Günter Harder

Lectures on Algebraic Geometry I

Sheaves, Cohomology of Sheaves, and Applications to Riemann Surfaces

2nd Edition

VIEWEG+TEUBNER

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Preface

I want to begin with a defense or apology for the title of this book. It is the first part of a two volume book. The two volumes together are meant to serve as an introduction into modern algebraic geometry. But about two thirds of this first volume concern homological algebra, cohomology of groups, cohomology of sheaves and algebraic topology. These chapters 1 to 4 are more an introduction into algebraic topology and homological algebra than an introduction into algebraic geometry. Only in the last Chapter 5 we will see some algebraic geometry. In this last chapter we apply the results of the previous sections to the theory of compact Riemann surfaces. Even this section does not look like an introduction into modern algebraic geometry, large parts of the material covered looks more like 19'th century mathematics. But historically the theory of Riemann surfaces is one of the roots of algebraic geometry.
We will prove the Riemann-Roch theorem and we will discuss the structure of the divisor class group. These to themes are ubiquitous in algebraic geometry. Finally I want to say that the theory of Riemann surfaces is also in these days a very active area, it plays a fundamental role in recent developments. The moduli space of Riemann surfaces attracts the attention of topologists, number theorists and of mathematical physicists. To me this seems to be enough justification to begin an introduction to algebraic geometry by discussing Riemann surfaces at the beginning.

Only in the second volume we will lay the foundations of modern algebraic geometry. We introduce the notion of schemes, I discuss the category of schemes, morphisms and so on. But as we proceed the concepts of sheaves, cohomology of sheaves and homological algebra, which we developed in this first volume, will play a predominant role. We will resume the discussion of the Riemann-Roch theorem and discuss the Picard group or jacobians of curves.

A few more words of defense. These books grew out of some series of lectures, which I gave at the university of Bonn. The first lectures I gave were lectures on cohomology of arithmetic groups and it was my original plan to write a book on the cohomology of arithmetic groups. I still have the intention to do so. Actually there exists a first version of such a book. It consists of a series of notes taken from a series of lectures I gave on this subject. Arithmetic groups Γ are groups of the form $\Gamma = \mathrm{SL}_n(\ \ ) \subset \mathrm{SL}_n(\ \ )$ or the symplectic group $\Gamma = \mathrm{Sp}_n(\ \ ) \subset \mathrm{Sp}_n(\ \ )$(See 5.2.24). These groups act on the symmetric spaces $X = G(\ \ )/K_\infty$ and the quotient spaces $\Gamma\backslash X$. The representations of the algebraic group G define sheaves $\widetilde{M}$ on this space and the cohomology groups $H^\bullet(\Gamma\backslash X,\widetilde{M})$ will be investigated in this third volume. Again the results in the first four chapters of the first volume will be indispensable.

But in this third volume we will also need some background in algebraic geometry. In some cases the quotient spaces $\Gamma\backslash X$ carry a complex structures, these are the Shimura varieties. Then it is important to know, that these quotients are actually quasi projective algebraic varieties and that they are defined over a much smaller field, namely a number field. To understand, why this is so, we interpret this spaces as parameter spaces of certain algebraic objects, i.e. they turn out to be " moduli spaces", especially the moduli spaces of abelian varieties. This last subject is already briefly touched in this first volume and will be resumed in the second and third volume.

Perhaps this is the right moment to confess that I consider myself as a number theorist. Number theory is a broad field and for the kind of questions, I am interested in, the methods and concepts algebraic geometry, cohomology of arithmetic groups, the theory of automorphic forms are essential. Therefore it is my hope that these three volumes together can serve as an introduction into an interesting branch of mathematics.

This book is addressed to students who have some basic knowledge in analysis, algebra and basic set theoretic topology. So a student at a German university can read it after the second year at the university.

I want to thank my former student Dr. J. Schlippe, who went through this manuscript many times and found many misprint and suggested many improvements. I also thank J. Putzka who "translated" the original Plain-Tex file into Latex and made it consistent with the demands of the publisher. But he also made many substantial suggestions concerning the exposition and corrected some errors.

Günter Harder Bonn, December 2007

Preface to the second edition

In the meantime the second volume of this book appeared and the publisher decided to prepare a second edition of this first volume.

For this new edition I corrected a few misprints and modified the exposition at some places. I also added a short section on moduli of elliptic curves with N-level structures. Here I followed closely the presentation of this subject in the Diploma thesis of my former student Christine Heinen.

This new paragraph anticipates some of the techniques of volume II. I originally planned to include it into the second Volume. Since I already had a section on moduli of elliptic curves with a differential and since the second volume became too long I abandoned this plan. Therefore, I was quite happy when I got the opportunity to include this section into the second edition of the first volume. It also helps a little bit to keep the balance between the two volumes. This moduli space and some generalizations of it will play a role in my book on "Cohomology of arithmetic groups".

Günter Harder Bonn, June 2011

Contents

Introduction

This first volume starts starts with a very informal introduction into category theory. It continues with an introduction into homological algebra. In view of the content of the third volume Chapter 2 is an introduction into homological algebra based on the example of cohomology of groups.

Chapter 3 introduces into the theory of sheaves. The role of sheaves is twofold: They allow us to formulate the concepts of manifolds as locally ringed spaces ($\mathcal{C}^\infty$-manifolds, complex manifolds, algebraic manifolds...); this is discussed in section 3.2. The concept of locally ringed space will be indispensable when we introduce the concept of schemes in the second volume.

The second role is played by the cohomology of sheaves which is covered in Chapter 4. My original notes gave only a very informal introduction into sheaf cohomology, but after a while I felt the desire to give a rather self contained account. So it happened that the introduction into sheaf cohomology became rather complete up to a certain level. I included spectral sequences, the cup product and the Poincaré duality of local systems on manifolds. I also discuss intersection products and the Lefschetz fixed point formula for some special cases. So it happened that Chapter 4 became very long and it has several subsections. Up to Chapter 4.7 the book may serve as an introduction into algebraic topology but with a strong focus on applications to algebraic geometry and to the cohomology of arithmetic groups. The discussion of singular homology is rather short.

In the final sections of Chapter 4 I discuss the analytic methods in the study of cohomology of manifolds. I discuss the de Rham isomorphism, which gives a tool to understand the cohomology of local systems. In analogy to that the Dolbeault isomorphism gives us an instrument to investigate the cohomology of holomorphic bundles on complex manifolds. Finally I explain the basic ideas of Hodge theory. Only in the section on Hodge theory I need to refer to some analytical results which are not proved in this book.

The last chapter 5 we apply these results and concepts to the theory of compact Riemann surfaces. In the first section of Chapter 5 we prove the theorem of Riemann-Roch. We want to make it clear that the hardest part in the proof of the theorem of Riemann-Roch is the finite dimensionality of some cohomology groups and this proof requires some difficult analysis. We also give some indications how these analytic results can be proved in our special case. From the theorem of Riemann-Roch it follows, that Riemann surfaces may be viewed as purely algebraic objects, we prove that they are smooth projective algebraic curves. At this point we see some concepts of commutative algebra entering the stage. They will be discussed in more detail in volume II. We discuss Abel's theorem which explains the structure of the divisor class group. It turns out that the group of divisor classes of degree zero is a complex torus with a principal polarization (Riemann Period relations), this says that it is an abelian variety over $\mathbb{C}$.

In the second section of Chapter 5 we discuss the meaning of this fact. We examine line bundles on these Jacobians and more general line bundles on abelian varieties. Especially we describe the spaces of sections of line bundles in terms of spaces of theta-series. We also explain in a very informal way the relationship to the moduli spaces of principally polarized abelian varieties. I also have a section on the theory of Jacobi-Theta-functions. This is the one dimensional case. It illustrates the connections to very old and classical mathematics. But in the back of my mind I see this also as a preparation for the book on cohomology of arithmetic groups. To say this differently, we see the connections between the moduli spaces of abelian varieties and the theory of modular forms.

This last chapter goes beyond homological algebra and algebraic topology. But it shows the enormous usefulness of these concepts. Chapter 5 can also be seen as a preparation for the second volume, which is an introduction into algebraic geometry. In the second half of Chapter 5 we discuss the structure of Jacobians, their Neron-Severi groups and the structure of endomorphism rings. These arguments and methods will appear again in the second volume, when we discuss the Jacobians of curves over arbitrary fields. In the last section of this first volume we give some outlook on celebrated results, which will also not be proved in the second volume, but for whose proof we provide some preparation.

1 Categories, Products, Projective and Inductive Limits

1.1 The Notion of a Category and Examples

I want to give a very informal introduction to the theory of categories. The main problem for a beginner is to get some acquaintance with the language and to get used to the abstractness of the subject. As a general reference I give the book [McL].

Definition 1.1.1. *A* **category** $\mathcal{C}$ *is*

(i) *a collection of objects* $\mathrm{Ob}(\mathcal{C})$.

We do not insist that this collection is a set. For me this means that we do not have the notion of equality of two objects. If we write $N \in \mathrm{Ob}(\mathcal{C})$ *then we mean that* N *is an object in the category* $\mathcal{C}$.

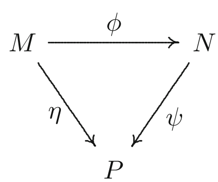
(ii) *To any two objects* $N,M \in \mathrm{Ob}(\mathcal{C})$ *there is attached a set* $\mathrm{Hom}_{\mathcal{C}}(N,M)$ *which is called the set of* **morphisms** *between these two objects.*

Usually we denote a morphism $\phi \in \mathrm{Hom}_{\mathcal{C}}(N,M)$ *by an arrow* $\phi : N \longrightarrow M$.

(iii) *For any three objects* N,M,P *we have the* **composition** *of morphisms*

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(N,M) \times \mathrm{Hom}_{\mathcal{C}}(M,P) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(N,P) \\ (\phi,\psi) & \longmapsto & \psi \circ \phi. \end{array}$$

If a morphism η *is a composition of* ϕ *and* ψ *then we denote this by a* **commutative diagram** *(or commutative triangle)*



We require that this composition is associative in the obvious sense (if we have four objects...). The reader should verify that this associativity can be formulated in terms of a tetrahedron all of whose four sides are commutative triangles. Here we use that the morphisms between objects form a set. In a set we know what equality between elements means.

(iv) *For any object* $N \in \mathrm{Ob}(\mathcal{C})$ *we have a distinguished element* $\mathrm{Id}_N \in \mathrm{Hom}_{\mathcal{C}}(N,N)$, *which is an identity on both sides under the composition.*

G. Harder, *Lectures on Algebraic Geometry I*, DOI 10.1007/978-3-8348-8330-8_1,

Everybody has seen the following categories

Example 1. *The category* **Ens** *of all sets where the arrows are arbitrary maps.*

Example 2. *The category* $\mathbf{Vect}_k$ *of vector spaces over a given field* k *where the sets of morphisms are the* k*-linear maps.*

Example 3. *The category* $\mathbf{Mod}_A$ *of modules over a ring* A *where the morphisms are* A*-linear maps. We also have the category of abelian groups* **Ab***, the category* **Groups** *of all groups where the morphisms are the homomorphisms of groups.*

Example 4. *The category* **Top** *of topological spaces where the morphisms are the continuous maps.*

I said in the beginning that we do not have the notion of equality of two objects M,N in a category. But we we can say that two objects $N,M \in \mathrm{Ob}(\mathcal{C})$ are **isomorphic.** This means that we can find two arrows $\phi : N \longrightarrow M$ and $\psi : M \longrightarrow N$ such that $\mathrm{Id}_N = \psi \circ \phi, \mathrm{Id}_M = \phi \circ \psi$. But in general it may be possible to find many such isomorphisms between the objects and hence we have many choices to identify them. Then it is better to refrain from considering them as equal.
For instance we can consider the category of finite dimensional vector spaces over a field k. Of course two such vector spaces are isomorphic if they have the same dimension. Since we may have many of these isomorphisms, we do not know how to identify them and therefore the notion of equality does not make sense.
But if we consider the category of framed finite dimensional $k-$vector spaces, i.e. vector spaces V equipped with a basis which is indexed by the numbers $1,2,\dots,n = \dim(V)$. Now morphisms which are linear maps which send basis elements to basis elements and which respect the ordering. Then the situation is different. We can say the objects form a set: If two such objects are isomorphic then the isomorphism is unique.

It is important to accept the following fact: The axioms give us a lot of flexibility, at no point we require that the elements in $\mathrm{Hom}_{\mathcal{C}}(N,M)$ are actual maps between sets (with some additional structure). Insofar all the above examples are somewhat misleading.
A simple example of a situation where the arrows are not maps is the following one:

Example 5. *We may start from a an ordered set* $\mathcal{I} = (I, \leq)$ *and we consider its elements as the objects of a category. For any pair* $i,j \in I$ *we say that* $\mathrm{Hom}_{\mathcal{I}}(i,j)$ *consists of one single element* $\phi_{i,j}$ *if* $i \leq j$ *and is empty otherwise. The composition is the obvious one obtained from the transitivity of the order relation.*

The reader may say that this is not a good example, because the $\phi_{i,j}$ can be considered as maps between the two sets $\{i\},\{j\}$ but that is the wrong point of view. To make this clear we can also construct a slightly different category $\mathcal{J}$ from our ordered set. We assume that the order relation satisfies $i \leq j$ and $j \leq i$ implies $i = j$ and hence we can define $i < j$ by $i \leq j$ and $i \neq j$. Then we may define the sets of morphisms as:

$$\mathrm{Hom}_{\mathcal{J}}(i,j) \text{ are finite sequences } \{i_0,i_1,\dots,i_n\} \text{ with } i_\nu < i_{\nu+1} \text{ and } i = i_0, j = i_n.$$

These sequences form a set. We leave it to the reader to verify that we have a composition and an identity. Now we may have many arrows between two objects $\{i\},\{j\}$ which are sets consisting of one element.

We may also do the following which may look strange at the first glance. If we have a category $\mathcal{C}$ we may revert the arrows and form the so called **opposite** category $\mathcal{C}^{\text{opp}}$ which has the same objects but where

$$\operatorname{Hom}_{\mathcal{C}^{\text{opp}}}(N,M) = \operatorname{Hom}_{\mathcal{C}}(M,N). \tag{1.1}$$

1.2 Functors

We need the notion of a **functor** F from one category $\mathcal{C}$ to another category $\mathcal{C}'$. A functor is a rule that transforms an object $N \in \operatorname{Ob}(\mathcal{C})$ into an object $F(N) \in \operatorname{Ob}(\mathcal{C}')$ and for any two objects $N,M \in \operatorname{Ob}(\mathcal{C})$ it provides maps

$$F_{N,M} : \operatorname{Hom}_{\mathcal{C}}(N,M) \longrightarrow \operatorname{Hom}_{\mathcal{C}'}(F(N),F(M)).$$

In other words: For any $\phi : N \longrightarrow M$ the functor produces an arrow

$$F_{N,M}(\phi) = F(\phi) : F(N) \longrightarrow F(M)$$

and this production should satisfy the obvious consistency conditions, namely respect identity elements and composition. Such an F together with the collection of maps between the sets of morphisms is called a **covariant** functor because direction of the arrows is preserved. We also have the notion of a **contravariant** functor from $\mathcal{C}$ to $\mathcal{C}'$ which turns the arrows backwards or what amounts to the same: it is a functor from the opposite category $\mathcal{C}^{\text{opp}}$ to $\mathcal{C}'$.
Any object X of a category defines functors from this category to the category **Ens**: We attach to it the covariant functor

$$h_X(Z) = \operatorname{Hom}_{\mathcal{C}}(X,Z).$$

If we have two objects Z,Z' and $\psi : Z \longrightarrow Z'$ then the composition produces $h_X(\psi) : \operatorname{Hom}_{\mathcal{C}}(X,Z) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(X,Z')$ which sends $\phi : X \longrightarrow Z$ to $\psi \circ \phi$. We may also put X into the second free place in the $\operatorname{Hom}_{\mathcal{C}}(\ ,\ )$ and consider $h^\circ_X(Z) = \operatorname{Hom}_{\mathcal{C}}(Z,X)$. This gives us a contravariant functor.

Example 6. *We have a contravariant functor from the category of vector spaces into itself: We send a vector space $V \in \operatorname{Ob}(\mathbf{Vect}_k)$ to its dual space $V^\vee = \operatorname{Hom}_k(V,k)$.*

Example 7. *A very clever example of a functor is the homology of a topological space (see [Ei-St]Chap. IV 8.4.1.): To any topological space X (i.e an object in the category* **Top***) we may attach the homology groups $H_0(X,\ ),H_1(X,\ ),\dots,H_i(X,\ ),\dots$ the indices run over all integers ≥ 0. These homology groups are abelian groups which depend functorially on the space X: A continuous map*

$$f : X \longrightarrow Y$$

between spaces induces a homomorphism between their homology groups

$$f_i : H_i(X,\ ) \longrightarrow H_i(Y,\ ) \qquad \text{for all indices } i.$$

This functor transforms a very complicated object -a topological space- into a simpler but not too simple object namely a family of abelian group. This can be used to prove that ^{n} *is not homeomorphic (not isomorphic in the category* **Top***) to* ^{m} *if* $n \neq m$. *To see this we remove the origin from* ^{n} *and from* ^{m} *and we will see that the resulting spaces will have non-isomorphic homology groups if* $n \neq m$. *(4.4.5). On the other hand if we had a homeomorphism between the two spaces we could arrange that it maps the origin to the origin. Hence we would get a homeomorphism between the modified spaces which then must induce isomorphisms on the homology groups and this is impossible.*
If I am right then these homology groups are historically the first examples where the concept of functors has been used.

We will see many more interesting functors in Chapter 2 on homological algebra.

1.3 Products, Projective Limits and Direct Limits in a Category

1.3.1 The Projective Limit

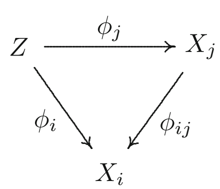
Let us assume that we have a category $\mathcal{C}$ and an ordered set $\mathcal{I} = (I, \leq)$. Furthermore we assume that to any $i \in I$ we have attached an object $X_i \in \mathrm{Ob}(\mathcal{C})$ and for any pair $i \leq j$ of indices we have an arrow $\phi_{ij} \in \mathrm{Hom}_{\mathcal{C}}(X_j, X_i)$. We assume that always $\phi_{ii} = \mathrm{Id}_{X_i}$ and for any triple $i \leq j \leq j'$ we have

$$\phi_{ij} \circ \phi_{jj'} = \phi_{ij'}. \tag{1.2}$$

We have seen in Example 7 that we may consider our ordered set $(I, \leq)$ as a category $\mathcal{I}$. Then we can summarize our assumptions by saying that $i \longrightarrow X_i$ is a contravariant functor from the category $\mathcal{I}$ to the category $\mathcal{C}$.
Such a family $(\{X_i\}_{i \in I}, \phi_{ij})$ is called a **projective system** or sometimes **inverse system** of objects in $\mathcal{C}$. For any object $Z \in \mathrm{Ob}(\mathcal{C})$ we define a set $\mathrm{Hom}_{\mathcal{C}}(Z, (\{X_i\}_{i \in I}, \phi_{ij}))$ which consists of families $\{\phi_i\}_{\{i \in I\}}$ of morphisms

$$\phi_i : Z \longrightarrow X_i$$

such that for any pair $i \leq j$ the diagram



commutes. It is clear that

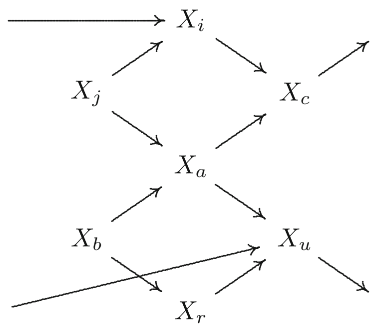
$$Z \longrightarrow \mathrm{Hom}_{\mathcal{C}}(Z, (\{X_i\}_{i \in I}, \phi_{ij}))$$

is a contravariant functor from $\mathcal{C}$ to **Ens** : A morphism $\phi : Z' \longrightarrow Z$ induces a map

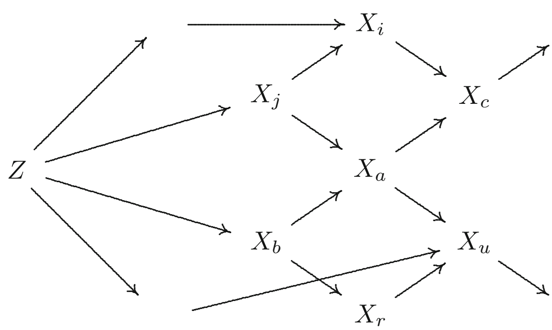
$$\mathrm{Hom}_{\mathcal{C}}(Z, (\{X_i\}_{i \in I}, \phi_{ij})) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(Z', (\{X_i\}_{i \in I}, \phi_{ij}))$$

which is induced by the composition.

We should think of $(\{X_i\}_{i\in I}, \phi_{ij})$ as a huge diagram



where we did not draw the compositions because they are redundant and make the picture complicated. Then an element $\phi \in \mathrm{Hom}_{\mathcal{C}}(Z, (\{X_i\}_{i\in I}, \phi_{ij}))$ is a system of arrows $\{\varphi_\nu : Z \longrightarrow X_\nu\}_{\nu\in I}$ into this diagram:



so that every diagram induced by a $i \le j$ commutes. Again we suppressed the compositions.
Question: In the special diagram, are the two arrows from Z to X_j and X_b arbitrary or are there constraints? If so, what kinds of constraints are there?

Definition 1.3.1 (Projective Limit)**.** *An object $P \in \mathcal{C}$ together with an element $\Phi \in \mathrm{Hom}_{\mathcal{C}}(P, (\{X_i\}_{i\in I}, \phi_{ij}))$ is called a* **projective limit** *of the system $(\{X_i\}_{i\in I}, \phi_{ij}))$ if for any $Z \in \mathrm{Ob}(\mathcal{C})$ the map*

$$\begin{array}{rcl} \mathrm{Hom}_{\mathcal{C}}(Z,P) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(Z, (\{X_i\}_{i\in I}, \phi_{ij})) \\ \psi & \longmapsto & \{\Phi_i \circ \psi\}_{\{i\in I\}} \end{array}$$

is a bijection. This is the so called **universal property** *of (P, Φ). The element Φ is called* **universal morphism***.*

In terms of our above diagrams this means that a projective limit P is an object that is squeezed between any Z and the diagram. Any ϕ from any Z into the diagram is obtained by first giving an arrow $Z \longrightarrow P$ and then composing it with the universal arrow Φ.
Such a projective limit may not exist in our category. But if it exists then this gives us a first example of a **representable functor**:

Starting from the functor $Z \longrightarrow \mathrm{Hom}_{\mathcal{C}}(Z,(\{X_i\}_{i\in I},\phi_{ij}))$ we find a P such that our functor is equivalent to the functor h_P° which we attached to P. More precisely we have a universal element $\Phi \in \mathrm{Hom}_{\mathcal{C}}(P,(\{X_i\}_{i\in I},\phi_{ij}))$ such that the equivalence of the functors is given by the universal property above (See also 1.3.4).

1.3.2 The Yoneda Lemma

We have a simple categorical argument which is called the Yoneda Lemma which shows that such a (P,Φ) - if it exists - is unique up to a **canonical isomorphism**. If we have a second pair (P',Φ') then we get from the universal property that Φ' is obtained from a uniquely defined morphism $\psi' : P' \longrightarrow P$ composed with Φ and conversely we get Φ from Φ' by composing with a unique $\psi : P \longrightarrow P'$. Finally the universal property yields that the composition $\psi' \circ \psi$ and $\psi \circ \psi'$ must be the identities.
So we can conclude: If a projective limit exists it is unique up to a canonical isomorphism and is denoted by

$$P = \varprojlim_{i\in I} X_i$$

This limit is also called the inverse limit because the arrow points backwards. We also should remember that the arrows in our system $\{X_i\}$ point from objects with a larger index to objects with smaller index. The universal morphism Φ is sometimes suppressed in the notation.
I will discuss some examples of projective limits which belong to the general education of anybody working in algebra or topology.

1.3.3 Examples

Example 8. *We consider the case where $\mathcal{C} = \mathbf{Ens}$ and the order relation on I is trivial, i.e. $i \le j$ if and only if $i = j$. Then there are no constraints between the maps*

$$\phi_i : Z \longrightarrow X_I.$$

We may take the product of these sets $P = \prod_{i\in I} X_i$ and the $\Phi_i : P \longrightarrow X_i$ are the usual set theoretic projections. Then $\{P,\Phi_i\}_{i\in I}$ is also the product in the categorical sense.

Example 9.

1. *We take the set of positive natural numbers $_+$ and we define as order relation the divisibility relation, i.e. $n \le m \Leftrightarrow n \mid m$. For any m we can define the quotient rings $/m$ and if $m \mid m'$ then we have the projection*

$$\phi_{m,m'} : \quad /m' \longrightarrow \quad /m \quad ,$$

and $\varphi_{m,m'}(x_{m'}) = x_m$ means that $x_{m'} \equiv x_m \mod m$. We can define a ring

$$\hat{} = \{(\dots,x_n,\dots)_{n\in \ +} \mid x_n \in \ /n \ , \quad x_{n'} = x_n \mod n' \textit{ if } n'|n\}$$

where addition and multiplication are taken componentwise, and we have the projection map

$$\hat{} \xrightarrow{\phi_n} \ /n \ ,$$

which is the projection to the n-th component. Then $(\hat{}\ ,\varphi_n)_{n\in \ +}$ is the projective limit in the category of rings.

2. *We may also look at the ordered set $\{p^n\}_{\{n=1,2,\ldots\}}$ where p is a prime. Then we get or $n \leq m$ the projective system*

$$/p^n \quad \longrightarrow \quad /p^m \quad \longrightarrow \ldots$$

and the projective limit

$$_p = \{(\ldots, x_n, \ldots) \mid x_m \equiv x_n \mod p^n \text{ if } n \leq m\}.$$

Each component x_n determines completely the x_m with $m \leq n$ but if we go backwards we get more and more refined information. We can put a topology onto $_p$, where a basis of open sets is given by the elements of the form $y + p^k \ _p$.

The ring $_p$ contains as a dense subring. It is a local ring without zero divisors, the unique maximal ideal is $p \ _p = (p)$. Its quotient field is the field $\mathbb{Q}_p$ of p-adic numbers. (See[Neu]Chap. II)

It follows from elementary number theory (The Chinese remainder theorem) that

$$\widehat{\ } = \prod_p \ _p.$$

This ring $\widehat{\ }$ is not integral, it has zero divisors.

Example 10. *It is not too difficult to see that in* **Ens** *projective limits exist. One simply forms the product*

$$X = \prod_{i \in I} X_i$$

and takes the subset of those elements $x = (\ldots, x_i, \ldots)_{i \in I}$ which satisfy $\phi_{ij}(x_j) = x_i$. This implies that also in such categories like the category of rings, the category of modules over a given ring products and projective limits exist.
But in the category of fields we even cannot form the product of two fields, because we cannot avoid zero divisors.

Example 11. *A very important example of a projective limit is the Galois group of a field k. We assume that we have constructed an algebraic closure $\overline{k}$ of k, this is a field with the following two properties*

(i) *Every $\alpha \in \overline{k}$ is algebraic over k, i.e. it satisfies a nontrivial equation*

$$\alpha^n + a_1 \alpha^{n-1} + \ldots + a_n = 0 \qquad \text{with } a_i \in k.$$

(ii) *The field $\overline{k}$ is algebraically closed.*

Such a field can always be constructed if we use the axiom of choice.
We have the set of finite normal extensions $k \subset K \subset \overline{k}$, this is an ordered set by inclusion. For any normal extension $k \subset k_1 \subset \overline{k}$ let $\mathrm{Gal}(k_1/k)$ be the group of automorphisms of k_1 whose restriction to k induces the identity. For a tower of finite normal extensions $k \subset K \subset L$ we have a surjective map

$$\mathrm{Gal}(L/k) \longrightarrow \mathrm{Gal}(K/k)$$

which is simply given by restriction. We can form the projective limit

$$\varprojlim_{K/k} \mathrm{Gal}(K/k)$$

of this system. It exists by the above remark. The restriction defines an isomorphism

$$\mathrm{Gal}(\overline{k}/k) \longrightarrow \varprojlim_{K/k} \mathrm{Gal}(K/k).$$

This is clear if we know that every automorphism $\sigma : K \longrightarrow K$ over k can be extended to an automorphism of the algebraic closure. (See [Neu]Chap. IV.1)

Example 12. *It is of course obvious that in the category* **Ensfin** *of finite sets we cannot have infinite products. But if we have a family $(\{X_i\}_{i\in I}, \phi_{ij})\}$ of finite sets we can form the product in* **Ens** *and we define a topology on this product. This should be the coarsest topology such that the projections*

$$p_i : \prod_{j\in I} X_j \longrightarrow X_i$$

become continuous. (On X_i we take the discrete topology, every subset is open). Hence we get a basis for the topology if we take finite intersections

$$\bigcap_{i\in E} p_i^{-1}(\{x_i\})$$

where E is finite and $x_i \in X_i$ a point.
It is not too difficult to prove that the product endowed with this topology becomes a compact space. The same holds if we take projective limits of finite sets (groups, rings,.....), these limits are compact topological spaces (groups, rings, ...). The resulting projective limits are called profinite sets (groups, rings,.....). For instance the ring

$$\widehat{\ } = \varprojlim_{m} \quad /m$$

is such a profinite ring. The Galois group $\mathrm{Gal}(\overline{k}/k)$ of a field k is a profinite group. The topology of this groups is called the **Krull topology.**

1.3.4 Representable Functors

I want to say a few words about representable functors. We discussed the example of projective limits. But the notion of representability for a functor is much more general. It may be applied to any contravariant or covariant functor which takes values in the category of sets.
If we have a contravariant functor $F : \mathcal{C} \longrightarrow$ **Ens** we may ask: Can we find an object X and an element $u \in F(X)$ such that for any $Z \in \mathrm{Ob}(\mathcal{C})$ we get a bijection

$$\mathrm{Hom}_{\mathcal{C}}(Z,X) \xrightarrow{\sim} F(Z)$$

which is given by the universal rule $\phi \longrightarrow F(\phi)(u)$?

If such an object X together with $u \in F(X)$ exists, then the Yoneda Lemma asserts that it is unique up to a canonical isomorphism. This means that the data provide a distinguished isomorphism between two solutions of the problem. The proof is basically the same as in the case of projective limits: If we have two such objects X,X' we have $\mathrm{Hom}_{\mathcal{C}}(X',X) \xrightarrow{\sim} F(X')$. Now the $u' \in F(X')$ provides a morphism in $\mathrm{Hom}_{\mathcal{C}}(X',X)$. Interchanging the two arguments gives us a morphism in the opposite direction. The compositions must be the identities.

1.3.5 Direct Limits

I begin with the simplest example. If we have a family $\{X_i\}_{i\in I}$ of sets then we can form the disjoint union

$$X = \bigsqcup_{i\in I} X_i.$$

This construction satisfies

$$\mathrm{Hom}_{\mathbf{Ens}}(\bigsqcup_{i\in I} X_i,Z) = \prod_{i\in I} \mathrm{Hom}_{\mathbf{Ens}}(X_i,Z). \tag{1.3}$$

Here is becomes clear that the formation of a disjoint union and a product are dual to each other. This means that the arrows are turned backwards. We formulate a principle:

The product *is constructed so that we know what the arrows* into it *are, the* disjoint union *so that we know what the arrows* from it *are.*

To describe inductive (or direct) limits we start again from an ordered set $(I, \leq)$. Now we consider a covariant functor which attaches to any i an $X_i \in Ob(\mathcal{C})$ and to any pair (i,j) with $i \leq j$ an element $\psi_{ij} \in \mathrm{Hom}_{\mathcal{C}}(X_i,X_j)$. So in contrast to the case of projective limits the arrows point from objects with a smaller index to objects with a larger index. Such a system (or functor) is called an **inductive system**.
This time we look at $\mathrm{Hom}_{\mathcal{C}}((\{X_i\}_{i\in I},\psi_{ij}),Z)$, these are now collections of morphisms $\psi_i : X_i \longrightarrow Z$ from the diagram to objects in $\mathcal{C}$. We say that an object L together with a map $\Psi = (\ldots,\Psi_i,\ldots) \in \mathrm{Hom}_{\mathcal{C}}((\{X_i\}_{i\in I},\phi_{ij}),L)$ is a direct limit of $(\{X_i\}_{i\in I},\psi_{ij})$ if

$$\mathrm{Hom}_{\mathcal{C}}(L,Z) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}((\{X_i\}_{i\in I},\psi_{ij}),Z), \tag{1.4}$$

where the bijection is given by the composition $\psi_i = \psi \circ \Psi_i$. If such a limit exists we write

$$L = \varinjlim_{i\in I} X_i.$$

It is clear that in the category **Ens** direct limits exist: Starting from an inductive system of sets $(\{X_i\}_{i\in I},\phi_{ij})$ we form the disjoint union $\bigsqcup_{i\in I} X_i$. We introduce an equivalence relation $\sim$ on this disjoint union. This equivalence relation will satisfy $x_i \sim x_j$ whenever $\phi_{ij}(x_i) = x_j$. This is not necessarily an equivalence relation, but we simply take the equivalence relation generated by the relation. Then it is not hard to see that the quotient of the disjoint union by this relation is a direct limit.

Definition 1.3.2. *An ordered set $(I, \leq)$ is called* **directed** *if for any two $i,j \in I$ we can find an element $l \in I$ such that $i < l,j < l$.*

If we have an inductive system of sets $(\{X_i\}_{i\in I},\phi_{ij})$ over a directed set, then the equivalence relation in our construction above can be described directly

$$x_i \sim x_j \Longleftrightarrow \exists\, l \in I \text{ s.t. } i \leq l, j \leq l \text{ and } \phi_{il}(x_i) = \phi_{jl}(x_j). \tag{1.5}$$

We may also look at the opposite case where the ordering relation on the set I is trivial, i.e. we have $i \leq j$ if and only if $i = j$. If we have an inductive system $(\{X_i\}_{i\in I},\phi_{ij})$ over such a set then the inductive limit should be called **a disjoint union**.
More examples of such direct limits will be constructed in Chapter 3 where we shall see that stalks of sheaves are direct limits. Generally we had projective limits as subsets of products, direct limits will be quotients of disjoint unions.
By the way in some sense this discussion of direct limits is superfluous: If we pass to the opposite category the direct limits become projective limits.

1.4 Exercises

Exercise 1. Do we have disjoint unions in the category $\mathbf{Vect}_k$? If so how does the disjoint union of two vector spaces look like.

Exercise 2.

(a) We may ask the same question for the category **Rings** of rings, for the category of commutative rings and for the category of groups.

(b) In any category we can consider diagrams of the form

$$\begin{array}{ccc} & \nearrow & B \\ A & & \\ & \searrow & C \end{array}$$

We can interpret this as an inductive system and we can ask whether the limit exists.

If for instance our category is the category of groups then the limit does exist and it is given by the almagamated product.

Exercise 3. Let us assume we have an index set $(I, \leq)$ and a projective system $(\{X_i\}_{i\in I},\phi_{ij})$ on it. Let us assume that the indexing set contains a maximal element m, i.e. $m \geq i$ for all elements $i \in I$. I claim that the projective limit exists. How does it look like? Can you formulate an analogous assertion for injective limits?

Exercise 4. Let us assume that we have a directed set $(I, \leq)$. We assume that we have an inductive system of rings $\{R_i\}_{i\in I}$. Does the direct limit exist? Hint: Forget the ring structure and consider the R_i as sets. Form the limit in the category of sets. Now you can reintroduce the ring structure on this limit by observing that any pair (or even finite set) of elements can be represented by elements in a suitable member R_i of the family.

Exercise 5. We have seen that we may interpret an ordered set $(A, \leq)$ as a category. What does it mean for such a category that the product of two elements exists?

2 Basic Concepts of Homological Algebra

In this chapter I want to explain the fundamental concepts of homological algebra. They play a fundamental role in algebraic geometry and in various other fields. I will do this for the specific case group (co-)homology.
This example will become important to us in the third volume of this book where we discuss the cohomology of arithmetic groups. But since in this particular case the basic principles of homological become very clear, I have chosen this example as introduction into the subject. The cohomology of sheaves, which can serve as a second example, will be discussed in Chapter 4. As a general reference for these two Chapters I can give the books [Ge-Ma] and [Go].

2.1 The Category $\mathbf{Mod}_\Gamma$ of Γ-modules

In the following Γ will always be a group. A Γ-module is an abelian group M together with an action of Γ: This means we have a map $\Gamma \times M \mapsto M,(\gamma,m) \longrightarrow \gamma m$, which satisfies $1_\Gamma m = m$, $(\gamma_1\gamma_2)m = \gamma_1(\gamma_2 m)$ and $\gamma(m_1+m_2) = \gamma m_1 + \gamma m_2$. These Γ-modules are the **objects** of the category of Γ-modules: If we write $M \in \mathrm{Ob}(\mathbf{Mod}_\Gamma)$, then this means that M is a Γ-module.
If $M_1,M_2 \in \mathrm{Ob}(\mathbf{Mod}_\Gamma)$, then we may consider the set

$$\begin{aligned}\mathrm{Hom}_{\mathbf{Mod}_\Gamma}(M_1,M_2) &= \mathrm{Hom}_\Gamma(M_1,M_2)\\ &= \Big\{\varphi : M_1 \longrightarrow M_2 \ \mid \varphi \text{ homomorphism of abelian groups}\\ &\qquad \varphi(\gamma m_1) = \gamma\varphi(m_1) \text{for all } \gamma \in \Gamma, m_1 \in M_1\Big\}\end{aligned}$$

On $\mathrm{Hom}_\Gamma(M_1,M_2)$ we have a natural structure of an abelian group: For any two elements $\varphi,\psi \in \mathrm{Hom}_\Gamma(M_1,M_2)$ we put $(\varphi+\psi)(m_1) = \varphi(m_1)+\psi(m_1)$.
Here we have another typical example of a category: We have a collection of objects – this collection is not a set in general – and for any two such objects we have a set of morphisms. (In our special case these sets of morphisms are abelian groups.) A certain bunch of axioms has to be satisfied: We have the identity $\mathrm{Id}_M \in \mathrm{Hom}_\Gamma(M,M)$, we have a composition $\mathrm{Hom}_\Gamma(M_1,M_2) \times \mathrm{Hom}_\Gamma(M_2,M_3) \longrightarrow \mathrm{Hom}_\Gamma(M_1,M_3)$ and Id_M is neutral with respect to this composition. (See the introduction in Chap. 1) In our special case this composition is bilinear.
The special category $\mathbf{Mod}_\Gamma$ has some extra features: Given $\varphi : M \longrightarrow N$ we can form the **kernel** and the **image**

$$\ker(\varphi) = \{m \mid \varphi(m) = 0\}, \mathrm{Im}\,(\varphi) = \{\varphi(m) \mid m \in M\}\,,$$

clearly these are also Γ-modules.

G. Harder, *Lectures on Algebraic Geometry I*, DOI 10.1007/978-3-8348-8330-8_2,

If $N \subset M$ is a Γ-submodule of M, then we may form the **quotient module**

$$M/N = M \mathrm{mod} N,$$

this is again a Γ-module. Finally, we have direct sums and direct products

$$\begin{aligned}\bigoplus_{i\in I} M_i &= \{(\dots m_i \dots)_{i\in I} \mid m_i \in M_i, \text{ almost all } m_i = 0\} \\ \prod_{i\in I} M_i &= \{(\dots m_i \dots)_{i\in I} \mid m_i \in M_i\}\end{aligned}$$

where the addition and the action of Γ are defined componentwise.
All these properties of imply that $\mathbf{Mod}_\Gamma$ is an abelian category. The notion of abelian categories can be axiomatized (see [Go]1.8).

Complexes of Γ-Modules

Definition 2.1.1. *If we have a sequence of maps between Γ-modules*

$$\dots \longrightarrow M_{\nu+1} \xrightarrow{d_{\nu+1}} M_\nu \xrightarrow{d_\nu} M_{\nu-1} \longrightarrow \dots$$

then this is called a **(homological) complex** *if $d_\nu \circ d_{\nu+1} = 0$ for all indices ν, i.e. if always $\mathrm{Im}\,(d_{\nu+1}) \subset \ker(d_\nu)$. The maps d_ν are the* **differentials** *of the complex. We often denote such a complex by $M_\bullet$ or $(M_\bullet, d_\bullet)$.*

Definition 2.1.2 (Exactness)**.** *The complex is called* **exact** *if we have $\mathrm{Im}\,(d_{\nu+1}) = \ker(d_\nu)$ for all indices ν.*

Definition 2.1.3 (Homology)**.** *We define the* **homology groups** *of such a complex as*

$$H_\nu(M_\bullet) = \frac{\ker(d_\nu : M_\nu \longrightarrow M_{\nu-1})}{\mathrm{Im}\,(d_{\nu+1} : M_{\nu+1} \longrightarrow M_\nu)}.$$

The elements in the kernel of d_ν are called **cycles** *(of degree ν), the elements in the image of $d_{\nu+1}$ are called* **boundaries** *(of degree ν).*

It is a tautology that

Lemma 2.1.4. *A complex is exact if and only if its homology groups are trivial.*

We can also consider complexes where the differentials raise the index by one then we write the indices ν as superscripts

$$\dots \longrightarrow M^{\nu-1} \xrightarrow{d^{\nu-1}} M^\nu \xrightarrow{d^\nu} M^{\nu+1} \longrightarrow \dots, \tag{2.1}$$

then this is a **cohomological** complex.
Very often we abbreviate and simply write $M^\bullet$ or $(M^\bullet, d^\bullet)$ for a (cohomological) complex.

Definition 2.1.5 (Cohomology)**.** *We define the* **cohomology groups** *of a cohomological complex by*

$$H^\nu(M^\bullet) = \frac{\ker(d^\nu : M^\nu \longrightarrow M^{\nu+1})}{\mathrm{Im}\,(d^{\nu-1} : M^{\nu-1} \longrightarrow M^\nu)}.$$

The elements in the kernel $Z^\nu(M) = \ker(d^\nu : M^\nu \longrightarrow M^{\nu+1})$ are called the **cocycles** *in degree ν and the elements in $B^\nu(M) = \mathrm{Im}\,(d_\nu M^{\nu-1} \longrightarrow M^\nu)$ are the* **coboundaries.**

Hence the cohomology is the group of cocycles modulo the coboundaries.
We abbreviate the graded direct sum over all cohomology groups by $H^\bullet(M^\bullet) = \bigoplus_\nu H^\nu(M^\bullet)$. Actually we may also view these cohomology groups as a complex of abelian groups with the differentials equal to zero. Again is clear that the following is true

Lemma 2.1.6. *A complex is exact if and only if its cohomology groups are trivial.*

Definition 2.1.7. *A map between two complexes*

$$\varphi^\bullet : M^\bullet \longrightarrow N^\bullet$$

is a sequence of maps $\phi^\nu : M^\nu \longrightarrow N^\nu$ *which commutes with the differentials.*

It is clear that such a map induces a map between the cohomology groups $H^\bullet(\varphi^\bullet) : H^\bullet(M^\bullet) \longrightarrow H^\bullet(N^\bullet)$.

Definition 2.1.8. *A* **(short) exact sequence** *is an exact complex*

$$0 \longrightarrow M' \xrightarrow{i} M \xrightarrow{p} M'' \longrightarrow 0,$$

i.e. i is injective, $\mathrm{Im}\,(i) = \ker(p)$ *and p is surjective, i.e. M'' is isomorphic to the quotient of M by the submodule $i(M') \simeq M'$.*

2.2 More Functors

2.2.1 Invariants, Coinvariants and Exactness

As I explained already in the first chapter a functor is a rule that produces in a functorial way an object in a target category from an object in the source category. If for instance the source category is $\mathbf{Mod}_\Gamma$ and the target category is the category $\mathbf{Ab}$ of abelian groups, then a functor

$$F : \mathbf{Mod}_\Gamma \longrightarrow \mathbf{Ab}$$

associates to any Γ-module $M \in \mathrm{Ob}(\mathbf{Mod}_\Gamma)$ an abelian group $F(M)$. Recall that functoriality means that for any $M_1, M_2 \in \mathrm{Ob}(\mathbf{Mod}_\Gamma)$ we have a map

$$F_{M_1,M_2} : \mathrm{Hom}_{\mathbf{Mod}_\Gamma}(M_1,M_2) \longrightarrow \mathrm{Hom}_{\mathbf{Ab}}(F(M_1),F(M_2)) \tag{2.2}$$

which sends Id_M to $\mathrm{Id}_{F(M)}$ and compositions into compositions. If we require in addition that this map is a homomorphism F_{M_1,M_2} between the abelian groups, then this functor is an **additive** functor between abelian categories.
There are two very simple functors between the category $\mathbf{Mod}_\Gamma$ and the category $\mathbf{Ab}$ of the abelian group

$$\begin{aligned} \text{Forget} : \mathbf{Mod}_\Gamma &\longrightarrow \mathbf{Ab} \\ \text{Trivial} : \mathbf{Ab} &\longrightarrow \mathbf{Mod}_\Gamma \end{aligned} \tag{2.3}$$

where the first factor "forgets" the Γ-module structure on the abelian group M and the second introduces the trivial Γ-action on an abelian group A, i.e. every element $\gamma \in \Gamma$ induces the identity on A. These two functors are so called exact functors.

Definition 2.2.1. *A functor are called an* **exact functor** *if it maps exact sequences into exact sequences.*

Homological algebra owes its existence to the fact that many important additive functors are not exact. Here comes the first example.

Definition 2.2.2. *If M is a Γ-module, we define the module of* **invariants** *by*

$$M^\Gamma = \{m \mid \gamma m = m \text{ for all } \gamma \in \Gamma\}.$$

It is an abelian group and hence we defined a functor

$$\text{Invariants} \quad \mathbf{Mod}_\Gamma \longrightarrow \mathbf{Ab}$$

from the category of Γ-modules to the category of abelian groups. If A is a trivial Γ-module, then $\mathrm{Hom}_{\mathbf{Mod}_\Gamma}(A,M) = \mathrm{Hom}_{\mathbf{Ab}}(A,M^\Gamma)$, and this property also characterizes the submodule M^Γ in M.

Definition 2.2.3. *The module M_Γ of* **coinvariants** *is defined as a quotient*

$$M \longrightarrow M_\Gamma$$

where M_Γ is a trivial Γ-module and for any Γ-module with trivial action by Γ we have

$$\mathrm{Hom}_{\mathbf{Mod}_\Gamma}(M,A) = \mathrm{Hom}_{\mathbf{Ab}}(M_\Gamma,A).$$

To give a different description of M_Γ we recall the notion of the **group ring** $R = \ [\Gamma]$ of our group Γ. It consists of all finite linear combinations

$$\sum_{\gamma\in\Gamma} n_\gamma \gamma \qquad n_\gamma \in \quad , \quad \text{almost all } n_\gamma = 0,$$

where we add componentwise (i.e. the additive group is the free abelian group over the set), and where we multiply

$$\left(\sum_\gamma n_\gamma \gamma\right) \cdot \left(\sum_\eta m_\eta \eta\right) = \sum_{\gamma,\eta} n_\gamma m_\eta \gamma\eta = \sum_\delta \left(\sum_{\gamma\eta=\delta} n_\gamma m_\eta\right) \delta. \tag{2.4}$$

This group ring contains the so called **augmentation ideal** I_Γ which is the kernel of the **augmentation map** $\epsilon : \ [\Gamma] \longrightarrow \quad$ which is defined by

$$\epsilon : \sum n_\sigma \sigma \mapsto \sum n_\sigma.$$

It is clear that this ideal is generated as a -module by elements of the form $1 - \gamma$.
For any Γ-module M the module $I_\Gamma M \subset M$ is a Γ-submodule, and it is also an easy exercise that

$$M_\Gamma = M/(I_\Gamma M)$$

has the desired property the module of coinvariants should have.
The following fact is the starting point of homological algebra:

Remark 1. In general the functors $M \longrightarrow M^\Gamma$ and $M \longrightarrow M_\Gamma$ are not exact.

To be more precise: If we start from a short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

of Γ-modules, then the sequence

$$0 \longrightarrow (M')^\Gamma \longrightarrow M^\Gamma \longrightarrow (M'')^\Gamma$$

is exact, but the last arrow is not surjective (in general).
A similar assertion holds for M_Γ. We only get an exact sequence

$$M'_\Gamma \longrightarrow M_\Gamma \longrightarrow M''_\Gamma \longrightarrow 0.$$

We say that $M \longrightarrow M^\Gamma$ is a **left exact** functor and $M \longrightarrow M_\Gamma$ is a **right exact** functor. The goal is to construct the so called **derived** functors which measure the deviation from exactness. We motivate this by an example.

2.2.2 The First Cohomology Group

I want to explain why the functor $M \longrightarrow M^\Gamma$ is not exact. Then I want to explain how this more or less automatically leads to the definition of the **derived functor**.
Let us start from an exact sequence of Γ-modules

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0.$$

We get an exact sequence of abelian groups

$$0 \longrightarrow (M')^\Gamma \longrightarrow M^\Gamma \longrightarrow (M'')^\Gamma.$$

We pick an element $m'' \in (M'')^\Gamma$, and we want to understand why this is not necessarily in the image of M^Γ. Of course we can find an element $m \in M$ which maps to m''. But there is no reason why this element should be invariant under Γ, the only thing we know is that for all $\gamma \in \Gamma$ the difference

$$m'_\gamma = m - \gamma m \in M'. \tag{2.5}$$

We get a map

$$\begin{array}{ccc} \Gamma & \longrightarrow & M' \\ \gamma & \longmapsto & m'_\gamma, \end{array}$$

and this map satisfies $m'_{\gamma_1} + \gamma_1 m'_{\gamma_2} = m'_{\gamma_1\gamma_2}$. A map $\Gamma \longrightarrow M'$ satisfying this relation is called a **1-cocycle** . On the set of all 1-cocycles we get a structure of an abelian group if we add the values and we denote by $Z^1(\Gamma, M')$, the abelian group of 1-cocycles. Our element m is in M^Γ if and only if the cocycle $m'_\gamma = m - \gamma m = 0$.
We notice that the choice of m is not unique, we may change $m \longrightarrow m + m'$ with $m' \in M'$. This is the only possible modification. Then we also modify the cocycle defined by m into $\gamma \mapsto m'_\gamma + m' - \gamma m'$. This leads to the definition of the group $B^1(\Gamma, M')$ of **1-coboundaries**. It is the group of those cocycles $\gamma \mapsto b_\gamma$ for which we can find a $m' \in M'$ such that $b_\gamma = m' - \gamma m'$ for all γ.

Hence we see: The element $m'' \in (M'')^\Gamma$ defines an element in $Z^1(\Gamma,M')$ which is well defined up to a coboundary. We introduce the first cohomology group (preliminary definition)

$$\check{H}^1(\Gamma,M') = Z^1(\Gamma,M')/B^1(\Gamma,M'), \tag{2.6}$$

and we have seen that any $m'' \in (M'')^\Gamma$ defines a class $\delta(m'') \in \check{H}^1(\Gamma,M')$ which is zero if and only if m'' is in the image of $M^\Gamma \longrightarrow (M'')^\Gamma$. It is clear that δ is a homomorphism, and that we have extended our exact sequence one step further

$$0 \longrightarrow (M')^\Gamma \longrightarrow M^\Gamma \longrightarrow (M'')^\Gamma \xrightarrow{\delta} \check{H}^1(\Gamma,M').$$

The next thing that can be checked easily is the functoriality of $M' \longrightarrow \check{H}^1(\Gamma,M')$. If we have a $\varphi \in \mathrm{Hom}_{\mathbf{Mod}_\Gamma}(M',N)$ then this induces a map

$$\check{\varphi}^{(1)} : \check{H}^1(\Gamma,M') \longrightarrow \check{H}^1(\Gamma,N),$$

and our above considerations also show that we get an even longer exact sequence

$$0 \longrightarrow (M')^\Gamma \longrightarrow M^\Gamma \longrightarrow (M'')^\Gamma \xrightarrow{\delta} \check{H}^1(\Gamma,M') \longrightarrow \check{H}^1(\Gamma,M) \longrightarrow \check{H}^1(\Gamma,M''),$$

the verification of exactness is left to the reader. But at the end it stops again: The last map needs not to be surjective.
We also see that this longer exact sequence depends functorially on the short exact sequence we started from. If we have a map between two exact sequences of Γ-modules

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'' & \longrightarrow & 0
\end{array}$$

then this induces a map between the two resulting exact sequences (in the sense of maps between complexes, i.e. all diagrams commute).
In principle we can try to extend our sequence beyond $\check{H}^1(\Gamma,M'')$. We pick an element in $\check{H}^1(\Gamma,M'')$ and try to lift it to an element in $\check{H}^1(\Gamma,M)$, and then we will see what the obstruction to this lifting will be. This will suggest a definition of a cohomology group $\check{H}^2(\Gamma,M')$. But actually there is a much more elegant way to define the cohomology functor which is also universal in the sense that it applies to many other cases. This will be done in section 2.3.

2.2.3 Some Notation

At this point we introduce some new notation, instead of M^Γ we also write $H^0(\Gamma,M)$ and $H_0(\Gamma,M)$ will be the same as M_Γ. This is a very suggestive notation if we use it for our exact sequence above.
Of course all this does not yet prove that $M \longrightarrow M^\Gamma$ is not exact in general. For instance, it could happen (in principle) that $\check{H}^1(\Gamma,M) = 0$ for all Γ and all M, or it could also be that $\check{H}^1(\Gamma,M') \longrightarrow \check{H}^1(\Gamma,M)$ is always injective. We will show in exercise 9 that for $\Gamma \neq \{1\}$ these functors are not trivial.

2.2.4 Exercises

Exercise 6. If A is a trivial Γ-module, then $\check{H}^1(\Gamma,A) = \mathrm{Hom}(\Gamma,A)$ where the last Hom is the Hom in the category of groups.

This shows that for suitable A the module $\check{H}^1(\Gamma,A) \neq 0$ if Γ is not equal to its commutator group $[\Gamma,\Gamma]$.
Let us now assume that $\Gamma' \subset \Gamma$ is a subgroup. We have the important functor from the category of Γ'-modules to the category of Γ-modules which is called **induction**. For any Γ'-module Y we define an abelian group

$$\mathrm{Ind}_{\Gamma'}^{\Gamma} Y = \{f : \Gamma \longrightarrow Y \mid f(\gamma'\gamma) = \gamma' f(\gamma) \text{ for all } \gamma' \in \Gamma', \gamma \in \Gamma\}, \tag{2.7}$$

and we define the action of Γ on $\mathrm{Ind}_{\Gamma'}^{\Gamma} Y$ by $(\gamma f)(\gamma_1) = f(\gamma_1\gamma)$. (Note that we do not have a support condition on the functions f, if the index of Γ' in Γ is infinite, then we may have infinitely many γ- $\bmod \Gamma'$ with $f(\gamma) \neq 0$.)
This is the **induced Γ-module** from the Γ'-module Y. It is very easy to check that for any Γ-module X we have an isomorphism (**Frobenius reciprocity**)

$$\mathrm{Hom}_{\Gamma}(X, \mathrm{Ind}_{\Gamma'}^{\Gamma} Y) \xrightarrow{\sim} \mathrm{Hom}_{\Gamma'}(X,Y)$$

which is given by $\varphi \mapsto \{x \mapsto \varphi(x)(1)\}$.

Exercise 7. We have a canonical (this means functorial in Y) isomorphism

$$\check{H}^1(\Gamma, \mathrm{Ind}_{\Gamma'}^{\Gamma} Y) \xrightarrow{\sim} \check{H}^1(\Gamma',Y).$$

This isomorphism is obtained from the following map on the level of cocycles: For any 1-cocycle $\{\gamma \mapsto f_\gamma\} \in Z^1(\Gamma, \mathrm{Ind}_{\Gamma'}^{\Gamma} Y)$ we define the 1-cocycle $\{\gamma' \mapsto \overline{f}_{\gamma'}\} \in Z^1(\Gamma',Y)$ by $\overline{f}_{\gamma'} = f_{\gamma'}(1)$. Show that this map sends coboundaries into coboundaries and induces an isomorphism on cohomology. (In the literature this and its generalisations run under the name *Lemma of Shapiro*)

Hint: We have to combine several little observations:

(i) We consider an 1-cocycle $\{\gamma \longmapsto f_\gamma\} \in Z^1(\Gamma, \mathrm{Ind}_{\Gamma'}^{\Gamma} Y)$, and we take into account that f_γ is actually a Y-valued function on Γ. Then the cocycle relation reads

$$f_{\gamma_1\gamma_2}(x) = f_{\gamma_1}(x) + (\gamma f_{\gamma_2})(x) = f_{\gamma_1}(x) + f_{\gamma_2}(x\gamma_1).$$

If we evaluate at $x = 1$ we get

$$f_{\gamma_2}(\gamma_1) = f_{\gamma_1\gamma_2}(1) - f_{\gamma_1}(1),$$

and this relation tells us that we only need to know the values $f_\gamma(1)$. Then the cocycle relation gives us the values of the f_γ at any $x \in \Gamma$.

(ii) If we have any function

$$\begin{aligned} h : \Gamma &\longrightarrow Y \\ h : \gamma &\longmapsto h_\gamma, \end{aligned}$$

we may put (think of h_γ as being $f_\gamma(1)$) $H_\gamma(x) = h_{x\gamma} - h_x$, then H_γ is a function on Γ with values in Y. If $\gamma \mapsto h_\gamma$ satisfies

$$h_{\gamma' x\gamma} - h_{\gamma' x} = \gamma'(h_{x\gamma} - h_x),$$

then $H_\gamma \in \mathrm{Ind}_{\Gamma'}^{\Gamma} Y$ and $\gamma \mapsto H_\gamma$ it is a 1-cocycle.

(iii) If we have a 1-cocycle $\gamma \mapsto f_\gamma$ in $Z^1(\Gamma, \mathrm{Ind}_{\Gamma'}^{\Gamma} Y)$, then $\gamma' \mapsto f_{\gamma'}(1)$ for $\gamma' \in \Gamma'$ is a one-cocycle in $Z^1(\Gamma',Y)$. Hence we have a map $Z^1(\Gamma, \mathrm{Ind}_{\Gamma'}^{\Gamma} Y) \longrightarrow Z^1(\Gamma',Y)$, and it is clear that this map sends coboundaries into coboundaries.

(iv) If we have a 1-cocycle $\gamma' \mapsto \overline{f}_{\gamma'}$ in $Z^1(\Gamma',Y)$, then we want to construct a 1-cocycle $\gamma \mapsto f_\gamma$ so that $\overline{f}_{\gamma'} = f_{\gamma'}(1)$. To do this we choose a system γ_i of representatives of $\Gamma' \backslash \Gamma$ where we choose the identity for the class Γ'.

For $\gamma = \gamma'\gamma_i$ we put $f_\gamma(1) = f_{\gamma'}(1)$ and apply (ii). The cocycle relation for $\gamma' \mapsto f_{\gamma'}(1)$ provides the decisive relation in (ii).This proves the surjectivity of our map between 1-cocycles in (iii).

(v) Finally, we have to check that $\gamma \mapsto f_\gamma$ is a coboundary if $\gamma' \mapsto \overline{f}\gamma'$ is a coboundary. We can write $f_{\gamma'}(1) = y - \gamma' y$ with $y \in Y$ and for all $\gamma' \in \Gamma'$. If we want to write $\gamma \mapsto f_\gamma$ as a boundary, i.e. $f_\gamma = c - \gamma c$, then this reads $f_\gamma(x) = c(x) - c(x\gamma)$, and evaluation at 1 yields $f_\gamma(1) = c(1) - c(\gamma)$. Hence we choose $c(1) = y$ and put $c(x) = y - f_x(1)$ and verify that this c bounds f_γ.

Exercise 8. Use the previous exercise to prove that for any group $\Gamma \neq \{1\}$ there is a Γ-module M s.t. $\check{H}^1(\Gamma,M) \neq 0$.

The group ring $[\Gamma]$ consists of linear combinations $\sum_{\gamma\in\Gamma} n_\gamma \gamma$ where we have a support condition: The coefficients $n_\gamma = 0$ for almost all γ. We add componentwise and the support condition allows us to define a product:

$$\Big(\sum_{\gamma\in\Gamma} n_\gamma\gamma\Big)\Big(\sum_{\gamma\in\Gamma} m_\gamma\gamma\Big) = \sum_\eta \Big(\sum_{\gamma,\gamma':\gamma\gamma'=\eta} n_\gamma m_{\gamma'}\Big)\eta$$

Exercise 9. The group ring $[\Gamma]$ is also a Γ-module by multiplication from the left. We get an exact sequence of Γ-modules

$$0 \longrightarrow I_\Gamma \longrightarrow [\Gamma] \longrightarrow \quad \longrightarrow 0.$$

If we apply the functor H_0 to this sequence and if we anticipate the left derived functor, we find the exact sequence of abelian groups

$$H_1(\Gamma, [\Gamma]) \longrightarrow H_1(\Gamma, \;) \longrightarrow I_\Gamma/I_\Gamma I_\Gamma \longrightarrow [\Gamma]/I_\Gamma \xrightarrow{\sim} \quad \longrightarrow 0$$
$$\|$$

Show that

$$\begin{array}{ccc} \Gamma & \longrightarrow & I_\Gamma/I_\Gamma I_\Gamma \\ \gamma & \longrightarrow & 1-\gamma \end{array}$$

induces an isomorphism $\Gamma/[\Gamma,\Gamma] = \Gamma_{\mathrm{ab}} \xrightarrow{\sim} I_\Gamma/I_\Gamma I_\Gamma$ ($[\Gamma,\Gamma]$ is the commutator subgroup). This suggests that $H_1(\Gamma, \;) = \Gamma_{\mathrm{ab}}$.

2.3 The Derived Functors

After these motivating considerations we explain the fundamental problem to be solved in homological algebra. We have the functor

$$M \longrightarrow M^{\Gamma} = H^0(\Gamma,M) \tag{2.8}$$

which is only left exact. We want to construct the **right derived functor**: This is a collection of functors $M \longrightarrow H^i(\Gamma,M)$ for $i = 0,1,2,\dots$, such that for any short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

we get a long exact sequence

$$\begin{array}{l} 0 \longrightarrow H^0(\Gamma,M') \to H^0(\Gamma,M) \to H^0(\Gamma,M'') \xrightarrow{\delta} \\ \quad \to H^1(\Gamma,M') \to H^1(\Gamma,M) \to H^1(\Gamma,M'') \longrightarrow \cdots \end{array} \tag{2.9}$$

which depends functorially on the exact sequence (see 2.3.4).
Finally we want this functor to be minimal (or universal) in the following sense:
If we have any other collection of functors $M \longrightarrow \widetilde{H}^i(\Gamma,M)$ for $i = 0,1,2\dots$ with $H^0(\Gamma,M) = \widetilde{H}^0(\Gamma,M)$, and the same properties as above, then we find a natural transformation $H^i(\Gamma,M) \longrightarrow \widetilde{H}^i(\Gamma,M)$, which is compatible with the connecting homomorphisms.
We want to indicate the main ideas how to construct these derived functors. The verification that the new construction of the H^1 gives the same result as our previous $\check{H}^1$ will be done in the exercises 2.4.3.
I want to explain a very simple principle that governs to the construction of these functors.

Definition 2.3.1. *We say that the sequence* **splits** *if one of the following equivalent assertions holds:*

(i) *We have a section to p. This is a Γ-module homomorphism $s : M'' \longrightarrow M$ for which $p \circ s = \mathrm{Id}_{M''}$.*

(ii) *The modules M splits, i.e. we have a Γ-submodule $\widetilde{M}''$ such that*

$$\begin{array}{ccc} M' \oplus \widetilde{M}'' & \xrightarrow{\sim} & M \\ (m',\widetilde{m}'') & \longmapsto & i(m') + \widetilde{m}''. \end{array}$$

(iii) *We have a Γ-module homomorphism $j : M \longrightarrow M'$ s.t. $j \circ i = \mathrm{Id}_{M'}$.*

A simple observation: If we have an exact sequence of Γ-modules

$$0 \longrightarrow M' \xrightarrow{i} M \xrightarrow{p} M'' \longrightarrow 0$$

then our functors H^0, H_0 will transform split exact sequences into split exact sequences, in other words if we restrict them to split exact sequences then they are exact.

2.3.1 The Simple Principle

This simple principle is based on the assumption that we have already constructed a derived functor $\{M \longrightarrow \widetilde{H^i}(\Gamma,M)\}$. Let us assume we have a class of $\mathcal{C}$ of Γ-modules which are acyclic for this functor. This means

Definition 2.3.2. *A module X is called* **acyclic for** $\widetilde{H}^i$ *if* $\widetilde{H^i}(\Gamma,X) = 0$ *for all* $i > 0$.

Definition 2.3.3. *An* **acyclic resolution** *of* $M \in \mathrm{Ob}(\mathbf{Mod}_\Gamma)$ *by objects in* $\mathcal{C}$ *is an exact sequence of* Γ*-modules*

$$0 \longrightarrow M \longrightarrow X^0 \longrightarrow X^1 \longrightarrow X^2 \longrightarrow \cdots$$

where the $X^\nu \in \mathcal{C}$.

Then we have a lemma, on which our simple principle is based:

Lemma 2.3.4. *If $\mathcal{C}$ is a class of acyclic objects for the derived functor $\{M \longrightarrow \widetilde{H}^i(\Gamma,M)\}$, and if*

$$0 \longrightarrow M \longrightarrow X^0 \longrightarrow X^1 \longrightarrow \cdots$$

is an acyclic resolution of M by objects in $\mathcal{C}$, then we have an isomorphism

$$\widetilde{H}^i(\Gamma,M) \simeq H^i((X^\bullet)^\Gamma).$$

Proof: By induction on i. For $i = 0$ we get the exact sequence

$$0 \longrightarrow M^\Gamma \longrightarrow (X^0)^\Gamma \longrightarrow (X^1)^\Gamma \longrightarrow \ldots$$

and

$$M^\Gamma \xrightarrow{\sim} \ker((X^0)^\Gamma \longrightarrow (X^1)^\Gamma) = H^0((X^\bullet)^\Gamma).$$

Now we cut the resolution into pieces. We get a short exact sequence

$$0 \longrightarrow M \longrightarrow X^0 \longrightarrow X^0/M \longrightarrow 0,$$

and we have a resolution by objects in $\mathcal{C}$

$$0 \longrightarrow X^0/M \longrightarrow Y^0 \longrightarrow Y^1 \longrightarrow \cdots$$

where $Y^{\nu-1} = X^\nu$. The first sequence yields a long exact sequence which is interrupted by many zeroes which come from the $\widetilde{H}^\bullet$-acyclicity of the X^0:

$$\begin{array}{l}
0 \longrightarrow M^\Gamma \longrightarrow (X^0)^\Gamma \longrightarrow (X^0/M)^\Gamma \longrightarrow \\
\longrightarrow \widetilde{H}^1(\Gamma,M) \longrightarrow 0 \longrightarrow \widetilde{H}^1(\Gamma,X^0/M) \xrightarrow{\sim} \\
\longrightarrow \widetilde{H}^2(\Gamma,M) \longrightarrow 0 \longrightarrow \widetilde{H}^3(\Gamma,X^0/M) \xrightarrow{\sim} \\
\longrightarrow \widetilde{H}^3(\Gamma,M) \longrightarrow 0 \longrightarrow \cdots
\end{array}$$

We check the case $i = 1$. Here we find $\widetilde{H}^1(\Gamma,M) \simeq (X^0/M)^\Gamma/(X^0)^\Gamma$, but $X^0/M \subset X^1$ is the kernel of $X^1 \longrightarrow X^2$, and $(X^0/M)^\Gamma = \ker((X^1)^\Gamma \longrightarrow (X^2)^\Gamma)$, and hence

$$\widetilde{H}^1(\Gamma,M) \simeq \frac{\ker((X^1)^\Gamma \longrightarrow (X^2)^\Gamma)}{\operatorname{im}((X^0)^\Gamma \longrightarrow (X^1)^\Gamma)} = H^1((X^\bullet)^\Gamma).$$

Hence we proved our assertion for $i = 1$ and then induction is clear. □

We want to apply this principle to construct the derived functors. But in some sense we are trapped: If we have not yet defined the derived functor, how can we know that certain objects are acyclic? This difficulty is resolved by the notion of **injective** modules.

Definition 2.3.5. *A* Γ*-module* I *is called* **injective** *if it has the following property: Whenever we have a diagram of* Γ*-modules*

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ {\scriptstyle\psi}\downarrow & & \\ I & & \end{array}$$

where $\ker(\varphi) \subset \ker(\psi)$*, then we can extend the diagram to a commutative diagram*

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ {\scriptstyle\psi}\downarrow & \swarrow{\scriptstyle\lambda} & \\ I & & \end{array}$$

Our assumption on φ,ψ is valid if φ is injective. If we want to check the injectivity of a module it clearly suffices to check diagrams with φ injective.
Injective modules have a very important property: Whenever we have a short exact sequence

$$0 \longrightarrow I \longrightarrow M \longrightarrow M' \longrightarrow 0,$$

and the module I is injective then the sequence splits. We simply apply the defining property of injective modules to

$$\begin{array}{ccccc} 0 & \longrightarrow & I & \longrightarrow & M \\ & & \downarrow{\scriptstyle \mathrm{Id}} & & \\ & & I & & \end{array}$$

Our simple observation above implies that we get exact sequences

$$0 \longrightarrow H^0(\Gamma,I) \longrightarrow H^0(\Gamma,M) \longrightarrow H^0(\Gamma,M') \longrightarrow 0 \tag{2.10}$$

and

$$0 \longrightarrow H_0(\Gamma,I) \longrightarrow H_0(\Gamma,M) \longrightarrow H_0(\Gamma,M') \longrightarrow 0. \tag{2.11}$$

whenever the module I on the left is injective. Since we require that the cohomology modules should measure the deviation from exactness and that they should be minimal in this respect, we expect them to vanish for injective modules. In other words we expect that injective modules should be acyclic, hence the injective modules provide a candidate for the a class $\mathcal{C}$. In view of our simple principle above we try to define the derived functors by using **injective resolutions**.
The following lemma is the starting point:

Lemma 2.3.6. *Every* Γ*-module* M *can be embedded into an injective module* I*.*

Sketch of the proof: First we consider the category **Ab** of abelian groups. This is the case $\Gamma = \{\mathrm{Id}\}$. One proves that the abelian group $\mathbb{Q}/\ $ is injective (this requires Zorn's lemma), then we see that every abelian group A can be embedded into a suitable product

$$A \longrightarrow \prod_{\alpha} \mathbb{Q}/\ \ .$$

If we have a Γ-module M we forget the Γ-module structure and embed it into an injective abelian group, i.e. $M \hookrightarrow J$. Now we get $\mathrm{Ind}_{\{1\}}^{\Gamma} M \hookrightarrow \mathrm{Ind}_{\{1\}}^{\Gamma} J$, and the module $\mathrm{Ind}_{\{1\}}^{\Gamma} J$ is injective in the category of Γ-modules. This follows from Frobenius reciprocity. Then we have achieved our goal since we have

$$M \hookrightarrow \mathrm{Ind}_{\{1\}}^{\Gamma} M \hookrightarrow \mathrm{Ind}_{\{1\}}^{\Gamma} J =: I.$$

Now the actual construction of the cohomology functor (the universal derived functor) becomes clear. We noticed that injective modules should be acyclic, i.e. $H^r(\Gamma,I) = 0$ for $r > 0$. But our Lemma 2.3.6 tells us that we can find an injective resolution of M, i.e.

$$0 \longrightarrow M \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

in short $0 \longrightarrow M \longrightarrow \mathcal{I}^\bullet$. Then our Lemma 2.3.4 tells us

$$H^\nu(\Gamma,M) \simeq H^\nu((I^\bullet)^\Gamma) = \frac{\ker\left((I^\nu)^\Gamma \longrightarrow (I^{\nu+1})^\Gamma\right)}{\mathrm{im}\left((I^{\nu-1})^\Gamma \longrightarrow (I^\nu)^\Gamma\right)},$$

should be taken as the definition of the cohomology.
Of course we have to investigate how these cohomology groups depend on the injective resolution and we have to show that $M \longrightarrow H^\bullet(\Gamma,M)$ is a functor.

2.3.2 Functoriality

If we have two Γ-modules M,N and a $\varphi \in \mathrm{Hom}_\Gamma(M,N)$ then we will construct a family of homomorphisms $H^\bullet(\phi^\bullet) : H^\bullet(\Gamma,M) \longrightarrow H^\bullet(\Gamma,N)$. We choose two injective resolutions $0 \longrightarrow M \longrightarrow I^\bullet$ and $0 \longrightarrow N \longrightarrow J^\bullet$, I claim that we can extend the map φ to a map between the complexes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \xrightarrow{i} & I^0 & \longrightarrow & I^1 & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle\varphi} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N & \xrightarrow{j} & J^0 & \longrightarrow & J^1 & \longrightarrow & \cdots
\end{array}$$

The existence of this extension is proved by induction on the degree. To get the first arrow $\varphi^0 : I^0 \longrightarrow J^0$ we apply the defining property of injective modules to get the arrow φ^0 in the diagram

$$\begin{array}{ccc}
M & \xrightarrow{i} & I^0 \\
{\scriptstyle j\circ\phi}\downarrow & \swarrow{\scriptstyle\varphi^0} & \\
J^0 & &
\end{array}$$

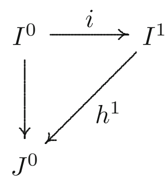
Then we construct φ^1 by the same principle and it is quite clear that at any step the existence of the vertical arrow follows directly from the defining property of injective modules (we only need that the J^ν are injective). This extension $\varphi^\bullet : I^\bullet \longrightarrow J^\bullet$ induces of course a map between the cohomology group

$$H^\bullet(\varphi^\bullet) : H^\bullet((I^\bullet)^\Gamma) \longrightarrow H^\bullet((J^\bullet)^\Gamma).$$

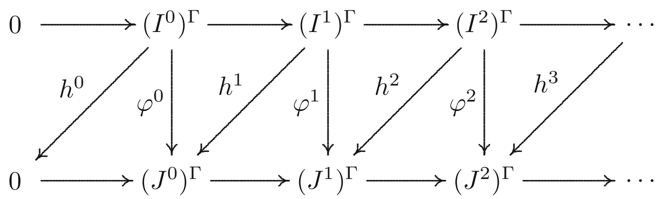
Now we have to worry what happens if we take two different extensions $\varphi^\bullet, \widetilde{\varphi^\bullet}$ of our map φ. I want to show that these two extensions induce the same map on the cohomology. To see this we can easily reduce to the case where $\varphi = 0$, and where $\varphi^\bullet$ is an arbitrary extension of $\varphi = 0$. Then I have to show that $\varphi^\bullet$ induces the zero map on the cohomology. I prove this by showing that under this assumption the map $\varphi^\bullet : I^\bullet \longrightarrow J^\bullet$ is actually **homotopic to zero**. This means that we can construct maps $h^\nu : I^\nu \longrightarrow J^{\nu-1}$ ($h^0 = 0$) such that

$$\varphi^\nu = d \circ h^\nu + h^{\nu+1} \circ d \tag{2.12}$$

To construct h^1 we observe that our assumption $\varphi = 0$ implies that the kernel of $I^0 \longrightarrow I^1$ is contained in the kernel of the vertical arrow $I^0 \longrightarrow J^0$. Since J^0 is injective we can construct $h^1 : I^1 \longrightarrow J^0$ which produces a commutative diagram



Now we modify the given vertical arrow $I^1 \longrightarrow J^1$ by subtracting the composition of h^1 and the horizontal arrow $I^0 \longrightarrow J^1$. To this modified arrow we can apply the previous argument and it becomes clear how to construct these h^ν by induction. Again the existence of an h^ν satisfying Equation 2.12 in any degree follows from the injectivity of the $J^{\nu-1}$ and the construction of the previous ones. But if we now apply our functor (invariants under Γ) we get



(We should have written ${\varphi^\bullet}^\Gamma, {h^\bullet}^\Gamma$ to be absolutely correct.) But now it is clear that $\varphi^\bullet$ induces zero in the cohomology. If we have a cycle $c_\nu \in (I^\nu)^\Gamma$ representing a given cohomology class then $\varphi^\nu(c_\nu) = d \circ h(c_\nu)$ and hence it represents the trivial class. If we apply this to a module M and the identity $\mathrm{Id} : M \longrightarrow M$ and two different resolutions of M, then we get a unique isomorphism between the resulting cohomology groups. In this sense the cohomology groups do not depend on the chosen resolution. Since the map $H^\bullet(\varphi^\bullet)$ does not depend on the choice of the extension of φ to the resolutions, the construction gives a unique family of homomorphisms $H^\bullet(\varphi^\bullet)$. This is functoriality.

2.3.3 Other Resolutions

If we start from an arbitrary resolution of our module M, say $0 \longrightarrow M \longrightarrow X^0 \longrightarrow X^1 \longrightarrow \cdots$ and if we also choose an injective resolution of M as above then our considerations in section 1.3.5 on direct limits show that we can construct a morphism of complexes of Γ-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{i} & X^0 & \longrightarrow & X^1 & \longrightarrow & \cdots \\ & & \downarrow \varphi & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M & \xrightarrow{j} & J^0 & \longrightarrow & J^1 & \longrightarrow & \cdots \end{array}$$

because we only need the injectivity of the $J^\bullet$. Therefore we get a canonical homomorphism

$$H^\bullet((X^\bullet)^\Gamma) \longrightarrow H^\bullet((J^\bullet)^\Gamma) = H^\bullet(\Gamma,M).$$

Our starting principle in section 2.3.1 says that this homomorphism will be an isomorphism if the modules X^ν are acyclic. But it is sometimes useful to consider such a resolution, even if it is not acyclic. It may be the case,that the cohomology groups $H^\bullet((X^\bullet)^\Gamma)$ are easier to understand than the the cohomology groups $H^\bullet(\Gamma,M)$ themselves. Then this homomorphism gives us some kind of approximation of the cohomology group. We will discuss this again in 4.6.1, the above homomorphism will be the edge homomorphism.

2.3.4 Injective Resolutions of Short Exact Sequences

Now we want to show that we get a long exact sequence in the derived functors if we start from a short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0. \tag{2.13}$$

We write our short exact sequence vertically and choose injective resolutions of the two modules M',M'' which we write horizontally. Imagine we have done this. Then we can write the direct sum in the middle and we get short vertical exact sequences. It will look like this:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M' & \xrightarrow{i'} & I'^0 & \longrightarrow & I'^1 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & & I'^0 \oplus I''^0 & & I'^1 \oplus I''^1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M'' & \xrightarrow{i''} & I''^0 & \longrightarrow & I''^1 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
\tag{2.14}
$$

The horizontal arrows in the middle are still missing. Now the injectivity of I'^0 allows an arrow Ψ from M to I'^0 which yields a commutative diagram

$$
\begin{array}{ccccc}
 & & 0 & & \\
 & & \downarrow & & \\
0 & \longrightarrow & M' & \longrightarrow & I'^0 \\
 & & \downarrow & \nearrow_{\Psi} & \\
 & & M & &
\end{array}
$$

Let p'' be the projection from M to M''. Then $\Psi \oplus (i'' \circ p'')$ gives us an injection

$$0 \longrightarrow M \longrightarrow I'^0 \oplus I''^0,$$

which we can fill into the diagram above. This yields a diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M' & \longrightarrow & I'^0 & \longrightarrow & U & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & I'^0 \oplus I''^0 & \longrightarrow & V & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M'' & \longrightarrow & I''^0 & \longrightarrow & W & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
\tag{2.15}
$$

We have $U \hookrightarrow I'^1$ and $W \hookrightarrow I''^1$ and again we construct as before an arrow

$$V \hookrightarrow I'^1 \oplus I''^1.$$

This goes on forever and is that we call an **injective resolution** of the exact sequence 2.13.

A Fundamental Remark

We have to be aware that in general the homomorphisms in the middle row

$$I'^\nu \oplus I''^\nu \longrightarrow I'^{\nu+1} \oplus I''^{\nu+1}$$

are **not** the direct sum of the two homomorphisms which are already given by the resolution of the extreme modules. We have to add a homomorphism

$$\Psi^\nu : I''^\nu \longrightarrow I'^{\nu+1} \tag{2.16}$$

to this direct sum (ν is of course an upper index and not an exponent). These Ψ^ν will satisfy a recursion relation: We will have

$$0 = \begin{cases} d'\Psi(m) + \Psi^0(i'' \circ p''(m)) & \text{for } \nu = 0 \\ d'\Psi^\nu(x''_\nu) + \Psi^{\nu+1}(d'' x''_\nu) & \text{for } \nu > 0 \end{cases}. \tag{2.17}$$

We will not be able to get $d'\Psi(m) = 0$ unless the sequence splits. Therefore we see that we will not be able to show that such a $\Psi^{\nu+1}$ can be chosen to be trivial if we do not have $d'(\Psi^\nu) = 0$. We come back to this point when we discuss the spectral sequence (see sections 4.6.4 and 4.6.6).

The Cohomology and the Long Exact Sequence

If we apply the functor $H^0(\Gamma, \ )$ to the double complex 2.15 we get the diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (I'^0)^\Gamma & \longrightarrow & (I'^1)^\Gamma & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (I'^0 \oplus I''^0)^\Gamma & \rightarrow & (I'^1 \oplus I''^1)^\Gamma & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (I''^0)^\Gamma & \longrightarrow & (I''^1)^\Gamma & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

where the horizontal complexes compute the cohomology of M',M and M'' respectively and where the vertical sequences are exact. From this we get a long exact sequence of the cohomology groups

$$\begin{array}{l}
0 \longrightarrow H^0(\Gamma,M') \rightarrow H^0(\Gamma,M) \rightarrow H^0(\Gamma,M'') \\
\xrightarrow{\ \delta\ } H^1(\Gamma,M') \rightarrow H^1(\Gamma,M) \rightarrow H^1(\Gamma,M'') \longrightarrow \cdots
\end{array}$$

I just give a few indication how we get the connecting homomorphisms δ. A class in $\xi'' \in H^p(\Gamma,M'')$ is represented by a cocycle $x'' \in (I''^p)^\Gamma$. This element can be lifted to an element $x \in (I'^p \oplus I''^p)^\Gamma$. This element is not necessarily a cocycle, under the coboundary operator it maps to an element $y \in (I'^{p+1} \oplus I'',^{p+1})^\Gamma$. This element can obviously be lifted to an element in $z \in I'^{p+1}$ which must be a cocycle. The class of this element is the image of $\xi'\prime$ under the connecting homomorphism.
Now we have constructed a derived functor using these injective resolutions. It is universal as one sees easily from the requirement that it vanishes on injective modules.

The Homology of Groups

Essentially the same strategy works for the construction of the left derived functor $M \longrightarrow H_i(\Gamma,M)$ for $i = 0,1,2,\ldots$, of the right exact functor $M \longrightarrow H_0(\Gamma,M) = M_\Gamma$. The defining property of injective modules implies that an injective module is always a direct summand if it sits in a bigger module. The dual notion is the notion of projective modules.

Definition 2.3.7 (Projective Module)**.** *A* Γ*-module* P *is called* **projective** *if for any diagram*

$$\begin{array}{ccccc} M & \xrightarrow{p} & N & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle i} & & \\ & & P & & \end{array}$$

where the top sequence is exact we can find a map $j : P \longrightarrow M$

$$\begin{array}{ccccc} M & \xrightarrow{p} & N & \longrightarrow & 0 \\ & {\scriptstyle j}\nwarrow & \uparrow{\scriptstyle i} & & \\ & & P & & \end{array}$$

so that $p \circ j = i$*.*

It is easily seen that free Γ-modules $\bigoplus_{i\in I}\ [\Gamma]$ are projective. Hence we find

(i) Every Γ-modules M has a projective resolution

$$\ldots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0.$$

(ii) Every projective Γ-module P which is a quotient of a Γ-module X is a direct summand, i.e. the sequence $0 \longrightarrow Y \longrightarrow X \longrightarrow P \longrightarrow 0$ splits.

The assertion (ii) implies that the sequence

$$0 \longrightarrow Y_\Gamma \longrightarrow X_\Gamma \longrightarrow P_\Gamma \longrightarrow 0$$

is still exact. Hence we should require $H_i(\Gamma,P) = 0$ for $i = 1,2,\ldots$. Now we may apply the same strategy as in the construction of the cohomology functor. For a module we choose a projective resolution $P_\bullet \longrightarrow M \longrightarrow 0$ and put

$$H_\bullet(\Gamma,M) = H_\bullet((P_\bullet)_\Gamma). \tag{2.18}$$

The same arguments as before show that this gives a universal left derived functor for the functor

$$M \longrightarrow M_\Gamma = H_0(\Gamma,M).$$

We get a long exact sequence where the arrows point in the opposite direction.

2.4 The Functors Ext and Tor

2.4.1 The Functor Ext

We may look at our previous constructions from a slightly more general point of view. The category of Γ-modules is the same as the category of R-modules where $R = R[\Gamma]$ is the group ring. We now consider the category $\mathbf{Mod}_R$ of modules over an arbitrary ring R.

To any pair of modules $M,N \in \mathrm{Ob}(\mathbf{Mod}_R)$ we can introduce the abelian group

$$\mathrm{Hom}_R(N,M) = \left\{ \varphi : N \longrightarrow M \mid \begin{array}{c} \varphi \text{ homomorphism of abelian groups} \\ \varphi(rm) = r\varphi(m) \text{ for all } r,m, \end{array} \right\}$$

and if we fix N then this becomes a functor from the category $\mathbf{Mod}_R$ to the category of abelian groups. But we may also fix M and vary N, then our functor becomes a contravariant functor:
A R-module homomorphism $\Psi : N_1 \longrightarrow N_2$ induces a map

$$\mathrm{Hom}_R(N_2,M) \longrightarrow \mathrm{Hom}_R(N_1,M)$$

by composition. (So far all our functors were covariant.) It is quite clear that the covariant functor $M \longrightarrow \mathrm{Hom}_R(N,M)$ is left exact and that the contravariant functor $N \longrightarrow \mathrm{Hom}_R(N,M)$ is right exact.
For a given N we may try to construct the right derived functor to $M \longrightarrow \mathrm{Hom}_R(N,M)$ and for given M we may try to construct a left derived functor to the functor $N \longrightarrow \mathrm{Hom}_R(N,M)$. The same principles as before tell us that the right derived functor should be zero on injective modules (same definition), and the left derived functor should be trivial on projective modules. Hence we choose an injective resolution of $0 \longrightarrow M \longrightarrow I^\bullet$ and define the right derived functor by

$$R\,\mathrm{Ext}^\bullet_R(N,M) = H^\bullet(\mathrm{Hom}_R(N,I^\bullet)). \tag{2.19}$$

I say again what this means: For a fixed R-module N and any exact sequence of R modules

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0,$$

we get a long exact sequence

$$\begin{array}{l} 0 \longrightarrow \mathrm{Hom}_R(N,M') \to \mathrm{Hom}_R(N,M) \to \mathrm{Hom}_R(N,M'') \longrightarrow \\ \longrightarrow R\,\mathrm{Ext}^1_R(N,M') \to R\,\mathrm{Ext}^1_R(N,M) \to R\,\mathrm{Ext}^1_R(N,M'') \longrightarrow \cdots \end{array} \tag{2.20}$$

But by construction these groups $R\,\mathrm{Ext}^i_R(N,M)$ are also functorial in N if we fix M, the functors $N \longrightarrow R\,\mathrm{Ext}^i(N,M)$ are contravariant.
Analogously we choose a projective resolution $P_\bullet \longrightarrow N \longrightarrow 0$ and define

$$L\,\mathrm{Ext}^\bullet_R(N,M) = H_\bullet(\mathrm{Hom}_R(P_\bullet,M)). \tag{2.21}$$

Our previous arguments show that we get a left derived functor which has all functorial properties. It is clear that the functors $M \longrightarrow \mathrm{Hom}_R(P,M)$ resp. $N \longrightarrow \mathrm{Hom}_R(N,I)$ are exact if P is projective (resp. I is injective). Hence we have in this case

$$R\,\mathrm{Ext}^i_R(P,M) = 0 \ , \ L\,\mathrm{Ext}^i_R(N,I) = 0.$$

We will indicate briefly how this implies that we have a functorial isomorphism

$$L\,\mathrm{Ext}^i_R(N,M) \simeq R\,\mathrm{Ext}^i_R(N,M). \tag{2.22}$$

To see this we choose two resolutions: $0 \longrightarrow M \longrightarrow I^\bullet$, $P_\bullet \longrightarrow N \longrightarrow 0$, and we form the double complex $\mathrm{Hom}_R(P_\bullet \longrightarrow N \longrightarrow 0, 0 \longrightarrow M \longrightarrow I^\bullet)$ which in full looks like

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathrm{Hom}_R(P_1,M) & \longrightarrow & \mathrm{Hom}_R(P_1,I^0) & \longrightarrow & \mathrm{Hom}_R(P_1,I^1) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathrm{Hom}_R(P_0,M) & \longrightarrow & \mathrm{Hom}_R(P_0,I^0) & \longrightarrow & \mathrm{Hom}_R(P_0,I^1) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathrm{Hom}_R(N,M) & \longrightarrow & \mathrm{Hom}_R(N,I^0) & \longrightarrow & \mathrm{Hom}_R(N,I^1) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Now the first vertical Complex computes the $L\,\mathrm{Ext}_R^\bullet(P,M)$ and the horizontal complex at the bottom computes $R\,\mathrm{Ext}_R^\bullet(P,M)$. All other vertical or horizontal complexes are exact. Then a simple diagram chase shows that the cohomology of the bottom horizontal complex and the first vertical complex are isomorphic.
We summarize

Lemma 2.4.1. *The functor in two variables* $\mathrm{Ext}_R^\bullet(N,M)$ *can be be computed from an an injective resolution of* M *or a projective resolution of* N. *The higher* $\mathrm{Ext}_R^i(N,M)$ *for* $i > 0$ *vanish if* M *is injective or if* N *is projective.*

2.4.2 The Derived Functor for the Tensor Product

Another functor in two variables is given by the tensor product. Here we have to be a little bit careful in case that our ring R is not commutative. We consider the categories $\mathbf{Mod}_R^L, \mathbf{Mod}_R^R$ of left and right R-modules.

Definition 2.4.2 (Tensor Product)**.** *The* **tensor product** *of a right* R*-module* N *and a left* R*-module* M *is an abelian group* $N \otimes_R M$ *together with a map*

$$
\begin{aligned}
\Psi : &\quad N \times M \longrightarrow N \otimes_R M \\
\Psi : &\quad (n,m) \longmapsto n \otimes m
\end{aligned}
$$

which has the following properties

(i) *It is a biadditive, i.e.*

$$
\begin{aligned}
\Psi(n_1 + n_2, m) &= (n_1 + n_2) \otimes m = n_1 \otimes m + n_2 \otimes m \\
\Psi(n + m_1 + m_2) &= m \otimes (m_1 + m_2) = n \otimes m_2 + n \otimes m_2.
\end{aligned}
$$

(ii) *For all* $r \in R, n \in N, m \in M$ *we have* $nr \otimes m = n \otimes rm$. *(This is the moment where we need that* N *is a right* R*-module and* M *is a left* R*-module).*

(iii) *This map is universal: If we have another* $\Psi' : N \times M \longrightarrow X$ *with an abelian group* X *which satisfies (i) and (ii) then we can find a* $\varphi : N \otimes_R M \longrightarrow X$ *such that* $\Psi' = \varphi \circ \Psi$.

It is easy to construct $N \otimes_R M$, we form the free abelian group which is generated by pairs $(n,m) \in N \times M$ and divide by the subgroup generated by elements of the form

$$\begin{array}{rcl} (n_1 + n_2,m) & - & (n_1,m) - (n_2,m) \\ (n,m_1 + m_2) & - & (n,m_1) - (n,m_2) \\ (nr,m) & - & (n,rm). \end{array}$$

If our ring R is commutative then we can give $N \otimes_R M$ the structure of an R-module: We simply define

$$r(n \otimes m) = nr \otimes m = n \otimes rm. \tag{2.23}$$

In this case of a commutative ring R we can assume that both variables N,M are left R-modules.
If we fix N then the functor $M \longrightarrow N \otimes_R M$ is a right exact functor but in general it is not exact. This means that for a short exact sequence $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ the sequence

$$N \otimes_R M' \longrightarrow N \otimes_R M \longrightarrow N \otimes_R M'' \longrightarrow 0$$

will be exact but the first arrow on the left will not be injective in general. We leave it as an exercise to the reader to verify the right exactness. In the section on flat morphisms of schemes we will discuss some examples which explain these phenomena (Volume 2). But if the module M'' is projective then the sequence stays exact if we tensorize by any N because the sequence can be split.
This allows us to construct the derived functor. We work with a projective resolution $P_\bullet \longrightarrow M \longrightarrow 0$, to define

$$R\,\mathrm{Tor}_\bullet^R(N,M) = H_\bullet(N \otimes_R P_\bullet). \tag{2.24}$$

This is a universal left derived functor of our functor above, it is clear that this is a functor in the two variables N,M.
We can also choose a projective resolution $Q_\bullet \longrightarrow N \longrightarrow 0$ define the functor

$$L\,\mathrm{Tor}_\bullet^R(N,M) = H_\bullet(Q_\bullet \otimes_R M). \tag{2.25}$$

Again it is not so difficult to prove that these two functors are indeed equivalent. To see this we consider the double complex defined by the two resolutions and the vertical and horizontal subcomplexes are acyclic in the "interior".

Again we summarize:

Lemma 2.4.3. *The functor in two variables* $\mathrm{Tor}_\bullet^R(N,M)$ *defined in that way can be computed by a projective resolution of* N *or a projective resolution of* M*. The higher* $\mathrm{Tor}_i^R(N,M)$ *vanish for* $i > 0$ *if one of the entries is a projective module.*

Definition 2.4.4. *A left R-module* M *is called* **flat** *if the functor* $N \longrightarrow N \otimes_R M$ *is exact.*

The following is obvious

Lemma 2.4.5. *The left R-module M is flat if and only if* $\mathrm{Tor}_i^R(N,M) = 0$ *for all $i > 0$ and all right R-modules N .*

Lemma 2.4.6. *The functors cohomology and homology of a group* Γ *are special cases of* $\mathrm{Ext}^\bullet$ *and* $\mathrm{Tor}^\bullet$*.*

We take for our ring the group ring $R = \ [\Gamma]$, and we observe: If   is the abelian group   with trivial Γ-action then

$$\mathrm{Hom}_{\ [\Gamma]}(\ ,M) = M^\Gamma,$$

and hence we see

$$\mathrm{Ext}^i_{\ [\Gamma]}(\ ,M) = H^i(\Gamma,M); \tag{2.26}$$

and

$$M \otimes_{\ [\Gamma]} \ = M_\Gamma,$$

hence

$$\mathrm{Tor}_\bullet^R(M,\ ) = H_\bullet(\Gamma,M). \tag{2.27}$$

We conclude this chapter with some extra remarks and some exercises. We observe that we can compute the cohomology of a group also from a projective resolution

$$\ldots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \quad \longrightarrow 0$$

Then Lemma 2.4.1 and our formula above implies $H^i(\Gamma,M) = H^i(\mathrm{Hom}_\Gamma(P_\bullet,M))$. We can construct some kind of natural projective resolution of . For our module P_0 we take the group ring $P_0 = \ [\Gamma]$ and the first arrow $P_0 \longrightarrow \ $ is the augmentation map. The group ring considered as an abelian group is the group of finitely supported maps $\mathrm{Map}_{\mathrm{fin}}(\Gamma,\ )$. We define

$$P_n := \mathrm{Map}_{\mathrm{fin}}(\Gamma^{n+1},\ ), \tag{2.28}$$

this becomes a projective Γ module if we define $(\sigma f)(\sigma_0,\ldots,\sigma_n) = f(\sigma^{-1}\sigma_0,\ldots,\sigma^{-1}\sigma_n)$. We define a boundary operator $d_n : P_n \longrightarrow P_{n-1}$ by

$$(d_n f)(\sigma_0,\ldots,\sigma_{n-1}) := \sum_{i,\tau}(-1)^i f(\sigma_0,\ldots,\tau,\ldots,\sigma_{n-1}), \tag{2.29}$$

where τ runs over Γ and is inserted at the $i-$th place. It is easy to check, that this gives a projective resolution.

2.4.3 Exercise

Exercise 10. Apply the previous paragraph to the case of a cyclic group $\Gamma = \ /n\ $. Let σ be a generator of the group. We have the exact sequence

$$0 \longrightarrow I_\Gamma \longrightarrow \ [\Gamma] \longrightarrow \quad \longrightarrow 0$$

and $I_\Gamma = \ [\Gamma](1-\sigma)$.

(a) In the case $n = 0$ (the infinite group) we have that I_Γ is a free module. This gives simple formulae for the cohomology and shows $H^\nu(\Gamma,M) = 0$ for $\nu \geq 2$.

(b) In the case of a finite group the map

$$\begin{array}{rcl} [\Gamma] & \longrightarrow & I_\Gamma \\ r & \longrightarrow & r(1-\sigma) \end{array}$$

has the kernel $[\Gamma](1+\ldots+\sigma^{n-1})$. Construct a "periodic" resolution from this and compute the cohomology.

Exercise 11. Compare our provisional cohomology groups $\check{H}^1(\Gamma,M)$ and the new ones. This is not so difficult. Use the following

(a) We observe that our new cohomology groups obviously satisfy: For a subgroup $\Gamma' \subset \Gamma$ and a Γ'-module

$$H^i(\Gamma, \mathrm{Ind}_{\Gamma'}^{\Gamma} Y) = H^i(\Gamma', Y).$$

(Choose an injective resolution of the Γ'-modules Y and ... .)

(b) We take $\Gamma' = \{1\}$. Then $H^1(\Gamma, \mathrm{Ind}_{\{1\}}^{\Gamma} M) = 0$. We constructed the sequence

$$0 \longrightarrow M \longrightarrow \mathrm{Ind}_{\{1\}}^{\Gamma} M \longrightarrow (\mathrm{Ind}_{\{1\}}^{\Gamma} M)/M \longrightarrow 0,$$

and we find

$$\left((\mathrm{Ind}_{\{1\}}^{\Gamma} M)/M\right)^{\Gamma} / (\mathrm{Ind}_{1}^{\Gamma} M)^{\Gamma} \simeq H^1(\Gamma, M).$$

But in Exercise 7 we proved that we also have $\check{H}^1(\Gamma, \mathrm{Ind}_{\{1\}}^{\Gamma} M) = 0$, the claim follows if we apply the exact sequence for $\check{H}$ to our exact seqence above.

Exercise 12. Let us consider the ring $R = k[X]/(X^2)$ where k is any field. Then the category of R-modules is the same as the category of k-vector spaces V together with an k-linear endomorphism $\alpha : V \longrightarrow V$ which satisfies $\alpha^2 = 0$. If $\dim_k V = 1$, then α must be zero. Compute $\mathrm{Ext}^1_R(k,k)$.

Does this ring a bell?

3 Sheaves

3.1 Presheaves and Sheaves

3.1.1 What is a Presheaf?

We start from a topological Space X and we define the category $\mathrm{Off}(X)$ of open sets: The objects are the open sets $U,V \subset X$ and the morphisms

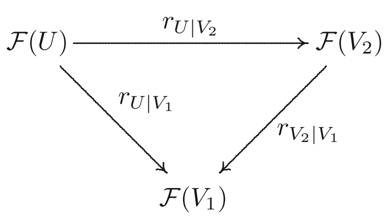
$$\mathrm{Hom}_{\mathrm{Off}(X)}(V,U) = \begin{cases} \emptyset & \text{if} \quad V \not\subset U \\ \{i\} & i \text{ is the inclusion if} \quad V \subset U. \end{cases} \tag{3.1}$$

Definition 3.1.1. *A* **presheaf** *on X with values in a category $\mathcal{C}$ is a contravariant functor from the category* $\mathrm{Off}(X)$ *with values in the category $\mathcal{C}$.*

We say again what this means: To any open set $U \subset X$ our presheaf $\mathcal{F}$ associates an object $\mathcal{F}(U) \in \mathrm{Ob}(\mathcal{C})$. Whenever we have an inclusion $V \xrightarrow{i} U$ we get a so-called **restriction morphism**

$$r_{U|V} : \mathcal{F}(U) \longrightarrow \mathcal{F}(V). \tag{3.2}$$

Of course we have $r_{U|U} = \mathrm{Id}$ and for $V_1 \subset V_2 \subset U$ we get a commutative diagram



which can be written awkwardly

$$r_{U|V_1} = r_{V_2|V_1} \circ r_{U|V_2}. \tag{Sh0}$$

If this functor $\mathcal{F}$ takes values in the category **Ab** of abelian groups (rings, modules over a ring, vector spaces, sets,...) we call it a presheaf of abelian groups (rings, modules over a ring, vector spaces, sets,...). For us the target category will always be of one these simpler categories. This means that the objects $\mathcal{F}(U)$ will be sets equipped with some kind of additional structure and the morphisms will be maps which respect this additional structure.

Under this assumption we know what the elements in $\mathcal{F}(U)$ are, they will be called the **sections** of $\mathcal{F}$ over U.

Sometimes it is a nagging question what $\mathcal{F}(\emptyset)$ should be. Usually we can take for $\mathcal{F}(\emptyset)$ a so called **final object** in the category, this is an object Ω such that for any other object $X \in \mathrm{Ob}(\mathcal{C})$ we have exactly one morphism from X to Ω. For the category of sets we can take any set consisting of just one element and for the category $\mathbf{Mod}_R$ we can take the zero module.

G. Harder, *Lectures on Algebraic Geometry I*, DOI 10.1007/978-3-8348-8330-8_3,

It is clear that presheaves with values in a given category $\mathcal{C}$ on X form a category $\mathcal{PS}_X$ by themselves. A morphism $\Psi \in \mathrm{Hom}_{\mathcal{PS}_X}(\mathcal{F},\mathcal{G})$ between two presheaves is a collection of morphisms

$$\Psi_U : \mathcal{F}(U) \longrightarrow \mathcal{G}(U), \tag{3.3}$$

which satisfies the obvious rule of consistency: whenever we have $V \subset U$ we get a commutative diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\Psi_U} & \mathcal{G}(U) \\ \big\downarrow r_{U|V} & & \big\downarrow r_{U|V} \\ \mathcal{F}(V) & \xrightarrow{\Psi_V} & \mathcal{G}(V) \end{array} \tag{3.4}$$

(If we were pedantic, we should also denote the $r_{U|V}$ differently ($r^{\mathcal{F}}_{U|V}$ or so).)
The category of presheaves (in a suitable target category $\mathcal{C}$) contains a (so called "full") subcategory, this is the category of sheaves. Before I can define sheaves I need:

3.1.2 A Remark about Products and Presheaf

Let us assume we have two indexing sets I,J and two families of objects $\{X_i\}_{i\in I}$, $\{Y_j\}_{j\in J}$ in a category with products. Assume that we have a map $\tau : J \longrightarrow I$ and in addition that for every $j \in J$ we have a morphism $f(j) : X_{\tau(j)} \longrightarrow Y_j$. Then we get for $j \in J$ a composition morphism

$$f(j) \circ p(\tau(j)) : \prod_{i\in I} X_i \longrightarrow Y_j$$

It is the definition of the product that this gives us a unique morphism

$$f_\tau : \prod_{i\in I} X_i \longrightarrow \prod_{j\in J} Y_j$$

which for any $j \in J$ produces a commutative diagram

$$\begin{array}{ccc} \prod_{i\in I} X_i & \xrightarrow{f_\tau} & \prod_{j\in J} Y_j \\ \big\downarrow & & \big\downarrow \\ X_{\tau(j)} & \xrightarrow{f(j)} & Y_j \end{array} \tag{3.5}$$

Hence morphisms from one product into another product can be obtained from maps between the indexing sets in the opposite direction and morphisms between the objects indexed by indices related by this map. We say that this arrow is *induced by the maps between the indexing sets and the maps between the objects.*

3.1.3 What is a Sheaf?

Now we explain the extra condition a presheaf has to satisfy if it wants to be a sheaf. We need that the target category $\mathcal{C}$ has products. For our purpose it is good enough if it is a category of rings or a category of modules.

Let $\mathcal{F}$ be a $\mathcal{C}$-valued presheaf on our space X. Let $U \subset X$ be open, let $U = \cup_{\alpha \in A} U_\alpha$ be an open covering. Then we get a diagram of maps

$$\mathcal{F}(U) \xrightarrow{p_0} \prod_{\alpha \in A} \mathcal{F}(U_\alpha) \underset{p_2}{\overset{p_1}{\rightrightarrows}} \prod_{(\alpha,\beta) \in A \times A} \mathcal{F}(U_\alpha \cap U_\beta), \tag{3.6}$$

where the arrows are given as follows: The arrow p_0 is induced by the maps

$$\mathcal{F}(U) \xrightarrow{r_{U|U_\alpha}} \mathcal{F}(U_\alpha)$$

and p_1,p_2 are induced by the two projections

$$\begin{array}{ccc} A \times A & \rightrightarrows & A \\ (\alpha,\beta) & \longmapsto & \alpha \\ (\alpha,\beta) & \longmapsto & \beta \end{array}$$

and the restriction maps $\mathcal{F}(U_\alpha) \longrightarrow \mathcal{F}(U_\alpha \cap U_\beta)$. If we assume that our target category is the category of sets, (abelian) groups, rings ... where the product is the simple-minded product then we can see what happens to $s \in \mathcal{F}(U)$: It is mapped to $\big(\ldots, r_{U|U_\alpha}(s), \ldots\big)_{\alpha \in A}$. For a section $(\ldots, s_\alpha, \ldots)_{\alpha \in A} \in \prod_{\alpha \in A} \mathcal{F}(U_\alpha)$ we have

$$\prod_{\alpha \in A} \mathcal{F}(U_\alpha) \ni (\ldots, s_\alpha, \ldots)_{\alpha \in A} \begin{array}{l} \xrightarrow{\quad p_1 \quad} \big(\ldots, r_{U_\alpha|U_\alpha \cap U_\beta}(s_\alpha), \ldots\big)_{(\alpha,\beta) \in A \times A} \\ \xrightarrow{\quad p_2 \quad} \big(\ldots, r_{U_\beta|U_\alpha \cap U_\beta}(s_\beta), \ldots\big)_{(\alpha,\beta) \in A \times A} \end{array}$$

In any case it is clear from condition Sh0 (see page 35) that the first arrow "equalizes" the two arrows p_1,p_2. This means that $p_1 \circ p_0 = p_2 \circ p_0$. Now we are ready to state the condition a sheaf has to satisfy. For simplicity we assume that our target category is one of the simple ones above.

Definition 3.1.2 (Sheaf)**.** *A presheaf $\mathcal{F}$ is a* **sheaf** *if and only if*

(Sh1) *The arrow p_0 is injective.*

(Sh2) *The image of p_0 is exactly the set of those elements where p_1,p_2 take the same values.*

We summarize the two conditions into

$$\mathcal{F}(U) \xrightarrow{\sim} \left(\prod_{\alpha \in A} \mathcal{F}(U_\alpha) \underset{p_2}{\overset{p_1}{\rightrightarrows}} \prod_{(\alpha,\beta) \in A \times A} \mathcal{F}(U_\alpha \cap U_\beta) \right) [p_1 = p_2]\,. \tag{3.7}$$

We will say that the above sequence is an **exact sequence of sets**.
Comment: In the case of an abstract target category $\mathcal{C}$ we would have to explain what injectivity of p_0 means and how we define the object $[p_1 = p_2]$ for a pair of morphisms $A \underset{p_2}{\overset{p_1}{\rightrightarrows}} B$. This is actually not so difficult.
Now we fix a target category $\mathcal{C}$. The sheaves with values in $\mathcal{C}$ form a "full" subcategory $\mathcal{S}_X$ of the category of presheaves with values in $\mathcal{C}$. This means that each sheaf is also a presheaf and for any two sheaves $\mathcal{F}$,$\mathcal{G}$ on X the sets of morphisms in the category of sheaves and in the category of presheaves are the same, i.e.

$$\mathrm{Hom}_{\mathcal{PS}_X}(\mathcal{F},\mathcal{G}) = \mathrm{Hom}_{\mathcal{S}_X}(\mathcal{F},\mathcal{G}).$$

3.1.4 Examples

Example 13. *On any topological Space X we have the sheaf $\mathcal{C}^0$ of continuous - or $\mathbb{C}$-valued functions.*

For any open set $U \subset X$ we put $\mathcal{C}^0(U)$ = ring of real (or complex) valued continuous functions on U. The restriction maps are given by the restriction of functions. The properties $(Sh1)$,$(Sh2)$ are obvious because the continuity of a function can be checked locally.

Example 14. *We can define the sheaf $U \longrightarrow {}_X(U)$ as the sheaf of locally constant - valued functions on U.*

Note that

$$U \longrightarrow \text{ constant } \quad \text{- valued functions on } U$$

would only define a presheaf because condition Sh2 will not be satisfied in general. This makes it clear what the general rule is: whenever we have a class of functions defined by certain properties then they provide a sheaf if these properties can be checked locally.
Of course we can replace by any abelian group A and define the sheaf

$$U \longrightarrow A_X(U) = \text{locally constant } A\text{-valued functions on } U.$$

We may look at these sheaves from a different point of view. We can put the discrete topology on A, and then we see that $A_X(U)$ is simply the abelian group of continuous functions on U with values in A. Sometimes we will write $\underline{A}$ instead of A_X.
If we have a point $p \in X$ then we can define the ring of germs of continuous functions in this point p.

Definition 3.1.3 (Germ)**.** *A* **germ** *of a continuous function at p is a continuous function $f : U_p \longrightarrow \mathbb{C}$ defined in an open neighborhood U_p of p modulo the following equivalence relation:*

$$(f : U_p \longrightarrow \mathbb{C}) \sim (g : V_p \longrightarrow \mathbb{C})$$

if and only if there is a neighborhood $W_p \subset U_p \cap V_p$ of p such that and $f|W_p = g|W_p$.

It is clear that the germs form a ring which is called $\mathcal{C}_{0,X,p}$. It is clear that this ring is the direct limit

$$\varinjlim_{U \ni p} \mathcal{C}^0(U) = \mathcal{C}^{0,X,p}$$

(See section 1.3.5).
This ring is a **local ring**, which means that it has a unique maximal ideal. This maximal ideal $\mathfrak{m}_p$ is the kernel of the evaluation at p. To see this one has to observe that a germ f which does not vanish at p also does not vanish in a small neighborhood of p and on this neighborhood we can define the continuous function $1/f$. This means that f is invertible in $\mathcal{C}^0_{X,p}$ and it follows that any ideal in $\mathcal{C}^0_{X,p}$ which is not contained in $\mathfrak{m}_p$ is the entire ring. Of course such a ring of germs is pretty big.
If we do the same thing with our sheaf ${}_X$ then it is clear that a germ at p is determined by its value at p. Hence in this case the ring of germs is simply ${}_{X,p} =$. This is not a local ring.

3.2 Manifolds as Locally Ringed Spaces

3.2.1 What Are Manifolds?

At this point I want to explain that the concept of sheaves gives us a better way to think of topological ($\mathcal{C}^0-$), differentiable ($\mathcal{C}^\infty-$) or complex manifolds. I hope the explanation will also be helpful for the understanding of the concept of sheaves.

Definition 3.2.1. *A* **topological manifold** X *is a Hausdorff space such that for each point* $p \in X$ *we can find an open neighborhood* U_p *of* p *which is homeomorphic to an open set in* $\ ^n$: $U_p \xrightarrow{\sim} U \subset \ ^n$.

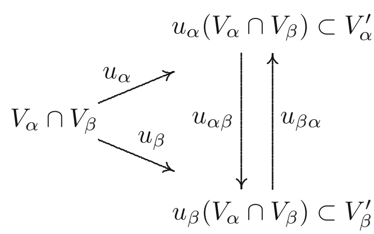
This is also called a $\mathcal{C}^0$-manifold, on this space we can define the sheaf $\mathcal{C}_X^0$ of germs of continuous functions with values in or $\mathbb{C}$.
A non-trivial theorem in algebraic topology asserts that two non-empty open sets $U \subset \ ^n$ and $V \subset \ ^m$ can only be homeomorphic if $n = m$ (see section 4.4.5). This allows us to speak of the dimension of the topological manifold provided it is connected.
I now recall the conventional definition of differentiable or complex manifolds.

Definition 3.2.2. *A* $\mathcal{C}^\infty$**-manifold** *of dimension* n *is a topological manifold* X *together with a* $\mathcal{C}^\infty$*-atlas: This is a family* $\{V_\alpha, u_\alpha\}_{\alpha \in A}$ *of open subsets such that*

(i) $X = \bigcup_{\alpha \in A} V_\alpha$

(ii) *The* u_α *are homeomorphisms* $u_\alpha : V_\alpha \xrightarrow{\sim} V'_\alpha$ *where the* V'_α *are open subsets in* $\ ^n$.

(iii) *If* $V_\alpha \cap V_\beta \neq \emptyset$ *then we get a diagram*



(3.8)

and we demand that $u_{\alpha\beta}, u_{\beta\alpha}$ *are* $\mathcal{C}^\infty$*-maps.*

The maps $u_\alpha : V_\alpha \xrightarrow{\sim} V'_\alpha$ *are called the* **local charts** *of the atlas. In this case it is easier to see that the dimension is well defined.*

We may define a **complex manifold** of dimension n in the same way. We demand that the V'_α are open in $\mathbb{C}^n$ and the $u_{\alpha\beta}, u_{\beta\alpha}$ are holomorphic maps. Of course it is clear that a complex manifold of dimension n also carries a structure of a $\mathcal{C}^\infty$-manifold of dimension $2n$.
Once we have the notion of $\mathcal{C}^\infty$-manifold (resp. complex manifold) we may define the sheaves of germs of $\mathcal{C}^\infty-$ (resp. holomorphic) functions:

Definition 3.2.3. *For* $U \subset X$ *and* $f : U \longrightarrow \mathbb{C}$, *we say that* f *is* $\mathcal{C}^\infty$ *(resp. holomorphic) if for any* $p \in U$ *and any* V_α *with* $p \in V_\alpha$ *the map*

$$\widetilde{f}_\alpha = f \circ u_\alpha^{-1} : u_\alpha(V_\alpha \cap U) \longrightarrow \mathbb{C}$$

is $\mathcal{C}^\infty$ *(resp. holomorphic).*

Let us denote these sheaves by $\mathcal{C}_X^\infty$ and $\mathcal{O}_X$ respectively.
After defining a $\mathcal{C}^\infty$- (resp. complex) manifold this way there is still a lot of talking about how to compare different atlases, what are equivalence classes of atlases, what are maximal atlases and so on.
With our definition we know what it means that a map $h : X \longrightarrow Y$ between two $\mathcal{C}^\infty$ (resp. complex) manifolds is a $\mathcal{C}^\infty$ (resp. holomorphic) map. Such a map should be continuous and then we use the atlases to formulate what else should be true, namely that the maps induced by the local charts should be $\mathcal{C}^\infty$ (resp. holomorphic). But we see that there is a different way of formulating that h is $\mathcal{C}^\infty$ (resp. holomorphic): Whenever we have open sets $U \subset X$,$V \subset Y$ such that $h(U) \subset V$, i.e. $h : U \longrightarrow V$ and a section $f \in \mathcal{C}_X^\infty(V)$ (resp. $f \in \mathcal{O}_X(V)$) then the composite $f \circ h$ is certainly continuous. It is not hard to check, that our map is $\mathcal{C}^\infty$ (resp. holomorphic) if and only if for any such pair U,V and any f the composite map $f \circ h$ is again $\mathcal{C}^\infty$ (resp. holomorphic), i.e. we get a map

$$\circ h : \mathcal{C}_Y^\infty(V) \longrightarrow \mathcal{C}_X^\infty(U) \quad (\text{resp.} \circ h : \mathcal{O}_Y(V) \longrightarrow \mathcal{O}_X(U)).$$

A better formulation is obtained if we introduce the sheaf (see the following sections on f_*,f^* and the adjointness formula) $h^*(\mathcal{C}_Y^\infty)$ on X: For any open subset $U \subset X$ the space of section $h^*(\mathcal{C}_Y^\infty)(U)$ consists of functions $f : U \longrightarrow \mathbb{C}$ which have the following property:
For any point $p \in U$ we can find a neighborhood U_p of p and an open set $V_{h(p)} \subset Y$ such that $h(U_p) \subset V_{h(p)}$ and we can find a section $\widetilde{f} \in \mathcal{C}_X^\infty(V_{h(p)})$ so that

$$f = \widetilde{f} \circ h.$$

Then we can say that a map $h : X \longrightarrow Y$ is $\mathcal{C}^\infty$ (resp. holomorphic) if it is continuous and induces a map

$$\begin{array}{rcl} \circ h : h^*(\mathcal{C}_Y^\infty) & \longrightarrow & \mathcal{C}_X^\infty \\ (\text{resp.} \quad \circ h : h^*(\mathcal{O}_Y) & \longrightarrow & \mathcal{O}_X). \end{array}$$

Of course the composition with h always induces a map

$$h^*(\mathcal{C}_Y^0) \longrightarrow \mathcal{C}_X^0$$

between the sheaves of continuous functions. A $\mathcal{C}^\infty$ resp. holomorphic map h has to respect the distinguished subsheaves which have been defined using the atlases.
I want to explain that these concepts of manifolds become much clearer if we follow Grothendieck and introduce the concept of **locally ringed spaces**. We turn the whole thing around and formulate a new definition of a $\mathcal{C}^\infty-$ (resp. complex) manifold:

Definition 3.2.4. *A $\mathcal{C}^\infty-$ (resp. complex)* **manifold** *is a topological space X together with a subsheaf $\mathcal{C}_X^\infty$ (resp. $\mathcal{O}_X$) in the sheaf of continuous functions such that for any point $p \in X$ we have a neighborhood U_p of p and a homeomorphism h between U_p and an open subset U' of $\ ^n$ (resp. $\mathbb{C}^n$) such that*

$$\begin{array}{rcl} (U_p,\mathcal{C}_X^\infty) & \simeq & (U_p',\mathcal{C}_{U_p'}^\infty) \\ (\textit{resp.} \ (U_p,\mathcal{O}_X) & \simeq & (U_p',\mathcal{O}_{U_p'})) \end{array}$$

where U_p is open in $\ ^n$ (resp. $\mathbb{C}^n$) and the sheaves are the sheaves of $\mathcal{C}^\infty$ (resp. holomorphic) functions on U_p' and where $\simeq$ means that the composition $\circ h$ induces an isomorphism between the subsheaves.

