Solid Mechanics and Its Applications

M. Reza Eslami

Finite Elements Methods in Mechanics

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This book is dedicated to my precious grandchildren: Arya, Roxan, Ava, and Lilia

Preface

The author is pleased to present *Finite Element Method in Mechanics*. This book will serve a wide range of readers, in particular, graduate students, Ph.D. candidates, professors, scientists, researchers in various industrial and government institutes, and engineers. Thus, the book should be considered not only as a graduate textbook, but also as a reference book to those working or interested in areas of finite element modelling of solid mechanics, heat conduction, and fluid mechanics.

The book is self-contained, so that the reader should not need to consult other sources while studying the topic. The necessary mathematical concepts and numerical methods are presented in the book and the reader may easily follow the subjects based on these basic tools. It is expected, however, that the reader should have some basic knowledge in the classical mechanics, theory of elasticity, and fluid mechanics.

The book contains 17 chapters, where the chapters cover the finite element modeling of all major areas of mechanics.

Chapter 1 presents the history of development of finite element method, where the key references are given and the progress of this science is discussed.

Chapter 2 is devoted to the basic mathematical concepts of finite element method. The method of calculus of variation is discussed and the distinction of boundary value problems versus the variational formulation is presented and several examples are given to make the reader familiar with the concepts of function and functional. The material then follows into the discussion of traditional Ritz and Galerkin methods. Numerical examples show the powerful nature of these numerical techniques.

Introduction to the finite element method is given in Chap. 3 with the discussion of elastic membrane. The subject of elastic membrane is selected because the height of membrane is approximated with finite element method. This gives a physical feeling for the finite element approximation to the reader.

Since a linear triangular element is employed to model the elastic membrane in Chap. 3, Chap. 4 discuss the one-, two-, and three-dimensional elements with linear and higher order approximation. The discussion gives a feeling to the reader that there is no limitation in the type of elements and the order of approximation, geometrically and mathematically. The subparametric, isoparametric, and superparametric elements are discussed and the natural coordinates are presented.

The finite element approximation of the field problems, harmonic and biharmonic, are given in Chap. 5.

Chapter 6 deals with the finite element approximation of the heat conduction equations. One-, two-, and three-dimensional conduction in solids are discussed and the transient heat conduction problems are presented. Both variational and Galerkin techniques are presented.

Up to this point, the reader learns how to obtain the element stiffness, capacitance, and force matrices for one element. He questions how a solution domain with many number of elements should be modeled and solved to obtain the required domain unknowns. A comprehensive treatment is given in Chap. 7 to give a proper tool to the reader to write his own computer program. Many numerical examples are solved to show the numerical scheme, and proper algorithms are given. The assembly of global matrices, bandwidth calculation, the method to apply the boundary conditions, and the Gauss elimination method are presented. The method of solution of the transient and dynamic finite element equations are then presented. The central difference method, the Houbolt Method, the Newmark Method, and the Wilson-θ method are presented. At this stage, the reader learns how to write his own computer problem. Now, he should learn how different problems of mechanics are formulated by the finite element approximation. These techniques are discussed in the following chapters.

Chapter 8 deals with the finite element approximation of beams. Static beam deflection equation, based on the Euler beam theory, is presented and the Galerkin and variational formulations are obtained. The axial, torsional, and lateral vibrations of beams are modeled. Finally, the vibrations of Timoshenko beam are presented.

Chapters 9 and 10 present the finite element formulations of elasticity problems based on Galerkin and variational formulations.

Torsion of prismatic bars and rods are given in Chap. 11 and quasistatic thermoelasticity theory is discussed in Chap. 12.

Chapter 13 is devoted to the finite element solution of viscous fluid mechanic problems. Derivation of the Navier-Stokes equations is presented and the finite element formulation of the two-dimensional fluid flow based on the velocity components and pressure are derived. In the following section, the vorticity transport model of the Navier-Stokes equations are obtained and the finite element formulations are derived. The method of solution of the resulting nonlinear finite element equation is presented.

Chapter 14 presents one-dimensional higher order elements. The local natural coordinate for the quadratic and cubic elements are derived and the Jacobian matrix is obtained. To describe the application, field problem for one-dimensional case is discussed and the element of the matrices are calculated. The chapter is completed with a discussion of layerwise theory for composite beams, where one-dimensional higher order element is used to discuss the problem.

The higher order element in two dimension in discussed in Chap. 15. The triangular element with quadratic and cubic shape functions are given in terms of the area element and the Jacobian matrix is calculated. The quadratic element is employed to obtain the element matrices for a two-dimensional field problem. In the following, the quadrilateral element is discussed and its application to the field problem is presented.

Chapter 16 presents the linear coupled thermoelasticity problems, and their method of solution by finite element method. This chapter is unique in the literature of finite element analysis of solid elastic continuum. The most general form of the three-dimensional classical coupled thermoelasticity equations are considered and the finite element formulations are presented.

Computer programs for three different types of problems are given in Chap. 17. The first program is related to the elastic membrane problem, where Poisson's equation is solved. This program may be used for any other application of Poisson's equation, such as the steady-state heat conduction, torsion of prismatic bars, inviscid incompressible fluid flow problems, and the pressure in porous media. The second computer program handles two-dimensional elasticity problems, and the third computer program presents three-dimensional transient heat conduction problems. The programs are written in C++ environment.

At the end of all the chapters, except Chap. 1, there are a number of problems for students to solve. Also, at the end of each chapter, there is a list of relevant references.

The book was prepared over some 40 years of teaching the graduate finite element course. During this long period of time, the results of classwork assignments and student research are carefully gathered and put into this volume of work. The author takes this opportunity to thank all his students who made possible to provide this piece of work.

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Contents

Chapter 1
Introduction and History

Abstract This section provides a brief history of the development of the finite element method. The general picture of the science of mechanics to formulate the most general rules, variational and boundary value problems, are presented. The approximate solution of both types of formulations are discussed and how these approximate solutions are transformed into the finite element method are presented. The key references in the development of finite element method are cited.

1.1 Introduction

Problems in classical mechanics are either formulated on the basis of the force summation method, or the variational method. The first method is more widely used in statement of the equilibrium condition of a system and derivations of the associated equilibrium equations. The concept of equilibrium of mechanical systems is expressed in a form that satifies Newton's law for static and dynamic systems. Applying Newton's law to a system results in a set of differential or partial differential equations describing the state of static and dynamic equilibrium of the system. The solution to these differential equations is necessary to determine the response of the system under the applied forces. The study and analysis of the system and its behavior under the applied forces depends upon the information that we obtain by solving the governing equations. A more exact and precise solution, provides better and more informative data about the system and helps in better understanding the behavior of the system. The second approach for formulating the classical problems of mechanics is the variational method. This method is based on the concept of energy and the principles governing the equilibrium of a system in nature. As a general rule of nature, a system is in a state of equilibrium if the associated functional is extremized within the constraint paths applied to the system. The state of the extremized functional always coincides with the equation stating the static or dynamic equilibrium condition of the system. On the other hand, while the variational principles of engineering

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systems are a fact of nature in regard to the state of equilibrium of a system, they reduce to the same governing differential equations of equilibrium and boundary conditions that we were able to obtain from the force summation method.

As an example of the foregoing discussion, consider the problems of solid mechanics and dynamics. The general principle governing the state of equilibrium of all the problems of this nature is the *principle of virtual work*. Accordingly, the *system* is in a state of equilibrium if the first variation of the work vanishes for all possible virtual paths. The expression for *general work* is called a *functional*. In the mathematical sense, the system is in a state of equilibrium if the functional is minimized within the given constraints. This leads to a class of classical extremum problems. However, the functional is a function of dependent functions. As the dependent functions vary, the functional varies too. Among all the classes of admissible functions, the functions which extremize the functional are the governing equilibrium equations of the system. We call the *function* the dependent function which governs the state of the equilibrium equation. The governing equations of equilibrium obtained from the force-summation method result in *boundary value problems*.

In solid mechanics, the general principle of virtual work is divided into two categories, the static and dynamic problems. The principle of virtual work reduces to the principle of minimum potential energy for the static problems and to Hamilton's principle for the dynamic problems. That is, the general expression for a functional in static problems is the total potential energy function of the system, and in dynamic problems is the Hamilton function. Now, one can employ the mathematical tools of the *calculus of variation* and apply them to either functions of static or dynamic problems. Minimizing the functional of the total potential energy of the static problems by the method of calculus of variations provides the boundary value problems associated with the static equilibrium of the system. An alternate method for arriving at the same boundary value problem is to set $\sum \vec{F} = 0$. The latter approach is more common in classical literature for obtaining the governing equations of a boundary value problem. By similar means, application of the Hamilton principle to dynamic problems and the minimization of the related functionals yield the boundary value problems of the system associated with $\sum \vec{F} = m\vec{a}$. It is, however, a common practice in engineering analysis to set up the governing differential equations of equilibrium of the system by the load summation method and try to solve the resulting equations by theoretical or numerical methods for obtaining the response of the system under the applied loads.

Either formulation of the engineering systems, variational or forced summation method, result in the governing differential equations of equilibrium. Traditionally, one has to consider the resulting differential equations and try to obtain a solution by either exact analytical or approximate numerical means. The exact analytical solutions to some classical problems with simple geometries and boundary conditions are well developed, but the majority of mathematical models describing the behavior of the physical systems are left analytically unsolved.

The necessity of understanding the behavior of mechanical systems under applied loads has forced alternate techniques for obtaining solutions of the problems. The alternate solution techniques include classical approximation methods and numerical methods.

Historically, numerical techniques were developed based on two different approaches, variational and weighted residual methods. In 1908, Ritz [1] developed a method where a series solution was assumed for the dependent function, and its constant coefficients were found such that the functional remained stationary with respect to constant coefficients. The traditional Ritz method was, therefore, developed as an approximate solution of the extremum problems, and was used whenever the functional of the problem was known. On the other hand, in 1915, Galerkin [2] developed another technique which was suitable for the approximate solution of the boundary value problems. According to this technique, a series solution was assumed for the dependent function and the residue of the governing equilibrium equation was made orthogonal with respect to the weighting functions. The weighting functions were selected to be the same functions in the series solution. The result yields a system of linear equations which are solved for the unknown constant coefficients of the series solution. The Galerkin method is one technique among three others, namely subdomain, collocation, and least squares, which are all developed as an approximate method of solution of the boundary value problems.

The method of finite differences is in the class of weighted residual methods and, as a special technique of collocation, considers the differential operators approximated by the Taylor series for a finite span of the variables. The first term of the series is retained and the differential operator is expressed in terms of the values of the function at the nodal points of a rectangular mesh. The efficiency of this method is proved whenever the solution domain is confined to a geometrically fine problem where the finite different meshes are either rectangular cells or sections of a circle. The residue of the governing equations are set to zero at the predefined nodal points of the meshed domain. The accuracy of the solution depends upon the proper number of nodal points in the solution domain.

All the mentioned approximate numerical techniques convert the governing system of differential equations of equilibrium into a set of linear equations to be solved for their unknowns. A more exact solution requires a larger number of terms, or nodal points, and consequently a larger system of equations and unknowns. Prior to the 1950s, the solutions of the set of equations for unknowns were restricted to a few, which could have been found through hand calculations. During the years 1954–1956, the concept of the computer was developed at the Massachusetts Institute of Technology and the concept was very rapidly turned into actual electronic computers. One of the very first applications for this new invention was the solution of a system of equations for unknowns. The approximate numerical techniques became very important, and the attention of scientific communities was rapidly turned toward this new concept of analysis. The old concepts of numerical methods became alive again, reborn as new concepts with the same basic mathematical principles, but implemented through new approaches.

A decade before the invention of the computer, Courant in 1943 [3] and Prager and Synge in 1947 [4] used the Ritz concept and applied it to the principle of minimum potential energy, formulating the elasticity problems in terms of finite elements. The proposed technique was not developed any further due to the limitations imposed on the solution of the systems of large numbers of equations and unknowns. In 1956, Turner et al. [5] proposed a numerical technique for solving complex structural problems using the structural stiffness method. This paper later became a basis for development of the finite element method. Szmelter [6] and Clough [7], in 1959 and 1960, respectively, applied the Ritz method to the two-dimensional elasticity problems and came up with the finite element method. Many papers were then published on the subject of finite elements using the Ritz method, modeling different problems of mechanics by the finite element method [8–13]. In 1967, Zienkiewicz [14] collected his extensive works and that of others into the first volume of a book on the finite element. The finite element method up to this time was primarily based on the Ritz method and was limited to problems which had a known expression for the functional. The fluid mechanics problems, which were based on Navier Stokes' equations, could not be analyzed by the finite element method, as the associated functional was not known at the time (and is still not known as of this writing). The fluid dynamic problems confined to the potential flow and creeping flow theories could be analyzed due to their known expressions for the functional.

Due to the lack of knowledge about the extremum principles of fluid dynamic problems, the finite element analysis of this class of problems was not developed until the late 1970s when the weighted residual approach was introduced to the finite element technique. Since the equations of motion of fluid flow were known, the weighted residual method, especially the Galerkin method, was employed to model the discretized solution domain of fluid flow problems. The finite element method based on the Galerkin technique was rapidly utilized to model all different problems of compressible and incompressible fluid flows. The energy equations for non-isothermal fluid flows were also treated through the convective heat transfer equations and the fluid and thermal boundary layer problems were modeled by the finite element method according to the weighted residual techniques.

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Chapter 2
Mathematical Foundations

Abstract This chapter presents basic mathematical concepts and tools for the development of finite element method. The concept of functional, associated with the variational formulation, and the function, associated with the boundary value problems, is discussed; the method of calculus of variation which transfers the variational formulation into the boundary value problem is also presented. A brief discussion of the numerical solution methods to handle the variational formulation and the boundary value problem is presented. Some numerical examples are given to show the convergence efficiency of the numerical methods.

2.1 Introduction

The fundamental problems of mechanics are governed by differential or partial differential equations which state the equilibrium and continuity conditions of the system. In particular cases, where the geometry, loading, and boundary conditions are simple, these governing differential equations are solved and the solutions are presented in the form of mathematical functions.

The problems of mechanics, in general, should satisfy the condition of the extremum of a functional at equilibrium condition. That is, a problem formulation is described by its proper differential equations of equilibrium as well as its associated energies at the extremal condition. While the formulation of problems based on the forced summation method and the variational method are entirely different, they are both related and yield identical results. That is, the equilibrium equations describing the system at equilibrium are associated with the minimum total work of the system.

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2.2 Statement of Extremum Principle

Engineering problems are usually formulated in terms of a system of equilibrium equations which may be in the form of algebraic, ordinary, or partial differential equations. For many systems, the equilibrium equations are equivalent to a known extremum problem. The reality for physical problems relies on the basic laws of nature wherein the state of equilibrium is associated with a specific physical law in nature.

The mathematical statement of the extremum principle is as follows; *a certain class of allowable functions* $\psi(t, x_i)$ $i = 1, 2, 3$ *is fixed between time intervals* $t_1 \leq t \leq t_2$ *and in space domain* $D(x_1, x_2, x_3)$*, and a means for associating a value* $\Phi(\psi)$ *with each function* ψ *is defined. For a particular function* ψ*, there exists a single function for* Φ *such that as* ψ *ranges through the class of allowable functions, the corresponding functional value of* Φ *varies. The class of functions* ψ *are called* **function** *and their corresponding* Φ *values are called* **functional**. *The extremum problem is to locate the function* ψ *in which its functional* Φ *remains stationary*.

In the statement of the extremum problems and the relationship between the functional and function, the following two questions arise:

1. Given a function $\psi(t, x_i)$, $i = 1, 2, 3$, associated with a boundary value problem, does an equivalent extremum problem exist? and if so, what is the class of allowable functions and what is the functional Φ?
2. Given an extremum problem, what is the equivalent function satisfying the boundary value problem?

We expect to answer the first question for the physical and engineering problems by the known extremum principle based on the laws of nature and physics. For mechanical problems, the examples are the law of entropy for thermodynamic problems, Hamilton's principle for the dynamic problems, and the law of minimum potential energy for the static problems. A more general treatment may be based on the principle of virtual work. The extremum values of entropy or work done under certain circumstances are always associated with the general equilibrium of the mechanical systems which are obtained by means of balance of forces, moments, and energies. There are no general mathematical treatments for obtaining the functional directly from its associated boundary value problem obtained through the balance of forces, moments, and energies.

The answer to the second question, on the other hand, is always positive and is based on the algorithm of *calculus of variations*. That is, it is always possible to find the associated boundary value problem of a given functional, using the method of calculus of variation. Applying this method, results in the boundary value problem expressing the state of equilibrium and natural boundary conditions, which are essentially obtained during the weak formulations of integral equations.

2.3 Method of Calculus of Variation

The general statement of the calculus of variation and the relationship between a functional and the function associated with its extremum is discussed in this section. In terms of physical problems, we try to obtain the equilibrium equation of a boundary value problem which governs the function $\psi(t, x_i)$, $i = 1, 2, 3$, through the minimization of its associated functional $\Phi(\psi)$.

Assume that the function $\psi(x_i)$, $i = 1, 2, 3$, is defined and is a function of the space variables x_1, x_2, and x_3 satisfy the equilibrium equation

$$L[\psi(x_1, x_2, x_3)] = 0. \tag{2.3.1}$$

The essential boundary condition which ψ has to satisfy is

$$B_i(\psi) = g_i \qquad i = 1, 2, .. \tag{2.3.2}$$

where L is a mathematical operation, B_i is a linear operation on ψ, and g_i is the non-homogeneous boundary condition applied on ψ. The associated functional of Eq. (2.3.1) is

$$\Phi = \Phi(\psi). \tag{2.3.3}$$

Now, consider a variational function u which meets the continuity conditions and satisfies the homogeneous boundary conditions

$$B_i(u) = 0 \qquad i = 1, 2, \ldots \tag{2.3.4}$$

Now, if ψ is the true solution of Eq. (2.3.1), $(\psi + \epsilon u)$ may be made to represent an arbitrary admissible function which satisfies the real non-homogeneous boundary conditions. For fixed u, the variational parameter ϵ may be changed to make a one-parameter family of admissible functions. Since u satisfies the homogeneous boundary conditions, it follows that

$$B_i[\psi + \epsilon u] = g_i \qquad i = 1, 2, \ldots \tag{2.3.5}$$

Substituting the family of admissible functions $\psi + \epsilon u$ in Eq. (2.3.3), the functional Φ may be changed by varying the variational parameter ϵ. The extremum of the functional Φ is obtained from the following rule:

$$\left(\frac{\partial \Phi[\psi + \epsilon u]}{\partial \epsilon}\right)|_{\epsilon=0} = 0. \tag{2.3.6}$$

This rule holds for every possible family of $(\psi + \epsilon u)$ for the arbitrary variational function u. The rule expressed in Eq. (2.3.6) is the basic and formal procedure of the method of calculus of variation. The basic approach in the treatment of Eq. (2.3.6) is integration by parts such that the arbitrary function u is factored out in the resulting equations. Since u is an arbitrary function, the remaining part is set to zero. This provides the boundary value problem and the associated natural boundary conditions.

To describe the method, a few general types of functional are considered, and, using the method of calculus of variation, their associated boundary value problems are obtained.

2.4 Function of One Variable, Euler Equation

Let us consider a functional Φ being a function of $y(x)$ and its first derivative $y'(x)$ as given below,

$$\Phi[y(x)] = \int_{x_1}^{x_2} F(x, y, y')dx. \tag{2.4.1}$$

The boundary conditions on $y(x)$ are assumed to be

$$y(x_1) = y_1 \qquad y(x_2) = y_2. \tag{2.4.2}$$

It is further assumed that the function F is continuous in the interval x_1 and x_2, and its derivative up to the first order exists and is continuous.

Among all functions $y(x)$ which satisfy the continuity conditions and the given boundary conditions, which we call the *class of admissible functions*, there is only one special function $y(x)$ which minimizes the functional Φ.

In order to determine $y(x)$, an arbitrary function $u(x)$ is selected such that it is continuous in the interval x_1 and x_2 along with its first derivative and satisfies the homogeneous boundary conditions

$$u(x_1) = u(x_2) = 0. \tag{2.4.3}$$

Now, the function $\bar{y}$ is constructed as

$$\bar{y}(x) = y(x) + \epsilon u(x) \tag{2.4.4}$$

where $\bar{y}(x)$ satisfies all the continuity conditions and the given boundary conditions. The parameter ϵ is a positive arbitrary variational parameter and is selected as sufficiently small so that the function $\bar{y}(x)$ is as close as possible to the function $y(x)$. Therefore, since $y(x)$ makes the functional at a relative minimum, for $\epsilon \neq 0$ the following inequality exists:

$$\Phi(y + \epsilon u) \geq \Phi(y). \tag{2.4.5}$$

The functional Φ is a function of ϵ and is at a relative minimum for $\epsilon = 0$. Calling $f(\epsilon) = \Phi(y + \epsilon u)$, $f(\epsilon)$ is a function of ϵ, and according to Eq. (2.4.5),

$$f(\epsilon) \geq f(0). \tag{2.4.6}$$

This suggests that $f(\epsilon)$ is at a relative minimum when $\epsilon = 0$, and since $f(\epsilon) = \Phi(y + \epsilon u)$ is differentiable, the necessary condition for $f(\epsilon)$ to be at a relative minimum is therefore

$$\frac{\partial f(\epsilon)}{\partial \epsilon}|_{\epsilon=0} = 0. \tag{2.4.7}$$

Introducing Eq. (2.4.4) into Eq. (2.4.1) yields

$$\Phi(y + \epsilon u) = \int_{x_1}^{x_2} F[x, (y + \epsilon u), (y' + \epsilon u')]dx. \tag{2.4.8}$$

Differentiating with respect to ϵ gives

$$\frac{\partial \Phi(y + \epsilon u)}{\partial \epsilon} = \int_{x_1}^{x_2} [\frac{\partial}{\partial \bar{y}} F(x, y + \epsilon u, y' + \epsilon u')u \quad + \frac{\partial}{\partial \bar{y}'} F(x, y + \epsilon u, y' + \epsilon u')u']dx. \tag{2.4.9}$$

Integrating the last integral by parts gives

$$\frac{\partial \Phi(y + \epsilon u)}{\partial \epsilon} = \int_{x_1}^{x_2} [\frac{\partial F(x, y + \epsilon u, y' + \epsilon u')}{\partial \bar{y}} - \frac{d}{dx} \frac{\partial F(x, y + \epsilon u, y' + \epsilon u')}{\partial \bar{y}'}] u dx + \frac{\partial F}{\partial \bar{y}'}(x, y + \epsilon u, y' + \epsilon u')u(x)|_{x_1}^{x_2}. \tag{2.4.10}$$

Setting $\epsilon = 0$ yields

$$\frac{\partial \Phi(y + \epsilon u)}{\partial \epsilon}|_{\epsilon=0} = \int_{x_1}^{x_2} [\frac{\partial F(x, y, y')}{\partial y} - \frac{d}{dx} \frac{\partial F(x, y, y')}{\partial y'}]u(x)dx + \frac{\partial F}{\partial y'} u|_{x_1}^{x_2} = 0. \tag{2.4.11}$$

The function $u(x)$ is an arbitrary function satisfying the homogeneous boundary conditions. The expression in Eq. (2.4.11) should be zero for all values of $u(x)$. This leads to the Euler equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} = 0 \tag{2.4.12}$$

subjected to the *natural boundary condition*

$$\frac{\partial F}{\partial y'} = 0 \quad \text{at} \quad x = x_1$$
$$\frac{\partial F}{\partial y'} = 0 \quad \text{at} \quad x = x_2. \tag{2.4.13}$$

Equation (2.4.12) is equivalent to the boundary value problem. Expanding the differential term yields

$$\frac{\partial F}{\partial y} - \frac{\partial^2 F}{\partial x \partial y'} - \frac{dy}{dx}\frac{\partial^2 F}{\partial y \partial y'} - \frac{d^2 y}{dx^2}\frac{\partial^2 F}{\partial y'^2} = 0. \tag{2.4.14}$$

Equation (2.4.14) is the necessary condition for a function $y(x)$ to minimize the functional $\Phi(y)$ given by Eq. (2.4.1).

2.5 Higher Order Derivatives

Consider a functional as a function of the variable x, function $y(x)$, and higher order derivatives of function $y^n(x)$, defined in the interval $[x_1, x_2]$ as

$$\Phi[y(x)] = \int_{x_1}^{x_2} F(x, y, y', ...y^n)dx. \tag{2.5.1}$$

The function $y(x)$ is (n) times differentiable with respect to x. The boundary conditions are given for the function and its derivatives up to the order $n - 1$, as

$$\begin{aligned} y(x_1) &= y_1 \quad .... \quad y^k(x_1) = y_1^k \\ y(x_2) &= y_2 \quad .... \quad y^k(x_2) = y_2^k \qquad k = 1, 2, \ldots, (n-1) \end{aligned} \tag{2.5.2}$$

where y_1, y_2, ..., y_1^k, and y_2^k are known functions on the boundary. That is, the boundary conditions are specified for the function itself and its derivatives up to order $(n - 1)$. Consider a variational function $u(x)$, and construct the function $\bar{y}(x)$ as

$$\bar{y}(x) = y(x) + \epsilon u(x). \tag{2.5.3}$$

The variational function $u(x)$ is an arbitrary function with the following properties:

1. The function $u(x)$ and its derivatives up to order n are continuous in the interval (x_1, x_2).
2. The function $u(x)$ and all of its derivatives up to order $(n - 1)$ satisfy the homogeneous boundary conditions.

$$u(x_1) = 0 \cdots u^k(x_1) = 0$$
$$u(x_2) = 0 \cdots u^k(x_2) = 0 \qquad k = 1, 2, \cdots (n-1).$$

Substitution of Eq. (2.5.3) in Eq. (2.5.1) yields

$$\Phi(y + \epsilon u) = \int_{x_1}^{x_2} F(x, y + \epsilon u, y' + \epsilon u', ...y^n + \epsilon u^n)dx. \tag{2.5.4}$$

The derivative of Φ with respect to ϵ gives

$$\frac{\partial \Phi}{\partial \epsilon} = \int_{x_1}^{x_2} (\frac{\partial F}{\partial \bar{y}} \frac{\partial \bar{y}}{\partial \epsilon} + \frac{\partial F}{\partial \bar{y'}} \frac{\partial \bar{y'}}{\partial \epsilon} + ... + \frac{\partial F}{\partial \bar{y}^n} \frac{\partial \bar{y}^n}{\partial \epsilon})dx. \tag{2.5.5}$$

Let $\epsilon = 0$, yields

$$\frac{\partial \Phi}{\partial \epsilon}|_{\epsilon=0} = \int_{x_1}^{x_2} (\frac{\partial F}{\partial y} u + \frac{\partial F}{\partial y'} u' + \frac{\partial F}{\partial y''} u'' + ... + \frac{\partial F}{\partial y^n} u^n)dx. \tag{2.5.6}$$

Using integration by parts, gives

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y'} u' dx = \frac{\partial F}{\partial y'} u(x)|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}(\frac{\partial F}{\partial y'})u dx \tag{2.5.7}$$

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y''} u'' dx = \frac{\partial F}{\partial y''} u'(x)|_{x_1}^{x_2} - \frac{d}{dx}(\frac{\partial F}{\partial y''})u(x)|_{x_1}^{x_2} + \int_{x_1}^{x_2} \frac{d^2}{dx^2}(\frac{\partial F}{\partial y''})u dx. \tag{2.5.8}$$

Higher order derivatives are similarly reduced to factors of $u(x)$ and a series of terms which should be evaluated at the boundaries $x = x_1$ and $x = x_2$. Since the arbitrary variational function $u(x)$ and all it's derivatives up to (n) are continuous in the interval $(x_1, \ x_2)$ and vanish at x_1 and x_2, Eq. (2.5.6) therefore reduces to

$$\frac{\partial \Phi(y + \epsilon u)}{\partial \epsilon}|_{\epsilon=0} = \int_{x_1}^{x_2} [\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} + \frac{d^2}{dx^2} \frac{\partial F}{\partial y''}$$
$$+ \ldots (-1)^n \frac{d^n}{dx^n} \frac{\partial F}{\partial y^n}]u(x)dx + \frac{\partial F}{\partial y'} u|_{x_1}^{x_2} + \frac{\partial F}{\partial y''} u'|_{x_1}^{x_2} - \frac{d}{dx}(\frac{\partial F}{\partial y''})u|_{x_1}^{x_2}$$
$$+ \ldots . \tag{2.5.9}$$

This equation is valid for all possible arbitrary functions $u(x)$ with the given properties. Therefore, if the integral equation should be zero for all possible functions $u(x)$, the following expressions must be identically equal to zero :

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} + \frac{d^2}{dx^2} \frac{\partial F}{\partial y''} + ...(-1)^n \frac{d^n}{dx^n} \frac{\partial F}{\partial y^n} = 0. \tag{2.5.10}$$

The natural boundary conditions are

$$\frac{\partial F}{\partial y'} - \frac{d}{dx}\,(\frac{\partial F}{\partial y''}) + \frac{d^2}{dx^2}\,(\frac{\partial F}{\partial y'''}) - \cdots + (-1)^{n-1}\frac{d^{n-1}}{dx^{n-1}}\,\frac{\partial F}{\partial y^n} = 0$$
$$\frac{\partial F}{\partial y''} - \frac{d}{dx}\,(\frac{\partial F}{\partial y'''}) + \cdots (-1)^{n-2}\frac{d^{n-2}}{dx^{n-2}}\,\frac{\partial F}{\partial y^n} = 0$$
$$\cdots\cdot$$
$$\cdots\cdot$$
$$\cdots\cdot$$
$$\frac{\partial F}{\partial y^{n-1}} = 0 \qquad \text{at} \;\; x = x_1, \;\; \text{and} \;\; x = x_2. \tag{2.5.11}$$

Since the function F is known, Eq. (2.5.10) results in the boundary value problem governing the function $y(x)$. This function minimizes the functional Φ given by Eq. (2.5.1). Equation (2.5.11) are the natural boundary conditions derived through the integrations by parts of the functional.

2.6 Minimization of Functions of Several Variables

Consider a function $u(x, y)$ defined in the domain D enclosed by the boundary curve C. The functional Φ is assumed to be proportional to the function u and it's first partial derivatives with respect to the variables x and y as defined:

$$\Phi[u(x, y)] = \int_D F(x,\; y,\; u,\; u_x,\; u_y)dxdy \tag{2.6.1}$$

where u_x and u_y are the partial derivatives of the function u with respect to x and y. We assume the functional F to be at least differentiable up to the second order and the extremizing function $u(x,\; y)$ differentiable up to the first order. We further assume that the class of admissible functions $u(x,\; y)$ has the following properties;

a-$u(x, y)$ is prescribed on boundary curve C.

b-$u(x, y)$ and it's partial derivatives with respect to x and y up to the order one, are continuous in the domain D.

We assume that the function $u(x, y)$ is the only function among the class of admissible functions which minimizes the functional Φ. Now, the variational function $g(x, y)$ with the following properties is considered

a-$g(x, y) = 0$ for all the boundary points on C.

b-$g(x, y)$ is continuous and differentiable in D.

We construct the variational function as

$$\bar{u}(x,\; y) = u(x,\; y) + \epsilon\, g(x,\; y). \tag{2.6.2}$$

Substituting in Eq. (2.6.1) and carrying out the partial derivatives gives

$$\frac{\partial \Phi}{\partial \epsilon} = \frac{\partial}{\partial \epsilon} \int_D F\left[x,\ y, (u+\epsilon g), (u+\epsilon g)_x, (u+\epsilon g)_y\right] dxdy. \tag{2.6.3}$$

Carrying out the integration and setting $\epsilon = 0$ yields

$$\frac{\partial \Phi}{\partial \epsilon}|_{\epsilon=0} = \int_D \left(g\frac{\partial F}{\partial u} + g_x \frac{\partial F}{\partial u_x} + g_y \frac{\partial F}{\partial u_y} \right) dxdy. \tag{2.6.4}$$

The subscripts indicate differentiating with respect to x or y. To evaluate the integral, consider the expressions

$$\begin{aligned} \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x} g\right) &= \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)g + \frac{\partial F}{\partial u_x} g_x \\ \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y} g\right) &= \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)g + \frac{\partial F}{\partial u_y} g_y. \end{aligned} \tag{2.6.5}$$

Substituting Eq. (2.6.5) in Eq. (2.6.4), the last two terms become

$$\begin{aligned} \int_D \left(\frac{\partial F}{\partial u_x} g_x + \frac{\partial F}{\partial u_y} g_y\right) dxdy &= \int_D \left[\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x} g\right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y} g\right)\right] dxdy \\ &- \int_D \left[\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)\right] g\ dxdy. \end{aligned} \tag{2.6.6}$$

Here, $\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)$ is called the total partial derivative with respect to x, and in performing the partial derivatives with respect to x, the variable y remains constant, that is

$$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) = \frac{\partial^2 F}{\partial x \partial u_x} + \frac{\partial^2 F}{\partial u_x \partial u} u_x + \frac{\partial^2 F}{\partial u_x^2} \frac{\partial u_x}{\partial x} + \frac{\partial^2 F}{\partial u_x \partial u_y} \frac{\partial u_y}{\partial x} \tag{2.6.7}$$

and

$$\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) = \frac{\partial^2 F}{\partial y \partial u_y} + \frac{\partial^2 F}{\partial u_y \partial u} u_y + \frac{\partial^2 F}{\partial u_y \partial u_x} \frac{\partial u_x}{\partial y} + \frac{\partial^2 F}{\partial u_y^2} \frac{\partial u_y}{\partial y}. \tag{2.6.8}$$

Now, using Green's integral theorem, the area integral is transferred into the line integral as

$$\int_D \left(\frac{\partial M}{\partial y} + \frac{\partial N}{\partial x}\right) dxdy = \int_C (Ndy - Mdx). \tag{2.6.9}$$

Using this rule, we obtain

$$\int_D [\frac{\partial}{\partial x} (\frac{\partial F}{\partial u_x} g) + \frac{\partial}{\partial y} (\frac{\partial F}{\partial u_y} g)] dxdy = \int_C (\frac{\partial F}{\partial u_x} dy - \frac{\partial F}{\partial u_y} dx) g. \quad (2.6.10)$$

The right-hand side of this equation is integrated over the boundary curve C. From Eqs. (2.6.10) and (2.6.6), we get

$$\int_D (\frac{\partial F}{\partial u_x} g_x + \frac{\partial F}{\partial u_y} g_y) dxdy = -\int_D [\frac{\partial}{\partial x} (\frac{\partial F}{\partial u_x}) + \frac{\partial}{\partial y} (\frac{\partial F}{\partial u_y})] g \, dxdy$$
$$+ \int_C (\frac{\partial F}{\partial u_x} dy - \frac{\partial F}{\partial u_y} dx) g. \quad (2.6.11)$$

Substituting Eq. (2.6.11) into Eq. (2.6.4), and letting the expression Eq. (2.6.4) be zero, we obtain

$$\int_D [\frac{\partial F}{\partial u} - \frac{\partial}{\partial x} (\frac{\partial F}{\partial u_x}) - \frac{\partial}{\partial y} (\frac{\partial F}{\partial u_y})] g \, dxdy$$
$$+ \int_C (\frac{\partial F}{\partial u_x} dy - \frac{\partial F}{\partial u_y} dx) g = 0. \quad (2.6.12)$$

Since the function $g(x, \ y)$ is arbitrary,

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x} \frac{\partial F}{\partial u_x} - \frac{\partial}{\partial y} \frac{\partial F}{\partial u_y} = 0 \quad \text{in D.} \quad (2.6.13)$$

This equation is called the Euler equation, which is the boundary value problem associated with the functional Eq. (2.6.1). The natural boundary condition is

$$\int_C (\frac{\partial F}{\partial u_x} dy - \frac{\partial F}{\partial u_y} dx) = 0 \quad \text{on C} \quad (2.6.14)$$

2.7 Cantilever Beam

Consider a cantilever beam of arbitrary cross-sectional area and length L and bending stiffness EI subjected to a bending force F, as shown in Fig. 2.1.

We will now write the potential energy function of the beam under the applied force F, and by minimization of this function, using the method of calculus of variation, we obtain the Euler equation for the equilibrium of the beam.

The potential energy of the beam is the sum of two parts, *internal strain energy*, and the *strain energy of the external forces*, as

$$V = U + \Omega \quad (2.7.1)$$

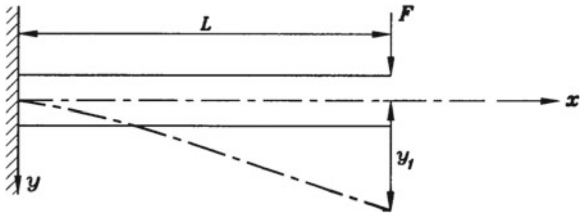

Fig. 2.1 A cantilever beam under transverse force F

where V is the total potential energy of the beam, U is the internal strain energy and Ω is the strain energy of the external forces.

From the strength of the material, it is recalled that the internal strain energy of a beam subjected to a bending force is obtained from the following relation

$$U = \int_0^L \frac{M^2 dx}{2EI} \tag{2.7.2}$$

where M is the bending moment distribution along the beam and dx is an element of the length of the beam the associated strain energy of which is dU. From the elementary beam theory, the bending moment M and the curvature $1/R$ are related by

$$M = \frac{EI}{R}. \tag{2.7.3}$$

Substituting M from Eq. (2.7.3) into Eq. (2.7.2) yields

$$U = \int_0^L \frac{EI}{2R^2}\, dx. \tag{2.7.4}$$

The radius of curvature R and the beam's elastic deflection equation are related as

$$R = \frac{(1 + y'^2)^{3/2}}{|y''|}. \tag{2.7.5}$$

Since for the small deformation, assumption $y' << 1$, y' is neglected compared to 1 in Eq. (2.7.5). Substituting the reduced form of Eq. (2.7.5) in Eq. (2.7.4) gives

$$U = \tfrac{1}{2} \int_0^L EI\, (y'')^2\, dx. \tag{2.7.6}$$

This equation is valid for an arbitrarily small deflection, provided that plastic deformation does not occur.

The potential energy of external force F is

$$\Omega = -F y_1 \tag{2.7.7}$$

where y_1 is the deflection under the force F in y-direction and the negative sign indicates that work is done on the system. The total potential energy from Eq. (2.7.1), after substituting from Eqs. (2.7.7) and (2.7.6), becomes

$$V = -F y_1 + \tfrac{1}{2} \int_0^L EI(y'')^2 \, dx. \tag{2.7.8}$$

In Eq. (2.7.8) y, the deflection function is a function of the variable x. As y takes on different values, the functional V varies. The equilibrium state of the beam occurs when the functional V has it's minimum value, and at this condition, $y(x)$ represents the deflection equation of the beam at equilibrium. The function $y(x)$ has to satisfy certain conditions. The function $y(x)$ and its first derivative $y'(x)$ have to be continuous over the interval $(0, L)$. Furthermore, the second derivative y'' must exist, and must be an integrable function. The function $y(x)$ must also satisfy the boundary conditions at the beam's boundary.

In this case, the boundary conditions at the clamped edge for a cantilever beam are $y(0) = y'(0) = 0$. These conditions are called the *essential boundary conditions* since they are physical constraints of the problem.

Now, we may apply the method of calculus of variation to obtain the minimum of the potential energy function and, thus, the equilibrium equation for the deflection of the beam. Let us define the variational function $\eta(x)$ and the variational parameter ϵ. The variational function $\eta(x)$ is arbitrary, and both itself and its derivative with respect to x are continuous functions between the interval $(0, L)$. The potential energy of the beam, corresponding to the deflection $\bar{y} = y + \epsilon\eta$, is

$$V(\bar{y}) = -F(y_1 + \epsilon\eta_1) + \tfrac{1}{2} \, EI \int_0^L (y'' + \epsilon\eta'')^2 dx \tag{2.7.9}$$

or

$$V(\bar{y}) = -F(y_1 + \epsilon\eta_1) + \tfrac{1}{2} \, EI[\int_0^L (y'')^2 dx + \int_0^L (\epsilon^2(\eta'')^2 dx$$
$$+ 2\int_0^L \epsilon y''(\eta'')^2 dx]$$

differentiating with respect to ϵ gives

$$\frac{\partial V}{\partial \epsilon} = -F\eta_1 + \tfrac{1}{2} EI[2\epsilon \int_0^L (\eta'')^2 dx + 2\int_0^L y'' \, \eta'' \, dx]$$

Setting $\epsilon = 0$ yields

$$\frac{\partial V}{\partial \epsilon}|_{\epsilon=0} = -F\eta_1 + EI\int_0^L y^{''}\ \eta^{''}\ dx = 0$$

where in the above equation ϵ is set equal to zero, and according to the rule of calculus of variation, the remaining expression is equal to zero. Two times integrations by parts give

$$-F\eta(L) + EIy^{''}\ \eta'|_0^L - EIy^{'''}\ \eta|_0^L + EI\int_0^L y^{IV}\ \eta\, dx = 0.$$

This may be written as

$$-F\eta(L) + EI[y^{''}(L)\eta'(L) - y^{'''}(L)\eta(L) + \int_0^L y^{IV}\ \eta\, dx] = 0. \tag{2.7.10}$$

In order that Eq. (2.7.10) vanishes for all the admissible functions $\eta(x)$, the following conditions must hold

$$y^{IV} = 0 \tag{2.7.11}$$

and

$$\begin{aligned} &F + EIy^{'''}(L) = 0 \qquad && y^{'''}(0) = 0 \\ &y^{''}(L) = 0 \qquad && EIy^{''}(0) = 0. \end{aligned} \tag{2.7.12}$$

The conditions Eq. (2.7.12) are known as the *natural boundary conditions*, since they are necessary to make the potential energy a minimum. Equation (2.7.11) is known as the Euler equation and is the equilibrium equation of the beam which minimizes the total potential energy equation under the given boundary conditions.

A general solution of Eq. (2.7.11) is

$$y = C_0 + C_1x + C_2x^2 + C_3x^3 \tag{2.7.13}$$

where $C_0,\ C_1,\ C_2$ and C_3 are the constants of integration. For the given essential boundary conditions, we have

$$C_0 = C_1 = 0. \tag{2.7.14}$$

The other two force boundary conditions related to the moment and shear force on $x = L$ give

$$C_2 = \frac{FL}{2EI} \qquad C_3 = \frac{-F}{6EI} \tag{2.7.15}$$

and therefore, the deflection equation of the beam becomes

$$y = \frac{FL}{2EI}\, x^2 - \frac{F}{6EI}\, x^3 \tag{2.7.16}$$

or

$$y = \frac{Fx^2}{2EI}\, (L - \frac{x}{3}) \tag{2.7.17}$$

2.8 Approximate Techniques

A system under equilibrium condition is considered, in which its equilibrium equation is described by the general form

$$L_{2m}[\psi] = f \tag{2.8.1}$$

where L_{2m} is a general type of mathematical operation of order $2m$ applied to the function ψ, and f is a known function of the given variables. The function ψ satisfies the general form of the boundary conditions given as

$$B_i[\psi] = g_i \qquad i = 1, 2, \ldots, 2m \tag{2.8.2}$$

where B_i is a linear mathematical operator describing the boundary conditions on ψ. Here, the known functions g_i are the given boundary conditions.

We further assume that the equilibrium equation Eq. (2.8.1) is associated with a variational problem such that the general expression for the functional Φ is

$$\Phi = \Phi(\psi). \tag{2.8.3}$$

An approximate solution of Eq. (2.8.1) may have the following linear form

$$\psi^* = \phi_0 + \sum_{j=1}^{n} C_j \phi_j \tag{2.8.4}$$

where the functions ϕ_j are linearly independent known functions of the variables in the solution domain D satisfying the homogeneous boundary conditions. Function ϕ_0 is a known function of the variables satisfying the nonhomogeneous boundary conditions, and the constants C_j are the undetermined parameters. With the above definitions, the functions ϕ_j in Eq. (2.8.4) satisfy the boundary conditions

$$\begin{aligned} &B_i[\phi_0] = g_i \quad i = 1, \ldots, 2m \\ &B_i[\phi_j] = 0 \quad i = 1, \ldots, 2m, \quad j = 1, \ldots, n. \end{aligned} \tag{2.8.5}$$

Thus, the function ψ of Eq. (2.8.1) satisfies all the boundary conditions for arbitrary values of the constant coefficients C_j.

We find the undetermined parameters C_j so that they make the functional Φ, related to system (2.8.1), stationary. In this case, a set of n simultaneous equations for the constants C_j must be obtained. This method may therefore be considered as a means for reducing a continuous equilibrium problem to an approximately equivalent equilibrium problem with n degrees of freedom. There are, however, two different approaches for finding the undetermined coefficients C_j: the weighted residual methods and the variational method. The weighted residual methods are based on four different techniques. These approaches are discussed in the following section.

2.8.1 *A: Weighted Residual Methods*

When the trial solution Eq. (2.8.4), which satisfies Eq. (2.8.5), is inserted into Eq. (2.8.1), the *residual equation* R is

$$R = f - L_{2m}[\psi^*] = f - L_{2m}[\phi_0 + \sum_{j=1}^{r} C_j \phi_j]. \tag{2.8.6}$$

For the exact solution, the residual R has to be identically zero. For a proper approximate solution, it should be restricted within a small tolerance. The classical weighted residual methods are as follows:

2.8.1.1 Collocation

The solution domain D is considered and n arbitrary points are selected inside the domain, usually with a known geometric pattern. The residual R of equation Eq. (2.8.6) is set equal to zero at n points in the domain D. That is

$$R = f - L_{2m}[\phi_0 + \sum_{j=1}^{n} C_j \phi_j] = 0. \tag{2.8.7}$$

This provides n simultaneous algebraic equations for the constants C_j. The locations of the points are arbitrary but, as mentioned, are usually such that D is covered more or less uniformly by a simple pattern.

2.8.1.2 Subdomain

The solution domain D is subdivided into n subdomains $D_i, \quad i = 1, 2, \ldots, n$, usually according to a simple pattern. Then, the integral of the residual Eq. (2.8.6) over each subdomain D_i is set equal to zero, as

$$\int_{D_i} R dD = 0 \quad i = 1, \ldots, n. \tag{2.8.8}$$

This equation provides a system of n algebraic equations to be solved for n constant coefficients C_j.

2.8.1.3 Galerkin

The complete solution domain is considered, and the residue R is made orthogonal with respect to the approximating functions ϕ_j over the whole domain as;

$$\int_{D} \phi_k R dD = 0 \quad k = 1, \ldots, n. \tag{2.8.9}$$

This equation provides a system of n algebraic equations for the n constant coefficients C_j.

