau}{S},\ \text{etc.}$$

$$\left\{\frac{d}{d\tau}\left[P_2(\tau)\right]Laplace\rightarrow\frac{P_2}{s};\frac{d}{d\tau}\left[P_1(\tau)\right]Laplace\rightarrow\frac{P_1}{s};\frac{d}{d\tau}\left[P_0(\tau)\right]Laplace\rightarrow\frac{P_0}{s}\right\}$$

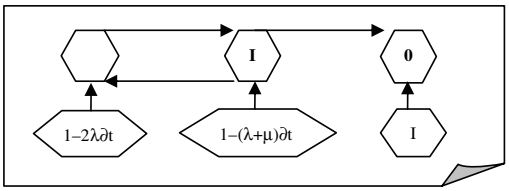
We have chosen the Laplacian operator (Laplace transformation) because if $P_2(0) = 1$ it is convenient to resolve the equations system:

$$\begin{bmatrix} (1-2.\lambda.\Delta\tau) & (\mu\times\Delta\tau) & 0 \\ (2\lambda.\Delta\tau) & (1-(\lambda+\mu)\Delta\tau) & 0 \\ 0 & (\lambda.\Delta\tau) & 1 \end{bmatrix}\times\begin{bmatrix} P_2(\tau) \\ P_1(\tau) \\ P_0(\tau) \end{bmatrix}\rightarrow\begin{bmatrix} \mu.\tau.\Delta\tau.P_1-\tau(2\lambda.\Delta\tau-1)P_2 \\ 2\tau.\lambda.\Delta\tau.P_2-\tau.\left[\Delta\tau(\mu+\lambda)-1\right]P_1 \\ \tau.P_0+\tau.\lambda.\Delta\tau.P_1 \end{bmatrix}$$

Knowing that:

$$\begin{bmatrix} (1-2.\lambda.\Delta\tau) & (\mu\times\Delta\tau) & 0 \\ (2\lambda.\Delta\tau) & (1-(\lambda+\mu)\Delta\tau) & 0 \\ 0 & (\lambda.\Delta\tau) & 1 \end{bmatrix}Laplace\rightarrow\begin{bmatrix} \dfrac{2\lambda.\Delta\tau-1}{s} & \dfrac{\mu.\Delta\tau}{s} & 0 \\ \dfrac{2\lambda.\Delta\tau}{s} & \dfrac{\mu.\Delta\tau+\lambda.\Delta\tau-1}{s} & 0 \\ 0 & \dfrac{\lambda.\Delta\tau}{s} & \dfrac{1}{s} \end{bmatrix}$$

$$\begin{bmatrix} \mu.\tau.\Delta\tau.P_1 - \tau\left(2\lambda.\Delta\tau - 1\right)P_2 \\ 2\tau.\lambda.\Delta\tau.P_2 - \tau\left[\Delta\tau\left(\mu+\lambda\right)-1\right]P_1 \\ \tau.P_0 + \tau.\lambda.\Delta\tau.P_1 \end{bmatrix} \text{ Laplace} \rightarrow \begin{bmatrix} \dfrac{\tau.P_2 + \mu.\tau.\Delta\tau.P_1 - 2\tau.\lambda.\Delta\tau.P_2}{s} \\ \dfrac{\tau.P_1 - \mu.\tau.\Delta\tau.P_1 - \tau.\lambda.\Delta\tau.P_1 + 2\tau.\lambda.\Delta\tau.P_2}{s} \\ \dfrac{\tau.P_0 + \tau.\lambda.\Delta\tau.P_1}{s} \end{bmatrix}$$



Figures 8.21. *Graphical illustration showing states of transition (when m = 2)*

Thus if $P_2(0) = 1$, we get the equations system (transformed by Laplace):

$$\left\{ \begin{array}{l} S\wp_2\left(S\right) - 1 = -2\times\lambda\times\wp_2\left(S\right) + \mu\times\wp_1\left(S\right) \\ S\wp_1\left(S\right) = 2\times\lambda\times\wp_2\left(S\right) - \left(\lambda+\mu\right)\times\wp_1\left(S\right) \\ S\wp_0\left(S\right) = \lambda\times\wp_1\left(S\right) \end{array} \right\} \quad [8.63]$$

This system is easy to resolve: As $\lambda\times\wp_1\left(S\right) \rightarrow \left(S\times\lambda\times\wp_1\right)/s$ the Laplacian $\wp_0\left(S\right)$ takes the following form:

$$\wp_0\left(S\right) = \frac{2\times\lambda^2}{S\left(S^2 + \left(3\times\lambda+\mu\right)\times S + 2\times\lambda^2\right)} \quad [8.64]$$

(λ_1 and λ_2) represent the respective roots of this equation.

$$\left(\lambda_1, \lambda_2\right) = \frac{1}{2}\times\left(-3\lambda+\mu\right) \pm \sqrt{\lambda^2 + \mu^2 + 6\lambda\times\mu} \quad [8.65]$$

Because the reliability expression is written as: $R(\tau) = 1 - P_0(\tau)$, we can therefore suppose:

$$R\left(\tau\right) = 1 - P_0\left(\tau\right) = \left(\frac{1}{\lambda_1 - \lambda_2}\right)\times\left(\lambda_1\times Exp^{\lambda_2\times\tau} - \lambda_2\times Exp^{\lambda_1\times\tau}\right) \quad [8.66]$$

The MTBF θ_{System} is presented as follows:

$$\theta_{system} = \frac{3}{2} \times \left(\frac{\lambda + \mu}{\lambda^2} \right) = \theta \left(\frac{\theta + 3 \times \tau}{2 \times \tau} \right) \quad [8.67]$$

Sometimes we are made to calculate an average repair time for a given system. To do this, consider a concrete example: Imagine a system composed of (m) pieces, which are all different from one another. λ_i represents the failure rate of the (ith) components to the (mth) component. The average time taken to repair a given component, for example the (ith) component, is expressed by τ_i. To estimate an average time to repair the system, we can postulate the expression ($\tau_{avg} = \tau_\mu$).

$$\tau_\mu = \left(\frac{\sum_{i=1}^{m} \lambda_i \times \tau_i}{\sum_{i=1}^{m} \lambda_i} \right) \; if \; \lambda = \sum_{i=1}^{m} \lambda_i \quad [8.68]$$

The MTBF (θ) can be written as:

$$MTBF \; \theta = \lambda^{-1} \rightarrow \tau_\mu = \frac{\sum_{i=1}^{m} \lambda_i \times \tau_i}{\lambda} = \theta \times \sum_{i=1}^{m} \lambda_i \times \tau_i \quad [8.69]$$

Numerical application: We can now deal with the calculation of the average failure rate. The results shown in the following figure confirm the speed of the failure rate in a Weibull law, as shown in Chapter 1 (volume 1).This exactly corresponds to a repair rate from a Weibull law, thus showing that the failure rates in fracture mechanics are not constant. For the last reliability analysis, we will present a predicted design project and trials. In mechanical reliability (of fracture), it is important to clearly distribute the stresses applied on each component. This is incidentally the conceptual aim at the time of distributing reliability constraints.

Well carried out experiments create history even from previous results (laboratories). Some trials are necessary to evaluate the reliability of components and of the system in its whole complexity. To do this, we must place the components in quasi-real conditions (or even better, in real conditions), aiming to respond to solicitations to make an appropriate decision on results.

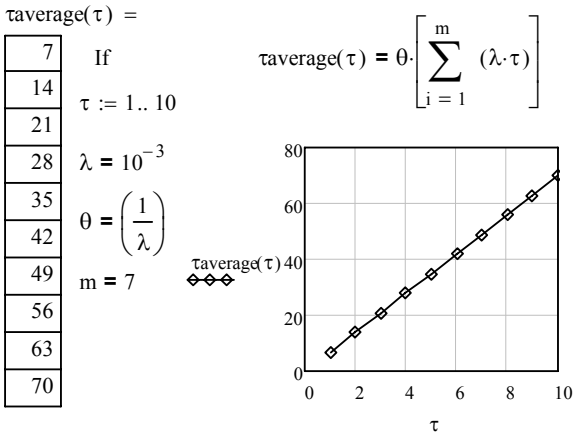


Figure 8.22. *Average failure rate*

8.8. Quality assurance in reliability

It is essential not to confuse quality with quality assurance. Declaring quality assurance goes beyond conventional fixed limits of product quality. Quality control is often defined by capability indicators (SPC) (see Chapters 2 and 3 of volume 3). Although reliability is related to quality control, using both mathematical and statistical tools, it is not safe to confuse the results and make them show what they do not necessarily show. Ultimately, the aims of mechanical reliability are:

1) to approve and measure the quality of components;

2) to qualify the means of production: manufacturing processes, tools of manufacture, and metrology;

3) to conduct essays of trials to requalify components and to validate previously held solutions.

8.8.1. *Projected analysis of reliability*

An existing system has a reliability R, which is dependent on the reliabilities of its components (in parallel, in a series, or a combination of the two).

Let R_{system} and $R_{component(1)}$, $R_{component(2)}$, …, $R_{component(m)}$ such that:

$$R_{system} = f\left\{\text{Reliability}_{component1}, \text{Reliability}_{component2}, ..., \text{Reliability}_{componentm}\right\} \qquad [8.70]$$

As indicated in the aims of reliability, it is always beneficial to aim to improve systems by their components and their design. It therefore goes without saying that new improved reliabilities must verify the condition:

$$\left\{ \begin{array}{l} f\left\{ \text{Reliability}_{component\ 1}^{improved}, \text{Reliability}_{component\ 2}^{improved}, ..., \text{Reliability}_{component\ m}^{improved} \right\} \succ \\ \quad ... \quad \succ R_{system}^{improved\ reliability} \left\{ or\ even \geq R_{system}^{improved\ reliability} \right\} \end{array} \right\} \qquad [8.71]$$

With the aim of achieving this, it is advisable to consider the archetypal reliability indicator, which is the failure rate (weakness or fracture, λ). Thus, for a system designed in a series, we can suppose:

$$f\left\{ \lambda_{component\ 1}^{improved} + \lambda_{component\ 2}^{improved} + ... + \lambda_{component\ \mathrm{m}}^{improved} \right\} \leq \lambda_{updated\ system}^{improvedreliability} \qquad [8.72]$$

The distribution of failure rate constraints is carried out technically by various mathematical approaches, including the weighting (weight) method. After mechanical design, we are more familiar with the MTBF (θ_i) of the designed system and therefore the failure rates ($\lambda = \Sigma\lambda_i$). According to the agreed safety importance for the new system, we allocate a weighting (weight) P_i.

$$P_i = P_{i\ updated\ system}^{improved\ reliability} = \left(\frac{\lambda_i}{\lambda} \right)_{system} \quad with \quad \Lambda = P_i \times \lambda_{specified}^{failure\ rate} \qquad [8.73]$$

Numerical application: With a total time of *normal functioning* equivalent to 62,000 hours, we can deduct the following:

Number of components assembled in a series	Piece 1	Piece 2	Piece 3	Piece 4	Piece 5	Piece 6	Piece 7	Total
Fracture time	68	56	54	35	25	15	5	258 hours

Table 8.1. *Data showing fracture time by individual pieces*

Solution

– $T_n = [68 \quad 56 \quad 54 \quad 35 \quad 25 \quad 15 \quad 5]$;

– TTPF = Total Time of Proper Functioning n = 1, 2, ..., 7;

– TTBF = Total time of fracture, when n = 7 components;

– TTBF = 62,000 hours (recorded data).

$$\sum_n T_n = 258.0000\ TTR = \sum_n T_n \ \ hours;\ \text{MTBF}_{\text{New}} = \frac{TTBF}{\sum_n T_n} = 240.3101\ hours$$

$$\lambda_{current} = \frac{\sum_n T_n}{TTBF}\ then\ \frac{\sum_n T_n}{TTBF} = 1.161 \times 10^{-3}\ \ hours$$

$$\theta_{new} = 68\ then\ \lambda_{new} = \frac{1}{\theta_{new}} = Failure\ rate_{new} = 0.0147 hours$$

Summary table of calculation formulae

Lifecycle or Time (hours) of functioning, MTBF	Current rate of failure, λ (read)	New MTBF θ_{new} (improved)	New rate of failure, λ
Current lifecycle	Current rate of failure, $\lambda \rightarrow \lambda$ = 258/62,000 hours =4.161 $\times 10^{-3}$ (hours)	Data	Formula
$MTBF = \theta_{system} =$ $= \frac{Lifecycle}{Fracture\ cycle}$ $\theta_{\text{current}}\text{Sys}$ = 62,000/258 = 240.3101 (hours)		θ_{new} = 68 hours	$\lambda_{system}^{new} = \frac{1}{\theta_{system}^{new}}$ $= \frac{1}{MTBF_{system}^{new}}$ λ = 1/68 = 0.0147 hours

Table 8.2. *Summary of calculation formulae*

If an $\text{MTBF}_{\text{cumulated}}$ = 62,000 hours, we can propose a table of new allocations, as follows:

$$P_i = \left\{ \frac{Current\ rate}{Sum\ of\ current\ rates} \right\} \quad [8.74]$$

$$Rate\ allocated = \left(\frac{P_i}{1/\theta} \right) \quad [8.75]$$

$$MTBF_{allocated} = \frac{1}{Rate_{allocated}} \quad [8.76]$$

Part	Fractures	Current rate	Weight	Allocated MTBF	Allocated rate	Allocated MTBF
n	n_i	n/MTBF$_{Cum}$	Pi =	1/Theta = 1/68	Weight × 1/θ= 1/68	1/Allocated rate
Part 1	68	0.00110	0.2636	0.014705882	3.8760E-03	2.5800E+02
Part 2	56	0.00090	0.2171	0.014705882	3.1920E-03	3.1329E+02
Part 3	54	0.00087	0.2093	0.014705882	3.0780E-03	3.2489E+02
Part 3	35	0.00056	0.1357	0.014705882	1.9950E-03	5.0126E+02
Part 4	25	0.00040	0.0969	0.014705882	1.4250E-03	7.0176E+02
Part 5	15	0.00024	0.0581	0.014705882	8.5499E-04	1.1696E+03
Part 6	5	0.00008	0.0194	0.014705882	2.8500E-04	3.5088E+03
n_i	**Σ = 258**	**Σ = 0,00416**	**Formula 1**	**0,014705882**	**Formula 2**	**Formula 3**

Table 8.3. *Calculation results: MTBF according to failures*

DISCUSSION.– Note that by improving the allocated MTBF, we have smaller rates of failure. This is the advantage of using the weighting (Weight) method.

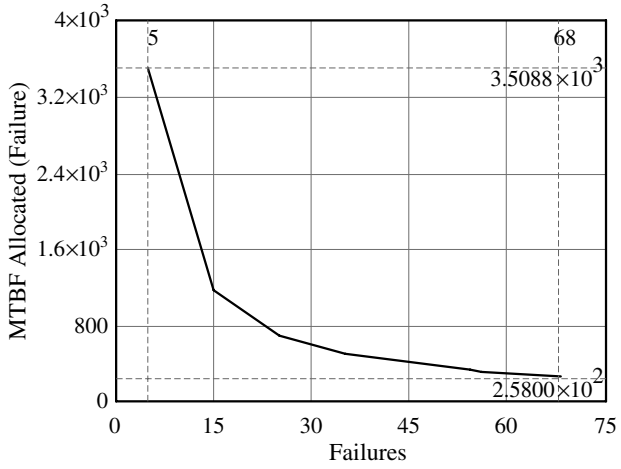


Figure 8.23. *Evolution of failing components before and after improvements*

In fracture mechanics through fatigue, fractures are often unexpected and sudden. Failures are characterized by catalectics. Their failure rates are variable because of impromptu deviations. From design to use, there is good reason to always properly consider the manufacturing quality, keeping in consideration the materials, environment, vibrations, temperature, etc.

Many laboratories collect exhaustive data with the aim of evaluating the failure rates of materials and structures based on behavioral knowledge. The cited example is the American military Standard, MIL-HDBK-217C, just to mention one. The deviation, which causes the failure (fracture) of a component, as we have often seen in this book, is characterized by a probability distribution (Weibull, Erlang, Birnbaum–Saunders, etc.). The parameters from every law are dependent on age. For example, the stress on a component is sometimes constant and sometimes variable.

The aim is always to try to keep this stress smaller or the same as the material's resistance. It is for this reason that calculating probability is nothing more than the reliability of the component at a certain "age". We will now present a simple applied example.

8.9. Birnbaum–Saunders distribution in crack spreading

Justification: We have chosen the Birnbaum–Saunders law because it fits in with reliability models for materials and structures prone to fatigue through cracking.

Starting hypothesis: Imagine the application of Birnbaum–Saunders on four mechanical components:

If $\tau = 0, 0.1, \ldots, 300$; $\eta = 10$; $\gamma_1 = 1$; $\gamma_2 = 2$; $\gamma_3 = 1.25$, and $\gamma_4 = 3$, we will now simulate and plot the curves of the necessary principle functions to calculate reliability.

$$f(\tau) = \left(\frac{1}{2\sqrt{\pi} \times \gamma_1^2 \times \eta \times \tau^2}\right) \times \left(\frac{\tau^2 - \eta^2}{\sqrt{\frac{\tau}{\eta}} - \sqrt{\frac{\eta}{\tau}}}\right) \times Exp\left\{-\frac{1}{\gamma_1^2} \times \left(\frac{\tau}{\eta} + \frac{\eta}{\tau} - 2\right)\right\}$$

$$g(\tau) = \left(\frac{1}{2\sqrt{\pi} \times \gamma_2^2 \times \eta \times \tau^2}\right) \times \left(\frac{\tau^2 - \eta^2}{\sqrt{\frac{\tau}{\eta}} - \sqrt{\frac{\eta}{\tau}}}\right) \times Exp\left\{-\frac{1}{\gamma_2^2} \times \left(\frac{\tau}{\eta} + \frac{\eta}{\tau} - 2\right)\right\}$$

$$h(\tau)=\left(\frac{1}{2\sqrt{\pi}\times\gamma_3^2\times\eta\times\tau^2}\right)\times\left(\frac{\tau^2-\eta^2}{\sqrt{\frac{\tau}{\eta}}-\sqrt{\frac{\eta}{\tau}}}\right)\times Exp\left\{-\frac{1}{\gamma_3^2}\times\left(\frac{\tau}{\eta}+\frac{\eta}{\tau}-2\right)\right\}$$

$$j(\tau)=\left(\frac{1}{2\sqrt{\pi}\times\gamma_4^2\times\eta\times\tau^2}\right)\times\left(\frac{\tau^2-\eta^2}{\sqrt{\frac{\tau}{\eta}}-\sqrt{\frac{\eta}{\tau}}}\right)\times Exp\left\{-\frac{1}{\gamma_4^2}\times\left(\frac{\tau}{\eta}+\frac{\eta}{\tau}-2\right)\right\}$$

8.9.1. ***Probability density and distribution function (Birnbaum–Saunders cumulative distribution through cracking)***

$$\mu1:=\left[\gamma1\cdot\left(1+\frac{\gamma^2}{2}\right)\right]=1.1666\qquad\mu2:=\left[\gamma2\cdot\left(1+\frac{\gamma^2}{2}\right)\right]=2.3332$$

$$\mu3:=\left[\gamma3\cdot\left(1+\frac{\gamma^2}{2}\right)\right]=1.4582\quad\mu4:=\left[\gamma4\cdot\left(1+\frac{\gamma^2}{2}\right)\right]=3.4998$$

$$F(\tau):=\Phi\left[\left(\frac{1}{\gamma1}\right)\cdot\left(\frac{\tau}{\eta}+\frac{\eta}{\tau}\right)\right]\quad\text{pnorm}(\tau,\mu1,\gamma1)$$

$$G(\tau):=\Phi\left[\left(\frac{1}{\gamma2}\right)\cdot\left(\frac{\tau}{\eta}+\frac{\eta}{\tau}\right)\right]\quad\text{pnorm}(\tau,\mu2,\gamma2)$$

$$H(\tau):=\Phi\left[\left(\frac{1}{\gamma3}\right)\cdot\left(\frac{\tau}{\eta}+\frac{\eta}{\tau}\right)\right]\quad\text{pnorm}(\tau,\mu3,\gamma3)$$

$$J(\tau):=\Phi\left[\left(\frac{1}{\gamma4}\right)\cdot\left(\frac{\tau}{\eta}+\frac{\eta}{\tau}\right)\right]\quad\text{pnorm}(\tau,\mu4,\gamma4)$$