In very simple words: A *so and so* manifold is a topological manifold on which we have a subsheaf of the sheaf of continuous functions which locally looks like the sheaf of *so and so* functions on some simple model space. In our examples the stalks are local rings, hence we get examples of so called **locally ringed** spaces.
It is not only so that we get a much clearer concept of $\mathcal{C}^\infty-$ or complex manifolds. It turns out that this concept allows generalizations to cases where we cannot work with atlases anymore. (see example 18)) and Chapter 6 in the second volume)
Let X be a $\mathcal{C}^\infty$-manifold of dimension d and $p \in M$ a point. We still have charts. By definition we can find a neighborhood U_p and sections $x_1,x_2,\ldots,x_d \in \mathcal{C}^\infty_X(U)$ such that the map $\underline{x} : U_p \longrightarrow \ {}^d$ which is given by $\underline{x}(q) = (x_1(q),x_2(q),\ldots,x_d(q))$ is a homeomorphism from U_p to an open subset $U' \subset \ {}^d$ and such that a function $f : U_p \longrightarrow$ is in $\mathcal{C}^\infty_X(U_p)$ if and only if $f \circ \underline{x}^{-1} : U' \longrightarrow$ is a $\mathcal{C}^\infty-$ function. Such a collection $x_1,x_2,\ldots,x_d$ will be called a **system of local coordinates** at p. We will then say that $f = f(x_1,x_2,\ldots,x_d)$ is a $\mathcal{C}^\infty$ function *in the variables* $x_1,x_2,\ldots,x_d$.
It is possible to define the category of locally ringed spaces.

Definition 3.2.5 (Locally Ringed Space)**.** *A* **locally ringed space** *is a topological space X together with a sheaf of rings whose stalks (see section 3.3.1) are local rings.*

To define the morphisms we start from continous maps $f : X \longrightarrow Y$ between the spaces. Then we use the functors f_*,f^* (see section 3.4) to formulate what happens between the sheaves. We will encounter these objects in the second volume Chapter 6.

3.2.2 Examples and Exercise

I want to discuss a couple of examples and exercises.

Example 15.

(a) *We define the structure of a complex space on the one dimensional projective space* $\mathbb{P}^1(\mathbb{C})$*. As a topological space this is the space of lines in* $\mathbb{C}^2$ *passing through the origin. This is also the space of all pairs* $(z_0,z_1) \neq (0,0)$ *of complex numbers divided by the equivalence relation*

$$(z_0,z_1) \sim (\lambda z_0,\lambda z_1), \quad \lambda \in \mathbb{C}^*.$$

We have the two open subsets U_0 *(respectively* U_1*) where the coordinate* $z_0 \neq 0$ *(respectively* $z_1 \neq 0$*.) On these open subsets we can normalize the non-zero coordinate to one and get bijections*

$$\begin{aligned} U_0 &\xrightarrow{\sim} \mathbb{C}, \quad U_1 \xrightarrow{\sim} \mathbb{C}, \\ (1,z) &\mapsto z, \quad (u,1) \mapsto u \end{aligned}$$

Now we define the sheaf $\mathcal{O}_{\ 1}$ *: For any open subset* $U \subset \mathbb{P}^1(\mathbb{C})$ *the sections of* $\mathcal{O}_{\ 1}(U)$ *consist of those* $\mathbb{C}$*-valued functions whose restrictions to* $U_0 \cap U$ *resp.* $U_1 \cap U$ *are holomorphic.*

(b) *Of course we can define the n-dimensional projective space* $\mathbb{P}^n(\mathbb{C})$*. Again it is the space of lines in* $\mathbb{C}^{n+1}$ *passing through the origin. We can identify this to the space*

$$\{(z_0,\cdots,z_n) \in \mathbb{C}^{n+1} \mid \quad \textit{not all} \quad z_i = 0\}/\mathbb{C}^*$$

where $\mathbb{C}^*$ acts diagonally. We define the subset

$$U_i = \{(z_0, \cdots, z_n) \in \mathbb{C}^{n+1} \mid z_i \neq 0\}/\mathbb{C}^*$$

and identify $U_i \widetilde{\longrightarrow} \mathbb{C}^n$ by the map $(z_0, \cdots, z_i, \cdots, z_n) \longmapsto \left(\frac{z_0}{z_i}, \cdots, \frac{z_n}{z_i}\right)$. The sheaf of holomorphic functions on $\mathbb{P}^n(\mathbb{C})$ is the sheaf of those functions whose restriction to the U_i is holomorphic.

Example 16. *We choose a lattice*

$$\Omega = \{n_1\omega_1 + n_2\omega_2 \mid n_1, n_2 \in \quad \}$$

in $\mathbb{C}$, where ω_1, ω_2 are linearly independent over . This lattice operates by translations in $\mathbb{C}$, we form the quotient $\mathbb{C}/\Omega$ as a topological space, the projection

$$\pi : \mathbb{C} \longrightarrow \mathbb{C}/\Omega.$$

is locally a homeomorphism. We define

$$\mathcal{O}_{\ /\Omega}(U) = \{f : U \longrightarrow \mathbb{C} \mid \pi^{-1}(U) \xrightarrow{f \circ \pi} \mathbb{C} \text{ is holomorphic}\}.$$

Then it is clear that this gives $\mathbb{C}/\Omega$ the structure of a complex manifold.

Example 17. *Let us assume that we have a holomorphic function $f : U \longrightarrow \mathbb{C}$ where $U \subset \mathbb{C}$ is open and contains the origin. We assume $f(0) = 0$. We consider f as a germ and we assume that its n-th iteration is the identity, i.e. $f(f(\ldots f(z))\ldots) = z$. We assume the f is of exact order n, i. e. no earlier iteration gives the identity. Of course $f(z) = \zeta z + a_1 z^2 \ldots$ where $\zeta = e^{\frac{2\pi i k}{m}}$ and $(k, m) = 1$. We can find a smaller open set $D \subset U$ such that $f(D) = D$. This defines a holomorphic action of the cyclic group $G = < f^\nu >$ of order n on D and we can form the quotient under this action. This is the space $D/G = B$. Let $\pi : D \longrightarrow B$ the projection map. We define a sheaf $\mathcal{O}_B$ on B: For any open set $V \subset B$ we define $\mathcal{O}_B(V)$ as the ring of holomorphic functions on the inverse image $\pi^{-1}(V) \subset D$ which are invariant under the action of the cyclic group G.*

Exercise 13. Prove that this sheaf defines a structure of a one dimensional complex manifold on B.
Hint: Consider the special case where $U = \mathbb{C}$ and $f(z) = \zeta z$ first. Of course the problem arises only in a neighborhood of the origin 0. There the stalk of the sheaf $\mathcal{O}_B$ is ring of power series in $w = z^n$ which have a strictly positive radius of convergence. Then return to the general case and prove that you can find a germ of a function $g(z) = z + b_2 z^2 + b_3 z^3 \ldots$ such that $f(g(z)) = g(\zeta z)$ and show that this reduces the problem to the first case.

Example 18. *Let us consider $\mathbb{C}^2$ and consider the following action of our cyclic group:*

$$f : (z_1, z_2) \mapsto (\zeta z_1, \zeta^{-1} z_2)$$

If we form the quotient $\pi : \mathbb{C}^2 \longrightarrow \mathbb{C}^2/G = B$ we can try to play the same game. Again we get the structure of a two dimensional complex variety except at the point $\pi(0) = 0$. Here we see that the germ of our sheaf $\mathcal{O}_B$ becomes a power series ring in $u = z_1^n, v = z_2^n, w = z_1 z_2$ and we have $uv = w^n$. This means u, v, w are not independent variables anymore. At the point 0 our space is **singular** *and not locally isomorphic to $(\mathbb{C}^2, \mathcal{O}_{\ 2})$. But our concepts of locally ringed spaces are strong enough to deal with this situation. Our example has the structure of a* **complex space** *which may have singular points.*

Example 19.

(a) *I want to give an idea of what a complex space might be. This is more subtle, and I need some difficult theorems from local complex analysis. We assume that* $U \subset \mathbb{C}^n$ *is an open subset and* $f_1(z_1, \cdots, z_n), \cdots, f_r(z_1, \cdots, z_n)$ *are holomorphic functions on* U*. Then we can consider the ideal* $I \subset \mathcal{O}_{\mathbb{C}^n}(U)$ *which is generated by these functions. We can look at the subset* Y *of common zeroes of the* f_i*, i.e.*

$$Y = \{\underline{z} = (z_1, \cdots, z_n) \mid f_i(\underline{z}) = 0 \text{ for all } i = 1 \cdots r\}$$

and this is of course also the set of common zeroes of all the $f \in I$*.*

Of course Y *is a topological space, for any open subset* $V \subset Y$ *we can look at the open sets* $U' \subset U$ *with* $U' \cap Y = V$*. For any such* U' *we form the quotient*

$$\mathcal{O}_{\mathbb{C}^n}(U')/(f_1, \cdots, f_r)$$

where $(f_1, \cdots, f_r)$ *is the* $\mathcal{O}_{\mathbb{C}^n}(U')$*-ideal generated by the* f_i*. We put*

$$\mathcal{O}_Y(V) = \varinjlim_{U' : U' \cap Y = V} \mathcal{O}_{\mathbb{C}^n}(U')/(f_1, \cdots, f_r).$$

Now it follows from deep theorems in local complex analysis that $V \longrightarrow \mathcal{O}_Y(V)$ *is in fact a sheaf (see [Gr-Re1], we can avoid this reference if we use the construction of quotient sheaves below). One checks that the stalk* $\mathcal{O}_{Y,y} = \varinjlim_{V : y \in V} \mathcal{O}_Y(V)$ *is a local ring and the pair* $(Y, \mathcal{O}_Y)$ *is in fact a locally ringed space. It can serve as a local model for a general complex space. I want to point out that we cannot interpret the rings* $\mathcal{O}_Y(V)$ *as rings of holomorphic functions on* Y*. We may for instance consider the case that* $U = \mathbb{C}$*, and we take the single function* $f(z) = z^2$*. Then* $Y = \{0\}$ *and the local ring is* $\mathbb{C}[z]/(z^2)$*. It contains nilpotent elements and cannot be interpreted as ring of holomorphic functions.*

But still our space $(Y, \mathcal{O}_Y)$ *is called a* **complex space** *(see [Gr-Re1], [Gr-Re2], Chap. I)).*

(b) *We say that our system of equations satisfies the* **Jacobi criterion** *in a point* $y \in Y$ *if the Jacobian matrix*

$$\left(\frac{\partial f_i}{\partial z_j}\right)_{i,j}(y) \qquad i = 1, \cdots, r, j = 1, 2 \cdots, n$$

has maximal rank r*. Then this is still true in a small open neighborhood of* y*. The theorem on implicit functions says that in a small neighborhood* $U_1 \subset \mathbb{C}^n$ *of* y *we can perform a change of coordinates* $u_i = g_i(z_1 \cdots z_n)$ *for* $i = 1, 2, \cdots, n$ *such that in the new coordinates our functions become* $f_1(u_1 \cdots u_n) = u_1, \cdots, f_r(u_1 \cdots u_n) = u_r$*. Hence we see that in this neighborhood*

$$Y \cap U_1 = \{(0, \cdots, 0, u_{r+1}, \cdots, u_n) \mid u_i \quad \text{suff. small}\},$$

and then $(Y, \mathcal{O}_Y)$ *is clearly an* $(n-r)$*-dimensional complex manifold in the neighborhood of* $y \in Y$*. In this case we do not have to invoke the above mentioned theorem.*

We can turn this around and say that a subset $Y \subset U$ is a d-dimensional submanifold of U if we can describe it locally as the common set of zeroes of $n-d$ holomorphic functions which satisfy the Jacobi criterion.

We come back to the situation in example 19 (a). We say that the ideal I defines a (smooth) submanifold of dimension d if the set of common zeroes Y is a submanifold of dimension d and if in addition at any point $y \in Y$ we can find $g_1, \dots, g_{n-d} \in I$ which satisfy the Jacobi criterion at the point y. In this situation the argument in example 19 (b) shows that these $g_1, \dots, g_{n-d}$ generate the ideal I if we restrict it to a small neighborhood of y.

(c) *A closed subset $Y \subset \mathbb{P}^n(\mathbb{C})$ is a d dimensional complex projective manifold if for any index i the intersection $Y \cap U_i$ is a d-dimensional complex submanifold of U_i.*

(d) *A homogeneous polynomial of degree k is a polynomial*

$$f(z_0, \cdots, z_n) = \sum a_{\nu_0 \cdots \nu_n} z_0^{\nu_0} \cdots z_n^{\nu_n}$$

where $a_{\nu_0 \cdots \nu_n} = 0$ unless $\sum \nu_i = k$. We cannot consider such a polynomial as a function on $\mathbb{P}^n(\mathbb{C})$. But of course it makes sense to speak of the zeroes of this polynomial on $\mathbb{P}^n(\mathbb{C})$. Therefore we may consider an ideal $I = \{f_1, \cdots, f_s\}$ which is generated by s homogeneous polynomials. We can look at the common set of zeroes

$$Y = \{\underline{z} = (z_0 \cdots z_n) \mid \underline{z} \neq 0, f_i(\underline{z}) = 0 \text{ for all } \}/\mathbb{C}^\times.$$

Such a set Y is called an **algebraic subset** *of $\mathbb{P}^n(\mathbb{C})$.*

If we restrict a homogeneous polynomial f to one of the open sets U_i above, then we can interpret it as a function on U_i because we can normalize the i-th coordinate of a point to one. Hence our ideal I defines an ideal I_i of holomorphic functions on each of the U_i.

Such a subset $Y \subset \mathbb{P}^n(\mathbb{C})$ is called a **smooth, projective** *(algebraic) variety of dimension d if each of the ideals I_i defines a smooth submanifold of dimension d in the sense of example 19(b). This definition is not yet very satisfactory because it needs input from analysis (the implicit function theorem), for a definition in purely algebraic terms I refer to volume 2.*

It can happen that we need more than $n-d$ homogeneous equations to describe a smooth projective variety of dimension d. Locally at a point y we can choose $n-d$ equations from our set of equations to describe Y but this subset may vary if the point moves around.

If we have such a complex d-dimensional submanifold $Y \subset \mathbb{C}^n$ then the coordinate functions $z_1, \dots, z_n$ are of course holomorphic functions on $\mathbb{C}^n$. Therefore they are also holomorphic after restriction to Y. If we have a point $y \in Y$ we may consider the functions

$$\widetilde{z}_i = z_i - z_i(y) \quad \text{for } i = 1, \dots, n$$

as holomorphic functions on Y. Then it follows from example 19(b) that we can pick d functions from this set - let us assume that these are $\widetilde{z}_1, \dots, \widetilde{z}_d$ - such that the remaining functions can be written locally as convergent power series in these, i.e.

$$\widetilde{z}_{d+j} = h_j(\widetilde{z}_1, \ldots, \widetilde{z}_d) \quad j = 1 \ldots n-d$$

Then the

$$\widetilde{z}_i = z_i - z_i(y) \qquad \text{for } i = 1, \ldots, n$$

are called a system of **local parameters** *at y. We could also call them (analytic)* **local coordinates**.

3.3 Stalks and Sheafification

3.3.1 Stalks

In our examples above we had the notion of a germ of a function at a point p. This notion can be extended to the more general classes of sheaves. Let us assume that we consider the category of (pre-)sheaves on X with values in some nice category (abelian groups, rings or sets). If we have a point $p \in X$ then we consider the set $\mathfrak{U}_p$ of open sets containing our point p. We define an ordering on this set

$$V \geq U \quad \text{if and only if} \quad V \subset U. \tag{sic!}$$

Then this is an inductive system which is also directed: to any U_1, U_2 we find a V with $U_1 \leq V$, $U_2 \leq V$.

Definition 3.3.1 (Stalk)**.** *If we have a (pre-)sheaf $\mathcal{F}$ on X we define the* **stalk** *in p by*

$$\mathcal{F}_p = \varinjlim_{U \in \mathfrak{U}_p} \mathcal{F}(U),$$

and this limit is simply the (abelian group, ring, set) of germs of sections.

It inherits the structure of an (abelian group, ring, set); this follows from this directedness and is discussed in the Exercise 4 in section 1.4.
An element $s_p \in \mathcal{F}_p$ is called a **germ of a section**. By definition it can always be represented by a section $s_U \in \mathcal{F}(U)$ where $U \in \mathfrak{U}_p$. If this is so we write $s_U|_p = s_p$ and we say that s_p is the restriction of s_U to the stalk at p.
Let s be a section over the open set U. If we have $s_p = 0$ at $p \in U$ then we find an open neighborhood V of p such that s restricted to this neighborhood is zero. Hence we can define the support of s:

Definition 3.3.2 (Support)**.** *The* **support** *of a section $s \in \mathcal{F}(U)$ is the closed subset of U where $s_p \neq 0$.*

These stalks help to clarify the difference of the notion of sheaves and presheaves. For any presheaf we can consider the map

$$\mathcal{F}(U) \longrightarrow \prod_{p \in U} \mathcal{F}_p, \tag{3.9}$$

which is given by restricting the sections to the stalks. Then we know:

Lemma 3.3.3.

(i) *This map is injective, if and only if our presheaf satisfies (Sh1).*

(ii) *If a presheaf $\mathcal{F}$ satisfies (Sh1) then it is a sheaf if and only if the following holds: A collection of germs $(\dots s_p \dots)_{p\in U}$ is the restriction of a section over U if for any p we find a $U_p \in \mathfrak{U}_p$ and a section $\widetilde{s}_p \in \mathcal{F}(U_p)$ such that $\widetilde{s}_p|_q = s_q$ for all $q \in U_p$.*

We leave the verification of this fact to the reader.

3.3.2 The Process of Sheafification of a Presheaf

We will show that to any presheaf $\mathcal{G}$ on a space X we can construct a sheaf $\mathcal{G}^{\#}$ together with a map $j : \mathcal{G} \longrightarrow \mathcal{G}^{\#}$ (in the category of presheaves) such that for any sheaf $\mathcal{F}$ we have

$$\mathrm{Hom}_{\mathcal{PS}_X}(\mathcal{G},\mathcal{F}) = \mathrm{Hom}_{\mathcal{S}_X}(\mathcal{G}^{\#},\mathcal{F}). \tag{3.10}$$

This can also be seen as another example of a representable functor. Our presheaf $\mathcal{G}$ defines a functor from the category $\mathcal{S}$ of sheaves (with values in the category of rings, abelian groups, sets) into the category of sets, namely the functor $\mathcal{F} \longrightarrow \mathrm{Hom}_{\mathcal{PS}_X}(\mathcal{G},\mathcal{F})$. Our sheaf $\mathcal{G}^{\#}$ is representing this functor. Hence by the Yoneda Lemma it is unique up to isomorphism.
To see that $\mathcal{G}^{\#}$ exists we use the stalks. It is possible to define $\mathcal{G}^{\#}$ quite directly, we define

$$\mathcal{G}^{\#}(U) = \left\{ (\dots s_p \dots) \in \prod_{p\in U} \mathcal{G}_p \;\middle|\; \begin{array}{c}\text{For any point } p \in U \;\exists \text{ open } U_p \\ p\in U_p\subset U \text{ and } \widetilde{s}_p\in\mathcal{G}(U_p), \text{ s. t.} \\ \widetilde{s}_p|q=s_q \text{ for all } q\in U\end{array} \right\}. \tag{3.11}$$

The reader should verify, that this defines indeed a sheaf, this sheaf has the same stalks as our original presheaf, we have a map $\mathcal{G} \longrightarrow \mathcal{G}^{\#}$ and it has the required property.
There exist some more abstract notions of sheaves on so called Grothendieck topologies, these are in some sense "spaces" which sometimes do not have points anymore. In such a case it is not possible to use the stalks, but still it is possible to construct $\mathcal{G}^{\#}$. Therefore I will give here another construction of $\mathcal{G}^{\#}$ which does not use stalks.
We consider coverings $\mathfrak{U} = \{U_i\}_{i\in I}$, $U = \bigcup_{i\in I} U_i$ of an open set U. We introduce the category of coverings. An arrow from a covering $\mathfrak{V} = \{V_\alpha\}_{\alpha\in A}$ to the covering $\mathfrak{U} = \{U_i\}_{i\in I}$ is a map

$$\tau : A \longrightarrow I$$

such that $\bigcup_{\alpha\in\tau^{-1}(i)} V_\alpha = U_i$. We write

$$\tau : \mathfrak{V} \longrightarrow \mathfrak{U}$$

for such a morphism. In general the arrow τ is not determined by the two coverings, but many constructions using this arrow will give results not depending on it.
We will say that such an arrow defines a **refinement** of $\mathfrak{U}$ by $\mathfrak{V}$. Sometimes we will say that $\mathfrak{V}$ is a **refinement** of $\mathfrak{U}$ if there is an arrow from $\mathfrak{V}$ to $\mathfrak{U}$.
The arrow τ defines a map between diagrams (see the general remark about maps between products at the beginning of this section)

$$\begin{array}{ccccc} \mathcal{G}(U) & \xrightarrow{p_0} & \prod_{i\in I}\mathcal{G}(U_i) & \overset{p_1}{\underset{p_2}{\rightrightarrows}} & \prod_{(i,j)\in I\times I}\mathcal{G}(U_i\cap U_j) \\ \downarrow\wr & & \downarrow & & \downarrow \\ \mathcal{G}(U) & \longrightarrow & \prod_{\alpha\in A}\mathcal{G}(V_\alpha) & \rightrightarrows & \prod_{(\alpha,\beta)\in A\times A}\mathcal{G}(V_\alpha\cap V_\beta) \end{array} \tag{3.12}$$

For any covering $\mathfrak{U}=\{U_i\}_{i\in I}$ of U we define

$$\mathcal{G}^{\mathfrak{U}}(U)[p_1=p_2] := \left\{ s\in \prod_{i\in I}\mathcal{G}(U_i) \mid p_1(s)=p_2(s) \right\}. \tag{3.13}$$

If $\mathfrak{V}$ is a refinement of $\mathfrak{U}$ then our map τ defines a map

$$\begin{array}{ccc} & & \mathcal{G}^{\mathfrak{U}}(U)[p_1=p_2] \\ & \nearrow & \\ \mathcal{G}(U) & & \downarrow \\ & \searrow & \\ & & \mathcal{G}^{\mathfrak{V}}(U)[p_1=p_2] \end{array}$$

It is not difficult to see that the vertical arrow does not depend on the choice of τ. Now we need the courage to believe that we can extend the notion of direct limit to this situation where we do not have an indexing set but a category which is directed because two coverings have always a common refinement. We put

$$\mathcal{G}^+(U) = \varinjlim_{\mathfrak{U}} \mathcal{G}^{\mathfrak{U}}(U)[p_1=p_2]. \tag{3.14}$$

We check that $\mathcal{G}^+$ is again a presheaf, and it satisfies condition (Sh1). Moreover if the original presheaf $\mathcal{G}$ satisfies already (Sh1) then $\mathcal{G}^+$ satisfies even (Sh2). Hence we see that $\mathcal{G}^{++}=\mathcal{G}^\#$ is always a sheaf. We have

$$i:\mathcal{G}\longrightarrow\mathcal{G}^\#,$$

and $\mathcal{G}^\#$ has the required universal property.

3.4 The Functors f_* and f^*

Given two topological spaces X,Y and a continuous map $f:X\longrightarrow Y$, we construct two functors f_*,f^* which transport sheaves on X to sheaves on Y and sheaves on Y to sheaves on X respectively. Let us denote by $\mathcal{S}_X$ (resp. $\mathcal{PS}_X$) the category of sheaves (resp. presheaves) on X with values in the category of abelian groups, rings or sets.

Definition 3.4.1. *If we have a sheaf $\mathcal{F}$ on X we define the sheaf $f_*(\mathcal{F})$ on Y by*

$$f_*(\mathcal{F})(V)=\mathcal{F}(f^{-1}(V))$$

for all open subsets $V\subset Y$. It is clear that $f_*(\mathcal{F})$ is a sheaf on Y, it is called the **direct image** *of $\mathcal{F}$.*

We also have the functor f^*, this is called the **inverse image** or sometimes the **pullback** of a sheaf. The functor f^* transforms sheaves on Y into sheaves on X. The idea is that the stalk of $f^*(\mathcal{G})$ in a point $x \in X$ is equal to the stalk of the original sheaf $\mathcal{G}$ in the point $y = f(x)$, i.e. $f^*(\mathcal{G})_x = \mathcal{G}_{f(x)}$. The actual construction is a little bit complicated. At first we define a presheaf $f'(\mathcal{G})$:
For $U \subset X$ we put

$$f'(\mathcal{G})(U) = \varinjlim_{V \supset f(U)} \mathcal{G}(V). \tag{3.15}$$

It is not difficult to verify that this is a presheaf and that for any covering $U = \cup_{i \in I} U_i$ we get an injective map

$$f'(\mathcal{G})(U) \longrightarrow \prod_{i \in I} f'(\mathcal{G})(U_i).$$

It satisfies (Sh1) but not necessarily (Sh2).

Definition 3.4.2. *We define by $f^*(\mathcal{G}) = f'(\mathcal{G})^{\#}$. the* **inverse image** *or* **pullback** *of a sheaf $\mathcal{G}$*

We recall that the stalks of the sheafification of a presheaf are equal to the stalks of the presheaf, hence we get

$$f^*(\mathcal{G})_x = \varinjlim_{x \in U} \varinjlim_{V \supset f(U)} \mathcal{G}(V) = \varinjlim_{V : f(x) \in V} \mathcal{G}(V) = \mathcal{G}_{f(x)}. \tag{3.16}$$

3.4.1 The Adjunction Formula

The functors f_*, f^* are **adjoint** functors. To be more precise: The functor f^* is **left adjoint** to f_*. This means that we have a functorial isomorphism

$$\mathrm{Hom}_{\mathcal{S}_X}(f^*(\mathcal{G}), \mathcal{F}) = \mathrm{Hom}_{\mathcal{S}_Y}(\mathcal{G}, f_*(\mathcal{F})). \tag{3.17}$$

Here "functorial" means that from morphisms $u : \mathcal{G}' \longrightarrow \mathcal{G}$ and $v : \mathcal{F} \longrightarrow \mathcal{F}'$, we get the obvious commutative diagrams.
It is not very difficult to verify the adjointness formula. From the construction of the sheafification we have $\mathrm{Hom}_{\mathcal{PS}_X}(f'\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{\mathcal{S}_X}(f^*\mathcal{G}, \mathcal{F})$. Hence a morphism ψ in $\mathrm{Hom}_{\mathcal{PS}_X}(f'\mathcal{G}, \mathcal{F})$ is a collection of $\psi_U : f'\mathcal{G}(U) \longrightarrow \mathcal{F}(U)$. It follows from the definition of $f'\mathcal{G}(U)$ and the properties of the direct limit that this is nothing else than a collection of maps

$$\psi_{U,V} : \mathcal{G}(V) \longrightarrow \mathcal{F}(U) \tag{3.18}$$

where U, V run over all open sets in X, Y which satisfy $f(U) \subset V$, and where the maps in this collection satisfy the obvious compatibilities. We will call $\psi_{U,V}$ the evaluation of ψ on U, V. Now we are allowed to evaluate on $U = f^{-1}(V)$ and we get a collection $\psi_{f^{-1}(V),V} = \phi_V : \mathcal{G}(V) \longrightarrow f_*(\mathcal{F})(V)$, i.e. an element in $\mathrm{Hom}_{\mathcal{S}_Y}(\mathcal{G}, f_*(\mathcal{F}))$. The other direction is also clear.

Remark 2. I find it always confusing and hard to memorize which functor is a left (right) adjoint of which. The question is whether f^* has to be placed into the source or the target of the Hom(,). Here is a simple rule that helps. We have to remember that f_* gives directly a sheaf while the construction of $f^*\mathcal{G}$ involves the process of sheafification and this uses direct limits. But as I explained in the chapter on categories direct limits are made so that we know what the maps *from* them are. Hence the free place on the left in Hom(,) is the place where $f^*\mathcal{G}$ belongs.

3.4.2 Extensions and Restrictions

We can consider the special case of an open subset $U \subset X$ and let $A = X \setminus U$ be its complement. Then we have the two inclusions $i : A \hookrightarrow X$, $j : U \hookrightarrow X$. For a sheaf $\mathcal{F}$ on X the sheaf $j^*(\mathcal{F})$ is very easy to understand since for an open set $V \subset U$ we have $j^*(\mathcal{F})(V) = \mathcal{F}(V)$. This is called the **restriction** of $\mathcal{F}$ to U. The operation $i^*(\mathcal{F})$ is much more delicate and will cause us some trouble (see section 4.4.1).
If we have a sheaf $\mathcal{G}$ on U then $j_*(\mathcal{G})$ is a delicate functor since it depends on the local topology in the neighborhood of boundary points (see section 4.1.2). It is not necessarily exact.
But for a sheaf $\mathcal{G}$ on A the $i_*(\mathcal{G})$ is easy to understand. Its stalks are zero outside of A and equal to the stalks of $\mathcal{G}$ on A. It is called the **extension by zero**.

3.5 Constructions of Sheaves

If we have a family of sheaves $\{\mathcal{F}_\alpha\}_{\alpha\in A}$ the we can define the product: For any open set $U \subset X$ we put

$$\left(\prod_{\alpha\in A} \mathcal{F}_\alpha\right)(U) := \prod_{\alpha\in A} \mathcal{F}_\alpha(U) \tag{3.19}$$

and it is easy to verify that this is again a sheaf. If our sheaves have values in the category of rings, modules, abelian groups etc. the product is again a sheaf with values in that category.
We have to be a little bit careful at this point. We can not say in general that the stalks of the product are isomorphic to the product of the stalks. But if the indexing set A is finite we check easily that for any $x \in X$

$$\left(\prod_{\alpha\in A} \mathcal{F}_\alpha\right)_x := \prod_{\alpha\in A} \mathcal{F}_{\alpha,x} \tag{3.20}$$

(See also [McL] for a detailed discussion). But if the $\mathcal{F}_\alpha$ are sheaves with values in the category of abelian groups and if we know in addition the for any $x \in X$ the stalks $\mathcal{F}_{\alpha,x} = 0$ for almost all $\alpha \in A$, then 3.20 is still true.
If the $\mathcal{F}_\alpha$ are abelian groups or modules we might be tempted to take the direct sum of sheaves. But this does not work in general. The naive definition gives only a presheaf because (Sh2) may be violated if the indexing set A is infinite.

Perhaps here is the right place to explain, that the sheaves on X with values in the category of abelian groups form an **abelian category** . First of all this says that for two such sheaves the set $\Psi \in \mathrm{Hom}_{\mathcal{S}_X}(\mathcal{F},\mathcal{G})$ is an abelian group: If we have two morphisms $\Psi = \{\Psi_U,\Phi_U\}_U,\Phi = \{\Phi_U,\Phi_U\}_U$ then $\Psi + \Phi = \{\Psi_U + \Phi_U\}_U$. This group structure is bilinear with respect to composition.
If we have a morphism $\Psi : \mathcal{F} \longrightarrow \mathcal{G}$ then we can define the **kernel** $\ker(\Psi)$ as the subsheaf $U \longrightarrow \ker(\Psi_U)$. This kernel has a categorical interpretation: For any other sheaf $\mathcal{A}$

$$\mathrm{Hom}_{\mathcal{S}_X}(\mathcal{A}, \ker(\Psi)) = \{\phi \in \mathrm{Hom}_{\mathcal{S}_X}(\mathcal{F},\mathcal{G})|\Psi \circ \phi = 0\}.$$

Now we may consider the presheaf

$$\mathcal{K}(U) = \mathcal{F}(U)/\ker(\Psi)(U). \tag{3.21}$$

It is fundamental that this presheaf is not necessarily a sheaf and this will be explained in detail in the next Chapter. It is not hard to verify the first sheaf condition (Sh1) but in general it does not satisfy the second condition (Sh2). Of course we can sheafify the presheaf $\mathcal{K}$ and we get the quotient sheaf

$$\mathcal{F}/\ker(\Psi) := \mathcal{K}^{\#} \tag{3.22}$$

This quotient has again a categorical interpretation and it is called the **coimage** of Ψ. We can also define the **image** of Ψ as a subsheaf of $\mathcal{G}$. It is simply $\mathrm{im}(\Psi)(U) = \Psi_U(\mathcal{K}^{\#}(U))$ and by construction it is isomorphic to the coimage. These two objects namely the coimage and image can be defined in a categorical context and it is one of the axioms for an abelian category that they should be canonically isomorphic (see [McL]).
In an abelian category we can define the notion of exact sequences but this will be discussed in the following chapter.

4 Cohomology of Sheaves

We consider sheaves with values in abelian groups. We can define the notion of an exact sequence of sheaves.

Definition 4.0.1 (Exact Sequence of Sheaves)**.** *A sequence of sheaves on a space X*

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

is **exact** *if for all points $x \in X$ the sequence of stalks is exact. It is easy to see that this is equivalent to*

(i) *For all open sets $U \subset X$ the sequence*

$$0 \longrightarrow \mathcal{F}'(U) \longrightarrow \mathcal{F}(U) \longrightarrow \mathcal{F}''(U)$$

is exact.

(ii) *For any $s'' \in \mathcal{F}''(U)$ we can find a covering $U = \bigcup_j U_j$ by open sets and $s_j \in \mathcal{F}(U_j)$ such that $s_j \mapsto s''|U_j$.*

It is the decisive point that the exactness of the sequence of sheaves does **not** imply that $\mathcal{F}(U) \longrightarrow \mathcal{F}''(U)$ is surjective. We can only find local liftings for an $s'' \in \mathcal{F}''(U)$.
Applied to $U = X$ this tells us that the functor of global sections $\mathcal{F} \longrightarrow \mathcal{F}(X)$ will not be exact in general. Hence we have to construct a right derived functor to it. As in Chapter 2 we introduce the notation $H^0(X,\mathcal{F})$ for $\mathcal{F}(X)$ and we want construct cohomology groups $H^1(X,\mathcal{F}), H^2(X,\mathcal{F}), \ldots$ which have functorial properties and such that any short exact sequence yields a long exact sequence

$$0 \longrightarrow \mathcal{F}'(X) \longrightarrow \mathcal{F}(X) \longrightarrow \mathcal{F}''(X) \longrightarrow H^1(X,\mathcal{F}') \longrightarrow \ldots \tag{4.1}$$

as in Chapter 2.
The following two examples are absolutely fundamental. In a nutshell we see everything that makes sheaf cohomology work. I also want to stress the almost perfect analogy between these two examples which will be explained in remark 3.

4.1 Examples

4.1.1 Sheaves on Riemann surfaces

In the previous section we introduced the notion of a complex manifold (see section 3.2.1). Here I want to consider a compact **Riemann surface** $(X,\mathcal{O}_X)$. This means that X is a compact connected complex manifold of dimension 1. For any $P \in X$ we find an open neighborhood U_P of P such that (see section 3.2)

$$(U_P,\mathcal{O}_X|U_P) \simeq (B,\mathcal{O}_B), \tag{4.2}$$

G. Harder, *Lectures on Algebraic Geometry I*, DOI 10.1007/978-3-8348-8330-8_4,

where $B = \{z \in \mathbb{C} \mid |z| < 1\}$ is the open unit disc and where $\mathcal{O}_B$ is the sheaf of holomorphic functions on B. We assume that the homeomorphism between the spaces maps P to the origin 0 in the disc.
The element $z \in \mathcal{O}_B(B)$ yields via the isomorphism an element $z_P \in \mathcal{O}_X(U_P)$. This element z_P vanishes at P. Obviously the stalk $\mathcal{O}_{X,P}$ of the sheaf $\mathcal{O}_X$ at P is the local ring of power series in z_P which have a strictly positive radius of convergence. The element z_P generates the maximal ideal $\mathfrak{m}_P$ of the stalk $\mathcal{O}_{X,P}$. Such an element is called a **uniformizer** or **uniformizing element** at P. Any power series

$$u_P = f(z_P) = az_P + bz_P^2 + \dots$$

which has a positive radius of convergence and with $a \neq 0$ can serve as an uniformizer as well.

Definition 4.1.1 (Meromorphic Function)**.** *A complex function*

$$g : U_P \backslash \{P\} \longrightarrow \mathbb{C}$$

is called **meromorphic** *on U_P if it is holomorphic and if we can find an integer n such that $z_P^n \cdot g = h$ extends to a holomorphic function on U_P. We say that g has a* **pole** *of order n at P if n is the smallest value for such an integer. We write* $\mathrm{ord}_P(g) = -n$ *and by definition $g \in z_P^{-n}\mathcal{O}_{X,P}$, but $g \notin z_P^{-n+1}\mathcal{O}_{X,P}$.*

If T is a finite subset of X and if $f : X \setminus T \longrightarrow \mathbb{C}$ is a holomorphic function then we say that f is meromorphic if its singularities at the points of T are at most poles (and not essential singularities). For any point $P \in T$ we have defined $\mathrm{ord}_P(f)$.

Definition 4.1.2. *We define the* **polar divisor** *of f by*

$$\mathrm{Div}_\infty(f) = \sum_{P \in T,\, \mathrm{ord}_P(f) < 0} \mathrm{ord}_P(f) P$$

which we consider as an an element in the **divisor group** $\mathrm{Div}(X)$*, this is the free abelian group generated by the points of X. Since X is compact it follows that f can only have a finite number of zeroes on $U = X \setminus T$ and this implies that $1/f$ is also holomorphic on some open set $U' = X \setminus T'$ where T' is finite and then $1/f$ is also meromorphic. We may also define the* **zero divisor** *of f as*

$$\mathrm{Div}_0(f) = -\mathrm{Div}_\infty(1/f)$$

and the **divisor** *of f as*

$$\mathrm{Div}(f) = \mathrm{Div}_0(f) + \mathrm{Div}_\infty(f).$$

Definition 4.1.3. *We have a homomorphism called the* **degree** *of a divisor ,*

$$\deg : \mathrm{Div}(X) \longrightarrow$$

which is given by $\deg : D = \sum n_P P \mapsto \sum n_P$.

Definition 4.1.4. *A divisor $D = \sum_P n_P P$ which is the divisor of a non-zero meromorphic function will be called a* **principal divisor***.*

We will see (see 5.1.4) that for a principal divisor $D = \mathrm{Div}(f)$ the degree $\deg(D) = \sum n_P = 0$. To any divisor $D = \sum_P n_P P$ we attach the sheaf $\mathcal{O}_X(D)$ which is defined by

$$\mathcal{O}_X(D)(U) = \{f \text{ meromorphic on } U \mid \mathrm{ord}_P(f) \geq -n_P \text{ for all } P \in U\}. \tag{4.3}$$

We could also say that $f \in z_P^{-n_P}\mathcal{O}_{X,P}$ for all P.

Definition 4.1.5. *A divisor $D = \sum n_P P$ is called* **effective** *if all $n_P \geq 0$, we could also call this a* **positive** *divisor and write $D \geq 0$.*

The definition of $\mathcal{O}_X(D)(U)$ can be reformulated: It consists of those meromorphic functions f on U for which the restriction $\mathrm{Div}(f)+D|U \geq 0$. If D is an effective divisor we have an inclusion of sheaves $\mathcal{O}_X \subset \mathcal{O}_X(D)$. We form the quotient sheaf $\mathbb{L}_D = \mathcal{O}_X(D)/\mathcal{O}_X$. It is clear that the stalk at P is $z_P^{-n_P}\mathcal{O}_{X,P}/\mathcal{O}_{X,P}$.
For any point P

$$z_P^{-n}\mathcal{O}_{X,P}/\mathcal{O}_{X,P} = \mathbb{L}_P^{(n)}$$

is the finite dimensional vector space of Laurent expansions at P of order $\leq n$, an element $\ell \in \mathbb{L}_P^{(n)}$ can be written as

$$\ell = \frac{a_n}{z_P^n} + \frac{a_{n-1}}{z_P^{n-1}} + \ldots + \frac{a_1}{z_P} \qquad \mathrm{mod} \quad \mathcal{O}_{X,P}. \tag{4.4}$$

If $a_n \neq 0$, we say that ℓ has a pole of order n. So the stalk of this sheaf at a point $P \in X$ is the vector space of all Laurent expansions of pole order $\leq n_P$. Especially the stalk is zero at points where $n_P = 0$ and therefore the sheaf $\mathbb{L}_D$ has only a finite number of non-zero stalks. It is called a **skyscraper sheaf**. We have the exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(D) \longrightarrow \mathbb{L}_D \longrightarrow 0. \tag{4.5}$$

It is clear that the space of sections $H^0(X,\mathbb{L}_D)$ is simply the direct sum of the stalks in the points P with $n_P > 0$. There is no interaction between the different points.
The question whether the sequence of global sections

$$0 \longrightarrow H^0(X,\mathcal{O}_X) \longrightarrow H^0(X,\mathcal{O}_X(D)) \longrightarrow H^0(X,\mathbb{L}_D) \longrightarrow 0$$

is exact amounts to whether a given collection of Laurent expansions at the finitely many points P with $n_P > 0$ can be realized by a meromorphic function on X. In general the answer is no and the discrepancy is controlled by the first cohomology group $H^1(X,\mathcal{O}_X)$ which we will define later. To be more precise we will construct a map

$$\delta : H^0(X,\mathbb{L}_D) \longrightarrow H^1(X,\mathcal{O}_X)$$

such that the extended sequence

$$0 \longrightarrow H^0(X,\mathcal{O}_X) \longrightarrow H^0(X,\mathcal{O}_X(D)) \longrightarrow H^0(X,\mathbb{L}_D) \longrightarrow H^1(X,\mathcal{O}_X) \tag{4.6}$$

becomes exact. The computation of $H^1(X,\mathcal{O}_X)$ is more or less equivalent to the Riemann-Roch Theorem which we will discuss in the chapter 5 on compact Riemann surfaces.

Exercise 14. Prove that in the case $X = \mathbb{P}^1(\mathbb{C})$ the above sequence of global sections is always exact.

Exercise 15. Prove that in the case $X = \mathbb{C}/\Omega$ the above sequence of global sections is not always exact.

4.1.2 Cohomology of the Circle

We consider the circle S^1 and the sheaf $\_$ which is defined by

$$\_(V) = \{f : V \longrightarrow \quad \mid f \quad \text{is locally constant}\}. \tag{4.7}$$

We pick a point $P \in S^1$ and let $U = S^1 \backslash \{P\}$. We define a sheaf $\ ^{(P)}$ on S^1 by

$$\_^{(P)}(V) = \_(U \cap V).$$

If $i : U \longrightarrow S^1$ is the inclusion then this is the sheaf $i_*(\_)$ (see 3.4.2). Clearly we have an inclusion $\_ \subset \_^{(P)}$ and for all $Q \neq P$ we have the equality of stalks

$$\__Q = \_^P_Q = \quad .$$

But in the point P we have

$$= \_P \hookrightarrow (\_^{(P)})_P = \quad \oplus$$

because on a little interval I_ε containing P we have $\_(I_\varepsilon) = \quad$ but $\_^{(P)}(I_\varepsilon) = \_(I_\varepsilon \cap U) = \quad \oplus \quad$. Hence we get an exact sequence of sheaves

$$0 \longrightarrow \_ \longrightarrow \_^{(P)} \longrightarrow \mathcal{S}_P \longrightarrow 0$$

where $\mathcal{S}_P$ is the skyscraper sheaf whose stalk at P is $\quad$ and zero elsewhere. We get the sequence of global sections

$$\begin{array}{ccccccc}
0 & \longrightarrow & H^0(S^1, _) & \xrightarrow{\sim} & H^0(S^1, _^{(P)}) & \longrightarrow & H^0(S^1, \mathcal{S}_P) \\
 & & \| & & \| & & \| \\
0 & \longrightarrow & & \xrightarrow{\sim} & & \longrightarrow &
\end{array}$$

and we see that the last arrow is not surjective. Again we need a non-zero $H^1(S^1, \ )$ to control the discrepancy.

We even can have an idea what this group $H^1(S^1, \ )$ should be. Intuitively we should think that the sheaf $\ ^{(P)}$ doubles the point P, so our circle becomes an interval I and it is at least plausible that

$$H^1(I, \ ) = H^1(S^1, \ ^{(P)}).$$

But the interval is contractible (see 4.4.1,4.4.24), and we will see that this implies $H^1(I, \ ) = 0$ (see 4.4.10). Hence we should expect (and we will prove this later) that

$$H^0(S^1, \mathcal{S}_P) \xrightarrow{\sim} H^1(S^1, \ ) \simeq \quad . \tag{4.8}$$

I want to stress another important point. We can ask whether $H^1(S^1, \ ) \simeq \quad$ is a canonical isomorphism. The answer is no!

This becomes clear if we recall that

$$H^0(S^1, \mathcal{S}_P) = (\quad \oplus \quad)/ \tag{4.9}$$

where $\quad$ is embedded diagonally. There is no way to distinguish between the two possibilities to identify $H^0(S^1, \mathcal{S}_P)$ to $\quad$.

But we can choose an orientation on S^1 (see page 62), this means that at each point we choose a direction (i.e. non zero tangent vector to S^1 up to a positive scalar) which varies continuously with the point. Then we have a distinction between the two intervals in the intersection $I_\varepsilon \cap U = U_\varepsilon^+ \cup U_\varepsilon^-$: We say that U_ε^+ is the interval which the chosen tangent vector at P points to. Then

$$\_(I_\varepsilon \cap U) = \_(U_\varepsilon^+) \oplus \_(U_\varepsilon^-) = \quad \oplus \quad , \tag{4.10}$$

and we now have a canonical identification $H^1(S, \quad) = \quad$ where we send (a,b) mod $\quad \mapsto a$.

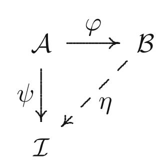
Remark 3. I want to stress the analogy between the two examples: The sheaves $\mathcal{O}_X$ and _ have a property in common: They are very rigid. This means that any section over a connected open subset U is determined by its restriction to an arbitrarily small non empty open subset $V \subset U$.
The analogy goes even further. If we consider the sheaf _ on a manifold M, then we can characterize _ as a subsheaf in the sheaf $\mathcal{C}_M^\infty$: It is the subsheaf of functions with zero derivatives. An analogous statement is true for $\mathcal{O}_X$. We can characterize $\mathcal{O}_X$ as the subsheaf in the sheaf of $\mathcal{C}^\infty$-functions annihilated by the **Cauchy-Riemann operator**.

4.2 The Derived Functor

4.2.1 Injective Sheaves and Derived Functors

We want to define a universal derived functor to the functor $\mathcal{F} \longrightarrow \mathcal{F}(X) = H^0(X,\mathcal{F})$. To do this we use the same ideas as in Chapter 2. We define the notion of an injective sheaf:

Definition 4.2.1. *A sheaf* $\mathcal{I}$ *is* **injective** *if in any diagram*



with $\ker(\varphi) \subset \ker(\psi)$ *we can find a map* $\eta : \mathcal{B} \longrightarrow \mathcal{I}$ *which makes this diagram commutative.*

It is rather easy to see that every sheaf $\mathcal{F}$ can be embedded into an injective sheaf. The following construction has been invented by Godement (see [Go]4.3). For any point $x \in X$ we embed the stalk $\mathcal{F}_x$ by an injection i_x into an injective abelian group I_x. We define the sheaf $\mathcal{I}$ by

$$\mathcal{I}(U) = \prod_{x \in U} I_x \tag{4.11}$$

and the restriction maps $\prod_{x\in U} I_x \longrightarrow \prod_{x \in V} I_x$ are induced by the inclusion $V \subset U$.
To prove the injectivity of $\mathcal{I}$ we consider our diagram above stalk by stalk and choose for each $x \in X$ an η_x such that the diagram

$$\begin{array}{ccc} \mathcal{A}_x & \xrightarrow{\varphi_x} & \mathcal{B}_x \\ {\scriptstyle \psi_x}\downarrow & \swarrow {\scriptstyle \eta_x} & \\ \mathcal{I}_x & & \end{array}$$

commutes. The collection of the η_x is a homomorphism from $\mathcal{B}$ to $\mathcal{I}$.
By construction this collection of i_x provides an embedding $i : \mathcal{F} \to \mathcal{I}$, for any open set $U \subset X$ the homomorphism $\eta_U : \mathcal{F}(U) \to \mathcal{I}(U)$ is induced by the maps $\mathcal{F}(U) \to \mathcal{F}_x \to I_x$.
Now it is obvious that we can find an injective resolution for any sheaf $\mathcal{F}$:

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}^0 \longrightarrow \mathcal{I}^1 \longrightarrow \dots \quad .$$

Consequently we define

$$H^\bullet(X,\mathcal{F}) = H^\bullet(\mathcal{I}^\bullet(X)). \tag{4.12}$$

The same arguments as in the previous section show that this defines a universal **right derived functor**.
The reader might (or should) be scared: How can we ever compute the cohomology of a sheaf if we use such *huge and bizarre* sheaves to define it?
Our strategy will be to exhibit classes of *smaller* sheaves which have the property that they are acyclic. One possibility to construct such sheaves is discussed in the following exercise.

Exercise 16. Let us assume that we have a sheaf of commutative rings $\mathcal{R}$ on X, the rings should have an identity, especially we have $1 \in \mathcal{R}(X)$. Let us assume that we have a so called partition of 1: For any covering $X = \bigcup_{i \in I} U_i$ we can find elements $h_i \in \mathcal{R}(X)$ such that $\mathrm{Supp}(h_i) \subset U_i$, for any point $x \in X$ we have only finitely many indices such that $h_{ix} \neq 0$ and finally $1 = \sum_i h_i$.
Show that sheaves $\mathcal{F}$ of $\mathcal{R}$-modules are acyclic.
Hint: Assume we have a short exact sequence

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

of $\mathcal{R}$-modules. Use the partition of unity to show that $\mathcal{F}(X) \longrightarrow \mathcal{F}''(X)$ is surjective. Then use the arguments above to show, that any sheaf of $\mathcal{R}$-modules has an injective resolution by sheaves of $\mathcal{R}$-modules.

We will see that on a $\mathcal{C}^\infty$-manifold M the sheaves of rings of $\mathcal{C}^\infty$-functions have a partition of unity. This will imply that for any $\mathcal{C}^\infty$-vector bundle (see 4.3.2) $\mathcal{E}$ and the sheaf $\mathcal{C}^\infty(\mathcal{E})$ of $\mathcal{C}^\infty$-sections in it

$$H^i(M,\mathcal{C}^\infty(\mathcal{E})) = 0 \text{ for all } i > 0. \tag{4.13}$$

4.2.2 A Direct Definition of H^1

We want to indicate briefly how we could approach the problem to define a right derived functor for $H^0(X,\mathcal{F})$ more directly. The reader should notice the analogy between this approach and the one used to define the first cohomology group in group cohomology (see section 2.2.2).
Let us assume we have an exact sequence of sheaves

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0.$$

We look at $\mathcal{F}(X) \longrightarrow \mathcal{F}''(X)$ and pick a section $s'' \in \mathcal{F}''(X)$. We want to find an $s \in \mathcal{F}(X)$ which maps to s''. Locally we can solve this problem. This means we can find a covering $X = \bigcup_{\alpha \in \mathcal{A}} U_\alpha$ and sections $s_\alpha \in \mathcal{F}(U_\alpha)$ which map to $s'' \mid U_\alpha$. But the s_α do not necessarily match: The difference

$$s'_{\alpha,\beta} = s_\alpha - s_\beta \mid U_\alpha \cap U_\beta \tag{4.14}$$

is a section in $\mathcal{F}'(U_\alpha \cap U_\beta)$ because it goes to zero in $\mathcal{F}''$. The collection $\{s'_{\alpha,\beta}\}_{(\alpha,\beta)\in\mathcal{A}\times\mathcal{A}}$ satisfies the **cocycle relation**, i.e. we have

$$s'_{\alpha,\beta} - s'_{\beta,\gamma} + s'_{\gamma,\alpha} \mid U_\alpha \cap U_\beta \cap U_\gamma = 0. \tag{4.15}$$

This suggests the definition of the group of 1-cocycles with respect to a covering $\mathfrak{U} = \{U_\alpha\}_{\alpha\in\mathcal{A}}$:

Definition 4.2.2. *The* **1-cocycles** *with respect to a covering $\mathfrak{U} = \{U_\alpha\}_{\alpha\in\mathcal{A}}$ are collections $(\dots,t_{\alpha,\beta},\dots) \in \prod_{(\alpha,\beta)\in\mathcal{A}\times\mathcal{A}} \mathcal{F}'(U_\alpha \cap U_\beta)$ which satisfy the cocycle relation (equation 4.15) above. They form a group which will be denoted by $Z^1(\mathfrak{U},\mathcal{F}')$.*

We may also define the group of coboundaries:

Definition 4.2.3. *An element $(\dots t_{\alpha,\beta} \dots)$ is a* **coboundary** *if we can find $s'_\alpha \in \mathcal{F}'(U_\alpha)$ such that $t'_{\alpha,\beta} = s'_\alpha - s'_\beta$. They form a group which will be denoted by $B^1(\mathfrak{U},\mathcal{F}')$.*

Definition 4.2.4 (Cohomology). *We define $H^1(X,\mathfrak{U},\mathcal{F}')$ to be the quotient*

$$H^1(X,\mathfrak{U},\mathcal{F}') = Z^1(\mathfrak{U},\mathcal{F}')/B^1(\mathfrak{U},\mathcal{F}').$$

Now it is clear that s'' defines an element $\delta(s'') \in H^1(S,\mathfrak{U},\mathcal{F}')$, and it is clear that s'' is in the image of $\mathcal{F}(X) \longrightarrow \mathcal{F}''(X)$ if and only if $\delta(s'') = 0$.
If we start from a different covering $\mathfrak{U}'$, then $\mathfrak{U}$ and $\mathfrak{U}'$ have a common refinement (see section 3.3.2) $\tau : \mathfrak{W} \longrightarrow \mathfrak{U}$, $\tau' : \mathfrak{W} \longrightarrow \mathfrak{U}'$. We get maps

$$\begin{array}{ccc} H^1(X,\mathfrak{U},\mathcal{F}') & & H^1(X,\mathfrak{U}',\mathcal{F}'). \\ & \searrow \quad \swarrow & \\ & H^1(X,\mathfrak{W},\mathcal{F}') & \end{array}$$

It is not difficult to see that these maps do not depend on the choice of of the arrows.
It is clear that these maps are compatible with δ and hence we get a **boundary operator**

$$\delta : \mathcal{F}''(X) \longrightarrow \varinjlim_{\mathfrak{U}} H^1(X,\mathfrak{U},\mathcal{F}') := \check{H}^1(X,\mathcal{F}'). \tag{4.16}$$

It is rather clear that we have a structure of an abelian group on the limit, the boundary operator is a homomorphism and the sequence

$$0 \longrightarrow H^0(X,\mathcal{F}') \longrightarrow H^0(X,\mathcal{F}) \longrightarrow H^0(X,\mathcal{F}'') \xrightarrow{\delta} \check{H}^1(X,\mathcal{F}') \longrightarrow \check{H}^1(X,\mathcal{F}) \longrightarrow \check{H}^1(X,\mathcal{F}'')$$

is exact.
Of course we need to compare this construction of cohomology groups with the other one using injective resolutions, this will be done in the exercise 17 below.

Definition 4.2.5. *A sheaf $\mathcal{F}$ on a space X is called* **flabby** *if for any open set $U \subset X$ the restriction map $\mathcal{F}(X) \longrightarrow \mathcal{F}(U)$ is surjective.*

This is a very strange property of a sheaf. For instance the continuous functions on a space almost never have this property.

Lemma 4.2.6. *Injective sheaves are flabby.*

Proof: To show this we consider an open subset $U \subset X$, we denote its inclusion by $j : U \longrightarrow X$. Let $A = X \setminus U$, let us denote the inclusion of the closed set by $i : A \longrightarrow X$. For any sheaf $\mathcal{F}$ we can take its restriction to A and extend this restriction again to X by using i_*. (Extension by zero: See section 3.4.2) We have a surjective homomorphism of sheaves $\mathcal{F} \longrightarrow i_* i^*(\mathcal{F})$ and this gives us an exact sequence of sheaves

$$0 \longrightarrow j_!(\mathcal{F}) \longrightarrow \mathcal{F} \longrightarrow i_* i^*(\mathcal{F}) \longrightarrow 0 \tag{4.17}$$

where of course $j_!(\mathcal{F})$ is just the kernel.
A short digression: We may give a direct definition of this kernel and call it again the extension of $\mathcal{F}|U$ to X by zero. To give this direct definition we recall the notion of the support of a section (see section 3.3.1) and notice that for any open set $V \subset X$ we have more or less by definition

$$j_!(\mathcal{F})(V) = \{s \in \mathcal{F}(V) \mid \text{the support of } s \text{ does not meet } V \cap A\}. \tag{4.18}$$

This means that this sheaf is a little bit delicate. By construction we have an inclusion $j_!(\mathcal{F})(V) \hookrightarrow \mathcal{F}$. In a sense the sheaf $j_!(\mathcal{F})$ "knows" the boundary points of U.
Now we come back to our original problem, we wanted to show that injective sheaves are flabby. We have an inclusion

$$\begin{array}{ccc} j_!\mathcal{I} & \longrightarrow & j_*\mathcal{I} \\ \downarrow & \swarrow \phi & \\ \mathcal{I} & & \end{array} \tag{4.19}$$

and since $\mathcal{I}$ is injective we find a homomorphism $\phi : j_*\mathcal{I} \longrightarrow \mathcal{I}$ which makes this diagram commute. If we have a section $s \in \mathcal{I}(U)$ then this is by definition the same as a section $s \in i_*(\mathcal{I})(X)$ and then $\phi(s) \in \mathcal{I}(X)$. It is clear from the diagram that $\phi(s)$ restricted to U is s. Moreover we see that our section $\phi(s)$ has support contained in the closure $\overline{U}$, the best we can expect. □

Exercise 17.

(a) Show that for a flabby sheaf $\mathcal{F}$ we have $\check{H}^1(X,\mathcal{F}) = 0$.

(b) Show that $\check{H}^1(X,\mathcal{I}) = 0$ for an injective sheaf. Show that this implies that for any sheaf $\mathcal{F}$

$$\check{H}^1(X,\mathcal{F}) = H^1(X,\mathcal{F}).$$

(c) Show that flabby sheaves are acyclic.

I discussed this construction of the first cohomology groups in detail, because here we can see how natural these constructions are. We meet a fundamental principle of homological algebra which is applied again and again:

Fundamental principle of homological algebra: *We want to lift a section $s'' \in H^0(X,\mathcal{F}'')$ to a section $s \in H^0(X,\mathcal{F})$. We localize the problem by choosing a covering for which we have local liftings. These are not unique and hence it can happen that they do not match on the intersections. These differences on the intersections yield a cocycle, and the class of this cocycle yields the obstruction to the global solution of the problem.*
We have seen how the same principle works in group cohomology (section 2.2.1). There we want to lift a Γ-invariant element $m'' \in (M'')^\Gamma$ to a Γ-invariant section $m \in M^\Gamma$. In this context localizing means that we drop the requirement that m should be $\Gamma-$invariant. Then we find a non unique lifting. The comparison of the local sections on the intersections of the open sets in the geometric situation corresponds here to the comparison of m with γm where γ runs through the group. This gives the cocycles $\gamma \mapsto m - \gamma m \in M'$.

This construction generalizes to higher cohomology groups. We can define the so called Čech cohomology by means of coverings. The cohomology defined by means of injective resolutions and the Čech cohomology coincide on reasonable spaces. We postpone this discussion.
At this point we make a short detour. Since we discussed H^1 in some detail it may be appropriate to discuss the *non-abelian* H^1, this means we discuss sheaves with values in non commutative groups and their first cohomology sets. This non-abelian cohomology plays an important role in the theory of bundles and I want to say some words about this subject.

4.3 Fiber Bundles and Non Abelian H^1

4.3.1 Fibrations

Fibre Bundle

I want to introduce the notion of fibre bundles.

Definition 4.3.1. *We consider maps between topological spaces $\pi : X \longrightarrow B$. If we have another such map $\pi' : X' \longrightarrow B$ then a* **map over** *B is a continuous map $f : X' \longrightarrow X$ for which $\pi \circ f = \pi'$.*

If $X' = B$ and $\pi' = \mathrm{Id}$ then a a map $f : B \longrightarrow X$ over B is also called **a section to** π.

We could also say that we have the category of spaces over B, this are spaces X together with a map $\pi : X \longrightarrow B$ and the morphisms are maps over B.

Definition 4.3.2 (Fibration)**.** *Let F (the* **fibre***) be a space and B (the* **base***) another space. A continuous map $\pi : X \longrightarrow B$ is called a* **(locally trivial) fibration** *with fibre F, if we can find a covering $B = \bigcup_{i \in I} U_i$ such that for any i we can find a homeomorphism Ψ_i over the base U_i*

$$\begin{array}{ccc} \pi^{-1}(U_i) & \xrightarrow[\sim]{\Psi_i} & U_i \times F \\ & \searrow \quad \swarrow {\scriptstyle \mathrm{pr}_1} & \\ & U_i & \end{array}$$

Locally in the base our space is a product of an open set in the base and the given fibre. We also say that $X \xrightarrow{\pi} B$ *is a* **fibre bundle** *with fibre* F*. The covering together with the maps* Ψ_i *is called a* **local trivialization**.

Definition 4.3.3. *If* $V \subset B$ *is open then a* **section** *to* π *over* V *is a continuous map* $s : V \longrightarrow X$ *for which* $\pi \circ s = \mathrm{Id}_V$*. We denote this set by* $\mathcal{C}(V)$ *and then the assignment* $V \longrightarrow \mathcal{C}(V)$ *defines the* **sheaf** $\mathcal{X}$ **of sections** *of the bundle* $X \longrightarrow B$.

It is important to consider fibers F which are not only topological spaces but also carry some extra structure.

Vector Bundles

For instance we can consider the case that F is a finite dimensional - or $\mathbb{C}$-vector space and where F, , $\mathbb{C}$ are equipped with the standard topology. For convenience we denote by a field which is either or $\mathbb{C}$. In this case we can make an additional assumption on our local trivialization. We assume that we have a covering $B = \bigcup_{i \in I} U_i$ and

$$\Psi_i : \pi^{-1}(U_i) \xrightarrow{\sim} U_i \times F \tag{4.20}$$

as before. But in addition we assume that for any pair i,j of indices the map

$$G_{ij} = (\Psi_j \mid U_i \cap U_j) \circ (\Psi_i^{-1} \mid U_i \cap U_j) : (U_i \cap U_j) \times F \longrightarrow (U_i \cap U_j) \times F \tag{4.21}$$

has the form $G_{ij}(u,x) = (u,g_{ij}(u)x)$ where $g_{ij}(u)$ is a **linear** automorphism of our vector space F.
It is clear that $u \mapsto g_{ij}(u)$ must be a continuous map from $U_i \cap U_j$ into the general linear group $G = Gl_n(\ )$. Moreover, it is obvious that we have a cocycle relation: For any triplet i,j,k of indices we have

$$g_{ij}(u) \cdot g_{jk}(u) = g_{ik}(u) \quad \text{for all} \quad u \in U_i \cap U_j \cap U_k. \tag{4.22}$$

Definition 4.3.4. *If this assumption (eq.* (4.22)*) is fulfilled, we say that* $\pi : X \longrightarrow B$ *is an* n*-dimensional* **vector bundle**.

I find this definition a little bit unsatisfactory because it needs the covering and the Ψ_i. We will give a second definition which I think is better. Of course our data allow us to introduce the structure of a vector space on each fibre $\pi^{-1}(b)$ such that the vector space structure "varies continuously with b". What do we mean by that? Our definition also implies for any i that we can find sections

$$e_1, \cdots ,e_n : U_i \longrightarrow \pi^{-1}(U_i) = U_i \times F,$$

such that in each point $u \in U_i$ the elements $e_1(u), \cdots ,e_n(u) \in \pi^{-1}(u)$ form a basis of this vector space. Now we can identify

$$\pi^{-1}(u) \xrightarrow{\sim} \ ^n$$

by sending $\sum a_\nu e_\nu(u) \longmapsto (a_1, \cdots a_n)$, and we get a map

$$\pi^{-1}(U_i) \longrightarrow U_i \times {}^n , \tag{4.23}$$

and the phrase "the vector space structure varies continuously with b" means that this is a homeomorphism.
This allows us to give a different formulation of the concept of a vector bundle. We can say that

Lemma 4.3.5 (Vector Bundle). *$\pi : X \longrightarrow B$ is an n-dimensional vector bundle if:*

(a) *For any $b \in B$ we have the structure of an n dimensional -vector space on the fibre $\pi^{-1}(b)$.*

(b) *For any $b \in B$ we can find a neighborhood V of b and sections.*

$$e_1, \cdots, e_n : V \longrightarrow \pi^{-1}(V)$$

such that these sections evaluated at any point $v \in V$ form a basis of $\pi^{-1}(v)$.

(c) *The map $\pi^{-1}(V) \longrightarrow V \times {}^n$, sending a point $x = \sum a_i\ e_i(v)$ over $v \in B$ to $(v, a_1, \cdots, a_n) \in V \times {}^n$, is a homeomorphism.*

Definition 4.3.6 (Local Trivialization). *If we have such a vector bundle $\pi : X \longrightarrow B$, and if we have an open set $V \subset B$ together with the sections*

$$e_i : V \longrightarrow \pi^{-1}(V), \qquad i = 1, \cdots, n$$

which form a basis at any point $v \in V$, then we call this a **local trivialization** *of a bundle.*

The sheaf $\mathcal{X}$ of sections into X has the natural structure a **module over the sheaf of continuous functions** $\mathcal{C}_B^0$: We can form the sum of two sections $s_1, s_2 \in \mathcal{X}(V)$ and multiply a section $s \in \mathcal{X}(V)$ by a section $f \in \mathcal{C}_B^0(V)$. This module is in fact locally free. If we have a trivialization $e_1, \ldots, e_n \in \mathcal{X}(V)$, then any section is of the form $s = \sum_i f_i e_i$, with $f_i \in \mathcal{C}_B^0(V)$. It is clear that we can define the concept of a **locally free module** over any locally ringed space.
On the other hand it is rather clear that a locally free module $\mathcal{E}$ over $\mathcal{C}_B^0$ also gives us a vector bundle. This observation is certainly not very deep but important.
If the base space B is a $\mathcal{C}^\infty$ manifold, then we can define what a $\mathcal{C}^\infty$ bundle is. In this case X is also a $\mathcal{C}^\infty$ manifold, we have the same assumptions on the fibres, the local sections are $\mathcal{C}^\infty$ and the map in c) is also $\mathcal{C}^\infty$.

4.3.2 Non-Abelian H^1

We know of course what it means that two vector bundles $X \longrightarrow B$ are isomorphic. Actually it is obvious that the vector bundles over a given base space form a category: A continuous map

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & X' \\ & \searrow \quad \nearrow & \\ & B & \end{array}$$

is a morphism of vector bundles, if φ restricted to the fibres is linear.
I want to explain the description of the set of isomorphism classes of vector bundles on B in terms of **non-abelian sheaf cohomology**. Given our vector bundle we select a covering $\mathfrak{V} = \{V_i\}_{i\in I}$ of B and local trivializations

$$e_{i,\nu} : V_i \longrightarrow \pi^{-1}(V_i), \qquad \nu = 1, \cdots ,n.$$

If we have an ordered pair (i,j) of indices, then we get a continuous map

$$g_{ij} : V_i \cap V_j \longrightarrow GL(n,\ \ )$$

such that

$$g_{ij}(v) \cdot (e_{i,\nu}(v)) = e_{j,\nu}(v). \tag{4.24}$$

To avoid misunderstandings: The topology on $GL(n,\ \ )$ is the standard topology (see Lemma 4.3.10)
This is clearly a one-cocycle, this means

$$\begin{aligned} g_{ij} \cdot g_{jk} &= g_{ik} \quad \text{on} \quad V_i \cap V_j \cap V_k \quad \text{and} \\ g_{ii} &= \mathrm{Id}\,. \end{aligned} \tag{4.25}$$

This suggests that we introduce the set of 1-cocycles with respect to our covering. We introduce the sheaf of germs of continuous maps form our space B to the group $G = GL(n,\ \ )$, we denote this sheaf by $\mathcal{C}^0(G)$. Then we define as before

$$Z^1(\mathfrak{V},\mathcal{C}^0(G)) = \{\underline{c} = (\cdots ,g_{ij}, \cdots) \in \prod_{i,j} \mathcal{C}(G)(V_i \cap V_j) \mid \underline{c} \text{ is a 1-cocycle}\}. \tag{4.26}$$

If we modify our local trivialization, then we modify the cocycle into $g'_{ij} = h_i g_{ij} h_j^{-1}$ on $V_i \cap V_j$, where $\underline{h} \in \prod \mathcal{C}^0(G)(V_i)$. This gives us an equivalence relation on $C^1(\mathfrak{V},\mathcal{C}^0(G))$ and dividing by this relation we get a set $H^1(B,\mathfrak{V},\mathcal{C}^0(G))$. Again we may change the covering, we can pass to common refinements and we end up with

$$H^1(B,\mathcal{C}^0(G)) = \varinjlim_{\mathfrak{V}} H^1(B,\mathfrak{V},\mathcal{C}^0(G)). \tag{4.27}$$

Since for $n > 1$ our sheaf $\mathcal{C}^0(G)$ takes values in the category of non-abelian groups, we cannot multiply cocycles and therefore we only get a set.
Now it follows from our considerations that:

Lemma 4.3.7. *The elements in $H^1(B,\mathcal{C}^0(G))$ are in one-to-one correspondence with the set of isomorphism classes of n-dimensional vector bundles on B.*

A completely analogous statement holds for $\mathcal{C}^\infty$ vector bundles.

4.3.3 The Reduction of the Structure Group

Orientation

We may introduce different kinds of additional structures on the fibres of a vector bundle $\pi : X \longrightarrow B$. For instance we may choose a Euclidian metric $< \ , \ >_b$ on the fibres which varies continuously with the point. Then we can choose local trivializations $e_1, \ldots ,e_n$ which are given by orthonormal basis vectors. If we compare two such local trivializations then our functions g_{ij} will be functions with values in the orthogonal group $O(n)$ and therefore it will correspond to an element in $H^1(B,\mathcal{C}^0(O(n)))$.

In such a situation we say that the additional structure induces a **reduction of the structure group**. In our special case above we have a reduction from $\mathrm{GL}_n(\quad)$ to the orthogonal group $O(n)$. (Compare [B-T] §6). If we have such a vector bundle with such a euclidian metric on it, we call it a **euclidian vector bundle**.
If we have a bundle of $\mathbb{C}$-vector spaces and we have a Hermitian metric on the fibres, which varies continuously, then we get by the same procedure a reduction of the structure group to $U(n)$. Such a bundle is called a **hermitian vector bundle**. A **euclidian (resp. hermitian) form** h is a family of euclidian (resp.hermitian) forms on the fibres, which gives the bundle the structure of a euclidian (resp. hermitan) bundle. If the bundle is $\mathcal{C}^\infty$ then we know what it means that h is $\mathcal{C}^\infty$.
Another such additional structure is an **orientation**. If we consider the highest exterior power $\Lambda^n(X/B)$, i.e we take the highest exterior power fibre by fibre, then we get a bundle of one dimensional vector spaces. On this bundle we have an action of the multiplicative group of positive real numbers $\ ^*_{>0}$. If we divide the bundle by this action then the quotient is a bundle $\widetilde{B} \longrightarrow B$ with fibres consisting of two points. If we can find a global section $s : B \longrightarrow \widetilde{B}$, then we say that $X \longrightarrow B$ is **orientable**. If we choose such a section then we say that $X \longrightarrow B$ is **oriented**.
If we have an orientation on B then we may choose local trivializations $e_1,\dots,e_n$ for which the ordered basis is positive with respect to the orientation. If we have done this then our g_{ij} will take values in the subgroup $Gl_n(\quad)^+$ of matrices with positive determinant and thus we have another case of the reduction of the structure group.
On a $\mathcal{C}^\infty$ manifold M we have the notion of the tangent bundle T_M. Locally on M we have coordinate functions $x_1,\dots,x_n$ so that any differentiable function is a differentiable function in the variables $x_1,\dots,x_n$ (see section 3.2). Then the vector fields $\partial/\partial x_1,\dots,\partial/\partial x_n$. provide a local trivialization of this tangent bundle. (See [Hir],[B-T] §6.)
(Actually I think there is no reason to look up a reference. A tangent vector Y at a point p is by definition a map $Y : \mathcal{C}^\infty_{M,p} \longrightarrow \quad$ which is $\quad$-linear and satisfies the Leibniz rule: We have $Y(fg) = f(p)Y(g) + g(p)Y(f)$ for all $f,g \in \mathcal{C}^\infty_{M,p}$. Such a Y is determined by its values on local coordinates $x_1,x_2,\dots x_n$. We define the tangent vectors $\frac{\partial}{\partial x_i}(x_j) = \delta_{ij}$. Then these $\frac{\partial}{\partial x_i}$ are also tangent vectors in the domain of validity of the local coordinates.)

Definition 4.3.8 (Riemannian Manifold, Oriented Manifold)**.** *If we have in addition a Euclidian metric on the tangent bundle then M is called a* **Riemannian manifold***. If we have chosen an orientation (if possible) then we call M* **oriented***.*

A caveat Of course we know what it means that M is a $\mathcal{C}^r$ manifold, here the local rings consist of functions which are only r-times differentiable. Then we loose a degree of differentiability if we define the tangent bundle, it is only a $\mathcal{C}^{r-1}$ manifold.

Local Systems

If B is a topological space and A an abelian group, then we attached to A the sheaf $\underline{A} = A_B$ of locally constant functions with values in A (see examples in 3.1.4).
We want to introduce the notion of **local A-systems** or **local systems** of A's.

Definition 4.3.9. *If $\mathcal{A}$ is a sheaf of abelian groups on B, then we call $\mathcal{A}$ a* **local A-system***, if for any point $b \in B$ we can find an open neighborhood V_b such that the restriction of $\mathcal{A}$ to V_b is isomorphic to A_{V_b}.*

This implies that for any point $b \in B$ the stalk $\mathcal{A}_b$ is isomorphic to A. At this point it is reasonable to assume that our space B is locally connected, i.e. for any point $b \in B$ and any open neighborhood V_b of b we can find a connected open neighborhood $U_b \subset V_b$ of b. If we have that $\mathcal{A} \mid V_b$ is isomorphic to A_{V_b} as above, and if we replace V_b by the connected open neighborhood U_b, then $\mathcal{A}(U_b) \simeq A$, and for any point $u \in U_b$ we get an isomorphism $\mathcal{A}(U_b) \longrightarrow \mathcal{A}_u$. If we now fix a covering $B = \bigcup V_i$, where the V_i are connected and we have isomorphisms

$$\Psi_i : \mathcal{A} \mid V_i \xrightarrow{\sim} A_{V_i},$$

then we may compare the Ψ_i on the intersections and we get

$$g_{ij} : V_i \cap V_j \longrightarrow \mathrm{Aut}(A)$$

which are locally constant (or continuous if $\mathrm{Aut}(A)$ is endowed with the discrete topology). Hence we see that the local A-systems are classified by the elements in $H^1(B,\underline{\mathrm{Aut}}(A))$ where $\underline{\mathrm{Aut}}(A)$ is the sheaf of locally constant functions in $\mathrm{Aut}(A)$.

Isomorphism Classes of Local Systems

We introduce the notion **local systems of vector spaces**. These are simply local systems where the group A has the additional structure of an $\ -$ or $\mathbb{C}-$vector space. Hence such a local system is a vector bundle

$$\pi : X \longrightarrow B$$

where each point $p \in B$ has an open connected neighborhood V, over which we have local sections $e_1, \ldots, e_n$, which are called **constant**. If we pass to a different connected open set V' over which we have constant sections $e'_1, \ldots, e'_n$, then on the intersection

$$e'_i = \sum a_{ij} e_j \tag{4.28}$$

where now the a_{ij} are locally constant functions on $V \cap V'$. Of course we can describe the set of isomorphism classes of local systems of vector spaces in terms of non-abelian cohomology. We consider the group $G_d = GL_n(\ )_d$ which is the general linear group but endowed with the discrete topology. It is clear that

Lemma 4.3.10. *The isomorphism classes of local systems of n-dimensional -vector spaces are given by $H^1(B,GL_n(\ )_d)$.*

These local systems of vector spaces are the same kind of objects as bundles with a flat connection (see also sections 4.10.1 and 4.10.2).

Principal G-bundels

Of course we can start from any topological group G, we can consider the sheaf of G-valued functions on B and we can look at the cohomology set $H^1(B,\mathcal{C}^0(G))$. This set classifies so called principal G-bundles.

Definition 4.3.11. *A bundle $P \longrightarrow B$ with a left action of G such that G acts simply transitively on the fibres is called a* **principal G-bundle**. *Then G is called the* **structure group** *of $P \longrightarrow B$. Two such principal bundles $P_1 \longrightarrow B$ and $P_2 \longrightarrow B$ are isomorphic if we have a a bundle isomorphism $\phi : P_1 \xrightarrow{\sim} P_2$ which commutes with the a ction of G. The* **trivial bundle** *is $G \times B \longrightarrow B$, with the left action of G on itself.*

Giving a local trivialization over an open set $U \subset B$ is the same as giving a section of the bundle over U.

4.4 Fundamental Properties of the Cohomology of Sheaves

4.4.1 Introduction

I will now state some results concerning the cohomology of sheaves. They are not so easy to prove. The proofs are sometimes a little bit sketchy, some steps are treated in the exercises.
If we have any space X and an abelian group A then we have defined the sheaf A_X of germs of locally constant A-valued functions: This is the **constant sheaf** attached to A. Sometimes – if it is clear what the underlying space is – we simply write $\underline{A}$. Then the underlining is made to distinguish the abelian group from the sheaf. (I am not sure whether this is actually necessary.)

Definition 4.4.1 (Cohomology of Sheaves)**.** *We define the cohomology of X with coefficients in A as*

$$H^\bullet(X,A) := H^\bullet(X,A_X).$$

If $A =$ then the cohomology groups $H^\bullet(X, \ )$ are equal to the ones defined by singular cochains, if the space X is reasonable. (This is a theorem, we come back to it later).
The first important result, which we will show, is that the cohomology of constant sheaves vanishes on certain contractible spaces. We begin by stating a special case which is also the starting point for the more general results:
If $D = \{(x_1, \dots, x_n) \in \ ^n \mid \Sigma x_i^2 \leq 1\}$ and $\overset{\circ}{D}$ the interior of D, then

$$H^i(D,A) = H^i(\overset{\circ}{D},A) = 0 \textit{ for } i \geq 1.$$

We will prove this later (see section 4.4.5). The following exercise treats the case $n = 1$.
Let us consider the following property $(\mathcal{E})$ of a sheaf $\mathcal{A}$ on the interval $X = [-1,1]$: For any open interval $I \subset [-1,1]$ the restriction map $\mathcal{A}([-1,1]) \longrightarrow \mathcal{A}(I)$ is surjective. (We only require that I is open in $[-1,1]$, i.e. it may contain the boundary points. Condition $(\mathcal{E})$ does not mean that $\mathcal{A}$ is flabby!)

Exercise 18. Show that the sheaves $\underline{A}_{[-1,1]}$ and injective sheaves have property $(\mathcal{E})$.

Exercise 19. If $\mathcal{A}$ has property $(\mathcal{E})$ and if we have an exact sequence

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0$$

then $\mathcal{F}(I) \longrightarrow \mathcal{G}(I)$ is surjective for any open interval in $[-1,1]$.

Exercise 20. If we have a sequence

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C} \longrightarrow 0$$

where $\mathcal{A}$ and $\mathcal{B}$ have property ($\mathcal{E}$) then $\mathcal{C}$ also has property ($\mathcal{E}$).

Exercise 21. For any sheaf $\mathcal{A}$ which satisfies ($\mathcal{E}$) we have $H^q([-1,1]),\mathcal{A}) = 0$ for all $q \geq 1$. Especially we have $H^q([-1,1],\underline{A}) = 0$ for any abelian group A.

This is some progress, I think we justified the computation in 4.1.2. But we will prove a stronger result which concerns a relative situation, i.e. the projection map $X \times I \longrightarrow X$. This stronger result will be provided by corollary 4.4.20 and the theorem 4.4.22, which say that the cohomology groups are invariant under homotopies. To get to this point we need to investigate a relative situation $f : X \longrightarrow Y$.

4.4.2 The Derived Functor to f_*

Given two spaces X,Y and a continuous map $f : X \longrightarrow Y$, we constructed the two functors f_*,f^* which transport sheaves on X to sheaves on Y and sheaves on Y to sheaves on X respectively (see section 3.4). Now we denote by $\mathcal{S}_X$ the category of sheaves on X with values in the category of abelian groups.

Definition 4.4.2 (Direct Image)**.** *If we have a sheaf $\mathcal{F}$ on X (with values in the category of abelian groups), then we defined the sheaf $f_*(\mathcal{F})$ on Y by*

$$f_*(\mathcal{F})(V) = \mathcal{F}(f^{-1}(V))$$

for all open subsets $V \subset Y$.

It is clear that $f_*(\mathcal{F})$ is a sheaf on Y. The functor f_* is left exact but not exact in general. We get our previous case if we take Y to be just one point, i.e. $Y = \{pt\}$. Then the stalk of $f_*(\mathcal{F})_{pt}$ in this point is simply $\mathcal{F}(X) = H^0(X,\mathcal{F})$.
Again we define a derived functor for f_* by the same method as before. We choose an injective resolution

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}^0 \longrightarrow \mathcal{I}^1 \longrightarrow \ldots$$

of $\mathcal{F}$, and we get a complex of sheaves on Y by taking the direct image

$$0 \longrightarrow f_*(\mathcal{I}^0) \longrightarrow f_*(\mathcal{I}^1) \longrightarrow \ldots.$$

This is now a complex of sheaves on the space Y. We define the sheaves (see section 3.5)

$$R^q f_*(\mathcal{F}) = \frac{\ker(f_*(\mathcal{I}^q) \longrightarrow f_*(\mathcal{I}^{q+1}))}{\mathrm{Im}\,(f_*(\mathcal{I}^{q-1}) \longrightarrow f_*(\mathcal{I}^q))}. \tag{4.29}$$

It is clear that the stalk of $R^q f_*(\mathcal{F})$ in a point y is simply the degree q-cohomology of the complex of stalks.
As before, we show that these sheaves do not depend on the choice of the resolution and that for any morphism

$$u : \mathcal{F} \longrightarrow \mathcal{G}$$

we get the derived maps

$$R^q u : R^q f_*(\mathcal{F}) \longrightarrow R^q f_*(\mathcal{G}).$$

Finally it is clear that $f_*(\mathcal{F}) = R^0 f_*(\mathcal{F})$, and that any short exact sequence of sheaves

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

leads to a long exact sequence

$$\begin{aligned} 0 &\longrightarrow f_*(\mathcal{F}') \longrightarrow f_*(\mathcal{F}) \longrightarrow f_*(\mathcal{F}'') \\ &\longrightarrow R^1 f_*(\mathcal{F}') \longrightarrow R^1 f_*(\mathcal{F}) \longrightarrow R^1 f_*(\mathcal{F}'') \longrightarrow R^2 f_*(\mathcal{F}') \longrightarrow \cdots \end{aligned} \tag{4.30}$$

The intuitive idea – which in some cases is right in some cases wrong – (see Theorem 4.4.17) is that the stalk of $R^q f_*(\mathcal{F})_y$ in a point y should be the cohomology of the fibre $f^{-1}(y) \subset X$ with coefficients in the restriction $i_y^*(\mathcal{F})$ of $\mathcal{F}$ to this fibre.
The following special case is very important (see section 3.4.2). It is clear that the following lemma is true.

Lemma 4.4.3. *Let $A \subset X$ be a closed subspace and let $i : A \longrightarrow X$ be the embedding of A into X. Then $i_*(\mathcal{F})$ is a sheaf on X. The stalk of $i_*(\mathcal{F})$ is given by*

$$i_*(\mathcal{F})_x = \begin{cases} \mathcal{F}_x & \text{if} \quad x \in A \\ 0 & \text{if} \quad x \notin A \end{cases}.$$

Hence it is clear that $\mathcal{F} \longrightarrow i_*(\mathcal{F})$ is an exact functor.

We also defined the functor f^*. This functor transforms sheaves on Y into sheaves on X.

Lemma 4.4.4. *The stalk of $f^*(\mathcal{G})$ in a point $x \in X$ is equal to the stalk of the original sheaf $\mathcal{G}$ in the point $y = f(x)$, i.e. $f^*(\mathcal{G})_x = \mathcal{G}_{f(x)}$ and so f^* is an exact functor.*

Since the exactness of sequences of sheaves can be checked stalkwise, it is clear that f^* is an exact functor. We know that these two functors are adjoint and I recall the adjointness formula

$$\mathrm{Hom}_{\mathcal{S}_X}(f^*(\mathcal{G}),\mathcal{F}) = \mathrm{Hom}_{\mathcal{S}_Y}(\mathcal{G},f_*(\mathcal{F})).$$

We want to discuss the consequences of existence of f^*,f_* and the adjointness formula for the cohomology and its functorial properties.

Lemma 4.4.5. *If $f : X \longrightarrow Y$ is continuous, and if $\mathcal{I}$ is an injective sheaf on X, then $f_*(\mathcal{I})$ is injective on Y.*

Proof: This follows directly from the adjointness formula and the exactness of f^*. □

4.4.3 Functorial Properties of the Cohomology

If we start from a sheaf $\mathcal{G}$ on the target space Y, and if we take an injective resolution

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{J}^0 \longrightarrow \mathcal{J}^1 \longrightarrow \mathcal{J}^2 \longrightarrow \ldots,$$

then we get a resolution

$$0 \longrightarrow f^*(\mathcal{G}) \longrightarrow f^*(\mathcal{J}^0) \longrightarrow f^*(\mathcal{J}^1) \longrightarrow f^*(\mathcal{J}^2) \longrightarrow \ldots.$$

As we have seen earlier in section 2.3.3, this gives us a map

$$H^q\left(0 \longrightarrow f^*(\mathcal{J}^0)(X) \longrightarrow f^*(\mathcal{J}^1)(X) \longrightarrow \ldots\right) \longrightarrow H^q(X,f^*(\mathcal{G})). \tag{4.31}$$

On the other hand we have a map between the complexes

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathcal{J}^0(Y) & \longrightarrow & \mathcal{J}^1(Y) & \longrightarrow & \mathcal{J}^2(Y) & \longrightarrow & \mathcal{J}^3(Y) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & f^*\mathcal{J}^0(Y) & \longrightarrow & f^*\mathcal{J}^1(Y) & \longrightarrow & f^*\mathcal{J}^2(Y) & \longrightarrow & f^*\mathcal{J}^3(Y) & \longrightarrow & \cdots \end{array} \tag{4.32}$$

this follows from the definition of f^*. Hence we get a functorial map

$$H^q(Y,\mathcal{G}) \longrightarrow H^q(X,f^*\mathcal{G}). \tag{4.33}$$

There is an especially important case of this: If $f : X \to Y$, and we consider the sheaf $\ _Y$ on Y, then we see easily that $f^*(\ _Y) = \ _X$. To see this we construct a homomorphism from $f'(\ _Y)$ to $\ _X$: For $U \subset X$, U open, we have

$$f'(\ )(U) = \varinjlim_{V \supset f(U)} \ (V).$$

For $V \supset f(U)$ we have $f^{-1}(V) \supset U$, and of course, we have maps

$$\ _Y(V) \longrightarrow \ _X(f^{-1}(V)) \longrightarrow \ _X(U),$$

and this provides a map

$$f'(\ _Y)(U) \longrightarrow \ _X(U).$$

This is a map from the presheaf $f'(\ _Y)$ to the sheaf $\ _X$, and this provides a unique map

$$f^*(\ _Y) \longrightarrow \ _X.$$

Looking at the stalks we see that this map is an isomorphism.
This yields the functoriality of the cohomology groups $H^q(X,\ )$. For any map $f : X \to Y$ we get

$$f^q : H^q(Y,\ ) \longrightarrow H^q(X,\ ).$$

There is another case: We always get a map $H^q(Y,f_*\mathcal{F}) \to H^q(X,f^*f_*\mathcal{F})$ and the adjointness provides the map $f^*f_*\mathcal{F} \longrightarrow \mathcal{F}$ which corresponds to the identity $f_*\mathcal{F} \to f_*\mathcal{F}$. The composition of these two maps yields a map $f^q : H^q(Y,f_*\mathcal{F}) \to H^q(X,\mathcal{F})$. For this map we have an easy theorem:

Theorem 4.4.6. *Let us assume that $f : X \to Y$ is continuous and $\mathcal{F}$ a sheaf on X. If the higher derived sheaves $R^q f_*(\mathcal{F}) = 0$ for $q \geq 1$, then we get an isomorphism*

$$f^q : H^q(Y, f_*(\mathcal{F})) \xrightarrow{\sim} H^q(X, \mathcal{F})$$

for all $q \geq 1$.

Proof: This is clear: We start from an injective resolution

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}^0 \longrightarrow \mathcal{I}^1 \longrightarrow \dots .$$

Then our assumption says that

$$0 \longrightarrow f_*(\mathcal{F}) \longrightarrow f_*(\mathcal{I}^0) \longrightarrow f_*(\mathcal{I}^1) \longrightarrow$$

is a resolution, and Lemma 4.4.5 implies that this resolution is injective. Hence

$$H^q(Y, f_*(\mathcal{F})) = H^q(0 \longrightarrow f_*(\mathcal{I}^0)(Y) \longrightarrow f_*(\mathcal{I}^1)(Y) \longrightarrow \dots) =$$
$$0 \longrightarrow \mathcal{I}^0(X) \longrightarrow \mathcal{I}^1(X) \longrightarrow \dots,$$

and this last complex computes the cohomology $H^q(X, \mathcal{F})$. □

One important consequence of this theorem is the case of an embedding

$$i : A \hookrightarrow X$$

where A is a closed subspace of X. In this case we have seen that i_* is an exact functor from sheaves on A to sheaves on X, hence $R^q i_*(\mathcal{F}) = 0$ for $q \geq 1$ and

$$H^q(A, \mathcal{F}) = H^q(X, i_*(\mathcal{F})).$$

If we want to apply the above theorem we have to understand how to compute the sheaves $R^q f_*(\mathcal{F})$. We want to show that under certain assumptions the stalks $R^q f_*(\mathcal{F})_y = H^q(f^{-1}(y), i_y^*(\mathcal{F}))$. A result of this kind is rather difficult to obtain, our goal is Theorem 4.4.17 (Proper base change). This theorem is very important and it also plays a fundamental role in algebraic geometry.
To get more precise informations which will allow us to compute cohomology groups in certain cases we have to make assumptions on our spaces.

4.4.4 Paracompact Spaces

In general the sheaves $R^q f_*(\mathcal{F})$ may be very difficult to compute. One possibility is to relate the stalks $R^q f_*(\mathcal{F})_y$ to the cohomology groups of the fibre $f^{-1}(y)$. This is possible if our spaces satisfy certain finiteness and separatedness properties.

Definition 4.4.7 (Locally Finite Covering)**.** *A covering $X = \bigcup_{\alpha \in \mathbf{A}} U_\alpha$ is called* **locally finite** *if for any point $x \in X$ we can find a neighborhood V_x of x such that V_x meets only finitely many of the U_α, i.e. the set of indices α for which $V_x \cap U_\alpha \neq \emptyset$ is finite.*

Definition 4.4.8. *A space X is called* **paracompact** *if it is Hausdorff and if for any open $U \subset X$ and any covering $U = \bigcup_{i \in I} U_i$ we can find a locally finite refinement of the covering. Recall that a refinement of the covering is another covering $U = \bigcup_{j \in J} W_j$ together with a map $\tau : J \to I$ such that for all $j \in J$ we have the inclusion $W_j \subset U_{\tau(j)}$. We call such a refinement a* **strong refinement** *if even the closures $\overline{W}_j$ are contained in $U_{\tau(j)}$.*

I claim:

Lemma 4.4.9. *If our space X is paracompact and locally compact then any covering $U = \bigcup_{i \in I} U_i$ of an open set U has a strong refinement which is locally finite.*

Proof: Since our space is Hausdorff and locally compact we know: For any point $x \in X$ and any open neighborhood V_x of x we can find an open neighborhood W_x such that its closure $\overline{W}_x$ is contained in V_x. Now it is clear how to get a strong locally finite refinement of a covering $U = \bigcup_{i \in I} U_i$: We can construct a strong refinement of the covering and after that we construct a locally finite refinement of this strong refinement. □

We have a simple criterion for paracompactness.

Definition 4.4.10. *We say that an open subset $U \subset X$ is* **exhaustible by compact subsets** *if we we can find an increasing sequence of compact subsets*

$$\emptyset = K_0 \subset \ldots \subset K_n \subset K_{n+1} \subset \ldots$$

s. t. $U = \bigcup K_n$ and for any n the compact set K_n is contained in the interior $\overset{\circ}{K}_{n+1}$ of the next one. We say that our space X is **exhaustible by compact sets**, *if the open subset $U = X$ has this property.*

Lemma 4.4.11. *A Hausdorff space X for which any open subset is exhaustible by compact subsets is paracompact.*

Proof: To see this we consider $U \subset X$ and a covering $U = \bigcup_{i \in I} U_i$ by open subsets. We choose an exhaustion by compact sets K_n as above. We choose inductively finite coverings of K_n. Assume we covered already K_{n-1}. For any $x \in K_n \setminus K_{n-1}$ we choose a V_x which

- has an empty intersection with K_{n-1}
- is contained in one of the covering sets U_i
- is contained in $\overset{\circ}{K}_{n+1}$.

We take a finite subcovering of the covering of K_n and we proceed. By construction the resulting covering is locally finite. □

It is not difficult to show that the following is true.

Lemma 4.4.12. *A Hausdorff space is paracompact if it is exhaustible by compact sets and if any open set U can be exhausted by a sequence of sets*

$$W_n \subset \overset{\circ}{W}_{n+1} \subset W_{n+1}$$

where the W_n are only closed subsets of X.

Proof: To see that this is true we observe that a closed subspace $A \subset X$ is exhaustible by compact sets. This implies that any covering of A by open sets has a locally finite refinement. (Same proof as for Lemma 4.4.11) Assume that we have a covering of U by open sets. We proceed as in the proof of Lemma 4.4.11 but now we construct locally finite coverings of the W_n (instead of finite ones) where we obey the same precautions as before. □

We come to a very technical lemma which says something about extension of sections. Assume that we have a closed embedding $i : A \hookrightarrow X$. For any sheaf $\mathcal{F}$ on X, we consider the sheaf $i^*(\mathcal{F})$ on A. Recall that this is the sheafification of the presheaf $V \longrightarrow i'(\mathcal{F})(V)$ (see Lemma 4.4.4) where V is open in A and

$$i'(\mathcal{F})(V) = \varinjlim_{U \supset V} (\mathcal{F}(U)). \tag{4.34}$$

If $\widetilde{s} \in \mathcal{F}(U)$ and if s is its image in $i^*(F)(U \cap A)$ then we say that s is the restriction of $\widetilde{s}$ to $A \cap U = V$.
Now we say that

Definition 4.4.13. *An embedding $i : A \hookrightarrow X$ is a* **nice embedding** *if for any open subset $V \subset A$ any section $s \in i^*(\mathcal{F})(V)$ can be extended into some neighborhood U of V in X.*

In other words this means it is in the image of $\mathcal{F}(U) \to i'(\mathcal{F})(V)$ for some U which satisfies $U \cap A \supset V$.
This condition can be reformulated by saying that $i'(\mathcal{F})$ is already a sheaf.

Lemma 4.4.14 (Extension of Sections)**.** *If X is paracompact and locally compact then any closed embedding $i : A \hookrightarrow X$ is nice.*

Proof: We start with $V \subset A$ and our section $s \in \mathcal{F}(V)$. We know from the definition of $i^*(\mathcal{F})$ that for any point $p \in V$ the image of s in the stalk $s_p \in i^*(\mathcal{F})_p$ is the restriction of a section $\widetilde{s}_p \in \mathcal{F}(U_p)$ where U_p is an open neighborhood of p in X. Hence we can find a covering $\bigcup_\alpha U_\alpha \supset V$ and sections $\widetilde{s}_\alpha \in \mathcal{F}(U_\alpha)$ such that $\widetilde{s}_\alpha$ maps to the restriction $s|U_\alpha \cap V$. We may assume that this covering is locally finite since our space is paracompact. Let $\{W_j\}_{j \in J}$ be a strong locally finite refinement of this covering. As usual we denote the map between the indexing sets by $\tau : J \to I$.
Let $q \in V$, we can find an open neighborhood V_q of q in X such that V_q meets only finitely many of the W_j and the U_α. We choose an open neighborhood $D_q \subset V_q$ which is contained in W_j for all those (finitely many) j for which $q \in W_j$ and also in all those finitely many U_α with $q \in U_\alpha$. We may also choose D_q so small that $D_q \cap W_j = \emptyset$ if $q \notin \overline{W}_j$ because D_q meets only finitely many of them anyway. It follows from the definition of i^* that we can take these D_q so small that we have $\widetilde{s}_\alpha|D_q = \widetilde{s}_\beta|D_q$ whenever $q \in U_\alpha \cap U_\beta$. Let $\widetilde{\widetilde{s}}_p \in \mathcal{F}(D_p)$ be the restriction of any of these $\widetilde{s}_\alpha$. I claim that these sections $\widetilde{\widetilde{s}}_p,\widetilde{\widetilde{s}}_q$ restrict to the same section over $D_p \cap D_q$ for any pair p,q. This is clear if $D_p \cap D_q = \emptyset$ so we may assume that $D_p \cap D_q \neq \emptyset$. If $D_p \subset W_j$ then $q \in \overline{W}_j$ because otherwise we have $D_q \cap W_j = \emptyset$ by construction and this implies $D_p \cap D_q = \emptyset$, a contradiction. We have $D_p \subset W_j \subset U_{\tau(j)}$. Since the W_j form a strong refinement of the U_α we even know that $\overline{W}_j \subset U_{\tau(j)}$. Hence $q \in U_{\tau(j)}$ and then we conclude that $D_q \subset U_{\tau(j)}$ again by construction. Consequently we have that D_p and D_q are contained in $U_{\tau(j)}$ and

this implies that the sections $\widetilde{\widetilde{s}}_p, \widetilde{\widetilde{s}}_q$ are restrictions of $\widetilde{s}_{\tau(j)}$. Hence the $\widetilde{\widetilde{s}}_p$ define a section $\widetilde{s}$ over $U = \bigcup D_q$ and this is the element we were looking for. □

I want to discuss a variant of this Lemma 4.4.14.

Lemma 4.4.15. *If we have a closed subset A which is locally compact and paracompact, we assume that we can find an subset $W \subset X$ with $W \supset A$ and such that $W \xrightarrow{\sim} W_0 \times A$ where W_0 is a topological space and we assume furthermore that the isomorphism sends $A \xrightarrow{\sim} \{w_0\} \times A$ for some point w_0 in W_0. Then the embedding $i : A \hookrightarrow X$ is nice.*

Proof: This can be shown by a slight modification of the proof of Lemma 4.4.14. We proceed as in the proof but we choose the open sets U_α to be of the form $U_\alpha = V_\alpha \times W_\alpha$ where V_α is open in V and W_α is a neighborhood of w_0 in W_0. Then we choose a strong locally finite refinement of the covering $V = \bigcup_{\alpha \in I} V_\alpha$. Let us denote this refinement by $V = \bigcup_{\beta \in J} Y_\beta$ and let $\tau : J \to I$ be the map for which $Y_\beta \subset V_{\tau(\beta)}$. This gives us a covering of V by open sets in X: We have $V \subset \bigcup_{\beta \in J} Y_\beta \times W_{\tau(\beta)}$. This covering now plays the role of the covering by the W_j in the proof of the Lemma 4.4.14. We proceed essentially in the same way as before. We choose neighborhoods D_q which satisfy $D_q \subset Y_\beta \times W_{\tau(\beta)}$ if $q = (q, w_0) \in Y_\beta \times W_{\tau(\beta)}$ and $D_q \cap Y_\beta \times W_{\tau(\beta)} = \emptyset$ if – here we have a slight modification – $q \notin \overline{Y}_\beta \times W_{\tau(\beta)}$. From here on the argument is the same. □

Lemma 4.4.16. *Let $i : A \hookrightarrow X$ be a nice embedding. If $\mathcal{I}$ is an injective sheaf on X then $i^*(\mathcal{I})$ is flabby and hence acyclic.*

Proof: Let $V \subset A$ be an open set and $s \in i^*(\mathcal{I})$. We find an open subset $U \subset X$ and a section $\widetilde{s} \in \mathcal{I}(U)$ which restricts to s. By Lemma 4.2.6 $\mathcal{I}$ is flabby, we can extend the section $\widetilde{s}$ to a section on X and the restriction of this extension to A extends s. □

These technical considerations will be applied to prove the following difficult theorem:

Theorem 4.4.17 (Proper Base Change)**.** *Let us assume that X is paracompact, that Y is locally compact and Hausdorff and that*

$$f : X \longrightarrow Y$$

is a proper map. Then for any sheaf $\mathcal{F}$ on X and any $y \in Y$ we have

$$R^q f_*(\mathcal{F})_y = H^q(f^{-1}(y), i_y^*(\mathcal{F})).$$

Recall that

Definition 4.4.18. *A map $f : X \to Y$ is called* **proper** *if the inverse image of a compact set in Y is again compact.*

Proof: We shall need a modification of the theorem, therefore we will also discuss to what extent we really need our assumptions.
Let $i_y : f^{-1}(y) \hookrightarrow X$ be the inclusion of the (closed and compact) fibre. Then we know from our assumptions that the embedding i_y is nice (Lemma 4.4.14). We formulate the following condition on our map f

(Cyl) For any open neighborhood of a fibre $U \supset f^{-1}(y)$ we find an open relatively compact neighborhood V_0 of y such that $f^{-1}(V_0) \subset U$.

We show that (Cyl) is valid under the assumption of the theorem. We consider the intersections

$$(X \setminus U) \cap f^{-1}\left(\overline{V}\right)$$

for all closures $\overline{V}$ of relatively compact open neighborhoods of y. Since $f^{-1}(\overline{V})$ is compact, the intersection is also compact. Since U is a neighborhood of $f^{-1}(y)$ we know that for $x \in X \setminus U$ we have $f(x) \neq y$. We may choose open neighborhoods $W_{f(x)}$,V_y s.t. their closure is compact and $f(W_{f(x)}) \cap V_y = \emptyset$, hence $f(x) \notin \overline{V}_y$. Hence $x \notin (X \setminus U) \cap f^{-1}(\overline{V}_y)$ and therefore

$$\bigcap_{V \ni y} (X \setminus U) \cap f^{-1}(\overline{V}) = \emptyset. \tag{4.35}$$

Now it follows from a standard argument on compact spaces that there must be a neighborhood V_0 of y with $f^{-1}(V_0) \subset U$.
The following considerations prove the assertion of the theorem under the following two assumptions

a) for all y the fibre $f^{-1}(y)$ is closed and the embedding is nice

b) The condition (Cyl) holds.

By definition we have

$$f_*(\mathcal{F})_y = \varinjlim_{V: y \in V} \mathcal{F}(f^{-1}(V))$$

and (Cyl) implies that

$$\varinjlim_{V: y \in V} \mathcal{F}(f^{-1}(V)) = \varinjlim_{U: f^{-1}(y) \subset U} \mathcal{F}(U).$$

Then the fact that the embedding of the fibre is nice yields

$$\varinjlim_{U: f^{-1}(y) \subset U} \mathcal{F}(U) = i_y^*(\mathcal{F})$$

and we conclude $f_*(\mathcal{F})_y = i_y^*(\mathcal{F})$.
This proves the theorem for $q = 0$. To prove it in general, we start from an injective resolution

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}^0 \longrightarrow \mathcal{I}^1 \longrightarrow \mathcal{I}^2 \longrightarrow \mathcal{I}^3 \longrightarrow \dots$$

on X. Then (we sometimes drop the brackets in f_*)

$$0 \longrightarrow f_*\mathcal{F} \longrightarrow f_*\mathcal{I}^0 \longrightarrow f_*\mathcal{I}^1 \longrightarrow f_*\mathcal{I}^2 \longrightarrow \dots$$

is a complex of injective sheaves. If we pass to the sequence of stalks at a point $y \in Y$, we get a complex of abelian groups

$$0 \longrightarrow f_*\mathcal{F}_y \longrightarrow f_*\mathcal{I}_y^0 \longrightarrow f_*\mathcal{I}_y^1 \longrightarrow f_*\mathcal{I}_y^2 \longrightarrow \dots,$$

and the cohomology of this complex is the stalk $R^q f_*(\mathcal{F})_y$. But this complex is equal to the complex

$$0 \longrightarrow i_y^*(\mathcal{F})(f^{-1}(y)) \longrightarrow i_y^*\mathcal{I}^0(f^{-1}(y)) \longrightarrow i_y^*\mathcal{I}^1(f^{-1}(y)) \longrightarrow i_y^*\mathcal{I}^2(f^{-1}(y)) \longrightarrow \dots,$$

and this is the complex of global sections of the complex of sheaves on $f^{-1}(y)$:

$$0 \longrightarrow i_y^*\mathcal{F} \longrightarrow i_y^*\mathcal{I}^0 \longrightarrow i_y^*\mathcal{I}^1 \longrightarrow i_y^*\mathcal{I}^2 \longrightarrow \dots$$

which is a flabby and hence acyclic resolution of $i_y^*\mathcal{F}$. Hence the cohomology of the above complex of global sections is $H^q(f^{-1}(y), i_y^*(\mathcal{F}))$. □

Corollary 4.4.19. *If X,Y and $f : X \longrightarrow Y$ are as in the theorem and if*

$$H^q(f^{-1}(y), i_y^*(\mathcal{F})) = 0 \quad \textit{for} \quad q \geq 1 \quad \textit{and all} \quad y \in Y,$$

then

$$f^q : H^q(Y, f_*\mathcal{F}) \xrightarrow{\sim} H^q(X, \mathcal{F})$$

is an isomorphism.

This is the combination of the Base Change Theorem (Theorem 4.4.17) and the Theorem 4.4.6.
The following corollary is not a direct consequence of the Proper Base Change Theorem.

Corollary 4.4.20. *If X is a Hausdorff space and if $p : X \times [0,1] \to X$ is the projection to the first factor then this projection induces isomorphisms in cohomology*

$$p^\bullet : H^\bullet(X, \_) \xrightarrow{\sim} H^\bullet(X \times [0,1], \_).$$

For any $t \in [0,1]$ the inclusion $x \mapsto x \times \{t\}$ induces an isomorphism in cohomology.

Proof: This is not a direct consequence of the Proper Base Change Theorem as it is stated since we do not make any assumption on X except that it is Hausdorff. But first of all our modified Lemma 4.4.15 implies that for any point x_0 in X the embedding $\{x_0\} \times I \hookrightarrow X \times I$ is nice. (We need that the fibre is closed so we can get away with the weaker assumption that points in X are closed.) Secondly it is clear that the condition (Cyl) in the proof of the Base change theorem is also fulfilled. This means that the proof is valid for the projection p.
The rest is clear since $p_* \_ = \_$, and since for $q \geq 0$

$$H^q(\{x\} \times [0,1], \_) = 0$$

by exercise 21.
The second assertion follows if we compose the inclusion with the projection. □

Definition 4.4.21. *Two maps $f,g : X \to Y$ are called* **homotopic** *if there is a map*

$$F : X \times [0,1] \longrightarrow Y$$

so that $F(x,0) = g(x)$, $F(x,1) = f(x)$.

Theorem 4.4.22 (The Homotopy Axiom)**.** *Let X be a Hausdorff space. If we have two homotopic maps $f,g : X \to Y$ then*

$$f^\bullet = g^\bullet : H^\bullet(Y, \_) \longrightarrow H^\bullet(X, \_).$$

Proof: Look at

$$X \underset{\text{bot}}{\overset{\text{top}}{\rightrightarrows}} X \times [0,1] \xrightarrow{F} Y$$

where the arrows are $x \mapsto (x,0)$, $x \mapsto (x,1)$. If we compose these arrows with F we get f,g. □

Definition 4.4.23. *A space X is called* **contractible to a point** $p \in X$ *if the two maps $f = \mathrm{Id}$ and the map g which maps all the points in X to the point p are homotopic.*

If we apply the homotopy axiom to this two maps we get

Lemma 4.4.24. *For a contractible Hausdorff space X we have*

$$H^i(X, \ ) = 0 \textit{ for all } i > 0.$$

It is clear that the space $\ ^n$ is contractible. The same thing holds for any open ball $B^n = \{(x_1, \ldots ,x_n) | \sum x_i^2 < 1\}$ and also for its closure.

4.4.5 Applications

We have the tools to compute cohomology groups of spheres and other simple spaces.

Cohomology of Spheres

We consider the sphere

$$S^n = \{(x_0,x_1, \ldots ,x_n) \in \ ^{n+1} | x_0^2 + x_1^2 + \ldots + x_n^2 = 1\}.$$

We cover it by the two balls $D_\pm$ which are defined by $x_n \geq 0$ or $x_n \leq 0$ respectively. We have the two inclusions $i_\pm : D_\pm \hookrightarrow S^n$. These balls are contractible, we have the sheaves $\ _{D_\pm}$ which we extend to the two sheaves $\underline{\ }_\pm = i_{\pm *}(\ _{D_\pm})$. We also have maps $\ _{S^n} \to \underline{\ }_\pm$ which on open sets $V \subset S^n$ are defined by the restriction $\ _{S^n}(V) \to \underline{\ }_\pm(V) = \ _{D_\pm}(V \cap D_\pm)$. This gives an inclusion $\ _{S^n} \hookrightarrow \underline{\ }_+ \oplus \underline{\ }_-$ which is an isomorphism in all the stalks which are not in the intersection of the two balls, i.e. which are not in the sphere S^{n-1}. In the points x in the intersection the inclusion is given by the diagonal $\underline{\ }_x = \ \subset (\underline{\ }_+ \oplus \underline{\ }_-)_x = \ \oplus \ $. From this we get an exact sequence of sheaves on S^n

$$0 \longrightarrow \underline{\ }_{S^n} \longrightarrow \underline{\ }_+ \oplus \underline{\ }_- \longrightarrow \underline{\ }_{S^{n-1}} \longrightarrow 0,$$

where the map $s : (\underline{\ }_+ \oplus \ _-)_x = \ \oplus \ \longrightarrow \ _{S^{n-1},x} = \ $ is the difference between the $+$ and $-$ component.
The cohomology of the two balls is trivial except in degree zero, hence we get

$$H^{\nu-1}(S^{n-1}, \underline{\ }) \xrightarrow{\sim} H^\nu(S^n, \underline{\ }) \tag{4.36}$$

if $\nu - 1 > 0$. In degree zero we find the exact sequence

$$0 \longrightarrow H^0(S^n, \underline{\ }) \longrightarrow H^0(D_+, \underline{\ }) \oplus H^0(D_-, \underline{\ }) \longrightarrow H^0(S^{n-1}, \underline{\ }) \longrightarrow H^1(S^n, \underline{\ }) \longrightarrow 0.$$

We can prove rather easily that

$$H^0(S^0, \_) = \quad \oplus \quad , \quad H^1(S^1, \_) \xrightarrow{\sim} \qquad (4.37)$$

and putting all this information together we get for $n > 0$

$$H^\nu(S^n, \_) \xrightarrow{\sim} \begin{cases} 0 & \text{for } \nu \neq 0,n \\ & \text{for } \nu = 0 \text{ or } \nu = n \end{cases} \qquad (4.38)$$

This is of course essentially the same calculation as the one in books on algebraic topology. In these books the two essential ingredients are homotopy and the so called Mayer-Vietoris sequence. Here the Mayer-Vietoris sequence is replaced by the construction of suitable exact sequences of sheaves.
This settles a question raised in the first Chapter (See 1.2 ,example 7): Is the dimension n of the space $\;^n$ a topological invariant? The answer is yes because we can read it off from the cohomology groups $H^\nu(\;^n \setminus \{p\}, \;)$, where $p \in \;^n$ is any point.

Orientations

Of course we have to be aware that the isomorphism $H^n(S^n, \;) \xrightarrow{\sim} \quad$ is not canonical (See also the example in section 4.1.2 at the beginning of this chapter). It depends on the choice of the homomorphism s above and it also depends on the choice of the isomorphism $H^{n-1}(S^{n-1}, \;) \xrightarrow{\sim} \quad$.

Definition 4.4.25. *We can say that we have a* **topological orientation** *on S^n, if we have chosen an isomorphism*

$$O_n : H^n(S^n, \;) \xrightarrow{\sim}$$

(See also sections 4.7.2,4.7.3) .
It is elementary that for $n > 0$ an orientation of the tangent bundle of the sphere (see 4.3.3) gives us a rule to choose a topological orientation. We pick a point $P \in S^n$. It is elementary that the choice of an orientation in the tangent space T_P at P defines a unique orientation of the sphere S^n. We choose a positively oriented orthonormal basis $\{e_1, \dots, e_{n-1}, e_n\}$ of tangent vectors in T_P.
It is clear that the intersection of S^n with the hyperplane spanned by $\{e_1, \dots, e_{n-1}, e_n\}$ is a sphere $S^{n-1} \subset S^n$ which contains P and whose tangent space at P is spanned by $\{e_1, \dots, e_{n-1}\}$. This sphere separates S^n into the two half spheres D_+, D_-, where D_+ is the half space where e_n points to. As before the two half spheres define sheaves $\_{}_+, \_{}_-$. The sheaf $\_{}_+ \oplus \_{}_- / \_$ on S^{n-1} is identified to $\_$ via the homomorphism s which in turn is fixed by the choice of e_n.
If now $n = 1$ then $S^0 = \{P, Q\}$. We have our exact sequence

$$\begin{aligned} H^0(S^0, (\_{}_+ \oplus \_{}_-)/\_) &= H^0(\{P\}, (\_{}_+ \oplus \_{}_-)/\_) \oplus H^0(\{Q\}, (\_{}_+ \oplus \_{}_-)/\_) \\ &= \quad \oplus \quad \xrightarrow{\delta} H^1(S^1, \;). \end{aligned}$$

The kernel of δ is the diagonal $\Delta = \{(x,x) | x \in \;\}$. We have $H^1(S^1, \;) = \quad \oplus \quad /\Delta$ and our rule will be:

Lemma 4.4.26. *The boundary operator δ maps the first summand $H^0(\{P\}, \_{}_+ \oplus \_{}_- / \_) = \quad$ isomorphically to $H^1(S^1, \;)$. The inverse of this isomorphism is our topological orientation induced by the given orientation.*

It is easy to verify that this identification does not depend on the choice of P, it only depends on the orientation.
For $n > 1$ we use the first part $\{e_1, \ldots ,e_{n-1}\}$ of the basis to put an orientation on S^{n-1}, this fixes a topological orientation on S^{n-1}. The homomorphism s is fixed by e_n. Therefore

Lemma 4.4.27. *For $n > 1$ the topological orientation on S^n is again given by the inverse of the boundary operator*

$$H^{n-1}(S^{n-1}, \quad) = \quad \xrightarrow{\delta} H^n(S^n, \quad)$$

on .

Compact Oriented Surfaces

Definition 4.4.28 (oriented surface)**.** *A two dimensional, compact, oriented manifold is called an* **oriented surface***.*

The simplest example is the 2-sphere S^2. If we have such a surface S we can construct a new one by the following construction: We pick two different points $p,q \in S$ and we choose two small neighborhoods D_p,D_q which are homeomorphic to a two dimensional disc. The boundaries $\partial\overline{D}_p,\partial\overline{D}_q$ can be identified to the oriented circle S^1. We form a cylinder $S^1 \times [0,1]$. We remove the interior of the two disks from the surface S and map $\partial(S^1 \times [0,1]) = S^1 \times \{0\} \cup S^1 \times \{1\}$ by taking the identity on each component to the boundaries of our two discs in $S \setminus D_p \cup D_q$.
Using this map we glue the cylinder to our surface, we add a so called handle. There is an obvious way to put an orientation onto the new surface if we have one on the old surface. It is a theorem in two dimensional topology that any oriented surface S can be obtained from the sphere by adding a certain number of handles.

Exercise 22. Let S be a compact oriented surface which has been obtained from the sphere by adding g handles. Show that $H^0(S,\_) = H^2(S,\_) = \quad$ and $H^1(S,\_) \xrightarrow{\sim} \quad ^{2g}$.

Hint: Construct a sequence of sheaves on S which is suggested by the process of adding a handle and proceed by induction.

We can also understand the cohomology of our oriented surface without such an explicit construction. This will be discussed in the section on Poincaré duality (see section 4.8.4).

4.5 Čech Cohomology of Sheaves

4.5.1 The Čech-Complex

For any space X, any sheaf $\mathcal{F}$ on X with values in the category of abelian groups and any open covering

$$\mathfrak{U} = \{U_i\}_{i \in I}, \quad X = \bigcup_{i \in I} U_i$$

of X, we will define the Čech-cohomology groups $\check{H}^q(X,\mathfrak{U},\mathcal{F})$, for $q = 0,1,2 \ldots$

To define these cohomology groups we introduce the so-called Čech complex. For any set of indices $(i_0,\ldots,i_q) \in I^{q+1}$ we define

$$U_{i_0\ldots i_q} = U_{i_0} \cap U_{i_1} \cap \ldots \cap U_{i_q}.$$

Then we put

$$C^q(X,\mathfrak{U},\mathcal{F}) = \prod_{(i_0,\ldots,i_q)\in I^{q+1}} \mathcal{F}(U_{i_0\ldots i_q}) \tag{4.39}$$

for $q = 0,1,\ldots$. We define a boundary map

$$d : C^q(X,\mathfrak{U},\mathcal{F}) \longrightarrow C^{q+1}(X,\mathfrak{U},\mathcal{F})$$

by the following formula

$$(d\,c)_{i_0\ldots i_{q+1}} = \sum_{\nu=0}^{q+1} (-1)^\nu \operatorname{res}(c_{i_0,\ldots,\widehat{i}_\nu,\ldots,i_{q+1}}). \tag{4.40}$$

Definition 4.5.1 (Čech Complex)**.** *The complex* $(C^\bullet(X,\mathfrak{U},\mathcal{F},d)$ *is called the* **Čech complex**.

We have to explain why formula (4.39) makes sense:
An element $c \in C^q(X,\mathfrak{U},\mathcal{F})$ is an element in a product and has components

$$c_{j_0,\ldots,j_q} \in \mathcal{F}(U_{j_0\ldots j_q}).$$

Hence dc will also have components which are indexed by elements in I^{q+2}. An element $(i_0,\ldots,i_{q+1}) \in I^{q+2}$ provides $q+2$ elements in I^{q+1} which are obtained by suppressing one of the components. By $(i_0,\ldots,\widehat{i}_\nu,\ldots,i_{q+1})$ we denote the element in I^{q+1} where we removed i_ν.
For all these $q+2$ possibilities we have the restriction associated to $U_{i_0\ldots i_{q+1}} \subset U_{i_0,\ldots,\widehat{i}_\mu,\ldots,i_{q+1}}$ which we simply denote by

$$\operatorname{res} : \mathcal{F}(U_{i_0,\ldots,\widehat{i}_\mu,\ldots,i_{q+1}}) \longrightarrow \mathcal{F}(U_{i_0,\ldots,i_{q+1}}).$$

Now it is clear that the formula gives the rule to compute the $(i_0\ldots i_{q+1})$-component of dc. We leave it as an exercise to prove that $d \circ d = 0$. Hence $(C^\bullet(X,\mathfrak{U},\mathcal{F}),d)$ is a complex of abelian groups.
Let us look at the beginning of our complex

$$0 \longrightarrow \prod_{i\in I} \mathcal{F}(U_i) \xrightarrow{d} \prod_{(i,j)\in I\times I} \mathcal{F}(U_i \cap U_j) \longrightarrow \ldots. \tag{4.41}$$

An element $c = (\ldots,c_i,\ldots)$ in the first term goes to zero if and only if

$$c_i|U_i \cap U_j = c_j|U_i \cap U_j \text{ for all } i,j \tag{4.42}$$

But since $\mathcal{F}$ is a sheaf this implies that this is the case if and only if c comes from a uniquely defined global section $s \in \mathcal{F}(X)$, i.e. $s_i = s|U_i$ for all i.