The main difference between the collocation and subdomain methods and the Galerkin method is that, in the collocation and subdomain methods, the solution domain is divided into a number of elements and nodal points, while in the Galerkin method the solution domain is considered as a whole. This is called the *traditional Galerkin method*.

2.8.1.4 Least Square

Similar to the Galerkin method, the complete solution domain is Considered, and the integral of the square of the residue is minimized with respect to the constant coefficients C_j as

$$\frac{\partial}{\partial C_k} \int_{D} R^2 d\ D = 0 \quad k = 1, \ldots, n. \tag{2.8.10}$$

This equation provides a system of n algebraic equations with n unknowns C_j, which may be solved for C_j. If L_{2m} is a linear mathematical operator, then Eq. (2.8.10) is simplified as

$$-2 \int_{D} R\ L_{2m}[\phi_k] dD = 0 \quad k = 1, \ldots, n \tag{2.8.11}$$

2.8.2 B: Stationary Functional Method

Let Φ be a functional such that the extremum problem for Φ is equivalent to the equilibrium problem. The *Ritz method* consists of treating the extremum problem directly by inserting the trial family Eq. (2.8.6) into Φ and setting

$$\frac{\partial \Phi}{\partial C_j} = 0 \quad j = 1, \ldots, n. \tag{2.8.12}$$

These n equations are solved for the constants C_j, and when multiplied by their corresponding functions ψ, represent an approximate solution to the extremum problem. It is an approximate solution, because it gives Φ a stationary value only for the class of functions ψ which are part of the trial family Eq. (2.8.5).

The most important step in the above discussion is the selection of the trial family Eq. (2.8.5). The purpose of the above criterion is merely to pick the *best* approximation from a given family.

2.9 Further Notes on the Ritz and Galerkin Methods

In discussion of the boundary-value and extremum problems, it was concluded that any boundary-value problem representing a mechanical system in equilibrium is associated with an equivalent extremum problem in which it's corresponding functional is in a relative minimum condition.

Now, let us assume that the potential energy of a mechanical system is represented by the following double integral

$$\Phi(\psi) = \int \int_D F(x,\ y,\ \psi,\ \psi_x,\ \psi_y) dx dy \tag{2.9.1}$$

subjected to the condition

$$\psi = \phi(s) \quad \text{on} \quad \Gamma \tag{2.9.2}$$

where Γ is the contour bounding the region D and the subscript in Eq. (2.9.1) indicates the derivative with respect to x or y. Let ψ be the exact solution to this problem, and $\Phi(\psi) = m$ the value of the minimum. If we can find a function $\bar{\psi}(x,\ y)$ which satisfies the boundary condition Eq. (2.9.2) and for which the value of functional $\Phi(\bar{\psi})$ is very close to m, then $\bar{\psi}$ is a good approximation for the minimum of functional Eq. (2.9.1). On the other hand, if we can find a minimizing sequence $\bar{\psi}$, i.e., a sequence of functions satisfying the condition Eq. (2.9.2), and for which $\Phi(\bar{\psi}_n)$ approaches m, it would be expected that such a sequence would converge to the solution.

Ritz proposed a classical method in which one can find $\bar{\psi}$, a function which minimizes the integral Eq. (2.9.1), systematically. To describe the Ritz method, let us assume ψ to be a function of the variables x and y with n coefficients $a_1,\ a_2,\ .....,a_n$.

$$\psi = \psi(x,\ y,\ a_1,\ a_2,\ .....,a_n). \tag{2.9.3}$$

This function is chosen in such a way that, regardless of the values of a_n, ψ satisfies the boundary condition Eq. (2.9.2). The Ritz method is then based on calculating the coefficients a_1 through a_n for which ψ of equation Eq. (2.9.3) minimizes the integral Eq. (2.9.1). Upon substitution of equations Eq. (2.9.3) into Eq. (2.9.1), and performing the necessary differentiation and integration, we find that Φ is converted into a function of the coefficients $a_1,\ a_2, ..., a_n$. That is, $\Phi = \Phi(a_1,\ a_2, ..., a_n)$. To minimize this function, Ritz proved that the coefficients a_n must satisfy the following system of equations:

$$\frac{\partial \Phi}{\partial a_k} = 0 \qquad k = 1,\ 2, \ldots, n. \tag{2.9.4}$$

Let us assume that the solution to Eq. (2.9.4) for n coefficients a_n is $\bar{a_1},\ \bar{a_2}, ..., \bar{a_n}$. Substituting this solution into Eq. (2.9.3) for ψ, we obtain

$$\bar{\psi}(x,\ y) = \bar{\psi}(x,\ y,\ \bar{a_1},\ \bar{a_2}, ..., \bar{a_n}) \tag{2.9.5}$$

for which $\bar{\psi}$ is now the minimum of integral Eq. (2.9.1).

Now, let us apply the Ritz method to obtain a close approximation to the actual minimum using a family of functions

$$\psi(x,\ y) = \psi_n(x,\ y,\ a_1,\ a_2, ..., a_n) \quad n = 1.\ 2, \ldots \tag{2.9.6}$$

Let $\bar{\psi_n}$ be the nth. approximation giving the last value for integral Φ in comparison with all the functions up to the nth. family. Since each successive family contains all the functions of the preceding, i.e., for each successive problem the class of admissible functions is broader, it is clear that the successive minimums are non-increasing,

$$\Phi(\bar{\psi_1}) \geq\ \Phi(\bar{\psi_2}) \geq \Phi(\bar{\psi_n}). \tag{2.9.7}$$

The Galerkin method is an approximate numerical technique which directly solves the boundary-value problems. This method is particularly suitable for nonlinear problems due to its fast rate of convergence. In order to describe the method, we assume a boundary value problem represented by the following differential equation:

$$L(y) = f(x) \tag{2.9.8}$$

subjected to homogeneous boundary conditions

$$\begin{aligned} y(x_1) &= 0 \\ y(x_2) &= 0 \end{aligned} \tag{2.9.9}$$

where L is a mathematical operator and $f(x)$ is a known function. Note that nonhomogeneous boundary conditions of

$$\begin{aligned} y(x_1) &= y_1 \\ y(x_2) &= y_2 \end{aligned} \tag{2.9.10}$$

can be transformed to the homogeneous conditions Eq. (2.9.9) with a proper change of variables.

Let us choose a set of continuous linearly independent functions $w_i(x)$ in the interval $(x_1 - x_2)$ that satisfy the boundary conditions Eq. (2.9.9), that is,

$$w_i(x_1) = w_i(x_2) = 0 \quad i = 1,\ 2, \ldots, n. \tag{2.9.11}$$

We seek the solution of the Eq. (2.9.8) in the form of

$$y_n = \sum_{i=1}^{n} a_i\ w_i(x) \tag{2.9.12}$$

where a_i's are constant coefficients to be determined.

Galerkin suggested that, in order to find the coefficients a_i, the following orthogonality condition must be satisfied by the functions $w_i(x)$ in the interval $(x_1,\ x_2)$

$$\int_{x_1}^{x_2} [L(\sum_{i=1}^{n} a_i\ w_i(x)) - f(x)]w_i(x)dx = 0 \qquad i = 1,\ 2, \ldots, n. \tag{2.9.13}$$

When the number of functions $w_i(x)$ tends to infinity, $(n \longrightarrow \infty)$, the solution tends to the exact solution. In order to solve for the coefficients a_i, the linear set of Eq. (2.9.13) has to be solved for the unknowns a_i (for discussion and solution of such a system of equations, one may refer to Kantrovich and Krylov [1]). In practical cases, a finite number of series Eq. (2.9.12) are considered from which, upon substitution in Eq. (2.9.13), a finite set of linear equations are obtained to solve for a_i.

The functions $w_i(x)$ are usually selected in polynomial or trigonometric forms as

$$\begin{aligned} &(x - x_1)(x - x_2) \qquad (x - x_1)^2(x - x_2) \qquad (x - x_1)^n(x - x_2) \\ &\sin \frac{n\ \pi(x - x_1)}{x_2 - x_1} \qquad n = 1,\ 2, \ldots \end{aligned} \tag{2.9.14}$$

It is obvious that the origin of the coordinate system can be transformed to x_1 and, thus, in Eq. (2.9.14) $x_1 = 0$.

The Galerkin method is a powerful tool for obtaining an approximate solution for the ordinary differential equations of any order n, systems of differential or partial differential equations.

2.10 Application of the Ritz Method

We will now apply the Ritz method to the solution of an ordinary differential equation of the second order [1]

$$L(y) = \frac{d}{dx}(py') - qy - f = 0 \tag{2.10.1}$$

under the homogeneous boundary conditions

$$y(0) = 0, \qquad y(L) = 0. \tag{2.10.2}$$

It may be verified that Eq. (2.10.1) is the minimum of the functional

$$\Phi(y) = \int_0^L [py'^2 + qy^2 + 2fy]dx \tag{2.10.3}$$

subjected to the boundary conditions Eq. (2.10.2). We furthermore assume that in the given interval the following inequalities are satisfied:

$$p(x) > 0 \qquad q(x) \geq 0 \qquad 0 \leq x \leq L. \tag{2.10.4}$$

Let us now take a series of linearly independent functions $\phi_k(x)$, $k = 1, 2, \ldots, n$, continuous in the interval $[0, L]$ together with their first derivatives and satisfying the conditions Eq. (2.10.2). Such a series of functions may be taken as, for example,

$$\begin{aligned} \phi_k &= \sin\frac{k\pi x}{L} \\ \phi_k &= (L - x)x^k \qquad k = 1, 2, \ldots, n. \end{aligned} \tag{2.10.5}$$

We now apply the Ritz method to obtain the minimum of the functional Eq. (2.10.3) using the series of linear combinations of the functions ϕ_k. We seek a solution in the form of

$$y_n = \sum_{k=1}^{n} a_k \phi_k. \tag{2.10.6}$$

Substituting y_n in Eq. (2.10.3), gives

$$\begin{aligned} \Phi(y_n) &= \int_0^L [py_n'^2 + qy_n^2 + 2fy_n]dx \\ &= \int_0^L [p(\sum_{k=1}^{n} a_k\phi_k')^2 + q(\sum_{k=1}^{n} a_k\phi_k)^2 + 2f(\sum_{k=1}^{n} a_k\phi_k)]dx. \end{aligned} \tag{2.10.7}$$

But

$$(\sum_{k=1}^{n} a_k \phi_k')^2 = \sum_{k=1}^{n}\sum_{s=1}^{n} a_k \; a_s \; \phi_k' \phi_s'. \tag{2.10.8}$$

Therefore

$$\Phi(y_n) = \int_0^L [p \sum_{k=1}^{n}\sum_{s=1}^{n} a_k a_s \phi_k' \phi_s' + q \sum_{k=1}^{n}\sum_{s=1}^{n} a_k a_s \phi_k \phi_s + 2f \sum_{k=1}^{n} a_k \phi_k] dx. \tag{2.10.9}$$

Calling

$$\alpha_{k,s} = \alpha_{s,k} = \int_0^L (p\phi_k' \phi_s' + q\phi_k \phi_s) dx$$
$$\beta_k = \int_0^L f \phi_k dx. \tag{2.10.10}$$

We have

$$\Phi(y_n) = \sum_{k=1}^{n}\sum_{s=1}^{n} \alpha_{k,s} a_k a_s + 2 \sum_{k=1}^{n} \beta_k a_k. \tag{2.10.11}$$

Taking the derivative with respect to a_s

$$\tfrac{1}{2} \frac{d\Phi(y_n)}{da_s} = \sum_{k=1}^{n} \alpha_{k,s} a_k + \beta_s = 0 \tag{2.10.12}$$

or

$$\tfrac{1}{2} \frac{d\Phi(y_n)}{da_s} = \sum_{k=1}^{n} \int_0^L (p\phi_k' \phi_s' + q\phi_k \phi_s) a_k dx + \int_0^L f \phi_s dx = 0. \tag{2.10.13}$$

Multiplying a_k through the parentheses

$$\tfrac{1}{2} \frac{d\Phi(y_n)}{da_s} = \int_0^L (\sum_{k=1}^{n} a_k p \phi_k' \phi_s' + \sum_{k=1}^{n} q \phi_k \phi_s a_k + f \phi_s) dx = 0 \tag{2.10.14}$$

or, finally

$$\int_0^L (p y_n' \phi_s' + q y_n \phi_s + f \phi_s) dx = 0 \qquad s = 1, 2, \ldots, n. \tag{2.10.15}$$

Equation (2.10.15) represents a set of n integral equations to be solved for a_k. Using the rule of integration by parts gives

$$\int_0^L p y_n' \phi_s' dx = [p y_n' \phi_s]_0^L - \int_0^L \frac{d}{dx}(p y_n')\phi_s dx. \tag{2.10.16}$$

The first term in the right-hand side of the above equation vanishes, as ϕ_s vanishes at 0 and L, and thus

$$\int_0^L p y_n' \phi_s' dx = -\int_0^L \frac{d}{dx}(p y_n')\phi_s dx. \tag{2.10.17}$$

Substituting in Eq. (2.10.15), yields

$$\int_0^L [\frac{d}{dx}(p y_n') - q y_n - f]\phi_s dx = 0 \tag{2.10.18}$$

or, finally

$$\int_0^L L(y_n)\phi_s dx = 0. \tag{2.10.19}$$

Noticed that application of the Ritz method in this case reduced the problem to that of the Galerkin method.

2.10.1 Non-homogeneous Boundary Conditions

In the previous section, we discussed application of the Ritz method to problems with homogeneous boundary conditions. Now, let us consider a problem with a general non-homogeneous boundary condition as

$$\begin{aligned} y(x_1) &= y_1 \\ y(x_2) &= y_2. \end{aligned} \tag{2.10.20}$$

For this case, we will take the solution in the form

$$y_n = \sum_{k=1}^{n} a_k \phi_k + \phi_0(x) \tag{2.10.21}$$

where $\phi_0(x)$ satisfies the given nonhomogeneous boundary conditions. Since the known functions $\phi_j(x)$ satisfy the homogeneous boundary conditions

$$\phi_k(x_1) = \phi_k(x_2) = 0 \quad k = 1, 2, \ldots, n \tag{2.10.22}$$

thus, the function ϕ_0 must satisfy the nonhomogeneous conditions as

$$\phi_0(x_1) = y_1$$
$$\phi_0(x_2) = y_2. \tag{2.10.23}$$

As an example, considering a linear approximation, function $\phi_0(x)$ has the following form:

$$\phi_0(x) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) + y_1 \tag{2.10.24}$$

where y_1 and y_2 are given values.

Example 1 Consider an ordinary differential equation such as

$$y'' + y + x = 0 \tag{2.10.25}$$

subject to the boundary conditions

$$y(0) = y(1) = 0. \tag{2.10.26}$$

It is required to find an approximate solution of the equation using the Ritz method.

Solution: The exact solution of the above differential equation, using the classical method for the solution of a differential equation with constant coefficients, is

$$y = \frac{\sin x}{\sin 1} - x. \tag{2.10.27}$$

Now, the approximate solution of Eq. (2.10.25) is found and compared with Eq. (2.10.27).

The corresponding expression for the functional of Eq. (2.10.25) is

$$I = \int_0^1 (y'^2 + y^2 - 2xy)dx. \tag{2.10.28}$$

Comparing Eq. (2.10.28) with Eq. (2.10.3) reveals that $p = 1, q = -1$, and $f = -x$.

The solution is approximated with one term of the series Eq. (2.10.5) as

$$y_1 = a_1\phi_1 = a_1 x(1 - x). \tag{2.10.29}$$

Substituting the approximate solution Eq. (2.10.29) in the expression for the functional, using Eq. (2.10.19), gives

$$\int_0^1 L(y_1)\phi_1 dx = \int_0^1 [-2a_1 + a_1 x(1 - x) + x]x(1 - x)dx = 0.$$

Multiplying and integrating gives

$$
\begin{aligned}
&[\frac{a_1}{5}\,x^5 - \frac{1+2a_1}{4}\,x^4 + \frac{1+3a_1}{3}\,x^3 - a_1x^2]_0^1 = 0\\
&\frac{a_1}{5} - \frac{1+2a_1}{4} + \frac{1+3a_1}{3} - a_1 = 0\\
&a_1 = \frac{5}{18}
\end{aligned}
$$

and thus, the solution is

$$y_1 = \frac{5}{18}\,x(1-x) \tag{2.10.30}$$

Example 2 Consider again the same problem as in Example (1), but with an approximate solution with two terms of the series being considered as

$$\phi_1 = x(1-x) \quad , \quad \phi_2 = x^2(1-x)$$

and

$$y_2 = x(1-x)(a_1 + a_2x).$$

Substituting in Eq. (2.10.19) gives

$$\int_0^1 L(y_2)\phi_1 dx = 0 \tag{2.10.31}$$

and

$$\int_0^1 L(y_2)\phi_2 dx = 0. \tag{2.10.32}$$

Substituting for y_2, ϕ_1, and ϕ_2 in Eqs. (2.10.31) and (2.10.32) yields

$$
\begin{aligned}
&\int_0^1 [y_2'' + x(1-x)(a_1 + a_2x) + x]x(1-x)dx = 0\\
&\int_0^1 [y_2'' + x(1-x)(a_1 + a_2x) + x]x^2(1-x)dx = 0
\end{aligned}
$$

or

$$
\begin{aligned}
&\int_0^1 [2(a_2 - a_1) - (6a_2 - 1 - a_1)x + (a_2 - a_1)x^2 - a_2x^3](x - x^2)dx = 0\\
&\int_0^1 [2(a_2 - a_1) + (a_1 + 1 - 6a_2)x + (a_2 - a_1)x^2 - a_2x^3](x^2 - x^3)dx = 0.
\end{aligned}
$$

Multiplying and integrating, yields

Table 2.1 Comparison of the exact solution with one-and two-term approximate solutions

	y	y_1	y_2
x=1/4	0.044	0.052	0.044
x=1/2	0.070	0.069	0.069
x=3/4	0.060	0.052	0.060

$$18a_1 + 9a_2 - 5 = 0$$
$$\frac{3}{20}\,a_1 + \frac{13}{105}\,a_2 - \frac{1}{20} = 0.$$

Solving for a_1 and a_2 gives

$$a_1 = \frac{71}{369}, \quad a_2 = \frac{7}{41}$$

or

$$y_2 = x(1-x)(\frac{71}{369} + \frac{7}{41}\,x). \tag{2.10.33}$$

Now, the exact solution of the differential equation Eq. (2.10.25) is compared with the one-term and two-term approximate solutions. The exact solution from Eq. (2.10.27) is called y, the one-term approximate solution from Eq. (2.10.30) is called y_1, and the two-term approximate solution from Eq. (2.10.33) is called y_2. All three solutions satisfy the given boundary conditions at $x = 0$ and $x = 1$. To compare the three solutions, their values at $x = 1/4$, $x = 1/2$, and $x = 3/4$ are calculated and shown in Table 2.1.

Comparing the results, it is seen that the error of the first approximation is about $\%15$ and that of the second approximation about $\%1$.

Example 3 Consider the Bessel differential equation

$$x^2y'' + xy' + (x^2 - 1)y = 0 \tag{2.10.34}$$

defined in the interval $1 \le x \le 2$. The boundary conditions at $x = 1$ and $x = 2$ are assumed as

$$y(1) = 1 \qquad\qquad y(2) = 2. \tag{2.10.35}$$

The exact solution of the assumed Bessel differential equation under the given boundary conditions is

$$y = 3.6072I_1(x) + 0.75195Y_1(x) \tag{2.10.36}$$

where I_1 and Y_1 are the modified Bessel functions of the first and second types and of order 1.

Now, the solution of Eq. (2.10.34) may be approximately obtained using the Galerkin method. Let us change the dependent function y to z by the transformation

Table 2.2 Comparison of the exact solution with a one-term approximate solution

x	y	y_1
1.3	1.4706	1.4703
1.5	1.7026	1.7027
1.8	1.9294	1.9297

$y = z + x$. Then, Eq. (2.10.34) transforms into the following form:

$$xz'' + z' + \frac{x^2 - 1}{x} z + x^2 = 0. \tag{2.10.37}$$

The boundary conditions in terms of the function z become $z(1) = z(2) = 0$. We assume the solution by one-term approximation $z = a_1\phi_1$, with $\phi_1 = (x-1)(2-x)$. Applying the Galerkin method to Eq. (2.10.37) gives

$$\int_1^2 [xz_1'' + z_1' + \frac{x^2 - 1}{x} z_1 + x^2]\phi_1 dx = 0. \tag{2.10.38}$$

Substituting for z_1 yields

$$\int_1^2 [-2a_1 x + (3 - 2x)a_1 + \frac{x^2 - 1}{x}(x - 1)(2 - x)a_1 + x^2] (x - 1)(2 - x)dx = 0. \tag{2.10.39}$$

Solving for a_1 gives

$$a_1 = 0.8110 \tag{2.10.40}$$

and the approximate solution of Eq. (2.10.34) with one-term approximation for y_1 becomes

$$y_1 = 0.8110(x - 1)(2 - x) + x \tag{2.10.41}$$

The exact solution Eq. (2.10.36) is compared with the one-term approximate solution Eq. (2.10.41) in the following at three different locations (Table 2.2).

It should be noted that a very close agreement is reached with even the one-term approximation.

2.11 Problems

1. When the equilibrium problems Eq. (2.8.1) and Eq. (2.8.2) are linear, the weighted-residual methods to the trial family Eq. (2.8.4) all lead to equations for the C_j having the following form:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1r} \\ a_{21} & a_{22} & \dots & a_{2r} \\ .. & .. & .. & .. \\ a_{r1} & a_{r2} & \dots & a_{rr} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_r \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_r \end{bmatrix}$$

Show that for collocation

$$a_{kj} = L_{2m}[\phi_j(P_k)] \qquad b_k = f(P_k) - L_{2m}[\phi_0(P_k)]$$

where the P_k are the r locations arbitrarily selected. Show that for the subdomain method

$$a_{kj} = \int_{D_k} L_{2m}[\phi_j]dD \qquad b_k = \int_{D_k} \{f - L_{2m}[\phi_0]\}dD$$

where the D_k are the r selected subdomains. Show that for the Galerkin method

$$a_{kj} = \int_D \phi_k\, L_{2m}[\phi_j]dD \qquad b_k = \int_D \phi_k\, (f - L_{2m}[\phi_0])dD$$

and for the least-square method

$$a_{kj} = \int_D L_{2m}[\phi_k]L_{2m}[\phi_j]dD \qquad b_k = \int_D L_{2m}[\phi_k](f - L_{2m}[\phi_0])dD.$$

Note that in every case the matrix A has to do with the characteristics of the system and that the matrix B is related to the *loading* in the domain and acting on the boundary.

2. Show that the equation applying to the unknown value of an approximate solution to Poisson's equation at a nodal point is the same by either the finite element or finite difference method (solve this problem after the introduction to the finite element method).
3. Employing the Galerkin method, solve the following differential equation:

$$y^{''''} + (Ax + B)y = C$$
$$y(x_1) = 0$$
$$y(x_2) = 0$$

where A, B and C are constants. Solve this problem first by taking $n = 1$ in Eq. (2.9.12), that is, take only one term of the series. Then, solve the problem by taking $n = 2$ and compare the results. Any numerical values may be assumed for the constants A, B, and C.
4. Consider a functional given in the form

$$\Phi[(u(x, y)] = \int_D F(x,\ y,\ u,\ u_x,\ u_y,\ u_{xx},\ u_{yy},\ u_{xy}) dxdy$$

where u_x and u_y are the first partial derivatives of the function u, and u_{xx}, u_{yy}, and u_{xy} are the second partial derivatives with respect to x and y. We assume the functional F to be at least differentiable up to the third order, and the extremizing function $u(x,\ y)$ differentiable up to the second order. We further assume that the class of admissible function $u(x,\ y)$ has the following properties;
a-$u(x,\ y)$ is prescribed on boundary curve C.
b-$u(x,\ y)$ and it's partial derivatives with respect to x and y up to the second order are continuous in the domain D.
Using the method of calculus of variations, obtain the associated Euler equation and the natural boundary conditions.

Further Readings

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Chapter 3
Finite Element of Elastic Membrane

Abstract To physically show the finite element approximation, an elastic membrane is selected. By the finite element method, we approximate the height of an elastic membrane. This provides a physical feeling of what the finite element approximation means. In addition, since the governing equation of an elastic membrane is Poisson's equation, many other mechanical problems governed by Poisson's equation may be treated by the detailed mathematical derivations of this chapter.

3.1 Introduction

Analysis of a finite element is described in this chapter. To give a geometric and physical explanation of the finite element method, an elastic membrane is selected. The governing equation of an elastic shallow membrane, which describes the height of the membrane due to the inside pressure, is Poisson's equation. According to the finite element technique, the domain of the membrane is described by a number of elements, and the height of the membrane over each element is approximated with a given shape function. The outcome of the finite element model of the elastic membrane is a geometric model, which is simple to imagine. As a result, the reader would receive a physical feeling for the finite element approximation. For example, if the membrane's domain is divided into a finite number of triangular elements, and the shape function over each element is selected as linear, it means that the smooth space surface for the height of the membrane is approximated with a flat plane over each triangular element.

Another important feature of this chapter is the derivation of the membrane functional. The related functional of the elastic membrane is obtained using the total potential energy of the elastic membrane. Now, since the equilibrium equation of the elastic membrane is similar to many other problems, such as conduction heat transfer or potential flow problems, we may thus write the functional of such problems from the membrane analogy. Otherwise, it is not possible to write the functional of the

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Solid Mechanics and Its Applications 216, DOI: 10.1007/978-3-319-08037-6_3,

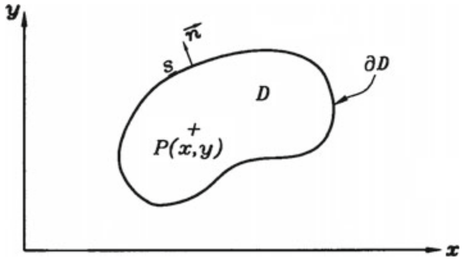


Fig. 3.1 Membrane domain in xy—plane

heat conduction problems right from the energy principals of thermodynamics. The variational formulation of heat and fluid flow problems are not yet obtained.

3.2 Poisson's Equation

The problem is to determine ϕ governed by equation

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=-f(x,\ y) \tag{3.2.1}$$

over D, where ϕ is an unknown function and $f(x, y)$ is a specified function of x and y. Furthermore, on the boundary ∂D, any of the following boundary conditions may be specified (Fig. 3.1).

$$\phi = f_1(s) \text{ on } \partial D, \quad \text{where } f_1 \text{ is specified} \tag{3.2.2}$$

$$\frac{\partial\phi}{\partial n} = f_2(s) \text{ on } \partial D, \quad \text{where } f_2 \text{ is specified} \tag{3.2.3}$$

$$\frac{\partial\phi}{\partial n} + h[\phi - f_3(s)] = 0 \text{ on } \partial D, \quad \text{where } f_3 \text{ and } h \text{ are specified.} \tag{3.2.4}$$

Any combination of Eq. (3.2.1) with the boundary conditions (3.2.2), (3.2.3), or (3.2.4) is called a boundary value problem (BVP). More specifically, if $f(x, y) = 0$ over D, Eq. (3.2.1) is called the plane *Dirichlet* problem. Note that Eq. (3.2.1) in this case, where $f = 0$, is called the Laplace equation. If $f(x, y) = cte$, then Eq. (3.2.1) is called Poisson's equation. Also, Eqs. (3.2.3) and (3.2.4) are called *Dirichlet* and *Neumann* boundary conditions, respectively.

3.2.1 Physical Examples

1. Steady state heat conduction: the governing differentiation equation is

$$k\nabla^2 T = Q \tag{3.2.5}$$

where T is the absolute temperature at any point, k is the coefficient of thermal conduction, and Q is the rate of energy generation per unit volume.

2. Torsion of the prismatic bar: the governing differential equation is

$$\nabla^2 \phi = -2G\theta \tag{3.2.6}$$

where ϕ is the stress function, G is the torsional rigidity, and θ is the angle of twist per unit length of the prismatic bar.
3. Pressure distribution in a porous media: the governing differential equation is

$$\nabla^2 p = 0 \tag{3.2.7}$$

where p is the pressure in the porous media.
4. Fluid dynamics: Consider a steady-state inviscid irrotational flow of an incompressible fluid. The potential function and the stream function are governed by the following equations:

$$\begin{aligned} \nabla^2 \phi &= 0, \quad \phi = f_1(s) \;\text{ on }\; \partial D \\ \nabla^2 \psi &= 0, \quad \psi = f_2(s) \;\text{ on }\; \partial D \end{aligned} \tag{3.2.8}$$

where ϕ is the potential function and ψ is the stream function. The velocity components in a two-dimensional fluid flow are related to the potential and stream functions as

$$\begin{aligned} u &= \frac{\partial \phi}{\partial x} \qquad & v &= \frac{\partial \phi}{\partial y} \\ u &= -\frac{\partial \psi}{\partial y} \qquad & v &= \frac{\partial \psi}{\partial x} \end{aligned} \tag{3.2.9}$$

where u is the velocity component along the x—axis and v is the velocity component along the y—axis.

The physical problems given in the above examples are all governed by Poisson's, or the Laplace, equation and their analytical solution may be obtained by the method of separation of variables. The final solution, however, requires calculations of the separation constants using the boundary conditions. Unless the solution domain does not have a regular shape, such as rectangular or circular, the integration constants can not be obtained. Using the finite element method, the numerical solution may be obtained for any type of boundary condition. Since the governing equation of the elastic membrane is identical with the above physical examples, all the mathematical and numerical formulations of the elastic membrane will be applicable to the above

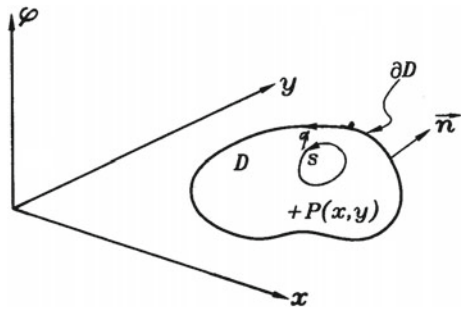


Fig. 3.2 A shallow membrane

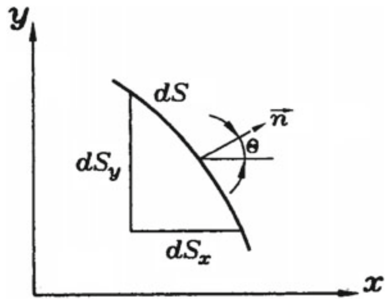


Fig. 3.3 Boundary of elastic membrane

examples. Thus, it is worth studying the mathematical and finite element formulations of the elastic membrane.

3.3 Weightless Elastic Membrane (Method I)

Consider a weightless elastic membrane based in an x–y plane, as shown in Fig. 3.2. Let $\phi(x, y)$ be the vertical displacement of the membrane and T be the stretched force per unit length of the membrane surface, which is assumed to be constant. Then, Tds is the net force for a length ds of the membrane boundary. The force Tds is tangent to the membrane surface. The vertical component of Tds for the shallow membrane, in ϕ-direction, is $Tds \times \partial\phi/\partial n$, where $\vec{n}$ is the unit outer normal vector to the boundary. This vertical force, integrated over the boundary of the membrane, should be balanced with the vertical force of the inside pressure under the membrane, as

$$\oint_{\partial D} T \frac{\partial \phi}{\partial n} \, dS + \int_D p dxdy = 0. \tag{3.3.1}$$

Recall from Green's theorem in plane and Fig. 3.3

$$\int_D (\vec{\nabla} \times \vec{A}) \cdot \vec{k}\, dxdy = \oint_{\partial D} \vec{A} \cdot d\vec{S}$$

$$d\vec{S} = dS_x \vec{i} + dS_y \vec{j}$$

$$d\vec{S} = (-dS \sin\theta)\vec{i} + (dS \cos\theta)\vec{j}$$

$$d\vec{S} = \left(-dS\frac{dy}{dn}\right)\vec{i} + \left(dS\frac{dx}{dn}\right)\vec{j}. \tag{3.3.2}$$

Note that

$$\frac{\partial\phi}{\partial n} = \frac{\partial\phi}{\partial x}\frac{dx}{dn} + \frac{\partial\phi}{\partial y}\frac{dy}{dn}. \tag{3.3.3}$$

Therefore

$$\oint_{\partial D} T\frac{\partial\phi}{\partial n}\, dS = \oint_{\partial D} T\left(-\frac{\partial\phi}{\partial y}\,\vec{i} + \frac{\partial\phi}{\partial x}\,\vec{j}\right) \cdot \left(-\frac{dy}{dn}\,\vec{i} + \frac{dx}{dn}\,\vec{j}\right) dS = \oint_{\partial D} T\ \vec{A} \cdot d\vec{S} \tag{3.3.4}$$

where

$$\vec{A} = -\frac{\partial\phi}{\partial y}\,\vec{i} + \frac{\partial\phi}{\partial x}\,\vec{j}. \tag{3.3.5}$$

Using Green's theorem in the plane, the line integral in Eq. (3.3.4) is converted to

$$\oint_{\partial D} T\frac{\partial\phi}{\partial n}\, dS = \oint_{\partial D} T\ \vec{A} \cdot d\vec{S} = \int_D T\ (\vec{\nabla} \times \vec{A}) \cdot \vec{k}\, dxdy = \int_D T\nabla^2\phi dxdy. \tag{3.3.6}$$

Substituting in Eq. (3.3.1), gives

$$\int_D (T\nabla^2\phi + p)dxdy = 0. \tag{3.3.7}$$

Since the above equation should hold for all values of the integrand, therefore

$$T\nabla^2\phi + p = 0. \tag{3.3.8}$$

3.4 Membrane Analysis (Method II)

Let ϕ be the vertical displacement of the membrane due to the inside pressure p, and T the membrane tension. The section of the membrane in ϕ–x is shown in Fig. 3.4. A strip of the membrane with width dy is considered. The tension force of the membrane across the strip width is Tdy. For a shallow membrane, where $\sin\alpha \approx \tan\alpha = \partial\phi/\partial x$, the net force in the vertical direction between the nodes of an element dx of the strip is

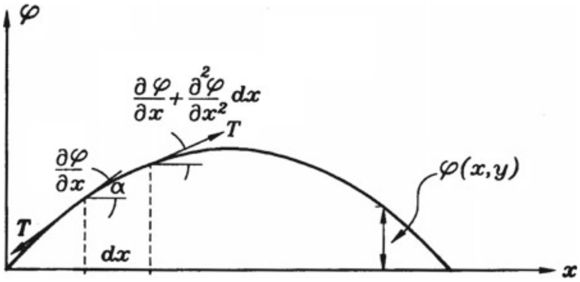


Fig. 3.4 Section of elastic membrane in ϕ–x plane

$$Tdy\left(\frac{\partial\phi}{\partial x}+\frac{\partial^2\phi}{\partial x^2}\,dx\right)-Tdy\left(\frac{\partial\phi}{\partial x}\right)=T\frac{\partial^2\phi}{\partial x^2}\,dxdy. \tag{3.4.1}$$

Now, a strip of width dx of the membrane in the y–ϕ plane is considered and the net vertical component of the force between the ends of an element dy of the strip is

$$Tdx\left(\frac{\partial\phi}{\partial y}+\frac{\partial^2\phi}{\partial y^2}\,dy\right)-Tdx\left(\frac{\partial\phi}{\partial y}\right)=T\frac{\partial^2\phi}{\partial y^2}\,dxdy. \tag{3.4.2}$$

Adding the vertical components of the forces of an element $dxdy$ of the membrane from Eqs. (3.4.1) and (3.4.2) results in

$$T\frac{\partial^2\phi}{\partial x^2}\,dxdy+T\frac{\partial^2\phi}{\partial y^2}\,dxdy. \tag{3.4.3}$$

These forces must be balanced with the vertical force of the inside pressure $pdxdy$, and thus, from a summation of the forces in the vertical direction, we have

$$T\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)+p=0. \tag{3.4.4}$$

Equation (3.4.4) may be solved by analytical or numerical methods. The boundary conditions on ϕ are of the form

$$\phi=f_1(S)\quad\text{on }\partial D \tag{3.4.5}$$

$$\frac{\partial\phi}{\partial n}=f_2(S)\quad\text{on }\partial D \tag{3.4.6}$$

where $f_1(S)$ and $f_2(S)$ are known functions on the boundary.

Equation (3.4.5) describes that the edge displacement of the membrane is fixed. In this case, the prescribed displacement on the edge is $f_1(S)$. Equation (3.4.6) is

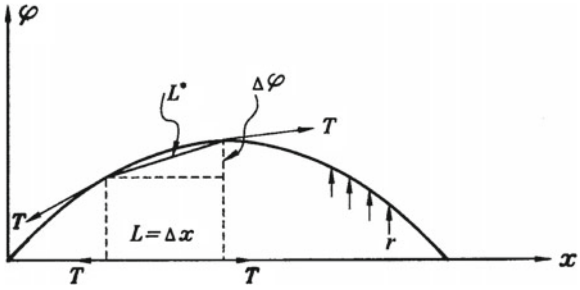


Fig. 3.5 Section of elastic membrane in ϕ–x plane

related to the edge of the membrane where force is prescribed. That is, $T\dfrac{\partial\phi}{\partial n}$ is equal to the vertical component of force per unit length along the membrane edge.

3.5 Strain Energy of Elastic Membrane

Recall that the potential energy associated with a conservative system in stable equilibrium is at a relative minimum. Let us use this law for the formulation of the extremum problem of a membrane. Consider an elastic weightless membrane under inside pressure. The deflection of the membrane is assumed to be small (shallow membrane assumption). Also, assume that the membrane tension T is constant. Take an elastic strip, as shown in Fig. 3.5. The change of potential energy is

$$\text{Change in potential energy} = \Delta V = \Delta U + \Delta\Omega \tag{3.5.1}$$

where

$$\begin{aligned}\Delta U &= \text{change in internal strain energy}\\ \Delta\Omega &= \text{change in potential energy of the external forces.}\end{aligned} \tag{3.5.2}$$

The change in internal strain energy can be calculated as

$$\Delta U = T(L^* - L) = T(\sqrt{\Delta\phi^2 + \Delta x^2} - \Delta x)$$
$$\Delta U = T\left[\left(\sqrt{1 + \left(\frac{\partial\phi}{\partial x}\right)^2}\right)\Delta x - \Delta x\right].$$

The radical in the above equation may be expanded by the binomial series as

$$(1+a)^n = 1 + na + \frac{n(n-1)a^2}{2!} + \frac{n(n-1)(n-2)}{3!}\,a^3 + \cdots \tag{3.5.3}$$

where, by comparison, $a = (\frac{\Delta\phi}{\Delta x})^2$ and $n = \frac{1}{2}$. Recalling the small deflection assumption, the higher order terms may be neglected, and therefore

$$\sqrt{1 + \left(\frac{\Delta\phi}{\Delta x}\right)^2} = 1 + \tfrac{1}{2}\left(\frac{\Delta\phi}{\Delta x}\right)^2 + \cdots . \tag{3.5.4}$$

Thus

$$\Delta U = \left\{\left[1 + \tfrac{1}{2}\left(\frac{\Delta\phi}{\Delta x}\right)^2\right]\Delta x - \Delta x\right\} T$$

$$\Delta U = \frac{T}{2}\left(\frac{\Delta\phi}{\Delta x}\right)^2 \Delta x \tag{3.5.5}$$

or

$$dU = \frac{T}{2}\left(\frac{d\phi}{dx}\right)^2 dx. \tag{3.5.6}$$

Extending the above result, which is the internal strain energy of a strip, to a differential element of a two-dimensional membrane yields

$$dU = \frac{T}{2}\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right] dxdy. \tag{3.5.7}$$

The potential energy of the external forces (inside pressure) is

$$d\Omega = -p\phi dxdy. \tag{3.5.8}$$

Substituting Eqs. (3.5.7) and (3.5.8) in (3.5.1), yields

$$V = \int_D \left\{\frac{T}{2}\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right] - p\phi\right\} dxdy. \tag{3.5.9}$$

Equation (3.5.9) is the expression for the functional associated with Poisson's equation. The importance of this expression springs from a situation in which the behavior of a system is defined by Poisson's equation and its energy principles are not known. As an example of such problems, one may refer to the conduction heat transfer in solids. While the variational formulation of the thermodynamic problems may be based on the entropy inequality, the detail formulations from the entropy inequality to the proper functional expression for the heat conduction problems are not yet developed in the literature. In fact, the conduction heat transfer in solids is directly derived from the first law of thermodynamics. From the membrane analogy, however, the expression (3.5.9) may be used for the functional of heat conduction problems in solids, solely because their boundary value problems are identical.

3.6 Application of Calculus of Variation

To show that the expression of the functional given in Eq. (3.5.9) is associated with Poisson's equation, we use the standard method of calculus of variations to minimize Eq. (3.5.9). The expression which minimizes the functional (3.5.9) should be Poisson's equation.

Consider a variational parameter ϵ and the variational function $u = u(x, y)$. The variational function $u(x, y)$ satisfies the homogeneous boundary conditions. The function $u(x, y)$ and its first partial derivatives with respect to x and y are continuous in the solution domain D. That is:

1. $u(x, y) = 0$ on $\partial\ D$.
2. $u(x, y)$, $u_{,x}$, and $u_{,y}$ are continuous in D.

Now, we may construct the function

$$\bar{\phi}(x, y) = \phi(x, y) + \epsilon u(x, y). \tag{3.6.1}$$

Substituting $\bar{\phi}$ in the expression of the functional from Eq. (3.5.9) gives

$$\begin{aligned}
V[\bar{\phi}] &= \int_D \left\{ \frac{T}{2}\left[\left(\frac{\partial(\phi + \epsilon u)}{\partial x} \right)^2 + \left(\frac{\partial(\phi + \epsilon u)}{\partial y} \right)^2 \right] - p\Big(\phi + \epsilon u\Big) \right\} dxdy \\
V[\bar{\phi}] &= \int_D \left\{ \frac{T}{2}\left[\left(\frac{\partial \phi}{\partial x} \right)^2 + 2\epsilon\ \frac{\partial \phi}{\partial x}\ \frac{\partial u}{\partial x} + \epsilon^2 \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial \phi}{\partial y} \right)^2 \right.\right. \\
&\quad \left.\left. + 2\epsilon\ \frac{\partial \phi}{\partial y}\ \frac{\partial u}{\partial y} + \epsilon^2 \left(\frac{\partial \phi}{\partial y} \right)^2 \right] - p\Big(\phi + \epsilon u\Big) \right\} dxdy \\
\frac{\partial V}{\partial \epsilon} &= \int_D \left\{ \frac{T}{2}\left[2\frac{\partial \phi}{\partial x}\ \frac{\partial u}{\partial x} + 2\epsilon \left(\frac{\partial u}{\partial x} \right)^2 + 2\frac{\partial \phi}{\partial y}\ \frac{\partial u}{\partial y} \right.\right. \\
&\quad \left.\left. + 2\epsilon \left(\frac{\partial u}{\partial y} \right)^2 \right] - pu \right\} dxdy
\end{aligned} \tag{3.6.2}$$

Now, let $\epsilon \longrightarrow 0$. Therefore,

$$\frac{\partial V}{\partial \epsilon}|_{\epsilon \to 0} = \int_D \left\{ T\left[\frac{\partial \phi}{\partial x}\ \frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial y}\ \frac{\partial u}{\partial y} \right] - pu \right\} dxdy = 0. \tag{3.6.3}$$

The above integral is divided into two integrals using the method of integrating by parts and the Gauss theorem as

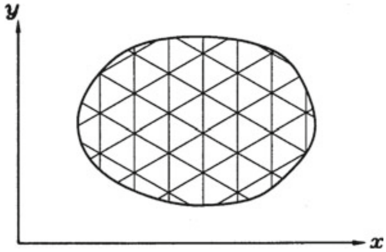


Fig. 3.6 Finite element representation of a continuum

$$\oint_{\partial D} uT\frac{\partial \phi}{\partial n}\,dS - \int_D u\left[\frac{\partial}{\partial x}\left(T\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(T\frac{\partial \phi}{\partial y}\right) + p\right]dxdy = 0. \quad (3.6.4)$$

Since u is an arbitrary function, we argue that the integrand of the above equation must vanish independently. Therefore,

$$T\frac{\partial \phi}{\partial n} = 0 \;\text{ on }\; \partial D, \quad \text{where } \phi \;\text{ is not specified}$$
$$\frac{\partial}{\partial x}\left(T\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(T\frac{\partial \phi}{\partial y}\right) = -p \;\text{ in } D. \quad (3.6.5)$$

The above equations are the necessary conditions for V to be minimum. Note that as a consequence of requiring V to be stationary, we are implicitly imposing the requirement that $\frac{\partial \phi}{\partial n} = 0$ on ∂D. Equation (3.6.5), which minimizes the functional of potential energy, is the equilibrium equation of the elastic membrane.

3.7 Introduction to the Finite Element Method

We begin by subdividing the domain of a typical problem into a number of arbitrary elements. Figure 3.6 illustrates the concept of subdividing the solution domain into a number of smaller triangular elements. Each of these subdivisions is called an *element*.

We will refer to the intersection of element sides as *nodal points*. In Fig. 3.6, the element sides are arbitrarily selected to be straight. Later on, we will note that both element geometry and the approximating function may be selected as nonlinear. That is, the sides of the element may be curves, the element may be selected to have four sides with irregular shape, and the approximating function may have a higher order form.

The Ritz method may be used to minimize the functional of energy. That is, we assume trial solution, or interpolating function, with undetermined coefficients associated with the nodal points displacement. Then, the principal of stationary potential

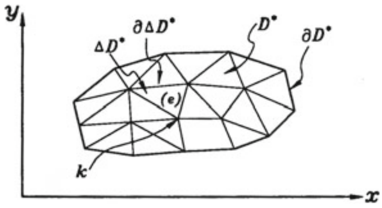


Fig. 3.7 Notation for the approximated continuum R^*

energy is applied to determine the coefficients such that the trial solution is close to the exact solution in each element.

To illustrate the concept of the finite element method in detail, let us consider the following example problem.

3.7.1 The Elastic Membrane

We seek the solution for the deflection (small) of an elastic weightless membrane from some (x, y) reference plane. The formulation of the associated boundary value problem and the equivalent extremum problem are given in the following:

3.7.2 Boundary value problem

Determine ϕ, the membrane deflection, such that

$$\nabla^2 \phi = -\frac{p}{T} \quad \text{in} \quad D \tag{3.7.1}$$

where $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$, p is the pressure under the membrane, and T is the traction force per unit length of the membrane. The boundary condition is $\phi = f_1(s)$ on ∂D, where f_1 is prescribed on the boundary as a function of arc length s.