8.9.2. *Graph plots for the four probability density functions and distribution functions*

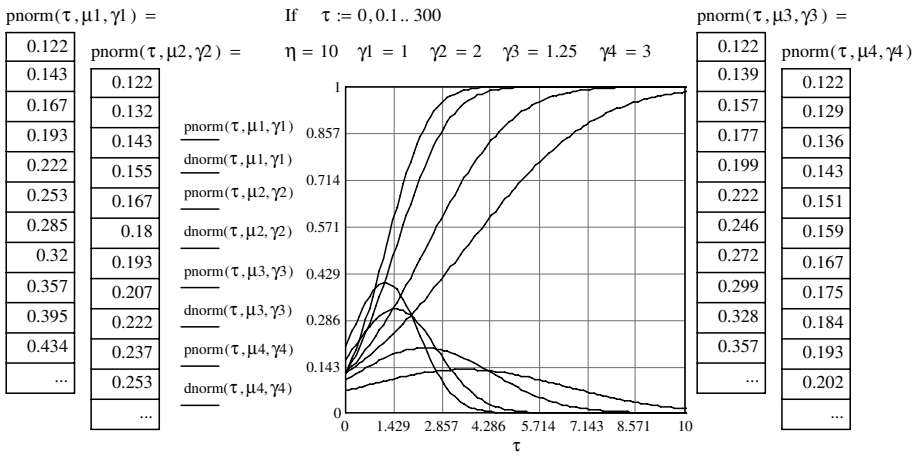


Figure 8.24. *Graphs to show the four probability density functions and distribution functions*

8.10. Reliability calculation for ages (τ) in hours of service, $R_i(\tau)$ = ?

Note that the appearance of the curve is very similar to a Gaussian. This leads us to write the following:

$$R(\tau) = 1 - \Phi\left\{\frac{R_{part\ 1}^{Fatigue} - \mu(\tau)}{\sigma(\tau)}\right\} = \Phi\left\{\frac{\mu(\tau) - R_{part\ 1}^{Fatigue}}{\sigma(\tau)}\right\} = \Phi\left\{\frac{\mu(\tau) - \tau \times 10^3}{\sigma(\tau) - \tau \times 10^3}\right\} \quad [8.77]$$

Averages (τ_i) in MPa	Service age (τ_i)	Standard deviation (τ_i) in MPa	Time in hours	$U = \left(\frac{\mu(\tau) - \tau \times 10^3}{\sigma(\tau) - \tau \times 10^3}\right)$	$R_i(\tau) = \Phi(U)$ = standard normal law (U)
1.1666	0.001	0.15	4000	0.735948052	0.769118844
2.3332	0.001	0.25	3600	0.378149254	0.64734014
1.4582	0.001	0.45	5600	0.80423301	0.789368791
3.4998	0.001	0.75	10000	0.702724324	0.758886217

Table 8.4. *Statistical calculation results*

This distribution law is characterized by its average $\mu_i(\tau_i)$ and its standard deviation $\sigma_i(\tau_i)$ in MPa (refer to Table 8.4 data and calculations). We must pay particular attention to the truthfulness of the distribution law. When distribution laws follow a Gaussian, resemble that of Gaussian, or can be returned to it, then it is easy to carry out reliability calculations on components, as previously shown.

We deemed that the Birnbaum–Saunders law presents a similar form to a Gaussian. In the opposite case, i.e. where laws are not normal and a reading $\Phi(U)$ cannot be taken, the calculations become a little more complex. We can therefore use the *moments method*, which we will now briefly recap. Imagine a mechanical assembly where the principal characteristic is dependent on x_{p1}, x_{p2}, …, x_{pm} of (m) pieces. The equation takes the following form:

$$F(p) = f\left(p_{part1}, p_{part2}, \ \ldots \ , p_{partm}\right) = f\left(p_1, p_2, \ \ldots \ , p_m\right) \qquad [8.78]$$

These statistics are measured, among others, by the averages of factors μ_{p1}, μ_{p2}, μ_{p3}, …, μ_{pm} and their respective standard deviations σ_{p1}, σ_{p2}, σ_{p3}, …, σ_{pm}.

$$\mu(p) = f\left(\mu_{p_1}, \mu_{p_2}, \ \mu_{p_3}, \ \ldots \ , \mu_{p_m}\right) = \sum_{i=1}^{m} \mu_{p_m} \qquad [8.79]$$

The standard deviation is presented as:

$$\sigma(p) = \sqrt{\sum_{i=1}^{m}\left(\frac{\partial f}{\partial p_i}\right)^2 \times \sigma_i^2(p) + 2\sum_{i<j}^{m}\left(\frac{\partial f}{\partial p_i}\right) \times \left(\frac{\partial f}{\partial p_j}\right) COVARIANCE\left(p_i, p_j\right)} \qquad [8.80]$$

Two distinct cases appear:

i) *p_i and p_j are independent*;

ii) *p_i and p_j are not independent*.

In the first case (i), the parameters are independent and we consider the characteristic (p_i) normal (or almost normal). If the functioning of the mechanical elements assembly is correct when this characteristic is between p_1 and p_2, the reliability is expressed as:

$$\left\{ \begin{array}{l} P(p) = F\left(p_1 \prec P \le p_1\right) = F\left(P \le p_2\right) - F\left(P \le p_1\right) \\ \text{or even by reading the standard normal law } \Phi(U) \end{array} \right\} \qquad [8.81]$$

$$\text{i.e. } R = \left\{ \begin{array}{l} \text{STANDARD.NORMAL.LAW} \\ \Phi\left(\dfrac{p_2 - \mu_p}{\sigma_p}\right) - \Phi\left(\dfrac{p_1 - \mu_p}{\sigma_p}\right) \end{array} \right\} \qquad [8.82]$$

In the second case (ii), the parameters are not independent and we consider the characteristic (p_i) normal. If the functioning of the mechanical elements assembly is *correct* for p_i and p_j, the standard deviation is expressed as:

$$\sigma(p) = \sqrt{\sum_{i=1}^{m} \left(\frac{\partial f}{\partial p_i}\right)^2 \times \sigma_i^2(p)} \qquad [8.83]$$

The reliability is read in the same way as for case (i).

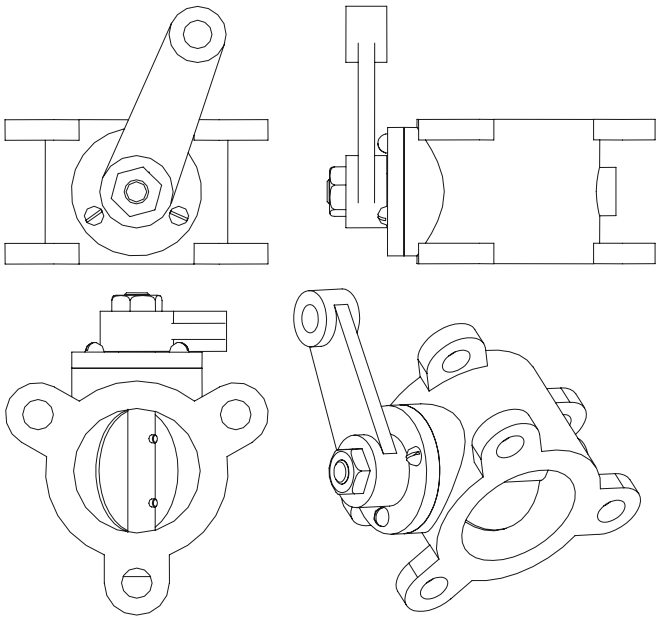


Figure 8.25. *Reliability of a crank-rod system*

Numerical application on a crank rod system: Imagine a classic crank rod system (C and R assembled in a series). The expressed force is:

$$C_{Crank/rod} = f \times \ell = f \times \left(\frac{d}{2}\right), \ \left[Nmm\right] \qquad [8.84]$$

The rod has an average stress value r = 7 MPa and a standard deviation σ_b = 1.25 MPa. The crank has an average stress value c = 4 MPa and a standard deviation σ_m = 1 MPa, which can be summarized as:

$$b = \overline{x}_{rod} = 7 \ ; \ \sigma_{rod} = 1.25 \ ; \ m = \overline{x}_{crank} = 4 \ ; \ \sigma_{crank} = 1.00$$

By considering the "independent dispersion" between the rod and the crank, in terms of fatigue, we get the reliability R_{system} for a time interval [τ_1 = 1 minute and τ_2 = 15 minutes].

Total average of the two components = Constant of time, in minutes

$$\overline{x}_\tau = \overline{x}_{rod} \times \overline{x}_{crank} = 7 \times 4 = 28.00 \,\text{min}$$

We can now calculate the total standard deviation, from the expression presented above:

$$\sigma_\tau = \sqrt{\sum_{i=1}^{m} \left(\frac{\partial f}{\partial \tau_i}\right)^2 \times \sigma_\tau^2} = \sqrt{\left(\frac{\partial f}{\partial b_{rod}}\right)^2 \times \sigma_{rod}^2 + \left(\frac{\partial f}{\partial m_{crank}}\right)^2 \times \sigma_{m_{crank}}^2} \qquad [8.85]$$

$$AN: \ \ \sigma_\tau = \sqrt{m^2 \times \sigma_{b_{rod}}^2 + b^2 \times \sigma_{m_{crank}}^2} = \sqrt{4^2 \times 1.25^2 + 7^2 \times 1^2} = 9.621$$

From τ_1 = 11 minutes to τ_2 = 45 minutes, we suppose that for a Gaussian distribution (*fatigue and cumulative use*) the calculation and reading of the crank rod system's reliability is shown in the equation [8.86] (Inventor).

$$\Re_{system}^{rod/crank} = \Phi\left(\frac{\tau_2 - \overline{x}_\tau}{\sigma_\tau}\right) - \Phi\left(\frac{\tau_1 - \overline{x}_\tau}{\sigma_\tau}\right) \ ; \ if \ \ \tau_2 = 45 \text{ minutes } and \ \ \tau_1 = 11 \text{ minutes}$$

$$\rightarrow \Re_{system}^{rod/crank} = \Phi\left(\frac{\tau_2 - \overline{x}_\tau}{\sigma_\tau}\right) - \Phi\left(\frac{\tau_1 - \overline{x}_\tau}{\sigma_\tau}\right) = \Phi\left(\frac{45 - 28}{9.621}\right) - \Phi\left(\frac{11 - 28}{9.621}\right) =$$

$$= \Phi\left(1.76698\right) - \Phi\left(-1.76698\right) = 0.9759345 - 0.0240655 = 0.951869$$

– GIVEN = STANDARD.NORMAL.LAW (1.97621) = 0.9759345 (Excel).

– AND = STANDARD.NORMAL.LAW (−1.97621) = 0.0240655 (Excel).

Verification is carried out on the above-mentioned mechanism, for a random recording between 11 and 45 minutes (sampling via fixed measures) during normal functioning. Ultimately, we can conclude that for cracking through fatigue, the crank rod system is reliable to ≈95%, according to the expressed hypothesis, i.e.:

– Independent statistical parameters of the crank and rod;

– Normal distribution (fatigue) or apparently normal (this happens because of cumulative use and material fatigue.

DISCUSSION.– Structural reliability currently constitutes a theoretical, methodological, and applied area of study, covering numerous needs in terms of quantitative estimation of risk for civil engineering structures. Structural reliability must be used alongside qualitative approaches to risk, supported by systematic analysis, which enlighten us and help us to choose the appropriate limit states to simulate.

This chapter only briefly discusses the huge variety of available approaches in structural reliability. The indicated references should allow the reader to satisfy a more sharpened curiosity.

Quantitative tools for estimating risk aim to evaluate the probabilities of harmful events occurring in civil engineering structures. These tools can be used when a structure's failure has been sufficiently described, for example through qualitative tools, which describe failures and their sequences. These allow us to express the failure in terms of limit state equations, by making physical sizes intervene. Then, once the probability laws for random variables in the limit state equation have been described on the basis of statistical inferences, we can obtain the sought after probabilities, using structural reliability methods. These are of course dependent on numerous hypotheses to do with probability laws and structural models: the probabilities are therefore conventional and do not express nor anticipate statistics of structural failures, which are fortunately infrequent. The information that we take from these probabilities helps in the process of decision making.

The principal local failure modes considered in oil platforms are: plasticization, buckling, physical fracture, deformation, speed of crack spreading, use, corrosion, erosion, invasion of water, and reversal for floating structures or structures in the process of floating for marine operations

8.11. Simulation methods in mechanical reliability of structures and materials: the Monte Carlo simulation method

This method was the object of a brief introduction in Chapter 7. We will now recap the essence of the method using calculations aimed at workshops and other supervised works. Consider $y = f(\tau_1, \tau_2, \ldots, \tau_n)$ the characteristic of an assembly or machine (gear clusters, hydraulic pump mechanisms, or structures assembled by welding, riveting, or bolting, and so on). This is in fact a value function $\tau_1, \tau_2, \ldots, \tau_n$ of n random parameters $T_1, T_2, \ldots, T_n$.

If we randomly draw a value in each n population, let $\tau_{1,1}, \tau_{2,1}, \ldots, \tau_{n1}$, we will obtain a value $y_1 = f(\tau_{1,1}, \tau_{2,1}, \ldots, \tau_{n,1})$ randomly drawn from the Y population. If we then repeat this operation N times, we will end up with a simulated sample of N values from the random variable Y. After this, it will be possible to determine the form and the parameters for the Y distribution.

To use the Monte Carlo (MC) simulation method, we must know how many random draws are necessary and how to do them in the $T_1, T_2, \ldots, T_n$ populations. The problem aims to determine the *number of necessary draws*, with a level of confidence to choose the number of observations necessary to represent the simulated distribution function with a given uncertainty.

By referring to the theory of conformity tests such as the (KS) Kolmogorov–Smirnov test (see Chapter 3), we can check the following expression:

$$\wp\left\{Maxi\left|F_{simulated}(x) - F_{real}(x)\right| \prec D_{(N,\alpha)}\right\} = \{1-\alpha\} \qquad [8.86]$$

– $D_{(N,\alpha)}$ = Level (threshold) of chosen confidence;

– $(1 - \alpha)$ = maximum uncertainty on N draws.

We have already presented this theory in Chapter 3, but we will summarize the essential content here:

$$D_{(N,\alpha)} = \left(K_\alpha / \sqrt{N}\right) \; hence \; N \geq \left(K_\alpha / D\right)^2 \qquad [8.87]$$

If, for example, we want to limit the uncertainty to ±1% (D = 0.001) in the distribution function simulation, and if we chose a 99% level of confidence (α = 0.001), there will need to be at least $N = (1.63/0.001)^2 = 1.63 \times 10^4$ draws.

By applying the values from Table 8.5, we can calculate N, the number of random draws:

$$N \geq \left(\frac{K_\alpha^2}{D} \right) = \left(1.63/1\%\right)^2 = 2.657 \times 10^4 \text{, at least.}$$

Controlled characteristics	Number $N \geq \left(K_\alpha / D\right)^2$ of draws, MC				
$(1 - \alpha)$	0.80	0.85	0.90	0.95	0.99
Risk α	0.20	0.15	0.10	0.05	0.01
K_α	1.07	1.14	1.22	1.36	1.63

Table 8.5. *Determining the number of necessary draws using the MC method*

We will now explain the manner in which the draws are generated. We have already shown in Chapters 2, 3, and 7 how to randomly draw values of p_i in a given statistical distribution law. On an interval p ∈ [0, 1] we can deduce the randomly drawn values τ_i whose distribution function is written as F(τ). It is sufficient to randomly draw one value p_i, in a law on $p \in [0, 1]$. Then we make it correspond to a value τ_i so that, F(τ_i) = p_i. Finally, we can see that the problem is simple and we can continue with randomly drawing values of p_i.

Imagine, for example, an operation that aims to randomly draw the (τ_i) in a Weibull distribution law with two parameters (then in a hypergeometric law and then in a Gamma (or Erlang) law and log–normal). The Weibull law has a form parameter of β = 2.5 and a scale parameter of η = 56, so we consider a *uniform* random number p_i, and suppose:

$$F\left(\tau_i\right) = 1 - Exp\left\{ -\left(\frac{\tau_i}{\mu} \right) \times \beta \right\} = p_i; \text{ hence}$$

$$p_i = \eta \left(-\ln\left(1 - p_i\right)^{\frac{1}{\beta}} \right) = 56 \times \sqrt[\left(5/2\right)]{-\ln\left(1 - p_i\right)} \qquad [8.88]$$

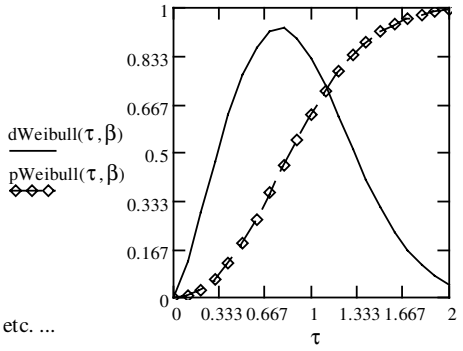
We can also use the software MathCAD to carry out the same operations, as shown in the following sections.

8.11.1. *Weibull law*

pWeibull(τ, β)	dWeibull(τ, β) =
0	0.000
$5.608 \cdot 10^{-3}$	0.126
0.026	0.293
0.064	0.467
0.119	0.630
0.19	0.767
0.272	0.866
0.361	0.920
0.454	0.929
0.546	0.896
0.632	0.828
0.71	0.734
0.778	0.626
...	...

Distribution of Weibull with two parameters (β,η)
dWeibull distribution function
and pWeibull (τ,β) cumulative distribution function

For β = 2.25 τ := 0, 0.1 .. 2



etc. ...

Figure 8.26. *Density and distribution functions (Results of Weibull calculations)*

rWeibull(m, β) =

Return the inverse cumulative probability distribution for probability p. For m > 0; m is an integer always greater than 0

Considering the random variables, when m > 0, and thus when m = 20 draws, we get the random table shown opposite: rWeibull (m, β)

Inverse function of the distribution function, when p is included between (0 and 1)

	0
0	1.247
1	0.758
2	1.021
3	0.484
4	1.282
5	0.621
6	1.081
7	1.474
8	1.335
9	0.138
10	1.399
11	1.993
12	0.815
13	0.740
14	1.297
15	0.904
16	1.596
17	0.534
18	0.828
19	0.407

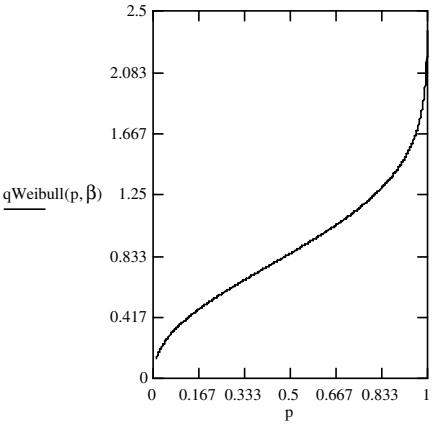


Figure 8.27. *Inverse distribution function (results from Weibull calculations)*

8.11.2. *Log-normal Law (of Galton)*

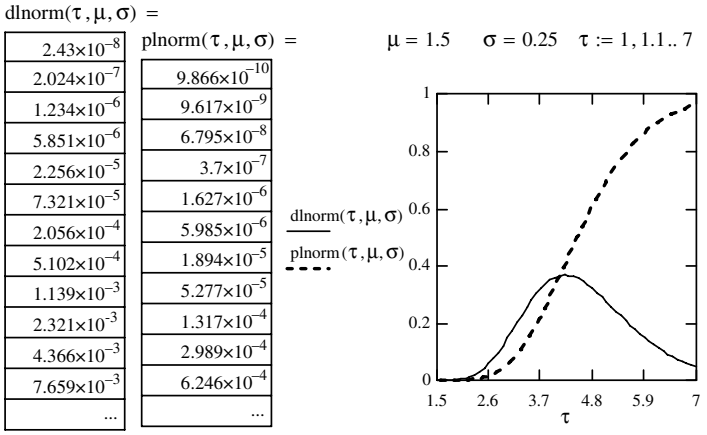


Figure 8.28. *Density and distribution functions (results from Galton calculations)*

8.11.3. *Exponential law*

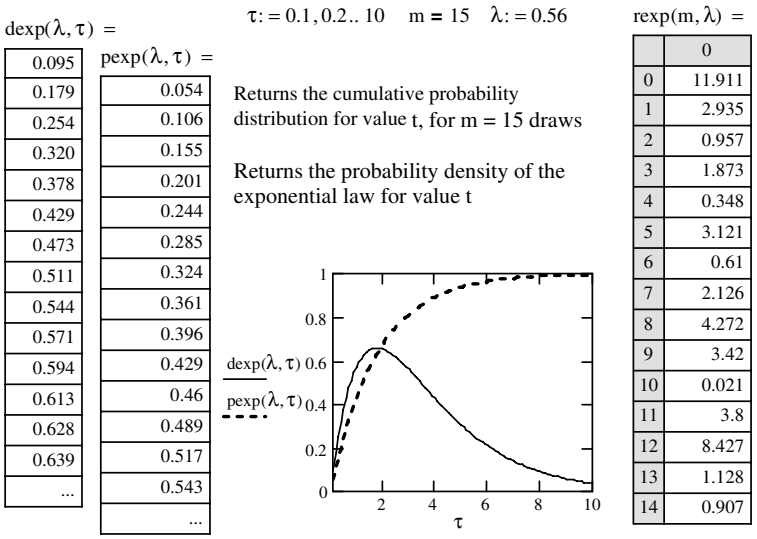


Figure 8.29. *Probability density and distribution functions (exponential)*

8.11.4. *Generation of random numbers*

Generation of random numbers can be done in different ways. We will now present a directed exercise (DE) based on a different approach to the previous ones. This DE can be used to create a vector of random numbers, which follows a uniform law, a normal law, or an exponential law on a given interval. This DE also includes two programs, which allow us to control the initial value to generate random numbers.