Definition 4.5.2 (Čech Cohomology)**.** *We define cycles $Z^q(X,\mathfrak{U},\mathcal{F})$ to be the kernel of d and boundaries are the elements $b \in C^q(X,\mathfrak{U},\mathcal{F})$ of the form $b = dc$ with $c \in C^{q-1}(X,\mathfrak{U},\mathcal{F})$. The boundaries form a subgroup $\mathcal{B}^q(X,\mathfrak{U},\mathcal{F})$ of $Z^q(X,\mathfrak{U},\mathcal{F})$ and now we define by*

$$\check{H}^q(X,\mathfrak{U},\mathcal{F}) = Z^q(X,\mathfrak{U},\mathcal{F})/\mathcal{B}^q(X,\mathfrak{U},\mathcal{F}).$$

the **Čech cohomology**.

We just saw

$$\check{H}^0(X,\mathfrak{U},\mathcal{F}) = \mathcal{F}(X). \tag{4.43}$$

Remark 4. In general these Čech cohomology groups do depend on $\mathfrak{U}$. Later on we shall see that under certain assumptions on the sheaves and on the space and the nature of the covering they will be independent of the covering.

We have the notion of a refinement of a covering (see 3.3.1). If $\tau : \mathfrak{V} \longrightarrow \mathfrak{U}$ is such a refinement, the map τ between the indexing sets yields a map between the Čech complexes $\tau^\bullet(C^\bullet(X,\mathfrak{U},\mathcal{F}),d) \longrightarrow (C^\bullet(X,\mathfrak{V},\mathcal{F}),d)$ and we get a map $\check{H}^\bullet(X,\mathfrak{U},\mathcal{F}) \longrightarrow \check{H}^\bullet(X,\mathfrak{V},\mathcal{F})$.

It is possible to show that on the level of cohomology this map does not depend on τ, but we do not need this fact here. Since the coverings form a category we can define the Čech cohomology groups of a space as direct limit

$$\varinjlim_{\mathfrak{U}} \check{H}^\bullet(X,\mathfrak{U},\mathcal{F}) =: \check{H}^\bullet(X,\mathcal{F}). \tag{4.44}$$

We can also look at the so called **alternating complex** $C^\bullet_{\text{alt}}(X,\mathfrak{U},\mathcal{F})$. It is defined as the subcomplex where the cochains satisfy

$$c_{i_0,\ldots,x,\ldots,x,\ldots,i_q} = 0 \tag{i}$$

and

$$c_{i_0,\ldots,x,\ldots,y,\ldots,i_q} = -c_{i_0,\ldots,y,\ldots,x,\ldots,i_q}. \tag{ii}$$

It is not too difficult to prove that $C^\bullet_{\text{alt}}(X,\mathfrak{U},\mathcal{F})$ is a subcomplex, i.e. the coboundary operator maps it into itself. It is a little bit more difficult to prove that

$$C^\bullet_{\text{alt}}(X,\mathfrak{U},\mathcal{F}) \longrightarrow C^\bullet(X,\mathfrak{U},\mathcal{F})$$

induces an isomorphism in cohomology. Sometimes it is easier to do computations using this smaller complex.

Exercise 23.

(a) Prove that

$$C^\bullet_{\text{alt}}(X,\mathfrak{U},\mathcal{F}) \longrightarrow C^\bullet(X,\mathfrak{U},\mathcal{F})$$

induces an isomorphism in cohomology.

(b) Consider the simplex

$$\Delta^{n+1} = \{(x_1,\ldots,x_{n+2}) \in \quad {}^{n+2} | x_i \geq 0, \sum x_i = 1\}.$$

Then we get a covering $\mathfrak{U}$ of Δ^{n+1} by the open sets

$$U_i = \{(x_1, \ldots, x_{n+2}) \in \Delta^{n+1} | x_i > 0\}.$$

Show that the cohomology groups

$$\check{H}^m(\Delta^{n+1}, \mathfrak{U}, \_) = \begin{cases} 0 & \text{if } m > 0 \\ & \text{if } m = 0. \end{cases}$$

(c) Now we remove the interior of Δ^{n+1} and we get the n-dimensional sphere

$$\partial\Delta = \{(x_1, \ldots, x_{n+2}) | \quad \text{at least one of the } x_i \text{ is zero}\} \simeq S^n.$$

Our covering of Δ^{n+1} induces a covering $\mathfrak{U}'$ on S^n.

$$U_i = \{(x_1, \ldots, x_{n+2}) \in S^n | x_i > 0\}.$$

Show that the Čech-cohomology groups $\check{H}^\bullet(S^n, \mathfrak{U}', \_)$ coincide with the cohomology groups computed by injective resolutions.

A rather elegant solution of this exercise can be obtained if we use the following Lemma whose proof I give for later references.

Lemma 4.5.3. *Let $\mathfrak{U}$ be a covering of an arbitrary space X and let us assume that in our covering $\mathfrak{U} = \{U_i\}_{i \in I}$ is a member $y \in I$ for which $U_y = X$. Then we have $H^q(X, \mathfrak{U}, \mathcal{F}) = 0$ for all $q \geq 1$.*

Proof: Let us assume we have a cocycle

$$c = (\ldots, c_{i_0, \ldots, i_q}, \ldots)_{(i_0, \ldots, i_q) \in I^{q+1}} \in Z^q(X, \mathfrak{U}, \mathcal{F}).$$

We construct a cochain $b \in C^{q-1}(X, \mathfrak{U}, \mathcal{F})$ by

$$b_{i_0, \ldots, i_{q-1}} = c_{y, i_0, \ldots, 1_{q-1}}.$$

We have to observe that

$$U_{i_0, \ldots, i_{q-1}} = U_{y, i_0, \ldots, i_{q-1}}.$$

Then

$$\begin{aligned} (db)_{i_0, \ldots, i_q} &= \Sigma(-1)^\nu b_{i_0, \ldots, \widehat{i_\nu}, \ldots, i_q} \\ &= \Sigma(-1)^\nu c_{y, i_0, \ldots, \widehat{i_\nu}, \ldots, i_q} \\ &= -(dc)_{y, i_0, \ldots, i_q} + c_{i_0, \ldots, i_q} = c_{i_0, \ldots, i_q}. \end{aligned}$$

□

To apply this to the exercise above we can consider the inclusion $U_0 \hookrightarrow \Delta^{n+1}$. The covering of Δ^{n+1} induces a covering of U_0 and these two coverings yield the same Čech complexes.

4.5.2 The Čech Resolution of a Sheaf

Remark 5 (Heuristic Remark)**.** Let $\mathcal{F}$ be a sheaf on X with coefficients in the category of abelian groups. Let us assume that we have a resolution of $\mathcal{F}$

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G}^0 \longrightarrow \mathcal{G}^1 \longrightarrow \dots \longrightarrow \mathcal{G}^n \longrightarrow \dots$$

A resolution of a sheaf may be very useful for the computation of the cohomology of $\mathcal{F}$. In Chapter 2 we showed: If the resolution above is acyclic then we can use it to compute the cohomology groups of $\mathcal{F}$, we have:

$$H^\bullet(X,\mathcal{F}) = H^\bullet(\mathcal{G}^\bullet(X)).$$

But even if a resolution is not acyclic it still may be helpful. For instance we still have a homomorphism

$$H^\bullet(\mathcal{G}^\bullet(X)) \longrightarrow H^\bullet(X,\mathcal{F})$$

which in general is neither injective nor surjective. But we have some kind of estimate for the deviation from being an isomorphism and in these estimates the cohomology groups $H^q(X,\mathcal{G}^p)$ will enter.

I want to put the Čech complex into this context. Let $\mathfrak{U} = \{U_\alpha\}_{\alpha\in \mathrm{A}}$ be a covering of our space X by open sets. We assume that this covering is locally finite, i.e. for any $x \in X$ we can find an open neighborhood V_x such that $V_x \cap U_\alpha = \emptyset$ for almost all $\alpha \in A$. Let $\mathcal{F}$ be a sheaf with values in the category of abelian groups. We give the indexing set A a total order and we denote by $\mathrm{A}^{q+1}_<$ the set of those sequences $\underline{\alpha} = (\alpha_0,\alpha_1,\dots,\alpha_q)$ where $\alpha_0 < \alpha_1 \dots < \alpha_q$. Again we put $U_{\underline{\alpha}} = U_{\alpha_0} \cap U_{\alpha_1} \cap \dots \cap U_{\alpha_q}$ and let $i_{\underline{\alpha}} : U_{\underline{\alpha}} \hookrightarrow X$ be the inclusion map. We restrict $\mathcal{F}$ to $U_{\underline{\alpha}}$ and take the direct image of this restriction, we obtain the sheaves $\mathcal{F}^*_{\underline{\alpha}} = i_{\underline{\alpha}*} i^*_{\underline{\alpha}}(\mathcal{F})$. I recall that these sheaves are defined by the rule $i_{\underline{\alpha}*} i^*_{\underline{\alpha}}(\mathcal{F})(V) = \mathcal{F}(V \cap U_{\underline{\alpha}})$. The stalk of this sheaf is equal to $\mathcal{F}_x$ if $x \in U_{\underline{\alpha}}$; it is zero if $x \notin \overline{U}_{\underline{\alpha}}$. It depends on the local structure of $\overline{U}_{\underline{\alpha}}$ in the boundary points $x \in \partial \overline{U}_{\underline{\alpha}}$. We have always a homomorphism $\mathcal{F}_x \longrightarrow \mathcal{F}^*_{\underline{\alpha}x}$. I allow myself to write $\mathcal{F}^*_\alpha$ for $\mathcal{F}^*_{\{\alpha\}}$.
Recall that we can define infnite products in the category of sheaves (see 3.5 and especially 3.20). Now we construct a resolution of our sheaf $\mathcal{F}$

$$0 \longrightarrow \mathcal{F} \longrightarrow \prod_{\alpha\in \mathrm{A}} \mathcal{F}^*_\alpha \longrightarrow \prod_{(\alpha,\beta)\in \mathrm{A}^2_<} \mathcal{F}^*_{(\alpha,\beta)} \longrightarrow \dots \longrightarrow \prod_{\underline{\alpha}\in \mathrm{A}^{q+1}_<} \mathcal{F}^*_{\underline{\alpha}} \longrightarrow \dots . \tag{4.45}$$

The first map is simply

$$\mathcal{F}_x \longrightarrow \prod_{\alpha\in \mathrm{A}} \mathcal{F}^*_{\alpha x}.$$

The boundary map

$$d : \prod_{\underline{\alpha}\in \mathrm{A}^{q+1}_<} \mathcal{F}^*_{\underline{\alpha}} \longrightarrow \prod_{\underline{\beta}\in \mathrm{A}^{q+2}_<} \mathcal{F}^*_{\underline{\beta}}$$

is given by the following rule: Let $s = (\dots s_{\underline{\alpha}} \dots) \in (\prod \mathcal{F}^*_{\underline{\alpha}})_x$, then

$$(ds)_{\underline{\beta}} = \sum_{i=0}^{q+1} (-1)^i s_{\beta_0 \dots \widehat{\beta}_i \dots \beta_{q+1}} \tag{4.46}$$

where we interpret $s_{\beta_0,\dots,\widehat{\beta}_i,\dots,\beta_{q+1}}$ as an element in $\mathcal{F}^*_{\underline{\beta},x}$. It is clear that this is a complex of sheaves.

Exercise 24.

(a) Prove that this complex of sheaves is exact.

Hint: We have to check exactness in the stalks. If $x \in X$ we know that we can find an element $\gamma \in$ A with $x \in U_\gamma$. Now we are in the same situation as in Lemma 4.5.3 abvove, except that we have modified the Čech complex since we have ordered the index set. But it is not difficult to adapt the Lemma to this situation here.

(b) Let E be a finite totally ordered set, i.e. $E = \{0,1,\ldots,n\}$. Let A be an abelian group, for any $r \in$ we define

$$C^r(A) = \bigoplus_{I \subset E,\ |I| = r+1} A,$$

by definition we have $C^r(A) = 0$ if $r \notin \{0,\ldots,n\} = E$. For a subset $I \subset E$ and $\alpha \in I$ we define $p(\alpha,I)$ as the position of α in I, i.e. $p(\alpha,I) = 0$ if α is the smallest element, $p(\alpha,I) = |I| - 1$ if α is the biggest one.

We define (co-)boundary operators

$$\begin{array}{rcl} d : C^r(A) & \longrightarrow & C^{r+1}(A) \\ \delta : C^r(A) & \longrightarrow & C^{r-1}(A) \\ \text{by} \qquad (da)_J & = & \displaystyle\sum_{\beta \in J} (-1)^{p(\beta,J)} a_{J \setminus \{\beta\}} \\ (\delta a)_J & = & \displaystyle\sum_{\beta \notin J} (-1)^{p(\beta,J \cup \{\beta\})} a_{J \cup \{\beta\}} \end{array}$$

where a_J is the J-th component of $a = (\ldots,a_J,\ldots) \in C^\bullet(A)$. We get two complexes

$$\begin{array}{ccl} 0 & \longrightarrow & C^0(A) \xrightarrow{d} \ldots \xrightarrow{d} C^r(A) \xrightarrow{d} C^{r+1}(A) \xrightarrow{d} \\ 0 & \longrightarrow & C^r(A) \xrightarrow{d} C^{r-1}(A) \xrightarrow{d} \ldots \xrightarrow{d} C^0(A) \longrightarrow 0. \end{array}$$

Show that these two complexes are exact using the ideas of the part (a) and the previous exercises

Let me come back to the heuristic remark above. I said that the complexes of sections $\mathcal{G}^\bullet(X)$ of a resolution contain some information concerning the cohomology of $\mathcal{F}$. Now we see that for the special case of the Čech resolution the resulting complex of global sections is the ordered Čech complex. We know that the ordered Čech complex gives us the same cohomology groups as the unordered Čech complex.
We see that coverings allow us to construct resolutions of sheaves. We already saw some other constructions providing resolutions of sheaves. If for instance we look back to our computation of the cohomology of the spheres (see section 4.4.5) then we see that our first short exact sequence is a resolution. We could extend this resolution by resolving __$_{S^{n-1}}$ and so on. Also the computation of the cohomology of a surface is obtained from a resolution of the sheaf _ on the surface.
This gives us the general idea that these resolutions in some sense provide a kind of cutting a space into simpler pieces. (See Exercise 4.4.14.)
In the following sections we discuss the technique of spectral sequences, we return to the Čech complex in 4.6.6..

4.6 Spectral Sequences

4.6.1 Introduction

The method of spectral sequences is designed to extract information on the cohomology of a sheaf from the cohomology of the sheaves in a resolution.
We consider a resolution of a sheaf $\mathcal{F}$:

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G}^0 \longrightarrow \mathcal{G}^1 \longrightarrow \dots \longrightarrow \mathcal{G}^n \longrightarrow \dots$$

We break the sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G}^0 \longrightarrow \mathcal{K} \longrightarrow 0$$

and we have seen in section 2.3.4 that we can find an injective resolution of this short exact sequence

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{G}^0 & \longrightarrow & \mathcal{K} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I^0 & \to & I^0 \oplus J^0 & \to & J^0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I^1 & \to & I^1 \oplus J^1 & \to & J^1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I^2 & \to & I^2 \oplus J^2 & \to & J^2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

We have the second *half* of the exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{G}^1 \longrightarrow \dots \longrightarrow \mathcal{G}^n \longrightarrow$$

and we can apply the same to this sequence. Proceeding in the same way forever, we get a diagram

$$
\begin{array}{ccccccccccccc}
 & & 0 & & 0 & & 0 & & & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{G}^0 & \longrightarrow & \mathcal{G}^1 & \longrightarrow & \cdots & \longrightarrow & \mathcal{G}^n & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & I^0 & \longrightarrow & I^{0,0} & \longrightarrow & I^{1,0} & \longrightarrow & \cdots & \longrightarrow & I^{p,0} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & I^1 & \longrightarrow & I^{0,1} & \longrightarrow & I^{1,1} & \longrightarrow & \cdots & \longrightarrow & I^{p,1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & I^2 & \longrightarrow & I^{0,2} & \longrightarrow & I^{1,2} & \longrightarrow & \cdots & \longrightarrow & I^{p,2} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & & & \vdots & &
\end{array}
$$

where all the I^ν and $I^{p,q}$ are injective, all squares commute.

Lemma 4.6.1. *This double complex of sheaves has two properties*

(a) *all horizontal sequences are exact.*

(b) *The vertical complexes $I^{\nu,\bullet}$ are injective resolutions of $\mathcal{G}^\nu$ and $I^\bullet$ is an injective resolution of $\mathcal{F}$.*

We apply the functor global sections to this diagram and get the **augmented** double complex $\widetilde{I^{\bullet\bullet}}(X)$

$$
\begin{array}{ccccccccccccc}
 & & 0 & & 0 & & 0 & & & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F}(X) & \longrightarrow & \mathcal{G}^0(X) & \longrightarrow & \mathcal{G}^1(X) & \longrightarrow & \cdots & \longrightarrow & \mathcal{G}^n(X) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & I^0(X) & \longrightarrow & I^{0,0}(X) & \longrightarrow & I^{1,0}(X) & \longrightarrow & \cdots & \longrightarrow & I^{p,0}(X) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & I^1(X) & \longrightarrow & I^{0,1}(X) & \longrightarrow & I^{1,1}(X) & \longrightarrow & \cdots & \longrightarrow & I^{p,1}(X) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & I^2(X) & \longrightarrow & I^{0,2}(X) & \longrightarrow & I^{1,2}(X) & \longrightarrow & \cdots & \longrightarrow & I^{p,2}(X) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & & & \vdots & &
\end{array}
$$

We replace the vertical complex on the left and the horizontal line on the top by zero and then we get the (non augmented) double complex $I^{\bullet,\bullet}$

$$
\begin{array}{ccccccccccccc}
 & & 0 & & 0 & & 0 & & & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & I^0(X) & \longrightarrow & I^{0,0}(X) & \longrightarrow & I^{1,0}(X) & \longrightarrow & \cdots & \longrightarrow & I^{p,0}(X) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & I^1(X) & \longrightarrow & I^{0,1}(X) & \longrightarrow & I^{1,1}(X) & \longrightarrow & \cdots & \longrightarrow & I^{p,1}(X) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & I^2(X) & \longrightarrow & I^{0,2}(X) & \longrightarrow & I^{1,2}(X) & \longrightarrow & \cdots & \longrightarrow & I^{p,2}(X) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & & & \vdots & &
\end{array}
$$

All squares commute and all vertical and horizontal sequences are complexes. We give a name to the differentials

$$
\begin{array}{lcll}
{}'d^{p,q} & : & I^{p,q}(X) \longrightarrow I^{p+1,q}(X) & \text{horizontal} \\
{}''d^{p,q} & : & I^{p,q}(X) \longrightarrow I^{p,q+1}(X) & \text{vertical.}
\end{array}
\tag{4.47}
$$

We get a simple complex $I^\bullet_{\mathrm{simp}}$ from $I^{\bullet,\bullet}$: We put

$$I^n_{\text{simp}}(X) = \bigoplus_{p+q=n} I^{p,q}(X) \tag{4.48}$$

and we define

$$\begin{aligned} d^n &: I^n_{\text{simp}}(X) \longrightarrow I^{n+1}_{\text{simp}}(X) \\ \text{by} \qquad d^n &= \sum_{p+q=n} {}'d^{p,q} + (-1)^p \, {}''d^{p,q} \, . \end{aligned} \tag{4.49}$$

It is clear that the commuting of the squares implies that

$$d^{n+1} \circ d^n = 0. \tag{4.50}$$

The following facts are more or less obvious from the construction in the previous part.

Lemma 4.6.2.

(a) The vertical complexes $(I^{p,\bullet}(X), {}''d)$ compute the cohomology of the sheaves $\mathcal{G}^p$, i.e.

$$H^q(X,\mathcal{G}^p) = H^q(I^{p,\bullet}(X), {}''d)$$

(b) The horizontal complexes $(I^{\bullet q}, {}'d)$ compute the cohomology of I^q and hence they are exact except in degree zero:

$$\begin{aligned} H^0(X,I^q) &= I^q(X) = H^0(I^{\bullet,q}(X), {}'d) \qquad \textit{and} \\ H^p(I^{\bullet,q}(X), {}'d) &= 0 \qquad\qquad\qquad\qquad \textit{for } p > 0. \end{aligned}$$

(c) The inclusion $I^\bullet(X) \hookrightarrow I^\bullet_{\text{simp}}(X)$ given by $x_q \mapsto (x_q,0,\cdots,0)$ induces an isomorphism

$$H^\bullet(I^\bullet(X)) \simeq H^\bullet(I^\bullet_{\text{simp}}(X))$$

and hence we have

$$H^\bullet(X,\mathcal{F}) = H^\bullet(I^\bullet_{\text{simp}}(X)).$$

The last assertion is not quite so obvious, it requires a little argument using (b). Let us look at a class which is represented by the cocycle $x = (x_{0,n}, \ldots, x_{n,0})$. The entries of the array are placed in our complex $I^\bullet_{\text{simp}}(X)$ like that:

$$\begin{matrix} 0 & \ldots & 0 & x_{n,0} \\ \vdots & \iddots & x_{n-1,1} & 0 \\ 0 & \iddots & \iddots & \vdots \\ x_{0,n} & 0 & \ldots & 0 \end{matrix} \tag{4.51}$$

The cocycle condition implies ${}'d^{n,0}(x_{n,0}) = 0$. Hence we find a $b = (0,\ldots,y_{n-1,0}) \in I^{n-1}_{\text{simp}}(X)$ such that ${}'d^{n-1,0}(y_{n-1,0}) = x_{n,0}$ and $x - d^{n-1}_{\text{simp}}(b)$ represents the same class but has its last component in the upper right corner equal to zero. Repeating this we can represent our element by a cocycle whose components are zero except the one in the lower left corner. This implies (c).

Summary: Starting from a resolution $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G}^\bullet$ of $\mathcal{F}$ we constructed a double complex $I^{\bullet\bullet}$ consisting of injective sheaves, such that the resulting simple complex $I^\bullet_{\text{simp}}(X)$ computes the cohomology groups of $H^\bullet(X,\mathcal{F})$. Of course we can compute these cohomology groups directly from an injective resolution of $\mathcal{F}$. Here we put a step in between, by resolving $\mathcal{F}$ by a complex $\mathcal{G}^\bullet$, which does not necessarily consist of acyclic sheaves and then we resolve this complex. This procedure may have an advantage: Let us assume that we find such a resolution $\mathcal{G}^\bullet$, where we have some information concerning $H^p(X,\mathcal{G}^q)$ (for instance some finiteness, vanishing in certain degress..). Then we will see in section 4.6.2 that this has consequences for the groups $H^{p+q}(X,\mathcal{F})$. We give some first indications how how this works.
We have the inclusion of the complex

$$\begin{array}{ccccccccc} \mathcal{G}^0(X) & \longrightarrow & \mathcal{G}^1(X) & \longrightarrow & \cdots & \longrightarrow & \mathcal{G}^n(X) & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & & & \downarrow & & \\ I^0_{\text{simp}}(X) & \to & I^1_{\text{simp}}(X) & \longrightarrow & \cdots & \longrightarrow & I^n_{\text{simp}}(X) & \longrightarrow & \cdots \end{array}$$

and hence we get from this construction a homomorphism

$$H^n(\mathcal{G}^\bullet(X)) \longrightarrow H^n(I^\bullet_{\text{simp}}(X)) = H^n(X,\mathcal{F}) \tag{4.52}$$

This is the so called **edge homomorphism.**
If the sheaves $\mathcal{G}^p$ are acyclic then section 2.3.1 tells us that this edge homomorphism is an isomorphism. This can also be seen by looking at the the double complex, the same argument which gave us (c) in the assertion above implies that the edge homomorphism is an isomorphism.
If the $\mathcal{G}^p$ are not acyclic then the edge homomorphism is neither injective nor surjective in general. But still we may get some information concerning the cohomology $H^n(X,\mathcal{F})$ from it. I recommend to the reader to solve the following exercise. It shows how these mechanisms work and it deals with the computation of $H^1(I^\bullet_{\text{simp}}(X))$. What I said above means that can get information on $H^n(X,\mathcal{F})$ in terms of the cohomology groups $H^q(X,\mathcal{G}^p)$ for $p+q=n$.
The cocycles are the elements $(x_{0,1},x_{1,0})$ which satisfy $'dx_{1,0}=0$, $''dx_{0,1}=0$ and $''dx_{1,0}+d'x_{0,1}=0$. Now a simple calculation solves the following exercise

Exercise 25.

(a) Show that the edge homomorphism in degree 1

$$H^1(\mathcal{G}^\bullet(X)) \longrightarrow H^1(X,\mathcal{F})$$

is injective. It provides an isomorphism to those classes which can be represented by cocycles with $x_{0,1}=0$.

(b) In other words: sending a class to $x_{0,1}$ induces a a homomorphism

$$H^1(X,\mathcal{F}) = H^1(I^\bullet_{\text{simp}}(X)) \longrightarrow H^1(X,\mathcal{G}^0)$$

and the kernel of this map is the image of the map in (a).

(c) The image of the map $H^1(X,\mathcal{F}) \longrightarrow H^1(X,\mathcal{G}^0)$ lands in the kernel of $H^1(X,\mathcal{G}^0) \longrightarrow H^1(X,\mathcal{G}^1)$ and we have a homomorphism

$$\ker\big(H^1(X,\mathcal{G}^0) \longrightarrow H^1(X,\mathcal{G}^1)\big) \longrightarrow H^2(\mathcal{G}^\bullet(X)).$$

(d) Show that we get even an exact sequence

$$0 \longrightarrow H^1(\mathcal{G}^\bullet(X)) \longrightarrow H^1(X,\mathcal{F}) \longrightarrow \ker\big(H^1(X,\mathcal{G}^0) \longrightarrow H^1(X,\mathcal{G}^1)\big) \longrightarrow H^2(\mathcal{G}^\bullet(X))$$

This extends to higher degrees, but the information we get is more complex. The basic point is that the double complex has two filtrations, these filtrations induce filtrations on the cohomology of the double complex. These filtrations are the **horizontal** filtration and the **vertical** filtration. These two filtrations induce filtrations on the cohomology of $I^\bullet_{\text{simp}}(X)$ and we have some information on the graded pieces of these filtrations. Actually we used already the horizontal filtration, essentially it provides the argument that proved Lemma 4.6.2 (c).

4.6.2 The Vertical Filtration

In the following discussion I start from a slightly more general situation. We forget the sheaf $\mathcal{F}$ and start from a complex of sheaves

$$0 \longrightarrow \mathcal{G}^0 \longrightarrow \mathcal{G}^1 \longrightarrow \dots \longrightarrow \mathcal{G}^n \longrightarrow \dots$$

we do not assume that it is exact. We want to construct an injective resolution of it. We adapt the approach we used when we constructed the double complex for the resolution $\mathcal{G}^\bullet$ of $\mathcal{F}$ in context of Lemma 4.6.1. The only difference is that we have to take the cohomology sheaves of this complex into account. Hence we do not make the assumption a) in Lemma 4.6.1.
We start at the left end of our complex and we break it

$$0 \longrightarrow Z(\mathcal{G}^0) \longrightarrow \mathcal{G}^0 \longrightarrow B(\mathcal{G}^1) \longrightarrow 0$$

and resolve this by the standard construction (Chapter 2.3.4). Observe that by definition $Z(\mathcal{G}^0) = H^0(\mathcal{G}^\bullet)$. Our injective resolution looks as follows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(\mathcal{G}^\bullet) & \longrightarrow & \mathcal{G}^0 & \longrightarrow & B(\mathcal{G}^1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I_H^{0,0} & \longrightarrow & I_H^{0,0} \oplus I_B^{1,0} & \longrightarrow & I_B^{1,0} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I_H^{0,1} & \longrightarrow & I_H^{0,1} \oplus I_B^{1,1} & \longrightarrow & I_B^{1,1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

For the following indices we always have the two short exact sequences

$$0 \longrightarrow B(\mathcal{G}^q) \longrightarrow Z(\mathcal{G}^q) \longrightarrow H^q(\mathcal{G}^\bullet) \longrightarrow 0$$

and

$$0 \longrightarrow Z(\mathcal{G}^q) \longrightarrow \mathcal{G}^q \longrightarrow B(\mathcal{G}^{q+1}) \longrightarrow 0.$$

We always resolve the first sequence by this method and then we use the resolution of the term in the middle for the left term in the second sequence, resolve the term on the right and then proceed by the standard construction to resolve the term in the middle. This goes on forever and we get an **injective resolution of the complex $\mathcal{G}^\bullet$** .

$$\begin{array}{ccccccccc}
 & & 0 & & & & 0 & & 0 & & \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{G}^0 & \longrightarrow & \cdots & \longrightarrow & \mathcal{G}^n & \longrightarrow & \mathcal{G}^{n+1} & \longrightarrow & \cdots \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I^{0,0} & \longrightarrow & \cdots & \longrightarrow & I^{n,0} & \longrightarrow & I^{n+1,0} & \longrightarrow & \cdots \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I^{0,1} & \longrightarrow & \cdots & \longrightarrow & I^{n,1} & \longrightarrow & I^{n+1,1} & \longrightarrow & \cdots \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \\
 & & \vdots & & & & \vdots & & \vdots & &
\end{array}$$

We call such a resolution an **adjusted** injective resolution.
We apply the functor global sections and we get a double complex $I^{\bullet,\bullet}(X)$. From this double complex we get the simple complex $I^\bullet_{\text{simp}}(X)$ and we are interested in the cohomology groups of this simple complex. On the double complex we define a filtration: We define $F^p(I^{\bullet,\bullet}(X))$ to be the subcomplex where the entries in the first $p-1$ vertical columns are zero. By we denote $F^p(I^\bullet_{\text{simp}}(X))$ we denote the resulting simple complex. The inclusion of complexes

$$F^p(I^\bullet_{\text{simp}}(X)) \hookrightarrow I^\bullet_{\text{simp}}(X)$$

induces a homomorphism in cohomology

$$H^n(F^p(I^\bullet_{\text{simp}}(X))) \longrightarrow H^n(I^\bullet_{\text{simp}}(X)).$$

and we define $F^p(H^n(I^\bullet_{\text{simp}}(X)))$ as the image of this homomorphism. This yields a filtration of the cohomology, we have

$$F^0(H^n(I^\bullet_{\text{simp}}(X))) = H^n(I^\bullet_{\text{simp}}(X))$$

$$\text{and } F^p(H^n(I^\bullet_{\text{simp}}(X))) = 0 \text{ for } \quad p > n.$$

Our final goal is to get some understanding of the quotients of the filtration

$$F^p(H^n(I^\bullet_{\text{simp}}(X)))/F^{p+1}(H^n(I^\bullet_{\text{simp}}(X))).$$

We have an exact sequence of complexes

$$0 \longrightarrow F^{p+1}(I^\bullet_{\text{simp}}(X)) \longrightarrow F^p(I^\bullet_{\text{simp}}(X)) \longrightarrow F^p(I^\bullet_{\text{simp}}(X))/F^{p+1}(I^\bullet_{\text{simp}}(X)) \longrightarrow 0.$$

The complex on the right is simply the vertical complex given by the p-th column. Hence we know

$$H^n(F^p(I^\bullet_{\text{simp}}(X))/F^{p+1}(I^\bullet_{\text{simp}}(X))) \xrightarrow{\sim} H^{n-p}(X,\mathcal{G}^p). \tag{4.53}$$

We rewrite the exact sequence in cohomology

$$\cdots \longrightarrow H^n(F^{p+1}(I^\bullet_{\text{simp}}(X))) \longrightarrow H^n(F^p(I^\bullet_{\text{simp}}(X))) \longrightarrow H^{n-p}(X,\mathcal{G}^p) \longrightarrow H^{n+1}(F^{p-1}(I^\bullet_{\text{simp}}(X)))$$

which yields an inclusion

$$H^n(F^p(I^\bullet_{\text{simp}}(X)))/\mathrm{Im}\,(H^n(F^{p+1}(I^\bullet_{\text{simp}}(X)))) \subset H^{n-p}(X,\mathcal{G}^p)$$

By definition we have a homomorphism

$$H^n(F^p(I^\bullet_{\text{simp}}(X)))/\mathrm{Im}\,(H^n(F^{p+1}(I^\bullet_{\text{simp}}(X)))) \longrightarrow F^p(H^n(I^\bullet_{\text{simp}}(X)))/F^{p+1}(H^n(I^\bullet_{\text{simp}}(X)))$$

which gives us (the first little piece of information)

E1 *The filtration steps $F^p(H^n(I^\bullet_{\text{simp}}(X)))/F^{p+1}(H^n(I^\bullet_{\text{simp}}(X)))$ are isomorphic to subquotients of $H^{n-p}(X,\mathcal{G}^p)$.*

We put $n-p=q$ and write $E_1^{p,q} = H^q(X,\mathcal{G}^p)$.
This assertion **E1** sometimes allows us to draw conclusions in the sense of the **Summary** above. If for instance we know that $H^q(X,\mathcal{G}^p)$ are finitely generated abelian groups, then we know that the cohomology groups of the total complex are also finitely generated abelian groups.
The next question is: How can we compute the subquotient of $E_1^{p,q}$ which is isomorphic to the subquotient $F^p(H^n(I^\bullet_{\text{simp}}(X)))/F^{p+1}(H^n(I^\bullet_{\text{simp}}(X)))$ of the cohomology?
A subquotient of E_1^{pq} is by definition of the form $Z_\infty^{p,q}/B_\infty^{p,q}$ where $B_\infty^{p,q} \subset Z_\infty^{p,q} \subset E_1^{pq}$, we have to compute these two submodules.
Our strategy will be to approximate these submodules $Z_\infty^{p,q}$ (resp. $B_\infty^{p,q}$) by a sequence of decreasing (resp. increasing) submodules, i.e. we will construct sequences of submodules

$$\begin{aligned} & Z_1^{p,q} \supset Z_2^{p,q} \supset \ldots \supset Z_\infty^{p,q} \\ \text{and} \quad & B_1^{p,q} \subset B_2^{p,q} \subset \ldots \subset B_\infty^{p,q} \end{aligned}$$

such that for large indices r we have $Z_r^{pq} = Z_\infty^{p,q}, B_r^{p,q} = B_\infty^{p,q}$.
The structure of a complex on $\mathcal{G}^\bullet$ induces a structure of a complex

$$H^q(X,\mathcal{G}^\bullet) = \quad \ldots \longrightarrow H^q(X,\mathcal{G}^{p-1}) \longrightarrow H^q(X,\mathcal{G}^p) \longrightarrow H^q(X,\mathcal{G}^{p+1}) \longrightarrow \ldots \quad . \tag{4.54}$$

We denote the boundary operators by $d_1^{p,q} : E_1^{p,q} \longrightarrow E_1^{p+1,q}$ and call the complex $(E_1^{p,q},d_1^{p,q})_{p,q}$ the E_1-term of our double complex. Since this is a complex we have the cocyles and coboundaries in it:

$$B_1^{p,q} \subset Z_1^{p,q} \subset E_1^{p,q}.$$

We want to show that

$$B_1^{p,q} \subset B_\infty^{p,q} \subset Z_\infty^{p,q} \subset Z_1^{p,q}.$$

First of all it is clear from the definition that $Z_\infty^{p,q}$ consists of those classes $\xi_{p,q} \in H^q(X,\mathcal{G}^p)$ which have a representative $x_{p,q} \in I^{p,q}(X)$, satisfying $''d^{p,q}(x_{p,q}) = 0$ which is the lower left entry of a cocycle

$$x = \begin{matrix} 0 & 0 & \cdots & 0 & x_{p+q,0} \\ \vdots & \vdots & \iddots & \iddots & 0 \\ 0 & 0 & x_{p+1,q-1} & \iddots & \vdots \\ 0 & x_{p,q} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \end{matrix}$$

i.e. given $x_{p,q}$ we want to place entries $x_{p+1,q-1},\ldots,x_{p+q,0}$ such that $dx = 0$. (We call this *Problem (C)* for a given $x_{p,q}$)
It is also clear that $\xi_{p,q} \in B_\infty^{p,q}$ if and only if we can find an element $y \in I_{\mathrm{simp}}^{n-1}(X)$ i.e. an element

$$y = \begin{matrix} 0 & \cdots & 0 & y_{n-1,0} \\ \vdots & \iddots & \iddots & 0 \\ 0 & \iddots & \iddots & \vdots \\ y_{0,n-1} & 0 & \cdots & 0 \end{matrix}$$

such that $x - d^{n-1}y \in F^{p+1}(I_{\mathrm{simp}}^n(X))$. (*Problem (B)*)
To solve *(C)* we we have at least to be able to fill the next spot, this is *Problem (C1)*. We analyze what the obstruction for solving *(C1)* is and then we try to solve *(C2)* under the assumption that we have solved *(C1)*. We proceed by induction.
To solve *(C1)* we have to find an $x_{p+1,q-1} \in I^{p+1,q-1}(X)$ for which

$$'dx_{p,q} = -''dx_{p+1,q-1}.$$

The element $'dx_{p,q}$ must be a cocycle for the vertical complex and therefore it represents a class in $H^q(X,\mathcal{G}^{p+1})$. We can find such an $x_{p+1,q-1}$ if and only if $'dx_{p,q}$ represents the trivial class. This means that the class $\xi_{p,q}$ goes to zero under

$$d_1^{p,q} : H^q(X,\mathcal{G}^p) \longrightarrow H^q(X,\mathcal{G}^{p+1})$$

i.e. it lies in the kernel $Z_1^{p,q} = \ker(d_1^{p,q})$ and this implies $Z_1^{p,q} \supset Z_\infty^{pq}$. Now we look at a class $\xi \in F^p(H^n(I_{\mathrm{simp}}^\bullet(X)))$ and represent it by a cocycle x as above.
Assume that the class $\xi_{p,q}$ represented by $x_{p,q}$ is in the image

$$\xi_{p,q} \in B_1^{p,q} = \mathrm{Im}\,(d_1^{p-1,q} : H^q(X,\mathcal{G}^{p-1}) \longrightarrow H^q(X,\mathcal{G}^p)).$$

Then this means that we can find an element $y_{p-1,q}$ which represents a class in $H^q(X,\mathcal{G}^{p-1})$ and therefore satisfies $''d^{p-1,q}(y_{p-1,q}) = 0$, and which maps to $x_{p,q}$ under $'d^{p-1,q}$. Then we can choose any element

$$y = \begin{matrix} 0 & \cdots & 0 & y_{p+q-1,0} \\ \vdots & \iddots & \iddots & 0 \\ 0 & y_{p-1,q} & \iddots & \vdots \\ 0 & 0 & \cdots & 0 \end{matrix} \tag{4.55}$$

then $d^{p+q-1}y$ is zero in the cohomology and $x - d^{p+q-1}y \in F^{p+1}(I^\bullet_{\text{simp}}(X))$.
We conclude that $B_1^{p,q} \subset B_\infty^{p,q} \subset Z_\infty^{pq} \subset Z_1^{pq}$, and we define

$$E_2^{p,q} = Z_1^{p,q}/B_1^{p,q} = H^p(H^q(X,\mathcal{G}^\bullet)).$$

In other words we define E_2^{pq} as the cohomology groups of the complex $(E_1^{p,q},d_1^{p,q})$. We get

E2 *The filtration steps $F^p(H^n(I^\bullet_{\text{simp}}(X)))/F^{p+1}(H^n(I^\bullet_{\text{simp}}(X)))$ are isomorphic to subquotients of $E_2^{p,q}$.*

(In view of applications we made some progress. I mentioned the applications to finiteness results at **(E1)**, now we get finiteness results if we only know that the $E_2^{p,q}$ are finitely generated.)
The decisive point is that we can proceed and define

$$d_2^{p,q} : E_2^{p,q} \longrightarrow E_2^{p+2,q-1}.$$

such that we get a complex

$$E_2^{p-2,q+1} \xrightarrow{d_2^{p-2,q+1}} E_2^{p,q} \xrightarrow{d_2^{p,q}} E_2^{p+2,q-1} \tag{4.56}$$

such that the cocyles and coboundaries of this complex satisfy $B_2^{p,q} \subset B_\infty^{p,q} \subset Z_\infty^{p,q} \subset Z_2^{p,q}$.
To construct this map we represent an element $\xi_{p,q} \in Z_1^{p,q}$ by a matrix

$$\begin{matrix} & & x_{p+1,q-1} \\ & & \downarrow \\ x_{p,q} & \longrightarrow & z_{p+1,q} \\ \downarrow & & \\ 0 & & \end{matrix} \quad \text{where } z_{p+1,q} = {'d^{p,q}}x_{p,q} = -{''d^{p+1,q-1}}(x_{p+1,q-1}).$$

This means that we encode the condition $\xi_{pq} \in Z_1^{pq}$ by giving the solution of (*C1*). But now we have to fill the next place (*C2*). We apply the horizontal boundary operator and we get

$$'d^{p+1,q-1}(x_{p+1,q-1}) = z_{p+2,q-1} \in I^{p+2,q-1}(X).$$

This element $z_{p+2,q-1}$ represents a class in $H^{q-1}(X,\mathcal{G}^{p+2})$ which is in the kernel $Z_1^{p+2,q-1}$ of $d_1^{p+2,q-1}$ because it is a boundary under the horizontal boundary operator. We can fill the next spot if the class of $z_{p+2,q-1}$ is zero. But since we made some choices, this is only a sufficient condition. We can modify the representatives for $\xi_{p,q}$ by boundaries which are the elements of the form

$$\begin{array}{ccc} & & z_{p+1,q-1} \\ & & \downarrow \\ {}'d(y_{p-1,q}) & \longrightarrow & 0 \end{array} \quad \text{where } {}''d(z_{p+1,q-1}) = 0$$

It is obvious that another choice modifies $z_{p+2,q-1}$ into $z_{p+2,q-1} +' d(z_{p+1,q-1})$ i.e. by an element in $B_1^{p+2,q-1}$. Hence we get a homomorphism

$$d_2^{p,q} : E_2^{p,q} \longrightarrow E_2^{p+2,q-1},$$

and we can solve $(C2)$ for the class $\xi_{p,q}$ if and only if it goes to zero under $d_2^{p,q}$. This also tells us that $Z_1^{p,q} \supset Z_2^{p,q} = \ker(d_2^{p,q}) \supset Z_\infty^{p,q}$. It is clear that we even get a complex

$$\ldots \longrightarrow E_2^{p-2,q+1} \xrightarrow{d_2^{p-2,q+1}} E_2^{p,q} \xrightarrow{d_2^{p,q}} E_2^{p+2,q-1} \longrightarrow \ldots \quad . \tag{4.57}$$

We want to show that $B_2^{p,q} = \mathrm{Im}\,(d_2^{p-2,q+1}) \subset B_\infty^{p,q}$. An element $x \in B_2^{p,q}$ is the boundary of an element in $Z_1^{p-2,q+1}$. This means that we can find an element $y \in I^{p+q-1}(X)$ which in its lower left corner is of the form

$$\begin{array}{ccc} & & y_{p-1,q} \\ & & \downarrow \\ y_{p-2,q+1} & \longrightarrow & z_{p-1,q+1} \\ \downarrow & & \\ 0 & & \end{array} \qquad \text{where } {}''d(y_{p-2,q+1}) = 0, \quad {}'d(y_{p-2,q+1}) = -{}''d(y_{p-1,q}) \tag{4.58}$$

and where ${}'d(y_{p-1,q}) = x_{p,q}$. Hence $x - dy$ represents a class in $F^{p+1}(H^n(I^\bullet_{\mathrm{simp}}(X)))$. Now we define $E_3^{p,q}$ as the cohomology of the complex, i.e.

$$E_3^{p,q} = Z_2^{p,q}/B_2^{p,q}$$

Now it is clear - and I will not give the formal proof - that this construction can be extended by induction to all r and we get

Lemma 4.6.3. *Starting from $E_1^{p,q} = H^q(X,\mathcal{G}^p)$ and $d_1^{p,q} : H^q(X,\mathcal{G}^p) \longrightarrow H^q(X,\mathcal{G}^{p+1})$ we can define a sequence of terms*

$$(E_r^{p,q},d_r) \quad d_r^{p,q} : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1} \qquad \textit{where } \; d_r^{p+r,q-r+1} \circ d_r^{p,q} = 0$$

such that at any level

$$E_{r+1}^{p,q} = \frac{\ker(d_r^{p,q} : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1})}{\mathrm{Im}\,(d_r^{p-r,q+r-1} : E_r^{p-r,q+r-1} \longrightarrow E_r^{p,q})}$$

and such that for all r the subquotients $F^p(H^n(I^\bullet_{\mathrm{simp}}(X)))/F^{p+1}(H^n(I^\bullet_{\mathrm{simp}}(X)))$ are isomorphic to subquotients of $E_r^{p,q}$. Since we are in the positive quadrant, i.e. $p,q \geq 0$ the sequence of modules E_r^{pq} becomes stationary after a while. This means that for $r \gg 0$ we have

$$E_r^{p,q} = F^p(H^n(I^\bullet_{\mathrm{simp}}(X)))/F^{p+1}(H^n(I^\bullet_{\mathrm{simp}}(X)))$$

Now we return to the situation in Lemma 4.6.1. In Lemma 4.6.2 we noticed that

$$H^\bullet(X,\mathcal{F}) \xrightarrow{\sim} H^n(I^\bullet_{\mathrm{simp}}(X)).$$

Therefore, we may replace in the statement above the cohomology of the double complex by the cohomology groups $H^\bullet(X,\mathcal{F})$ and the induced filtration. Usually one summarizes the assertion in Lemma 4.6.3 by saying:

Lemma 4.6.4 (Spectral Sequence). *We have a* **spectral sequence** *with E_1 term $E_1^{pq} = H^q(X,\mathcal{G}^p)$ which converges to $H^n(X,\mathcal{F})$ and this is abbreviated by*

$$(H^q(X,\mathcal{G}^q),d_1) \Rightarrow H^n(X,\mathcal{F}).$$

If we happen to know the E_2 term we also write this for the E_2 term

$$(E_2^{p,q},d_2) \Rightarrow H^n(X,\mathcal{F})$$

4.6.3 The Horizontal Filtration

Assume that we have a complex $\mathcal{G}^\bullet$ which starts in degree zero and that we have an adjusted injective resolution $\mathcal{G}^\bullet \longrightarrow I^{\bullet,\bullet}$ (see Lemma 4.6.1). We change the notation and give the index q to the vertical complexes. (We want a certain consistency therefore we arrange things so that p is the index for the horizontal filtration.)
We can apply to it the same method which we applied for the vertical filtration. Let ${}'F^p(I^{\bullet,\bullet})$ be the subcomplex where the entries in the first $p-1$ lines are zero. Now we use the specific form of the adjusted injective resolution. The horizontal complex ${}'F^p(I^{\bullet,\bullet}(X))/{}'F^{p+1}(I^{\bullet,\bullet}(X))$ is of the form

$$I_B^{q-1,p}(X) \oplus I_H^{q-1,p}(X) \oplus I_B^{q,p}(X) \longrightarrow I_B^{q,p}(X) \oplus I_H^{q,p}(X) \oplus I_B^{q+1,p}(X) \longrightarrow$$
$$I_B^{q+1,p}(X) \oplus I_H^{q+1,p}(X) \oplus I_B^{q+2,p}(X) \longrightarrow$$

where the differential is always zero on the first two summands and is the identity isomorphism between the third term in degree q and the first term in degree $q+1$. This makes it easy to compute the cohomology. We get

$$H^q({}'F^p(I^{\bullet,\bullet})(X)/({}'F^{p+1}(I^{\bullet,\bullet})(X)) = I_H^{q,p}(X) \tag{4.59}$$

and this is the ${}'E_1$ term of the spectral sequence. The differential

$${}'d_1^{p,q} : I_H^{q,p}(X) \longrightarrow I_H^{q,p+1}(X)$$

is the differential which obtained from the differential in the resolution $H^q(\mathcal{G}^\bullet) \longrightarrow I_H^{q,\bullet}$ and then taking the induced complex on the global sections. Hence we see that the E_2 term of the spectral sequence is

$${}'E_2^{p,q} \xrightarrow{\sim} H^p(X,H^q(\mathcal{G}^\bullet)).$$

From now on the reasoning is exactly the same as in the case of the vertical filtration we get a spectral sequence which converges to $H^\bullet(I^\bullet_{\text{simp}}(X))$. The differential d_2 now goes

$$d_2 : {}'E_2^{p,q} \longrightarrow{}' E_2^{p+2,q-1}$$

Two Special Cases

a) We may look at the computation in context of Lemma 4.6.2 from a slightly different point of view. We start from the resolution

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G}^0 \longrightarrow \mathcal{G}^1 \longrightarrow \dots \longrightarrow \mathcal{G}^n \longrightarrow \dots \tag{4.60}$$

from which we constructed the double complex. Then conditions a) and b) in Lemma 4.6.1. are valid and our complex is acyclic (if we include $\mathcal{F}$).

We now consider consider the horizontal filtration by subcomplexes $'F^q(I^\bullet_{\text{simp}}(X)) \subset I^\bullet_{\text{simp}}(X)$ where the entries in the first $q-1$ horizontal lines are zero. If we apply the same arguments to this horizontal filtration we get something that we have done already. Since the $I^\bullet$ are acyclic our arguments yield that

$$H^\bullet('F^q(I^\bullet_{\text{simp}}(X))/'F^{q+1}(I^\bullet_{\text{simp}}(X))$$

vanishes except we are in degree zero and

$$H^0('F^q(I^\bullet_{\text{simp}}(X))/'F^{q+1}(I^\bullet_{\text{simp}}(X)) = H^q(X,\mathcal{F}).$$

Hence we see that for this filtration

$$'E_1^{p,q} = \begin{cases} 0 & p \neq 0 \\ H^n(X,\mathcal{F}) & p = 0 \end{cases}. \tag{4.61}$$

We do not have any non trivial differentials. Hence we see again that the double complex computes the cohomology $H^\bullet(X,\mathcal{F})$ and we see that in this special case the horizontal filtration is not of interest, we recovered the results in Lemma 4.6.2. This is only true since we assumed a) in Lemma 4.6.1.

b) We have also a special situation where the vertical filtration is uninteresting. Let us assume that the sheaves $\mathcal{G}^q$ are acyclic for the functor global sections. Its $E_1^{q,p}$ term is as always $H^p(X,\mathcal{G}^q)$ but this is zero for $p > 0$ (remember p and q have been interchanged). Hence we have only the $H^0(X,\mathcal{G}^p)$. The differentials are given by $d : H^0(X,\mathcal{G}^q) \longrightarrow H^0(X,\mathcal{G}^{q+1})$ and his gives us the $E_1^{q,p}$ term as

$$E_1'^{pq} = \begin{cases} H^q(\mathcal{G}^\bullet(X)) & \text{for } p = 0 \\ 0 & \text{for } p > 0 \end{cases} \tag{4.62}$$

The higher differentials are zero.

So we find that under our assumption above the vertical filtration tells us that the complex $I^\bullet_{\text{simp}}(X)))$ computes the cohomology of the complex $\mathcal{G}^\bullet(X)$ and the horizontal filtration shows that we have a spectral sequence

$$H^p(X,H^q(\mathcal{G}^\bullet)),d_2) \Rightarrow H^n(\mathcal{G}^\bullet(X)). \tag{4.63}$$

Applications of Spectral Sequences

The method of the spectral sequence has many applications. We will apply spectral sequences at many places lateron in this book.
Here we give some indications how such applications can look like.

a) We start from a sheaf $\mathcal{F}$ and a resolution

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G}^0 \longrightarrow \mathcal{G}^1 \longrightarrow \dots \longrightarrow \mathcal{G}^n \longrightarrow \dots$$

One typical applications provides finiteness results. I indicated this already during the discussion of the spectral sequence. For instance if we can show that $H^q(X,\mathcal{G}^p) = E_1^{p,q}$ or $E_2^{p,q}$ are finitely generated abelian groups or finite dimensional vector spaces, then we can conclude that the same is true for the target groups (vector spaces) $H^n(X,\mathcal{F})$.

b) Another typical application concerns Euler characteristics. If we know that the cohomology groups $H^n(X,\mathcal{F})$ are finite dimensional vector spaces over a field k which vanish for $n \gg 0$ then we define the **Euler characteristic**

$$\chi(X,\mathcal{F}) = \sum_\nu (-1)^\nu \dim_k(H^\nu(X,\mathcal{F})).$$

It is of course clear that

$$\chi(X,\mathcal{F}) = \sum_n \sum_{p+q=n} (-1)^{p+q} \dim_k(E_\infty^{p,q}).$$

But if we have for a certain level r that the total dimension of the $E_r^{p,q}$ is finite then it follows from simple principles in linear algebra that

$$\sum_{p,q} (-1)^{p+q} \dim_k(E_r^{p,q}) = \sum_{p,q} (-1)^{p+q} \dim_k(E_{r+1}^{p,q}).$$

Then we can conclude

$$\chi(X,\mathcal{F}) = \sum_{p,q} (-1)^{p+q} \dim_k(E_r^{p,q}).$$

If already the $H^q(X,\mathcal{G}^q)$ have finite total dimension then

$$\chi(X,\mathcal{F}) = \sum_{p,q} (-1)^{p+q} \dim_k(H^q(X,\mathcal{G}^p)).$$

c) There are interesting cases where one knows the structure of the groups $E_r^{p,q}$ for some r and one also knows that the $d_r^{p,q}$ are zero. Then we have $E_r^{p,q} = E_{r+1}^{p,q}$. It can happen that the differentials on this level vanish again and that this goes on forever. Then we say that the spectral sequence **degenerates at level** E_r. In such a case the $E_r^{p,q} = E_\infty^{p,q}$ are equal to the subquotient in the filtration on the target. If for instance the cohomology groups are finite dimensional vector spaces then we can compute the dimensions of the cohomology $\dim_k(H^n(X,\mathcal{F})) = \sum_{p,q:p+q=n} \dim_k(E_r^{p,q})$.

There are important cases where we have degeneration at level E_1 and E_2. But it also happens in some cases that the computation of the higher differentials becomes an extremely difficult task.

d) A very important application is the following. We start from a space X and a sheaf $\mathcal{F}$ of abelian groups on it. Furthermore we assume that we have a continuous map $f : X \longrightarrow Y$. We know of course that $H^0(X,\mathcal{F}) = H^0(Y,f_*(\mathcal{F}))$.

We can compute the cohomology groups $H^n(X,\mathcal{F})$ and the derived sheaves $R^p f_*(\mathcal{F})$ from an injective resolution

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}^0 \longrightarrow \mathcal{I}^1 \longrightarrow \mathcal{I}^2 \longrightarrow \dots . \tag{4.64}$$

If we apply f_* to this resolution, then we get a complex of sheaves on Y:

$$0 \longrightarrow f_*(\mathcal{F}) \longrightarrow f_*(\mathcal{I}^0) \longrightarrow f_*(\mathcal{I}^1) \longrightarrow f_*(\mathcal{I}^2) \longrightarrow \dots .$$

Now we choose an adjusted resolution of the complex $f_*(\mathcal{I}^\bullet)$ and apply the global sections functor

$$\begin{array}{ccccccccc}
 & & 0 & & & & 0 & & 0 & & \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & f_*(\mathcal{I}^0)(Y) & \longrightarrow & \cdots & \longrightarrow & f_*(\mathcal{I}^n)(Y) & \longrightarrow & f_*(\mathcal{I}^{n+1})(Y) & \longrightarrow & \cdots \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{I}^{0,0}(Y) & \longrightarrow & \cdots & \longrightarrow & \mathcal{I}^{n,0}(Y) & \longrightarrow & \mathcal{I}^{n+1,0}(Y) & \longrightarrow & \cdots \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{I}^{0,1}(Y) & \longrightarrow & \cdots & \longrightarrow & \mathcal{I}^{n,1}(Y) & \longrightarrow & \mathcal{I}^{n+1,1}(Y) & \longrightarrow & \cdots \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \\
 & & \vdots & & & & \vdots & & \vdots & &
\end{array}$$

We know that the sheaves $f_*(\mathcal{I})$ are in fact acyclic for the functor global sections (Lemma 4.4.5) and hence we are in the special case 4.6.3 b). The vertical filtration tells us that the complex $\mathcal{I}^\bullet_{\text{simp}}(Y)$ computes the cohomology of $f_*(\mathcal{I}^\bullet)(Y) = \mathcal{I}^\bullet(X)$ and this is $H^\bullet(X,\mathcal{F})$. The horizontal filtration gives us a spectral sequence which converges to $H^n(X,\mathcal{F})$ where the E_2 term is $H^p(Y,R^q f_*(\mathcal{F}))$, i.e.

$$\left(H^p\left(Y,R^q f_*(\mathcal{F})\right),d_2\right) \Rightarrow H^n(X,\mathcal{F})$$

4.6.4 The Derived Category

We consider complexes of sheaves on X

$$\mathcal{G}^\bullet = \ldots \longrightarrow \mathcal{G}^\nu \longrightarrow \mathcal{G}^{v+1} \longrightarrow \cdots$$

where we may have positive and negative degrees ν. Sometimes we assume that our complexes satisfy some boundedness conditions, this means that the entries are zero for large negative degrees or large positive degrees or even both.
We introduce the sheaves of cocycles $Z(\mathcal{G}^\nu) = \ker(\mathcal{G}^\nu \longrightarrow \mathcal{G}^{\nu+1})$ and the sheaves of coboundaries $B(\mathcal{G}^\nu) = \mathrm{im}(\mathcal{G}^{\nu-1} \longrightarrow \mathcal{G}^\nu)$ and the cohomology sheaves

$$H^\nu(\mathcal{G}^\bullet) = \frac{\ker(\mathcal{G}^\nu \longrightarrow \mathcal{G}^{\nu+1})}{\mathrm{im}(\mathcal{G}^{\nu-1} \longrightarrow \mathcal{G}^\nu)}.$$

If we have two such complexes we have an obvious notion of a morphism

$$\mathcal{G}_1^\bullet \xrightarrow{\psi} \mathcal{G}_2^\bullet.$$

It is clear that ψ induces a morphism of sheaves between the cohomology sheaves

$$H^\nu(\mathcal{G}_1^\bullet) \longrightarrow H^\nu(\mathcal{G}_2^\bullet).$$

Now it is possible to construct a new category from this, namely **the derived category** $D(\mathcal{S}_X)$. It is defined as a "quotient" category of the category of complexes: A morphism

$$\psi : \mathcal{G}_1^\bullet \longrightarrow \mathcal{G}_2^\bullet$$

is declared to be an isomorphism if it induces an isomorphism on the cohomology sheaves. Such morphisms are called quasi-isomorphisms. A quasi-isomorphism ψ induces an invertible morphism in the derived category and this inverse is not necessarily induced by a morphism in the category of complexes.
This means that the objects are the complexes of sheaves but the sets of morphisms $\mathrm{Hom}_{D(\mathcal{S}_X)}(\mathcal{G}_1^\bullet,\mathcal{G}_2^\bullet)$ become complicated.

If we have quasi-isomorphisms $\mathcal{A}^\bullet \xrightarrow{\phi} \mathcal{G}_1^\bullet$ and $\mathcal{G}_2^\bullet \xrightarrow{\psi} \mathcal{B}^\bullet$ and if we have a morphism $\Phi : \mathcal{A}^\bullet \longrightarrow \mathcal{B}^\bullet$ then we get an element

$$\psi \circ \Phi \circ \phi^{-1} \in \mathrm{Hom}_{D^+(\mathcal{S}_X)}(\mathcal{G}_1^\bullet,\mathcal{G}_2^\bullet)$$

This construction of the derived category will not be discussed in further detail here. (See for instance [Ge-Ma], Chap. III and IV.)
In the following discussion I want to consider the subcategory $D^+(\mathcal{S}_X)$ of complexes which have non zero entries only in positive degrees.
We have a new way to speak of resolutions. If we have a sheaf $\mathcal{F}$ we can view it as a complex

$$\mathcal{F}[0] : \quad 0 \longrightarrow \mathcal{F} \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

where the sheaf sits in degree zero. If we have a resolution

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G}^0 \longrightarrow G^1 \longrightarrow \cdots \longrightarrow \mathcal{G}^n \longrightarrow$$

we write this as morphism $\psi : \mathcal{F}[0] \longrightarrow \mathcal{G}^\bullet$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{F} & \longrightarrow & 0 & \longrightarrow & & & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & \mathcal{G}^0 & \longrightarrow & \mathcal{G}^1 & \longrightarrow & \cdots & \longrightarrow & \mathcal{G}^n & \longrightarrow & \cdots
\end{array}$$

and the fact that $\mathcal{G}^\bullet$ is a resolution translates into the fact that ψ is an isomorphism in the derived category.

We can introduce the notion of a resolution of a complex. This is a double complex

$$\begin{array}{ccccccccccc}
 & & 0 & & & & 0 & & 0 & & \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{G}^0 & \longrightarrow & \cdots & \longrightarrow & \mathcal{G}^n & \longrightarrow & \mathcal{G}^{n+1} & \longrightarrow & \cdots \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{A}^{0,0} & \longrightarrow & \cdots & \longrightarrow & \mathcal{A}^{n,0} & \longrightarrow & \mathcal{A}^{n+1,0} & \longrightarrow & \cdots \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{A}^{0,1} & \longrightarrow & \cdots & \longrightarrow & \mathcal{A}^{n,1} & \longrightarrow & \mathcal{A}^{n+1,1} & \longrightarrow & \cdots \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \\
 & & \vdots & & & & \vdots & & \vdots & &
\end{array}$$

where the vertical complexes are exact.

We can drop the line on the top and consider the double complex of sheaves $\mathcal{A}^{\bullet\bullet}$ and we can pass to the simple complex $\mathcal{A}^\bullet_{\text{simp}}$. Of course we have a morphism

$$r : \mathcal{G}^\bullet \longrightarrow \mathcal{A}^\bullet_{\text{simp}}$$

and I leave it as an exercise to the reader to prove that this is an isomorphism in the derived category. (See Lemma 4.6.2)

If we now take an injective resolution $\mathcal{G}^\bullet \longrightarrow \mathcal{I}^{\bullet,\bullet}$ then $r : \mathcal{G}^\bullet \longrightarrow \mathcal{I}^\bullet_{\text{simp}}$ gives us an isomorphism of our given complex with a complex whose components $\mathcal{I}^\nu$ are injective.

It is also rather clear that this construction is functorial in $\mathcal{G}^\bullet$, if we have a morphism between two complexes $\psi : \mathcal{G}^\bullet_1 \longrightarrow \mathcal{G}^\bullet_2$ then this extends to a morphism of the injective resolutions

$$\widetilde{\Psi} : \mathcal{I}_1^{\bullet,\bullet} \longrightarrow \mathcal{I}_2^{\bullet,\bullet}$$

and if we pass to the simple complexes we get a diagram

$$\begin{array}{ccc}
\mathcal{G}^\bullet & \xrightarrow{r_1} & \mathcal{I}^\bullet_{1,\text{simp}} \\
\Psi \downarrow & & \downarrow \widetilde{\Psi}^\bullet \\
\mathcal{G}^\bullet_2 & \xrightarrow{r_2} & \mathcal{I}^\bullet_{2,\text{simp}}
\end{array}$$

this extension is unique by the definition of the derived category.

Once we have the notion of the derived category we find a new concept of what a derived functor should be. I explain this in the context of sheaves on spaces and the global section functor, but it works in a much more general context.

If we have a complex of sheaves $\mathcal{G}^\bullet$ on our space X , then we choose an injective resolution $\mathcal{G}^\bullet \longrightarrow \mathcal{I}^{\bullet\bullet}$ and use the isomorphism $r : \mathcal{G}^\bullet \longrightarrow \mathcal{I}^\bullet_{\text{simp}}$. Now $I^\bullet_{\text{simp}}(X)$ is a complex of abelian groups and can be viewed as an object in the derived category of abelian groups. The functor $\mathcal{G}^\bullet \longrightarrow I^\bullet_{\text{simp}}(X)$ is now the derived functor

$$\mathcal{R}^\bullet H^0_X : D^+(\mathcal{S}_X) \longrightarrow D^+(Ab)$$

from the derived category of sheaves on X to the derived category of abelian groups. We apply it to our sheaf $\mathcal{F}$. We view it as a complex $\mathcal{F}[0]$ and consider $\mathcal{R}^\bullet H^0_X(\mathcal{F}[0])$. This is a complex of abelian groups and we recover the cohomology groups $H^\bullet(X,\mathcal{F})$ as the cohomology groups of the complex $\mathcal{R}^\bullet H^0_X(\mathcal{F}[0])$.
We may of course apply this also to the case of a continous map $f : X \longrightarrow Y$ and a sheaf $\mathcal{F}$ on X. We take an injective resolution $\mathcal{F} \longrightarrow I^\bullet$, we view this as an isomorphism $\mathcal{F}[0] \xrightarrow{\sim} I^\bullet$ in the derived category and define

$$\mathcal{R}^\bullet f_*(\mathcal{F}) = f_*(I^\bullet),$$

this is now a sheaf in the derived category of sheaves on Y. Our "old" derived functor is now simply the cohomology of this complex.

Philosophical remark: Our "old" derived functors transform a sheaf $\mathcal{F}$ on X into a collection of sheaves $\{Rf_*^\nu(\mathcal{F})\}_{\nu=0,1\dots} = R^\bullet f_*(\mathcal{F})$. We can view this as a complex where all the differentials are zero. Certainly this is an object of different nature. The "new" derived functor $\mathcal{R}^\bullet f_*(\,)$ sends objects in the derived category of sheaves on X into the derived category of sheaves on Y, so the nature of the object does not change: Complexes go to complexes. Hence $\mathcal{R}^\bullet f_*(\mathcal{F}[0])$ is a "higher level object", it contains more information then just the cohomology groups $R^\bullet f_*(\mathcal{F})$.

In section 2.3.1 I explained that we may - after defining the derived functor by using injective resolutions - compute it from acyclic resolutions. The same is of course true in the context of derived categories.

Lemma 4.6.5. *If we have a resolution of our complex $\mathcal{G}^\bullet \longrightarrow \mathcal{A}^{\bullet,\bullet}$ as above and if the $\mathcal{A}^{pq}$ are acyclic for the functor $H^0(X,\quad)$ then we have*

$$\mathcal{R}^\bullet H^0_X(G^\bullet) \xrightarrow{\sim} H^0(X,\mathcal{A}^{\bullet,\bullet}) = \mathcal{A}^{\bullet,\bullet}(X).$$

Proof: To see this we choose an injective resolution $\mathcal{G}^\bullet \longrightarrow \mathcal{I}^{\bullet,\bullet}$ The definition of injective sheaves allows us to construct a commutative diagram of complexes

$$\begin{array}{ccc} \mathcal{G}^\bullet & \longrightarrow & \mathcal{A}^{\bullet,\bullet} \\ \downarrow & & \downarrow \\ \mathcal{G}^\bullet & \longrightarrow & \mathcal{I}^{\bullet,\bullet} \end{array}$$

which then induces a homomorphism of complexes

$$\mathcal{A}^\bullet(X)_{\text{simp}} \longrightarrow \mathcal{I}^\bullet(X)_{\text{simp}}$$

which must be an isomorphism in the derived category. To see this we look at the vertical filtration (see section 4.6.2) and find that we get the same $E_1^{p,q}$ term, namely $H^q(X,\mathcal{G}^p)$, on both sides. Here we used the acyclicity of the $\mathcal{A}^{\bullet,\bullet}$ resolution. Now the rest follows from a simple argument of functoriality: We get an isomorphism for the $E_\infty^{p,q}$ and hence the homomorphism must be an isomorphism. □
Especially we see

Corollary 4.6.6. *If $\mathcal{A}^\bullet$ is a complex of acyclic sheaves then*

$$\mathcal{R}^\bullet H^0_X(\mathcal{A}^\bullet) \xrightarrow{\sim} \mathcal{A}^\bullet(X)$$

The Composition Rule

The concept of derived categories allows a very elegant formulation of the content of the theory of spectral sequences. I want to explain this in a special case but it will be clear what happens in more general situations.
I recall the situation in b) on page 96. We start from a continuous map $f : X \longrightarrow Y$ between two topological spaces. We consider the abelian category of sheaves $\mathcal{F}$ on X with values in the category of abelian groups. We have the functors $\mathcal{F} \longrightarrow H^0(X,\mathcal{F}) = H^0_X(\mathcal{F})$ and f_*. It is clear that $H^0_X(\ )$ is the composition of f_* and $H^0_Y(\ )$. We want to understand the resulting relation between the derived functor of $H^0_X(\ )$ and the derived functors of f_* and $H^0_Y(\ )$.
We introduced the higher direct images $R^\bullet f_*(\mathcal{F})$ as the derived functor of the direct image functor f_* and this ist just a collection of sheaves on Y which are indexed by degrees.
But we also defined the derived functor

$$\mathcal{R}^\bullet f_* : D(S_X) \longrightarrow D(S_Y)$$

which sends a complex of sheaves on X to an object in the derived category of sheaves on Y. It sends a sheaf $\mathcal{F}$ to $\mathcal{R}^\bullet f_*(\mathcal{F}) = f_*(I^\bullet)$ (see above) and the cohomology of this object gives us the derived sheaves $R^\bullet f_*(\mathcal{F})$.
We apply the derived functor $\mathcal{R}^\bullet H^0_Y$ to $f_*(\mathcal{I}^\bullet)$. Lemma 4.4.5 tells us that the complex of sheaves $f_*(I^\bullet)$ consists of injective sheaves. Therefore we can conclude that

$$\mathcal{R}^\bullet H^0_Y(f_*(I^\bullet)) = f_*(I^\bullet)(Y).$$

(See Remark 6 below.) Since $f_*(I^\bullet) = \mathcal{R}^\bullet f_*(\mathcal{F})$ and $f_*(I^\bullet)(Y) = I^\bullet(X) = \mathcal{R}^\bullet H^0_X(\mathcal{F})$ and we get the composition rule

$$\mathcal{R}^\bullet H^0_Y \circ \mathcal{R}^\bullet f_*(\mathcal{F}) \simeq \mathcal{R}^\bullet H^0_X(\mathcal{F}). \tag{4.65}$$

Here it is of course clear and important to notice, that this rule does not only apply to sheaves $\mathcal{F}$ on X but actually we should apply it to complexes of sheaves, i.e. to objects in the derived category $D^+(\mathcal{S}_X)$. So the more conceptual way to write the composition rule is

$$\mathcal{R}^\bullet H^0_Y \circ \mathcal{R}^\bullet f_*(\mathcal{G}^\bullet) \simeq \mathcal{R}^\bullet H^0_X(\mathcal{G}^\bullet).$$

Remark 6. We should observe that we used Lemma 4.4.5, hence we knew that the sheaves in the complex were injective. But we should be aware that in the next step we only used that the sheaves $f_*(I^\bullet)$ are acyclic. (See Lemma 4.6.5.)

This gives us a general principle, which also applies in other situations:
If we pass to the derived category then the derived functor of a composition is the composition of the derived functors provided the first functor sends injective object into acyclic objects.

This may for instance be applied to the following situation: Let Γ be a group and Γ' a normal subgroup in it. For any Γ-module M the module $M^{\Gamma'}$ is a Γ/Γ'-module and clearly we have that $M^{\Gamma} = (M^{\Gamma'})^{\Gamma/\Gamma'}$. In other words we have $H^0(\Gamma, \ )$ is the composite of $H^0(\Gamma', \ )$, which sends Γ-modules to Γ/Γ' -modules, and $H^0(\Gamma/\Gamma', \ )$, which sends Γ/Γ'-modules to abelian groups. It is not hard to see that in this case we also get the composition rule for the derived functors.
Of course this formulation of the content of the concept of spectral sequences is very elegant. Actually it says more than point c) on page 96 but sometimes it may be necessary to go back to the formulation involving the $E_r^{p,q}$. The point is somehow that the objects in the derived category contain much more information, but certainly also some extra information, which is of no interest for us.
By the way, if we consider the case of group cohomology then we get a spectral sequence with E_2 term

$$H^p(\Gamma/\Gamma', H^q(\Gamma', M)) \Rightarrow H^{p+q}(\Gamma, M).$$

Exact Triangles

In the derived category of an abelian category (for instance $D(\mathcal{S}_X), D(Ab)$) we do not have the notion of a short exact sequence. The reason is basically that a short exact sequence leads to a long exact sequence in cohomology. I recall the discussion in the fundamental remark on page 26. We saw that in our injective resolution the differentials in the middle are not necessarily the direct sums of the differentials of the two outer resolutions, we need to add homomorphisms $\Psi^\nu : I''^\nu \longrightarrow I'^{\nu+1}$.
Now we may look at these Ψ^ν from a different point of view. We introduce the translation operator T which transforms the complex $I'^\bullet$ into $I'^\bullet[1]$, this is our original complex shifted by 1. In other words the p-th component of $I'^\bullet[1]$ is equal to the $(p-1)$-th component of $I'^\bullet$. Then we see that the recursion relations for the Ψ^ν simply say: The negative sum of the Ψ^ν defines a morphism of complexes

$$-\Psi^\bullet : I''^\bullet \longrightarrow I'^\bullet[1].$$

This tells us that the resolution of an exact sequence has the structure of an exact triangle. In this situation this says that we get a complex of complexes

$$\ldots \longrightarrow I'^\bullet \longrightarrow I^\bullet \longrightarrow I''^\bullet \longrightarrow I'^\bullet[1] \longrightarrow I^\bullet[1] \longrightarrow \ldots$$

and this is abbreviated by

$$\begin{array}{ccc} I'^\bullet & \longrightarrow & I^\bullet \\ & \nwarrow \quad \swarrow & \\ & I''^\bullet & \end{array}$$

where the map $I''^\bullet \longrightarrow I'^\bullet$ has degree one.
We can now easily define what a **triangle** in a derived category of an abelian category should be. It consists of three complexes $X^\bullet, Y^\bullet, Z^\bullet$ and a sequence of morphisms

$$\ldots \longrightarrow Z^\bullet[-1] \longrightarrow X^\bullet \longrightarrow Y^\bullet \longrightarrow Z^\bullet \longrightarrow X^\bullet[1] \longrightarrow \ldots$$

such that the compositum of two consecutive arrows is zero (in the derived category, i.e. induces zero in cohomology). We call such a triangle **exact** if we get a long exact sequence in cohomology after taking the cohomology of the complexes.

4.6.5 The Spectral Sequence of a Fibration

This spectral sequence is especially useful if we can apply it in combination with base change.

Definition 4.6.7. *We say that the map*

$$f : X \longrightarrow Y$$

is a **cohomological fibration for the sheaf** $\mathcal{F}$ *if the sheaves* $\mathcal{R}^q f_*(\mathcal{F})$ *form local systems on* Y *(see Definition 4.3.9) whose stalk in* y *is given by* $H^q(f^{-1}(y), i_y^*(\mathcal{F}))$.

If our space Y is locally connected this means we have base change, i.e.

$$\mathcal{R}^q f_*(\mathcal{F})_y \simeq H^q(f^{-1}(y), i_y^*(\mathcal{F})), \tag{4.66}$$

and for any point y we can find a connected neighborhood V_y such that $\mathcal{R}^q f_*(\mathcal{F})$ restricted to V_y is isomorphic to the sheaf of locally constant sections into $\mathcal{R}^q f_*(\mathcal{F})_y$.
The intuitive meaning of this notion is that $\mathcal{R}^q f_*(\mathcal{F})$ is **the system of cohomology groups of the fibres**.
To produce examples of such cohomological fibrations we consider maps $f : X \longrightarrow Y$ which are locally trivial fibrations with some fibre F. (See section 4.3.1.) Furthermore we assume that for any local trivialization

$$\begin{array}{ccccc} \Psi_i : & f^{-1}(U_i) & \xrightarrow{\sim} & U_i \times F, & \\ & & \searrow \quad \swarrow \mathrm{pr}_1 & & \\ & & U_i & & \end{array}$$

the restriction of $\mathcal{F}$ to $f^{-1}(U_i)$ is isomorphic to a pullback with respect to the projection $\mathrm{pr}_F : U_i \times F \longrightarrow F$ of a sheaf on the fibre F. For the following discussion we assume that X,Y are Hausdorff and that Y is locally contractible, i.e. each point $y \in Y$ has arbitrarily small contractible (see Lemma 4.4.24) neighborhoods. Then it is clear:

Lemma 4.6.8. *Under these assumptions* $f : X \longrightarrow Y$ *is a cohomological fibration for* $\mathcal{F}$.

The assumption on the local structure of the sheaves is certainly satisfied if the sheaf $\mathcal{F}$ is isomorphic to the sheaf $A_X = \underline{A}$ for an abelian group A.
If we assume in addition that our space Y is pathwise connected and if we pick a base point $y_0 \in Y$, then we will also show (see in 4.8.1) that our local system is basically the same object as a representation of the fundamental group (see [Hat], Chap. I) $\pi_1(Y,y_0)$ on $H^q(f^{-1}(y_0),\underline{A})$. Especially for a simply connected base space Y (see [Hat], loc. cit.) we even have

$$\mathcal{R}^q f_*(\underline{A}) = \underline{H^q(f^{-1}(y),\underline{A})}. \tag{4.67}$$

I want to discuss some special cases where this spectral sequence for a fibration becomes very useful.

Sphere Bundles an Euler Characteristic

Let us consider a fibre space

$$\pi : X \longrightarrow Y$$

(see section 4.3) where the fibre F is homeomorphic to a sphere S^{n-1}. Then we have the E_2-term in the spectral sequence

$$E_2^{p,q} = H^p(Y, R^q\pi_*(\_)), \tag{4.68}$$

and it is clear that $R^0\pi_*(\_) = \_$. We have $R^q\pi_*(\_) = 0$ for $q \neq 0, n-1$ and $R^{n-1}\pi_*(\_)$ is a local system where the stalks are isomorphic to . We say that this fibration by spheres is orientable if the local system is trivial, and we say that the fibration is oriented if we fix an isomorphism

$$R^{n-1}\pi_*(\_) \xrightarrow{\sim} \_.$$

Now we consider the E_2-term of the spectral sequence. It looks like

$$\begin{array}{cccc}
0 & 0 & \cdots & 0 \\
H^0(Y,_) & H^1(Y,_) & \cdots & H^p(Y,_) \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & & \vdots \\
0 & 0 & \cdots & 0 \\
H^0(Y,R^{n-1}\pi_*(_)) & H^1(Y,R^{n-1}\pi_*(_)) & \cdots & H^p(Y,R^{n-1}\pi_*(_)) \\
0 & 0 & \cdots & 0
\end{array} \tag{4.69}$$

and the differential operator d_2 is given by an arrow that points 2 steps to the right and one step up. Thus it is zero (unless we have $n - 1 = 1$) and stays zero for a while. So the terms $E_2^{p,q} = E_3^{p,q} \cdots$ stay constant for a while until we come to the differential d_n which sends

$$d_n^{p,n-1} : H^p(Y, R^{n-1}\pi_*(\_)) \longrightarrow H^{p+n}(Y, R^0\pi_*(\_)), \tag{4.70}$$

and now the $E_{n+1}^{p,q}$ may be different from $E_n^{p,q}$. After that the spectral sequence degenerates. Therefore we get an exact sequence

$$\begin{array}{l}
H^{p-1}(Y,R^{n-1}\pi_*(_)) \xrightarrow{d_n^{p-1,n-1}} H^{p+n-1}(Y,R^0\pi_*(_)) \longrightarrow H^{p+n-1}(X,_) \longrightarrow \\
\qquad \longrightarrow H^p(Y,R^{n-1}\pi_*(_)) \xrightarrow{d_n^{p,n-1}} H^{p+n}(Y,R^0\pi_*(_)) \rightarrow \cdots
\end{array} \tag{4.71}$$

which is the so called **Gysin sequence**. It contains relevant information concerning the fibration. If for instance, one of the differentials $d_n^{p-1,n-1}$ is not zero and not surjective, then the map

$$H^{p+n-1}(Y, R^0\pi_*(\_)) = H^{p+n-1}(Y,\_) \longrightarrow H^\bullet(X,\_)$$

is not injective. From this we can conclude that under this assumption the fibration cannot have a section

$$s : Y \longrightarrow X$$

to π because the composition $\pi \circ s$ would induce the identity on $H^\bullet(Y,\_)$.

If the bundle is oriented, then $R^{n-1}\pi_*(\_) = \_$, and we have a canonical generator $e \in R^{n-1}\pi_*(\_)$. This gives a class in $H^0(Y,R^{n-1}\pi_*(\_))$ which is mapped to a class

$$e = d_n^{0,n-1}(e_0) \in H^n(Y,\_), \tag{4.72}$$

and this class is the so called Euler class of the fibration. If it is non zero, then the bundle has no section.
Let M be a compact, oriented $\mathcal{C}^\infty$-manifold of dimension n, we assume that we have a Riemannian metric on the tangent bundle. (See 4.8.2.) We consider the bundle of tangent vectors of length 1, this is denoted by $S(T_M) \longrightarrow M$ and it gives us an example of a sphere bundle, with spheres of dimension $n-1$. Then the above class $e = d_n^{0,n-1}(e_0) \in H^n(M,\_) =$ (see 4.8.5) In 4.9.1 is called the **Euler class**, it is a also a number. We will give some indications is equal to the **Euler characteristic**

$$\chi(M) = \sum_{\nu=0}^{\nu=n} (-1)^\nu \dim H^\nu(M,\mathbb{Q}).$$

We conclude:

Lemma 4.6.9. *If a compact, oriented manifold M has non zero Euler characteristic $\chi(M)$, then the bundle $S(T_M) \longrightarrow M$ has no section. Hence a continous vector field (i.e. a section in TM) must have zeroes.*

4.6.6 Čech Complexes and the Spectral Sequence

I return to the Čech resolutions constructed from coverings $\mathfrak{U} = \{U_\alpha\}_{\alpha\in\mathrm{A}}$ (see page 81):

$$\mathcal{F}^\bullet_{\mathfrak{U}} = 0 \longrightarrow \mathcal{F} \longrightarrow \Pi_{\alpha\in\mathrm{A}}\mathcal{F}^*_\alpha \longrightarrow \Pi_{(\alpha,\beta)\in\mathrm{A}^2_<}\mathcal{F}^*_{\alpha_\beta} \longrightarrow \cdots \tag{4.73}$$

In view of our previous discussion this means that we have an isomorphism in the derived category

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{F} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \prod_{\alpha\in\mathrm{A}} \mathcal{F}^*_\alpha & \longrightarrow & \prod_{(\alpha,\beta)\in\mathrm{A}^2_<} \mathcal{F}^*_{\alpha,\beta} & \longrightarrow & \prod_{(\alpha,\beta,\gamma)\in\mathrm{A}^3_<} \mathcal{F}^*_{\alpha,\beta,\gamma} & \longrightarrow & \cdots
\end{array}$$

and hence these two complexes have isomorphic derived functors.
The sheaves $\mathcal{F}^*_{\underline{\alpha}}$ are concentrated on the closed subsets $\overline{U_{\underline{\alpha}}} = \overline{U_{\alpha_0} \cap \ldots \cap U_{\alpha_p}}$ and our resolution is acyclic if and only if the sheaves $\mathcal{F}^*_{\underline{\alpha}}$ on $\overline{U}_\alpha$ are acyclic. In this case we say that the covering $\mathfrak{U}$ **provides an $\mathcal{F}$-acyclic** resolution.
We consider the vertical filtration (see 4.6.2 and 4.6.3). We get for our E_1-term

$$E_1^{p,q} = H^q(X, \prod_{\underline{\alpha}\in\mathrm{A}^{p+1}_<} \mathcal{F}^*_{\underline{\alpha}}) = \prod_{\underline{\alpha}\in\mathrm{A}^{p+1}_<} H^q(\overline{U}_{\underline{\alpha}},\mathcal{F}^*_{\underline{\alpha}}) = H^q(X,\mathcal{F}^\bullet_{\mathfrak{U}}). \tag{4.74}$$

The $E_2^{p,0}$ term is the Čech cohomology and the edge homomorphism yields a homomorphism

$$\check{H}^p(X,\mathfrak{U},\mathcal{F}) \longrightarrow H^p(X,\mathcal{F}).$$

Clearly we have

Lemma 4.6.10. *If the covering provides an $\mathcal{F}$-acyclic resolution then the edge homomorphism is an isomorphism or – in other words – the Čech complex computes the cohomology of $\mathcal{F}$.*

In general this edge homomorphism needs not to be injective because we may have a non-trivial differential

$$d_2 : E_2^{p-2,1} \longrightarrow E_2^{p,0} = \check{H}^p(X,\mathfrak{U},\mathcal{F}).$$

But for $p = 1$ this differential is zero and it follows that the edge homomorphism

$$\check{H}^1(X,\mathfrak{U},\mathcal{F}) \longrightarrow H^1(X,\mathcal{F})$$

is injective (see Exercise 25). But of course it may be non surjective, its image is the kernel of

$$H^1(X,\mathcal{F}) \longrightarrow \prod_{\alpha\in A} H^1(X,\mathcal{F}_\alpha) = \prod_{\alpha\in A} H^1(\overline{U}_\alpha,\mathcal{F}_\alpha).$$

I want to consider a special case. We cover our space X by two open sets $X = U \cup V$, then our resolution becomes very short:

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_U \oplus \mathcal{F}_V \longrightarrow \mathcal{F}_{U\cap V} \longrightarrow 0$$

where $\mathcal{F}_U = i_{U*}i_U^*(\mathcal{F})$ and so on. Then our spectral sequence has only two columns: We have as E_1-term

$$\begin{array}{ccccc}
H^q(U,\mathcal{F}) \oplus H^q(V,\mathcal{F}) & \longrightarrow & H^q(U\cap V,\mathcal{F}) & \longrightarrow & 0 \\
\vdots & & \vdots & & \vdots \\
\underbrace{H^0(U,\mathcal{F}) \oplus H^0(V,\mathcal{F})} & \longrightarrow & \underbrace{H^0(U\cap V,\mathcal{F})} & \longrightarrow & 0 \\
p=0 & & p=1 & &
\end{array}$$

and the horizontal boundary operator is taking the difference of the restriction maps. Then we see that the spectral sequence degenerates on E_2 level and we get a long exact sequence

$$H^{q-1}(U\cap V,\mathcal{F}) \longrightarrow H^q(X,\mathcal{F}) \longrightarrow H^q(U,\mathcal{F}) \oplus H^q(V,\mathcal{F}) \longrightarrow H^q(U\cap V,\mathcal{F}) \longrightarrow \dots$$

which is called the **Mayer-Vietoris sequence**. It is of course nothing else than the long exact sequence obtained from the short exact sequence which is given by the resolution. With a slight modification we used this Mayer-Vietoris sequence already when we computed the cohomology of spheres (see section 4.4.4).

Definition 4.6.11. *A* **CW-complex** *is a space which is obtained by successive* **attachment of cells**.

By this we mean the following:
We start with a point, this is the simplest CW-complex. If Y is already a CW-complex, and if

$$f : S^{n-1} \longrightarrow Y$$

is a continuous map, then we construct a new space $X = D^n \cup_f Y$ which is again a CW-complex. To construct this new space X we consider S^{n-1} as the boundary of D^n and $X = D^n \cup_f Y$ is obtained from the disjoint union $D^n \sqcup Y$ by identifying $x \in S^{n-1}$ to $f(x) \in Y$. This process is called **attaching an n-cell** to Y.
We can relate the cohomology of the spaces X and Y. If we consider a tubular neighborhood T of the boundary sphere (i.e. $\{(x_1, \cdots, x_n) \mid 1 - \varepsilon < \sum x_i^2 \leq 1\}$) then $V = T \cup_f Y$ is open in X and clearly the inclusion $Y \hookrightarrow V$ is a homotopy equivalence. The open ball $\overset{\circ}{D^n} = U$ is also open in X and we have a covering

$$X = U \cup V.$$

The open set U is acyclic and $U \cap V$ is homotopy equivalent to S^{n-1}. Thus our spectral sequence yields for $q \geq 1$

$$\ldots \longrightarrow H^{q-1}(S^{n-1},\_) \longrightarrow H^q(X,\_) \longrightarrow H^q(Y,\_) \longrightarrow H^q(S^{n-1},\_) \longrightarrow \ldots \qquad (4.75)$$

This tells us that we have some control how the cohomology of Y changes if we attach an n-cell. More precisely we can say that we can compute the cohomology of X if we already know the cohomology of Y and if we understand the boundary operator on the E_1 term:

$$H^{n-1}(Y,\_) \longrightarrow H^{n-1}(S^{n-1},\_).$$

There is a very prominent example where this method of computing the cohomology is especially successful. We consider the n-dimensional complex projective space $\mathbb{P}^n(\mathbb{C})$ (see Example 15 in section 3.2.2).

Exercise 26.

(a) Show that the topological space $\mathbb{P}^n(\mathbb{C})$ is obtained from $\mathbb{P}^{n-1}(\mathbb{C})$ by attaching a $2n$-cell.

(b) Show that

$$H^\bullet(\mathbb{P}^n(\mathbb{C}),\_) = \bigoplus_{i=0}^{n} \quad e_i$$

where $e_i \in H^{2i}(\mathbb{P}^n(\mathbb{C}),\_)$ is a free generator.

A Criterion for Degeneration

Let us assume that our complex of sheaves has the following property: For any index q we can find a splitting of the short exact sequance

$$0 \longrightarrow Z(\mathcal{G}^q) \longrightarrow \mathcal{G}^q \longrightarrow B(\mathcal{G}^{q+1}) \longrightarrow 0. \qquad (4.76)$$

If we now construct our adjusted resolution, then we can achieve that the vertical differentials

$$\begin{array}{ccc} I_Z^{q,p} & \oplus & I_B^{q+1,p} \\ & \downarrow & \\ I_Z^{q,p+1} & \oplus & I_B^{q+1,p+1} \end{array} \tag{4.77}$$

are the direct sum of the differentials of the resolutions of $Z(\mathcal{G}^q)$ and $B(\mathcal{G}^{q+1})$ (see 2.3.4). This means that we get for the adjusted resolution of the complex

$$\begin{array}{ccccc} I^{q,p} & = & I_B^{q,p} \oplus I_H^{q,p} \oplus I_B^{q+1,p} & = & I_Z^{q,p} \oplus I_B^{q+1,p} \\ \downarrow & & \downarrow & & \downarrow \\ I^{q,p+1} & = & I_B^{q,p+1} \oplus I_H^{q,p+1} \oplus I_B^{q+1,p+1} & = & I_Z^{q,p+1} \oplus I_B^{q+1,p+1} \end{array}$$

and the rightmost vertical arrow can be taken as the direct sum of the arrows in the resolution of $Z(\mathcal{G}^p)$ and $B(\mathcal{G}^{p+1})$.

I claim that this implies that the two spectral sequences for $I^\bullet_{\mathrm{simp}}(X)$ degenerate on E_2-level.

We consider the horizontal filtration. The E_2-term is given by $H^p(X,H^q(\mathcal{G}^\bullet))$. An element in this group is represented by the element $\xi_{q,p} \in I_H^{q,p}(X)$ which is mapped to zero under the vertical boundary map

$$\begin{array}{c} I_H^{q,p}(X) \\ \Big\downarrow \\ I_H^{q,p+1}(X). \end{array}$$

But if we view it as an element $x_{q,p} = (0,x_{q,p},0)$ in $I^{q,p}(X)$, then it is mapped to an element

$$\eta_{q,p+1} = (\eta_{q,p+1},0,0) \in I_B^{q,p+1}(X) \subset I_B^{q,p+1}(X) \oplus I_H^{q,p+1}(X) \oplus I_B^{q+1,p+1}(X).$$

We look at the boundary map

$$\begin{array}{ccc} I^{q-1,p+1}(X) & \longrightarrow & I^{q,p+1}(X) \\ \| & & \| \\ I_B^{q-1,p+1}(X) \oplus I_H^{q-1,p+1}(X) \oplus I_B^{q,p+1}(X) & \longrightarrow & I_B^{q,p+1}(X) \oplus I_H^{q,p+1}(X) \oplus I_B^{q+1,p+1}(X) \end{array}$$

and we see that our element $\eta_{q,p+1}$ is the image of the element

$$\widetilde{\eta}_{q-1,p+1} = (0,0,\eta_{q,p+1}) \ \in I^{q-1,p+1}(X)$$

under this boundary map. Now our assumption on the existence of the splitting implies that this element goes to zero under the vertical differential, because this vertical differential respects the decomposition

$$I^{q-1,p+1}(X) = I_Z^{q-1,p+1}(X) \oplus I_B^{q,p+1}(X).$$

But then the element

$$\widetilde{\xi}_{q,p} = \xi_{q,p} + (-1)^{p-1}\widetilde{\eta}_{q-1,p+1}$$

is a cocycle. This implies that

$$E_\infty^{p,q} = E_2^{p,q},$$

and this is the degeneration of the spectral sequence.

The argument for the vertical filtration is essentially the same.

We even get more. We know that the E_2 term is a step in the filtration and hence

$$H^p(X,H^q(\mathcal{G}^\bullet)) \xrightarrow{\sim} {}'F^p H^n(X,\mathcal{G}^\bullet)/{}'F^{p+1}H^n(X,\mathcal{G}^\bullet).$$

But we just constructed a homomorphism

$$i_{p,q} : H^p(X,H^q(\mathcal{G}^\bullet)) \longrightarrow H^n(X,\mathcal{G}^\bullet)$$

because to any class $\xi_{q,p}$ we constructed a cocycle $\widetilde{\xi}_{q,p}$ in $I^n_{\mathrm{spl}}(X)$. Hence we even get

Lemma 4.6.12. *We have a splitting*

$$H^n(X,\mathcal{G}^\bullet) \simeq \bigoplus_{p+q=n} H^p(X,H^q(\mathcal{G}^\bullet)). \tag{4.78}$$

This splitting is not canonical because it may depend on the choice of the splitting $\mathcal{G}^p = Z(\mathcal{G}^p) \oplus B(\mathcal{G}^{p+1})$ since this choice influences the correction term η. But the images of the $H^p(X,H^q(\mathcal{G}^\bullet))$ are well defined modulo the horizontal filtration.

An Application to Product Spaces

We consider a product space $Z = X \times Y$, let p_1,p_2 be the projections to the first and second factor. We assume that Y has a finite covering $Y = \bigcup_{\alpha\in I} U_\alpha$ by open sets which is $\underline{\ \ }$-acyclic (see 4.6.6). We get a covering of Z by the open set $p_2^{-1}(U_\alpha)$, with respect to this covering we consider the Čech resolution of the sheaf $\underline{\ \ }$ on Z:

$$0 \longrightarrow \underline{\ \ } \longrightarrow \prod_\alpha \underline{\ \ }_\alpha \longrightarrow \prod_{(\alpha,\beta)} \underline{\ \ }_{\{\alpha,\beta\}} \longrightarrow$$

as on page 81. We abbreviate the notation and denote the Čech complex simply by $\underline{A}^\bullet$. Then

$$\begin{array}{ccccccc} & & 0 & & & & \\ & & \downarrow & & & & \\ & & \underline{\ \ } & & & & \\ & & \downarrow & & & & \\ 0 & \longrightarrow & \underline{A}^0 & \longrightarrow & \underline{A}^1 & \longrightarrow & \cdots \end{array}$$

is an isomorphism in the derived category and we get $H^\bullet(Z,\underline{\ \ }) \simeq H^\bullet(Z,\underline{A}^\bullet)$.

Now it follows from our assumptions on the covering that the sheaves $\underline{A}^p$ are acyclic for the projection map p_1 to the factor X. We have that

$$p_{1,*}(\underline{A}^p) = \underline{\prod_{\underline{\alpha}} \underline{\ \ }_{\underline{\alpha}}(X)} = \underline{\prod_{\underline{\alpha}} \quad (U_{\alpha_0} \cap \ldots \cap U_{\alpha_r})}$$

i.e. $p_{1,*}(\underline{A}^\bullet)$ is the complex of locally constant sheaves on X associated to the abelian groups $\prod_\alpha \quad (U_{\alpha_0} \cap \cdots U_{\alpha_r})$. This is a complex of finitely generated free _ modules, we denote it by $B^\bullet$. Then we know $H^\bullet(Z,\_) = H^\bullet(X,\underline{B}^\bullet)$.
We apply the previous observation. Since the complex $B^\bullet$ is a complex of finitely generated free modules, we can conclude that the quotient $A^p/Z(A^p)$ is also free and therefore we can split off the boundaries. Hence we know that the spectral sequence degenerates, and we get an isomorphism

$$K : \bigoplus_{p+q=n} H^p(X,H^q(Y,\_)) \xrightarrow{\sim} H^n(Z,\_).$$

This isomorphism may depend on the splitting because this splitting influences the choice of the correction term η above.
Under our assumptions the modules $H^q(Y,\ )$ are finitely generated abelian groups. This allows us to write these groups as quotient of two finitely generated free abelian groups, i.e. we have an exact sequence

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow H^q(Y,\ ) \longrightarrow 0$$

where M_1,M_2 are finitely generated and free. Now we have obviously

$$H^p(X,M_i) = H^p(X,\ ) \otimes M_i,$$

and hence we get an exact sequence

$$H^p(X,\ ) \otimes M_1 \longrightarrow H^p(X,\ ) \otimes M_2 \longrightarrow H^p(X,H^q(X,\ )) \longrightarrow$$
$$\longrightarrow H^{p+1}(X,\ ) \otimes M_1 \longrightarrow H^{p+1}(X,\ ) \otimes M_2.$$

This yields a short exact sequence. We observe that the first arrow on the left yields a cokernel

$$H^q(X,\ ) \otimes M_2/M_1 = H^q(X,\ ) \otimes H^p(Y,\ ), \tag{4.79}$$

and the last arrow on the right has the kernel $\mathrm{Tor}^1\,(H^{q+1}(X,\ ),H^p(Y,\ ))$ (see section 2.4.3), and hence our short exact sequence will be

$$0 \longrightarrow H^q(X,\ ) \otimes H^p(Y,\ ) \longrightarrow H^q(X,H^p(Y,\ )) \longrightarrow \mathrm{Tor}^1\,(H^{q+1}(X,\ ),H^p(Y,\ )) \longrightarrow 0.$$

If we make the further assumption that $H^\bullet(X,\ )$ is finitely generated the module on the right is finite. Then the restriction of K to the tensor products gives us a homomorphism

$$\bigoplus_{p+q=n} H^p(X,\ ) \otimes H^q(Y,\ ) \longrightarrow H^n(X \times Y,\ ), \tag{4.80}$$

which is injective and has a finite cokernel.

This is the so called **Künneth homomorphism**. This homomorphism does not depend on the choice of the splitting. To see that this is the case we assume that our space X has a finite -acyclic covering $\{V_\beta\}_{\beta\in B}$ by open sets. In this case we can consider our locally constant sheaves $\underline{A}^p$ on X and take their Čech resolution provided by $\{V_\beta\}_{\beta\in B}$. Taking sections we get a double complex in which the (p,q) component is

$$\prod_{\underline{\beta}\in B_<^{p+1}} \prod_{\alpha\in A_<^{q+1}} ((V_{\underline{\beta}} \times Y) \cap (X \times U_{\underline{\alpha}})),$$

and where the vertical and horizontal boundary operators are induced from the boundary operators in the Čech complexes. But then it is clear: If we have cocycles

$$\xi^p \in \prod_{\underline{\beta}\in B_<^{p+1}} (V_{\underline{\beta}}), \qquad \eta^q \in \prod_{\underline{\alpha}\in A<^{q+1}} (U_{\underline{\alpha}}),$$

then we can define

$$\xi^p \eta^q = (\cdots \xi^p_{\underline{\beta}} \eta^q_{\underline{\alpha}} \cdots), \tag{4.81}$$

and this is a cocycle for the resulting simple complex which computes $H^\bullet(Z,\ )$. Hence we see that we do not need the correction in 4.6.12 which shows that the class does not depend on the splitting.

In the next section I discuss products in a more general context and then we will see that K does not depend on the choice of the covering.

We apply the same reasoning to the vertical filtration. A slightly different argument gives us another construction of the canonical homomorphism

$$K : \bigoplus_{p+q=n} H^p(X,\ ) \otimes H^q(Y,\ ) \longrightarrow H^n(X \times Y,\ ).$$

We may interchange the role of X,Y this means we study the spectral sequence attached to the map $p_2 : X \times Y \longrightarrow Y$. Now we assume that X also has a -acyclic covering by open sets. Then the E_2 term is $H^q(Y,H^p(X,\ ))$. We get homomorphisms

$$K : \bigoplus_{p+q=n} H^p(Y,\ ) \otimes H^q(X,\ ) \longrightarrow H^n(X \times Y,\ ).$$

If we compute the cohomology of the two spaces starting from Čech coverings, and if we interchange the two spaces, then the two simple complexes resulting from the double complexes are actually isomorphic. We simply have to reflect along the diagonal. But we have to observe the sign convention in the definition of the differentials. This forces us to put signs. This eventually results in the formula: If we look at the two product maps

$$H^p(X,\ ) \otimes H^q(Y,\ ) \xrightarrow{\ i_1\ } H^n(X \times Y,\ ),$$
$$H^q(Y,\ ) \otimes H^p(X,\ ) \xrightarrow{\ i_2\ } H^n(X \times Y,\ ),$$

then we have

$$i_1(\alpha \otimes \beta) = (-1)^{pq} i_2(\beta \otimes \alpha). \tag{4.82}$$

4.6.7 The Cup Product

We want to discuss products in a more general context. We start with a commutative ring R with identity and we consider sheaves of R-modules on topological spaces. If we have two such sheaves $\mathcal{F}$,$\mathcal{G}$ on a space X, then we can consider the tensor product sheaf $\mathcal{F} \otimes_R \mathcal{G}$ on X. It is plausible that this should be defined as the sheaf attached to the presheaf

$$U \longrightarrow \mathcal{F}(U) \otimes_R \mathcal{G}(U)$$

(see 3.3.1), and it is really not too hard to show that the stalk of this sheaf is given by

$$(\mathcal{F} \otimes_R \mathcal{G})_x = \mathcal{F}_x \otimes_R \mathcal{G}_x$$

for all points $x \in X$.
Now we consider two spaces X,Y and the two projections p_1,p_2 from $X \times Y$ to X and Y respectively. If now $\mathcal{F}$ and $\mathcal{G}$ are sheaves of R-modules on X and Y respectively, then we can define the exterior tensor product

$$\mathcal{F}\widehat{\otimes}_R\mathcal{G} = p_1^*(\mathcal{F}) \otimes_R p_2^*(\mathcal{G}) \tag{4.83}$$

as a sheaf on $X \times Y$.
We want to construct a R-module homomorphism

$$m : \bigoplus_{i+j=n} H^i(X,\mathcal{F}) \otimes H^j(Y,\mathcal{G}) \longrightarrow H^n(X \times Y,\mathcal{F}\widehat{\otimes}_R\mathcal{G}).$$

It is not so entirely obvious how this can be done. We start in the obvious manner and take injective resolutions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & I^2 & \longrightarrow & \cdots \end{array}$$

and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{G} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & J^0 & \longrightarrow & J^1 & \longrightarrow & J^2 & \longrightarrow & \cdots \end{array}$$

Then the resulting morphism of complexes

$$\mathcal{F}\widehat{\otimes}_R\mathcal{G} \longrightarrow (I^\bullet\widehat{\otimes}_R J^\bullet)_{\text{simp}}$$

needs not to be an isomorphism in the derived category. In other words, the simple complex of sheaves on the right hand side is not necessarily exact because the tensor product is not exact.
Therefore it seems to be reasonable to assume that one of the sheaves is flat and admits a flat acyclic resolution, say

$$\begin{array}{ccccccc} & & 0 & & & & \\ & & \downarrow & & & & \\ & & \mathcal{G} & & & & \\ & & \downarrow & & & & \\ 0 & \longrightarrow & \mathcal{A}^0 & \longrightarrow & \mathcal{A}^1 & \longrightarrow & \cdots \end{array}$$

where flat means of course that the stalks $\mathcal{A}^i_x$ are flat R-modules. Then we find that the double complex

$$\begin{array}{ccccccccc}
\mathcal{F}\widehat{\otimes}_R\mathcal{G} & \longrightarrow & \mathcal{F}\widehat{\otimes}_R\mathcal{A}^0 & \rightarrow & \mathcal{F}\widehat{\otimes}_R\mathcal{A}^1 & \rightarrow & \mathcal{F}\widehat{\otimes}_R\mathcal{A}^2 & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
I^0\widehat{\otimes}_R\mathcal{G} & \longrightarrow & I^0\widehat{\otimes}_R\mathcal{A}^0 & \rightarrow & I^0\widehat{\otimes}_R\mathcal{A}^1 & \rightarrow & I^0\widehat{\otimes}_R\mathcal{A}^2 & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
I^1\widehat{\otimes}_R\mathcal{G} & \longrightarrow & I^1\widehat{\otimes}_R\mathcal{A}^0 & \rightarrow & I^1\widehat{\otimes}_R\mathcal{A}^1 & \rightarrow & I^1\widehat{\otimes}_R\mathcal{A}^2 & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\vdots & & \vdots & & \vdots & & \vdots & &
\end{array}$$

has exact rows and exact columns and hence we get a resolution of $\mathcal{F}\widehat{\otimes}_R\mathcal{G}$ by the simple complex $(I^\bullet\widehat{\otimes}_R\mathcal{A}^\bullet_{\text{simp}})$ which we write down

$$0 \longrightarrow \mathcal{F}\widehat{\otimes}_R\mathcal{G} \longrightarrow I^0\widehat{\otimes}_R\mathcal{A}^0 \longrightarrow I^1\widehat{\otimes}_R\mathcal{A}^0 \oplus I^0\widehat{\otimes}_R\mathcal{A}^1 \longrightarrow \dots \ .$$

Hence we get a map (edge homomorphism)

$$m_0 : H^n((I^\bullet\widehat{\otimes}_R\mathcal{A}^\bullet)_{\text{simp}}(X \times Y)) \longrightarrow H^n(X \times Y,\mathcal{F}\widehat{\otimes}_R\mathcal{G}).$$

We have a morphism between double complexes

$$I^\bullet(X) \otimes_R \mathcal{A}^\bullet(Y) \longrightarrow (I^\bullet \otimes_R \mathcal{A}^\bullet)(X \times Y) \tag{4.84}$$

and this induces a homomorphism in cohomology

$$m' : \bigoplus_{i+j=n} H^i(I^\bullet(X)) \otimes H^j(\mathcal{A}^\bullet(Y)) \longrightarrow H^n((I^\bullet\widehat{\otimes}_R\mathcal{A}^\bullet)(X \times Y)), \tag{4.85}$$

the composition $m' \circ m_0 = m$ is the map which we want to construct. We notice that neither m_0 nor m' needs to be an isomorphism (See 4.80)

We want to show that this product does not depend on the resolution. To see that this is so we first consider exact sequences

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0$$

of sheaves on X. Since we assumed that $\mathcal{G}$ is flat we get an exact sequence

$$0 \longrightarrow \mathcal{F}_1\widehat{\otimes}_R\mathcal{G} \longrightarrow \mathcal{F}_2\widehat{\otimes}_R\mathcal{G} \longrightarrow \mathcal{F}_3\widehat{\otimes}_R\mathcal{G} \longrightarrow 0,$$

and we get two exact sequences

$$\dots \longrightarrow H^{i-1}(X,\mathcal{F}_2) \longrightarrow H^{i-1}(\mathcal{F}_3) \xrightarrow{\delta} H^i(X,\mathcal{F}_1) \longrightarrow \dots,$$

and

$$\dots H^{i-1+j}(X \times Y,\mathcal{F}_2\widehat{\otimes}_R\mathcal{G}) \xrightarrow{\delta} H^{i-1+j}(X \times Y,\mathcal{F}_3\widehat{\otimes}_R\mathcal{G}) \longrightarrow H^{i+j}(X \times Y,\mathcal{F}_1\widehat{\otimes}_R\mathcal{G}) \dots .$$

Now the formula

$$m(\delta(\xi) \otimes \eta) = \delta(m(\xi \otimes \eta)) \tag{4.86}$$

for $\xi \in H^{i-1}(X,\mathcal{F}_3)$ and $\eta \in H^j(Y,\mathcal{G})$ is obvious by construction. If we now take the resolution of $\mathcal{F}$ and break it

$$0 \longrightarrow \mathcal{F} \longrightarrow I^0 \longrightarrow \mathcal{R}_1 \longrightarrow 0{,}0 \longrightarrow \mathcal{R}_1 \longrightarrow I^1 \longrightarrow \mathcal{R}_2 \longrightarrow 0, \ldots$$

and this reduces the proof of the uniqueness the map m to the assertion that

$$m : H^0(X,\mathcal{F}) \otimes H^j(Y,\mathcal{G}) \longrightarrow H^j(X \times Y,\mathcal{F}\widehat{\otimes}_R\mathcal{G})$$

is independent of the resolution. But this is obvious because in this case m is the following map: Any element $s \in H^0(X,\mathcal{F})$ induces a morphism

$$m(s) : p_2^*(\mathcal{G}) \longrightarrow p_1^*(\mathcal{F}) \otimes_R p_2^*(\mathcal{G})$$

which is given by multiplication and clearly

$$m(s \otimes \xi) = m(s)^j(\xi) \tag{4.87}$$

for all $\xi \in H^j(X,\mathcal{G})$.
Now it is clear that the general considerations fit into the context of our earlier discussion of the Künneth-formula and the cup product in the previous section:
If we consider spaces X,Y which have a nice acyclic covering, then the acyclic resolutions

$$0 \longrightarrow \quad \longrightarrow \prod_{\alpha} \_{\{\alpha\}} \longrightarrow \prod_{(\alpha,\beta)} \_{\{\alpha,\beta\}} \longrightarrow \cdots$$

are resolutions by free -modules and therefore they are also flat. Since we have

$$\_{X} \widehat{\otimes} \ \_{Y} = \_{X\times Y},$$

we see that the above considerations generalize the previous ones.
We may take $X = Y$, and we consider the product

$$H^p(X, \ ) \otimes H^q(X, \ ) \longrightarrow H^{p+q}(X \times X, \ ).$$

Now we consider the diagonal $X \xrightarrow{\Delta} X \times X$, and we can consider the restriction.

Definition 4.6.13. *We can define the* **cup product** *of the two classes by*

$$\Delta^* i(\alpha \otimes \beta) =: \alpha \cup \beta,$$

Now we have seen – at least for reasonable spaces – that the cohomology groups

$$H^\bullet(X,\_) = \bigoplus_p H^p(X,\_)$$

carry the additional structure of a graded anticommutative algebra. We want to determine the structure of this algebra in some special cases.

4.6.8 Example: Cup Product for the Comology of Tori

Let us consider an n-dimensional vector space V over   and let $\Gamma \subset V$ be a lattice, i.e. a free submodule of rank n such that V/Γ becomes a compact space. We can choose a basis $e_1, \cdots, e_n$ of Γ, this is also a basis for V and we get an isomorphism

$$V/\Gamma \simeq (\ \ /\ \ )^n = (S^1)^n. \tag{4.88}$$

The Künneth formula yields a homomorphism

$$H^\bullet(S^1,\_) \otimes \cdots \otimes H^\bullet(S^1,\_) \longrightarrow H^\bullet((S^1)^n,\_).$$

Since the cohomology groups $H^\nu(S^1,\_)$ are free, it follows that this is indeed an isomorphism, hence $H^\bullet((S^1)^n,\_)$ is a free module of rank 2^n over . Especially we get $H^n((S^1)^n,\_) =$. It remains to determine the structure as a graded algebra.
We give an orientation to the circles: The basis vector e_i can be viewed as a tangent vector at 0 of the i-th component circle $\{0\} \times \dots S^1 \times \{0\} \subset (S^1)^n$ and this tangent vector gives the positive orientation of this component. Now we notice that $H^1(\ \ /\ \ ,\_) =$ (see section 4.1.2). We consider the cohomology in degree p. If we have a class $\xi \in H^p(V/\Gamma,\_)$, then we can attach to it an alternating p-linear map $\varphi_\xi \in \mathrm{Hom}^p_{\mathrm{alt}}(\Gamma,\ \ )$. To define this element we have to give the value $\varphi_\xi(\gamma_1, \cdots, \gamma_p)$ for any p-tuple $\underline{\gamma} = (\gamma_1, \cdots, \gamma_p)$ of elements in Γ. We take these elements and construct a homomorphism

$$\alpha_{\underline{\gamma}} : \ \ ^p/\ \ ^p \longrightarrow V/\Gamma$$

which is given by

$$\alpha_{\underline{\gamma}}(x_1, \cdots, x_p) = x_1\gamma_1 + \cdots + x_p\gamma_p. \tag{4.89}$$

The class $\alpha^*_{\underline{\gamma}}(\xi) \in H^p(\ \ ^p/\ \ ^p,\_) =$, and this is our definition

$$\varphi_\xi(\gamma_1, \cdots, \gamma_p) = \alpha^*_{\underline{\gamma}}(\underline{\xi}). \tag{4.90}$$

We see rightaway that this value is zero if $\gamma_1 \cdots \gamma_p$ are linearly dependent because then the image of $\alpha_{\underline{\gamma}}$ is an $(S^1)^{p'}$ with $p' < p$.
We have to show that the map φ_ξ is p-linear. This is easily reduced to the following special case: We consider $\ \ ^{p+1}/\ \ ^{p+1}$, and we consider the three inclusions $i_1, i_2, \Delta : \ \ ^p/\ \ ^p \longrightarrow \ \ ^{p+1}/\ \ ^{p+1}$ given by

$$\begin{array}{rcccc} i_1 & : & (x_1, \dots, x_p) & \longmapsto & (x_1, 0, x_2, \dots, x_p) \\ i_2 & : & (x_1, \dots x_p) & \longmapsto & (0, x_1, x_2, \dots, x_p) \\ \Delta & : & (x_1, \dots, x_p) & \longrightarrow & (x_1, x_1, x_2, \dots, x_p) \end{array}$$

and for a class $\xi \in H^{p+1}(\ \ ^{p+1}/\ \ ^{p+1},\_)$ we have to show that

$$i_1^*(\xi) + i_2^*(\xi) = \Delta^*(\xi). \tag{4.91}$$

Both sides are linear in ξ and hence we have to check this equality for classes

$$\xi_1 \in H^1(\ \ /\ \ ,\_) \otimes H^0(\ \ /\ \ ,\ \ ) \otimes H^{p-1}(\ \ ^{p-1}/\ \ ^{p-1},\_)$$

and

$$\xi_2 \in H^0(\ \ /\ \ ,\_) \otimes H^1(\ \ /\ \ ,\_) \otimes H^{p-1}(\ \ ^{p-1}/\ \ ^{p-1},\_),$$

and then it is obviously true. This gives us a homomorphism of graded modules

$$\alpha : H^\bullet(V/\Gamma,\underline{\ \ }) \longrightarrow \mathrm{Hom}^\bullet_{\mathrm{alt}}(\Gamma,\ \ ).$$

It is a well known elementary fact that the right hand side has the structure of an anticommutative graded algebra where the product is given by

$$(\varphi \wedge \psi)(\gamma_1 \cdots \gamma_m) = \sum_t (-1)^{\varepsilon(t)} \varphi(\gamma_{i_1} \cdots \gamma_{i_p}) \cdot \psi(\gamma_{j_1} \cdots \gamma_{j_q}) \tag{4.92}$$

where φ is a p-form, ψ is a q-form $m = p + q$, the summation is over all partions of the set $\{1, \ldots ,m\}$ into a set of p elements and a set of q elements and $(-1)^{\varepsilon(t)}$ is the obvious sign.
Perhaps it is not so much of a surprise that:

Lemma 4.6.14. *The homomorphism*

$$\alpha : H^\bullet(V/\Gamma,\underline{\ \ }) \longrightarrow \mathrm{Hom}^\bullet_{\mathrm{alt}}(\Gamma,\ \ )$$

is an isomorphism of graded algebras.

To verify this we write $V/\Gamma = (\ \ /\ \ )^n$, and we have the following basis for the cohomology: We look at ordered subsets $\underline{i} = i_1 < i_2 \cdots < i_p$ of $\{1, \cdots n\}$ and form

$$1 \otimes \cdots \otimes 1 \otimes e_{i_1} \otimes \cdots e_{i_2} \cdots e_{i_p} \otimes 1 \cdots \otimes 1 = \xi_{\underline{i}}$$

where $e_{i_\nu} \in H^1(\ \ /\ \ ,\underline{\ \ })$ is the generator determined by the orientation. The e_i can be viewed as basis elements for Γ at the same time, then

$$\varphi_{\xi_{\underline{i}}}(e_{j_1}, \cdots ,e_{j_p}) = \begin{cases} 1 & \text{if } i_1 = j_1 \cdots i_p = j_p \\ 0 & \text{else} \end{cases},$$

and clearly

$$\xi_{\underline{i}} \cup \xi_{\underline{i}'} = \begin{cases} 0 & \text{if } \underline{i} \text{ and } \underline{i}' \text{ are not disjoint} \\ (-1)^{\varepsilon(\underline{i},\underline{i}')} \xi_{\underline{i} \cup \underline{i}'} & \text{else} \end{cases}. \tag{4.93}$$

This proves the assertion.

A Connection to the Cohomology of Groups

At this point it seems to be reasonable to explain the relationship between the group cohomology $H^\bullet(\Gamma,\ \ )$, which is discussed in Chapter 2 and the cohomology groups $H^\bullet(V/\Gamma,\underline{\ \ })$.
To any Γ-module M we can attach a sheaf $\widetilde{M}$ on V/Γ. This is simple: For an open subset $U \subset V/\Gamma$ we consider the inverse image $\widetilde{U} \subset V$ under the projection and put

$$\widetilde{M}(U) = \{f : \widetilde{U} \longrightarrow M | f \text{ locally constant and } f(u + \gamma) = \gamma f(u) \text{ for all } u \in \widetilde{U},\gamma \in \Gamma\}.$$

It is clear that for any point $x \in V/\Gamma$ we can find a contractible neighborhood U_x such that for any connected component of $\widetilde{U}_x$ the projection to U_x is a homeomorphism. Hence $\widetilde{M}(U_x) \simeq M$, where the identification depends on the choice of such a component. Furthermore it is quite clear that $H^0(V/\Gamma,\widetilde{M}) = M^\Gamma$, so it should not be such a surprise that in fact $H^\bullet(\Gamma,M) = H^\bullet\left(V/\Gamma,\widetilde{M}\right)$.