3.7.3 Extremum problem

We are to determine ϕ, the membrane deflection, such that

$$V = \int_D \left\{ \frac{T}{2} \left[\left(\frac{\partial \phi}{\partial x} \right)^2 + \left(\frac{\partial \phi}{\partial y} \right)^2 \right] - \phi p \right\} dx dy \tag{3.7.2}$$

is a minimum with respect to the given boundary condition.

The emphasis is the solution of the extremum problem by approximate technique. We assume the trial function ϕ^*, for ϕ, over each element in D^* of the finite element representation. Figure 3.7 shows the finite element representation of the problem solution domain.

We require ϕ^* to be sufficiently smooth such that a corresponding value of $V = V[\phi^*]$, as given by (3.7.2), will be a minimum. Also, ϕ^* must satisfy the forced boundary conditions. It is further required that the approximating function ϕ^* be close to the exact solution ϕ at the nodal points in D^*. Essentially, the original problem with infinite degrees of freedom is reduced to one with a finite number of degrees of freedom. If the elements in D^* are smaller (with a larger number of elements in the solution domain), or if the interpolating function is of a higher order, then our goal of matching ϕ^* to ϕ should become ever closer, at least in principle. Practical difficulties, such as round-off error and computation time, enforce the user to make judicious choices regarding the element size and the degree of approximating shape function selected to model the problem.

In Fig. 3.7, let a typical node in D^* be denoted by k and let the approximating value of ϕ at node k be denoted by ϕ_k^*. Further, let a typical element in D^* be named the base element (e). Then, the approximate potential energy associated with the problem is

$$V^* = \sum_{e=1}^{r} (V^*)^e \tag{3.7.3}$$

where r is the number of elements in D^* and

$$(V^*)^e = \int_{\Delta D^e} \left\{ \frac{T}{2} \left[\left(\frac{\partial \phi^*}{\partial x} \right)^2 + \left(\frac{\partial \phi^*}{\partial y} \right)^2 \right] - p\phi^* \right\} dxdy \tag{3.7.4}$$

where ΔD^e is the subdomain of D^* associated with the base element (e).

A necessary condition for V^* to be a minimum is

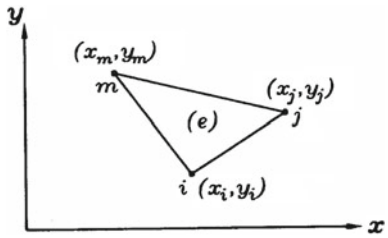
$$\frac{\partial V^*}{\partial \phi_k^*} = 0 \quad k = 1, 2, \ldots, n \tag{3.7.5}$$

where n is the number of nodal points in D^*. Equation (3.7.5) implies that the nodal point values of the interpolating function ϕ^* play the role of generalized coordinates for the approximate solution over D^e.

Consider a typical base element (e), as shown in Fig. 3.8. The asterisk of V^* and ϕ^* is dropped for simplicity and it is assumed that these quantities are approximate values unless noted otherwise.

We assume a simple linear interpolating function for ϕ in the base element (e). The assumed approximating function for ϕ represents, in the geometrical sense, a flat plane surface over ΔD^*. We can then express ϕ in terms of values of ϕ on the nodal points i, j, and m. The coordinates of the nodal points are (x_i, y_i), (x_j, y_j),

Fig. 3.8 Notation for a typical base element (*e*)



and (x_m, y_m), respectively. The linear interpolation function for ϕ in (e) is assumed as

$$\phi = a_1 + a_2 x + a_3 y. \tag{3.7.6}$$

We solve for $a_1,\ a_2$, and a_3 by requiring

$$\begin{aligned}
\phi &= \phi_i \quad \text{at } x = x_i \quad y = y_i \\
\phi &= \phi_j \quad \text{at } x = x_j \quad y = y_j \\
\phi &= \phi_m \quad \text{at } x = x_m \quad y = y_m.
\end{aligned}$$

The resulting solution for the a_1, a_2, and a_3 is

$$\begin{aligned}
a_1 &= \frac{1}{2\Delta}\,(a_i\phi_i + a_j\phi_j + a_m\phi_m) \\
a_2 &= \frac{1}{2\Delta}\,(b_i\phi_i + b_j\phi_j + b_m\phi_m) \\
a_3 &= \frac{1}{2\Delta}\,(c_i\phi_i + c_j\phi_j + c_m\phi_m)
\end{aligned} \tag{3.7.7}$$

where

$$2\Delta = 2(\text{area of the base element } (e)) = \det\begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{vmatrix}$$

and

$$\begin{cases} a_i = x_j y_m - x_m y_j \\ b_i = y_j - y_m \\ c_i = x_m - x_j \end{cases} \qquad \begin{cases} a_j = x_m y_i - x_i y_m \\ b_j = y_m - y_i \\ c_j = x_i - x_m \end{cases} \qquad \begin{cases} a_m = x_i y_j - x_j y_i \\ b_m = y_i - y_j \\ c_m = x_j - x_i. \end{cases} \tag{3.7.8}$$

Substituting Eqs. (3.7.7) in (3.7.6) and rearranging the terms in products of nodal point values of ϕ yields

$$\phi = \frac{1}{2\Delta}\,\{(a_i + b_i\ x + c_i\ y)\phi_i + (a_j + b_j x + c_j y)\phi_j + (a_m + b_m x + c_m y)\phi_m\} \tag{3.7.9}$$

let

$$N_i = \frac{a_i + b_i x + c_i y}{2\Delta} \quad , \quad N_j = \frac{a_j + b_j x + c_j y}{2\Delta} \quad , \quad N_m = \frac{a_m + b_m x + c_m y}{2\Delta}. \tag{3.7.10}$$

Then, Eq. (3.7.9) for the element (e) may be written as

$$\phi^{(e)} = N_i \phi_i + N_j \phi_j + N_m \phi_m. \tag{3.7.11}$$

Note that N_i, N_j, and N_m are functions of geometry only and are called shape functions. Introducing matrix notation, we write the interpolating function for the element (e) as

$$\phi^{(e)} = \langle N_i \quad N_j \quad N_m \rangle \begin{Bmatrix} \phi_i \\ \phi_j \\ \phi_m \end{Bmatrix}^{(e)} = \langle N \rangle^{(e)} \{\phi\}^{(e)} \tag{3.7.12}$$

where $\{\phi\}^{(e)}$ implies the degrees of freedom vector, or generalized coordinates vector, associated with the element (e) for the approximate solution ϕ. From Eqs. (3.7.4) (asterisk dropped)

$$V = \sum_{e=1}^{r} \int_{D(e)} \left\{ \frac{T}{2} \left[\left(\frac{\partial \phi}{\partial x} \right)^2 + \left(\frac{\partial \phi}{\partial y} \right)^2 \right] - p\phi \right\} dxdy. \tag{3.7.13}$$

We substitute the interpolating function from Eq. (3.7.12) in Eq. (3.7.13) and perform the differentiation over the element area. Furthermore, the interpolating function ϕ is locally dependent upon the generalized coordinates $\langle \phi_i \quad \phi_j \quad \phi_m \rangle^{(e)}$ associated with a typical element (e).

Although the sum in Eq. (3.7.13) is taken over the complete set of elements in D, the entire set need not be considered in any one computation given by Eq. (3.7.5). Due to the local dependence of the interpolating function for ϕ, and noting that only a finite number of elements have a common nodal point, only the collection of elements which share the nodal point k need be considered in the eventual summation implied by Eq. (3.7.13). We shall call this collection of elements the finite element neighborhood of nodal point k.

Definition The finite element neighborhood, η_k, associated with a nodal point k, is the set of finite elements such that the generalized coordinate, ϕ_k, occurs in the interpolating function (for ϕ) associated with each element in η_k. As an example, for the linear interpolating function considered here, the finite element neighborhood of node k is shown shaded in Fig. 3.9.

In order to apply Eq. (3.7.13), we must be able to compute $\partial V / \partial \phi_i$ for the element e. Our eventual goal is to let $i \to k$ in order to generate a set of n linear algebraic equations, which can be solved for ϕ_k on n generated nodal points. The differentiation

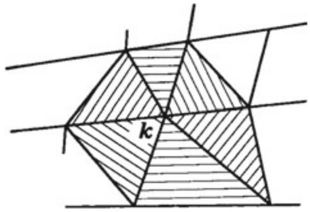


Fig. 3.9 Finite element neighborhood of nodal point k

of the total potential energy of the element (e) with respect to ϕ_i is

$$\frac{\partial V^e}{\partial \phi_i} = \int_{D(e)} \left\{ T \left[\frac{\partial \phi}{\partial x} \frac{\partial}{\partial \phi_i} \left(\frac{\partial \phi}{\partial x} \right) + \frac{\partial \phi}{\partial y} \frac{\partial}{\partial \phi_i} \left(\frac{\partial \phi}{\partial y} \right) \right] - p \frac{\partial \phi}{\partial \phi_i} \right\} dxdy. \tag{3.7.14}$$

Substituting Eq. (3.7.11) in (3.7.14) and performing the differentiation yields

$$\frac{\partial V^e}{\partial \phi_i} = \int_{D(e)} \left\{ \frac{T}{4\Delta^2} \left(\langle b_i \; b_j \; b_m \rangle \begin{Bmatrix} \phi_i \\ \phi_j \\ \phi_m \end{Bmatrix}^e b_i + \langle c_i \; c_j \; c_m \rangle \begin{Bmatrix} \phi_i \\ \phi_j \\ \phi_m \end{Bmatrix}^e c_i \right) - pN_i \right\} dxdy. \tag{3.7.15}$$

We note that the scalar product is involved in the first two terms of the integrand of Eq. (3.7.15), where the coefficients b and c are constants. Thus

$$\frac{\partial V^e}{\partial \phi_i} = \frac{T}{4\Delta} (\langle b_i b_i \;\; b_i b_j \;\; b_i b_m \rangle + \langle c_i c_i \;\; c_i c_j \;\; c_i c_m \rangle) \begin{Bmatrix} \phi_i \\ \phi_j \\ \phi_m \end{Bmatrix}^{(e)} - \int_{D(e)} pN_i dxdy. \tag{3.7.16}$$

If p is assumed to be constant in the element (e), then the last term of Eq. (3.7.16) becomes

$$-\frac{p}{2\Delta} \int_{D(e)} (a_i + b_i x + c_i y) dxdy = -\frac{p}{2} (a_i + b_i \bar{x} + c_i \bar{y}) = -\frac{p\Delta}{3}$$

where $\bar{x}$, $\bar{y}$ are the coordinates of the centroid of element (e). Then, Eq. (3.7.16) may be written as

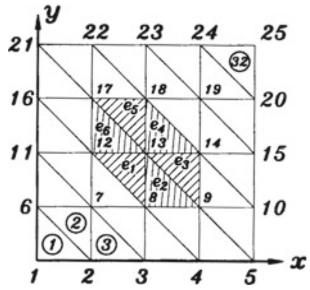


Fig. 3.10 Example solution domain

$$\frac{\partial V^e}{\partial \phi_i} = \frac{T}{4\Delta}(\langle b_i b_i \ b_i b_j \ b_i b_m \rangle + \langle c_i c_i \ c_i c_j \ c_i c_m \rangle)^{(e)} \begin{Bmatrix} \phi_i \\ \phi_j \\ \phi_m \end{Bmatrix}^{(e)} - \frac{p\Delta}{3} \tag{3.7.17}$$

where, in this example, T and p are constants in $D(e)$. It may be noted, however, that a simple representation of T and p as variables over D^e may be achieved by letting T and p be constant within any one element, but different from one element to the next. If the mean values of T and p are used within any one element, then a step function representation of the variables $T(x, y)$, $p(x, y)$ is obtained. It can also be noted that the quantities b, c, and Δ are expressed in terms of the coordinates i, j, and m of the nodal points of element (e).

Let us consider an example to describe the application of Eq. (3.7.17). Consider, for simplicity, a square solution domain D. We subdivide the domain and number the nodal points as shown in Fig. 3.10. Let us select node i in Eq. (3.7.17) to be node 13 in Fig. 3.10. Then, the finite element neighborhood of node 13, based on our previous assumptions, is shown shaded in Fig. 3.10 and

$$\frac{\partial V}{\partial \phi_i} = \sum_{e=1}^{NE} \frac{\partial V^{(e)}}{\partial \phi_i} = 0 \tag{3.7.18}$$

The sum in this equation should actually be taken over the elements e_1 through e_6, as labeled in Fig. 3.10. Thus

$$\frac{\partial V}{\partial \phi_{13}} = 0 = \sum_{e_1}^{e_6} \left[\frac{T}{4\Delta} \ (\langle b_i \ b_j \ b_m \rangle b_i + \ \ \langle c_i \ c_j \ c_m \rangle c_i) \begin{Bmatrix} \phi_i \\ \phi_j \\ \phi_m \end{Bmatrix} - \frac{p\Delta}{3} \right]^{e_r} \quad r = 1, 2, \ldots 6. \tag{3.7.19}$$

We let $i = 13$ for each element in e_r. To assign the remaining (j, m) for each element in e_r, we proceed in a Counter-clockwise direction around the element, always starting with node 13. Thus,

$$
\begin{aligned}
&\text{for} \quad e_1, \quad i = 13 \quad j = 12, \quad m = 8\\
&\text{for} \quad e_2, \quad i = 13 \quad j = 8, \quad m = 9\\
&\text{for} \quad e_3, \quad i = 13 \quad j = 9, \quad m = 14\\
&\text{for} \quad e_4, \quad i = 13 \quad j = 14, \quad m = 18\\
&\text{for} \quad e_5, \quad i = 13 \quad j = 18, \quad m = 17\\
&\text{for} \quad e_6, \quad i = 13 \quad j = 17, \quad m = 12
\end{aligned} \tag{3.7.20}
$$

and thus

$$
\begin{aligned}
\frac{\partial V}{\partial \phi_{13}} = 0 = {} & \frac{T^{(1)}}{4\Delta^{(e_1)}} \left(\langle b_{13}, b_{12}, b_8 \rangle b_{13} + \langle c_{13}, c_{12}, c_8 \rangle c_{13}\right)^{e_1} \\
& \times \begin{Bmatrix} \phi_{13} \\ \phi_{12} \\ \phi_8 \end{Bmatrix} + \frac{T^{(2)}}{4\Delta^{(e_2)}} \left(\langle b_{13}, b_8, b_9 \rangle b_{13} + \langle c_{13}, c_8, c_9 \rangle c_{13}\right)^{e_2} \begin{Bmatrix} \phi_{13} \\ \phi_8 \\ \phi_9 \end{Bmatrix} \\
& + \cdots \\
& + \frac{T^{(6)}}{4\Delta^{(e_6)}} \left(\langle b_{13}, b_{17}, b_{12} \rangle b_{13} + \langle c_{13}, c_{17}, c_{12} \rangle c_{13}\right)^{e_6} \begin{Bmatrix} \phi_{13} \\ \phi_{17} \\ \phi_{12} \end{Bmatrix} \\
& - \frac{p}{3} \left(\Delta^{e_1} + \Delta^{e_2} + \cdots + \Delta^{e_6}\right).
\end{aligned} \tag{3.7.21}
$$

Note that in Eqs. (3.7.21) the parameters b_s and c_s are constants and depend on the coordinates of the element nodal points. It is simply verified that $(b_{13})^{e_1} \neq (b_{13})^{e_2}$.

We may call the coefficients of ϕ_s by k_s. Then, substituting Eq. (3.7.21) in the set of Eq. (3.7.5) results in a system of linear 25 × 25 equations for 25 unknowns ϕ_k. The 13th equation of this system is

$$
\begin{bmatrix} \\ \\ k_{13,8} \; \cdot \; k_{13,13} \; \cdot \; k_{13,18} \\ \\ \\ \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \cdot \\ \phi_8 \\ \cdot \\ \phi_{13} \\ \cdot \\ \phi_{18} \\ \cdot \\ \phi_{25} \end{Bmatrix} = \frac{p}{3} \begin{Bmatrix} \\ \\ \Delta^{e_1} + \Delta^{e_2} + \cdots + \Delta^{e_6} \\ \\ \\ \end{Bmatrix} \tag{3.7.22}
$$

or

$$
[K]\{\phi\} = \{F\}. \tag{3.7.23}
$$

It is seen that the columns 1–7 and 19–25 of this equation are zeros. In general, the coefficients in any row s of the coefficients matrix $[K]$ may be nonzero only when they correspond to coefficients of unknowns at nodal points associated with the elements in the finite element neighborhood of node s. It follows that the coupling of the elements of the unknown vector are accomplished by nonzero elements of $[K]$, and this in turn implies the nature of the connectivity of the finite element representation of the solution domain of the problem. Thus, the resulting system of finite element equilibrium equations is banded in nature. The bandwidth of the resulting stiffness matrix depends upon the nodal points numbering system. The more efficient solution of the resulting equations is when the bandwidth of the stiffness matrix is the smallest.

An alternate approach to derive equations similar to (3.7.22) is based on the concepts employed in structural mechanics. Let us consider applying Eq. (3.7.5) for each of the generalized coordinates (vertical displacements of the membrane) ϕ_i, ϕ_j, ϕ_m, associated with the nodes of the typical element e. Then, employing matrix notation, and referring to Eq. (3.7.17), we can write,

$$\frac{T}{4\Delta}\begin{bmatrix} b_ib_i + c_ic_i & b_ib_j + c_ic_j & b_ib_m + c_ic_m \\ b_jb_i + c_jc_i & b_jb_j + c_jc_j & b_jb_m + c_jc_m \\ b_mb_i + c_mc_i & b_mb_j + c_mc_j & b_mb_m + c_mc_m \end{bmatrix}\begin{Bmatrix} \phi_i \\ \phi_j \\ \phi_m \end{Bmatrix} \to \frac{p\Delta}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}. \tag{3.7.24}$$

The above equation represents the force balance, or equilibrium equation, of an element of the membrane in terms of vertical displacements of its resultant nodes and forces, or pressure. The coefficients such as $T/(4\Delta) \times (b_i\ b_i + c_i\ c_i)$ may be regarded as *stiffness coefficients*. Let us rewrite Eq. (3.7.24) as

$$\begin{bmatrix} k_{ii} & k_{ij} & k_{im} \\ k_{ji} & k_{jj} & k_{jm} \\ k_{mi} & k_{mj} & k_{mm} \end{bmatrix}\begin{Bmatrix} \phi_i \\ \phi_j \\ \phi_m \end{Bmatrix} \to \begin{Bmatrix} f_i \\ f_j \\ f_m \end{Bmatrix}. \tag{3.7.25}$$

Now, consider the case where the displacements $\phi_i = \phi_m$ are fixed and are zero. If the membrane element is given a unit of vertical displacement at node $\underline{j}$, then we can interpret k_{ij} to be the reaction at the fixed node $\underline{i}$ due to the unit displacement, $\phi_j = 1$, of node j. The other coefficients can be interpreted by a similar line of reasoning. Thus, the square matrix of Eq. (3.7.24) or (3.7.25), essentially represents the response characteristics or stiffness of the membrane element.

If we assemble a group of elements together to represent a membrane model, then the elements consequently share nodes in common. It follows that the stiffness of the assembled structure may be obtained by simply adding together the appropriate stiffness coefficients of individual elements according to the way in which they are interconnected. That is, those coefficients which multiply the displacement at a common node would be summed. In effect, this procedure corresponds to adding the stiffness of springs in parallel (connected to a common displacing point) in order to obtain the overall spring stiffness associated with the point.

The above structurally motivated procedure, which can be used to obtain Eq. (3.7.25), is referred to as the *direct stiffness method*. The development of this seemingly obvious procedure (along with the structural finite element concept) for obtaining the response characteristics of an approximated continuum is responsible in large part for the present success of the finite element method.

3.7.4 Boundary Conditions

To account for boundary conditions of the Dirichlet type, $\phi = f_1(s)$, we simply set the values of ϕ associated with the boundary nodal points equal to an appropriate mean value. The mean value for a point r may be obtained by averaging $f_1(s)$ over a segment of the boundary on either side of the point. Thus, a step function representation of $f_1(s)$ over the boundary is achieved.

A variety of computational schemes have been used for rearranging the equations so that the specified values of ϕ essentially become part of the constant vectors on the right side of Eq. (3.7.23). If we consider the membrane problem, then by partitioning the stiffness matrix, one separates the original equation into two sub-equations such that the ϕ values for the interior nodes are unknown for one set of equations, and the components (reactions) of the F vector associated with the boundary nodes are unknown for the second set. Solving the second set and back substituting into the first set yields the solution for reactions at fixed boundary nodes and displacement of the interior nodes of the membrane.

More complicated boundary conditions of the form

$$T\,\frac{\partial \phi}{\partial n} + k\phi = q \tag{3.7.26}$$

may be represented by an appropriate addition of terms to the functional of Eq. (3.7.2).

The appropriate functional representing potential energy of the membrane with boundary conditions (3.7.26) is

$$V = \int_D \left\{ \frac{T}{2}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2\right] - p\phi \right\} dxdy + \oint_{\partial D} \frac{k}{2}\,\phi^2 dS - \oint_{\partial D} q\phi dS. \tag{3.7.27}$$

The additional boundary integral terms may be regarded as being the potential energy derived from the membrane edge being supported by a continuous elastic foundation along ∂D and also being loaded by a distributed load q. The edge is then free to move vertically subjected to the resulting force balance requirement of Eq. (3.7.26).

Equation (3.7.26), which is a force balance relation for the membrane, may be regarded as a flux balance relation in the convection boundary condition for steady state heat transfer problems. One may simply let $\phi \rightarrow T$, the temperature, $-k/T \rightarrow h/k^*$, where h and k^* are the film coefficient and conductivity, respectively, of a body immersed in a fluid of constant temperature.

Let us consider a finite element model to represent the condition (3.7.26). Refer to Fig. 3.11 for notation. Essentially, we seek to apply the requirement

$$\frac{\partial V}{\partial \phi_r} = 0 \tag{3.7.28}$$

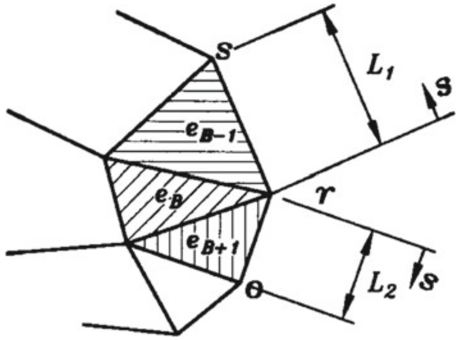


Fig. 3.11 Notation for boundary elements and nodes

which is similar to Eq. (3.7.5). We note, however, that the finite element neighborhood of node r (shaded) includes two boundary segments, o–r and r–s. This node is common in the elements e_{B-1}, e_B, and e_{B+1}. These elements contribute to V as previously discussed, and the effect on V of the segments o–r and r–s can be handled readily. If we consider the segment r–s, and assume ϕ to vary linearly along it, we can write

$$\phi = \phi_r + \frac{s}{L_1}\,(\phi_s - \phi_r) \tag{3.7.29}$$

which is consistent with our previous assumption. We substitute Eq. (3.7.29) in the line integral terms of Eq. (3.7.27) and apply Eq. (3.7.28) to obtain

$$-\frac{qL_1}{2} + \frac{kL_1}{3}\left(\phi_r + \frac{\phi_s}{2}\right). \tag{3.7.30}$$

Similarly, from the segment o–r, also in the finite element neighborhood of node r, we obtain the additional contribution to Eq. (3.7.28) as

$$-\frac{qL_2}{2} + \frac{kL_2}{3}\left(\phi_r + \frac{\phi_o}{2}\right). \tag{3.7.31}$$

In conclusion, the coefficients of ϕ_r and ϕ_s appearing in Eqs. (3.7.29) and (3.7.30) are added to the appropriate locations in the coefficients matrix corresponding to Eq. (3.7.25).

If a problem involves an obvious axis of symmetry in the solution domain, then we may take advantage and reduce the solution domain. In this case, along the axis of symmetry of the solution domain $\frac{\partial \phi}{\partial n} = 0$. This condition is met as a natural boundary condition resulting from the minimization of the area integral of Eq. (3.7.27). An axis of symmetry of the solution domain may be handled as a boundary of the reduced solution domain. It should be noted that a geometrical axis of symmetry is not necessarily an axis of symmetry for a solution.

3.8 Problems

1. Show that

$$\int_D \left[T \left(\frac{\partial \phi}{\partial x} \frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial y} \frac{\partial u}{\partial y} \right) - pu \right] dxdy$$
$$= \oint_{\partial D} uT \frac{\partial \phi}{\partial n} \, dS - \int_D u \left[\frac{\partial}{\partial x} \left(T \frac{\partial \phi}{\partial x} \right) + \frac{\partial}{\partial y} \left(T \frac{\partial \phi}{\partial y} \right) + p \right] dxdy.$$

2. Consider a square elastic shallow membrane of dimensions 1×1 under lateral pressure $p = 5$ with the membrane surface tension $T = 10$. The given data are in dimensionless form. The membrane sides along the x and y-directions are divided into two equal sections providing eight triangular elements. If the boundary conditions are zero along the sides of the membrane, calculate the height of the middle node.
3. What is the minimum band width of the stiffness matrix of an elastic membrane if the sides along the x and y-directions are each divided into ten divisions?

Further Readings

1. Oden JT (1972) Finite elements of nonlinear continua. McGraw-Hill, New York
2. Langhaar HL (1962) Energy methods in applied mechanics. John Wiley, New York
3. Fung YC (1965) foundations of solid mechanics. Prentice Hall, New York

Chapter 4
Elements and Local Coordinates

Abstract The presentation of the finite element method for the elastic membrane problem, a triangular element with straight sides and linear shape function to approximate the elevation of the elastic membrane due to lateral pressure, was employed. In this chapter, we correct ourself that there is no limitation as far as the geometry of the element and the order of approximation of the shape function is concerned. In one, two, and three dimensional problems elements with higher order geometries and approximating shape functions may be used to prepare a finite element model. Also, since it is always more efficient to employ the local coordinates for the integration purpose of the element stiffness and force matrices, the local and global coordinate systems are discussed and the Jacobian matrix is explained.

4.1 Introduction

In the previous chapter, two-dimensional triangular elements with straight edges were considered to approximate the deflection of the membrane by the finite element method. The first question that one may ask is, what about the membrane with curved edges?, And, is it possible to approximate the membrane elevation with higher order curves, rather than a flat surface confined to a triangle, to improve the accuracy? The answers to these questions are positive. Elements with curved edges to model and fit the boundary of the solution region can be used in finite element analysis. Also, higher order polynomials can be employed to approximate the dependent function in the element. These techniques are simply incorporated into finite element modeling without significant complications in the solution procedure. The following sections show the common curved and higher order elements in finite element analysis. It is, however, customary to consider polynomials up to the third order degree as the shape functions. Polynomials of orders four and higher are seldom used to model an element, due to the possibility of occurrence of roots of the polynomial within the element.

M. R. Eslami, *Finite Elements Methods in Mechanics*,
Solid Mechanics and Its Applications 216, DOI: 10.1007/978-3-319-08037-6_4,

4.2 Subparametric, Isoparametric, and Superparametric Elements

An element in the finite element model is characterized by its two principle features, *the degree and order of approximating function* describing the dependent function in the element (order of shape function), and the *order of geometric description* of the element. An element in one-, two-, and three-dimensional space is not necessarily made of straight sides. Curved side elements for modeling the solution regions with curved boundaries are frequently used in finite element modeling. By the *order of geometric description* of the element, we mean the degree of polynomial describing the curved sides of the element.

An element is called *isoparametric* when the order of approximating shape function and the order of geometric description of the element are equal. The element is *subparametric* when the geometric order is less than the order of the approximating shape function of the element, and is *superparametric* when the geometric order is larger than the order of the approximating shape function.

Consider, for example, a one-dimensional element. Let u describe the displacement of the element. The nodal values for displacement are denoted by U_i and the nodal coordinates are represented by X_i. The interpolation function for the element shape function u and element geometric description s are

$$u = \sum_{i=1}^{m} N_i U_i \quad s = \sum_{i=1}^{n} N_i' X_i \tag{4.2.1}$$

where N_i and N_i' are the shape functions describing the dependent function u and the geometric configuration s of the element in one-dimensional space. Note that the element's geometry is not necessarily along straight axis x, and may have a curved form in the x–y system. The two expressions of Eq. (4.2.1) for u and s give the values of u and s within the element in terms of the nodal values U_i and X_i. The type of element is defined as follows:

1. An element is isoparametric when $m = n$. That is, the same order of polynomials is selected to describe both element geometry and the approximating shape function.
2. The element is subparametric when $n < m$.
3. The element is superparametric when $n > m$.

This concept is extended to two- and three-dimensional space, and elements can be selected of any type. The three types of element in two dimensions are shown in Fig. 4.1. The symbol ◯ denotes a nodal point for displacement and • denotes a nodal point describing the side geometry. Figure 4.1a shows an isoparametric element, Fig. 4.1b a subparametric, and Fig. 4.1c a superparametric.

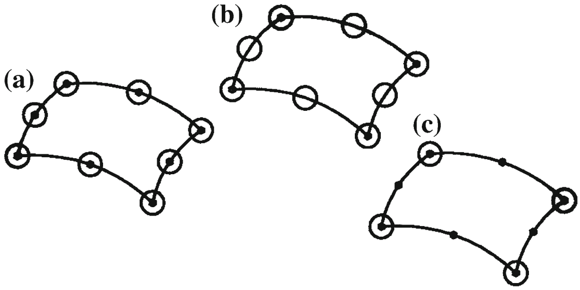
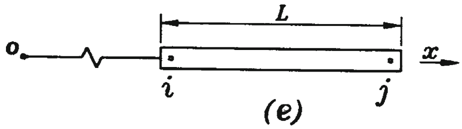


Fig. 4.1 Subparametric, isoparametric, and superparametric elements

Fig. 4.2 One-dimensional simplex element



4.3 One-Dimensional Elements

4.3.1 Straight Linear Element

This is the simplest element to approximate a one-dimensional element, where the element's geometry is a straight line and the shape function is linear. The element is shown in Fig. 4.2. If the dependent function is called ϕ and the variable is x, the shape function is

$$\phi^{(e)} = \alpha_1 + \alpha_2 x. \tag{4.3.1}$$

The constants α_1 and α_2 are found in terms of ϕ_i and ϕ_j. This element is called a *one-dimensional simplex element*, or isoparametric element of the first order.

4.3.2 Straight Quadratic Element

To improve the accuracy of the solution, the shape function in the base element (e) can be approximated with a second order shape function as FIX

$$\phi^{(e)} = \alpha_1 + \alpha_2 x + \alpha_3 x^2. \tag{4.3.2}$$

The three constants α_1 to α_3 are found in terms of ϕ_i, ϕ_j, and ϕ_k located at three nodal points i, j, and k, where node j is at equal distance from nodes i and k. The element is shown in Fig. 4.3.

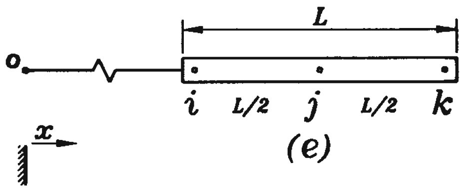


Fig. 4.3 Quadratic element

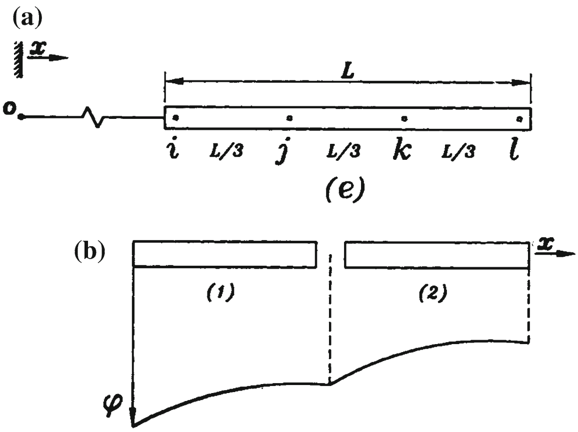


Fig. 4.4 Cubic element, C^0-continuous

4.3.3 Straight Cubic Element

To further improve the accuracy, a cubic element can be assumed such as

$$\phi^{(e)} = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3. \tag{4.3.3}$$

The four constants α_1 to α_4 are found according to the nodal point data on the element. There are two ways to evaluate the constants α. The first is to consider four nodal points in the element, as shown in Fig. 4.4a. The constants α_1 to α_4 are then calculated in terms of ϕ_i, ϕ_j, ϕ_k, and ϕ_l. This approximation, similar to element types 4.3.1 and 4.3.2 of Sect. 4.3, results in a continuous variation of the dependent function ϕ in the solution domain, but the slope of ϕ between two adjacent elements is not necessarily equal, as shown in Fig. 4.4b. Another technique for finding the coefficients α is to define a straight element with two end nodal points i and j, but with nodal values of ϕ and $d\phi/dx$ (or $\phi_{,x}$), as shown in Fig. 4.5a. This type of element approximation insures the continuity of function and its derivative with respect to x between any two adjacent elements, as shown in Fig. 4.5b, and is thus called C^1-continuous element.

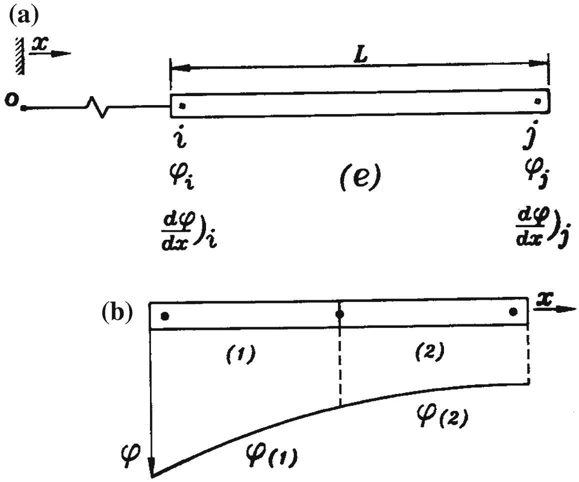


Fig. 4.5 Cubic element, C^1-continuous

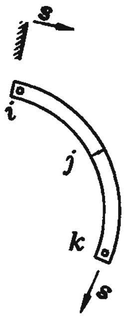


Fig. 4.6 Curved quadratic element

4.3.4 Curved Quadratic Element

The one-dimensional element may geometrically be selected as a curved element of a second order polynomial. In this case, the shape function can also be approximated by the same polynomial, as shown in Fig. 4.6.

The shape function in terms of s, measured along the arc, is

$$\phi^{(e)} = \alpha_1 + \alpha_2 s + \alpha_3 s^2 \tag{4.3.4}$$

where α_1 to α_3 are found in terms of ϕ_i, ϕ_j, and ϕ_k. Since the element curve equation is also described by a second order polynomial, the element is called *second order isoparametric element*.

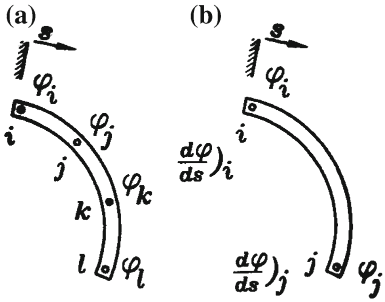


Fig. 4.7 Curved cubic element

4.3.5 Curved Cubic Element

This element is similar to the previous element except that the curve equation and the shape function are both described by a cubic polynomial. This element is shown in Fig. 4.7. The shape function in terms of s, measured along the arc length, is

$$\phi^{(e)} = \alpha_1 + \alpha_2 s + \alpha_3 s^2 + \alpha_4 s^3. \tag{4.3.5}$$

The constant coefficients α_1 to α_4 are found in terms of ϕ on the nodal points. Both types of approximations described in Sect. 4.3.3 are allowable. Two nodal points are selected at the ends of the element, and the other two may be selected either in between the ends at equal distances, describing the function ϕ, or selected at the same end points and describing $\phi_{,s}$. The latter element provides C^1-continuous element, where both the function and its derivative are continuous between any two adjacent elements.

4.4 Two-Dimensional Elements

4.4.1 Linear Triangular Element

This is the simplest element in two-dimension, where a triangular element with straight sides and three nodal points are considered, as shown in Fig. 4.8. The shape function for the dependent function ϕ is

$$\phi^{(e)} = \alpha_1 + \alpha_2 x + \alpha_3 y. \tag{4.4.1}$$

This is called a *simplex element* and is of first order isoparametric type.

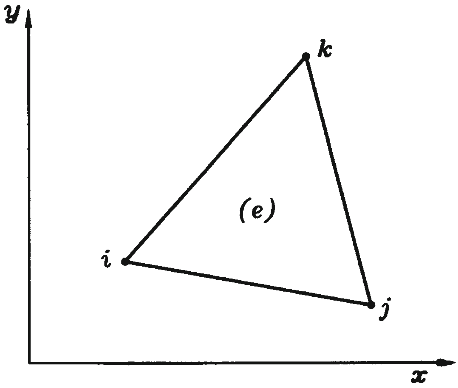


Fig. 4.8 Two-dimensional simplex element

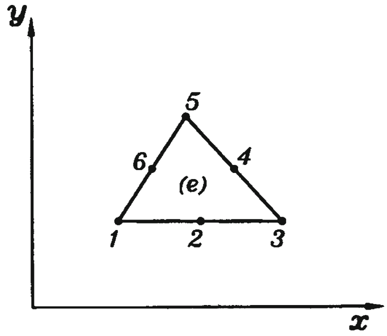


Fig. 4.9 Two-dimensional quadratic element

4.4.2 Quadratic Element

The element is considered triangular with straight sides. Six nodal points are defined on the element, three on the apex, and three in the middle of sides, as shown in Fig. 4.9. The shape function is assumed quadratic, as

$$\phi^{(e)} = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy + \alpha_5 x^2 + \alpha_6 y^2. \tag{4.4.2}$$

The six constants α_1 to α_6 are found in terms of ϕ_1 to ϕ_6 at six nodal points on the element.

4.4.3 Cubic Element

The element is triangular with straight sides. Ten nodal points are defined on the element, three on the apex, six on the sides at equal distances on each side, and one at the centroid of the element, as shown in Fig. 4.10. The shape function is assumed to be a third order polynomial as given

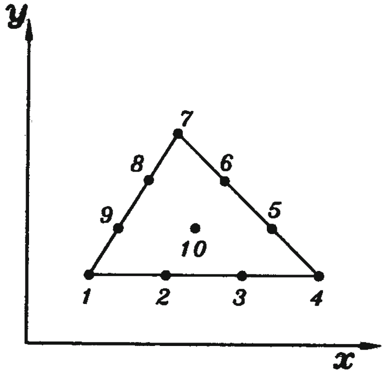


Fig. 4.10 Two-dimensional cubic element

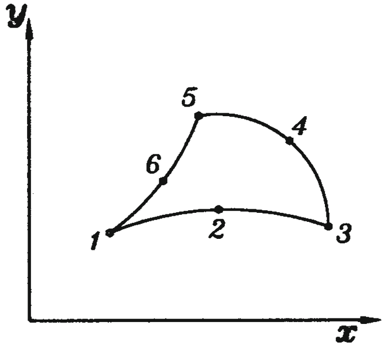


Fig. 4.11 Two-dimensional curved quadratic element

$$\phi^{(e)} = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy + \alpha_5 x^2 + \alpha_6 y^2 + \alpha_7 xy^2 + \alpha_8 x^2 y + \alpha_9 x^3 + \alpha_{10} y^3. \tag{4.4.3}$$

The constants α_1 to α_{10} are found in terms of ϕ_1 to ϕ_{10}, at ten nodal points on the element.

4.4.4 Curved Quadratic Element

The element is triangular with curved sides described by a second order Polynomial, and the shape function is also considered second order. The element is shown in Fig. 4.11 and the shape function is the same as Eq. (4.4.2).

This is a second order isoparametric element in two-dimension.

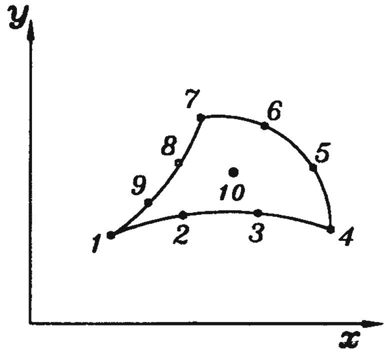


Fig. 4.12 Two-dimensional curved cubic element

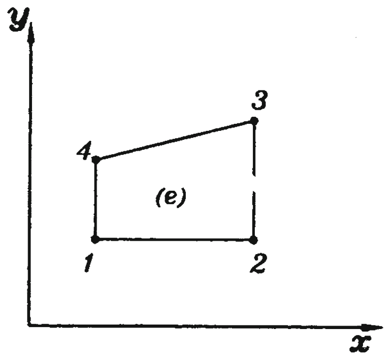


Fig. 4.13 Two-dimensional quadrilateral element

4.4.5 Curved Cubic Element

The same shape function as in Eq. (4.4.3) is used for the triangular element with sides described by a cubic polynomial, as shown in Fig. 4.12.

The nodal points are equally distanced along the curved sides and the values of the constants α_1 to α_{10} are found in terms of ϕ_1 to ϕ_{10}. This is a third order isoparametric element.

4.4.6 Quadrilateral Element

The elements in two-dimension may be selected quadrilateral with straight or curved sides. The simplest form of this element is shown in Fig. 4.13 with straight sides.

The shape function is approximated by a second order polynomial as

$$\phi^{(e)} = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy. \tag{4.4.4}$$

The four constants α_1 to α_4 are found in terms of ϕ_1 to ϕ_4, the values of the shape function on the nodal points.

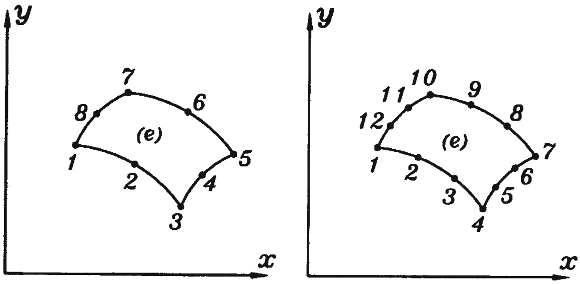
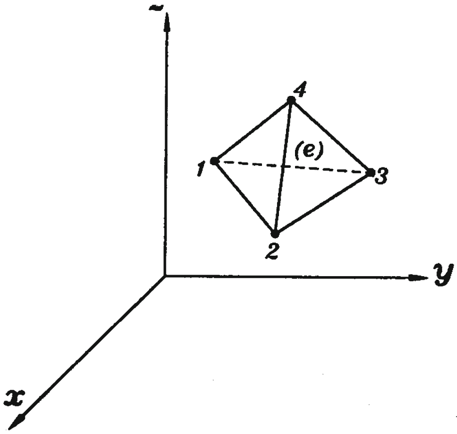


Fig. 4.14 Two-dimensional curved quadrilateral element

Fig. 4.15 Three-dimensional simplex element



It is interesting to note that the advantage of this element over the triangular simplex element is that a solution domain approximated with a quadrilateral element contains half of triangular elements while the approximation function is second order, rather than linear. In another words, approximation with a quadrilateral element will cut the element numbers by half and improve the accuracy of the solution by considering a higher order polynomial, compared to the simplex element.

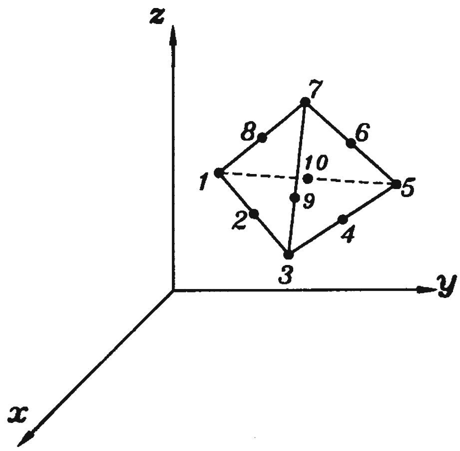
The other types of the quadrilateral elements with higher order shape functions are shown in Fig. 4.14. The sides may be straight or curved.

4.5 Three-Dimensional Elements

4.5.1 Linear Tetrahedral Element

A tetrahedral element of four nodes and flat side planes in a three-dimensional coordinates system is considered, as shown in Fig. 4.15. The shape function in the xyz system is selected linear as

Fig. 4.16 Three-dimensional quadratic element



$$\phi^{(e)} = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 z. \tag{4.5.1}$$

The values of α_1 to α_4 are calculated in terms of four values of ϕ on the nodal points. This is a first order isoparametric element in three-dimension.

4.5.2 Quadratic and Cubic Elements

The element is tetrahedral with flat side planes, but with 10 nodes at the corners and middle of the sides, as shown in Fig. 4.16. The approximating shape function is

$$\begin{aligned}\phi^{(e)} =& \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 z + \alpha_5 xy + \alpha_6 xz + \alpha_7 yz \\ &+ \alpha_8 x^2 + \alpha_9 y^2 + \alpha_{10} z^2.\end{aligned} \tag{4.5.2}$$

The constants α_1 to α_{10} are found in terms of the values of ϕ at 10 nodal points on the base element (e).

The cubic element is similar to Fig. 4.16 but with four nodes at four corners and 12 nodes on six sides, two nodes equally spaced at each side. The shape function is similar to Eq. (4.5.2) with six more terms related to the extra cubic terms.

4.5.3 Quadratic and Cubic Curved Isoparametric Elements

Quadratic and cubic curved isoparametric elements in three-dimensions are considered geometrically to fit the solution domain with curved boundaries. The quadratic element has ten nodes, shown by • in Fig. 4.17, and the cubic element has 16 to

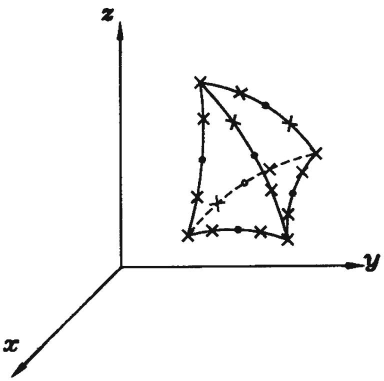


Fig. 4.17 Three-dimensional curved element

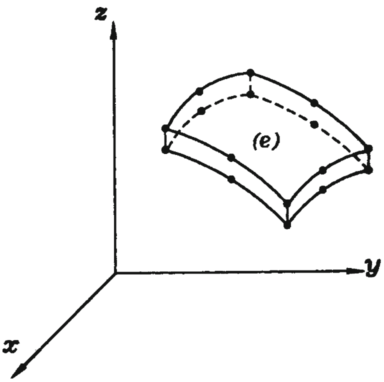


Fig. 4.18 Six sides elements

19 nodes, where the element with 16 nodes is shown by $\times$ in Fig. 4.17. The shape function for the quadratic element is similar to Eq. (4.5.2), and the shape function of the cubic element is similar to Eq. (4.5.2), with six to nine more cubic terms.