8.11.4.1. *Uniform law*

As was developed in Chapter 7, we will continue to present the numerical results for some common distribution laws.

– Let n, the number of random centered variables $\rightarrow n = 1.56 \times 10^3$.

– Let l_0, the upper and lower extremities $\rightarrow l_0 = 0$.

– Let ens_hist, the number of groups for the histogram $\rightarrow$ ens_hist = 33.

– Let N, vector of random centered variables $\rightarrow N$ = runif (n, lo, hi).

We are using the MathCAD software, but here is the conventional way of writing the frequency distribution, from our program:

$$\mathbf{hr}:=\frac{\text{upper}-\text{lower}}{\text{ens\_hist}}$$

lower: = floor(min(N)) upper: = ceil(max(N))

j: = 0..ens_hint $\text{int}_j := \text{lower} + \text{hr}\cdot j$

f: = hist(int, N) $\text{int} := \text{int} + \text{hr}\cdot\frac{1}{2}$

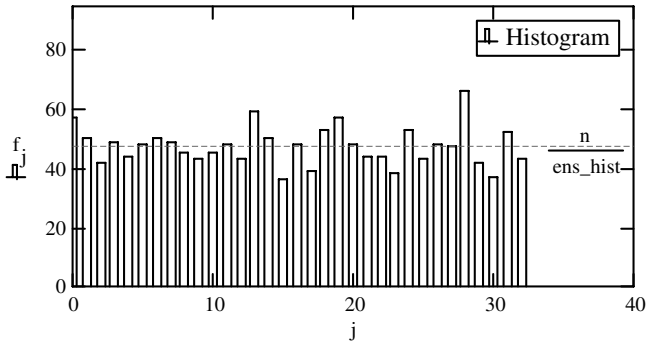


Figure 8.30. *Histogram showing a uniform law (through MC simulation)*

8.11.4.2. *Normal law*

– Let n, the number of random centered variables $\rightarrow n = 1 \times 10^3$.

– Let μ, the upper and lower extremities $\rightarrow \mu = 2.5462$.

– Let σ, the upper and lower extremities $\rightarrow \sigma = 0.5462$.

– Let ens_hist, the number of groups for the histogram $\rightarrow$ ens_hist = 33.

– Let N, vector of random centered variables $\rightarrow N = \mathbf{r}\text{norm}(n, \mu, \sigma)$.

Enter the number of groups for the histogram : ens_hist = 33

Vectors of random centred variables : $N = \text{rnorm}(n, \mu, \sigma)$

Frequency distribution: $\text{lower} = \text{floor}(\min(N))$ $\text{upper} = \text{ceil}(\max(N))$

$$hr = \frac{\text{upper} - \text{lower}}{\text{ens\_hist}} \qquad int_j = \text{lower} + hr \cdot j$$

$$j := 0 .. \text{ens\_hist} \qquad f = \text{hist}(int, N) \qquad int = int + \frac{1}{2} \cdot hr$$

Normal adjustment function $F(x) = n \cdot hr \cdot \text{dnorm}(x, \mu, \sigma)$

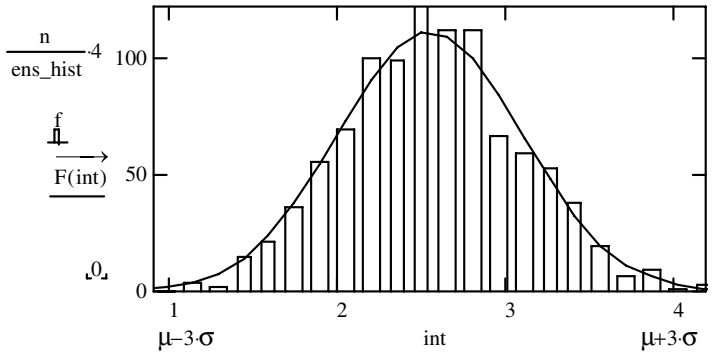


Figure 8.31. *Histogram showing normal law (using MC simulation)*

8.11.4.3. *Exponential law*

– Let n, the number of random centered variables $\rightarrow n = 1.56 \times 10^3$.

– Let α, the rate $\rightarrow \alpha = 2.5462$.

– Let ens_hist, the number of groups for the histogram → ens_hist = 33.

– Let *N*, vector of random centered variables →*N* = rexp (*n, α)*.

$$\text{lower} = \text{floor}(\min(N)) \quad \text{upper} = \frac{\text{ceil}(\max(N))}{2} \quad f = \text{hist}(\text{int}, N) \quad hr = \frac{\text{upper} - \text{lower}}{\text{ens\_hist}}$$

Frequency distribution $\text{int}_j = \text{lower} + hr \cdot j \quad \text{int} = \text{int} + 0.5 \cdot hr \quad j := 0 .. \text{ens\_hist}$

Exponential adjustment function: $F(x) = n \cdot hr \cdot \text{dexp}(x, \alpha)$

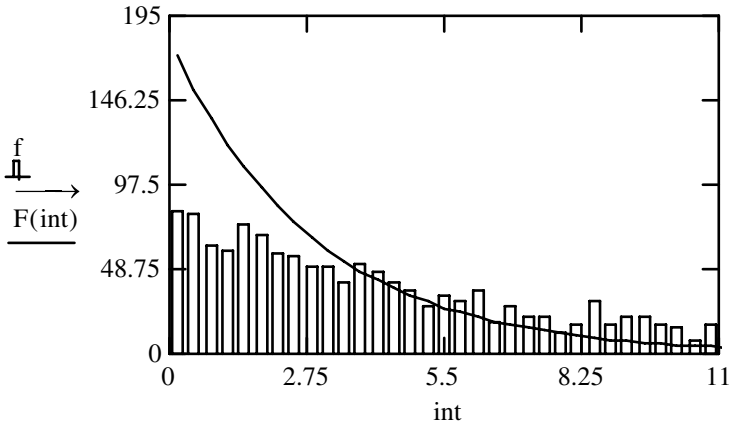


Figure 8.32. *Histogram showing exponential law (using MC simulation)*

8.11.4.4. *Control of initial value*

The following program allows us to generate the same set of random numbers following a normal law at each iteration by resetting the initial value.

SameNormal(n, μ, σi) :=
  n = 1×10^3
  μ = 2.5462
  σ = 0.5462
  for i ∈ 0 .. 8
    Seed(1)
    nums ← rnorm(n, μ, σ)
    $M^{\langle i \rangle}$ ← nums
  M

The exact same set of random numbers will be generated each time: Each column includes the same numbers in the same order. Also, this program allows us to control and repeat the generation process of a new set of random numbers from a given initial value. In Chapter 7 (MC simulation), we presented the theoretical content of this.

$$\text{lower} = \text{floor}\left(\min\left(M^{\langle sel-1\rangle}\right)\right) \qquad \text{upper} = \text{ceil}\left(\max\left(M^{\langle sel-1\rangle}\right)\right)$$

$$hr = \frac{\text{upper} - \text{lower}}{\text{ens\_hist}} \qquad j := 0 .. \text{ens\_hist}$$

$$int_j = \text{lower} + hr \cdot j \qquad int = int + 1 \cdot hr$$

$$\text{Ajust}(x) = n \cdot hr \cdot \text{dnorm}(x, \mu, \sigma) \qquad Q = \text{hist}\left(int, M^{\langle sel-1\rangle}\right)$$

$\text{SameNormal}(n, \mu, \sigma) =$

	0	1	2	3	4	5	6	7	8
0	2.306	2.306	2.306	2.306	2.306	2.306	2.306	2.306	2.306
1	2.175	2.175	2.175	2.175	2.175	2.175	2.175	2.175	2.175
2	2.288	2.288	2.288	2.288	2.288	2.288	2.288	2.288	2.288
3	2.027	2.027	2.027	2.027	2.027	2.027	2.027	2.027	2.027
4	1.625	1.625	1.625	1.625	1.625	1.625	1.625	1.625	1.625
5	2.57	2.57	2.57	2.57	2.57	2.57	2.57	2.57	2.57
6	2.48	2.48	2.48	2.48	2.48	2.48	2.48	2.48	2.48
7	2.85	2.85	2.85	2.85	2.85	2.85	2.85	2.85	2.85
8	3.743	3.743	3.743	3.743	3.743	3.743	3.743	3.743	3.743
9	2.988	2.988	2.988	2.988	2.988	2.988	2.988	2.988	2.988
10	3.084	3.084	3.084	3.084	3.084	3.084	3.084	3.084	3.084
11	3.017	3.017	3.017	3.017	3.017	3.017	3.017	3.017	3.017
12	3.046	3.046	3.046	3.046	3.046	3.046	3.046	3.046	3.046
13	2.914	2.914	2.914	2.914	2.914	2.914	2.914	2.914	2.914
14	1.976	1.976	1.976	1.976	1.976	1.976	1.976	1.976	1.976
15	2.584	2.584	2.584	2.584	2.584	2.584	2.584	2.584	...

Table 8.6. *Random numbers generated according to a Gaussian*

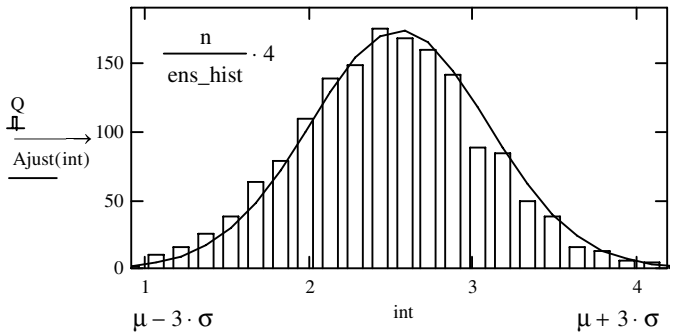


Figure 8.33. *Histogram showing normal law (MC simulation at the fifth draw)*

8.11.4.5. *Estimating Monte Carlo probability*

This DE uses random number generators from the software MathCAD to estimate quantity probabilities with unknown distributions.

Question: Demonstrate in an empirical manner that the average of a logistical distribution 133 sized sample with parameters L and S is located on the interval [a, b].

L = 1; S = 0.33; and the interval [a, b] = [0.90, 1.2535] c.-to-d. a = 0.90 and b = 1.2535

We define the MC sampling parameters as follows:

– Number of samples to collect; N Samples = 1.54×10^3.

– Number of points of data per sample; S Size = 133.

We carry out the sample and collect the sample statistics:

When *i:=0*, …, (N Samples − 1) we will get Mean s_i = mean(rlogis(S Size, L, S))

Then estimate the desired probability

$$\left\{ \begin{array}{l} Success = \sum_{i=1}^{N_{Samples}} \left\{ a \le Means_{i-1} \le b \right\} \ then \\ \Rightarrow \ \text{Prob} = \sum_{i=1}^{N_{Samples}} \frac{Success}{N_{Samples}} = 0.971 \cong 1 \end{array} \right\} \qquad [8.89]$$

8.12. Elements of safety via the couple: resistance and stress (R, S)

We have seen in Chapter 1 that when the load imposed on a component of a mechanism is unknown, it is represented by a stress distribution that is either variable or constant through time (according to the lifecycle). The resistance opposing this load is therefore unknown. This is explained by the variability of intrinsic materials, the sizing (rating), and other thermal and environmental factors. The phenomenon (R, S) can be described by a resistance probability distribution, which is itself variable according to the lifecycle. We are then in a position to calculate (estimate) the reliability of the piece (mechanism and/or structure) in connection with the lifecycle (longevity). The probability of a piece having at the very most an equal stress σ(MPa) is written as:

$$P(\sigma \le \tau) = \int_{-\infty}^{\tau} g(\sigma) d\sigma \qquad [8.90]$$

The probability that the piece resists is written as:

$$P(\tau \le R \le \tau + d\tau) = f(\tau) d\tau \qquad [8.91]$$

The basic probability is therefore written as:

$$dR = f(\tau) d\tau \times \int_{-\infty}^{\tau} g(\sigma) d\sigma \qquad [8.92]$$

If we proceed to the sum of the two probabilities (R, S), the explanation is justified by the fact that the piece must resist the load for all values of (τ). Therefore:

$$R = \int_{-\infty}^{+\infty} f(\tau) \times \left(\int_{-\infty}^{\tau} g(\sigma) d\sigma \right) d\tau \qquad [8.93]$$

It is equally possible to present the resistance equation (R) as follows:

$$R = \int_{-\infty}^{+\infty} g(\tau) \times \left(\int_{\tau}^{\infty} f(\sigma) d\rho \right) d\tau \qquad [8.94]$$

Here is the classic illustration of the above:

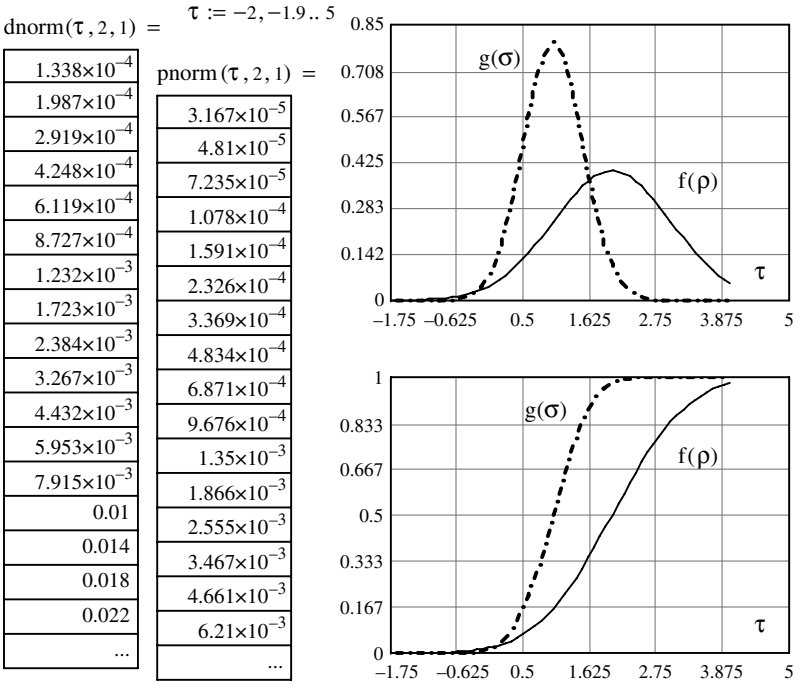


Figure 8.34. *Probability distribution of resistances f(ρ) and stresses g(σ)*

The literature proposes some expressions dedicated to calculating the reliability of components according to the applied stress and its resistance. The following are some examples of common distributions:

***Normal law*:** R = Resistance and S = Stress:

$$\left\{ \begin{array}{l} R(\tau) = \Phi\left(\dfrac{\overline{x}_\rho - \overline{x}_\sigma}{\sqrt{\sigma_\rho^2 + \sigma_\sigma^2}} \right) \textit{ when} \\ \left\{ \begin{array}{c} R \to \textit{Average} \quad = \overline{x}_\rho \\ R \to \text{Standard deviation} = \sigma_\rho \end{array} \right\} \textit{ and } \left\{ \begin{array}{c} S \to \textit{Average} \quad = \overline{x}_\sigma \\ S \to \text{Standard deviation} = \sigma_\sigma \end{array} \right\} \end{array} \right\} \qquad [8.95]$$

***Galton law (log–normal)*:** R = Resistance and S = Stress:

$$\left\{ \begin{array}{l} R(\tau) = \Phi\left(\dfrac{\overline{x}_\rho - \overline{x}_\sigma}{\sqrt{\sigma_\rho^2 + \sigma_\sigma^2}} \right) \text{ when} \\ \left\{ \begin{array}{l} R \to \text{Average} = \overline{x}_\rho \\ R \to \text{Standard deviation} = \sigma_\rho \end{array} \right\} \text{ and } \left\{ \begin{array}{l} S \to \text{Average} = \overline{x}_\sigma \\ S \to \text{Standard deviation} = \sigma_\sigma \end{array} \right\} \end{array} \right\} \quad [8.96]$$

***Exponential law with an average μ*:** R = Resistance and S = Stress:

$$\left\{ \begin{array}{l} R(\tau) = 1 - Exp\left(-\dfrac{\overline{x}}{\mu} - \dfrac{\sigma^2}{2\mu^2} \right) \text{ when} \\ \left\{ \begin{array}{l} R \to \text{Average} = \overline{x} \\ R \to \text{Standard deviation} = \sigma \end{array} \right\} \text{ and } \{S \to \text{Average} = \mu\} \end{array} \right\} \quad [8.97]$$

***Exponential law with an average μ_c*:** R = Resistance and S = Stress:

$$\left\{ \begin{array}{l} R(\tau) = 1 - Exp\left(\dfrac{\mu_\rho}{\mu_\rho + \mu_c} \right) \text{ when} \\ \{R \to \text{Average} = \mu_\rho\} \text{ and } \{S \to \text{Average} = \mu_c\} \end{array} \right\} \quad [8.98]$$

We can use the Mellin transformation either numerically or graphically.

8.13. Reliability trials

– Testing hypothesis trials;

– Determination trials (precise estimation or through intervals).

Determination trials try to find the statistical distribution form. The estimation can be precise or by intervals (see Chapters 2 and 3, Volume 1). Hypothesis test trials allow us to refuse or accept an established hypothesis with the risk (or probability) of wrongly rejecting the established hypothesis. These trials are generally designed for quality control (see Chapter 2 and 3, Volume 3). The hypothesis is normally quite simple: it allows us to confirm or invalidate if the measuring results, relative to the lifecycle, let us conclude that the average lifecycle exceeds the previously imposed value.

The hypothesis is sometimes double. In other words, the test is designed to accept a portion of material with a high probability when the MTBF is large enough, and to reject the portion with a strong probability when the MTBF is sufficiently weak. This is generally done once we accept a portion of components with a probability equivalent to (1 α = 0.95). This is simply understood by using the following example: If the hypothesis H_I has a satisfied MTBF = θ_I = 36,000 hours it will be accepted, and will be rejected with a probability (1 β = 0.90) when the hypothesis H_{II} has a satisfied MTBF = θ_{II} = 3,600 hours. We have already approached this subject in the chapter in Volume 3 dedicated to quality control.