Actually this can be derived from the spectral sequence of the fibration $V \longrightarrow V/\Gamma$, we have to exploit the fact that V is contractible.
This is a special instance of the cohomology theory of arithmetic groups, which will be discussed (this is at least my plan) in the third volume of this book.

4.6.9 An Excursion into Homotopy Theory

We want to discuss briefly an application of the spectral sequence which is not directly related to the goals of this book, but which is certainly important and beautiful.
For a pathwise connected space X together with a base point x_0 one defines the homotopy groups $\pi_n(X,x_0)$, on the other hand we have the singular homology groups $H_i(X,\ \ )$ which are also not discussed here (except in the chapter on cohomology of manifolds (see 4.8.6). As a general reference I refer to [Hat]. We always have the so called **Hurewicz homomorphism**

$$\pi_n(X,x_0) \longrightarrow H_n(X,\ \ ). \tag{4.94}$$

A famous theorem of W. Hurewicz asserts:

Theorem 4.6.15 (W. Hurewicz)**.** *Let X be pathwise connected with base point x_0. Let $n > 0$ be an integer. For $n > 1$ let us assume that $\pi_1(X,x_0) = 1$ and*

$$H_i(X,\ \ ) = 0 \qquad \textit{for} \qquad 1 < i < n.$$

Then the Hurewicz homomorphism $\pi_n(X,x_0) \longrightarrow H_n(X,\ \ )$ is an isomorphism. For $n = 1$ we get an isomorphism

$$\pi_1(X,x_0)_{ab} = \pi_1(X,x_0)/[\pi_1(X,x_0),\pi_1(X,x_0)] \xrightarrow{\sim} H_1(X,\ \ ).$$

Here $\pi_1(X,x_0)_{ab}$ is the abelianized fundamental group, i.e. the maximal abelian quotient.

We cannot prove this theorem here, since we neither defined the homotopy groups nor the homology groups. But for any abelian group A we can also define the singular cohomology group $H^n_{\mathrm{sing}}(X,A)$ and for reasonable spaces we have

$$H^n_{\mathrm{sing}}(X,A) \simeq H^n(X,\underline{A}),$$

i.e. the singular cohomology with coefficients in A is isomorphic to sheaf cohomology. Now the universal coeffient theorem implies (see [Hat], Chap. 3) that

$$\mathrm{Hom}(H_i(X,\ \ ),\mathbb{Q}/\ \ ) \simeq H^i(X,\underline{\mathbb{Q}/\ \ }),$$

where we have to exploit the fact that $\mathbb{Q}/\ \ $ is injective in the category of abelian groups. Hence we can reformulate the Hurewicz theorem:
An element $[\varphi] \in \pi_n(X,x_0)$ is represented by a map of pointed spaces (the basepoints are pt and x_0) $\varphi : (S^n,pt) \longrightarrow (X,x_0)$. Such a map induces a map

$$\varphi^* : H^n(X,\underline{\mathbb{Q}/\ \ }) \longrightarrow H^n(S^n,\underline{\mathbb{Q}/\ \ }) = \mathbb{Q}/\ \ .$$

The map $\xi \mapsto \varphi^*(\xi)$ defines a homomorphism

$$H^n(X,\underline{\mathbb{Q}/\ \ }) \longrightarrow \mathrm{Hom}(\pi_n(X,x_0),\mathbb{Q}/\ \ ).$$

(the dual of the Hurewicz map).

Theorem 4.6.16 (Dual of Hurewicz Theorem). *If $H^i(X,\underline{\mathbb{Q}/\;}) = 0$ for $0 < i < n$, and $\pi_1(X,x_0) = 1$ if $n > 1$, then this map is an isomorphism.*

Indication of proof: We introduce the space $(\Sigma X,x_0)$ of continuous path starting at x_0, i.e. the space of all σ

$$\begin{array}{rcl} \sigma : [0,1] & \longrightarrow & X \\ \sigma(0) & = & x_0. \end{array}$$

The open neighborhoods of a path σ are given by those paths which stay in an open neighborhood of the image $\sigma([0,1])$. Then this space is contractible and we have a map

$$\begin{array}{cccc} e & : & (\Sigma X,x_0) & \longrightarrow & X \\ e & : & \sigma & \longmapsto & \sigma(1). \end{array}$$

This map is a cohomological fibration, the fibre over x_0 is the loop space $\Omega(X,x_0)$. In sense of the definition 4.3.9 we get a local system $\mathcal{H}^\bullet(\Omega(X,x_0,\underline{\mathbb{Q}/\;}))$ whose fibres over x_0 are given by $H^\bullet(\Omega(X,x_0,\underline{\mathbb{Q}/\;}))$. Hence have a spectral sequence

$$H^\bullet\left(X,\mathcal{H}^\bullet(\Omega(X,x_0),\underline{\mathbb{Q}/\;})\right) \Rightarrow H^\bullet(\Sigma X,\underline{\mathbb{Q}/\;}).$$

We prove the Hurewicz theorem by induction on n. The key is the observation that $H^i(\Sigma X,\underline{\mathbb{Q}/\;}) = 0$ for $i > 0$ since ΣX is contractible.
If $n = 1$, then we consider the E_2-term in our spectral sequence in degree one

$$\begin{array}{ccc} H^0(X,\mathcal{H}^0(\Omega(X,x_0),\underline{\mathbb{Q}/\;}) & H^1(X,\mathcal{H}^0(\Omega(X,x_0),\underline{\mathbb{Q}/\;}))) & H^2(X,\mathcal{H}^0(\Omega(X,x_0),\underline{\mathbb{Q}/\;}))) \\ H^0(X,\mathcal{H}^1(\Omega(X,x_0),\underline{\mathbb{Q}/\;}) & H^1(X,\mathcal{H}^1(\Omega(X,x_0),\underline{\mathbb{Q}/\;}))) & * \\ \vdots & \vdots & \ddots \end{array}$$

Since $H^1(\Sigma X,\underline{\mathbb{Q}/\;}) = 0$ we see that the two $E_\infty^{01},E_\infty^{10}$ must become zero. It follows that the term $H^1(X,\mathcal{H}^0(\Omega(X,x_0)) = 0$ because the differentials going into it and out of it are zero. We also see that

$$H^0(X,\mathcal{H}^1(\Omega(X,x_0),\underline{\mathbb{Q}/\;})) \longrightarrow H^2(X,\mathcal{H}^0(\Omega(X,x_0),\underline{\mathbb{Q}/\;}))$$

is an isomorphism, but this we will not need.
The local system $\mathcal{H}^0(\Omega(X,x_0),\underline{\mathbb{Q}/\;})$ is easy to compute, it is a module under the fundamental group $\Gamma = \pi_1(X,x_0)$. I recall the definition of the universal covering space $\widetilde{X} \longrightarrow X$, by definition this is the space of path-connected components of $\Omega(X,x_0)$. The fundamental group $\pi(X,x_0) = \Gamma$ is the group of automorphisms of $\widetilde{X} \longrightarrow X$. Then it is easy to see that $\mathcal{H}^0(\Omega(X,x_0))$ is the local system given by the Γ-module $\mathcal{C}(\Gamma,\mathbb{Q}/\;)$ of all $\mathbb{Q}/\;$-valued functions on Γ where Γ acts by translations. This module contains the constant functions and hence we get an exact sequence of sheaves on X

$$0 \longrightarrow \underline{\mathbb{Q}/\;} \longrightarrow \mathcal{H}^0(\Omega(X,x_0)) \longrightarrow \mathcal{M} \longrightarrow 0,$$

where $\mathcal{M}$ is the quotient sheaf. We get a long exact sequence in cohomology

$$0 \to H^0(X,\underline{\mathbb{Q}/\ \ }) \to H^0\left(X,\mathcal{H}^0(\Omega(X,x_0),\underline{\mathbb{Q}/\ \ })\right) \to H^0(X,\mathcal{M}) \to H^1(X,\mathbb{Q}/\ \ ) \to 0.$$

For the local systems over X the sections $H^0(X,\ ))$ are simply the invariants under Γ. We get an exact sequence

$$(\mathbb{Q}/\ \ )^\Gamma \xrightarrow{\sim} (\mathcal{C}(\Gamma,\mathbb{Q}/\ \ ))^\Gamma \longrightarrow \mathcal{M}^\Gamma \xrightarrow{\sim} H^1(X,\mathbb{Q}/\ \ ) \longrightarrow 0,$$

the last zero is just our first observation above.
Now it follows from our results on the cohomology of groups that (See Exercise 2.2.4)

$$\mathcal{M}^\Gamma \simeq H^1(\Gamma,\mathbb{Q}/\ \ ) = \mathrm{Hom}(\Gamma,\mathbb{Q}/\ \ ),$$

and hence we proved the result for $n = 1$.
For $n > 1$ we apply the same method. Now we know that $\Omega(X,x_0)$ is pathwise connected, because we also assumed the vanishing of the fundamental group. Hence we see that

$$\mathcal{H}^0(\Omega(X,x_0),\underline{\mathbb{Q}/\ \ }) \simeq \underline{\mathbb{Q}/\ \ }.$$

Then we find many zeroes in the bottom row of the spectral sequence and the local system of cohomology groups $\mathcal{H}^i(\Omega(X,x_0),\underline{\mathbb{Q}/\ \ })$ will be constant. This shows that the E_2-term in the spectral sequence looks as follows

$$\begin{array}{ccccc}
H^0\left(X,\mathcal{H}^0(\Omega(X,x_0),\underline{\mathbb{Q}/\ \ })\right) & 0 & \cdots & 0 & H^n\left(X,\mathcal{H}^0(\Omega(X,x_0),\underline{\mathbb{Q}/\ \ })\right) \\
H^0\left(X,\mathcal{H}^1(\Omega(X,x_0),\underline{\mathbb{Q}/\ \ })\right) & * & \cdots & * & * \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
H^0(X,\mathcal{H}^{n-1}(\Omega(X,x_0),\underline{\mathbb{Q}/\ \ })) & * & \cdots & * & *
\end{array} \tag{4.95}$$

the zeroes in the first line a forced by our assumption. Again we exploit the fact that ΣX is contractible. I claim that $\mathcal{H}^i(\Omega(X,x_0),\underline{\mathbb{Q}/\ \ }) = 0$ for $i < n-1$. This follows by induction on i, for $i = 1$ the differential ends up on the first row and hence is zero. Then $\mathcal{H}^1(\Omega(X,x_0),\underline{\mathbb{Q}/\ \ }) = 0$, and this put zeroes into the second line. Then we continue and this argument breaks down at $i = n-1$. Hence we conclude that our spectral sequence looks as follows

$$\begin{array}{ccccc}
H^0\left(X,\mathcal{H}^0(\Omega(X,x_0),\underline{\mathbb{Q}/\ \ })\right) & 0 & \cdots & 0 & H^n\left(X,\mathcal{H}^0(\Omega(X,x_0),\underline{\mathbb{Q}/\ \ })\right) \\
0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 \\
H^0(X,\mathcal{H}^{n-1}(\Omega(X,x_0),\underline{\mathbb{Q}/\ \ })) & * & \cdots & * & *
\end{array} \tag{4.96}$$

the lines from $i = 1$ to $i = n-2$ are filled with zeroes. The differential

$$d_n^{0,n-1} : H^0\left(X,\mathcal{H}^{n-1}(\Omega(X,x_0),\underline{\mathbb{Q}/\ \ })\right) \longrightarrow H^n\left(X,\mathcal{H}^0(\Omega(X,x_0),\underline{\mathbb{Q}/\ \ })\right)$$

must be an isomorphism. (It is a little bit similar to filling a Sudoku puzzle).
This implies that

$$\mathcal{H}^{n-1}(\Omega(X,x_0),\underline{\mathbb{Q}/\ \ })_{x_0} \simeq H^n(X,\underline{\mathbb{Q}/\ \ })$$

i.e.

$$H^{n-1}(\Omega(X,x_0),\underline{\mathbb{Q}/\ }) \xrightarrow{\sim} H^n(X,\underline{\mathbb{Q}/\ }).$$

Now we have the exact sequence for homotopy groups which say that

$$\pi_{i-1}(\Omega(X,x_0),\underline{\mathbb{Q}/\ }) \simeq \pi_i(X,x_0),$$

and the Hurewicz theorem follows. □

It is quite amusing to consider the special case of $X = S^n$. In this case we find

$$H^i(\Omega S^n,\mathbb{Q}/\ \ ) = \begin{cases} \mathbb{Q}/ & \text{for } i = k\,(n-1) \\ 0 & \text{else} \end{cases} \tag{4.97}$$

4.7 Cohomology with Compact Supports

4.7.1 The Definition

Let X be a locally compact space and $\mathcal{F}$ a sheaf of abelian groups on X. If we have a section $s \in H^0(X,\mathcal{F})$ then its support $\mathrm{Supp}(s) = |s|$ is the set of $x \in X$ with $s_x \neq 0$. It is always closed. If we have an open subset $U \subset X$ and a section $s \in \mathcal{F}(U)$ then its support $|s|$ is closed in U but not necessarily in X.
We can define the submodule $H^0_c(X,\mathcal{F})$ of sections with compact support. This yields a left exact functor and we define the cohomology with compact supports as the right derived functor of $H^0_c(X,\mathcal{F})$. In accordance with our general principles we choose an injective resolution

$$0 \longrightarrow \mathcal{F} \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \dots$$

of $\mathcal{F}$ and define

$$H^i_c(X,\mathcal{F}) = H^i(H^0_c(X,I^\bullet)).$$

The cohomology with compact supports has properties which are quite different from those of the ordinary cohomology. For instance it does not satisfy the homotopy axiom. We will see in the section on the cohomology of manifolds that it is dual to the ordinary cohomology. Of course on a compact space X we have $H^\bullet_c(X,\mathcal{F}) = H^\bullet(X,\mathcal{F})$. If we have open sets $U \subset V \subset X$ then we have natural maps

$$H^i_c(U,\mathcal{F}) \longrightarrow H^i_c(V,\mathcal{F}) \longrightarrow H^i_c(X,\mathcal{F}),$$

here we see that the restriction maps which we had in the theory of sheaves are turned backwards. On the other hand if we have a map $f : X \longrightarrow Y$ then we will not be able to define a map from $H^i_c(Y,\underline{\ \ })$ to $H^i_c(X,\underline{\ \ })$ unless the map is proper.
Let us assume that $U \overset{i}{\hookrightarrow} X$ is an open subset of our space, and let us assume that its closure is compact, then its boundary $\partial\overline{U} = \overline{U} \setminus U$ is also compact. Let $\mathcal{F}$ be a sheaf on U. We define two new sheaves on X: The direct image $i_*(\mathcal{F})$ where

$$i_*(\mathcal{F})(V) = \mathcal{F}(U \cap V)$$

and the extension by zero

$$i_!(\mathcal{F})(V) = \{s \in \mathcal{F}(V \cap U) \mid |s| \text{ does not meet } V \cap \partial\overline{U}\}$$

One checks easily that $i_!(\mathcal{F})$ has the stalks

$$\begin{aligned} i_!(\mathcal{F})_x &= \mathcal{F}_x \quad \text{if} \quad x \in U \\ i_!(\mathcal{F})_y &= 0 \quad \text{if} \quad y \notin U. \end{aligned}$$

We have a morphism of sheaves $i_!(\mathcal{F}) \longrightarrow i_*(\mathcal{F})$ which is an isomorphism in all stalks except the ones on the boundary $\partial\overline{U}$.

Proposition 4.7.1. *If X is a locally compact space and $i : U \hookrightarrow X$ an open subset with compact closure and if $\mathcal{F}$ is a sheaf of abelian groups on U then*

$$H_c^\bullet(U,\mathcal{F}) = H^\bullet(X,i_!(\mathcal{F})).$$

Proof: This is almost clear from the definition. We choose an injective resolution of the sheaf $\mathcal{F}$ on U

$$0 \longrightarrow \mathcal{F} \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \dots$$

and we notice that

$$i_!(I^\bullet)(X) = H_c^0(U,I^\bullet).$$

□

4.7.2 An Example for Cohomology with Compact Supports

The Cohomology with Compact Supports for Open Balls

Now we consider the sheaf $\_$ on the open ball $D^n \subset \ ^n$. We want to compute $H_c^\bullet(D^n,\_)$. To do this we embed $D^n \stackrel{i}{\hookrightarrow} \overline{D}^n$. On $\overline{D}^n$ we have an exact sequence

$$0 \longrightarrow i_!(\_) \longrightarrow i_*(\_) \longrightarrow i_*(\_)/i_!(\_) \longrightarrow 0.$$

The sheaf $i_*(\_)$ is $\_{}_{\overline{D}^n}$ and the sheaf $i_*(\_)/i_!(\_)$ is concentrated on $\overline{D}^n \setminus D^n = S^{n-1}$ and on this space it is simply $\_{}_{S^{n-1}}$. We write the long exact sequence in cohomology, exploit our Proposition 4.7.1 and get

$$\dots \longrightarrow H_c^\nu(D^n,\_) \longrightarrow H^\nu(\overline{D}^n,\_) \longrightarrow H^\nu(S^{n-1},\_) \longrightarrow H_c^{\nu+1}(D^n,\_).$$

We have $H^\nu(\overline{D}^n,\_) = 0$ for $\nu > 0$ and hence we get for $\nu = 0$

$$\begin{array}{ccccccc} 0 & \longrightarrow H^0(\overline{D}^n,\_) & \longrightarrow H^0(S^{n-1},\_) & \longrightarrow H_c^1(D^n,\_) & \longrightarrow 0 \\ & \| & & & \end{array}$$

and for $\nu \geq 0$

$$H^\nu(S^{n-1},\_) \xrightarrow{\sim} H_c^{\nu+1}(D^n,\_).$$

Our computation of the cohomology groups of spheres yields

$$H_c^\nu(D^n,\_) \xrightarrow{\sim} \begin{cases} & \text{for } \nu = n \\ 0 & \text{for } \nu \neq n\,. \end{cases}$$

Again we have to discuss these nagging questions of orientation. We can say that a **topological orientation** is an isomorphism

$$O_n : H^d_c(D^n, \_) \longrightarrow \quad .$$

If we have a homeomorphism $f : D^n \xrightarrow{\sim} D^n$ then it induces necessarily an isomorphism $f^* : H^d_c(D^n, \_) \xrightarrow{\sim} H^d_c(D^n, \_)$ which can only be multiplication by ± 1. We say that f preserves the orientation, if it induces the identity on $H^d_c(D^n, \_)$. If we take for instance the homeomorphism that sends $(x_1, x_2, \ldots, x_n) \mapsto (x_1, x_2, \ldots, -x_n)$, i.e. we change the sign of the last coordinate then we get multiplication by -1 on $H^d_c(D^n, \_)$.
But again we have a rule to determine a topological orientation from an orientation on the tangent space $\ ^n$ of D^n at the origin. The tangent bundle of $\ ^n$ is trivial, hence we get an orientation at any point. This orientation provides an orientation on S^{n-1} by the following rule: A basis of tangent vectors $\{e_1, \ldots, e_{n-1}\}$ at some point P on S^{n-1} is positively oriented if $\{\mathfrak{n}_P =$ outward normal vector, $e_1, \ldots, e_{n-1}\}$ is positively oriented. If $n > 1$ this orientation gives a topological orientation on S^{n-1}. If $n > 1$ then the isomorphism

$$H^{n-1}(S^{n-1}, \_) \xrightarrow{\sim} H^n_c(D^n, \_)$$

provides the desired topological orientation.
If $n = 1$ we choose the orientation given by $\frac{\partial}{\partial x}$. We have the exact sequence

$$0 \longrightarrow i_!(\_) \longrightarrow i_*(\_) \longrightarrow \_{\{+1\}} \oplus \_{\{-1\}} \longrightarrow 0$$

which provides the long exact sequence

$$0 \longrightarrow \quad \longrightarrow \quad \oplus \quad \longrightarrow H^1(I, i_!(\_)) \longrightarrow 0$$

and our convention is that we identify $H^1(I, i_!(\_)) = \quad$ via the first summand (corresponding to the point $+1 \in S^0$).
We can look at this rule to fix orientations from a slightly different point of view. Let $n > 1$. If we pick a point $P \in S^{n-1}$ we can find a small open ball U_P around P, which is diffeomorphic to D^{n-1}. We take the same orientation on U_P as above. Then we have $i_P : U_P \hookrightarrow S^{n-1}$ and the inclusion $i_{P,!}(\_) \hookrightarrow \_$. Hence we have

Lemma 4.7.2. *The quotient sheaf* $\_/i_{P,!}(\_)$ *has no cohomology in degree* $\neq 0$. *Hence we get an isomorphism*

$$H^{n-1}(S^{n-1}, i_{P,!}(\_)) \xrightarrow{\sim} H^{n-1}(S^{n-1}, \_).$$

Then we can define the topological orientation on S^{n-1} by the topological orientation which we have on $H^{n-1}_c(U_P, \_)$.
We want to consider a relative situation. Let us assume that we have a diagram

$$\begin{array}{ccc} X & \xhookrightarrow{\ i\ } & \overline{X} \\ \downarrow \pi & \swarrow \overline{\pi} & \\ Y & & \end{array}$$

and we want to assume this is some kind of fibration by n-dimensional balls. By this we mean that locally in Y we can trivialize our diagram

$$\begin{array}{ccc} V\times D^n & \overset{i}{\hookrightarrow} & V\times \overline{D}^n \\ \big\downarrow & \swarrow & \\ V & & \end{array}$$

If we choose a covering $Y=\cup V_\alpha$ such that we have trivializations over the V_α, then we get identifications (see 4.3.1): For $v\in V_\alpha\cap V_\beta$ we have homeomorphisms

$$g_{\alpha,\beta}(v):(D^n,\overline{D}^n)\longrightarrow(D^n,\overline{D}^n) \tag{4.98}$$

which means $g_{\alpha,\beta}(v)$ is a homeomorphism of $\overline{D}^n$ which maps the interior to the interior and the boundary to the boundary.
We call this fibration oriented if the $g_{\alpha,\beta}(v)$ preserve the orientation, and if we selected a consistent orientation on the fibres. We consider the sheaf $\_$ on X and its extension $i_!(\_)$ to $\overline{X}$. We want to compute the cohomology $H^\bullet(\overline{X},i_!(\_))$. We apply the spectral sequence for a fibration (see section 4.6.6), and we have the E_2-term $H^p(Y,R^q\pi_*(i_!(\_)))$. Our computation in the previous section yields

$$R^q\pi_*(i_!(\_))=\begin{cases}0 & q\neq n\\ \_ & q=n\end{cases}$$

(remember that we have the orientation) and consequently the spectral sequence degenerates and

$$H^{p+n}(\overline{X},i_!(\_))=H^p(Y,R^n\pi_*(i_!\_))=H^p(Y,\_). \tag{4.99}$$

Formulae for Cup Products

We want to explain some formulae for cup products of certain classes. These formulae will be important later, they help us to understand the intersection product of cycles. It is technically convenient to replace the open (resp. closed) ball D^n (resp. $\overline{D}^n$) by the open (resp. closed) box

$$B^n=\{(x_1,\cdots,x_n)\mid |x_i|<1\}\subset\overline{B}^n=\{(x_1,\cdots,x_n)\mid |x_i|\leq 1\}$$

the pairs $(D^n,\overline{D}^n)$ and $(B^n,\overline{B}^n)$ are homeomorphic. Of course $B^0=\overline{B}^0=\{(0)\}$ is a point.
Let us assume that we have two numbers d_1,d_2 with $d_1+d_2\geq n$. We consider products

$$\overline{B}^{d_1}\times B^{n-d_1}\qquad\text{and}\qquad B^{n-d_2}\times\overline{B}^{d_2},$$

and we consider embeddings

$$i_1:\begin{array}{ccc}\overline{B}^{d_1}\times B^{n-d_1} & \longrightarrow & \overline{B}^n\\ ((x_1,\cdots,x_{d_1}),(y_1,\cdots,y_{n-d_1})) & \longmapsto & (x_1,\cdots,x_{d_1},y_1,\cdots,y_{n-d_1})\end{array}$$

and

$$i_2:\begin{array}{ccc}B^{n-d_2}\times\overline{B}^{d_2} & \longrightarrow & \overline{B}^n\\ ((y_1,\cdots,y_{n-d_2}),(x_1,\cdots,x_{d_2})) & \longmapsto & (y_1,\cdots,y_{n-d_2},x_1,\cdots,x_{d_2}).\end{array}$$

We also have the projections

$$\begin{array}{rcl}\pi_1 : \overline{B}^{d_1} \times B^{n-d_1} & \longrightarrow & \overline{B}^{d_1}, \\ \pi_2 : B^{n-d_2} \times \overline{B}^{d_2} & \longrightarrow & \overline{B}^{d_2}.\end{array}$$

We can apply the results from the previous page. We have the sheaves $i_{1,!}(\ )$ and $i_{2,!}(\ )$ on $\overline{B}^n$, and clearly equation 4.99 give us

$$\begin{array}{rcl}H^{n-d_1}(\overline{B}^n, i_{1,!}(\_)) & \xrightarrow{\sim} & \\ H^{n-d_2}(\overline{B}^n, i_{2,!}(\_)) & \xrightarrow{\sim} & \end{array}$$

We select orientations as given by the ordering of the y-coordinates and on a y coordinate we orient from -1 to $+1$. (See 4.7.2.) We want to consider the cup product

$$H^{n-d_1}(\overline{B}^n, i_{1,!}(\_)) \times H^{n-d_2}(\overline{B}^n, i_{2,!}(\_)) \xrightarrow{\cup} H^{2n-d_1-d_2}(\overline{B}^n, i_{1,!}(\_) \otimes i_{2,!}(\_)).$$

The tensor product of the two sheaves is easy to compute. We have an embedding

$$\overline{B}^{d_1+d_2-n} \times B^{2n-d_1-d_2} \xrightarrow{i_{1,2}} \overline{B}^n$$

$$((x_1, \dots, x_{d_1+d_2-n}), (y_1, \cdots, y_{2n-d_1-d_2})) \longmapsto$$
$$(y_1, \dots, y_{n-d_2}, x_1, \dots, x_{d_1+d_2-n}, y_{n-d_2+1}, \dots, y_{2n-d_1-d_2})$$

and an isomorphism provided by the multiplication on the stalks

$$i_{1,!}(\_) \otimes i_{2,!}(\_) \xrightarrow{\sim} i_{1,2,!}(\_).$$

We choose the orientation on $B^{2n-d_1-d_2}$ which is given by the ordering of the coordinates. Then all the cohomology groups in

$$H^{n-d_1}(\overline{B}^n, i_{1,!}(\_)) \times H^{n-d_2}(\overline{B}^n, i_{2,!}(\_)) \longrightarrow H^{2n-d_1-d_2}(\overline{B}^n, i_{1,2,!}(\_)).$$

are identified to . Now I claim:

Proposition 4.7.3. *Under these identifications the cup product is given by the multiplication* $\ \times\ \longrightarrow\ $.

Proof: The following argument may be considered as somewhat sketchy, I ask the reader to fill the gaps. First of all we can restrict to the case $d_1 + d_2 = n$, and now we have enough flexibility to reduce to the case $d_1 = n-1$, $d_2 = 1$. In this case the embedding is $i_{1,2} = j_n$ where $j_n : B^n \hookrightarrow \overline{B}^n$ is the standard embedding. Then we recall that

$$H^1_c(B^1, \_) = H^1(\overline{B}^1, i_!(\_))$$

can be computed from the exact sequence of sheaves on $\overline{B}^1 = [-1,1]$

$$0 \longrightarrow i_!(\_) \longrightarrow \_ \longrightarrow \_/i_!(\_) \longrightarrow 0$$

where $\_/i_!(\_)$ is the skyscraper with stalk   on $\{-1,1\}$. Our rule of identification was

$$= H^0(\{1\}, \_) = H^1(\overline{B}^1, i_!(\_)).$$

Now we apply the principles developed in 4.6.7. We have $\overline{B}^n = \overline{B}^{n-1} \times [-1,1]$ and $i_{1,!}(\_) = \_{\overline{B}^{n-1}} \widehat{\otimes} i_!(\_)$. Then our little exact sequence provides a new exact sequence

$$0 \longrightarrow i_{1,!}(\_) \longrightarrow \_{\overline{B}^n} \longrightarrow \_{\overline{B}^{n-1}} \widehat{\otimes} \; \_/i_!(\_) \longrightarrow 0.$$

The canonical generator ϵ_1 (i.e. the element 1) in the first factor $H^1(\overline{B}^n, i_{1,!}(\ \ ))$ is the image of $1 \widehat{\otimes} 1 \in H^0(\overline{B}^{n-1}, \_) \widehat{\otimes} H^0(\{1\}, \_)$ under the boundary map

$$H^0(\overline{B}^{n-1}, \_) \widehat{\otimes} H^0(\{-1,1\}, \_) \longrightarrow H^1(\overline{B}^n, i_{1,!}(\ \ )).$$

Now we have to multiply this generator with the canonical generator ϵ_2 in the second factor $H^{n-1}(\overline{B}^n, i_{2,!}(\_))$. Recall that $i_2 : B^{n-1} \times \overline{B}^1 = B^{n-1} \times [-1,1] \hookrightarrow \overline{B}^n$. Then this canonical generator is $\epsilon_2 = \epsilon' \widehat{\otimes} 1 \in H^{n-1}(\overline{B}^{n-1}, j_{n-1,!}(\_)) \widehat{\otimes} H^0(B^1, \_)$, where $\epsilon' \in H^{n-1}(\overline{B}^{n-1}, j_{n-1,!}(\_))$ is the generator provided by the standard orientation. We get an exact sequence

$$0 \longrightarrow i_{1\ 2,!}(\_) \longrightarrow j_{n-1,!}(\_) \widehat{\otimes} \_ \longrightarrow j_{n-1,!}(\_) \widehat{\otimes} (\_/i_!(\_)) \longrightarrow 0.$$

The product $m(\epsilon_1 \otimes \epsilon_2) \in H^n(\overline{B}^n, i_{1\ 2,!}(\_)) = H^n_c(B^n, \_)$ (see 4.6.7) is the image under the boundary operator

$$H^{n-1}(\overline{B}^n, j_{n-1,!}(\_) \widehat{\otimes} \_/i_!(\_)) \xrightarrow{\delta} H^n(\overline{B}^n, i_{1\ 2,!}(\_))$$

of the element $m(\epsilon' \otimes 1_{\{1\}})$ where $1_{\{1\}}$ is the canonical generator in $H^0(\{1\}, \_)$. We have shown that this is the canonical generator with respect to the standard orientation in $H^n(\overline{B}^n, i_{1\ 2,!}(\_)) = H^n(\overline{B}^n, j_{n,!}(\_))$. This proves the assertion of Proposition 4.7.3. □

We have a diagram

$$\begin{array}{ccc} H^{n-1}(\partial \overline{B}^n, \_) & \xrightarrow{\quad\delta\quad} & H^n(\overline{B}^n, j_{n,!}(\_)) \\ \downarrow & & \downarrow \\ H^{n-1}(S^{n-1}, \_) & \xrightarrow{\quad\delta\quad} & H^n(\overline{D}^n, j_{n,!}(\_)) \end{array}$$

where the vertical arrows are isomorphisms which are induced by your favorite orientation preserving homeomorphism $(\overline{B}^n, \partial \overline{B}^n) \xrightarrow{\sim} (\overline{D}^n, S^{n-1})$. The embedding $\overline{B}^{n-1} \times \{1\} \subset \partial \overline{B}^n$ provides an isomorphism

$$H^{n-1}(\overline{B}^{n-1}, j_{n-1,!}(\_)) \xrightarrow{\sim} H^{n-1}(\partial \overline{B}^n, \_) \xrightarrow{\sim} H^{n-1}(S^{n-1}, \_).$$

These isomorphisms map the canonical generator in $H^{n-1}(\overline{B}^{n-1}, j_{n-1,!}(\_))$ to the canonical generator in $H^{n-1}(S^{n-1}, \_)$ which in turn is mapped by δ to the canonical generator in $H^n(\overline{D}^n, j_{n,!}(\_))$.

4.7.3 The Fundamental Class

Let M be a connected $\mathcal{C}^0$-manifold of dimension n. If we have a point $p \in M$ then we can find a neighborhood V_p of p which is homeomorphic to an open ball $D \subset \ ^n$. Then

we have $H^d_c(V_p,\_) \xrightarrow{\sim}$ but this isomorphism is not canonical. If we have two points p,q and two small open neighborhoods V_p,V_q of these points, then we have no consistent way to identify $H^d_c(V_p,\_)$ and $H^d_c(V_q,\_)$. But if we choose a path $\gamma : [0,1] \longrightarrow M$ which starts at p and ends at q, then we get an identification along the path (see the discussion of this argument in the following section on local systems). We say that M is orientable if for any two points this identification does not depend on the path. If M is orientable then we can choose a generator in $H^d_c(V_p,\_)$ for all p which is consistent with the above identification along paths. Once we have chosen such generators we call the manifold oriented.
If our manifold has a differentiable structure, then we have another notion of orientation on M (see sections 4.1.2, 4.3.3,4.4.5). In this case it is easy to see that the two concepts of being oriented coincide.
In the next chapter we will prove that for a connected and oriented $\mathcal{C}^\infty$-manifold M of dimension n, any point $p \in M$ and any open ball $D_p \subset M$ containing p the map $H^d_c(D_p,\_) \longrightarrow H^d_c(M,\_)$ is an isomorphism. This is the starting point to get Poincaré duality.

Definition 4.7.4. *The image of the generator in* $H^d_c(D_p,\_)$ *is called* **fundamental class** *of* M*.*

This class does not depend on the point p.

4.8 Cohomology of Manifolds

4.8.1 Local Systems

I want to study the cohomology of local coefficient systems on $\mathcal{C}^\infty$-manifolds. (See 4.3.3.) In the following let M be a $\mathcal{C}^\infty$-manifold of dimension d. At this point we do not assume that M is compact, but we make some kind of finiteness assumption: We want to assume that M is countable at infinity.

Definition 4.8.1. *A manifold M is called* **countable at infinity** *if we can find an increasing sequence of relatively compact open sets $W_n \subset W_{n+1}$ where $\overline{W}_n \subset W_{n+1}$ for all n and which exhausts the manifold M, i.e. $\bigcup W_n = M$.*

This condition is close to the paracompactness of M.
Let M be connected, let $\mathcal{V}$ be a local system on it (see 4.3.3). We know that the stalks at two different points x,y are always isomorphic to each other but in general we do not have the possibility to identify them in a consistent way. This is explained by the following argument which everybody has seen during the discussion of the principle of analytic continuation in theory of complex functions:

Since M is connected, we can choose a **path** $\gamma : [0,1] \longrightarrow M$ with $\gamma(0) = x$ and $\gamma(1) = y$. We cover the path by finitely many sufficiently small open sets U_i, on which $\mathcal{V}$ is trivial. This gives us a subdivision $0 = t_0 < t_1 < \ldots < t_n = 1$ such that the $\gamma[t_i,t_{i+1}]$ are entirely in one of the covering sets U_ν and hence we can identify $\mathcal{V}_{\gamma(t_i)} = \mathcal{V}_{\gamma(t_{i+1})} = \mathcal{V}(U_\nu)$. This sequence of identifications yields an identification

$$\Psi_\gamma : \mathcal{V}_x \xrightarrow{\sim} \mathcal{V}_y.$$

This identification depends on the path, but it is not difficult to see that it only depends on the homotopy class $[\gamma]$ of the path.
If we choose a base point $x_0 \in M$ and consider closed paths which start and end at x_0. We can compose such paths. Then it is a fundamental fact that the homotopy classes of these closed paths form a group under composition. This is the **fundamental group** $\pi_1(M,x_0)$ (see for instance [Hat]). We get a representation of the fundamental group

$$\begin{aligned} \rho : \pi_1(M,x_0) &\longrightarrow \mathrm{Aut}(\mathcal{V}_{x_0}) \\ \rho : [\gamma] &\longmapsto (\Psi_{[\gamma]} : \mathcal{V}_{x_0} \longrightarrow \mathcal{V}_{x_0}). \end{aligned}$$

It is not hard to see that the local system can be reconstructed from this representation: We consider the set of pairs $([\gamma],v)$ where $[\gamma]$ is a homotopy class of paths from x_0 to x and $v \in \mathcal{V}_{x_0}$. The stalk of $\mathcal{V}$ at a point $x \in M$ will be this set divided by the equivalence relation

$$([\gamma],v) \sim ([\gamma_1],v_1)$$

if and only if

$$\rho([\gamma_1]^{-1} \circ [\gamma])(v_1) = v. \tag{4.100}$$

We can express this by saying that we have an equivalence of categories:

Abelian groups V together with an action of $\pi_1(M,x_0)$

and

local systems $\mathcal{V}$ whose stalk at x_0 is isomorphic to V.

If we have a local system $\mathcal{V}$ on our manifold M then under certain assumptions we can construct a dual local system $\mathcal{V}^\vee$. We want to study

$$H^i(M,\mathcal{V}) \quad \text{and} \quad H^i_c(M,\mathcal{V}^\vee)$$

and we will – again under certain assumptions – construct a duality between $H^i(M,\mathcal{V})$ and $H_c^{d-i}(M,\mathcal{V}^\vee)$, where d is the dimension of M. This will be Poincaré duality.

4.8.2 Čech Resolutions of Local Systems

We want to assume our d-dimensional $\mathcal{C}^\infty$-manifold M from now on to be paracompact.

Lemma 4.8.2. *On such a manifold M we can find a countable covering $M = \bigcup_{\alpha \in \mathrm{A}} U_\alpha$ by open sets which has the following two properties:*

(1) *The covering is locally finite, i.e. to any point $p \in M$ we can find an open neighborhood V_p containing p such that we have only finitely many $\alpha \in \mathrm{A}$ such that $U_\alpha \cap V_p \neq \emptyset$.*

(2) *For any finite set $\alpha_0, \ldots, \alpha_q \in \mathrm{A}$ the pair of spaces $(\overline{U}_{\underline{\alpha}},U_{\underline{\alpha}})$ is homeomorphic to the pair $(\overline{D}^d,D^d)$= (closed d-dimensional ball, open d-dimensional ball).*

Before I can outline the proof I need a definition.

Definition 4.8.3. *A* **partition of unity** *is a family* $\{h_i\}_{i\in I}$ *of positive* $\mathcal{C}^\infty$*−functions which has the following properties:*

(i) *The support of any* h_i *is small so that we can find an open set* $U_i \subset M$ *which is* $\mathcal{C}^\infty$*-isomorphic to an open ball* $D \subset \ {}^d$ *and* $\mathrm{Supp}(h_i) \subset U_i$.

(ii) *For any point* $x \in M$ *there are only finitely many indices* $j \in I$ *with* $h_j(x) \neq 0$.

(iii) *We have*

$$\sum_{i\in I} h_i = 1.$$

The construction of such a partition of unity is standard and quite easy.

Proof: I want to explain briefly why we can find such a covering.
We can use the paracompactness to introduce a Riemannian metric on M. To do this we construct a partition of unity on M. We can construct a Riemannian metric g_i on each of the U_i simply by transporting the standard metric on the ball by means of the diffeomorphism. We multiply this metric by h_i, then it extends to a quadratic form on the tangent bundle of M which is positive definite on the support of h_i and zero outside. Adding up these metrics gives the desired Riemannian metric.
Now we invest some differential geometry. Any point $x \in M$ has an open neighborhood V_x which is diffeomorphic to a ball (see [B-K], Prop. 6.4.6.) and which has the property that it is geodesically convex: Any two points $y,z \in V_x$ can be joined by a unique geodesic which lies in V_x (see [B-K], 6.4.6). To find this we may simply take a small ball $B(x,\varepsilon) = V_x$, these are all those points which have distance $< \varepsilon$ from x. The closure of such a ball is diffeomorphic to a closed ball in ^{d}, the boundary $\overline{\partial B(x,\varepsilon)}$ is a sphere. It is a smooth hypersurface in M.
Now we come back to the construction of a covering with the required properties. We assume that M is countable at infinity. We can exhaust it by a sequence of relatively compact open sets W_n which in addition have the property that $\overline{W}_n \subset W_{n+1}$.
We start at an index n and cover $\overline{W}_n$ by a finite family of such small balls as above. We require that these balls are contained in W_{n+1}. Now we proceed with $\overline{W}_{n+1}$ but we require in addition that these balls have empty intersection with $\overline{W}_{n-1}$. Then it is clear that the union of all these families provides a covering $\{U_\alpha\}_{\alpha\in\mathrm{A}}$ of M. The intersections $U_{\underline{\alpha}} = U_{\alpha_0} \cap \ldots \cap U_{\alpha_q}$ are diffeomorphic to open balls in ^{d}. We will not verify that $(\overline{U}_{\underline{\alpha}},U_{\underline{\alpha}})$ satisfies (2).
For us it is enough to know the following condition is true:

(locbound) *For any point* x *in the boundary* $x \in \partial U_{\underline{\alpha}} = \overline{U}_{\underline{\alpha}} \setminus U_{\underline{\alpha}}$ *the intersection* $B(x,\varepsilon) \cap \overline{U_{\underline{\alpha}}}$ *is contractible.*

Each point in the intersection is joined to x by a unique geodesic lying in this intersection. □

I propose to call such a covering also a **convex** covering. (See also [B-T], they call these coverings good covers.)
We give the indexing set A a total order, in other words we identify it to $\{0,\ldots,n\}$ or .
We consider $(q+1)$-tuples $\underline{\alpha} = (\alpha_0,\ldots,\alpha_q) \in \mathrm{A}^{q+1}$ where the indices are increasing. Let us denote this subset of indices by $\mathrm{A}^{q+1}_<$. For such an $\underline{\alpha} = (\alpha_0,\ldots,\alpha_q)$ we have the inclusion

$$i_{\underline{\alpha}} : U_{\underline{\alpha}} \hookrightarrow M$$

and starting from our local system $\mathcal{V}$ we form the sheaf (see page 81)

$$\mathcal{V}^*_{\underline{\alpha}} = i_{\underline{\alpha}*} i^*_{\underline{\alpha}}(\mathcal{V}).$$

The sheaf $\mathcal{V}^*_{\underline{\alpha}}$ has non-zero stalks only in the points $x \in \overline{U_{\underline{\alpha}}}$ and in such a point the stalk is equal to $\mathcal{V}_x$. Here we need that for any point $x \in \overline{U_{\underline{\alpha}}}$ and a small ball $B(x,\varepsilon)$ that

$$\mathcal{V}(B(x,\varepsilon)) = \mathcal{V}^*_{\underline{\alpha}}(B(x,\varepsilon) \cap \overline{U}_{\underline{\alpha}}).$$

Outside of $\overline{U_{\underline{\alpha}}}$ the sheaf has been mowed.
Now we consider the Čech resolution of our sheaf $\mathcal{V}$ (see section 4.5.2 and 4.6.6):

$$0 \longrightarrow \mathcal{V} \longrightarrow \prod_{\alpha \in \mathrm{A}} \mathcal{V}^*_{\alpha} \longrightarrow \prod_{(\alpha,\beta) \in \mathrm{A} \times \mathrm{A}_<} \mathcal{V}^*_{(\alpha,\beta)} \longrightarrow \ldots \longrightarrow \prod_{\underline{\alpha} \in \mathrm{A}^{q+1}_<} \mathcal{V}^*_{\underline{\alpha}} \longrightarrow \ldots .$$

This is now an acyclic resolution (section see 4.5.2) since all the sheaves $\mathcal{V}^*_{\underline{\alpha}}$ are acyclic by the homotopy axiom.
Hence we see that the cohomology groups $H^\nu(M,\mathcal{V})$ can be computed from the complex of global sections (see 4.6.10)

$$\begin{array}{ccccc}
\prod_{\alpha \in \mathrm{A}} \mathcal{V}^*_{\alpha}(M) \longrightarrow & \prod_{(\alpha\beta) \in \mathrm{A} \times \mathrm{A}_<} \mathcal{V}^*_{(\alpha,\beta)}(M) \longrightarrow \ldots \longrightarrow & \prod_{\alpha \in \mathrm{A}^{q+1}_<} \mathcal{V}^*_{\underline{\alpha}}(M) \longrightarrow \\
\| & \| & \| \ldots \\
\prod_{\alpha \in \mathrm{A}} \mathcal{V}(U_\alpha) \longrightarrow & \prod_{(\alpha,\beta) \in \mathrm{A} \times \mathrm{A}_<} \mathcal{V}(U_\alpha \cap U_\beta) \longrightarrow \ldots \longrightarrow & \prod_{\alpha \in \mathrm{A}^{q+1}_<} \mathcal{V}(U_{\underline{\alpha}})) \longrightarrow \ldots
\end{array}$$

which is the Čech complex attached to the resolution.

4.8.3 Čech Coresolution of Local Systems

We introduce the **Čech coresolution**. To do this we consider the sheaves

$$i_{\underline{\alpha}!} i^*_{\underline{\alpha}}(\mathcal{V}) = \mathcal{V}^!_{\underline{\alpha}}.$$

These sheaves are zero outside of the open sets $U_{\underline{\alpha}}$ and on these sets they coincide with $\mathcal{V}$.
For any $\underline{\alpha}$ we define a morphism $\iota_{\underline{\alpha}} : \mathcal{V}^!_{\underline{\alpha}} \longrightarrow \mathcal{V}$. To do this we choose an open set $U \subset M$ and a section $s \in \mathcal{V}^!_{\underline{\alpha}}(U)$. This is by definition a section $s \in \mathcal{V}(U \cap U_{\underline{\alpha}})$ whose support $|s| = W$ is closed in $\overline{U}_{\underline{\alpha}}$ and therefore also in U. Hence $U \setminus W$ is open. But W is also open since $\mathcal{V}$ is a local system. Hence we have a disjoint decomposition into open subsets $U = W \cup (U \setminus W)$ and $\mathcal{V}(U) = \mathcal{V}(W) \oplus \mathcal{V}(U \setminus \overline{W})$. Our morphism $\iota_{\underline{\alpha}}$ is now defined by $\iota_{\underline{\alpha}}(s) \mapsto (s,0)$.
Hence we can define a complex of sheaves

$$\ldots \longrightarrow \prod_{\underline{\alpha} \in \mathrm{A}^{q+1}_<} \mathcal{V}^!_{\underline{\alpha}} \longrightarrow \ldots \longrightarrow \prod_{(\alpha,\beta) \in \mathrm{A} \times \mathrm{A}_<} \mathcal{V}^!_{\alpha,\beta} \longrightarrow \prod_{\alpha \in \mathrm{A}} \mathcal{V}^!_{\alpha} \longrightarrow \mathcal{V} \longrightarrow 0$$

where the boundary operator is given by

$$(ds_x)_{\alpha_0,\dots,\alpha_q} = \sum_\beta (-1)^{\varepsilon(\beta,\underline{\alpha})} s_{x,\alpha_0,\dots,\beta,\dots,\alpha_q}$$

where β runs over those indices which do not occur in $\underline{\alpha}$ and where $\varepsilon(\beta,\underline{\alpha})$ gives us the position of β with respect to the ordering, where $s_{x,\alpha_0,\dots,\beta,\dots,\alpha_q}$ is an element in the stalk $\mathcal{V}^!_{\alpha_0,\dots,\beta,\dots,\alpha_q,x}$. The last homomorphism on the right is simply summation $\sum s_{x,\alpha}$. Again it is clear that this is an exact complex of sheaves (see Exercise 24).
On these open sets our sheaves $\mathcal{V}^!_{\underline{\alpha}}$ are isomorphic to a constant sheaf. We now assume that M is oriented, then we get for the cohomology with compact supports (see 4.7.2)

$$H^\nu(M,\mathcal{V}^!_{\underline{\alpha}}) = \begin{cases} 0 & \text{for } \nu \neq d \\ \mathcal{V}(U_{\underline{\alpha}}) & \text{for } \nu = d \end{cases},$$

and $\mathcal{V}(U_{\underline{\alpha}}) \simeq V$.
With a grain of salt we may consider this as an acyclic coresolution for the right exact functor

$$\mathcal{V} \longrightarrow H^d_c(M,\mathcal{V}),$$

it is called coresolution because all the arrows point in opposite directions. Of course we have to show that the functor is right exact, this is the source for Poincaré duality.
I claim that we have $H^\nu_c(M, \prod_{\underline{\alpha}\in A^{q+1}_<} \mathcal{V}^!_\alpha) = 0$ if $\nu \neq d$ and

$$H^d_c(M, \prod_{\underline{\alpha}\in A^{q+1}_<} \mathcal{V}^!_\alpha) = \bigoplus_{\underline{\alpha}\in A^{q+1}_<} H^d(M,\mathcal{V}^!_\alpha). \tag{4.101}$$

To see this we take the injective resolution constructed by Godement for the $\mathcal{V}^!_{\underline{\alpha}}$ (see 4.2.1). Then the product of the sheaves in the resolution gives a resolution of the product:

$$0 \longrightarrow \prod_{\underline{\alpha}\in A^{q+1}_<} \mathcal{V}^!_{\underline{\alpha}} \longrightarrow \prod_{\underline{\alpha}\in A^{q+1}_<} I^0_{\underline{\alpha}} \longrightarrow \prod_{\underline{\alpha}\in A^{q+1}_<} I^1_{\underline{\alpha}} \longrightarrow \dots$$

and to compute the cohomology with compact support we look at the resulting complex of global sections with compact support. But since any compact set meets only finitely many of the open sets U_α we see that

$$H^0_c(M, \prod_{\underline{\alpha}\in A^{q+1}_<} I^q_{\underline{\alpha}}) = \bigoplus_{\underline{\alpha}\in A^{q+1}_<} H^0_c(M,I^q_{\underline{\alpha}}).$$

To see this we have to take into account that the stalks of the sheaves $I^q_{\underline{\alpha}}$ are zero outside $U_{\underline{\alpha}}$ which is clear from the construction. Then the claim follows.
We apply the functor H^d_c to our coresolution and get a complex

$$\dots \longrightarrow \bigoplus_{\underline{\alpha}\in A^{q+1}_<} H^d(M,\mathcal{V}^!_{\underline{\alpha}}) \longrightarrow \dots \longrightarrow \bigoplus_{\alpha\in A} H^d(M,\mathcal{V}^!_\alpha) \longrightarrow 0. \tag{4.102}$$

We introduce degrees on this complex by giving the degree i to $\bigoplus_{\underline{\alpha}\in A^{i+1}_<} H^d_c(M,\mathcal{V}^!_{\underline{\alpha}})$. Hence the complex becomes a homological complex: the degree of the boundary operator is -1. Furthermore I want to make the additional assumption:

(Bound) *The number of indices α for which U_α contains a given x is not only finite but even bounded independently of x.*

This has the consequence that our complex of sheaves is bounded, i.e. it is trivial for large i (and for $i < 0$ anyway).

Theorem 4.8.4. *Under the assumption* **(Bound)** *the cohomology of this complex is the cohomology with compact supports*

$$H_c^{d-i}(M,\mathcal{V}) = H^i\left(\ldots \longrightarrow \bigoplus_{\underline{\alpha}\in \mathrm{A}_<^{q+1}} H^d(M,\mathcal{V}^!_{\underline{\alpha}}) \longrightarrow \ldots \longrightarrow \bigoplus_{\alpha\in\mathrm{A}} H^d(M,\mathcal{V}^!_\alpha) \longrightarrow 0\right).$$

Proof: We use the same arguments which we used when we proved that we can compute cohomology groups by acyclic resolutions. We break the complex of sheaves into pieces

$$\ldots \longrightarrow \prod_{\underline{\alpha}\in\mathrm{A}_<^{q+1}} \mathcal{V}^!_{\underline{\alpha}} \longrightarrow \ldots \longrightarrow \prod_{\alpha,\beta\in\mathrm{A}_<^2} \mathcal{V}^!_{(\alpha,\beta)} \longrightarrow \mathcal{G} \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{G} \longrightarrow \prod_{\alpha\in\mathrm{A}} \mathcal{V}^!_\alpha \longrightarrow \mathcal{V} \longrightarrow 0.$$

The second short complex gives us a long exact sequence if we apply the cohomology with compact supports. Since the sheaf in the middle has only cohomology with compact supports in degree d we get

$$H_c^{i-1}(M,\mathcal{V}) \simeq H_c^i(M,\mathcal{G}) \quad \text{for} \quad i \neq d-1,d$$

and

$$0 \to H_c^{d-1}(M,\mathcal{V}) \longrightarrow H_c^d(M,\mathcal{G}) \longrightarrow \bigoplus_{\alpha\in\mathrm{A}} H^d(M,\mathcal{V}^!_\alpha) \longrightarrow H_c^d(M,\mathcal{V}) \longrightarrow H_c^{d+1}(M,\mathcal{G}) \to 0.$$

At first we want to conclude that $H_c^m(M,\mathcal{V}) = H_c^m(M,\mathcal{G}) = 0$ for $m > d$. If not we would have $H_c^m(M,\mathcal{G}) \neq 0$ for some $m > d$. But $\mathcal{G}$ sits in a short exact sequence

$$0 \longrightarrow \mathcal{G}_1 \longrightarrow \prod_{\alpha,\beta\in\mathrm{A}_<^2} \mathcal{V}^!_{\alpha,\beta} \longrightarrow \mathcal{G} \longrightarrow 0$$

and $\mathcal{G}_1$ is the end of the complex

$$\ldots \longrightarrow \prod_{(\alpha,\beta,\gamma,\delta)\in\mathrm{A}_<^4} \mathcal{V}^!_{(\alpha,\beta,\gamma,\delta)} \longrightarrow \prod_{(\alpha,\beta,\gamma)\in\mathrm{A}_<^3} \mathcal{V}^!_{(\alpha,\beta,\gamma)} \longrightarrow \mathcal{G}_1 \longrightarrow 0.$$

We would get $H_c^{m+1}(M,\mathcal{G}_1) \neq 0$ and applying the same procedure again and again we get a contradiction, because the complex is finite to the left. Hence we get in degree d

$$0 \longrightarrow H_c^{d-1}(M,\mathcal{V}) \longrightarrow H_c^d(M,\mathcal{G}) \longrightarrow \bigoplus_{\alpha\in\mathrm{A}} H^d(M,\mathcal{V}^!_\alpha) \longrightarrow H_c^d(M,\mathcal{V}) \longrightarrow 0.$$

Induction on the length of the complex gives us that the complex

$$0 \longrightarrow \ldots \longrightarrow \bigoplus_{\underline{\alpha}\in \mathrm{A}^{q+1}_<} H^d(M,\mathcal{V}^!_{\underline{\alpha}}) \longrightarrow \ldots \longrightarrow H^d(M,\mathcal{V}^!_{(\alpha,\beta)}) \longrightarrow 0$$

computes the cohomology $H_c^{d-\bullet}(M,\mathcal{G})$ and the theorem follows. □

During the proof we saw:

Corollary 4.8.5. *Under the assumption of the theorem we have*

$$H_c^m(M,\mathcal{V}) = 0 \text{ for } m > d,$$

this implies the right exactnes of $H_c^d(M,\mathcal{V})$.

4.8.4 Poincaré Duality

In this section we assume that our manifold M is oriented. Let R be a commutative ring with identity. We assume that we have a local system $\mathcal{V}$ on M which has values in the category of finitely generated projective R-modules. We can also consider the dual local system $\mathcal{V}^\vee = \mathrm{Hom}_R(\mathcal{V},R)$. Our assumptions imply that $\mathcal{V}^{\vee\vee} \xrightarrow{\sim} \mathcal{V}$. We assume that we have a convex covering which satisfies **(Bound)**. We compute the cohomology $H^\bullet(M,\mathcal{V})$ and the cohomology with compact support $H_c^\bullet(M,\mathcal{V}^\vee)$ by means of the two complexes which we obtain from a convex covering. We write the complexes

$$\begin{array}{ccccccccccc}
0 & \xrightarrow{d} & \prod_{\alpha\in \mathrm{A}} \mathcal{V}(U_\alpha) & \xrightarrow{d} & \prod_{(\alpha,\beta)\in \mathrm{A}^2_<} \mathcal{V}(U_\alpha\cap U_\beta) & \xrightarrow{d} & \ldots & \xrightarrow{d} & \prod_{\underline{\alpha}\in \mathrm{A}^{q+1}_<} \mathcal{V}(U_{\underline{\alpha}}) & \xrightarrow{d} & \ldots \\
 & & \| & & \| & & & & \| & & \\
0 & \xrightarrow{d} & X^0 & \xrightarrow{d} & X^1 & \xrightarrow{d} & \ldots & \xrightarrow{d} & X^q & \xrightarrow{d} & \ldots
\end{array}$$

and

$$\begin{array}{ccccccccccc}
\ldots & \xrightarrow{\delta} & \bigoplus_{\underline{\alpha}\in \mathrm{A}^{q+1}_<} \mathcal{V}^\vee(U_{\underline{\alpha}}) & \xrightarrow{\delta} & \ldots & \xrightarrow{\delta} & \bigoplus_{(\alpha.\beta)\in \mathrm{A}^2_<} \mathcal{V}^\vee(U_\alpha\cap U_\beta) & \xrightarrow{\delta} & \bigoplus_{\alpha\in \mathrm{A}} \mathcal{V}^\vee(U_\alpha) & \xrightarrow{\delta} & 0 \\
 & & \| & & & & \| & & \| & & \\
\ldots & \xrightarrow{\delta} & Y^q & \xrightarrow{\delta} & \ldots & \xrightarrow{\delta} & Y^2 & \xrightarrow{\delta} & Y^1 & \xrightarrow{\delta} & 0
\end{array}$$

where we made the identification $\mathcal{V}^\vee(U_{\underline{\alpha}}) = H_c^d(M,\mathcal{V}^{\vee,!}_{\underline{\alpha}})$. We define a pairing

$$< , >: Y^q \times X^q \longrightarrow R$$

which is given by the formula for $s = (\ldots,s_{\underline{\alpha}},\ldots) \in \prod \mathcal{V}(U_{\underline{\alpha}})$ and $t = (\ldots,t_{\underline{\alpha}},\ldots) \in \bigoplus \mathcal{V}^\vee(U_{\underline{\alpha}})$ we define

$$< s,t >= \sum_{\underline{\alpha}} s_{\underline{\alpha}} \cdot t_{\underline{\alpha}} \tag{4.103}$$

where $s_{\underline{\alpha}} \cdot t_{\underline{\alpha}}$ is the pairing induced by the pairing on the coefficient systems. The expression makes sense because t has only finitely many non-zero entries. We have

$$< ds,t >=< s,\delta t > \tag{4.104}$$

for $s \in X^q, t \in Y^{q+1}$. Since these complexes compute the cohomology and the cohomology with compact supports respectively we get a pairing

$$H^q(M,\mathcal{V}) \times H_c^{d-q}(M,\mathcal{V}^\vee) \longrightarrow R. \tag{4.105}$$

We want to discuss the properties of this pairing. We need a finiteness condition:

Definition 4.8.6. *We say that M is of* **finite cohomological type***, if for any coefficient system $\mathcal{V}$ we can find a finite subset $\mathbf{F} \subset \mathrm{A}$ such that the projection map*

$$\begin{array}{ccccccccc}
0 & \xrightarrow{d} & \prod_{\alpha\in\mathrm{A}} \mathcal{V}(U_\alpha) & \xrightarrow{d} & \prod_{(\alpha,\beta)\in\mathrm{A}^2_<} \mathcal{V}(U_\alpha\cap U_\beta) & \xrightarrow{d} \cdots \xrightarrow{d} & \prod_{\underline{\alpha}\in\mathrm{A}^{q+1}_<} \mathcal{V}(U_{\underline{\alpha}}) & \xrightarrow{d} & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \xrightarrow{d} & \prod_{\alpha\in\mathbf{F}} \mathcal{V}(U_\alpha) & \xrightarrow{d} & \prod_{(\alpha,\beta)\in\mathbf{F}^2_<} \mathcal{V}(U_\alpha\cap U_\beta) & \xrightarrow{d} \cdots \xrightarrow{d} & \prod_{\underline{\alpha}\in\mathbf{F}^{q+1}_<} \mathcal{V}(U_{\underline{\alpha}}) & \xrightarrow{d} & \cdots
\end{array}$$

induces an injection in the cohomology of the two complexes.

Under this assumption it follows that the cohomology groups $H^q(M,\mathcal{V})$ are of of finite type over R, i.e. they are of the form: submodule of a finitely generated free R module divided by the image of a finitely generated free R-module.

Lemma 4.8.7. *A manifold M is certainly of finite cohomological type if the following is true*

(a) *The manifold M contains an open submanifold N whose closure $\overline{N}$ is compact and whose boundary ∂N is a submanifold of codimension 1.*

(b) *The inclusion $\overline{N} \hookrightarrow M$ is a homotopy equivalence.*

This situation occurs if we consider the cohomology of arithmetic groups.
Of course a compact manifold is always of finite cohomological type.

Definition 4.8.8. *For any R-module A we define the* **torsion subgroup** *A_{tors} to be the submodule of those elements x which are anihilated by a non zero divisor in R.*

In the following theorem we write A/Tors for A/A_{tors}.

Theorem 4.8.9 (Poincaré Duality)**.** *Let M be a manifold of finite cohomological type. We also assume that we have a convex covering which satisfies* **(Bound)***. We assume that $\mathcal{V}$ is a local system of finite dimensional vector spaces over a field k. Then the pairing*

$$H^q(M,\mathcal{V}) \times H_c^{d-q}(M,\mathcal{V}^\vee) \longrightarrow k$$

is non-degenerate for all q.
The vector spaces $H_c^{d-q}(M,\mathcal{V}^\vee)$ and $H^q(M,\mathcal{V})$ are finite dimensional, the cohomology $H^q(M,\mathcal{V})$ vanishes for $q > d$.
If R is a discrete valuation ring (or more generally a Dedekind ring) and if $\mathcal{V}$ is a local system of free R-modules of finite rank then the pairing

$$H^q(M,\mathcal{V})/\mathrm{Tors} \times H_c^{d-q}(M,\mathcal{V}^\vee)/\mathrm{Tors} \longrightarrow R$$

is non-degenerate.

Here I mean non degenerate in the strong sense: For any element ξ on one side, which is not a proper multiple of another element, we can find an element η on the other side such that $<\xi,\eta>=1$.

Proof: Our basic ring R is a field k or a discrete valuation ring. We compute the cohomology by means of the two complexes $X^\bullet$ and $Y^\bullet$ respectively. We have the pairing

$$<\,,\,>: Y^q \times X^q \longrightarrow k.$$

These spaces may be of infinite dimension. We say that a linear form

$$\lambda : X^q = \prod_{\alpha \in \mathrm{A}^q_<} H^0(M,\mathcal{V}^*_\alpha) \longrightarrow R$$

is **continuous** if it factors over a quotient $\prod_{\alpha\in E_\lambda} H^0(M,\mathcal{V}^*_\alpha)$ where E_λ is a finite subset of $\mathrm{A}^q_<$. Then it is clear that Y^q is the space of continuous linear forms on X^q. It is also clear that X^q is the space of all linear forms on Y^q.

In X^q (resp. Y^q) we have the subspaces of cocycles and of coboundaries

$$\begin{aligned} B^q(X^\bullet) &\subset Z^q(X^\bullet) \subset X^q \\ B^q(Y^\bullet) &\subset Z^q(Y^\bullet) \subset Y^q. \end{aligned}$$

Since $B^q(X^\bullet) = d(X^{q-1})$ by definition we find that

$$\begin{aligned} Z^q(Y^\bullet) &= \{y \in Y^q | \delta y = 0\} \\ &= B^q(X^\bullet)^\perp \\ &= \{y \in Y^q | < B^q(X^\bullet),y >= 0\} \end{aligned}$$

and by the same argument we find that

$$Z^q(X^\bullet) = B^q(Y^\bullet)^\perp.$$

We now assume that our ground ring is a field k. The spaces X^q,Y^q are in perfect duality. If they were finite dimensional we could conclude that for any subspace W of one of them we have $(W^\perp)^\perp = W$. This is always true for subspaces $W \subset Y^q$. We also know that always $W \subset (W^\perp)^\perp$. If $y \notin W$ then we have a finite set $\mathbf{F} \subset \mathrm{A}$ of indices such that y is already in $\bigoplus_{\underline{\alpha}\in \mathbf{F}^{q+1}_<} \mathcal{V}^!(U_{\underline{\alpha}})$. This is a finite dimensional subspace of Y^q. Thus we can find an $x \in \prod_{\underline{\alpha}\in F^{q+1}_<} \mathcal{V}(U_{\underline{\alpha}})$ with $< W,x >= 0$ and $< y,x >\neq 0$. Hence $y \notin W^{\perp\perp}$.

But for subspaces $W \subset X^q$ the same argument is only true for closed subspaces, which means

$$W = \{x \in X^q \mid \lambda(x) = 0 \text{ for continuous linear forms } \lambda \text{ which vanish on } W\}.$$

We want to consider the case $W = B^q(X^\bullet)$. Here we use our assumption. Let $x \in Z^q(X^\bullet)$ but $x \notin B^q(X^\bullet)$, then we can find a finite subset $\mathbf{F} \subset \mathrm{A}$ such that the projection $x_\mathbf{F}$ of x to $\prod_{\underline{\alpha}\in F^{q+1}_<} \mathcal{V}(U_{\underline{\alpha}})$ is not in the image of the boundary map $d^{q-1}_\mathbf{F} : \prod_{\underline{\alpha}\in F^q_<} \mathcal{V}(U_{\underline{\alpha}}) \longrightarrow \prod_{\underline{\alpha}\in F^{q+1}_<} \mathcal{V}(U_{\underline{\alpha}})$. We find a $y \in \bigoplus_{\underline{\alpha}\in \mathbf{F}^{q+1}_<} \mathcal{V}^\vee(U_{\underline{\alpha}})$ which vanishes on the image of $d^{q-1}_\mathbf{F}$ but not on $x_\mathbf{F}$, i.e. $< y,\mathrm{Im}\,(d^{q-1}_\mathbf{F}) >= 0$ and $< y,x >\neq 0$. This element y is of course also in Y^q and it vanishes on $B^q(X^\bullet)$ but not on x.

This proves that under our finiteness assumption we have

$$\begin{aligned} B^q(X^\bullet)^{\perp\perp} &= B^q(X^\bullet) \\ B^q(Y^\bullet)^{\perp\perp} &= B^q(Y^\bullet). \end{aligned}$$

But then it is obvious that the pairing

$$H^q(M,\mathcal{V}) \times H_c^{d-q}(M,\mathcal{V}^\vee) \longrightarrow k$$

is non-degenerate.

The next two statements follow easily: The vector spaces $H^q(M,\mathcal{V})$ are finite dimensional because M is of finite cohomological type. By duality it follows that $H_c^{d-q}(M,\mathcal{V}^!)$ are finite dimensional. On the other hand $H_c^\nu(M,\mathcal{V}^!) = 0$ for $\nu < 0$ (definition) and $\nu > d$ (Corollary 4.8.5).

Now we come to the second half of the theorem, we assume that $\mathcal{V}$ is a local system of free R-modules of finite rank where R is a discrete valuation ring. Let K be the quotient field of R, let (π) be the maximal ideal of R, let $k = R/(\pi)$ be the residue field. We will apply the first half of the theorem twice, we can consider the local systems $\mathcal{V} \otimes K = \mathcal{V}_K$ and $\mathcal{V} \otimes k = \mathcal{V}/\pi\mathcal{V}$.

We get an exact sequence of local systems

$$0 \longrightarrow \mathcal{V} \xrightarrow{\times\pi} \mathcal{V} \longrightarrow \mathcal{V}/\pi\mathcal{V} \longrightarrow 0.$$

From this short exact sequence we get two exact sequences in cohomology and in cohomology with compact supports which suitably interpreted give short exact sequences

$$0 \longrightarrow H^q(M,\mathcal{V}) \otimes k \longrightarrow H^q(M,\mathcal{V} \otimes k) \longrightarrow H^{q+1}(M,\mathcal{V})[\pi] \longrightarrow 0 \qquad \text{(mod)}$$

and

$$0 \longrightarrow H_c^{d-q}(M,\mathcal{V}^\vee) \otimes k \longrightarrow H_c^{d-q}(M,\mathcal{V}^\vee \otimes k) \longrightarrow H_c^{d+1-q}(M,\mathcal{V}^\vee)[\pi] \longrightarrow 0 \qquad (\text{mod}_c)$$

where $[\pi]$ means kernel under multiplication by π.

Since M is of finite cohomological type we know that $H^q(M,\mathcal{V})$ is a finitely generated R-module. I claim that this also implies that $H_c^{d-q}(M,\mathcal{V}^\vee)$ is finitely generated. It follows from our exact sequence and the corollary above that $H_c^{d-q}(M,\mathcal{V}^\vee) \otimes k$ is finitely generated. We lift generators to $H_c^{d-q}(M,\mathcal{V}^\vee)$ and then these lifts generate a submodule U of $H_c^{d-q}(M,\mathcal{V}^\vee)$. If we already knew that $H_c^{d-q}(M,\mathcal{V}^\vee)$ is finitely generated, then the lemma of Nakayama [Ei] would imply that these lifted generators generate $H_c^{d-q}(M,\mathcal{V}^\vee)$. Hence we have to show that in the exact sequence

$$0 \longrightarrow U \longrightarrow H_c^{d-q}(M,\mathcal{V}^\vee) \longrightarrow W \longrightarrow 0$$

we have $W = 0$.

If we tensorize by k we get $W = \pi W$, this means that W is infinitely divisible. Now we observe that $H_c^{d-q}(M,\mathcal{V}^\vee) \otimes K = H_c^{d-q}(M,\mathcal{V}^\vee \otimes K)$ and $H^q(M,\mathcal{V}) \otimes K = H^q(M,\mathcal{V} \otimes K)$, these vector spaces are finite dimensional and dual to each other. We can find elements $v_1,\ldots,v_s \in H^q(M,\mathcal{V})$ (resp. $w_1,\ldots,w_s \in H_c^{d-q}(M,\mathcal{V}^\vee)$) whose images in $H^q(M,\mathcal{V}) \otimes K$ (resp. $H_c^{d-q}(M,\mathcal{V}^\vee)$) form a K-basis. If we evaluate these basis elements by the pairing we get an $(s \times s)$ matrix whose determinant is in R and non zero. Let $\widetilde{U}$ be the lattice

generated by the images of $w_1,\dots,w_s$ in $H_c^{d-q}(M,\mathcal{V}^\vee)/\mathrm{Tors}$. If $\widetilde{U}_1 \supset \widetilde{U}$ is a larger R-module then we can find a smallest integer $a = a(\widetilde{U}_1,\widetilde{U})$ such that $\pi^a\widetilde{U}_1 \subset \widetilde{U}$. If now $W \neq 0$ then this implies that we can find larger lattices $\widetilde{U} \subset \widetilde{U}_1$ such that $a(\widetilde{U}_1,\widetilde{U})$ becomes arbitrarily large. If we now replace the elements $w_1,\dots,w_s \in H_c^{d-q}(M,\mathcal{V}^\vee)$ by a basis of $\widetilde{U}_1$ then we can again form the evaluation matrix as above but its determinant gets multiplied by π^{-b} where b goes to infinity if $\geq a(\widetilde{U}_1,\widetilde{U})$ goes to infinity. But this determinant must still be in R, this contradicts $W \neq 0$.
We get two more exact sequences

$$\begin{aligned} &0 \longrightarrow H^q(M,\mathcal{V})_{\mathrm{tors}} \longrightarrow H^q(M,\mathcal{V}) \longrightarrow H^q(M,\mathcal{V})/\mathrm{Tors} \longrightarrow 0 \\ \text{and}\qquad &0 \longrightarrow H_c^{d-q}(M,\mathcal{V}^\vee)_{\mathrm{tors}} \longrightarrow H_c^{d-q}(M,\mathcal{V}^\vee) \longrightarrow H_c^{d-q}(M,\mathcal{V}^\vee)/\mathrm{Tors} \longrightarrow 0. \end{aligned}$$

The two modules on the right are free of finite rank, the two ranks are equal. We have the R-valued pairing between the modules in the middle, this pairing vanishes on the two torsion submodules. This gives us the pairing

$$H^q(M,\mathcal{V})/\mathrm{Tors} \times H_c^{d-q}(M,\mathcal{V}^\vee)/\mathrm{Tors} \longrightarrow R \tag{Poin}$$

and this is the pairing which we want to show to be non degenerate. To say it again this means that for two bases $v_1,\dots,v_s \in H^q(M,\mathcal{V})/\mathrm{tors}$ and $w_1,\dots,w_s \in H_c^{d-q}(M,\mathcal{V}^\vee)$ the evaluation matrix has as determinant an element in $R^\times$.
Since any finitely generated module over R is the direct sum of its torsion submodule and a free module we can tensorize our two sequences above by k and get exact sequences

$$0 \longrightarrow H^q(M,\mathcal{V})_{\mathrm{tors}} \otimes k \longrightarrow H^q(M,\mathcal{V}) \otimes k \longrightarrow H^q(M,\mathcal{V})/\mathrm{Tors} \otimes k \longrightarrow 0$$

and

$$0 \longrightarrow H_c^{d-q}(M,\mathcal{V}^\vee)_{\mathrm{tors}} \otimes k \longrightarrow H_c^{d-q}(M,\mathcal{V}^\vee) \otimes k \longrightarrow H_c^{d-q}(M,\mathcal{V}^\vee)/\mathrm{Tors} \otimes k \longrightarrow 0.$$

Combining this with our two sequences (mod), (mod$_c$) we see that for the cohomology groups with coefficients we get filtrations

$$H^q(M,\mathcal{V})_{\mathrm{tors}} \otimes k \subset H^q(M,\mathcal{V}) \otimes k \subset H^q(M,\mathcal{V} \otimes k)$$

and

$$H_c^{d-q}(M,\mathcal{V}^\vee)_{\mathrm{tors}} \otimes k \subset H_c^{d-q}(M,\mathcal{V}^\vee) \otimes k \subset H_c^{d-q}(M,\mathcal{V}^\vee \otimes k).$$

In both filtrations the quotient of the rightmost module by the previous one is

$$H^{q+1}(M,\mathcal{V})[\pi] \text{ resp. } H_c^{d+1-q}(M,\mathcal{V}^\vee)[\pi].$$

We know already that the pairing

$$H^q(M,\mathcal{V} \otimes k) \times H_c^{d-q}(M,\mathcal{V}^\vee \otimes k) \longrightarrow k$$

is non degenerate. We also know that for the orthogonal complements of the leftmost modules we have

$$H^q(M,\mathcal{V})\otimes k \subset (H_c^{d-q}(M,\mathcal{V}^\vee)_{\mathrm{tors}}\otimes k)^\perp$$
$$H_c^{d-q}(M,\mathcal{V}^\vee)\otimes k \subset (H^q(M,\mathcal{V})_{\mathrm{tors}}\otimes k)^\perp.$$

This implies that for each of the two modules

$$H^{q+1}(M,\mathcal{V})[\pi] \text{ resp. } H_c^{d+1-q}(M,\mathcal{V}^\vee)[\pi]$$

a certain quotient of this module is in perfect duality with

$$H_c^{d-q}(M,\mathcal{V}^\vee)_{\mathrm{tors}}\otimes k \text{ resp. } H^q(M,\mathcal{V})_{\mathrm{tors}}\otimes k.$$

This gives us two sets of inequalities

$$\dim_k(H^{q+1}(M,\mathcal{V})[\pi]) \geq \dim_k(H_c^{d-q}(M,\mathcal{V}^\vee)_{\mathrm{tors}}\otimes k)$$

and

$$\dim_k(H_c^{d+1-q}(M,\mathcal{V}^\vee)[\pi]) \geq \dim_k(H^q(M,\mathcal{V})_{\mathrm{tors}}\otimes k).$$

For a finitely generated torsion R-module A we have $\dim_k(A\otimes k) = \dim_k(A[\pi])$. This implies that

$$\sum_q \bigl(\dim_k(H^{q+1}(M,\mathcal{V})[\pi]) + \dim_k(H_c^{d+1-q}(M,\mathcal{V}^\vee)[\pi])\bigr)$$
$$= \sum_q \left(\dim_k(H_c^{d-q}(M,\mathcal{V}^\vee)_{\mathrm{tors}}\otimes k) + \dim_k(H^q(M,\mathcal{V})_{\mathrm{tors}}\otimes k)\right).$$

Hence we see that in our inequalities we have in fact equalities. But this in turn implies that our inclusions above are even equalities

$$H^q(M,\mathcal{V})\otimes k = (H_c^{d-q}(M,\mathcal{V}^\vee)_{\mathrm{tors}}\otimes k)^\perp$$
$$H_c^{d-q}(M,\mathcal{V}^\vee)\otimes k = (H^q(M,\mathcal{V})_{\mathrm{tors}}\otimes k)^\perp.$$

Then it follows that the pairing

$$H^q(M,\mathcal{V})/\mathrm{Tors}\otimes k \times H_c^{d-q}(M,\mathcal{V}^\vee)/\mathrm{Tors}\otimes k \longrightarrow k$$

is non degenerate. But this is the original pairing (Poin) $\mod \pi$. If this reduction $\mod \pi$ is non degenerate then also (Poin) must be non degenerate. □

I want to keep the following byproduct of the above proof:

Corollary 4.8.10. *The non degenerate pairing*

$$H^q(M,\mathcal{V}\otimes k)\times H_c^{d-q}(M,\mathcal{V}^\vee\otimes k) \longrightarrow k$$

induces non degenerate pairings

$$H^{q+1}(M,\mathcal{V})[\pi]\times H_c^{d-q}(M,\mathcal{V}^\vee)_{tors}\otimes k \longrightarrow k$$
$$H_c^{d+1-q}(M,\mathcal{V}^\vee)[\pi]\times H^q(M,\mathcal{V})_{tors}\otimes k \longrightarrow k.$$

The extension of the theorem from discrete valuation rings to Dedekind rings is rather clear, if one knows enough about Dedekind rings. They will be discussed in the second volume.
At this point it is tempting to ask, whether these pairings are given by a cup product. We should be aware, that this does not make sense because the cohomology with compact support is not the cohomology of a sheaf (see 4.7.1). Only after a suitable compactification of M we we have such an interpretation. We come back to this point in 4.8.7.

4.8.5 The Cohomology in Top Degree and the Homology

We assume that M is of finite cohomological type and oriented. We start from a local system $\mathcal{V}$ and we assume that we obtained it form an action of the fundamental group $\pi = \pi_1(M,x_0)$ on an abelian group (or R-module) V. We do not make any further assumption. We have $\mathcal{V}_{x_0} = V$. We want to compute the cohomology with compact support in top degree. We will see that this can be expressed completely in terms of the action of π on V. Let I_π be the augmentation ideal, we introduced the module of coinvariants $V/I_\pi = V_\pi$. Our aim is to show that $H^d_c(M,\mathcal{V}) \xrightarrow{\sim} V_\pi$. But recall that this quotient is $H_0(\pi,V)$. (See page 28.) This makes it plausible that at least on manifolds the cohomology with compact supports behaves like homology.
We start from our complex

$$\ldots \longrightarrow \bigoplus_{(\alpha,\beta)\in \mathrm{A}^2_<} H^d(M,\mathcal{V}^!_{(\alpha,\beta)}) \longrightarrow \bigoplus_{\alpha\in \mathrm{A}} H^d(M,\mathcal{V}_\alpha) \longrightarrow H^d_c(M,\mathcal{V}) \longrightarrow 0.$$

Let α_0 be an index such that $x_0 \in U_{\alpha_0}$. I claim that the map

$$H^d(M,\mathcal{V}^!_{\alpha_0}) \simeq V \longrightarrow H^d_c(M,\mathcal{V})$$

is surjective and induces an isomorphism

$$V/I_\pi V \xrightarrow{\sim} H^d_c(M,\mathcal{V}).$$

Let α be any other index. We can choose a sequence $\alpha_0,\alpha_1,\ldots,\alpha_r = \alpha$ of indices such that $U_{\alpha_i \cap U_{\alpha_{i+1}}} \neq \emptyset$ for all i. For any pair of consecutive indices α_i,α_{i+1} we restrict the boundary operator

$$\delta : \bigoplus_{(\alpha,\beta)\in \mathrm{A}^2_<} H^d(M,\mathcal{V}^!_{(\alpha,\beta)}) \longrightarrow \bigoplus_{\gamma\in \mathrm{A}} H^d(M,\mathcal{V}^!_\gamma)$$

to the direct summand

$$H^d(M,\mathcal{V}^!_{(\alpha_i,\alpha_{i+1})})$$

(we assume $\alpha_i < \alpha_{i+1}$ otherwise we interchange the indices). It is clear that the image of this restriction in the target module lies in the submodule

$$H^d(M,\mathcal{V}^!_{\alpha_i}) \bigoplus H^d(M,\mathcal{V}^!_{\alpha_{i+1}}).$$

We have a natural isomorphism

$$\Psi_{\alpha_i,\alpha_{i+1}} : H^d(M,\mathcal{V}^!_{\alpha_i}) \xrightarrow{\sim} H^d(M,\mathcal{V}^!_{\alpha_{i+1}}).$$

which is the composition of the isomorphisms

$$H^d(M,\mathcal{V}^!_{\alpha_i\alpha_{i+1}}) \xrightarrow{\sim} H^d(M,\mathcal{V}^!_{\alpha_i})$$
$$H^d(M,\mathcal{V}^!_{\alpha_i\alpha_{i+1}}) \xrightarrow{\sim} H^d(M,\mathcal{V}^!_{\alpha_{i+1}})$$

which are induced by the inclusions $U_{\alpha_i} \cap U_{\alpha_{i+1}} \hookrightarrow U_{\alpha_i}$ and $U_{\alpha_i} \cap U_{\alpha_{i+1}} \hookrightarrow U_{\alpha_{i+1}}$. It is clear from the definition that the image of δ restricted to $H^d(M,\mathcal{V}^!_{\alpha_i\alpha_{i+1}})$ is the submodule

$$(H^d(M,\mathcal{V}^!_{\alpha_i}), -\Psi_{\alpha_i\alpha_{i+1}}(H^d(M,\mathcal{V}^!_{\alpha_i}))),$$

and hence we see that this submodule is in the kernel of

$$H^d(M,\mathcal{V}^!_{\alpha_i}) \bigoplus H^d(M,\mathcal{V}^!_{\alpha_{i+1}}) \longrightarrow H^d_c(M,\mathcal{V}).$$

Now our chain of indices gives us by composition an isomorphism

$$\Psi_{\alpha_0,\alpha_1} \ldots,_{\alpha_r} : H^d(M,\mathcal{V}^!_{\alpha_0}) \longrightarrow H^d(M,\mathcal{V}^!_{\alpha})$$

and it is clear that the elements

$$(H^d(M,\mathcal{V}^!_{\alpha_0}), -\Psi_{\alpha_0,\alpha_1\ldots,\alpha_r}(H^d(M,\mathcal{V}^!_{\alpha_0})))$$

lie in the kernel of

$$H^d(M,\mathcal{V}^!_{\alpha_0}) \oplus H^d(M,\mathcal{V}^!_{\alpha}) \longrightarrow H^d_c(M,\mathcal{V}).$$

From this it follows that the summand $H^d(M,\mathcal{V}^!_{\alpha_0}) \simeq V$ is mapped surjectively to $H^d_c(M,\mathcal{V})$.

Now we assume that our chain of indices comes back, i.e. $\alpha = \alpha_r = \alpha_0$. Then we can construct a path

$$\gamma : [0,1] \longrightarrow M$$

with $\gamma(0) = \gamma(1) = x_0$ which is obtained by joining x_0 inside of U_{α_0} to a point in $U_{\alpha_0} \cap U_{\alpha_1}$, this point to a point in $U_{\alpha_1} \cap U_{\alpha_2}$ and so on and finally joining the point in $U_{\alpha_{r-1}} \cap U_{\alpha_{\alpha_r}}$ to x_0. The homotopy class of this path is uniquely determined by the chain of indices.

Then it is clear from the construction of the local system from the action $\rho : \pi \longrightarrow \mathrm{Aut}(V)$ that $\Psi_{\alpha_0,\alpha_1} \ldots,_{\alpha_r} = \rho([\gamma])$. Hence we see that all elements of the form $(\mathrm{Id} - \rho([\gamma]))\mathcal{V}$ lie in the kernel of

$$H^d(M,\mathcal{V}^!_{\alpha_0}) \longrightarrow H^d_c(M,\mathcal{V})$$

and the surjective map factors

$$\begin{array}{ccc} \mathcal{V} & \longrightarrow & H^d_c(M,\mathcal{V}) \cdot \\ & \searrow \quad \nearrow & \\ & V/I_\pi V & \end{array}$$

But now it follows that

$$V/I_\pi V \xrightarrow{\sim} H^d_c(M,\mathcal{V})$$

must be an isomorphism, because the group $\bigoplus_{(\alpha,\beta)\in \mathrm{A}^2_<} H^d_c(M,\mathcal{V}^!_{(\alpha,\beta)})$ is generated by its direct summands.

Especially we see again that under our assumptions above

$$H^d_c(M,\_) = \ .$$

4.8.6 Some Remarks on Singular Homology

We can also define the **singular homology groups** $H_i(M,\mathcal{V})$. To do this we consider continous maps

$$\sigma : \Delta_q \longrightarrow M$$

where $\Delta_q = \{(t_0, \ldots ,t_q) \in {}^{q+1}_{+} | \sum t_i = 1\}$ is the q-dimensional **standard simplex**. We can consider the pull back $\sigma^*(\mathcal{V})$ of our local system and since Δ_q is contractible, we have

$$\sigma^*(\mathcal{V})(\Delta_q) = \sigma^*(\mathcal{V})_p$$

where p is any point in our simplex. We form linear combinations

$$\sum m_\sigma \cdot \sigma$$

where $m_\sigma \in \sigma^*(\mathcal{V})(\Delta_q)$. These linear combinations form an abelian group $C_q(M,\mathcal{V})$. We define a boundary operator

$$\partial_q : C_q(M,\mathcal{V}) \longrightarrow C_{q-1}(M,\mathcal{V}).$$

To do this we observe that we have face maps

$$\begin{aligned} \tau_i : & \qquad \Delta_{q-1} \longrightarrow \Delta_q \\ \tau_i : & \qquad (t_0, \ldots ,t_q) \longrightarrow (t_0, \ldots ,t_{i-1},0,t_i, \ldots ,t_q) \end{aligned}$$

and we put

$$\partial_1(m_\sigma \sigma) = \sum (-1)^i m_\sigma \cdot \sigma \circ \tau_i$$

where we use the fact that

$$(\sigma \circ \tau_i)^*(\mathcal{V})(\Delta_{q-1}) = \sigma^*(\mathcal{V})(\Delta_q).$$

An easy computation yields $\partial_{q-1} \circ \partial_q = 0$ hence we get the chain complex with coefficients in $\mathcal{V}$

$$\ldots \longrightarrow C_q(M,\mathcal{V}) \longrightarrow C_{q-1}(M,\mathcal{V}) \longrightarrow \cdots \longrightarrow C_0(M,\mathcal{V}) \longrightarrow 0$$

and by definition the homology groups of this complex are the homology groups of M with coefficients in $\mathcal{V}$:

$$H_q(M,\mathcal{V}) = H_q(C_\bullet(M,\mathcal{V})).$$

It is clear what $H_0(M,\mathcal{V})$ is: we see that $C_0(M,\mathcal{V})$ is the group of linear combinations

$$\sum_{x \in M} m_x \cdot x$$

where $m_x \in \mathcal{V}_x$. Of course we see that $m_x x - m_y y$ is a boundary if we can find a path $\gamma : [0,1] \longrightarrow M$ with $\gamma(0) = x$, $\gamma(1) = y$ and $[\gamma] m_x = m_y$. Hence it is clear that

$$H_0(M,\mathcal{V}) = V/I_\pi V \simeq H_c^d(M,\mathcal{V}).$$

This suggests that for a manifold M and a local system $\mathcal{V}$ on it we have the equality

$$H_i(M,\mathcal{V}) \simeq H_c^{d-i}(M,\mathcal{V}). \tag{4.106}$$

We will come back to this point in the third volume, see also [Hat], Chap. 3 3.3.

4.8.7 Cohomology with Compact Support and Embeddings

If we want to understand the cohomology $H_c^\bullet(M,\mathcal{V})$ it is sometimes very useful to embed M into a compact space. Let us consider an open embedding

$$i : M \hookrightarrow \overline{M},$$

where $\overline{M}$ is compact. Then we can consider the sheaf $i_!(\mathcal{V})$, and we know

$$H_c^\bullet(M,\mathcal{V}) = H^\bullet(\overline{M},i_!(\mathcal{V})).$$

We may also consider the direct image $i_*(\mathcal{V})$. Here we have to be careful because the functor i_* is not exact in general. But if we assume that our local system is acyclic with respect to i_*, then we know that

$$H^\bullet(M,\mathcal{V}) = H^\bullet(\overline{M},i_*(\mathcal{V})).$$

Especially we may have the situation that $\overline{M}$ is an oriented manifold with boundary and M is the interior of $\overline{M}$. Then it is clear that $M \hookrightarrow \overline{M}$ is a homotopy equivalence and a local system $\mathcal{V}$ on M extends to a local system on $\overline{M}$, which we denote by $\overline{\mathcal{V}}$. Under these circumstances we have

$$H^\bullet(M,\mathcal{V}) = H^\bullet(\overline{M},i_*(\mathcal{V})) = H^\bullet(\overline{M},\overline{\mathcal{V}}).$$

If now $\mathcal{V}$ is a local system of free R-modules of finite rank and $\mathcal{V}^\vee = \mathrm{Hom}(\mathcal{V},R)$ the dual system, then we have the Poincaré pairing

$$H_c^q(M,\mathcal{V}) \times H^{d-q}(M,\mathcal{V}^\vee) \longrightarrow H_c^d(M,R) \simeq R$$

which we may also write as

$$H^q(\overline{M},i_!(\mathcal{V})) \times H^{m-q}(\overline{M},i_*(\mathcal{V}^\vee)) \longrightarrow R.$$

It should not be too much of a surprise that this pairing can also be expressed in terms of the cup product.
We start with the observation that both sheaves $i_!(\mathcal{V})$ and $i_*(\mathcal{V}^\vee)$ have flat acyclic resolutions. In this situation we defined the product (see 4.6.10)

$$H^q(\overline{M},i_!(\mathcal{V})) \times H^{d-q}(\overline{M},i_*(\mathcal{V}^\vee)) \longrightarrow H^d(\overline{M},i_!(\mathcal{V})\widehat{\otimes} i_*(\mathcal{V}^\vee)),$$

and we have the evaluation pairing

$$i_!(\mathcal{V})\widehat{\otimes} i_*(\mathcal{V}^\vee) \longrightarrow i_!(R).$$

Now the cup product composed with the evaluation provides a pairing

$$H^q(\overline{M},i_!(\mathcal{V})) \times H^{d-q}(\overline{M},i_*(\mathcal{V}^\vee)) \longrightarrow H^d(\overline{M},i_!(R)) = H_c^d(M,R) = R.$$

We complement 4.8.9 by stating

This pairing is equal to the Poincaré duality pairing (4.107)

To see this we apply the same idea as in section 4.6.7 and reduce the comparison of the two pairings to the case where one of the factors is in degree zero.

We compute the cohomology groups from Čechresolutions. Our situation is a little bit different from the previous one since now our manifold has a boundary. But we may put a Riemannian metric on M as before and at first we cover a neighborhood of ∂M by small open "half-balls" with center on the boundary. Then the complement of the union of these balls is compact, and we cover it by small balls whose closure does not hit the boundary. Let us denote this covering by $\{U_\alpha\}_{\alpha\in\mathrm{A}}$.
For any $\alpha_0\cdots\alpha_q$ we consider $U_{\alpha_0}\cap\cdots\cap U_{\alpha_q}=U_{\underline{\alpha}}$, and we remove the boundary points from it (if there are any) and call the result $\overset{\circ}{U}_{\underline{\alpha}}$. Then

$$i_{\underline{\alpha}}:\overset{\circ}{U}_{\underline{\alpha}}\longrightarrow\overline{M}$$

is the inclusion, and we define

$$i_{\underline{\alpha},!}i^*_{\underline{\alpha}}(\mathcal{V})=\mathcal{V}^!_\alpha. \tag{4.108}$$

Now we compute our cohomology groups from the Čech resolution and the coresolution as before. We have

$$\cdots\longrightarrow\prod_{\alpha,\beta}\mathcal{V}^{\vee,!}_{(\alpha,\beta)}\longrightarrow\prod\mathcal{V}^{\vee,!}_\alpha\longrightarrow i_!(\mathcal{V}^\vee)\longrightarrow 0,$$

and

$$0\longrightarrow\mathcal{V}\longrightarrow\prod_\alpha\mathcal{V}^*_\alpha\longrightarrow\prod_{(\alpha,\beta)}\mathcal{V}^*_{(\alpha,\beta)}\longrightarrow\cdots.$$

Of course it is clear that the Poincaré pairing

$$H^0(\overline{M},i_*(\mathcal{V}))\times H^d(\overline{M},i_!(\mathcal{V}^\vee))\longrightarrow R$$

is given by the cup product. Then we proceed by induction on the degree. We break the two resolutions

$$\begin{array}{ccccccccc}0&\longrightarrow&\mathcal{G}&\longrightarrow&\bigoplus\mathcal{V}^{\vee,!}_\alpha&\longrightarrow&i_!(\mathcal{V}^\vee)&\longrightarrow&0\\0&\longrightarrow&\mathcal{V}&\longrightarrow&\prod_\alpha\mathcal{V}^*_\alpha&\longrightarrow&\mathcal{H}&\longrightarrow&0\end{array}$$

and we get the following pieces of long exact sequences

$$0\longrightarrow H^{d-1}(\overline{M},i_!(\mathcal{V}^\vee))\xrightarrow{\delta^\vee}H^d(\overline{M},\mathcal{G})\longrightarrow\cdots$$

and

$$H^0(\overline{M},\mathcal{H})\xrightarrow{\delta}H^1(\overline{M},i_*(\mathcal{V}))\longrightarrow 0.$$

The pairing $i_!(\mathcal{V}^\vee)\times i_*(\mathcal{V})\longrightarrow i_!(R)$ induces a pairing

$$\Big(\bigoplus\mathcal{V}^{\vee,!}_\alpha\Big)\times\prod_\alpha\mathcal{V}^*_\alpha\longrightarrow i_!(R)$$

(see section 4.8.4), and this induces a pairing

$$\mathcal{G} \times \mathcal{H} \longrightarrow i_!(R).$$

If we look at the definition of the Poincaré pairing of two classes $\xi \in H^{d-1}(\overline{M},i_!(\mathcal{V}^\vee))$ and $\eta = \delta(\psi) \in H^1(\overline{M},i_*(\mathcal{V}))$, then we have

$$\langle \xi,\eta \rangle = \langle \delta^\vee \xi,\psi \rangle. \tag{4.109}$$

But the right hand side is also the cup product of the classes $\delta^\vee \xi \in H^d(\overline{M},\mathcal{G})$ and $\psi \in H^0(\overline{M},\mathcal{H})$ this means we have $\langle \delta^\vee \xi,\psi \rangle = \delta^\vee \xi \cup \psi$. The cup product satisfies the rule

$$\delta^\vee \xi \cup \psi = \xi \cup \delta\psi = \xi \cup \eta. \tag{4.110}$$

Puting the equalities together we find $\langle \xi,\eta \rangle = \xi \cup \eta$ for $\xi \in H^{d-1}(\overline{M},i_!(\mathcal{V}^\vee))$ and $\eta = \delta(\psi) \in H^1(\overline{M},i_*(\mathcal{V}))$. The general case follows by the same argument inductively.

4.8.8 The Fundamental Class of a Submanifold

The homology groups can be defined for any space X and they provide a covariant functor from spaces to abelian groups: If we have a continuous map $f : X \longrightarrow Y$, then we get a homomorphism

$$f_{*,i} : H_i(X,\ \ ) \longrightarrow H_i(Y,\ \ )$$

for all degrees i.
This suggests that we should also have this kind of functoriality for the cohomology with compact supports on an oriented manifold M.
I want to discuss a special case where we see this functoriality. We consider a connected oriented manifold M and an oriented submanifold $N \subset M$, let m,n be the dimensions of M and N respectively. Let us denote the inclusion map by i. We allow that N has several connected components, but the dimensions of the components should be all the same. We choose an auxiliary Riemannian metric. This Riemannian metric splits the tangent bundle of M along N into $T_M = T_N \oplus T_{M/N}$, where $T_{M/N}$ is the **normal bundle**. We choose the orientation of the normal bundle $T_{M/N}$ such that the chosen orientation on T_M is the one obtained from the above direct sum decomposition and the orientations on the summands.
Let $\mathcal{V}$ be a local system on M, let $\mathcal{V}'$ be its restriction to N. If we consider the homology groups then we get get - directly from the definition - a homomorphism

$$H_i(N,\mathcal{V}') \longrightarrow H_i(M,\mathcal{V}).$$

Now let us accept 4.106, which says that on our manifolds the homology groups are isomorphic to cohomology groups with compact support, then we get

Corollary 4.8.11. *We have a natural homomorphism*

$$H_c^j(N,\mathcal{V}') \longrightarrow H_c^{m-n+j}(M,\mathcal{V}).$$

Proof: I want to construct this homomorphism directly.

Using the exponential map we can construct a tubular neighborhood $i_N : T_{M/N}(\epsilon) \subset M$ (see [Sp], p.465). We have the projection $\pi : T_{M/N}(\epsilon) \longrightarrow N$ where the fibres $\pi^{-1}(b)$ can be identified to small open balls in $T_{M/N,b}$. By $\overline{T_{M/N}(\epsilon)}$ we denote the closure of $T_{M/N}(\epsilon)$ in M. This gives us a fibration by open and closed balls as in 4.7.2.
Let $\mathcal{V}_N$ be the restriction of our local system $\mathcal{V}$ to the open subset $T_{M/N}(\epsilon)$, then we put $\mathcal{V}_N^! = i_{N,!}(\mathcal{V}_N)$, we have an inclusion $\mathcal{V}_N^! \hookrightarrow \mathcal{V}^!$ and therefore a homomorphism

$$H_c^\bullet(T_{M/N}(\epsilon),\mathcal{V}_N) = H_c^\bullet(\overline{T_{M/N}(\epsilon)},\mathcal{V}_N^!) = H^\bullet(M,\mathcal{V}_N^!) \longrightarrow H_c^\bullet(M,\mathcal{V}).$$

We are in the situation of 4.7.2 and get

$$H_c^{j+m-n}(\overline{T_{M/N}(\epsilon)},\mathcal{V}_N^!) \xrightarrow{\sim} H_c^j(N,R^{m-n}\pi_*(\mathcal{V}_N^!)),$$

and since obviously $\mathcal{V}' = R^{m-n}\pi_*(\mathcal{V}_N^!)$ we constructed our homomorphism. □

Now we assume that N is compact and that $\mathcal{V} =$. Then we get

$$H_c^j(N,\ \ ) \longrightarrow H_c^{m-n+j}(M,\ \ ),$$

and if $\pi_0(N)$ is the set of connected components of N we have the map

$$\bigoplus_{\pi_0(N)} \quad = H^0(N,\ \ ) \longrightarrow H_c^{m-n}(M,\ \ ).$$

If N is connected and compact then the image of 1 under this map is a class $[N] \in H_c^{m-n}(M,\ \ )$. It is called **the fundamental class** of N in M.
Let ω be a cohomology class on M which sits in the complementary degree $n = \dim N$, then we can restrict it by the inclusion map i to N. If N is connected and compact then $i^*(\omega) \in H^n(N,\ \ ) =$. Then we get

$$i^*(\omega) = [N] \cup \omega \in H^m(M,\ \ ) = \quad . \qquad (4.111)$$

This is essentially Proposition 4.7.3.

4.8.9 Cup Product and Intersections

Let us assume we have two oriented compact submanifolds N_1,N_2 of codimensions d_1,d_2 in our oriented manifold M. We get two classes $[N_1],[N_2]$ in the cohomology with compact support, they sit in degrees d_1,d_2. We want to understand the cup product of these two classes. Now we put $m = \dim M$.
We assume that our two submanifolds **intersect transversally**. This means that in any point p of $N_1 \cap N_2$ the intersection of the two tangent spaces $T_{N_1,p} \cap T_{N_2,p}$ has dimension $c := m - d_1 - d_2$. This implies that the intersection $N_1 \cap N_2$ is again a compact submanifold of codimension $d_1 + d_2$. It may have several connected components. We write

$$N_1 \cap N_2 = \bigcup C_j$$

where the C_j are the connected components. For any point $p \in C_j$ we get an exact sequence of tangent spaces

$$0 \longrightarrow T_{C_j,p} \longrightarrow T_{N_1,p} \oplus T_{N_2,p} \longrightarrow T_{M,p} \longrightarrow 0$$

where the arrow from the direct sum to the tangent space of M is given by: First component minus second component. This gives us an isomorphism

$$\bigwedge^{c}(T_{C_j,p}) \otimes \bigwedge^{m}(T_{M,p}) \xrightarrow{\sim} \bigwedge^{m-d_1}(T_{N_1,p}) \otimes \bigwedge^{m-d_2}(T_{N_2,p})$$

and this puts an orientation O_j on C_j for all j. Let $[C_j]$ be the fundamental class of the manifold C_j equipped with the orientation O_j. I claim

$$\sum [C_j] = [N_1] \cup [N_2] \tag{4.112}$$

We can look at the special case where $d_1 + d_2 = n$. In this case the cup product lands in $H^m_c(M,\_)$ and hence it is a number. If we keep the assumption of transversality then the intersection is a finite number of points. Now the tangent space of a point has always a canonical orientation. If now $c \in N_1 \cap N_2$ then we define

$$m(c) = \begin{cases} 1 & \text{if the orientation } O_c \text{ is canonical} \\ -1 & \text{if not} \end{cases} \tag{4.113}$$

Our formula becomes

$$[N_1] \cup [N_2] = \sum_{c \in N_1 \cap N_2} m(c) \tag{4.114}$$

It is purely local problem to verify these formulae. According to Equation 4.111 we have to restrict the class $[N_2]$ to N_1. We recall the construction of $[N_2]$, this class was the image of a class in $H^{m-d_2}_c(TN_2,\_) = H^{m-d_2}(M,\_^{!}_{N_2})$. If we restrict this class we get a class in $H^{m-d_2}(N_1,\_ \otimes \_^{!}_{N_2})$, the rest is clear. We could also refer directly to Proposition 4.7.3.

Lemma 4.8.12 (The Degree of Maps)**.** *Let us assume that M_1,M_2 are two compact and oriented manifolds of the same dimension d. Let $f : M_1 \longrightarrow M_2$ be a $\mathcal{C}^\infty$ map which has the following property: There is a point $x \in M_2$ such that the inverse image $f^{-1}(x)$ is finite and that for all $y \in f^{-1}(x)$ the derivative $D_{f,y} : T_{M_1,y} \longrightarrow T_{M_2,x}$ is an orientation preserving isomorphism. Under these conditions we have that the restriction map*

$$f^{(d)} : H^d(M_2,\_) = \quad \longrightarrow H^d(M_1,\_) =$$

is the multiplication by the cardinality $|f^{-1}(x)|$ of the fibre.

Proof: To see this we choose a small open ball $x \in D$ such that $f^{-1}(D)$ is a union of disjoint balls D_y around y, such that $f : D_y \longrightarrow D_x$ is a diffeomorphism. We get a commutative diagram

$$\begin{array}{ccc} H^d(M_2,\_) & \longrightarrow & H^d(M_1,\_) \\ \uparrow & & \uparrow \\ H^d_c(D_x,\_) & \longrightarrow & \sum_{y \in f^{-1}(x)} H^d_c(D_y,\_). \end{array}$$

The rest is clear, the generator $1 \in H_c^d(D_x,\underline{\ })$ is mapped to the generator in $H^d(M_2,\underline{\ })$ under the upwards arrow and to $(1,1,\ldots,1)$ under the horizontal arrow. The element $1 \in H_c^d(D_x,\underline{\ })$ is mapped to $|f^{-1}(x)|$ under the upwards arrow. □

The number $|f^{-1}(x)|$ is called the **degree of the map** f, we denote it by $\deg(f)$.

Of course this degree can be defined for $f : M_1 \longrightarrow M_2$, we simply define it by $f^{(d)}(\xi) = \deg(f)\xi$, where $0 \neq \xi \in H^d(M_2,\underline{\ })$ and both cohomology groups are identified to   via the orientations. We may ask to what extend this degree is always-without our assumption above- the number of points in a fibre $f^{-1}(x)$. We discuss a case, where this is true, but we have to count the points in the fibres with multiplicities.

Let us assume that we have a point $x \in M_2$ such that $f^{-1}(x)$ is a finite set. Then we can find a neighborhood $x \in V_x$, which is an open ball and neighborhoods W_y of the points $y \in f^{-1}(x)$, which are also open balls, such that $f : W_y \longrightarrow V_x$ and all these maps are proper. Then we get for any y a homomorphism

$$H_c^d(V_x, \ ) = \quad \longrightarrow H_c^d(W_y, \ ) = \quad ,$$

which given by multiplication by an integer $e(y)$. This integer may be zero or negative. Then the same argument as the one in the proof of the above lemma yields

$$\sum_{y \in f^{-1}(x)} e(y) = \deg(f). \tag{4.115}$$

4.8.10 Compact oriented Surfaces

Let S be a compact oriented 2-dimensional manifold, these objects are also called **(compact oriented) surfaces**. We have seen that for any ring R

$$H^0(S,\underline{R}) = R \qquad \text{and} \qquad H^2(S,\underline{R}) = R \tag{4.116}$$

and the only unknown cohomology sits in degree one. For any prime p we have the exact sequence of sheaves

$$0 \longrightarrow \underline{\ } \longrightarrow \underline{\ } \longrightarrow \underline{\mathbb{F}}_p \longrightarrow 0$$

and in the resulting long exact sequence we find the piece

$$0 \longrightarrow H^1(S,\underline{\ }) \longrightarrow H^1(S,\underline{\ }) \longrightarrow H^1(S,\underline{\mathbb{F}}_p) \longrightarrow 0.$$

We have zeroes at both ends because $H^0(S,\underline{\ }) \longrightarrow H^0(S,\underline{\mathbb{F}}_p)$ (resp. $H^2(S,\underline{\ }) \longrightarrow H^2(S,\underline{\ })$) is surjective (resp. injective). This implies that $H^1(S,\underline{\ })$ is torsion free. Since we also know that these cohomology groups are finitely generated we conclude that $H^1(S,\underline{\ })$ is free of some rank.

Now we have the Poincaré or cup product duality pairing

$$H^1(S,\underline{\ }) \times H^1(S,\underline{\ }) \longrightarrow \quad ,$$

which is non degenerate and alternating. A well known result from elementary algebra tells us that we can find a basis $e_1,\ldots,e_g,f_1,\ldots,f_g$ of $H^1(S,\underline{\ })$ such that the duality pairing is given by

$$e_i \cup f_j = \delta_{ij}, \qquad e_i \cup e_j = 0, \qquad f_i \cup f_j = 0. \tag{4.117}$$

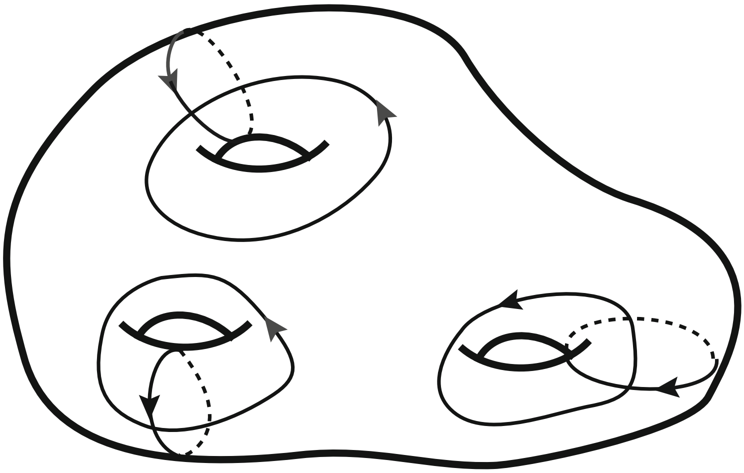


Figure 4.1 An example of a compact surface of genus 3

The number g is called the genus of the surface, the rank of $H^1(S,\underline{\ })$ is $2g$. For a surface of genus 3 we can draw the following picture.
We see three pairs of 1-cycles. They form a basis in homology. But we also may view these cycles as submanifolds isomorphic to S^1 which are oriented by the arrows. These submanifolds have fundamental classes e_1,f_1,e_2,f_2,e_3,f_3 in $H^1(S,\underline{\ })$ and if we numerate them in the right way we have the above values of the intersection pairing.

4.8.11 The Cohomology Ring of $\mathbb{P}^n(\mathbb{C})$

We are now able to determine the structure of the cohomology ring $H^\bullet(\mathbb{P}^n(\mathbb{C}),\underline{\ })$ (see Exercise 26). The fundamental class of any hyperplane $L \simeq \mathbb{P}^{n-1}(\mathbb{C}) \subset \mathbb{P}^n(\mathbb{C})$ gives us a multiple of the generator $ae_1 \in H^2(\mathbb{P}^n(\mathbb{C}),\underline{\ })$. (Since we are dealing with complex manifolds, all manifolds have a canonical orientation.) Now we can put n such hyperplanes in general position such that they intersect transversally and $L_1 \cap \dots \cap L_n$ is a point. The fundamental class of a point is the generator in $H^{2n}(\mathbb{P}^n(\mathbb{C}),\underline{\ })$. We conclude that $a^n e_1 \cup e_1 \dots e_1$ is this generator, it follows that $a = 1$, e_1 is the fundamental class of the hyperplane and

$$H^\bullet(\mathbb{P}^n(\mathbb{C}),\underline{\ }) = \quad [e_1]/(e_1^{n+1}). \tag{4.118}$$

4.9 The Lefschetz Fixed Point Formula

Let M be a connected, compact and oriented manifold of dimension d. Let $f : M \longrightarrow M$ be a continuous map. It induces endomorphisms $f^\nu : H^\nu(M,\underline{\mathbb{Q}}) \longrightarrow H^\nu(M,\underline{\mathbb{Q}})$. The Lefschetz fixed point formula gives us an expression for the alternating sum of traces (the **Lefschetz number** of f)

$$\operatorname{tr}\left(f^\bullet | H^\bullet(M,\underline{\mathbb{Q}})\right) = \sum_{\nu=0}^{d} (-1)^\nu \operatorname{tr}(f^\nu | H^\nu(M,\underline{\mathbb{Q}})) \tag{4.119}$$

in terms of local data at the fixed points of f on M. I formulate a precise version in the case of isolated fixed points and give some indications how this is proved. Actually it is a rather formal consequence of our previous considerations on the cup product, the Poincaré duality and the Künneth formula.

We consider the graph $\Gamma_f = \{(x,f(x)) \subset M \times M | x \in M\}$ of f. It is a submanifold of $M \times M$ and it is isomorphic to M via the projection to the first coordinate. We give it the orientation of M. Hence it defines a cohomology class $[\Gamma_f] \in H^d(M \times M,\underline{\mathbb{Q}})$. The fixed point formula will come out if we compute the cup product of the class $[\Gamma_f]$ and the class of the diagonal $\Delta = \Gamma_{\mathrm{Id}}$ in two different ways.

We apply the Künneth homomorphism (see page 109), since we have rational coefficients we get an isomorphism

$$H^d(M \times M,\underline{\mathbb{Q}}) \xrightarrow{\sim} \bigoplus_{\nu=0}^{d} \left(H^\nu(M,\underline{\mathbb{Q}}) \times H^{d-\nu}(M,\underline{\mathbb{Q}})\right).$$

The cup product yields a non degenerate pairing $H^\nu(M,\underline{\mathbb{Q}}) \times H^{d-\nu}(M,\underline{\mathbb{Q}}) \longrightarrow \mathbb{Q}$, hence we get isomorphisms

$$H^\nu(M,\underline{\mathbb{Q}}) \times H^{d-\nu}(M,\underline{\mathbb{Q}}) \xrightarrow{\sim} \mathrm{End}(H^\nu(M,\underline{\mathbb{Q}})),$$

which are given by

$$u^{(\nu)} \otimes u^{(d-\nu)} \mapsto \left\{v^{(\nu)} \mapsto (u^{(d-\nu)} \cup v^{(\nu)})u^{(\nu)}\right\}.$$

It is a formal consequence of our definitions that in fact

$$[\Gamma_f] = \sum_{\nu=0}^{d} f^\nu \qquad \in \qquad \bigoplus_{\nu} \mathrm{End}(H^\nu(M,\underline{\mathbb{Q}})).$$

The diagonal $\Delta \subset M \times M$ is the graph of the identity. A little bit of linear algebra shows that the cup product of the classes $f^p \in H^p(M,\underline{\mathbb{Q}}) \otimes H^{d-p}(M,\underline{\mathbb{Q}})$ and $\mathrm{Id}^{d-p} \otimes \mathrm{Id}^p$ is given by

$$f^p \cup \mathrm{Id}^{n-p} = (-1)^p \,\mathrm{tr}(f^p). \tag{4.120}$$

We conclude that

$$[\Gamma_f] \cup [\Delta] = \mathrm{tr}(f^\bullet | H^\bullet(M,\mathbb{Q})). \tag{4.121}$$

Now we compute the cup product by interpreting it as an intersection number (see section 4.8.9). The points in the intersection of the two graphs are exactly the fixed points of our map, i.e. $\mathrm{Fix}(f) = \{x \in M | f(x) = x\}$. Here we assume that the fixed points of f are isolated, i.e. that f has finitely many fixed points and the graphs Γ_f and Δ intersect transversally. We have the two derivatives which send the tangent space $T_{M,x}$ at x to the tangent space at (x,x): The first one $D_{\mathrm{Id},f,x}$ sends a tangent vector $t \in T_{M,x}$ to $(t,D_{f,x}(t))$ and the second one $D_{\mathrm{Id},\mathrm{Id},x}$ does $t \mapsto (t,t)$.

Transversality means that we get a direct sum decomposition

$$T_{M\times M,(x,x)} = D_{\mathrm{Id}\,,f,x}(T_{M,x}) \oplus D_{\mathrm{Id}\,,\,\mathrm{Id}\,,x}(T_{M,x}).$$

We have to compare the given orientation on $T_{M\times M,(x,x)}$ to the orientation obtained from the orientation on the direct sum $D_{f,x}(T_{M,x}) \oplus D_{\mathrm{Id}\,,x}(T_{M,x})$, as I explained in section 4.8.9 this induces a sign $m(x)$. The derivative D_f of f at the fixed point x induces an endomorphism of the tangent space $T_{M,x}$ and the assumption that x is isolated implies $\det(\mathrm{Id}\, - D_{f,x}|T_{M,x}) \neq 0$. Now it is easy to see that this sign is equal to

$$s_f(x) = \mathrm{sign}(\det(\mathrm{Id}\, - D_{f,x}|T_{M,x})). \tag{4.122}$$

Hence we proved the fixed point formula for an f with isolated fixed points

$$\mathrm{tr}(f^\bullet|H^\bullet(M,\mathbb{Q})) = \sum_{x\in \mathrm{Fix}(f)} s_f(x). \tag{4.123}$$

Actually it is not difficult to derive a more general fixed point formula for cohomology with coefficients in a local system. Let M be as above and $\mathcal{V}$ a local system of finite dimensional vector spaces over some field k on M. A differentiable map $f : M \longrightarrow M$ gives us a homomorphism $f^q : H^q(M,\mathcal{V}) \longrightarrow H^q(M,f^*(\mathcal{V}))$. (See 4.4.3.) If we now have as an extra datum a homomorphism of sheaves $g : f^*(\mathcal{V}) \longrightarrow \mathcal{V}$, then we get a composition

$$(f^q,g) : H^q(M,\mathcal{V}) \longrightarrow H^q(M,\mathcal{V}).$$

Again we can define the Lefschetz number

$$\mathrm{tr}((f^\bullet,g)|H^\bullet(M,\mathcal{V})) = \sum_{\nu=0}^{d} (-1)^\nu \,\mathrm{tr}\big((f^\nu,g)|H^\nu(M,\mathcal{V})\big). \tag{4.124}$$

Now at a fixed point $x \in M$ our g gives us an endomorphism $g(x) : \mathcal{V}_x = f^*(\mathcal{V})_x \longrightarrow \mathcal{V}_x$. Then we get under the same assumption of transversality the formula

$$\mathrm{tr}((f^\bullet,g)|H^\bullet(M,\mathcal{V})) = \sum_{x\in \mathrm{Fix}(f)} s_f(x)\,\mathrm{tr}(g(x)). \tag{4.125}$$

The proof is essentially the same as in the case of trivial coefficients.

4.9.1 The Euler Characteristic of Manifolds

I recall the situation on page 104, I want to give some brief indications what is going on. Let us assume that we have a $\mathcal{C}^\infty$-vector field X on M, this is simply a $\mathcal{C}^\infty$-section in the tangent bundle. It follows from the theory of differential equations, that we can find a one parameter group $g_t = \exp(tX)$ of diffeomorphisms $g_t : M \longrightarrow M$ such that

$$\left(\frac{d}{dt} g_t\right)_{t=0}(x) = X(x) \;\text{ for all }\; x \in M. \tag{4.126}$$

We assume that this vector field has only isolated zeroes and that these zeroes are non degenerate.

Definition 4.9.1. *If m_0 is a zero of the vector field, then we can choose local coordinates at x_0 such that $X = \sum f_i \frac{\partial}{\partial x_i}$, where all the f_i vanish at m_0. A zero is called* **non degenerate** *if the matrix*

$$\Big(\frac{\partial^2 f}{\partial x_i \partial x_j}(m_0)\Big)_{ij}$$

has non zero determinant. In this case we define the index $\mathrm{ind}(X)(m_0)$ *of the vector field at m_0 to be $(-1)^n$ times the sign of the determinant of this matrix.*

If we now apply the Lefschetz fixed point formula to g_t for sufficiently small values of t, then the fixed points are exactly the zeroes of X and a simple calculation in local coordinates shows that for a fixed point m_0 we have the equality $\mathrm{ind}(X)(m_0) = s_{g_t}(m_0)$. On the other hand it is clear that the diffeomorphism g_t is homotopic to the identity. Therefore the sum of the alternating traces of g_t on the cohomology is equal to the Euler characteristic of M. Hence we get

Theorem 4.9.2 (Lefschetz fixed point formula for the identity)**.** *If M is a compact oriented $\mathcal{C}^\infty$-manifold and if X is a $\mathcal{C}^\infty$-vector field with only isolated non degenerate zeroes, then*

$$\chi(M) = \sum_{m_0 \in \text{zeroes of } X} \mathrm{ind}(X)(m_0).$$

This formula should be interpreted as the Lefschetz fixed point formula for the map $f = \mathrm{Id}$. If we try to carry over the computation of section 4.9 to this situation, then the graph of Id is the diagonal Δ and clearly we have $[\Delta] \cup [\Delta] = \chi(M)$. But now we have the problem that we can not interpret the value of the cup product as an intersection number, at least we can not interpret it as a finite sum of contributions over fixed points. If we find a vector field with isolated non degenerate zeroes, then we use it to deform the first factor in the product $[\Delta] \cup [\Delta]$ and replace it by the graph of g_t. The fundamental class of the graph Γ_{g_t} is equal to $[\Delta]$, but now we may apply section 4.9.