4.5.4 Six Sides Elements (Parallelepiped)

Elements with six sides with straight or curved side planes and edges are considered in three-dimensions, as shown in Fig. 4.18a, b. Up to 19 nodes are selected on these types of elements to approximate a full cubic shape function. The sides and edges may be straight or curved. The efficiency of these elements, considering the computation time, is higher compared to the tetrahedral elements. Using the six side elements reduces the total number of elements in the solution domain, while the degree of the approximation polynomial is higher compared to the tetrahedral element.

4.6 Global and Local Coordinates

To solve a problem by the finite element method, the first step is to sketch the problem geometry in a global coordinates system and divide it into a number of arbitrary elements. To evaluate the finite element matrices associated with the base element (e), it is always preferable to use a local coordinates system referring to that particular element to avoid complications incurred by the use of a global coordinates system. Since the local coordinates system is somehow fixed to the base element (e), its origin is different from the origin of the global system and is usually rotated through some defined angle. In addition, there are different analytical methods to evaluate the integrals of the members of finite element matrices in the local coordinate system. These analytical integration methods are extremely important in finite element analysis and significantly reduce the computation time. For this important reason, the local coordinates system and the analytical integration techniques related to the local coordinates system are essential tools in finite element analysis.

Let us consider the global coordinates system to be shown by the xyz system and the local coordinate system fixed to the base element (e) by $\xi\eta\zeta$. The relation between two coordinates systems are continuous and differentiable and are known as [1]

$$\begin{aligned} x &= x(\xi, \eta, \zeta) \\ y &= y(\xi, \eta, \zeta) \\ z &= z(\xi, \eta, \zeta). \end{aligned} \tag{4.6.1}$$

This coordinate transformation is unique such that it maps a point in the xyz system (domain D) to a unique point in the $\xi\eta\zeta$ system (domain $\hat{D}$). A line differential is mapped from the xyz system to the $\xi\eta\zeta$ system according to the following rule:

$$\begin{Bmatrix} dx \\ dy \\ dz \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} \begin{bmatrix} d\xi \\ d\eta \\ d\zeta \end{bmatrix}. \tag{4.6.2}$$

The square matrix on the right-hand side is called the transformation matrix, or the Jacobian matrix, and is

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \zeta} \end{bmatrix}. \tag{4.6.3}$$

Thus, a line element in the xyz system is transformed to a line element in the $\xi\eta\zeta$ system as

$$\begin{Bmatrix} dx \\ dy \\ dz \end{Bmatrix} = [J] \begin{Bmatrix} d\xi \\ d\eta \\ d\zeta \end{Bmatrix}. \quad (4.6.4)$$

The inverse transformation is possible and unique and is

$$\begin{Bmatrix} d\xi \\ d\eta \\ d\zeta \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} dx \\ dy \\ dz \end{Bmatrix}. \quad (4.6.5)$$

This condition, which implies the existence of an inverse transformation, requires that $|J| > 0$ every where in both domains.

Now, consider a function $\phi^{(e)}(\xi, \eta, \zeta)$ defined in the base element (e) in local coordinates and described in terms of the shape function $N_i^{(e)}(\xi, \eta, \zeta)$ and the nodal values ϕ_i as

$$\phi^{(e)}(\xi, \eta, \zeta) = N_i^{(e)}(\xi, \eta, \zeta)\phi_i \quad i = 1, 2, \ldots r \quad (4.6.6)$$

where r is the total number of nodes per element (e), and ϕ_i is the value of ϕ on nodal point i. The transformation law between two coordinate systems xyz and $\xi\eta\zeta$ are

$$x_i = \beta_{ij} N_j(\xi, \eta, \zeta) \quad i = 1, 2, 3 \; j = 1, 2, \ldots r \quad (4.6.7)$$

where β_{ij} are the coordinates (x, y, z) of node j in D. Equation (4.6.7) is the explicit form of Eq. (4.6.1), which maps $\hat{D}$ to D. The transformation of the shape functions and their derivatives are done similarly as

$$N_j(x, y, z) = \hat{N}_j[\xi(x, y, z), \eta(x, y, z), \zeta(x, y, z)]$$
$$\frac{\partial N_j}{\partial x_i} = \frac{\partial \hat{N}_j}{\partial \xi}\frac{\partial \xi}{\partial x_i} + \frac{\partial \hat{N}_j}{\partial \eta}\frac{\partial \eta}{\partial x_i} + \frac{\partial \hat{N}_j}{\partial \zeta}\frac{\partial \zeta}{\partial x_i} \quad i = 1, 2, 3. \quad (4.6.8)$$

The integration of a scalar function $\phi(x, y, z)$ in D is related to its equivalent value in $\hat{D}$ as

$$\int_D \phi(x, y, z) dD = \int_{\hat{D}} \hat{\phi}(\xi, \eta, \zeta)|J(\xi, \eta, \zeta)| d\hat{D} \quad (4.6.9)$$

where

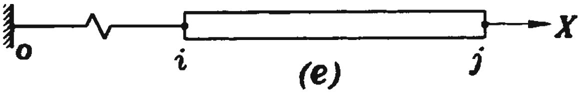


Fig. 4.19 One-dimensional simplex element

$$\hat{\phi}(\xi,\eta,\zeta)=\phi[x_j\hat{N}_j(\xi,\eta,\zeta),\,y_j\hat{N}_j(\xi,\eta,\zeta),\,z_j\hat{N}_j(\xi,\eta,\zeta)],\quad j=1,2,\ldots r \tag{4.6.10}$$

and $|J(\xi,\eta,\zeta)|$ is the determinant of the Jacobian matrix.

4.7 Local Coordinates in One-Dimension

It is a general rule in finite element analysis to derive the element matrices in the local coordinates. The element matrices are then multiplied by the proper rotation matrix, transformed to the global coordinates system, and assembled. The reason for selecting the local coordinates is the ease of calculation and integration of the members of the element matrices.

Consider a one-dimensional simplex element (e), as shown in Fig. 4.19. The shape function is

$$\phi^{(e)}=\alpha_1+\alpha_2 x. \tag{4.7.1}$$

The boundary conditions of the element are

$$\begin{aligned} x&=x_i \qquad & x&=x_j\\ \phi&=\phi_i \qquad & \phi&=\phi_j. \end{aligned} \tag{4.7.2}$$

Substituting in Eq. (4.7.1) gives

$$\begin{aligned} \phi_i&=\alpha_1+\alpha_2 x_i\\ \phi_j&=\alpha_1+\alpha_2 x_j. \end{aligned} \tag{4.7.3}$$

Solving for α_1 and α_2 yields

$$\alpha_1=\frac{\phi_i x_j-\phi_j x_i}{x_j-x_i}\quad \alpha_2=\frac{\phi_j-\phi_i}{x_j-x_i}. \tag{4.7.4}$$

Substituting in Eq. (4.7.1), yields

$$\phi^{(e)}=\frac{x_j-x}{L}\,\phi_i+\frac{x-x_i}{L}\,\phi_j \tag{4.7.5}$$

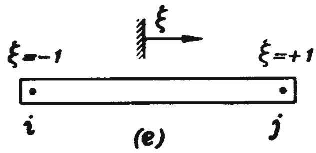


Fig. 4.20 A one-dimensional element with natural coordinates

where $L = x_j - x_i$. Calling

$$N_i = \frac{x_j - x}{L} \quad N_j = \frac{x - x_i}{L} \tag{4.7.6}$$

the shape function reduces to

$$\phi^{(e)} = N_i\phi_i + N_j\phi_j = \langle\, N\,\rangle^{(e)}\{\phi\} \tag{4.7.7}$$

where the matrix of the shape function is

$$\langle\, N\,\rangle^{(e)} = \langle\, N_i \quad N_j\,\rangle. \tag{4.7.8}$$

The shape functions N_i and N_j were defined in terms of the global coordinate variable x. A local coordinate s may be defined such that at $x = x_i$, $s = 0$ and at $x = x_j$, $s = L$. Thus, since $x = x_i + s$,

$$N_i = 1 - \frac{s}{L} \quad N_j = \frac{s}{L}. \tag{4.7.9}$$

The range of variation of s is $0 \le s \le L$.

Since the variable s has a dimension of length, it is further necessary to define a dimensionless variable (Fig. 4.20). Calling the dimensionless variable

$$\xi = \frac{2s}{L} - 1 \tag{4.7.10}$$

and measuring it from the center of the element (e), it is seen that the range of variation of ξ between nodes i and j is

$$-1 \le \xi \le +1. \tag{4.7.11}$$

The shape functions in terms of the dimensionless variable ξ are

$$N_i = \tfrac{1}{2}\,(1 - \xi) \quad N_j = \tfrac{1}{2}\,(1 + \xi). \tag{4.7.12}$$

Since the variable is ultimately changed from x to ξ, a proper Jacobian must be considered in evaluation of the integrals of the members of force and stiffness matrices. An integral in the global coordinate system may be evaluated in the local coordinate system using the Jacobian of transformation as

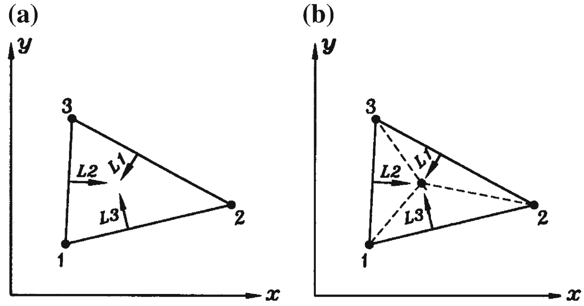


Fig. 4.21 Area coordinates

$$\int f(x)dx = \int f(\xi)|J|d\xi \tag{4.7.13}$$

where the Jacobian in this case is

$$dx = \frac{\partial x}{\partial \xi}\, d\xi = |J|d\xi$$
$$|J| = \frac{L}{2}, \tag{4.7.14}$$

for example

$$\int N_i(x)dx = \int_{x_i}^{x_j} \frac{x_j - x}{L} dx = \int N_i(\xi)|J|d\xi = \int_{-1}^{+1} \frac{1}{2}(1-\xi)\frac{L}{2}d\xi = \frac{L}{2}. \tag{4.7.15}$$

4.8 Local Coordinates in Two-Dimensions

A popular coordinate system in triangular elements in two -dimensions is the area coordinates. Using this coordinates system, the integration of the shape functions and their derivatives may be obtained by analytical methods [2].

Consider the triangular element (e) with nodes 1, 2, and 3, as shown in Fig. 4.21a. The area of the element is

$$A = \frac{bh}{2}. \tag{4.8.1}$$

An arbitrary point P in the element makes the triangle A_1 the area of which is, see Fig. 4.21b

$$A_1 = \frac{l_{23} h_1}{2} \tag{4.8.2}$$

where l_{23} is the length of side 23 and h_1 is the height of the point P relative to l_{23}. The ratio

$$L_1 = \frac{A_1}{A} \tag{4.8.3}$$

is considered as a coordinate of the arbitrary point P. The other coordinates of the point P are

$$L_2 = \frac{A_2}{A} \quad L_3 = \frac{A_3}{A} \tag{4.8.4}$$

where A_2 and A_3 are the areas of the triangles of apex P and bases l_{13} and l_{12}, respectively. The coordinates of any point in the element (e) are defined by the coordinates L_1, L_2, and L_3, as shown in Fig. 4.21c. Since

$$L_1 + L_2 + L_3 = 1 \tag{4.8.5}$$

thus, two of the three coordinates L_1, L_2, and L_3 are independent. These new variables are called *area coordinates*, and have some interesting properties.

It is easily verified that the area coordinates L_1, L_2, and L_3 are identical to the linear shape functions in the triangular elements as

$$L_1 = N_1 \quad L_2 = N_2 \quad L_3 = N_3. \tag{4.8.6}$$

Due to their definition, their value is one at the node indicated by its subscript and zero at the other nodes, i.e.,

$$L_1 = \begin{cases} 1 & \text{on node 1} \\ 0 & \text{on nodes 2 and 3.} \end{cases} \tag{4.8.7}$$

The coordinate transformation between two systems (x, y) and (L_1, L_2, L_3) are

$$\begin{aligned} x &= L_1 X_1 + L_2 X_2 + L_3 X_3 \\ y &= L_1 Y_1 + L_2 Y_2 + L_3 Y_3 \end{aligned} \tag{4.8.8}$$

where (X_1, Y_1), (X_2, Y_2), and (X_3, Y_3) are the coordinates of nodes 1, 2, and 3, respectively. Either of the variables L_1, L_2, and L_3 may be eliminated in terms of the others using Eq. (4.8.5), leaving only two independent variables.

The significance of using the area coordinates is the capability for analytical evaluation of the element integrals. This is an important factor in preparation of the finite element model. If the element integrals are left to be evaluated numerically, a substantial computation time will be required. It is always preferable to evaluate the element integrations by analytical methods to save computation time for other essential steps of the finite element solution. When area coordinates are used, the following rules are known for evaluation of the element integrals [3]

$$\int_{\mathcal{L}} L_1^a L_2^b d\mathcal{L} = \frac{a!b!}{(a+b+1)!}\,\mathcal{L} \tag{4.8.9}$$

$$\int_A L_1^a L_2^b L_3^c dA = \frac{a!b!c!}{(a+b+c+2)!}\,2A. \tag{4.8.10}$$

Equation (4.8.9) is used to evaluate the line integrals over the edge of triangular elements and $\mathcal{L}$ is the length of the edge of the element under consideration. Some examples of these integral formulas are given in the following:

$$\int_A N_i N_j dA = \int_A L_1^1 L_2^1 L_3^0 dA = \frac{1!1!0!}{(1+1+0+2)!}\,2A = \frac{A}{12}$$

$$\int_A N_i^2 dA = \int_A L_1^2 L_2^0 L_3^0 dA = \frac{2!0!0!}{(2+0+0+2)!}\,2A = \frac{A}{6}.$$

4.9 Volume Integral

Similar to the area coordinates, the volume coordinates in a tetrahedral element are defined as the ratio of volumes. The coordinates of an arbitrary point P in the base element (e) are defined by L_1, L_2, L_3, and L_4, where

$$L_1 = \frac{V_1}{V}, \quad L_2 = \frac{V_2}{V}, \quad L_3 = \frac{V_3}{V}, \quad L_4 = \frac{V_4}{V} \tag{4.9.1}$$

where V is the total volume of the element, V_1 is the volume of the tetrahedral made of apex P and side opposite to node 1, and so on. From Eq. (4.9.1),

$$L_1 + L_2 + L_3 + L_4 = 1. \tag{4.9.2}$$

Three of the four variables L_1, L_2, L_3, and L_4 are independent. It is further verified that

$$L_1 = N_1, \quad L_2 = N_2, \quad L_3 = N_3, \quad L_4 = N_4. \tag{4.9.3}$$

The coordinate transformation law between the conventional coordinates x, y, z and the volume coordinates L_1, L_2, L_3 and L_4 are

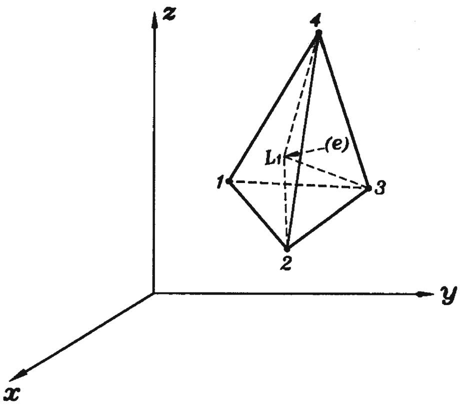


Fig. 4.22 Volume coordinates

$$
\begin{aligned}
x &= L_1 X_1 + L_2 X_2 + L_3 X_3 + L_4 X_4 \\
y &= L_1 Y_1 + L_2 Y_2 + L_3 Y_3 + L_4 Y_4 \\
z &= L_1 Z_1 + L_2 Z_2 + L_3 Z_3 + L_4 Z_4.
\end{aligned}
\tag{4.9.4}
$$

The advantage of using volume coordinates in tetrahedral elements is the simple analytical integration formula of the element matrices. The formula for the volume integral is [2]

$$
\int_V L_1^a L_2^b L_3^c L_4^d dV = \frac{a!b!c!d!}{(a+b+c+d+3)!} 6V. \tag{4.9.5}
$$

The volume coordinate integral rule is a simple means by which to evaluate the necessary matrix integral calculations analytically. This formula eliminates the need for numerical integration of the elements of matrices when preparing the finite element model of a problem. For example,

$$
\begin{aligned}
\int_V N_1 N_2 dV &= \int_V \left(\frac{a_1 + b_1 x + c_1 y + d_1 z}{6V}\right)\left(\frac{a_2 + b_2 x + c_2 y + d_2 z}{6V}\right) dxdydz \\
&= \int_V L_1 L_2 dV = \frac{V}{20}.
\end{aligned}
\tag{4.9.6}
$$

4.10 Problems

1. Use Eq. (4.3.3) and obtain the coefficients α_1 to α_4 in terms of the nodal values for the C^0 and C^1-continuous shape functions.
2. Obtain the coefficients α_1 to α_6 of Eq. (4.4.2) in terms of the coordinates of nodes 1 to 6.

3. For a three-dimensional simplex element, find the coefficients α_1 to α_4 from Eq. (4.5.1) in terms of the nodal values ϕ_1 to ϕ_4.
4. Write the C^0 and C^1-continuous shape functions of a one-dimensional element in terms of the local coordinate ξ. Then, find the Jacobian matrix of the transformed coordinate.
5. Prove that for a triangular simplex element, as shown in Fig. 4.21, $L_1 = N_1$, $L_2 = N_2$, and $L_3 = N_3$, and that they are 1 at the node they represent and zero at the other nodes.
6. Verify the coordinate transformation law of a simplex triangular element given by Eq. (4.8.8) and find the Jacobian of the transformation.
7. Prove that for a tetrahedral simplex element, as shown in Fig. 4.22, $L_1 = N_1$, $L_2 = N_2$, $L_3 = N_3$, and $L_4 = N_4$, and that they are 1 at the node they represent and zero at the other nodes.
8. Verify the coordinate transformation law of a simplex tetrahedral element given by Eq. (4.9.4) and find the Jacobian of the transformation.

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Chapter 5
Field Problems

Abstract Many problems in mechanics are governed by harmonic and biharmonic partial differential equations. This chapter presents detail derivation of the finite element matrices for harmonic and biharmonic problems. The finite element matrices for harmonic equations in Cartesian and cylindrical coordinates, axisymmetric condition, are derived.

5.1 Introduction

Field problems are related to that class of problems in which the governing partial differential equation is Poisson's equation. Many problems in mechanics are governed by Poisson's partial differential equation. Problems of conduction heat transfer in solids, potential flow problems, diffusion of pressure in porous media, and elastic torsion of prismatic bars are examples of field problems. Their equilibrium conditions are all governed by Poisson's equation, in which the dependent variables are temperature, potential or stream functions, pressure, and stress function, respectively. The general treatment and the basic mathematical formulations are similar to the elastic membrane. This chapter attempts to generalize the finite element formulation of field problems such that the formulations can be easily adopted to any similar problems.

5.2 Governing Equations

The general form of Poisson's partial differential equation describing a time-dependent function in the xyz-coordinates system is

$$\frac{\partial}{\partial x}\,(k_{xx}\,\frac{\partial \phi}{\partial x}) + \frac{\partial}{\partial y}\,(k_{yy}\,\frac{\partial \phi}{\partial y}) + \frac{\partial}{\partial z}\,(k_{zz}\,\frac{\partial \phi}{\partial z}) + Q = a\,\frac{\partial \phi}{\partial t} \quad \text{in } D \qquad (5.2.1)$$

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Solid Mechanics and Its Applications 216, DOI: 10.1007/978-3-319-08037-6_5,

where k_{xx}, k_{yy}, k_{zz}, and a are the material properties which may be either constants or functions of space variables. The function Q is the heat generation per unit volume per unit time, and may be a constant or a function of space variables.

The equilibrium equation (5.2.1) is exposed to the initial and boundary conditions described as:

The initial condition:

$$\phi(x, y, z, t) = \phi_0(x, y, z, 0) \quad \text{in} \;\; D \text{ for } \; t = 0. \tag{5.2.2}$$

The boundary conditions:

$$\begin{aligned} &\phi(x, y, z, t) = \phi_s \quad \text{on } \partial D \text{ for } t > 0 \\ &k_{xx} \frac{\partial \phi}{\partial x} l + k_{yy} \frac{\partial \phi}{\partial y} m + k_{zz} \frac{\partial \phi}{\partial z} n + q + h(\phi - \phi_\infty) = 0 \\ &\quad \text{on } \partial D \text{ for } t > 0 \end{aligned} \tag{5.2.3}$$

where ∂D is the boundary of the solution domain D. Here, $\phi_0(x, y, z, 0)$ is the known value of ϕ at $t = 0$, ϕ_s is the known value of ϕ on the boundary, l, m, and n are the cosine directions of a normal unit vector $\vec{n}$ at the boundary. The parameters q, h, and ϕ_∞ are assumed to be known constants.

The functional associated with Poisson's equation (5.2.1) is written in analogy with the elastic membrane problem as

$$\begin{aligned} \chi = &\int_V \tfrac{1}{2} \, [k_{xx} \, (\frac{\partial \phi}{\partial x})^2 + k_{yy} \, (\frac{\partial \phi}{\partial y})^2 + k_{zz} \, (\frac{\partial \phi}{\partial z})^2 - 2(Q - a \, \frac{\partial \phi}{\partial t})\phi] dV \\ &+ \int_{S_1} q\phi dS + \int_{S_2} \tfrac{1}{2} \, h(\phi - \phi_\infty)^2 dS. \end{aligned} \tag{5.2.4}$$

It is simply verified that the minimum of the functional (5.2.4), using the method of calculus of variations, is Eq. (5.2.1) subjected to the boundary conditions (5.2.3).

Defining the gradient matrix $\langle g \rangle$ as

$$\langle g \rangle = \langle \frac{\partial \phi}{\partial x} \;\; \frac{\partial \phi}{\partial y} \;\; \frac{\partial \phi}{\partial z} \rangle \tag{5.2.5}$$

and the conduction coefficient matrix as

$$[k] = \begin{bmatrix} k_{xx} & 0 & 0 \\ 0 & k_{yy} & 0 \\ 0 & 0 & k_{zz} \end{bmatrix} \tag{5.2.6}$$

the functional (5.2.4) can be written in matrix form as

$$\chi = \int_V \tfrac{1}{2}\,[\{g\}^T[k]\{g\} - 2(Q - a\,\frac{\partial \phi}{\partial t})\phi]dV$$
$$+ \int_{S_1} q\phi dS + \int_{S_2} \tfrac{1}{2}\,h(\phi - \phi_\infty)^2 dS. \tag{5.2.7}$$

To find the finite element solution to the field problems, the solution domain is discretized into NE number of elements with total NN number of nodes. The function ϕ is approximated for each element, and the functional is the total sum of the functionals of each element (e) as

$$\chi = \sum_{e=1}^{NE} \chi^{(e)} = \sum_{e=1}^{NE} \left(\int_{V(e)} \tfrac{1}{2}\,\{g^{(e)}\}^T[k]\{g^{(e)}\}dV \right.$$
$$\left. - \int_{V(e)} \left(Q - a\,\frac{\partial \phi}{\partial t}\right)^{(e)} \phi^{(e)} dV \right)$$
$$+ \sum_{e=1}^{NE} \left(\int_{S_1(e)} q^{(e)}\phi^{(e)} dS + \int_{S_2(e)} \tfrac{1}{2}\,h(\phi^{(e)} - \phi_\infty)^2 dS \right). \tag{5.2.8}$$

Here, S_1 and S_2 are the boundaries of the solution domain where the following boundary conditions are specified:

$$k\,\frac{\partial \phi}{\partial n} = q'' \quad \text{on} \quad S_1$$
$$k\,\frac{\partial \phi}{\partial n} = h(\phi - \phi_\infty) \quad \text{on} \quad S_2. \tag{5.2.9}$$

Denoting the shape function for approximation of ϕ in the base element (e) by $\langle N^{(e)} \rangle$, the function ϕ in terms of its nodal values in element (e) is approximated as

$$\phi^{(e)} = \langle N^{(e)}(x, y, z) \rangle \{\Phi(t)\}^{(e)} \tag{5.2.10}$$

where $\{\Phi\}^{(e)}$ is the matrix of nodal values of ϕ. It is noted that the space and time variables are considered in separate functions. This is called the *Kantrovich* approximation. For an element of p nodal points

$$\langle \Phi \rangle^{(e)} = \langle \Phi_1 \; \Phi_2 \; \cdots \; \Phi_p \rangle$$

Substituting Eq. (5.2.10) in (5.2.5), the gradient matrix in terms of the nodal values is

$$\{g^{(e)}\} = \begin{Bmatrix} \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \\ \frac{\partial \phi}{\partial z} \end{Bmatrix}^{(e)} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \cdots & \frac{\partial N_p}{\partial x} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \cdots & \frac{\partial N_p}{\partial y} \\ \frac{\partial N_1}{\partial z} & \frac{\partial N_2}{\partial z} & \cdots & \frac{\partial N_p}{\partial z} \end{bmatrix}^{(e)} \begin{Bmatrix} \Phi_1 \\ \Phi_2 \\ \cdot \\ \cdot \\ \cdot \\ \Phi_p \end{Bmatrix}. \tag{5.2.11}$$

Calling

$$[B]^{(e)} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \cdots & \frac{\partial N_p}{\partial x} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \cdots & \frac{\partial N_p}{\partial y} \\ \frac{\partial N_1}{\partial z} & \frac{\partial N_2}{\partial z} & \cdots & \frac{\partial N_p}{\partial z} \end{bmatrix}^{(e)}. \tag{5.2.12}$$

Then

$$\{g\}^{(e)} = [B]^{(e)}\{\Phi\}. \tag{5.2.13}$$

Substituting in the expression for functional of element (e), yields

$$\begin{aligned}\chi^{(e)} &= \int_{V(e)} \tfrac{1}{2}\,\{\Phi\}^T[B]^T[k][B]\{\Phi\}dV - \int_{V(e)} Q\{\Phi\}^T\{N\}dV \\ &+ \int_{V(e)} a\{\Phi\}^T\{N\}\langle N\rangle \left\{\frac{\partial \Phi}{\partial t}\right\} dV + \int_{S_1(e)} q\{\Phi\}^T\{N\}dS \\ &+ \int_{S_2(e)} \tfrac{1}{2}\,h\{\Phi\}^T\{N\}\langle N\rangle\{\Phi\}dS - \int_{S_2(e)} h\phi_\infty\{\Phi\}^T\{N\}dS \\ &+ \int_{S_2(e)} \tfrac{1}{2}\,h\phi_\infty^2 dS. \end{aligned} \tag{5.2.14}$$

Since the matrix $\{\Phi\}$ is constant, it is taken out of the integral sign. Applying the Ritz law yields

$$\frac{\partial \chi}{\partial\{\Phi\}} = \frac{\partial}{\partial\{\Phi\}} \sum_{e=1}^{NE} \chi^{(e)} = 0. \tag{5.2.15}$$

The differentiation of the functional with respect to $\{\Phi\}$ may be applied to the functional of each individual element and the resulting expressions summed over the entirety of elements in the solution domain. The differentiation follows as

$$\frac{\partial \chi^{(e)}}{\partial \{\Phi\}^{(e)}} = \left(\int_{V(e)} [B]^T [k][B] dV\right) \{\Phi\} + \left(\int_{V(e)} a\{N\}\langle N\rangle dV\right) \left\{\frac{d\Phi}{dt}\right\}$$
$$- \int_{V(e)} Q\{N\} dV + \int_{S_1(e)} q\{N\} dS + \left(\int_{S_2(e)} h\{N\}\langle N\rangle dS\right) \{\Phi\}$$
$$- \int_{S_2(e)} h\phi_\infty \{N\} dS. \tag{5.2.16}$$

This equation may be written as

$$\frac{\partial \chi^{(e)}}{\partial \{\Phi\}^{(e)}} = [c]^{(e)} \{\dot{\Phi}\}^{(e)} + [k]^{(e)} \{\Phi\}^{(e)} + \{f\}^{(e)} \tag{5.2.17}$$

where the definitions of each of the above matrices are

$$[c]^{(e)} = \int_{V(e)} a\{N\}\langle N\rangle dV$$
$$[k]^{(e)} = \int_{V(e)} [B]^T [k][B] dV + \int_{S_2(e)} h\{N\}\langle N\rangle dS$$
$$\{f\}^{(e)} = -\int_{V(e)} Q\{N\} dV + \int_{S_1(e)} q\{N\} dS - \int_{S_2(e)} h\phi_\infty \{N\} dS. \tag{5.2.18}$$

Substituting Eq. (5.2.17) in (5.2.15) and adding up the differentiated functional of all the elements in the solution domain gives

$$\frac{\partial \chi}{\partial \{\Phi\}} = \sum_{e=1}^{NE} [c]^{(e)} \{\dot{\Phi}\}^{(e)} + [k]^{(e)} \{\Phi\}^{(e)} + \{f\}^{(e)} = 0 \tag{5.2.19}$$

or, in general,

$$[C]\left\{\dot{\Phi}\right\} + [K]\{\Phi\} = \{F\} \tag{5.2.20}$$

where

$$[C] = \sum_{e=1}^{NE} [c]_G^{(e)}$$
$$[K] = \sum_{e=1}^{NE} [k]_G^{(e)}$$
$$[F] = \sum_{e=1}^{NE} \{f\}_G^{(e)}. \tag{5.2.21}$$

The matrices $[C]$, $[K]$, and $\{F\}$ are called the capacitance, stiffness, and force matrices, respectively. The index G means that the matrices are transformed into the global coordinate system. The relationship between the element matrices in the local and global coordinate systems are given as

$$\begin{aligned}
[c]_G^{(e)} &= [R]^T [c]_L^{(e)} [R] \\
[k]_G^{(e)} &= [R]^T [k]_L^{(e)} [R] \\
\{f\}_G^{(e)} &= [R]^T \{f\}_L^{(e)}
\end{aligned} \tag{5.2.22}$$

where matrix $[R]$ is the rotation matrix, transforming the local coordinates into global coordinates.

5.3 Axisymmetric Field Problems

Consider Poisson's equation in two-dimensional axisymmetric problems in rz-plane as

$$k_r \left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} \right) + k_z \frac{\partial^2 \phi}{\partial z^2} = p \tag{5.3.1}$$

where z is along the axis of symmetry and r is the radial direction. Considering the base element (e) in the rz-plane, as shown in Fig. 5.1, the shape function for the dependent function ϕ may be written as

$$\phi = N_i \Phi_i + N_j \Phi_j + N_k \Phi_k \tag{5.3.2}$$

where

$$\begin{aligned}
N_i &= \frac{1}{2A} (a_i + b_i r + c_i z) \\
N_j &= \frac{1}{2A} (a_j + b_j r + c_j z) \\
N_k &= \frac{1}{2A} (a_k + b_k r + c_k z).
\end{aligned} \tag{5.3.3}$$

The constants a, b, and c are defined in terms of the nodal coordinates and A is the area of the element (e). Using the Galerkin method, the residue of Eq. (5.3.1) is made orthogonal with respect to the shape functions N_s, $s = i, j, k$ as

$$\int_{V(e)} \left(k_r \frac{\partial^2 \phi}{\partial r^2} + \frac{k_r}{r} \frac{\partial \phi}{\partial r} + k_z \frac{\partial^2 \phi}{\partial z^2} - p \right) N_s dV \qquad s = i, j, k. \tag{5.3.4}$$

Equation (5.3.4) may be written as

$$\int_{V(e)} [\frac{k_r}{r} \frac{\partial}{\partial r} (r \frac{\partial \phi}{\partial r}) + k_z \frac{\partial^2 \phi}{\partial z^2} - p] N_s dV \qquad s = i, j, k. \tag{5.3.5}$$

The terms with the second order partial derivatives are subjected to weak formulation using the Gauss integral theorem. From the rule of differentiation

$$\frac{1}{r} \frac{\partial}{\partial r} \left[N_s r \frac{\partial \phi}{\partial r} \right] = \frac{1}{r} \frac{\partial N_s}{\partial r} r \frac{\partial \phi}{\partial r} + \frac{1}{r} N_s \frac{\partial}{\partial r} (r \frac{\partial \phi}{\partial r})$$

or

$$\frac{1}{r} N_s \frac{\partial}{\partial r} (r \frac{\partial \phi}{\partial r}) = \frac{1}{r} \frac{\partial}{\partial r} \left[N_s r \frac{\partial \phi}{\partial r} \right] - \frac{\partial N_s}{\partial r} \frac{\partial \phi}{\partial r}. \tag{5.3.6}$$

Similarly, since

$$\frac{\partial}{\partial z} [N_s \frac{\partial \phi}{\partial z}] = N_s \frac{\partial^2 \phi}{\partial z^2} + \frac{\partial N_s}{\partial z} \frac{\partial \phi}{\partial z}$$

then

$$N_s \frac{\partial^2 \phi}{\partial z^2} = \frac{\partial}{\partial z} (N_s \frac{\partial \phi}{\partial z}) - \frac{\partial N_s}{\partial z} \frac{\partial \phi}{\partial z}. \tag{5.3.7}$$

Substituting Eqs. (5.3.6) and (5.3.7) in (5.3.5) gives

$$\int_{V(e)} \left[-k_r \frac{\partial N_s}{\partial r} \frac{\partial \phi}{\partial r} - k_z \frac{\partial N_s}{\partial z} \frac{\partial \phi}{\partial z} - N_s p \right] dV + \int_{V(e)} [\frac{k_r}{r} \frac{\partial}{\partial r} (N_s r \frac{\partial \phi}{\partial r}) + k_z \frac{\partial}{\partial z} (N_s \frac{\partial \phi}{\partial z})] dV. \tag{5.3.8}$$

The second volume integral of Eq. (5.3.8) is transformed into the surface integral using the Gauss integral theorem as

$$\int_{\Gamma(e)} \left[\frac{k_r}{r} (N_s r \frac{\partial \phi}{\partial r}) \cos\theta + k_z N_s \frac{\partial \phi}{\partial z} \sin\theta \right] d\Gamma \tag{5.3.9}$$

where θ is the angle of a unit outer normal vector to the element boundary with r-axis. Simplifying expression (5.3.9) and substituting in Eq. (5.3.8) yields

$$\int_{V(e)} \left[k_r \frac{\partial N_s}{\partial r} \frac{\partial \phi}{\partial r} + k_z \frac{\partial N_s}{\partial z} \frac{\partial \phi}{\partial z} \right] dV - \int_{V(e)} N_s p dV - \int_{\Gamma(e)} N_s \left(k_r \frac{\partial \phi}{\partial r} \cos\theta + k_z \frac{\partial \phi}{\partial z} \sin\theta \right) d\Gamma. \tag{5.3.10}$$

Substituting for ϕ its value from Eq. (5.3.2) yields

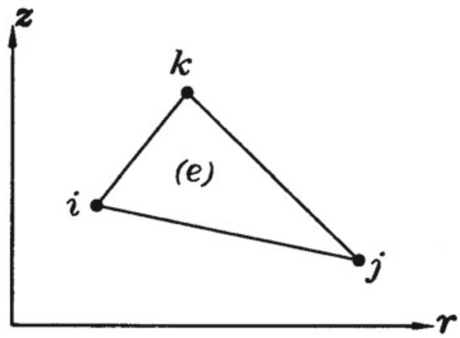

Fig. 5.1 An element in axisymmetric plane

$$\left[\int_{V(e)}\left(k_r\,\frac{\partial N_s}{\partial r}\,\frac{\partial N_t}{\partial r}+k_z\,\frac{\partial N_s}{\partial z}\,\frac{\partial N_t}{\partial z}\right)dV\right]\Phi_t-\int_{V(e)}N_s p dV$$
$$-\left[\int_{\Gamma(e)}N_s\left(k_r\,\frac{\partial N_t}{\partial r}\,\cos\theta+k_z\,\frac{\partial N_t}{\partial z}\,\sin\theta\right)d\Gamma\right]\Phi_t\quad s,t=i,j,k. \tag{5.3.11}$$

In matrix form, Eq. (5.3.11) for element (e) is

$$[k]^{(e)}\{\Phi\}^{(e)}-\{f_p\}^{(e)}-\{f_b\}^{(e)} \tag{5.3.12}$$

where

$$[k]_{st}^{(e)}=\int_{V(e)}\left(k_r\,\frac{\partial N_s}{\partial r}\,\frac{\partial N_t}{\partial r}+k_z\,\frac{\partial N_s}{\partial z}\,\frac{\partial N_t}{\partial z}\right)dV$$
$$\{f_p\}^{(e)}=\int_{V(e)}N_s p dV \qquad s,t=i,j,k$$
$$\{f_b\}^{(e)}=\int_{\Gamma(e)}N_s\left(k_r\,\frac{\partial N_t}{\partial r}\,\cos\theta+k_z\,\frac{\partial N_t}{\partial z}\,\sin\theta\right)d\Gamma. \tag{5.3.13}$$

Assembling all the elements of the solution domain, Eq. (5.3.12) is added up for the elements and becomes

$$[K]\{\Phi\}=\{F_p\}+\{F_b\} \tag{5.3.14}$$

where the force matrix $\{F_p\}$ is due to the force p distributed in the solution domain, and $\{F_b\}$ is the contribution of the boundary forces.

For a simplex element shown in Fig. 5.1, using the set of shape functions (5.3.3), the elements of the stiffness matrix are

$$[k]^{(e)}=\frac{2\pi\bar{r}k_r}{4A}\begin{bmatrix}b_ib_i & b_ib_j & b_ib_k\\ b_jb_i & b_jb_j & b_jb_k\\ b_kb_i & b_kb_j & b_kb_k\end{bmatrix}+\frac{2\pi\bar{r}k_z}{4A}\begin{bmatrix}c_ic_i & c_ic_j & c_ic_k\\ c_jc_i & c_jc_j & c_jc_k\\ c_kc_i & c_kc_j & c_kc_k\end{bmatrix} \tag{5.3.15}$$

where

$$\bar{r} = \frac{R_i + R_j + R_k}{3}. \tag{5.3.16}$$

The elements of the force matrix are

$$\{f_p\}^{(e)} = \int_{V(e)} p\{N\}dV = 2\pi p \int_{A(e)} \begin{Bmatrix} rN_i \\ rN_j \\ rN_k \end{Bmatrix} dA \tag{5.3.17}$$

where $dV = 2\pi r dA$. Replacing the shape functions with the area coordinates and considering the rule of coordinate transformation

$$r = N_i R_i + N_j R_j + N_k R_k = L_1 R_i + L_2 R_j + L_3 R_k. \tag{5.3.18}$$

Equation (5.3.17) is rewritten as

$$\{f_p\}^{(e)} = 2\pi p \int_{A(e)} \begin{Bmatrix} L_1(L_1 R_i + L_2 R_j + L_3 R_k) \\ L_2(L_1 R_i + L_2 R_j + L_3 R_k) \\ L_3(L_1 R_i + L_2 R_j + L_3 R_k) \end{Bmatrix} dA. \tag{5.3.19}$$

The integration, using the rule of area coordinates, yields

$$\{f_p\}^{(e)} = \frac{2\pi p A}{12} \begin{Bmatrix} 2R_i + R_j + R_k \\ R_i + 2R_j + R_k \\ R_i + R_j + 2R_k \end{Bmatrix}. \tag{5.3.20}$$

The distribution of p in element (e) was assumed to be uniform. In the case of variable p inside the element (e), proper integration technique must be considered to take care of the variation of p.

5.4 Biharmonic Field Problems

Many problems of mechanics are governed by the biharmonic operator. The lateral deflection of plates, the Airy stress function of the two-dimensional elasticity, and the potential theory of thermoelasticity are examples of such problems. The finite element solution of such problems may be approached by the variational formulation of the biharmonic operator, employing the Ritz method. For such an approach, the functional of the biharmonic operator must be considered.

Consider a biharmonic field problem. The functional of a biharmonic equation is

$$\chi(\phi) = \int\int_D \left[\left(\nabla^2 \phi \right)^2 - 2f(x, y)\phi \right] dxdy \tag{5.4.1}$$

where ϕ is the solution of a biharmonic field problem. We first show that the expression of the functional given in the above equation is associated with $\nabla^4\phi = f(x, y)$. To prove this assertion, we use the standard method of calculus of variations to minimize the expression of the functional. The expression which minimizes the functional should be $\nabla^4\phi = f(x, y)$. We consider the expression

$$\chi(\phi) = \int\int_D \left[\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} \right)^2 - 2f\phi \right] dxdy. \tag{5.4.2}$$

Expanding the first term

$$\chi(\phi) = \int\int_D \left[\left(\frac{\partial^2\phi}{\partial x^2} \right)^2 + \left(\frac{\partial^2\phi}{\partial y^2} \right)^2 + 2\frac{\partial^2\phi}{\partial x^2}\frac{\partial^2\phi}{\partial y^2} - 2f\phi \right] dxdy. \tag{5.4.3}$$

Consider a variational parameter ϵ and the variation function $u = u(x, y)$. The variation function $u(x, y)$ must satisfy the same conditions of the class of admissible functions of ϕ.

The function $u(x, y)$ and its first and second partial derivatives with respect to x and y are continuous in the solution domain D. That is:

1. $u(x, y) = 0$ on ∂D
2. $u(x, y),\ u_{,x},\ u_{,y},\ u_{,xx},\ u_{,yy}, u_{,xy}$ are continuous in D.

Now, we may construct the function

$$\phi^*(x, y) = \phi(x, y) + \epsilon u(x, y). \tag{5.4.4}$$

Substituting ϕ^* in the expression of the functional gives

$$\begin{aligned} \chi(\phi^*) = \int\int_D \Bigg\{ & \left[\left(\frac{\partial^2(\phi + \epsilon u)}{\partial x^2} \right) \right]^2 + \left[\left(\frac{\partial^2(\phi + \epsilon u)}{\partial y^2} \right) \right]^2 \\ & + 2\left[\frac{\partial^2(\phi + \epsilon u)}{\partial x^2} \frac{\partial^2(\phi + \epsilon u)}{\partial y^2} \right] - 2f(\phi + \epsilon u) \Bigg\} dxdy. \end{aligned} \tag{5.4.5}$$

To apply the method of calculus of variation, the derivative of the functional with respect to the variational parameter ϵ is

$$\begin{aligned} \frac{\partial\chi(\phi^*)}{\partial\epsilon} = \int\int_D \Bigg[& 2\epsilon\left(\frac{\partial^2 u}{\partial x^2} \right)^2 + 2\frac{\partial^2 u}{\partial x^2}\frac{\partial^2\phi}{\partial x^2} + 2\epsilon\left(\frac{\partial^2 u}{\partial y^2} \right)^2 + 2\frac{\partial^2 u}{\partial y^2}\frac{\partial^2\phi}{\partial y^2} \\ & + 2\left(\frac{\partial^2\phi}{\partial x^2}\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2}\frac{\partial^2\phi}{\partial y^2} + 2\epsilon\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} \right) - 2uf(x, y) \Bigg] dxdy. \end{aligned} \tag{5.4.6}$$

Setting the variational parameter equal to zero yields

$$\frac{\partial \chi(\phi)}{\partial \epsilon}|_{\epsilon \to 0} = \int\int_D \left[\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 u}{\partial y^2}\frac{\partial^2 \phi}{\partial x^2} - uf(x, y) \right] dxdy. \tag{5.4.7}$$

This expression is simplified to

$$\frac{\partial \chi(\phi)}{\partial \epsilon}|_{\epsilon \to 0} = \int\int_D [\nabla^2 \phi \nabla^2 u - uf(x, y)] dxdy. \tag{5.4.8}$$

The first term of the integral may be written as

$$\begin{aligned} \int\int_D [\nabla^2 \phi \nabla^2 u] dxdy = & \int\int_D [\frac{\partial}{\partial x}(\nabla^2 \phi \frac{\partial u}{\partial x}) + \frac{\partial}{\partial y}(\nabla^2 \phi \frac{\partial u}{\partial y})] dxdy \\ & - \int\int_D [\frac{\partial}{\partial x}(u \ \frac{\partial \nabla^2 \phi}{\partial x}) + \frac{\partial}{\partial y}(u \ \frac{\partial \nabla^2 \phi}{\partial y})] dxdy \\ & + \int\int_D u \ [\frac{\partial^2 \nabla^2 \phi}{\partial x^2} + \frac{\partial^2 \nabla^2 \phi}{\partial y^2}] dxdy. \end{aligned} \tag{5.4.9}$$

Now, using the Green integral theorem, the area integral is transferred into the line integral as

$$\int\int_D (\frac{\partial M}{\partial y} + \frac{\partial N}{\partial x}) dxdy = \int_{\partial D} (Ndy - Mdx). \tag{5.4.10}$$

Using this rule, the first and second area integrals transform into the line integral around the boundary of solution domain ∂D as

$$\begin{aligned} - \int\int_D [\frac{\partial}{\partial x}(\nabla^2 \phi \frac{\partial u}{\partial x}) + \frac{\partial}{\partial y}(\nabla^2 \phi \frac{\partial u}{\partial y})] dxdy = & \int_{\partial D} \nabla^2 \phi (\frac{\partial u}{\partial x} dy - \frac{\partial u}{\partial y} dx) \\ = & \int_{\partial D} \nabla^2 \phi \frac{\partial u}{\partial n} \ ds \end{aligned} \tag{5.4.11}$$

$$\begin{aligned} - \int\int_D [\frac{\partial}{\partial x}(u \ \frac{\partial \nabla^2 \phi}{\partial x}) + \frac{\partial}{\partial y}(u \ \frac{\partial \nabla^2 \phi}{\partial y})] dxdy = & - \int_{\partial D} u \ [\frac{\partial \nabla^2 \phi}{\partial x} \ dy - \frac{\partial \nabla^2 \phi}{\partial x}] \ dx \\ = & - \int_{\partial D} u \ \frac{\partial}{\partial n} \ \nabla^2 \phi ds. \end{aligned} \tag{5.4.12}$$

Substituting Eqs. (5.4.9), (5.4.11) and (5.4.12) into (5.4.8) for $\frac{\partial \chi(\phi)}{\partial \epsilon}|_{\epsilon \longrightarrow 0}$, and letting this equation be zero, we obtain

$$\frac{\partial \chi(\phi)}{\partial \epsilon}|_{\epsilon \to 0} = \int\int_D \left[\nabla^2\phi\nabla^2 u - uf(x,y)\right] dxdy$$
$$= \int\int_D u\ \nabla^4\phi dxdy + \int_{\partial D} \nabla^2\phi \frac{\partial u}{\partial n}\ ds - \int_{\partial D} u\ \frac{\partial}{\partial n}\ \nabla^2\phi ds. \tag{5.4.13}$$

Since $u(x, y)$ is an arbitrary function, we argue that the integral of the above equation must vanish independently. Therefore, the resulting boundary value problem is

$$\nabla^4\phi - f(x,y) = 0 \tag{5.4.14}$$

Or

$$\frac{\partial^4\phi}{\partial x^4} + 2\frac{\partial^4\phi}{\partial x^2\partial y^2} + \frac{\partial^4\phi}{\partial y^4} = f(x,y) \tag{5.4.15}$$

and the natural boundary conditions are

$$\frac{\partial}{\partial n}\ \nabla^2\phi = 0 \text{ and } \nabla^2\phi = 0 \quad \text{ on } \partial D. \tag{5.4.16}$$

5.5 Finite Element of Biharmonic Formulation

The functional of a biharmonic equation is

$$\chi = \int_D [(\nabla^2\phi)^2 - 2f(x,y)\phi]dxdy. \tag{5.5.1}$$

The solution domain is divided into NE number of arbitrary elements, and the Ritz law is applied to the functional of each element for the generalized degrees of freedom q_i (Fig. 5.2). The function distribution in the base element (e) is approximated as

$$\phi^{(e)} = N_i q_i \tag{5.5.2}$$

and the Ritz law is

$$\frac{\partial \chi}{\partial q_i} = \frac{\partial \sum_{e=1}^{NE} \chi^{(e)}}{\partial q_i} = 0. \tag{5.5.3}$$

The derivative of the functional of the base element, $\chi^{(e)}$, with respect to the nodal degree of freedom is

$$\frac{\partial \chi^{(e)}}{\partial q_i} = \int_{D(e)} \left(\nabla^2\phi\right) \frac{\partial \nabla^2\phi}{\partial q_i} dxdy - \int_{D(e)} \frac{\partial}{\partial q_i}(f(x,y)\phi)dA. \tag{5.5.4}$$

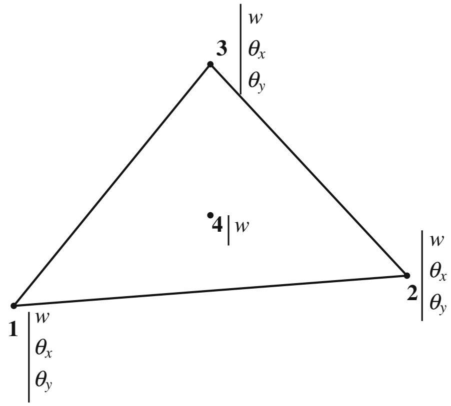

Fig. 5.2 Degrees of freedom of the base element (e)

The first term is

$$\int_{D(e)} \left(\nabla^2\phi\right) \frac{\partial\left(\nabla^2\phi\right)}{\partial q_i} dxdy$$
$$= \left\{ \int_{D(e)} \left(\frac{\partial^2 N_i}{\partial x^2} + \frac{\partial^2 N_i}{\partial y^2}\right)\left(\frac{\partial^2 N_i}{\partial x^2} + \frac{\partial^2 N_i}{\partial y^2}\right) dxdy \right\} q_i = [k]\{q\} \quad (5.5.5)$$

and the second term is

$$-\int_{D(e)} f(x, y) N_i dxdy = -\{f\}. \quad (5.5.6)$$

With the sum of the above equations, the final form of the finite element equilibrium equation when all element equations are assembled is

$$[K]\{Q\} = \{F\}. \quad (5.5.7)$$

This equation is solved for the unknown values of the nodal degrees of freedom at the nodal points.