H_I is the null hypothesis, α is a type-one or supplier's risk and θ_I is the Level of Acceptable Quality (LAQ). All are used as a level of reputed classic quality.

REMARK.– Works are essentially dedicated to quality control and reliability. We have already mentioned that these two disciplines share their statistical–mathematical tools, but each interpretation is slightly different. Statistical distribution laws remain the same, however. By using the examples presented here, we are able to prove the truthfulness of this fact again.

H_{II} represents the alternative hypothesis where β is the type II risk or the user's risk. Θ_{II} indicates the LAQ or the level of reliability, which will be accepted. We have mentioned the difference in interpretation between the results of the two major disciplines involved in risk: reliability and quality control.

As with quality control, the type of reliability trial is classified according to the type of trail. To do this, beforehand we must fix the sample size, i.e. the number of components we want to test (quality control). The same thing is done for reliability trials, the lifecycle or the number of imposed fractures before the end of the trial. We can use a *truncated trial* when the lifecycle is required, and a *censored trial* when the number of fractures is required.

Occasionally we wait for the trial's submitted sample plan to fail before ending the trial. This happens in cases where damage occurs (catastrophes, destructive trials, etc.). Progressive trials are integral in double hypothesis tests. The sample size is not fixed, hence sequential decision-making following weaknesses (fractures or partial failure). Often, the decision is to do with the continuation of the trial, its closure (often the categorical refusal of the portion of components), or even the stopping of the trial to accept the portion. We will discuss this in Chapters 2 and 3 in Volume 3 on quality control, but now we will present a similar work with a reliability perspective and finally end with educational examples.

8.13.1. *Controlling risks and efficiency in mechanical reliability*

Following trial plans in simple sampling, and by obeying a power distribution law (Weibull, exponential, Poisson, binominal, Gamma, Erlang, Birnbaum–Saunders, etc.), we can use a classic exponential law for our calculations. We can then take an enlightened decision by judging the MTBF(θ) of the pieces in use.

Hypothesis I: Imagine an MTBF(θ) = 1,810 hours. The cumulated duration trial is represented by τ = 3,600 hours. Following Poisson's law (law of power) we can postulate the expression of the average number of fractures, which will appear through physical use.

$$Average = \mu(\kappa) = \left(\frac{\tau_f}{\theta}\right) = \left(\frac{3600}{1810}\right) = 1.99 \qquad [8.99]$$

If κ = (1, 2, 3, …, 10) and m = 10 we can calculate:

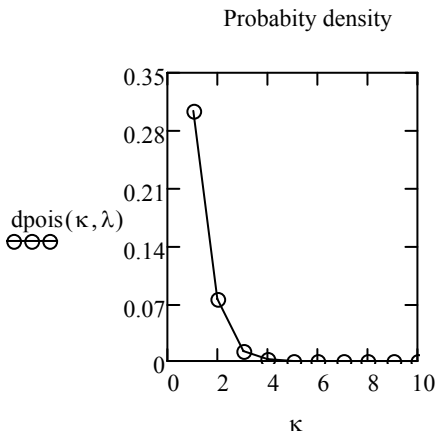
$$\theta = \left(\frac{3600}{1810}\right) = 1.99 \; ; \lambda = \left(\frac{1}{\theta}\right) = 0.503 \; ; \; P\{\lambda, \kappa\} = \left(\frac{\lambda^{\kappa}}{\kappa!}\right) \times Exp\{-\lambda\}$$

$\text{dpois}(\kappa, \lambda)$ =

0.304
0.076
0.013
$1.61 \cdot 10^{-3}$
$1.619 \cdot 10^{-4}$
$1.357 \cdot 10^{-5}$
$9.746 \cdot 10^{-7}$
$6.125 \cdot 10^{-8}$
$3.422 \cdot 10^{-9}$
$1.72 \cdot 10^{-10}$

$\text{ppois}(\kappa, \lambda)$ =

0.909
0.985
0.998
1
1
1
1
1
1
1



$\text{rpois}(m, \lambda)$ =

	0
0	0
1	0
2	0
3	0
4	1
5	1
6	0
7	0
8	1
9	0

Figure 8.35. *Probability density and distribution function from Poisson's law*

The *rpois*(m, λ) function sends back random values according to which Poisson's law can be justified to simulate the process. The functions *dpois*(κ, λ) and *ppois*(κ, λ) are functions of probability density and distribution when $\kappa = 0$ to 10 and l = 0.503 = 0.5.

For an exponential distribution law, we normally derive the following:

Null hypothesis → true hypothesis H_I	Alternative hypothesis → true hypothesis H_{II}
α is a risk from I → therefore $(1 - \alpha)$	β is a risk from II → therefore β
α	$(1 - \beta)$
⇓ DECISION ⇓	
Accept H_I	*Refuse* H_{II}

Table 8.7. *Accepting and rejecting plans*

Let H_I be a null hypothesis in which θ = 360 hours. Let another alternative hypothesis H_{II} of MTBF θ_2 = 180 hours. τ_{cum} is the cumulated duration of the trial. The acceleration zone of the trial is located between [0, 2] in line with the observed characteristic, in this case the number of fractures (κ). We can therefore suppose:

$$\begin{cases} P\left(\kappa \le \dfrac{f}{\theta_1}\right) = Exp^{-\frac{\tau_f}{\theta_1}} \displaystyle\sum_{\kappa=0}^{f} \frac{\left(\dfrac{\tau_f}{\theta_1}\right)^{\kappa}}{\kappa!} = 1-\alpha \\ P\left(\kappa \le \dfrac{f}{\theta_1}\right) = Exp^{-\frac{\tau_f}{\theta_2}} \displaystyle\sum_{\kappa=0}^{f} \frac{\left(\dfrac{\tau_f}{\theta_2}\right)^{\kappa}}{\kappa!} = \beta \end{cases} \qquad [8.100]$$

Numerical application according to exponential law:

$$P(\kappa,\theta 1) = e^{\frac{-\tau}{\theta 1}} \cdot \left(\sum_{\kappa=0}^{f} \frac{\frac{\tau}{\theta 1}}{\kappa!} \right) = 1 - \alpha \quad \text{et} \quad P(\kappa,\theta 2) = e^{\frac{-\tau}{\theta 2}} \cdot \left(\sum_{\kappa=0}^{f} \frac{\frac{\tau}{\theta 2}}{\kappa!} \right) = \beta$$

$$Risk\{I\}(\alpha);\ \alpha = 0.33\ we\ get\ (1-\alpha) = 0.67\ So\ P(\kappa,\theta_1) = 0.67 \rightarrow \text{Accept}$$

$$Risk\{II\}(\beta);\beta = 0.90\ we\ get\ \beta = 0.90\ So\ P(\kappa,\theta_2) = 0.90 \rightarrow \text{Refuse}$$

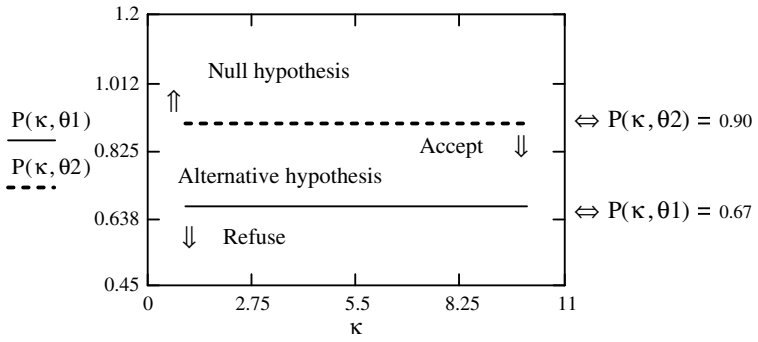


Figure 8.36. *Exponential law – accepting or rejecting plan*

In a total time of service $\tau = 460$ hours and for an MTBF θ evolving between 10 and 620 hours, we calculate the acceptance probability of the plan, which is in fact simply the curve of efficiency Eff(θ). The efficiency curve Eff(θ) is presented as follows:

$$\theta = 10,\ldots, 620 \text{ hours};\ \tau = 460 \text{ hours} \rightarrow \text{Eff}(\theta) = Exp\left\{-\frac{\tau}{\theta}\right\} \times \left[\frac{1}{2}\left(\frac{\tau}{\theta}\right)^2 + 1 + \left(\frac{\tau}{\theta}\right)\right]$$

Eff(θ) = Acceptance probability of plan = Efficiency of trial plan

	0
0	0
1	0
2	1.741×10^{-14}
3	2.843×10^{-13}
4	3.083×10^{-12}
5	2.411×10^{-11}
6	1.447×10^{-10}
7	6.985×10^{-10}
8	2.814×10^{-9}
9	9.735×10^{-9}
10	...

θ is the MTBF, θ1 is the acceptable MTBF and θ2 is the tolerated MTBF

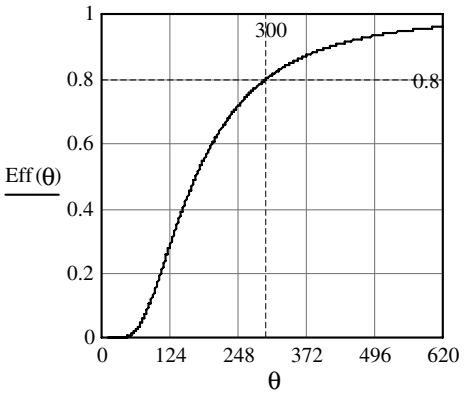


Figure 8.37. *Efficiency curve of a sampling plan in reliability*

Comments: As there are four possibilities to consider, first we will present those involving plan acceptances:

$$\text{If } \theta 1\text{: 300 hours} \quad \tau 1\text{: 460 hours} \quad \text{Eff}(\theta 1) = e^{\frac{-\tau 1}{\theta 1}} \cdot \left[\frac{1}{2}\left(\frac{\tau 1}{\theta 1} \right)^2 + 1 + \left(\frac{\tau 1}{\theta 1} \right) \right]$$

If a risk (α) $\alpha = 20\% \rightarrow \text{Eff}(\theta_1) = (1 - \alpha)$ then the hypothesis H_1 will be accepted at $(1 - \alpha) = 0.8$.

$$\text{If } \theta 2\text{: 562 hours} \quad \tau 2\text{: 460 hours} \quad \text{Eff}(\theta 2) = e^{\frac{-\tau 1}{\theta 1}} \cdot \left[\frac{1}{2}\left(\frac{\tau 2}{\theta 2} \right)^2 + 1 + \left(\frac{\tau 2}{\theta 2} \right) \right]$$

If a risk (β) $\rightarrow \beta = \text{Eff}(\theta_2) \rightarrow$ as $= \text{Eff}(\theta_2) = 0.950$ then β the hypothesis H_1 will be accepted at (β) = 0.95.

The two other possibilities reject the plan (refer Table 8.7):

– We refuse the plan at the risk (α, type I risk, = 20%) in hypothesis H_I.

– We refuse the plan at the risk ($1 - \beta$, type II result, = 5%) in hypothesis H_I.

Finally, there is a ratio (δ) of discrimination in the plan which is expressed as $\delta(\theta) = \left(\theta_2 / \theta_1\right)$. Also, θ_0 corresponds to the acceptance probability $\text{Prob}_{accep} = (1/2)$ with a *balancing point of the plan* = 0.5.

8.13.2. *Truncated trials*

A trial is said to be truncated if we stop a trial after a trial period (τ) and in this trial we have not had to resort to component replacements. The cumulated duration is presented as:

$$\tau_{cum} = \tau \times (n - \kappa) + \sum_{i=1}^{\kappa} \tau_i ; \; \tau_{cum} = \tau \times n \; (\textit{With replacements}) \qquad [8.101]$$

$$\tau cum := \tau \cdot (n - \kappa) + \sum_{i=1}^{\kappa} \tau_i \qquad \tau cum1 := \tau \cdot (n)$$

$\tau cum =$

	0
0	15
1	17
2	19
3	21
4	23
5	25
6	27
7	29
8	31
9	33
10	15

When

$\tau = 100$

$n = 7$

$\kappa = 5$

$i := 1..10$

$\tau_i =$

1	
2	
3	
4	
5	
6	
7	
8	
9	
0	

$\tau cum1 =$

	0
0	0
1	7
2	14
3	21
4	28
5	35
6	42
7	49
8	56
9	63
10	0

8.13.3. *Censored trials*

If the trial needs its defective components replaced, the trial's lifecycle will be calculated by the relationship shown in [8.102]. This is a censored trial, with or without replacement. The respective expressions are presented as:

$$\left\{ \begin{array}{l} \tau_{cum} = \tau_r \times (n-r) + \sum_{i=1}^{r} \tau_i; \\ \tau_{cum} = \tau_r \times (n-r+1) + \sum_{i=1}^{r-1} \tau_i \ \ (replacement) \end{array} \right\} \qquad [8.102]$$

If $\tau = 100$ hours, $n = 7$, $r = 5$, i = 1, …, 10, the cumulated time τ_{cum} will be:

$$\left\{ \begin{array}{l} \tau_{cum}^{censored\ 1} = \tau_r \times (n-r) + \sum_{i=1}^{r} \tau_i = 215 \text{ hours } (without\ replacement) \\ \tau_{cum}^{censored\ 2} = \tau_r \times (n-r+1) + \sum_{i=1}^{r-1} \tau_i = 310 \text{ hours } (with\ replacement) \end{array} \right\}$$

8.13.4. *Trial plan*

To show a trial plan, without regard to the type of trial, it is worth making two principle hypotheses:

H_I for an MTBF $\theta = \theta_I$ and H_{II} for an MTBF $\theta = \theta_{II}$

Then we can associate respective acceptation probabilities to them:

$P_{rob}(\theta_I) = (1 - \alpha)$ and $P_{rob}(\theta_{II}) = \beta$

Finally, the plan is defined by the solution from the Pearson test.

$$\left\{ \frac{\chi^2_{(2\kappa;\, 1-\beta)}}{\chi^2_{(2\kappa;\, \alpha)}} \le \left(\frac{\theta_1}{\theta_2} \right) = \left(\frac{MTBF_1}{MTBF_2} \right) \text{ and } Eff\left(\theta\right) = \frac{\theta_1 \times \chi^2_{(2\kappa;\, \alpha)}}{2} \right\} \qquad [8.103]$$

When κ = the number of fractures $Eff(\theta)$, the cumulated trial duration and $\chi^2_{(2\kappa;\, \alpha)}$ is the Chi-squared test variable (K Pearson) with 2κ degrees of freedom, whose distribution function is (α). To summarize, we suppose the following:

– ***Censored trial*:** r = κ fractures, the acceptation criterion is $\tau_{acc} \ge Eff(\theta)$.

– ***Truncated trial*:** the cumulated duration $\tau_{cum} = Eff(\theta)$, the acceptation criterion $\tau_{acc} = \kappa - 1$.

Numerical application: Imagine a plan characterized by θ_1 = 3,001 hours, θ_2 = 301 hours, α = 0.05, and β = 0.95. Calculate the acceleration factor of the trial and check the hypothesis by justifying it with the Pearson test.

Solution: $Factor_{acceleration} = \left(\frac{\theta_1}{\theta_2} \right) = \left(\frac{3001}{301} \right) hours = 9.970 hours \approx 10 hours$

$$\text{If } \left\{ \begin{array}{l} \kappa = 5; dll = \nu = 2\kappa = 10; \alpha = 0.05 \Rightarrow qchisq\left(\alpha, \nu\right) = \chi^2_{(2\kappa,\alpha)} = \chi^2_{(10,0.05)} = 3.9403 \\ \kappa = 5; dll = \nu = 2\kappa = 10; \alpha = 0.95 \Rightarrow qchisq\left(\alpha, \nu\right) = \chi^2_{(2\kappa,\alpha)} = \chi^2_{(10,0.95)} = 18.3070 \end{array} \right\}$$

$$\text{We therefore suppose: } \left\{ \frac{\chi^2_{(2\kappa;\, 1-\beta)}}{\chi^2_{(2\kappa;\, \alpha)}} = \frac{18.3070}{3.9403} = 4.6461 \right\}$$

$$\text{If } \left\{ \begin{array}{l} \kappa = 7; Df = \nu = 2\kappa = 14; \alpha = 0.05 \Rightarrow qchisq\left(\alpha, \nu\right) = \chi^2_{(2\kappa,\alpha)} = \chi^2_{(14,0.05)} = 6.5706 \\ \kappa = 7; Df = \nu = 2\kappa = 14; \alpha = 0.95 \Rightarrow qchisq\left(\alpha, \nu\right) = \chi^2_{(2\kappa,\alpha)} = \chi^2_{(14,0.95)} = 23.6848 \end{array} \right\}$$

$$\text{Then we suppose: } \left\{ \frac{\chi^2_{(2\kappa;\ 1-\beta)}}{\chi^2_{(2\kappa;\ \alpha)}} = \frac{23.6848}{6.5706} = 3.6047 \right\}$$

$$\text{The comparison allows } \left\{ \frac{\chi^2_{(2\kappa;\ 1-\beta)}}{\chi^2_{(2\kappa;\ \alpha)}} = 3.6047 \right\} \text{ and } \left\{ \frac{\chi^2_{(2\kappa;\ 1-\beta)}}{\chi^2_{(2\kappa;\ \alpha)}} = 4.6461 \right\}$$

$$\text{Then we keep } \left\{ Eff\left(\theta\right) = \frac{\theta_1 \times \chi^2_{(2\kappa;\ \alpha)}}{2} = \frac{3001 \times 6.5706}{2} = 9.8592 \times 10^3 \ hours \right\}$$

For (κ) fractures and a cumulated trial duration *Eff(θ)* we can calculate, using the Pearson test, χ^2 to 2κ df (degrees of freedom) of function (α) to finally end up with the following corresponding condition:

$$Eff\left(\theta\right) = \theta_1 \frac{\chi^2_{(2\kappa;\ \alpha)}}{2} \simeq 9859 \ hours$$

As we have sufficiently presented this in quality control (see Chapters 2 and 3 volume 3), we will now move on to an example of progressive trial plans, involving the power distribution law (Exponential Law). Imagine the parameters that characterize the plan, i.e. the MTBF θ_1 and θ_2, the risks α and β. The aim is that for each fracture (κ) of noted reliability, we will calculate the cumulated trial duration $\tau_{cumaccept}(\kappa)$. This will allow us to take an enlightened decision: accept the portion and the cumulated trial duration $\tau_{cumreject}(\kappa)$ allowing the justified decision to reject the portion. Thus for the κth fracture we will reason out as follows:

– If the cumulated trial duration is $\tau_{cumF}(\kappa) \geq \tau_{cumaccept}(\kappa)$, we will stop the trial to eventually accept the portion.

– If the cumulated trial duration is $\tau_{cumF}(\kappa) < \tau_{cumrejet}(\kappa)$, we will stop the trial to eventually reject the portion.

– If the cumulated trial duration is $\tau_{cumrejet}(\kappa) \leq \tau_{cumF}(\kappa) < \tau_{cumaccept}(\kappa)$, we will continue the trial.