4.10 The de Rham and the Dolbeault Isomorphism

4.10.1 The Cohomology of Flat Bundles on Real Manifolds

Let M be a $\mathcal{C}^\infty$-manifold and let $\mathcal{V}$ be a local system consisting of finite dimensional - or $\mathbb{C}$-vector spaces. (See 4.3.3.) Let us denote the dimension of M by m, let n be the dimension of the vector spaces in the local system. Locally on small connected open subsets $U \subset M$ we have a trivialization of $\mathcal{V}$ by constant sections $e_1, \cdots, e_n$ and

$$\mathcal{V}(U) = \left\{ \sum_{i=1}^{n} a_i e_i \mid a_i \in \quad \right\}. \tag{4.127}$$

We define

$$\mathcal{V}_\infty(U) = \left\{ \sum_{i=1}^{n} f_i e_i \mid f_i \in \mathcal{C}^\infty(U) \right\}, \tag{4.128}$$

and this gives us the sheaf of $\mathcal{C}^\infty$-sections in $\mathcal{V}$.
Let Ω^p_M be the sheaf of $\mathcal{C}^\infty$-p-forms on M. We can define a differential

$$d : \mathcal{V}_\infty(U) \longrightarrow \mathcal{V}_\infty(U) \otimes \Omega^1_M(U)$$

by

$$d : \sum_{i=1}^n f_i e_i \longmapsto \sum_{i=1}^n e_i \otimes df_i.$$

If we pass to another open set U' and if we choose a trivialization $e'_1, \cdots, e'_n$ over U' then we get expressions

$$e_i = \sum a_{ij} e'_j$$

over $U \cap U'$ where the a_{ij} are locally constant. Therefore it is clear that the definition of the differential does not depend on the choice of the constant sections. Thus we see that we can define a global differential

$$d : \mathcal{V}_\infty \longrightarrow \mathcal{V}_\infty \otimes \Omega^1_M.$$

It is clear from the definition that for any open set $U_1 \subset M$

$$\mathcal{V}(U_1) = \{s \in \mathcal{V}_\infty(U_1) \mid ds = 0\}.$$

We can extend our differential to forms of higher degree

$$s : \mathcal{V}_\infty \otimes \Omega^p_M \longrightarrow \mathcal{V}_\infty \otimes \Omega^{p+1}_M$$

by

$$d(\sum s_i \otimes \omega_i) = \sum s_i \otimes d\omega_i + \sum ds_i \wedge \omega_i,$$

where ds_i is of the form $ds_i = \sum g_{ij} \otimes \omega'_j$ and hence $ds_i \wedge d\omega_i = \sum_i g_{ij} \otimes \omega'_j \wedge \omega_i$. It is well known that $dd = 0$. We recall some rules for the exterior derivatives of differential forms: In local coordinates we have $d(f(x_1,x_2,\dots,x_d)dx_1 \wedge \dots \wedge dx_p) = df \wedge dx_1 \wedge \dots \wedge dx_p$ and from this we get easily $d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^p \omega_1 \wedge d\omega_2$ where $p = \deg(\omega_1)$.
We get the so called **de Rham complex** of sheaves

$$0 \longrightarrow \mathcal{V} \longrightarrow \mathcal{V}_\infty \longrightarrow \mathcal{V}_\infty \otimes \Omega^1_M \longrightarrow \dots \longrightarrow \mathcal{C}^\infty(\mathcal{V}) \otimes \Omega^m_M \longrightarrow 0.$$

We introduce the notation $\mathcal{V}_\infty \otimes \Omega^p_M = \Omega^p_\infty(\mathcal{V})$. A form $\omega \in \Omega^p_\infty(\mathcal{V})(U)$ is called *closed* if $d\omega = 0$.

Definition 4.10.1. *If we have a $\mathcal{C}^\infty$-vector bundle $\mathcal{E}$ over M (see 4.3.1, here the g_{ij} have to be $\mathcal{C}^\infty$-functions), then we may consider differentials*

$$d : \mathcal{C}^\infty(\mathcal{E}) \longrightarrow \mathcal{C}^\infty(\mathcal{E}) \otimes \Omega^1_M$$

which satisfy

$$\begin{aligned} d(s_1 + s_2) &= ds_1 + ds_2 \\ d(fs_1) &= f ds_1 + s_1 \otimes df \end{aligned}$$

for local sections s_1,s_2 and local $\mathcal{C}^\infty$-functions f. Such differentials are called **connections** *on $\mathcal{E}$. A connection is called a* **flat connection** *if*

$$\begin{aligned} d(ds_1) &= d\left(\sum s_i \otimes \omega_j\right) \\ &= \sum s_j \otimes d\omega_j + ds_i \wedge \omega_j = 0. \end{aligned}$$

We saw that starting from a local coefficient system $\mathcal{V}$ we have a canonical flat connection on $\mathcal{V}_\infty$. But in turn, if we have a flat connection, then we can attach a local system $\mathcal{E}_0$ to $\mathcal{E}$ by defining

$$\mathcal{E}_0(U) = \{s \in \mathcal{C}^\infty(\mathcal{E})(U) \mid ds = 0\}.$$

It is of course clear that the flatness of the connection is necessary for the construction of the de Rham complex.
It is not very hard to see that

Lemma 4.10.2. *The de Rham complex is exact.*

This follows from the well-known Lemma of Poincaré which says the following:

Lemma 4.10.3 (Poincaré)**.** *A closed p-form ω of degree $p > 0$ on a convex open set $U \subset {}^m$ can be written as $d\psi = \omega$ with $\psi \in \Omega_\infty^{p-1}(U)$. (See [B-T], Chap. I, §6.)*

We can apply this here because our local system $\mathcal{V}$ is locally trivial. Therefore the de Rham complex gives us a resolution of the sheaf $\mathcal{V}$.
I claim that this resolution is also acyclic, we have

$$H^i(M,\Omega_\infty^p(\mathcal{V})) = 0 \qquad \text{for } i \geq 1 \text{ and all } p \geq 0. \tag{4.129}$$

To see that this is the case we apply Exercise 16. We have (see 4.8.2) a partition of unity for the sheaf $\mathcal{C}_M^\infty$. If we have any $\mathcal{C}^\infty$-vector bundle $\mathcal{E}$ on M then the sheaf $\mathcal{C}^\infty(\mathcal{E})$ is a sheaf of $\mathcal{C}_M^\infty$ modules. Then our Exercise 16 yields that the higher cohomology groups $\mathcal{C}^\infty(\mathcal{E})$ vanish.
We apply the functor global sections and then the resulting complex of global differential forms computes the cohomology (see section 2.3.1). Hence we get the famous

Theorem 4.10.4 (de Rham)**.** *We have an isomorphism*

$$H^i(M,\mathcal{V}) \cong H^i(\Omega_\infty^\bullet(\mathcal{V})(M)) = \frac{\{\omega \in \Omega_\infty^p(\mathcal{V})(M) | d\omega = 0\}}{\{d\psi | \psi \in \Omega_\infty^{p-1}(\mathcal{V})(M)\}}.$$

This is called **de Rham Isomorphism***.*

If we consider the same complex but with sections which have compact support, then we get (see second example below)

$$0 \longrightarrow (\mathcal{V}_\infty)_c(M) \longrightarrow \Omega_\infty^1(\mathcal{V})_c(M) \longrightarrow \dots \longrightarrow \Omega_\infty^m(\mathcal{V})_c(M) \xrightarrow{\int} \mathcal{V}(M) \longrightarrow 0.$$

The argument in Exercise 16 applies as well to the cohomology with compact supports and we get by the same token

$$H_c^i(M,\mathcal{V}) = H^i(\Omega_{\infty,c}^\bullet(\mathcal{V})(M)), \tag{4.130}$$

provided we have the appropriate form of the Lemma of Poincaré (see the example below).

Example 20.

(a) *If for instance we take the trivial system $\underline{\ }$ on $M = {}^m$ then a closed form ω of degree $p > 0$ on ^{m} can be written as $d\psi$ with $\psi \in \Omega_\infty^{p-1}({}^m)$. If $p = 0$ then a closed form is a constant function $f = c \neq 0$, then we can not write it as $f = d\psi$, because the space of forms of degree -1 is zero. Thus we get $H^0({}^m,\underline{\ }) =$ and $H^i({}^m,\underline{\ }) = 0$ for $i > 0$. (See 4.4.24.)*

(b) *If we consider the cohomology with compact supports then a closed form in degree zero which has compact support must vanish. Hence we get* $H_c^0(\ \ ^m,\_) = 0$*. But if we have a form* ω *with compact support on* $\ \ ^m$ *which is of degree* m *then we may not be able to find a* ψ *with compact support such that* $\omega = d\psi$*. If we could do so we would have*

$$\int_{\mathbb{R}^m} \omega = \int_D \omega = \int_{\partial D} \psi = 0 \tag{4.131}$$

where D *is a big closed ball which contains the supports of* ω *and* ψ*. Hence we get a surjective linear form*

$$\begin{aligned} \mathrm{int} : H_c^d(\ \ ^m,\_) &\longrightarrow \\ [\omega] &\longmapsto \int_{\mathbb{R}^m} \omega. \end{aligned} \tag{4.132}$$

It is easy to see that a form ω *with compact support for which in addition* $\int_{\mathbb{R}^m} \omega = 0$ *can be written as* $\omega = d\psi$ *with* $\psi \in \Omega_c^{d-1}(\ \ ^m)$*. We get that the above map* int *is an isomorphism. If we take the entire de Rham complex with compact supports*

$$0 \longrightarrow (\_{}_\infty)_c(\ \ ^m) \longrightarrow \Omega^1_\infty(\_)_c(\ \ ^m) \longrightarrow \ldots \longrightarrow \Omega^d_\infty(\_)_c(\ \ ^m) \xrightarrow{\int} \quad \longrightarrow 0,$$

then it is easy to see that it is exact in degrees $< d$*, i.e. we have a Lemma of Poincaré for forms with compact support except in the top degree. Comparing this to (a) above gives us the simplest version of Poincaré duality.*

The Product Structure on the de Rham Cohomology

We want to discuss the product structure of the cohomology in the context of the de Rham isomorphism. If we have two manifolds M and N, then the resolutions of the sheaf $\_$ by the two de Rham complexes are flat (comp. the discussion in 4.6.7.). If we consider the product $M \times N$ and the two projections p_1,p_2, then we have a homomorphism of complexes

$$p_1^*(\Omega^\bullet_M) \otimes_{\mathbb{R}} p_2^*(\Omega^\bullet_N) = \Omega^\bullet_M \widehat{\otimes}_{\mathbb{R}} \Omega^\bullet_N \longrightarrow \Omega^\bullet_{M\times N}$$

which is given by the exterior multiplication of the differential forms. Hence it is clear that the product

$$\begin{aligned} H^p(M,\_) \times H^q(N,\_) &\longrightarrow H^{p+q}(M\times N,\_) \\ (\alpha,\beta) &\longrightarrow \alpha \widehat{\otimes}_{\mathbb{R}} \beta \end{aligned}$$

is induced by the exterior multiplication of the differential forms which represent the classes α,β.

Especially it becomes clear that the cup product on $H^\bullet(M,\_)$ is induced by the structure of an exterior algebra on the differential forms.

If we have a local system $\mathcal{V}$ of finite dimensional -vector spaces and its dual $\mathcal{V}^\vee$, then we have the evaluation $e : \mathcal{V} \otimes \mathcal{V}^\vee \longrightarrow$ and we get a pairing

$$H_c^i(M,\mathcal{V}) \times H^{m-i}(M,\mathcal{V}^\vee) \longrightarrow H_c^m(M,\mathcal{V}\otimes\mathcal{V}^\vee) \longrightarrow H_c(M,\_).$$

Theorem 4.10.5. *If M is a connected and oriented manifold, then we get the Poincaré duality pairing which on two classes $[\omega] \in H_c^i(M,\mathcal{V})$ and $[\eta] \in H^{m-i}(M,\mathcal{V}^\vee)$, which are represented by ω and η, is given by*

$$[\omega] \times [\eta] \longrightarrow \int_M e(\omega \wedge \eta).$$

If we take this as definition for the Poincaré duality pairing it is not so clear why it is non degenerate. We come back to this point in section 4.11.

The de Rham Isomorphism and the fundamental class

The de Rham isomorphism also provides a different way of looking at the notion of the fundamental class and the formulae for the cup product (see 4.8.8, 4.8.9). Let us consider an open ball $D^m \subset M$ in our connected, oriented manifold of dimension m. We assume it to be oriented. If we remove the origin p from D, then we have a diffeomorphism

$$D^m \setminus \{0\} \simeq (0,1) \times S^{m-1}$$

which is given by

$$(x_1,\cdots,x_m) \longrightarrow \left(\sqrt{x_1^2+\cdots+x_m^2}, \left(\frac{x_1}{\sqrt{x_1^2+\cdots+x_m^2}},\cdots,\frac{x_m}{\sqrt{x_1^2+\cdots+x_m^2}}\right)\right) = (r,\varphi).$$

On the oriented sphere S^{m-1} we have a unique differential form ω in degree $m-1$ which is invariant under the orthogonal group $\mathrm{SO}(m)$, and which satisfies

$$\int_{S^{m-1}} \omega = 1.$$

Now we choose a $\mathcal{C}^\infty$-function $h(r)$ which is identically equal to one if r is close to zero and identically equal to zero if r is close to one. This provides the differential form

$$h(r)\omega = \psi$$

on $D^m \setminus \{0\}$. If we take its **exterior derivative**

$$d\psi = \frac{\partial h(r)}{\partial r} \cdot dr \wedge \omega = \widetilde{\omega}, \tag{4.133}$$

then $\widetilde{\omega}$ is a form on $D^m \setminus \{0\}$ which vanishes identically in a small open ball around zero and near the boundary of D^m. Therefore we can extend it to a differential form on M and clearly we have

$$\int_M \widetilde{\omega} = 1.$$

Thus we constructed a form which represents the canonical generator in $H_c^m(M,\ )$, it is also the fundamental class of the submanifold $\{p\}$.

Proposition 4.10.6. *We see that we can represent the fundamental class of our connected oriented manifold M by a differential form which has its support in a* shell *around an arbitrary point, here I mean by a shell the difference set between a larger small ball and a smaller small ball around p.*

Let us assume that M is an oriented manifold and $N \subset M$ is an oriented submanifold, let n,m be their respective dimensions. We have a similar interpretation of the fundamental class (see 4.8.8) of N by differential forms which have their support in a bundle of shells. We construct a tubular neighborhood $T_{M/N}(\epsilon)$ of N such that we have the projection

$$\pi : T_{M/N}(\epsilon) \longrightarrow N$$

and such that locally in N we have

$$\begin{array}{ccc} \pi^{-1}(V) & \xrightarrow{\sim} & V \times D^{m-n} \\ \downarrow & & \\ V. & & \end{array}$$

On $V \times D^{m-n}$ we construct a $m-n$-form $\widetilde{\omega}_V$ which is the pullback of a form $\widetilde{\omega}$ on D^{m-n} which is constructed as above.
Now we choose a covering $N = \bigcup_{i\in I} V_i$ which is locally finite and which trivializes $\pi : TN(\epsilon) \longrightarrow N$, and we choose a partition of unity $1 = \sum h_i$ with $\mathrm{Supp}(h_i) \subset V_i$. On each $\pi^{-1}(V_i)$ we construct $\widetilde{\omega}_i$ and we put

$$\widetilde{\omega}_N = \sum h_i \widetilde{\omega}_i.$$

For any point in $x \in TN(\epsilon)$ we have

$$(d\widetilde{\omega}_N)_x = \sum (dh_i)_x \wedge \widetilde{\omega}_i - d(\Sigma h_i)_x \wedge \widetilde{\omega}_i = 0,$$

and we see that $\widetilde{\omega}_N \in \Omega^{m-n}(M)$ is a closed form. It is clear that this form represents the fundamental class

$$[N] \in H^{m-n}(M, \quad).$$

If now N_1,N_2 are two oriented submanifolds in M, and if we assume that one of them is compact, then we have the two classes

$$[N_1] = [\widetilde{\omega}_{N_1}], [N_2] = [\widetilde{\omega}_{N_2}]$$

where one of the forms has compact support. We just saw that

$$[N_1] \cup [N_2] = \widetilde{\omega}_{N_1} \wedge \widetilde{\omega}_{N_2}.$$

If now these two submanifolds are of complementary dimension, and if they intersect transversally, then it is easy to see that

$$\int_M \widetilde{\omega}_{N_1} \wedge \widetilde{\omega}_{N_2} = \sum_{c \in N_1 \cap N_2} \int_{D(c)} \widetilde{\omega}_{N_1} \wedge \widetilde{\omega}_{N_2}$$

where $D(c)$ is a small ball containing the local support of $\widetilde{\omega}_{N_1} \wedge \widetilde{\omega}_{N_2}$. It is easy to verify that these contributions from the points are equal to $m(c)$ (see (4.113) for the definition of $m(c)$).

4.10.2 Cohomology of Holomorphic Bundles on Complex Manifolds

Let M be a complex manifold (see 3.2) of complex dimension m. From our discussion in section 4.3.2 it is rather clear what a holomorphic vector bundle $\mathcal{E}$ of rank n on M is. This is of course a bundle $\pi : \mathcal{E} \longrightarrow M$ of $\mathbb{C}$-vector spaces for which the transition functions $g_{ij} : V_i \cap V_j \longrightarrow GL(n,\mathbb{C})$ are holomorphic. It follows from our general principles in 4.3.3 that the holomorphic vector bundles are classified by $H^1(M,GL_n(\mathcal{O}_M))$ where $GL_n(\mathcal{O}_M)$ is the sheaf of holomorphic functions from M to $GL_n(\mathbb{C})$.
To such a holomorphic vector bundle $\mathcal{E}$ we have the sheaf of germs of holomorphic sections, which will be denoted by the same letter. This sheaf will be a locally free $\mathcal{O}_M$-module and in turn a locally free $\mathcal{O}_M$-module gives a holomorphic vector bundle. Of course we can forget the complex structure, we also have the sheaf $\mathcal{O}_{M_\infty}$ of $\mathcal{C}^\infty$ sections on M. If we speak about the $\mathcal{C}^\infty$-manifold $(M,\mathcal{O}_{M_\infty})$ we also denote it by M_∞.
Now we can define the sheaf $\mathcal{E}_\infty$ of $\mathcal{C}^\infty$-sections in the bundle, we have the inclusion of sheaves $\mathcal{E} \hookrightarrow \mathcal{E}_\infty$.
The following discussion will show that considering the pair $(\mathcal{E},\mathcal{E}_\infty)$ is completely analogous to the concept of local systems $(\mathcal{V},\mathcal{V}_\infty)$ (see Remark 3).

The Tangent Bundle

We pick a point $p \in M$ and an open neighborhood U_p of p such that

$$(U_p,\mathcal{O}_{M|U_p}) \simeq (D_p,\mathcal{O}_{D_p}),$$

where D_p is an open ball in $\mathbb{C}^m$ whose center is $p = (0,\dots,0)$. The tangent bundle T_M is of course a holomorphic bundle which over U_p can be trivialized by the derivations $\frac{\partial}{\partial z_1},\dots,\frac{\partial}{\partial z_m}$. We write the complex coordinates by their real and imaginary parts

$$(z_1,\dots,z_m) = (x_1 + iy_1,\dots,x_m + iy_m).$$

Then the tangent bundle T_{M_∞} of the $\mathcal{C}^\infty$-manifold M_∞ has a basis – locally at p – which is given by

$$\frac{\partial}{\partial x_1},\frac{\partial}{\partial y_1},\dots,\frac{\partial}{\partial x_m},\frac{\partial}{\partial y_m}.$$

These sections are only sections in T_{M_∞}. This bundle of $2m$-dimensional real vector spaces has the structure of a bundle of m-dimensional complex vector spaces where locally the multiplication by i is given by

$$I : \begin{cases} \frac{\partial}{\partial x_1} \longmapsto \frac{\partial}{\partial y_1}; & \frac{\partial}{\partial y_1} \longmapsto -\frac{\partial}{\partial x_1} \\ \qquad\vdots & \qquad\vdots \\ \frac{\partial}{\partial x_m} \longmapsto \frac{\partial}{\partial y_m}; & \frac{\partial}{\partial y_m} \longmapsto -\frac{\partial}{\partial x_m}. \end{cases}$$

We have a privileged orientation on the underlying $\mathcal{C}^\infty$-manifold which is determined by requiring that $dx_1 \wedge dy_1 \wedge \dots dx_m \wedge dy_m$ is positive.
We can take the tensor product

$$T_{M_\infty} \otimes_{\mathbb{R}} \mathbb{C} = T_M, \tag{4.134}$$

and get a bundle of $2m$-dimensional complex vector spaces. On this bundle of complex $2m$-dimensional vector spaces we still have the linear transformation I above and $T_{M,}$ decomposes into two eigenspaces which are the eigenspaces with eigenvalues i and $-i$ for I:

$$T_{M_\infty,} \quad = T^{1,0}_{M,} \;\oplus T^{0,1}_{M,} \;, \tag{4.135}$$

where $T^{1,0}_{M,}$ is the eigenspace for the eigenvalue i for I and $T^{0,1}_{M,}$ is the eigenspace for the eigenvalue $-i$ for I. It is easy to see that locally on M the bundle $T^{1,0}_{M,}$ has the basis (fibre by fibre)

$$\begin{aligned}\frac{\partial}{\partial z_1} &= 1 \otimes \frac{\partial}{\partial x_1} - i \otimes \frac{\partial}{\partial y_1}\\ &\vdots\\ \frac{\partial}{\partial z_m} &= 1 \otimes \frac{\partial}{\partial x_m} - i \otimes \frac{\partial}{\partial y_m}.\end{aligned}$$

This provides a structure of a holomorphic vector bundle on $T^{1,0}_{M,}$, the local trivialization is given by the above basis. We say that I induces a **complex structure**. The composition map $T_M \longrightarrow T_M \otimes \mathbb{C} \longrightarrow T^{1,0}_{M,}$ induces an isomorphism of complex vector bundles. The composition $T_M \longrightarrow T_M \otimes \mathbb{C} \longrightarrow T^{0,1}_{M,}$ is an antilinear isomorphism.
Here we apply some very simple principles of linear algebra which can be confusing and their application requires some care.
If we have a $\mathbb{C}$-vector space V, we may define the complex conjugate space $\overline{V}$. Its underlying abelian group is V but the scalar multiplication

$$\mathbb{C} \times \overline{V} \longrightarrow \overline{V}$$

is given by

$$(z,v) \longmapsto \overline{z} \cdot v,$$

where the dot on the right hand side denotes the scalar multiplication of $v \in V$ by $\overline{z} \in \mathbb{C}$. Hence we see that the identity map $\mathrm{Id} : V \longrightarrow \overline{V}$ is antilinear.
If we consider our complex vector space V over $\mathbb{C}$ as a real vector space together with a linear transformation I with $I^2 = -\mathrm{Id}$, then we can extend I to a linear transformation on $V \otimes_{\mathbb{R}} \mathbb{C}$ and decomposes into the eigenspace $V^{1,0}$ and $V^{0,1}$ of I with eigenvalues $\pm i$. The vector spaces V,$\overline{V}$ considered as real vector spaces are isomorphic by the identity map. In the following diagram the compositions of the horizontal maps

$$\begin{array}{ccccc} V & \longrightarrow & V \otimes_{\mathbb{R}} \mathbb{C} & \xrightarrow{\mathrm{pr}_{1,0}} & V^{1,0} \\ \big\downarrow \mathrm{Id} & & & & \\ \overline{V} & \longrightarrow & V \otimes_{\mathbb{R}} \mathbb{C} & \xrightarrow{\mathrm{pr}_{0,1}} & V^{0,1} \end{array}$$

are isomorphisms of $\mathbb{C}$-vector spaces.
The thing that may cause confusion is the following fact: On $V \otimes_{\mathbb{R}} \mathbb{C}$ we have the complex conjugation on the coefficients which may also be denoted by $v \longrightarrow \overline{v}$. vfill

Then we get obviously

$$\overline{V^{1,0}} = V^{0,1}, \tag{4.136}$$

but now putting a bar on $V^{1,0}$ has a different meaning, we get a different underlying set in contrast to our convention above.
On the other hand we can say that we constructed canonical isomorphisms

$$V \xrightarrow{\sim} V^{1,0}$$
$$\overline{V} \xrightarrow{\sim} V^{0,1}$$

which allow us to identify V to $V^{1,0}$ and $\overline{V}$ to $V^{0,1}$. If we insert the map given by complex conjugation on the right end of our diagram above, then we get a commutative diagram and the inconsistency in notation dissolves.
Here I want to introduce a simplification in the notation. Instead of $V^{0,1}$,$V^{1,0}$ I will write $V^{0,1}$,$V^{1,0}$. The double superscript indicates already that these spaces lie in the complexification of a tensor product of a complex vector space over with $\mathbb{C}$, so the subscript is redundant.

The Bundle $\Omega^{p,q}_M$

We can form the dual bundle Ω^1_M of T_M. Attached to this bundle we have the sheaf of $\mathcal{C}^\infty$-sections in this bundle which is denoted by $\Omega^1_{M_\infty}$. We have a decomposition

$$\Omega^1_{M_\infty,\ } = \Omega^1_{M_\infty} \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{\sim} \Omega^{1,0}_M \oplus \Omega^{0,1}_M. \tag{4.137}$$

The sheaf Ω^1_M is locally generated by $dz_1, \ldots ,dz_m$, we have

$$< dz_\nu, \frac{\partial}{\partial x_\mu} - i \otimes \frac{\partial}{\partial y_\mu} > = 2\delta_{\nu\mu} \tag{4.138}$$
$$< dz_\nu, \frac{\partial}{\partial x_\mu} + i \otimes \frac{\partial}{\partial y_\mu} > = 0.$$

We can define the fibres of $\Omega^1_{M_\infty,\ }$ at a point p simply as

$$\Omega^1_{M_\infty,\ ,p} = \mathrm{Hom}_{\mathbb{R}}(T_{M,p},\mathbb{C}) \tag{4.139}$$

and then $\Omega^{1,0}_{M,p} = \{\omega | \omega(It_p) = i\omega(t_p)\}$ for all tangent vectors $t_p \in T_{M,p}$, in other words

$$\Omega^{1,0}_{M,p} = \mathrm{Hom}\ \ (T_{M,p},\mathbb{C}). \tag{4.140}$$

Analogously we have that $\Omega^{0,1}_{M,p}$ are the antilinear 1-forms. If we have a local section $\omega \in \Omega^{1,0}_M(U)$, then the complex conjugate $\overline{\omega}$ is given by

$$\overline{\omega}(t_p) = \overline{\omega(t_p)}, \tag{4.141}$$

where $t_p \in T_{M,p}$ is a tangent vector at the point $p \in U$.
Again we can form the complex of differential forms

$$\ldots \longrightarrow \Omega^{k-1}_{M_\infty,} \quad \longrightarrow \Omega^{k}_{M_\infty,} \quad \longrightarrow \Omega^{k+1}_{M_\infty,} \quad \longrightarrow \ldots .$$

The sheaf (vector bundle) of k-forms decomposes:

$$\Omega^k_{M_\infty,} \quad = \bigoplus_{p+q=k} \Omega^{p,q}_M, \tag{4.142}$$

where
$$\Omega^{p,q}_M = \bigwedge\nolimits^p \Omega^{1,0}_{M_\infty,} \quad \otimes \bigwedge\nolimits^q \Omega^{0,1}_{M_\infty,} \quad . \tag{4.143}$$

Locally a (p,q) form can be written as

$$\omega = \sum_{\alpha,\beta} f_{\alpha,\beta} dz_{\alpha_1} \wedge \ldots \wedge dz_{\alpha_p} \wedge d\overline{z}_{\beta_1} \wedge \ldots \wedge d\overline{z}_{\beta_q}, \tag{4.144}$$

where the $f_{\alpha,\beta}$ are complex valued $\mathcal{C}^\infty$-functions on U (the open set where we have these local coordinates). We get a decomposition of the exterior differential operator

$$d : \Omega^k_{M,} \quad \longrightarrow \Omega^{k+1}_{M,}$$

as $d = \frac{1}{2}(d' + d'')$, where

$$d'\omega = \sum_\gamma \frac{\partial f_{\alpha,\beta}}{\partial z_\gamma} dz_\gamma \wedge dz\alpha_1 \ldots \wedge dz_{\alpha_p} \wedge d\overline{z}_{\beta_1} \wedge \ldots \wedge d\overline{z}_{\beta_0} \tag{4.145}$$

and
$$d''\omega = (-1)^p \sum_\delta \frac{\partial f_{\alpha,\beta}}{\partial \overline{z}_\delta} dz_{\alpha_1} \wedge \ldots \wedge dz_{\alpha_p} \wedge d\overline{z}_\delta \wedge d\overline{z}_{\beta_1} \wedge \ldots \wedge d\overline{z}_{\beta_q}$$

The factor $\frac{1}{2}$ is explained by the fact that $\frac{\partial}{\partial z_\nu}, \frac{\partial}{\partial, \overline{z}_\mu}$ and $dz_\nu, d\overline{z}_\mu$ are not exactly dual bases of each other. We have

$$\begin{aligned} d' :& \Omega^{p,q}_M \longrightarrow \Omega^{p+1,q}_M \\ d'' :& \Omega^{p,q}_M \longrightarrow \Omega^{p,q+1}_M. \end{aligned}$$

Now we come back to our holomorphic vector bundle $\mathcal{E}$. We can embed the sheaf $\mathcal{E}$ of holomorphic sections into the sheaf of $\mathcal{C}^\infty$-sections, we write

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_\infty = \Omega^{0,0}_M(\mathcal{E}).$$

As in the case of local systems we can characterize the subsheaf of holomorphic sections by a differential equation. We define the operator

$$d'' : \mathcal{E}_\infty \longrightarrow \Omega^{0,1}_M(\mathcal{E}_\infty) = \mathcal{E}_\infty \otimes \Omega^{0,1}_M.$$

To do this we write a local section on U in $\mathcal{E}_\infty$ in the form

$$s = \sum_i f_i s_i,$$

where the f_i are $\mathcal{C}^\infty$-functions and the s_i form a basis of the holomorphic sections. Then we put

$$d''s = \sum_{i,\nu} \frac{\partial f_i}{\partial \overline{z}_\nu} s_i \otimes d\overline{z}_\nu. \tag{4.146}$$

This is well-defined because (just as in the case of local systems where the corresponding s_i were constant) we put $d''s_i = 0$. This is consistent with the change of trivializations because holomorphic functions f are characterized by $\frac{\partial f}{\partial \overline{z}_\nu} = 0$. As in the case of local systems we get a complex of sheaves

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_\infty \otimes \Omega_M^{0,0} \longrightarrow \mathcal{E}_\infty \otimes \Omega_M^{0,1} \longrightarrow \ldots \longrightarrow \mathcal{E}_\infty \otimes \Omega_M^{0,m} \longrightarrow 0. \tag{4.147}$$

We need an analogon of the Lemma of Poincaré, this is the

Lemma 4.10.7 (Dolbeault)**.** *The complex* (4.147) *of sheaves is exact.*

For a proof I refer to [Gr-Ha], Chap. 0, section 2.
Combined with our previous observation, namely that the sheaves $\Omega_M^{p,q}(\mathcal{E})$ are acyclic, this gives us an acyclic resolution of the sheaf $\mathcal{E}$. From our general principles we get

Theorem 4.10.8 (Dolbeault Isomorphism)**.** *We have an isomorphism*

$$H^i(M,\mathcal{E}) \simeq H^i(\Omega_M^{0\bullet}(\mathcal{E})(M)), \tag{4.148}$$

which is called **Dolbeault isomorphism***.*

Again we get the consequence

Theorem 4.10.9 (Dolbeault)**.** *The cohomology groups $H^k(M,\mathcal{E})$ of a holomorphic vector bundle on a compact connected complex manifold M vanish for $k >$ $\dim(M)$.*

Since M is a complex manifold we mean of course by $\dim(M)$ its complex dimension, this is half the dimension of the underlying $\mathcal{C}^\infty$-manifold.

4.10.3 Chern Classes

Definition 4.10.10. *A* **holomorphic line bundle** $\mathcal{L}$ *on a compact complex manifold M is a holomorphic vector bundle of rank 1. The isomorphism classes of these line bundles form a group under the tensor product and this group is the first cohomology $H^1(M,\mathcal{O}_M^*)$ (see section 4.3.3).*

We have a homomorphism from the sheaf of holomorphic functions $\mathcal{O}_M$ to $\mathcal{O}_M^*$ which is given by the exponential function

$$\begin{aligned} \mathcal{O}_M(U) &\longrightarrow \mathcal{O}_M^*(U) \\ f &\longmapsto e^{2\pi i f}, \end{aligned}$$

and this is a surjective homomorphism of sheaves. The kernel is the sheaf of locally constant -valued functions , thus we get an exact sequence of sheaves

$$0 \longrightarrow \_ \longrightarrow \mathcal{O}_M \longrightarrow \mathcal{O}_M^* \longrightarrow 1.$$

This leads to the exact sequence in cohomology

$$\ldots \longrightarrow H^1(M,\mathcal{O}_M) \longrightarrow H^1(M,\mathcal{O}_M^*) \xrightarrow{\delta} H^2(M,\_) \longrightarrow \ldots. \tag{4.149}$$

Definition 4.10.11. *The group* $H^1(M,\mathcal{O}_M^*)$ *is called the* **Picard group** *of* M*, the kernel of the connecting homomorphism is denoted by* $\mathrm{Pic}^0(M)$.

Definition 4.10.12. *If we have a line bundle* $\mathcal{L}$*, and its isomorphism class corresponds to* $[\mathcal{L}] \in H^1(M,\mathcal{O}_M^*)$*, then the image under* δ *is called the* **(first) Chern class** $c_1(\mathcal{L})$ *of* $\mathcal{L}$*, i.e.*

$$\delta([\mathcal{L}]) = c_1(\mathcal{L}) \in H^2(M,\ ).$$

We want to give a geometric interpretation of this class. We assume that our holomorphic bundle has a non zero section $s \in H^1(M,\mathcal{L})$ which has an additional property, namely it defines a **smooth divisor**. By this I mean the following: for any open set $U \subset M$ over which our bundle becomes trivial we select a nowhere vanishing section $1_U \in H^0(U,\mathcal{L})$. Our section s can be written as

$$s = f_U \cdot 1_U$$

where f_U is a holomorphic function. Now we require that the differential df_U is non zero in all the points where f_U – and therefore s – is zero. The implicit function theorem implies that the set of zeroes of s is a complex submanifold $Y \subset M$ which is of complex codimension one. This is our smooth divisor.
Since we are in the complex case, we know that M and Y have natural orientations, and this also defines a relative orientation (see 4.8.8). In this situation we attached a fundamental class $[Y] \in H^2(M,\_)$ to Y.

Proposition 4.10.13. *Under these conditions we have the equality*

$$[Y] = c_1(\mathcal{L}).$$

Proof: Let $p \in Y$ be any point and a neighborhood U_p of the point $p \in M$ such that we have an isomorphism

$$(U_p,\mathcal{O}_{U_p}) \simeq (B,\mathcal{O}_B),$$

where $B \subset \mathbb{C}^m$ is an open polydisc, say

$$B = \{(z_1,\cdots,z_m) \mid |z_i| < 1\}.$$

Then it follows easily from the theorem on implicit functions that we can assume that $Y \cap U_p = \{(0,z_2,\ldots,z_m)\}$ and that the bundle $\mathcal{L}|U_p$ is generated by z_1. We find a covering of a tubular neighnorhood

$$Y \subset \bigcup_{\alpha \in \mathrm{A}} U_\alpha = TY$$

where the U_α are of the above form $U_\alpha = V_\alpha \times D_\alpha, V_\alpha \subset Y$, the coordinate z_α on the disk generates the bundle on U_α.
We shrink this neighborhood TY slightly to a neighborhood $T_\epsilon Y$ by making the discs a little bit smaller. We can achieve that the closure of the smaller neighborhood is contained in the larger neighborhood. We get a covering of M if we include $U_0 = M \setminus \overline{T_\epsilon Y}$ into our covering family. By assumption we can trivialize the bundle on each of these open sets U_α. On U_0 we trivialize the bundle by using the section s. From this we get our 1-cocycle $g_{\alpha\beta} \in \mathcal{O}_M^*(U_\alpha \cap U_\beta)$. We introduce an auxiliary Riemannian metric and we construct a refined covering by convex sets (see section 4.8.2): For any $p \in Y$ we choose a convex neighborhood whose closure is contained in a U_α. For any point $m \notin Y$ we choose a convex neighborhood whose closure does not meet Y. Let the indexing set of this second covering be Γ. With a slight change of notation we write

$$M = \bigcup_{\alpha \in \mathrm{A}} U_\alpha \ \cup \ \bigcup_{\gamma \in \Gamma} U_\gamma.$$

The bundle is trivialized on the covering sets, the trivialization is simply the restriction. We get a 1-cocycle by restriction.
Since the covering sets are convex we can find $h_{\alpha\beta} = \frac{1}{2\pi i} \log g_{\alpha\beta}$ on $U_\alpha \cap U_\beta$, and we choose $h_{\gamma\gamma'} = 0$ if $\gamma, \gamma' \in \Gamma$. Then we get the _-valued 2-cocycle

$$c_{\alpha\beta\delta} = h_{\alpha\beta} - h_{\beta\delta} + h_{\alpha\delta} \qquad\qquad \text{on} \quad U_\alpha \cap U_\beta, \cap U_\delta \tag{4.150}$$

and this 2-cocycle represents our class $c_1(\mathcal{L})$. But we notice that $c_{\alpha\beta\delta} = 0$ if all three open sets lie in the complement of Y. This means that $c_{\alpha\beta\delta} \neq 0$ implies that at least one of the indices lies in A. Consequently

$$\overline{U_\alpha \cap U_\beta \cap U_\delta} \subset TY \qquad\qquad \text{if} \quad c_{\alpha\beta\delta} \neq 0.$$

Now we consider the sheaf $i_!(\ \ )$ on M where $i: \ TY \longrightarrow M$ is the inclusion. We just saw that our 2-cocycle takes its values in $i_!(\_)$, and we conclude that $c_1(\mathcal{L})$ is the image of the class $c_1^Y(\mathcal{L}) \in H^2(M, i_!\ \ )$, which is the class represented by our cocycle. But we know that

$$H^2(M, i_!\ \ ) = H^0(Y, R^2\pi_*(i_!\ \ ))$$

(see 4.8.8). Since we have a relative orientation we have $R^2\pi_*(i_!(\ \ )) = \_$ on Y and by definition

$$[Y] = 1_Y = \text{constant} \quad \text{-valued function 1.}$$

We want to show that $c_1^Y(\mathcal{L}) = [Y]$. This can be checked locally in the points p on Y. This means that for any point $p \in Y$ we consider the neighborhood $U_\alpha = V_\alpha \times D_\alpha$ which contains this point. We have to show that the restriction to the disk at p,

$$c_1^Y(\mathcal{L}) \in H^0(\{p\}, R^2 i_!(\_)),$$

is the canonical generator.
We cover the disc $D_\alpha = \{z \mid |z| < 1\}$ by open sets. The first one is $V_0 = \{z \mid |z| < r\}$ where $r < 1$ but close to one. We set

$$V_1 = \{z_1 \mid \mathrm{Re}\,(z_1) > \varepsilon, z_1 \in D\}$$

where $\varepsilon > 0$ is small and

$$V_2 = e^{\frac{2\pi i}{3}}\ V_1 \quad , \quad V_3 = e^{\frac{4\pi i}{3}}\ V_1.$$

This yields a covering of D_α. We compute the 1-cocycle by the recipe given in our discussion above. On the V_i with $i = 1,2,3$ the constant function 1 trivializes the bundle. On V_0 it is the function z. We get

$$\begin{aligned} & g_{ij} = 1^q \quad \text{if} \quad 1 \leq i,j \leq 3 \\ \text{and} \qquad & g_{0i} = z \quad \ \text{for} \quad i = 1,2,3. \end{aligned}$$

We have to write these $g_{\alpha\beta}$ as $e^{2\pi i h_{\alpha\beta}}$ with some function $h_{\alpha\beta}$ on $V_{\alpha\beta}$. Of course we take $h_{ij} = 0$ for $1 \leq i,j \leq 3$. To define the h_{0j} we take a path γ from 1 to a point $z \in V_0 \cap V_j$ which goes counterclockwise around zero, and

$$h_{0j} = \frac{1}{2\pi i} \int_\gamma \frac{d\zeta}{\zeta}.$$

We have to compute the differences

$$\begin{aligned} &c_{0ij} = h_{0i} - h_{0j} + h_{ij} \\ \text{and get} \quad &c_{012} = c_{123} = 0 \\ \text{but} \quad &c_{013} = 1. \end{aligned}$$

Now it is clear that this 2-cocycle with values in $i_!(\ )$ yields the positive generator in

$$H^2\left(D_\alpha, i_!(\ )\right) = H^0\left(\{p\}, R^2\pi_* i_!(\ )\right),$$

and this proves the proposition. □

Let us assume that $\dim M = d$ and let us assume that $\mathcal{L}_1,\dots,\mathcal{L}_d$ are line bundles. We assume that each of these line bundle has a section $s_i \in H^0(M,\mathcal{L})$ which defines a smooth divisor $Y_i = [s_i = 0]$ and let us assume that these smooth divisors intersect transversally (see section 4.8.9). This has the consequence that the intersections $Y_1 \cap Y_2 \dots \cap Y_k = Z_k$ are smooth complex submanifolds. Let us consider a point p in the intersection of all the Y_i and local trivializations $t_i \in H^0(U_p,\mathcal{L}_i)$ of the line bundles at p. Then locally at p we have $s_i = f_i t_i$, where f_i is holomorphic at p and $f_i(p) = 0$. Then our transversality assumption implies that $f_1, f_2, \dots, f_d$ is a system of local coordinates at p. The point is isolated in the intersection. We can invoke our formula (4.113). This leads us to the following proposition.

Proposition 4.10.14. *Under the assumptions from above the Chern class is a class in* $H^{2m}(M,\ )$ *and hence a number. This number is the cardinality of the intersection*

$$c_1(\mathcal{L}_1) \cup c_1(\mathcal{L}_2) \cup \dots \cup c_1(\mathcal{L}_d) = |Y_1 \cap Y_2 \cap \dots \cap Y_d|.$$

Of course we may always form the above cup product of d Chern classes of line bundles and we call the result the *intersection number* of the line bundles. We may even take one line bundle $\mathcal{L}$ and call $c_1(\mathcal{L})^d$ the **d-fold or total selfintersection number** of the line bundle. We will indicate later (see section 5.3.1) that on projective smooth varieties this cup product can always be interpreted as an intersection number of smooth divisors.

The Line Bundles $\mathcal{O}_{\ ^n(\ )}(k)$

I want to outline the construction of a familiy of line bundles $\mathcal{O}_{\ ^n(\ )}(k)$ on $\mathbb{P}^n(\mathbb{C})$. I begin with the construction of $\mathcal{O}_{\ ^n(\ )}(1)$. We consider the coordinate functions $z_i : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}$ as linear forms on $\mathbb{C}^{n+1}$. Starting from these linear forms we construct the bundle $\mathcal{O}_{\ ^n(\ )}(1)$. This bundle becomes trivial when we restrict it to one of the open subsets U_i and over this subset z_i is a trivializing section, i.e. it is nowhere zero. For any pair i,j of indices we have the two trivializing sections z_i,z_j on $U_i \cap U_j$. They are related by the equation $z_i = (z_i/z_j)z_j$ and $z_i/z_j = g_{ij}$ is a holomorphic nowhere vanishing function on $U_i \cap U_j$. These quotients define the transition functions (see section 4.3.1) defining the bundle $\mathcal{O}_{\ ^n(\ )}(1)$. It is clear that z_i defines in fact a global section in $H^0(\mathbb{P}^n(\mathbb{C}),\mathcal{O}_{\ ^n(\ )}(1))$ and this section defines a smooth divisor $[z_i = 0]$ and this is the hyperplane at infinity for those people who live in U_i. Hence we see that the Chern class of the bundle $\mathcal{O}_{\ ^n(\ )}(1)$ is the fundamental class of an arbitrary hyperplane in $\mathbb{P}^n(\mathbb{C})$.

In view of our considerations in section 4.8.11 this means that the Chern class $c_1(\mathcal{O}_{\ ^n(\ )}(1))$ is a generator in $H^2(\mathbb{P}^n(\mathbb{C}),\ )$.
Hence we have

Lemma 4.10.15. *The other bundles are simply the tensor products*

$$\mathcal{O}_{\ ^n(\ )}(1)(n) = \mathcal{O}_{\ ^n(\ )}(1)^{\otimes n}$$

and their Chern classes are given by n times the generator.

4.11 Hodge Theory

4.11.1 Hodge Theory on Real Manifolds

In this section I describe some very powerful analytical tools which provide insight into the structure of cohomology groups. They are based on the construction of certain linear elliptic differential operators (Laplace operators) which arise if we try to write down an inverse for the operators d,d',d'' in the de Rham or Dolbeault complexes. We need some results on elliptic linear differential operators which we do not prove here. (See for instance [Wel], Chap. IV.)
We go back to the situation where we have an oriented $\mathcal{C}^\infty$-manifold M, and a local system of finite dimensional   - or $\mathbb{C}$-vector spaces $\mathcal{V}$ on M. Let m be the dimension of M.
We have the de Rham complex

$$0 \longrightarrow \mathcal{V} \longrightarrow \mathcal{V}_\infty \longrightarrow \mathcal{V}_\infty \otimes \Omega^1_M \longrightarrow \ldots \longrightarrow \mathcal{V}_\infty \otimes \Omega^m_M \longrightarrow 0.$$

If we take global sections and if we drop the first term the resulting complex computes the cohomology groups $H^\nu(M,\mathcal{V})$.
We have seen that we can construct a Riemannian metric $<\ ,\ >$ on M and using the same method we construct a Euclidean (or Hermitian) metric $<\ ,\ >_h$ on $\mathcal{V}_\infty$. The metric on the tangent bundle provides a metric on the bundle of differential forms Ω^p_M. This allows us to define a scalar product on the sections $f \in \mathcal{V}_\infty \otimes \Omega^p_M(M)$: It is clear that the metric on $\mathcal{V}_\infty$ and the Riemannian metric together give us a metric on the tensor product of fibres $\mathcal{V}_x \otimes \Omega^p_{Mx}$ at any point x. Hence we get a function $x \mapsto < v_{1x} \otimes \omega_{1x}, v_{1x} \otimes \omega_{2x} >$ on M for any two $v_1 \otimes \omega_1, v_2 \otimes \omega_2 \in \mathcal{V}_\infty \otimes \Omega^p_M(M)$. Since our manifold M is oriented and Riemannian we have a unique section $\omega_{\text{top}} \in \Omega^m_M(M)$ which has length one at each point and is positive with respect to the orientation. Hence we can integrate

$$< v \otimes \omega_1, v_2 \otimes \omega_2 >= \int_M < v_{1x}, v_{2x} >_h < \omega_{1x}, \omega_{2x} > \omega_{\text{top}}. \tag{4.151}$$

Here we have to assume that the integral converges. This is certainly so if M is compact. Otherwise we have to introduce the notion of integrable sections.
There is another way to describe this scalar product.

Definition 4.11.1. *We have the* **Hodge-$*$-operator** *on the bundle of forms*

$$* : \Omega^p_M \longrightarrow \Omega^{m-p}_M$$

which is defined pointwise by the requirement

$$\omega_1 \wedge *\omega_2 =< \omega_1, \omega_2 > \cdot \omega_{\text{top}}.$$

It is straightforward that

$$**=(-1)^{p(m-p)}.$$

The Euclidian metric h gives us an isomorphism $i_h : \mathcal{V}_\infty \xrightarrow{\sim} \mathcal{V}_\infty^\vee$ defined by the rule $i_h(w)(v) = < v,w >_h$. This is not an isomorphism between the local systems $\mathcal{V}$ and $\mathcal{V}^\vee$. Using this isomorphism we define an operator

$$*_h : \mathcal{V}_\infty \otimes \Omega^p(M) \longrightarrow \mathcal{V}_\infty^\vee \otimes \Omega^{m-p}(M)$$

by the formula

$$*_h(v \otimes \omega) = i_h(v) \otimes *\omega. \tag{4.152}$$

We define an operator going in the opposite direction by

$$*_h^\vee(v^\vee \otimes \omega) = i_h^{-1}(v^\vee) \otimes *\omega. \tag{4.153}$$

Then we have as before

$$*_h *_h^\vee = (-1)^{p(m-p)}. \tag{4.154}$$

Let us denote by $e_h : \mathcal{V}_\infty \otimes \mathcal{V}_\infty \longrightarrow \mathcal{C}_M^\infty$ (resp. $e_0 : \mathcal{V} \otimes \mathcal{V}^\vee \longrightarrow \underline{\ \ }$) the evaluation maps defined by h (resp. the canonical pairing). They define $e_h : \mathcal{V}_\infty \otimes \Omega^p \wedge \mathcal{V}_\infty \otimes \Omega^{m-p} \longrightarrow \Omega^m$ (resp. $e_0 : \mathcal{V}_\infty \otimes \Omega^p \wedge \mathcal{V}_\infty^\vee \otimes \Omega^{m-p} \longrightarrow \Omega^m$).
Hence we get for our scalar product for two sections $v_1 \otimes \omega_1, v_2 \otimes \omega_2 \in \mathcal{V}_\infty \otimes \Omega_M^p(M)$ the formula

$$\begin{aligned} < v_1 \otimes \omega_1, v_2 \otimes \omega_2 > &= \int_M e_h(v_1 \otimes v_2)\omega_1 \wedge *\omega_2 \\ &= \int_M e_0((v_1 \otimes \omega_1) \wedge *_h(v_2 \otimes \omega_2)). \end{aligned}$$

Now it becomes clear that we can define an adjoint operator to the exterior derivative

$$\delta : \mathcal{V}_\infty \otimes \Omega_M^p(M) \longrightarrow \mathcal{V}_\infty \otimes \Omega_M^{p-1}(M)$$

we simply put

$$\delta = (-1)^{m(p+1)+1} *_h^\vee d *_h . \tag{4.155}$$

We have to verify that for $v_1 \otimes \omega_1 \in \mathcal{V}_\infty \otimes \Omega_M^{p-1}(M), v_2 \otimes \omega_2 \in \mathcal{V}_\infty \otimes \Omega_M^p(M)$ we have

$$< d(v_1 \otimes \omega_1), v_2 \otimes \omega_2 > = < v_1 \otimes \omega_1, \delta(v_2 \otimes \omega_2) > .$$

To see this we perform a simple calculation

$$\begin{aligned} &< d(v_1 \otimes \omega_1), v_2 \otimes \omega_2 > \\ &\quad = \int_M e_h(d(v_1 \otimes \omega_1) \wedge v_2 \otimes *\omega_2) \\ &\quad = \int_M e_0(d(v_1 \otimes \omega_1) \wedge *_h v_2 \otimes \omega_2) \\ &\quad = \int_M e_0(d(v_1 \otimes \omega_1 \wedge *_h v_2 \otimes \omega_2)) - (-1)^{p-1} v_1 \otimes \omega_1 \wedge d(*_h v_2 \otimes \omega_2)). \end{aligned}$$

From this moment on we assume that M is compact. Since the pairing e_0 is constant we get

$$\int_M e_0(d(\omega_1 \otimes v_1 \wedge *_h(v_2 \otimes \omega_2))) = \int_M de_0((\omega_1 \otimes v_1 \wedge *_h(v_2 \otimes \omega_2))) = 0$$

and hence

$$\begin{aligned} < dv_1 \otimes \omega_1, v_2 \otimes \omega_2 > &= (-1)^p \int_M \omega_1 \otimes v_1 \wedge d *_h v_2 \otimes \omega_2 \\ &= (-1)^{p+(m-p+1)(p-1)} \int_M < v_1 \otimes \omega_1 \wedge *_h *_h^\vee d(*_h v_2 \otimes \omega_2) \\ &=< \omega_1 \otimes v_1, \delta v_2 \otimes \omega_2 > . \end{aligned}$$

Definition 4.11.2. *We define the* **Laplace operator**

$$\Delta = d\delta + \delta d$$

which sends p-forms to p-forms.

It is clear that this is a linear operator of second order and it is elliptic (see [Wel], Chap. IV). I do not give the definition of elliptic operators here because for the conclusions we draw from ellipticity we refer to books which also give the definition.
From the theory of elliptic operators we get a result, which we formulate a little bit informally (see also 4.11.3).

Theorem 4.11.3. *We have a "decomposition" into eigenspaces*

$$\mathcal{V}_\infty \otimes \Omega^p_M(M) = \sum_\lambda \mathcal{V}_\infty \otimes \Omega^p_M(M)(\lambda),$$

where

$$\mathcal{V}_\infty \otimes \Omega^p_M(M)(\lambda) = \{\omega \in \mathcal{V}_\infty \otimes \Omega^p_M(M) \mid \Delta\omega = \lambda\omega\}.$$

All the eigenspaces have a finite dimension and the eigenvalues tend to infinity, i.e. for any finite interval $[0,T]$ we have only finitely many eigenvalues λ. The sign $\sum_\lambda$ means that any ω can be written as

$$\omega = \sum_\lambda \omega_\lambda$$

where the convergence is uniform on M and stays uniform if we apply a finite number of derivatives.

Definition 4.11.4. *The set of eigenvalues is called the* **spectrum** *of the operator, the eigenvalues are positive as one sees from the equality*

$$< \omega, \Delta\omega >=< d\omega, d\omega > + < \delta\omega, \delta\omega > .$$

The forms which are annihilated by Δ, i.e. which satisfy $\Delta\omega = 0$, are called **harmonic forms**.

Once we believe this we can compute the cohomology very easily.

Proposition 4.11.5. *The operators d and δ respect the decomposition into eigenspaces, they send eigenspaces into eigenspaces with the same eigenvalue.*

Let $\omega \in \mathcal{V}_\infty \otimes \Omega^p_M(M)$ be a closed form. Write ω in the form $\omega = \omega_0 + \omega'$, where ω_0 is the harmonic component, i.e. the component of ω in the eigenspace to $\lambda = 0$. Then

$$d\omega = d\omega_0 + d\omega' = 0$$

and hence $d\omega_0 = 0$ and $d\omega' = 0$. But

$$\omega' = \sum_{\lambda \neq 0} \omega'_\lambda,$$

where $d\omega'_\lambda = 0$ for all λ. Hence we get for $\lambda \neq 0$

$$\omega'_\lambda = \frac{1}{\lambda}\Delta\omega'_\lambda = \frac{1}{\lambda}(d\delta + \delta d)\omega'_\lambda = d\frac{1}{\lambda}\delta\omega'_\lambda$$

and therefore

$$\omega' = d(\sum_{\lambda \neq 0} \frac{1}{\lambda}\delta\omega'_\lambda).$$

This means that ω_0 represents the same cohomology class as ω. Hence we have that

Theorem 4.11.6. *The harmonic forms satisfy $d\omega = \delta\omega = 0$. Sending a harmonic form to its cohomology class provides an isomorphism*

$${}^{p}(\mathcal{V}_\infty \otimes \Omega^p_M)(M) = \{\omega \in \mathcal{V}_\infty \otimes \Omega^p_M(M) | \Delta\omega = 0\} \xrightarrow{\sim} H^p(M,\mathcal{V}).$$

It is clear how this follows from Theorem 4.11.3. We observe that Δ is a positive operator. We have

$$< \Delta\omega,\omega > = < d\omega,d\omega > + < \delta\omega,\delta\omega > \quad \geq 0.$$

If $\Delta\omega = 0$ then we conclude

$$0 = < \delta\omega,\delta\omega > = < d\omega,d\omega >$$

this implies the first assertion. Since harmonic forms are closed they define cohomology classes. If ω is harmonic and $\omega = d\Psi$ then $< \omega,\omega > = < \omega,d\Psi > = < \delta\omega,\Psi > = 0$ and hence $\omega = 0$. The map from harmonic forms to cohomology is injective. The surjectivity has been shown above. For a complete proof see [Wel] Chap IV, Thm. 5.2.

We can give some indications how Theorem 4.11.6. can be proved without using Theorem 4.11.3. Since we introduced the scalar product on $\mathcal{V}_\infty \otimes \Omega^p_M(M)$ we may take the completion with respect to this scalar product, and we get the Hilbert space

$$L^2(\mathcal{V}_\infty \otimes \Omega^p_M(M)) = \mathcal{V}_\infty \otimes \Omega_{(2)}(M)$$

of quadratically integrable differential forms with values on $\mathcal{V}_\infty$.

If we have a closed form $\omega \in \mathcal{V}_\infty \otimes \Omega^p_M(M)$, then we can modify it by a form $d\psi$ and we can try to minimize the square of the L^2-norm

$$\|\omega + d\psi\|_2^2 = \int_M \langle \omega + d\psi, \omega + d\psi \rangle.$$

We look at the limes inferior of all the real numbers $\|\omega+d\psi\|_2^2$ where ψ varies. We can find a sequence $\omega + d\psi_n = \omega_n$ such that $\|\omega_n\|_2^2$ converges to this infimum. Since the unit ball in our Hilbert space is weakly compact, we can find a weakly convergent subsequence, i.e. we may assume that ω_n converges weakly to a form $\omega_0 \in L^2(\mathcal{V}_\infty \otimes \Omega^p_M(M))$.We would like to prove that ω_0 is a $\mathcal{C}^\infty$-form, that it is harmonic and that this form represents the given class, i.e. $\omega_0 = \omega + d\psi_0$.
Assume that we know that ω_0 is a harmonic form. This means that it is $\mathcal{C}^\infty$ and satisfies $d\omega_0 = \delta\omega_0 = 0$. Then this implies

$$\begin{aligned} \langle d\omega_0, \eta \rangle &= \langle \omega_0, \delta\eta \rangle = 0 \\ \langle \delta\omega_0, \psi \rangle &= \langle \omega_0, d\psi \rangle = 0 \end{aligned}$$

for all $\psi \in \mathcal{V}_\infty \otimes \Omega^{p-1}_M(M), \eta \in \mathcal{C}^\infty(\mathcal{V}) \otimes \Omega^{p+1}_M(M)$. The point is that the equalities

$$< \omega_0, \delta\eta > = 0 \quad \text{and} < \omega_0, d\psi >= 0$$

make sense for all $\omega_0 \in L^2(\mathcal{V}_\infty \otimes \Omega^p_M(M))$. And in our case they are true because

$$\langle \omega_0, \delta\eta \rangle = \lim_{n \longrightarrow \infty} \langle \omega + d\psi_n, \delta\eta \rangle = \langle d\omega + dd\psi_n, \eta \rangle = 0$$

(this is the definition of weak convergence) and the second one follows from the minimality of the norm $\|\omega_0\|_2^2$.
This means that ω_0 is a so called **weak solution** of the differential equations $d\omega = \delta\omega = 0$. The really deep input from analysis is that the validity of the two equations

$$\langle \omega_0, \delta\eta \rangle = \langle \omega_0, d\psi \rangle = 0$$

for all η, ψ implies that ω_0 must be indeed $\mathcal{C}^\infty$ and then it follows that ω_0 must be harmonic ([Wel]).
The rest is easy. We need to know that ω_0 still represents the given cohomology class. This follows from Poincaré duality. We consider the dual local system $\mathcal{V}^\vee$. We have the non degenerate pairing

$$H^p(M, \mathcal{V}) \times H^{m-p}(M, \mathcal{V}^\vee) \longrightarrow$$

which in terms of differential forms is given by integration over M. Hence we see that for any cohomology class $[\omega'] \in H^{m-p}(M, \mathcal{V}^\vee)$ which is represented by a $\mathcal{C}^\infty - (m-p)$-form ω' that

$$[\omega] \cup [\omega'] = \int_M \operatorname{tr}(\omega \wedge \omega') = \int \operatorname{tr}((\omega + d\psi_n) \wedge \omega'),$$

and weak convergence gives that this integral is equal to

$$\int \operatorname{tr}(\omega_0 \wedge \omega') = [\omega_0] \cup [\omega']. \tag{4.156}$$

Theorem 4.11.6 has some consequences, for instance the finite dimensionality of the space of harmonic forms implies

Corollary 4.11.7. *For a compact oriented $\mathcal{C}^\infty$-manifold the cohomology $H^p(M,\mathcal{V})$ has finite dimension for any local system of finite dimensional - or $\mathbb{C}$-vector spaces.*

Of course it follows already from the de Rham isomorphism that

Corollary 4.11.8. *For a compact manifold M and a local system $\mathcal{V}$ of real (or complex) vector spaces $H^p(M,\mathcal{V}) = 0$ for $p > \dim(M)$.*

Then it is clear from the construction that

Corollary 4.11.9. *The operator $*_h$ induces an isomorphism*

$$j_h^p : \quad {}^p(M,\mathcal{V}) \xrightarrow{\sim} \quad {}^{m-p}(M,\mathcal{V}^\vee)$$

which depends of course on the choice of the metrics. We have the duality pairing

$$H^p(M,\mathcal{V}) \times H^{m-p}(M,\mathcal{V}^\vee) \xrightarrow{\cup} \quad .$$

If we identify the cohomology groups to the spaces of harmonic forms, then we find that for a non zero $\omega \in H^p(M,\mathcal{V})$ we get $\omega \cup j_h^p(\omega) > 0$ and this implies of course again, that the Poincaré pairing is non-degenerate.

All these consequences were known to us, they even hold for more general local systems. But in the next section where we discuss the analogous situation of holomorphic bundles on complex manifolds the proofs really require some analysis. For instance the proofs for the finite dimensionality of certain cohomology groups need analytic methods. It can be obtained from the theory of elliptic operators or one uses methods from the theory of topological vector spaces.
Finally I want to mention that the results of Hodge Theory allow an interpretation in language of derived categories.

Corollary 4.11.10. *The de Rham complex computes the cohomology, it is a complex of infinite dimensional vector spaces. The harmonic forms provide a subcomplex where all the differentials are zero and this subcomplex also computes the cohomology. Hence we see that the de Rham complex is isomorphic to its cohomology in the derived category of -vector spaces.*

4.11.2 Hodge Theory on Complex Manifolds

Now we consider a compact complex manifold M. We introduce a Hermitian metric h on the tangent bundle T_M.

Some Linear Algebra

I have to recall some simple facts from linear algebra which concern these metrics. Therefore I start from a complex vector space V of finite dimension m. In the following I view V as a real vector space of dimension $2m$ which is endowed with a linear transformation $I : V \longrightarrow V$ which satisfies $I^2 = -\operatorname{Id}$. The structure as a $\mathbb{C}$-vector space is regained if we define scalar multiplication of $v \in V$ by i by $v \mapsto I(v)$.

If we have a Hermitian form h on V then we can write

$$h(v_1,v_2) = \operatorname{Re} h(v_1,v_2) + i \cdot \operatorname{Im} h(v_1,v_2)$$

and it is clear that

$$\begin{aligned} \operatorname{Re} h : V \times V \longrightarrow \quad & \text{is symmetric} \\ \operatorname{Im} h : V \times V \longrightarrow \quad & \text{is alternating.} \end{aligned}$$

Since

$$h(Iv_1,Iv_2) = h(iv_1,iv_2) = h(v_1,v_2)$$

we see that both components satisfy

$$\begin{aligned} \operatorname{Re} h(Iv_1,Iv_2) &= \operatorname{Re} h(v_1,v_2) \\ \operatorname{Im} h(Iv_1,Iv_2) &= \operatorname{Im} h(v_1,v_2), \end{aligned}$$

in other words: I is an isometry for the real part and for the imaginary part. But we may also recover h from either part. We simply write

$$h(v_1,Iv_2) = \operatorname{Re} h(v_1,Iv_2) + i \operatorname{Im} h(v_1,Iv_2)$$

and since

$$h(v_1,Iv_2) = -ih(v_1,v_2)$$

this yields

$$h(v_1,v_2) = -\operatorname{Im} h(v_1,Iv_2) + i \operatorname{Re} h(v_1,Iv_2)$$

and from this we get

$$\begin{aligned} \operatorname{Re} h(v_1,v_2) &= -\operatorname{Im} h(v_1,Iv_2) \\ \operatorname{Im} h(v_1,v_2) &= \quad \operatorname{Re} h(v_1,Iv_2). \end{aligned}$$

Hence we see that a sesquilinear form h on V (this is a Hermitian form without the requirement that it should be positive definite) is the same thing as a symmetric form or an alternating form

$$\operatorname{Re} h : V \times V \longrightarrow \qquad\qquad \operatorname{Im} h : V \times V \longrightarrow$$

for which I is an isometry.

Proposition 4.11.11. *The form h is Hermitian (positive definite) if and only if* $\operatorname{Re} h$ *is Euclidean.*

We complexify V and extend $\mathrm{Re}\,h$ to a bilinear form

$$\mathrm{Re}\,h \quad : V \quad \times V \quad \longrightarrow \mathbb{C}.$$

We have the decomposition

$$V \quad = V^{1,0} \oplus V^{0,1}$$

into $\pm i$-eigenspaces for I and it is clear that $V^{1,0}$,$V^{0,1}$ are isotropic with respect to $\mathrm{Re}\,h$, i.e.

$$\mathrm{Re}\,h \quad (V^{1,0},V^{1,0}) = \mathrm{Re}\,h \quad (V^{0,1},V^{0,1}) = \{0\}.$$

This follows from the definition of the $V^{1,0}$,$V^{0,1}$ as eigenspaces for I with eigenvalue $\pm i$. But the pairing

$$\mathrm{Re}\,h \quad : V^{1,0} \times V^{0,1} \longrightarrow \mathbb{C}$$

will be not trivial in general. If for instance the form h is positive definite then this pairing is a perfect duality.
We have an isomorphism of complex vector spaces

$$j : V \longrightarrow V^{1,0}$$

which is obtained by the embedding of V into V followed by the projection. Under this isomorphism we send

$$j : v \longmapsto \frac{1}{2}(v - Iv \otimes i)$$

and we can recover the Hermitian form h from $\mathrm{Re}\,h$ by the formula

$$h(v_1,v_2) = \frac{1}{2}\mathrm{Re}\,h \quad (j(v_1),\overline{j(v_2)})$$

where $\overline{}$ is of course the antilinear isomorphism from V^{10} to V^{01} introduced by complex conjugation on the factor $\mathbb{C}$ in the tensor product V .
We introduce a so-called **Hodge structure** on the pair (V,I). This is a homomorphism

$$h_{\mathcal{D}} : \mathbb{C}^{\times} \longrightarrow \mathrm{GL}_{\mathbb{R}}(V)$$

and it is defined as

$$h_{\mathcal{D}}(z) = h_{\mathcal{D}}(a + bi) = a \cdot \mathrm{Id} + b \cdot I.$$

It is clear that this map is a homomorphism. With respect to the Euclidean metric on V it has the property that

$$< h_{\mathcal{D}}(z)v_1,h_{\mathcal{D}}(z)v_2 > = z\overline{z} \cdot < v_1,v_2 > ,$$

it is not an isometry but a similitude.
If we complexify the space to V then it is obvious that

$$V^{1,0} = \{v | h_{\mathcal{D}}(z)v = zv\}$$
$$V^{0,1} = \{v | h_{\mathcal{D}}(z)v = \overline{z}v\}.$$

The action of $\mathbb{C}^\times$ commutes with complex conjugation: We have $\overline{h_{\mathcal{D}}(z)v} = h_{\mathcal{D}}(z)\overline{v}$ on V . We can extend this action of $\mathbb{C}^\times$ to the exterior powers $\bigwedge^n V$ and $(\bigwedge^n V)$ simply by

$$h_{\mathcal{D}}(z)(v_1 \wedge \ldots \wedge v_n) = h_{\mathcal{D}}(z)v_1 \wedge \ldots \wedge h_{\mathcal{D}}(z)v_n$$

and it is clear that we can characterize the subspace

$$V^{p,q} = \bigwedge^p V^{1,0} \otimes \bigwedge^q V^{0,1} \subset \bigwedge^{p+q} V$$

as

$$\bigwedge^p V^{1,0} \otimes \bigwedge^q V^{0,1} = \left\{\omega \in \bigwedge^{p+q} V \ | h_{\mathcal{D}}(z)\omega = z^p \overline{z}^q \omega\right\}.$$

Of course if we extend $\mathrm{Re}\, h$ to a bilinear form on $\bigwedge^n V$ by

$$\begin{aligned}\mathrm{Re}\, h \ (\varphi,\psi) &= \mathrm{Re}\, h \ (v_1 \wedge \ldots \wedge v_n, w_1 \wedge \ldots \wedge w_n) \\ &= \det(\mathrm{Re}\, h \ (v_i,w_j))\end{aligned}$$

then we have

$$\mathrm{Re}\, h \ (h_{\mathcal{D}}(z)\varphi, h_{\mathcal{D}}(z)\psi) = (z\overline{z})^n \mathrm{Re}\, h \ (\varphi,\psi).$$

This implies for the $*$-operator that

$$* : \bigwedge^p V^{1,0} \otimes \bigwedge^q V^{0,1} \xrightarrow{\sim} \bigwedge^{m-p} V^{1,0} \otimes \bigwedge^{m-q} V^{0,1}.$$

This must be so because the product $v_{p,q} \wedge *w_{p,q}$ is in top degree and

$$h_{\mathcal{D}}(z)v_{p,q} \wedge *w_{p,q} = (z\overline{z})^m v_{p,q} \wedge *w_{p,q}$$

We can extend our Hermitian form h to a Hermitian form on $\bigwedge^p V^{1,0} \otimes \bigwedge^q V^{0,1}$ by

$$h(\varphi,\psi) = \mathrm{Re}\, h \ (\varphi,\overline{\psi}).$$

Kähler Manifolds and their Cohomology

Now we come back to our compact complex manifold M of dimension m, we assume that we have introduced a Hermitian metric $< , >_h$ on T_M. This introduces a Hermitian metric on Ω^1_M. We have the decomposition

$$\Omega^1_{M,} \ = \Omega^1_{M_\infty} \otimes \mathbb{C} = \Omega^{1,0}_M \bigoplus \Omega^{0,1}_M$$

and

$$\begin{aligned}\bigwedge^n \Omega_{M,} \ &= \bigoplus_{p+q=n} \bigwedge^p \Omega^{1,0}_M \otimes \bigwedge^q \Omega^{0,1}_M \\ &= \bigoplus_{p+q=n} \Omega^{p,q}_M.\end{aligned}$$

We have the Euclidean metric $\operatorname{Re} h$ on $\Omega^1_{M,\infty}$ and it induces a star operator

$$* : \bigwedge^n \Omega^1_{M_\infty} \longrightarrow \bigwedge^{2m-n} \Omega^1_{M_\infty}.$$

We have seen in the above section on linear algebra that we should extend this antilinearily to

$$* : \bigwedge^n \Omega^1_{M,} \longrightarrow \bigwedge^{2m-n} \Omega^1_{M,}$$

and that this operator sends

$$* : \Omega^{p,q}_M \longrightarrow \Omega^{m-p,m-q}_M.$$

We define the scalar product on the sections $\Omega^{p,q}_M(M)$ by

$$< \omega_1,\omega_2 >= \int_M \omega_1 \wedge *\omega_2.$$

Now we are able to define the adjoint operators to d' and d'', we put

$$\begin{aligned} \delta' &= - * d' * \\ \delta'' &= - * d'' * . \end{aligned}$$

The sign factor simplifies because our manifold has an even dimension when we consider it as a real manifold. Of course we have to verify the adjointness formulas

$$\begin{aligned} < d'\omega_1,\omega_2 > &=< \omega_1,\delta'\omega_2 > \\ < d''\omega_1,\omega_2 > &=< \omega_1,\delta''\omega_2 > . \end{aligned}$$

To see this we observe that it is enough to check the case where $\omega_1 \in \Omega^{p-1,q}(M),\omega_2 \in \Omega^{r,s}(M)$.[1] Let us consider the first case. We see that both sides are zero unless $p = m-r,q = m-s$. So we assume that this is the case. Now we perform the same calculation as in the real case where at certain places we have to replace d' by d and then again d by d'. We observe that

$$\begin{aligned} < d'\omega_1,\omega_2 > &=< d\omega_1,\omega_2 > \\ &=< \omega_1,\delta\omega_2 > \\ &=< \omega_1,\delta'\omega_2 > . \end{aligned}$$

This allows us to define the Laplace operators

$$\Delta' = d'\delta' + \delta'd' \quad \text{and} \quad \Delta'' = d''\delta'' + \delta''d''.$$

We want to compare these operators to the real Laplacian. Here we find

$$\begin{aligned} \Delta &= (d' + d'')(\delta' + \delta'') + (\delta' + \delta'')(d' + d'') \\ &= \Delta' + \Delta'' + \delta''d' + d'\delta'' + \delta'd'' + d''\delta'. \end{aligned}$$

This is not so good unless we know that the mixed contributions disappear. This is indeed the case if our metric h satisfies a certain condition, which I now want to explain.
Our metric h on the tangent bundle has its imaginary part

$$\operatorname{Im} h : T_{M_\infty} \times T_{M_\infty} \longrightarrow \quad .$$

Hence the imaginary part defines a 2-form ω_h on the manifold. If we complexify the tangent bundle and if we observe that I is an isometry for ω_h, then we see that ω_h is a form of type (1,1) because it must be zero on $T^{1,0} \otimes T^{1,0}$ and $T^{0,1} \otimes T^{0,1}$. This is the so-called **Kähler form** of the metric. Kähler discovered the following

Theorem 4.11.12. *If the form ω_h is closed, i.e. $d\omega_h = 0$, then the sum of the mixed terms is zero and we have*

$$\Delta = \frac{1}{2}\Delta' = \frac{1}{2}\Delta''.$$

I will not prove this theorem here. (See [We2], Chap. II, Thm. II) But in our later discussion of the special case of Riemann surfaces – in this case we have automatically $d\omega_h = 0$ – I will carry out the necessary calculations in this special case.

Definition 4.11.13. *A complex manifold is called* **Kähler manifold** *if it is equipped with a Hermitian metric for which $d\omega_h = 0$.*

The Theorem 4.11.12 has the following important consequences:

Theorem 4.11.14. *Let M be a compact Kähler manifold*

(a) *The operators Δ',Δ'' respect the decomposition, in any degre k we have*

$$\Omega^k_{M_\infty,}\ (M) = \bigoplus_{p+q=k} \Omega^{p,q}_M(M),$$

and then $\Delta = \sum_{p,q} \Delta^{p,q}$ where $\Delta^{p,q} : \Omega^{p,q}_M(M) \longrightarrow \Omega^{p,q}_M(M)$.

(b) *The harmonic forms $\omega \in \Omega^k_{M_\infty,}\ (M)$ are sums of harmonic forms*

$$\omega = \sum_{p+q=k} \omega_{p,q},$$

and

$$\Delta\omega = 0 \qquad \Longleftrightarrow \qquad \Delta'\omega = \Delta''\omega = 0.$$

(c) *A form ω is harmonic if and only if it satisfies all the equations*

$$d'\omega = d''\omega = 0$$
$$\delta'\omega = \delta''\omega = 0$$

This follows by the same positivity argument which we used in the real case.
This provides us the famous Hodge decomposition.

Theorem 4.11.15 (Hodge decomposition)**.** *Let M a compact complex Kähler manifold, then we have the decomposition*

$$H^n(M,\mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(M,\mathbb{C}).$$

of the cohomology.

The Cohomology of Holomorphic Vector Bundles

We have methods to compute the cohomology of a holomorphic bundle $\mathcal{E}$ on compact Kähler manifolds M which are analogous to the methods in section 4.11.1, where we computed the cohomology of local systems. We choose a positive definite Hermitian metrics on the tangent bundle and on the bundle $\mathcal{E}$ itself. If $\mathcal{E}^\vee = \mathrm{Hom}(\mathcal{E},\mathcal{O}_M)$ is the dual bundle, then the Hermitian metric h on the bundle $\mathcal{E}$ provides an antilinear isomorphism $i_h : \mathcal{E} \longrightarrow \mathcal{E}^\vee$ which is defined by $v \mapsto \{w \mapsto < w,v >_h\}$. We consider the Dolbeault complex

$$0 \longrightarrow \mathcal{E}_\infty(M) \xrightarrow{d''} \mathcal{E}_\infty \otimes \Omega_M^{0,1}(M) \xrightarrow{d''} \mathcal{E}_\infty \otimes \Omega_M^{0,2}(M) \longrightarrow \ldots .$$

As in section 4.11.1 we define the operators

$$*_h : \mathcal{E}_\infty \otimes \Omega^{p,q} \longrightarrow \mathcal{E}_\infty^\vee \otimes \Omega^{m-p,m-q} \qquad \text{and}$$
$$*_h^\vee : \mathcal{E}_\infty \otimes \Omega^{p,q} \longrightarrow \mathcal{E}_\infty^\vee \otimes \Omega^{m-q,m-p}$$

by

$$*_h(v \otimes \omega) = i_h(v) \otimes *\omega \qquad \text{and} \qquad *_h^\vee (v^\vee \otimes \omega) = (i_h)^{-1}(v^\vee) \otimes *\omega.$$

Again we introduce a scalar product on the sections $\mathcal{E}_\infty \otimes \Omega^{p,q}(M)$ by

$$\begin{aligned} < s_1 \otimes \omega_1, s_2 \otimes \omega_2 > &= \int_M e_h(s_1,s_2)\omega_1 \wedge *\omega_2 \\ &= \int_M e_0(s_1 \otimes \omega_1 \wedge *_h(s_2 \otimes \omega_2)) \end{aligned}$$

We can construct the adjoint operator

$$\delta'' : \mathcal{E}_\infty \otimes \Omega_M^{0,q}(M) \longrightarrow \mathcal{E}_\infty \otimes \Omega_M^{0,q-1}(M)$$

for d'', it is given by

$$\delta'' = - *_h^\vee d'' *_h$$

and we have the Laplacian

$$\Delta'' = \delta'' d'' + d'' \delta''.$$

We get in analogy with section 4.11.6.

Theorem 4.11.16. *The cohomology groups are given by*

$$H^p(M,\mathcal{E}) = \quad {}^p(\mathcal{E}_\infty \otimes \Omega_M^{0,p}(M)) = \left\{\omega \in \mathcal{E}_\infty \otimes \Omega_M^{0,p}(M) \mid \Delta''\omega = 0\right\}.$$

Especially we can conclude that these groups are finite dimensional.

The finite dimensionality is fundamental and there is no easy way to get it . We will give proofs in the special case of tori in 4.11.3,(this is not so difficult) and we will prove it for Riemann surfaces in the next chapter.

Serre Duality

We apply this to the holomorphic line bundle Ω^m of differentials of highest degree. Its cohomology is computed from the complex

$$0 \longrightarrow \Omega^m(M) \longrightarrow \Omega^{m,0}(M) \longrightarrow \Omega^{m,1}(M) \longrightarrow \ldots \longrightarrow \Omega^{m,m-1}(M) \longrightarrow \Omega^{m,m}(M) \longrightarrow 0.$$

We want to compute $H^m(M,\Omega^m)$, this is the space of harmonic forms for Δ'' in $\Omega^{m,m}(M)$. The star operator sends $\Omega^{m,m}(M)$ to $\Omega^{0,0}(M)$ and the Δ''-harmonic forms to the Δ'-harmonic forms. But the Δ'-harmonic sections in this sheaf are the antiholomorphic functions on M. Since M is compact we can conclude that these are the constants. It follows that

Theorem 4.11.17 (Serre)**.** *On a compact, connected complex manifold M of dimension m we have* $\dim H^m(M,\Omega^m) = 1$ *and we have a canonical isomorphism*

$$H^m(M,\Omega^m) \xrightarrow{\sim} \mathbb{C},$$

which is induced by

$$\omega \longmapsto \int_M \omega.$$

This isomorphism does not depend on the choice of the metric. The cup product induces a pairing

$$H^p(M,\mathcal{E})\times H^{m-p}(M,\mathcal{E}^\vee\otimes\Omega^m) \longrightarrow H^m(M,\mathcal{E}\otimes\mathcal{E}^\vee\otimes\Omega^m) \longrightarrow H^m(M,\Omega^m) \xrightarrow{\sim} \mathbb{C}. \quad (4.157)$$

and this pairing is non degenerate.

This is §3 Thm.4. in [Se2] and is called Serre duality. We can also deduce this theorem from 4.11.16 by the same argument which we used to show that Hodge Theory for local systems implies Poincaré duality: We get the antilinear isomorphism $j_h^p : \ {}^p(M,\mathcal{E}) \xrightarrow{\sim} {}^{m-p}(M,\mathcal{E}^\vee \otimes \Omega^m)$ induced by $*_h$ and then we exploit the positive definteness of the scalar product.

In his paper [Se2] also J.P. Serre gives a proof of the finite dimensionality of the cohomology groups $H^p(M,\mathcal{E})$ which is not based on Hodge Theory but uses results on topological vector spaces instead. This proof of Serre is more in the spirit of our discussion of the cohomology of manifolds in section 4.8. There we started from convex coverings and used the fact that the Čech complex computes the cohomology. In this approach it is central that constant sheaves on contractible spaces are acyclic. We have to find a substitute for this in complex analysis, we indicate briefly how this works.

In the local theory of several complex variables one introduces certain simple types of domains, for instance polycylinders $P = \{(z_1,\ldots,z_n) \in \mathbb{C}^n \mid |z_i| < r_i\}$ or balls $D = \{(z_1,\ldots,z_n) \in \mathbb{C}^n \mid \sum |z_i|^2 < r\}$. These are so called **domains of holomorphy** or **Stein manifolds** (see [Gr-Re1]). These domains play a similar role as our convex balls in the convex covering of a manifold (see 4.8.2). They satisfy certain finiteness conditions and they are acyclic for so-called **coherent sheaves**, i.e. for a coherent sheaf we have $H^p(D,\mathcal{F}) = 0$ for $p > 0$. These are the famous theorems A and B which go back to Oka and Cartan. (See [Gr-Re1],Chap. A. §2, Chap. III. §3.)

Now we can try to compute the cohomology $H^p(M,\mathcal{E})$ by starting from suitable Čech complexes obtained by coverings by open sets which are domains of holomorphy. Then we encounter the problem that in contrast to the case of local systems the spaces of sections $H^0(U_{\underline{\alpha}},\mathcal{E})$ are of infinte dimension. They have to be endowed with topologies, they become **Frechet spaces**. We pass to a refinement of the covering and then certain linear operators will be compact, which then eventually leads to finite dimensionality. For details I refer to the paper [Se2] by J.P. Serre.

4.11.3 Hodge Theory on Tori

We have a special case where the two main theorems of Hodge Theory (Theorem 4.11.6 and Theorem 4.11.14) are easy to prove. We consider a lattice $\Gamma \subset {}^n$ (see 4.6.8), and we consider the compact $\mathcal{C}^\infty$-manifold

$$M = {}^n/\Gamma.$$

For any point $u \in M$ we have a canonical identification $T_xM = {}^n$. In the following we take $u = 0$. If we take the standard Euclidean metric $< \ , \ >$ on ^{n} then we get a Riemannian metric on M. If $x_1, \dots, x_n$ are the coordinates on ^{n}, then the differential forms can be written as

$$\omega = \sum f_{i_1 \cdots i_p} \, dx_{i_1} \wedge \cdots \wedge dx_{i_p}.$$

I want to consider complex valued differential forms, i.e. the $f_{i_1 \cdots i_p}$ are complex valued $\mathcal{C}^\infty$-functions.
A basically simple and straightforward computation yields a formula for the Laplace operator:

$$\Delta\omega = \sum \left(-\sum \frac{\partial^2 f_{i_1 \cdots i_p}}{\partial x_i^2} \right) dx_{i_1} \wedge \cdots \wedge dx_{i_p}. \tag{4.158}$$

Now we consider the **dual lattice**

$$\Gamma^\vee = \{\varphi \in {}^n \mid \langle \varphi,\Gamma \rangle \subset \quad \},$$

then for $\varphi \in \Gamma^\vee$ the function

$$e_\varphi(x) = e^{2\pi i \langle \varphi, x \rangle} \tag{4.159}$$

on M is an eigenfunction for the Laplacian

$$\Delta e_\varphi(x) = 4\pi^2 \langle \varphi,\varphi \rangle e_\varphi(x).$$

Now any $\mathcal{C}^\infty$-function on M has a Fourier expansion

$$f = \sum_{\varphi \in \Gamma^\vee} a_\varphi e_\varphi(x), \tag{4.160}$$

where the absolute values $|a_\varphi|$ tend to zero very rapidly. Consequently any differential form can be written as convergent infinite sum

$$\omega = \sum_{\varphi} \omega_{\varphi}, \tag{4.161}$$

where $\Delta\omega_{\varphi} = +4\pi^2\langle\varphi,\varphi\rangle\omega_{\varphi}$. This is the **decomposition** in 4.11.3. It has the required property: There are only finitely many φ which satisfy $4\pi^2\langle\varphi,\varphi\rangle \leq T$. We apply our arguments in the *proof* of Theorem 4.11.6. The harmonic forms are the constant forms

$$\omega_0 = \sum a_{i_1\cdots i_p} dx_{i_1} \wedge \cdots \wedge dx_{i_p},$$

where $a_{i_1\cdots i_p} \in \mathbb{C}$. We conclude that the cohomology ring $H^{\bullet}(M,\mathbb{C})$ is the exterior algebra of the complexified dual tangent space $\mathrm{Hom}(T_0M,\mathbb{C})$. This agrees with 4.6.8, but the result over there is slightly sharper because it gives the structure over .
If we consider a complex torus

$$M = \mathbb{C}^n/\Gamma$$

where Γ is a lattice of rank $2n$, then M is a complex manifold and the tangent space is the complex vector space $\mathbb{C}^n$ in any point of M. On this tangent space we introduce the standard Hermitian metric

$$\sum_{\nu=1}^{n} z_\nu \overline{z}_\nu = h(z,z).$$

Again we perform a simple computation and find

$$\Delta' = \Delta'' : \sum_{\underline{\alpha},\underline{\beta}} f_{\underline{\alpha}\underline{\beta}} dz_{\underline{\alpha}} \wedge d\overline{z}_{\underline{\beta}} = \omega \longmapsto \sum_{\underline{\alpha},\underline{\beta}} \left(-\sum_{\nu} \frac{\partial^2 f_{\underline{\alpha}\underline{\beta}}}{\partial z_\nu \partial \overline{z}_\nu} \right) dz_{\underline{\alpha}} \wedge d\overline{z}_{\underline{\beta}}. \tag{4.162}$$

We have the dual lattice

$$\Gamma^{\vee} = \{\varphi \in \mathbb{C}^n \mid \mathrm{Re}\, h(\varphi,\gamma) \in \quad \text{for all } \gamma \in \Gamma\},$$

and we can expand $\mathcal{C}^{\infty}$-functions

$$f(z) = \sum a_{\varphi\in\Gamma^{\vee}} e^{2\pi i \mathrm{Re}\, h\langle\varphi,z\rangle}. \tag{4.163}$$

Now we argue as before. We have the Dolbeault complex

$$0 \longrightarrow \mathcal{O}_M \longrightarrow \mathcal{C}^{\infty}(M) \xrightarrow{d''} \Omega^{01}_M(M) \longrightarrow \cdots,$$

and we have the adjoint δ''. The operator $\Delta'' = \delta''d'' + d''\delta''$ has the form above, we can decompose into eigenspaces. If we take global sections, we find that

$$\begin{aligned} H^{\bullet}(M,\mathcal{O}_M) &= \quad {}^{\bullet}(M,\Omega^{0,\bullet}_M(M)) \\ &= \{\omega \in H^0(M,\Omega^{0,\bullet}_M(M)) \mid \Delta''\omega = 0\}, \end{aligned}$$

and again the harmonic forms are the constants.
We conclude that

$$H^{\bullet}(M,\mathcal{O}_M) = \mathrm{Hom}^{\bullet,\mathrm{alt}}(\overline{T_0(M)},\mathbb{C}). \tag{4.164}$$

This result will be used in the next chapter.

5 Compact Riemann surfaces and Abelian Varieties

5.1 Compact Riemann Surfaces

5.1.1 Introduction

Definition 5.1.1. *A* **compact Riemann surface** *is a compact complex manifold of dimension 1.*

Let S be such a surface. It has a canonical orientation (see section 4.10.2). On pages 77 and 146 we have seen that the cohomology groups of such a surface are given by

$$\begin{aligned} H^0(S, \ \ ) &= \\ H^1(S, \ \ ) &= \ ^{2g} \\ H^2(S, \ \ ) &= \end{aligned}$$

and in addition the Poincaré duality gives us an alternating perfect pairing

$$< , >_{\text{cup}}: H^1(S, \ \ ) \times H^1(S, \ \ ) \longrightarrow \ \ .$$

The number g is called **genus** of the surface. The genus g is also a measure for the complexity of the Riemann surface. We will show that a Riemann surface S of genus $g = 0$ is isomorphic to the so called **Riemann sphere** $\mathbb{P}^1(\mathbb{C})$ (see section 3.2.2 Example 15 a) and section 5.1.6). In section 4.4.5 we showed that $H^1(\mathbb{P}^1(\mathbb{C}), \ \ ) = 0$ and therefore $\mathbb{P}^1(\mathbb{C})$ has genus zero.
Of course it is clear that a holomorphic function on a compact Riemann surface S must be constant. We will work very hard to show that on any compact Riemann surface we can find a nonconstant meromorphic function (see section 4.1.1 and Corollary 5.1.13). We will achieve this goal in Corollary 5.1.13, when we prove the theorem of Riemann-Roch. Once we have a nonconstant meromorphic function f we can cover S by the two open sets U_0 (resp. U_1) where f (resp. $f^{-1} = 1/f$) is holomorphic. We get holomorphic maps

$$\begin{aligned} f : U_0 &\longrightarrow \mathbb{C} \\ w &\longmapsto f(w) \end{aligned}$$

and

$$\begin{aligned} f^{-1} : U_1 &\longrightarrow \mathbb{C} \\ u &\longrightarrow f^{-1}(u) \end{aligned}$$

and it is clear that these two maps provide a surjective map $S \to \mathbb{P}^1(\mathbb{C})$ which is also denoted by f (see section 3.2). It will turn out that this map has finite fibres and the number of points in the fibres (counted with the right multiplicities) is equal to the degree of the polar divisor (see section 4.1.1,5.1.7). This kind of maps will become a decisive tool for the understanding of Riemann surfaces (see section 5.1.7).

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5.1.2 The Hodge Structure on $H^1(S,\mathbb{C})$

We study the cohomology with coefficients in $\mathbb{C}$. I want to change the notation slightly. On our Riemann surface Ω^1_S will be the sheaf of holomorphic 1−forms. The sheaves of $\mathcal{C}^\infty$ differential forms will be denoted by $\Omega^\bullet_{S_\infty}$. We consider the de Rham complex

$$0 \longrightarrow \mathbb{C} \longrightarrow \Omega^0_{S_\infty} \longrightarrow \Omega^1_{S_\infty} \longrightarrow \Omega^2_{S_\infty} \longrightarrow 0.$$

Then

$$H^i(S,\mathbb{C}) = H^i(\Omega^\bullet_{S_\infty}(S)).$$

We recall our results from section 4.10.2. We have a complex structure on the tangent bundle T_S this is a linear transformation $I : T_S \longrightarrow T_S$ which satisfies $I^2 = -Id$. We get a decomposition

$$T_{S,} \quad = T^{1,0}_S \oplus T^{0,1}_S.$$

This provides a decomposition of the complex of differential forms, which only effects 1-forms:

$$0 \longrightarrow \mathbb{C} \longrightarrow \Omega^{0,0}_S \xrightarrow{d'+d''} \Omega^{1,0}_S \oplus \Omega^{0,1}_S \xrightarrow{d'+d''} \Omega^{1,1}_S \longrightarrow 0. \tag{5.1}$$

The sheaf $\Omega^{1,0}_S$ contains the sheaf Ω^1_S of holomorphic 1-forms. (See the section on the cohomology of holomorphic vector bundles in section 4.11.2 applied to $\mathcal{E} = \Omega^p_S$.)
In local coordinates at a point p we have

$$df = d'f + d''f = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \overline{z}}d\overline{z}, \tag{5.2}$$

and for a 1-form

$$\omega = f dz + g d\overline{z} = \omega' + \omega''$$

we get

$$d\omega = d''\omega' + d'\omega'' = (\frac{\partial f}{\partial \overline{z}} - \frac{\partial g}{\partial z})dz \wedge d\overline{z}. \tag{5.3}$$

Especially we see that a (1,0)-form $\omega = f dz$ is holomorphic if and only if $d\omega = 0$ or $d''\omega = 0$. We introduce a Hermitian form h on the tangent bundle T_S (see section 4.11.2). As I explained in general discussion such a Hermitian metric is the same as a Euclidian metric $\mathrm{Re}\, h = h_0$ on the tangent bundle T_{S_∞} which satisfies $h_0(x,y) = h_0(Ix,Iy)$ for any two tangent vectors $x,y \in T_{S_\infty,p}$ at a point p. This induces a metric on the dual bundle T^*_S which we will denote by $< , >$.
If we pick a point $p \in S$ and a local coordinate z at p then it identifies a neighborhood U_p to a disc around zero in $\mathbb{C}$. The differential dz is a generator of the $\mathcal{O}_S(U_p)$ module of holomorphic differentials $\Omega^1(U_p)$. In the neighborhood U_p of p our Hermitian metric is given by a strictly positive function (see in section 4.11.2)

$$u \longrightarrow \mathrm{Re}\, h \quad < dz(u),d\overline{z}(u) > \tag{5.4}$$

which we simply denote by $< dz,d\overline{z} >$. Since we can view U_p as an open disc in $\mathbb{C}$ we have $dz = dx + idy$ and

$$< dz,d\overline{z} > =< dx,dx > + < dy,dy >= 2 < dx,dx > \tag{5.5}$$

because the complex structure I which sends dx to dy is an isometry. The metric and the orientation give us a distinguished form ω_{top} in degree 2 which is positive with respect to he orientation and has length 1 with respect to the metric. It is given by

$$\begin{aligned}\omega_{\text{top}} &= i\frac{dz \wedge d\overline{z}}{< dz,d\overline{z} >} = i\frac{(dx + idy) \wedge (dx - idy)}{< dx,dx > + < dy,dy >} \\ &= i(-i)\frac{dx \wedge dy}{< dx,dx >} = \frac{dx \wedge dy}{< dx,dx >}.\end{aligned} \tag{5.6}$$

Of course $d\omega_{\text{top}} = 0$ and hence we see that our Riemann surface is a Kähler manifold (see Theorems 4.11.12 and 4.11.14). Now it is rather easy to check that the Hodge-$*$-operator does the following

$$\begin{aligned} *: \quad & f \longrightarrow i\overline{f}\frac{dz \wedge d\overline{z}}{< dz,d\overline{z} >} \\ *: \quad & f dz \longrightarrow i\overline{f}d\overline{z} \\ *: \quad & g d\overline{z} \longrightarrow -i\overline{g}dz \\ *: \quad & dz \wedge d\overline{z} \longrightarrow -i < dz,d\overline{z} > . \end{aligned} \tag{5.7}$$

We can introduce the adjoint operators δ',δ'' (see 4.11.12) and define the Laplacian

$$\begin{aligned}\Delta &= (d' + d'')(\delta' + \delta'') \\ &= d'\delta' + \delta'd' + d''\delta'' + \delta''d'' + d'\delta'' + \delta''d' + d''\delta' + \delta'd'' \\ &= \Delta' + \Delta'' + \quad \text{extra terms.}\end{aligned} \tag{5.8}$$

The extra terms add up to zero because the metric is a Kähler metric. (See 4.11.12.) I stated this result without proof in the general case, therefore I will carry out the calculation for our special situation. On the forms of degree 0 or 2 this is rather clear. If we consider for instance an $f \in \Omega_S^{0,0}(S)$ then

$$\begin{aligned}\delta''d'f &= \delta''\frac{\partial f}{\partial z}dz = *d'' * \frac{\partial f}{\partial z}dz \\ &= -i * d''\overline{\frac{\partial f}{\partial z}}d\overline{z} = 0\end{aligned} \tag{5.9}$$

and the same principle works for the other combinations. But for forms of degree one we have to work a little bit. Let us consider the case $\omega = f dz$. Then we see easily that two of the four terms vanish, this we see by looking at the degree:

$$\delta''d'\omega = d'\delta''\omega = 0. \tag{5.10}$$

For the other two terms we have to compute.

$$\begin{aligned}\delta' d'' f dz &= \delta' \frac{\partial f}{\partial \overline{z}} d\overline{z} \wedge dz \\ &= - * d' * \frac{\partial f}{\partial \overline{z}} dz \wedge d\overline{z}\end{aligned} \tag{5.11}$$

$$\begin{aligned}*id' \frac{\partial \overline{f}}{\partial z} < dz,d\overline{z} > &= *i \left(\frac{\partial^2 \overline{f}}{\partial z^2} < dz,d\overline{z} > + \frac{\partial \overline{f}}{\partial z} \frac{\partial}{\partial z} < dz,d\overline{z} > \right) \wedge dz \\ &= \quad \left(\frac{\partial^2 f}{\partial \overline{z}^2} < dz,d\overline{z} > + \frac{\partial f}{\partial \overline{z}} \overline{\frac{\partial}{\partial z} < dz,d\overline{z} >} \right) d\overline{z}.\end{aligned} \tag{5.12}$$

Since $< dz,d\overline{z} >$ is positive and therefore real, we have

$$\overline{\frac{\partial}{\partial z} < dz,d\overline{z} >} = \frac{\partial}{\partial \overline{z}} < dz,d\overline{z} > . \tag{5.13}$$

Now we treat the second term:

$$\begin{aligned}d''\delta' f dz &= -d'' * d' * f dz = -d'' * (-i) d' \overline{f} d\overline{z} \\ &= -id'' * \frac{\partial \overline{f}}{\partial z} dz \wedge d\overline{z} = -d'' \frac{\partial f}{\partial \overline{z}} < dz,d\overline{z} > \\ &= - \left(\frac{\partial^2 f}{\partial \overline{z}^2} < dz,d\overline{z} > + \frac{\partial f}{\partial \overline{z}} \frac{\partial}{\partial \overline{z}} < dz,d\overline{z} > \right) d\overline{z}.\end{aligned} \tag{5.14}$$

Hence we see that the two terms add up to zero and

$$(\delta' d'' + d''\delta') f dz = 0. \tag{5.15}$$

We apply our general theorem (Theorem 4.11.14) in the section on Hodge Theory to this case. We are mainly interested the first cohomology group. We get that it is given by the harmonic forms in degree one and these harmonic forms are sums of harmonic forms in the degrees (1,0) and (0,1). I will give the proof of the following Theorem for our special case. A reader, who is willing to take the general results on the theory of elliptic operators for granted or knows how to prove them, should skip these proofs.

Theorem 5.1.2 (Hodge Decomposition for Compact Riemann Surfaces)**.**
A form $\omega = f dz \in \Omega^1_{S_\infty}(S)$ is harmonic if and only if $d'\omega = d''\omega = \delta'\omega = \delta''\omega = 0$. Two of these equations are automatically fulfilled, the other two are equivalent to ω being holomorphic. We get the Hodge decomposition

$$H^1(S,\mathbb{C}) = H^0(S,\Omega^1_S) \oplus \overline{H^0(S,\Omega^1_S)}.$$

The $\mathbb{C}$ vector-space $H^1(S,\mathbb{C}) = H^1(S,\quad) \otimes_{\mathbb{R}} \mathbb{C}$ has the complex conjugation on it as an antilinear map and $\overline{H^0(S,\Omega^1_S)}$ is the complex conjugate of $H^0(S,\Omega^1_S)$ under this complex conjugation.(See section 4.11.2, especially the discussion concerning the formation of the complex conjugate space of a $\mathbb{C}$-vector space.) Then this implies

$$\dim H^0(S,\Omega^1_S) = g. \tag{5.16}$$

Proof: (Theorem 5.1.2) I want to give an indication how this consequence of the general Hodge Theory can be proved in this special situation. Only the last assertion has to be proved, we have to show that any cohomology class can be represented as the sum of a holomorphic and an antiholomorphic form. Of course there is no problem in degree zero and degree 2. So we look at the case $p = 1$ and start from a 1-form

$$\omega = \omega^{1,0} + \omega^{0,1}$$

which is closed and represents a cohomology class $[\omega] \in H^1(S,\mathbb{C})$. We have seen that we can construct a weakly convergent sequence $\omega_n = \omega + d\psi_n$ (see page 167) such that the weak limit $\omega_0 \in \Omega^1_{(2)}(S)$ satisfies

$$\int_S \omega_0 \wedge d\psi = 0$$

for all $\psi \in \mathcal{C}^\infty(S)$. We also have

$$\int_S \omega_0 \wedge \delta\eta = 0$$

for all $\eta \in \Omega^{1,1}_{S_\infty}(S)$ because this is true for all $\omega + d\psi_n$. Now we decompose

$$\omega_0 = \omega_0^{1,0} + \omega_0^{0,1}.$$

I claim that even

$$\int_S \omega_0^{1,0} \wedge d''\psi = \int_S \omega_0^{1,0} \wedge \delta'\eta = 0$$

for all $\psi \in \mathcal{C}^\infty(S)$,$\eta \in \Omega^{1,1}_{S_\infty}(S)$. We have

$$\int_S \omega_0^{1,0} \wedge d''\psi + \int_S \omega_0^{0,1} \wedge d'\psi = 0 \tag{+,d}$$

and

$$\int_S \omega_0^{1,0} \wedge \delta'\eta + \int_S \omega_0^{0,1} \wedge \delta''\eta = 0 \tag{+,δ}$$

for all ψ,η. We take $\eta = *\overline{\psi}$ and then we get from our local formulae (see equation 5.7)

$$\delta''\eta = -id'\psi$$
$$\delta'\eta = id''\psi$$

and the second line becomes

$$i\int_S \omega_0^{1,0} \wedge d''\psi - i\int_S \omega_0^{0,1} \wedge d'\psi = 0. \tag{$-$,d}$$

Subtracting $i(+,d)$ from $(-,d)$ we find

$$\int_S \omega_0^{1,0} \wedge d''\psi = \int_S \omega_0^{0,1} \wedge d'\psi = 0$$

for all $\psi \in \Omega^{0,0}_{S_\infty}(S)$. These two orthogonality relations do not involve the Hermitian metric anymore.
We want to conclude that these orthogonality relations imply that $\omega_0^{1,0}$ is itself a holomorphic 1-form. The holomorphicity is a local property of $\omega_0^{1,0}$. We choose a point p and a neighborhood U_p such that we can identify $(U_p,\mathcal{O}_{U_p})$ with the disc $(D,\mathcal{O}_D)$. Let z be the coordinate function on D. Our differential form can be written $\omega_0^{1,0} = f(z)dz$ and since the restriction of f to D must be square integrable, we have

$$\int_D |f(z)|^2 i \; \frac{dz \wedge d\overline{z}}{\langle dz,d\overline{z}\rangle}$$

Since the function $< dz,d\overline{z} >$ is bounded and bounded away from zero the square integrability condition is equivalent to

$$\int_D |f(z)|^2 i \; dz \wedge d\overline{z} < \infty.$$

Now we exploit the orthogonality relation $\langle \omega_0^{1,0},d''\psi\rangle = 0$ for $\mathcal{C}^\infty$-functions ψ with compact support in D, we have

$$\int_D \omega_0^{1,0} \wedge d''\psi = 0$$

for all compactly supported $\psi \in \Omega^0_\infty(D)$
We introduce polar coordinates and write

$$f(z) = f(r,\varphi) = \sum_{m\in} a_m(r)e^{im\varphi}$$

and $a_m(r)$ is square integrable on $[0,1]$ with respect to rdr. Square integrability means that

$$\sum_m \int_0^1 |a_m(r)|^2 rdr < \infty$$

We can choose our function ψ, and we take

$$\psi(z) = b(r)e^{-in\varphi}$$

where $b(r)$ is $\mathcal{C}^\infty$ on $[0,1)$ and has compact support in $[0,1)$. Then an easy computation shows

$$\frac{\partial}{\partial \overline{z}}\psi(z) = \frac{1}{2}\; e^{i\varphi}\left(\frac{\partial}{\partial r}b(r) + \frac{n}{r}b(r)\right)e^{-in\varphi} = \frac{1}{2}\left(\frac{\partial}{\partial r}b(r) + \frac{n}{r}b(r)\right)e^{-i(n-1)\varphi}.$$

Consequently our assumption implies

$$\int_0^1 a_{n-1}(r)\left(\frac{\partial}{\partial r}b(r) + \frac{n}{r}b(r)\right) rdr = 0$$

for all such choices of $b(r)$ and n. But now we know enough elementary analysis to show that this implies that for all $b(r)$

$$\int_0^1 \left(-r\frac{\partial}{\partial r}a_{n-1}(r) + na_{n-1}(r)\right) b(r)dr = 0$$

and therefore we can conclude that

$$a_{n-1}(r) = c_{n-1}r^{n-1}$$

with some constant c_{n-1}. It follows that $a_n(r) = 0$ for $n < 0$ since $\int_0^1 r^{2n} r\, d = \infty$. Hence we get

$$f(z) = \sum_{n=0}^{\infty} c_n r^n e^{in\varphi}$$

and

$$\sum |c_n|^2 \cdot \frac{1}{(2n+2)^2} < \infty.$$

This is good enough to show that f is holomorphic on the disc. This finishes the proof of Theorem 5.1.2. □

The cup product $< , >_{\text{cup}}$: $H^1(S,\ ) \times H^1(S,\ ) \longrightarrow H^2(S,\ ) =\ $ extends to a bilinear pairing $< , >_{\text{cup}}$: $H^1(S,\mathbb{C}) \times H^1(S,\mathbb{C}) \longrightarrow H^2(S,\mathbb{C}) = \mathbb{C}$ and we know (see section 4.10.1) that this pairing is given by

$$< [\omega_1],[\omega_2] >_{\text{cup}} = \int_S \omega_1 \wedge \omega_2$$

where ω_1,ω_2 are closed forms which represent the classes $[\omega_1]$,$[\omega_2]$ in the cohomology.

Lemma 5.1.3. *With respect to the pairing* $< , >_{\text{cup}}$ *the two subspaces* $H^0(S,\Omega^1_S)$,$\overline{H^0(S,\Omega^1_S)}$ *are maximal isotropic spaces and hence the cup product induces a perfect bilinear pairing*

$$< , >_{\text{cup}}: H^0(S,\Omega^1_S) \times \overline{H^0(S,\Omega^1_S)} \longrightarrow \mathbb{C}.$$

5.1.3 Cohomology of Holomorphic Bundles

For any holomorphic vector bundle on $\mathcal{E}$ on S we consider the Dolbeault complex

$$0 \longrightarrow \mathcal{E} \xrightarrow{d''} \Omega^{0,0}_S(\mathcal{E}) \xrightarrow{d''} \Omega^{0,1}_S(\mathcal{E}) \longrightarrow 0. \tag{5.17}$$

We changed our notation slightly and write $\Omega^{p,q}_S(\mathcal{E})$ instead of $\mathcal{E}_\infty \otimes \Omega^{p,q}$. The cohomology groups of $\mathcal{E}$ are computed from the complex

$$0 \longrightarrow \Omega^{0,0}_S(\mathcal{E})(S) \xrightarrow{d''} \Omega^{0,1}_S(\mathcal{E})(S) \longrightarrow 0. \tag{5.18}$$

Now choose in addition a Hermitian metric $< , >_h$ on the bundle $\mathcal{E}$ and on T_S.
The metrics on $\mathcal{E}$ and on T_S provide an adjoint operator $\delta'' : \Omega^{0,1}_\infty(\mathcal{E})(S) \longrightarrow \Omega^{0,0}(\mathcal{E})(S)$ and now Hodge Theory implies (see section 4.11.2 and consequence (c) of Theorem 4.11.14.):

Theorem 5.1.4. *We have that:*

$$H^0(S,\mathcal{E}) = \{s \in \Omega_S^{0,0}(\mathcal{E}) | d''s = 0\}$$
$$H^1(S,\mathcal{E}) = \{s \in \Omega_S^{0,1}(\mathcal{E}) | \delta''s = 0\}.$$

and the cohomology groups are finite dimensional.

Again I stress that the proof of this finite dimensionality even in this one dimensional case needs some input from analysis. Either we use the theory of elliptic operators or some results from the theory of topological vector spaces. (See [Fo] or [Se2] for the second method.)
Actually for the cohomology in degree zero $H^0(S,\mathcal{E})$ the finite dimensionality is not difficult but the finite dimensionality of $H^1(S,\mathcal{E})$ is by no means easy. To make this book more selfcontained I will outline a proof of these finite dimensionality results for our special case of compact Riemann surfaces. The reader who is willing to believe the general results on elliptic operators may skip up to page 191.
The equality $H^0(S,\mathcal{E}) = \{s \in \Omega_S^{0,0}(\mathcal{E}) | d''s = 0\}$ is tautological. We prove the finite dimensionality. We proceed by induction on the rank of the bundle. Let us assume that we have a non zero section $s \in H^0(S,\mathcal{E})$. Then we show

Lemma 5.1.5. *To this non zero section s we can attach a line subbundle $\mathcal{L} \subset \mathcal{E}$ such that $\mathcal{E}/\mathcal{L}$ is again a vector bundle and $s \in H^0(S,\mathcal{L})$.*

Proof: The section s provides a map

$$\begin{aligned} \mathcal{O}_S &\longrightarrow \mathcal{E} \\ f &\longmapsto fs \end{aligned}$$

for any holomorphic function f on some open subset $U \subset S$. This yields indeed a line subbundle $\mathcal{L}'$ but it is not yet the one we want. If we are at a point $x \in S$ where $s(x) = 0$, then we can choose a neighborhood U_x and a local trivialization of $\mathcal{E}$ by local sections $e_1, \ldots, e_n$ which are nowhere zero on U_x. Our section s can be written as

$$s = \sum_{i=1}^{n} f_i e_i$$

with f_i holomorphic at x and $f_i(x) = 0$ for all $i = 1, \ldots, n$. This implies that the set of zeroes of s is a finite subset of S. But since $\dim S = 1$ we have a local uniformizer $\pi_x \in \mathfrak{m}_x \subset \mathcal{O}_{S,x}$ and $f_i = \pi_x^{n_i} h_i$ where $h_i \in \mathcal{O}_{S,x}^*$. Let m be the minimum of the n_i. Then

$$\pi_x^{-m} s = \sum \pi_x^{-m} f_i e_i$$

extends to a section in $\mathcal{E}$ which is defined over U_x. This section defines a subbundle $\mathcal{L}^{(x)} \subset \mathcal{E}|_{U_x}$. But this line subbundle coincides with the above bundle $\mathcal{L}'$ if we restrict to the complement of the point x. Hence we see that we can glue the $\mathcal{L}'$ and the $\mathcal{L}^{(x)}$ to a line bundle $\mathcal{L}$ on S. We have $\mathcal{L} \subset \mathcal{E}$, the quotient $\mathcal{E}/\mathcal{L}$ is a vector bundle of smaller rank and $s \in H^0(S,\mathcal{L})$. This reduces the proof of the finite dimensionality to the case of line bundles. But if we have a line bundle $\mathcal{L}$ and a section $s \neq 0$, then we look again at the inclusion $\mathcal{O}_S \longrightarrow \mathcal{L}$ induced by the section, and we get an exact sequence

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}/\mathcal{O}_S \longrightarrow 0$$

and now $\mathcal{L}/\mathcal{O}_S$ is a skyscraper sheaf (see section 4.1.1). Since $H^0(S,\mathcal{O}_S) = \mathbb{C}$ and obviously $\dim H^0(S,\mathcal{L}/\mathcal{O}_S) < \infty$, we are through.
The proof of the second assertion in Theorem 5.1.4 is much more difficult. We begin by the observation that the duality pairing gives us a linear map

$$\Psi : H^1(S,\mathcal{E}) \longrightarrow H^0(S,\mathcal{E}^\vee \otimes \Omega^1)^\vee.$$

Since we are already in the highest degree we have

$$\Omega^{0,1}(\mathcal{E})(S)/d''(\Omega^{0,0}(\mathcal{E})(S)) = H^1(S,\mathcal{E})$$

and the linear map is induced by the map also called Ψ

$$\Psi : \Omega^{0,1}(\mathcal{E})(S) \longrightarrow H^0(S,\mathcal{E}^\vee \otimes \Omega^1)^\vee$$

$$\omega_1 \longmapsto \left\{ \omega \mapsto \int_S e_0(\omega_1 \wedge \omega) \right\}$$

where $\omega_1 \wedge \omega \in \Omega^{1,1}(\mathcal{E}^\vee \otimes \mathcal{E})(S)$ and e_0 is the evaluation of the duality pairing. If $\omega \neq 0$ then we can find an ω_1 such that $\Psi(\omega_1)(\omega) \neq 0$, this implies that the homomorphism Ψ is surjective. Once we have shown that it is injective, then the proof is finished.
We have the operator

$$d'' : \Omega_S^{0,0}(\mathcal{E})(S) \longrightarrow \Omega_S^{0,1}(\mathcal{E})(S)$$

and the image of d'' lands in the kernel of Ψ. The pairing

$$\Omega^{0,1}_{(2)}(\mathcal{E})(S) \times \Omega^{1,0}_{(2)}(\mathcal{E}^\vee)(S) \longrightarrow \mathbb{C},$$

which is given by

$$(\omega_1,\omega) \longmapsto \int_S e_0(\omega_1 \wedge \omega)$$

is a duality between the two Hilbert spaces. We see that Ψ extends to a continuous linear map

$$\Psi : \Omega^{0,1}_{(2)}(\mathcal{E})(S) \longrightarrow H^0(S,\mathcal{E}^\vee \otimes \Omega^1)^\vee$$

By definition the closure of the space spanned by the $d''f$ is the orthogonal complement of its orthogonal complement. This last subspace spanned by those $\omega \in \Omega^{1,0}(\mathcal{E}^\vee)_{(2)}(S)$ which satisfy

$$\int_S e_0(d''f \wedge \omega)$$

for all $f \in \Omega^{0,0}(\mathcal{E})(S)$. This means that ω is a weak solution for the equation $d''\omega = 0$. Then the same reasoning as in the proof of Lemma 5.1.2 shows that ω must indeed be holomorphic. Hence we see that the closure of the space spanned by the $d''f$ is in fact the kernel of Ψ, the $d''f$ span a dense subspace.
Now we have to solve the differential equation $d''f = \omega$ for a given element $\omega \in \ker(\Psi) \subset \Omega^{0,1}(\mathcal{E})(S)$. We try to solve the equation locally, we choose a small disc D such that the bundle becomes trivial over D.
Let us assume we have a form $\omega \in \Omega^{0,1}(\mathcal{E})(D)$ and let us assume that this form is square integrable.

Then I claim that we can find an $\eta \in \Omega^{0,0}(\mathcal{E})(D)$ such that $d''\eta = \omega$ and that we can bound the L^2-norm of η:

$$\|\eta(z)\|_{2,D} \leq C\|\omega\|_{2,D}.$$

It is clear that the validity of the L^2-estimates does not depend on the Hermitian metric. These two facts together allow us to restrict to the case where D is the unit disc, where $\mathcal{E}|D \xrightarrow{\sim} \mathcal{O}_D$ and the metric is the trivial metric. Then we have to show: If $f : D \longrightarrow \mathbb{C}$ is a $\mathcal{C}^\infty$ -function on D which is square integrable, i.e.

$$\|f\|_2^2 = \int_D |f(z)|^2 i dz \wedge d\overline{z} < \infty,$$

then we can find a $\mathcal{C}^\infty$-function u on D which satisfies

$$\frac{\partial u}{\partial \overline{z}} = f$$

and

$$\|u\|_2 \leq C\|f\|_2.$$

The point is that we can write down an explicit solution for this differential equation:

$$u(z) = \frac{1}{2\pi i}\int_D \frac{f(\zeta)}{\zeta - z} d\zeta \wedge d\overline{\zeta}.$$

(I thank Ingo Lieb for showing me the following argument.) I claim that this is a solution which fulfills the required estimates. It is easy to see that this function is $\mathcal{C}^\infty$. If $z_0 \in D$, then we can find a $\mathcal{C}^\infty$-function χ on D which is one on a small neighborhood of z_0 and zero on a small neighborhood of the boundary of D. Then

$$u(z) = \frac{1}{2\pi i}\int_D \frac{\chi(\zeta)f(\zeta)}{\zeta - z} d\zeta \wedge d\overline{\zeta} + \frac{1}{2\pi i}\int_D \frac{(1-\chi(\zeta))f(\zeta)}{\zeta - z} d\zeta \wedge d\overline{\zeta}.$$

The second summand is holomorphic at z_0 and hence annihilated by $\partial/\partial\overline{z}$. The first summand can be written as an integral over $\mathbb{C}$ and a substitution yields that we have

$$\frac{1}{2\pi i}\int_D \frac{\chi(\zeta)f(\zeta)}{\zeta - z} d\zeta \wedge d\overline{\zeta} = \frac{1}{2\pi i}\int \frac{\chi(\zeta+z)f(\zeta+z)}{\zeta} d\zeta \wedge d\overline{\zeta}.$$

We can differentiate under the integral sign because $\chi \cdot f$ has compact support and the singularity disappears if we change to polar coordinates. This proved that $u(z)$ is differentiable and that $\frac{\partial u(z)}{\partial \overline{z}}$ does not depend on χ.
Now it is an amusing exercise to show that for $f = 1$ we have $u(z) = \overline{z}$ and from this it follows easily that u satisfies the differential equation for all f.
We have to prove the estimate. I think it is also very easy to see that the integral

$$\frac{1}{2\pi i}\int_D \frac{1}{|\zeta - z|} d\zeta \wedge d\overline{\zeta}$$

is bounded by a constant not depending on z. We may work with polar coordinates around the point z.