5.6 Finite Element Solution

The shape function in terms of the local coordinates may be considered to be of the following form:

$$\phi = a_1 + a_2\xi + a_3\eta + a_4\xi^2 + a_5\xi\eta + a_6\eta^2 + a_7\xi^3 + a_8\xi^2\eta + a_9\xi\eta^2 + a_{10}\eta^3. \quad (5.6.1)$$

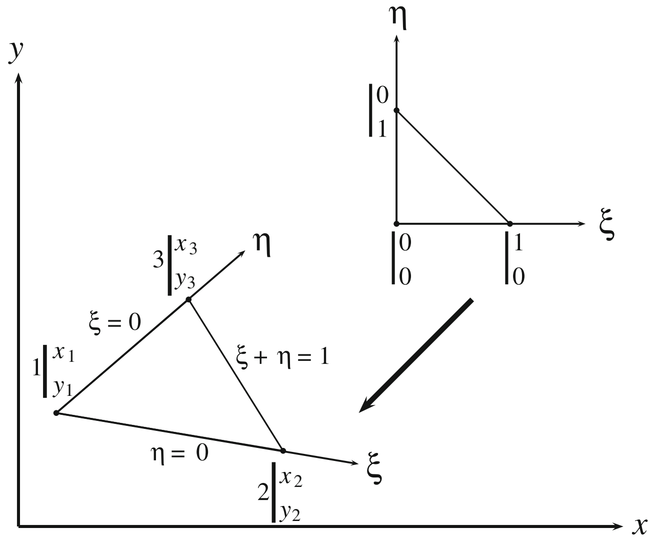

Fig. 5.3 Element and the natural coordinates

The coordinate transformation law between the global coordinates (x, y) and the local coordinates (ξ, η), as shown in Fig. 5.3, is

$$\begin{aligned} x &= (1 - \xi - \eta)x_1 + \xi(x_2) + \eta(x_3) \\ y &= (1 - \xi - \eta)y_1 + \xi(y_2) + \eta(y_3). \end{aligned} \tag{5.6.2}$$

The inverse transformation is

$$\begin{aligned} \xi &= -\frac{x_1 y_3 - x_1 y + x_3 y + y_1 x - y_1 x_3 - x y_3}{x_2 y_3 - x_2 y_1 - x_1 y_3 - y_2 x_3 + y_2 x_1 + y_1 x_3} \\ \eta &= \frac{-x_1 y + x_2 y - x_2 y_1 - y_2 x + y_2 x_1 + y_1 x}{x_2 y_3 - x_2 y_1 - x_1 y_3 - y_2 x_3 + y_2 x_1 + y_1 x_3}. \end{aligned} \tag{5.6.3}$$

The nodal degrees of freedom for a C^1-continuous element are selected to be ϕ, θ_x, and θ_y as given

$$\begin{aligned} \phi =& a_1 + a_2\xi + a_3\eta + a_4\xi^2 + a_5\xi\eta + \text{‘}a_6\eta^2 \\ &+ a_7\xi^3 + a_8\xi^2\eta + a_9\xi\eta^2 + a_{10}\eta^3 \\ \theta x =& \frac{\partial\phi}{\partial\xi} = a_2 + 2a_4\xi + a_5\eta + 3a_7\xi^2 + 2a_8\xi\eta + a_9\eta^2 \end{aligned} \tag{5.6.4}$$

$$\theta_y = \frac{\partial\phi}{\partial\eta} = a_3 + a_5\xi + 2a_6\eta + a_8\xi^2 + 2a_9\xi\eta + 3a_{10}\eta^2 \tag{5.6.5}$$

where the coefficients a_i in terms of the nodal degrees of freedom at nodes 1, 2, 3, and 4 are

$$
\begin{aligned}
a_1 &= \phi_1 \\
a_2 &= \theta_{x1} \\
a_3 &= \theta_{y1} \\
a_4 &= -3\phi_1 - 2\theta_{x1} + 3\phi_2 - \theta_{x2} \\
a_5 &= -3\theta_{x1} - 13\phi_1 - 3\theta_{y1} - 7\phi_2 + 2\theta_{x2} - \theta_{y2} - \theta_{x3} - 7\phi_3 \\
&\quad + 2\theta_{y3} + 27\phi_4 \\
a_6 &= -3\phi_1 - 2\theta_{y1} + 3\phi_3 - \theta_{y3} \\
a_7 &= \theta_{x1} + 2\phi_1 - 2\phi_2 + \theta_{x2} \\
a_8 &= 2\theta_{y1} + 3\theta_{x1} + 13\phi_1 + 7\phi_2 - 2\theta_{x2} + 2\theta_{y2} + \theta_{x3} + 7\phi_3 - 2\theta_{y3} - 27\phi_4 \\
a_9 &= 13\phi_1 + 2\theta_{x1} + 3\theta_{y1} + 7\phi_2 - 2\theta_{x2} + \theta_{y2} + 2\theta_{x3} + 7\phi_3 - 2\theta_{y3} - 27\phi_4 \\
a_{10} &= \theta_{y1} + 2\phi_1 - 2\phi_3 + \theta_{y3}.
\end{aligned} \tag{5.6.6}
$$

The constants a_1 to a_{10} are substituted into Eq. (5.6.1) and ϕ is written in terms of the shape functions N_1 to N_{10} as

$$
\begin{aligned}
\phi &= N_1\phi_1 + N_2\theta_{x1} + N_3\theta_{y1} + N_4\phi_2 + N_5\theta_{x2} + N_6\theta_{y2} + N_7\phi_3 + N_8\theta_{x3} \\
&\quad + N_9\theta_{y3} + N_{10}\phi_4
\end{aligned} \tag{5.6.7}
$$

where the shape functions in terms of the local coordinates are

$$
\begin{aligned}
N_1 &= -3\xi^2 - 13\xi\eta - 3\eta^2 + 2\xi^3 + 13\xi^2\eta + 1 + 13\eta^2\xi + 2\eta^3 \\
N_2 &= -2\xi^2 + 2\eta^2\xi + \xi^3 + \xi - 3\xi\eta + 3\xi^2\eta \\
N_3 &= -3\eta^2\xi + \eta + \eta^3 - 3\xi\eta + 2\xi^2\eta - 2\eta^2 \\
N_4 &= 7\eta^2\xi - 2\xi^3 - 7\xi\eta + 7\xi^2\eta + 3\xi^2 \\
N_5 &= \xi^3 - 2\xi^2\eta - \xi^2 - 2\eta^2\xi + 2\xi\eta \\
N_6 &= \eta^2\xi - \xi\eta + 2\xi^2\eta \\
N_7 &= 7\eta^2\xi - 2\eta^3 - 7\xi\eta + 7\xi^2\eta + 3\eta^2 \\
N_8 &= \xi^2\eta + 2\eta^2\xi - \xi\eta \\
N_9 &= -2\eta^2\xi - \eta^2 + \eta^3 + 2\xi\eta - 2\xi^2\eta \\
N_{10} &= 27\xi\eta - 27\eta^2\xi - 27\xi^2\eta.
\end{aligned} \tag{5.6.8}
$$

Employing these shape functions for the base element (e), the stiffness and force matrices of elements are constructed using Eqs. (5.5.5) and (5.5.6), and the final finite element equilibrium Eq. (5.5.7) is assembled using a proper algorithm. A method for assembling the global stiffness and force matrices is presented in Chap. 7.

5.7 Problems

1. Consider a one-dimensional field problem along the x-axis. Using a one-dimensional simplex element, obtain the elements of the capacitance, stiffness, and force matrices.
2. Consider a solution domain made of two straight one-dimensional elements with three nodes. With the assumptions of Problem 1, obtain the global matrices.
3. Reconsider Problem 2 and assume that the coordinates of nodes are 1, 3, and 5, respectively. For $Q = +10$, $q = +20$, $h = 15$, and $\phi_\infty = 70$, when the one-dimensional element has a circular Cross-section with diameter $d = 0.5$, calculate the global matrices.
4. Consider a two-dimensional field problem in xy-coordinates. Obtain the elements of the stiffness and force matrices associated with a triangular simplex element. It is assumed that the $h = q = 0$, and $Q \neq 0$.
5. Obtain the elements of the capacitance matrix of Problem 4.
6. Solve Problem 4 for a three-dimensional field problem with a tetrahedral simplex element.
7. An element in an axisymmetric plane is assumed. With a linear shape function, obtain:

   a. the elements of the shape function in the r–z plane.
   b. with the linear shape function, obtain the members of the stiffness matrix.
   c. obtain the members of the force matrix.

8. Reconsider Problem 7 and assume that the coordinates of the nodal points i, j, and k of the element (e) in the r–z plane are (1,2), (2,3), and (4,2), respectively. Compute the members of the stiffness and force matrices, where $k_r = k_z = 20$, and $p = 50$.
9. Consider the functional

$$I = \int\int_D \left\{ (\nabla\Phi)^2 - 2(1-\nu)\left[\frac{\partial^2\Phi}{\partial x^2}\frac{\partial^2\Phi}{\partial y^2} - \left(\frac{\partial^2\Phi}{\partial x \partial y}\right)^2 \right] - 2f(x,y)\Phi \right\} dxdy$$
$$- 2\int_{\partial D} p(s)\Phi ds + 2\int_{\partial D} m(s)\frac{\partial\Phi}{\partial n} ds.$$

   Use the method of calculus of variation to obtain the natural boundary conditions. The associated boundary value problem has already been discussed and is the biharmonic operator on Φ.

Further Readings

1. Zienkiewicz OC, Cheung YK (1965) Finite elements in the solution of field problems. Eng J 220:507–510
2. Segerlind LJ (1984) Applied finite element analysis. Wiley, New York
3. Kantorovich LV, Krylov VI (1964) Approximate methods of higher analysis (trans: Benster CD Interscience Publishers, New York) P. Noordhoff-Groningen, The Netherlands

Chapter 6
Conduction Heat Transfer in Solids

Abstract Heat conduction problem in solid continuum is one of the major fields in mechanics. This chapter presents detail derivations of the finite element matrices of one, two, and three-dimensional conduction problems. Both variational and Galerkin methods are employed to derive the finite element formulations. Derivation of the capacitance matrix for the transient heat conduction problems is carried out for the one, two, and three-dimensional cases.

6.1 Introduction

Heat transfer in solid bodies by conduction is governed by the equation obtained by the first law of thermodynamics for the balance of heat. This equation, for the steady state conduction and the homogeneous and isotropic materials, is Poisson's equation. For unsteady heat conduction, the time rate of change of the temperature is added. Due to the analogy of the governing equation of heat conduction with an elastic membrane, we may write the expression for the functional. Otherwise, a straight approach for deriving the expression of the functional from thermodynamic principles is not yet formulated. Therefore, the Ritz method may be used to derive the finite element formulations of conduction heat transfer in solids. This is primarily possible due to the mathematical analogy of the governing equation of heat conduction with that of the elastic membrane. That is, there is no physical justification for an expression of the functional of heat conduction problems from the point of view of thermodynamics.

In this chapter, the finite element formulations of the heat conduction problems in a solid continuum are obtained based on both variational and weighted residual methods. The variational derivations are based on the Ritz method, and the weighted residual derivations are based on the Galerkin method. It is interesting to note how the weak formulations in the Galerkin method provide a powerful means to derive all possible boundary conditions in Galerkin formulations. As one of the first historical

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Solid Mechanics and Its Applications 216, DOI: 10.1007/978-3-319-08037-6_6,

works on application of the finite element method to heat conduction problems, one may refer to the paper presented by Wilson and Nickell [1],

6.2 Galerkin Formulations

From the first law of thermodynamics, the thermal energy balance in a solid continuum is [2]

$$-(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}) + Q = \rho c \frac{\partial T}{\partial t} \tag{6.2.1}$$

where q_i are the components of heat flux, Q is the rate of energy generation per unit volume per unit time, T is the absolute temperature, ρ is the mass density, and c is the specific heat of the material under consideration. From Fourier's law of heat conduction in anisotropic solids, the heat flux components are related to the gradients of temperature as [3]

$$\begin{aligned} q_x &= -(k_{xx}\frac{\partial T}{\partial x} + k_{xy}\frac{\partial T}{\partial y} + k_{xz}\frac{\partial T}{\partial z}) \\ q_y &= -(k_{yx}\frac{\partial T}{\partial x} + k_{yy}\frac{\partial T}{\partial y} + k_{yz}\frac{\partial T}{\partial z}) \\ q_z &= -(k_{zx}\frac{\partial T}{\partial x} + k_{zy}\frac{\partial T}{\partial y} + k_{zz}\frac{\partial T}{\partial z}) \end{aligned} \tag{6.2.2}$$

where k_{ij} are the coefficients of heat conduction in a solid continuum of general anisotropic properties, and are assumed to be constant with the temperature variations. Substituting Eqs. (6.2.2) in (6.2.1) results in the thermal equilibrium equation of a solid continuum. The initial and the general boundary conditions are one, or a combination, of the following

$$\begin{aligned} &T(x, y, z, 0) = T_0(x, y, z) \quad \text{at } t = 0 \\ &T(x, y, z, t) = T_s \quad \text{on } S_1 \quad \text{and } t > 0 \\ &q_x l + q_y m + q_z n = -q^{''} \quad \text{on } S_2 \text{ and } t > 0 \\ &q_x l + q_y m + q_z n = h(T - T_\infty) \quad \text{on } S_3 \text{ and } t > 0 \\ &q_x l + q_y m + q_z n = \sigma \epsilon T^4 - \alpha q_r \quad \text{on } S_4 \text{ and } t > 0 \end{aligned} \tag{6.2.3}$$

where $T_0(x, y, z)$ is the known initial temperature, T_s is the known specified temperature, $q^{''}$ is the known heat flux on the boundary, h and T_∞ are the convection coefficient and ambient temperature, σ is the Steffan-Boltzman constant, ϵ is the radiation coefficient of the boundary surface, α is the boundary surface absorption coefficient, and q_r is the rate of thermal flux reaching the boundary surface per unit area. The cosine directors of the unit outer normal vector to the boundary in x, y, and z-directions are shown by l, m, and n, respectively.

Now, we may consider a solution domain divided into a number of elements. The base element (e) with r–nodes is considered. With a Kantrovich approximation for the time and space domain, the temperature distribution in the base element (e) is approximated as

$$T^{(e)}(x, y, z, t) = \sum_{i=1}^{r} N_i(x, y, z)T_i(t) \tag{6.2.4}$$

or, in matrix form

$$T^{(e)}(x, y, z, t) = \langle N(x, y, z)\rangle\{T(t)\}^{(e)}. \tag{6.2.5}$$

The gradient matrix of temperature is

$$\begin{Bmatrix} \frac{\partial T}{\partial x} \\ \frac{\partial T}{\partial y} \\ \frac{\partial T}{\partial z} \end{Bmatrix}^{(e)} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \cdots & \frac{\partial N_r}{\partial x} \\ \cdot & \cdot & \cdot\cdot & \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \cdots & \frac{\partial N_r}{\partial y} \\ \cdot & \cdot & \cdot\cdot & \\ \frac{\partial N_1}{\partial z} & \frac{\partial N_2}{\partial z} & \cdots & \frac{\partial N_r}{\partial z} \end{bmatrix}^{(e)} \begin{Bmatrix} T_1(t) \\ T_2(t) \\ \cdot \\ \cdot \\ T_r(t) \end{Bmatrix}^{(e)} \tag{6.2.6}$$

or

$$\begin{Bmatrix} \frac{\partial T}{\partial x} \\ \frac{\partial T}{\partial y} \\ \frac{\partial T}{\partial z} \end{Bmatrix}^{(e)} = [B]^{(e)}\{T(t)\}^{(e)} \tag{6.2.7}$$

where

$$[B]^{(e)} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \cdots & \frac{\partial N_r}{\partial x} \\ \cdot & \cdot & \cdot\cdot & \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \cdots & \frac{\partial N_r}{\partial y} \\ \cdot & \cdot & \cdot\cdot & \\ \frac{\partial N_1}{\partial z} & \frac{\partial N_2}{\partial z} & \cdots & \frac{\partial N_r}{\partial z} \end{bmatrix}^{(e)} . \tag{6.2.8}$$

Using the approximation of Eq. (6.2.5) and applying the Galerkin method to the equilibrium Eq. (6.2.1) yields

$$\int_{V(e)} (\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} - Q + \rho c \frac{\partial T}{\partial t}) N_i dV = 0 \quad i = 1, 2, \ldots r. \tag{6.2.9}$$

Now, we may use the Gauss integral theorem to apply integration by parts to selected terms. This is a very important stage of the preparation of the finite element model of the problem and is called the *weak formulation*. The weak formulation is used for two important reasons: to obtain the natural boundary conditions, and to lower the order of differentiation of the terms of highest differentiation order for the use of lower order shape functions. The first reason is important since a clever formulation is the one which provides a model with all possible boundary conditions. These possible boundary conditions may be obtained through proper weak formulations applied to the selected terms, although not necessarily the one with the highest differentiation order.

The weak formulation of the heat flux gradients gives

$$\int_{V(e)} (\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}) N_i dV = \int_{S(e)} (\vec{q} \cdot \vec{n}) N_i dS - \int_{V(e)} \langle \frac{\partial N_i}{\partial x} \quad \frac{\partial N_i}{\partial y} \quad \frac{\partial N_i}{\partial z} \rangle \begin{Bmatrix} q_x \\ q_y \\ q_z \end{Bmatrix} dV \tag{6.2.10}$$

where $S(e)$ is the boundary of the base element (e). Substituting Eq. (6.2.10) in (6.2.9) and rearranging the terms gives

$$\int_{V(e)} \rho c \frac{\partial T}{\partial t} N_i dV - \int_{V(e)} \langle \frac{\partial N_i}{\partial x} \quad \frac{\partial N_i}{\partial y} \quad \frac{\partial N_i}{\partial z} \rangle \begin{Bmatrix} q_x \\ q_y \\ q_z \end{Bmatrix} dV = \int_{V(e)} Q N_i dV - \int_{S(e)} (\vec{q} \cdot \vec{n}) N_i dS \quad i = 1, 2, \cdots r. \tag{6.2.11}$$

According to the boundary conditions given by Eq. (6.2.3), the last surface integral is decomposed into four integrals over S_1 through S_4 as

$$\begin{aligned} &\int_{V(e)} \rho c \frac{\partial T}{\partial t} N_i dV - \int_{V(e)} \langle \frac{\partial N_i}{\partial x} \frac{\partial N_i}{\partial y} \frac{\partial N_i}{\partial z} \rangle \begin{Bmatrix} q_x \\ q_y \\ q_z \end{Bmatrix} dV \\ &= \int_{V(e)} Q N_i dV - \int_{S_1} (\vec{q} \cdot \vec{n}) N_i dS + \int_{S_2} q'' N_i dS \\ &\quad - \int_{S_3} h(T - T_\infty) N_i dS - \int_{S_4} (\sigma \epsilon T^4 - \alpha q_r) N_i dS \qquad i = 1, 2, \cdots r. \end{aligned} \tag{6.2.12}$$

Note that the signs of the integrals in Eq. (6.2.12) depend upon the direction of the heat input. The positive sign is defined when the heat is given to the body, and the negative when the heat is removed from the body. That is, q'' is defined to be positive in Eq. (6.2.12), since we have assumed that the heat flux is given to the body. On the other hand, we have assumed negative convection, which means the heat is removed from the S_3 boundary by convection. Similarly, the S_4 boundary is assumed to radiate to the ambient, as the sign of this integral is considered negative. From Eq. (6.2.2)

$$\begin{Bmatrix} q_x \\ q_y \\ q_z \end{Bmatrix}^{(e)} = -\begin{bmatrix} k_{xx} & k_{xy} & k_{xz} \\ k_{yx} & k_{yy} & k_{yz} \\ k_{zx} & k_{zy} & k_{zz} \end{bmatrix}^{(e)} \begin{Bmatrix} \frac{\partial T}{\partial x} \\ \frac{\partial T}{\partial y} \\ \frac{\partial T}{\partial z} \end{Bmatrix}^{(e)}. \tag{6.2.13}$$

Substituting from Eq. (6.2.7) in (6.2.13), gives

$$\begin{Bmatrix} q_x \\ q_y \\ q_z \end{Bmatrix}^{(e)} = -[k][B]\{T\}^{(e)}. \tag{6.2.14}$$

Upon substitution of Eq. (6.2.14) in (6.2.12) and adding up and assembling the resulting matrices for all the elements in the solution domain, the finite element equilibrium equation is obtained as

$$[C]\{\dot{T}\} + [[K_c] + [K_h] + [K_r]]\{T\} = \{F_Q\} + \{F_s\} + \{F_q\} + \{F_h\} + \{F_r\} \tag{6.2.15}$$

where the element of each of the matrices for the base element (e) is

$$[c]^{(e)} = \int_{V(e)} \rho c\{N\}\langle N\rangle dV$$

$$[k_c]^{(e)} = \int_{V(e)} [B]^T[k][B]dV$$

$$[k_h]^{(e)} = \int_{S_3(e)} h\{N\}\langle N\rangle dS$$

$$[k_r]^{(e)}\{T\} = \int_{S_4(e)} \sigma\epsilon T^4\{N\}dS$$

$$\{f_s\}^{(e)} = \int_{S_1(e)} (\vec{q}\cdot\vec{n})\{N\}dS$$

$$\{f_Q\}^{(e)} = \int_{V(e)} Q\{N\}dV$$

$$\{f_q\}^{(e)} = \int_{S_2(e)} q''\{N\}dS$$
$$\{f_h\}^{(e)} = \int_{S_3(e)} hT_\infty\{N\}dS$$
$$\{f_r\}^{(e)} = \int_{S_4(e)} \alpha q_r\{N\}dS \tag{6.2.16}$$

where $[c]$ is called the capacitance matrix, $[k_c]$, $[k_h]$, and $[k_r]$ are the conductive, convective, and the radiative stiffness matrices, respectively. The force matrices are the heat generation matrix $\{f_Q\}$, the heat flux matrix $\{f_q\}$, the convective matrix $\{f_h\}$, and the radiative force matrix $\{f_r\}$. It is important to note that the stiffness terms $[k_h]^{(e)}$ and $[k_r]^{(e)}$ appear only for the elements which are exposed to these two types of boundary conditions. Similarly, the force matrices $\{f_q\}^{(e)}$, $\{f_h\}^{(e)}$, and $\{f_r\}^{(e)}$ are considered only for the elements which are experiencing such boundary conditions.

6.3 Variational Formulations

The partial differential equation of heat conduction in a solid continuum based on the first law of thermodynamics was given by Eq. (6.2.1). Considering Fourier's law of the form

$$q_x = -k_x \frac{\partial T}{\partial x}$$
$$q_y = -k_y \frac{\partial T}{\partial y}$$
$$q_z = -k_z \frac{\partial T}{\partial z} \tag{6.3.1}$$

the first law of thermodynamics for the conduction of heat transfer in a solid continuum becomes

$$\frac{\partial}{\partial x}(k_x \frac{\partial T}{\partial x}) + \frac{\partial}{\partial y}(k_y \frac{\partial T}{\partial y}) + \frac{\partial}{\partial z}(k_z \frac{\partial T}{\partial z}) + Q = \rho c \frac{\partial T}{\partial t} \tag{6.3.2}$$

The initial and boundary conditions are

$$T(x, y, z, 0) = T_0(x, y, z) \tag{6.3.3}$$

$$T(x, y, z, t) = T_s$$
$$k_x \frac{\partial T}{\partial x} l + k_y \frac{\partial T}{\partial y} m + k_z \frac{\partial T}{\partial z} n + q = h(T - T_\infty). \tag{6.3.4}$$

The associated functional of Eq. (6.3.2) is obtained in analogy with the membrane problem as

$$\chi = \int_V \frac{1}{2} \left\{ k_x \, (\frac{\partial T}{\partial x})^2 + k_y \, (\frac{\partial T}{\partial y})^2 + k_z \, (\frac{\partial T}{\partial z})^2 - 2(Q - \rho c \, \frac{\partial T}{\partial t})T \right\} dV$$
$$+ \int_{S_1} qT dS + \int_{S_2} \frac{1}{2} \, h(T - T_\infty)^2 dS. \tag{6.3.5}$$

It is easily verified that the extremum of Eq. (6.3.5) reduces to Eq. (6.3.2). The finite element approximation is followed by applying the Ritz method to the functional (6.3.5). Using the Kantrovich approximation to separate the time domain from the space domain in the base element (e), the temperature is approximated as

$$T^{(e)}(x, y, z, t) = \langle N(x, y, z)\rangle^{(e)} \{T(t)\}^{(e)} \tag{6.3.6}$$

where

$$\langle N\rangle^{(e)} = \langle N_1 \;\, N_2 \;\; \cdots \, N_r\rangle \tag{6.3.7}$$

and r is the total number of nodes in the base element (e). The total functional of the solution domain is the sum of the functional of each of its subdomain elements (e) as

$$\chi = \sum_{e=1}^{NE} \chi^{(e)} \tag{6.3.8}$$

The Ritz method is

$$\frac{\partial \chi}{\partial T_i} = \sum_{e=1}^{NE} \frac{\partial \chi^{(e)}}{\partial T_i} = 0 \qquad i = 1, 2, \cdots NN \tag{6.3.9}$$

where NE and NN are the total number of the elements and nodal points in the solution region, respectively. The differentiation of the functional of base element (e) with respect to the nodal temperature T_i is

$$\frac{\partial \chi^{(e)}}{\partial T_i} = \int_{V(e)} \left\{ k_x \, \frac{\partial T^{(e)}}{\partial x} \, \frac{\partial}{\partial T_i} \, (\frac{\partial T^{(e)}}{\partial x}) + k_y \, \frac{\partial T^{(e)}}{\partial y} \, \frac{\partial}{\partial T_i} \, (\frac{\partial T^{(e)}}{\partial y}) \right.$$
$$\left. + k_z \, \frac{\partial T^{(e)}}{\partial z} \, \frac{\partial}{\partial T_i} \, (\frac{\partial T^{(e)}}{\partial z}) - (Q - \rho c \, \frac{\partial T^{(e)}}{\partial t}) \, \frac{\partial T^{(e)}}{\partial T_i} \right\} dV$$
$$+ \int_{S_1(e)} q \, \frac{\partial T^{(e)}}{\partial T_i} dS + \int_{S_2(e)} h(T^{(e)} - T_\infty) \, \frac{\partial T^{(e)}}{\partial T_i} \, dS. \tag{6.3.10}$$

Differentiating the terms of Eq. (6.3.10) yields

$$\frac{\partial T^{(e)}}{\partial x} = \langle \frac{\partial N_1}{\partial x} \ \frac{\partial N_2}{\partial x} \ \cdots \ \frac{\partial N_r}{\partial x} \rangle^{(e)} \{T\}^{(e)}$$
$$\frac{\partial}{\partial T_i} \left(\frac{\partial T^{(e)}}{\partial x}\right) = \frac{\partial N_i}{\partial x}$$
$$\frac{\partial T^{(e)}}{\partial T_i} = N_i$$
$$\frac{\partial T^{(e)}}{\partial t} = \langle N \rangle^{(e)} \left\{ \frac{\partial T}{\partial t} \right\}$$
$$\left\{ \frac{\partial T^{(e)}}{\partial t} \right\} = \langle \frac{\partial T_1}{\partial t} \ \frac{\partial T_2}{\partial t} \ \cdots \ \frac{\partial T_r}{\partial t} \rangle^T = \langle N \rangle^{(e)} \left\{ \frac{\partial T}{\partial t} \right\} \tag{6.3.11}$$

Substitution of Eqs. (6.3.11) in (6.3.10) yields

$$\frac{\partial \chi^{(e)}}{\partial T_i} = [k_1]^{(e)}\{T\}^{(e)} - \{f\}^{(e)} + [k_2]^{(e)}\{T\}^{(e)} + [c]^{(e)}\{\dot{T}\}^{(e)} \tag{6.3.12}$$

where

$$[k_1]_{ij}^{(e)} = \int_{V(e)} \left(k_x \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + k_y \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} + k_z \frac{\partial N_i}{\partial z} \frac{\partial N_j}{\partial z} \right) dV$$
$$[k_2]_{ij}^{(e)} = \int_{S_2(e)} h N_i N_j dS$$
$$[c]_{ij}^{(e)} = \int_{V(e)} \rho c N_i N_j dV$$
$$\{f\}_i^{(e)} = \int_{V(e)} Q N_i dV - \int_{S_1(e)} q N_i dS + \int_{S_2(e)} h T_\infty N_i dS. \tag{6.3.13}$$

The sign of the force matrices is selected positive when heat is given to the body.

An alternative approach is the matrix representation of the element matrices, which is more convenient to use. Defining the gradient matrix $\{g\}$ as

$$\{g\} = \left\{ \begin{array}{c} \frac{\partial T}{\partial x} \\ \frac{\partial T}{\partial y} \\ \frac{\partial T}{\partial z} \end{array} \right\}. \tag{6.3.14}$$

It follows from Eq. (6.3.6) that, for the base element (e), the matrix $\{g\}$ is

$$\{g\}^{(e)} = \begin{Bmatrix} \frac{\partial T}{\partial x} \\ \frac{\partial T}{\partial y} \\ \frac{\partial T}{\partial z} \end{Bmatrix}^{(e)} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \cdots & \frac{\partial N_r}{\partial x} \\ \cdot & \cdot & \cdot\cdot & \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \cdots & \frac{\partial N_r}{\partial y} \\ \cdot & \cdot & \cdot\cdot & \\ \frac{\partial N_1}{\partial z} & \frac{\partial N_2}{\partial z} & \cdots & \frac{\partial N_r}{\partial z} \end{bmatrix}^{(e)} \begin{Bmatrix} T_1 \\ T_2 \\ \cdot \\ \cdot \\ T_r \end{Bmatrix}^{(e)} = [B]^{(e)}\{T\}^{(e)}. \tag{6.3.15}$$

Substituting in the functional of element (e) given by Eq. (5.2.7), gives

$$\begin{aligned} \chi^{(e)} = {} & \{T\}^T \left(\int_{V(e)} \tfrac{1}{2} [B]^T [k][B] dV \right) \{T\} \\ & - \{T\}^T \int_{V(e)} Q\{N\} dV + \{T\}^T \left(\int_{V(e)} \rho c \{N\}\langle N\rangle dV \right) \{\dot{T}\} \\ & + \{T\}^T \left(\int_{S_1(e)} q\{N\} dS \right) + \{T\}^T \left(\int_{S_2(e)} \tfrac{1}{2} h\{N\}\langle N\rangle dS \right) \{T\} \\ & + \int_{S_2(e)} \tfrac{1}{2} h T_\infty^2 dS - \{T\}^T \left(\int_{S_2(e)} h T_\infty \{N\} dS \right) \end{aligned} \tag{6.3.16}$$

where the matrix $[k]$ is

$$[k]^{(e)} = \begin{bmatrix} k_x & 0 & 0 \\ 0 & k_y & 0 \\ 0 & 0 & k_z \end{bmatrix}^{(e)}.$$

Applying the Ritz method, the following matrix forms are obtained for the element matrices

$$\begin{aligned} [k]^{(e)} &= \int_{V(e)} [B]^T [k][B] dV + \int_{S_2(e)} h\{N\}\langle N\rangle dS \\ [c]^{(e)} &= \int_{V(e)} \rho c \, \{N\}\langle N\rangle dV \\ \{f\}^{(e)} &= + \int_{S_2(e)} h T_\infty \{N\} dS + \int_{V(e)} Q\{N\} dV - \int_{S_1(e)} q\{N\} dS. \end{aligned} \tag{6.3.17}$$

The matrix form of the element matrices, as given by Eq. (6.3.17), are more convenient to use.

Substitution of Eqs. (6.3.12) in (6.3.9), and performing the summations over all the elements in the solution domain, yields

$$\frac{\partial \chi}{\partial T_i} = \sum_{e=1}^{NE} \frac{\partial \chi^{(e)}}{\partial T_i}$$
$$= \sum_{e=1}^{NE} \left([c]^{(e)} \{\dot{T}\}^{(e)} + [[k_1]^{(e)} + [k_2]^{(e)}]\{T\}^{(e)} - \{f\}^{(e)} \right) = 0. \qquad (6.3.18)$$

This equation results in the general equilibrium finite element equation of the system as

$$[C]\{\dot{T}\} + [K]\{T\} = \{F\} \qquad (6.3.19)$$

where

$$[C] = \sum_{e=1}^{NE} [R]^T [c]^{(e)} [R]$$
$$[K] = \sum_{e=1}^{NE} [R]^T [[k_1]^{(e)} + [k_2]^{(e)}][R]$$
$$\{F\} = \sum_{e=1}^{NE} [R]^T \{f\}^{(e)} \qquad (6.3.20)$$

where $[R]$ is the proper rotation matrix for transferring the matrices from the local element coordinates to the problem global coordinates. The coordinate transformation is

$$[c]_g^{(e)} = [R]^T [c]_l [R]$$
$$[k]_g^{(e)} = [R]^T [k]_l [R]$$
$$\{f\}_g^{(e)} = [R]^T \{f\}_l. \qquad (6.3.21)$$

Equation (6.3.19) is the finite element equilibrium equation describing the transient three-dimensional temperature distribution in a solid continuum. For a steady-state condition, matrix $[C]$ vanishes and the solution is obtained from the following equation:

$$[K]\{T\} = \{F\}. \qquad (6.3.22)$$

The elements of the stiffness and force matrices depend upon the order of approximation, element geometry, and the problem dimensions. In the following sections, the one-, two-, and three-dimensional problems are discussed.

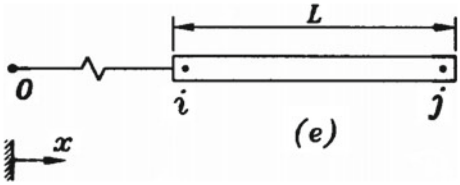

Fig. 6.1 A one-dimensional straight element

6.4 One-Dimensional Conduction

Consider a one-dimensional conductive heat transfer in a solid. The examples are conduction heat transfer in extended surfaces in the form of rods, and heat transfer in bars. If the bar is straight, or the elements can be approximated as a straight element, its geometry is as shown in Fig. 6.1. Assuming a linear element, two nodes i and j are sufficient to describe the approximation for the temperature distribution along the element. The linear temperature approximation is

$$T^{(e)} = N_i T_i + N_j T_j = \langle N_i \quad N_j \rangle \begin{Bmatrix} T_i \\ T_j \end{Bmatrix}^{(e)} \tag{6.4.1}$$

where

$$N_i = \frac{x_j - x}{L} \qquad N_j = \frac{x - x_i}{L}. \tag{6.4.2}$$

Thus

$$\langle B \rangle^{(e)} = \langle -\frac{1}{L} \quad \frac{1}{L} \rangle^{(e)} \tag{6.4.3}$$

Since

$$[k] = [k_x].$$

Using the variational formulation, the first part of the stiffness matrix is

$$[k_1]^{(e)} = \int_{V(e)} [B]^T [k][B] dV = \int_{V(e)} \begin{Bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{Bmatrix} [k_x] \langle -\frac{1}{L} \quad \frac{1}{L} \rangle dV.$$

For a bar of constant cross section A, $dV = Adx$ and

$$[k_1]^{(e)} = \frac{kA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

The second part of the stiffness matrix is related to the convection on the boundary of the element as

$$[k_2]^{(e)} = \int_{S_2(e)} h\{N\}\langle N\rangle dS = \int_{S_2(e)} h \begin{Bmatrix} N_i \\ , N_j \end{Bmatrix} \langle N_i \quad N_j\rangle dS$$

or

$$[k_2]^{(e)} = \int_{S_2(e)} h \begin{bmatrix} N_i^2 & N_i N_j \\ N_j N_i & N_j^2 \end{bmatrix} dS.$$

This integral should be evaluated on the portion of the element surface where the convective heat transfer is specified, assuming the peripheral surface of the element is exposed to convective heat transfer to the ambient at $(h,\ T_\infty)$. Since $dS = Pdx$, thus,

$$[k_2]^{(e)} = \int_{S_2(e)} h \begin{bmatrix} N_i^2 & N_i N_j \\ N_j N_i & N_j^2 \end{bmatrix} Pdx$$

where P is the periphery of the element. Carrying out the integration gives

$$[k_2]^{(e)} = \frac{PhL}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

The total stiffness matrix of the element (e) is then

$$[k]^{(e)} = \frac{kA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{PhL}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

The force matrix is the sum of three matrices. The first part of the force matrix is related to the heat generation within the element at the rate Q as

$$\{f_1\}^{(e)} = \int_{V(e)} Q \begin{Bmatrix} N_i \\ N_j \end{Bmatrix} dV = \int_{V(e)} Q \begin{Bmatrix} N_i \\ N_j \end{Bmatrix} Adx.$$

For constant Q,

$$\{f_1\}^{(e)} = \frac{QAL}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}.$$

The second part of the force matrix is the radiative part appearing wherever a surface of the element is exposed to the radiative heat flux $+q''$. Let us assume that the cross section at node j is exposed to the radiation q''. Thus,

$$\{f_2\}^{(e)} = \int_{S_1(e)} q \begin{Bmatrix} N_i \\ N_j \end{Bmatrix} dS = \int_{S_1(e)} q'' \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} dA = q'' A_j \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

where A_j is the area of cross-section of node j. For the constant cross-section element, $A_j = A$.

The third part of the force matrix is the convective force defined on the surface of the element, assuming an element with peripheral surface exposed to the convective heat transfer. If heat is transferred by convection to the ambient, this force matrix becomes

$$\begin{aligned}\{f_3\}^{(e)} &= -\int_{S_2(e)} hT_\infty \begin{Bmatrix} N_i \\ N_j \end{Bmatrix} dS = -\int_{S_1(e)} hT_\infty \begin{Bmatrix} N_i \\ N_j \end{Bmatrix} P dx \\ &= -\tfrac{hT_\infty PL}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}.\end{aligned}$$

The total force matrix of the element (e) is then

$$\{f\}^{(e)} = \frac{QAL}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + q'' A \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} - \frac{hT_\infty PL}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}.$$

Note that the sign of the heat flux force is assumed positive, which means the body is receiving heat flux q'' on its boundary S_1. The convective heat transfer on the S_2 boundary is assumed negative.

6.5 Two-Dimensional Conduction

Two-dimensional heat conduction problems in the $x - y$ coordinates may be modeled by the linear triangular elements as shown in Fig. 6.2. The linear approximation for the temperature distribution in the base element (e) is

$$T^{(e)} = a_1 + a_2 x + a_3 y \tag{6.5.1}$$

where a_1, a_2, and a_3 are constants to be determined in terms of the nodal coordinates. At nodes i, j, and k, the nodal temperatures are T_i, T_j, and T_k, respectively. Substituting these conditions in Eq. (6.5.1) gives

$$\begin{aligned} T_i &= a_1 + a_2 x_i + a_3 y_i \\ T_j &= a_1 + a_2 x_j + a_3 y_j \\ T_k &= a_1 + a_2 x_k + a_3 y_k. \end{aligned} \tag{6.5.2}$$

Solving for a_1, a_2, and a_3 yields

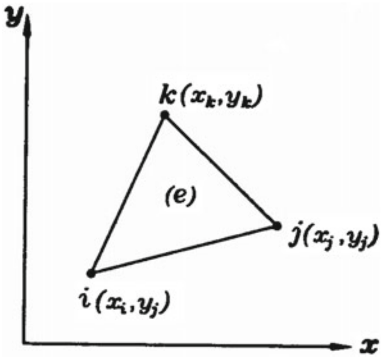

Fig. 6.2 Two-dimensional triangular element

$$
\begin{aligned}
a_1 &= \frac{1}{2A}\left[(x_j y_k - x_k y_j)T_i + (x_k y_i - x_i y_k)T_j + (x_i y_j - x_j y_i)T_k\right] \\
a_2 &= \frac{1}{2A}\left[(y_j - y_k)T_i + (y_k - y_i)T_j + (y_i - y_j)T_k\right] \\
a_3 &= \frac{1}{2A}\left[(x_k - x_j)T_i + (x_i - x_k)T_j + (x_j - x_i)T_k\right].
\end{aligned} \tag{6.5.3}
$$

Substituting in Eq. (6.5.1), gives

$$
T^{(e)} = N_i T_i + N_j T_j + N_k T_k = \langle N_i \quad N_j \quad N_k \rangle \begin{Bmatrix} T_i \\ T_j \\ T_k \end{Bmatrix}
$$

or

$$
T^{(e)} = \langle N \rangle^{(e)} \{T\}^{(e)} \tag{6.5.4}
$$

where

$$
\begin{aligned}
N_i &= \frac{a_i + b_i x + c_i y}{2A} \\
N_j &= \frac{a_j + b_j x + c_j y}{2A} \\
N_k &= \frac{a_k + b_k x + c_k y}{2A}.
\end{aligned} \tag{6.5.5}
$$

Here,

$$
2A = 2 \times \text{area of element } (e) = \begin{vmatrix} 1 & 1 & 1 \\ x_i & x_j & x_k \\ y_i & y_j & y_k \end{vmatrix} \tag{6.5.6}
$$

and

$$\begin{cases} a_i = x_j y_k - x_k y_j \\ b_i = y_j - y_k \\ c_i = x_k - x_j \end{cases} \quad \begin{cases} a_j = x_k y_i - x_i y_k \\ b_j = y_k - y_i \\ c_j = x_i - x_k \end{cases} \quad \begin{cases} a_k = x_i y_j - x_j y_i \\ b_k = y_i - y_j \\ c_k = x_j - x_i \end{cases}. \tag{6.5.7}$$

The stiffness and force matrices are calculated using Eq. (6.3.17). According to definition, the submatrices of the stiffness matrix are

$$[B]^{(e)} = \begin{bmatrix} \dfrac{\partial N_i}{\partial x} & \dfrac{\partial N_j}{\partial x} & \dfrac{\partial N_k}{\partial x} \\ \dfrac{\partial N_i}{\partial y} & \dfrac{\partial N_j}{\partial y} & \dfrac{\partial N_k}{\partial y} \end{bmatrix} = \frac{1}{2A^{(e)}} \begin{bmatrix} b_i & b_j & b_k \\ c_i & c_j & c_k \end{bmatrix} \tag{6.5.8}$$

and

$$[k] = \begin{bmatrix} k_x & 0 \\ 0 & k_y \end{bmatrix}. \tag{6.5.9}$$

Thus, the first part of the stiffness matrix, for an element of unit thickness, is

$$[k_1]^{(e)} = \int_{V(e)} [B]^T [k][B] dV$$

$$= \int_{V(e)} \frac{1}{4A^2} \begin{bmatrix} b_i & c_i \\ b_j & c_j \\ b_k & c_k \end{bmatrix} \begin{bmatrix} k_x & 0 \\ 0 & k_y \end{bmatrix} \begin{bmatrix} b_i & b_j & b_k \\ c_i & c_j & c_k \end{bmatrix} dxdy. \tag{6.5.10}$$

This, upon integration, reduces to the following matrices:

$$[k_1]^{(e)} = \frac{k_x}{4A} \begin{bmatrix} b_i b_i & b_i b_j & b_i b_k \\ b_j b_i & b_j b_j & b_j b_k \\ b_k b_i & b_k b_j & b_k b_k \end{bmatrix}^{(e)} + \frac{k_y}{4A} \begin{bmatrix} c_i c_i & c_i c_j & c_i c_k \\ c_j c_i & c_j c_j & c_j c_k \\ c_k c_i & c_k c_j & c_k c_k \end{bmatrix}^{(e)}. \tag{6.5.11}$$

The second part of the stiffness matrix is related to elements that are exposed to convective heat transfer, and is

$$[k_2]^{(e)} = \int_{S_2(e)} h\{N\}\langle N\rangle dS = \int_{S_2(e)} h \begin{bmatrix} N_i N_i & N_i N_j & N_i N_k \\ N_j N_i & N_j N_j & N_j N_k \\ N_k N_i & N_k N_j & N_k N_k \end{bmatrix} dS. \tag{6.5.12}$$

As an example, if the heat is transformed by convection through the $ij-$side of the element, then $N_k = 0$ and

$$[k_2]^{(e)} = \frac{hS_{ij}}{6} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

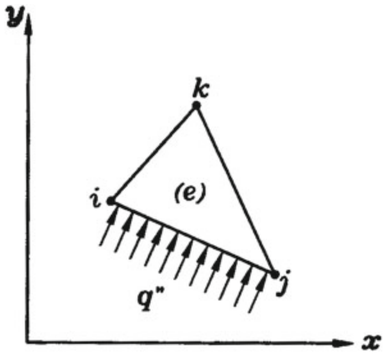

Fig. 6.3 Convection heat transfer on side ij

where S_{ij} is the area of side ij of the element (e) as shown in Fig. 6.3.