In Chapters 1 and 2 of volume 3, we have developed trial plans for quality control. We will use the same methodology to show that trial durations $\tau_{cumaccept}(\kappa)$, $\tau_{cumF}(\kappa)$, and $\tau_{cumrejet}$ develop in a linear way according to the number of fractures (κ).

$$\left\{ \begin{array}{l} \tau_{accept}\left(\kappa\right) = s \times \kappa - h_1 \\ \tau_{accept}\left(\kappa\right) = s \times \kappa - h_2 \end{array} \right\} \text{ with} \qquad [8.104]$$

$$\left\{ \begin{array}{l} h_1 = MTBF_1 \times \left(\frac{a}{f-1}\right) = \theta_1 \times \left(\frac{a}{f-1}\right) \\ h_2 = MTBF_2 \times \left(\frac{a}{f-1}\right) = \theta_1 \times \left(\frac{b}{f-1}\right) \end{array} \right\} \qquad [8.105]$$

$$\left\{ \begin{array}{l} s = \theta_1 \times \left(\frac{g}{f-1}\right) because\ f = \left(\frac{\theta_1}{\theta_2}\right) = \text{inverse of discrimination ratio } \left(\frac{1}{\delta}\right) \\ g = \ln\left(f\right);\ a = \ln\left(\frac{\beta}{1-\alpha}\right)\ and\ b = \ln\left(\frac{1-\beta}{\alpha}\right);\ \textit{see Chapters 12 and 13} \end{array} \right\} \qquad [8.106]$$

We can now see how h_1, h_2, and s are proportional to the MTBF θ_1. This allows us to establish universal plans exclusively according to α, β, and f, whose critical durations are written as follows:

Coefficients (a), (b), α, and β from a progressive plan							*Coeff.* ***f and S'***	
β%	α%	**1**	**2**	**5**	**10**	**20**	***f***	***S'***
1	**a**	−4.595	−4.585	−4.554	−4.500	−4.382	1.1	0.953
	b	4.595	3.902	2.986	2.293	1.599	1.2	0.912
2	**a**	−3.902	−3.892	−3.861	−3.807	−3.689	1.5	0.811
	b	4.585	3.892	2.976	2.282	1.589	2.0	0.693
5	**a**	−2.986	−2.976	−2.944	−2.890	−2.773	2.5	0.611
	b	4.554	3.861	2.944	2.251	1.558	3.0	0.549
10	**a**	−2.293	−2.282	−2.251	−2.251	−2.079	4.0	0.462
	b	4.500	3.807	2.890	2.197	1.504	5.0	0.402
20	**a**	−1.599	−1.589	−1.558	−1.504	−1.386	6.0	0.358
	b	4.382	3.689	2.079	2.079	1.386	7.0	0.324

Table 8.8. *Coefficients (a), (b), α, and β; f and S' from a progressive plan*

$$\left\{ \tau'_{accept}(\kappa) = \frac{\tau_{accept}(\kappa)}{\theta_1} \text{ and } \tau'_{reject}(\kappa) = \frac{\tau'_{reject}(\kappa)}{\theta} \right\} \quad [8.107]$$

$$\left\{ \begin{array}{l} \tau'_{accept}(\kappa) = \left(\frac{a}{f-1} + \frac{\ln f}{f-1}\kappa \right) = -h'_1 + S' \times \kappa \\ \tau'_{reject}(\kappa) = \left(-\frac{b}{f-1} + \frac{\ln f}{f-1}\kappa \right) = -h'_2 + S' \times \kappa \end{array} \right\} \quad [8.108]$$

Example of numerical application in mechanical reliability:

Let: (α) = 2%, (β) = 5%, f = 3, and an MTBF = θ_1 = 3

The principle calculation formulae are:

$$\textit{Calculation formulae} \rightarrow \left\{ \begin{array}{l} a = \ln\left(\frac{\beta}{1-\alpha}\right) = \ln\left(\frac{5\%}{1-2\%}\right) \\ b = \ln\left(\frac{1-\beta}{\alpha}\right) = \ln\left(\frac{5\%}{1-2\%}\right) \\ S' = \left(\frac{\ln f}{f-1}\right) = \ln\left(\frac{5\%}{1-2\%}\right) \end{array} \right\}$$

Solutions (numerical, graphical, and tabulated): Following the previous formulae, we can calculate the plan's coefficients, which are discussed in the following sections.

8.13.5. *Coefficients for the trial's acceptance plan*

$$\textit{Calculation results} \rightarrow \left\{ \begin{array}{l} a = \ln\left(\frac{\beta}{1-\alpha}\right) = \ln\left(\frac{5\%}{1-2\%}\right) = -2.9755 \\ b = \ln\left(\frac{1-\beta}{\alpha}\right) = \ln\left(\frac{5\%}{1-2\%}\right) = 3.8607 \\ S' = \left(\frac{\ln f}{f-1}\right) = \ln\left(\frac{5\%}{1-2\%}\right) = 0.5493 \end{array} \right\}$$

For the failures (weaknesses) of pieces 1 to 7, i.e. $\kappa = (1, 2, \ldots, 7)$, the cumulated acceptance duration with $T_{accep}(\kappa) = \tau'_{accep}(\kappa)$ will be calculated as:

$$\left\{\tau'_{accept}(\kappa) = \left(-\frac{a}{f-1} + \frac{\ln f}{f-1}\kappa\right)\right\} \text{ and } \left\{\tau'_{accept}(\kappa) = \frac{\tau_{accept}(\kappa)}{\theta_1}\right\} \quad [8.109]$$

τaccept(κ) =	Taccept(κ) =
2.0371	0.6790
2.5864	0.8621
3.1357	1.0452
3.6850	1.2283
4.2343	1.4114
4.7836	1.5945
5.3329	1.7776

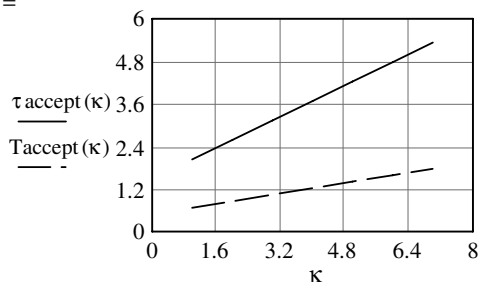


Figure 8.38. *Trial's acceptance plan*

8.13.6. *Trial's rejection plan (in the same conditions)*

By analogy of the above, for the failures $\kappa = (1 \text{ to } 7)$, we will have the cumulated duration of rejection, with $T_{reject}(\kappa) = \tau'_{reject}(\kappa)$, as follows:

$$\left\{\tau'_{reject}(\kappa) = \left(-\frac{b}{f-1} + \frac{\ln f}{f-1}\kappa\right)\right\} \text{ and } \left\{\tau'_{reject}(\kappa) = \frac{\tau'_{reject}(\kappa)}{\theta}\right\} \quad [8.110]$$

τreject(κ) =	Treject(κ) =
–1.3811	–0.4604
–0.8318	–0.2773
–0.2824	–0.0941
0.2669	0.0890
0.8162	0.2721
1.3655	0.4552
1.9148	0.6383

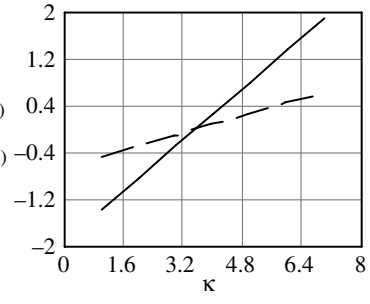


Figure 8.39. *Trial's rejection plan*

By referring to Table 8.8 it is easy to proceed with the same calculations and graphs. For our part, we prefer numerical calculations to avoid precision errors.

Graphical summary of a progressive plan in reliability:

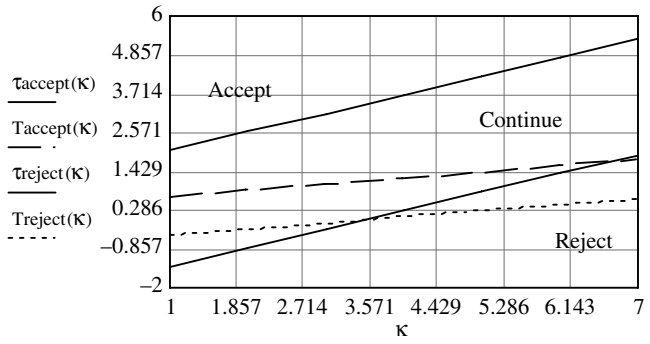


Figure 8.40. *Experiment summary of a progressive trial plan*

In the case of rejection, the minimum trial duration is never negative. Consequently, the decision to reject the plan is only taken after the number of fractures does not equal the first value (whole positive number > 0) or even equal to the ratio $b/\ln(f)$ of the cumulated duration, as shown in the following equation:

$$\tau_{reject}\left(1^{st}\ value\right) = \theta_1\left(-\frac{f}{f-1}+\frac{\ln f}{f-1}\right) \qquad [8.111]$$

In the case of acceptance, the minimum trial duration is expressed as follows: First, a cumulated duration equal to the first fracture (weakness) must be reached. Then we can decide on acceptance where the minimum cumulated trial duration (before definitive acceptance) is written as follows:

$$\tau_{accept}\left(1^{st}\ value\right) = \theta_1\left(-\frac{a}{f-1}+\frac{\ln f}{f-1}\right) \qquad [8.112]$$

In mechanical reliability (electromagnetic, hydraulic, and electronic, to name just a few of the many areas of engineering) the components (portion of material) are often tested for an average MTBF in the interval of [θ_1 to θ_2]. We then *suspect* that the trial duration can be "*self-perpetuating*" and become quite *costly* in terms of calculation time. To proceed according to a time that corresponds to the MTBF(θ), we recommend the following:

$$MTBF = \theta = \left(\frac{\ln f}{f-1}\right)\times\theta_1 = \left(S'\times\theta_1\right);\ \left(S'\right) is\ tabulated \qquad [8.113]$$

Following what we have just said, it is worth predicting the halt of the trial. This is conventionally known as *truncation*. Normally, we conduct a trial truncation after three times the number of fractures. As a matter of course, it would be wise to proceed according to the simple sampling plan. This is advantageous because we can stop the trial as soon as a cumulated duration *Eff(θ)* in connection with the number of fractures $\geq(\nu = \kappa - 1)$ has been reached.

For the first ($\kappa = 1$) and f = 5, ($\theta_1 = 3$) when $\alpha = 2\%$ and $\beta = 5\%$

$$a = \ln\left(\frac{\beta}{1-\alpha}\right) = -2.9755;\ b = \ln\left(\frac{1-\beta}{\alpha}\right) = 3.8607;\ S' = \ln\left(\frac{\ln(f)}{f-1}\right) = 0.5493$$

$$\tau_{accept}\left(\kappa = 1\right) = \theta_1\left(-\frac{a}{f-1} + \frac{\ln f}{f-1}\right) = 3.4387 \leftarrow \left(Si\ \theta_1 = 3\right) otherwise = 1.1462\theta_1$$

$$\tau_{reject}\left(\kappa = 1\right) = \theta_1\left(-\frac{f}{f-1} + \frac{\ln f}{f-1}\right) = -6.2929 \leftarrow \left(si\ \theta_1 = 3\right) otherwise = -2.0976\theta_1$$

$$\kappa_{reject} \geq \left(\frac{b}{\ln(f)}\right) = \left(\frac{3.8607}{\ln(5)}\right) = \left(\frac{3.8607}{1.6094}\right);\ \kappa_{reject} \to 2.39884428974400398$$

Carry over $\kappa = 3$in which: $\tau_{reject}\left(\kappa = 1\right) = \theta_1\left(-\frac{f}{f-1} + \frac{\ln f}{f-1}\right) = -2.0976\times\theta_1$

This is also verified by the Pearson test discussed in section 8.12.7. The plan in simple sampling that corresponds to this test is discussed in the following.

8.13.7. *Trial plan in reliability and K Pearson test χ^2*

$$\left\{\begin{array}{l}\kappa = 7(dof), \nu = 2\kappa;\ \alpha = 80\%,\ qchisq(\alpha,\nu) = \chi^2_{2\kappa;\alpha} = \chi^2_{2\kappa;80\%} = 18.1508\\ \kappa = 7(dof), \nu = 2\kappa;\ \alpha = 2.5\%,\ qchisq(\alpha,\nu) = \chi^2_{2\kappa;\alpha} = \chi^2_{2\kappa;2.5\%} = 5.6287\end{array}\right\}$$

$$\left(\frac{\chi^2_{2\kappa;0.80}}{\chi^2_{2\kappa;0.05}}\right)=\frac{18.1508}{5.6287}=3.3247\le 4 \text{ with the risk of } \beta=2.5\%$$

$$When\ \left(\kappa=r=7\right)=Eff\left(\theta\right)=\theta_1\frac{\left(\chi^2_{2\kappa;0.05}\right)}{2}=\theta_1\left(\frac{5.6287}{2}\right)=2.8144\times\theta_1$$

Lifecycle calculations are the basis for the majority of reliability calculations. The methods are varied and targeted depending on the case being studied. We do not apply an exponential distribution model to cumulative use phenomena because this law is hardly relevant. In addition, precision reliability is costly in terms of calculation time, for example in offshore structures. Data collection is time-consuming and sometimes it may take decades for the sample to have a representative significance of reality. For a constant failure rate of 100 components, more than 100,000 hours will be necessary for 10 components (welded structures for example) to fail. Without calculations, this is the equivalent of more than 10 years.

– *What should we do to confront with this situation?*

– *Make do with simulations or carry out often costly experiments?*

For educational purposes, the problem does not have consequential responsibilities. For industrial needs, we must obtain true experimental data for a significant *true reliability model.* To overcome the problem of *slow trials*, it is a pragmatic idea to carry out accelerated trials. In this case, it is advisable to correctly pose the reliability problem, i.e. the applied stresses. We will attempt to go beyond the standard stresses caused by normal use. Because of this, we will *break* the normal rules of material resistance (RDM). This is the same case in terms of carrying out accelerated trials to reduce lifecycles. This is possible as long as we do not push too far on increasing the applied stresses. We will study (or present) three distinct cases relevant to engineering techniques to support what we have just said.

Accelerated trials are one of the most common approaches for obtaining a reliability law or the failure rate of systems or components. These trials reduce the lifecycles of components by accelerating the damage at the origin of weakness. Thus, the stress levels felt by the component are of notable severity and provide a rapid large quantity of data. Apart from in the electronic domain, there are very few case studies dedicated to accelerated trials in mechanics. However, the technical literature is becoming increasingly focused on accelerated trials in continued witnessed environments.

8.14. Reliability application on speed reducers (gears)

The lifecycle Log(N) of a cluster of gear wheels is composed of d_{p1} and d_{p2} respective to the initial ∅ of entry and exit. (**l**) is the width of teeth in (mm),

$M_2 = M_{exit}$ is the couple of forces (moments or torque in Nm) measured at the exit of movement. The angle of pressure α = 20° (according to ISO) or α = 14.5° (according to AGMA-USA). Q (doesn't deliberately appear in the function 8.114) is a function of the reduction relationship dependent on the initial ∅ of the cluster. The factors ρ and φ are parameters intrinsic to the materials used in solid gears (steel) where the lifecycle is expressed by:

$$Log(\mathrm{N}) = \rho \times \left(\varphi - Log \frac{8 \times \mathrm{M}_{exit}}{d_{p1} \times d_{p2} \times \ell \times \Re \times Sin(\alpha)} \right) \quad [8.114]$$

$$\left\{ \begin{array}{l} \Re = \left(\dfrac{2 \times dp_2}{dp_1 + dp_2} \right) \leftarrow ratio\ for\ gears\ in\ exterior\ contact \\ ratio\ for\ gears\ in\ interior\ contact \rightarrow \Re = \left(\dfrac{2 \times dp_2}{dp_1 - dp_2} \right) \end{array} \right\} \quad [8.115]$$

Consider the case of the gear cluster comprising of teeth with a width $\mathbf{l}_1$ led by a moment M_1. By analogy, we have the characteristics of the exit pinion with width L_2 led by a moment M_2, and we can then suppose the acceleration factor ξ_{acc}:

$$\xi_{acc} = \left(\frac{N_1}{N_2} = \frac{\omega_1}{\omega_2} \right) = \left(\frac{\ell_1 \times M_2}{\ell_2 \times M_1} \right)^{\rho} \quad [8.116]$$

Imagine a material SAE1020 (USA) of factor $\rho \cong 7$. The moment M_1 = 64.16794 lbf.in (7.25 Nm) and the width $\mathbf{l}_1$ = 0.492126 inch (1.25 cm), which leads another pinion (exit) with moment M_2 = 73.90376 lbf.in (8.35 Nm) and with width $\mathbf{l}_2$ = 0.492126 inch (1.25 cm). Then:

Calculate the acceleration factor of the reliability trial and plot the graphical evolution according to the material.

Solution: *Let: $\mathbf{l}_1 = \mathbf{l}_2$ = 0.492126 inch (=1.25 cm); M_1 = 64.16794 lbf.in (= 7.25 Nm) M_2 = 73.90376 lbf.in (8.35 Nm); $\rho \cong 7$ and $\varphi \cong 3.18$ (according to material dynamics SAE standard steel 1020, c-to-d. 0.20% carbon). The angle of pressure α = 14.5° according to AGMA USA (or 20° according to ISO, USA). Following the acceleration factor $\xi_{acc}(\rho)$ we postulate:*

$$\xi_{acc}(\rho) = \left(\frac{N_1}{N_2} = \frac{\omega_1}{\omega_2} \right) = \left(\frac{\ell_1 \times M_2}{\ell_2 \times M_1} \right)^{\rho} = \left(\frac{2 \times 8.35}{2 \times 7.25} \right)^{7} = 2.688$$

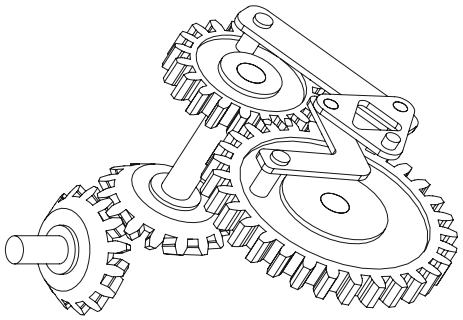


Figure 8.41. *Reliability case study of gears*

The expression for the calculation of the "lifecycle" of the gear cluster (pinions with exterior contact) is presented as follows by the literature on machine elements (speed reducers):

$$\log(N) = \rho \times \left(\varphi - \log \frac{8 \times M_{exit}}{d_{p1} \times d_{p2} \times \ell_{avg} \times \Re \times Sin(\alpha)} \right) \rightarrow N = ? \qquad [8.117]$$

$$\log(N) = 7. \left(3.18 - \log \frac{8 \times 73.90376}{4.724409 \times 4.724409 \times 0.492126 \times 1.25 \times 14.5°} \right) = 8.164612720$$

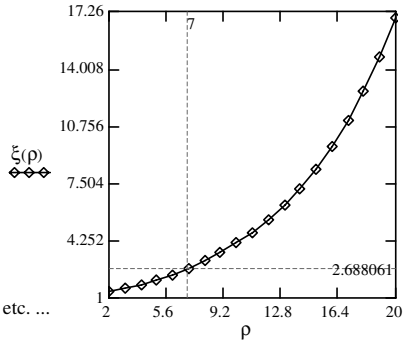
$$\Rightarrow N = \left(According\ to\ AGMA \leftarrow 3.51436 \times 10^3 \right) \equiv \left(According\ to\ ISO \leftarrow 9.727715 \times 10^7 \right)$$

$\xi(\rho)$ =

1.326468
1.527725
1.759518
2.026479
2.333945
2.688061
3.095904
3.565627
4.106619
4.729691
5.447299
6.273786
...

Facteur Acceleration factor of reliability trial according to material

When $\rho := 2..20$ $\quad \xi(\rho) = \left(\frac{Width1 \cdot MExit}{Width2 \cdot MEntry} \right)^{\rho}$



etc. ...

Figure 8.42. *Acceleration factor of trial, evolution according to material*

8.14.1. *Applied example on hydraulic motors*

Imagine an oil pressure hydraulic motor, the lifecycle of which (in normal functioning) is presented as follows:

$$f_{lifecycle}(\tau) = \left(P^{\alpha} \times n^{\beta}\right) \times \xi;\ [hours\ or\ life\ cycle] \qquad [8.118]$$

where:

ξ is a factor (adimensional) which takes account of the material;

P is the oil pressure of the motor (entry, c.-to-d. before use) in MPa (Psi);

n is the RPM of the motor in turns/minute;

α and β are factors given by the motor's manufacturer.

In this expression, from the literature, (α = −3.2535) and (β = −1.0945) are deliberately chosen negative factors to emphasize the effect of acceleration in reliability trials. The respective pressures and rotation speed are of the following type.