To get the L^2-estimate we start from

$$|u(z)|^2 = \frac{1}{4\pi^2}\left|\int_D \frac{f(\zeta)}{\zeta - z} d\zeta \wedge d\overline{\zeta}\right|^2 \leq \frac{1}{4\pi^2}\left|\int_D \frac{|f(\zeta)|}{|\zeta - z|^{1/2}} \cdot \frac{1}{|\zeta - z|^{1/2}} d\zeta \wedge d\overline{\zeta}\right|^2.$$

This is the square of the scalar product of two L^2-functions, and we get by Cauchys integral formula that the right hand side is

$$\leq \frac{1}{4\pi^2}\left(\int_D \frac{|f(\zeta)|^2}{|\zeta - z|} d\zeta \wedge d\overline{\zeta}\right)\left(\int_D \frac{1}{|\zeta - z|} d\zeta \wedge d\overline{\zeta}\right).$$

We mentioned already that the second factor is bounded by a constant not depending on z and consequently we get

$$\frac{1}{2\pi i}\int_D |u(z)|^2 dz \wedge d\overline{z} \leq C \int_D \int_D \frac{|f(\zeta)|^2}{|z - \zeta|} d\zeta \wedge d\overline{\zeta} \cdot dz \wedge d\overline{z},$$

and if we change integration and use our above estimate a second time, then we get

$$\|u\|_2^2 \leq C' \cdot \|f\|_2^2.$$

Inside of our Hilbert space $\Omega^{0,0}_{(2)}(\mathcal{E})(D)$ we can consider the holomorphic square integrable functions. I claim that this subspace is closed and even better:

Lemma 5.1.6. *Any weakly convergent sequence of holomorphic functions η_n with bounded $\Omega^{0,0}_{(2)}(\mathcal{E})(D)$-norm converges locally uniformly to a holomorphic function on D.*

Proof: This is an immediate consequence of Cauchys integral formula. We pick a point $Q \in D$ and we put three concentric discs around Q:

$$Q \in D_1 \subset D_2 \subset D_3 \subset D$$

each of them is slightly bigger than the previous one. If we have a circle $\Gamma \subset D_3 \setminus D_2$ then we get from Cauchys formula for $z \in D_2$

$$\eta_n(z) = \frac{1}{2\pi i}\int_\Gamma \eta_n(\zeta)\frac{1}{\zeta - z} d\zeta.$$

Now we integrate over all Γ_r between D_2 and D_3 and consider $z \in D_1$. We get

$$\eta_n(z) = \frac{c}{2\pi i}\int_{D_3 \setminus D_2} \eta_n(\zeta)\frac{1}{\zeta - z} d\zeta \wedge d\overline{\zeta},$$

where c is a constant depending on the width of the annulus. We can read this as a scalar product, since the sequence η_n is weakly convergent to η we see that η_n converges pointwise to the function

$$\widetilde{\eta} : z \longmapsto \frac{c}{2\pi i}\int_{D_3 \setminus D_2} \eta(\zeta)\frac{1}{\zeta - z} d\zeta \wedge d\overline{\zeta},$$

which is holomorphic on D_1. But now the Cauchy formula also gives us that the η_n are equicontinous and then it follows that the convergence $\eta_n \longrightarrow \widetilde{\eta}$ is locally uniform and that $\eta = \widetilde{\eta}$. □

Lemma 5.1.7. *For any open set $U \subset S$ (this will be a disc D or the entire S) we have a decomposition*

$$\Omega^{0,0}_{(2)}(\mathcal{E})(U) = \widetilde{\Omega^{0,0}_{(2)}(\mathcal{E})}(U) \oplus \mathcal{E}_{(2)}(U)$$

where the second summand is the closed space of the holomorphic square integrable sections and the first summand is its orthogonal complement. For $U = S$ the second summand is of course the finite dimensional subspace $H^0(S,\mathcal{E}^\vee \otimes \Omega^1)$.

We choose an $\omega \in \ker(\Psi)$, and we have seen that we can find a sequence of functions $\psi_n \in \widetilde{\Omega^{0,0}(\mathcal{E})}(S)$ such that

$$\|d''\psi_n - \omega\|_2 \longrightarrow 0.$$

I claim that the sequence of L^2-norms $\{\|\psi_n\|\}_{n\in}$ is bounded. To see this we cover S by a finite family of discs

$$S = \bigcup_{\alpha\in A} D_\alpha.$$

We restrict the members of the family $\{\psi_n\}_{n\in}$ to the discs D_α and call the restrictions $\psi_n^{(\alpha)}$. Now we have an orthogonal decomposition of the restriction

$$\Omega^{0,0}_{(2)}(\mathcal{E})(D_\alpha) = \widetilde{\Omega^{0,0}_{(2)}(\mathcal{E})}(D_\alpha) \oplus \mathcal{E}_{(2)}(D_\alpha) \tag{5.19}$$

and accordingly we have $\psi_n^{(\alpha)} = \psi_n^{(\alpha,\prime)} + \psi_n^{(\alpha,\text{hol})}$. We get $d''\psi_n^{(\alpha)} = d''\psi_n^{(\alpha,\prime)}$. From previous results (see page 187 f.) we know that we have an $\eta_n^{(\alpha)} \in \Omega^{0,0}_S(\mathcal{E})(D_\alpha)$ for which $d''\eta_n^\alpha = d''\psi_n^\alpha$ and

$$\|\eta_n^\alpha\|_{2,D_\alpha} \leq C\|d''\psi_n^\alpha\|_{2,D_\alpha}.$$

We have $\eta_n^\alpha = \psi_n^{(\alpha,\prime)} + h_n$ with $h_n \in \mathcal{E}_{(2)}(\mathcal{E})(D_\alpha)$. The orthogonality of the above decomposition implies that $\|\psi_n^{(\alpha,\prime)}\|$ stays bounded. We get

$$\|\psi_n^{(\alpha,\prime)}\|_{2,D_\alpha} \leq \|\eta_n^\alpha\|_{2,D_\alpha}. \tag{5.20}$$

Hence we see: If the sequence $\{\|\psi_n\|_2\}_n$ is unbounded, then there must be an α such that sequence $\|\psi_n^{(\alpha,\text{hol})}\|_{2,D_\alpha}$ is unbounded.
Now we consider the sequence of functions $\psi_n/\|\psi_n\|_2$. We can extract a subsequence which is weakly convergent. On any D_α this sequence has the same limit as $\psi_n^{(\alpha,\text{hol})}/\|\psi_n\|_2$, hence it converges to a holomorphic function. This function must be zero because our ψ_n where chosen from the orthogonal complement of the holomorphic sections. It follows from Lemma 5.1.6 that the sequence $\psi_n/\|\psi_n\|$ converges uniformly to zero. This cannot be the case because the L^2-norm of the members of the sequence is one. We get a contradiction.
So we see that the sequence of norms $\|\psi_n\|$ is bounded. Now we do what we always do at this point. We extract a weakly convergent subsequence. If ψ is the limit of this subsequence we found the element which satisfies

$$\int_S (\psi \wedge d''\eta) = \omega \wedge \eta = 0 \tag{5.21}$$

for all sections $\eta \in \Omega_S^{1,0}(S)$ which have compact support. Now an argument similar to the one in the proof of Lemma 5.1.2 yields that ψ is $\mathcal{C}^\infty$ and that we have $\omega = d''\psi$. This finishes the proof of Theorem 5.1.4. □

We have seen that Ψ is an isomorphism, so as a byproduct we get:

Lemma 5.1.8. *The duality pairing in section 4.11.2*

$$H^1(S,\mathcal{E}) \times H^0(S,\mathcal{E}^\vee \otimes \Omega_S^1) \longrightarrow \mathbb{C}$$

is non degenerate.

If our vector bundle is the structure sheaf $\mathcal{O}_S$ then this gives us a canonical identification

$$H^1(S,\mathcal{O}_S^1) \xrightarrow{\sim} H^0(S,\Omega_S^1)^\vee.$$

5.1.4 The Theorem of Riemann-Roch

On the Picard Group

Recall that the Picard group $\mathrm{Pic}(S)$ of a compact Riemann surface is the group of isomorphism classes of holomorphic line bundles on the Riemann surface (See 4.10.3. This group is isomorphic to $H^1(S,\mathcal{O}_S^*)$.

The exact sequence of sheaves (see 4.10.3 and section 4.3.3)

$$0 \longrightarrow \quad \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_S^* \longrightarrow 1$$

on our Riemann surface provides a long exact cohomology sequence

$$0 \longrightarrow \quad \longrightarrow \mathbb{C} \longrightarrow \mathbb{C}^\times \longrightarrow H^1(S, \ ) \longrightarrow H^1(S,\mathcal{O}_S) \to H^1(S,\mathcal{O}_S^*) \longrightarrow H^2(S, \ ) \longrightarrow 0$$

(We have seen that $H^2(S,\mathcal{O}_S) = 0$ because the Dolbeault complex stops in degree 1.) In 4.1.1 we definied a sheaf $\mathcal{O}_S(D)$ to any divisor D. It is clear that we have:

Lemma 5.1.9. *For any divisor D then $\mathcal{O}_S(D)$ is a line bundle and we have that $\mathcal{O}_S(D) \otimes \mathcal{O}_S(D_1) \xrightarrow{\sim} \mathcal{O}_S(D + D_1)$ and that $\mathcal{O}_S(D) \xrightarrow{\sim} \mathcal{O}_S$ if and only if D is principal. Hence we get*

$$\mathrm{Div}(S)/\text{ principal divisors } \xrightarrow{\sim} \mathrm{Pic}(S).$$

If we have a line bundle $\mathcal{L}$ and a divisor D we define $\mathcal{L}(D) = \mathcal{L} \otimes \mathcal{O}_S(D)$. For a line bundle $\mathcal{L}$, which we can view as an element $[\mathcal{L}]$ in $H^1(S,\mathcal{O}_S^*)$ we define the degree $\deg(\mathcal{L}) = \delta([\mathcal{L}]) \in H^2(S, \ ) = \ $. (We have a canonical orientation on S.) This degree has various properties.

Lemma 5.1.10. *If we have a line bundle $\mathcal{L}$ on S and a point P then*

$$\deg(\mathcal{L}(P)) = \deg(\mathcal{L}) + 1$$

Proof: This is a special case of Proposition 4.10.13. □

Exercises

Exercise 27. If we choose a Hermitian metric h on our line bundle $\mathcal{L}$, if we pick a point P and a neighborhood U_P and a local section $s \in \mathcal{L}(U_p)$ which is a generator for all points in U_P then we can form the expression

$$d''d' \log h(s,s) = \omega_h.$$

This is a (1,1)-form on S which is closed and it does not depend on the choice of the generator s. Show that the class of this form in $H^2(S,\mathbb{C})$ is the Chern class

$$c_1(\mathcal{L}) = \deg(\mathcal{L}).$$

Exercise 28. I refer to the proof of Lemma 5.1.10: We consider the holomorphic 1-form $z_P^{-1} dz_P$ on the annulus $U_1 \cap U_2$. It is clear that we can extend this form to a $\mathcal{C}^\infty$-form η on the disc $U_2 = D_P$ (we simply multiply it by a $\mathcal{C}^\infty$-function which is one on the annulus and zero in a neighborhood of P). If we consider $d''\eta$ we get a (1,1) form on the disc D_P which has compact support because it vanishes on the annulus. Hence it defines a class in $H^2_c(D_P,\mathbb{C})$, this maps to $H^2(S,\mathbb{C})$.
Show that this class is the class $\delta([\mathcal{O}_S(P)])$. This way we can construct a form of type (1,1) which represents the degree. This form can be written as a boundary on any open set in S which misses a small disc around P.

Exercise 29. Let us assume that we have an arbitrary compact complex manifold X and a divisor $D \subset X$ which is locally given on the open sets of a covering $\mathfrak{U} = \{U_\alpha\}$ by one equation $f_\alpha = 0$. We choose a Hermitian metric on X. We choose our covering in such a way that we place small balls around the points on D and choose a finite subcovering $\mathfrak{U} = \{U_\alpha\}$ of D. Then we supplement it by an open set U_0 which is the set of points having distance $> \varepsilon$ from D.
Construct a (1,1)-form ω_D which has support in the complement of U_0 and which represents the class $c_1(\mathcal{O}_X(D))$. Show that this form is a boundary outside of the support of D.

Exercise 30. If we have divisors $D_1, D_2, \dots, D_d$ ($d = \dim X$) which intersect nicely then we can consider the intersection number $D_1 \cdot D_2 \cdot \ldots \cdot D_d$.
Show that this intersection number can also be computed by the integral

$$\int_X \omega_{D_1} \wedge \omega_{D_2} \wedge \ldots \wedge \omega_{D_d}.$$

Exercise 31. Of course we can attach to any line bundle $\mathcal{L}$ its Chern class $c_1(\mathcal{L}) \in H^2(X,\ )$. If we have d such bundles $\mathcal{L}_1, \dots, \mathcal{L}_d$ we can compute their intersection number and we can take the cup product of their Chern classes which gives an element in $H^{2d}(X,\ ) =$. Exercise 30 gives us the equality of these numbers

$$\mathcal{L}_1 \cdot \ldots \cdot \mathcal{L}_d = c_1(\mathcal{L}_1) \cup \ldots \cup c_1(\mathcal{L}_d)$$

Show the equality of these numbers without using the de Rham isomorphism.

The Theorem of Riemann-Roch

Lemma 5.1.10 from above implies:
If we have a line bundle $\mathcal{L}$ and a non-zero section $s \in H^0(S,\mathcal{L})$ then on a suitably small open set U we can write $s = f\,t$ where t is a local generator and f is holomorphic. This function f defines a divisor on U, it is the divisor of its zeroes (see section 4.1.1). Since we can do this everywhere we get a divisor $D = \mathrm{Div}(s)$ and it is clear that $\mathcal{L} \sim \mathcal{O}_S(D)$. Then it follows from Lemma 5.1.10 that

$$\deg(\mathcal{L}) = \deg(D). \tag{5.22}$$

Hence we can conclude that the degree of a line bundle which has non zero sections must be ≥ 0.
If D is the divisor of a meromorphic function f then this function defines a section in $\mathcal{O}(D)$ and f^{-1} defines a section in $\mathcal{O}(-D)$ and consequently we must have

$$\deg(\mathrm{Div}(f)) = 0. \tag{5.23}$$

We also can conclude that

Corollary 5.1.11. *A line bundle of degree zero has a non zero section if and only if it is trivial.*

We can formulate the

Theorem 5.1.12 (Theorem of Riemann-Roch)**.** *If $\mathcal{L}$ is a line bundle on a compact Riemann surface S then*

$$\dim \quad H^0(S,\mathcal{L}) - \dim \quad H^1(S,\mathcal{L}) = \deg(\mathcal{L}) + 1 - g.$$

We have

$$\dim \quad H^1(S,\mathcal{L}) = \dim \quad H^0(S,\mathcal{L}^{-1} \otimes \Omega^1_S).$$

Furthermore we have $\deg(\Omega^1_S) = 2g-2$ *and consequently* $\dim \quad H^1(S,\mathcal{L}) = 0$ *if* $\deg(\mathcal{L}) \geq 2g-1$.

Proof: This is now more or less obvious. We proved the finite dimensionality of the cohomology groups H^0 and H^1 in the previous section. We write $\chi(\mathcal{L})$ for the left hand side.
We have the isomorphism $H^0(S,\Omega^1_S) \xrightarrow{\sim} H^1(S,\mathcal{O}_S)$. This implies that

$$\dim \quad H^1(S,\mathcal{O}_S) = g \tag{5.24}$$

and hence the assertion is true for $\mathcal{L} = \mathcal{O}_S$. If we want to prove it for our given sheaf $\mathcal{L}$ we pick a point P and consider the exact sequence

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}(rP) \longrightarrow \mathcal{L}(rP)/\mathcal{L} \longrightarrow 0 \tag{5.25}$$

for a large value of r. Then the dimension of the space of sections of the skyscraper sheaf becomes large and this space of sections is mapped to the finite dimensional $H^1(S,\mathcal{L})$. This implies that eventually $H^0(S,\mathcal{L}(rP))$ will be non zero. But then a non zero section gives us an inclusion $\mathcal{O}_S \hookrightarrow \mathcal{L}(rP)$ with a skyscraper quotient $\mathcal{S}$ (see proof of Lemma 5.1.5). We have the exact sequence

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{L}(rP) \longrightarrow \mathcal{S} \longrightarrow 0 \tag{5.26}$$

and a glance at the resulting exact sequence yields that

$$\chi(\mathcal{L}(rP)) - \chi(\mathcal{O}_S) = \dim H^0(S,\mathcal{S}). \tag{5.27}$$

This is also the degree of $\mathcal{L}(rP)$ (Lemma 5.1.10 iterated). Hence we have proved the first formula for $\mathcal{L}(rP)$. Then the same argument applied backwards proves it for $\mathcal{L}$.
It remains to prove the formula for the degree of Ω^1_S. To get this we apply the first formula in the theorem to the sheaf Ω^1_S. We get

$$\dim \ H^0(S,\Omega^1_S) - \dim \ H^1(S,\Omega^1_S) = \deg(\Omega^1_S) + 1 - g. \tag{5.28}$$

The left hand side is equal $g-1$ and the theorem is proved. □

I would like to stress again that the real difficulty in the proof of the Riemann-Roch Theorem is to show that $H^1(S,\mathcal{O}_S)$ is finite dimensional. In the course of this proof we saw:

Corollary 5.1.13. *For any line bundle $\mathcal{L}$ on a compact Riemann surface S and for any point $P \in S$, we can find an integer $r > 0$ such that* $\dim H^0(S,\mathcal{L}(rP)) > 0$. *Even more precisely, for $r >> 0$ we have* $\dim H^0(S,\mathcal{L}((r+1)P)) = \dim H^0(S,\mathcal{L}(rP))+1$. *This implies that we can find a meromorphic function which has a first order zero or a first order pole at a given point.*

5.1.5 The Algebraic Duality Pairing

At this point we have proved a very strong finiteness result: Any line bundle $\mathcal{L}$ on a compact Riemann surface S has a very simple acyclic resolution, we take an effective divisor $D = \Sigma n_p P$ with sufficiently large degree and then

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}(D) \longrightarrow \mathbb{L}_D \longrightarrow 0 \tag{5.29}$$

is an acyclic resolution of $\mathcal{L}$ (see section 4.1.1). We get the exact sequence

$$0 \longrightarrow H^0(S,\mathcal{L}) \longrightarrow H^0(S,\mathcal{L}(D)) \longrightarrow H^0(S,\mathbb{L}_D) \longrightarrow H^1(S,\mathcal{L}) \longrightarrow 0. \tag{5.30}$$

We have seen that $\dim H^1(S,\mathcal{L}) = \dim H^0(S,\mathcal{L}^{-1} \otimes \Omega^1_S)$ but we can prove a stronger result. We construct a new non degenerate bilinear pairing

$$H^1(S,\mathcal{L}) \times H^0(S,\mathcal{L}^{-1} \otimes \Omega^1_S) \longrightarrow \mathbb{C}. \tag{5.31}$$

To get this pairing we represent an element $\xi \in H^1(S,\mathcal{L})$ as the image under the boundary map. We lift it to an element

$$\widetilde{\xi} = (\ldots \widetilde{\xi}_P \ldots)_{P\in|D|} \in H^0(S,\mathbb{L}_D)$$

where $|D|$ is the support of D. We choose small discs D_P around these P such that we can trivialize the bundle $\mathcal{L}$ over these discs by non vanishing sections $t_P \in H^0(D_P,\mathcal{L})$. Then the components $\widetilde{\xi}_P$ can be written as

$$\widetilde{\xi}_P = \left(\frac{b_{-n}}{z_P^n} + \cdots + \frac{b_{-1}}{z_P}\right) t_P$$

where z_P is a local parameter at P. If now $\eta \in H^0(S,\mathcal{L}^{-1} \otimes \Omega_S^1)$, we can write the restriction of η to D_p in the form

$$\eta\,|_{D_P} = t_P^{-1} \cdot f(z_P) dz_P,$$

and we can consider the product

$$\widetilde{\xi}_P \eta = \left(\frac{b_{-n}}{z_P^n} + \cdots + \frac{b_{-1}}{z_P}\right) \cdot f(z_P) dz_P = \left(\frac{a_{-n}}{z_P^n} + \cdots \frac{a_{-1}}{z_P} + \cdots\right) dz_P = \omega_P,$$

this is a holomorphic 1-form on $D_p \setminus \{P\}$ which may have a pole (i.e. it is a meromorphic 1-form). To such a meromorphic 1-form we attach its residue at P, it is given by

$$\mathrm{Res}_P \left(\frac{a_{-n}}{z_P^n} + \cdots \frac{a_{-1}}{z_P} \cdots\right) dz_P = a_{-1}. \tag{5.32}$$

It is not clear a priori that this residue is well defined but everybody who still wants to continue reading this book should know the formula

$$a_{-1} = \frac{1}{2\pi i} \int_\Gamma \omega_P \tag{5.33}$$

where Γ is a path in $D_P \backslash \{P\}$ which winds counterclockwise around P just once. The integral on the right hand side is defined independently of the choice of a generator. Then we define

$$\langle \xi,\eta \rangle = \sum_P \mathrm{Res}_P(\widetilde{\xi}_P \eta) = \sum_P \mathrm{Res}_P(\omega_P). \tag{5.34}$$

We have to show that the value of this pairing does not depend on the choice of the lifting. If we replace $\widetilde{\xi}$ by $\widetilde{\xi} + f$ where $f \in H^0(S,\mathcal{L}(D))$, then $f\eta = \omega$ is a meromorphic 1-form on S, it is an element in $H^0(S,\Omega_S^1(D))$. For such a form it is clear that the sum of the residues vanishes. We simply observe that we can take the D_P so small that they do not intersect and for the path Γ_P we take their boundaries with counterclockwise orientation. Then

$$\begin{aligned} \sum_P \mathrm{Res}_P(\omega) &= \frac{1}{2\pi i} \sum \int_{\Gamma_P} \omega \\ &= \frac{1}{2\pi i} \int_{S \backslash \cup D_P} d\omega = 0. \end{aligned} \tag{5.35}$$

This proves that we get a well defined pairing

$$H^1(S,\mathcal{L}) \times H^0(S,\mathcal{L}^{-1} \otimes \Omega_S^1) \longrightarrow \mathbb{C}.$$

But it is also clear that any non zero element $\eta \in H^0(S,\mathcal{L}^{-1} \otimes \Omega_S^1)$ induces a non zero linear form on $H^1(S,\mathcal{L})$. To see that this is so we simply compute this linear form on $\mathbb{L}_D$, and then it is obviously non zero. This implies that the map $H^0(S,\mathcal{L}^{-1} \otimes \Omega_S^1) \longrightarrow H^1(S,\mathcal{O}_S)^\vee$ is injective and hence an isomorphism because the spaces have the same dimension.

This non degenerate pairing is called the **algebraic duality pairing**, in this special case it was certainly known to Riemann. It expresses the fact that the existence of holomorphic differentials on a Riemann surface of higher genus provides an obstruction for a collection of Laurent expansions $\widetilde{\xi} \in H^0(S,\mathcal{O}_S(D)/\mathcal{O}_S)$ to come from a meromorphic function (see Exercise 15). I tried to find the following proposition in [Rie], *Theorie der Abelschen Funktionen*:

Proposition 5.1.14 (Riemann)**.** *Such a $\widetilde{\xi}$ comes from a meromorphic function if and only if for all holomorphic differentials ω we have*

$$\sum_P \mathrm{Res}_P(\widetilde{\xi}\omega) = 0.$$

But I could not dig it out!
Later in section 5.1.9 we will compare this algebraic duality pairing with the (analytic) Serre duality pairing.

5.1.6 Riemann Surfaces of Low Genus

If the genus of the Riemann surface S is equal to zero and if P is any point, then it follows from the Theorem of Riemann-Roch and Serre duality that

$$\dim H^0(S,\mathcal{O}_S(P)) = 2,$$

and we conclude that we can find a meromorphic function f which is holomorphic everywhere except at the point P and at P it has a simple pole.
We saw already that this function gives us a map

$$f : S \longrightarrow \mathbb{P}^1(\mathbb{C}),$$

I claim that this map is an isomorphism between Riemann surfaces. To see this we observe that there is exactly one point – namely the point P – which goes to infinity. If $U = S \setminus \{P\}$, then we get for the restriction

$$\begin{aligned} f : U &\longrightarrow \mathbb{C} = \{(z,1) \mid z \in \mathbb{C}\} \subset \mathbb{P}^1(\mathbb{C}) \\ f : u &\longmapsto (f(u),1) \end{aligned}$$

(see Example 15 a)). For any $c \in \mathbb{C}$ we know that the polar divisor of $f - c$ is $-P$. Hence the zero divisor is of degree one and is equal to Q where $f(Q) = c$. Since S is compact it follows that this map is a homeomorphism.
We still have to show that it is biholomorphic. For any point Q we can find a neighborhood $D_Q \subset S$ such that

$$(D_Q,\mathcal{O}_{D_Q}) \simeq (D,\mathcal{O}_D)$$

where D is the unit disc in $\mathbb{C}$. Let z be the resulting uniformizing element. Under the map f this neighborhood is mapped to an open set $f(D_Q)$ which contains $f(Q) \in \mathbb{P}^1(\mathbb{C})$. We choose a uniformizing element z_Q, this is a holomorphic function defined in a neighborhood of $f(Q)$ which has a first order zero at $f(Q)$. Then $z_Q \circ f$ is a holomorphic function on a smaller disc $D'_Q \subset D_Q$ and hence a power series in z. Since the function $z_Q \circ f$ is injective we can conclude that

$$z_Q \circ f = az + \text{terms of higher order}$$

with $a \neq 0$. Now $\mathcal{O}_{\;\;^1,f(Q)}$ is the ring of convergent power series in z_Q and $\mathcal{O}_{S,Q}$ is the ring of convergent power series in z. We see that the map

$$\mathcal{O}_{\;\;^1,f(Q)} \longrightarrow \mathcal{O}_{S,Q}$$

(see Example 15 a)) is an isomorphism and this proves our assertion.

We can also give examples of Riemann surfaces of genus one. If $\Omega \subset \mathbb{C}$ is a lattice, then the quotient $S = \mathbb{C}/\Omega$ is a compact Riemann surface. It is homeomorphic to $\;\;^2/\;\;^2$, and hence we have $H^1(S,\;\;) = \;\;^2$ (see section 4.6.8) and hence we see that S has genus 1. We know that the space of holomorphic differentials is of dimension one and clearly the form $\omega = dz$ is a generator.
If in turn S is a compact Riemann surface of genus one, then we may do the following: We pick a point $s_0 \in S$ and we consider the following space

$$\widetilde{S} = \{(s,\gamma) \mid s \in S, \gamma \text{ homotopy class of a path starting in } s_0 \text{ and ending in } s\}.$$

We have the projection

$$\pi : \widetilde{S} \longrightarrow S$$

and locally this projection is a homeomorphism. (This construction can be done for any connected Riemann surface, then $\widetilde{S}$ is the so called universal cover of S.) It is also clear that we have a structure of a Riemann surface on $\widetilde{S}$. We choose a non zero holomorphic 1-form ω. Now we can construct a holomorphic map h from $\widetilde{S}$ to $\mathbb{C}$. We simply send

$$h : \widetilde{s} = (s,\gamma) \longmapsto \int_\gamma \omega \tag{5.36}$$

where we choose a differentiable path in the homotopy class. I leave it as an exercise to the reader to show that this map is an isomorphism between $\widetilde{S}$ and $\mathbb{C}$. It is also not difficult to show that $h^{-1}(s_0) = \Omega$ is a lattice in $\mathbb{C}$ (we will fill this gap in at the end of 5.1.11) and that the map factorizes over an isomorphism

$$\begin{array}{ccc} \widetilde{S} & \xrightarrow{h} & \mathbb{C} \\ \pi\downarrow & & \downarrow\pi \\ S & \xrightarrow{\widetilde{h}} & \mathbb{C}/\Omega. \end{array}$$

This makes it clear that all compact Riemann surfaces of genus 1 are of the form $\mathbb{C}/\Omega$.

5.1.7 The Algebraicity of Riemann Surfaces

From a Riemann Surface to Function Fields

We are now able to show that compact Riemann surfaces may be considered as purely algebraic objects. More precisely we can say that compact Riemann surfaces are the same objects as smooth, connected, projective curves over $\mathbb{C}$. It will be discussed in the second volume of this book what this exactly means.

It is clear that the meromorphic functions on S form a field $K = \mathbb{C}(S)$. We will show that this field is finitely generated over $\mathbb{C}$ and it is of trancendence degree 1 (See [Ei], Chap. II and Appendix 1). We will see that we can reconstruct the Riemann surface S from its function field. We will also show, that for any function field K of transcendence degree one over $\mathbb{C}$ we can construct a unique Riemann surface S such that $K = \mathbb{C}(S)$. Finally we will see that we have a so called antiequivalence of categories. If we have two compact Riemann surfaces S_1,S_2 then the non constant holomorphic maps $f : S_1 \longrightarrow S_2$ are in one-to-one correspondence with the homomorphisms ${}^tf : \mathbb{C}(S_2) \longrightarrow \mathbb{C}(S_1)$, which are the identity on $\mathbb{C}$.

Example 21. *If we consider the Riemann sphere $S = S^2 = \mathbb{P}^1(\mathbb{C})$ (see Example 15 a) or sections 5.1.1 and 5.1.6), then $\mathbb{C}(S^2) = \mathbb{C}(z)$ is the rational function field in one variable. It is the quotient field of the polynomial ring $\mathbb{C}[z]$ which is the ring of meromorphic functions which are holomorphic on $U_0 = \mathbb{P}^1(\mathbb{C}) \setminus \{\infty\}$.*

We have seen 5.1.13 that for point $P \in S$ and for $n \gg 0$ we can find a non-constant function $f \in H^0(S,\mathcal{O}_S(nP))$.
As I explained in the introduction to this chapter, a non-constant meromorphic function f on S provides a surjective map $f : S \longrightarrow \mathbb{P}^1(\mathbb{C})$. We put $U_0 = S \setminus \{P\}$, then f is a holomorphic function on U_0. Let U_1 be the complement of the set of zeroes of f.

If we have a point $s \in S$ where f is holomorphic, then the differential df is holomorphic at this point. If it is non-zero at s then we know from the theorem of implicit functions that f yields a biholomorphic map from a neighborhood of s to a neighborhood of $f(s)$.

Definition 5.1.15. *We say that a surjective map $f : S \longrightarrow \mathbb{P}^1(\mathbb{C})$ given by a nonconstant meromorphic function f on S is* **unramified** *or* **not ramified** *in a point $s \in S$, where it is holomorphic, if df is not zero at this point s.*
If f has a pole at s then we replace f by $g = \frac{1}{f}$ and we say that f unramified at s if $dg \neq 0$, i.e. the function g is unramified at s. In terms of f this can be reformulated: The differential df has a pole of second order.

Definition 5.1.16. *A map f as above is called* **unramified at a point** *$x \in \mathbb{P}^1(\mathbb{C})$ if it is unramified in all points of the fibre $f^{-1}(x)$.*

It is clear that the set of points where f is ramified is finite. If f is unramified at $x \in \mathbb{P}^1(\mathbb{C})$ then we can apply Lemma 4.8.9 and get that the degree of f is equal to the cardinality of the fibre $f^{-1}(x)$. This cardinality is also the degree of the zero divisor of $f - x$, if x is a finite point, (i.e. $x \neq \infty$) or the degree of the polar divisor if x is the point at infinity. This makes it clear that the degree of f is equal to the zero divisor of $f - c$ for any finite point $c \in \mathbb{P}^1(\mathbb{C})$. By definition this zero divisor is

$$\mathrm{Div}_0(f - c) = \sum_{y:f(y)=c} e(y)y.$$

A straightforward computation shows, that these numbers $e(y)$ are the same numbers as the numbers defined subsequently to Lemma 4.8.9. We conclude

$$\sum_{y:f(y)=c} e(y) = \deg(f) \tag{5.37}$$

for all

$$c \in \mathbb{P}^1(\mathbb{C}) \tag{5.38}$$

We may change the coordinates on $\mathbb{P}^1(\mathbb{C})$ by sending z to $\frac{az+b}{cz+d}$ where the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible with coefficients in $\mathbb{C}$. For any two different points $p_1,p_2 \in \mathbb{P}^1(\mathbb{C})$ we can find such a fractional linear transformation A, which sends these two points to 0 and ∞. This allows us to assume that our map is unramified at 0 and ∞. We give it a new name and write

$$\pi : S \longrightarrow \mathbb{P}^1(\mathbb{C}).$$

Let V_0 (resp. V_1) be the complement of 0 (resp. ∞), let $U_0 = \pi^{-1}(V_0)$, $U_1 = \pi^{-1}(V_1)$. For any set $V \subset S$ (or in $\mathbb{P}^1(\mathbb{C})$) which is the complement of a finite number of points we define $\mathcal{O}_S^{\mathrm{mer}}(V)$ to be the ring of those holomorphic functions on V which have at most poles in the points $S \setminus V$. For $V_0,V_1 \subset \mathbb{P}^1(\mathbb{C})$ these rings are polynomial rings in one variable, we write

$$\begin{aligned} \mathcal{O}^{\mathrm{mer}}_{{}^1(\ \ )}(V_0) &= \mathbb{C}[z] \\ \mathcal{O}^{\mathrm{mer}}_{{}^1(\ \ )}(V_1) &= \mathbb{C}[z^{-1}]. \end{aligned}$$

We may also consider the rings $\mathcal{O}_S^{\mathrm{mer}}(U_0)$, $\mathcal{O}_S^{\mathrm{mer}}(U_1)$ and these two rings are modules for $\mathcal{O}^{\mathrm{mer}}_{{}^1(\ \ )}(V_0)$ and $\mathcal{O}^{\mathrm{mer}}_{{}^1(\ \ )}(V_0)$ respectively. Our function f is now $z \circ \pi$.

Proposition 5.1.17.

1. *The modules $\mathcal{O}_S^{\mathrm{mer}}(U_0)$ (resp.$\mathcal{O}_S^{\mathrm{mer}}(U_1)$) over $\mathcal{O}^{\mathrm{mer}}_{{}^1(\ \ )}(V_0)$ (resp. over $\mathcal{O}^{\mathrm{mer}}_{{}^1(\ \ )}(V_1)$) are finitely generated.*

2. *If $\{\alpha_1, \dots ,\alpha_t\} \subset V_0$ is a finite subset and $V_0' = V_0 \setminus \{\alpha_1, \dots ,\alpha_t\}$ and $U_0' = \pi^{-1}(V_0')$ then*
$$\mathcal{O}_S^{\mathrm{mer}}(U_0') = \mathcal{O}_S^{\mathrm{mer}}(U_0) \cdot \mathcal{O}^{\mathrm{mer}}_{{}^1(\ \ )}(V_0').$$

3. *The functions in $\mathcal{O}_S^{\mathrm{mer}}(U_0)$ (resp.$\mathcal{O}_S^{\mathrm{mer}}(U_1)$) separate the points in U_0 (resp. U_1), i.e. for $x \neq y \in U_0$ we find an $f \in \mathcal{O}_S^{\mathrm{mer}}(U_0)$ for which $f(x)(\neq f(y)$.*

Proof: We show that $\mathcal{O}_S^{\mathrm{mer}}(U_0)$ is a finitely generated $\mathcal{O}^{\mathrm{mer}}_{{}^1(\ \ )}(V_0)$ module. We consider the divisor $D_\infty = \sum_{P \in \pi^{-1}(\infty)} P$, it is the divisor of poles of the function f pulled back to S. (Here we use that π is unramified at ∞, actually this is only technical). For $n > 0$ we consider the vector spaces $H^0(S,\mathcal{O}_S(nD_\infty))$. They form an increasing sequence of vector spaces exhausting $\mathcal{O}_S^{\mathrm{mer}}(U_0)$ if $n \longrightarrow \infty$. The dimension of these spaces is given by the Theorem of Riemann-Roch: If $n \gg 0$ then

$$\dim \ H^0(S,\mathcal{O}_S(nD_\infty)) = n \deg(D_\infty) + 1 - g.$$

We observe that the multiplication by z yields a linear map

$$\times z : H^0(S,\mathcal{O}_S(nD_\infty)) \longrightarrow H^0(S,\mathcal{O}_S((n+1)D_\infty))$$

and I claim that this map becomes surjective if we divide the space on the right hand side by the subspace $H^0(S,\mathcal{O}_S(nD_\infty))$. We pick a function $h \in H^0(S,\mathcal{O}_S((n+1)D_\infty))$ its polar divisor is of the form $D = \sum_{P\in\pi^{-1}(\infty)} m_P P$ with $m_P \le n+1$. If even $m_P \le n$ for all n then this function is in the subspace, which we divide out. Now we observe that it follows from our assumption $n \gg 0$ that

$$H^0(S,\mathcal{O}_S(nD_\infty)) \longrightarrow H^0(S,\mathcal{O}_S(nD_\infty)/\mathcal{O}_S((n-1)D_\infty))$$

is surjective. Therefore we can find a function $f \in H^0(S,\mathcal{O}_S(nD_\infty))$ which has an n-th order pole at a given point P where $m_P = n+1$ and has at most an $(n-1)$-th order pole at all the other points in $\pi^{-1}(\infty)$. For a suitable combination $h - azf$ the number of m_P which are equal to $n+1$ drops by one and our assertion follows by induction. Our claim implies that the $\mathcal{O}^{\mathrm{mer}}_{1(\ )}(V_0)$-module $\mathcal{O}_S^{\mathrm{mer}}(U_0)$ is generated by $H^0(S,\mathcal{O}_S(n_0D_\infty))$ for some sufficiently large n_0 and a) follows.
Now the second part is is not difficult anymore. Let f_1 be a meromorphic function in $\mathcal{O}_S^{\mathrm{mer}}(U_0')$. We can find a function $h \in \mathcal{O}_{\ 1(\ )}(V_0')$ which has a zero in the points $\alpha_1,\dots,\alpha_t$ and nowhere else (take the inverse of a function which has poles in exactly these points). If we pull it back to U_0 it has zeroes in all points in the fibres $\pi^{-1}(\alpha_i)$, i.e. in all points in $U_0 \setminus U_0'$, and nowhere else. Hence $f_1 \cdot h^N$ will be holomorphic in all points of $U_0 \setminus U_0'$ and this means $f_1 h^N \in \mathcal{O}_S^{\mathrm{mer}}(U_0')$. The last assertion *3* is just another simple Riemann-Roch exercise. □

Now we consider the function field $K = \mathbb{C}(S)$ of meromorphic functions. It is clear that the function field of the Riemann sphere $\mathbb{C}(\mathbb{P}^1(\mathbb{C})) = \mathbb{C}(z)$ is the rational function field in one variable. The assertion in the second part in Proposition 5.1.17 implies that any meromorphic function h on S can be written as a quotient $h = g/F$ where $g \in \mathcal{O}_S^{\mathrm{mer}}(U_0)$ and F is a meromorphic function in $\mathbb{C}(\mathbb{P}^1(\mathbb{C}))$. Therefore we can conclude: If $y_1,\dots y_d$ is a set of generators of the $\mathcal{O}^{\mathrm{mer}}_{1(\ )}(V_0)$-module $\mathcal{O}_S^{\mathrm{mer}}(U_0)$ then $\mathbb{C}(S)$ is generated by these elements as a $\mathbb{C}(\mathbb{P}^1(\mathbb{C}))$-vector space. It follows that $\mathbb{C}(S)$ is a finite extension of $\mathbb{C}(z) = \mathbb{C}(\mathbb{P}^1(\mathbb{C}))$.
I summarize into a theorem whose first part is proved by the above considerations:

Theorem 5.1.18. *The field of meromorphic functions on a compact Riemann S surface is a finite extension of a rational function field $\mathbb{C}(f)$, where f is any nonconstant meromorphic function on S. The choice of such a function f yields a holomorphic map $f : S \longrightarrow \mathbb{C}(\mathbb{P}^1(\mathbb{C}))$, which induces the inclusion $\mathbb{C}(f) \hookrightarrow \mathbb{C}(S)$. We have the equality of degrees*

$$[\mathbb{C}(S) : \mathbb{C}(f)] = \deg(f)$$

Proof: It remains to prove the equality of the degrees. We invoke the theorem of the primitive element: We can find an $\theta \in \mathbb{C}(S)$ such that $\mathbb{C}(S) = \mathbb{C}(h)[\theta]$ and θ is a zero of the irreducible polynomial

$$P[X] := a_n(z)X^n + a_{n-1}(z)X^{n-1} + \cdots + a_0(z) \in \mathbb{C}(\mathbb{P}^1(\mathbb{C}))[X]$$

where the $a_i(z)$ are polynomials in z, we have $a_n(z) \neq 0, a_0(z) \neq 0$, and $n = [\mathbb{C}(S) : \mathbb{C}(z)]$. Then the $\mathbb{C}(z)$ vector space K has the basis $1,\theta,\dots,\theta^{n-1}$. We can express the above generators $y_1,\dots,y_d$ as linear combinations

$$y_i = \sum_{\nu=0}^{n-1} a_{i,\nu}(z)\theta^\nu,$$

where the coefficients $a_{i,\nu}(z)$ are in $\mathbb{C}(z)$. If we remove a finite number of points $\{\alpha_1,\dots,\alpha_t\} \subset V_0$ as above, then we may assume that

a) the coefficients $a_{i,\nu}(z) \in \mathcal{O}^{\mathrm{mer}}_{{}^1(\ \ )}(V_0')$
b) the coefficient $a_n(z)$ does not vanish on V_0'
c) at all points α in V_0' the polynomial $a_n(\alpha)X^n + a_{n-1}(\alpha)X^{n-1} + \cdots + a_0(\alpha) \in \mathbb{C}[X]$ has n different roots.

Then our proposition above yields

$$\mathcal{O}_S^{\mathrm{mer}}(U_0') = \mathcal{O}_S^{\mathrm{mer}}(V_0')[\theta].$$

Now we consider the fibre $\pi^{-1}(\alpha)$, it is clear that θ is holomorphic in the points $\beta \in \pi^{-1}(\alpha)$and the values $\theta(\beta)$ are roots of the polynomial $a_n(\alpha)X^n + a_{n-1}(\alpha)X^{n-1} + \cdots + a_0(\alpha)$. Our theorem is proved if we can show that

$$\beta \mapsto \theta(\beta)$$

is a bijection between the points in the fibre and the roots.
Since θ separates the points in the fibre, the map is injective. Hence we have to prove that it is surjective. We introduce another Riemann surface namely

$$U_0'' = \{(\alpha,w) | (\alpha,w) \in V_0' \times \mathbb{C}, a_n(\alpha)w^n + a_{n-1}(\alpha)w^{n-1} + \cdots + a_0(\alpha) = 0\}.$$

This is a Riemann surface because for any point $(\alpha,\beta) \in U_0''$ we can find a small disk D_α around α and a holomorphic function $w_\beta : D_\alpha \longrightarrow \mathbb{C}$ such that $u \mapsto (u,w_\beta(u))$ is a homeomorphism from D_α to a neighborhood $D_{\alpha,\beta}$ of (α,β) in U_0''. (This is of course the implicit function theorem, we have $P'[\beta] \neq 0$ if we evaluate at $z = \alpha$.)
A heuristic formulation: For $u \in D_\alpha$ the root of $a_n(u)X^n + a_{n-1}(u)X^{n-1} + \cdots + a_0(u) = 0$ which is "close" to β is given by a holomorphic function $w_\beta(u)$ in u.
Our aim is to show that U_0'' is connected. We have the inclusion $U_0' \subset U_0''$ as an open subset and also the complement is open. Then the connectedness implies $U_0' =\subset U_0''$. Assume it is not. Now let U_0' any connected component and π be the projection from U_0' to V_0'. For any $\alpha \in V_0'$ we divide the set of roots of $a_n(\alpha)X^n + a_{n-1}(\alpha)X^{n-1} + \cdots + a_0(\alpha) = 0$ into the subset $\pi^{-1}(\alpha)$ and its complement $\pi^{-1}_{\mathrm{not}}(\alpha)$. We choose an starting point α. We choose a small disk D_α around α as above and we have the holomorphic functions $u \mapsto w_\beta(u)$.
We consider the polynomials

$$Q[X] = \prod_{\beta \in \pi^{-1}(\alpha)} (X - w_\beta(u)) = X^d + b_1(u)X^{d-1} + \cdots + b_0(u)$$

$$Q_{\mathrm{not}}[X] = \prod_{\beta \notin \pi^{-1}(\alpha)} (X - w_\beta(u)) = X^{d_1} + c_1(u)X^{d_1-1} + \cdots + c_0(u).$$

Clearly we have $P[X] = Q[X]Q_{\mathrm{not}}[X]$, this holds over D_α. Our considerations above imply that the numbers d,d_1 are locally constant and hence constant. Now we will show that the holomorphic functions $u \mapsto b_i(u), u \mapsto c_i(u)$, which are defined on the disk D_α extend to holomorphic and even meromorphic functions on V_0'. It is clear that this yields $U_0' = U_0''$ because we assumed that $P[X]$ is irreducible.

But it is clear that these functions extend. We can cover V_0' by disks D_{α_ν} such that we have the local roots $w_\beta^{(\nu)}$. Then we have the coefficients $c_i^{(\nu)}(u)$,$b_i^{(\nu)}(u)$ on D_{α_ν}. But if we have two such disks D_{α_ν},D_{α_μ} we clearly have that the restrictions of the coefficients $c_i^{(\nu)}$,$b_i^{(\nu)}$ and $c_i^{(\mu)}$,$b_i^{(\mu)}$ to $D_{\alpha_\nu} \cap D_{\alpha_\mu}$ are equal. Therefore any of these coefficients defines a holomorphic function on V_0'. It remains to show that they are meromorphic. Let us first consider one of the point $\alpha_i \in V_0$, which has been removed. The roots are the the zeroes of $a_n(z)X^n + a_{n-1}(z)X^{n-1} + \cdots + a_0(z)$. The coefficient $a_n(z)$ may have a zero at α_i, we write $a_n(z) = (z-\alpha_i)^{n_i}\widetilde{a}_n(z)$, where $\widetilde{a}_n(\alpha_i) \neq 0$. We multiply the polynomial by $(z-\alpha_i)^{n(n_i-1)}$, then our highest coefficient becomes $(z-\alpha_i)^{n_i n}\widetilde{a}_n(z)$. We make a substitution and put $Y = (z-\alpha_i)^{n_i}X$ then we get the polynomial in Y

$$\widetilde{a}_n(z)Y^n + \widetilde{a}_{n-1}(z)Y^{n-1} + \cdots + \widetilde{a}_0(z)$$

where the $\widetilde{a}_i(z)$ are holomorphic in α_i. We can find a small disk D_{α_i} around α_i and a number $c > 0$ such that this polynomial in Y has n different roots and such that $|\widetilde{a}_n(z)| > c > 0$. Then it is elementary to show, that the n roots of this polynomial in Y stay bounded in the punctured disk. This implies that for all roots of the old polynomial in X we have that $(z-\alpha_i)^{n_i}w_\beta(z)$ stays bounded in the pinctured disk. This implies that $(z-\alpha_i)^m b_i(z)$,$(z-\alpha_i)^m c_i(z)$ stay bounded in the punctured disk, provided m is sufficiently large. But then we know that these coefficients have at most a pole in α_i. It remains the point at infinity, but here we carry out the same argument on V_1. □

The reconstruction of S from K

We explain how we can reconstruct S from K. To do this we will use in an ad hoc manner some arguments from commutative algebra which will be explained in a more systematic way in chapter 7 in the second volume of this book. As a general reference I recommend the books [Ei], [Neu] and [A-McD], the book of M. Atiyah and I. G. MacDonald contains in its Chap. 9 the briefest exposition of the results which we will need in this section.

The finiteness of $\mathcal{O}_S^{\text{mer}}(U_0)$ as an $\mathcal{O}_{\mathbb{P}^1}^{\text{mer}}(V_0)$-module implies by a standard argument of commutative algebra that any element $h \in \mathcal{O}^{\text{mer}}(U_0)_S$ is **integral over** $\mathcal{O}_{\mathbb{P}^1}^{\text{mer}}(V_0)$. This means by definition that any element $h \in \mathcal{O}^{\text{mer}}(U_0)$ satisfies an equation of the form

$$h^n + a_1 h^{n-1} + \ldots + a_n = 0,$$

where the $a_i \in \mathcal{O}_{1(\ )}^{\text{mer}}(V_0)$ and $n > 0$. (See [Ei],Chap. I.4 , [Neu], Chap. I. 2.)
But if in turn $h \in \mathbb{C}(S)$ is integral over $\mathcal{O}_{1(\ )}^{\text{mer}}(V_0)$ then it must be holomorphic on U_0, to see this we simply look at the possible order of a pole. We conclude

Proposition 5.1.19. *The ring $\mathcal{O}_S^{\text{mer}}(U_0)$ is the* **integral closure** *of $\mathcal{O}_{\mathbb{P}^1}^{\text{mer}}(V_0)$ in K and this means that it consists of all the elements in K which are integral over $\mathcal{O}_{\mathbb{P}^1}^{\text{mer}}(V_0)$. (See [Neu], I.2.)*

The principal observation is that a point $P \in S$ defines a subring $\mathcal{O}_P^{\text{mer}} \subset K$, it is the ring of meromorphic functions which are regular at P. This ring is a **valuation ring** (see [Ei], II.11.7, [Neu], II. §3) with quotient field K and this means:

a) For any $f \in K$ we have f or f^{-1} is in $\mathcal{O}_P^{\text{mer}}$.

In addition to this we know that the ring satisfies a second condition

b)The ring $\mathcal{O}_P^{\mathrm{mer}}$ is not equal to K and it contains the constant functions $\mathbb{C}$.

Such a ring $\mathcal{O}_P^{\mathrm{mer}}$ has a unique maximal ideal which consists of the elements

$$\begin{aligned}\mathfrak{m}_P &= \{f \in \mathcal{O}_P^{\mathrm{mer}} | f^{-1} \notin \mathcal{O}_P^{\mathrm{mer}}\} \\ &= \{f \in K | f \text{ vanishes at } P\}.\end{aligned} \tag{5.39}$$

This maximal ideal is non zero and it is generated by any function which has a first order zero at P. This means that the ring is even a **discrete valuation** ring (see, [Neu], §3). The elements which are not in the maximal ideal are invertible. The property (a) implies that the quotient field of such a ring is K.

We consider the set $\mathrm{Val}(K)$ of subrings of K, which satisfy the conditions a) and b). Our next aim is to show

Theorem 5.1.20. *The map $P \mapsto \mathcal{O}_P^{\mathrm{mer}}$ gives a bijection $S \xrightarrow{\sim} \mathrm{Val}(K)$.*

The proof of this theorem will take a while. To start we forget the Riemann surface and consider any field K over $\mathbb{C}$ which is a finite extension of a rational function field $\mathbb{C}(x)$. We can write $K = \mathbb{C}(x)[y]$ where y satisfies an irreducible polynomial equation

$$y^n + a_1(x)y^{n-1} \cdots a_n(x) = 0$$

with $a_i(x) \in \mathbb{C}(x)$. We study the set $\mathrm{Val}(K)$.

Proposition 5.1.21. *a)All $A \in \mathrm{Val}(K)$ are discrete valuation rings, i.e. the maximal ideal $\mathfrak{m}_A$ is always a principal ideal.*
b)The composition $\mathbb{C} \longrightarrow A \longrightarrow A/\mathfrak{m}_A$ is an isomorphism and this means that the residue field is canonically isomorphic to $\mathbb{C}$. This also means that we can evaluate an $f \in A$ at A and the result is $f(A) = f \mod \mathfrak{m}_A$.
c) Furthermore for any $f \in K$ the set of A such that $f \notin A$ is finite.

Before entering the proof of this proposition I want to discuss the fundamental consequences of this fact. If we have any non zero element $f \in K$ and an element $A \in \mathrm{Val}(K)$, then we know that either $f \in A$ or $f^{-1} \in A$. In the first case we say that f is **regular** at A. If $f \notin$ we say that has a pole at A. If f is regular, and it π_A is a generator of the maximal ideal, then we can write $f = \pi_A^n u$ with $u \in A^\times$ and we say that f has a zero of order n at A. If $f \notin A$ the we say that f has a pole of order n at A, if f^{-1} has a zero of order n. We also denote this number by $\mathrm{ord}_{\mathfrak{p}}(f)$.
Of course any A is determined by its maximal ideal $\mathfrak{m}_A = \{f \in K | f \in A, f^{-1} \notin A\}$ and these maximal ideals are traditionally also denoted by $\mathfrak{p}, \mathfrak{q}, \ldots$. We will freely switch between these notations.
Finally I want to say that now we have a completely algebraic notion of the divisor of an element $f \in K^\times$, it is simply $\mathrm{Div}(f) = \sum_{\mathfrak{p}} \mathrm{ord}_{\mathfrak{p}}(f)\mathfrak{p}$. The sum is finite because we may apply c) to f and f^{-1}.

To prove the proposition we have to invest a little bit of commutative algebra. If $K_0 = \mathbb{C}(x)$ then an $A \in \mathrm{Val}(K_0)$ contains $\mathbb{C}[x]$ or $\mathbb{C}[x^{-1}]$. Let us assume that $A \supset \mathbb{C}[x]$. The maximal ideal $\mathfrak{m}_A$ intersected with $\mathbb{C}[x]$ gives us a non zero prime ideal in $\mathbb{C}[x]$. It is an elementary fact that the non zero prime ideals in $\mathbb{C}[x]$ are of the form $(x-\alpha)$. This implies that the elements of $\mathrm{Val}(K_0)$ are in one-to-one correspondence with the points in $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$: For any $\alpha \in \mathbb{C}$ we have the ring

$$A_\alpha = \left\{ f = \frac{P(x)}{Q(x)} \middle| \, Q(\alpha) \neq 0 \right\}, \tag{5.40}$$

and for ∞ we have

$$A_\infty = \left\{ f = \frac{P(x)}{Q(x)} \middle| \, \deg(P) \leq \deg(Q) \right\} \tag{5.41}$$

where P,Q are polynomials and $f(A_a) = f(a)$. Clearly these valuation rings are discrete valuation rings. We also saw that we have a map $\mathrm{Val}(K) \longrightarrow \mathrm{Val}(\mathbb{C}(x))$ which is defined by the intersection $A \mapsto A \cap \mathbb{C}(x)$.

Proposition 5.1.22. *The rings* $\mathbb{C}[x]$ *and* $\mathbb{C}[x^{-1}]$ *are* **Dedekind rings** *(see [Neu], Chap. I.§3, [Ei], II, §11).*

We consider the integral closures B_0 (resp. B_∞) of $\mathbb{C}[x]$ (resp. $\mathbb{C}[x^{-1}]$) in the field K. Then the theory of **Dedekind rings** (see [Neu], Chap. I.§8) implies that these integral closures are finitely generated modules over $\mathbb{C}[x]$ (resp. $\mathbb{C}[x^{-1}]$).
Since the polynomial rings have unique factorization, it follows that these modules are even free of rank $[K : \mathbb{C}(x)]$. This fact has the following consequence:

Lemma 5.1.23. *If* $\mathfrak{p}$ *is a non zero prime ideal in* B_0 *then* $\mathfrak{p}$ *is maximal and* $B_0/\mathfrak{p} = \mathbb{C}$. *The ring*

$$B_{0,\mathfrak{p}} = \left\{ \frac{f}{g} \middle| \, f,g \in B_0, g \notin \mathfrak{p} \right\}$$

is a discrete valuation ring.

To see that this is so we consider $\mathfrak{p}_0 = \mathfrak{m}_A \cap \mathbb{C}[x]$. It is clear that $\mathfrak{p}_0$ is non zero. Then $B_0/\mathfrak{p}$ is an integral domain and a finite dimensional vector space over $\mathbb{C} = \mathbb{C}[x]/\mathfrak{p}_0$. This implies that $B_0/\mathfrak{p} = \mathbb{C}[x]/\mathfrak{p}_0 = \mathbb{C}$. The fact that $B_{0,\mathfrak{p}}$ is a discrete valuation ring is perhaps the fundamental result for Dedekind rings, we refer to [Ei] Chap.II. 11, [Neu],Chap. I. §11, Prop. 11.5 or Chap. 7 in the second volume of this book.

Now we pick an $A \in \mathrm{Val}(K)$ and assume $A \supset \mathbb{C}[x]$. (Otherwise it contains the other ring.) I claim that this implies $A \supset B_0$. To see this we take an $f \in B_0$ and write down the polynomial equation

$$f^n + a_1(x) f^{n-1} + \cdots + a_n(x) = 0$$

with $a_i(x) \in \mathbb{C}[x]$. If now $f \notin A$ then f^{-1} lies in the maximal ideal $\mathfrak{m}_A$ of A and our polynomial equation yields

$$-a_1(x) f^{-1} \cdots - a_n(x) f^{-n} = 1$$

which gives a contradiction. Now $A \supset B_0$, we consider the prime ideal $\mathfrak{p} = \mathfrak{m}_A \cap B_0$ and and $\mathfrak{p}$ must be maximal. Then

$$A = \left\{ \frac{f}{g} \middle| f,g \in B_0, g \notin \mathfrak{p} \right\}. \tag{5.42}$$

We define $\mathrm{Val}_0(K)$ to be the set of $A \in \mathrm{Val}(K)$ which contain $\mathbb{C}[x]$ and hence B_0. Our considerations above make it clear that we can identify

$$\mathrm{Val}_0(K) \xrightarrow{\sim} \{\text{ non zero prime ideals in } B_0\} \xrightarrow{\sim} \{\mathbb{C}\text{−linear homomorphisms from } B_0 \text{ to } \mathbb{C}\}$$

This proves the second assertion in Proposition 5.1.21 above. Now we consider the prime ideals $\mathfrak{p} \in B_0$ which lie over a given prime ideal $(x-\alpha) = (x-\alpha)\mathbb{C}[x]$, i.e. for which $\mathfrak{p} \cap \mathbb{C}[x] = (x-\alpha)$. The ring $B_{0,\mathfrak{p}}$ is a discrete valuation ring, its maximal ideal is generated by an element $\pi_{\mathfrak{p}}$. Then we get integers $e_{\mathfrak{p}}$ so that

$$(x-\alpha)B_{0,\mathfrak{p}} = (\pi_{\mathfrak{p}}^{e_{\mathfrak{p}}})$$

this are the ramification indices. Again we have a result from commutative algebra:

The number of prime ideals lying over $(x-\alpha)$ is finite and the projections define an isomorphism

$$B_0/(x-\alpha)B_0 \xrightarrow{\sim} \prod_{\mathfrak{p} \supset (x-\alpha)} B_{0,\mathfrak{p}}/(\pi_{\mathfrak{p}}^{e_{\mathfrak{p}}})$$

is an isomorphism. See [Neu], Chap I,§8.
Actually this almost clear at this point.
It is also clear that the complements of $\mathrm{Val}_0(K)$ and $\mathrm{Val}_\infty(K)$ in $\mathrm{Val}(K)$ are finite because it is rather obvious that there are only finitely many prime ideals in B_0 (resp. B_∞) which lie above (x) (resp.(x^{-1})). This implies the finiteness assertion in Proposition 5.1.21 if we apply our consideration to $x = f$ and therefore the proposition is proved.

Definition 5.1.24. *We define a topology on* $\mathrm{Val}(K)$. *The open sets $U \subset \mathrm{Val}(K)$ are defined as the complements of finite sets, and of course we have to add the empty set. This topology is called the* **Zariski topology**.
We can define the sheaf of meromorphic functions. For any open set $U \subset \mathrm{Val}(K)$ we put

$$\mathcal{O}(U) = \bigcap_{A \in U} A,$$

this is the ring of functions which are regular on U and meromorphic on S. This gives $(\mathrm{Val}(K), \mathrm{Zar}, \mathcal{O})$ *the structure of a locally ringed space.*

If we take any $f \in K$ which is not constant, i.e. $f \notin \mathbb{C}$, then D_f is the set of points where f is regular. Then

$$\mathcal{O}(D_f) = \text{the integral closure of } \mathbb{C}[f] \text{ in } K. \tag{5.43}$$

This equality follows from the fact that a Dedekind ring is the intersection of the discrete valuation rings in the quotient field which contain it. (See [Neu], Chap. I, Theorem 3.3. or look at the divisor $h \in K^\times$.)

A brief comment: This object $(\mathrm{Val}(K), \mathrm{Zar}\,, \mathcal{O})$ is almost what is called a smooth, projective, connected curve over $\mathbb{C}$ in modern algebraic geometry. The only thing missing is the so called **generic point**. This generic point is simply the field K. We can just drop the assumption $A \neq K$ for our valuation rings and put $\widetilde{\mathrm{Val}}(K) = \mathrm{Val}(K) \cup \{K\}$. We define the Zariski topology on $\widetilde{\mathrm{Val}}(K)$, the open sets are the complements of finite subsets in $\mathrm{Val}(K)$ and the empty set. We define the sheaf as before and now $(\widetilde{\mathrm{Val}}(K), \mathrm{Zar}\,, \mathcal{O})$ is a locally ringed space and this is now really a smooth, connected, projective curve. The stalks of the structure sheaf are discrete valuation rings in the closed points and the stalk in $\{K\}$ is K.

Back to the Riemann surface: Now we assume again that K is the field of meromorphic functions on our compact Riemann surface S. We observed earlier that we have a map

$$S \longrightarrow \mathrm{Val}(K)$$

and we want to show that this is a bijection. Here it is clear that we have to use the compactness of the Riemann surface.
We pick a valuation ring $A \subset K$, let

$$\mathfrak{m}_A = \{f \in A \mid f^{-1} \notin A\} \tag{5.44}$$

be its maximal ideal. Our goal is to show that there is a unique point $P \in S$ such that $A = \mathcal{O}_P^{\mathrm{mer}}$. We will show that this point P is the common zero of the $f \in \mathfrak{m}_A$ and it also can be characterized as the unique point where all the elements of A are regular.
We pick a generator $f \in \mathfrak{m}_A$ and consider the intersection $A \cap \mathbb{C}[f]$ then $\mathfrak{m}_A \cap \mathbb{C}[f] = (f)$, because the principal ideal (f) is maximal and

$$B = A \cap \mathbb{C}(f) = \left\{ \tfrac{g}{h} \mid g,h \in \mathbb{C}[f], h \notin (f) \right\}.$$

We consider the diagram

$$\begin{array}{ccc} S & \xrightarrow{\ \pi\ } & \mathbb{P}^1(\mathbb{C}) \\ \downarrow & & \downarrow \\ \mathrm{Val}(K) & \xrightarrow{\ \widetilde{\pi}\ } & \mathrm{Val}(\mathbb{C}(f)) \end{array}$$

induced by f. As before $V_0 = \mathbb{P}^1(\mathbb{C}) \setminus \{\infty\}$ and $U_0 = \pi^{-1}(V_0)$. Then our ring B consists of those meromorphic functions on $\mathbb{P}^1(\mathbb{C})$ which are regular in 0. Our ring $A \in \widetilde{\pi}^{-1}(0)$. It suffices to show that the map $\pi^{-1}(0) \longrightarrow \widetilde{\pi}^{-1}(0)$ is surjective. The integral closure A_1 of B in K is a free module of rank $[K : \mathbb{C}(f)]$. The points in the fibre are exactly the $A' \in \mathrm{Val}(K)$, which contain A_1 and as before we have

$$A_1/A_1 f = \bigoplus_{\mathfrak{p}} A_1/\mathfrak{p}^{e_\mathfrak{p}}, \tag{5.45}$$

where the $\mathfrak{p}$ are the maximal ideals in A_1, they are in one-to-one correspondence to the elements in $\widetilde{\pi}(0)$. Since $\dim\ (A_1/\mathfrak{p}^{e_\mathfrak{p}}) = e_\mathfrak{p}$ we get $[K : \mathbb{C}(x)] = \sum_\mathfrak{p} e_\mathfrak{p}$. For the zero divisor of the function f on the Riemann surface S we have $\sum_{z \in pi^{-1}(0)} e_z z$ and it is clear that $e(z) = e(\mathfrak{p})$ if z maps to $\mathfrak{p}$. From 5.1.7 follows that the degree of the divisor on the Riemann surface is also $[K : \mathbb{C}(x)]$, this implies the equality of the fibres.

The Recovery of the Analytic Topology: The set S has some further structure, it has a topology and a sheaf of complex valued functions on it. We want to reconstruct this structure starting from K. In principle we have done this during the proof of 5.1.18. The detailed exposition may be a little bit boring, so the reader could skip this section Our Riemann surface is also a locally ringed space, and it is clear that the map

$$(S,\mathcal{O}_S) \longrightarrow (\mathrm{Val}(K), \mathrm{Zar}\,,\mathcal{O}_S^{\mathrm{mer}})$$

is a morphism between locally ringed spaces. This is of course not an isomorphism because on the left hand side we have many more functions, the ring $\mathcal{O}_{S,P}$ is much larger than $\mathcal{O}_{S,P}^{\mathrm{mer}} = A$ if P maps to A.
We still go one step further. Again we forget the compact Riemann surface S, and we start from a function field

$$K = \mathbb{C}(x)[y]$$

where

$$0 = y^n + a_1(x)y^{n-1} + \ldots + a_n(x).$$

We put $S = \mathrm{Val}(K)$, on this set we have the Zariski topology and our sheaf $\mathcal{O}_S$ with respect to the Zariski topology. We want to construct a finer topology on S. then S together with this finer topology will be called S_{an}. Of course the identity $S_{\mathrm{an}} \longrightarrow S$ will now be continuous. Furthermore we want to construct a sheaf $\mathcal{O}_S^{\mathrm{an}}$ of $\mathbb{C}$-valued functions on S_{an} such that we get a locally ringed space and such that $(S_{\mathrm{an}},\mathcal{O}_S^{\mathrm{an}})$ will be a compact Riemann surface.
Finally we can restrict meromorphic functions $f \in \mathcal{O}_S(U)$ to the open sets in S_{an}, and this will induce a morphism of locally ringed spaces

$$(S_{\mathrm{an}},\mathcal{O}_S^{\mathrm{an}}) \longrightarrow (S, \mathrm{Zar}\,,\mathcal{O}_S).$$

We come to the construction of the analytic topology. For any open subset $U \subset S$ we have the ring $\mathcal{O}_S(U)$, and we can interpret $\mathcal{O}_S(U)$ as ring of $\mathbb{C}$-valued functions on U. We introduce the coarsest topology on U for which all these functions are continuous.
If we have two different points $A,B \in S$, then it is clear that we cannot have $A \supset B$ or $B \subset A$. Hence we can find an $f \in A$ for which $f \notin B$. Since we can add a constant, we can assume $f \notin \mathfrak{m}_A$. Then $g = 1/f \in A$ but $g \notin \mathfrak{m}_A$ and $g \in \mathfrak{m}_B$. In other words, the element g is regular at A and at B and $g(A) \neq 0$ and $g(B) = 0$. Hence we have $A,B \in D_f$ and from the definition of the analytic topology follows that we can find neighborhoods of A and B whose intersection is empty and we have proved that our analytic topology is Hausdorff.

We want to describe a neighborhood of a point $A \in S$, and we want to show that A has neighborhoods isomorphic to a disc in $\mathbb{C}$.

This is of course clear if $K = \mathbb{C}(x)$, in this case we could identify

$$\mathrm{Val}(K) = \mathbb{P}^1(\mathbb{C}) \tag{5.46}$$

and the analytic topology is of course the usual topology on $\mathbb{P}^1(\mathbb{C})$.

We reduce the general case to this one. We have our point $A \in S$. We choose an element $f \in \mathfrak{m}_A$ which generates the ideal. Again we consider the integral closure $\mathcal{O}(D_f)$ of $\mathbb{C}[f]$ in K. We have

$$\begin{array}{ccc} \mathbb{C}[f]\cdot f & \subset & \mathbb{C}[f] \\ \cap & & \cap \\ (f) & \subset & \mathcal{O}(D_f). \end{array}$$

Since $\mathbb{C}[f]$ is principal it follows that the $\mathbb{C}[f]$-module $\mathcal{O}(D_f)$ is free of rank $n = [K : \mathbb{C}(f)]$

$$\mathcal{O}(D_f) = \bigoplus_i \mathbb{C}[f]y_i = \mathbb{C}[f,y_1,y_2\dots y_n] \tag{5.47}$$

where the elements $f,y_1,\dots,y_n$ satisfy some polynomial identities

$$P(f,y_1,\dots,y_n) = 0,$$

with some polynomials $P(F,Y_1,\dots,Y_n)$ from the polynomial ring $\mathbb{C}[F,Y_1,\dots Y_n]$. If I is the ideal generated by all these polynomials then we get an isomorphism

$$\mathbb{C}[F,Y_1,\dots Y_n]/I \xrightarrow{\sim} \mathcal{O}(D_f). \tag{5.48}$$

We introduce the evaluation map

$$\begin{aligned} E : D_f &\longrightarrow \mathbb{C}^{n+1} \\ E : u &\longmapsto (f(u),y_1(u),\dots,y_n(u)). \end{aligned} \tag{5.49}$$

Then the elements of $\mathcal{O}(D_f)$ separate the points in D_f because the points correspond to the maximal ideals of $\mathcal{O}(D_f)$. Therefore the evaluation map is injective. The image consists of those points $(z_0,z_1,\dots,z_n) \in \mathbb{C}^{n+1}$ which satisfy $P(z_0,z_1,\dots,z_n) = 0$ for all elements $P \in I$.

Our point A is mapped to an element $(0,a_1,\dots,a_n) = (f(A),y_1(A),\dots,y_n(A))$. We have a finite set of distinct points $A = A_0,A_1,\dots,A_m$ in S for which $f(A_1) = \dots = f(A_m) = 0$. We can find an $r > 0$ such that for all i,α,β we have $|y_i(A_\beta) - y_i(A_\alpha)| > 2r$ whenever these two numbers are not equal. We consider the open set $U \subset S$ which is defined by the requirement

$$U = \{B \mid |y_i(B) - y_i(A)| < r \text{ for all } 1 = 1,2,\dots n\}.$$

We consider the projection to the first coordinate

$$\begin{aligned} p : U &\longrightarrow \mathbb{C} \\ B &\longmapsto f(B) \end{aligned}$$

and this projection is by construction a homeomorphism to the image. Now we observe that we can write any of our y_i in the form

$$y_i = y_i(A) + \gamma_i f + R_i,$$

where $\gamma_i \in \mathbb{C}$ and $R_i = f^2 g_i/h_i$ where $g_i,h_i \in \mathcal{O}(D_f)$ and $h_i(A) \neq 0$. We represent these elements by polynomials $G_i,H_i \in \mathbb{C}[F,Y_1,\dots Y_n]$ and then we know that the ideal I above contains elements of the form

$$L_i = H_i(F,Y_1,\dots,Y_n)(Y_i - y_i(A)) - H_i(F,Y_1,\dots,Y_n)\gamma_i F - F^2 G(F,Y_1,\dots,Y_n)$$

for $i = 1,\dots,n$. The independent variables are F and the Y_i for $i = 1\dots,n$, and the partial Jacobi matrix $\left(\frac{\partial L_i}{\partial Y_i}\right)(A)_{i,j}$ is a diagonal matrix with non zero entries on the diagonal and therefore it has maximal rank. Hence we can conclude from the theorem of implicit functions that for a suitably small $\varepsilon > 0$ we can construct an inverse to the projection p above

$$\begin{aligned} q : D(\varepsilon) &\longrightarrow U \\ z &\longmapsto (z,y_1(z),\dots,y_n(z)), \end{aligned}$$

where now $y_1(z),\dots,y_n(z)$ are convergent power series and q identifies $D(\varepsilon)$ to an open neighborhood $U(\varepsilon)$ of A in S. On this open neighborhood we can define the sheaf $\mathcal{O}_S^{\mathrm{an}}(U(\varepsilon))$ of holomorphic functions, this is simply the sheaf of holomorphic functions on our small disc. Hence we constructed the structure of a compact Riemann surface $(S_{\mathrm{an}},\mathcal{O}_S^{\mathrm{an}})$ and clearly the identity map

$$(S_{\mathrm{an}},\mathcal{O}_S^{\mathrm{an}}) \longrightarrow (S, \mathrm{Zar}\,,\mathcal{O}_S)$$

is a morphism of locally ringed spaces.

One word concerning the notation. Here we think that the algebraic object $(S, \mathrm{Zar}\,,\mathcal{O}_S)$ is given first and to denote the analytic object we put the sub- and superscripts and write $S_{\mathrm{an}},\mathcal{O}_S^{\mathrm{an}}$. In the beginning of this section we did the opposite. There the Riemann surface was given and we had to introduce the sub- and superscripts Zar, mer.

Connection to Algebraic Geometry

Finally I want to say a few words about the connection to algebraic geometry. I come back to the description of

$$\mathcal{O}(D_f) = \mathbb{C}[f,y_1,\dots,y_n] = \mathbb{C}[F,Y_1,\dots,Y_n]/I.$$

We described the image of D_f under the evaluation map as a set of solutions of polynomial equations

$$Y = E(D_f) = \{(a_0,a_1,\dots,a_n) \mid P(a_0,a_1,\dots,a_n) = 0 \quad \text{for all} \quad P \in I\},$$

and this means (by definition) that this image is an affine algebraic variety over $\mathbb{C}$. I claim that for any point $B = (a_0,\dots,a_n) \in Y$ we can pick an index i such that $y_i - a_i = \widetilde{y}_i$ is a local parameter: In a small neighborhood the other coordinates of a point $b \in Y$ can be expressed as holomorphic functions in $\widetilde{y}_i$. We simply apply our arguments above to B. Therefore our variety is in fact one dimensional and smooth (see Example 19).

Actually we can say even more. Since $\mathcal{O}(D_f)$ is the integral closure of $\mathbb{C}[f]$ in the function field we know that the elements $\widetilde{y}_i$ satisfy an equation

$$\widetilde{y}_i^{n_i} + a_1\widetilde{y}_i^{n_i-1} + \ldots + a_{n_i} = 0$$

where the coefficients $a_i \in \mathbb{C}[f|$. We may assume that this polynomial is irreducible. We must have $a_{n_i}(A) = 0$. It is not entirely obvious but true that the previous coefficient a_{n_i-1} does not vanish at A. We can conclude that for the points B in our small neighborhood of A the polynomial $Y^{n_i} + a_1(B)Y^{n_i-1} \ldots + a_{n_i}(B)$ has exactly one root which is close to one. This means in classical terms that $\widetilde{y}_i^{n_i}$ is an algebraic function in the variable $z = f(B)$, it is a root of the polynomial which is distinguished and depends analytically on z.

Of course a few points are missing, namely, the points in $S \setminus D_f$. But we can find an element $g \in K$ which is regular at these missing points. We have a second evaluation map which identifies

$$D_g \xrightarrow{\sim} Y_1 \subset \mathbb{C}^{m+1}$$

and $\mathcal{O}(D_g) = \mathbb{C}[g,u_1, \cdots ,u_m]$. In Y we have the open subset Y_g where g is regular and in Y_1 the open subset $Y_{1,f}$ where f is regular and these two open sets are identified to $D_f \cap D_g$ under the evaluation maps.

We have to say in terms of the two data what the regular functions on $D_f \cap D_g$ are. I claim that

$$\mathcal{O}_S(D_g \cap D_f) = \mathbb{C}[g,u_1, \ldots ,u_m,f,y_1, \ldots ,y_n], \tag{5.50}$$

and this means that the regular functions on $D_g \cap D_f$ can be written as sums of products of elements in $\mathcal{O}(D_f)$ and $\mathcal{O}(D_g)$. If $h \in \mathcal{O}_S(D_g \cap D_f)$, then this function may have poles in $T_f \cup T_g$ where $T_f = S \setminus D_f$, $T_g = S \setminus D_g$. We want to modify h by sums of products u_1u_2 where u_1 has poles only in T_f and u_2 has poles only in T_g. Let us pick a point $t \in T_f$ with $t \notin T_g$ and $s \in T_g$, $s \notin T_f$ such that h has a pole at t. If such a pair (s,t) does not exist there is nothing to prove. We produce a function u_1 which has a pole at t and nowhere else. This is possible by the Theorem of Riemann-Roch (Theorem 5.1.4). We produce a function u_2 which has a pole at s and nowhere else but which in addition has a simple zero at t. Then $u_1u_2^m$ has a simple pole at t for a suitable choice of m. Now we can modify h by subtracting a suitable power of u_1u^m,

$$h - \gamma \cdot (u_1u^m)^k,$$

such that the pole order of h at t drops. This means that the total pole order at points in $T_f \setminus (T_f \cap T_g)$ drops. We repeat this process until h does not have any pole in the set $T_f \setminus (T_f \cap T_g)$, and then the modified function has only poles in T_g. Then we achieved our goal.

I summarize: *Our space S together with the sheaf $\mathcal{O}_S$ is covered by two affine varieties D_f,D_g (or affine schemes) and the ring of regular functions on the intersection $D_f \cap D_g$ is generated by the regular functions on D_f and D_g. With a corn of salt this means that we constructed a separated scheme.*

For this see [Ha], II. 4. or in the second volume of this book. Actually it is even projective this will be discussed in the second volume, too.

Elliptic Curves

We have a brief look at the case of Riemann surfaces of genus one. We have seen (5.1.6) that they are of the form $S = \mathbb{C}/\Omega$ where Ω is a lattice in $\mathbb{C}$. Notice that in this description the surface has a distinguished point $O \in S$, which is the image of $0 \in \mathbb{C}$ and the addition on $\mathbb{C}$ induces on S the structure of a complex analytic abelian group.

Definition 5.1.25. *A compact Riemann surface of genus one with a distinguished point O is called an* **elliptic curve***.*

Definition 5.1.26. *The meromorphic functions on $S = \mathbb{C}/\Omega$ are called* **elliptic functions.**

It is not so difficult to produce meromorphic functions on S, in a first semester course on function theory it is taught that we have the two special meromorphic functions (see for instance [La], Chap. I,§2 and Chap. III for the following)

$$\wp(z) = \frac{1}{z^2} + \sum_{\substack{\omega\in\Omega \\ \omega\neq 0}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right) \tag{5.51}$$

$$\wp'(z) = -2 \sum_{\omega\in\Omega} \frac{1}{(z-\omega)^3} \tag{5.52}$$

on S. The functions $\wp$,$\wp'$are called **Weierstraß $\wp$,$\wp'$-function.** It is clear that $\wp \in H^0(\mathbb{C}/\Omega, \mathcal{O}(2O))$,$\wp' \in H^0(\mathbb{C}/\Omega, \mathcal{O}(3O))$, hence we have that the seven functions

$$1,\wp,\wp',\wp^2,\wp\wp',\wp^3,\wp^2 \in H^0(\mathbb{C}/\Omega, \mathcal{O}(6O))$$

and the theorem of Riemann-Roch implies, that this space has dimension 6. Hence the 7 functions are linearily dependent. A simple computation gives, that they are related by an equation

$$\wp'(z)^2 = 4\wp(z)^3 - g_2(\Omega)\wp(z) - g_3(\Omega) \tag{5.53}$$

where the coefficients $g_2(\Omega)$, $g_3(\Omega)$ can be expressed in terms of the lattice, they are given by the convergent series

$$g_2(\Omega) = 60 \sum_{\substack{\omega\neq 0 \\ \omega\in\Omega}} \frac{1}{\omega^4} \tag{5.54}$$

$$g_3(\Omega) = 140 \sum_{\substack{\omega\neq 0 \\ \omega\in\Omega}} \frac{1}{\omega^6}.$$

The functions $g_2(\Omega)$ and $g_3(\Omega)$ can be viewed as functions on the space of lattices they are called modular forms.

Furthermore we know:

Lemma 5.1.27. *These two functions $\wp$ and their derivation $\wp'$ generate the field of meromorphic functions on S. We get an embedding into the projective space, we map*

$$z \mapsto (\wp'(z),\wp(z),1) = \left(1,\frac{\wp(z)}{\wp'(z)},\frac{1}{\wp'(z)}\right) \in \mathbb{P}^2(\mathbb{C}).$$

This map provides an analytic isomorphism

$$S \xrightarrow{\sim} \{(x,y,u) \in (\mathbb{C}^3 \setminus 0)/\mathbb{C}^\times | y^2u - 4x^3 - g_2(\Omega)xu^2 - g_3(\Omega)u^3 = 0\} \subset \mathbb{P}^2(\mathbb{C}).$$

This description of an elliptic curve as a projective curve is called the **Weierstraß normal form** of the elliptic curve. We can think of S as being an object in analytic complex geometry, the right hand side is an algebraic object. The point $\infty = (x,y,u) = (0,1,0)$ corresponds to $O \in \mathbb{C}/\Omega$.
Our covering of the projective variety by the two affine varieties looks as follows: We have the subset, where we have $u \neq 0$ (here only the point O is missing) and $y \neq 0$ (Which are the missing points in this case?). The rings of meromorphic functions on the two pieces are $\mathbb{C}[x,y,1]$ and $\mathbb{C}[\frac{x}{y},1,\frac{1}{y}]$.

On the left hand side we have the structure of an analytic group (multiplication and taking the inverse are holomorphic maps). This gives us a group structure on the right hand side. This group structure is given by the classical addition theorems for the Weierstraß $\wp$-function. Therefore we can say that S gets the structure of an algebraic group (see 5.3.1). The neutral element for this group structure is the point O. We come back to this point in 5.2.8.
Here we see that the genus is only a very weak invariant for a Riemann surface. If we have two elliptic curves $S_1 = \mathbb{C}/\Omega_1, S_2 = \mathbb{C}/\Omega_2$ then we may ask whether we can find a holomorphic map $f : S_1 \longrightarrow S_2$, which is not constant. It is not hard to see that we can find such a map if and only if we can find a complex number $\alpha \neq 0$ such that $\alpha\Omega_1 \subset \Omega_2$. We can find a holomorphic isomorphism, if and only if $\alpha\Omega_1 = \Omega_2$. Hence we see that the elliptic curves are parameterized by the space of lattices $\Omega \in \mathbb{C}$ modulo the equivalence relation $\Omega_1 \sim \Omega_2$ if and only if $\Omega_1 = \alpha\Omega_2$. This is the **moduli space** of elliptic curves (see also 5.2.5). It has the structure of a one dimensional complex variety.

5.1.8 Géométrie Analytique et Géométrie Algébrique - GAGA

Definition 5.1.28. *An analytic sheaf $\mathcal{E}^{\mathrm{an}}$ on S_{an} is called a* **coherent sheaf** *if it is a sheaf of $\mathcal{O}_S^{\mathrm{an}}$–modules, and if for any point $P \in S$ we can find an open neighborhood D_P and finitely many sections $t_1,\ldots,t_s$ such that for any $Q \in D_P$ these sections generate the $\mathcal{O}_{S,Q}^{\mathrm{an}}$–module $\mathcal{E}_Q^{an}$.*

We have the same notion for Zariski sheaves on S and clearly any coherent Zariski sheaf $\mathcal{E}$ provides a coherent analytic sheaf $\mathcal{E}^{\mathrm{an}} = \mathcal{E} \otimes_{\mathcal{O}_S^{\mathrm{mer}}} \mathcal{O}_S^{\mathrm{an}}$. The following simple observation is important and holds in both cases

Lemma 5.1.29. *Assume we have sections $u_1,\ldots,u_r \subset \mathcal{E}(V_P)$ for an open neighborhood V_P of P (in either topology). Assume that the images of these sections in the stalk $\mathcal{E}_P$ generate the stalk as $\mathcal{O}_{S,P}$-module. Then these sections also generate the stalks in all points of an open neighborhood $V_P' \subset V_P$.*

I think this is rather obvious.
Remark: If we apply the Lemma of Nakayama ([Ei], [A-McD]), then we see that we only need that the images of the $\widetilde{u}_i$ in $\mathcal{E}_P \otimes \mathcal{O}_{S,P}/\mathfrak{m}_P$ generate the $\mathcal{O}_{S,P}/\mathfrak{m}_P$-vector space.
Now we encounter the simplest case of the so called GAGA–principle. In our situation this principle says that this construction provides an equivalence of categories.

Definition 5.1.30. *A coherent sheaf $\mathcal{E}$ is called a* **torsion sheaf** *if for all points P the stalk $\mathcal{E}_P^{\mathrm{an}}$ (or $\mathcal{E}_P = \mathcal{E}_P^{\mathrm{mer}}$) consists of torsion elements, i.e. each element is annihilated by a non zero element in the local ring.*

We will see further down that any coherent $\mathcal{E}$ sheaf has a maximal torsion subsheaf and the quotient by this torsion subsheaf is locally free.

Proposition 5.1.31. *For any coherent sheaf $\mathcal{E}^{an}$ on S we can find a unique coherent Zariski sheaf $\mathcal{E}$ such that*

$$\mathcal{E}^{an} = \mathcal{E} \otimes_{\mathcal{O}_S^{mer}} \mathcal{O}_S^{an}.$$

For any pair $\mathcal{F}$,$\mathcal{G}$ of coherent Zariski sheaves the map

$$\mathrm{Hom}_S(\mathcal{F},\mathcal{G}) \longrightarrow \mathrm{Hom}_{S^{an}}(\mathcal{F}^{an},\mathcal{G}^{an})$$

is a bijection.

Proof: Starting from a coherent sheaf $\mathcal{E}^{an}$ we have to construct $\mathcal{E} = \mathcal{E}^{\mathrm{mer},}$ such that this sheaf gives back $\mathcal{E}^{\mathrm{an}}$ under the process above.

The strategy is simple: For a Zariski open subset $U = S \setminus T$, we have to say what $\mathcal{E}(U)$ should be. We have to say what it means for a section $s \in \mathcal{E}(U)$ to have at most poles in T. Then we simply say that $\mathcal{E}(U)$ consists of those sections of $\mathcal{E}^{\mathrm{an}}(U)$, which have at most poles at the finitely many missing points in T. Finally we have to prove that we have enough sections to generate the vector space $\mathcal{E}_P^{\mathrm{an}} \otimes \mathcal{O}_{S,P}/\mathfrak{m}_P$.

At first we prove our result for torsion sheaves. We pick a point $P \in S$, an open neighborhood D_P and sections $t_1,t_2,\ldots,t_s \in \mathcal{E}^{an}(D_P)$ which generate the stalks in the neigborhood. Their image in the stalk is annihilated by a non zero element $f \in \mathcal{O}_P$ (the local ring is integral). But then this element f can be extended into a small neighborhood D_P such that it is non zero at any point $Q \in D_P$ where $Q \neq P$. Hence we see that the stalks $\mathcal{E}_Q^{\mathrm{an}} = 0$ for all $Q \neq P$ in a small neighborhood of P. Since S is compact we can conclude that torsion sheaves are the skyscraper sheaves. Now we observe that for any point P and any positive integer $r > 0$ we have the equality

$$\mathcal{O}_{S,P}^{\mathrm{an}}/(\mathfrak{m}_P^{\mathrm{an}})^r = \mathcal{O}_{S,P}^{\mathrm{mer}}/(\mathfrak{m}_P)^r,$$

and therefore analytic and Zariski torsion sheaves are the same objects.

We come to the general case. Since $\mathcal{O}_{S,P}^{\mathrm{an}}$ is a discrete valuation ring, we can find generators $u_1,\ldots,u_m$ such that the stalk $\mathcal{E}_{S,P}^{\mathrm{an}}$ is the direct sum

$$\mathcal{E}_{S,P}^{\mathrm{an}} = \bigoplus_i \mathcal{O}_{S,P}^{\mathrm{an}} u_i.$$

We apply Lemma 5.1.29 and write the u_i as restrictions of some $\widetilde{u}_i$ which are defined in a neighborhood D_P. Now some of the u_i are torsion elements and these elements define a torsion subsheaf if we restrict to this neighborhood. If we still shrink this neighborhood further then this torsion subsheaf has support in P and the quotient is free. This happens in a small neighborhood of an arbitrary point P and shows us that we can define a finite skyscraper sheaf $\mathcal{E}^{\mathrm{an}}_{\mathrm{tor}S} \subset \mathcal{E}^{\mathrm{an}}$ such that the quotient $\mathcal{E}^{\mathrm{an}}/\mathcal{E}_{\mathrm{tors}} = \mathcal{E}'^{\mathrm{an}}$ is locally free. But if we have a locally free sheaf $\mathcal{E}'^{\mathrm{an}}$ and a section s which is defined in a punctured disc $\dot{D}_P = D_P \setminus \{P\}$, then we know what it means that s has at most a pole at P. We express $s = \sum g_i \widetilde{u}_i$, where g_i is holomorphic on $\dot{D}_P$. Then s has at most a pole if the g_i have at most a pole in P. We say that $s \in \mathcal{E}_P(\dot{D}_P)$ has at most a pole at P if its image in $\mathcal{E}'_P(\dot{D}_P)$ has at most a pole.

Hence for any Zariski open subset $U \subset S$ we can define the $\mathcal{O}_S(U)$–module of meromorphic sections $\mathcal{E}(U) = \mathcal{E}^{\mathrm{mer}}(U)$, these are the analytic sections which have at most poles in the finitely many missing points. Now we need to prove that the sections in $\mathcal{E}(U)$ generate the stalk at any point $Q \in U$. We saw before that it suffices to show that these sections generate $\mathcal{E}^{\mathrm{an}} \otimes \mathcal{O}_{S,Q}/\mathfrak{m}_Q$. Once we have shown this, it is clear that $\mathcal{E} \otimes_{\mathcal{O}_S} \mathcal{O}^{\mathrm{an}}_S = \mathcal{E}^{\mathrm{an}}$.

To prove this surjectivity we consider the case of a locally free sheaf $\mathcal{E}$ first. We pick a point $P \in S$, we can form the sheaf $\mathcal{E}^{\mathrm{an}} \otimes \mathcal{O}_S(rP) = \mathcal{E}^{\mathrm{an}}(rP)$, this is the larger sheaf, where the sections are allowed to have a pole of order $\leq r$ at P (compare the proof of the Theorem of Riemann-Roch (Theorem 5.1.12)). Then we get the exact sequence

$$0 \longrightarrow H^0(S,\mathcal{E}^{\mathrm{an}}) \longrightarrow H^0(S,\mathcal{E}^{\mathrm{an}}(rP)) \longrightarrow H^0(S,\mathcal{E}^{\mathrm{an}}(rP)/\mathcal{E}^{an}) \longrightarrow H^1(S,\mathcal{E}^{\mathrm{an}}).$$

Since we have $\dim H^1(S,\mathcal{E}^{an}) < \infty$ (Theorem 5.1.4) we conclude that for $r \gg 0$ the space $H^0(S,\mathcal{E}^{\mathrm{an}}(rP)/\mathcal{E}^{\mathrm{an}})$ has a non zero section. We apply Lemma 5.1.5 and conclude that we can find a line subbundle $\mathcal{L} \subset \mathcal{E} \otimes \mathcal{O}_S(rP)$ such that $\mathcal{E} \otimes \mathcal{O}_S(rP)/\mathcal{L}$ is again locally free. Hence we get an exact sequence of locally free sheaves

$$0 \longrightarrow \mathcal{L} \otimes \mathcal{O}_S(-rP) \longrightarrow \mathcal{E}^{\mathrm{an}} \longrightarrow \mathcal{E}^{\mathrm{an}}/(\mathcal{L} \otimes \mathcal{O}_S(-rP)) \longrightarrow 0.$$

The rank of the quotient bundle is the rank of $\mathcal{E}^{\mathrm{an}}$ minus 1. We conclude that $\mathcal{E}^{\mathrm{an}}$ has a filtration by locallay free subsheaves $0 \subset \mathcal{E}^{\mathrm{an}}_1 \subset \mathcal{E}^{\mathrm{an}}_2 \subset \ldots \subset \mathcal{E}^{an}$ such that the successive quotients are line bundles. From this we can conclude easily, that $H^1(S,\mathcal{E}^{\mathrm{an}}(rP)) = 0$ provided $r \gg 0$. We simply write the exact sequences resulting from the filtration. Then we pick any point $Q \in U$ and a second point $P \in T$. We have the locally free submodule $\mathcal{E}^{\mathrm{an}} \otimes \mathcal{O}_S(-Q) \subset \mathcal{E}^{\mathrm{an}}$ consisting of those sections which vanish at Q. We get an exact sequence

$$0 \longrightarrow \mathcal{E}^{\mathrm{an}}(rP) \otimes \mathcal{O}_S(-Q) \longrightarrow \mathcal{E}^{\mathrm{an}}(rP) \longrightarrow \mathcal{E}^{\mathrm{an}}/\mathcal{E}^{\mathrm{an}} \otimes \mathcal{O}_S(-Q) \longrightarrow 0.$$

For $r \gg 0$ we know that

$$H^1\left(S,\mathcal{E}^{\mathrm{an}}(rP) \otimes \mathcal{O}_S(-Q)\right) = 0,$$

we conclude that the map

$$H^0\left(S,\mathcal{E}^{\mathrm{an}}(rP)\right) \longrightarrow H^0\big(S,\mathcal{E}^{\mathrm{an}}/(\mathcal{E}^{\mathrm{an}} \otimes \mathcal{O}_S(-Q))\big)$$

is surjective. By definition we have $H^0(S,\mathcal{E}^{\mathrm{an}}(rP)) \subset \mathcal{E}(U)$ and hence we see that we find enough sections to generate $H^0(S,\mathcal{E}^{\mathrm{an}}/\mathcal{E}^{\mathrm{an}} \otimes \mathcal{O}_S(-Q))$, we get from Lemma 5.1.29 and the remark following that

$$\mathcal{E}^{\mathrm{an}} = \mathcal{E} \otimes_{\mathcal{O}_S^{\mathrm{mer}}} \mathcal{O}_S^{\mathrm{an}}.$$

The same argument also works if $\mathcal{E}^{\mathrm{an}}$ has torsion because the H^1 of a torsion sheaf vanishes.
We observe that for the global sections we have $H^0(S,\mathcal{E}^{\mathrm{an}}) = H^0(S,\mathcal{E})$. Now we also know that for two coherent sheaves we can define the coherent sheaf $\mathrm{Hom}(\mathcal{F}^{\mathrm{an}},\mathcal{G}^{\mathrm{an}})$ (resp.$\mathrm{Hom}(\mathcal{F},\mathcal{G})$ and then

$$\begin{aligned}\mathrm{Hom}_{S_{\mathrm{an}}}(\mathcal{F}^{\mathrm{an}},\mathcal{G}^{\mathrm{an}}) &= H^0(S_{\mathrm{an}}, \mathrm{Hom}(\mathcal{F}^{\mathrm{an}},\mathcal{G}^{\mathrm{an}}))\\ &= H^0(S, \mathrm{Hom}(\mathcal{F},\mathcal{G})) = \mathrm{Hom}_S(\mathcal{F},\mathcal{G}),\end{aligned}$$

this gives us the last statement in Proposition 5.1.31. □

What we have seen here is a special case of a general principle which is called the GAGA-principle (see the headline of this section). In a very rough form it says that compact complex manifolds are in fact algebraic, provided they have enough meromorphic functions. Especially a complex subvariety $Y \subset \mathbb{P}^n(\mathbb{C})$ is always algebraic (Theorem of Chow [Ch]). In such a case the coherent algebraic and the coherent analytic sheaves form equivalent categories (see [Se1]). We will come back to this principle in the second half of this chapter.

5.1.9 Comparison of Two Pairings

We apply Theorem 5.1.4 to the case $\mathcal{E} = \mathcal{O}_{\mathcal{S}}$ and get

$$H^1(S,\mathcal{O}_S) \simeq \overline{H^0(S,\Omega^1_S)}. \tag{5.55}$$

We have the Hodge decomposition of $H^1(S,\mathbb{C}) \xrightarrow{\sim} H^0(S,\Omega^1_S) \oplus \overline{H^0(S,\Omega^1_S)}$ (5.1.2). If we compute the cohomology $H^1(S,\mathbb{C})$ using the de Rham complex then the cup product

$$H^1(S,\mathbb{C}) \times H^1(S,\mathbb{C}) \longrightarrow \mathbb{C}$$

on the cohomology is given by integrating cup products of representing forms. (See 4.10.1.) If we consider the above decomposition, the two summands are isotropic and we get the $\mathbb{C}$-linear pairing

$$H^0(S,\Omega^1_S) \times \overline{H^0(S,\Omega^1_S)} \longrightarrow \mathbb{C}$$

which is given by

$$(\omega_1,\overline{\omega}_2) \longmapsto < \omega_1,\overline{\omega}_2 > = \int \omega_1 \wedge \overline{\omega}_2.$$

The combination of the isomorphism above and the pairing yields a $\mathbb{C}$-bilinear pairing

$$H^0(S,\Omega_S^1) \times H^1(S,\mathcal{O}_S) \longrightarrow \mathbb{C}.$$

We will call this pairing the **analytic pairing**. In section 5.1.5 we constructed the Serre duality pairing between these two vector spaces, this pairing is defined in purely algebraic terms.

Theorem 5.1.32. *The analytic pairing is $-2\pi i$ times the Serre duality pairing.*

Proof: To see this we need some simple considerations which in principle concern the comparison between Čech cohomology of sheaves and the cohomology groups obtained by injective (or acyclic) resolutions, for instance the de Rham resolution.

We pick a point $P \in S$ and an $n \gg 0$ such that the map

$$H^0(S,\mathcal{O}_S(nP)/\mathcal{O}_S) \longrightarrow H^1(S,\mathcal{O}_S)$$

becomes surjective. We choose an element $\eta \in H^1(S,\mathcal{O}_S)$ and we lift it to an element $\xi \in H^0(S,\mathcal{O}_S(nP)/\mathcal{O}_S)$. We choose a disc D_P around P and a local coordinate z_P which is zero at P. Now we represent an element ξ by a Laurent series

$$f(z) = \frac{a_n}{z^n} + \ldots + a_0 + a_1 z + \ldots .$$

We cover S by two open sets U_1,U_2, where $U_1 = D_p$ and U_2 is the complement of a smaller closed disc $\overline{D_p(\varepsilon)}$ around P, hence $U_1 \cap U_2$ is an annulus. We have that $f \in \mathcal{O}_S(U_1 \cap U_2)$ and it defines a 1-cocycle for the covering $S = U_1 \cup U_2$. This cocycle maps to η under the edge homomorphism (see Lemma 4.6.10). Now we proceed and use the de-Rham resolution, we get a diagram

$$\begin{array}{ccc}
\mathcal{O}_S(U_1) \oplus \mathcal{O}_S(U_2) & \longrightarrow & \mathcal{O}_S(U_1 \cap U_2) \\
\downarrow & & \downarrow \\
\Omega_S^{0,0}(U_1) \oplus \Omega_S^{0,0}(U_2) & \longrightarrow & \Omega_S^{0,0}(U_1 \cap U_2) \\
\downarrow d'' & & \downarrow d'' \\
\Omega_S^{0,1}(U_1) \oplus \Omega_S^{0,1}(U_2) & \longrightarrow & \Omega_S^{0,1}(U_1 \cap U_2) \\
\downarrow d'' & & \downarrow d'' \\
\vdots & & \vdots
\end{array}$$

We send f to $\Omega_S^{0,0}(U_1 \cap U_2)$ and I claim that we may write $f|U_1 \cap U_2$ as the restriction of a $\mathcal{C}^\infty$-function h_1 on $U_1 = D_p$. To see this we simply multiply the function f, which is actually defined on the punctured disk, by a $\mathcal{C}^\infty$-function which is identically equal to 1 on the annulus and which is identically zero in a neighborhood of zero. This C^∞-function on the disc is holomorphic on the annulus, but if we go into the interior it certainly loses this property. This means that $d''h_1 = \psi$ is an element in $\Omega_\infty^{0,1}(U_1)$ which has compact support and therefore it can be extended by zero to S. Then $\psi \in \Omega_\infty^{0,1}(S)$, it is closed and it represents our given class in $H^1(S,\mathcal{O}_S)$ via the Dolbeault isomorphism.

The integral $\int_S \omega \wedge \psi$ for a holomorphic 1-form ω on S gives the value of the analytic pairing between ξ and ω. We compute this integral. We observe that

$$\int_S \omega \wedge \psi = \int_{D_p} \omega \wedge \psi$$

and the integrand has compact support in D_p. We choose a circle $\partial D_p(r)$ which lies in the annulus, we still have

$$\int_{D_p} \omega \wedge \psi = \int_{D_p(r)} \omega \wedge \psi.$$

But now we write again $\psi = d''h_1$ and we have $\omega \wedge dh_1 = \omega \wedge (d'h_1 + d''h_1) = \omega \wedge d''h_1$. Therefore

$$\begin{aligned}\int_{D_p(r)} \omega \wedge d''h_1 &= \int_{D_p(r)} \omega \wedge dh_1 = -\int_{D_p(r)} d(h_1\,\omega) \\ &= -\int_{\partial D_p(r)} h_1\omega = -\int_{\partial D_p(r)} f\omega \\ &= -2\pi i\,\mathrm{Res}_P(f\omega).\end{aligned}$$

and now the right hand side is by definition the value of the Serre duality pairing multiplied by $2\pi i$. □

Since our pairings are non degenerate we conclude that we have two different ways of producing an identification $H^1(S,\mathcal{O}_S) \xrightarrow{\sim} H^0(S,\Omega^1_S)^\vee$ which differ by a factor $-2\pi i$. We could call the one produced by the cup product the **analytic identification** and the other one the **algebraic identification**. We will mostly use the analytic identification.

5.1.10 The Jacobian of a Compact Riemann Surface

Let S be a compact Riemann surface of genus g. We defined the Picard group of S $\mathrm{Pic}(S) = H^1(S,\mathcal{O}_S^*)$ to be the group of isomorphism classes of holomorphic line bundles on S. Our exact sequence in section 5.1.4 provides the homomorphism

$$H^1(S,\mathcal{O}_S^*) = \mathrm{Pic}(S) \longrightarrow H^2(S,\ \ ) =$$

Definition 5.1.33. *The kernel of* $\mathrm{Pic}(S) \longrightarrow\ \ $ *is denoted by* $\mathrm{Pic}^0(S)$ *and it is called the* **Jacobian** *of the curve and sometimes we write* $J = \mathrm{Pic}^0(S)$.

The exact sequence yields

$$\mathrm{Pic}^0(S) = H^1(S,\mathcal{O}_S)/H^1(S,\ \ ). \tag{5.56}$$

Here we divide a g-dimensional $\mathbb{C}$-vector space by a free -module of rank $2g$, I claim that we are in fact dividing by a lattice, i.e. the submodule is in fact discretely embedded. To see this we recall the Hodge decomposition (Lemma 5.1.2) and get inclusions

$$H^1(S,\ \ ) \hookrightarrow H^1(S,\ \ ) \hookrightarrow H^1(S,\mathbb{C}) = H^0(S,\Omega^1_S) \oplus \overline{H^0(S,\Omega^1_S)}.$$

Since $H^1(S,\ ) = H^1(S,\ ) \otimes\ $, we see that $H^1(S,\ )$ is a lattice in $H^1(S,\ )$. On the other hand it is clear that the projection of $H^1(S,\ )$ to any of the two summands in the decomposition of $H^1(S,\mathbb{C})$ is an isomorphism since the summands are complex conjugate. This implies that the inclusions followed by the projection

$$H^1(S,\ ) \hookrightarrow \overline{H^0(S,\Omega^1_S)} = H^1(S,\mathcal{O}_S) \tag{5.57}$$

maps $H^1(S,\ )$ isomorphically to a lattice Γ in $H^1(S,\mathcal{O}_S)$. We want to denote this isomorphism by

$$j : H^1(S,\ ) \xrightarrow{\sim} \Gamma.$$

Of course it is clear that the multiplication of line bundles in $\mathrm{Pic}^0(S)$ induces the addition on $H^1(S,\mathcal{O}_S)/\Gamma$ and hence we see that the quotient

$$J = \mathrm{Pic}^0(S) = H^1(S,\mathcal{O}_S)/\Gamma$$

has a natural structure of a connected, compact complex-analytic group of dimension g. Such a group is called a **complex torus**. Hence we summarize

Theorem 5.1.34. *The Jacobian of a compact Riemann surface of genus g has the structure of a complex torus of dimension g.*

5.1.11 The Classical Version of Abel's Theorem

In the previous section we described the group of line bundles Pic^0 in terms of the cohomology group $H^1(S,\mathcal{O}_S^*)$. Our main tool was the exact sequence

$$0 \longrightarrow H^1(S,\ ) \longrightarrow H^1(S,\mathcal{O}_S) \longrightarrow H^1(S,\mathcal{O}_S^*) \longrightarrow H^2(S,\ )$$

which allowed us to define the degree of the line bundle and gave us the description

$$\mathrm{Pic}^0(S) = H^1(S,\mathcal{O}_S)/H^1(S,\ ).$$

Now we recall (see section 5.1.4) that the group of line bundles $\mathrm{Pic}(S)$ may also be described as the group of divisor classes

$$\mathrm{Div}(S)/\text{ principal divisors } \xrightarrow{\sim} \mathrm{Pic}(S).$$

For a divisor $D = \sum_P n_P P$ the degree of the line bundle is

$$\delta(\mathcal{O}_S(D)) = \deg(\mathcal{O}_S(D)) = \deg(D) = \sum n_P$$

and by composition we get the isomorphism

$$\mathrm{Div}^0(S)/\text{principal divisors } \xrightarrow{\sim} H^1(S,\mathcal{O}_S)/H^1(S,\ ).$$

We want to compute this isomorphism. If $\mathcal{O}_S(D) \in \mathrm{Pic}^0(S)$, how can we compute the corresponding element in $H^1(S,\mathcal{O}_S)/H^1(S,\ )$?

We reformulate our problem slightly. The analytic pairing gives us an identification $H^1(S,\mathcal{O}_S) \xrightarrow{\sim} H^0(S,\Omega^1_S)^\vee$, the Poincaré duality gives an identification $H_1(S,\ ) \xrightarrow{\sim} H^1(S,\ )$ (see section 4.8.6). The resulting embedding $i_1 : H_1(S,\ ) \hookrightarrow H^0(S,\Omega^1_S)^\vee$ is obtained by the following rule: We represent a homology class $[c]$ by a cycle c and to this class we attach the linear form

$$\varphi_c : \omega \longmapsto \int_c \omega. \tag{5.58}$$

Then the homomorphism $[c] \mapsto \varphi_c$ is our embedding i_1. Hence our problem is to compute the isomorphism

$$\mathrm{Div}^0(S)/\text{ principal divisors } \xrightarrow{\sim} H^0(S,\Omega^1_S)^\vee/H_1(S,\ ).$$

Let D be a divisor of degree zero. We write it as $\sum_{i=1}^n P_i - \sum_{i=1}^n Q_i$. We find $\mathcal{C}^\infty$-maps $\sigma_i : [0,1] \longrightarrow S$ with $\sigma_i(0) = P_i, \sigma_i(1) = Q_i$. We identify $[0,1]$ to the standard 1-simplex Δ_1 (see section 4.8.6) by $t \mapsto (t,1-t)$. Then $\mathfrak{z}_D = \sum_{i=1}^n \sigma_i$ is a 1-chain whose boundary is $\partial\mathfrak{z}_D = D$. This means this 1-chain provides a map

$$\begin{aligned} \varphi_{D,\mathfrak{z}_D} &: H^0(S,\Omega^1_S) \longrightarrow \mathbb{C} \\ \varphi_{D,\mathfrak{z}_D} &: \omega \mapsto \int_{\mathfrak{z}_D} \omega. \end{aligned} \tag{5.59}$$

If we have a second 1-cycle $\mathfrak{z}'_D$ which also satisfies $\partial\mathfrak{z}'_D = D$ then $\mathfrak{z}'_D = \mathfrak{z}_D + c_D$ where c_D is a closed 1-cycle, i.e. $\partial c_D = 0$. Hence we see that

$$\varphi_{D,\mathfrak{z}_D} - \varphi_{D,\mathfrak{z}'_D} \in H_1(S,\ ) \subset H^0(S,\Omega^1_S)^\vee.$$

Hence we see that D defines a well-defined element $\widetilde{\varphi}_D \in H^0(S,\Omega^1_S)^\vee/H_1(S,\ )$.

Theorem 5.1.35 (Theorem of Abel)**.** *The isomorphism*

$$\mathrm{Div}^0(S)/\text{ principal divisors } \xrightarrow{\sim} H^0(S,\Omega^1_S)^\vee/H_1(S,\ ).$$

is given by $[D] \mapsto \widetilde{\varphi}_D$.

Proof: To prove this it suffices to consider the case of two points P,Q on our Riemann surface S which lie in a small disc D_P. This is clear because our map $D \longrightarrow \varphi_D$ is a homomorphism from the group of divisors of degree zero to $H^0(S,\Omega^1_S)^\vee/H_1(S,\ )$ and these divisors generate the group of divisors of degree zero.
We assume that our local coordinate z is zero at P and 1 at Q. We want to compute the class of the line bundle $\mathcal{O}_S(Q-P)$ in $H^1(S,\mathcal{O}_S)/H^1(S,\ )$. To be more precise we want to find a representative of this class in $\Omega^{0,1}_S(S)$ and identify it as a linear form on the space of holomorphic differentials.
We draw the straight path γ from P to Q in our disc and we cover S by $U_2 = S\backslash\gamma$ and $U_1 = D_P$. The meromorphic function $\frac{z}{z-1}$ trivializes our bundle on U_1 and the constant function 1 trivializes it on U_2. Hence the holomorphic function $\frac{z}{z-1}$ on $U_1 \cap U_2 = D\backslash\gamma$ defines a Čech cocycle with values in $\mathcal{O}^*_S$ and its image in $H^1(S,\mathcal{O}^*_S)$ is the class of $\mathcal{O}_S(Q-P)$. I claim that we can define the function $\log\frac{z}{z-1}$ on $D\backslash\gamma$. This is so because we

can write down integrals $\int_a^z \frac{dz}{z}$ and $\int_a^z \frac{dz}{z-1}$ along a path from a point a to z which avoids γ. The values of these integrals depend not only on z but also on the homotopy class of the path. But the multivaluedness drops out if we take the difference of the integrals, which then gives us the function $\log \frac{z}{z-1}$. The element $\frac{1}{2\pi i} \log \frac{z}{z-1} \in \mathcal{O}_S(D\backslash\{\gamma\})$ is a 1-cocycle with values in $\mathcal{O}_S$. It defines a class in $\xi_{P,Q} \in H^1(S,\mathcal{O}_S)$ which maps to the class of $\mathcal{O}_S(Q-P)$ in $H^1(S,\mathcal{O}_S^*)$. This class $\xi_{P,Q}$ can be represented by a closed form of type (0,1). To find such a form we shrink the set U_2 a little bit to a set U_2' so that it is the complement of a little neighborhood N of γ.

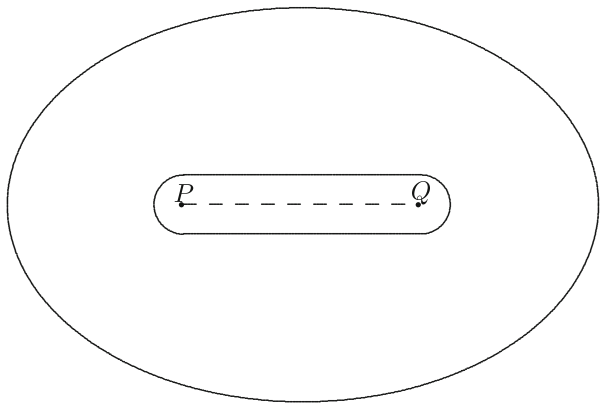


Figure 5.1 In the picture above this neighborhood is the "cigar" containing the path from P to Q and U_2' is the complement of the "cigar". This "cigar" is obtained by drawing half circles of radius $\epsilon > 0$ around P,Q and then joining the endpoints by straight lines parallel to γ. The boundary is a $\mathcal{C}_1$-manifold.

By the same argument as in section 5.1.9 we extend the restriction of $\frac{1}{2\pi i} \log \frac{z}{z-1}$ to $U_2' \cap D_P$ to a $\mathcal{C}_\infty$-function h on D_P and put $\mu = d''h$. This form μ has compact support in U, hence it can be extended to a (0,1)–form on S which then represents $\xi_{P,Q} \in H^1(S,\mathcal{O}_S)$ (a special case of the argument in Theorem 5.1.32). Again we have that the pairing of this class with a holomorphic 1-form ω is given by $\int_S \omega \wedge \mu$. To compute this integral we observe that $\omega \wedge \mu$ has support in the neighborhood N of γ, hence it suffices to integrate over this neighborhood. But now we can write $\omega \wedge \mu = \omega \wedge dh$ and our integral becomes $-\int_{\partial N} \omega \wedge h$ where $h = \frac{1}{2\pi i} \log \frac{z}{z-1}$ on the boundary of N.
Letting this neighborhood shrink to γ the values of $\log \frac{z}{z-1}$ differ by $2\pi i$ on the two sides of our path γ. Hence we get that

$$< \xi_{P,Q},\omega >= \int_\gamma \omega \tag{5.60}$$

and this is Abel's theorem in the case that our divisor is $Q-P$, and P,Q close to each other. □

Of course the theorem of Abel tells us that on a Riemann surface of genus zero any divisor of degree zero is principal because there are no holomorphic differentials. This we know already. We can exploit this fact to construct isomorphisms $f : S \xrightarrow{\sim} \mathbb{P}^1(\mathbb{C})$ for any Riemann surface S of genus zero. We simply take two points $P \neq Q$ and find a meromorphic function f with divisor $\mathrm{Div}(f) = P - Q$. This function is such an isomorphism.

This last argument also shows:

Lemma 5.1.36. *On a Riemann surface of genus $g > 0$ a divisor of the form $P - Q$ with $P \neq Q$ is never principal.*

The theorem of Abel is the source for the so called self-duality of the Jacobian, which will be discussed in detail later in section 5.2.3. We pick a point P_0 and consider the morphism

$$\begin{aligned} i_{P_0} : S &\longrightarrow J \\ P &\longmapsto (P) - (P_0). \end{aligned} \tag{5.61}$$

This is clearly a holomorphic map. We just saw that for Riemann surfaces S of genus $g > 0$ this map is injective. We want to explain how Abels theorem gives us its differential. The tangent space of J at any point is $H^1(S,\mathcal{O}_S)$ and hence we see that the space of holomorphic 1-forms on J is $H^0(S,\Omega_S)$. Therefore i_{P_0} yields a $\mathbb{C}$-linear map between the spaces of holomorphic 1-forms

$$i_{P_0}^* : H^0(J,\Omega_J^1) = H^0(S,\Omega_S^1) \longrightarrow H^0(S,\Omega_S^1) \tag{5.62}$$

and I claim that this map must be the identity. If $\omega \in H^0(J,\Omega_J^1)$ is a holomorphic 1-form and if $X \in T_P$ is a tangent vector at P_0 we compute $i_{P_0}^*(\omega)_P(X)$. We choose a local coordinate z at P, then we may assume that $X = \frac{\partial}{\partial z}$. Then $i_{P_0}^*(P + h \cdot \frac{\partial}{\partial z})$ is the linear form

$$\omega \longmapsto \int_{P_0}^{P+h\frac{\partial}{\partial z}} \omega = \int_{P_0}^{P} \omega + \int_{P}^{P+h\frac{\partial}{\partial z}} \omega,$$

and this yields

$$i_{P_0}^*(\omega)_P(X) = i_{P_0}^*(\omega)_P(\frac{\partial}{\partial z}) = \omega_P(\frac{\partial}{\partial z})$$

and hence $i_{P_0}^*(\omega) = \omega$.

We see that i_{P_0} gives us a holomorphic embedding of the curve into its Jacobian. This map also induces a homomorphism between the Picard groups

$$i_{P_0}^* : \mathrm{Pic}(J) \longrightarrow \mathrm{Pic}(S).$$

We will define a subgroup $\mathrm{Pic}^0(J)$ (see Proposition 5.2.3) and we will prove that the restriction

$${}^t i_{P_0} : \mathrm{Pic}^0(J) \longrightarrow \mathrm{Pic}^0(S)$$

is an isomorphism. This is the so called self duality of J.
In section 5.1.6. I stated the theorem that Riemann surfaces S of genus 1 are of the form $S = \mathbb{C}/\Omega$. To get this description I stated that the universal cover $\widetilde{S}$ is the complex plane and I gave it as an exercise to verify this. Our considerations above solve this exercise. For surfaces of genus $g = 1$ that map i_{P_0} is an isomorphism.

This has an important consequence: If we pick a point $P_0 \in S$, then we get a group structure on the Riemann surface by transporting the group structure from J to S. The point P_0 will then be the neutral element for this group structure.
The sublattice $\Gamma^\vee = H_1(S,\ \ ) \subset H^0(S,\Omega^1_S)^\vee$ is called the **period lattice**. Recall that it consists of linear forms on $H^0(S,\Omega^1_S)$ and these linear forms are given by the **period integrals**

$$\gamma \mapsto \omega \mapsto \int_\gamma \omega$$

where γ is a closed 1-cycle representing a homology class.
These period integrals are historically the origin of the whole theory of Riemann surfaces. Let us consider the special case of an elliptic curve, which we write in a slightly modified Weierstraß form

$$y^2 = x(x-1)(x-\lambda) = x^3 - (1+\lambda)x^2 + \lambda x,$$

we assume that $\lambda \notin [-\infty,1]$. In the complex plane we choose a straight path $\bar\gamma_1$ from 0 to 1, this is our intervall $[0,1]$ and a straight path $\bar\gamma_2$ from 1 to λ. Now we produce closed cycles in our elliptic curve. We start at zero and go to one. For any x we choose a square root $y(x) = \sqrt{x(x-1)(x-\lambda)}$ which varies continuously with x. At 1 we turn back, but now we take the other root. The path $x \mapsto y(x)$ for $0 \le x \le 1/2$ and $x \mapsto -y(x)$ for $1/2 \le x \le 1$ gives us a closed path γ_1. If we project it to the x-plane then we get $\bar\gamma_1$ going from zero to 1 and back. We can do the same thing with $\bar\gamma_2$. The differential $\omega = \frac{dx}{y}$ is holomorphic and we get two period integrals

$$\omega_1(\lambda) = \int_{\gamma_1} \omega = 2\int_0^1 \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}$$

$$\omega_2(\lambda) = \int_{\gamma_2} \omega = 2\int_1^\lambda \frac{dx}{\sqrt{x(x-1)(x-\lambda)}},$$

where the notation is traditional but a little bit sloppy. For further information on this subject we recommend [Hu] Chapter 9.

5.1.12 Riemann Period Relations

The cup product $<\ ,\ >_\cup$ defines a non degenerate alternating pairing on our lattice Γ. On the other hand we have $\Gamma \otimes \ \ = \Gamma_\mathbb{R} \xrightarrow{\sim} H^1(S,\mathcal{O}_S)$ and this identification provides a complex structure I on $\Gamma_\mathbb{R}$, namely the one which is induced by the multiplication by i on $H^1(S,\mathcal{O}_S)$.
We will show that the complex structure I is an isometry for the extension of $<\ ,\ >_\cup$ to $\Gamma_\mathbb{R}$. To get this information we show that the pairing is the imaginary part of a Hermitian form h on $(\Gamma_\mathbb{R},I)$ (see 4.11.2). We define this form h and show that $-\mathrm{Im}\, h = <\ ,\ >_\cup$. In addition it will turn out that h is positive definite.

We define a Hermitian scalar product on $\overline{H^0(S,\Omega^1_S)}$. If we have two antiholomorphic forms $\overline{\omega}_1,\overline{\omega}_2 \in \overline{H^0(S,\Omega^1_S)}$ we put

$$h < \overline{\omega}_1, \overline{\omega}_2 > = -2i \int_S \overline{\omega}_1 \wedge \omega_2. \tag{5.63}$$

If we write locally $\overline{\omega}_1 = f_1 d\overline{z}$ and $\overline{\omega}_2 = f_2 d\overline{z}$ then the integrand becomes

$$f_1(dx - idy) \wedge \overline{f}_2(dx + idy) = 2if_1\overline{f}_2 dx \wedge dy,$$

hence we see that h is a positive definite Hermitian form.