The force matrix due to the internal energy generation is

$$\{f_1\}^{(e)} = \int_{V(e)} Q\{N\}dV = \frac{QA^{(e)}}{3} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \tag{6.5.13}$$

where the thickness of the element is assumed to be a unit. The second part of the force matrix is due to the heat flux $q^{''}$. For positive heat flux $+q^{''}$, this force matrix is

$$\{f_2\}^{(e)} = \int_{S_1(e)} q\{N\}dS = \int_{S_1(e)} q^{''} \begin{Bmatrix} N_i \\ N_j \\ N_k \end{Bmatrix} dS. \tag{6.5.14}$$

As an example, if the side ij of the element (e) is exposed to the heat flux $+q^{''}$, the related force matrix is

$$\{f_2\}^{(e)} = \int_{S_1(e)} q^{''} \begin{Bmatrix} N_i \\ N_j \\ 0 \end{Bmatrix} dS = \frac{q^{''} S_{ij}}{2} \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix}.$$

The third part of the force matrix, due to a negative convection, where heat is removed from the surface $S_2(e)$, is

$$\{f_3\}^{(e)} = -\int_{S_2(e)} hT_\infty\{N\}dS = -\int_{S_2(e)} hT_\infty \begin{Bmatrix} N_i \\ N_j \\ N_k \end{Bmatrix} dS$$

which again reduces to either one of the following matrices, depending on the side of the element (e) exposed to free convection:

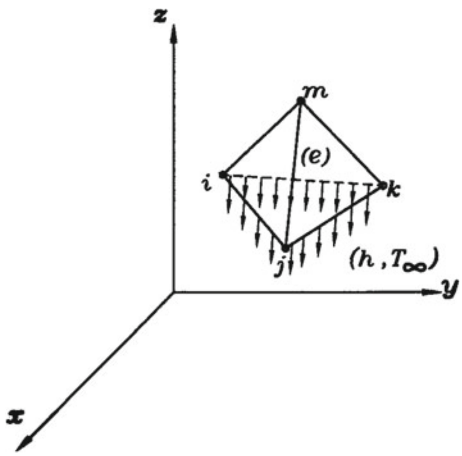

Fig. 6.4 A three-dimensional simplex element

$$\{f_3\}^{(e)} = \begin{cases} -\dfrac{hT_\infty S_{ij}}{2}\begin{Bmatrix}1\\1\\0\end{Bmatrix} & \text{for } ij \text{ side} \\ -\dfrac{hT_\infty S_{kj}}{2}\begin{Bmatrix}0\\1\\1\end{Bmatrix} & \text{for } kj \text{ side} \\ -\dfrac{hT_\infty S_{ik}}{2}\begin{Bmatrix}1\\0\\1\end{Bmatrix} & \text{for } ik \text{ side.} \end{cases}$$

The total force matrix is

$$\{f\}^{(e)} = \{f_1\}^{(e)} + \{f_2\}^{(e)} + \{f_3\}^{(e)} \tag{6.5.15}$$

6.6 Three-Dimensional Conduction

Consider a three-dimensional solid continuum subjected to a thermal gradient. The heat is transferred into the body by means of the energy generation within the body, or through the boundary surfaces by convection, radiation, or conduction. The heat is then transferred through the body by conduction.

To model the finite element formulations of the problem, a linear approximation for the temperature distribution within the three-dimensional element (e) may be assumed. A four-side base element (e) is considered in the xyz-system, as shown in Fig. 6.4. Four nodes i, j, k, and m are considered, and the temperature distribution within the element (e) is assumed to be linear as

$$T^{(e)} = a_1 + a_2 x + a_3 y + a_4 z. \tag{6.6.1}$$

The four constants a_1 through a_4 are evaluated in terms of the nodal temperatures and the coordinates as

$$\begin{aligned} T_i &= a_1 + a_2 x_i + a_3 y_i + a_4 z_i \\ T_j &= a_1 + a_2 x_j + a_3 y_j + a_4 z_j \\ T_k &= a_1 + a_2 x_k + a_3 y_k + a_4 z_k \\ T_m &= a_1 + a_2 x_m + a_3 y_m + a_4 z_m \end{aligned} \tag{6.6.2}$$

Solving the system of eq. (6.6.2) for a's yields

$$T^{(e)} = N_i T_i + N_j T_j + N_k T_k + N_m T_m = \langle N \rangle^{(e)} \{T\}^{(e)} \tag{6.6.3}$$

where

$$\begin{aligned} N_i &= \frac{a_i + b_i x + c_i y + d_i z}{6V} \qquad & N_j &= \frac{a_j + b_j x + c_j y + d_j z}{6V} \\ N_k &= \frac{a_k + b_k x + c_k y + d_k z}{6V} \qquad & N_m &= \frac{a_m + b_m x + c_m y + d_m z}{6V} \end{aligned} \tag{6.6.4}$$

where the constant coefficients a's, b's, c's, and d's are given in terms of the nodal coordinates. For node i, they are

$$\begin{aligned} a_i &= \begin{vmatrix} x_j & y_j & z_j \\ x_k & y_k & z_k \\ x_m & y_m & z_m \end{vmatrix} \qquad & b_i &= \begin{vmatrix} 1 & y_j & z_j \\ 1 & y_k & z_k \\ 1 & y_m & z_m \end{vmatrix} \\ c_i &= \begin{vmatrix} x_j & 1 & z_j \\ x_k & 1 & z_k \\ x_m & 1 & z_m \end{vmatrix} \qquad & d_i &= \begin{vmatrix} x_j & y_j & 1 \\ x_k & y_k & 1 \\ x_m & y_m & 1 \end{vmatrix}. \end{aligned} \tag{6.6.5}$$

The other coefficients for nodes j, k, and m are found in a similar manner. The element volume is

$$V^{(e)} = \frac{1}{6} \begin{vmatrix} 1 & x_i & y_i & z_i \\ 1 & x_j & y_j & z_j \\ 1 & x_k & y_k & z_k \\ 1 & x_m & y_m & z_m \end{vmatrix}. \tag{6.6.6}$$

The stiffness matrix is calculated using Eq. (6.6.3) for temperature. The first part of the stiffness matrix is

$$[k_1]^{(e)} = \int_{V(e)} [B]^T [k][B] dV \tag{6.6.7}$$

which, upon substitution, is

$$[k_1]^{(e)} = \frac{k_x}{36V}\begin{vmatrix} b_i b_i & b_i b_j & b_i b_k & b_i b_m \\ b_j b_i & b_j b_j & b_j b_k & b_j b_m \\ b_k b_i & b_k b_j & b_k b_k & b_k b_m \\ b_m b_i & b_m b_j & b_m b_k & b_m b_m \end{vmatrix}$$

$$+ \frac{k_y}{36V}\begin{vmatrix} c_i c_i & c_i c_j & c_i c_k & c_i c_m \\ c_j c_i & c_j c_j & c_j c_k & c_j c_m \\ c_k c_i & c_k c_j & c_k c_k & c_k c_m \\ c_m c_i & c_m c_j & c_m c_k & c_m c_m \end{vmatrix}$$

$$\times \frac{k_z}{36V}\begin{vmatrix} d_i d_i & d_i d_j & d_i d_k & d_i d_m \\ d_j d_i & d_j d_j & d_j d_k & d_j d_m \\ d_k d_i & d_k d_j & d_k d_k & d_k d_m \\ d_m d_i & d_m d_j & d_m d_k & d_m d_m \end{vmatrix} \tag{6.6.8}$$

The second part of the stiffness matrix is related to the convection of those elements which are exposed to this type of boundary conditions and is

$$[k_2]^{(e)} = \int_{S_2(e)} h\{N\}\langle N\rangle dS = \int_{S_2(e)} h \begin{vmatrix} N_i N_i & N_i N_j & N_i N_k & N_i N_m \\ N_j N_i & N_j N_j & N_j N_k & N_j N_m \\ N_k N_i & N_k N_j & N_k N_k & N_k N_m \\ N_m N_i & N_m N_j & N_m N_k & N_m N_m \end{vmatrix} dS \tag{6.6.9}$$

As an example, when the triangular side ijk is exposed to free heat convection, $N_m = 0$ and the integration of Eq. (6.6.9), using the area coordinates, reduces to

$$[k_2]^{(e)} = \frac{hA_{ijk}}{12}\begin{vmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}.$$

The force matrix is calculated in a similar manner as one and two-dimensional problems. The first part of the force matrix, due to the thermal energy generation per unit volume per unit time, is

$$\{f_1\}^{(e)} = \int_{V(e)} Q\{N\}dV = \int_{V(e)} Q \begin{Bmatrix} N_i \\ N_j \\ N_k \\ N_m \end{Bmatrix} dV = \frac{QV^{(e)}}{4}\begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix}. \tag{6.6.10}$$

The second part of the force matrix related to the radiation $+q''$ to the surface of the element (e) is

$$\{f_2\}^{(e)} = \int_{S_1(e)} q''\{N\}dS = \int_{S_1(e)} q'' \begin{Bmatrix} N_i \\ N_j \\ N_k \\ N_m \end{Bmatrix} dS.$$

This is reduced to one of the following forms, depending on the boundary surface exposed to the heat flux $+q''$:

$$\{f_2\}^{(e)} = \begin{cases} \frac{q'' A_{ijk}}{3} \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{Bmatrix} & \text{for } ijk \text{ side} \\ \frac{q'' A_{jkm}}{3} \begin{Bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{Bmatrix} & \text{for } jkm \text{ side} \\ \frac{q'' A_{kmi}}{3} \begin{Bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{Bmatrix} & \text{for } kmi \text{ side} \\ \frac{q'' A_{mij}}{3} \begin{Bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{Bmatrix} & \text{for } mij \text{ side} \end{cases}. \tag{6.6.11}$$

The third part of the force matrix related to the free convection heat transfer on the boundary of the element (e), when heat is transferred by convection to the ambient, is

$$\{f_3\}^{(e)} = -\int_{S_2(e)} hT_\infty\{N\}dS = \int_{S_2(e)} hT_\infty \begin{Bmatrix} N_i \\ N_j \\ N_k \\ N_m \end{Bmatrix} dS. \tag{6.6.12}$$

If the side *ijk* of the base element (e) is exposed to the free convection to the ambient, the related force matrix is

$$\{f_3\}^{(e)} = -\frac{hT_\infty A_{ijk}}{3} \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{Bmatrix}.$$

The total force matrix is the sum of all element forces, and is

$$\{f\}^{(e)} = \{f_1\}^{(e)} + \{f_2\}^{(e)} + \{f_3\}^{(e)}. \tag{6.6.13}$$

6.7 Transient Heat Conduction

The finite element equilibrium equation for transient heat conduction problems was derived in Sect. 3, and is

$$[C]\{\dot{T}\} + [K]\{T\} = \{F\} \tag{6.7.1}$$

where the capacitance matrix $[c]$ for the base element (e) was defined as

$$[c]^{(e)} = \int_{V(e)} \rho c \{N\}\langle N\rangle dV. \tag{6.7.2}$$

This matrix is now calculated for the one-, two-, and three-dimensional simplex elements.

For the one-dimensional linear element (e), with two nodes i and j and length L, the matrix $[c]$ reduces to

$$[c]^{(e)} = \rho c \int_0^L \begin{Bmatrix} (1-\frac{x}{L}) \\ (\frac{x}{L}) \end{Bmatrix} \langle (1-\frac{x}{L}) \ \ (\frac{x}{L})\rangle A dx \tag{6.7.3}$$

which, after integration, gives

$$[c]^{(e)} = \left(\frac{\rho c A L}{6}\right)^{(e)} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

For two-dimensional elements with linear shape functions, the matrix $[c]$ is

$$[c]^{(e)} = \int_{V(e)} \rho c \begin{bmatrix} N_i N_i & N_i N_j & N_i N_k \\ N_j N_i & N_j N_j & N_j N_k \\ N_k N_i & N_k N_j & N_k N_k \end{bmatrix} dA \tag{6.7.4}$$

which, after integration, gives

$$[c]^{(e)} = \left(\frac{\rho c A}{12}\right)^{(e)} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

For the three-dimensional elements with linear shape functions, the matrix $[c]$ is

$$[c]^{(e)} = \int_{V(e)} \rho c \begin{bmatrix} N_i N_i & N_i N_j & N_i N_k & N_i N_m \\ N_j N_i & N_j N_j & N_j N_k & N_j N_m \\ N_k N_i & N_k N_j & N_k N_k & N_k N_m \\ N_m N_i & N_m N_j & N_m N_k & N_m N_m \end{bmatrix} dV \tag{6.7.5}$$

which, after integration, yields

$$[c]^{(e)} = \left(\frac{\rho c V}{20}\right)^{(e)} \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}.$$

Once the capacitance, stiffness, and force matrices are obtained for the problem under consideration and the matrices are all assembled for the total elements in the solution region, the finite element equilibrium equation is well-defined, and is solved. Since this equation is an ordinary differential equation in terms of time for the transient problems, a numerical scheme must be employed to integrate the system of equations numerically in the time domain and solve the system for each time step. The most appropriate technique is the finite difference method in the time domain. The total transient time is divided into a number of time increments, and, for each time increment, the equation is solved in terms of its known matrices at an earlier time interval. There are a variety of finite difference numerical schemes for solving the time-dependent problems. The direct integration method for solving the transient problems is discussed in Chap. 7.

6.8 Problems

1. A one-dimensional simplex element with nodes i, j and circular cross-section under heat flux q'' at the cross-sections i and j and convective heat transfer to the ambient form at its periphery is considered. Find the stiffness and force matrices.
2. A one-dimensional heat transfer in a rod is considered. The rod is divided into two elements e_1 and e_2 with lengths l_1 and l_2 and constant cross-sectional area A and periphery p. Element e_1 is exposed to heat transfer to the ambient at (h, T_∞), where the element e_2 is under exposed heat flux q'' from its periphery. The rod is thermally insulated from its ends. Find the stiffness and force matrices of the rod.
3. Reconsider Problem 2, in which the initial temperature of the rod at $t = 0$ is assumed to be a constant T_0. At $t > 0$, the thermal boundary conditions given in the problem are assumed to be applied. Calculate the capacitance matrix of the rod, if the heat is generated at the rate of Q per unit volume and unit time.
4. Problem 2 is considered with the numerical values $l_1 = 2$ cm, $l_2 = 3$ cm, with the first node of the element e_1 at the coordinates $x = 0$. It is assumed that $h = 20$ Kcal/min/cm^2, $T_\infty = 25^o$C, $q'' = 50$ Kcal/min/cm^2. Calculate the members of the stiffness and force matrices of the rod.

5. Consider a two-dimensional heat conduction in xy-coordinates. Obtain the elements of the stiffness and force matrices associated with a triangular simplex element with nodes i, j and k. It is assumed that the sides ij and jk are exposed to heat convection at h, T_∞ and side ik is exposed to a heat flux q''. The heat is assumed to be generated at a constant rate Q.
6. Obtain the elements of the capacitance matrix of Problem 5.
7. A heat transfer domain of rectangular shape with node coordinates A at (0,0), B at (4,0), C at (0,3), and D at (4,3) centimeters is considered. Heat is generated at the rate $Q = 50$ K cal / sec / cm^3 in the domain. Connecting nodes A and D, two triangular elements are obtained. Find the problem stiffness and force matrices if the sides AB and AC transfer the heat to the ambient at h, T_∞, where the other two sides are thermally insulated. Take $h = 20$ K cal / min / cm^2 and $T_\infty = 10\ ^oC$.
8. Reconsider Problem 7, in which the initial temperature of the domain at $t = 0$ is $T_0 = 25\ ^oC$. At $t > 0$, the boundary conditions are as given. Calculate the capacitance, stiffness, and force matrices, when $\rho = 7800$ kg/cm^3 and $c = 20$.

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Chapter 7
Computer Methods

Abstract Once the theoretical derivations to obtain the finite element matrices for a base element (e) is completed, the element matrices are assembled to form the finite element equilibrium equation. The computer methods to solve this final equation includes the matrix assembling, bandwidth calculation, application of the boundary conditions, solution algorithm, and preparation of the output results. This chapter briefly explains the computer techniques to perform the required operations and through a number of numerical examples show the details of the mathematical concepts. That is, it puts the mathematical concepts into the computer algorithms. At the end of the chapter, a number of classical methods of solution of dynamic finite element equations are discussed.

7.1 Introduction

In this chapter, brief discussions of the computational methods are presented. The structures of the finite element computer programs are similar in a number of subprograms. All programs written to solve a problem with the finite element method have three major sections: the input section, the section dealing with the mathematical operations, and the output section. The section on mathematical operations should be able to solve a banded matrix subjected to a number of boundary conditions. This process needs to establish and assemble the global stiffness matrix from the input data, transform the global stiffness matrix to a banded matrix, apply the boundary conditions, employ a solution technique to solve the resulting system of equations, back substitution, calculate the nodal required information, and finally, prepare the output data. This strategy is for a simple static finite element problem. The solution of the dynamic problems depends upon the type of dynamic analysis. If a finite element program is considered as a system of integrated subsystems, then each module is responsible for a definite task, and may be used to perform that special job. A computer program developed to solve a two-dimensional elasticity problem may be

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easily modified and adapted to solve a two-dimensional heat conduction problems, as many subprograms are about identical in both cases. Therefore, one would be well justified in investing some time in writing a finite element program, as well as trying to separate it into different modules, each responsible for a definite task. A main program can then gather these modules into a more complete computer program for a special problem. Another more general main program can put these special programs into a more general purpose finite element computer program.

7.2 Assembly of the Global Matrices

The element matrices are of finite dimensions and are related to the total number of nodes per element and the degrees of freedom per node. These matrices are assembled into the global matrices the dimensions of which depend upon the total number of nodes of the problem and the degrees of freedom per node. Thus, the global matrices have large dimensions and are obtained as the sum of the element matrices of much smaller dimensions. This means the assembly process of the element matrices is not just a simple addition, but rather a process of expansion and summation.

Consider a solution domain discretized into NN nodes. Furthermore, consider only one degree of freedom per node, called u. An element (e) of the solution domain may have the nodes, say, l, m, and n, and the element unknown matrix

$$\langle u\rangle^{(e)} = \langle u_l \;\; u_m \;\; u_n\rangle^{(e)}. \tag{7.2.1}$$

The global unknown matrix, which includes the unknowns u_l, u_m, and u_n of the element (e), is

$$\langle U\rangle = \langle u_1 \;\; u_2 \cdots u_l \cdots u_m \cdots u_n \cdots u_{NN}\rangle. \tag{7.2.2}$$

The global unknown matrix contains u_1 to u_{NN} in increasing order of numbers, where u_l, u_m, and u_n are placed in $\langle U\rangle$ in sequence of their subscript number. The question is where their associated stiffness matrix coefficients are placed in the global stiffness matrix $[K]$?. An algorithm is needed to do this job with a system of referencing related to the subscript number of unknown u_s, $s = 1, 2, \ldots NN$. To establish the algorithm, consider the following example.

Example 1 A two-dimensional problem in the x–y coordinates is considered. The solution domain is assumed to be a rectangle which is divided into two elements, as shown in Fig. 7.1.

The node numbers are shown in the figure. The unknown function at the nodes is u. The global unknown matrix is

$$\langle U\rangle = \langle u_1 \;\; u_2 \;\; u_3 \;\; u_4\rangle. \tag{7a}$$

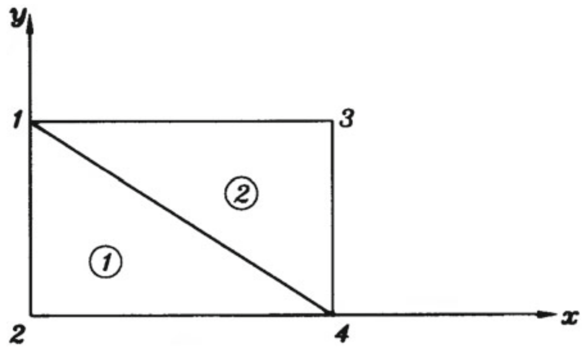


Fig. 7.1 A rectangular domain with two elements

The element unknown matrices for the elements (1) and (2) are

$$\langle u\rangle^{(1)} = \langle u_1\ u_2\ u_4\rangle. \tag{7b}$$

$$\langle u\rangle^{(2)} = \langle u_1\ u_4\ u_3\rangle. \tag{7c}$$

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}^{(1)} \begin{Bmatrix} u_1 \\ u_2 \\ u_4 \end{Bmatrix} \Rightarrow \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix}^{(1)}. \tag{7d}$$

and

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}^{(2)} \begin{Bmatrix} u_1 \\ u_4 \\ u_3 \end{Bmatrix} \Rightarrow \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix}^{(2)}. \tag{7e}$$

The global stiffness matrix is 4×4, while the element stiffness matrix is 3×3. The global stiffness matrix is the sum of the element stiffness matrices. This means that the element stiffness matrices should be first expanded into a 4×4 matrix, and then summed over in the global stiffness matrix. The expansion process is proportional to the subscript number of the unknown matrices $\langle u\rangle^{(1)}$ and $\langle u\rangle^{(2)}$. This is

$$\begin{bmatrix} k_{11} & k_{12} & 0 & k_{13} \\ k_{21} & k_{22} & 0 & k_{23} \\ 0 & 0 & 0 & 0 \\ k_{31} & k_{32} & 0 & k_{33} \end{bmatrix}^{(1)} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} \Rightarrow \begin{Bmatrix} f_1 \\ f_2 \\ 0 \\ f_3 \end{Bmatrix}^{(1)}. \tag{7f}$$

Table 7.1 Matrix CON(i, j)

	Nodes		
Element	l	m	n
1	1	2	4
2	1	4	3

$$\begin{bmatrix} k_{11} & 0 & k_{13} & k_{12} \\ 0 & 0 & 0 & 0 \\ k_{31} & 0 & k_{33} & k_{32} \\ k_{21} & 0 & k_{23} & k_{22} \end{bmatrix}^{(2)} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} \Rightarrow \begin{Bmatrix} f_1 \\ 0 \\ f_3 \\ f_2 \end{Bmatrix}^{(2)}. \tag{7g}$$

It should be noted that in the expansion process, the columns and rows of the element matrices are displaced according to the sequence number of the global unknown matrix $\langle U \rangle$. The process of summation is now possible, and is

$$\begin{bmatrix} k_{11}^{(1)} + k_{11}^{(2)} & k_{12}^{(1)} & k_{13}^{(2)} & k_{13}^{(1)} + k_{12}^{(2)} \\ k_{21}^{(1)} & k_{22}^{(1)} & 0 & k_{23}^{(1)} \\ k_{31}^{(2)} & 0 & k_{33}^{(2)} & k_{32}^{(2)} \\ k_{31}^{(1)} + k_{21}^{(2)} & k_{32}^{(1)} & k_{23}^{(2)} & k_{33}^{(1)} + k_{22}^{(2)} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} f_1^{(1)} + f_1^{(2)} \\ f_2^{(1)} \\ f_3^{(2)} \\ f_3^{(1)} + f_2^{(2)} \end{Bmatrix}. \tag{7h}$$

Example 1 describes the process of summation of the element stiffness and force matrices into the global matrices in two steps. The first step is to expand the element matrices to the dimensions of the global matrices, and the second step is the summation of the element matrices to form the global matrices. The final form of the finite element equation is

$$[K]\{U\} = \{F\}. \tag{7.2.3}$$

where

$$[K] = \sum_{e=1}^{NE} [k]^{(e)} \tag{7.2.4}$$

$$\{F\} = \sum_{e=1}^{NE} \{f\}^{(e)}. \tag{7.2.5}$$

where NE is the total number of the elements in the solution domain. The system of expansion of the element matrices into the global matrices was based on the subscript number of the associated u_s in the element unknown matrix. The subscript numbers of the nodal points around the element (e) are real numbers compared with the locations in the global matrices. If we form a connectivity matrix of the elements, it may be used to perform the assembly operations for the stiffness and force matrices. To show the process, consider the example problem.

The nodal points around element (1) are (1, 2, 4) and those around element (2) are (1, 4, 3). Saving these numbers in a matrix called CON(i, j), we have Table 7.1.

This matrix is constructed during the input data entry, and it is all known when the input data is completed. This matrix is called the *connectivity matrix*, and gives the information regarding the node numbers around each of the elements in the solution domain. The index i ranges from 1 to NE and refers to the element, and the index j defines the node number around element (i) with the consideration of the degrees of freedom per node. For triangular elements, j takes on three numbers, and for the four-side elements, j takes on four numbers.
(a) When we have one degree of freedom per node, the unknown matrix is

$$\langle U \rangle = \langle u_1 \;\; u_2 \;\; u_3 \;\; u_4 \rangle. \tag{7.2.6}$$

For element (1)

$$\begin{aligned} \langle U \rangle^{(1)} &= \langle u_1 \;\; u_2 \;\; u_4 \rangle \\ \mathrm{CON}(1, j) &= \langle 1 \;\; 2 \;\; 4 \rangle. \end{aligned}$$

For element (2)

$$\begin{aligned} \langle U \rangle^{(2)} &= \langle u_1 \;\; u_4 \;\; u_3 \rangle \\ \mathrm{CON}(2, j) &= \langle 1 \;\; 4 \;\; 3 \rangle. \end{aligned}$$

(b) When we have two degrees of freedom per node, such as u and v, the unknown matrix is

$$\langle U \rangle = \langle u_1 \;\; v_1 \;\; u_2 \;\; v_2 \;\; u_3 \;\; v_3 \;\; u_4 \;\; v_4 \rangle.$$

For element (1)

$$\begin{aligned} \langle U \rangle^{(1)} &= \langle u_1 \;\; v_1 \;\; u_2 \;\; v_2 \;\; u_4 \;\; v_4 \rangle \\ \mathrm{CON}(1, j) &= \langle 1 \;\; 2 \;\; 3 \;\; 4 \;\; 7 \;\; 8 \rangle. \end{aligned}$$

For element (2)

$$\begin{aligned} \langle U \rangle^{(2)} &= \langle u_1 \;\; v_1 \;\; u_4 \;\; v_4 \;\; u_3 \;\; v_3 \rangle \\ \mathrm{CON}(2, j) &= \langle 1 \;\; 2 \;\; 7 \;\; 8 \;\; 5 \;\; 6 \rangle. \end{aligned}$$

The expansion of the element stiffness and force matrices $[k]^{(e)}$ and $\{f\}^{(e)}$ into the global matrices $[K]^{(e)}$ and $\{F\}^{(e)}$ is done using the $\mathrm{CON}(i, j)$ matrix as

$$\begin{aligned} k_{ij}^{(e)} &\to K_{IJ}^{(e)} \\ f_i^{(e)} &\to F_I^{(e)}, \end{aligned}$$

where for each element (e)

$$\text{CONE}(i) = \text{CON}(e, i)$$
$$\text{CONE}(j) = \text{CON}(e, j).$$

and

$$I = \text{CONE}(i) \quad i = 1, 2, \ldots n_{de}$$
$$J = \text{CONE}(j) \quad j = 1, 2, \ldots n_{de}.$$

where n_{de} is the total number of degrees of freedom of element (e). Therefore, the expansion process is done as

$$\begin{aligned} K_{IJ}^{(e)} &= K_{CONE(i),CONE(j)}^{(e)} \equiv k_{ij}^{(e)} \\ F_I^{(e)} &= F_{CONE(i)}^{(e)} \equiv f_i^{(e)}. \end{aligned} \tag{7.2.7}$$

On this basis, an algorithm may be written as.

Algorithm

- Initialize matrices $[K]$ and $\{F\}$.

- For each element (e)

$$K_{IJ} = K_{IJ} + k_{IJ}^{(e)} \quad i = 1, 2, \ldots n_{de}$$
$$F_I = F_I + f_I^{(e)} \quad j = 1, 2, \ldots n_{de}.$$

where

$$I = \text{CONE}(i)$$
$$J = \text{CONE}(j).$$

This simple algorithm assembles the stiffness and force matrices for all elements in the solution domain.

7.3 Bandwidth Calculation

Due to the nature of finite element approximation, the resulting global mass and stiffness matrices are banded. The reason is very simple, as each element is only connected to a few number of elements around its sides and nodes. It is very important to recognize the bandwidth of the mass and stiffness matrices and use a numerical solution algorithm which employs the bandwidth of the matrices [1]. This is in contrast to the standard Gauss elimination method, which constructs the solution procedure based on $n \times n$ matrices.

To determine the bandwidth of the stiffness matrix, we may consider the member K_{IJ} of the global stiffness matrix. The diagonal members of the global matrix are located on locations $I = J$. For the upper triangle of the stiffness matrix (where $J > I$), the band of an element K_{IJ} of the global matrix is $b_{IJ}^{(e)} = J - I$, or

$$b_{IJ}^{(e)} = J - I = \text{CONE}(j) - \text{CONE}(i) \quad J > I. \tag{7.3.1}$$

Now, when the element stiffness matrices $k_{ij}^{(e)}$ are assembled, their members, which are located at far distance from the diagonal, are the band of the global matrix for that element and for the rows under consideration. Thus, the element bandwidth for row I, when the element stiffness matrix $k_{ij}^{(e)}$ is assembled, is the maximum value of $b_{IJ}^{(e)}$ for all non-zero terms of that row, and is

$$\begin{aligned} i &= 1, 2, \ldots n_{de} \\ I &= \text{CONE}(i) \\ j &= 1, 2, \ldots n_{de} \\ b_I^{(e)} &= Max\left(\text{CONE}(j) - I\right). \end{aligned} \tag{7.3.2}$$

When all the elements are assembled, the bandwidth for row I of the global matrix is

$$\begin{aligned} e &= 1,\ NE \\ b_I &= Max(b_I^{(e)}). \end{aligned} \tag{7.3.3}$$

for all elements (e). The final bandwidth of the global matrix is the maximum b_I of all the rows as

$$\begin{aligned} I &= 1,\ NN \\ b &= Max(b_I). \end{aligned} \tag{7.3.4}$$

where NE and NN are the total number of elements and nodal points in the solution region, respectively.

With a similar method, the element band height $h_J^{(e)}$, band height of each column of the global stiffness matrix h_J, and the total band height of the stiffness matrix h is obtained. The element band height $h_J^{(e)}$ of column J of $[K^{(e)}]$ is the maximum of $h_{IJ}^{(e)}$ of all non-zero terms of column J, as obtained from the following algorithm

$$\begin{aligned} j &= 1, 2, \ldots n_{de} \\ J &= \text{CONE}(j) \\ i &= 1, 2, \ldots n_{de} \\ h_J^{(e)} &= Max\,(J - \text{CONE}(i))\,. \end{aligned} \tag{7.3.5}$$

The band height h_J of column J of the global stiffness matrix $[K]$, when all elements are assembled, is

$$h_J = Max(h_J^{(e)}). \tag{7.3.6}$$

and the total band height h of the global stiffness matrix is

$$\begin{aligned} J &= 1,\ NN \\ h &= Max(h_J). \end{aligned} \tag{7.3.7}$$

The band height of each column is needed for the skyline method of solution, as discussed later in this chapter.

7.4 Boundary Conditions

The finite element equilibrium equation for static problems has the general form

$$[K]\{X\} = \{F\}. \tag{7.4.1}$$

Before solving this system of linear equations, the boundary conditions must be applied. It is very important to note that the boundary conditions must be well defined such that once the forces are applied to the model, the model stays stationary. An ill-defined boundary condition results in the rigid body motion of the model. The sign of rigid body motion appears in the calculation procedure, as the determinant of the stiffness matrix becomes zero.

Let us define the boundary conditions by

$$\{X\}_i = \{\bar{X}\}_i. \tag{7.4.2}$$

where $\{X\}_i$ are the degrees of freedom in which their boundary values are defined by $\{\bar{X}\}_i$. The boundary conditions (7.4.2) must be introduced in Eq. (7.4.1) and the equilibrium equation (7.4.1) be modified and solved for the rest of the degrees of freedom. The solution will be influenced by the given boundary conditions. There are several techniques for introducing the boundary conditions into the equilibrium equation [2].

Method 1: Large Number on Diagonal Term

The global stiffness and force matrices are assembled before the boundary conditions are considered. Once the assembly process is completed, the boundary conditions are introduced. For each known boundary condition $X_i = \bar{X}_i$, its associated term on the diagonal of the stiffness matrix K_{ii} is replaced by $K_{ii} + \beta$, where β is a very large number compared to the other terms of K_{ij}. Since β is an arbitrary large number, $K_{ii} + \beta \approx \beta$, and thus, its corresponding unknown Y_i in the same row is essentially very small and negligible compared to the other unknowns. Keep in mind that at the nodes where their nodal values are known, their reaction forces are unknown. The right-hand side of equation i is $-K_{ii}\bar{X}_i$. The term $K_{ji}\bar{X}_i$ is subtracted from the right-hand side of all the other equations, such that equation j before and after modification is

- before modification $K_{j1}X_1 + K_{j2}X_2 + \cdots K_{ji}\bar{X}_i + \cdots K_{jn}X_n = F_j$
- after modification $K_{j1}X_1 + K_{j2}X_2 + \cdots K_{ji}Y_i + \cdots K_{jn}X_n = F_j - K_{ji}\bar{X}_i$. (7.4.3)

Note that since Y_i is negligibly small, $K_{ji}Y_i$ is negligible compared to the other terms of the equation. In addition, since the term $K_{ij}\bar{X}_i$ is moved to the right-hand side of the equation, the j—equation is thus identical before and after modification. Therefore, the right-hand side of all j—equations (except equation i) are modified by $F_j^* = F_j - K_{ji}\bar{X}_i$. The i—equation before and after modification is

- before modification $K_{i1}X_1 + K_{i2}X_2 + \cdots K_{ii}\bar{X}_i + \cdots K_{in}X_n = R_i$
- after modification $K_{i1}X_1 + K_{i2}X_2 + \cdots \beta Y_i + \cdots K_{in}X_n = -K_{ii}\bar{X}_i$. (7.4.4)

Note that we purposely named the force at the node where its nodal displacement is known by R_i. The value of R_i, the reaction force, is unknown. That is, at a node, either the force is known or its nodal displacement. Subtracting the second of Eq. (7.4.4) from the first equation gives

$$K_{ii}\bar{X}_i - \beta Y_i = R_i + K_{ii}\bar{X}_i.$$

or

$$R_i = -\beta Y_i. \tag{7.4.5}$$

where R_i is the reaction force associated with degree of freedom X_i.

The advantage of this technique is that the original symmetric stiffness matrix remains symmetric after modifications to include the boundary conditions. The method is very simple to apply.

Method 2: Wiping Rows and Columns

Consider the boundary condition $X_i = \bar{X}_i$ defined at node i. To apply the boundary conditions, the row and column of the stiffness matrix associated with the node i are all set equal to zero, except K_{ii}, which is set equal to 1. The force matrix is then properly modified as

$$\begin{aligned} F_j &= F_j - K_{ji}\bar{X}_i \quad j = 1, 2, \ldots n \qquad i \neq j \\ F_i &= \bar{X}_i \\ K_{ij} &= K_{ji} = 0 \quad j = 1, 2, \ldots n \qquad i \neq j \\ K_{ii} &= 1. \end{aligned} \tag{7.4.6}$$

The matrix representation of Eq. (7.4.6) is

$$\begin{bmatrix} K_{11} & \cdot\cdot & K_{1,i-1} & 0 & K_{1,i+1} & \cdot\cdot & K_{1n} \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & \cdot \\ K_{i-1,1} & \cdot\cdot & K_{i-1,i-1} & 0 & K_{i-1,i+1} & \cdot\cdot & K_{i-1,n} \\ 0 & & 0 & 1 & 0 & & 0 \\ K_{i+1,1} & \cdot\cdot & K_{i+1,i-1} & 0 & K_{i+1,i+1} & \cdot\cdot & K_{i+1,n} \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & \cdot \\ K_{n,1} & \cdot\cdot & K_{n,i-1} & 0 & K_{n,i+1} & \cdot\cdot & K_{nn} \end{bmatrix} \begin{Bmatrix} X_1 \\ \cdot \\ \cdot \\ X_{i-1} \\ X_i \\ X_{i+1} \\ \cdot \\ \cdot \\ X_n \end{Bmatrix} = \begin{Bmatrix} F_1 - K_{1i}\bar{X}_i \\ \cdot \\ \cdot \\ F_{i-1} - K_{i-1.i}\bar{X}_i \\ \bar{X}_i \\ F_{i+1} - K_{i+1.i}\bar{X}_i \\ \cdot \\ \cdot \\ F_n - K_{ni}\bar{X}_i \end{Bmatrix}. \tag{7.4.7}$$

The matrix equation (7.4.7) is solved for all the unknowns $\{X\}$ except X_i, the boundary value of which is given as $\bar{X}_i$. In this method, the reaction forces are not directly obtained.

Example 2 A two-dimensional problem is considered as shown in Fig. 7.2. The solution domain is assumed to be a rectangle which is divided into four elements.

The nodes are numbered as is shown in the figure. The unknown function at node i is u_i. The global unknown vector is

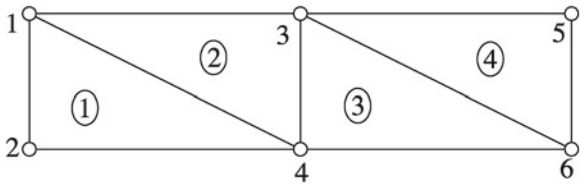


Fig. 7.2 A rectangular domain with four elements

$$\langle U \rangle = \langle u_1 \;\; u_2 \;\; u_3 \;\; u_4 \;\; u_5 \;\; u_6 \rangle.$$

The local unknown vectors of the elements are

$$\begin{aligned}
\langle u \rangle^{(1)} &= \langle u_1 \;\; u_2 \;\; u_4 \rangle \\
\langle u \rangle^{(2)} &= \langle u_1 \;\; u_4 \;\; u_3 \rangle \\
\langle u \rangle^{(3)} &= \langle u_3 \;\; u_4 \;\; u_6 \rangle \\
\langle u \rangle^{(4)} &= \langle u_3 \;\; u_6 \;\; u_5 \rangle.
\end{aligned}$$

The general form of equilibrium equations at the elements level are

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}^{(e)} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix}^{(e)}.$$

The connectivity matrix, CON(i,j), for these four elements is (Table 7.2).
The global stiffness matrix is 6×6, and is constructed by the sum of the element stiffness matrices. Now, we proceed by assembling the local stiffness matrix and force vector of each element into the global stiffness matrix and force vector of the problem, respectively.
For element one:

$$\mathrm{CONE}(i) = \mathrm{CON}(1, i) = \langle 1 \;\; 2 \;\; 4 \rangle.$$

$$\begin{array}{llll}
k_{11}^{(1)} \rightarrow K_{11} & k_{12}^{(1)} \rightarrow K_{12} & k_{13}^{(1)} \rightarrow K_{14} & f_1^{(1)} \rightarrow F_1 \\
k_{21}^{(1)} \rightarrow K_{21} & k_{22}^{(1)} \rightarrow K_{22} & k_{23}^{(1)} \rightarrow K_{24} & f_2^{(1)} \rightarrow F_2\,. \\
k_{31}^{(1)} \rightarrow K_{41} & k_{32}^{(1)} \rightarrow K_{42} & k_{33}^{(1)} \rightarrow K_{44} & f_3^{(1)} \rightarrow F_4
\end{array}$$

The global stiffness matrix and force vector become

Table 7.2 Matrix CON(i, j)

	Nodes		
Element	l	m	n
1	1	2	4
2	1	4	3
3	3	4	6
4	3	6	5

$$\left[K_{IJ}\right] = \left[K_{IJ}\right] + \left[k_{IJ}^{(e)}\right] = \begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & 0 & k_{13}^{(1)} & 0 & 0 \\ k_{21}^{(1)} & k_{22}^{(1)} & 0 & k_{23}^{(1)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ k_{31}^{(1)} & k_{32}^{(1)} & 0 & k_{33}^{(1)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$F_I = F_I + f_I^{(e)} = \left\{ f_1^{(1)} \; f_2^{(1)} \; 0 \; f_3^{(1)} \; 0 \; 0 \right\}^T.$$

For element two:

$$\text{CONE}(i) = \text{CON}(2, i) = \langle 1 \;\; 4 \;\; 3 \rangle.$$

$$\begin{array}{llll} k_{11}^{(2)} \to K_{11} & k_{12}^{(2)} \to K_{14} & k_{13}^{(2)} \to K_{13} & f_1^{(2)} \to F_1 \\ k_{21}^{(2)} \to K_{41} & k_{22}^{(2)} \to K_{44} & k_{23}^{(2)} \to K_{43} & f_2^{(2)} \to F_4 \, . \\ k_{31}^{(2)} \to K_{31} & k_{32}^{(2)} \to K_{34} & k_{33}^{(2)} \to K_{33} & f_3^{(2)} \to F_3 \end{array}$$

The global stiffness matrix and force vector become

$$[K_{IJ}] = [K_{IJ}] + \left[k_{IJ}^{(e)}\right] = \begin{bmatrix} k_{11}^{(1)} + k_{11}^{(2)} & k_{12}^{(1)} & k_{13}^{(2)} & k_{13}^{(1)} + k_{12}^{(2)} & 0 & 0 \\ k_{21}^{(1)} & k_{22}^{(1)} & 0 & k_{23}^{(1)} & 0 & 0 \\ k_{31}^{(2)} & 0 & k_{33}^{(2)} & k_{32}^{(2)} & 0 & 0 \\ k_{31}^{(1)} + k_{21}^{(2)} & k_{32}^{(1)} & k_{23}^{(2)} & k_{33}^{(1)} + k_{22}^{(2)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$F_I = F_I + f_I^{(e)} = \left\{ f_1^{(1)} + f_1^{(2)} \; f_2^{(1)} \; f_3^{(2)} \; f_3^{(1)} + f_2^{(2)} \; 0 \; 0 \right\}^T.$$

For element three:

$$\text{CONE}(i) = \text{CON}(3, i) = \langle 3 \;\; 4 \;\; 6 \rangle.$$

$$\begin{array}{llll} k_{11}^{(3)} \to K_{33} & k_{12}^{(3)} \to K_{34} & k_{13}^{(3)} \to K_{36} & f_1^{(3)} \to F_3 \\ k_{21}^{(3)} \to K_{43} & k_{22}^{(3)} \to K_{44} & k_{23}^{(3)} \to K_{46} & f_2^{(3)} \to F_4 \\ k_{31}^{(3)} \to K_{63} & k_{32}^{(3)} \to K_{64} & k_{33}^{(3)} \to K_{66} & f_3^{(3)} \to F_6. \end{array}$$

The global stiffness matrix and force vector become

$$[K_{IJ}] = [K_{IJ}] + \left[k_{IJ}^{(e)}\right] = \begin{bmatrix} k_{11}^{(1)} + k_{11}^{(2)} & k_{12}^{(1)} & k_{13}^{(2)} & k_{13}^{(1)} + k_{12}^{(2)} & 0 & 0 \\ k_{21}^{(1)} & k_{22}^{(1)} & 0 & k_{23}^{(1)} & 0 & 0 \\ k_{31}^{(2)} & 0 & k_{33}^{(2)} + k_{11}^{(3)} & k_{32}^{(2)} + k_{12}^{(3)} & 0 & k_{13}^{(3)} \\ k_{31}^{(1)} + k_{21}^{(2)} & k_{32}^{(1)} & k_{23}^{(2)} + k_{21}^{(3)} & k_{33}^{(1)} + k_{22}^{(2)} + k_{22}^{(3)} & 0 & k_{23}^{(3)} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_{31}^{(3)} & k_{32}^{(3)} & 0 & k_{33}^{(3)} \end{bmatrix}.$$

$$F_I = F_I + f_I^{(e)} = \left\{ f_1^{(1)} + f_1^{(2)} \;\; f_2^{(1)} \;\; f_3^{(2)} + f_1^{(3)} \;\; f_3^{(1)} + f_2^{(2)} + f_2^{(3)} \;\; 0 \;\; f_3^{(3)} \right\}^T.$$

For element four:

$$\text{CONE}(i) = \text{CON}(4, i) = \langle 3 \;\; 6 \;\; 5 \rangle.$$

$$\begin{array}{llll} k_{11}^{(4)} \to K_{33} & k_{12}^{(4)} \to K_{36} & k_{13}^{(4)} \to K_{35} & f_1^{(4)} \to F_3 \\ k_{21}^{(4)} \to K_{63} & k_{22}^{(4)} \to K_{66} & k_{23}^{(4)} \to K_{65} & f_2^{(4)} \to F_6\,. \\ k_{31}^{(4)} \to K_{53} & k_{32}^{(4)} \to K_{56} & k_{33}^{(4)} \to K_{55} & f_3^{(4)} \to F_5 \end{array}$$

The final forms of the global stiffness matrix and the force vector become

$$[K_{IJ}] = [K_{IJ}] + \left[k_{IJ}^{(e)}\right]$$
$$= \begin{bmatrix} k_{11}^{(1)} + k_{11}^{(2)} & k_{12}^{(1)} & k_{13}^{(2)} & k_{13}^{(1)} + k_{12}^{(2)} & 0 & 0 \\ k_{21}^{(1)} & k_{22}^{(1)} & 0 & k_{23}^{(1)} & 0 & 0 \\ k_{31}^{(2)} & 0 & k_{33}^{(2)} + k_{11}^{(3)} + k_{11}^{(4)} & k_{32}^{(2)} + k_{12}^{(3)} & k_{13}^{(4)} & k_{13}^{(3)} + k_{12}^{(4)} \\ k_{31}^{(1)} + k_{21}^{(2)} & k_{32}^{(1)} & k_{23}^{(2)} + k_{21}^{(3)} & k_{33}^{(1)} + k_{22}^{(2)} + k_{22}^{(3)} & 0 & k_{23}^{(3)} \\ 0 & 0 & k_{31}^{(4)} & 0 & k_{33}^{(4)} & k_{32}^{(4)} \\ 0 & 0 & k_{31}^{(3)} + k_{21}^{(4)} & k_{32}^{(3)} & k_{23}^{(4)} & k_{33}^{(3)} + k_{22}^{(4)} \end{bmatrix}.$$

$$F_I = F_I + f_I^{(e)}$$
$$= \{ f_1^{(1)} + f_1^{(2)} \;\; f_2^{(1)} \;\; f_3^{(2)} + f_1^{(3)} + f_1^{(4)} \;\; f_3^{(1)} + f_2^{(2)} + f_2^{(3)} \;\; f_3^{(4)} \;\; f_3^{(3)} + f_2^{(4)} \}^T.$$

Example 3 Calculate the bandwidth of the stiffness matrix for the previous example.