The normal and habitual parameters used for the motor are P_1= 18 MPa with an RPM, n_1 = 1,360 tr/min. However, to estimate the duration of the lifecycle in reliability, we have tested the motor with deliberately "exaggerated" parameters: P_2 = 36 MPa with an RPM, n_2 = 3,200 tr/min. The aim of the research is to avoid an experiment with "true" functioning conditions. We will now estimate an acceleration factor of the trial. In hydraulic reliability, the acceleration factor F_{acc} is expressed by the relationship of speeds or stresses. Therefore, we maintain:

$$F_{acc}(trial) = \left(\frac{P_1}{P_2}\right)^{\alpha} \times \left(\frac{RPM_1}{RPM_2}\right)^{\beta} = \left(\frac{18}{36}\right)^{3.2535} \times \left(\frac{1360}{3200}\right)^{1.0945} = 24.3293$$

By increasing the RPM2 from 1,360 to 3,200 we can observe the evolution of the trial's acceleration factor, with the log–log scale:

$$F_{acc}(Accelerated\ trial) = \left(\frac{P_1}{P_2}\right)^{\alpha} \times \left(\frac{RPM_{1\ Const}}{RPM_{2accelerated}}\right)^{\beta} \qquad [8.119]$$

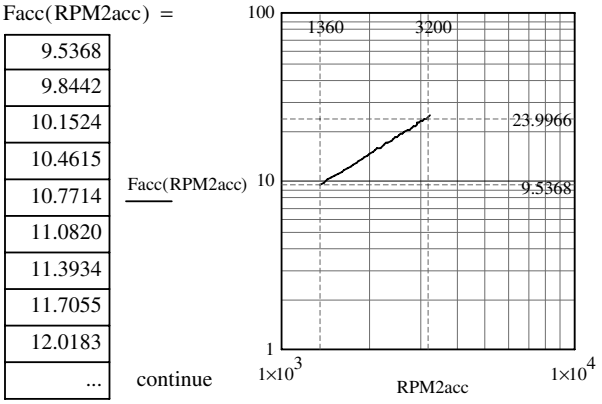


Figure 8.43. *Acceleration factor in mechanical reliability*

8.14.1.1. *Lifecycle of the bearing*

Imagine a number of machines with motors. Each motor turns at an RPM n = constant (rotations/min). Normally, we intervene in the bearings after about 6,000 hours of use in the motors (if the bearings are correctly assembled and well maintained). We have unfortunately noticed a malfunctioning, so the following rule is adopted: "change the bearings after the theoretical lifecycle" F(τ), expressed by:

$$F(\tau) = 18 \times \left(10^6 / n\right) \text{ in hours of normal use} \qquad [8.120]$$

We know the type of accelerated trial to use for a reliability plan. To optimize the interventions at a workshop level, we have predicted an accelerated plan running from 960 to 3,600 tr/min, to be able to observe the maintenance of the bearings without dismantling them. Calculate the acceleration factor F_{acc} and plot the appropriate curve.

8.14.1.2. *Numerical and graphical solutions*

When RPM_{entry} = 960 tr/min; $\alpha = 10^6$

$$F_{acc}(trial) = 18 \times \left(\frac{\alpha}{RPM}\right) = 18 \times \left(\frac{10^6}{960}\right) = 1.875 \times 10^4$$

As previously explained, we vary the RPM (accelerate) of the trial from 960 to 3,200 tr/min and we observe the acceleration factor (using a semi-log scale):

$$F_{acc}\left(trial\right) = 18\left(\frac{\alpha}{RPM_{entry}/RPM_{accelerated}}\right) = 18\left(\frac{RPM_{accelerated}\times\alpha}{RPM_{entry}}\right) \quad [8.121]$$

Facc(RPM2acc) =

2.55×10^7
2.63×10^7
2.70×10^7
2.78×10^7
2.85×10^7
2.92×10^7
3.00×10^7
3.08×10^7
3.15×10^7
...

Facc(RPMacce)

etc. ...

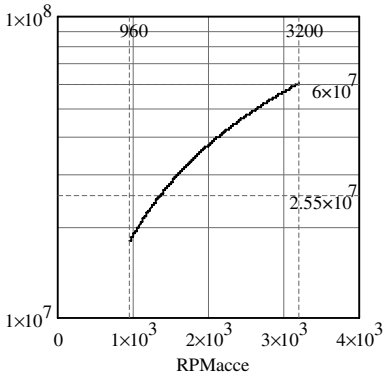


Figure 8.44. *Acceleration factor (bearings)*

8.14.1.3. *Comments*

Whether we are dealing with a speed reducer, a pneumatic motor, hydraulics, a transistor, mechanical components, electric, or others, the reliability calculation is based on data collected from the experiment. Yet it has already been discussed that this collection is costly in time and means. Thus to try to present significant results in the limits of reduced time we have to resort to accelerated trials. The danger of this type of trial for the industry is that the true representativeness of the experiment is questionable. Because of this, we distort the truthfulness of the data from the reliability calculation, and thus of safety.

8.15. Reliability case study in columns under stress of buckling

Problem outline from an RDM and reliability perspective: Imagine a column submitted to buckling according to specifications from the AISC norms.

The fixed ends *x* and *y* are axes of deviated flexion (buckling) under the load (P). Buckling according to [8.45] is written:

$$\text{Buckling} = \left(\frac{K_x \times L_x}{r_x}\right) \textbf{(fig. 8.45-1)} \text{ and backling} = \left(\frac{K_y \times L_y}{r_y}\right) \textbf{(fig. 8.45-2)} \quad [8.122]$$

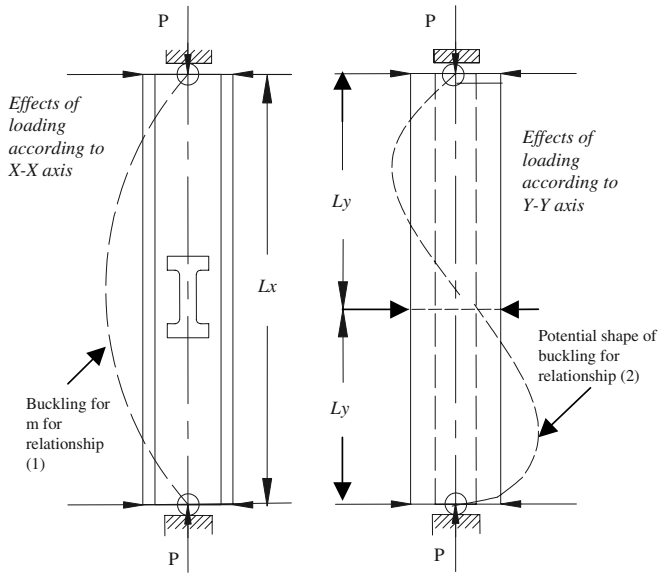


Figure 8.45. *Reliability of columns submitted to buckling*

Calculation parameters RDM and reliability:

RDM data			
Parameters Factors	Denomination	ISO units in SI	Imperial measurements
K	Factor of effective length of column (adimensional)	1	1
L_x according to X–X	Distance from the fixed end to the middle of the column according to x–x (see Figure 8.45)	7.62 m or 7,620 mm	25 ft
L_y according to Y–Y	Distance from the fixed end to the middle of the column according to y–y (see Figure 8.45)	3.078 m or 3,048 mm	10 inch
P	Load applied	533,786.6 N	120 × E3 lb.f
F_y	Failure threshold	827.3709 MPa = 8.273709E8 Pa	32×10^3 in Psi
E	Elasticity modulus	68,947.57 MPa = 6.894757E10 Pa	10×10^6 in Psi
NORMAL STATISTICS: PROBABILISTIC DATA FOR $n = 25$			
μ	Arithmetic average		
σ	Standard deviation		

Table 8.9. *Initial data for the buckling column case study*

8.15.1. *RDM solution*

a) According to AISC standards, the load (see tables) according to the weak control supervision. We can read the load value P and calculate K_x L_y:

– If P = 1.2 × 10^5 *lbf* (or 533,786.6 N), we calculate K_x Ly = 10 ft =3.078 m = 3,048 mm.

– The area of section A = 11.5 $inch^2$ = 7,419.34 mm^2.

– Radius of gyration according to axis Y–Y = r_y = 1.98 inch = 50,292 mm.

– Radius of gyration relationship according to the axis Y–Y and X–X → Rr_x r_y = 1.16.

– Radius of gyration according to axis X–X = r_x = → Rr_x r_y (r_y) = 4.277 inch = 108.6358 mm.

b) Calculation for the slenderness relationship of the column: C_{column}

$$C_c = \sqrt{\frac{2\pi^2 \times E}{F_y}} = 78.54$$

c) We can use the technical literature to suppose that for a fracture threshold we calculate local buckling F_{a_local} and elastic buckling $F_{a\_elastic}$ as follows:

$$F_{a\_Elastic}\left(K,L,r\right) = \left(12 \times \pi^2 \times E\right) \Big/ 23 \times \left(\frac{K \times L}{r}\right)^2 \qquad [8.123]$$

$$F_{a\_local} X\left(K,L,r,C_c\right) = \frac{\dfrac{2 \times C_c^2 - \left(\dfrac{K \times L}{r}\right)^2}{2 \times C_c^2} \times F_y}{\dfrac{5}{3} + \dfrac{3 \times \left(\dfrac{K \times L}{r}\right)}{8 \times C_c} - \dfrac{\left(\dfrac{K \times L}{r}\right)^3}{8 \times C_c^3}} \qquad [8.124]$$

Numerical applications in the two principle units

$$F_{a\_Elastic} X\left(K,L,r\right) = 1.047 \times 10^4 \text{ psi} = 7.218811 \times 10^7 Pa = 72.18811 \text{ Mpa}$$

$$F_{a\_local} X\left(K,L,r,C_c\right) X = 1.006 \times 10^4 psi = 6.936126 \times 10^7 Pa = 69.36126 \text{ Mpa}$$

The series of columns used to determine whether we should opt for a local or elastic fitting allows us to better control the intensity of the stress constraint F_{axisX} along axis X–X.

$$According\ to\ X = if\left\{F_{alocX} \prec "local"; elastic\right\}\ According\ to\ X = "local"$$

$$F_{aX} = if\left\{According\ to\ X = "local", F_{alocX}, F_{elaX}\right\} \rightarrow F_{aX} = 1.006 \times 10^4\ psi$$

d) Calculation: of the local and elastic buckling constraint F_{alocY} and F_{aelaY} according to the axis Y–Y:

$$Elastic\ buckling: F_{aelaY} = F_{aela}\left(K, L_y, r_y\right) \rightarrow F_{aelaY} = 1.184 \times 10^4\ psi$$

$$Local\ buckling: F_{alocY} = F_{aloc}\left(K, L_y, r_y, C_c\right) \rightarrow F_{aelaY} = 1.184 \times 10^4\ psi$$

The series of columns used to determine whether we should opt for a local or elastic fitting allows us to better control the intensity of the stress constraint F_{axisX} along axis Y–Y.

$$According\ to\ Y = \ if\ \left\{F_{alocY} \prec F_{aelaY}, "local", "elastic"\right\}\ According\ to\ Y = "local"$$

$$F_{aY} = if\left\{According\ to\ Y = "local", F_{alocY}, F_{aelaY}\right\} \rightarrow F_{aY} = 1.184 \times 10^4\ psi$$

e) Determining the failure threshold F_a by using the smallest stress (fracture threshold) according to axes: X–X and Y–Y.

$$F_a = if\left\{F_{aX} \prec F_{aY}; F_{aX}, F_{aY}\right\} therefore \rightarrow F_a = 1.006 \times 10^4\ psi$$

f) Calculation of "true stress" (Pascalian model of simple elasticity): f_a

$$F_{a\_elastic}\left(K, L, r\right) = \frac{P}{A} = 1.043 \times 10^4\ psi = 71.91232\ MPa = 7.191232 \times 10^7\ Pa$$

Comment: We retain the solution F_a from the series of columns to either confirm or reject the selection. We must bear in mind that the true stress intensity is smaller than the admissible stress (choice of material).

Here is our conditioned reasoning:

$$selection = if\left\{F_a \succ f_a\ , "OK", "NG"\right\}\ or\ even\ selection = "NG"$$

– The percentage value of acceptance (upper) is expressed with the sign (+).

– The percentage value of refusal (lower) is expressed with the sign (−).

$$\text{Our result: } (\% = \pm ?) = \left(\frac{F_a - f_a}{F_a} \right) = (-)3.747\%$$

Conclusion: If the selection of the column is not acceptable or if the column is significantly over-designed, we proceed with a re-conceptualization starting from step (a), by repeating all calculations.

8.15.2. *Problem outline and probabilistic solution (reliability and error)*

Beforehand, we suppose that the distribution law in about 30 experiments is "Gaussian". We do not have any proof but we base this judgment on the equality of the averages from the experiments grouped in Table 8.10. To create this table, enter two vectors of data (15 values of stress constraint) to analyze $F_{elastic}$ and F_{local}.

Statistics

Felastic =

	0
0	70.010
1	71.750
2	72.150
3	70.250
4	70.000
5	72.000
6	70.000
7	71.500
8	70.600
9	72.188
10	72.189
11	70.000
12	72.000
13	72.188
14	71.000

Flocal =

	0
0	69.362
1	69.850
2	72.560
3	69.850
4	71.750
5	70.515
6	71.954
7	70.956
8	71.857
9	72.456
10	69.956
11	71.950
12	72.358
13	72.455
14	69.985

PROBLEM: Reject or accept:
Null hypothesis H0:

Enter (a) a significant threshold

$\alpha = 0.01 \quad n1 = 15 \text{ and } n2 = 15$

Averages: m1 and m2

$m1 = \text{mean}(\text{Felastic}) = 71.188 \text{ MPa}$

$m2 = \text{mean}(\text{Flocal}) = 71.188 \text{MPa}$

Standard deviations: s1 and s2

$$s1 = \text{stdev}(\text{Felastic}) \cdot \sqrt{\frac{n1}{n1 - 1}} = 0.946$$

$$s2 = \text{stdev}(\text{Local}) \cdot \sqrt{\frac{n2}{n2 - 1}} = 1.158$$

Degrees of freedom: Df

$Df = (\nu = n_1 + n_2 - 2 = 28)$

Table 8.10. *Statistics of the problem*

8.15.2.1. *Standard error of distance, estimated s*

$$s = \sqrt{\frac{(n_1 - 1) \times s_1^2 + (n_2 - 1) \times s_2^2}{\nu} \times \left(\frac{1}{n_1} + \frac{1}{n_2} \right)} \rightarrow therefore\ s = 0.386 \qquad [8.125]$$

8.15.2.2. *Statistical test, t_statistic*

$$t_1 = \left(\frac{m_1 - m_2}{s} \right) \rightarrow therefore\ t_1 = 1.899 \times 10^{-3}$$

$$P\{|T| \succ |t|\} :\rightarrow P = 2 \times \{1 - pt\}(|t_1|, \nu)\ \ therefore\ \ P = 0.998 \qquad [8.126]$$

At the threshold of [α/2], the percentage point T is written as:

$$T = \left| qt\left(\frac{\alpha}{2},\ \nu \right) \right| \rightarrow\ therefore\ T = 2.763 \qquad [8.127]$$

8.15.2.3. *Rejecting the null hypothesis if |t| > T*

As $t_1 = 1.899 \times 10^{-3}$ and T = 2.763 then = $|t_1| < T$. Therefore, the normal hypothesis of Gaussian distribution is not strongly rejected. Having demonstrated the normality of the distribution, we can justifiably apply Cornell's theory to calculate the reliability index and the fracture probability of the buckling column. Cornell's reliability index ($\beta_{Cornell}$) is in this case calculated following relationship [5.4] (see Chapter 5) as follows:

$$\beta_{Cornell}^{F_{elastic}} = \left\{ \frac{M_{F_{elastic}}}{\sigma_{F_{elastic}}^2} \right\} = 79.464\ \ and\ \ \beta_{Cornell}^{F_{local}} = \left\{ \frac{M_{F_{local}}}{\sigma_{F_{local}}^2} \right\} = 53.064 \qquad [8.128]$$

Following relationship [5.12] (see Chapter 5) we get [8.129]:

$$P_{f_{Cornell}^{F_{elastic}}} = \Phi\left(-\beta_{Cornell}^{F_{elastic}} \right) =\ and\ \ P_{f_{Cornell}^{F_{local}}} = \Phi\left(-\beta_{Cornell}^{F_{local}} \right) \qquad [8.129]$$

Numerical calculations:

$$P_{Cornell-elastic} = \left\{ \frac{m_1 \times 10^{-3}}{S_1^2} \right\} = 0.079464$$

$$P_{fracture-elastic} = \Phi\left\{-\beta_{Cornell-elastic}\right\} = 0.4683316234$$

$$\beta_{Cornell-local} = \left\{\frac{m_2 \times 10^{-3}}{S_2^2}\right\} = 0.053064$$

$$P_{local\ fracture} = \Phi\left\{-\beta_{Cornell-local}\right\} = 0.4788404574$$

The failure rate $\lambda(\tau)$ is written according to the relationship [5.21] shown below in [8.130]:

$$\lambda(\tau) = \lim_{\Delta\tau \to 0}\left(\frac{\Re(\tau) - \Re(\tau + \Delta\tau)}{\Delta\tau \times \Re(\tau)}\right) = \frac{dF(\tau)}{d\tau}\frac{1}{d\tau} = -\frac{dLog\Re(\tau)}{d\tau} = \frac{f(\tau)}{\Re(\tau)} \quad [8.130]$$

8.16. Adjustment of least squared for nonlinear functions

8.16.1. *Specific case study 1: a Weibull law with two parameters*

Particular attention: As mentioned in Chapter 2, volume 1, and due to *convergence* issues, we now present a case study relevant to the Weibull law using the Minerr method to adjust the least squared (LS) for nonlinear functions. The Minerr method of course returns the last values calculated when the maximum number of iterations to a solution have been reached without which convergence is achieved.

Problem outline: Imagine a Weibull distribution law with two parameters relative to the data on the roughness of a plate manufactured in 6061 steel (see volume 1, Chapter 2, Figure 2.6 and Table 2.2). Imagine the two vectors A and B respectively face A and face B of the manufactured plate by down milling in opposition (see volume 1, Chapter 2). The data from the Weibull law with two parameters express the roughness of the piece in 6061. Calculate the parameters that optimize the Weibull law with two parameters (η, β) and plot the Weibull curve by demonstrating the curve speed.

Solution: The Weibull law with two parameters is thus written as:

$$Weibull\left(A, \eta, \beta\right) = \eta \times \beta \times A^{\beta-1} \times Exp\left\{-\eta \times A^{\beta}\right\} \quad [8.131]$$

We estimate the initial value of the respective parameters (η, β) = (0.85, 1.15) hence Weibull (A, η, β) = 0.415 (according to our calculation results). With the help of the Levenberg–Marquardt method (a program on MathCAD), we can

considerably reduce the steps involved in the calculations of the problem. This method allows us to calculate the *sum of residual value squares* [*SSE(η,β)*]. The equation to reduce the resolution group using *Minerr* method is written as follows:

$$resid(\eta,\beta) = B - \overrightarrow{Weibull(A,\eta,\beta)} \quad Given \quad 0 = \text{resid}(\eta,\beta) \qquad [8.132]$$

A =

	0
0	0.1453
1	0.3691
2	0.9582
3	0.7018
4	0.8917
5	1.1546
6	1.2719
7	1.4609
8	1.6571
9	1.8390
10	2.0290
11	2.2190

Correspondence of the function (Weibull law density with two unknown parameters):

This law is considered *a priori*

n := length(B) – 1

i := 1, ..., n

n is the number of sampling measures from the roughness of two faces of a plate (6061)

B =

	0
0	0.1546
1	0.1178
2	0.5435
3	0.5064
4	0.6063
5	0.6228
6	0.5690
7	0.4536
8	0.4382
9	0.3164
10	0.2925
11	0.1955

Table 8.11. *Experimental data from roughness measures*

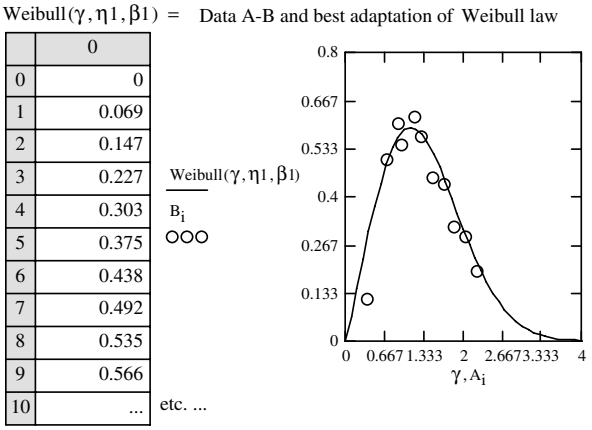
The parameters used to optimize adjustment are thus calculated:

$$\begin{pmatrix}\eta_1\\ \beta_1\end{pmatrix} = Minerr(\eta,\beta) \; therefore \; \begin{pmatrix}\eta_1\\ \beta_1\end{pmatrix} = \begin{pmatrix}0.423\\ 2.110\end{pmatrix} \qquad [8.133]$$

The sum of squares [SSE = *Sum Squared Error*] is implicitly reduced by this method:

$$SSE(\eta,\beta) = \sum resid(\eta,\beta)^2 = 0.807 \qquad [8.134]$$

If (γ = 0, 0.1, …, 5), we can plot the graph from the results obtained using Levenberg–Marquardt optimization method.