Now we take two cohomology classes $\xi, \eta \in H^1(S, \quad)$. Using the de Rham isomorphism, we can represent them by differential forms which we can decompose

$$\begin{aligned} \omega_\xi &= \overline{\omega_\xi^{0,1}} + \omega_\xi^{0,1} \\ \omega_\eta &= \overline{\omega_\eta^{0,1}} + \omega_\eta^{0,1}. \end{aligned} \tag{5.64}$$

The cup product pairing is given by integrating the representing differential forms

$$\begin{aligned} < \xi, \eta >_\cup &= \int_S \omega_\xi \wedge \omega_\eta \\ &= -\int_S \omega_\eta^{0,1} \wedge \overline{\omega_\xi^{0,1}} + \int_S \omega_\xi^{0,1} \wedge \overline{\omega_\eta^{0,1}} \\ &= -\mathrm{Im}\, h(\omega_\xi^{0,1}, \omega_\eta^{0,1}). \end{aligned} \tag{5.65}$$

We have the isomorphism

$$j : H^1(S, \quad) \xrightarrow{\sim} \Gamma \subset H^1(S, \mathcal{O}_S) \xrightarrow{\sim} \overline{H^0(S, \Omega_S^1)}$$

and it is clear the the classes $j(\xi)$ (resp. $j(\eta)$) are represented by $\omega_\xi^{0,1}$ (resp. $\omega_\eta^{0,1}$). We can transport the cup product pairing via j to Γ, then we get the famous

Theorem 5.1.37 (Riemann period relations)**.** *The restriction of the imaginary part of the Hermitian form h to Γ is the cup product times -1. Especially we can conclude that the values of* $\mathrm{Im}\, h$ *on* $\Gamma \times \Gamma$ *are integers and this form is a perfect pairing.*

5.2 Line Bundles on Complex Tori

5.2.1 Construction of Line Bundles

The presentation of the material in this section is greatly inspired by the work of David Mumford ([Mu1], [Mu2]).
The period relations are of great importance, because they allow the construction of line bundles on J. The positivity of the form h will ensure that these bundles will be **ample** and this means roughly that high positive powers of this bundle have many sections (see below section 5.2.4). To explain this construction of line bundles we consider a more general situation.

Let V be a complex vector space of dimension g and let $\Gamma \subset V$ be a lattice in V, this means that Γ is a free -module of rank $2g$ which sits in V as a discrete submodule. The quotient $A = V/\Gamma$ is a compact complex analytic variety which also carries the structure of a complex analytic abelian group, it is a complex torus (section 5.1.10). We have $\Gamma_{\mathbb{R}} \xrightarrow{\sim} V$ as real vector space and as usual we denote by I the complex structure on $\Gamma_{\mathbb{R}}$ induced by this isomorphism.

We change our point of view slightly. Our starting point is a free abelian group Γ of rank $2g$ on which we have an alternating 2-form

$$\langle \ , \ \rangle : \Gamma \times \Gamma \longrightarrow \quad .$$

A second datum is a complex structure $I : \Gamma_{\mathbb{R}} \longrightarrow \Gamma_{\mathbb{R}}$ which is an isometry for the pairing, i.e. $\langle Ix,Iy \rangle = \langle x,y \rangle$ for all $x,y \in \Gamma_{\mathbb{R}}$. In the sequel I will say that $\langle \ , \ \rangle$ and I are **compatible**.
We put $V = (\Gamma_{\mathbb{R}},I)$ and consider it as a complex vector space. Then $A = V/\Gamma$ is our complex torus. Let H on $V = (\Gamma_{\mathbb{R}},I)$ be the Hermitian form obtained from $(\langle \ , \ \rangle,I)$ (see pages 169 f.).
The pairing $\langle \ , \ \rangle$ allows us to construct certain line bundles

$$\mathcal{L}_H(\langle \ , \ \rangle,\eta,\varphi) = \mathcal{L}((\langle \ , \ \rangle,I),\eta,\varphi)$$

which depend on additional data φ and η where $\varphi \in \mathrm{Hom}(\Gamma,\mathbb{C})$ and where η is a map

$$\eta : \Gamma/2\Gamma \longrightarrow \tfrac{1}{2}\ \ /\ $$

which satisfies the compatibility relation

$$\tfrac{1}{2}\langle \gamma_1,\gamma_2 \rangle \ + \eta(\gamma_1+\gamma_2) - \eta(\gamma_1) - \eta(\gamma_2) = 0 \mod \tag{5.66}$$

for all $\gamma_1,\gamma_2 \in \Gamma$. We say that η is **adapted** to the alternating form $\langle \ , \ \rangle$.

These data allow us to construct a line bundle. We consider an open connected neighborhood U of a point $x \in A$ which is so small that the connected components U_α in the inverse image of U map isomorphically to U under the projection

$$p : V \longrightarrow A.$$

For any two such components $U_\alpha,U_\beta \subset p^{-1}(U)$ there is exactly one element $\gamma \in \Gamma$ such that $\gamma + U_\alpha = U_\beta$. We define a sheaf $\mathcal{L}_H(\langle \ , \ \rangle,\eta,\varphi) = \mathcal{L}(\langle \ , \ \rangle,\eta,\varphi)$ whose sections over U are the holomorphic functions

$$f : p^{-1}(U) \longrightarrow \mathbb{C}$$

which satisfy the transformation rule

$$f(z+\gamma) = f(z)e^{\pi(H(z,\gamma)+\frac{1}{2}H(\gamma,\gamma))+2\pi i(\varphi(\gamma)+\eta(\gamma))}. \tag{5.67}$$

The reader should notice that $e^{2\pi i\eta(\gamma)}$ is well defined and is equal to ± 1.

I claim that giving such a function is the same as giving a holomorphic function on any of the connected components U_α in $p^{-1}(U)$ and then extending it to the other components by the transformation rule. To see this we have to check consistency which means we have to verify that

$$f(z+\gamma_1+\gamma_2) = f((z+\gamma_1)+\gamma_2).$$

We compute both sides:

$$f(z+\gamma_1+\gamma_2) = f(z)e^{\pi(H(z,\gamma_1+\gamma_2)+\frac{1}{2}H(\gamma_1+\gamma_2,\gamma_1+\gamma_2))+2\pi i(\varphi(\gamma_1+\gamma_2)+\eta(\gamma_1+\gamma_2))} \tag{5.68}$$
$$= f(z)e^{\pi(H(z,\gamma_1)+H(z,\gamma_2)+\frac{1}{2}(H(\gamma_1,\gamma_1)+H(\gamma_2,\gamma_2)+H(\gamma_1,\gamma_2)+H(\gamma_2,\gamma_1)))+2\pi i(\varphi(\gamma_1)+\varphi(\gamma_2)+\eta(\gamma_1+\gamma_2))}.$$

For the other side we get

$$f((z+\gamma_1)+\gamma_2) = f(z+\gamma_1)e^{\pi(H(z+\gamma_1,\gamma_2)+\frac{1}{2}H(\gamma_2,\gamma_2))+2\pi i(\varphi(\gamma_2)+\eta(\gamma_2))} \tag{5.69}$$
$$= f(z)\cdot e^{\pi(H(z,\gamma_1)+\frac{1}{2}H(\gamma_1,\gamma_1)+H(z,\gamma_2)+H(\gamma_1,\gamma_2)+\frac{1}{2}H(\gamma_2,\gamma_2))+2\pi i(\varphi(\gamma_1)+\varphi(\gamma_2)+\eta(\gamma_1)+\eta(\gamma_2))}.$$

The exponential factors are equal because their quotient is

$$e^{\pi(\frac{1}{2}(H(\gamma_2,\gamma_1)-H(\gamma_1,\gamma_2)))+2\pi i(\eta(\gamma_1+\gamma_2)-\eta(\gamma_1)-\eta(\gamma_2))} \tag{5.70}$$
$$= e^{2\pi i(\frac{1}{2}\operatorname{Im} H(\gamma_2,\gamma_1)+\eta(\gamma_1+\gamma_2)-\eta(\gamma_1)-\eta(\gamma_2))}.$$

and this is equal to 1 since we required equation (5.66) for η. The result of this computation can be formulated differently: The map

$$\begin{aligned} C_H(\varphi,\eta) : \Gamma &\longrightarrow \mathcal{O}_V(V) \\ \gamma &\longmapsto \pi\left(H(z,\gamma)+\tfrac{1}{2}H(\gamma,\gamma)\right)+2\pi i\left(\varphi(\gamma)+\eta(\gamma)\right) \end{aligned} \tag{5.71}$$

is a 1-cocycle on Γ with values in the holomorphic functions on V modulo the constant functions which have values in $2\pi i$. This shows that

$$\mathcal{L}_H(\langle\ ,\ \rangle,\eta,\varphi)(U) \simeq \mathcal{O}(U_\alpha)$$

for any component in $U_\alpha \subset p^{-1}(U)$. This means that $\mathcal{L}_H(\langle\ ,\ \rangle,\eta,\varphi)$ is a (holomorphic) line bundle. Now we see why the integrality of $\operatorname{Im} H$ on $\Gamma\times\Gamma$ is so important.

The data H and I determine each other, therefore we may either suppress the subscript H or the I in the notation. On the other hand it follows from the considerations on pages 169 ff. that the pair of -bilinear forms $<\ ,\ >$,H determines the complex structure I which is not directly visible in the definition of the line bundle. Hence it may be sometimes useful to keep the H.

In section 4.6.8 we have shown that the second cohomology group is

$$H^2(A,\ ) = H^2(V/\Gamma,\ ) = \operatorname{Hom}(\Lambda^2\Gamma,\ ). \tag{5.72}$$

We have the exact sequence

$$0 \longrightarrow H^1(A,\ \ ) \longrightarrow H^1(A,\mathcal{O}_A) \longrightarrow H^1(A,\mathcal{O}_A^*) \longrightarrow H^2(A,\ \ ) \tag{5.73}$$

and we leave it as an exercise to the reader to verify that

$$c_1(\mathcal{L}_H(\langle\ ,\ \rangle,\eta,\varphi)) = \operatorname{Im} H|\Gamma \times \Gamma. \tag{5.74}$$

It is not too difficult to show that for a given $\langle\ ,\ \rangle$ we can find an η. It is not unique, but it is easy to see that for two choices η_H,η'_H we can find a homomorphism $\delta : \Gamma \to \mathbb{C}$ so that $\delta(\Gamma) \subset \frac{1}{2}\ \ $ and $\delta(\gamma) = \eta_H(\gamma) - \eta'_H(\gamma) \in\ \ $. Then it is clear that

$$\mathcal{L}_H(\langle\ ,\ \rangle,\eta,\varphi) \xrightarrow{\sim} \mathcal{L}_H(\langle\ ,\ \rangle,\eta',\varphi+\delta).$$

Our next aim is to show that this construction gives us all line bundles on A. More precisely we want to give a description of the Picard group (see section 5.1.10) $\operatorname{Pic}(A)$ in terms of these data $<\ ,\ >$, I, η, φ. The bilinear form gives us the Chern class and once the bilinear form is fixed the φ will give the line bundles with a given Chern class. In any case it is clear that we have:

Proposition 5.2.1. *The group of Chern classes of line bundles is the kernel of the homomorphism*

$$H^2(A,\ \ ) \longrightarrow H^2(A,\mathcal{O}_A).$$

We have seen that $H^2(A,\ \ ) = \operatorname{Hom}(\Lambda^2\Gamma,\ \ )$ and it is an easy exercise in linear algebra to show that an element

$$c \in H^2(A,\ \ ) = \operatorname{Hom}(\Lambda^2\Gamma,\ \ )$$

goes to zero in $H^2(A,\mathcal{O}_A)$ if and only if the extension

$$c_{\mathbb{R}} : (\Gamma \otimes\ \ ) \wedge (\Gamma \otimes\ \ ) \longrightarrow$$

satisfies $c_{\mathbb{R}}(Ix,Iy) = c_{\mathbb{R}}(x,y)$, i.e. the complex structure is an isometry. It is of course clear that these alternating forms c which satisfy $c_{\mathbb{R}}(Ix,Iy) = c_{\mathbb{R}}(x,y)$ form a finitely generated subgroup $\operatorname{NS}(A)$ of $\operatorname{Hom}(\Lambda^2\Gamma,\ \ )$.

Definition 5.2.2. *This group* $\operatorname{NS}(A)$ *is called the* **Neron-Severi group**.

We should be aware that this group $\operatorname{NS}(A)$ can be trivial, actually this is the case for a generic choice of the complex structure on $\Gamma_{\mathbb{R}}$.

But for the classes c in the Neron-Severi group we gave an explicit construction of line bundles with Chern class c. We can take any $\mathcal{L}(c,\eta,\varphi)$.

Proposition 5.2.3. *The homomorphism from the subgroup generated by the* $\mathcal{L}(c,\eta,\varphi)$ *to the Neron-Severi group* $\operatorname{NS}(A)$ *is surjective.*

To get the group of all line bundles we return to its description as $H^1(A,\mathcal{O}_A^*)$ and put:

$$\operatorname{Pic}^0(A) = \ker\left(\delta : \operatorname{Pic}(A) \longrightarrow H^2(A,\ \ )\right)$$

From our familiar exact sequence we get

$$\mathrm{Pic}^0(A) = H^1(A,\mathcal{O}_A)/H^1(A,\ \ ).$$

Again we get from section 4.6.8 that $H^1(A,\ \ ) = H^1(\mathbb{C}^g/\Gamma,\ \ ) = \mathrm{Hom}(\Gamma,\ \ )$. To compute $H^1(A,\mathcal{O}_A)$ we consider the Dolbeault complex

$$0 \longrightarrow \Omega^0_M(\mathcal{O}_A)(A) \longrightarrow \Omega^{0,1}_M(\mathcal{O}_A)(A) \longrightarrow \Omega^{0,2}_M(\mathcal{O}_A)(A) \longrightarrow \dots \tag{5.75}$$

The tangent bundle of A is trivial. Using the translations we can identify the tangent space at any point to $T_{A,0} \simeq V$, the tangent space at zero.

Hence the bundle of differentials is also trivial and at any point

$$\Omega^1_{A,x} = \mathrm{Hom}\ \ (V,\mathbb{C}). \tag{5.76}$$

Of course the bundle $\Omega^{0,1}_A$ is also trivial and if we give a basis to V and write $z = (\dots,z_\alpha,\dots) \in \mathbb{C}^g = V$ then the global sections $\Omega^{0,1}_\infty(A)$ are given by

$$\omega = \sum f_\alpha d\overline{z}_\alpha \tag{5.77}$$

where f_α is a $\mathcal{C}_\infty$-function on A. We apply the principles of Hodge theory: we choose a positive definite Hermitian form on the tangent bundle, which we get from a Hermitian form on $V = T_{A,0}$. We choose it in such a way that the basis vectors above form an orthonormal basis. Then it is an easy computation to show that

$$\Delta''\omega = (d''\delta'' + \delta''d'')\omega = \left(\sum_\beta \frac{\partial^2 f_\alpha}{\partial z_\beta \partial \overline{z}_\beta}\right) d\overline{z}_\alpha. \tag{5.78}$$

In section 4.11.3 we proved that

$$^{0,1}\left(\Omega^{0,1}_\infty(A)\right) = \left\{\sum_\alpha c_\alpha d\overline{z}_\alpha \,\middle|\, c_\alpha \in \mathbb{C}\right\} \tag{5.79}$$

and that we get an isomorphism

$$\left\{\sum c_\alpha d\overline{z}_\alpha \,\middle|\, c_\alpha \in \mathbb{C}\right\} \xrightarrow{\sim} H^1(A,\mathcal{O}_A). \tag{5.80}$$

It does not depend on the metric, it is induced from the embedding of the translation invariant differential form into the space of all differential forms.

We consider the -vector space $V^\vee = \mathrm{Hom}_\mathbb{R}(V,\ \ )$. On this vector space we define a complex structure by $I\phi(Iv) = \phi(v)$, i.e $I\phi(v) = -\phi(Iv)$. Then we have

$$V^\vee \longrightarrow \mathrm{Hom}(V \otimes_\mathbb{R} \mathbb{C},\mathbb{C}) = \mathrm{Hom}(V^{1,0},\mathbb{C}) \oplus \mathrm{Hom}(V^{0,1},\mathbb{C}) \tag{5.81}$$

and the composition with the two projections is bijective. The projection to the second summand is $\mathbb{C}$-linear, i.e. $I\phi(v) = i\phi(v)$. This means that we have $\mathbb{C}$-linear isomorphisms

$$V^\vee \xrightarrow{\sim} \mathrm{Hom}(V^{0,1},\mathbb{C}) \xrightarrow{\sim} H^1(A,\mathcal{O}_A). \tag{5.82}$$

To get the group $\mathrm{Pic}^0(A)$ we have to divide the group $H^1(A,\mathcal{O}_A)$ by the subgroup $H^1(A,\ \ )$ and this means that we have to divide $V^\vee$ by $\Gamma^\vee = \mathrm{Hom}(\Gamma,\ \ ) \subset V^\vee$. Hence we get an isomorphism

$$c : \mathrm{Pic}^0(A) \xrightarrow{\sim} V^\vee/\Gamma^\vee \xrightarrow{\sim} \mathrm{Hom}(\Gamma,\mathbb{C})/(\mathrm{Hom}\ \ (V^{1,0},\mathbb{C}) + \mathrm{Hom}(\Gamma,\ \ )). \tag{5.83}$$

Here I recall that $\mathrm{Hom}\ \ (V^{1,0},\mathbb{C}) = \mathrm{Hom}\ \ (V,\mathbb{C}) = \{\phi \mid \phi(Iv) = i\phi(v)\}$.

We want to invert this isomorphism. We constructed the line bundles $\mathcal{L}_H(\langle\ ,\ \rangle,\eta,\varphi)$ where $\varphi \in \mathrm{Hom}(\Gamma,\mathbb{C})$. We denote the restrictions of φ to $V^{1,0}$ and $V^{0,1}$ respectively by $\varphi_{1,0},\varphi_{0,1}$ and hence $\varphi = (\varphi_{1,0},\varphi_{0,1})$.
It is clear from the construction that we have

Lemma 5.2.4.

(a) *The two line bundles $\mathcal{L}_H(\langle\ ,\ \rangle,\eta,\varphi)$ and $\mathcal{L}_H(\langle\ ,\ \rangle,\eta,\varphi')$ are isomorphic if*

$$\varphi - \varphi' = (\psi,0).$$

(b) *The bundles $\mathcal{L}_H(\langle\ ,\ \rangle,\eta,\varphi)$ and $\mathcal{L}_H(\langle\ ,\ \rangle,\eta,\varphi')$ are actually the same line bundles if*

$$\varphi - \varphi' \in \mathrm{Hom}(\Gamma,\ \ ).$$

Proof: To see (a) we observe that $e^{2\pi i\psi(z)}$ is holomorphic on V and multiplication by this function provides an isomorphism between $\mathcal{L}_H(\langle\ ,\ \rangle,\eta,\varphi)$ and $\mathcal{L}_H(\langle\ ,\ \rangle,\eta,\varphi')$.

The assertion b) is obvious because $e^{2\pi i\varphi(\gamma)} = e^{2\pi i\varphi'(\gamma)}$ for all $\gamma \in \Gamma$.

□

If now the alternating form $\langle\ ,\ \rangle = O$ is the trivial nullform then we choose $\eta_O = 0$. We find

$$\mathcal{L}(O,0,\varphi) \otimes \mathcal{L}(O,0,\varphi') = \mathcal{L}(O,0,\varphi + \varphi'),$$

i.e. our construction of line bundles yields a homomorphism

$$\mathrm{Hom}(\Gamma,\mathbb{C}) \longrightarrow \mathrm{Pic}^0(A)$$

which by the previous Lemma factors through $\mathrm{Hom}\ \ (V,\mathbb{C}) + \mathrm{Hom}(\Gamma,\ \ )$. Hence our construction yields a homomorphism

$$d : \mathrm{Hom}(\Gamma,\mathbb{C})/(\mathrm{Hom}\ \ (V,\mathbb{C}) + \mathrm{Hom}(\Gamma,\ \ )) \longrightarrow \mathrm{Pic}^0(A).$$

The remaining part of the proof follows from Proposition 5.2.5.
I leave it as an exercise to the reader to show:

Proposition 5.2.5. *The two homomorphisms c,d defined in the proof of Lemma 5.2.4 are inverse to each other.*

Corollary 5.2.6. *If $A = V/\Gamma$ is a complex torus then the group $\mathrm{Pic}^0(A)$ has again the structure of a complex torus and is canonically isomorphic to $\overline{V}^\vee/\Gamma^\vee$.*

Definition 5.2.7. *This torus is called the* **dual torus** *and is denoted by* $A^\vee$.

Our considerations also imply that the bundles with a given Chern class form a principal homogeneous space under

$$\mathrm{Hom}\ \ (\Gamma,\mathbb{C})/(\mathrm{Hom}\ \ (V,\mathbb{C}) \oplus \mathrm{Hom}(\Gamma,\ \ )). \tag{5.84}$$

But this description requires a choice of an η adapted to $\langle\ ,\ \rangle$. We have seen that changing η can be corrected by the modification of the linear form φ.
Now it is clear that all line bundles $\mathcal{L}$ on A are of the form $\mathcal{L}_H(\langle\ ,\ \rangle,\eta,\varphi) = \mathcal{L}(\langle\ ,\ \rangle,I,\eta,\varphi)$.

The Poincaré Bundle

We see that the line bundles on A with Chern class zero are parameterized by the points of the dual torus $A^\vee$. We want to make this statement more precise. We construct a line bundle $\mathcal{N}$ on $A \times A^\vee$ which has the following property: For any point $y \in A^\vee$ the isomorphism class of the line bundle $\mathcal{N}$ restricted to $A \times \{y\} \xrightarrow{\sim} A$ is the isomorphism class corresponding to the point $y \in A^\vee = \mathrm{Pic}^0(A)$.

We know what we have to do: we have to construct the right line bundle on

$$A \times A^\vee = V/\Gamma \times V^\vee/\Gamma^\vee = (V \oplus V^\vee)/(\Gamma \oplus \Gamma^\vee).$$

To do this we have to find the right alternating form. Starting from the non degenerate pairing (evaluation)

$$\begin{aligned} \Gamma \times \Gamma^\vee &\longrightarrow \\ (\gamma,\psi) &\longmapsto \psi(\gamma) \end{aligned} \tag{5.85}$$

we get the **tautological alternate pairing**

$$e : (\Gamma \oplus \Gamma^\vee) \times (\Gamma \oplus \Gamma^\vee) \longrightarrow\ \ , \tag{5.86}$$

which is defined by

$$e\,\langle(\gamma_1,\psi_1),(\gamma_2,\psi_2)\rangle = \psi_2(\gamma_1) - \psi_1(\gamma_2).$$

Of course it is clear that I is an isometry for this alternating form on $(\Gamma \oplus \Gamma^\vee) \otimes\ \ $. Now we have to find an η which is adapted to e. An easy calculation shows that we can take $\eta((\gamma,\psi)) = \frac{1}{2}\psi(\gamma)$. We consider the line bundle $\mathcal{L}(e,\eta,0)$ on $A \times A^\vee$.

We write down the Hermitian form on $V \oplus V^\vee$:

$$H((z,w),(z_1,w_1)) = -Iw_1(z) + w(Iz_1) + i(w_1(z) - w(z_1)) \tag{5.87}$$

and hence we get for the local sections of our bundle $\mathcal{L}(e,\eta,0)$ on $A \times A^\vee$.

$$f(z+\gamma,w+\psi) = f(z,w)e^{\pi(-I\psi(z)+w(I\gamma)+i(\psi(z)-w(\gamma))+\pi i\psi(\gamma)}. \tag{5.88}$$

If we now fix the second variable w and restrict the bundle to $A \times \{w\}$ then we get

$$\begin{aligned} f((z+\gamma,w)) &= f(z,w)e^{\pi(w(I\gamma)+i(-w(\gamma))} \\ &= f(z,w)e^{\pi i((-i\otimes Iw-w)(\gamma))}. \end{aligned} \tag{5.89}$$

This is now the bundle $\mathcal{L}(O,0,-i\otimes Iw-w)$. To get its isomorphism class we have to project $-i\otimes Iw-w$ to Hom $(V^{0,1},\mathbb{C})$. This projection is clearly the projection of $-2w$ and hence we see that the restriction is isomorphic to the line bundle which corresponds to $-w$ under the homomorphism $V^\vee \longrightarrow \mathrm{Pic}^0(A)$.

If we exchange the roles of A and $A^\vee$and fix the variable z and restrict to $\{z\}\times A^\vee$ then we get the line bundle

$$f((z,w+\psi)) = f(z,w)e^{\pi(-I\psi(z)+i\psi(z))} \tag{5.90}$$

on $A^\vee$. This is clearly the line bundle on $A^\vee$ which corresponds to z under the homomorphism $V^{\vee\vee}=V\to \mathrm{Pic}^0(A^\vee)\to A$.

Definition 5.2.8. *The bundle $\mathcal{L}(e,\eta,0)$ is called the* **Poincaré bundle** *and gets the new name $\mathcal{N}$.*

Proposition 5.2.9. *The Poincaré bundle realizes isomorphisms*

$$\begin{aligned} A^\vee &\xrightarrow{\sim} \mathrm{Pic}^0(A) \\ w &\longmapsto \mathcal{N} \mid A\times\{w\} \end{aligned}$$

and

$$\begin{aligned} A &\xrightarrow{\sim} \mathrm{Pic}^0(A^\vee) \\ z &\longmapsto \mathcal{N} \mid \{z\}\times A^\vee. \end{aligned}$$

Universality of $\mathcal{N}$

We briefly discuss another property of the bundle $\mathcal{N}$, which is called universality, we will skip some details. Let us assume that we have a line bundle $\mathcal{L}$ on $A\times T$ where T is a complex analytic variety. We assume that T is connected, and we also assume that $\mathcal{L}\mid A\times t_0$ is in $\mathrm{Pic}^0(A)$ for some point t_0. Now we can define a map

$$\psi : T \longrightarrow A^\vee = \mathrm{Pic}^0(A)$$

which is defined by

$$\mathcal{L}\mid A\times\{t\} \simeq \mathcal{N}_{\psi(t)}.$$

I claim that ψ is indeed an analytic map, and that in addition for any point $t_0\in T$ we can find a neighborhood V of t_0 such that

$$\mathcal{L}\mid A\times V \simeq (\mathrm{Id}\times\psi^*)(\mathcal{N})\mid A\times V.$$

We introduce the following notation: If we have two line bundles $\mathcal{L}_1,\mathcal{L}_2$ on $X \times T$ then we write

$$\mathcal{L}_1 \sim_T \mathcal{L}_2$$

if these bundles are isomorphic locally in T. This means that for any point $t \in T$ we can find a neighborhood V such that

$$\mathcal{L}_1 \mid X \times V \simeq \mathcal{L}_2 \mid X \times V.$$

Hence we can find a line bundle $\mathcal{M}$ on T such that

$$\mathcal{L}_1 \simeq \mathcal{L}_2 \otimes p_2^*(\mathcal{M}),$$

where p_2 is the projection to T. We can reformulate the claim

Proposition 5.2.10. *Let T be a connected complex manifold (or even only a connected complex space). For any line bundle $\mathcal{L}$ on $A\times T$, which satisfies $\mathcal{L} \mid A\times\{t_0\}$ for some point $t_0 \in T_0$, we have a unique holomorphic map $\psi : T \longrightarrow A^\vee$ such that $\mathcal{L} \sim_T (\mathrm{Id} \times\psi)^*(\mathcal{N})$.*

This looks very plausible but in fact it not so easy. I will gives a somewhat sketchy argument why this is true. The assertion is local in T, hence we can restrict our attention to an open neighborhood U of a given point $t_0 \in T$. We assume that we have local coordinates $u_1, \dots ,u_n$. We introduce a relative Dolbeault-complex, this will be the family of Dolbeault complexes along the fibres of the projection $A \times U \longrightarrow U$. To define this complex we observe that in any point $(x,u) \in A \times U$ we have the space $T^A_{x,u}$ of vertical tangent vectors along $A\times\{u\}$. We also choose a neighborhood V_x of x and assume that we have local coordinates $z_1, \dots ,z_d$, we actually take the linear coordinates in a connected component of the inverse image of U in $V = \mathbb{C}^g$. We define $\Omega^{0p}_{A\times U/U}$ to be the sheaf of forms, which on this neighborhood are given by

$$\omega = \sum f_{i_1,\dots,i_p}(z,u)d\bar{z}_{i_1} \wedge \dots \wedge d\bar{z}_{i_p}.$$

where the coefficients $f_{i_1,\dots,i_p}(z,u)$ are $\mathcal{C}^\infty$ and holomorphic in the variables u_i, evaluated at (x,u) these are multilinear alternating forms of type $(0,p)$ on $T^A_{x,u}$.
Now it is clear that we get a complex

$$0 \longrightarrow \mathcal{O}_{A\times U} \longrightarrow \Omega^{00}_{A\times U/U} \xrightarrow{d''} \Omega^{01}_{A\times U/U} \longrightarrow \dots .$$

Now we have to use a relative lemma of Dolbeault, which gives us that this complex is a resolution and we have to use a relative version of Exercise 16 to prove that this is indeed an acyclic resolution for the functor p_{2*} of $\mathcal{O}_{A\times U}$.
Now it is rather clear that our line bundle $\mathcal{L}$ is given by a cohomology class in $H^1(A \times U,\mathcal{O}_{A\times U})$. This class can now be represented (locally) by an element

$$\omega = \sum_i f_i(x,u)d\bar{z}_i,$$

where $f_i(z,u)$ is $\mathcal{C}^\infty$ and holomorphic in the $u_1, \dots ,u_n$. Now we use the same arguments as in 4.11.3, 5.1.1 and conclude that this form defines the same class as

$$\omega^0 = \sum_i f_i^0(x,u) d\bar{z}_i,$$

where $f_i^0(x,u) = f_i^0(u)$ is the constant Fourier-coefficient of $f_i(z,u)$, i.e. it does not depend on x (we assumed that the coordinates $z_1,\dots,z_d$ are linear) and since it is given by an integral in the x-direction is holomorphic in u. But now we know that the coefficients $f_i^0(u)$ are local coordinates for $\mathcal{L} \mid A \times \{u\}$ considered as a point in $A^\vee$ and this finishes the proof.
I should say that this is not the "right" proof of the lemma above. It is much more natural to prove it starting from the finiteness results for coherent sheaves in complex analytic geometry. These finiteness results imply so called semi-continuity theorems for the cohomology of coherent sheaves and these give a much more transparent proof of the lemma. These finiteness results are very deep (See [Gr-Re2], Chap. 10)
We will encounter a similar situation in 5.3.1, where we discuss the Picard group on certain products $X \times Y$. In the second volume we will analogous statements in the context of algebraic geometry. In that case the truth of the assertion will be a consequence of the construction, and we will need the full strength of the finiteness results in algebraic geometry. In the context of algebraic geometry the finiteness results are easier to prove.

5.2.2 Homomorphisms Between Complex Tori

If we have two such tori

$$V_1/\Gamma_1 = A_1 \qquad\qquad V_2/\Gamma_2 = A_2$$

then an analytic homomorphism $\phi : A_1 \to A_2$ is of course the same thing as a $\mathbb{C}$-linear map $\phi : V_1 \to V_2$ which maps the lattice Γ_1 into Γ_2. We may also view ϕ as an element $\phi : \Gamma_1 \to \Gamma_2$ which after extension to a linear map $\Gamma_1 \otimes \quad \to \Gamma_2 \otimes \quad$ respects the complex structures on $\Gamma_1 \otimes \quad = V_1$, $\Gamma_2 \otimes \quad = V_2$.

We summarize:

Proposition 5.2.11. *The module* $\mathrm{Hom}(A_1,A_2)$ *is a submodule of* $\mathrm{Hom}(\Gamma_1,\Gamma_2)$. *It consists of those elements which after extension to commute with the complex structures.*

(This looks rather innocent, but it is not. The reader should look at the discussion on the last page of the book in section 5.3.4.) A homomorphism $\varphi : A_1 \to A_2$ also induces a homomorphism between the Picard groups

$$\varphi^* : \mathrm{Pic}(A_2) \longrightarrow \mathrm{Pic}(A_1)$$

which is induced by the pull back of line bundles.

We can restrict this homomorphism to the groups $\mathrm{Pic}^0(A_2) = A_2^\vee$ and $\mathrm{Pic}^0(A_1) = A_1^\vee$ and denote this restriction by

$$\varphi^\vee : A_2^\vee \longrightarrow A_1^\vee.$$

A priori this is a homomorphism between abstract groups, but from the explicit description of the isomorphism $\mathrm{Pic}^0(A_i) \xrightarrow{\sim} A_i^\vee$ it becomes clear:

Proposition 5.2.12. *The element $\varphi^\vee$ is a homomorphism of complex tori. This homomorphism – viewed as an element in* $\mathrm{Hom}(\Gamma_2^\vee,\Gamma_1^\vee)$ *– is simply the adjoint of the element* $\varphi \in \mathrm{Hom}(\Gamma_1,\Gamma_2)$*. Especially we see that the function $\varphi \mapsto \varphi^\vee$ is additive.*

Proof: To see that this is true, we consider an element $x \in \mathrm{Pic}^0(A_2)$. We gave an explicit construction of a line bundle $\mathcal{L}_x$ corresponding to x. We choose a linear map

$$\lambda_x : \Gamma_2 \longrightarrow \mathbb{C},$$

which after extension to $\Gamma_2 \otimes \mathbb{C}$ and restriction to $\overline{V}_2$ maps to x.

For an open set $V \subset A_2$ the space of sections is

$$\mathcal{L}_x(V) = \left\{ f : \pi^{-1}(V) \longrightarrow \mathbb{C} \middle| \; f \text{ is holomorphic and } f(z+\gamma) = f(z)e^{2\pi i \lambda_x(\gamma)} \right\}$$

and the fibre of $\mathcal{L}_x$ in a point $y \in H^1(S,\mathcal{O}_S)/\Gamma$ is given by

$$(\mathcal{L}_x)_y = \left\{ f : \pi^{-1}(y) \longrightarrow \mathbb{C} \middle| f(y+\gamma) = f(y)e^{2\pi i \lambda_x(\gamma)} \right\}.$$

If now $\varphi : A_1 \longrightarrow A_2$ and if $y_1 \in A_1$, then

$$\varphi^*(\mathcal{L}_x)_{y_1} = (\mathcal{L}_x)_{\varphi(y_1)}.$$

If we consider the diagram

$$\begin{array}{ccccc} \Gamma_1 & \xrightarrow{\varphi} & \Gamma_2 & \xrightarrow{\lambda_x} & \mathbb{C} \\ \downarrow & & \downarrow & & \\ V_1 & \xrightarrow{\varphi} & V_2 & & \end{array}$$

then we see that $\varphi^*(\mathcal{L})$ is the line bundle defined by the composition

$$\lambda_x \circ \varphi : \Gamma_1 \longrightarrow \mathbb{C}$$

and this proves the desired formula. □

We may also consider the induced map

$$\overline{\varphi}^* : \mathrm{NS}(A_2) \longrightarrow \mathrm{NS}(A_1).$$

This homomorphism is easy to describe: An element $e \in \mathrm{NS}(A_2)$ is an alternating form $e : \Gamma_2 \times \Gamma_2 \longrightarrow$ and $\overline{\varphi}^*(e)$ is simply the form on $\Gamma_1 \times \Gamma_1$ induced by φ, i.e.

$$\overline{\varphi}^*(e)\langle \gamma_1,\gamma_1' \rangle = e\langle \varphi(\gamma_1),\varphi(\gamma_2) \rangle.$$

Therefore we get

Lemma 5.2.13. *The function $\phi \mapsto \overline{\phi}^*$ is quadratic, i.e. we have*

$$\overline{\phi + \psi}^* = \overline{\phi}^* + \overline{\psi}^* + < \phi,\psi >$$

where $(\phi,\psi) \longrightarrow < \phi,\psi >$ is a bilinear map

$$\mathrm{Hom}(A_1,A_2) \times \mathrm{Hom}(A_1,A_2) \longrightarrow \mathrm{Hom}(\mathrm{NS}(A_2),\mathrm{NS}(A_1)).$$

The Neron Severi group and $\mathrm{Hom}(A,A^\vee)$.

We come to another interpretation of the Neron-Severi group. An element $e \in \mathrm{NS}(A)$ defines a homomorphism

$$\begin{aligned} \Phi(e) : \Gamma &\longrightarrow \Gamma^\vee \\ \gamma &\longmapsto \{\gamma' \mapsto e\langle\gamma,\gamma'\rangle\}\,. \end{aligned}$$

The condition that I is an isometry for the extension $e_\mathbb{R}$ to $\Gamma_\mathbb{R}$ implies that $\Phi(e)$ extends to a $\mathbb{C}$-linear homomorphism

$$\widetilde{\Phi}(e) : V \longrightarrow V^\vee$$

We have the inclusions $\Gamma \subset V$ and $\Gamma^\vee \subset V^\vee$ and it is clear that $\widetilde{\Phi}(e)$ maps Γ into $\Gamma^\vee$ and induces $\Phi(e)$ on the lattices.

Therefore we see that we have a canonical homomorphism

$$\Phi : \mathrm{NS}(A) \longrightarrow \mathrm{Hom}(A,A^\vee).$$

Any element $\phi : \Gamma \to \Gamma^\vee$ has a transpose

$$\phi^\vee : \Gamma^{\vee\vee} = \Gamma \longrightarrow \Gamma^\vee.$$

We can define the alternating elements $\mathrm{Hom}_{\mathrm{alt}}(A,A^\vee)$ to be the elements which satisfy $\phi^\vee = -\phi$ and it is an easy exercise in linear algebra to show that our above map Φ provides an isomorphism

$$\Phi : \mathrm{NS}(A) \longrightarrow \mathrm{Hom}_{\mathrm{alt}}(A,A^\vee). \tag{5.91}$$

If we have a homomorphism $\phi : A_1 \to A_2$ between two complex tori and consider the induced homomorphism $\phi^* : \mathrm{NS}(A_2) \to \mathrm{NS}(A_1)$, then we get a homomorphism $\Phi_1 \circ \phi^* \circ \Phi_2^{-1} : \mathrm{Hom}_{\mathrm{alt}}(A_2,A_2^\vee) \to \mathrm{Hom}_{\mathrm{alt}}(A_1,A_1^\vee)$ and it is straightforward from the definition that this homomorphism sends

$$\psi \longmapsto {}^t\phi \circ \psi \circ \phi. \tag{5.92}$$

The inverse of this homomorphism Φ is given by the map that sends an alternating element ϕ to the form

$$e_\phi\langle\gamma,\gamma'\rangle = \phi(\gamma')(\gamma). \tag{5.93}$$

We have another homomorphism $\Psi : \mathrm{Hom}(A,A^\vee) \to \mathrm{NS}(A)$. To get this homomorphism we start from the line bundle $\mathcal{N}$ on $A \times A^\vee$. For any $\phi : A \to A^\vee$ we get an embedding $i_\phi : A \to A \times A^\vee$ by $z \mapsto (z,\phi(z))$. We get a bundle $i_\phi^*(\mathcal{N})$ on A and its Chern class is $\Psi(\phi)$. The resulting form

$$< \gamma_1,\gamma_2 >_\phi = \phi(\gamma_1)(\gamma_2) - \phi(\gamma_2)(\gamma_1) = e_\phi < \gamma_1,\gamma_2 > - e_{\phi^\vee} < \gamma_1,\gamma_2 > \tag{5.94}$$

on Γ is alternating. It depends only on the alternating component of ϕ and for alternating ϕ the map $\phi \mapsto <\ ,\ >_\phi$ is twice the inverse of Φ.

The construction of Ψ starting from a line bundle

We want to give a different construction of the homomorphism Φ which works with the line bundles themselves rather than with their Chern classes.

To our element $e \in \mathrm{NS}(A)$ we choose a line bundle $\mathcal{L}$ with $c_1(\mathcal{L}) = e$, in other words we choose an adapted η and a $\varphi : \Gamma \to \mathbb{C}$ and consider the line bundle

$$\mathcal{L} = \mathcal{L}(e,\eta,\varphi).$$

Any element $x \in A$ induces a translation $T_x : y \mapsto x + y$ on A and we can consider the line bundle $T_x^*(\mathcal{L}) \otimes \mathcal{L}^{-1}$. To compute this line bundle we choose an element $\widetilde{x}$ in the fibre $p^{-1}(x)$. Let H_e the attached Hermitian form then the fibre of $T_x^*(\mathcal{L})$ at a point z is equal to the fibre of $\mathcal{L}$ at $x + z$ and therefore it is given by the functions which satisfy

$$f(\widetilde{z} + \widetilde{x} + \gamma) = f(\widetilde{z} + \widetilde{x})e^{\pi\left(H_e(\widetilde{z}+\widetilde{x},\gamma)+\frac{1}{2}H_e(\gamma,\gamma)\right)+2\pi i(\varphi(\gamma)+\eta_{H_e}(\gamma))} \tag{5.95}$$

for all $\widetilde{z} \in p^{-1}(z)$.

Comparing this to the fibre of $\mathcal{L}$ at z yields that the fibre of $T_x^*(\mathcal{L}) \otimes \mathcal{L}^{-1}$ is given by the functions

$$f(\widetilde{z} + \gamma) = f(\widetilde{z})e^{\pi H_e(\widetilde{x},\gamma)}. \tag{5.96}$$

This line bundle is obtained from the linear form $\varphi_{\widetilde{x}} : \gamma \mapsto H_e(\widetilde{x},\gamma)$, in other words it is isomorphic to $\mathcal{L}_O(\varphi_{\widetilde{x}})$. An easy calculation shows that this linear form is of the type $(0,\varphi_{1,0})$, in other words it is trivial on the first component in the decomposition

$$\mathrm{Hom}\ \ (\Gamma \otimes \mathbb{C},\mathbb{C}) = \mathrm{Hom}\ \ (V \oplus \overline{V},\mathbb{C}). \tag{5.97}$$

The same calculation shows that the linear form $\psi_{\widetilde{x}} : \gamma \mapsto H(\gamma,\widetilde{x})$ is of type $(\psi_{0,1},0)$. Therefore we do not change the isomorphism class of the line bundle if we replace $\varphi_{\widetilde{x}}$ by

$$\phi_{\widetilde{x}}(\gamma) = H(\widetilde{x},\gamma) - H(\gamma,\widetilde{x}) = 2i\mathrm{Im}\, H_e(\widetilde{x},\gamma) = 2ie < \widetilde{x},\gamma > . \tag{5.98}$$

Hence we see that $T_x^*(\mathcal{L}) \otimes \mathcal{L}^{-1} \xrightarrow{\sim} \mathcal{L}(O,0,e < \widetilde{x},\_ >)$ where $e < \widetilde{x},\_ >$ is a linear map from Γ to $\mathbb{C}$.

Therefore it is clear that we have

Lemma 5.2.14. *The map*

$$x \mapsto T_x^*(\mathcal{L}) \otimes \mathcal{L}^{-1}$$

from A to $A^\vee$ is a homomorphism and this homomorphism is equal to $\Phi(e)$.

This new description of Φ has the advantage that it is constructed in terms of the bundles rather than in terms of the Chern classes. It is of course important that this homomorphism depends only on the Chern class of the line bundle $\mathcal{L}$.

Definition 5.2.15. *An element $c = c_1(\mathcal{L})$ of the Neron-Severi group is called* **rationally non degenerate** *if the alternating pairing $c\ \ : \Gamma\ \ \times \Gamma\ \ \longrightarrow \mathbb{Q}$ is non degenerate.*

Proposition 5.2.16. *Let $c = c_1(\mathcal{L})$ be a rationally non degenerate element in the Neron-Severi group. Then the induced homomorphism $\phi_c : \Gamma \to \Gamma^\vee$ is injective and the image $\phi_c(\Gamma) \subset \Gamma^\vee$ has finite index. From our description of the complex tori it is immediately clear that the kernel of*

$$x \longmapsto T_x^*(\mathcal{L}) \otimes \mathcal{L}^{-1}$$

is canonically isomorphic to $\Gamma^\vee/\phi_c(\Gamma)$. Hence we have an isomorphism

$$\ker(\phi_c) \xrightarrow{\sim} \Gamma^\vee/\phi_c(\Gamma).$$

I want to mention, that the order of this index is a perfect square. This follows easily, if we believe that we can find a basis $e_1,\dots,e_g,f_1,\dots,f_g$ such that $e(e_\nu,f_\nu) = -e(f_\nu,g_\nu) = n_i$ and all other values give zero. Then we get as a basis for the dual module $\Gamma^\vee$ the elements $e_1/n_1,\dots,e_g/n_g,f_1/n_1,\dots,f_g/n_g$ and $[\Gamma^\vee : \Gamma] = \prod n_\nu^2$. The number $|\prod_\nu n_\nu|$ is called the **Pfaffian** $\mathrm{Pf}(e)$ of e.

5.2.3 The Self Duality of the Jacobian

We specialize these considerations to the Jacobian J of our Riemann surface S. We resume our considerations in section 5.1.10. We saw that

$$J = H^1(S,\mathcal{O}_S)/\Gamma = \overline{H^0(S,\Omega^1_S)}/\Gamma$$

where we identify $H^1(S,\mathcal{O}_S) = \overline{H^0(S,\Omega^1_S)}$ by means of the Dolbeault isomorphism. The submodule Γ is the image of $H^1(S,\ )$ under the homomorphism $j : H^1(S,\ ) \to \Gamma$. The complex structure I on $\Gamma_{\mathbb{R}} \xrightarrow{\sim} H^1(S,\mathcal{O}_S)$ is the one induced from $H^1(S,\mathcal{O}_S)$.
On this module Γ we have the privileged alternating form given by the cup product $e_0 : \Gamma \times \Gamma \to\ $. It provides an isomorphism

$$\varphi_{e_0} : \Gamma \longrightarrow \Gamma^\vee.$$

The Riemann period relations (see section 5.1.12) say that the complex structure on $\Gamma_{\mathbb{R}}$ is an isometry for e_0. Hence we get an isomorphism

$$\begin{array}{cccc} j_{e_0} : & J & \longrightarrow & J^\vee \\ & \| & & \| \\ & H^1(S,\mathcal{O}_S)/\Gamma & \longrightarrow & H^0(S,\Omega^1_S)^\vee/\Gamma^\vee \end{array}$$

The isomorphism or – what is the same – the class $e_0 \in \mathrm{NS}(J)$ is called the **canonical polarization** of J (see Definition 5.2.21). It is an additional datum attached to the complex torus.
At the end of the discussion of Abel's theorem we discussed the embedding

$$i_{P_0} \quad : \quad S \longrightarrow J$$

which provided a homomorphism

$$\begin{array}{cccc} {}^t i_{P_0} : & \mathrm{Pic}^0(J) & \longrightarrow & \mathrm{Pic}^0(S) \\ & \| & & \| \\ & J^\vee & \longrightarrow & J \end{array}$$

and now it is clear from these computations that ${}^t i_{P_0}$ is the inverse of the canonical polarization.
The polarization j_{e_0} induces an isomorphism $\mathrm{Hom}(J,J) \xrightarrow{\sim} \mathrm{Hom}(J,J^\vee)$ and combining this with the isomorphism (5.91) gives us an isomorphism

$$\mathrm{NS}(J) \xrightarrow{\sim} \mathrm{End}_{\mathrm{sym}}(J,J) \tag{5.99}$$

where the subscript sym refers to the pairing e_0. Using this isomorphism we can interpret the induced morphism $\overline{\varphi}^*$ as endomorphism $\overline{\varphi}^*$ of $\mathrm{End}_{\mathrm{sym}}(J,J)$ and it is clear from the definition that

$$\varphi^*(\psi) = {}^t\varphi\psi\varphi \tag{5.100}$$

5.2.4 Ample Line Bundles and the Algebraicity of the Jacobian

The Kodaira Embedding Theorem

Let us assume that we have an alternating form $e = <\ ,\ >: \Gamma\times\Gamma \longrightarrow$ and a compatible complex structure I. So far it did not play any role that the Hermitian form H attached to this form e was positive definite. We want to discuss the implication of the positivity and we will see that it implies that sufficiently high powers of this bundle will have many sections.
Before I discuss this implication of the positivity I want to place this positivity into a general context. I refer to the section 4.11.2 on Kähler manifolds. There we attached a 2-form ω_h to any (positive definite) Hermitian form h on the tangent bundle. In our case here the tangent bundle of $A = V/\Gamma$ is trivial and isomorphic to $\Gamma \otimes$ at the origin. Then our 2-form ω_h on A is invariant by translation and at the origin it is our form e. It is clear that ω_h is closed, it defines a class $[\omega_h] \in H^2(A,\ )$ and of course

$$[\omega_h] = e.$$

If now in addition the Hermitian form H obtained from the alternating form and the complex structure is positive definite, then $<\ ,\ >$ gives us a Kähler metric on $A = V/\Gamma$ whose class is integral.
I want to formulate the famous embedding theorem of Kodaira. Before I can do this I have to make a short comment on the coordinate free definition of the projective space.

Definition 5.2.17. *If V is any $\mathbb{C}$-vector space of finite dimension, then we define $\mathbb{P}(V)$ to be the space of linear hyperplanes $H \subset V$.*

We have to say what the holomorphic functions in a neighborhood of a point $H \in \mathbb{P}(V)$ are. This point is defined as the set of zeroes of a linear form λ_H. If $v_0 \in V \setminus H$ and $v \in V$ then then $\lambda \mapsto \lambda(v)/\lambda(v_0)$ defines a function on the set of those $\lambda \in V^\vee$ for which $\lambda(v_0) \neq 0$, hence for those λ in a small neighborhood of λ_H. We choose a basis $v_1,v_2,\dots,v_{n-1}$ of H, then we define $x_i(\lambda) = \lambda(v_i)/\lambda(v_0)$. These functions vanish at $\lambda_H = H$.

Definition 5.2.18. *The local ring of germs of holomorphic functions at H is now defined as the ring of power series in the x_i which have a strictly positive radius of convergence. In other words these $x_i(\lambda)$ form a system of local coordinates at H.*

Now we see that

$$\mathbb{P}(V) = (V^\vee \setminus \{0\})/\mathbb{C}^\times \tag{5.101}$$

and if we choose a basis for $V^\vee$ then we get back our previous definition. We have the **tautological line bundle** whose fibre over H is simply the line λ with $\lambda(H) = 0$. It is easy to see that this gives us the bundle $\mathcal{O}_{\ (V)}(-1)$. The dual bundle is $\mathcal{O}_{\ (V)}(1)$ and we have a canonical isomorphism

$$H^0(\mathbb{P}(V),\mathcal{O}_{\ (V)}(1)) \xrightarrow{\sim} V.$$

Now we state the famous embedding theorem of Kodaira.

Theorem 5.2.19 (Kodaira Embedding Theorem)**.** *Let X be a compact complex manifold. Let us assume that we have a Hermitian metric h on T_X whose corresponding class ω_h is closed and defines an integral class in $H^2(X,\ )$. Then we can find a line bundle $\mathcal{L}$ on X whose Chern class $c_1(\mathcal{L}) = [\omega_h]$. For $n \gg 0$ we have that $H^q(X,\mathcal{L}^{\otimes n}) = 0$ for all $q > 0$ and for any $x \in X$ we can find a section $s \in H^0(X,\mathcal{L}^{\otimes n})$ which does not vanish at x. Then we get a holomorphic map*

$$\begin{aligned} \Theta_n(\mathcal{L}) : X &\longrightarrow \mathbb{P}(H^0(X,\mathcal{L}^{\otimes n})) \\ x &\longmapsto H_x = \{s \in H^0(X,\mathcal{L}^{\otimes n}) | s(x) = 0\}, \end{aligned}$$

which for suitably large values of n is an embedding, i.e. it defines an isomorphism between X and a smooth closed complex submanifold Y of $\mathbb{P}(H^0(X,\mathcal{L}^{\otimes n}))$.

This theorem will not be proved here, for a proof see [Se1]
We have a tautological example for this theorem.

Example 22. *If our manifold X is the projective space $\mathbb{P}^n(\mathbb{C})$ itself and the bundle is $\mathcal{L} = \mathcal{O}_{\ ^n(\ )}(1)$, then we can take $n = 1$ and already this bundle provides an embedding. A closer look shows that this embedding is simply the identity.*

In the general case it is clear from the construction that the restriction by $\Theta_n(\mathcal{L})$ of the bundle $\mathcal{O}_{\ (H^0(X,\mathcal{L}^{\otimes n}))}(1)$ to X is our bundle $\mathcal{L}^{\otimes n}$. We will give a few more comments on this theorem when we discuss the Theorem of Lefschetz. We apply this theorem to our complex tori.

Corollary 5.2.20. *The class ω_h is given by $e =< , >$ and the Hermitian metric h is given by $H = H_{<,>}$, hence H has to be positive definite. Any bundle $\mathcal{L} = \mathcal{L}_H(\langle\ ,\ \rangle,\eta,\varphi)$ is of the type as in the theorem and provides a projective embedding.*

Definition 5.2.21. *If we can find such a compatible alternating form e on (V,I) for which the attached Hermitian form is positive definite, then we say that our complex torus is an* **abelian variety**. *The class e is called a* **polarization** *of $A = V/\Gamma$. Two polarizations e_1,e_2 are considered to be equivalent if we can find integers $n,m > 0$ such that $ne_1 = me_2$. If the alternating form e is non degenerate, then we will call it a* **principal polarization**. *The canonical polarization on a Jacobian J of a Riemann surface is principal.*

We will almost prove the above theorem of Kodaira in the special case of abelian varieties (see Theorem 5.2.35). This will be done by showing that the bundles have a lot of sections. After that we will make it more precise what a projective embedding is.

The Spaces of Sections

We want to compute the space of global sections in our line bundles $\mathcal{L}(\langle\ ,\ \rangle,\eta,\varphi)$. To do this we give a different description of these bundles: we modify the cocycle $C_H(z,\gamma)$ by a boundary.
To get this modification we choose a sublattice $G \subset \Gamma$ of rank g such that Γ/G is free and the alternating form $<\ ,\ >$ is trivial on G. This is possible because our form is alternating. Then our Hermitian form H restricted to G takes real values and is symmetric. Since we have $G \oplus IG = V$ we can extend this restriction to a symmetric $\mathbb{C}$-bilinear form h on V. Since H is $\mathbb{C}$-linear in the first variable we have

$$H(z,\gamma) = h(z,\gamma) \text{ for all } \gamma \in G. \tag{5.102}$$

For simplicity I want to assume that the restrictions of η and φ to G are trivial. Actually we can assume this without loss of generality. To see this we apply Lemma 5.2.4. The function $\eta|G$ satisfies $\eta(g_1+g_2) = \eta(g_1)+\eta(g_2)$ for $g_1,g_2 \in G$. We can construct a linear form $\varphi' : \Gamma \longrightarrow \frac{1}{2}$ such that $\eta|G = \varphi'|G \mod 2$. Now we modify η by φ' such that $\eta(\gamma) = 0$ for all $\gamma \in G$. Once we have done this we also modified φ to φ_1. We can restrict the form $\varphi_1 : \Gamma \longrightarrow \mathbb{C}$ to G and extend this φ to a linear $\mathbb{C}$-form ψ on V. We have seen that $\mathcal{L}(<\ ,\ >,\eta,\varphi) \simeq \mathcal{L}(<\ ,\ >,0,\varphi_1-\psi)$ and hence we may also assume that φ restricted to G is trivial.

We look at our 1-cocycle $\mod 2\pi i$

$$\gamma \mapsto \pi(H(z,\gamma) + \tfrac{1}{2}H(\gamma,\gamma)) + 2\pi i(\varphi(\gamma)+\eta(\gamma)) =: C_H(z,\gamma). \tag{5.103}$$

Proposition 5.2.22. *This cocycle is uniquely determined by Γ, $<\ ,\ >$, the complex structure I, η and φ.*

We change our notation slightly and denote the resulting bundle by $\mathcal{L}(C_H,\eta,\varphi)$. Now we consider global sections in this bundle and this means that we consider holomorphic functions which satisfy

$$f(z+\gamma) = f(z)e^{C_H(z,\gamma)+2\pi i(\varphi(\gamma)+\eta(\gamma))}. \tag{5.104}$$

We modify these functions and consider

$$\widetilde{f}(z) = f(z)\cdot e^{-\frac{\pi}{2}h(z,z)}. \tag{5.105}$$

These functions can be considered as sections in a new bundle $\mathcal{L}(C_{\mathrm{hol}},\eta,\varphi)$ which is isomorphic to the given one but which is described by a different 1-cocycle. If we put

$$C_{\mathrm{hol}}(z,\gamma) = \pi(H(z,\gamma) - h(z,\gamma)) + \tfrac{\pi}{2}\left(H(\gamma,\gamma) - h(\gamma,\gamma)\right) \tag{5.106}$$

then the sections of the bundle $\mathcal{L}(C_{\mathrm{hol}},\eta,\varphi)$ are functions which satisfy

$$\widetilde{f}(z+\gamma) = \widetilde{f}(z)\cdot e^{C_{\mathrm{hol}}(z,\gamma)+2\pi i(\varphi(\gamma)+\eta(\gamma))}. \tag{5.107}$$

This new 1-cocycle has the disadvantage that it depends on the choice of G but it has several advantages:

1. We have $H(z,\gamma) = h(z,\gamma)$ for all $z \in V$, $\gamma \in G$ and $\delta(\gamma) = 0$, $\varphi(\gamma) = 0$ for all $\gamma \in G$. Hence we see that $\widetilde{f}(z+\gamma) = \widetilde{f}(z)$ for all $\gamma \in G$, the function $\widetilde{f}$ is periodic with respect to the sublattice G.
2. We will show that the cocycle depends "holomorphically" on I and this means that we can view the abelian varieties together with the bundles as a holomorphic family.

Further down we will give a rather explicit description of the space of sections of these line bundles. Before I carry out this computation in detail I want to explain how we can view the variable I as a variable in a complex variety and what it means, that the cocycle depends holomorphically on I.

5.2.5 The Siegel Upper Half Space

We explain how we can view the complex structures as points in a complex variety, this variety will be the Siegel upper half space. In accordance with our previous definitions we say

Definition 5.2.23. *A* **principally polarized abelian variety** *is a triplet* $A = (\Gamma, \langle\ ,\ \rangle, I)$ *where*

1. Γ *is a free -module of rank* $2g$ *and* $\langle\ ,\ \rangle$ *is a skew symmetric form*
$$\langle\ ,\ \rangle : \Gamma \times \Gamma \longrightarrow$$
which is non degenerate over . This means that we can write our lattice
$$\Gamma = \bigoplus_{\nu=1}^{g} \quad e_\nu \oplus \quad f_\nu$$
where $\langle e_\nu, f_\nu\rangle = -1 = -\langle f_\nu, e_\nu\rangle$ *and where all other* $\langle\ ,\ \rangle$ *between basis elements are zero.*
2. *The element* I *is a complex structure on* $\Gamma_{\mathbb{R}}$*, we have* $I^2 = -\operatorname{Id}$ *and it respects the alternating form* $\langle\ ,\ \rangle_{\mathbb{R}} : \Gamma_{\mathbb{R}} \times \Gamma_{\mathbb{R}} \longrightarrow$ *.*
3. *On the complex vector space* $V = (\Gamma_{\mathbb{R}}, I)$ *we can define a Hermitian form* H_I *on* V *by*
$$\operatorname{Im} H_I(x,y) = \langle x,y\rangle$$
for all $x,y \in \Gamma_{\mathbb{R}}$*. It is part of our assumption that this form is positive definite.*

Clearly these data provide a complex torus $A = V/\Gamma$. We want to explain that these data can be viewed as points in a complex manifold. The datum that varies is the element I, we want to show that we can interpret these I as points on a complex manifold. We extend the scalars to $\mathbb{C}$, we extend the form $\langle\ ,\ \rangle$ bilinearly to $\Gamma \quad = \Gamma \otimes \mathbb{C}$. If such an element I is given, then Γ decomposes
$$\Gamma \quad = \Gamma^{1,0} \oplus \Gamma^{0,1},$$

where $\Gamma^{1,0}$ is the eigenspace for I with eigenvalue i and $\Gamma^{0,1}$ is the eigenspace with eigenvalue $-i$. Hence we see that I defines a subspace $\Gamma^{0,1}$, which is maximal isotropic, i.e. all scalar products of two elements in $\Gamma^{0,1}$ are zero. We introduce the **Grassmann variety** $\mathbf{Gr}_g$ of maximal isotropic subspaces (with respect to $< , >$) in Γ . This is a set. These subspaces have dimension g. We can define the structure of a complex manifold on $\mathbf{Gr}_g$: Let $X \subset \Gamma$ be such a maximal isotropic subspace. We can find a second maximal isotropic subspace Y_0 such that $\Gamma \quad = X \oplus Y_0$. We say that Y_0 is in **general position** (or **in opposition**) to X. We choose a basis $\{x_1, x_2, \ldots, x_g\}$ of X and $\{y_1, y_2, \ldots, y_g\}$ of Y_0 such that

$$< x_\nu, y_\mu > = \delta_{\nu,\mu}.$$

If now Y is any maximal isotropic subspace which is in opposition to X, then it has a unique basis of the form

$$\widetilde{y}_\nu = y_\nu + \sum \tau_{\nu,\mu} x_\mu.$$

An easy computation shows that a subspace generated by elements $\widetilde{y}_1, \widetilde{y}_2, \ldots, \widetilde{y}_g$ of the above form is isotropic if and only if the $\tau_{\nu,\mu}$ are symmetric, i.e. $\tau_{\nu,\mu} = \tau_{\mu,\nu}$. Hence we see that the $\tau_{\nu,\mu}$ with $\nu \leq \mu$ can serve as local coordinates for a complex structure on $\mathbf{Gr}_g$ in a neigborhood of the point Y_0. The point Y_0 has coordinates $\tau_{\nu,\mu} = \delta_{\nu,\mu}$. These local coordinates are valid on the set of those maximal isotropic subspaces which are in opposition to X. If we have an element $Y \in \mathbf{Gr}_g$, which is not in opposition to X, then we choose another X'. It is elementary to check that the two complex structures restricted to those Y, which are in opposition to X and X' are the same.
Of course our maximal isotropic sublattices G for which Γ/G is free yield points $G \otimes \mathbb{C}$ in the Grassmannian. These are the **integral points** in the Grassmannian. In the second volume we will learn that the Grassmannian is actually a "projective scheme over the integers ". Then the set of our G above will be the -valued points of this scheme.

Definition 5.2.24. *The* **symplectic group** $\mathrm{Sp}_g(\ )$ *is the group of linear transformations in* $\mathrm{GL}_{2g}(\ )$ *which leave the alternating form* $< , >$ *invariant.*

For any commutative ring R with identity the group $\mathrm{Sp}_g(R)$ is the corresponding subgroup of $\mathrm{GL}_{2g}(R)$. This means that Sp_g is an algebraic group over . It is elementary that $\mathrm{Sp}_g(\ )$ acts transitively on the set of all sublattices G as above. The group $\mathrm{Sp}_g(\mathbb{C})$ acts transitively on $\mathbf{Gr}_g$.
The stabilizer P_z of a point $z \in \mathbf{Gr}_g$ is a **parabolic subgroup** of $\mathrm{Sp}_g(\mathbb{C})$. It is a special type of parabolic subgroup, it is maximal and a so called Siegel parabolic subgroup. Since $\mathrm{Sp}_g(\mathbb{C})$ acts transitively on $\mathbf{Gr}_g$, the Siegel parabolic subgroups are conjugate to each other. We say that two Siegel parabolic subgroups P_{z_1}, P_{z_2} are in opposition to each other, if the two corresponding maximal isotropic subspaces Z_1, Z_2 satisfy $Z_1 \cap Z_2 = \{0\}$, or if they span Γ .
To any element I we can attach a point in $\mathbf{Gr}_g$. Actually we have two choices – namely we can attach $\Gamma^{1,0}$ or $\Gamma^{0,1}$ to I – but in our situation we choose

$$I \longrightarrow \Gamma^{0,1} = \{u \in \Gamma \quad | \, Iu = -i \otimes u\}.$$

On $\mathbf{Gr}_g$ we have complex conjugation, it interchanges the two spaces in the decomposition and sends the element I to $-I$. This means that the two parabolic subgroups (the stabilizers of $\Gamma^{1,0}$ and $\Gamma^{0,1}$) are in opposition.

If in turn we have a point $z \in \mathbf{Gr}_g$, and the corresponding parabolic subgroup P_z, and if P_z and $P_{\overline{z}}$ are in opposition, then we get a decomposition

$$\Gamma \quad = W \oplus \overline{W}$$

where $W = z$. Then we can consider the automorphism J which acts by multiplication by i on W and $-i$ on $\overline{W}$. Clearly this defines a complex structure on $\Gamma_{\mathbb{R}}$: The elements of $\Gamma_{\mathbb{R}}$ are the elements of the form $\gamma = w + \overline{w}$ and $I\gamma = w \otimes i + \overline{w} \otimes (-i) = w \otimes i + \overline{w \otimes i}$. We conclude that:

Proposition 5.2.25. *We have a bijection*

$$\{I \mid I^2 = -\mathrm{Id}, \langle Ix, Iy \rangle = \langle x, y \rangle\} \xrightarrow{\sim} \mathbf{Gr}_g^0$$

where $\mathbf{Gr}_g^0$ is the set of points z for which z and $\overline{z}$ are in opposition. This induces a complex structure on the set of all I.

On $\mathbf{Gr}_g^0$ we have an action of $\mathrm{Sp}_g(\ )$ by conjugation, we want to determine the orbits. Recall that we know:

Proposition 5.2.26. *An element I defines a Hermitian form H_I on the complex vector space $(\Gamma_{\mathbb{R}}, I)$ and the stabilizer of the element I is the unitary group $U_I \subset \mathrm{Sp}_g(\ )$ of the Hermitian form.*

This Hermitian form H_I has a signature (p,q) with $p + q = g$ and $H_I \simeq U(p,q)$. Now it is an easy – or perhaps better – a well known theorem that:

Theorem 5.2.27. *The orbits under $\mathrm{Sp}_g(\ )$ on $\mathbf{Gr}_g$ are given by the signatures (p,q) of the Hermitian forms H_I.*

Especially we have the open orbit $\ _g \subset \mathbf{Gr}_g^0$ where the form H_I is positive definite. This is the orbit which is hit by the principally polarized abelian varieties. It is elementary to show that $\mathrm{Sp}_g(\ )$ acts transitively on $\ _g$.

We see that two such principally polarized abelian varieties $(\Gamma, <\ ,\ >, I)$ and $(\Gamma, <\ ,\ >, I')$ are isomorphic if we can find an automorphism of $(\Gamma, <\ ,\ >)$ which sends I to I'. The group of these automorphisms is the symplectic group $G(\ ) = \mathrm{Sp}_g(\ )$ and this gives us a hint that we can formulate a theorem, which roughly says:

Theorem 5.2.28. *Abelian varieties with a principal polarization are parameterized by $G(\ )\backslash\ _g$.*

I stated this result because I want to give a first idea what a so called **moduli space** is. In general moduli spaces are complex spaces (later on they will be algebraic varieties or even schemes), whose points classify objects of given type. In our case above the objects are principally polarized abelian varieties of dimension g and the moduli space is the above quotient. It is in fact a complex space and to any of its points we construct in a certain natural way an isomorphism class of a principally polarized abelian variety and any isomorphism class corresponds to a unique point. Hence the set of isomorphism classes of principally polarized abelian varieties has in a certain natural way the structure of a complex space. It would be better if we could attach to any point $z \in G(\ )\backslash\ _g$ in a canonical way an abelian variety A_z and not only an isomorphism class. This abelian variety should vary "holomorphically" with z. This touches a subtle point in the theory of moduli spaces. We come back to this point in volume II.

Actually it turns out that now we are asking to much, we discuss this in the following section.

Elliptic curves with level structure

I want to invite the reader to a short excursion. We want to make the above consideration more precise for the case $g = 1$, this means for elliptic curves. We return to the situation discussed in5.2.8. There we explained that elliptic curves can be written as $\mathcal{E} = \mathbb{C}/\Omega$ where the **period lattice** $\Omega = \ \omega_1 \oplus \ \omega_2$ and where $\omega_1,\omega_2 \in \mathbb{C}$ are linearly independent over . If we have a second lattice Ω_1 then $\mathcal{E}$ and $\mathcal{E}_1 = \mathbb{C}/\Omega_1$ are isomorphic if and only if we can find an $\alpha \in \mathbb{C}$ such that $\alpha\Omega = \Omega_1$.
The real vector space $\mathbb{C} = \ ^2$ has an orientation: the ordered basis $\{1,i\}$ is positively oriented. For a given lattice $\Omega = \ \omega_1 \oplus \ \omega_2$ we can require that the ordered basis $\{\omega_1,\omega_2\}$ is positively oriented. This means that $\frac{\omega_2}{\omega_1} = \tau = x + iy$ has positive imaginary part, i.e. $y > 0$, in other words τ is an element in the **upper half plane**

$$= \{\tau = x + iy \mid y > 0\}.$$

It is clear from above that $\mathbb{C}/\Omega \xrightarrow{\sim} \mathbb{C}/ \ 1 \oplus \ \tau$. We may choose another oriented basis for our lattice. We get these basis if we take a matrix

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\ )$$

and the new basis is given by $\{a\tau + b, c\tau + d\}$. Then put $\tau' = \gamma\tau = \frac{a\tau+b}{c\tau+d}$ and clearly we get an isomorphism

$$i_\gamma : \mathbb{C}/ \ 1 \oplus \ \tau \xrightarrow{\sim} \mathbb{C}/ \ 1 \oplus \ \tau'$$

which given by multiplication by $\alpha = \frac{1}{c\tau+d}$.
Let us put $\Gamma = \mathrm{SL}_2(\ )$. The group Γ acts on by $(\gamma,\tau) \mapsto \frac{a\tau+b}{c\tau+d}$ and we established a bijection between the set of isomorphism classes of elliptic curves and the points in $\mathrm{SL}_2(\mathcal{Z})\backslash$, and this is our theorem 5.2.28 for $g = 1$.
The following facts can be found in [La] or any other book on elliptic functions. The action of Γ on the upper half plane is properly discontinuous, and this means that

For any $\tau \in$ we can find an open neighborhood V_τ of τ such that for all $\gamma \in \Gamma$ we have $\gamma V_\tau \cap V_\tau = \emptyset$ unless $\gamma\tau = \tau$. For any τ the group $\Gamma_\tau = \{\gamma | \gamma\tau = \tau\}$ is a finite cyclic group.

A point τ is called a fixed point if there is a $\gamma \in \Gamma, \gamma \neq \pm\mathrm{Id}$ such that $\gamma\tau = \tau$. The fixed points form two orbits under Γ : We have the two fixed points $\tau = i$, $\rho = \frac{1+i\sqrt{3}}{2}$, we take the positive root. The set of fixed points consists of the orbits of these two points.
We can define the structure of a complex space on $\Gamma\backslash$, it is clear what the holomorphic functions in a neighborhood of a point $z \in \Gamma\backslash$ are: Choose a τ which lies above z, choose a neighborhood V_τ as above, which is invariant under Γ_τ. Let W_z be the image of V_τ. Then the holomorphic functions on W_z are the holomorphic functions on V_τ which are invariant under Γ_τ. If we use the arguments from 3.2.2 and example 17 and exercise 13 then it is even clear that the quotient is a (non compact) Riemann surface.

We want to make the assertion of the above theorem 5.2.28 more precise. We try to attach to any point $z \in \Gamma\backslash$ an actual elliptic curve $\mathcal{E}_z$, not only an isomorphism class. This seems to be easy: We pick a point $\tau \in$ which projects to z and choose the elliptic curve $\mathbb{C}/\ 1\oplus\ \tau$ and try $\mathcal{E}_z = \mathbb{C}/\ 1\oplus\ \tau$. What happens if we choose another point τ' projecting to z? We find a γ with $\gamma\tau = \tau'$ and identify the two elliptic curves by the rule given above. At this point we encounter a fundamental problem. The element γ is never unique, we always can replace it by $-\gamma$ and this gives another isomorphism $-i_\gamma$ between our two elliptic curves. This tells us that **there is no consistent choice of $\mathcal{E}_z$.**

We have a remedy. We choose an integer $N \geq 3$ and consider the homomorphism $\Gamma \longrightarrow \mathrm{SL}_2(\ /N\ )$ it turns out to be surjective and its kernel is denoted by $\Gamma(N)$, the principal congruence subgroup mod N. It is an easy lemma that $\Gamma(N)$ does not contain elements of finite order different from Id. Therefore, it is clear that $\Gamma(N)$ acts fixed point free on .

Any elliptic curve $\mathcal{E} = \mathbb{C}/\Omega$ has the endomorphism $N\,\mathrm{Id} : \mathcal{E} \longrightarrow \mathcal{E}, z \mapsto Nz$, it has the kernel $\mathcal{E}[N] = \frac{1}{N}\Omega/\Omega \xrightarrow{\sim}\ /N\ \oplus\ /N$. THis allows us to introduce a new kind of object, namely elliptic curves with some extra structure, so called $N-$ level structures. This are pairs $(\mathcal{E},\{e_1,e_2\})$, where $e_1,e_2 \in \mathcal{E}[N]$ and where these two points generate $E[N]$, in other words they provide an isomorphism $\mathcal{E}[N] \xrightarrow{\sim}\ /N\ \oplus\ /N$, whose inverse is given by $(a,b) \mapsto ae_1 + be_2$. The elements in $\mathcal{E}[N]$ are called N-division points.
To such an elliptic curve with N-level structure we can attach a topological invariant. We observe that we have an alternating pairing $<\ ,\ >: \Omega\times\Omega \longrightarrow$ which is defined by the rule $< \omega_1,\omega_2 > \mapsto 1$ (recall that we have the orientation on $\mathbb{C}$), this can also be interpreted as an intersection of the two homology classes (See also 4.6.8)) provided by ω_1,ω_2. If we now have our two N-division points e_1,e_2 we can lift them to points in $\frac{1}{N}\Omega$:

$$\widetilde{e}_1 = \frac{a}{N} + \frac{b}{N}\tau, \widetilde{e}_2 = \frac{c}{N} + \frac{d}{N}\tau,$$

and because they generate $\mathcal{E}[N]$ the number $ad - bc$ must be prime to N. Actually it is clear that the quantity $< e_1,e_2 >_N := ad - bc \mod N$ is well defined and an element in $(\ /N\ )^\times$. This is our topological invariant attached to $(\mathcal{E},\{e_1,e_2\})$.
We resume the discussion from above, we want to make the assertion of theorem 5.2.28 more precise, but now for elliptic curves with N-level structure. We consider the action of $\Gamma(N)$ on , the action is fixed point free, the quotient $\Gamma(N)\backslash$ is a Riemann surface and the projection $\pi :\ \longrightarrow \Gamma(N)\backslash$ is an unramified covering. We pick an $a \in (\ /N\ )^\times$. For any $z \in \Gamma(N)\backslash$ we pick a point τ projecting to it and $(\mathbb{C}/\ 1 +\ \tau, \{\frac{a}{N}, \frac{\tau}{N}\})$ is an elliptic curve with N-level structure. If we pick another point $\tau' \in \pi^{-1}(z)$ then we find a unique $\gamma \in \Gamma(N)$ such that $\gamma(\tau) = \tau'$ and $\imath_\gamma$ provides an isomorphism between the two curves with level structure. We can say that we constructed a curve $\mathcal{E}_z$ with N-level structure for any $z \in \Gamma(N)\backslash$. On the other hand it is clear that $(\mathbb{C}/\ 1 +\ \tau, \{\frac{a}{N}, \frac{\tau}{N}\})$ and $(\mathbb{C}/\ 1 +\ \tau', \{\frac{a}{N}, \frac{\tau}{N}\})$ are isomorphic if and only if we find a $\gamma \in \Gamma(N)$ such that $\gamma(\tau) = \gamma'$.
We form the product $\Gamma(N)\backslash\ \times (\ /N\ )^\times$ let p_1 be the projection to the first coordinate. Consider

$$\widetilde{\mathcal{E}}_N = \{(u,(z,a)) | (z,a) \in \Gamma(N)\backslash\ \times (\ /N\ )^\times, u \in \mathcal{E}_z\}.$$

We have an obvious complex structure on this set, the holomorphic coordinates are local lifts from z to τ and from u to $w \in \mathbb{C}$. Hence it is a surface, we have the projection

$$\pi_N : \widetilde{\mathcal{E}}_N \longrightarrow \Gamma(N)\backslash \quad \times (\quad /N \quad)^\times .$$

We have two sections 4.3.1 e_1,e_2 to π_N whose values at any $z \times a$ are given by

$$e_1(z,a) = \frac{a}{N}, e_2(z,a) = \frac{\tau}{N} \in \mathbb{C}/ \quad 1 + \quad \tau.$$

This object $\widetilde{\pi}_N : \widetilde{\mathcal{E}}_N \longrightarrow \Gamma(N)\backslash \quad \times (\quad /N \quad)^\times$, together with the two sections e_1,e_2 can be viewed as the "universal elliptic curve" with N-level structure. By this we mean the following:
Let us consider a morphism $p : X \longrightarrow S$ between two complex spaces such that for any $x \in S$ the fiber $\pi^{-1}(x) = X_x$ is a smooth curve of genus one. Let us assume in addition that we have a holomorphic section $O : S \longrightarrow X$ to p. Then the fibers are elliptic curves and $p : X \longrightarrow S$ is called a family of elliptic curves. Especially we know that these fibers come with a group structure. If we now have two sections $f_1 : S \longrightarrow X, f_2 : S \longrightarrow X$ to p, such that for any $x \in S$ the two elements $f_1(x), f_2(x) \in X_s[N]$ and are a pair of generators, then we say that $(p : X \longrightarrow S, \{f_1, f_2\})$ is a family of elliptic curves with N-level structure.
Now we can state a result which a much more precise version of theorem 5.2.28

Theorem 5.2.29. *Let $(p : X \longrightarrow S, \{f_1, f_2\})$ be a family of elliptic curves with N-level structure. Then there exists a unique holomorphic maps Φ, Ψ which provide a commutative diagram*

$$\begin{array}{ccc} \Psi : X & \longrightarrow & \widetilde{\mathcal{E}}_N \\ \downarrow & & \downarrow \\ \Phi : S & \longrightarrow & \Gamma(N)\backslash \quad \times (\quad /N \quad)^\times \end{array} \tag{5.108}$$

such that for any point $x \in S$ the restriction $\Psi_x : X_x \longrightarrow (\widetilde{\mathcal{E}}_N)_{\Phi(s)}$ is an isomorphism and maps $f_i(x)$ to $e_i(\Phi(x))$.

This is of course highly plausible, essentially we have to show that the period lattice depends holomorphically on the variable $x \in S$. We do not give a detailed proof this fact here, we come back to this kind of problem in Volume II, 9.6.2. A similar problem is discussed in this Volume I in 5.2.10.

In some cases we can give a rather explicit description of $\Gamma(N)\backslash$ and the universal elliptic curve with N-level structure over it. I include this discussion in the second edition of this volume I, because I wanted to present this in volume II, but finally there was some lack of space (and energy).
Let $N \geq 3$, we consider the elliptic curve with N-level structure $\mathcal{E} = (\mathbb{C}/ \quad 1 + \quad \tau, \{\frac{a}{N}, \frac{\tau}{N}\})$. We have a minor problem of notation: We have a group structure on $\mathcal{E}$ and we denote the addition of two points $P,Q \in \mathcal{E}$ by $P \oplus Q$ and $m \bullet P = P \oplus P \oplus \cdots \oplus P$. We do this because we want to keep the usual notation $D = n_1 P_1 + n_2 P_2 + \cdots + n_m P_m$ for a divisor. The theorem of Abel (see 5.1.35) says that a divisor is principal, if its degree $\deg(D) = \sum n_i = 0$ and if $n_1 \bullet P_1 \oplus n_2 \bullet P_2 \cdots \oplus n_r \bullet P_r = 0$. We put $\frac{1}{N} = r, \frac{\tau}{N} = s$. For any $z \in \mathcal{E}$ we consider the divisor

$$D_z = z + z \oplus r + z \oplus 2 \bullet r + \cdots + z \oplus (N-1) \bullet r.$$

The divisor $D_z - D_0$ is principal if and only if z is a N division point. The space of sections $H^0(\mathcal{E},\mathcal{O}_{\mathcal{E}}(D_z))$ has dimension N and of course it contains the constants. The divisors D_z are invariant under translations by the cyclic group $< r >$ generated by r, therefore, we have an action of $< r >$ on $H^0(\mathcal{E},\mathcal{O}_{\mathcal{E}}(D_z))$ for any z, we denote this action by $\rho(r)$, i.e. $\rho(r) = T_r$.
We pick $z = 0$. For any $a = 1,2,\ldots,(N-1)$ the divisor $D_{a\cdot s} - D_0$ is principal, we choose a meromorphic function f_a having this divisor. It is unique up to a scalar, we put $f_0 = 1$. It is clear that $f_a \in H^0(\mathcal{E},\mathcal{O}_{\mathcal{E}}(D_0))$ is an eigenvector under the translation by r, and more precisely

$$T_r(f_a)(z) = f_a(z+r) = e^{\frac{2\pi i a}{N}} f_a.$$

(This relation follows from the properties of the Weierstrass σ -function and the formulae in [La] Chap. 18 §1)
We normalize the choice of these f_a. We pick an eigenvector f_{N-1}. Using this eigenvector we define an action of the cyclic group $< s >$ on $H^0(\mathcal{E},\mathcal{O}_{\mathcal{E}}(D_0))$. We simply put $\rho(s)(f) = T_s(f)f_{N-1}$, keeping track of the polar part of the divisors we see that $\rho(s)(f) \in H^0(\mathcal{E},\mathcal{O}_{\mathcal{E}}(D_0))$. Clearly $\rho(s)^N f_{N-1} = \alpha f_0 = \alpha$. If we modify the choice of f_{N-1} by a factor β then $\alpha \mapsto \alpha\beta^N$ and hence may assume that $\alpha = 1$. Then f_{N-1} is unique up a $N-$th root of unity.
We get a group $H[N]$ of automorphisms of $H^0(\mathcal{E},\mathcal{O}_{\mathcal{E}}(D_0))$, it is the group generated by $\rho(r)$,$\rho(s)$, both of order N. Under the action of $\rho(r)$ we have a decomposition into eigenspaces

$$H^0(\mathcal{E},\mathcal{O}_{\mathcal{E}}(D_0)) = \oplus_{0\le a\le N-1}\mathbb{C}f_a$$

and $\rho(s)f_a = f_{a-1}$. We have commutation rule

$$[\rho(s),\rho(r)] = \rho(s)\rho(r)\rho(s)^{-1}\rho(r)^{-1} = e^{\frac{2pii}{N}}\,\mathrm{Id}\,.$$

The group $H[N]$ is called the *Heisenberg group.*
We get an holomorphic map (see 5.2.7)

$$z \mapsto (f_0(z),f_1(z),\ldots,f_{N-1}(z),\Phi_N : \mathcal{E} \hookrightarrow \mathbb{P}^{N-1}(\mathbb{C}),$$

and this is in fact an embedding of $\mathcal{E}$ into the projective space. This embedding is canonical, it is determined by the elliptic curve together with its level structure. What remains is to find are the equations defining $\mathcal{E}$ as a curve in the projective space.
This means we introduce independent variables $X_0,X_1,\cdots,X_{N-1}$, and define the action of the Heisenberg group such that $X_i \mapsto f_i$ becomes an $H[N]$ isomorphism. For $k > 0$ we look at the linear map from the homogenous polynomials of degree k

$$\mathbb{C}[X_0,X_1,\cdots,X_{N-1}][k] \longrightarrow H^0(\mathcal{E},\mathcal{O}_{\mathcal{E}}(kD_0))$$

given by sending X_i to f_i. It has a kernel I_k and the elements in these kernels yield the equations defining $\mathcal{E}$. It follows from general finiteness results (See also Volume II) that finitely many of these equations suffice to describe $\mathcal{E}$. The difficult problem is to find these equations, but in the cases $N = 3,N = 4$, the action of the Heisenberg is very helpful and we can write down these equations explicitly.

The following is taken from the Bonn Diploma thesis of Christine Heinen ([Hei], in which she carries out following computations in detail. We begin with the case $N = 3$. The space of homogenous polynomials of degree 3 in X_0,X_1,X_2 has dimension 10 and the dimension of $H^0(\mathcal{E},\mathcal{O}_{\mathcal{E}}(3D_0))$ has dimension 9 by Riemann-Roch. Hence we must have an non trivial polynomial F of degree 3 that goes to zero. We still have the action of $H[3]$. The monomials are eigenvectors under the action of $< r >$, the 4 monomials X_0^3,X_1^3,X_2^3 and $X_0X_1X_2$ have eigenvalue 1, then we have 3 monomials having eigenvalue $\zeta = e^{\frac{2pii}{3}}$ and three monomials having eigenvalue ζ^2. If we look at F and the action of $< r >$ on it, then it is a sum of three terms which are eigenvectors with eigenvalues 1,$e^{\frac{2pii}{3}}$,$e^{\frac{4pii}{3}}$ and each of these terms vanishes on $\mathcal{E}$. But a simple inspection of pole orders shows that the terms with eigenvalue different from 1 must be identically zero, hence our polynomial only involves monomials with eigenvalue one. We conclude that our polynomial is of the form

$$F = aX_0^3 + bX_1^3 + cX_2^3 - dX_0X_1X_2.$$

Again looking at pole orders yields that $abc \neq 0$ and this implies immediately that F is unique up to a scalar. But we still have the action of $< s >$. This cyclic group permutes the x_i and then it is easy to see that we must have $a = b = c$ and hence we can assume $a = b = c = 1$ and our equation defining the curve becomes

$$F = X_0^3 + X_1^3 + X_2^3 - dX_0X_1X_2.$$

We compute the coordinates of the origin and the two points r,s. Recall that we still have not yet pinned down f_2, it is only determined up to a third root of unity. By construction we have

$$1 = f_0(z) = f_2(z + \frac{2\tau}{3})f_2(z + \frac{\tau}{3})f_2(z),\ \ f_1(z) = f_2(z + \frac{\tau}{3})f_2(z)$$

and $z \mapsto (\frac{1}{f_2(z)},f_2(z + \frac{\tau}{3}),1)$. Evaluation at $z = 0$ yields $0 \mapsto (0,f_2(\frac{\tau}{3}),1)$ and we have the equation $f_2(\frac{\tau}{3})^3 + 1 = 0$. Since we still have option to multiply f_2 by a third root of unity we can normalize $f_2(\frac{\tau}{3}) = -1$. With this choice of f_2 the origin becomes

$$O = (0,-1,1).$$

Now we really pinned down the embedding.

Remark: We could have chosen this normalization right from the beginning, then we have the problem to show that the above number $\alpha = 1$.

The computation of the coordinates of r,s is easy. We have to evaluate $(f_0(z),\ f_1(z),\ f_2(z))$ at $z = \frac{1}{3},\frac{\tau}{3}$. In the first case we have to observe that f_1,f_2 have a pole, hence we evaluate $(\frac{1}{f_1(z)},1,\frac{f_2(z)}{f_1(z)})$ at $z = \frac{1}{3}$. We know that the f_1,f_2 are eigenvectors under the translation by $< r >$ with eigenvalue ρ,ρ^2 and hence

$$\frac{f_2(z)}{f_1(z)}_{z=r} = \rho\frac{f_2(z)}{f_1(z)}_{z=0},$$

and this last ratio is -1 as follows from the coordinates of the origin. The second point is easier we have $(f_0(s),f_1(s),f_2(s)) = (1,0,-1)$. Therefore we see that under the embedding the two chosen 3-division points go to

$$r \mapsto (0,1,-\rho),\ \ s \mapsto (1,0,-1).$$

We have still the parameter d and it is at least plausible that $d = d(\tau)$ is a holomorphic function in the variable τ. We want our curve to be smooth (see example 19) an easy calculation shows that this means $d^3 \neq 27$. If on the other hand d with $d^3 \neq 27$ is given then we may consider object the curve $\mathcal{E}$ defined by the equation $X_0^3 + X_1^3 + X_2^3 - dX_0X_1X_2 = 0$, it is smooth. It contains the point $O = (0,-1,1)$, we choose this as the origin and hence $(\mathcal{E},O)$ is now an elliptic curve. It also contains $e_1 = (0,1,-\rho)$,$e_2 = (1,0,-1)$, they form a system of generators of the 3-division points, we have $< e_1,e_2 >_3 = 1$. Hence we can say that the object

$$\widetilde{\mathcal{E}} = \{\mathcal{E} := X_0^3 + X_1^3 + X_2^3 - dX_0X_1X_2 = 0, O = (0,-1,1), e_1 = (0,1,-\rho), e_2 = (1,0,-1)\}$$

is a family of elliptic curves with 3-level structure with $< e_1,e_2 >_3 = 1$ over the Riemann surface $X(3) := \mathbb{C} \setminus \{3,3\rho,3\rho^2\} = \mathbb{P}^1(\mathbb{C}) \setminus \{\infty,3,3\rho,3\rho^2\}$. If we apply theorem 5.2.29 then the map

$$\Phi : X(3) \longrightarrow \Gamma(3)\backslash \quad \times \{1\}$$

is obviously the inverse of the map $\tau \mapsto d(\tau)$. We see that the restriction of $\widetilde{\mathcal{E}}_3$ to $\Gamma(3)\backslash \quad \times \{1\}$ is canonically isomorphic to $\widetilde{\mathcal{E}} \longrightarrow X(3)$.

We have a brief look at the case $N = 4$. Again we put $r = \frac{1}{4}, s = \frac{\tau}{4}$. We choose $f_3(z)$ such that $f(2s) = -1$. Then $f_0(z) = \rho^4(s)(f_3(z)) = f(z+3s)f(z+2s)f(z+s)f(z) = c$ is a non zero complex number. Our embedding $\mathcal{E} \hookrightarrow \mathbb{P}^3$ is given by

$$z \mapsto (f_0(z), f_1(z), f_2(z), f_3(z) = (c, f(z+2s)f(z+s)f(z), f(z+s)f(z), f(z))$$

We introduce the indeterminates X_0,X_1,X_2,X_3 as before we choose $k = 2$ and consider the linear map

$$S_2 = \mathbb{C}[X_0,X_1,X_2,X_3][2] \longrightarrow H^0(\mathcal{E},\mathcal{O}_{\mathcal{E}}(2D_0))$$

We observe that the space of homogenous polynomials of degree 2 has dimension 18, whereas $H^0(\mathcal{E},\mathcal{O}_{\mathcal{E}}(2D_0))$ has dimension 16. There must be a kernel $\mathcal{I}[2]$, whose dimension is ≥ 2. This kernel must be invariant under the action of the Heisenberg group. If we decompose the action of $H[4]$ on S_2 we easily see that it decomposes into 3 non isomorphic modules of rank 2 and a 4 dimensional module which is given by

$$W = (\mathbb{C}(X_0^2 + cX_2^2) \oplus \mathbb{C}(X_1^2 + cX_3^2)) \bigoplus (\mathbb{C}X_0X_2 \oplus \mathbb{C}X_1X_3).$$

A simple inspection of the other eigenspaces shows that we must have $\mathcal{I}[2] \subset W$. It can not be the entire space W hence we see easily that it is spanned by two elements

$$X_0^2 + cX_2^2 - bX_1X_3,\ cX_1^2 + c^2X_3^2 - bX_0X_2$$

these are eigenvectors for $\rho(r)$ with eigenvalues $1,-1$ respectively. We get the second expression if we apply $\rho(s)$ to the first one. Here b is just another non zero complex number. Hence we have three unknown numbers namely c,b and the value $f_3(s) = a$. We have the three points O,r,s on the curve, they have the coordinates

$$O = (0,-a,a,1),\ \ r = (0,a,\zeta^3 a,1)\ \ s = (c,0,-a,a)$$

and hence they satisfy the above equations. This yields the two relations $b + ac = 0$, $c = -a^2$ and hence $b = a^3$. Therefore, we see that our curve is given by the two equations

$$X_0^2 - a^2X_2^2 - a^3X_1X_3 = 0, \ -X_1^2 + a^2X_3^2 - aX_0X_2 = 0,$$

coordinates of the origin and the two 4-division points are given above. Again we have to prove that the parameter a is a holomorphic function $\tau \mapsto a(\tau)$ it is invariant under the action $\Gamma(4)$. A straightforward computation shows (example 19 b), (apply the Jacobi criterion) that a has to avoid the five values $0, \pm 2, \pm 2i$. We define $X(4) := \mathbb{C} \setminus \{\pm 2, \pm 2i, 0\} = \mathbb{P}^1(\mathbb{C}) \setminus \{\pm 2, \pm 2i, 0, \infty\}$ and we have the curve

$$\widetilde{\mathcal{E}} := \{ \begin{matrix} X_0^2 - a^2X_2^2 - a^3X_1X_3 = 0, \ -X_1^2 + a^2X_3^2 - aX_0X_2 = 0, \\ O = (0, -a, a, 1), \ r = (0, a, \zeta^3 a, 1) \ \ s = (c, 0, -a, a) \end{matrix} \}$$

is a family of elliptic curves over $X(4)$ with $< r,s >_4 = 1$ and this family is isomorphic to the restriction of $\widetilde{\mathcal{E}}_4$ restricted to $\Gamma(4)\backslash \quad \times \{1\}$. Especially we get again an isomorphism $\Phi : X(4) \xrightarrow{\sim} \Gamma(4)\backslash \quad$.

The following considerations anticipate some of the concepts from volume II, actually they are complementary to the discussion of moduli spaces in Volume II, 9.6.2. They provide an example for the general principle that some objects, which belong to complex analysis, can be considered as objects in the realm of abstract algebra and algebraic geometry.

We start from a commutative ring A with identity, let $S = \mathrm{Spec}(A)$ be the set of prime ideals endowed with the Zariski topology (See Volume II, Chap. 6). This space S will be the replacement of our complex space S above. We have the notion of an elliptic curve over S (see Volume II, 9.6.2). Again we choose an integer $N \geq 3$, we assume that $\frac{1}{N} \in A$. If this ring is an algebraically closed field k and if $\mathcal{E}$ is an elliptic curve over k then we still know that the group of N-division points $\mathcal{E}[N](k) \xrightarrow{\sim} \quad /N \quad \oplus \quad /N \quad$. Therefore, we know what an elliptic curves $p : \mathcal{E} \longrightarrow S$ with N- level structure is: This means that we have two sections e_1, e_2 to p which lie in $\mathcal{E}[N](S)$, and which at any point $s \in S$ generate the N-division points in $\mathcal{E}[N](\overline{k(s)})$, where $\overline{k(s)}$ is an algebraic closure of the residue field $k(s)$. Again we have a alternating pairing, which now takes values in the group μ_N of N-th roots of unity (See Volume II, 7.5.7) and denoted by

$$\mathcal{E}[N](S) \times \mathcal{E}[N](S) \longrightarrow \mu_N, (e,f) \mapsto w(e,f).$$

This pairing is related to our old pairing by the relation

$$w(e,f) = e^{<e,f>_N \frac{2\pi i}{N}}$$

We can translate the above arguments, which essentially prove theorem 5.2.29, into the context of algebraic geometry. The essential tool is provided by the semicontinuity theorems and reasoning is based on the same strategy that is used in Volume II, 9.6.2.

We can write down the "universal" elliptic curve with 3-level structure. Consider the ring $\quad [\rho] = \quad [T]/(T^2 + T + 1)$, i. e. we adjoin a third root of unity. Then we adjoin an indeterminate, let us call it Y, and we invert $Y^3 - 27$, so we get a ring

$$A_3 = \quad [\rho, Y, \frac{1}{Y^3 - 27}].$$

Over this ring we write down our curve with 3-level structure

$$\widetilde{\mathcal{E}} = \{\mathcal{E} = X_0^3 + X_1^3 + X_2^3 - YX_0X_1X_2 = 0, O = (0, -1, 1), e_1 = (0, 1, -\rho), e_2 = (1, 0, -1)\},$$

we have $w(e_1,e_2) = \rho$.
If we now have an elliptic curve $(p : \mathcal{E} \longrightarrow \mathrm{Spec}(A),\{t_1,t_2\})$ with 3-level structure over any ring A (with identity and $\frac{1}{3} \in A$), we assume that A contains a primitive third root of unity, call it ρ_1 and assume that $w(t_1,t_2) = \rho_1$. We use these sections to write down the divisors D_{at_2} as above (See also Volume II,9.6.2 where we use the zero sections to write down divisors). We apply the semicontinuity theorems and see that $D_{at_2} - D_0$ are principal, i.e. divisors of a function f_a. (Here is a minor technical point, the semicontinuity theorems only yield that they locally trivial in the base. This means that for any point $x \in S$ we find an open neighborhood V_x such that the restriction of $D_{at_2} - D_0$ to $p^{-1}(V_x)$ is the divisor of a function $f_a^{(x)}$. For $a = 2$ we normalized $f_2^{(x)}(t_1) = -1$, and hence these $f_2^{(x)}$ fit together on the different open sets). We can proceed as in the complex analytic case and find

Theorem 5.2.30. *For any commutative ring with identity and $\frac{1}{3} \in A$ and any elliptic curve $(p : \mathcal{E} \longrightarrow \mathrm{Spec}(A),\{t_1,t_2\})$ with 3-level structure we find a ring homomorphism $\Phi : A_3 \longrightarrow A$ such that $\rho \mapsto w(t_1,t_2)$ and with $\Phi(Y) = d$ we have*

$$(p : \mathcal{E} \longrightarrow \mathrm{Spec}(A),\{t_1,t_2\}) =$$

$$\{\mathcal{E} = X_0^3 + X_1^3 + X_2^3 - dX_0X_1X_2 = 0, O = (0,-1,1), t_1 = (0,1,-\rho_1), t_2 = (1,0,-1)\}$$

Basically the same reasoning provides an explicit universal curve with 4-level structure. We define $[i] = [T]/(T^2+1)$, we adjoin the indeterminate Y and define the ring

$$A_4 = [i,\frac{1}{2},Y,\frac{1}{Y-Y^5}].$$

Over this ring we write a curve with 4-level structure

$$\widetilde{\mathcal{E}} := \{\begin{matrix} X_0^2 - Y^2X_2^2 - Y^3X_1X_3 = 0, & -X_1^2 + Y^2X_3^2 - YX_0X_2 = 0, \\ O = (0,-Y,Y,1), & r = (0,Y,-iY,1) \quad s = (c,0,-Y,Y) \end{matrix}\}.$$

Then we get again

Theorem 5.2.31. *For any commutative ring with identity and $\frac{1}{2} \in A$ and any elliptic curve $(p : \mathcal{E} \longrightarrow \mathrm{Spec}(A),\{t_1,t_2\})$ with 4-level structure we find a ring homomorphism $\Phi : A_4 \longrightarrow A$ such that $i \mapsto w(t_1,t_2)$ and with $\Phi(Y) = a$ we have*

$$\mathcal{E} := \{\begin{matrix} X_0^2 - a^2X_2^2 - a^3X_1X_3 = 0, & -X_1^2 + a^2X_3^2 - aX_0X_2 = 0, \\ O = (0,-a,a,1), & r = (0,a,-ia,1) \quad s = (c,0,-a,a) \end{matrix}\}.$$

We can consider the problem of finding a universal elliptic curve for any integer $N \geq 3$. We consider commutative rings R with identity and a homomorphism $[\frac{1}{N}] \longrightarrow R$. We consider elliptic curves $\mathcal{E}$ over R which come with a N-level structure. If two such elliptic curves with N-level structure are isomorphic then the isomorphism is unique and hence these objects form a set $M_N(R)$. We can ask the question whether this functor is representable by a ring or better by an affine scheme (see 1.3.4) The above arguments show that for $N = 3, N = 4$ the functor $R \mapsto M_N(R)$ is representable by an affine scheme of finite type. From here it it is not too difficult to show that this functor is representable for all values of $N \geq 3$ by a scheme of finite type over $[\frac{1}{N}]$. The representing affine scheme is called the moduli scheme (or moduli space), in our case we also denote it by M_N.

We come back to the beginning of this excursion and recall that we actually wanted to understand the case $N = 1$, i.e. elliptic curves with no level structure and to construct a moduli space M_1. We have studied this problem also in Volume II 9.6.2. where we consider elliptic curves endowed with a nowhere vanishing differential.
We have an obvious action of the group $\mathrm{GL}_2(\ \ /N\ \ )$ on the level structures of an elliptic curve with N-level structure, hence we get an action of this group on M_N. So we get an action of this group on M_N, and we ask whether we can form the quotient $M_N/\mathrm{GL}_2(\ \ /N\ \ )$ and this quotient can be our moduli space M_1. It it explained in Volume II, 9.6.2 that this can not work.
The way out of this dilemma is to define more complicated objects, these will be the stacks. The stack $M_1^{(3)}/\operatorname{Spec}(\ \ [\frac{1}{3}]$ will simply be the object M_3 together with action of $\mathrm{GL}_2(\ \ /3\ \ )$, we just do not form the quotient. Accordingly the stack $M_1^{(4)}/\operatorname{Spec}(\ \ [\frac{1}{4}]$ is M_4 together with action of $\mathrm{GL}_2(\ \ /4\ \ )$. We can also construct $M_1^{(12)}/\operatorname{Spec}(\ \ [\frac{1}{6}]$. We get a diagram

$$\begin{array}{ccccc} M_1^{(3)} & \longleftarrow & M_1^{(12)} & \longrightarrow & M_1^{(4)} \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec}(\ \ [\frac{1}{3}]) & \leftarrow & \operatorname{Spec}(\ \ [\frac{1}{6}]) & \rightarrow & \operatorname{Spec}(\ \ [\frac{1}{4}]) \end{array} \tag{5.109}$$

and the stack $M_1/\operatorname{Spec}(\ \ )$ is - in a certain sense - this diagram of schemes together with the group actions.

The end of the excursion

Here is the end of the excursion. We return to the discussion before theorem 5.2.28. We had chosen a basis $e_1, \dots e_g, f_g, \dots, f_1$ for Γ. If we have selected an element I, we write

$$f_\nu = \sum (x_{\nu,\mu} + y_{\nu,\mu} I) e_\mu, \tag{5.110}$$

we put $\tau_{\nu,\mu} = x_{\nu,\mu} + y_{\nu,\mu} \cdot i$.

The element I gives the decomposition $\Gamma\ \ = \Gamma^{1,0} \oplus \Gamma^{0,1}$. We attached the space $\Gamma^{0,1} \in \mathbf{Gr}_g$ to I, and we want to write "coordinates" for this point. In view of our considerations above we choose $X = G\ \ $. We observe that

$$\Gamma\ \ = G\ \ \oplus \Gamma^{0,1}. \tag{5.111}$$

The map $V \longrightarrow \Gamma\ \ /\Gamma^{0,1}$ is an isomorphism by construction and hence

$$f_\nu - \sum \tau_{\nu,\mu} \otimes e_\mu \in \Gamma^{0,1}. \tag{5.112}$$

We mentioned already that $\Gamma^{0,1}$ is maximal isotropic if and only if the matrix $Z = (\tau_{\nu,\mu})$ is symmetric and our Hermitian form is positive definite if and only if the real part of this matrix is positive definite. Therefore the $\tau_{\nu,\mu}$ are the holomorphic coordinates for the possible choices of I.

Hence we get a new description of $\ _g$ it can be identified to the points in the **Siegel half space**

$$\ _g = \{Z|Z = X + iY\}$$

where Z is symmetric and Y is positive definite.

5.2.6 Riemann-Theta Functions

We consider the cocycle $\gamma \longrightarrow C_{\mathrm{hol}}(z,\gamma)$ and I want to explain that this cocycle depends holomorphically on I. To be more precise we can fix an element $\gamma \in \Gamma$ and consider this cocycle as a function in the variables z and I. Then we want to show that this cocycle is holomorphic in both variables.

We have the two forms

$$H : \Gamma_{\mathbb{R}} \times \Gamma_{\mathbb{R}} \longrightarrow \mathbb{C}$$
$$h : \Gamma_{\mathbb{R}} \times \Gamma_{\mathbb{R}} \longrightarrow \mathbb{C},$$

where H is Hermitian with respect to the element I and where h is linear in both variables with respect to I. Now we extend these forms to $\Gamma\ \ = \Gamma_{\mathbb{R}} \otimes \mathbb{C}$ bilinearly, i.e. we have

$$H\ \ (\gamma \otimes z,\delta \otimes w) = zw\ H(\gamma,\delta) \tag{5.113}$$
$$h\ \ (\gamma \otimes z,\delta \otimes w) = zw\ h(\gamma,\delta)$$

for $\gamma,\delta \in \Gamma$ and $z,w \in \mathbb{C}$.
We observe that the inclusion $\Gamma_{\mathbb{R}} \longrightarrow \Gamma\ \ $ induces an isomorphism

$$\Gamma_{\mathbb{R}} \longrightarrow \Gamma\ \ /\Gamma^{0,1},$$

and this map is $\mathbb{C}$-linear if we give $\Gamma_{\mathbb{R}}$ the complex structure where multiplication by i is given by I.

We can decompose $\Gamma\ \ = G\ \ \oplus \Gamma^{0,1}$, and hence we can write any element $\gamma \in \Gamma\ \ $ as a sum

$$\gamma = \gamma_G + \gamma_{0,1} = p_G(\gamma) + p_{0,1}(\gamma). \tag{5.114}$$

Now we consider the expression $H\ \ (z,\gamma) - h\ \ (z,\gamma)$, and we observe that this depends only on $\gamma \mod G\ \ $ in the second variable in (5.102). On the other hand we see: If $z \in \Gamma$ is in $\Gamma^{0,1}$, then

$$H\ \ (z,\gamma) = h\ \ (z,\gamma) = 0. \tag{5.115}$$

This is clear because $z = \delta + I\delta \otimes i$ with some $\delta \in \Gamma_{\mathbb{R}}$ and

$$H\ \ (\delta + I\delta \otimes i,\gamma) = H(\delta,\gamma) + i\ H(I\delta,\gamma) = H(\delta,\gamma) - H(\delta,\gamma) = 0, \tag{5.116}$$

and the same holds for $h\ \ $. Hence we conclude that $H\ \ (z,\gamma) - h\ \ (z,\gamma)$ defines a bilinear form

$$\Gamma\ \ /\Gamma^{0,1} \times \Gamma\ \ /G\ \ \longrightarrow \mathbb{C}.$$

We can express this form in terms of the original alternating form:

Lemma 5.2.32. *We write as above* $\gamma = \gamma_G + \gamma_{0,1}$*, then I claim that*

$$H \quad (z,\gamma) - h \quad (z,\gamma) = 2i\langle z,\gamma_{0,1}\rangle$$

where $\langle\ ,\ \rangle$ *is of course the bilinear extension of* $\langle\ ,\ \rangle$ *to* Γ .

Proof: To see that this is the case it suffices to show that

$$H \quad (z,\gamma_{0,1}) - h \quad (z,\gamma_{0,1}) = 2i\langle z,\gamma_{0,1}\rangle \quad .$$

In this case $\gamma_{0,1} = \delta + I\delta \otimes i$ and because h is bilinear with respect to I, we get

$$h \quad (z,\delta + I\delta \otimes i) = 0$$

as above. Hence we have to show that

$$H \quad (z,\gamma_{0,1}) = 2i\langle z,\gamma_{0,1}\rangle \quad .$$

We may assume that $z \in \Gamma_{\mathbb{R}}$ and again we write $\gamma_{0,1} = \delta + I\delta \otimes i$. Then

$$H \quad (z,\delta + I\delta \otimes i) = H(z,\delta) + i \cdot H(z,I\delta) = i\ H(z,I\delta) + H(z,\delta).$$

We invoke our formulae for H and get

$$\begin{aligned} i(\langle z,\delta\rangle + i\langle z,I\delta\rangle) - \langle z,I\delta\rangle + i\langle z,\delta\rangle &= 2i\langle z,\delta\rangle - 2\langle z,I\delta\rangle \\ &= 2\left(\langle z,\delta\otimes i\rangle \quad - \langle z,I\delta\rangle\right) \\ &= 2i\left(\langle z,\delta\rangle + \langle z,I\delta\otimes i\rangle \quad \right) = 2i\langle z,\gamma_{0,1}\rangle \quad . \end{aligned}$$

□

This function $(z,I) \mapsto 2i\langle z,\gamma_{0,1}\rangle$ is now clearly holomorphic in the variables z,I. We get for our 1-cocycle

$$C_{\text{hol}}(z,\gamma) + 2\pi i(\eta(\gamma) + \varphi(\gamma)) = 2\pi i\langle z,\gamma_{0,1}\rangle \quad + \pi i\langle \gamma,\gamma_{0,1}\rangle \quad + 2\pi i(\eta(\gamma) + \varphi(\gamma)). \quad (5.117)$$

I want to give an indication how we can describe the space of sections in the d'th power in the bundle defined by this cocycle.
Recall that we have a basis $e_1,\ldots,e_g,f_g,\ldots,f_1$ as in the beginning of section 5.2.5 and that G is spanned by the $e_1,\ldots,e_g$. Let $C_{\text{hol}}(z,\gamma)$ be the cocycle obtained from these data. Our basis $e_1,\ldots,e_g$ of G is also a $\mathbb{C}$-basis of V. We choose a positive integer d. We look for solutions of

$$\widetilde{f}(z+\gamma) = \widetilde{f}(z)e^{d(C_{\text{hol}}(z,\gamma)+2\pi i(\varphi(\gamma)+\eta(\gamma)))}, \tag{5.118}$$

these are the sections of the bundle $\mathcal{L}(C_{\text{hol}},\eta,\varphi)^{\otimes d}$. The periodicity of $\widetilde{f}$ with respect to G means that $\widetilde{f}(z_1+n_1,\ldots,z_g+n_g) = \widetilde{f}(z_1,\ldots,z_g)$ for all $n_1,\ldots,n_g \in \quad ^g$. We introduce the new variables

$$u_\nu = e^{2\pi i z_\nu}, \qquad\qquad q_{\nu,\mu} = e^{2\pi i \tau_{\nu,\mu}}.$$

We have the symmetry relation $q_{\nu,\mu} = q_{\mu,\nu}$. We define $h(u_1,\dots,u_g) = \widetilde{f}(z_1,\dots,z_g)$.

In these new variables we have a different description of our complex torus. The holomorphic map

$$\Pi : (z_1,\dots,z_g) \longrightarrow (u_1,\dots,u_g)$$

yields an isomorphism

$$\Pi : V/G \xrightarrow{\sim} (\mathbb{C}^\times)^g$$

and by definition $h \circ \Pi = f$. Then we get a biholomorphic map

$$\widetilde{\Pi} : V/\Gamma \xrightarrow{\sim} (\mathbb{C}^\times)^g/Q, \tag{5.119}$$

where Q is the free abelian subgroup generated by the arrays $\{(q_{\nu,1},\dots,q_{\nu,g})\}_{\nu=1,\dots,g}$. We rewrite the transformation rule in terms of these new variables. We can write an element $\gamma = \sum_{j=1}^g n'_j e_j + \sum_{\mu=1}^g n_\mu f_\mu$. If we pass to the variables u_ν then the first summand does not contribute and can be ignored. Then

$$z + \gamma = \left(z_1 + n'_1 + \sum_\nu n_\nu \tau_{\nu 1}, \dots, z_g + n'_g + \sum_\nu n_\nu \tau_{\nu,g} \right).$$

Such a translation by γ has the following effect on the new variables

$$L_\gamma : u_\mu \longrightarrow u_\mu \cdot \prod_\nu q_{\nu\mu}^{n_\nu}.$$

We obtain

$$h\left(u_1 \prod_\nu q_{\nu 1}^{n_\nu}, \dots, u_g \prod_\nu q_{\nu g}^{n_\nu} \right) = h(u_1,\dots,u_g) e^{d(\pi C_{\mathrm{hol}}(z,\gamma) + 2\pi i \varphi(\gamma))}. \tag{5.120}$$

We compute the exponential factors on the right hand side. The relation (5.112) says that

$$\gamma_{0,1} = \sum n_\mu (f_\mu - \sum \tau_{\mu,\nu} \otimes e_\nu).$$

If $z = \sum z_\mu e_\mu = \sum (x_\mu + y_\mu I) e_\mu$, then it follows that

$$H(z,\gamma) - h(z,\gamma) = 2i \sum z_\mu n_\mu,$$

and

$$\begin{aligned} H(\gamma,\gamma) - h(\gamma,\gamma) &= 2i \langle \gamma, p_{0,1}(\gamma) \rangle \\ &= 2i \left\langle \sum_\mu n_\mu f_\mu, \sum_\mu n_\mu \left(f_\mu - \sum_\nu \tau_{\mu\nu} \otimes e_\nu \right) \right\rangle \\ &= -2i \sum_{\nu,\mu} n_\nu n_\mu \tau_{\nu\mu}. \end{aligned} \tag{5.121}$$

We conclude that for $z = \sum z_\mu e_\mu$ and $\gamma = \sum n_\mu f_\mu$

$$\begin{aligned} C_{\text{hol}}(z,\gamma) &= \pi(H(z,\gamma) - h(z,\gamma)) + \tfrac{\pi}{2}(H(\gamma,\gamma) - h(\gamma,\gamma)) \\ &= 2i\pi \sum z_\mu n_\mu - \pi i \sum_{\nu,\mu} n_\nu n_\mu \tau_{\nu,\mu}. \end{aligned} \tag{5.122}$$

Then our recursion formula for sections in $\mathcal{L}(dC_{\text{hol}},d\eta,d\varphi)$ becomes

$$h\left(u_1 \prod_\nu q_{\nu,1}^{n_\nu}, \ldots, u_g \prod_\nu q_{\nu,g}^{n_\nu}\right) = h(u_1,\ldots,u_g) \prod_{\nu=1}^{g} u_\nu^{dn_\nu} (\prod_{\lambda,\kappa=1}^{g} q_{\lambda,\kappa}^{-n_\lambda dn_\kappa}) e^{2\pi d(\varphi(\gamma)+\eta(\gamma))}. \tag{5.123}$$

Now we expand the function h into a Laurent series

$$h(u_1,\ldots,u_g) = \sum a_{m_1,\ldots,m_g} u_1^{m_1} \ldots u_g^{m_g}. \tag{5.124}$$

Our transformation rule for an element $\gamma = \sum_\nu n_\nu f_\nu$ yields the following recursion:

$$a_{m_1,\ldots,m_g} \prod_{\nu,\mu} q_{\nu,\mu}^{n_\nu m_\mu} = a_{m_1-dn_1,\ldots,m_g-dn_g} (\prod_{\lambda,\kappa} q_{\lambda,\kappa}^{-n_\lambda dn_\kappa}) e^{2\pi i d(\varphi(\gamma)+\eta(\gamma))}. \tag{5.125}$$

From this we conclude that the coefficients $a_{\nu_1,\ldots,\nu_g}$ for $0 \le \nu_i \le d-1$ determine the rest of the coefficients. On the other hand we can choose values for the coefficients $a_{\alpha_1},\ldots,a_{\alpha_g}$ arbitrarily for the indices $0 \le \alpha_i \le d-1$ and then we define the other coefficients by

$$a_{\alpha_1-dn_1,\ldots,\alpha_g-dn_g} = a_{\alpha_1,\ldots,\alpha_g} (\prod_{\nu\kappa} q_{\nu,\kappa}^{n_\nu\alpha_\kappa + dn_\nu n_\kappa}) e^{2\pi i d(\varphi(\gamma)+\eta(\gamma))}. \tag{5.126}$$

Now we make the fundamental observation that the positive definiteness of our matrix Y above implies an estimate

$$|a_{\alpha_1-dn_1,\ldots,\alpha_g-dn_g}| < e^{-c(n_1^2+\ldots n_g^2)} \tag{5.127}$$

with some constant $c > 0$ depending on $Y = (y_{\nu,\kappa})$, where $y_{\nu,\kappa} = \text{Im}\,(\tau_{\nu,\kappa})$. To see this we rewrite $q_{\nu,\kappa}^{n_\nu\alpha_\kappa+dn_\nu n_\kappa} e^{2\pi i d(\varphi(\gamma)+\eta(\gamma))}$ in terms of the $\tau_{\nu,\mu}$. We compute the absolute value of this factor. Observe that the factor

$$\prod_{\nu,\kappa} |q_{\nu,\kappa}^{dn_\nu n_\kappa}| = e^{-\pi(\sum_{\mu\kappa} y_{\mu,\kappa} n_\nu n_\kappa)}. \tag{5.128}$$

This gives an estimate of the form above for this term because Y is positive definite. The other contributions are of the form $e^{L(n_1,\ldots,n_g)}$, where L is linear.
This implies that the Laurent series will be convergent for all $u_1,\ldots,u_g \in \mathbb{C}^*$ and we conclude:

Proposition 5.2.33. *We can write down explicitly all sections in a line bundle of the form $\mathcal{L}(C_{\text{hol}},\eta,\varphi)^{\otimes d}$ on $A = V/\Gamma$ as infinite Laurent series. These series converge for all $(u_1,u_2,\ldots,u_g) \in (\mathbb{C}^\times)^g$ and are determined by its coefficients $a_{\nu_1,\ldots,\nu_g}$ for $0 \le \nu_i \le d-1$. These coefficients can be given arbitrarily, i.e. the dimension of the space of sections in the line bundle $\mathcal{L}(C_{\text{hol}},\eta,\varphi)^{\otimes d}$ is d^g.*

These sections are called **Riemann-Theta functions**.

We only considered line bundles, which are powers of a a line bundle attached to a principal polarization. Not all abelian varieties (See 5.2.21) admit a principal polarization. Nevertheless the same considerations apply to arbitrary abelian varieties. With a little bit more effort in linear algebra it is not difficult to show:

Theorem 5.2.34. *Let $A = V/\Gamma$ be a complex torus. If $e \in \mathrm{NS}(A)$ is an alternating form on Γ for which the corresponding Hermitian form H_e is positive definite, then*

$$\dim H^0\left(V/\Gamma, \mathcal{L}_{H_e}, (\eta_{H_e}, \varphi)\right) = \mathrm{Pf}(<\,,>).$$

We want to return to our Jacobian J. There we have the cup product pairing on $H^1(S,\ \ ) \simeq \Gamma$. If H_0 is the corresponding Hermitian form then we can form the line bundle $\mathcal{P} = \mathcal{L}_{H_0}(e_0, \eta, \varphi)$ with an arbitrary φ and suitable η_{H_e}. Our theorem yields

$$\dim \quad H^0(J, \mathcal{P}) = 1. \tag{5.129}$$

If we take powers of this line bundle then $\det(r <\,,>) = r^{2g}$ and it follows that

$$\dim \quad H^0(J, \mathcal{P}^{\otimes r}) = r^{2g}. \tag{5.130}$$

5.2.7 Projective embeddings of abelian varieties

This can be used to construct an embedding into the projective space:

Theorem 5.2.35 (Lefschetz)**.** *If we take $r = 3$ then the morphism*

$$\begin{aligned} \Theta : J &\longrightarrow \mathbb{P}\left(H^0(J, \mathcal{P}^{\otimes 3})\right) \\ x &\longmapsto H_x = \left\{ s \in H^0(J, \mathcal{P}^{\otimes 3}) | s(x) = 0 \right\} \end{aligned}$$

is everywhere defined and yields an embedding of J into the projective space.