Solution: The bandwidth of the system matrices will be calculated by the algorithm described for the bandwidth calculation. In a computer code, determining the bandwidth of the system matrices, prior to assembling local element matrices, is beneficial in allocating an appropriate amount of storage volume to the system matrices. The algorithm loops over each element of the FE model and calculates the corresponding bandwidth. The final bandwidth is then the maximum of all those calculated.

For element one:

$$i = 1, \quad I = CONE(1) = 1 \quad \begin{cases} j = 1, \ J = CONE(1) = 1 \\ j = 2, \ J = CONE(2) = 2 \\ j = 3, \ J = CONE(3) = 4 \end{cases}$$

$$b_I^{(e)} = Max(\ |J - I|\ ) \quad \rightarrow \quad b_1^{(1)} = Max(\ 0,\ 1,\ 3\ ) = 3$$

$$i = 2, \quad I = CONE(2) = 2 \quad \begin{cases} j = 1, \ J = CONE(1) = 1 \\ j = 2, \ J = CONE(2) = 2 \\ j = 3, \ J = CONE(3) = 4 \end{cases}$$

$$b_I^{(e)} = Max(\ |J - I|\ ) \quad \rightarrow \quad b_2^{(1)} = Max(\ 1,\ 0,\ 2\ ) = 2$$

$$i = 3, \quad I = CONE(3) = 4 \quad \begin{cases} j = 1, \ J = CONE(1) = 1 \\ j = 2, \ J = CONE(2) = 2 \\ j = 3, \ J = CONE(3) = 4 \end{cases}$$

$$b_I^{(e)} = Max(\ |J - I|\ ) \quad \rightarrow \quad b_4^{(1)} = Max(\ 3,\ 2,\ 0\ ) = 3.$$

For element two:

$$i = 1, \quad I = CONE(1) = 1 \quad \begin{cases} j = 1, \ J = CONE(1) = 1 \\ j = 2, \ J = CONE(2) = 4 \\ j = 3, \ J = CONE(3) = 3 \end{cases}$$

$$b_I^{(e)} = Max(\ |J - I|\ ) \quad \rightarrow \quad b_1^{(2)} = Max(\ 0,\ 3,\ 2\ ) = 3$$

$$i = 2, \quad I = CONE(2) = 4 \quad \begin{cases} j = 1, \ J = CONE(1) = 1 \\ j = 2, \ J = CONE(2) = 4 \\ j = 3, \ J = CONE(3) = 3 \end{cases}$$

$$b_I^{(e)} = Max(\ |J - I|\ ) \quad \rightarrow \quad b_4^{(2)} = Max(\ 3,\ 0,\ 1\ ) = 3$$

$$i = 3, \quad I = CONE(3) = 4 \quad \begin{cases} j = 1, \ J = CONE(1) = 1 \\ j = 2, \ J = CONE(2) = 2 \\ j = 3, \ J = CONE(3) = 4 \end{cases}$$

$$b_I^{(e)} = Max(\ |J - I|\ ) \quad \rightarrow \quad b_3^{(2)} = Max(\ 3,\ 0,\ 1\ ) = 3.$$

For element three:

$$i = 1, \quad I = CONE(1) = 3 \quad \begin{cases} j = 1, \; J = CONE(1) = 3 \\ j = 2, \; J = CONE(2) = 4 \\ j = 3, \; J = CONE(3) = 6 \end{cases}$$

$$b_I^{(e)} = Max(\,|J - I|\,) \;\; \rightarrow \;\; b_3^{(3)} = Max(\,0,\; 1,\; 3\,) = 3$$

$$i = 2, \quad I = CONE(2) = 4 \quad \begin{cases} j = 1, \; J = CONE(1) = 3 \\ j = 2, \; J = CONE(2) = 4 \\ j = 3, \; J = CONE(3) = 6 \end{cases}$$

$$b_I^{(e)} = Max(\,|J - I|\,) \;\; \rightarrow \;\; b_4^{(3)} = Max(\,1,\; 0,\; 2\,) = 2$$

$$i = 3, \quad I = CONE(3) = 6 \quad \begin{cases} j = 1, \; J = CONE(1) = 3 \\ j = 2, \; J = CONE(2) = 4 \\ j = 3, \; J = CONE(3) = 6 \end{cases}$$

$$b_I^{(e)} = Max(\,|J - I|\,) \;\; \rightarrow \;\; b_6^{(3)} = Max(\,3,\; 1,\; 0\,) = 3.$$

For element four:

$$i = 1, \quad I = CONE(1) = 3 \quad \begin{cases} j = 1, \; J = CONE(1) = 3 \\ j = 2, \; J = CONE(2) = 6 \\ j = 3, \; J = CONE(3) = 5 \end{cases}$$

$$b_I^{(e)} = Max(\,|J - I|\,) \;\; \rightarrow \;\; b_3^{(4)} = Max(\,0,\; 3,\; 2\,) = 3$$

$$i = 2, \quad I = CONE(2) = 6 \quad \begin{cases} j = 1, \; J = CONE(1) = 3 \\ j = 2, \; J = CONE(2) = 6 \\ j = 3, \; J = CONE(3) = 5 \end{cases}$$

$$b_I^{(e)} = Max(\,|J - I|\,) \;\; \rightarrow \;\; b_6^{(4)} = Max(\,3,\; 0,\; 2\,) = 3$$

$$i = 3, \quad I = CONE(3) = 5 \quad \begin{cases} j = 1, \; J = CONE(1) = 1 \\ j = 2, \; J = CONE(2) = 2 \\ j = 3, \; J = CONE(3) = 4 \end{cases}$$

$$b_I^{(e)} = Max(\,|J - I|\,) \;\; \rightarrow \;\; b_5^{(4)} = Max(\,2,\; 1,\; 0\,) = 2.$$

The bandwidth for row I of the global matrix is

$$b_I = Max(b_I^{(e)}) \quad \begin{cases} b_1 = Max(\,b_1^{(1)},\; b_1^{(2)},\; b_1^{(3)},\; b_1^{(4)}\,) = 3 \\ b_2 = Max(\,b_2^{(1)},\; b_2^{(2)},\; b_2^{(3)},\; b_2^{(4)}\,) = 2 \\ b_3 = Max(\,b_3^{(1)},\; b_3^{(2)},\; b_3^{(3)},\; b_3^{(4)}\,) = 3 \\ b_4 = Max(\,b_4^{(1)},\; b_4^{(2)},\; b_4^{(3)},\; b_4^{(4)}\,) = 2 \\ b_5 = Max(\,b_5^{(1)},\; b_5^{(2)},\; b_5^{(3)},\; b_5^{(4)}\,) = 2 \\ b_6 = Max(\,b_6^{(1)},\; b_6^{(2)},\; b_6^{(3)},\; b_6^{(4)}\,) = 3 \end{cases}$$

The final bandwidth b of the global matrix is the maximum b_I of all rows

$$b = Max(b_I) = Max(\ b_1\ , b_2\ , b_3\ , b_4\ , b_5\ , b_6\ ) = 3$$

Example 4 Consider the following system of equations;

$$\begin{cases} 2X_1 + 2\overline{X}_2 + X_3 - 2X_4 = 2 \\ 4X_1 + 6\overline{X}_2 + 5X_3 + X_4 = R_2 \\ X_1 - \overline{X}_2 + X_3 + X_4 = 4 \\ 2X_1 + \overline{X}_2 + X_3 + X_4 = 5 \end{cases}$$

Impose the prescribed condition of $\overline{X}_2 = 4$ to the system of equations once by the technique of *large number on diagonal term* and once by the technique of *wiping rows and columns*. Solve the resultant system of equations by the Gauss elimination method to obtain the remaining unknowns.

(a) Solution by the technique of *large number on diagonal term*: To impose the known boundary condition $\overline{X}_2 = 4$ on the system of equations, the associated diagonal term K_{22} is replaced by a very large number $\beta = \underline{10^{12}}$. In addition, the right-hand side of the 2nd equation, R_2, is replaced by $-K_{22}\overline{X}_2$, and the term $k_{j2}\overline{X}_2$ is subtracted from all the other jth equations. The system of equations after modification take the following form:

$$\begin{cases} 2X_1 + 2Y_2 + X_3 - 2X_4 = 2 - 2 \times 4 = -6 \\ 4X_1 + 10^{12}Y_2 + 5X_3 + X_4 = -6 \times 4 = -24 \\ X1 - Y_2 + X_3 + X_4 = 4 - (-1) \times 4 = 8 \\ 2X_1 + Y_2 + X_3 + X_4 = 5 - 1 \times 4 = 1 \end{cases}$$

Now, the Gauss elimination method is implemented to obtain the unknowns of the above system of equations. To produce zero terms in the first column, the first equation is multiplied by $-4/2 = -2$ and the result is added to the second equation, the first equation is multiplied by $-1/2$ and added to the third equation, and finally, the first equation is multiplied by $-2/2 = -1$ and added to the fourth equation. The result is

$$\begin{cases} 2X_1 + 2Y_2 + X_3 - 2X_4 = -6 \\ 0 + 10^{12}Y_2 + 3X_3 + 5X_4 = -12 \\ 0 - 2Y_2 + \frac{1}{2}X_3 + 2X_4 = 11 \\ 0 - Y_2 + 0 + 3X_4 = 7 \end{cases}$$

Now, the second equation is multiplied by $2/10^{12} = 2 \times 10^{-12}$ and added to the third equation, and multiplied by $1/10^{12} = 10^{-12}$ and added to the fourth equation to give

$$\begin{cases} 2X_1 + 2Y_2 + X_3 - 2X_4 = -6 \\ 0 + 10^{12}Y_2 + 3X_3 + 5X_4 = -12 \\ 0 + 0 + \frac{1}{2}X_3 + 2X_4 = 11 \\ 0 + 0 + 0 + 3X_4 = 7 \end{cases}$$

The last step is to produce a zero term in the third column and below the fourth equation, which is already zero. Now, the system of equations is upper triangularized and is ready to solve. The backward solution from the fourth equation for X_4 gives

$$X_4 = \frac{7}{3}.$$

This value of X_4 is substituted in the third equation to obtain X_3 as

$$\frac{1}{2}X_3 = 11 - 2X_4 = 11 - 2 \times \frac{7}{3} \rightarrow X_3 = \frac{38}{3}.$$

By substituting X_4 and X_3 in the second equation, Y_2 is obtained as

$$10^{12}Y_2 = -12 - 3X_3 - 5X_4 = -12 - 3 \times \frac{38}{3} - 5 \times \frac{7}{3} \rightarrow Y_2 = -\frac{185}{3} \times 10^{-12}.$$

Finally, by substituting X_4, X_3 and Y_2 in the first equation, X_1 is obtained as

$$\begin{aligned} 2X_1 = -6 - 2Y_2 - X_3 + 2X_4 &= -6 - 2 \times \frac{-115}{3} \times 10^{-12} - \frac{38}{3} \\ &+ 2 \times \frac{7}{3} \rightarrow X_1 = -7. \end{aligned}$$

The unknown value in the right-hand side of the 2nd equation, R_2, is obtained as

$$R_2 = -\beta Y_2 = -10^{12} \times \frac{-185}{3} \times 10^{-12} = \frac{185}{3}.$$

(b) Solution by the technique of *wiping rows and columns*: To impose the known boundary condition $\overline{X}_2 = 4$ on the system of equations, all entries of the row and column of the stiffness matrix associated with the index $i = 2$ are set equal to zero, except $K_{ii} = K_{22}$, which is set equal to 1.

$$\begin{aligned} &K_{2j} = K_{j2} = 0 \quad \text{for} \quad j = 1, 3, 4 \ \ (j \neq 2) \\ &K_{22} = 1. \end{aligned}$$

Accordingly, the force matrix is modified as

$$\begin{aligned} &F_j = F_j - K_{j2}\overline{X}_2 \quad \text{for} \quad j = 1, 3, 4 \ \ (j \neq 2) \\ &F_2 = \overline{X}_2. \end{aligned}$$

The system of equations after these modifications takes the following form:

$$\begin{cases} 2X_1 + 0 + X_3 - 2X_4 = 2 - 2 \times 4 = -6 \\ 0 + \overline{X}_2 + 0 + 0 = 4 \\ X1 - 0 + X_3 + X_4 = 4 - (-1) \times 4 = 8 \\ 2X_1 - 0 + X_3 + X_4 = 5 - 4 \times 1 = 1 \end{cases}$$

Now, similar to the previous case, the Gauss elimination method is implemented to obtain the unknowns of the above system of equations. The first column has already a zero term in the second equation. To cancel the non-zero terms in the first column, the first equation is multiplied by $-1/2$ and the result is added to the third equation, and the first equation is multiplied by $-2/2 = -1$ and added to the fourth equation. The result is

$$\begin{cases} 2X_1 + 0 + X_3 - 2X_4 = -6 \\ 0 + \overline{X}_2 + 0 + 0 = 4 \\ 0 - 0 + \frac{1}{2}X_3 + 2X_4 = 11 \\ 0 - 0 + 0 + 3X_4 = 7 \end{cases}$$

The system of equations is now upper triangularized and is ready for implementing the backward solution. From the fourth solution, X_4 is obtained as

$$X_4 = \frac{7}{3}.$$

By substituting the value of X_4 in the third equation, X_3 is obtained as

$$\frac{1}{2}X_3 = 11 - 2X_4 = 11 - 2 \times \frac{7}{3} \to X_3 = \frac{38}{3}.$$

From the second equation, we have $\overline{X}_2 = 4$, as is expected. Finally, the first equation gives the value of X_1 after substitution of values of X_3 and X_4 as

$$2X_1 = -6 - X_3 + 2X_4 = -6 - \frac{38}{3} + 2 \times \frac{7}{3} \to X_1 = -7.$$

The unknown value in the right-hand side of the 2nd equation, R_2, is obtained as

$$R_2 = 4X_1 + 6\overline{X}_2 + 5X_3 + X_4 = 4 \times (-7) + 6 \times 4 + 5 \times \frac{38}{3} + \frac{7}{3} = \frac{185}{3}.$$

7.5 Gauss Elimination

The Gauss elimination method is widely used to solve systems of equations obtained through finite element modeling. The standard method of Gauss elimination is developed to solve a system of equations with a constant coefficient matrix of $n \times n$ size, or a square matrix. This method is partially modified to handle the rectangular banded matrices, which are naturally obtained for finite element problems. Gauss elimination is a direct method of solution of the system of equations. An alternative to this method is the iteration and relaxation algorithm, which is sometimes employed to solve a system of equations obtained by finite element modeling.

To describe the method, consider the general system of three linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= c_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= c_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= c_3. \end{aligned} \tag{7.5.1}$$

As the first step, the first equation is multiplied through by $-a_{21}/a_{11}$ and added to the second equation to replace the second equation. Similarly, the first equation is multiplied through $-a_{31}x_1/a_{11}$ and added to the third equation to replace the third equation. The final set of equations become

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= c_1 \\ a'_{22}x_2 + a'_{23}x_3 &= c'_2 \\ a'_{32}x_2 + a'_{33}x_3 &= c'_3. \end{aligned} \tag{7.5.2}$$

where a' and c' are the new coefficients resulting from the multiplications and additions. The result of the first step operation was elimination of the variable x_1 from the second and third equations. Now, as the second step, we try to eliminate x_2 from the third equation. Multiplying through the second equation by $-a'_{32}/a'_{22}$ and adding it to the third equation yields

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= c_1 \\ a'_{22}x_2 + a'_{23}x_3 &= c'_2 \\ a''_{33}x_3 &= c''_3. \end{aligned} \tag{7.5.3}$$

where a''_{33} and c''_3 are obtained from the arithmetic operations. The system of equations (7.5.3) is now triangularized and is ready to be solved. The solution begins from the last equation, in which

$$x_3 = c''_3/a''_{33}. \tag{7.5.4}$$

The value of x_3 from Eq. (7.5.4) is substituted into the second of Eq. (7.5.3) and is used to solve for x_2. Finally, the values of x_3 and x_2 are substituted into the first of Eq. (7.5.3) and x_1 is obtained. The solution procedure will begin once the system of equations are triangularized, as seen in Eq. (7.5.3).

The operation of triangularization may be followed through a series of matrix multiplications. Consider the following matrix

$$[D] = [A|C] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & c_1 \\ a_{21} & a_{22} & a_{23} & c_2 \\ a_{31} & a_{32} & a_{33} & c_3 \end{bmatrix}. \tag{7.5.5}$$

where the broken line indicates the matrix partitioning. Defining the matrices [3]

$$[S_1] = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{a_{21}}{a_{11}} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad [S_2] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{a_{31}}{a_{11}} & 0 & 1 \end{bmatrix}$$

$$[S_3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{a'_{32}}{a'_{22}} & 1 \end{bmatrix} \tag{7.5.6}$$

it is verified that the following matrix multiplication provides the triangularization from (7.5.3) :

$$[S_3][S_2][S_1][D] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & c_1 \\ 0 & a'_{22} & a'_{23} & c'_2 \\ 0 & 0 & a''_{33} & c''_3 \end{bmatrix}. \tag{7.5.7}$$

Once the matrix of coefficients is triangularized, the solution follows from the last equation, and proceeds upward by proper back substitution.

Example 5 Obtain the solution of the following system of four equations:

$$\begin{aligned} 2x_1 + 5x_2 - 3x_3 + x_4 &= 6 \\ 5x_1 + x_2 - 2x_3 - 5x_4 &= 15 \\ -3x_1 - 2x_2 + 6x_3 + 2x_4 &= 12 \\ x_1 - 5x_2 + 2x_3 + 3x_4 &= 7. \end{aligned}$$

Solution: Following the Gauss elimination method, the first equation is multiplied by $-5/2$ and the result is added to the second equation, then multiplied by 3/2 and added to the third equation, and finally, multiplied by $-1/2$ and added to the fourth equation, to give

$$\begin{aligned} 2x_1 + 5x_2 - 3x_3 + x_4 &= 6 \\ 0 - \frac{23}{2}x_2 + \frac{11}{2}x_3 - \frac{15}{2}x_4 &= 0 \\ 0 + \frac{11}{2}x_2 + \frac{3}{2}x_3 + \frac{7}{2}x_4 &= 21 \\ 0 - \frac{15}{2}x_2 + \frac{7}{2}x_3 + \frac{5}{2}x_4 &= 4. \end{aligned}$$

Now, the second equation is multiplied by 11/23 and is added to the third equation, and then multiplied by $-15/23$ and added to the fourth equation, to give

$$\begin{aligned}
2x_1 + 5x_2 - 3x_3 + x_4 &= 6 \\
-\frac{23}{2}x_2 + \frac{11}{2}x_3 - \frac{15}{2}x_4 &= 0 \\
+\frac{95}{23}x_3 - \frac{2}{23}x_4 &= 21 \\
-\frac{2}{23}x_3 + \frac{170}{23}x_4 &= 4.
\end{aligned}$$

Finally, the third equation is multiplied by 2/95 and is added to the fourth Equation, yielding

$$\begin{aligned}
2x_1 + 5x_2 - 3x_3 + x_4 &= 6 \\
-\frac{23}{2}x_2 + \frac{11}{2}x_3 - \frac{15}{2}x_4 &= 0 \\
+\frac{95}{23}x_3 - \frac{2}{23}x_4 &= 21 \\
+\frac{16146}{2185}x_4 &= \frac{422}{95}.
\end{aligned}$$

The backward solution from the fourth equation for x_4 is

$$x_4 = \frac{211}{351} = 0.601.$$

This value of x_4 is set in the third equation to obtain x_3 as

$$\frac{95}{23} - \frac{2}{23}\,\frac{211}{23} = 21 \rightarrow\ x_3 = 5.097$$

Substituting x_4 and x_3 in the second equation, x_2 is obtained as

$$-\frac{23}{2}x_2 + 107.8 - 99.9 = 0 \rightarrow\ x_2 = 2.046$$

And, finally, substituting x_4, x_3, and x_2 in the first equation, x_1 is calculated as

$$2x_1 + 5(2.046) - 3(5.097) + 0.601 = 6 \rightarrow\ x_1 = 5.23$$

The matrix multipliers $[S]$ for this problem are

$$[S_1] = \begin{bmatrix} 1 & 0\,0\,0 \\ -\frac{5}{2} & 1\,0\,0 \\ 0 & 0\,1\,0 \\ 0 & 0\,0\,1 \end{bmatrix} \quad [S_2] = \begin{bmatrix} 1\,0\,0\,0 \\ 0\,1\,0\,0 \\ \frac{3}{2}\,0\,1\,0 \\ 0\,0\,0\,1 \end{bmatrix}$$

$$[S_3] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 0 & 1 \end{bmatrix} \quad [S_4] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{11}{23} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[S_5] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\frac{15}{23} & 0 & 1 \end{bmatrix} \quad [S_6] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{2}{95} & 1 \end{bmatrix}$$

The process of diagonalization of the coefficient matrix is done through the following matrix multiplication:

$$[S_6][S_5][S_4][S_3][S_2][S_1][D] = \begin{bmatrix} 2 & 5 & -3 & 1 & 6 \\ 0 & -11.5 & 5.5 & -7.5 & 0 \\ 0 & 0 & 4.1304 & 0.0869 & 21 \\ 0 & 0 & 0 & 7.39 & 4.442 \end{bmatrix}$$

The process of Gauss elimination is to transform a square matrix into an upper triangular form, as described. This operation can be done in a series of matrix multiplications. Consider the square stiffness matrix [K]. To transform the matrix [K] into an upper triangular form, the matrix multiplications are performed as [4]

$$[L_{n-1}^{-1}]\cdots[L_2^{-1}][L_1^{-1}][K] = [U]. \tag{7.5.8}$$

where $[U]$ is the final upper triangular matrix. The matrix $[L_i]$ is called the Gauss multiplying matrix, and is defined as

$$[L_i^{-1}] = \begin{bmatrix} 1 & & & & & & & 0 \\ & \cdot & & & & & & \\ & & \cdot & & & & & \\ & & & \cdot & & & & \\ & & & & 1 & & & \\ & & & & -l_{i+1,i} & \cdot & & \\ & & & & -l_{i+2,i} & & \cdot & \\ & 0 & & & \cdot & & & \cdot \\ & & & & \cdot & & & & \cdot \\ & & & & -l_{ni} & & & & 1 \end{bmatrix}. \tag{7.5.9}$$

where the Gauss multiplying factors are

$$l_{i+j,i} = \frac{k_{i+j,i}^{(i)}}{k_{ii}^{(i)}}. \tag{7.5.10}$$

The right superscript (i) in Eq. (7.5.10) indicates that an element of the matrix

$$[L_{i-1}^{-1}]\cdots[L_2^{-1}][L_1^{-1}][K]. \tag{7.5.11}$$

is used.

Reverting Eq. (7.5.8), yields

$$[K]=[L_1][L_2]\cdots[L_{n-1}][U]. \tag{7.5.12}$$

where

$$[L_i]=\begin{bmatrix} 1 & & & & & & & \\ & \cdot & & & & 0 & & \\ & & \cdot & & & & & \\ & & & \cdot & & & & \\ & & & & 1 & & & \\ & & & & l_{i+1,i} & \cdot & & \\ & & & & l_{i+2,i} & & \cdot & \\ & 0 & & & \cdot & & & \cdot \\ & & & & \cdot & & & & \cdot \\ & & & & l_{ni} & & & & 1 \end{bmatrix}. \tag{7.5.13}$$

Calling

$$[L]=[L_1][L_2]\cdots[L_{n-1}]. \tag{7.5.14}$$

the matrix triangularization is obtained by

$$[K]=[L][U]. \tag{7.5.15}$$

where the Gauss multiplying matrix is

$$[L]=\begin{bmatrix} 1 & & & & & & \\ l_{21} & 1 & & & & & \\ l_{31} & l_{32} & 1 & & & & \\ l_{41} & l_{42} & l_{43} & 1 & & & \\ \cdot & & & & \cdot & & \\ \cdot & & & & & \cdot & \\ \cdot & & & & & 1 & \\ l_{n1} & & & & & l_{n,n-1} & 1 \end{bmatrix}. \tag{7.5.16}$$

Example 6 Obtain the Gauss multiplying matrices of Example 5.

Solution: The coefficient matrix is

$$[A]^{(1)} = \begin{bmatrix} 2 & 5 & -3 & 1 \\ 5 & 1 & -2 & -5 \\ -3 & -2 & 6 & 2 \\ 1 & -5 & 2 & 3 \end{bmatrix}$$

For $i = 1$,

$$l_{1+j,1} = \frac{A^{(1)}_{1+j,1}}{A^{(1)}_{11}} \rightarrow \quad l_{2,1} = \frac{A^{(1)}_{21}}{A^{(1)}_{11}} = \frac{5}{2} \quad l_{3,1} = \frac{A^{(1)}_{31}}{A^{(1)}_{11}} = -\frac{3}{2}$$

$$l_{4,1} = \frac{A^{(1)}_{41}}{A^{(1)}_{11}} = \frac{1}{2}$$

Thus,

$$[l_1] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{5}{2} & 1 & 0 & 0 \\ -\frac{3}{2} & 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 & 1 \end{bmatrix} \qquad [l_1^{-1}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{5}{2} & 1 & 0 & 0 \\ \frac{3}{2} & 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 0 & 1 \end{bmatrix}$$

Matrix $[A]^{(2)}$ is

$$[A^{(2)}] = [l_1^{-1}][A^{(1)}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{5}{2} & 1 & 0 & 0 \\ \frac{3}{2} & 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 5 & -3 & 1 \\ 5 & 1 & -2 & -5 \\ -3 & -2 & 6 & 2 \\ 1 & -5 & 2 & 3 \end{bmatrix}$$

Thus,

$$[A^{(2)}] = \begin{bmatrix} 2 & 5 & -3 & 1 \\ 0 & -11.5 & 5.5 & -7.5 \\ 0 & 5.5 & 1.5 & 3.5 \\ 0 & -7.5 & 3.5 & 2.5 \end{bmatrix}$$

For $i = 2$,

$$l_{2+j,2} = \frac{A^{(2)}_{2+j,2}}{A^{(2)}_{22}} \rightarrow \quad l_{3,2} = \frac{A^{(2)}_{32}}{A^{(2)}_{22}} = -\frac{11}{23} \quad l_{4,2} = \frac{A^{(2)}_{42}}{A^{(2)}_{22}} = \frac{15}{23}$$

Thus,

$$[l_2] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\frac{11}{23} & 1 & 0 \\ 0 & \frac{15}{23} & 0 & 1 \end{bmatrix} \qquad [l_2^{-1}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{11}{23} & 1 & 0 \\ 0 & -\frac{15}{23} & 0 & 1 \end{bmatrix}$$

Matrix $[A]^{(3)}$ is now

$$[A^{(3)}] = [l_2^{-1}][A^{(2)}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{11}{23} & 1 & 0 \\ 0 & -\frac{15}{23} & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 5 & -3 & 1 \\ 0 & -\frac{23}{2} & \frac{11}{2} & -\frac{15}{2} \\ 0 & \frac{11}{2} & \frac{3}{2} & \frac{7}{2} \\ 0 & -\frac{15}{2} & \frac{7}{2} & \frac{5}{2} \end{bmatrix}$$

Thus,

$$[A^{(3)}] = \begin{bmatrix} 2 & 5 & -3 & 1 \\ 0 & -\frac{23}{2} & \frac{11}{2} & -\frac{15}{2} \\ 0 & 0 & \frac{95}{23} & -\frac{2}{23} \\ 0 & 0 & -\frac{2}{23} & \frac{170}{23} \end{bmatrix}$$

For $i = 3$,

$$l_{3+j,3} = \frac{A^{(3)}_{3+j,3}}{A^{(3)}_{33}} \rightarrow \quad l_{4,3} = \frac{A^{(3)}_{43}}{A^{(3)}_{33}} = -\frac{2}{95}$$

and

$$[l_3] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{2}{95} & 1 \end{bmatrix} \qquad [l_3^{-1}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{2}{95} & 1 \end{bmatrix}$$

Therefore,

$$[U] = [l_3^{-1}][A^{(3)}] = [l_3^{-1}][l_2^{-1}][l_1^{-1}][A]$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{2}{95} & 1 \end{bmatrix} \begin{bmatrix} 2 & 5 & -3 & 1 \\ 0 & -\frac{23}{2} & \frac{11}{2} & -\frac{15}{2} \\ 0 & 0 & \frac{95}{23} & -\frac{2}{23} \\ 0 & 0 & -\frac{2}{23} & \frac{170}{23} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 5 & -3 & 1 \\ 0 & -\frac{23}{2} & \frac{11}{2} & -\frac{15}{2} \\ 0 & 0 & \frac{95}{23} & -\frac{2}{23} \\ 0 & 0 & 0 & \frac{16146}{2185} \end{bmatrix}$$

The matrix $[L]$ is then

$$[L] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{5}{2} & 1 & 0 & 0 \\ -\frac{3}{2} & -\frac{11}{23} & 1 & 0 \\ \frac{1}{2} & \frac{15}{23} & -\frac{2}{95} & 1 \end{bmatrix}$$

Matrix $[U]$ is an upper triangular matrix. If we define a diagonal matrix $[D]$, in which its diagonal elements are $d_{ii} = U_{ii}$, then we can write

$$[U] = [D][\widehat{S}]. \tag{7.5.17}$$

Substituting for $[U]$ from Eq. (7.5.17) into (7.5.15) gives

$$[K] = [L][D][\widehat{S}]. \tag{7.5.18}$$

When the matrix $[K]$ is symmetric, its decomposition is unique, and we obtain

$$[\widehat{S}] = [L]^T. \tag{7.5.19}$$

which, substituting into Eq. (7.5.18), gives

$$[K] = [L][D][L]^T. \tag{7.5.20}$$

where $[L]^T$ is the transposition of $[L]$. Equation (7.5.20) is the factorization of matrix $[K]$ into $[L][D][L]^T$ form.

Now, consider the system of equations

$$[K]\{X\} = \{F\}. \tag{7.5.21}$$

The load matrix $\{F\}$ is reduced to obtain the matrix $\{V\}$ as

$$[L]\{V\} = \{F\}. \tag{7.5.22}$$

where

$$\{V\} = [L_{n-1}^{-1}] \cdots [L_2^{-1}][L_1^{-1}]\{F\}. \tag{7.5.23}$$

Substituting the factorized form of $[K]$ from Eq. (7.5.20) into (7.5.21) gives

$$[L][D][L]^T\{X\} = \{F\} = [L]\{V\}. \tag{7.5.24}$$

or

$$[D][L]^T\{X\} = \{V\}. \tag{7.5.25}$$

The solution for the unknown matrix $\{X\}$ from Eq. (7.5.25) is now obtained as

$$[L]^T\{X\} = [D]^{-1}\{V\}. \tag{7.5.26}$$

The steps of solution and the related matrices are shown in Example 7.

Example 7 Find the solution of the system of equations of Example 5 with steps of triangularization and the related matrices.

Solution: From the solution of Example 6

$$[L] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{5}{2} & 1 & 0 & 0 \\ -\frac{3}{2} & -\frac{11}{23} & 1 & 0 \\ \frac{1}{2} & \frac{15}{23} & -\frac{2}{95} & 1 \end{bmatrix} \qquad [L]^T = \begin{bmatrix} 1 & \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \\ 0 & 1 & -\frac{11}{23} & \frac{15}{23} \\ 0 & 0 & 1 & -\frac{2}{95} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now,

$$[L^{-1}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2.5 & 1 & 0 & 0 \\ 0.3043 & 0.4783 & 1 & 0 \\ 1.1368 & -0.6421 & 0.0211 & 1 \end{bmatrix}$$

and

$$[L^T]^{-1} = [L^{-1}]^T = \begin{bmatrix} 1 & -2.5 & 0.3043 & 1.1368 \\ 0 & 1 & 0.4783 & -0.6421 \\ 0 & 0 & 1 & 0.0211 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The matrix $[V]$ from Eq. (7.5.23) is

$$\{V\} = [L^{-1}]\{F\} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2.5 & 1 & 0 & 0 \\ 0.3043 & 0.4783 & 1 & 0 \\ 1.1368 & -0.6421 & 0.0211 & 1 \end{bmatrix} \begin{Bmatrix} 6 \\ 15 \\ 12 \\ 7 \end{Bmatrix} = \begin{Bmatrix} 6 \\ 0 \\ 21 \\ 4.44 \end{Bmatrix}$$

The diagonal matrix $[D]$ from the solution of Example 2 is

$$[D] = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -\frac{23}{2} & 0 & 0 \\ 0 & 0 & \frac{95}{23} & 0 \\ 0 & 0 & 0 & \frac{16146}{2185} \end{bmatrix}$$

The inverse of the diagonal matrix is

$$[D^{-1}] = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{2}{23} & 0 & 0 \\ 0 & 0 & \frac{23}{95} & 0 \\ 0 & 0 & 0 & \frac{2185}{16146} \end{bmatrix}$$

Then

$$[D^{-1}]\{V\} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{2}{23} & 0 & 0 \\ 0 & 0 & \frac{23}{95} & 0 \\ 0 & 0 & 0 & \frac{2185}{16146} \end{bmatrix} \begin{Bmatrix} 6 \\ 0 \\ 21 \\ 4.44 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 0 \\ 5.084 \\ 0.601 \end{Bmatrix}$$

And, finally,

$$\{X\} = [L^T]^{-1}[D^{-1}]\{V\} = \begin{bmatrix} 1 & -2.5 & 0.3043 & 1.1368 \\ 0 & 1 & 0.4783 & -0.6421 \\ 0 & 0 & 1 & 0.0211 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 3 \\ 0 \\ 5.084 \\ 0.601 \end{Bmatrix} = \begin{Bmatrix} 5.23 \\ 2.046 \\ 5.097 \\ 0.601 \end{Bmatrix}$$

The algorithm of Gauss elimination based on the foregoing discussion is now presented [4].

Algorithm

$$d_{11} = k_{11}$$

$$\text{for} \quad j = 2, \cdots n$$

$$g_{m_j,j} = k_{m_j,j}$$

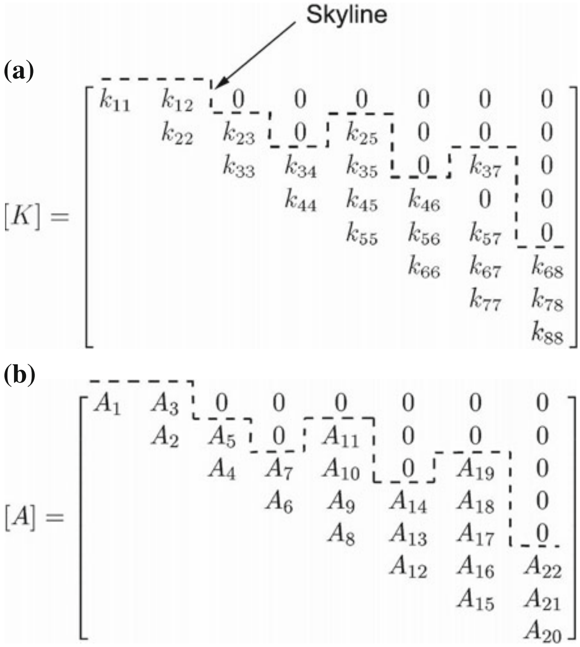
$$g_{ij} = k_{ij} - \sum_{r=m_m}^{i-1} l_{ri} g_{rj} \qquad i = m_j + 1, \ldots j - 1.$$

where

$$l_{ij} = \frac{g_{ij}}{d_{ii}} \qquad i = m_j, \ldots j - 1$$

$$d_{jj} = k_{jj} - \sum_{r=m_j}^{j-1} l_{rj} g_{rj}.$$

where $m_m = Max(m_i, m_j)$, m_i is the row number of the first non-zero element in column i, and m_j is the row number of the first non-zero element in column j (see Sect. 7.6).



(c)

$$DK(I) = < 1 \ \ 2 \ \ 4 \ \ 6 \ \ 8 \ \ 12 \ \ 15 \ \ 20 \ \ 23 >$$

Fig. 7.3 Skyline representation of a banded matrix

7.6 Skyline Method, Static Problems

The stiffness matrix obtained in a finite element problem is always a banded matrix. Therefore, the Gauss elimination solution method has to be partially modified to handle the banded matrix, rather than a full square matrix. The resulting algorithm will substantially reduce the required storage capacity of the standard Gauss elimination method, in addition to major cuts in unnecessary arithmetic operations for zero members which are located outside of the band of the matrix. The zero members inside the band of the matrix, however, are considered, and their necessary arithmetic operations during the elimination process must be carried out.

The skyline method is basically developed to eliminate some of the zero members inside the band of the stiffness matrix. The results are further cut in unnecessary arithmetic operations and the improvement of the computation time. In addition, the rectangular banded matrix $[K]$ is transformed into a one-dimensional array $\{A\}$, resulting in further improvement in storage capacity.

To describe the skyline method, consider a typical stiffness matrix $[K]$, as shown in Fig. 7.3. It is assumed that the matrix $[K]$ is symmetric and, therefore, only the upper half of the matrix is shown [4]. Figure 7.3 shows that only the upper half of the matrix is considered, including the diagonal members. The skyline is drawn on the top of each non-zero element of columns. The elements above the skyline on each column are, thus, zero. The zero elements under the skyline are considered in the calculations. By inspecting Fig. 7.3, it is clear that some of the zero members which are considered in the banded matrix are excluded from the arithmetic operations in the skyline algorithm. As for the storage strategy, a numbering system must be adopted to save the elements under the skyline in the one-dimensional array $A(I)$. Knowing the height of the skyline of each column, we can save the location of the diagonal members of $[K]$ in the one-dimensional array $\langle DK \rangle$, as shown in Fig. 7.3. The first location of $\langle DK \rangle$ is always 1, and the last location number is one number larger than the last column location in $[K]$ (number 23 in Fig. 7.3 shows that the last non-zero member of matrix $[K]$ in column 8 is located at 22). The value $m_i,\ i = 1, \ldots n$, is defined as the row number of the first non-zero element in column i. The variables $m_i,\ i = 1, \ldots n$ will thus define the skyline of the matrix. The variable $(i - m_i)$ is the column height of the i–th column. The half-bandwidth of the stiffness matrix, m_B, is equal to $Max(i - m_i),\ i = 1, 2, \ldots n$. The values of m_i for Fig. 7.3 are $m_1 = 1$, $m_2 = 1$, $m_3 = 2$, $m_4 = 3$, $m_5 = 2$, $m_6 = 4$, $m_7 = 3$, and $m_8 = 6$. From Fig. 7.3, it is noted that $DK(I)$ is equal to the sum of the column heights up to the $(i - 1)$st column plus I. With known reference system $DK(I)$, the location of each element of matrix $[K]$ in the array $\langle A \rangle$ is known, and with a proper referencing system and algorithm, the Gauss elimination method is used to solve the system of equations in which their coefficient matrix is stored in a one-dimensional array $\langle A \rangle$.

The skyline algorithm is used in the computer program ADINA [5]. Also, a listing of the solver using the skyline method for the symmetric matrix $[K]$ is available in reference [4].

7.7 Solution of Transient Problems

A general form of the transient finite element equation is

$$[C]\{\dot{X}\} + [K]\{X\} = \{F\}. \tag{7.7.1}$$

where $[C]$ is the damping matrix. An example of this equation is the transient heat conduction problem. The direct integration method may be used to integrate Eq. (7.7.1) in the time domain. Two states of $\{X\}$, separated by time increment Δt and denoted by $\{X\}_t$ and $\{X\}_{t+\Delta t}$, are considered. According to the trapezoidal rule, the two vectors are related as

$$\{X\}_{t+\Delta t} = \{X\}_t + \left[(1 - \beta)\{\dot{X}\}_t + \beta\{\dot{X}\}_{t+\Delta t}\right](\Delta t). \tag{7.7.2}$$

where β is a constant which may be selected by the analyst. Equation (7.7.1) is written at times t and $t + \Delta t$, the first equation is multiplied by $(1 - \beta)$ and the second equation is multiplied by β to give

$$
\begin{aligned}
(1-\beta)\left([C]\{\dot{X}\}_t + [K]\{X\}_t\right) &= (1-\beta)\{F\}_t \\
\beta\left([C]\{\dot{X}\}_{t+\Delta t} + [K]\{X\}_{t+\Delta t}\right) &= \beta\{F\}_{t+\Delta t}.
\end{aligned}
\tag{7.7.3}
$$

These two equations are added and Eq. (7.7.2) is used to eliminate the time derivatives of $\{X\}$. The resulting equation is solved for $\{X\}_{t+\Delta t}$, which yields

$$
\left(\frac{1}{\Delta t}\,[C] + \beta[K]\right)\{X\}_{t+\Delta t} = \left(\frac{1}{\Delta t}\,[C] - (1-\beta)[K]\right)\{X\}_t + (1-\beta)\{F\}_t + \beta\{F\}_{t+\Delta t}.
\tag{7.7.4}
$$

In the derivation of Eq. (7.7.3), it is assumed that the matrices $[C]$ and $[K]$ are constant by time. This equation is used to calculate the vector $\{X\}$ at time $t + \Delta t$ in terms of its value at time t. The solution starts with the known initial value of $\{X\}$ at $t = 0$ ($\{X\}_0$). With a proper time increment Δt, the value of $\{X\}$ at $t = \Delta t$ is obtained using the known value $\{X\}_0$. The process is then repeated for the next Δt, and so on. We may march through the time as long as required.

The stability of the solution algorithm depends on the numerical value of Δt, which is inversely proportional to the stability parameter β. For $\beta < 1/2$, the largest value of Δt for the solution to be stable is [6]

$$
\Delta t_{cr} = \frac{2}{(1-2\beta)w_{max}}.
\tag{7.7.5}
$$

where w_{max} is the largest eigenvalue of the problem. It is important to note that the value of β, which provides an unconditionally stable algorithm, does not prove the accuracy of the solution. The accuracy measure depends upon the numerical value of Δt.

For $\beta \geq 1/2$, the algorithm is unconditionally stable, regardless of the value of Δt. Different values of β are associated with various numerical schemes. The value $\beta = 0$ provides the forward finite difference method, which is a conditionally stable algorithm. The Crank-Nicolson, or trapezoidal rule, is associated with $\beta = 1/2$, which is an unconditionally stable method. The value of $\beta = 2/3$ is the Galerkin algorithm and is an unconditionally stable scheme. For $\beta = 1$, we get the unconditionally stable standard backward finite difference method. For $\beta = 0$, the algorithm is called *explicit*, and for $\beta > 0$, it is called *implicit*.

It is usually preferable to select $\beta = 1/2$. In this case, some unwanted oscillations may appear in the solution. The oscillations may be reduced by using a smaller value for Δt, or numerically damped by selecting a value for β larger than $1/2$. For nonlinear problems, the only choice for β which provides an unconditionally stable solution is $\beta = 1$. In this case, the analyst must be very careful in obtaining the accuracy of the solution [6].

7.8 Solution of Dynamic Problems

When the applied loads or the initial conditions of a system are functions of time and their variations by time are rapid enough to excite the inertia of the system, then the response of the system is obtained through the dynamic equations. For dynamic problems, the finite element equation is of general form

$$[M]\{\ddot{X}\} + [C]\{\dot{X}\} + [K]\{X\} = \{F\}. \tag{7.8.1}$$

where $[M]$, $[C]$, and $[K]$ are the mass, damping, and stiffness matrices, $\{F\}$ is the force matrix, and $\{X\}$ is the unknown matrix. The matrices $\{\ddot{X}\}$ and $\{\dot{X}\}$ are the acceleration and velocity matrices. By the finite element approximation, we have been able to transform the differential operator of space into a system of algebraic equations, while the time variable remains in the equation. The final equilibrium equation (7.8.1) is a system of second order differential equations in terms of time. To solve this system, several classical methods are developed which may be used to integrate the dynamic equations. These methods are described as given.