	0
0	0
1	0.069
2	0.147
3	0.227
4	0.303
5	0.375
6	0.438
7	0.492
8	0.535
9	0.566
10	...

Figure 8.46. *Adaptation of Weibull law with two parameters*

The sum squared error (zero if a true solution existed) is written as:

$$\frac{SSE(\eta_1, \beta_1)}{n-2} = 4.933605 \times 10^{-3} \quad [8.135]$$

The same number is reduced by the solver by checking the internal error variable, *ERR* (from the MathCAD software). In the case of Levenberg–Marquardt, ERR represents the square root of SSE, the calculation of which can be written as:

$$\left(\frac{ERR^2}{n-2}\right) = 4.933605 \times 10^{-3} \quad [8.136]$$

We can also directly perform a reduction with the help of the SSE equation and the *Minimize* function from the MathCAD software, whose (combined gradient):

$$\begin{pmatrix} \eta_2 \\ \beta_2 \end{pmatrix} = \text{Minimize}\left(SSE, \eta, \beta\right) = \{?\} \quad [8.137]$$

The so-called Quasi-Newton methods with a combined gradient (MathCAD) generate similar results:

$$\begin{pmatrix} \eta_2 - \eta_1 \\ \beta_2 - \beta_1 \end{pmatrix} = \begin{pmatrix} -1.774 \times 10^{-9} \\ -4.876 \times 10^{-9} \end{pmatrix} \quad alors \quad \left\{\frac{SSE(\eta_2, \beta_2)}{n-2}\right\} = 4.933605 \times 10^{-3} \quad cqfd$$

Assessment of perfect compatibility:

$$\left\{\frac{SSE(\eta_2,\beta_2)}{n-2}=4.933605\times10^{-3}\equiv\frac{SSE(\eta_1,\beta_1)}{n-2}=4.933605\times10^{-3}\right\}$$

Comment: We notice that there is a perfect adequacy in our results. We recommend optimization through this approach. Previously in 1995, we presented a paper [GRO 95] on 44 measures of crack length (Ncycles to fracture through cracking) on a cross-joined structure, with the aim of finding the minimal error. The results concluded a Weibull distribution with the following two parameters:

$$\varepsilon=\left(\frac{ERR^2}{n-2}\right)_{Min}=\sum_{i=1}^{44}\left(N_{\text{theoritical cycles}}\ 1-N_{\text{experimental cycles}}\ \right)^2 \qquad [8.138]$$

8.17. Conclusion

The collection and analysis of results in reliability hardly ever come from simple simulations when it involves true safety. The decision to proceed according to any particular method must be based on the exactness of an event, such as the date and precise reason for fracture, the implicated materials, etc. Physical characteristics (e.g. length of crack) must never be an obstacle in terms of true measurement (microscopy, gauge, ultrasound, etc.).

The existing reliability models do not pose any problem. The major issue lies in the *accuracy* of the model used to calculate reliability indicators. In this work, we have presented many methods. It is advisable to remain prudent on the adaptability of each method involved in experiments subject to analysis.

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Appendix

NOTE.– Freely available software (including spreadsheet packages – e.g. Excel) offer access to the common statistical tables and distribution laws. Hence, it would be pointless to give these tables, on paper, here.

Euler's Gamma function as a (β) Weibull function:

Law $\Gamma(x)$ is a second-order Euler Gamma function:

$$\left\{ a=\Gamma\left(1+\frac{1}{\beta}\right) \text{ and } b=\sqrt{\Gamma\left(1+\frac{2}{\beta}\right)-\Gamma^2\left(1+\frac{1}{\beta}\right)} \right\}$$

Table of values (a) and (b) of the Eulerian function as a Weibull f(β) are shown in Table A.1.

β	a	b	β	a	b
0.20	120.00	1901	1.20	0.941	0.780
0.25	24.000	199.0	1.40	0.911	0.660
0.30	9.2600	56.60	1.60	0.897	0.574
0.40	3.3233	10.50	1.80	0.889	0.511
0.50	2.0000	4.470	2.00	0.886	0.463
0.60	1.5000	2.650	2.50	0.887	0.380
0.70	1.266	1.850	3.00	0.893	0.324
0.80	1.133	1.430	0.35	0.900	0.285
0.90	1.052	1.180	4.00	0.906	0.255
1.00	1.000	1.000	5.00	0.918	0.210

Table A.1. *Values of the coefficients of the Eulerian Gamma function (a and b)*

Kolmogorov–Smirnov (KS) test:

(N) Size of the sample	Significance level α for **D** = maximum $[f_0(x) - s_n(x)]$				
	.20	**.15**	**.10**	**.05**	**.01**
1	.900	.925	.950	.975	.995
2	.684	.726	.776	.842	.929
3	.565	.597	.642	.708	.828
4	.494	.525	.564	.624	.733
5	.446	.474	.510	.565	.669
6	.410	.436	.470	.521	.618
7	.381	.405	.438	.486	.577
8	.358	.381	.411	.457	.543
9	.339	.360	.388	.432	.514
10	.322	.342	.368	.410	.490
11	.307	.326	.352	.391	.468
12	.295	.313	.338	.375	.450
13	.284	.302	.325	.361	.433
14	.274	.292	.314	.349	.418
15	.266	.283	.304	.338	.404
16	.258	.274	.295	.328	.392
17	.250	.266	.286	.318	.381
18	.244	.259	.278	.309	.371
19	.237	.252	.272	.301	.363
20	.231	.246	.264	.294	.356
25	.210	.220	.240	.270	.320
30	.190	.200	.220	.240	.290
35	.180	.190	.210	.230	.270
>35	1.07	1.22	1.22	1.36	1.63

Table A.2. *Values of ($D_n.\alpha$) in the Kolmogorov–Smirnov Test*

Student's *t*-distribution table:

Uppermost critical values of the distribution t with ν degrees of freedom (*DoF*) at the reliability threshold of $\alpha/2 = 5\%$							
ν	$\tau_{\alpha/2}(\nu)$	ν	$\tau_{\alpha/2}(\nu)$	ν	$\tau_{\alpha/2}(\nu)$	ν	$\tau_{\alpha/2}(\nu)$
1	12.706	26	2.056	51	2.008	76	1.992
2	4.303	27	2.052	52	2.007	77	1.991
3	3.182	28	2.048	53	2.006	78	1.991
4	2.776	29	2.045	54	2.005	79	1.990
5	2.571	30	2.042	55	2.004	80	1.990
6	2.447	31	2.040	56	2.003	81	1.990
7	2.365	32	2.037	57	2.002	82	1.989
8	2.306	33	2.035	58	2.002	83	1.989
9	2.262	34	2.032	59	2.001	84	1.989
10	2.228	35	2.030	60	2.000	85	1.988
11	2.201	36	2.028	61	2.000	86	1.988
12	2.179	37	2.026	62	1.999	87	1.988
13	2.160	38	2.024	63	1.998	88	1.987
14	2.145	39	2.023	64	1.998	89	1.987
15	2.131	40	2.021	65	1.997	90	1.987
16	2.120	41	2.020	66	1.997	91	1.986
17	2.110	42	2.018	67	1.996	92	1.986
18	2.101	43	2.017	68	1.995	93	1.986
19	2.093	44	2.015	69	1.995	94	1.986
20	2.086	45	2.014	70	1.994	95	1.985
21	2.080	46	2.013	71	1.994	96	1.985
22	2.074	47	2.012	72	1.993	97	1.985
23	2.069	48	2.011	73	1.993	98	1.984
24	2.064	49	2.010	74	1.993	99	1.984
25	2.060	50	2.009	75	1.992	100	1.984

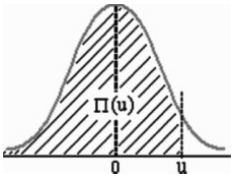
Table A.3. *Student's t-distribution table*

Binomial distribution:

Significance level for one-direction test						
d_f	**.10**	**.05**	**.025**	**.01**	**.005**	**.000**
1	3.078	6.314	12.706	31.821	63.657	636.619
2	1.886	2.920	4.303	6.965	9.925	31.598
3	1.638	2.353	3.182	4.541	5.841	12.941
4	1.533	2.132	2.776	3.747	4.604	8.610
5	1.476	2.015	2.571	3.365	4.032	6.859
6	1.440	1.943	2.447	3.143	3.707	5.959
7	1.415	1.895	2.365	2.998	3.499	5.405
8	1.397	1.860	2.306	2.896	3.355	5.041
9	1.383	1.833	2.262	2.821	3.250	4.781
10	1.372	1.812	2.228	2.764	3.169	4.587
11	1.363	1.796	2.201	2.718	3.106	4.437
12	1.356	1.782	2.179	2.681	3.055	4.318
13	1.350	1.771	2.160	2.650	3.012	4.221
14	1.345	1.761	2.145	2.624	2.977	4.140
15	1.341	1.753	2.131	2.602	2.947	4.073
16	1.337	1.746	2.120	2.583	2.921	4.015
17	1.333	1.740	2.110	2.567	2.898	3.965
18	1.330	1.734	2.101	2.552	2.878	3.922
19	1.328	1.729	2.093	2.539	2.861	3.883
20	1.325	1.725	2.086	2.528	2.845	3.850
21	1.323	1.721	2.080	2.518	2.831	3.819
22	1.321	1.717	2.074	2.508	2.819	3.792
23	1.319	1.714	2.069	2.500	2.807	3.767
24	1.318	1.711	2.064	2.492	2.797	3.745
25	1.316	1.708	2.060	2.485	2.787	3.725
26	1.315	1.706	2.056	2.479	2.779	3.707
27	1.314	1.703	2.052	2.473	2.771	3.690
28	1.313	1.701	2.048	2.467	2.763	3.674
29	1.311	1.699	2.045	2.462	2.756	3.659
30	1.310	1.697	2.042	2.457	2.750	3.646
40	1.303	1.684	2.021	2.423	2.704	3.551
60	1.296	1.671	2.000	2.390	2.660	3.460
120	1.289	1.658	1.980	2.358	2.617	3.373
X	1.282	1.645	1.960	2.326	2.576	3.291

Table A.4. *Binomial distribution table*

Normal distribution table:



Distribution function ***P*** of the reduced centered normal law (*u*). Probability of finding a value less than at (u) → P(−u) = 1 − P(u). Example: For Φ(1.73) = 0.95818, we choose the row 1.7 and the column 0.03, and read → (1.7 + 0.03 = 1.73)

u	**0.00**	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	**0.93319**	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899
2.3	0.98928	0.98956	0.98983	0.99010	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.99180	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.99430	0.99446	0.99461	0.99477	0.99492	0.99506	0.99520
2.6	0.99534	0.99547	0.99560	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.99720	0.99728	0.99736
2.8	0.99744	0.99752	0.99760	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.99900
3.1	0.99903	0.99906	0.99910	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.99940	0.99942	0.99944	0.99946	0.99948	0.99950
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	0.99960	0.99961	0.99962	0.99964	0.99965
3.4	0.99966	0.99968	0.99969	0.99970	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976
3.5	0.99977	0.99978	0.99978	0.99979	0.99980	0.99981	0.99981	0.99982	0.99983	0.99983
3.6	0.99984	0.99985	0.99985	0.99986	0.99986	0.99987	0.99987	0.99988	0.99988	0.99989
3.7	0.99989	0.99990	0.99990	0.99990	0.99991	0.99991	0.99992	0.99992	0.99992	0.99992
3.8	0.99993	0.99993	0.99993	0.99994	0.99994	0.99994	0.99994	0.99995	0.99995	0.99995
3.9	0.99995	0.99995	0.99996	0.99996	0.99996	0.99996	0.99996	0.99996	0.99997	0.99997

Table A.5. *Gaussian normal law table*

Values of probabilities associated with Z in a normal law:

The probability in a direction rejects the null hypothesis (HO) if the value is less than that given by the value of the RCRV, Z (reduced centered random variable). Second column (Z↓) for the first decimal. The row (Z→) provides the second decimal to be added to the first one

↓ Z →	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641
.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	2.810	.2776
.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	2.483	.2451
.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0038	.0036
2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007

Table A.6. *Probability values associated with Z in a normal law*

Karl Pearson or χ^2 law:

Based on the number of degrees of freedom (which we can read in the first column) and the risk of error (α) (on the first row), we find the value of the difference χ^2 that is likely to be surpassed. The value of χ^2 for which the probability of a value less than χ^2 depending on the number n of degrees of freedom is $\alpha = F_n(\chi^2)$.

$n\backslash\alpha$	0.005	0.010	0.025	0.050	0.100	0.250	0.500	0.750	0.900	0.950	0.975	0.990	0.995
1	0.0000	0.0002	0.0010	0.0039	0.0158	0.102	0.455	1.32	2.71	3.84	5.02	6.63	7.88
2	0.0100	0.0201	0.0506	0.103	0.211	0.575	1.39	2.77	4.61	5.99	7.38	9.21	10.6
3	0.0717	0.115	0.216	0.352	0.584	1.21	2.37	4.11	6.25	7.81	9.35	11.3	12.8
4	0.207	0.297	0.484	0.711	1.06	1.92	3.36	5.39	7.78	9.49	11.1	13.3	14.9
5	0.412	0.554	0.831	1.15	1.61	2.67	4.35	6.63	9.24	11.1	12.8	15.1	16.7
6	0.676	0.872	1.24	1.64	2.20	3.45	5.35	7.84	10.6	12.6	14.4	16.8	18.5
7	0.989	1.24	1.69	2.17	2.83	4.25	6.35	9.04	12.0	14.1	16.0	18.5	20.3
8	1.34	1.65	2.18	2.73	3.49	5.07	7.34	10.2	13.4	15.5	17.5	20.1	22.0
9	1.73	2.09	2.70	3.33	4.17	5.90	8.34	11.4	14.7	16.9	19.0	21.7	23.6
10	2.16	2.56	3.25	3.94	4.87	6.74	9.34	12.5	16.0	18.3	20.5	23.2	25.2
11	2.60	3.05	3.82	4.57	5.58	7.58	10.3	13.7	17.3	19.7	21.9	24.7	26.8
12	3.07	3.57	4.40	5.23	6.30	8.44	11.3	14.8	18.5	21.0	23.3	26.2	28.3
13	3.57	4.11	5.01	5.89	7.04	9.30	12.3	16.0	19.8	22.4	24.7	27.7	29.8
14	4.07	4.66	5.63	6.57	7.79	10.2	13.3	17.1	21.1	23.7	26.1	29.1	31.3
15	4.60	5.23	6.26	7.26	8.55	11.0	14.3	18.2	22.3	25.0	27.5	30.6	32.8
16	5.14	5.81	6.91	7.96	9.31	11.9	15.3	19.4	23.5	26.3	28.8	32.0	34.3
17	5.70	6.41	7.56	8.67	10.1	12.8	16.3	20.5	24.8	27.6	32.0	33.4	35.7
18	6.26	7.01	8.23	9.39	10.9	13.7	17.3	21.6	26.0	28.9	31.5	34.8	37.2
19	6.84	7.63	8.91	10.1	11.7	14.6	18.3	22.7	27.2	30.1	32.9	36.2	38.6
20	7.43	8.26	9.56	10.9	12.4	15.5	19.3	23.8	28.4	31.4	34.2	37.6	40.0
21	8.03	8.90	10.3	11.6	13.2	16.3	20.3	24.9	29.6	32.7	35.5	38.9	41.4
22	8.64	9.54	11.0	12.3	14.0	17.2	21.3	26.0	30.8	33.9	36.8	40.3	42.8
23	9.26	10.2	11.7	13.1	14.8	18.1	22.3	27.1	32.0	35.2	38.1	41.6	44.2
24	9.89	10.9	12.4	13.8	15.7	19.0	23.3	28.2	33.2	36.4	39.4	43.0	45.6
25	10.5	11.5	13.1	14.6	16.5	19.9	24.3	29.3	34.4	37.7	40.6	44.3	46.9
26	11.2	12.2	13.8	15.4	17.3	20.8	25.3	30.1	35.6	38.9	41.9	45.6	48.3
27	11.8	12.9	14.6	16.2	18.1	21.7	26.3	31.5	36.7	40.1	43.2	47.0	49.6
28	12.5	13.6	15.3	16.9	18.9	22.7	27.3	32.6	37.9	41.3	44.5	48.3	51.0
29	13.1	14.3	16.0	17.7	19.8	23.6	28.3	33.7	39.1	42.6	45.7	49.6	52.3
30	13.8	15.0	16.8	18.5	20.6	24.5	29.3	34.8	40.3	43.8	47.0	50.9	53.7
40	20.7	22.2	24.4	26.5	29.1	33.7	39.3	45.6	51.8	55.8	59.3	63.7	66.8
50	28.0	29.7	32.4	34.8	37.7	42.9	49.3	56.3	63.2	67.5	71.4	76.2	79.5
60	35.5	37.5	40.5	43.2	46.5	52.3	59.3	67.0	74.4	79.1	83.3	88.4	92.0
70	43.3	45.4	48.8	51.7	55.3	61.7	69.3	77.6	85.5	90.5	95.0	100.4	104.2
80	51.2	53.5	57.2	60.4	64.3	71.1	79.3	88.1	96.6	101.9	106.6	112.4	116.3
90	59.2	61.8	65.6	69.1	73.3	80.6	89.3	98.6	107.6	113.1	118.1	124.1	128.3
100	67.3	70.1	74.2	77.9	82.4	90.1	99.3	109.1	118.5	124.3	129.6	135.8	140.2

Table A.7. χ^2 *law*

For large values of n, the law of probability of χ^2 tends toward a normal law of average n and variance $2n$. If n is sufficiently large, the random variable $\left\{\left(\sqrt{2\chi^2}\right)-\left(\sqrt{2n-1}\right)\right\}$ more-or-less obeys a reduced centered normal law. If the calculated value of χ^2 is greater than the value shown at the predetermined confidence threshold, we reject the null hypothesis H_0.