I want to comment on this theorem without proving it, its proof will be discussed in the second volume in the section on Jacobians. I give an outline of the steps which have to be carried out.

1. At first we need to know that for any $x \in J$ we can find a section $s \in H^0(J, \mathcal{P}^{\otimes 3})$ which does not vanish at this point.
2. Secondly we have to prove that for any pair of points $x \neq y$ we can find a section which vanishes at x but not at y.
3. Finally we need to know the following: If we pick a point x and a section s_0 which does not vanish at x then the ratios s/s_0 are function on J which are defined in a suitable neighborhood of x. Then we have to show that we can find sections $s_1, \ldots, s_g$ which vanish at x such that the differentials $d(s_1/s_0), \ldots, d(s_g/s_0)$ generate the dual tangent space. This implies that the local ring $\mathcal{O}_{J,x}$ is the ring of convergent power series in $s_1/s_0, \ldots, s_g/s_0$.

(It is in fact not too difficult to prove these assertions with our present knowledge. In the definition of $\mathcal{P}$ we have the freedom of choosing the element φ, let us take $\varphi = 0$. We consider the set of zeroes of a non trivial section $s \in H^0(J,\mathcal{P})$. Since two such sections are proportional this is well defined, locally it is described by one equation, hence it is a divisor $\Theta_{\mathcal{P}}$. It is the so called Θ-divisor . We know that choosing another value of φ amounts to translating $\mathcal{P}$ by an element $x \in J$. If we know choose $x, -x \in J$ or x_1,x_2,x_3 such that $x_1 + x_2 + x_3 = 0$, then we have isomorphisms

$$T_x(\mathcal{P}) \otimes T_{-x}(\mathcal{P}) \xrightarrow{\sim} \mathcal{P}^{\otimes 2}, T_{x_1}(\mathcal{P}) \otimes T_{x_2}(\mathcal{P}) \otimes T_{x_3}(\mathcal{P}) \xrightarrow{\sim} \mathcal{P}^{\otimes 3}.$$

The factors on the left hand side have a one dimensional space of sections, which vanish on a translate of $\Theta_{\mathcal{P}}$. This allows us to construct sections in $H^0(J,\mathcal{P}^{\otimes 3})$ for which we know the set of zeroes. This is good enough to prove 1) and 2), the point 3) is a little bit more delicate.)

If all this is shown then it is clear that

Lemma 5.2.36. *The image of J under the map Θ is a complex analytic submanifold in $Y \subset \mathbb{P}\left(H^0(J,\mathcal{P}^{\otimes 3})\right)$ and*

$$\Theta : J \xrightarrow{\sim} Y$$

is in fact an analytic isomorphism.

Now we use the classical Theorem of Chow which says that a smooth and closed submanifold of $\mathbb{P}^n(\mathbb{C})$ is in fact a smooth projective algebraic variety (see [Ch],[Se1] and section 5.1.7.) Hence we can define the image $\Theta(J) = Y$ as the set of common zeroes of a finite number of homogeneous polynomials $\{F_1,F_2,\dots,F_t\}$ in $n+1$ variables. Furthermore for any point $x \in Y$ we take a linear form L which does not vanish at x and then the functions $F_i/L^{\deg F_i}$ generate the ideal $\mathcal{I}_{Y,x}$ of germs of holomorphic functions which vanish on Y in a neighborhood of x. Then the pair $(J,\mathcal{P}^{\otimes 3}) \xrightarrow{\sim} (Y,\mathcal{O}_{\;\;^{n}}(1)|Y)$ becomes an object in algebraic geometry. To make this precise we have to say a few words about the comparison between algebraic and analytic geometry.

As in the case of Riemann surfaces (see section 5.1.8) we define a new topology on Y, namely the Zariski topology. If we have a homogeneous polynomial $f(z_0,\dots,z_n)$ then we can look at the set $V(f) \subset \mathbb{P}^n(\mathbb{C})$ where it vanishes and the set $D(f) \subset \mathbb{P}^n(\mathbb{C})$ where it does not vanish. These sets $D(f)$ form a basis for the Zariski topology on $\mathbb{P}^n(\mathbb{C})$, i.e. the Zariski open subsets in $\mathbb{P}^n(\mathbb{C})$ are unions of sets of the form $D(f)$. The Zariski open subsets in Y are the intersections of Zariski open subsets in $\mathbb{P}^n(\mathbb{C})$ with Y. As in the case of Riemann surfaces we know that the identity map $Y_{\mathrm{an}} \longrightarrow Y_{\mathrm{Zar}}$ is continuous.

If now $U \subset \mathbb{P}^n(\mathbb{C})$ is a Zariski open subset we say that a holomorphic function $f : U \to \mathbb{C}$ is meromorphic if for any point $y \in U$ we can find homogeneous polynomials g,h of the same degree, such that $h(y) \neq 0$ and such that $f = g/h$ on the open set $U \cap D(h)$. We put as before

$$\mathcal{O}^{\mathrm{mer}}_{\;^{n}(\;\;)}(U) = \{f : U \longrightarrow \mathbb{C} | f \text{ is meromorphic}\}. \tag{5.131}$$

We can do the same thing with Y and define the sheaf $\mathcal{O}_Y^{\mathrm{mer}}$. As in the case of Riemann surfaces the identity map

$$(Y_{\mathrm{an}},\mathcal{O}_Y) \longrightarrow (Y_{\mathrm{Zar}},\mathcal{O}_Y^{\mathrm{mer}})$$

is a morphism between locally ringed spaces.

A meromorphic function on Y is an element in some $\mathcal{O}_Y^{\mathrm{mer}}(U)$ where $U \neq \emptyset$ is Zariski open in Y. Now Y was special, namely it was the image of J under Θ. Hence it is connected as a topological space and from this it follows easily that the intersection of two non empty Zariski open sets is again non empty. This allows us to define the field $\mathbb{C}(Y) = \mathbb{C}(J)$ of meromorphic functions on J. We state without proof:

Theorem 5.2.37. *The field of meromorphic functions on J is a finitely generated extension of $\mathbb{C}$ of trancendence degree d.*

As in the case of Riemann surfaces we can define coherent sheaves of $\mathcal{O}_Y$-modules (resp. $\mathcal{O}_Y^{\mathrm{mer}}$-modules) on Y_{an} (resp. Y_{Zar}). In both cases this are sheaves of modules under the structure sheaf which locally are finitely generated.

It is the content of Serre's paper "Géométrie algébrique et géométrie analytique" (in short GAGA) that these two categories are equivalent. In simple words: To any coherent $\mathcal{O}_{\ ^n,\mathrm{an}}$-sheaf $\mathcal{F}_{\mathrm{an}}$ on $\mathbb{P}^n(\mathbb{C})$ we can find a unique subsheaf $\mathcal{F}$ of $\mathcal{O}_{\ ^n}$-modules (i.e. $\mathcal{F}(U)$ is an $\mathcal{O}_{\ ^n}$-module for any $U \subset \mathbb{P}^n$, Zariski open) such that $\mathcal{F}^{\mathrm{an}} = \mathcal{F} \otimes_{\mathcal{O}_{\ ^n}} \mathcal{O}_{\ ^n,\mathrm{an}}$.

A first consequence of the GAGA-principle is that the sheaf $\mathcal{I}_{Y,\mathrm{an}}$ which defines the analytic subspace Y is the extension of a sheaf of ideals $\mathcal{I}_Y \subset \mathcal{O}_{\ ^n}$, and this is of course the statement of Chow's theorem.
The sheaf $\mathcal{I}_{Y,an}$ is a coherent sheaf (see section 5.1.8) and the clue to the GAGA-principle is the following theorem.

Theorem 5.2.38. *For any coherent sheaf $\mathcal{F}^{an}$ on $\mathbb{P}^n(\mathbb{C})$ we can find an integer $r > 0$ such that $H^q(\mathbb{P}^n(\mathbb{C}), \mathcal{F}^{\mathrm{an}} \otimes \mathcal{O}_{\ ^n(\ )}(r)) = 0$ for $q > 0$ and the sections $H^0(\mathbb{P}^n(\mathbb{C}), \mathcal{F}^{\mathrm{an}} \otimes \mathcal{O}_{\ ^n(\ )}(r))$ generate the stalks $\mathcal{F}_x^{an}$ at all points x.*

Once we have this result, then the general results from GAGA can be proved by a strategy which generalizes the arguments in section 5.1.8.
If we now consider line bundles on J we have the freedom to look at them as complex analytic bundles or as line bundles on the projective varieties $(J,\mathcal{O}_J)$, i.e. as bundles with respect to the Zariski topology. Hence we will not make any distinction between these two kinds of line bundles, we identify

$$\begin{aligned} \mathrm{Pic}_{\mathrm{Zar}}(J) &= \mathrm{Pic}(J) = \mathrm{Pic}(J^{\mathrm{an}}) \\ H^1_{\mathrm{Zar}}(J,\mathcal{O}_J^*) &= H^1(J,\mathcal{O}_{J,\mathrm{an}}^*) \end{aligned} \tag{5.132}$$

where actually $H^1(J,\mathcal{O}_{J,\mathrm{an}}^*)$ was exactly what we called $H^1(J,\mathcal{O}_J^*)$ before.
Mutatis mutandis these considerations apply to arbitrary abelian varieties, i.e. for complex tori $A = V/\Gamma$, for which we can find an $e \in \mathrm{NS}(A)$ with H_e positive definite.

I anticipate a few concepts that will be explained in more detail in the second volume. We can define regular maps between projective varieties, this are of course holomorphic maps, which preserve the subsheaves of meromorphic functions. Actually the GAGA-principle tells us that this is automatically true for holomorphic maps. We can define the product $X \times Y$ of two projective varieties. This allows us to define abelian varieties as projective algebraic varieties X which are connected and which have a product map $m : X \times X \longrightarrow X$, which puts a group structure on X. Forming the inverse must also be a regular map, this is probably automatically true, once we defined m. Hence we see that the notion of an abelian variety is a completely algebraic concept.

5.2.8 Degeneration of Abelian Varieties

At this point we achieved also something else. We can consider the $\tau_{\nu,\mu}$ as complex analytic variables and our considerations show that we can consider our abelian varieties as a holomorphic family of abelian varieties. We have the new description $V/\Gamma \xrightarrow{\sim} (\mathbb{C}^\times)^g/Q$. (See 5.119.)Of course we have the constraint that the imaginary part $\mathrm{Im}\,(\tau_{\nu,\mu})$ must be positive definite. This gives the constraint for the free abelian subgroup Q: The (symmetric) matrix

$$(-\log|q_{\nu,\mu}|)_{\nu,\mu}$$

has to be positive definite. This is an open subset $\mathcal{S}_g$ in the complex variety of symmetric (g,g) matrices with coefficients in $\mathbb{C}^\times$. For any $Q \in \mathcal{S}_g$ we constructed a projective embedding of $\mathbb{C}^\times/Q$ by Theta functions. Of course we may consider the graded ring of Theta functions

$$\bigoplus_{r=0}^{\infty} H^0(\mathbb{C}^\times/Q,\mathcal{P}^{\otimes r}).$$

It can be shown that this ring is finitely generated, it is generated by the sections in $H^0((\mathbb{C}^\times/Q,\mathcal{P}^{\otimes r})$ with $r = 0,1,2,3$. Then we can consider a "free" graded ring

$$\mathbb{C}[X_1,Y_1,\ldots Y_{2^g},Z_1,\ldots,Z_{3^g}]$$

where X_1 sits in degree one, the Y_i sit in degree 2 and the Z_i sit in degree 3. We can construct a surjective homomorphism from this graded ring

$$\mathbb{C}[X_1,Y_1,\ldots,Y_1,\ldots Y_{2^g},Z_1,\ldots,Z_{3^g}] \longrightarrow \bigoplus_{r=0}^{\infty} H^0(\mathbb{C}^\times/Q,\mathcal{P}^{\otimes r})$$

by sending the X_1,Y_ν,Z_μ to a basis of sections in $H^0(\mathbb{C}^\times/Q,\mathcal{P}^{\otimes r})$ for $r = 1,2,3$ respectively. The kernel of this homomorphism consists the relations satisfied by linear combinations of products of Theta functions. These relations can be explicitely given (See [B-L]], Chap. 7). Moreover the coefficients of these linear combinations depend on Q and we can write the relations in such a way that these coefficients depend holomorphically on Q.
Now it is an interesting question to ask: What happens if some of the $q_{\nu,\mu}$ tend to zero? We say that the abelian variety degenerates and it is of great importance to understand this degeneration process. The point is that this degeneration can be given an arithmetic meaning. It is also important if we want to construct a compactification of the moduli space (see [Fa-Ch]). We discuss this process in detail in the case of genus one in the following section.

The Case of Genus 1

I want to discuss these constructions in the special case of curves of genus one. We can assume that the Jacobian is of the form

$$J = \mathbb{C}/\{1,\tau\}$$

where $\tau \in \mathbb{C}$, $\mathrm{Im}\,(\tau) > 0$ and where $\{1,\tau\} = \Gamma$ is the -lattice generated by the elements $1,\tau$. Any alternating pairing is determined by its value on the basis elements. We have a canonical generator defined by

$$\langle 1,\tau \rangle = -1.$$

All other alternating pairings are of the form $d\langle\ ,\ \rangle$ with some integer d. In this case it is clear that $\langle\ ,\ \rangle$ is the imaginary part of a hermitian form H on $\mathbb{C}$. If $y = \mathrm{Im}\,(\tau)$, then this form is given by

$$H(z_1,z_2) = \frac{1}{y} z_1 \overline{z}_2. \tag{5.133}$$

It is positive definite, this explains the minus sign. We consider maps η

$$\eta : \Gamma/2\Gamma \longrightarrow \frac{1}{2}\quad /$$

which satisfies the compatibility relation

$$\frac{1}{2}\langle \gamma_1,\gamma_2 \rangle\ + \eta(\gamma_1 + \gamma_2) - \eta(\gamma_1) - \eta(\gamma_2) \equiv 0 \mod \tag{5.134}$$

for all $\gamma_1,\gamma_2 \in \Gamma$. We say that η is **adapted to the alternating form** $<\ ,\ >$. Now we consider line bundles $\mathcal{L}(d\langle\ ,\ \rangle,d\eta,0)$ on $\mathbb{C}/\Gamma$ which are defined by the following rule: For an open subset $U \subset \mathbb{C}/\Gamma$ and its inverse image $\pi^{-1}(U) = V \subset \mathbb{C}$ we have

$$\mathcal{L}(d\langle\ ,\ \rangle,d\eta,0)(U) = \Big\{ f : V \to \mathbb{C} \mid f \text{ holomorphic, } f(z+\gamma) = e^{d\left(\frac{\pi}{y} z\overline{\gamma} + \frac{\pi}{2y}\gamma\overline{\gamma} + 2\pi i \eta(\gamma)\right)} f(z) \Big\}.$$

If d is even the η term drops out.

Let us look at the case $d = 1$ first. In this case we must have a non trivial η. One possibility is to take

$$\eta(1) = \eta(\tau) = \eta(1+\tau) = \frac{1}{2},$$

and there are three other choices, namely, taking the value $\frac{1}{2}$ on exactly one of the elements $\{1,\tau,1+\tau\}$ and zero on the two others.

I want to stick to the first choice, it is in a sense the most canonical. We investigate the line bundle $\mathcal{L}(<\ ,\ >,\eta,0)$. We have to look at functions which satisfy

$$f(z+\gamma) = e^{\frac{\pi}{y} z\overline{\gamma} + \frac{\pi}{2y}\gamma\overline{\gamma} + 2\pi i \eta(\gamma)} f(z). \tag{5.135}$$

The factor in front has to be interpreted as a 1-cocycle $\Gamma \mapsto (\mathcal{O}\ \ (\mathbb{C}))^{\times}$. We modify this cocycle. We choose the submodule $G \subset \Gamma$ to be the module generated by 1. Now η is not trivial on the vector 1. This forces us to make some minor modifications. For a local section f of our line bundle we put

$$\widetilde{f}(z) = e^{-\frac{\pi}{2y} z^2} f(z) \tag{5.136}$$

and then we find

$$\widetilde{f}(z+1) = -\widetilde{f}(z). \tag{5.137}$$

Here we have to take into account that $\eta(1) = \frac{1}{2}$. Now a simple computation shows

$$\begin{aligned}\widetilde{f}(z+n\tau) &= e^{-\frac{\pi}{2y}(z+n\tau)^2} f(z+n\tau) \\ &= e^{-\frac{\pi}{2y}(z+n\tau)^2} e^{\frac{\pi}{y}zn\overline{\tau}+\frac{\pi}{2y}n^2\tau\overline{\tau}+\pi in} f(z) \\ &= e^{-2\pi inz} e^{-\pi in^2\tau+\pi in} \widetilde{f}(z).\end{aligned} \tag{5.138}$$

We introduce variables $u = e^{\pi iz}$ and $p = e^{\pi i\tau}$. Then our first relation above says that our function $\widetilde{f}$ has a Laurent expansion

$$\widetilde{f}(u) = \sum_{m=1 \mod 2} a_m u^m. \tag{5.139}$$

The second relation gives a recursion for the coefficients a_m which says

$$\widetilde{f}(up^n) = (-1)^n u^{-2n} \cdot p^{-n^2} \widetilde{f}(u). \tag{5.140}$$

This means for the expansion

$$a_m u^n p^{mn} = (-1)^n u^{-2n} p^{-n^2} (\ldots + a_{m+2n} u^{m+2n} \ldots), \tag{5.141}$$

and hence for any choice n,m

$$a_{m+2n} = (-1)^n p^{mn+n^2} a_m. \tag{5.142}$$

Since the coefficients with an even index are vanishing, we see that the coefficient a_1 determines all the others. We put it equal to one and then we get

$$\widetilde{f}(u) = \sum_{m\in} (-1)^m \cdot p^{m+m^2} u^{1+2m}. \tag{5.143}$$

Since we have $\operatorname{Im}(\tau) > 0$ we have $|p| < 1$ and hence our power series converges for all $u \in \mathbb{C}^\times$. This function is one of the **Jacobi Theta functions**. We change the notation and write

$$\vartheta_{0,0}(u,p) = \sum_{m\in} (-1)^m p^{m+m^2} u^{1+2m}. \tag{5.144}$$

We have seen that in modern language this Theta function is a section in a line bundle on the Riemann surface $J = \mathbb{C}/\{1,\tau\}$.
We can ask ourselves whether we have a different description of this line bundle. Clearly it is of degree one. Hence it should be of the form $\mathcal{O}_J(P)$ with some point $P \in \mathbb{C}/\{1,\tau\}$. The bundle $\mathcal{O}_J(P)$ has a non trivial section which vanishes at P. Hence we see that our $\vartheta_{0,0}(u,p)$ must vanish for some value of u. A simple computation yields

$$\vartheta_{0,0}(1,p) = 0, \tag{5.145}$$

and hence we conclude

$$\mathcal{L}(\langle\ ,\ \rangle,0,\eta) \simeq \mathcal{O}_J(O) \tag{5.146}$$

where $O \in \mathbb{C}/\{1,\tau\}$ is the zero element.

Now it becomes clear what the other choices of η will give. In section 5.2.1 I explained that different choices of η can be compensated by changing φ. In this case we can consider $\varphi : \Gamma \to \mathbb{C}$ such that $\varphi(\Gamma) \subset \frac{1}{2}\ $, of course what matters is the resulting homomorphism (see Lemma 5.2.4)

$$\overline{\varphi} : \Gamma/2\Gamma \longrightarrow \tfrac{1}{2}\ \ /\ \ .$$

We have three non zero such homomorphisms and

$$\mathcal{L}(\langle\ ,\ \rangle,0,\eta') = \mathcal{L}(\langle\ ,\ \rangle,\varphi,\eta)$$

if $\eta' = \overline{\varphi}+\eta$. We could carry out the same calculations and get three more Theta functions

$$\begin{aligned} \vartheta_{0,1}(u,p) &= \sum_{m\in} (-1)^m p^{m^2} u^{2m} \\ \vartheta_{1,0}(u,p) &= \sum_{m\in} p^{m^2+m} u^{2m+1} \\ \vartheta_{1,1}(u,p) &= \sum_{m\in} p^{m^2} u^{2m} \end{aligned} \tag{5.147}$$

and they correspond to the linear forms with $\overline{\varphi}_{i,j}(1) = \frac{i}{2} \mod\ $, $\overline{\varphi}_{i,j}(\tau) = \frac{j}{2} \mod\ $. These give the four Jacobi Theta functions. The kernel of $\varphi_{i,j}$ defines a 2-torsion point $P_{i,j} \in J$, and we must have that $\vartheta_{i,j}$ is a non zero section in $H^0\left(J,\mathcal{L}(\langle\ ,\ \rangle,\varphi_{i,j},\eta\right)$ and

$$\mathcal{L}(\langle\ ,\ \rangle,\varphi_{i,j},\eta) = \mathcal{O}_J(P_{i,j}). \tag{5.148}$$

In the section 5.2.6 we learned how to write down sections in $H^0\left(J,\mathcal{L}(dC_{\mathrm{hol}},d\eta,0)\right)$. We know from the Riemann-Roch Theorem that this space of sections has dimension d. We get the same result from the recursion formulae, we always can choose d coefficients and they define all the others. For sections in $H^0(J,\mathcal{L}(d\langle\ ,\ \rangle,d\eta,0))$ the recursion is

$$\begin{aligned} \widetilde{f}(-u) &= (-1)^d \widetilde{f}(u) \\ \widetilde{f}(up^m) &= (-1)^{dm} u^{-2md} p^{-dm^2} \widetilde{f}(u). \end{aligned} \tag{5.149}$$

If we expand $\widetilde{f}$ into a Laurent series in u the coefficients with even (resp. odd) indices vanish if d is odd (resp. even).

Remark 7. The reader should notice that the recursion defines a line bundle on $\mathbb{C}^\times/\langle p^2\rangle$ and not on $\mathbb{C}^\times/\langle p\rangle$. The ratio of two sections $\widetilde{f}/\widetilde{g}$ satisfies $\widetilde{f}/\widetilde{g}(u) = \widetilde{f}/\widetilde{g}(-u)$ and is therefore a meromorphic function in u^2 which is invariant under multiplication by p^2. Hence it is a meromorphic function on $\mathbb{C}^\times/\langle p^2\rangle$. This has to be so because we have the isomorphism $\mathbb{C}/\langle 1,\tau\rangle \xrightarrow{\sim} \mathbb{C}^\times/\langle p^2\rangle$ which is given by $z \to e^{2\pi i z} = u^2$.

If $d = 2$ then

$$\theta_1 = \vartheta_{0,0}^2 = \ldots - \Big(\sum_{m\in} p^{2m^2+2m}\Big)u^0 + \Big(\sum_{m\in} p^{2m^2}\Big)u^2 + \ldots$$

is a section and we find a second section where the coefficient $a_0 = 1$ and $a_2 = 0$, namely

$$\theta_2 = \sum_{m\in} p^{2m^2} u^{4m}. \tag{5.150}$$

(We will sometimes suppress the variables u,p in our notation).We consider $d = 3$, we have already two sections, namely, θ_1^3 and $\theta_1\theta_2$, and we can write a third section

$$\theta_3 = \sum_{m\in} (-1)^m p^{3m^2+m} u^{1+6m}. \tag{5.151}$$

We are now in exactly the same situation as in the discussion of the Weierstraß normal form (5.1.7). We have the sections

$$\begin{aligned} &\theta_1 \in H^0(J,\mathcal{O}_J(O)) && \subset H^0(J,\mathcal{O}_J(3O)) \\ &\theta_2 \in H^0(J,\mathcal{O}_J(2O)) && \subset H^0(J,\mathcal{O}_J(3O) \\ \text{and} \quad &\theta_3 \in H^0(J,\mathcal{O}_J(3O)). \end{aligned}$$

We must have linear relations among the monomials θ_3^2, $\theta_3\theta_2\theta_1$, $\theta_3\theta_1^3$, θ_2^3, $\theta_2^2\theta_1^2$, $\theta_2\theta_1^4$, θ_1^6.

Now we take into account that our curve depends on a parameter τ and hence on $e^{\pi i\tau} = p$, the coefficients of the relations must be holomorphic functions in the variable p. We look at specific sections in our line bundle for $d = 2$. We have the two division points $\frac{1}{2}$, $\frac{\tau}{2}$ and $\frac{1+\tau}{2}$ in $\mathbb{C}/\{1,\tau\}$, we call them $P_{1,0}$, $P_{0,1}$ and $P_{1,1}$ respectively. Then $P_{0,0} = O$. For $z \in \mathbb{C}$ we put $u(z) = e^{\pi iz}$. The ratios

$$\xi_{\nu,\mu}(z) = \xi_{\nu,\mu}(z,p) = \frac{\vartheta_{\nu,\mu}^2(u(z),p)}{\vartheta_{0,0}^2(u(z),p)} \tag{5.152}$$

are meromorphic functions on $\mathbb{C}/\{1,\tau\}$ and $\mathrm{Div}(\xi_{\nu,\mu}) = 2P_{\nu,\mu} - 2O$.

We choose one of these points, say $P_{1,0}$, we put $x = \xi_{1,0}$ and we consider the function

$$r(z,p) = x(z,p)\,(x(z,p) - x(P_{0,1},p)) \cdot (x(z,p) - x(P_{1,1},p))\,. \tag{5.153}$$

We have $\mathrm{Div}\,(x - x(P_{0,1})) = P_{0,1} + Q - 2O$, but since this divisor is principal we have $Q = P_{0,1}$. The same argument holds for the third factor. Therefore this function has divisor $2P_{0,1} + 2P_{1,0} + 2P_{1,1} - 6O$. The function

$$y = y(z) = \frac{\vartheta_{1,0}(u(z),p)\vartheta_{0,1}(u(z),p)\vartheta_{1,1}(u(z),p)}{\vartheta_{0,0}^3(u(z),p)} \tag{5.154}$$

has the divisor $P_{0,1} + P_{1,0} + P_{1,1} - 3O$ and hence

$$\mathrm{Div}(y^2) = 2P_{0,1} + 2P_{1,0} + 2P_{1,1} - 6O. \tag{5.155}$$

We get $\mathrm{Div}(y^2) = \mathrm{Div}(r(z,p))$ and conclude that

$$y^2 = a \cdot x\,(x - x(P_{0,1}))\,(x - x(P_{1,1}))\,. \tag{5.156}$$

Our division points are $P_{1,0} = \frac{1}{2}$, $P_{1,1} = \frac{\tau+1}{2}$ and $P_{0,1} = \frac{\tau}{2}$. A simple calculation shows that $\vartheta_{1,0}(P_{1,0}) = 0$. Since $x = \frac{\vartheta_{1,0}^2}{\vartheta_{0,0}^2}$ and we have

$$\begin{aligned}
\vartheta_{1,0}\left(u(\tfrac{\tau}{2}),p\right) &= \sum_{m\in} p^{m^2+m} e^{\frac{\pi i\tau}{2}(2m+1)} = \sum_{m\in} p^{m^2+m} e^{\pi i\tau m} e^{\frac{\pi i\tau}{2}} \\
&= \left(\sum_{m\in} p^{m^2+2m}\right) e^{\frac{\pi i\tau}{2}} = \left(\sum_{m\in} p^{m^2}\right) e^{-\frac{\pi i\tau}{2}} \\
\vartheta_{1,0}(u(\tfrac{\tau+1}{2}),p) &= \sum_{m\in} p^{m^2+m} e^{\pi i\frac{\tau+1}{2}(2m+1)} = \left(\sum_{m\in} p^{m^2+m} e^{\pi i\tau m}\cdot e^{\pi i m}\right) e^{\frac{\pi i}{2}} e^{\frac{\pi i}{2}} \\
&= \left(\sum_{m\in} (-1)^m p^{m^2+2m}\right)\cdot e^{\frac{\pi i\tau}{2}}\cdot e^{\frac{\pi i}{2}} = \left(\sum_{m\in} (-1)^{m+1} p^{m^2}\right) e^{-\frac{\pi i\tau}{2}} e^{\frac{\pi i}{2}}.
\end{aligned} \tag{5.157}$$

The same calculation for $\vartheta_{0,0}$ yields

$$\begin{aligned}
\vartheta_{0,0}(u(P_{1,0}),p) &= \left(\sum_{m\in} p^{m^2+m}\right) e^{\frac{\pi i}{2}} \\
\vartheta_{0,0}(u(P_{0,1}),p) &= \left(\sum_{m\in} (-1)^m p^{m^2+2m}\right) e^{\frac{\pi i\tau}{2}} \\
\vartheta_{0,0}(u(P_{1,1}),p) &= \left(\sum_{m\in} p^{m^2+2m}\right) e^{\frac{\pi i\tau}{2}} e^{\frac{\pi i}{2}}.
\end{aligned} \tag{5.158}$$

We get

$$y^2 = a\cdot x\left(x - \frac{\left(\sum_{m\in}\ p^{m^2}\right)^2}{\left(\sum_{m\in}\ (-1)^{m+1}p^{m^2}\right)^2}\right)\left(x - \frac{\left(\sum_{m\in}\ (-1)^{m+1}p^{m^2}\right)^2}{\left(\sum_{m\in}\ p^{m^2}\right)^2}\right). \tag{5.159}$$

We can compute the factor a. We look at the leading term in the expansion for

$$x = \left(\frac{\vartheta_{1,0}}{\vartheta_{0,0}}\right)^2 \qquad\text{and}\qquad y = \frac{\vartheta_{1,0}\vartheta_{1,1}\vartheta_{0,1}}{\vartheta_{0,0}^3}$$

at $z = 0$, i.e. $u = 1$. If

$$\vartheta_{0,0}(u(z),p) = \beta z \qquad +\text{higher order terms} \tag{5.160}$$

then

$$x = \frac{\vartheta_{1,0}(1,p)^2}{\beta^2 z^2} \qquad +\text{higher order terms} \tag{5.161}$$
$$y = \frac{\vartheta_{1,0}(1,p)\vartheta_{1,1}(1,p)\vartheta_{0,1}(1,p)}{\beta^3 z^3} \qquad +\text{higher order terms.}$$

Hence we get

$$a = \frac{\vartheta_{1,1}(1,p)^2 \cdot \vartheta_{0,1}(1,p)^2}{\vartheta_{1,0}(1,p)^4} = \tfrac{1}{16}\left(1 - 12p^2 + 66p^2 - 232p^4 + \ldots\right). \tag{5.162}$$

This power series is a square, we check easily that

$$a = \left(\tfrac{1}{4}(1 - 6p^2 + 15p^4 + \ldots)\right)^2$$

and we substitute y by $\frac{4y}{1-6p^2+15p^4+\ldots}$ and get

$$y^2 = x\left(x - \frac{\left(\sum_{m\in} \ p^{m^2}\right)^2}{\left(\sum_{m\in} \ (-1)^{m+1}p^{m^2}\right)^2}\right)\left(x - \frac{\left(\sum_{m\in} \ (-1)^{m+1}p^{m^2}\right)^2}{\left(\sum_{m\in} \ p^{m^2}\right)^2}\right). \tag{5.163}$$

Now it follows from a simple calculation that

$$\frac{\left(\sum_{m\in} \ p^{m^2}\right)^2}{\left(\sum_{m\in} \ (-1)^{m+1}p^{m^2}\right)^2} + \frac{\left(\sum_{m\in} \ (-1)^{m+1}p^{m^2}\right)^2}{\left(\sum_{m\in} \ p^{m^2}\right)^2} = 2 + 64p^2 + 512p^4 + 2816p^6 + \ldots = \lambda(q) \tag{5.164}$$

is a power series in p^2, we get a family of curves which depend on a parameter p

$$y^2 = x\left(x^2 - \left(2 + 64p^2 + 512p^4 + 2816p^6 + \ldots\right)x + 1\right). \tag{5.165}$$

This is now an equation for an elliptic curve. The distinguished point is the point at infinity. The projective curve is given by

$$\widetilde{\mathcal{E}}_p : y^2 v = x\left(x^2 - \left(2 + 64p^2 + 512p^4 + \ldots\right)xv + v^2\right). \tag{5.166}$$

We make this a little bit more explicit. We write the expansions

$$\vartheta_{0,0}(u,p) = \left(u - u^{-1}\right)\left(1 - \left(u^{-2} + u^2 + 1\right)p^2 + \left(u^{-4} + u^4 + u^{-2} + u^2 + 1\right)p^6 - \ldots\right) \tag{5.167}$$

and

$$\vartheta_{1,0}(u,p) = \left(u + u^{-1}\right)\left(1 + u^{-2} - 1 + u^2\right)p^2 + \left(u^{-4} + u^4 - u^{-2} - u^2 + 1\right)p^6 + \ldots\Big). \tag{5.168}$$

The other two series are of the form ($\epsilon = 0{,}1$)

$$1+\sum_{m=1}^{\infty}(-1)^{m\epsilon}(u^{-m}+u^m)p^{m^2}.$$

If we multiply them together we get only even powers of u and p.

We see that in the expressions for $\vartheta_{0,1}(u,p)$, $\vartheta_{1,1}(u,p)$ and in the second factors of $\vartheta_{0,0}(u,p)$, $\vartheta_{1,0}(u,p)$ we only have even powers of p and the coefficients of the powers of p and are always polynomials in u^2, u^{-2} whose degree is the summation index and whose coefficients are ± 1. (This is good enough for convergence for any choice $u \in \mathbb{C}^{\times}$ and p with $|p|<1$.) We introduce new functions

$$y_1(u,p) = \frac{(u-u^{-1})^3}{4(u+u^{-1})}\, y(u,p). \tag{5.169}$$

$$x_1(u,p) = \frac{(u-u^{-1})^2}{(u+u^{-1})^2} x(u,p) \tag{5.170}$$

and for them get the expressions

$$\begin{aligned} y_1(u,p) =& \frac{(u-u^{-1})^3}{4(u+u^{-1})} \frac{\vartheta_{1,0}(u,p))\vartheta_{0,1}(u,p)\vartheta_{1,1}(u,p)}{\vartheta_{0,0}^3(u,p)} \frac{\vartheta_{1,0}^2(1,p)}{\vartheta_{0,1}(1,p)\vartheta_{1,1}(1,p)} \\ =& 1+\big(-4(u^{-4}+u^4)+u^{-2}+u^2+6\big)p^2 \\ &+\big(-4(u^{-6}+u^6)+3\big(u^{-4}+u^4\big)+24(u^{-2}+u^2)+38\big)p^4+\dots \\ =& 1+\sum_{m=1}^{\infty} Y_m(u^2,u^{-2})p^{2m} \end{aligned} \tag{5.171}$$

and

$$\begin{aligned} x_1(u,p) =& \frac{(u-u^{-1})^2}{(u+u^{-1})^2} \frac{\vartheta_{1,0}^2(u,p)}{\vartheta_{0,0}^2(u,p)} \\ &= 1+4\left(u^2+u^{-2}\right)p^2+ \\ &\left(3\left(u^{-6}+u^6\right)+4\left(u^{-4}+u^4\right)+9\left(u^{-2}+u^2\right)+8\right)p^4+\dots \\ &= 1+\sum_{m=1}^{\infty} X_m(u^2,u^{-2})p^{2m}. \end{aligned} \tag{5.172}$$

where $Y_m(u^2,u^{-2})$,$X_m(u^2,u^{-2})$ are polynomials in u^2, u^{-2}, invariant under $u \to u^{-1}$ with integer coefficients. We have an estimate for the degree of the Y_m, the absolute values of the coefficients of the Y_m: They can be estimated by $C\sqrt{m}$ for some constant C.

We introduce new variables $q=p^2$, $w=u^2$, recall that this now means $q=e^{2\pi i\tau}$,$w=e^{2\pi i z}$. We rewrite all the occuring expressions in the variables q and w, i.e.

$$\widetilde{\lambda}(q)=\lambda(p), \qquad \widetilde{x}_1(w,q)=x_1(u,p), \qquad \widetilde{y}_1(w,q)=y_1(u,p),$$
$$\widetilde{X}_m(w,w^{-1})=X_m(u^2,u^{-2}), \qquad \widetilde{Y}_m\left(w,w^{-1}\right)=Y_m(u^2,u^{-2}),$$

We rewrite our original functions $x(u,p)$, $y(u,p)$ in the new variables and get

$$\widetilde{x}(w,q) = \frac{w+w^{-1}+2}{w+w^{-1}-2}\,\widetilde{x}_1(w,q) = \frac{w+w^{-1}+2}{w+w^{-1}-2}(1+\sum_{m=1}^{\infty}\widetilde{X}_m(w,w^{-1})q^m)$$

$$\widetilde{y}(w,q) = \frac{4(w+1)}{(w-1)(w+w^{-1}-2)}\,\widetilde{y}_1(w,q) = \frac{4(w+1)}{(w-1)(w+w^{-1}-2)}(1+\sum_{m=1}^{\infty}\widetilde{Y}_m(w,w^{-1})q^m)$$

Now we know that $\widetilde{x}(w,q)$,$\widetilde{y}(w,q)$ satisfy the equation

$$\widetilde{y}(w,q)^2 = \widetilde{x}(w,q)\left(\widetilde{x}(w,q)^2 - \left(2+64p^2+512p^4+2816p^6+\ldots\right)\widetilde{x}(w,q)+1\right)$$

and we have proved

Proposition 5.2.39. *For any point q in the punctured disc we get an elliptic curve $\widetilde{\mathcal{E}}_q$, the map*

$$\begin{array}{lll} \mathbb{C}^\times \longrightarrow \widetilde{\mathcal{E}}_q & & \\ w \longmapsto (\widetilde{x}(w,q),\widetilde{y}(w,q),1) & & \text{if } w \neq 1 \\ w \longmapsto \left(\dfrac{\widetilde{x}(w,q)}{\widetilde{y}(w,q)},1,\dfrac{1}{\widetilde{y}(w,q)}\right) & & \text{if } w \text{ is near } 1 \end{array}$$

provides an isomorphism of complex analytic groups

$$\mathbb{C}^\times/<q> \xrightarrow{\sim} \widetilde{\mathcal{E}}_q.$$

I write this map (and other similar ones) in a more suggestive form

$$w \longmapsto (\widetilde{x}(w,q),\widetilde{y}(w,q),1) = \left(\frac{\widetilde{x}(w,q)}{\widetilde{y}(w,q)},1,\frac{1}{\widetilde{y}(w,q)}\right).$$

But we may also consider the product of the punctured disc $\dot{D}$ with the projective plane $\mathbb{P}^2(\mathbb{C})$. The homogeneous coordinates of the plane are (x,y,v). Then we have constructed a family of elliptic curves – namely

$$\begin{array}{ccc} \widetilde{\mathcal{E}} : \widetilde{y}^2 v = \widetilde{x}^3 - \widetilde{\lambda}(q)\widetilde{x}^2 v + \widetilde{x}v^2 & \hookrightarrow & \dot{D}\times\mathbb{P}^2(\mathbb{C}) \\ & \searrow^{p_0} & \downarrow p_1 \\ & & \dot{D}. \end{array} \tag{5.173}$$

Here $\widetilde{\mathcal{E}}$ is a smooth complex variety and the fibre over the point $q \in \dot{D}$ is our elliptic curve $\widetilde{\mathcal{E}}_q$. Now we discuss the degeneration of the curve. What happens if $q \to 0$?
Of course we can extend our diagram to

$$\begin{array}{ccc} \overline{\widetilde{\mathcal{E}}} : \widetilde{y}^2 v = \widetilde{x}^3 - \widetilde{\lambda}(q)\widetilde{x}^2 v + \widetilde{x}v^2 & \hookrightarrow & D\times\mathbb{P}^2(\mathbb{C}) \\ & \searrow^{p_0} & \downarrow p_1 \\ & & D. \end{array} \tag{5.174}$$

We define $\overline{\widetilde{\mathcal{E}}}$ by the same equation. We can put $q = 0$, then we get the curve

$$\mathcal{E}_0 : \widetilde{y}^2 v = \widetilde{x}^3 - 2\widetilde{x}^2 v + \widetilde{x} v^2 = \widetilde{x}(\widetilde{x} - v)^2, \tag{5.175}$$

this is not an elliptic curve, because it has a singularity. We see that we have a morphism from the projective line $\mathbb{P}^1(\mathbb{C})$ to $\mathcal{E}_0$:

$$f : t \longmapsto \left(t^2, t(t^2-1), 1\right) = \left(\frac{t}{t^2-1}, 1, \frac{1}{t(t^2-1)}\right). \tag{5.176}$$

Proposition 5.2.40. *The morphism*

$$f : \mathbb{P}^1(\mathbb{C}) \setminus \{\pm 1\} \longrightarrow \mathcal{E}_0 \setminus \{(1,0,1)\}$$

is biholomorphic. The points $t = \pm 1$ both map to $(1,0,1)$.

The point $(1,0,1)$ is a so called **double point** on $\mathcal{E}_0$.. In a small neighborhood of $(1,0,1)$ the curve looks like

$$y^2 = (x-1)^2 \text{ and this is } (y - (x-1))(y + (x-1))$$

and this are two crossing straight lines.
Therefore we can say that for $q \to 0$ the elliptic curve degenerates into a rational curve with an ordinary double point. The curve $\mathcal{E}_0$ is called the special fiber. It looks as if it has genus zero, but a closer look shows that the singular point raises the genus back to 1. We consider smaller discs $D(r) = \{q \mid |q| < r\}$. A germ of a holomorphic section in $\widetilde{\mathcal{E}}$ is a holomorphic map $s : \dot{D}(r) \to \widetilde{\mathcal{E}}$ defined on some $\dot{D}(r)$ such that $s(q) \in \widetilde{\mathcal{E}}_q$ for all $q \in \dot{D}$, i.e. $p_0 \circ s = \mathrm{Id}$. As usual two germs are considered as equal if they are equal on their common domain of definition. We say that such a section is meromorphic if it extends to a holomorphic section from D to $\mathbb{P}^2(\mathbb{C})$. These meromorphic sections define a group $\widetilde{\mathcal{E}}(\dot{D}_0)$, where $\dot{D}_0$ is a notation for the "germ of the punctured disc". It is clear how we get such germs. We consider non zero Laurent series

$$f(q) = \sum_{n > -N} a_n q^n$$

which are convergent on some $\dot{D}(r)$. They form a field $\mathcal{O}(\dot{D}_0)^\times$. We define a map

$$\begin{aligned} \Theta : \mathcal{O}(\dot{D}_0)^\times &\longrightarrow \widetilde{\mathcal{E}}(\dot{D}) \\ \text{by} \qquad f(q) &\longmapsto \{q \mapsto (\widetilde{x}(f(q),q), \widetilde{y}(f(q),q), 1)\} \\ &= \left\{ q \mapsto \left(\frac{\widetilde{x}(f(q),q)}{\widetilde{y}(f(q),q)}, 1, \frac{1}{\widetilde{y}(f(q),q)}\right) \right\}. \end{aligned} \tag{5.177}$$

Now it is clear that the Laurent series $\{q^n\}_{n\in}$ go to the identity. I claim (without proof, it is rather clear anyway):

Lemma 5.2.41. *The map Θ induces an isomorphism*

$$\Theta : \mathcal{O}(\dot{D}_0)^\times / \langle q \rangle \xrightarrow{\sim} \widetilde{\mathcal{E}}(\dot{D})$$

Any $f \in \mathcal{O}(\dot{D}_0)^\times$ is modulo $\langle q \rangle$ of the form

$$f(q) = a_0 + a_1 q + a_2 q^2 + a_3 q^3 + \dots$$

where $a_0 \neq 0$. We define a homomorphism which can be considered as evaluation at 0

$$\Theta_0 : \widetilde{\mathcal{E}}(\dot{D}) \longrightarrow \widetilde{\mathcal{E}}_0 \tag{5.178}$$

which sends

$$\begin{aligned} f(q) \longmapsto (\widetilde{x}(f(0),0), \widetilde{y}(f(0),0),1) &= \left(\frac{\widetilde{x}(f(0),0)}{\widetilde{y}(f(0),0))}, 1, \frac{1}{\widetilde{y}(f(0),0)} \right) \\ &= \left(\frac{a_0 + a_0^{-1} + 2}{a_0 + a_0^{-1} - 2}, \frac{4(a_0+1)}{(a_0-1)(a_0 + a_0^{-1} - 2)}, 1 \right) \\ &= \left(\frac{a_0 + a_0^{-1} + 2}{4(a_0+1)}, 1, \frac{(a_0-1)(a_0 + a_0^{-1} - 2)}{4(a_0+1)} \right) \end{aligned}$$

This also defines a morphism $\mathbb{P}^1(\mathbb{C}) \to \widetilde{\mathcal{E}}_0$, where now the two points 0, ∞ are mapped to the singular point. We see that the evaluation map Θ_0 induces a biholomorphic isomorphism $\mathbb{C}^\times \xrightarrow{\sim} \widetilde{\mathcal{E}}_0 \backslash \{(1,0,1)\}$, and this allows us to put a group structure on $\mathcal{E}_0 \backslash \{(1,0,1)\}$, such that Θ_0 becomes a homomorphism.

The Algebraic Approach

The point of the previous consideration is, that the objects which we constructed are essentially purely algebraic objects. To explain this I need to anticipate some of the concepts of the second volume. The holomorphic functions on the "germ of discs" are simply the power series with some positive radius of convergence, the meromorphic functions are the Laurent series, also with some positive radius of convergence. But we can ignore convergence, we may consider the ring of all (formal) power series and the field of all Laurent series in the variable q. But if we do not care about convergence, then there is no reason why the coeffients should be complex numbers. We can consider the rings of formal power series and formal Laurent series with coefficients in   , these are the rings $[[q]]$ and $[[q]]\left[\frac{1}{q}\right]$. For reasons which will become clear in a moment, we will need that 2 is invertible in our rings, so we enlarge   to $R = \ [\frac{1}{2}]$, i.e. we consider the rings

$$A = R[[q]] \qquad \text{and} \qquad B = R[[q]]\left[\frac{1}{q}\right].$$

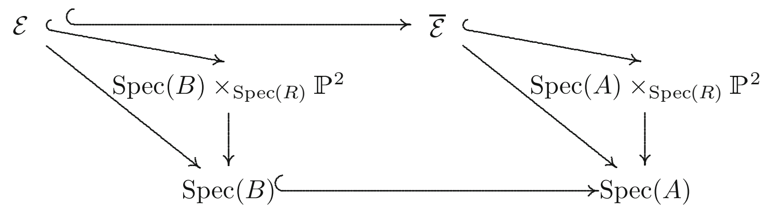
The units in A are the power series whose constant term is $\pm 2^n$ and the units in B are of the form $q^n(\pm 2^n + \sum_{k>0} a_k q^k)$. Now we can attach some kind of geometric objects

$$\mathrm{Spec}(A) = \mathrm{Spec}\,(\ [[q]]) \qquad \text{and} \qquad \mathrm{Spec}(B) = \mathrm{Spec}\left(\ [[q]]\left[\frac{1}{q}\right]\right)$$

to these rings. The reader may think of them as an infinitesimally small (punctured) disc, they are "affine schemes".
We have the scheme $\mathbb{P}^2/\,\mathrm{Spec}(\ )$. I do not give the definition, but I formulate its essential property. One may think of $\mathbb{P}^n/\,\mathrm{Spec}(\ )$ as a covariant functor from the category of commutative rings S with unit element to the category of sets. For any such ring S the set $\mathbb{P}^n(S)$ is the set of $(n+1)$-tuples

$$\Big\{(a_0,a_1,\dots a_n) \mid \exists\, r_0,r_1,\dots,r_n \in S \text{ such that } \sum_i r_i a_i = 1\Big\}/\simeq,$$

where the equivalence relation identifies two such tuples $(a_0,a_1,\dots a_n)$, $(b_0,b_1,\dots b_n)$ if and only if there is a unit $c \in S^\times$ such that $a_i = cb_i$ for all $i = 0,1,\dots,n$. If $S = \mathbb{C}$, then this is our earlier definition, if S is a local ring, i.e. it has a unique maximal ideal ($\neq S$), then the condition about the existence of the r_i says that at least one of the entries is not in the maximal ideal. Now I want to explain that our constructions above provide a diagram



(5.179)

the meaning of this will be explained now. We consider the categories of rings over A and over B. This means that we consider rings S with identity together with a homomorphism $i : A \to S$ (this is a ring over A) or together with a morphism $j : B \to S$.

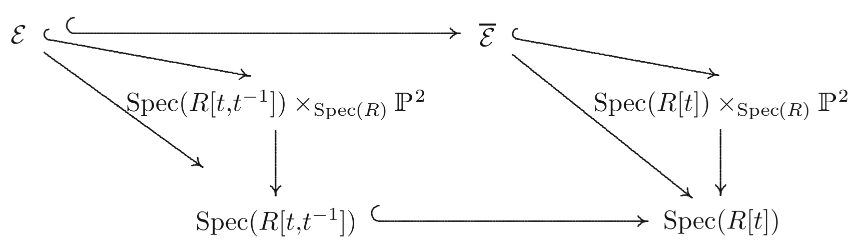
Question: What is the relation between these two kind of rings? Is any ring over A automatically also ring over B?

Now we define $\mathcal{E}$ and $\overline{\mathcal{E}}$ as subfunctors of the projective plane: We put

$$\mathcal{E}(S) = \Big\{(x,y,v) \in \mathbb{P}^2(S) \mid y^2v = x^3 - i(\widetilde{\lambda}(q))x^2v + xv^2\Big\} \tag{5.180}$$

$$\text{and} \qquad \overline{\mathcal{E}}(S) = \Big\{(x,y,v) \in \mathbb{P}^2(S) \mid y^2v = x^3 - j(\widetilde{\lambda}(q))x^2v + xv^2\Big\}.$$

Of course we can replace both rings by smaller rings. We consider the polynomial ring $R[t]$ we embed it into A by sending t to $\lambda(q) - 2 = 64q + 512q^2 \dots$ and then $R\left[t,\frac{1}{t}\right]$ embeds into B. (Here we need that 2 is invertible.) Then we can write the same diagrams over these smaller rings:



(5.181)

and this now means that $\mathcal{E}$, $\overline{\mathcal{E}}$ now define functors from the category of rings over $R[t]$ (resp. $R\left[t,\frac{1}{t}\right]$). It turns out, that $\mathcal{E} \to R\left[t,\frac{1}{t}\right]$ is in fact an elliptic curve, if we remove the point $t = -4$. We will discuss this example in the second volume. Now the reader may wonder, why we work with the large rings, whereas the situation with the small rings seems to be much simpler.

If we work over the rings A,B then we still can write down sections, or in other words we can describe the groups of values $\mathcal{E}(A)$ and $\overline{\mathcal{E}}(B)$. If we consider power series

$$f(q) = \sum_{n \geq N} a_n q^n$$

whose lowest order term $a_k q^k$ has a coefficient $a_k \in R^\times$, then we may substitute this power series for w into $\widetilde{x}(w,q)$, $\widetilde{y}(w,q)$ and we get a point

$$(\widetilde{x}(f(q),q),\widetilde{y}(f(q),q),1) = \left(\frac{\widetilde{x}(f(q),q)}{\widetilde{y}(f(q),q)}, 1, \frac{1}{\widetilde{y}(f(q),q)}\right) \tag{5.182}$$

in $\mathcal{E}(A)$. Actually we have to look a little bit closer to this process of substitution. We can multiply by a power of q, this does not change the point in the projective space. So we assume that $k = 0$ and we get

$$\begin{aligned} \widetilde{x}(f(q),q) &= \frac{f(q)+f(q)^{-1}+2}{f(q)+f(q)^{-1}-2}\,\widetilde{x}_1(f(q),q), \\ \widetilde{y}(f(q),q) &= \frac{4(f(q)+1)}{(f(q)-1)(f(q)+f(q)^{-1}-2)}\,\widetilde{y}_1(f(q),q). \end{aligned} \tag{5.183}$$

Of course there is no problem substituting $f(q)$ for w into $x_1(w,q)$, $y_1(w,q)$, but the factor in front may cause trouble. We attach to the power series the point in projective space with coordinates

$$\big(f(q)+f(q)^{-1}+2)(f(q)-1)\widetilde{x}_1(f(q),q),\, 4(f(q)+1)\widetilde{y}_1(f(q),q),(f(q)+f(q)^{-1}+2)(f(q)-1)\big)\,.$$

To get a point in the projective space we must be able to find r_0,r_1,r_2 which combine the entries in the coordinate vector to one, and it is an amusing exercise to verify that this is the case under our assumptions.

We will come back to this in the second volume.

5.3 Towards the Algebraic Theory

5.3.1 Introduction

During our discussion of the Jacobian J of a Riemann surface S and the description of the Picard group of J we made use of transcendental methods. We worked in the category of (compact) complex manifolds. Especially we described the complex manifold J as quotient $J = \mathbb{C}^g/\Gamma$ where then $\Gamma = H_1(J,\ )$.
On the other hand we have seen that the Riemann surface S can be viewed as the set of points of a non singular projective curve over the complex numbers. We also have stated the result that the Jacobian J of S is a projective algebraic variety. We also know that J has the structure of an algebraic group (see 5.2.7). We have a holomorphic line bundle $\mathcal{P}$, which has the property that $T_x(\mathcal{P}) \otimes \mathcal{P}^{-1}$ is not trivial unless x is the neutral element and $\mathcal{P}^{\otimes 3}$ provides a projective embedding. In this section we will aim at an algebraic formulation of our central results, starting from these algebraic data. We still use the transcendental arguments in the proofs.

To illustrate what I mean I consider an endomorphism $\phi : J \longrightarrow J$. If we look at it in the trancendental context, we know that it is an endomorphism $\phi : \Gamma \to \Gamma$. Hence we can define the trace $\mathrm{tr}(\phi)$, our lattice is free of rank $2g$. But we will show that this trace can also be expressed in terms of intersection numbers of certain line bundles obtained from $\mathcal{P}$ and ϕ, this is then an algebraic definition of $\mathrm{tr}(\phi)$.
It is the content of Chapter 10 in the second volume that the main results of the present chapter here can be formulated and proved in purely algebraic terms. This implies that we can replace the ground field $\mathbb{C}$ by an arbitrary field k.
Our starting objects in this section will be a compact Riemann surface S, its Jacobian J and its dual $J^\vee$. On J we have the canonical polarization e_0 given by the Riemann period relations, it defines a line bundle $\mathcal{P}$ whose class in the Neron-Severi group is e_0. It also yields the isomorphism $j_{e_0} : J \xrightarrow{\sim} J^\vee$.
The key to an algebraic approach to understand J and the Riemann surface S itself is the investigation of the Picard group of varieties of the form $S \times S$, $S \times J$, $J \times J$ and $J \times J^\vee$.
Let X be any smooth, projective, connected variety over $\mathbb{C}$ (see Example 15 a)). We use the above mentioned principles from GAGA. Then we have

$$0 \longrightarrow H^1(X,\ \ ) \longrightarrow H^1(X,\mathcal{O}_X) \longrightarrow H^1(X,\mathcal{O}_X^*) \longrightarrow H^2(X,\ \ ) \longrightarrow \dots \tag{5.184}$$

and from here

$$0 \longrightarrow \mathrm{Pic}^0(X) \longrightarrow \mathrm{Pic}(X) \xrightarrow{\delta} H^2(X,\ \ ). \tag{5.185}$$

Of course this sequence makes only sense in the analytic context. The class $\delta(\mathcal{L})$ is the Chern class of $\mathcal{L}$ and the subgroup generated by the Chern classes is called the **Neron-Severi group** $\mathrm{NS}(X)$.

The Algebraic Definition of the Neron-Severi Group

If X is any smooth projective algebraic variety then the group $H^2(X,\ \ )$ is of course a transcendental object, it needs the concept of continuity in its definition. But if we believe in GAGA then the group $\mathrm{Pic}(X)$ is defined in the context of algebraic geometry. We also can give an algebraic definition of subgroups which are close to $\mathrm{Pic}^0(X)$.
For instance we can define the subgroup $\mathrm{Pic}^{0,0}(X)$ of those line bundles which are algebraically equivalent to zero:

Definition 5.3.1. *We say that a line bundle $\mathcal{L}$ on X is* **algebraically equivalent to zero** *if we can find a connected projective algebraic variety T over $\mathbb{C}$ and a line bundle $\widetilde{\mathcal{L}}$ on $X \times T$ such that there are two points t_1, t_0 on T for which $\mathcal{L}_{t_1} = \widetilde{\mathcal{L}}|X \times t_1 \xrightarrow{\sim} \mathcal{L}$ and $\mathcal{L}_{t_0} = \widetilde{\mathcal{L}}|X \times t_0 \xrightarrow{\sim} \mathcal{O}_X$.*

Naively speaking this means that we can deform our bundle into the trivial bundle. It is of course clear that during such a deformation process the Chern classes of the bundles stay constant. This means that the group $\mathrm{Pic}^{0,0}(X)$ of lines bundles algebraically equivalent to zero is always contained in $\mathrm{Pic}^0(X)$.
If we divide $\mathrm{Pic}(X)$ by this subgroup we get a modified **Neron-Severi group**

$$\mathrm{NS}_{\mathrm{alg}}(X) = \mathrm{Pic}(X)/\,\mathrm{Pic}^{0,0}(X) \tag{5.186}$$

which is defined in the context of algebraic geometry.
Our results (for instance in section 5.2.1) imply that for abelian varieties A over $\mathbb{C}$ we have in fact $\mathrm{Pic}^0(A) = \mathrm{Pic}^{0,0}(A)$ and thus we have an algebraic definition of $\mathrm{NS}(A)$ for abelian varieties.
Hence we see that the Neron-Severi group $\mathrm{NS}(X) \subset H^2(X,\ \ )$, which is only defined in the analytic context, is a quotient of $\mathrm{NS}_{\mathrm{alg}}(X)$ which can be defined in the context of algebraic geometry. If in the following sections we formulate a result, then we say that we have an algebraic result, if we can state in terms of elements of $\mathrm{NS}_{\mathrm{alg}}(X)$. For a first example see our construction in 5.2.2. This does not mean that the proof is purely algebraic.

The Algebraic Definition of the Intersection Numbers

At this point I want to outline how we can define in purely algebraic terms the intersection numbers of line bundles on a smooth connected projective variety $X \subset \mathbb{P}^n(\mathbb{C})$. We put $d = \dim X$.
Let $\mathcal{L}_1, \ldots, \mathcal{L}_d$ be line bundles on X, let $c_1(\mathcal{L}_1), \cdots, c_1(\mathcal{L}_d)$ be their Chern classes. We can form the cup product $c_1(\mathcal{L}_1) \cup \cdots \cup c_d(\mathcal{L}_d) \in H^{2d}(X,\ \ ) = \ \ $, and the result is a number. We have already seen that under certain favorable circumstances we can interpret this number as the number of points in the intersection of d smooth divisors (see Proposition 4.10.14)

$$c_1(\mathcal{L}_1) \cup \cdots \cup c_1(\mathcal{L}_d) = Y_1 \cap \cdots \cap Y_d. \tag{5.187}$$

I want to explain that it is always possible to interpret this cup product of Chern classes as intersection numbers. I have to appeal to some theorems in projective algebraic geometry (Theorem of Bertini) which will be discussed in more detail in the second volume.
Our projective space $\mathbb{P}^n(\mathbb{C})$ has the line bundle $\mathcal{O}_{\ \ n(\ \ )}(1) = \mathcal{H}$ on it (see page 163). We will show that for any bundle $\mathcal{L}$ on X we can find an integer $k > 0$ and a non zero section $s \in H^0(X, \mathcal{L} \otimes \mathcal{H}^{\otimes k})$ such that $[s = 0]$ is a smooth divisor (see section 4.10.3). We take our bundle $\mathcal{L}_1$ and choose sections $s_1 \in H^0(X, \mathcal{L}_1 \otimes \mathcal{H}^{\otimes k})$ and $t_1 \in H^0(X, \mathcal{H}^{\otimes k})$ which both provide a smooth divisor on X.
If we look at the cup product of the Chern classes, we find the equality

$$c_1(\mathcal{L}_1) \cup \ldots \cup c_1(\mathcal{L}_d) = c_1(\mathcal{L}_1 \otimes \mathcal{H}^{\otimes k}) \cup c_1(\mathcal{L}_2) \cup \cdots \cup c_1(\mathcal{L}_d) - c_1(\mathcal{H}^{\otimes k}) \cup c_1(\mathcal{L}_2) \cup \cdots \cup c_1(\mathcal{L}_d).$$

Now the two divisors $[s_1 = 0]$ and $[t_1 = 0]$ are again smooth projective varieties. They are perhaps not connected but their connected components $Z_1, \ldots, Z_\nu, \ldots$ are also smooth projective varieties by the Theorem of Chow. We can restrict the remaining line bundles $\mathcal{L}_2, \ldots, \mathcal{L}_d$ to these components.
Now we assume by induction that we have an algebraic definition of the intersection number of $\mathcal{L}'_2, \ldots, \mathcal{L}'_d$ of $d-1$ line bundles on smooth projective varieties of dimension $d-1$. Then the above argument gives us an algebraic definition for the intersection number of d line bundles on X.
Here we have to observe that in view of our result in Proposition 4.10.14 we know that this definition does not depend on the choices of n and of the sections s_i and t_1 because the intersection numbers are also given by the cup product. But in the context of algebraic geometry, when cohomology groups are not available, then we have to work a little bit more to show this independence. In other words: We propose a definition of

an intersection product $\mathrm{NS}_{\mathrm{alg}}(X)^d \longrightarrow$, but to see, that it is well defined we need topology.

Proposition 5.3.2. *If we have d line bundles $\mathcal{L}_1,\dots,\mathcal{L}_d$ on a smooth projective algebraic variety X of dimension d. Assume that $\mathcal{L}_1$ is not the trivial bundle, that $H^0(X,\mathcal{L}_1) \neq 0$ and that $\mathcal{L}_2,\dots,\mathcal{L}_d$ are ample. Then we know that their intersection product*

$$c_1(\mathcal{L}_1) \cup \cdots \cup c_1(\mathcal{L}_d) > 0.$$

This is clear if $d = 1$, because a non zero section $s \in H^0(S,\mathcal{L}_1)$ must have a zero, because $\mathcal{L}_1$ is not trivial. The rest follows by induction. We can choose an integer $k > 0$ such that $\mathcal{L}_2^{\otimes k}$ has a section t such that $t = 0$ defines a smooth divisor Y on X. We can choose t such that $[s = 0] \not\subseteq [t = 0] = Y$. (This is again the Theorem of Bertini.) Now we need that the restriction of $\mathcal{L}_1$ to Y is again not trivial, then we can apply induction. We have to show that $[s = 0] \cap [t = 0] \neq \emptyset$. Assume this is not the case. We know by definition that $[s = 0]$ is an algebraic subset. Any section $t_1 \in H^0(X,\mathcal{L}_2^{\otimes k})$ defines a holomorphic function t_1/t on $[s = 0]$. Since we can assume that $\mathcal{L}_2^{\otimes k}$ provides an embedding we can achieve that $t_1(x) = 0$, $t_1(y) \neq 0$ where x,y are two arbitrarily given points on $[s = 0]$. On the other hand we know that $[s = 0]$ is compact, hence any such function restricted to $[s = 0]$ has a maximum for its absolute value on $[s = 0]$. Here we encounter a little difficulty. Since we can not assume that $[s = 0]$ is smooth, we can not apply the maximum principle from the theory of functions. But in fact it can be shown that it also holds for arbitrary algebraic subsets $Z \subset X$: A bounded holomorphic function on Z is constant on the connected components of Z. If we accept this fact, then we have proved the above proposition.

The Study of some Special Neron-Severi groups

If X is equal to one of our four varieties $S \times S$, $S \times J$, $J \times J$ and $J \times J^\vee$ and if we write it as a product $X = Y \times Z$. We want to study the Picard group $\mathrm{Pic}(Y \times Z)$, its subgroup $\mathrm{Pic}^0(Y \times Z)$ and its Neron Severi quotient $\mathrm{NS}(Y \times Z) = \mathrm{Pic}(Y \times Z)/\mathrm{Pic}^0(Y \times Z)$. We apply the considerations from 4.6.7 to $X = Y \times Z$. Clearly we have a morphism $\mathcal{O}_Y \widehat{\otimes} \mathcal{O}_Z \longrightarrow \mathcal{O}_{Y\times Z}$. Moreover our sheaves are a of -vector spaces so the injective resolutions are also flat. Hence we get a homomorphism

$$\begin{aligned} H^1(Y,\mathcal{O}_Y) \otimes H^0(Z,\mathcal{O}_Z) \oplus H^0(Y,\mathcal{O}_Y) \otimes H^1(Z,\mathcal{O}_Z) = \\ H^1(Y,\mathcal{O}_Y) \oplus H^1(Z,\mathcal{O}_Z) \longrightarrow H^1(Y \times Z,\mathcal{O}_{Y\times Z}) \end{aligned} \tag{5.188}$$

and this homomorphism is in fact an isomorphism. This implies

$$\mathrm{Pic}^0(X) = \mathrm{Pic}^0(Y \times Z) \xrightarrow{\sim} \mathrm{Pic}^0(Y) \oplus \mathrm{Pic}^0(Z). \tag{5.189}$$

We say that Pic^0 *is linear.*

We will not use that 5.188 is an isomorphism and we will not prove it. But I want to make a few comments. It is easy to see that is injective: We simply choose points $y_0 \in Y, z_0 \in Z$ and restrict the classes $H^1(Y \times Z, \mathcal{O}_{Y\times Z})$ to $H^1(\{y_0\} \times Z, \mathcal{O}_Z)$ and $H^1(Y \times \{z_0\}, \mathcal{O}_Y)$. The composition of arrows $H^1(Y,\mathcal{O}_Y) \oplus H^1(Z,\mathcal{O}_Z) \longrightarrow H^1(Y \times Z, \mathcal{O}_{Y\times Z}) \longrightarrow H^1(\{y_0\} \times Z, \mathcal{O}_Z) \oplus H^1(Y \times \{z_0\}, \mathcal{O}_Y)$ is an isomorphism. This shows the injectivity. The surjectivity is more difficult. It becomes easy if we accept the following result, which seems to be very plausible.

Let p_2 be the projection from $X = Y \times Z \longrightarrow Z$. Then $R^q p_{2,*}(\mathcal{O}_{Y\times Z})$ is a free coherent sheaf on Z and for any point $z \in Z$ we have

$$R^q p_{2,*}(\mathcal{O}_{Y\times Z}) \otimes (\mathcal{O}_{Z,z}/\mathfrak{m}_z) = R^q p_{2,*}(\mathcal{O}_{Y\times Z}) \otimes \mathbb{C} \xrightarrow{\sim} H^q(p_2^{-1}(z), \mathcal{O}_{Y\times\{z\}}) \xrightarrow{\sim} H^q(Y, \mathcal{O}_Y).$$

This result looks rather innocent and has the flavour of a base change theorem 4.4.17. It again related to the deep finiteness results in complex analytic geometry (See the discussion in 5.2.1). It is a consequence of the so called semi continuity theorem. These results will be proved in volume II in the context of algebraic geometry, they are much more difficult to prove in complex analytic geometry.

If we accept this fact then we apply the spectral sequence and get for the E_2 term See 4.6.3, d))

$$(H^p(Z, R^q f_*(\mathcal{O}_{Y\times Z})), d_2) \Rightarrow H^n(Y \times Z, \mathcal{O}_{Y\times Z})$$

If $n = 1$ then we get two steps in the filtration, namely $H^1(Y, p_{2,*}(\mathcal{O}_{Y\times Z}) = H^1(Y, \mathcal{O}_Y)$ and $H^0(Z, R^1 p_{2,*}(\mathcal{O}_{Y\times Z})$. It follows that the dimension $H^1(Y \times Z, \mathcal{O}_{Y\times Z})$ is less or equal to sum of the dimensions of $H^1(Y, \mathcal{O}_Y)$ and $H^1(Z, \mathcal{O}_Z)$ and this combined with the injectivity proves the assertion.

We may also derive the isomorphism 5.188 from the results in 5.2.1 if one of the factors is an abelian variety. There we gave a hint how such a semicontinuity can be proved under certain assumptions.

The isomorphism 5.189 is called the theorem of the square and will be proved in the context of algebraic geometry in volume II.

We are more interested in the Neron Severi group. We recall the notation $\Gamma \simeq H^1(S, \ )$ and then

$$\begin{aligned} H^2(J, \ ) &= \mathrm{Hom}(\Lambda^2\Gamma, \ ) \\ H^2(S \times S, \ ) &= H^2(S, \ ) \oplus \left(H^1(S, \ ) \otimes H^1(S, \ )\right) \oplus H^2(S, \ ) \\ &= \quad \oplus (\Gamma \otimes \Gamma) \oplus \quad . \\ H^2(S \times J, \ ) &= H^2(S, \ ) \oplus \left(H^1(S, \ ) \otimes H^1(J, \ )\right) \oplus H^2(J, \ ) \\ &= \quad \oplus (\Gamma \otimes \Gamma^\vee) \oplus \mathrm{Hom}(\Lambda^2\Gamma, \ ). \end{aligned}$$

We have to find out what the Neron-Severi group will be in our four cases. I claim that we have a submodule $\mathrm{NS}'(Y \times Z) \subset H^1(Y, \ ) \otimes H^1(Z, \ )$ such that we get a direct sum decomposition into three summands

$$\mathrm{NS}(Y \times Z) = \mathrm{NS}(Y) \oplus \mathrm{NS}'(Y \times Z) \oplus \mathrm{NS}(Z).$$

To see this we observe that we have pullbacks $p_1^*(\mathcal{L}_1)$, $p_2^*(\mathcal{L}_2)$ of line bundles on the two factors, which have Chern classes $(c_1,0,0)$, $(0,0,c_2)$ with respect to the above decomposition. On the other hand we can choose points $y_0 \in Y$ and $z_0 \in Z$ and restrict a bundle $\mathcal{L}$ on $Y \times Z$ to $y_0 \times Z$, $Y \times z_0$. The Chern classes of these restrictions do not depend on the selected points (because Y, Z are connected) and if we modify $\mathcal{L}$ by the product of the inverses of the pullbacks we get a bundle whose Chern class is $(0,c_2,0)$.

The first and the third summand are considered as less interesting at this point since they are filled up by the Chern classes of line bundles which are pull backs from the two factors. We are interested in the summand in the middle.
We have the morphisms

$$S \times S \xrightarrow{\mathrm{Id} \times i_{P_0}} S \times J \xrightarrow{i_{P_0} \times \mathrm{Id}} J \times J \xrightarrow{\mathrm{Id} \times j_{e_0}} J \times J^\vee.$$

This induces a sequence of isomorphisms between the $H^1 \otimes H^1$ component of the second cohomology groups

$$\Gamma \otimes \Gamma \longleftarrow \Gamma \otimes \Gamma^\vee \longleftarrow \Gamma^\vee \otimes \Gamma^\vee \longleftarrow \Gamma^\vee \otimes \Gamma^\vee$$

where the isomorphism is always the tensor product of the identity and the Poincaré duality. It is clear that this sequence of isomorphisms also induces homomorphisms between the corresponding subgroups $\mathrm{NS}'(Y \times Z)$ and we have:

Proposition 5.3.3. *With the obvious notation we get a sequence of isomorphisms*

$$\mathrm{NS}'(S \times S) \longleftarrow \mathrm{NS}'(S \times J) \longleftarrow \mathrm{NS}'(J \times J) \longleftarrow \mathrm{NS}'(J \times J^\vee).$$

Proof: To see that this is indeed the case we recall that the Neron-Severi group is always the kernel of

$$H^2(X,\ \ ) \longrightarrow H^2(X,\mathcal{O}_X).$$

In our situation we have to apply the Künneth formula and look at the kernel of

$$H^1(Y,\ \ ) \oplus H^1(Z,\ \ ) \longrightarrow H^1(Y,\mathcal{O}_Y) \oplus H^1(Z,\mathcal{O}_Z)$$

and then the claim follows because the maps

$$H^1(J^\vee,\mathcal{O}_{J^\vee}) \xrightarrow{\sim} H^1(J,\mathcal{O}_J) \longrightarrow H^1(S,\mathcal{O}_S)$$

are isomorphisms. □

There is a slightly different way of looking at this proposition. We have seen that we have to study the alternating 2-forms on $\Gamma \oplus \Gamma$, $\Gamma \oplus \Gamma^\vee$, $\Gamma^\vee \oplus \Gamma^\vee$. If our two summands are Γ_1,Γ_2 then we denote the space of those alternating 2-forms which are trivial on $\Gamma_1 \times \Gamma_1,\Gamma_2 \times \Gamma_2$ by $\mathrm{Alt}_2'(\Gamma_1 \oplus \Gamma_2,\ \ )$. We get an isomorphism

$$\Gamma_1^\vee \otimes \Gamma_2^\vee \xrightarrow{\sim} \mathrm{Alt}_2'(\Gamma_1 \oplus \Gamma_2,\ \ )$$

which sends an element $\psi_1 \otimes \psi_2 = \Psi$ to the alternating form

$$e_\Psi : ((\gamma_1,\gamma_2),(\gamma_1',\gamma_2')) \longmapsto \psi_1(\gamma_1)\psi_2(\gamma_2') - \psi_1(\gamma_1')\psi_2(\gamma_2).$$

To get the Neron-Severi group we have to look at those alternating forms which after tensorization by   are compatible with the complex structure. Then we have to translate it back into a condition for $\Psi \in \Gamma_1^\vee \otimes \Gamma_2^\vee$.
If for instance $\Gamma_1 = \Gamma^\vee$ and $\Gamma_2 = \Gamma$ then the we get $\Gamma \otimes \Gamma^\vee = \mathrm{End}(\Gamma)$. Then it is obvious that the alternating form Ψ is compatible with the complex structure on $(\Gamma \oplus \Gamma^\vee)_\mathbb{R}$ if and only if the corresponding element in $\mathrm{End}(\Gamma)$ is compatible with the complex structure on $\Gamma_\mathbb{R}$. Hence we have shown

$$\mathrm{NS}'(J \times J^\vee) \xrightarrow{\sim} \mathrm{End}(J). \tag{5.190}$$

To get this identification we did not use the polarization.
Now we consider the case $J \times J$. In this case we have to look at $\Gamma \otimes \Gamma$ and this is the module of bilinear forms on $\Gamma^\vee$ and via Poincaré duality this is the same as the module of bilinear forms on Γ. Following the identifications we see that an element $\gamma_1 \otimes \gamma_2 \in \Gamma \otimes \Gamma$ gives us the bilinear form $(\eta_1,\eta_2) \mapsto e_0 < \gamma_1,\eta_1 > e_0 < \gamma_2,\eta_2 >$.
Now it is an easy exercise that under the identification

$$\Gamma \otimes \Gamma \xrightarrow{\sim} \Gamma \otimes \Gamma^\vee \xrightarrow{\sim} \mathrm{End}(\Gamma)$$

the element $\mathrm{Id} \in \mathrm{End}(\Gamma)$ corresponds to the polarization form $e_0 \in \Gamma \otimes \Gamma \xrightarrow{\sim} \Gamma^\vee \otimes \Gamma^\vee$.
This element e_0 therefore defines an element E_0 in $\mathrm{Alt}_2'(\Gamma \oplus \Gamma,\ )$ which is given by

$$E_0\langle(\gamma_1,\gamma_2),(\gamma_1',\gamma_2')\rangle = e_0\langle\gamma_1,\gamma_2'\rangle - e_0\langle\gamma_2,\gamma_1'\rangle. \tag{5.191}$$

This alternating form is the Chern class of the line bundle $\mathcal{L}(E_0,0,0) = \mathcal{N}$ on $J \times J$.

More generally it is now obvious that under the identification

$$\Gamma \otimes \Gamma \xrightarrow{\sim} \Gamma \otimes \Gamma^\vee \xrightarrow{\sim} \mathrm{End}(\Gamma)$$

an element $\varphi \in \mathrm{End}(\Gamma)$ corresponds to the bilinear form E_φ given by

$$E_\varphi < \gamma_1,\gamma_2 > = E_0 < \gamma_1,\varphi(\gamma_2) > . \tag{5.192}$$

We can summarize this discussion and say

Theorem 5.3.4. *We have a canonical identification*

$$\mathrm{NS}'(J \times J) \xrightarrow{\sim} \mathrm{End}(J)$$

which is given by the map which sends an element $\psi \in \mathrm{End}(J)$ to the class of the line bundle $(\mathrm{Id} \times \psi)^*(\mathcal{N})$.

This should be seen in conjunction with our earlier result

$$\mathrm{NS}(J) \xrightarrow{\sim} \mathrm{End}_{\mathrm{sym}}(J).$$

The previous theorem is already close to an algebraic statement. But the bundle $\mathcal{N}$ has been constructed in the transcendental context, we described it in terms of the 2-cocycle obtained from the alternating form E_0. I want to point out that we have a construction of our line bundle $\mathcal{N}$ on $J \times J$ in algebraic terms using only the bundle $\mathcal{P}$: We consider the product and three maps

$$J \times J \begin{array}{c} \xrightarrow{p_1} \\ \xrightarrow{m} \\ \xrightarrow{p_2} \end{array} J$$

and we put

$$\mathcal{N} = m^*(\mathcal{P}) \otimes p_1^*(\mathcal{P})^{-1} \otimes p_2^*(\mathcal{P})^{-1}. \tag{5.193}$$

It is quite clear that this bundle has a Chern class, the class E_0. But it is also clear that this bundle does not depend on the choice of $\mathcal{P}$: If we modify $\mathcal{P}$ by a line bundle $\mathcal{L}$ which has Chern class zero then this amounts to changing the linear form in the construction. But this change cancels in the construction of $\mathcal{N}$, this means that $\mathcal{N}$ is a canonical bundle on $J \times J$.

We have seen the construction of a similar bundle – which also was called $\mathcal{N}$ – in section 5.2.1. This was called the Poincaré bundle and it lives on $J \times J^\vee$. This bundle can be constructed for any complex torus A and is an analytic object. Now the polarization bundle $\mathcal{P}$ provides the isomorphism

$$\varphi_{\mathcal{P}} : J \longrightarrow J^\vee = \mathrm{Pic}^0(J) \tag{5.194}$$

which is given by

$$x \longmapsto T_x(\mathcal{P}) \otimes \mathcal{P}^{-1}.$$

We get an isomorphism

$$\mathrm{Id} \times \varphi_{\mathcal{P}} : J \times J \xrightarrow{\sim} J \times J^\vee$$

and we can take the pullback of the Poincaré bundle by this map. Of course we then get the above bundle $\mathcal{N}$ on $J \times J$. Since we think of J as the Jacobian of a curve and therefore the polarization is canonical, we allow ourselves to give the two bundles on $J \times J$ and $J \times J^\vee$ the same name.

5.3.2 The Structure of End(J)

The Rosati Involution

Since we have an inclusion $\mathrm{End}(J) \subset \mathrm{Hom}(\Gamma,\Gamma)$, we know that $\mathrm{End}(J)$ is a finitely generated algebra over .

For any $\varphi \in \mathrm{End}(J)$ there is an endomorphism

$$\varphi^* : \mathrm{Pic}(J) \longrightarrow \mathrm{Pic}(J)$$

of the Picard group which is given by the pull back of line bundles. We denote by ${}^t\varphi$ the restriction of φ^* to the subgroup $\mathrm{Pic}^0(J)$. We use the canonical polarization of the Jacobian and get the transposed

$${}^t\varphi : J \longrightarrow J.$$

I want to point out that these assertions make sense in the context of algebraic geometry. We have seen that the group $\mathrm{NS}'(J \times J)$ has an algebraic definition and this is also the case for $\mathrm{End}(J)$. We have seen that ${}^t\varphi$ corresponds to the transpose of $\varphi : \Gamma \longrightarrow \Gamma$ with respect to e_0.

Definition 5.3.5. *The map* $\varphi \longrightarrow {}^t\varphi$ *is called the* **Rosati involution** *(with respect to the standard polarization).*

Proposition 5.3.6. *The Rosati involution has the properties*

$$\begin{aligned} {}^t(\varphi + \psi) &= {}^t\varphi + {}^t\psi \\ {}^t(\varphi\psi) &= {}^t\psi {}^t\varphi. \end{aligned}$$

It is of course clear that φ^* also induces an endomorphism

$$\overline{\varphi}^* : \mathrm{NS}(J) \longrightarrow \mathrm{NS}(J).$$

We use the identification (equation (5.99)) combined with the selfduality and get

$$\mathrm{NS}(J) \simeq \mathrm{End}_{\mathrm{sym}}(J), \tag{5.195}$$

it is clear that the selfduality turns alternating homomorphisms from J to $J^\vee$ into symmetric endomorphisms of J. We saw in equation (5.91) that

$$\overline{\varphi}^* : \psi \longrightarrow {}^t\varphi\psi\varphi.$$

We know that $\varphi \longrightarrow \overline{\varphi}^*$ is quadratic, this means that we can consider $\varphi_1 + \varphi_2$ and then

$$(\overline{\varphi_1 + \varphi_2})^* = \overline{\varphi}_1^* + \overline{\varphi}_2^* + \langle \varphi_1, \varphi_2 \rangle$$

where $\langle \varphi_1, \varphi_2 \rangle : \mathrm{NS}(J) \longrightarrow \mathrm{NS}(J)$ depends bilinearily on the two variables.

To any $\varphi \in \mathrm{End}(J)$ we can define $tr(\varphi)$ and $\deg(\varphi)$ simply as the trace and the determinant of φ considered as endomorphism of Γ, i. e.

$$\begin{aligned} \mathrm{tr}(\varphi) &= \mathrm{tr}(\varphi : \Gamma \longrightarrow \Gamma) \\ \deg(\varphi) &= \det(\varphi : \Gamma \longrightarrow \Gamma). \end{aligned} \tag{5.196}$$

These functions have the obvious properties

$$\begin{aligned} \mathrm{tr}({}^t\varphi) &= \mathrm{tr}(\varphi) \\ \det({}^t\varphi) &= \det(\varphi) \\ \det(\varphi_1\varphi_2) &= \det(\varphi_1)\det(\varphi_2). \end{aligned} \tag{5.197}$$

We have the following fundamental result:

Theorem 5.3.7 (Positivity of the Rosati-Involution)**.** *For any* $\varphi \in \mathrm{End}(J)$, $\varphi \neq 0$ *we have*

$$\mathrm{tr}(\varphi {}^t\varphi) > 0.$$

Proof: At first we give a transcendental proof which uses the lattice Γ. We consider $\Gamma_{\mathbb{R}}$ as a $\mathbb{C}$-vector space and choose an orthonormal basis (with respect to the Hermitian form) $\{e_1,e_2,\cdots,e_g\}$. Then we put $f_i = Ie_i$ and $\{e_1,f_1,\ldots,e_g,f_g\}$ is an basis for $\Gamma_{\mathbb{R}}$. Our alternating form will have the values $\langle e_i,f_i\rangle = -1 = -\langle f_i,e_i\rangle$ and all other values are zero. Then it is clear that for any endomorphism ψ we have

$$\mathrm{tr}(\psi) = -\sum\langle\psi(e_i),f_i\rangle + \sum\langle\psi(f_i),e_i\rangle.$$

If $\psi = {}^t\varphi\varphi$ then we get

$$\begin{aligned}\mathrm{tr}({}^t\varphi\varphi) &= -\sum_{i=1}^{g}\langle\varphi(e_i),\varphi(f_i)\rangle + \sum_{i=1}^{g}\langle\varphi(f_i),\varphi(e_i)\rangle \\ &= -2\sum_{i=1}^{g}\langle\varphi(e_i),\varphi(Ie_i)\rangle.\end{aligned}$$

Since φ commutes with I, the last sum is equal to

$$-2\sum\langle\varphi(e_i),I\varphi(e_i)\rangle = +\sum 2h_{\langle\ ,\ \rangle}(\varphi(e_i),\varphi(e_i)).$$

The terms are ≥ 0 and since at least one of the $\varphi(e_i) \neq 0$ ($\varphi(f_i) = \varphi(Ie_i) = \varphi(e_i)!$) we conclude that the sum must be strictly positive. □

A Trace Formula

Our definitions of the degree and of the trace are given in terms of the lattice Γ. Hence they are transcendental and the positivity of the Rosati involution does not make sense in algebraic geometry at this point.
Therefore we have to give a definition of the degree and the trace in algebraic terms. For the degree this is easy. We consider

$$\begin{array}{ccc} J & \xrightarrow{\varphi} & J \\ \uparrow & & \uparrow \\ H^1(S,\mathcal{O}_S)/\Gamma & \xrightarrow{\varphi} & H^1(S,\mathcal{O}_S)/\Gamma. \end{array} \tag{5.198}$$

and then we see easily:

Proposition 5.3.8. *The degree of φ is non zero if and only if the morphism φ is finite. If the degree of φ is non zero, then we have*

$$\deg(\varphi) = \textit{number of points in } \varphi^{-1}(0).$$

Proof: This is rather clear: The determinant of φ is equal to the index of $\varphi(\Gamma)$ in Γ. If $\Gamma' \subset H^1(\mathcal{O}_S,\ )$ is the inverse image of Γ, then $\Gamma'/\Gamma \simeq \Gamma/\varphi(\Gamma)$ and this proves that the order of the kernel $\varphi^{-1}(0)$ is also equal to $\det(\varphi)$. □

It is also clear that the derivative of φ induces an isomorphism of the tangent spaces at at the points in $\varphi^{-1}(0)$ and zero. We apply Lemma 4.8.12.
The morphism φ also induces an inclusion of the field of meromorphic functions on J into itself $\mathbb{C}(J) \hookrightarrow \mathbb{C}(J)$. The subfield is the field of invariants under Γ'/Γ. Since we also know that the meromorphic functions on J separate the points in $\varphi^{-1}(0)$ it follows from Galois theory that

$$\deg(\varphi) = \text{ degree of the extension } \mathbb{C}(J) \hookrightarrow \mathbb{C}(J). \tag{5.199}$$

From the definition of the degree as a determinant it follows that

$$\deg(\varphi + n\,\mathrm{Id}) = a_0(\varphi) + \cdots a_{2n-1}(\varphi)n^{2g-1} + n^{2g}, \tag{5.200}$$

and then by definition the **trace** is given by

$$\mathrm{tr}(\varphi) = a_{2n-1}(\varphi). \tag{5.201}$$

The point of this formula is that $\deg(\varphi + n\,\mathrm{Id})$ is a polynomial in n of degree $2g$ and highest coefficient $= 1$. We want to derive such an expression for the degree $\deg(\varphi + n\,\mathrm{Id})$ from its algebraic definition, namely as the number of points in a fibre.

The decisive point in the following considerations will be, that for an element $\psi \in \mathrm{End}(J)$ the element $\overline{\psi}^*(e_0) \in \mathrm{NS}(J)$ contains relevant information on the endomorphism ψ.

In a first step we will show that we can express the degree of the endomorphism ψ in terms of this class. The g-th power with respect to the cup product is an element in $H^{2g}(J,\ ) =\ $. Hence it is a number. But from the point of view of algebraic geometry we think of $\overline{\psi}^*(e_0)$ as an element in $\mathrm{NS}(J)$, which can be represented by a line bundle. The g-fold selfintersection of this line bundle in the context of algebraic geometry (see pages 273 f.) is also a number. We explained in the sections 4.8.9 and 5.3.1 that these two numbers are the same.
The element e_0 is an alternating form on $\Gamma^\vee$. If $\dim J = g$, then we can raise this element into the gth power in the cohomology ring and we have seen in 4.6.8 that this means that we have to take its gth exterior power

$$e_0^g = e_0 \wedge e_0 \ldots \wedge e_0 \in \mathrm{Hom}^{2g}_{\mathrm{alt}}(\Gamma^\vee,\ ) \simeq\ . \tag{5.202}$$

This is the selfintersection number of the class e_0. Further down we will compute it, but here we do not need it. Actually in the following consideration we can replace e_0 by any polarization. Its image under $\overline{\psi}^*$ is given by

$$\overline{\psi}^*(e_0)(\gamma_1,\gamma_2) = e_0(\psi(\gamma_1),\psi(\gamma_2)), \tag{5.203}$$

and it is an elementary exercise in linear algebra that we have

$$\overline{\psi}^*(e_0)^g = \deg(\psi) \cdot e_0^g, \tag{5.204}$$

where $\det(\psi)$ is of course the determinant of the endomorphism ψ on the free module Γ which is of rank $2g$. Since $e_0^g > 0$ we found an algebraic formula for the degree of ψ. We apply this to the endomorphism $\psi + n\,\mathrm{Id}$.
We consider the map

$$\overline{(\psi + n\,\mathrm{Id})}^* : \mathrm{NS}(J) \longrightarrow \mathrm{NS}(J).$$

If we invoke the identification $\mathrm{NS}(J) \simeq \mathrm{End}_{\mathrm{sym}}(J)$ then for $\phi \in \mathrm{End}_{\mathrm{sym}}(J)$ we have $(\psi + n\,\mathrm{Id})^*(\phi) = ({}^t\psi + n\,\mathrm{Id})\phi(\psi + n\,\mathrm{Id})$ and hence this map is

$$\overline{(\psi + n\,\mathrm{Id})}^* = \overline{\psi}^* + n\langle\psi, \mathrm{Id}\rangle + n^2 \cdot \mathrm{Id}\,. \tag{5.205}$$

From this we get the formula for the degree

$$\begin{aligned}\deg(\psi + n\,\mathrm{Id}) \cdot e_0^g &= ((\psi + n\,\mathrm{Id})(e_0))^g \\ &= (\psi^*(e_0) + n\langle\psi, \mathrm{Id}\rangle(e_0) + n^2 \cdot e_0)^g \\ &= \ldots + gn^{2g-1}e_0^{g-1} \wedge \langle\psi, \mathrm{Id}\rangle(e_0) + n^{2g} \cdot e_0^g\end{aligned} \tag{5.206}$$

and hence we get the formula

$$\mathrm{tr}(\psi) \cdot e_0^g = ge_0^{g-1} \wedge \langle\psi, \mathrm{Id}\rangle(e_0), \tag{5.207}$$

and this gives us the trace as a cup product of classes in the cohomology evaluated on the fundamental cocycle. Hence we found a formula for the trace in algebraic terms, since we can represent the Chern classes by bundles and then we interpret the cup product in terms of intersection numbers.
Let us assume that the endomorphism ψ is a product of the form $\psi = {}^t\varphi\varphi$. Then

$$\begin{aligned}e_0\langle(\psi + \mathrm{Id})\gamma_1, (\psi + Id)\gamma_2\rangle &= (\psi + \mathrm{Id})^* e_0\langle\gamma_1, \gamma_2\rangle \\ &= \psi^* e_0\langle\gamma_1, \gamma_2\rangle + e_0\langle\gamma_1, \gamma_2\rangle + e_0\langle\gamma_1, \psi\gamma_2\rangle + e_0\langle\psi\gamma_1, \gamma_2\rangle\end{aligned}$$

and the sum of the last two terms is $\langle\psi, \mathrm{Id}\rangle(e_0)\langle\gamma_1, \gamma_2\rangle$. Hence we get

$$\langle\psi, \mathrm{Id}\rangle(e_0)\langle\gamma_1, \gamma_2\rangle = 2e_0\langle\varphi\gamma_1, \varphi\gamma_2\rangle,$$

and this means that

$$\langle\psi, \mathrm{Id}\rangle(e_0) = \varphi^*(e_0).$$

This gives us the formula

$$\mathrm{tr}({}^t\varphi\varphi) \cdot e_0^g = 2ge_0^{g-1} \wedge \varphi^*(e_0). \tag{5.208}$$

We return to the interpretation in terms of algebraic geometry. We know that e_0 is the class of an ample line bundle $\mathcal{P}$ and we have seen that the highest intersection numbers of line bundles are equal to the highest cup product of their Chern classes. Hence we can say that

$$\mathrm{tr}({}^t\varphi\varphi) \cdot \mathcal{P}^g = 2g\mathcal{P}^{g-1} \cdot \varphi^*(\mathcal{P}). \tag{5.209}$$

This formula gives us an algebraic approach to the positivity of the Rosati involution. I claim that the right hand side must be positive if $\varphi \neq 0$. We apply Proposition 5.3.2. We may replace $\mathcal{P}$ by a translate $T_x(\mathcal{P})$ because this does not change the Chern class and hence it does not change the value of the intersection product. We have a non zero section $s \in H^0(J,\mathcal{P})$, the set $[s = 0]$ is not empty. We may assume that $\varphi(J) \not\subset [s = 0]$ because we can modify $\mathcal{P}$ by a translation. Hence we see that $H^0(J,\varphi^*(\mathcal{P}))$ has a non trivial section. This section must have zeroes: As in the proof of Proposition 5.3.2 we show that $\varphi(J) \cap [s = 0] \neq \emptyset$. Since $\varphi \neq 0$, we can find points $x_1,x_2 \in J$, for which $\varphi(x_1) \neq \varphi(x_2)$. We can find sections $t \in H^0(J,\mathcal{P}^{\otimes k})$ for $k \gg 0$ and then $t/s^{\otimes k}$ defines nonconstant holomorphic functions on $\varphi(J)$, and hence on J. This is not possible because $\varphi(J)$ has strictly positive dimension.

Finally we want to give a formula for the trace of an endomorphism in terms of intersection numbers of two divisors on the surface $S \times S$. We return to our bundle

$$\mathcal{N} = m^*(\mathcal{P}) \otimes p_1^*(\mathcal{P})^{-1} \otimes p_1^*(\mathcal{P})^{-1}$$

on $J \times J$. It has Chern class zero when restricted to $e \times J$ and $J \times e$ and its Chern class is E_0. If we pick an element $\psi \in \mathrm{End}(J)$, then we can consider the bundle $(\mathrm{Id} \times \psi)^*(\mathcal{N})$ on $J \times J$. We have the inclusion $i_{P_0} \times i_{P_0} : S \times S \longrightarrow J \times J$ and get the line bundle

$$(i_{P_0} \times i_{P_0})^* \circ (\mathrm{Id} \times \psi)^*(\mathcal{N}) = \mathcal{L}_\psi \tag{5.210}$$

on $S \times S$. The Chern class of this line bundle sits in $\mathrm{NS}'(S \times S) \subset H^1(S,\ ) \otimes H^1(S,\ ) = \Gamma \otimes \Gamma$. Of course this homomorphism $\psi \mapsto \mathcal{L}_\psi$ realizes the isomorphism

$$\mathrm{NS}'(S \times S) \xrightarrow{\sim} \mathrm{End}(J)$$

which we gave in Proposition 5.3.3.

Now we can state the famous

Theorem 5.3.9 (Trace formula)**.**

$$\Delta \cdot \mathcal{L}_\psi = -\,\mathrm{tr}(\psi).$$

Proof: This is a rather formal consequence of the definitions. The following computations have been indicated in our discussion of the Lefschetz fixed point formula (see section 4.9). We have seen that the intersection product of two divisors is equal to the cup product of the Chern classes evaluated on the fundamental class. The cup product of the classes $\xi_1 \otimes \eta_1,\ \xi_2 \otimes \eta_2 \in \Gamma \otimes \Gamma \subset H^2(S_1 \times S_1,\ )$ is given by $-\xi_1\xi_2 \cup \eta_1\eta_2$ where now $\xi_1\xi_2 \in H^2(S,\ ) \otimes H^0(S,\ )$ and $\eta_1\eta_2 \in H^0(S,\ ) \otimes H^2(S,\ )$.
Since we have the alternating 2-form, we can choose as a standard basis on Γ a -basis $u_1,\dots,u_g,v_1,\dots,v_g$ such that

$$e_0\langle u_i,v_i\rangle = -e_0\langle v_i,u_i\rangle = 1$$

and all other products are zero. Under this identification the element

$$E = \sum_i u_i \otimes v_i - \sum_i v_i \otimes u_i \in \Gamma \otimes \Gamma$$

becomes the identity element in $\Gamma^\vee \otimes \Gamma = \mathrm{Hom}(\Gamma,\Gamma)$: The element E applied to an element $\gamma \in \Gamma$ yields

$$E(\gamma) := \sum_i \langle u_i,\gamma\rangle v_i - \sum_i \langle v_i,\gamma\rangle u_i.$$

Then it is clear that $E(u_i) = u_i$ and $E(v_i) = v_i$. Then the Chern class of $\mathcal{L}_\psi$ is given by

$$c_1(\mathcal{L}_\psi) = \sum_i u_i \otimes \psi(v_i) - \sum_i v_i \otimes \psi(u_i)$$

and

$$\begin{aligned} E \cup c_1(\mathcal{L}_\psi) &= \left(\sum_i u_i \otimes v_i - \sum_i v_i \otimes u_i\right) \cup \left(\sum_i u_i \otimes \psi(v_i) - \sum_i v_i \otimes \psi(u_i)\right) \\ &= \sum_i \langle u_i,v_i\rangle \cdot \langle v_i,\psi(u_i)\rangle + \sum_i \langle u_i,v_i\rangle \cdot \langle u_i,\psi(v_i)\rangle \\ &= \sum_i \langle v_i,\psi(u_i)\rangle - \sum_i \langle u_i,\psi(v_i)\rangle = -\operatorname{tr}(\psi). \end{aligned}$$

This ends the proof of the trace formula. □

The Fundamental Class $[S]$ of S under the Abel Map

Let us consider the Abel map $i_{p_0} : S \longrightarrow J$ which induces a map on the first cohomology

$$\begin{array}{ccc} H^1(J,\ ) & \xrightarrow{i^*_{P_0}} & H^1(S,\ ) \\ \| & & \| \\ \Gamma^\vee & \longrightarrow & \Gamma \end{array} \tag{5.211}$$

which we identified as the inverse of the polarization map. It induces a map

$$\begin{array}{ccc} H^2(J,\ ) & \longrightarrow & H^2(S,\ ) \\ \| & & \| \\ \Lambda^2\Gamma^\vee & \longrightarrow & \end{array} \tag{5.212}$$

and this map is by definition the evaluation by the dual form

$$\varphi \wedge \psi \longmapsto e^\vee(\varphi,\psi).$$

This linear form on $H^2(J,\ )$ is the fundamental class $[S]$ of the Riemann surface (See page 144) in $H_2(J,\ )$ or in $H^{2g-2}(J,\ )$. We want to give a formula for this class in terms of the intersection product of the polarization class.

If we choose a basis $e_1,\cdots,e_g,f_1\cdots f_g$ on Γ such that $\langle e_i,f_i\rangle = 1 = -\langle f_i,e_i\rangle$ and all other pairings give zero, then the $-f_1,\cdots,-f_g,e_1,\cdots,e_g$ are the elements of the dual basis if we identify Γ and $\Gamma^\vee$ by the polarization map. Then the form e_0 is given by $\sum_i e_i \wedge f_i$ again and our form in $H^{2g-2}(J,\ )$ which is the fundamental class of S is given by

$$\sum_i \quad e_1 \wedge f_i \cdots \widehat{e_i \wedge f_i} \wedge \cdots e_g \wedge f_g,$$

i.e. the factor $e_i \wedge f_i$ is left out.

The polarization class $e_0^\vee \in \Lambda^2\Gamma^\vee$ itself maps to g in $H^2(S,\ )$, and it is clear that

$$(e_0^\vee)^{g-1} = \left(\sum_i e_i \wedge f_i\right) \wedge \cdots \wedge \left(\sum_i e_i \wedge f_i\right) = (g-1)![S] \tag{5.213}$$

$$\text{and} \qquad (e_0^\vee)^g = g!. \tag{5.214}$$

Now recall the formula for the trace

$$\mathrm{tr}(\psi) = g\frac{\mathcal{P}^{g-1} \cup \langle \mathrm{Id}\,, \psi\rangle(\mathcal{P})}{\mathcal{P}^g}.$$

We have seen that $\mathcal{P}^{g-1} = (g-1)![S]$ and $\mathcal{P}^g = g!$, hence we get

$$\mathrm{tr}(\psi) = [S] \cup \langle \mathrm{Id}\,, \psi\rangle(\mathcal{P}) \tag{5.215}$$

5.3.3 The Ring of Correspondences

We have the isomorphism $\mathrm{Pic}(S \times S)/\left(p_1^*(\mathrm{Pic}(S)) + p_2^*(\mathrm{Pic}(S))\right) \simeq \mathrm{End}(J)$. We want to explain how we can define a ring structure on the left hand side directly.

If we have an irreducible divisor $D \subset S \times S$ we can look at it as a so called correspondence: To any point $z \in S$ we can consider the points $(z,z_i) \in D$ and call the points z_i counted with multiplicity as the points corresponding to z. We can form the free group of these divisors and mod out by the divisors of the form $S \times D'$ or $D'' \times S$ where D' (resp. D'') is a divisor in the first or second factor, let us call this $\mathcal{R}$. After we have done this, we can introduce a product on this group: We choose suitable representatives D_1, D_2 of two elements and consider the divisors in $D_1 \times S$, $S \times D_2$ on $S \times S \times S$. Now we take the intersection – this makes sense if we made a careful choice – and project this intersection to the two outer factors.

This induces a ring structure on $\mathcal{R}$ with identity which is given by the class of the diagonal. It is clear that this ring has an involution which is obtained by interchanging the two factors.
We can also define a trace: For any $[D] \in \mathcal{R}$ we choose a representative D for which $D \mid z_0 \times S$ and $D \mid S \times z_0$ are both in $\mathrm{Pic}^0(S)$. Then we put

$$-\mathrm{tr}([D]) = \Delta \cdot D. \tag{5.216}$$

Now it is clear that

$$-\mathrm{tr}({}^t[D] * [D]) = ({}^t[D] \times [D]) \cdot \Delta = D \cdot D. \tag{5.217}$$

We will show in the second volume that this last number is strictly negative if $D \neq 0$. This is of course the positivity of the Rosati involution.
We know that $\mathcal{R} \simeq \mathrm{End}(J)$, our considerations show that we can define this ring of correspondences without reference to the Jacobian.

5.3.4 An Algebraic Substitute for the Cohomology

I think that I convinced the reader that the cohomology groups $H^1(S,\ \ ) = \Gamma$, $H^1(J,\ \ ) = \Gamma^\vee$ play a fundamental role in understanding the structure of S and J. Therefore we should have a substitute for these cohomology groups in the algebraic context. This will be explained in volume 2. Here we give an indication how we can get an algebraic definition of cohomology groups, if we enlarge the coefficient ring to a larger ring.
We have

$$J = H^1(S,\mathcal{O}_S)/H^1(S,\ \ ) = H^1(S,\mathcal{O}_S)/\Gamma.$$

The module Γ does not make sense in the context of algebraic geometry. Now we consider the endomorphism

$$n\,\mathrm{Id} : J \longrightarrow J,$$

and we consider the kernel

$$J[n] = \ker(n\,\mathrm{Id} : J \longrightarrow J). \tag{5.218}$$

This kernel is obviously isomorphic to

$$\tfrac{1}{n}\,\Gamma/\Gamma \simeq (\ \ /n\ \ )^{2g}. \tag{5.219}$$

But this kernel has an algebraic definition. We consider J as a projective variety over $\mathbb{C}$ which has the structure of an abelian algebraic group and then the kernel of $n\,\mathrm{Id}$ is a finite algebraic group over $\mathbb{C}$.
Once we have done this, we observe that we have for $n \mid n_1$ a map $J[n] \longrightarrow J[n_1]$, and we can define

$$\mathrm{Tors}(J) = \varinjlim_n J[n], \tag{5.220}$$

where the ordering on is given by divisibility. Of course it is clear that

$$\mathrm{Tors}(J) = \Gamma \otimes \mathbb{Q}/\ \ , \tag{5.221}$$

this is the group of **torsion points** and we conclude:

Proposition 5.3.10. *Even if the module Γ cannot be defined in terms of algebraic geometry, the module*

$$\Gamma \otimes \mathbb{Q}/$$

is an algebraic geometric object.

We can pass to the dual module, we consider $\mathrm{Hom}(\mathrm{Tors}(J),\mathbb{Q}/\ \ )$. It is an elementary fact that

$$\mathrm{Hom}(\mathbb{Q}/\ \ ,\mathbb{Q}/\ \ ) \simeq \varprojlim_n\ \ /n\ \ \ = \widehat{\ \ },$$

and therefore we get

$$\mathrm{Hom}(\mathrm{Tors}(J),\mathbb{Q}/\ \ ) = \Gamma \otimes \widehat{\ \ }. \tag{5.222}$$

Now we can define the so called Tate module.

Definition 5.3.11. *The* **Tate module** *is defined as*

$$T(J) = \varprojlim_{n} J[n],$$

where now for $n \mid n_1$ the map $J[n_1] \longrightarrow J[n]$ is given by multiplication by n_1/n.

(See example 9.) The Chinese remainder theorem yields

$$\mathbb{Q}/\ \ = \bigoplus_{\ell:\ell\text{prime}} \mathbb{Q}_\ell/\ \ {}_\ell \tag{5.223}$$

where ℓ runs over the primes, $\mathbb{Q}_\ell$ is the ℓ-adic completion and $\ {}_\ell$ is the ring of ℓ-adic integers. This yields a decomposition

$$\mathrm{Tors}(J) = \bigoplus_{\ell} \mathrm{Tors}(J)_\ell$$

where $\mathrm{Tors}(J)_\ell = \lim\ J[\ell^\alpha]$ and dually $\widehat{\ } = \prod_\ell \ {}_\ell$ and

$$T(J) = \prod_{\ell} T_\ell(J).$$

Definition 5.3.12. *For any prime ℓ we define the $\ell-$***adic cohomology** *groups of our Riemann surface S by*

$$H^0(S,\ {}_\ell) = \ {}_\ell$$
$$H^1(S,\ {}_\ell) = \mathrm{Hom}(T_\ell(J),\ {}_\ell)$$
$$H^2(S,\ {}_\ell) = \ {}_\ell.$$

Now we are back at the opening line of this chapter, the only difference is that the coefficients   are replaced by $\ {}_\ell$.

In section 5.1.7 we worked very hard to show that a compact Riemann surface S is the same kind of object as a projective algebraic curve $C \subset \mathbb{P}^n(\mathbb{C})$. Such a curve can be defined as the set of common zeroes of a set of algebraic equations. We can interpret our results above by saying that the $\ell-$adic cohomology groups are in fact attached to the algebraic curve C. Why is this of any relevance?

Let us assume that the defining equations of our Riemann surface can be chosen in such a way that the coefficients are in $\mathbb{Q}$. For example we may assume that our curve is embedded into $\mathbb{P}^2(\mathbb{C})$ and defined as the set of zeroes of the somewhat famous homogenous equation

$$x^n + y^n + z^n = 0 \text{ where } n \text{ is an integer} \geq 1$$

For any subfield $K \subset \mathbb{C}$ we may consider the set $C(K)$ of K-valued points, this is simply the set of solutions of the equations for which the coordinates are in K, i.e. the points on the curve, which lie in $\mathbb{P}^n(K)$. Then $S = C(\mathbb{C})$. It clear that the automorphisms of $\mathbb{C}$ map the curve $C(\mathbb{C}) = S \subset \mathbb{P}^n(\mathbb{C})$ into itself. Since these automorphisms are not continuous (except the identity and the complex conjugation), they do not induce automorphisms on the cohomology groups $H^\nu(S,\ \ )$. But the it can be shown that they induce automorphisms on the cohomology groups $H^\nu(S,\ {}_\ell) = H^\nu(C,\ {}_\ell)$. We see this action in the case of $H^1(S,\ {}_\ell)$ because these automorphisms induce automorphisms on the group of torsion points and hence on the Tate module. Actually it is clear that the torsion points have coordinates in $\bar{\mathbb{Q}}$, hence this action factorizes over the action of the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ (see Example 11).

This means that the cohomology groups $H^{\bullet}(S,\ \ _{\ell})$ have a much richer structure than the plain cohomology groups $H^{\bullet}(S,\ \ )$. They are modules for the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.
Of course we already noticed that Hodge theory provides an additional structure on the cohomology: The tensor product $H^1(S,\ \ ) \otimes \mathbb{C}$ contains the distinguished maximal isotropic subspace of holomorphic differentials $H^0(S,\Omega^1)$.
To illustrate the importance of this fact I formulate a result of Faltings, we anticipate some definitions from volume II.
An abelian variety A is defined over a number field $K \subset \bar{\mathbb{Q}}$, if we can find a projective embedding and a defining set of equations, which have coefficients in K. Then it is clear that we can find a finite extension $K \subset L \subset \bar{Q}$, such that all endomorphisms are defined over L. We have an embedding

$$\mathrm{End}(A) \otimes \ \ _{\ell} \hookrightarrow \mathrm{End}(T_{\ell}(A)).$$

Since the endomorphisms are defined over L we know that $\mathrm{End}(A) \otimes \ \ _{\ell}$ commutes with the action of the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/L)$. The theorem of Faltings asserts (see [Fa])

Under the above hypothesises $\mathrm{End}(A) \otimes \ \ _{\ell}$ *is the commutant of the action of the Galois group, i.e. consists of those elements, which commute with the action of the Galois group.*

This theorem can be used to decide questions of the following kind. Assume that we have two projective, smooth and irreducible curves $C_1(\mathbb{C}) = S_1, C_2(\mathbb{C}) = S_2$ and let us assume that the defining equations have coefficients in $\mathbb{Q}$. Can we decide whether there exists a non constant holomorphic map $f : S_1 \longrightarrow S_2$ or more generally whether we can find a third curve S which has non constant holomorphic maps $f_1 : S \longrightarrow S_1$ and $f_2 : S \longrightarrow S_2$.
In principle we get an answer from Hodge theory. We apply the considerations on page 274 to the product of our two curves. Then we see that we have to find out whether we can find $\mathbb{C}$-linear maps $\phi : H^0(S_1,\Omega^1_{S_1})^{\vee} \longrightarrow H^0(S_1,\Omega^1_{S_2})^{\vee}$ which map the lattice $\Gamma_1^{\vee}$ into $\Gamma_2^{\vee}$(see 5.2.3). But this may be difficult to decide, because we have to compute the period lattices and hence we have to compute the period integrals (See 5.1.11). Therefore we see that finding such a ϕ means that we have to find certain linear relations with rational coefficients among the period integrals of the two Riemann surfaces. This is difficult if not impossible. For instance we have no way to decide whether two irrational numbers, which may be obtained from certain integrals, are linearly dependend over $\mathbb{Q}$. One way to establish such relations is to transform these period integrals into other ones by making clever substitutions. But this throws us back to our original problem.
On the other hand we may also try to compare the two actions of the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on $H^1(S_1,\ \ _{\ell})$ and $H^1(S_2,\ \ _{\ell})$. We may try to decide whether there are non trivial homomorphisms between these two Galois modules. Now number theory provides at least theoretically some tools to decide this question. But then the theorem of Faltings implies that we also can find a curve S which has non trivial holomorphic maps to S_1 and S_2.
I do not know, whether this the right place to formulate a final exercise, I will come back to it in the second volume, there are also places in the literature, where it is solved:

Exercise 32. Is there a non constant holomorphic map between the two elliptic curves

$$y^2 + y = x^3 - x^2 \text{ and } y^2 + y = x^3 - x^2 - 10x - 20$$

The above result concerning the endomorphism rings is also the key to Faltings' proof of the Mordell conjecture. This Mordell conjecture says:

If C is a smooth, irreducible projective curve over some number field $K \subset \mathbb{C}$ (this means that $C(\mathbb{C})$ is a compact Riemann surface and the defining equation can taken with coefficients in K) and if the genus is greater than one, then the number K rational points $\#C(K)$ is finite.

Finally I want to mention that Wiles' proof of Fermat's last theorem [Wi] is based on the understanding of the action of the Galois group on ℓ-adic cohomology groups. In this case Wiles studies the action on the first cohomology of elliptic curves defined over $\mathbb{Q}$.

Bibliography

[A-McD] Atiyah, M. F.; Macdonald, I. G. *Introduction to commutative algebra* Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont. 1969 ix+128 pp 1995

[B-L] Birkenhake, Christina; Lange, H. *Complex abelian varieties.* Second edition. Grundlehren der Mathematischen Wissenschaften, 302. Springer-Verlag, Berlin, 2004. xii+635 pp.

[B-T] Bott, R ; Tu, Loring W. *Differential forms in algebraic topology.* Graduate Texts in Mathematics, 82. Springer-Verlag, New York-Berlin, 1982. xiv+331 pp.

[B-K] Buser, P.; Karcher, H. *Gromov's almost flat manifolds.* Astérisque, 81. Société Mathématique de France, Paris, 1981. 148 pp.

[Ca] Cartan, H. *Variétés analytiques complexes et cohomologie.* Colloque sur les fonctions de plusieurs variables, tenu à Bruxelles, 1953, pp. 41–55

[Ca-Se] Cartan,H.; Serre, J.-P. *Un théoréme de finitude concernant les variétés analytiques compactes.* C. R. Acad. Sci. Paris 237, (1953). 128–130

[Ca-Ei] Cartan, H.; Eilenberg, S. *Homological algebra.* With an appendix by David A. Buchsbaum. Princeton Landmarks in Mathematics. Princeton University Press, Princeton NJ, 1999. xvi+390 pp.

[Ch] Chow, W.-L. *On compact complex analytic varieties.* Amer. J. Math. 71, (1949). 893–914

[Ei-St] Eilenberg, S.; Steenrod, N. *Foundations of algebraic topology.* Princeton University Press, Princeton, New Jersey, 1952. xv+328 pp

[Ei] Eisenbud, D. *Commutative algebra. With a view toward algebraic geometry.* Graduate Texts in Mathematics, 150. Springer-Verlag, New York, xvi+785 pp.

[Fa] Faltings, G. Endlichkeitssätze für abelsche Varietäten über Zahlkörpern. Invent. Math. 73 (1983), no. 3, 349–366.

[Fa-Ch] Faltings, G.; Chai, C.-L. *Degeneration of abelian varieties.* With an appendix by David Mumford. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) 22. Springer-Verlag, Berlin, 1990. xii+316 pp.

[Fo] Forster, O. *Lectures on Riemann surfaces.* Graduate Texts in Mathematics, 81. Springer-Verlag, New York-Berlin, 1981. viii+254 pp.

G. Harder, *Lectures on Algebraic Geometry I*, DOI 10.1007/978-3-8348-8330-8,

[Ge-Ma] Gelfand, S. I.; Manin, Y. I. *Homological algebra, Algebra, V,* 1–222, Encyclopaedia Math. Sci., 38, Springer, Berlin, 1994

[Go] Godement, R. *Topologie algébrique et théorie des faisceaux.* Publications de l'Institut de Mathèmatique de l'Université de Strasbourg, XIII.Actualités Scientifiques et Industrielles, No. 1252. Hermann, Paris, 1973. viii+283 pp.

[Gr-Ha] Griffiths, Ph.; Harris, J. *Principles of Algebraic Geometry.* John Wiley & Sons Inc, 1994. 814 pp.

[Gr-Re1] Grauert, H.; Remmert, R. *Theory of Stein spaces.* Grundlehren der Mathematischen Wissenschaften 236. Springer-Verlag, Berlin-New York, 1979. xxi+249 pp.

[Gr-Re2] Grauert, H.; Remmert, R. *Coherent analytic sheaves.* Grundlehren der Mathematischen Wissenschaften 265. Springer-Verlag, Berlin, 1984. xviii+249 pp.

[Ha] Hartshorne, R. *Algebraic geometry.* Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977. xvi+496 pp.

[Hat] Hatcher, A. *Algebraic topology.* Cambridge University Press, Cambridge, 2002. xii+544 pp.

[Hei] Heinen, Ch. *Eine Konstruktion des feinen Modulschemas für elliptische Kurven mit voller n-Struktur* Diplomarbeit Bonn, 1986

[Hir] Hirsch, M. W. *Differential topology.* Graduate Texts in Mathematics, No. 33. Springer-Verlag, New York-Heidelberg, 1976. x+221 pp.

[Hi] Hitchin, N. *Global differential geometry.* Mathematics unlimited—2001 and beyond, 577–591, Springer, Berlin, 2001.

[Ho] Hodge, W. V. D. *The Theory and Applications of Harmonic Integrals.* Cambridge University Press, Cambridge,England; Macmillan Company, New York, 1941. ix+281 pp.

[Hu] Husemöller, D. *Elliptic Curves* Graduate Texts in Mathematics, 111 Springer-Verlag, New York, 1987. xvi+350 pp.

[Iv] Iversen, B. *Cohomology of sheaves.* Universitext. Springer-Verlag, Berlin, 1986. xii+464 pp.

[La] Lang, S. *Elliptic Functions* Addison-Wesley Publishing Company, INC.

[McL] Mac Lane, S. *Categories for the working mathematician.* Second edition. Graduate Texts in Mathematics, 5. Springer-Verlag, New York, 1998. xii+314 pp.

[Mu1] Mumford, D. *Abelian varieties.*
Tata Institute of Fundamental Research Studies in Mathematics, No. 5 Published for the Tata Institute of Fundamental Research, Bombay; Oxford University Press, London 1970 viii+242 pp.

[Mu2] Mumford, D. *Tata lectures on theta. I-II-III*
Progress in Mathematics, 28, 43, 97. Birkhäuser Boston, Inc., Boston, MA

[Neu] Neukirch, J. *Algebraic number theory.*
Grundlehren der Mathematischen Wissenschaften 322. Springer-Verlag, Berlin, 1999

[Pa] Palais, R. S. *Seminar on the Atiyah-Singer index theorem.*
With contributions by M. F. Atiyah, A. Borel, E. E. Floyd, R. T. Seeley, W. Shih and R. Solovay. Annals of Mathematics Studies, No. 57 Princeton University Press, Princeton, N.J. 1965 x+366 pp

[Rie] Riemann, B. *Gesammelte mathematische Werke*
Springer-Verlag, Berlin,Heidelberg, New York, 1990

[Se1] Serre, J.-P. *Géométrie algébrique et géométrie analytique.*
Ann. Inst. Fourier, Grenoble 6 (1955–1956), 1–42.

[Se2] Serre, J.-P. *Un théoréme de dualité*
Comment. Math. Helv. 29, (1955). 9–26

[Sp] Spivak, M. *A comprehensive introduction to Differential Geometry, vol I*
Publish or Perish, Inc., Berkeley, 1979

[We1] Weil, A. *On the projective embedding of Abelian varieties.*
Algebraic geometry and topology. A symposium in honor of S. Lefschetz, pp. 177–181. Princeton University Press, Princeton, N. J., 1957.

[We2] Weil, A. *Introduction à l'étude des variétés kählériennes.*
Publications de l'Institut de Mathématique de l'Université de Nancago, VI. Actualités Sci. Ind. no. 1267 Hermann, Paris 1958 175 pp.

[Wel] Wells, R. O., Jr. *Differential analysis on complex manifolds.*
Second edition. Graduate Texts in Mathematics, 65. Springer-Verlag, New York-Berlin, 1980. x+260 pp.

[Wi] Wiles, A *Modular elliptic curves and Fermat's Last Theorem*
Annals of Mathematics, 142 (1995), 443-551

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G. Harder, *Lectures on Algebraic Geometry I*, DOI 10.1007/978-3-8348-8330-8,