d_f	.99	.98	.95	.90	.80	.70	.50	.30	.20	.10	.05	.02	.01	.001
1	.00016	.00063	.0039	.016	.064	.15	.46	1.07	1.64	2.71	3.84	5.41	6.64	10.83
2	.02	.04	.10	.21	.45	.71	1.39	2.41	3.22	4.60	5.99	7.82	9.21	13.82
3	.12	.18	.35	.58	1.00	1.42	2.37	3.66	4.64	6.25	7.82	9.84	11.34	16.27
4	.30	.43	.71	1.06	1.65	2.20	3.36	4.88	5.99	7.78	9.49	11.67	13.28	18.46
5	.55	.75	1.14	1.61	2.34	3.00	4.35	6.06	7.29	9.24	11.07	13.39	15.09	20.52
6	.87	1.13	1.64	2.20	3.07	3.83	5.35	7.23	8.56	10.64	12.59	15.03	16.81	22.46
7	1.24	1.56	2.17	2.83	3.82	4.67	6.35	8.38	9.80	12.02	14.07	16.62	18.48	24.32
8	1.65	2.03	2.73	3.49	4.59	5.53	7.34	9.52	11.03	13.36	15.51	18.17	20.09	26.12
9	2.09	2.53	3.32	4.17	5.38	6.39	8.34	10.66	12.24	14.68	16.92	19.68	21.67	27.88
10	2.56	3.06	3.94	4.86	6.18	7.27	9.34	11.78	13.44	15.99	18.31	21.16	23.21	29.59
11	3.05	3.61	4.58	5.58	6.99	8.15	10.34	12.90	14.63	17.28	19.68	22.62	24.72	31.26
12	3.57	4.18	5.23	6.30	7.81	9.03	11.34	14.01	15.81	18.55	21.03	24.05	26.22	32.91
13	4.11	4.76	5.89	7.04	8.63	9.93	12.34	15.12	16.98	19.81	22.36	25.47	27.69	34.53
14	4.66	5.37	6.57	7.79	9.47	10.82	13.34	16.22	18.15	21.06	23.68	26.87	29.14	36.12
15	5.23	5.98	7.26	8.55	10.31	11.72	14.34	17.32	19.31	22.31	25.00	28.26	30.58	37.70
16	5.81	6.61	7.96	9.31	11.15	12.62	15.34	18.42	20.46	23.54	26.30	29.63	32.00	39.29
17	6.41	7.26	8.67	10.08	12.00	13.53	16.34	19.51	21.62	24.77	27.59	31.00	33.41	40.75
18	7.02	7.91	9.39	10.86	12.86	14.44	17.34	20.60	22.76	25.99	28.87	32.35	34.80	42.31
19	7.63	8.57	10.12	11.65	13.72	15.35	18.34	21.69	23.90	27.20	30.14	33.69	36.19	43.82
20	8.26	9.24	10.85	12.44	14.58	16.27	19.34	22.78	25.04	28.41	31.41	35.02	37.57	45.32
21	8.90	9.92	11.59	13.24	15.44	17.18	20.34	23.86	26.17	29.62	32.67	36.34	38.93	46.80
22	9.54	10.60	12.34	14.04	16.31	18.10	21.24	24.94	27.30	30.81	33.92	37.66	40.29	48.27
23	10.20	11.29	13.09	14.85	17.19	19.02	22.34	26.02	28.43	32.01	35.17	38.97	41.64	49.73
24	10.86	11.99	13.85	15.66	18.06	19.94	23.34	27.10	29.55	33.20	36.42	40.27	42.98	51.18
25	11.52	12.70	14.61	16.47	18.94	20.87	24.34	28.17	30.68	34.38	37.65	41.57	44.31	52.62
26	12.20	13.41	15.38	17.29	19.82	21.79	25.34	29.25	31.80	35.56	38.88	42.86	45.64	54.05
27	12.88	14.12	16.15	18.11	20.70	22.72	26.34	30.32	32.91	36.74	40.11	44.14	46.96	55.48
28	13.56	14.85	16.93	18.94	21.59	23.65	27.34	31.39	34.03	37.92	41.34	45.42	48.28	56.89
29	14.26	15.57	17.71	19.77	22.48	24.58	28.34	32.46	35.14	39.09	42.56	46.69	49.59	58.30
30	14.95	16.31	18.49	20.60	23.36	25.51	29.34	33.53	36.25	40.26	43.77	47.96	50.89	59.70

Table A.8. *Pearson's χ^2 table: probability below H_0 such that $\chi^2 \geq$ Chi squared*

The critical value for the *chi-square* χ^2 distribution is found at the intersection of a column, corresponding to a given probability, and a row, corresponding to the degrees of freedom. For instance, the critical value of χ^2 with four degrees of freedom for the probability 0.25 is equal to 5.38527. This means that the surface under the density curve of χ^2 with four degrees of freedom on the left of the value 5.38527 is equal to 0.25 (to 25% of the surface).

D_f/Area	**.995**	**.990**	**.975**	**.950**	**.900**	**.750**	**.500**	.250	**.100**	**.050**	**.025**	**.010**	**.005**
1	0.00004	0.00016	0.00098	0.00393	0.01579	0.10153	0.45494	1.32330	2.70554	3.84146	5.02389	6.63490	7.87944
2	0.01003	0.02010	0.05064	0.10259	0.21072	0.57536	1.38629	2.77259	4.60517	5.99146	7.37776	9.21034	10.59663
3	0.07172	0.11483	0.21580	0.35185	0.58437	1.21253	2.36597	4.10834	6.25139	7.81473	9.34840	11.34487	12.83816
4	0.20699	0.29711	0.48442	0.71072	1.06362	1.92256	3.35669	**5.38527**	7.77944	9.48773	11.14329	13.27670	14.86026
5	0.41174	0.55430	0.83121	1.14548	1.61031	2.67460	4.35146	6.62568	9.23636	11.07050	12.83250	15.08627	16.74960
6	0.67573	0.87209	1.23734	1.63538	2.20413	3.45460	5.34812	7.84080	10.64464	12.59159	14.44938	16.81189	18.54758
7	0.98926	1.23904	1.68987	2.16735	2.83311	4.25485	6.34581	9.03715	12.01704	14.06714	16.01276	18.47531	20.27774
8	1.34441	1.64650	2.17973	2.73264	3.48954	5.07064	7.34412	10.21885	13.36157	15.50731	17.53455	20.09024	21.95495
9	1.73493	2.08790	2.70039	3.32511	4.16816	5.89883	8.34283	11.38875	14.68366	16.91898	19.02277	21.66599	23.58935
10	2.15586	2.55821	3.24697	3.94030	4.86518	6.73720	9.34182	12.54886	15.98718	18.30704	20.48318	23.20925	25.18818
11	2.60322	3.05348	3.81575	4.57481	5.57778	7.58414	10.34100	13.70069	17.27501	19.67514	21.92005	24.72497	26.75685
12	3.07382	3.57057	4.40379	5.22603	6.30380	8.43842	11.34032	14.84540	18.54935	21.02607	23.33666	26.21697	28.29952
13	3.56503	4.10692	5.00875	5.89186	7.04150	9.29907	12.33976	15.98391	19.81193	22.36203	24.73560	27.68825	29.81947
14	4.07467	4.66043	5.62873	6.57063	7.78953	10.16531	13.33927	17.11693	21.06414	23.68479	26.11895	29.14124	31.31935
15	4.60092	5.22935	6.26214	7.26094	8.54676	11.03654	14.33886	18.24509	22.30713	24.99579	27.48839	30.57791	32.80132
16	5.14221	5.81221	6.90766	7.96165	9.31224	11.91222	15.33850	19.36886	23.54183	26.29623	28.84535	31.99993	34.26719
17	5.69722	6.40776	7.56419	8.67176	10.08519	12.79193	16.33818	20.48868	24.76904	27.58711	30.19101	33.40866	35.71847
18	6.26480	7.01491	8.23075	9.39046	10.86494	13.67529	17.33790	21.60489	25.98942	28.86930	31.52638	34.80531	37.15645
19	6.84397	7.63273	8.90652	10.11701	11.65091	14.56200	18.33765	22.71781	27.20357	30.14353	32.85233	36.19087	38.58226
20	7.43384	8.26040	9.59078	10.85081	12.44261	15.45177	19.33743	23.82769	28.41198	31.41043	34.16961	37.56623	39.99685
21	8.03365	8.89720	10.28290	11.59131	13.23960	16.34438	20.33723	24.93478	29.61509	32.67057	35.47888	38.93217	41.40106
22	8.64272	9.54249	10.98232	12.33801	14.04149	17.23962	21.33704	26.03927	30.81328	33.92444	36.78071	40.28936	42.79565
23	9.26042	10.19572	11.68855	13.09051	14.84796	18.13730	22.33688	27.14134	32.00690	35.17246	38.07563	41.63840	44.18128
24	9.88623	10.85636	12.40115	13.84843	15.65868	19.03725	23.33673	28.24115	33.19624	36.41503	39.36408	42.97982	45.55851
25	10.51965	11.52398	13.11972	14.61141	16.47341	19.93934	24.33659	29.33885	34.38159	37.65248	40.64647	44.31410	46.92789
26	11.16024	12.19815	13.84390	15.37916	17.29188	20.84343	25.33646	30.43457	35.56317	38.88514	41.92317	45.64168	48.28988
27	11.80759	12.87850	14.57338	16.15140	18.11390	21.74940	26.33634	31.52841	36.74122	40.11327	43.19451	46.96294	49.64492
28	12.46134	13.56471	15.30786	16.92788	18.93924	22.65716	27.33623	32.62049	37.91592	41.33714	44.46079	48.27824	50.99338
29	13.12115	14.25645	16.04707	17.70837	19.76774	23.56659	28.33613	33.71091	39.08747	42.55697	45.72229	49.58788	52.33562
30	13.78672	14.95346	16.79077	18.49266	20.59923	24.47761	29.33603	34.79974	40.25602	43.77297	46.97924	50.89218	53.67196

Table A.9. *Table of critical values for the chi-square distribution from the χ^2 test*

Graphs

There are many ways to graphically represent the distribution laws – in our case, the exponential law and the log–normal law. The graph for the Weibull distribution with two and three parameters is shown as follows:

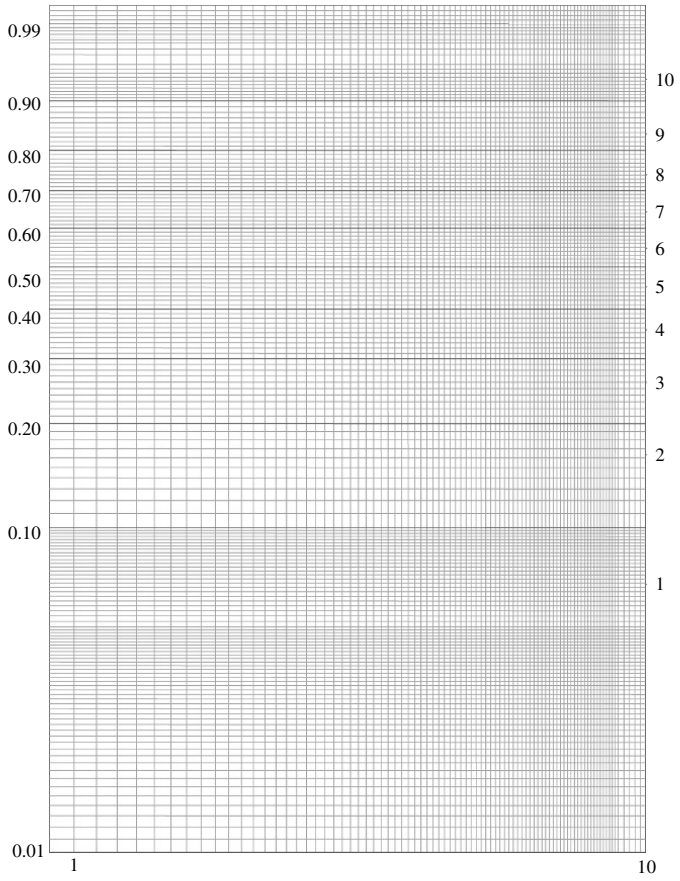


Figure A.1. *Graph of 3-parameter Weibull distribution*

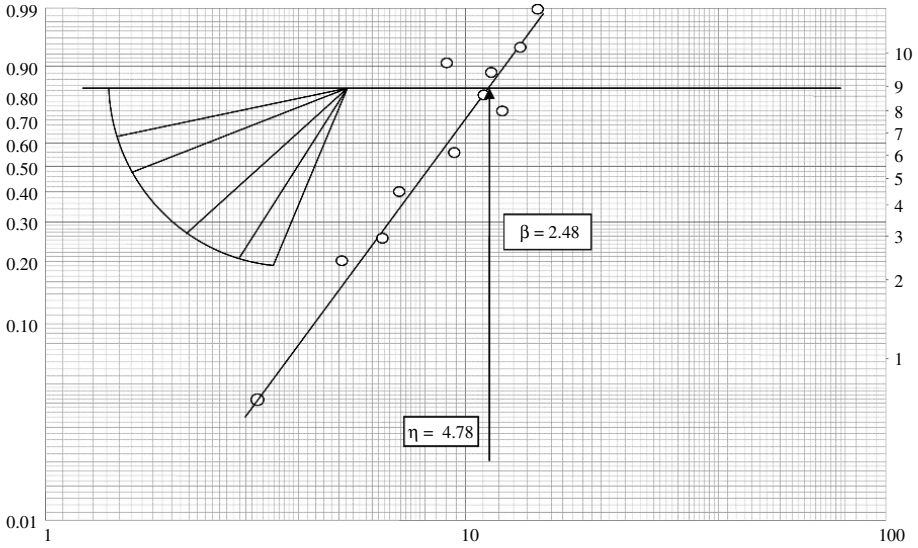


Figure A.2. *Allan plait graph of 2-parameter Weibull distribution*

Reliability

The diagrammatic representation of M. Hohenbichler method demonstrates a high degree of precision in certain cases where the analytical expression of ϕ_m is known. The basics on the formulations are to be found in the book written by H. Hohenbichler and R. Rackwitz [HOH 87]. Comparison of the FORM method and the exact solutions of the multi-normal integral for equiprobable and equicorrelated components of the various β and ρ are usually in the figure from the paper by M. Hohenbichler and R. Rackwitz, which can easily be found in reference [HOH 82].

Main regulations, codes, and technical norms:

Rabaska Study Project of the impact on the environment (Rabaska Project: Environmental impact study). Volumes 2 and 3 (see English version)

CANADA-USA-EUROPE. (Appendix K 18, January 2006)

The Rabaska deepwater terminal is designed in accordance with the regulations, codes, and norms of Quebec and Canada, and with the American and International norms and recommendations listed below.

Reference/Acronym	Title	Origin
CSA ACNOR	Canadian Standards Association *Association Canadienne de Normalisation*	Canada (E) Canada (F)
CNRC	*Conseil national de recherches Canada* (Canadian National Research Council)	Canada
BNQ	*Bureau de normalisation du Québec* (Quebec Standards Bureau)	Canada-Québec
CISC	*Institut canadien de la construction en acier* (Canadian Institute for Steel Construction)	Canada
CGA (ACG, French)	*Association canadienne du gaz* (Canadian Gas Association)	Canada
ACI	American Concrete Institute	USA
AGA	American Gas Association	USA
ANSI	American National Standards Institute	USA
API	American Petroleum Institute	USA
ASCE	American Society of Civil Engineers	USA
ASME	American Society of Mechanical Engineers	USA
ASTM	American Society for Testing Materials	USA
BS	British Standards	UK
AFNOR	*Association Française de Normalisation* (French Standards Association)	France

Table A.10. *References/Acronyms*

Earthquake resistance

The terminal's anti-earthquake design also conforms to the prescriptions of Canada's national building code: based on one earthquake which does not disrupt **normal service** (Operating Basis Earthquake) and another which provokes a **safe shutdown of the terminal**. The following table presents the various norms regarding how these two reference earthquakes are to be quantified.

Norm-Ref	**Operating basis earthquake**	**Safe shutdown earthquake**
CSA Z276-01	SSN (*Séisme de service normal*) = Operating Basis Earthquake Earthquake with a 10% probability of being exceeded in 50 years, equating to a recurrence period of 475 years.	SAS (*Séisme d'arrêt sécuritaire*) = Safe shutdown earthquake Earthquake with a probability of being exceeded no greater than 0.1% per year, equating to a recurrence period of 1,000 years.
NFPA 59A-2001	OBE (Operating Basis Earthquake) Earthquake with a 10% probability of being exceeded in 50 years, equating to a recurrence period of 475 years.	SSE (Safe Shutdown Earthquake) Earthquake with a probability of being exceeded not greater than 1% per 50 years, equating to a recurrence period of 4,975 years.
EN 1473:1997	OBE (Operating Basis Earthquake) Earthquakes with a recurrence period equal to 475 years.	SSE (Safe Shutdown Earthquake) Earthquakes with a recurrence period equal to 10,000 years.
Rabaska	Application of the 3 norms. Recurrence period of 475 years	Application of the European norm. Recurrence period of 10,000 years
Emergency shutdown system		
CSA Z276	The Canadian standard prescribes: – An emergency shutdown system (ESS) to isolate or close off sources of flammable liquids and shut down machinery that would prolong or aggravate the emergency situation. The control may be manual or automatic. – That the installation be put in safety mode in case of a failure in the electricity or air supply.	
NFPA 59A	Idem CSA.	
EN 1473 Europe	The norm describes the control systems and defines the role of emergency shut down (ESD). The presence of an emergency electrogenic group is required.	
Rabaska	Application of the prescriptions of all 3 norms.	

Table A.11. *Relevant international norms (Canada and elsewhere)*

Regulations, codes, and norms (Canada, Quebec, and USA):

Title	Edition	Reference
Normes: Ouvrages routiers: Tome III – Ouvrage d'art et Tome VIII – Matériaux		Ministère des Transports du Québec
Règlement sur les mécaniciens de machines fixes	2004	M-6, r.1
Handbook of Steel Construction (8th Ed)	2004	CISC
Code national du bâtiment du Canada, (modifié pour le Québec)	2005	CNBC
Code national de la plomberie du Canada	1995	CNRC-NRC
Galvanisation à chaud des objets de forme irrégulière	2003	CAN/CSA-G164-FM92
Règles de calcul des charpentes en bois	2001	CAN/CSA-O86-F01
Règles de calcul aux états limites des charpentes en acier	2001	CAN/CSA-S16-F01
Code canadien sur le calcul des ponts routiers	2005	CAN/CSA-S6-00
Commentary on CAN/CSA-S6-00 Canadian Highway Bridge – Design Code	2000	CSA-S6.1-00
Règles de calculs aux états limites des charpentes en acier	1994	CSA S16.1
Strength design in aluminium	2005	CAN3-S157-05
Béton: constituants et exécution des travaux/Essais concernant le béton	2000	CSA A23.1-F00/ A23.2-F00
Calcul des ouvrages en béton	2004	CSA A23.3-04
Acier de construction	2004	CSA G40.21- F004
Métaux d'apport et matériaux associés pour le soudage à l'arc	2001	CSA W48 - F01
Construction soudée en acier (soudage à l'arc)	2003	CSA W59-F03
Exigences générales, critères de calcul, conditions environnementales et charges.	2004	CSA S471-F04
Barres d'acier en billettes pour l'armature du béton	2002	CSA-G30.18-M92
Metric Building Code Requirement for Structural Concrete	2002	ACI 318M
Guide for the design and construction of fixed offshore concrete structures.	1984	ACI 357 R-84
State-of-the-art report on concrete structures for the Arctic.	1997	ACI 375.1R91
Planning and Designing and Constructing Fixed Offshore Structures in Ice Environments.	1982	API Bulletin 2N
Welding of Pipe Lines and Related Facilities (19th ed) and Errata	1999 2001	API 1104
Flanged Safety Relief Valves (5th Ed)	2002	API 526

Table A.12. *Regulations, codes, and norms (Canada, USA, and Europe)*

Title	Edition	Reference
Steel Gate Valves – Flanged and Butt Welding Ends (11th ed)	2001	API 600
Corrosion Resistant, Bolted Bonnet Gate Valves – Flanged and Butt-Welding Ends (6th ed)	2001	API 603
Minimum Design Loads for Buildings & Other Structures	2002	ASCE 7
Unified Inch Screw threads (UN and UNR Thread Form)	1989 R2001	ASME B1.1
Pipe Threads, General Purpose (inch)	1983 R2001	ASME B1.20.1
Safety Standard for Mechanical Power Transmission Apparatus	2000	ASME B15.1
Forged Fittings, socket-welding and Threaded	2001	ASME B16.11
Butt welding Ends	1997	ASME B16.25
Surface Texture, Surface Roughness, Waviness, and Lay	2002	ASME B46.1
NDT Procedures	Latest	ASME Section 5
Quality Control Standard for Control Valve Seat Leakage	2003	ASME/FCI-70-2
Appropriate Materials Standards	Latest	ASTM
Rules for the design, construction and inspection of offshore structures. Det norske Veritas, Hovik, Norway	1977	Det norske Veritas
Systèmes de management de la qualité – Exigences	2000	ISO 9001
Systèmes de management de la qualité – Principes essentiels et vocabulaire		ISO 9000
Unités SI et recommandations pour l'emploi de leurs multiples et de certaines autres unités	2000	ISO 1000

Table A.12 (continued). *Regulations, codes, and norms (Canada, USA, and Europe)*

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