7.8.1 The Central Difference Method

The dynamic equation (7.8.1) is a set of ordinary differential equations with constant coefficients. Any classical method of solution of the system of ordinary differential equations may be used to solve this system. These techniques essentially assume a small enough time increment and write the incremental form of Eq. (7.8.1). The resulting incremental form is then solved by a proper method by marching through the time. Through this method, we have essentially used the finite difference method in the time domain. A simple finite difference method is to assume a central difference and, with a proper time increment Δt, approximate the velocity and acceleration matrix as [3]

$$\{\dot{X}\}_t = \frac{1}{2\Delta t}\left(\{X\}_{t+\Delta t} - \{X\}_{t-\Delta t}\right) \tag{7.8.2}$$

$$\{\ddot{X}\}_t = \frac{1}{\Delta t^2}\left(\{X\}_{t+\Delta t} - 2\{X\}_t + \{X\}_{t-\Delta t}\right). \tag{7.8.3}$$

Now, we may write Eq. (7.8.1) at time t as

$$[M]\{\ddot{X}\}_t + [C]\{\dot{X}\}_t + [K]\{X\}_t = \{F\}_t. \tag{7.8.4}$$

Substituting $\{\ddot{X}\}_t$ and $\{\dot{X}\}_t$ from Eqs. (7.8.3) and (7.8.2) in (7.8.4) and rearranging the terms to give

$$\left(\frac{1}{\Delta t^2}[M] + \frac{1}{2\Delta t}[C]\right)\{X\}_{t+\Delta t} = \{F\}_t - \left([K] - \frac{2}{\Delta t^2}[M]\right) \times \{X\}_t - \left(\frac{1}{\Delta t^2}[M] - \frac{1}{2\Delta t}[C]\right)\{X\}_{t-\Delta t}. \tag{7.8.5}$$

From this equation, $\{X\}_{t+\Delta t}$ is solved in terms of the known values of $\{X\}_t$ and $\{X\}_{t-\Delta t}$. To solve this equation at time increment Δt, we must know $\{X\}_t$ and $\{X\}_{t-\Delta t}$. Since the solution starts from the known initial conditions, it is thus assumed that at $t = 0$, $\{X\}_0, \{\dot{X}\}_0$ and $\{\ddot{X}\}_0$ are known. Note that with the known initial values of $\{X\}_0$ and $\{\dot{X}\}_0$, the initial value of $\{\ddot{X}\}_0$ can be used from Eq. (7.8.1) as

$$\{\ddot{X}\}_0 = [M]^{-1}\left(\{F\}_0 - [C]\{\dot{X}\}_0 - [K]\{X\}_0\right). \tag{7.8.6}$$

With the known initial values of $\{X\}_0, \{\dot{X}\}_0$, and $\{\ddot{X}\}_0$, the value of $\{X\}_{-\Delta t}$ is obtained using Eq. (7.8.2) and (7.8.3) as

$$\{X\}_{-\Delta t} = \{X\}_0 - \Delta t\{\dot{X}\}_0 + \frac{\Delta t^2}{2}\{\ddot{X}\}_0. \tag{7.8.7}$$

Once the vector $\{X\}_{-\Delta t}$ is calculated in terms of the initial conditions, time is advanced an increment Δt and the solution procedure is repeated using Eq. (7.8.5).

The selection of time increment Δt for the convergence of the solution is very important. A proper value for Δt must be selected with two important features. This value must be small enough to insure the solution convergency, and truly exhibit the vibrational characteristics and behavior of the system. The system will not exhibit any of its natural behavior, such as its vibrating behavior, for time increments which are larger than Δt. The analytical solution of a continuous system provides a set of infinite eigenvalues corresponding to natural frequencies of the dynamical system. Not all of these infinite natural frequencies are important in the engineering problem. In fact, to motivate and excite the higher natural frequencies and mode shapes of a continuous system requires a very severe and short time period of dynamical excitations. Therefore, a solution for the dynamic behavior of a continuous system for engineering purposes needs consideration up to a finite number of natural frequencies and mode shapes. The selection of time increment Δt must be based on this highest natural frequency of the system. The foregoing discussion suggests that the selection of time increment Δt must be some how related to the mass and stiffness matrices. The natural frequencies of a continuous system, which is discretized by finite element modeling, depends upon its mass and stiffness matrices. A criteria for selecting a proper value for the time increment is [4]

$$\Delta t < \Delta t_{cr} = \frac{T_n}{\pi}. \tag{7.8.8}$$

where T_n is the period of lowest natural frequency of vibration of the finite element assemblage. The calculation of T_n is not easy. A finite element model with total $N = NN \times DOF/Node$ degrees of freedom, where NN is the total number of nodes, has N eigenvalues where T_n of Eq. (7.8.8) must be the smallest of all time periods associated with the eigenvalues. It is virtually mathematically impossible, and physically insignificant, to obtain all eigenvalues of a finite element model. As an example, a finite element model may easily have a total of 10^6 degrees of freedom, where the best and most efficient eigenvalue calculation algorithms may determine the eigenvalues of a matrix of the order of 10^2. Finally, the physical significance of the natural frequencies of a continuous system are, say, up to the first 20 lowest natural frequencies. For this reason, the best strategy for selecting a proper value for the time increment is usually by trial and error. We must select a value for Δt and solve the problem. If the solution converges, for the selected Δt, repeat the solution for another Δt which is either twice or half of the previous Δt and check the solution for its consistency with the previous solution.

7.8.2 The Houbolt Method

The Houbolt method, similar to the central difference method, assumes an approximation for the velocity and acceleration matrices in terms of the displacement matrix. The approximations are [7]

$$\{\dot{X}\}_{t+\Delta t} = \frac{1}{6\Delta t}\left(11\{X\}_{t+\Delta t} - 18\{X\}_t + 9\{X\}_{t-\Delta t} - 2\{X\}_{t-2\Delta t}\right). \quad (7.8.9)$$

$$\{\ddot{X}\}_{t+\Delta t} = \frac{1}{\Delta t^2}\left(2\{X\}_{t+\Delta t} - 5\{X\}_t + 4\{X\}_{t-\Delta t} - \{X\}_{t-2\Delta t}\right). \quad (7.8.10)$$

These approximations are based on two backward-difference formulas with errors of $(\Delta t)^2$. Now, consider Eq. (7.8.1) written at time $(t + \Delta t)$ as

$$[M]\{\ddot{X}\}_{t+\Delta t} + [C]\{\dot{X}\}_{t+\Delta t} + [K]\{X\}_{t+\Delta t} = \{F\}_{t+\Delta t}. \quad (7.8.11)$$

Substituting Eqs. (7.8.9) and (7.8.10) in Eq. (7.8.11) and keeping the unknown matrix at time $(t + \Delta t)$ at the left-hand side, gives

$$\begin{aligned}\left(\frac{2}{\Delta t^2}[M] + \frac{11}{6\Delta t}[C] + [K]\right)\{X\}_{t+\Delta t} &= \{F\}_{t+\Delta t} + \left(\frac{5}{\Delta t^2}[M] + \frac{3}{\Delta t}[C]\right)\\ &\times \{X\}_t - \left(\frac{4}{\Delta t^2}[M] + \frac{3}{2\Delta t}[C]\right)\{X\}_{t-\Delta t}\\ &+ \left(\frac{1}{\Delta t^2}[M] + \frac{1}{3\Delta t}[C]\right) \times \{X\}_{t-2\Delta t}.\end{aligned} \quad (7.8.12)$$

To start the procedure for the time increment Δt, in addition to the known initial conditions $\{X\}_0$, $\{\dot{X}\}_0$, and $\{\ddot{X}\}_0$, we need to have $\{X\}_{-\Delta t}$ and $\{X\}_{-2\Delta t}$. This means that a special starting procedure is required. One method may be based on the integration of Eq. (7.8.1) for the solution of $\{X\}_{-\Delta t}$ and $\{X\}_{-2\Delta t}$, using a central difference technique.

7.8.3 The Newmark Method

In this technique, the velocity and displacement matrices at time $t + \Delta t$ are approximated in terms of their values at time t as [8]

$$\{\dot{X}\}_{t+\Delta t} = \{\dot{X}\}_t + [(1-\alpha)\{\ddot{X}\}_t + \alpha\{\ddot{X}\}_{t+\Delta t}]\Delta t \tag{7.8.13}$$

$$\{X\}_{t+\Delta t} = \{X\}_t + \{\dot{X}\}_t \Delta t + [(\frac{1}{2} - \beta)\{\ddot{X}\}_t + \beta\{\ddot{X}\}_{t+\Delta t}](\Delta t)^2. \tag{7.8.14}$$

where the coefficients α and β are parameters which determine the accuracy and stability of the numerical technique. Rewriting Eq. (7.8.1) for time $(t + \Delta t)$, we get

$$[M]\{\ddot{X}\}_{t+\Delta t} + [C]\{\dot{X}\}_{t+\Delta t} + [K]\{X\}_{t+\Delta t} = \{F\}_{t+\Delta t}. \tag{7.8.15}$$

Equation (7.8.14) is solved for $\{\ddot{X}\}_{t+\Delta t}$. The result is then substituted for $\{\ddot{X}\}_{t+\Delta t}$ in Eq. (7.8.13) to obtain an expression for $\{\dot{X}\}_{t+\Delta t}$. Now, two expressions are obtained for $\{\dot{X}\}_{t+\Delta t}$ and $\{\ddot{X}\}_{t+\Delta t}$ in terms of $\{X\}_{t+\Delta t}$ and other matrices, which are substituted in Eq. (7.8.15), and the resulting expression is solved for $\{X\}_{t+\Delta t}$ to give the following relation:

$$[\widehat{K}]\{X\}_{t+\Delta t} = \{\widehat{F}\}. \tag{7.8.16}$$

where

$$\begin{aligned}
[\widehat{K}] &= \left[\frac{1}{\beta(\Delta t)^2}[M] + \frac{\alpha}{\beta(\Delta t)}[C] + [K]\right] \\
\{\widehat{F}\} &= \{F\}_{t+\Delta t} + [M]\left(\frac{1}{\beta(\Delta t)^2}\{X\}_t + \frac{1}{\beta(\Delta t)}\{\dot{X}\}_t + (\frac{1}{2\beta} - 1)\{\ddot{X}\}_t\right) \\
&\quad - [C]\left(\{\dot{X}\}_t + (\Delta t)(1-\alpha)\{\ddot{X}\}_t + \frac{\alpha}{\beta(\Delta t)}\{X\}_t + \frac{\alpha}{\beta}\{\dot{X}\}_t - \frac{\alpha(\Delta t)}{\beta}\left(\frac{1}{2} - \beta\right)\{\ddot{X}\}_t\right).
\end{aligned} \tag{7.8.17}$$

With the given initial conditions $\{X\}_0$, $\{\dot{X}\}_0$, and $\{\ddot{X}\}_0$, Eq. (7.8.16) is used to march through the time and obtain $\{X\}$ at an increment advanced in time. The velocity and acceleration matrices at time $(t + \Delta t)$ are obtained from Eqs. (7.8.13) and (7.8.14).

The Newmark time marching algorithm is the name of a family of methods which are obtained for different numerical values for α and β. As an example, the method for $\alpha = 1/2$ and $\beta = 1/4$ is called the *average acceleration method*, which is unconditionally stable.

7.8.4 The Wilson-θ Method

Based on the Wilson-θ method, the acceleration between times t and $t + \Delta t$ is approximated by a linear function [9]. Figure 7.4 shows the variation of the acceleration between times t and $(t + \theta \Delta t)$, where $\theta \geq 1.0$. For $\theta = 1$, the method reduces to the linear acceleration method. The method becomes unconditionally stable for $\theta \geq 1.37$. A value of $\theta = 1.4$ is usually considered in the numerical solutions.

Denoting τ as the time increment, where $0 \leq \tau \leq \theta \Delta t$, the acceleration matrix at time $t + \tau$ is approximated as

$$\{\ddot{X}\}_{t+\tau} = \{\ddot{X}\}_t + \frac{\tau}{\theta \Delta t} \left(\{\ddot{X}\}_{t+\theta \Delta t} - \{\ddot{X}\}_t\right). \tag{7.8.18}$$

Integrating Eq. (7.8.18) twice with respect to time gives

$$\{\dot{X}\}_{t+\tau} = \{\dot{X}\}_t + \tau \{\ddot{X}\}_t + \frac{\tau^2}{2\theta \Delta t} \left(\{\ddot{X}\}_{t+\theta \Delta t} - \{\ddot{X}\}_t\right) \tag{7.8.19}$$

$$\{X\}_{t+\tau} = \{X\}_t + \tau \{\dot{X}\}_t + \frac{\tau^2}{2} \{\ddot{X}\}_t + \frac{\tau^3}{6\theta \Delta t} \left(\{\ddot{X}\}_{t+\theta \Delta t} - \{\ddot{X}\}_t\right). \tag{7.8.20}$$

Evaluating Eqs. (7.8.19) and (7.8.20) at time $\tau = \theta \Delta t$ gives

$$\{\dot{X}\}_{t+\theta \Delta t} = \{\dot{X}\}_t + \frac{\theta \Delta t}{2} \left(\{\ddot{X}\}_{t+\theta \Delta t} + \{\ddot{X}\}_t\right) \tag{7.8.21}$$

$$\{X\}_{t+\theta \Delta t} = \{X\}_t + \theta \Delta t \{\dot{X}\}_t + \frac{(\theta \Delta t)^2}{6} \left(\{\ddot{X}\}_{t+\theta \Delta t} + 2\{\ddot{X}\}_t\right). \tag{7.8.22}$$

Equations (7.8.21) and (7.8.22) are solved for $\{\ddot{X}\}_{t+\theta \Delta t}$ and $\{\dot{X}\}_{t+\theta \Delta t}$ in terms of $\{X\}_{t+\theta \Delta t}$ as

$$\{\ddot{X}\}_{t+\theta \Delta t} = \frac{6}{(\theta \Delta t)^2} \left(\{X\}_{t+\theta \Delta t} - \{X\}_t\right) - \frac{6}{\theta \Delta t} \{\dot{X}\}_t - 2\{\ddot{X}\}_t \tag{7.8.23}$$

$$\{\dot{X}\}_{t+\theta \Delta t} = \frac{3}{\theta \Delta t} \left(\{X\}_{t+\theta \Delta t} - \{X\}_t\right) - 2\{\dot{X}\}_t - \frac{\theta \Delta t}{2} \{\ddot{X}\}_t. \tag{7.8.24}$$

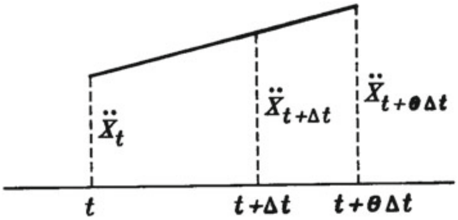
Now, the equilibrium equation (7.8.1) is written at time $t + \theta \Delta t$ as

$$[M]\{\ddot{X}\}_{t+\theta \Delta t} + [C]\{\dot{X}\}_{t+\theta \Delta t} + [K]\{X\}_{t+\theta \Delta t} = \{\bar{F}\}_{t+\theta \Delta t}. \tag{7.8.25}$$

where $\{\bar{F}\}_{t+\theta \Delta t}$ is an average force during the time t and $t + \theta \Delta t$, and is obtained as

$$\{\bar{F}\}_{t+\theta \Delta t} = \{F\}_t + \theta \left(\{F\}_{t+\Delta t} - \{F\}_t\right). \tag{7.8.26}$$

Fig. 7.4 Variation of acceleration



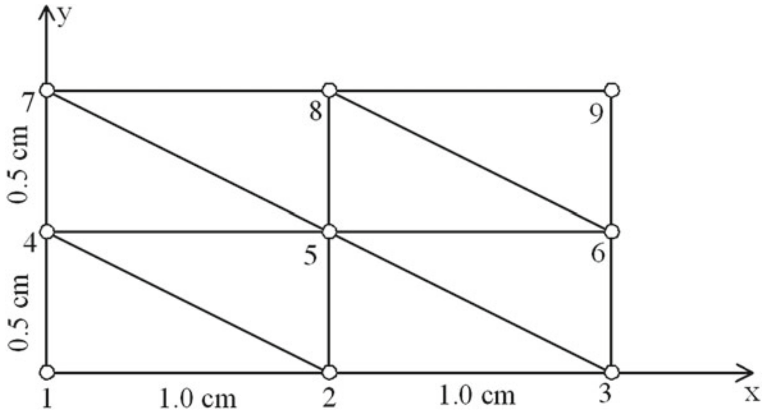


Fig. 7.5 Elastic membrane in xy-plane

Substituting Eqs. (7.8.23) and (7.8.24) in Eq. (7.8.25), an expression is obtained for $\{X\}_{t+\theta\Delta t}$ which can be solved in terms of the known values of $\{X\}_t$. The result is then substituted in Eq. (7.8.23) to obtain $\{\ddot{X}\}_{t+\theta\Delta t}$. Finally, the value of $\{\ddot{X}\}_{t+\theta\Delta t}$ is substituted in Eqs. (7.8.18), (7.8.19), and (7.8.20), and these expressions are used for $\tau = \Delta t$ to calculate $\{\ddot{X}\}_{t+\Delta t}$, $\{\dot{X}\}_{t+\Delta t}$, and $\{X\}_{t+\Delta t}$.

7.9 Problems

1. Use the solution algorithm to obtain the solution of the system of equations of Example 5.
2. Consider an elastic membrane in the xy-plane with surface tension T and pressure P, as shown in Fig. 7.5. Use the algorithm given in Sect. 7.2 to generate the global stiffness and force matrices. The element dimension along the x-direction is 1 cm and in the y-direction is 0.5 cm.
3. Calculate the band width of the stiffness matrix of the elastic membrane of Problem 2.
4. Consider the same elastic membrane. The boundary conditions of nodes 1, 2, 3, 6, 9, 8, 7, and 4 are $\phi = 0$. Use method 1 of the boundary conditions to obtain the final form of the equilibrium equation of node 5.

5. Reconsider Problem 3 and use method 2 of the boundary conditions to obtain the final form of the equilibrium equation of node 5.
6. A system of linear equations is assumed of the form

$$x + y + z = 2$$
$$2x - y + 2z = 7$$
$$x + y - 3z = -6.$$

Use the Gauss elimination technique to obtain the upper triangular matrix with the help of matrix multipliers $[L]$ defined by Eq. (7.5.16).
7. Decompose the matrix $[K]$ of the above problem into $[L][D][L]^T$, defined by Eq. (7.5.20).
8. Find the solution to the system of equations in Problem 6 by the method given by the equation

$$\{X\} = [L^T]^{-1}[D^{-1}]\{V\}$$

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Chapter 8
Finite Element of Beams

Abstract The finite element derivations for a base element of beams and bars under different types of behaviors are discussed in this chapter. The static lateral defection and axial, torsional, and lateral vibrations of beams and bars are studied and the members of mass, stiffness, and force matrices are derived. The chapter concludes with a discussion of the Timoshenko beam and the derivations of the element matrices.

8.1 Introduction

Beams are one of the basic elements in many practical structural design problems. The classical flexural beam theory is based on the Love-Kirchhoff assumption, where it is assumed that the plane sections of the beam remain plane after determination. This simplifying assumption implies that the transverse shear strains are zero. In addition, the normal lateral stress and strain are assumed to be zero. Such a beam theory is referred to as the Euler beam. A more advanced theory assumes non-zero transverse shear deformation and includes the rotary inertia. The mathematical modeling of the beam with these additional assumptions, which is still categorized as the flexural beam theory, is called the Timoshenko beam theory. Whenever a more precise beam analysis is required, the Timoshenko beam theory may be used.

In this Chapter, the finite element analysis of beams based on both the Euler beam and the Timoshenko beam are discussed. The variational formulation of the Euler beam is written on the basis of membrane analogy.

8.2 Euler Beam, Variational Formulation

The deflection equation of the beam based on the Love-Kirchhoff hypothesis is

$$EI\,\frac{d^2y}{dx^2} = M(x) \tag{8.2.1}$$

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Solid Mechanics and Its Applications 216, DOI: 10.1007/978-3-319-08037-6_8,

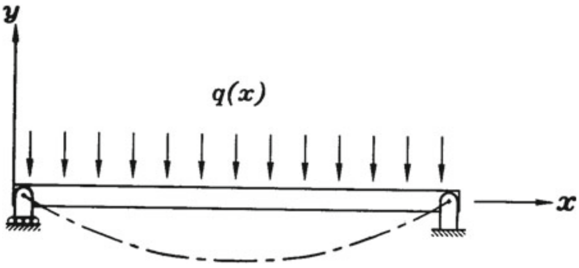


Fig. 8.1 A beam under lateral distributed force $q(x)$

where y is the beam's lateral deflection, which is a function of the x-coordinate along the beam, $M(x)$ is the bending moment acting on the cross-section of the beam and is a function of x, E is the modulus of elasticity, and I is the moment of inertia of the cross-section. A typical beam is shown in Fig. 8.1. The analogy of Eq. (8.2.1) with the deflection of the two-dimensional shallow elastic membrane, suggests the expression of the functional of the beam as

$$V = \int_0^L \left[\frac{EI}{2} \left(\frac{dy}{dx} \right)^2 + M(x)y \right] dx. \tag{8.2.2}$$

Dividing the beam into a number of elements and considering the base element (e), the shape function for the deflection may be written as (Fig. 8.2)

$$y^{(e)}(x) = N_i Y_i + N_j Y_j = \langle N \rangle^{(e)} \{Y\}^{(e)} \tag{8.2.3}$$

where

$$N_i = \frac{x_j - x}{L} \quad N_j = \frac{x - x_i}{L} \tag{8.2.4}$$

where $L = x_j - x_i$. The total functional of the beam is the sum of the functionals of each of the elements of the beam as

$$V = \sum_{e=1}^{NE} (V)^{(e)} \tag{8.2.5}$$

where the functional of element (e) is

$$(V)^{(e)} = \int_{x_i}^{x_j} \left[\frac{EI}{2} \left(\frac{dy}{dx} \right)^2 + M(x)y \right] dx. \tag{8.2.6}$$

Applying the Ritz method, we get

$$\frac{\partial (V)^{(e)}}{\partial Y_m} = \int_{x_i}^{x_j} \left[EI \left(\frac{dy}{dx} \right) \frac{\partial}{\partial Y_m} \left(\frac{dy}{dx} \right) + M(x) \frac{\partial y}{\partial Y_m} \right] dx \quad m = i, j \tag{8.2.7}$$

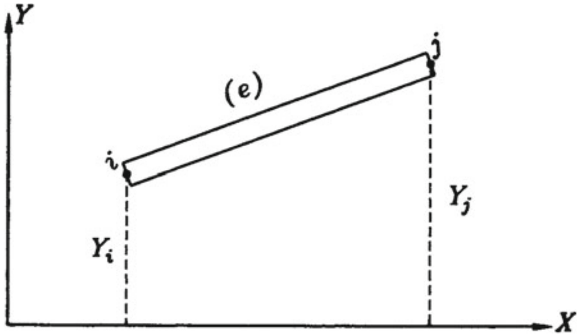


Fig. 8.2 An element of the beam

Substituting for $y^{(e)}$ from Eq. (8.2.3) in Eq. (8.2.7), and carrying out the integration, gives

$$\frac{\partial (V)^{(e)}}{\partial Y_m} = \int_{x_i}^{x_j} \left[EI \langle \frac{dN_i}{dx} \quad \frac{dN_j}{dx} \rangle \frac{dN_m}{dx} \begin{Bmatrix} Y_i \\ Y_j \end{Bmatrix} + M(x) N_m \right] dx \quad m = i, j. \tag{8.2.8}$$

The element stiffness and force matrices are then calculated as

$$[k]^{(e)} = \frac{EI}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{8.2.9}$$

$$\{f\}^{(e)} = -\frac{L}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} M_i \\ M_j \end{Bmatrix} \tag{8.2.10}$$

where the distribution of $M(x)$ between nodes i and j is approximated by the linear shape functions as $M(x) = N_i(x)M_i + N_j(x)M_j$. The elements are assembled to form the general equilibrium equation as

$$[K]\{Y\} = \{F\}. \tag{8.2.11}$$

With proper boundary conditions, the unknown matrix $\{Y\}$ is determined.

8.3 Euler Beam, Galerkin Formulation

The deflection equation of the beam is

$$EI \frac{d^2 y}{dx^2} = M(x). \tag{8.3.1}$$

Dividing the beam into a number of elements, the deflection of the base element (e) may be approximated with a linear shape function as

$$y^{(e)} = \langle N_i \quad N_j \rangle^{(e)} \begin{Bmatrix} Y_i \\ Y_j \end{Bmatrix} \tag{8.3.2}$$

where N_i and N_j are given by Eq. (8.2.4). Applying the Galerkin method yields

$$\int_0^L \left(EI \frac{d^2 y^{(e)}}{dx^2} - M^{(e)}(x) \right) N_m dx = 0 \qquad m = i, j \tag{8.3.3}$$

The weak formulation of the first term is

$$\int_0^L EI \frac{d^2 y^{(e)}}{dx^2} \, N_m dx = EI \frac{dy^{(e)}}{dx} \, N_m |_0^L - \int_0^L EI \frac{dN_m}{dx} \, \frac{dy^{(e)}}{dx} \, dx \quad m = i, j. \tag{8.3.4}$$

Substituting Eq. (8.3.4) in Eq. (8.3.3), gives

$$EI \frac{dy^{(e)}}{dx} \, N_m |_0^L - \int_0^L \left(EI \frac{dN_m}{dx} \, \frac{dy^{(e)}}{dx} + M^{(e)}(x) \, N_m \right) dx = 0 \quad m = i, j. \tag{8.3.5}$$

Substituting for element deflection from Eq. (8.3.2), we get

$$\begin{aligned} \frac{d}{dx} \{N\} &= \frac{d}{dx} \begin{Bmatrix} 1 - \frac{x}{L} \\ \frac{x}{L} \end{Bmatrix} = \frac{1}{L} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \\ \frac{dy}{dx} &= \frac{d}{dx} \langle N \rangle \{Y\} = \frac{1}{L} \langle -1 \quad 1 \rangle \begin{Bmatrix} Y_i \\ Y_j \end{Bmatrix} \end{aligned} \tag{8.3.6}$$

Using Eq. (8.3.6), the element stiffness and force matrices are

$$\begin{aligned} [k]^{(e)} \{Y\} &= \int_0^L EI \frac{d}{dx} \{N\} \frac{dy}{dx} \, dx = \int_0^L \frac{EI}{L^2} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} < -1 \quad 1 > \begin{Bmatrix} Y_i \\ Y_j \end{Bmatrix} dx \\ &= \frac{EI}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} Y_i \\ Y_j \end{Bmatrix} \\ \{f\}_1^{(e)} &= - \int_0^L \{N\} \, M^{(e)}(x) \, dx = - \int_0^L \{N\} < N > \begin{Bmatrix} M_i \\ M_j \end{Bmatrix} dx \\ &= - \frac{L}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} M_i \\ M_j \end{Bmatrix} \\ \{f\}_2^{(e)} &= EI \{N\} \frac{dy}{dx} \, |_0^L = EI \{N\} \theta |_0^L = EI \begin{Bmatrix} -\theta_0 \\ \theta_L \end{Bmatrix} \end{aligned} \tag{8.3.7}$$

where a linear distribution is assumed for $M^{(e)}(x)$ between nodes i and j. Adding up the stiffness and force matrices, yields the finite element equilibrium equation as

$$[K]\{Y\} = \{F\}_1 + \{F\}_2 \tag{8.3.8}$$

where the definition of the stiffness and force matrices for the base element (e) are given in Eq. (8.3.7). The force matrix is divided into two parts, $\{F\}_1$ and $\{F\}_2$. The force $\{F\}_1$ is related to the bending moment distribution along the element (e), while $\{F\}_2$ is related to the kinematical bounding conditions at the ends of the beam. The kinematic force $\{f\}_2^{(e)}$ contains the slopes of the element at nodes i and j. Due to the continuity of beam, these slopes cancel each other out between any two adjacent elements except the very first and last nodes of the beam or anywhere in the beam where the slope is defined as a given boundary condition. Both matrices are, however, of force type, and thus, the force and kinematical conditions are both placed in the force matrix.

8.4 Axial Vibration of Bars and Beams

Consider a straight bar of cross section A. The bar is subjected to an axial dynamic force $p(t)$. As a result, the axial displacement u is produced in the bar which is a function of position x along the bar and time t, $u = u(x, t)$. The differential equation for the axial vibration of the bar is

$$E\frac{\partial^2 u}{\partial x^2} = \rho\frac{\partial^2 u}{\partial t^2} + p(t) \tag{8.4.1}$$

where E is the elastic modulus and ρ is the mass density. Note that the axial strain in the bar is

$$\epsilon_x = \frac{\partial u}{\partial x} \tag{8.4.2}$$

The kinematical boundary conditions are either of the followings:

$$\begin{aligned} u &= \bar{u} \quad \text{at } x = \bar{L} \\ \epsilon_x &= \frac{\partial u}{\partial x} = \bar{\epsilon_x} \quad \text{at } x = \bar{L} \end{aligned} \tag{8.4.3}$$

where $\bar{L}$ is the position in which the boundary conditions are specified. The first boundary condition is of kinematical type with specified displacement, and the second boundary condition is of the force type, in which $\bar{\epsilon}_x$ is defined and may be related to the applied force on the boundary. The force boundary condition is obtained using Hooke's law in one-dimension as

$$\sigma_x = E\epsilon_x \tag{8.4.4}$$

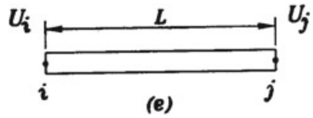


Fig. 8.3 An element of rod

or

$$\bar{p}(t) = A\sigma_x = AE\epsilon_x = AE\bar{\epsilon}_x. \tag{8.4.5}$$

Consider the base element (e) with nodes i and j and length L, as shown in Fig. 8.3. The element shape function is assumed to be linear as

$$u^{(e)} = \langle N_i \quad N_j \rangle \begin{Bmatrix} U_i(t) \\ U_j(t) \end{Bmatrix} \tag{8.4.6}$$

where $U_i(t)$ and $U_j(t)$ are the axial displacements of nodes i and j, and

$$N_i = \frac{x_j - x}{L} \qquad N_j = \frac{x - x_i}{L}. \tag{8.4.7}$$

Applying the Galerkin method, we have

$$\int_{V(e)} \left(E\,\frac{\partial^2 u}{\partial x^2} - \rho\,\frac{\partial^2 u}{\partial t^2} - p \right) N_m dV = 0 \quad m = i, j \tag{8.4.8}$$

The weak formulation of the first term yields

$$\int_{V(e)} \frac{\partial^2 u}{\partial x^2}\, N_m dV = \frac{\partial u}{\partial x}\, N_m A|_0^L - \int_0^L A\frac{\partial u}{\partial x}\,\frac{dN_m}{dx} dx$$

$$\int_{V(e)} \frac{\partial^2 u}{\partial x^2}\, N_m dV = A\epsilon_x N_m|_0^L - \int_0^L A \left\langle \frac{dN_i}{dx} \quad \frac{dN_j}{dx} \right\rangle \begin{Bmatrix} U_i(t) \\ U_j(t) \end{Bmatrix} \times \frac{dN_m}{dx} dx \quad m = i, j \tag{8.4.9}$$

where A is the cross sectional area of the element (e). Thus, the elements of the stiffness matrix are

$$[k_{ij}]^{(e)} = \int_0^L EA \begin{bmatrix} \frac{dN_i}{dx}\,\frac{dN_i}{dx} & \frac{dN_i}{dx}\,\frac{dN_j}{dx} \\ \frac{dN_j}{dx}\,\frac{dN_i}{dx} & \frac{dN_j}{dx}\,\frac{dN_j}{dx} \end{bmatrix} dx. \tag{8.4.10}$$

The mass matrix from Eq. (8.4.9) is

$$\int_0^L \rho\ AN_m \ddot{u} dx = \int_0^L \rho\ AN_m \langle N_i \quad N_j \rangle \begin{Bmatrix} \ddot{U}_i \\ \ddot{U}_j \end{Bmatrix} dx. \tag{8.4.11}$$

Thus, the elements of the mass matrix are

$$[m_{ij}]^{(e)} = \int_0^L \rho\ A \begin{bmatrix} N_i N_i & N_i N_j \\ N_j N_i & N_j N_j \end{bmatrix} dx. \tag{8.4.12}$$

The force matrices from Eqs. (8.4.8) and (8.4.9) are

$$\begin{aligned} \{f_1\}^{(e)} &= EA\epsilon_x \begin{Bmatrix} N_i \\ N_j \end{Bmatrix} |_0^L \\ \{f_2\}^{(e)} &= \int_0^L Ap \begin{Bmatrix} N_i \\ N_j \end{Bmatrix} dx \end{aligned} \tag{8.4.13}$$

where $\{f_1\}$ is the force matrix related to the kinematical boundary conditions and $\{f_2\}$ is the axial force matrix related to the applied force on the beam.

Using the shape functions given by Eq. (8.4.7), the stiffness matrix becomes

$$[k]^{(e)} = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

The mass matrix becomes

$$[m]^{(e)} = \frac{\rho AL}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

and the force matrices are

$$\{f_1\}^{(e)} = EA\epsilon_x \begin{Bmatrix} N_i \\ N_j \end{Bmatrix} |_0^L = \sigma_x A \begin{Bmatrix} N_i \\ N_j \end{Bmatrix} |_0^L$$

Evaluating at nodes 0 and L, yields

$$\{f_1\}^{(e)} = \left[F_x(L) \begin{Bmatrix} N_i(L) \\ N_j(L) \end{Bmatrix} - F_x(0) \begin{Bmatrix} N_i(0) \\ N_j(0) \end{Bmatrix} \right] = \begin{Bmatrix} -F_x(0) \\ F_x(L) \end{Bmatrix}$$

and

$$\{f_2\}^{(e)} = \frac{Ap(t)L}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Once the displacement is obtained, axial stress is calculated as

$$\sigma_x = E\epsilon_x = E\,\frac{\partial u}{\partial x} = E\left\langle \frac{dN_i}{dx} \quad \frac{dN_j}{dx} \right\rangle^{(e)} \begin{Bmatrix} U_i(t) \\ U_j(t) \end{Bmatrix}$$

or

$$\sigma_x = \frac{E}{L}\,[U_j(t) - U_i(t)]^{(e)}$$

8.5 Torsional Vibration of Bars and Beams

Consider a beam under time-dependent torsional moment $T(x, t)$. Calling the angle of twist by θ, the equation of torsional vibration of the beam is

$$G\frac{\partial^2\theta}{\partial x^2} + \frac{T(x,t)}{J} = \rho\frac{\partial^2\theta}{\partial t^2} \tag{8.5.1}$$

where G is the shear modulus, ρ is the mass density, J is the polar moment of inertia of the cross-section, and $T(x, t)$ is the applied torque. The dependent function θ is defined as the angle of twist per unit length of the beam.

The boundary conditions on θ are defined as

$$\begin{aligned} \theta &= \bar{\theta}_0 \qquad & x &= 0 \\ \theta &= \bar{\theta}_L \qquad & x &= L \end{aligned} \tag{8.5.2}$$

where L is the length of the bar. The initial conditions are

$$\begin{aligned} \theta(x, 0) &= \theta_0 \\ \frac{\partial\theta}{\partial t}(x, 0) &= \dot{\theta}_0 \end{aligned} \tag{8.5.3}$$

To apply the Galerkin finite element method, the beam is divided into NE number of elements and NN nodes. The base element (e) may be considered, and with a linear approximation for θ, the shape function is

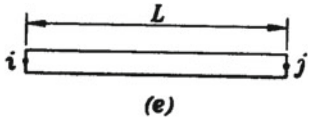
$$\theta^{(e)} = \langle N_i \quad N_j \rangle \begin{Bmatrix} \Theta_i(t) \\ \Theta_j(t) \end{Bmatrix}. \tag{8.5.4}$$

The element with two end nodal points is shown in Fig. 8.4. The Galerkin approximation to Eq. (8.5.1) for the base element (e) yields

$$\int_{V(e)} \left(G\frac{\partial^2\theta}{\partial x^2} + M(x,t) - \rho\frac{\partial^2\theta}{\partial t^2} \right) N_m dV = 0 \quad m = i, j \tag{8.5.5}$$

where $M(x, t) = T(x, t)/J$. The weak formulation of the first term gives

Fig. 8.4 An element of beam



$$\int_{V(e)} G\,\frac{\partial^2\theta}{\partial x^2}\,N_m dV = G\,\frac{\partial\theta}{\partial x}\,N_m A|_0^L - \int_0^L GA\,\frac{\partial\theta}{\partial x}\,\frac{dN_m}{dx}\,dx. \tag{8.5.6}$$

Substituting for $\theta^{(e)}$ from Eq. (8.5.4), gives

$$\int_{V(e)} G\,\frac{\partial^2\theta}{\partial x^2}\,N_m dV = G\,\frac{\partial\theta}{\partial x}\,N_m A|_0^L - \int_0^L GA\left\langle \frac{dN_i}{dx}\;\frac{dN_j}{dx}\right\rangle \frac{dN_m}{dx}\,dx \times \begin{Bmatrix} \Theta_i(t) \\ \Theta_j(t) \end{Bmatrix} \quad m = i, j \tag{8.5.7}$$

where A is the cross-sectional area of the element. Substituting Eq. (8.5.7) in Eq. (8.5.5) gives

$$\int_0^L GA\left\langle \frac{dN_i}{dx}\;\frac{dN_i}{dx}\right\rangle \begin{Bmatrix} \Theta_i(t) \\ \Theta_j(t) \end{Bmatrix} \frac{dN_m}{dx}\,dx + \int_0^L A\rho\langle N_i \quad N_j\rangle N_m dx \times \begin{Bmatrix} \ddot{\Theta}_i(t) \\ \ddot{\Theta}_j(t) \end{Bmatrix} - \int_0^L AM(x,t)N_m dx = GA\,\frac{\partial\theta}{\partial x}\,N_m|_0^L \quad m = i, j. \tag{8.5.8}$$

The term on the right-hand side of Eq. (8.5.8) cancels out between each of the two adjacent elements, due to the natural continuity of θ, except for the first and last nodal points of the beam. This term contributes to the force matrix. Adding up Eq. (8.5.8) for all the elements in the solution domain, results in the finite element equilibrium equation as

$$[M]\{\ddot{\Theta}\} + [K]\{\Theta\} = \{F\} \tag{8.5.9}$$

where the mass, stiffness, and force matrices for the base element (e) are

$$[k_{ij}]^{(e)} = \int_0^L GA \begin{bmatrix} \frac{dN_i}{dx}\,\frac{dN_i}{dx} & \frac{dN_i}{dx}\,\frac{dN_j}{dx} \\ \frac{dN_j}{dx}\,\frac{dN_i}{dx} & \frac{dN_j}{dx}\,\frac{dN_j}{dx} \end{bmatrix} dx \tag{8.5.10}$$

$$[m_{ij}]^{(e)} = \int_0^L A\rho \begin{bmatrix} N_i\,N_i & N_i\,N_j \\ N_j\,N_i & N_j\,N_j \end{bmatrix} dx \tag{8.5.11}$$

$$\{f\}^{(e)} = GA\,\frac{\partial\theta}{\partial x}\left\{\begin{matrix} N_i \\ N_j \end{matrix}\right\}\Bigg|_0^L + \int_0^L AM(x,t)N_m dx. \tag{8.5.12}$$

The stiffness matrix, after substituting for the shape functions and integrating, yields

$$[k]^{(e)} = \frac{GA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

The mass matrix becomes

$$[m]^{(e)} = \frac{\rho AL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

The force matrix, due to the weak formulation, is

$$\{f_1\}^{(e)} = GAN_m\frac{\partial\theta}{\partial x}|_0^L = GA\left\{\begin{matrix} N_i \\ N_j \end{matrix}\right\}\left\langle\begin{matrix} \frac{dN_i}{dx} & \frac{dN_j}{dx} \end{matrix}\right\rangle\left\{\begin{matrix} \Theta_i \\ \Theta_j \end{matrix}\right\}|_0^L$$

or

$$\{f_1\}^{(e)} = GA\begin{bmatrix} N_i\dfrac{dN_i}{dx} & N_i\dfrac{dN_j}{dx} \\ N_j\dfrac{dN_i}{dx} & N_j\dfrac{dN_j}{dx} \end{bmatrix}\left\{\begin{matrix} \Theta_i \\ \Theta_j \end{matrix}\right\}|_0^L$$

Substituting for the shape functions gives

$$\{f_1\}^{(e)} = \frac{GA}{L}\begin{bmatrix} -(1-\frac{x}{L}) & (1-\frac{x}{L}) \\ -\frac{x}{L} & \frac{x}{L} \end{bmatrix}\left\{\begin{matrix} \Theta_i \\ \Theta_j \end{matrix}\right\}|_0^L$$

Finally, the force matrix, due to the weak formulation, becomes

$$\{f_1\}^{(e)} = \frac{GA}{L}\left\{\begin{matrix} \Theta_i - \Theta_j \\ \Theta_j - \Theta_i \end{matrix}\right\}$$

Force, due to the applied torque for constant M in the element, is

$$\{f_2\}^{(e)} = \int_0^L AM(x,t)N_m dx = \frac{AML}{2}\left\{\begin{matrix} 1 \\ 1 \end{matrix}\right\}$$

8.6 Lateral Vibration of Beams

Consider a beam of general cross-section subjected to the lateral dynamic force. The equation for the lateral vibration of the beam is

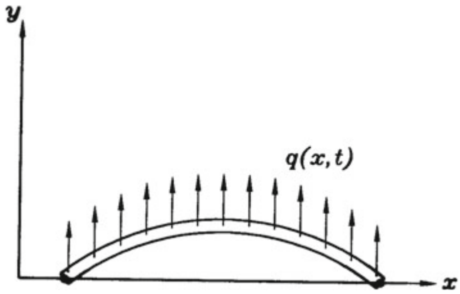


Fig. 8.5 Lateral vibration of beam

$$-\frac{\partial^2}{\partial x^2}\left[EI(x)\,\frac{\partial^2 y(x,t)}{\partial x^2}\right] + q(x,t) = m(x)\,\frac{\partial^2 y(x,t)}{\partial t^2} \tag{8.6.1}$$

where E is the modules of elasticity, $I(x)$ is the cross-sectional area moment of inertia about the z-axis and passing through the center of the cross-section of the beam, $m(x)$ is the mass per unit length, and $q(x,t)$ is the lateral force being a function of position and time, as shown in Fig. 8.5. The lateral deflection of the beam is $y(x,t)$, which is a function of position x and time t. The boundary conditions are:

Clamped edge at $x = 0$

$$y(0,t) = 0 \qquad \frac{\partial y(0,t)}{\partial x} = 0 \tag{8.6.2}$$

Simply supported edge at $x = 0$

$$y(0,t) = 0 \qquad EI(x)\,\frac{\partial^2 y(0,t)}{\partial x^2} = 0 \tag{8.6.3}$$

Free edge at $x = 0$

$$EI(x)\,\frac{\partial^2 y(0,t)}{\partial x^2} = 0 \qquad \frac{\partial}{\partial x}\left[EI(x)\,\frac{\partial^2 y(0,t)}{\partial x^2}\right] = 0. \tag{8.6.4}$$

The initial conditions are the specified initial deflection $y_0(x)$ and velocity $\dot{y}_0(x)$, as

$$y(x,0) = y_0(x) \tag{8.6.5}$$

$$\dot{y}(x,0) = \dot{y}_0(x). \tag{8.6.6}$$

To find the finite element solution, the solution domain is divided into NE elements and NN nodes. Let us consider the case of a beam with constant cross-sectional area, where $I(x) = I$ and $m(x) = \rho$. The base element (e) is considered and the Galerkin method is used, employing the equilibrium Eq. (8.6.1), as

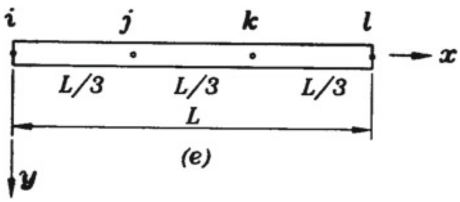


Fig. 8.6 An element of beam with third order polynomial, C^0-continuous

$$\int_V \left[EI \frac{\partial^4 y}{\partial x^4} + \rho \frac{\partial^2 y}{\partial t^2} - q(x,t) \right] N_m dV = 0 \tag{8.6.7}$$

where $dV = Adx$. Since the equilibrium equation has a term with a fourth order partial derivative, the linear shape function for y is node suitable as it all vanishes. A second order polynomial may be appropriate to approximate the lateral deflection of the beam, provided that the first term is considered for two times weak formulations. The weak formulations of the first term of Eq. (8.6.7) is

$$\begin{aligned}\int_0^L \frac{\partial^4 y}{\partial x^4} N_m dx &= \frac{\partial^3 y}{\partial x^3} N_m \,|_0^L - \int_0^L \frac{\partial^3 y}{\partial x^3} \frac{dN_m}{dx} dx \\ &= \frac{\partial^3 y}{\partial x^3} N_m \,|_0^L - \frac{\partial^2 y}{\partial x^2} \frac{dN_m}{dx} \,|_0^L + \int_0^L \frac{\partial^2 y}{\partial x^2} \frac{d^2 N_m}{dx^2} dx.\end{aligned} \tag{8.6.8}$$

Substituting Eq. (8.6.8) in Eq. (8.6.7), gives

$$\begin{aligned}&EIA \int_0^L \frac{\partial^2 y}{\partial x^2} \frac{d^2 N_m}{dx^2} dx + \rho A \int_0^L N_m \frac{\partial^2 y}{\partial t^2} dx \\ &\quad = A \int_0^L q(x,t) N_m dx - EIA \frac{\partial^3 y}{\partial x^3} N_m |_0^L + EIA \frac{\partial^2 y}{\partial x^2} \frac{dN_m}{dx} \,|_0^L.\end{aligned} \tag{8.6.9}$$

Considering a third order polynomial for the shape function of the element (e), as shown in Fig. 8.6. The shape function for a C^0-continuous third order element may be written as

$$y^{(e)} = N_i Y_i + N_j Y_j + N_k Y_k + N_l Y_l \tag{8.6.10}$$

where the shape functions are

$$N_i = 1 - \frac{11}{2} \frac{x}{L} + 9 \left(\frac{x}{L}\right)^2 - \frac{9}{2} \left(\frac{x}{L}\right)^3$$
$$N_j = 9 \frac{x}{L} - \frac{45}{2} \left(\frac{x}{L}\right)^2 + \frac{27}{2} \left(\frac{x}{L}\right)^3$$

$$N_k = -\frac{9}{2}\,\frac{x}{L} + 18\,\left(\frac{x}{L}\right)^2 - \frac{27}{2}\,\left(\frac{x}{L}\right)^3$$
$$N_l = \frac{x}{L} - \frac{9}{2}\,\left(\frac{x}{L}\right)^2 + \frac{9}{2}\,\left(\frac{x}{L}\right)^3 \tag{8.6.11}$$

Substituting Eq. (8.6.10) in Eq. (8.6.9), the stiffness matrix for the element (e) is

$$\begin{aligned}[k]^{(e)}\{Y(t)\} &= EIA\int_0^L \frac{d^2y}{dx^2}\,\frac{d^2N_m}{dx^2}dx \\ &= \left(EIA\int_0^L \left\{\frac{d^2N_s}{dx^2}\right\}\left\langle\frac{d^2N_m}{dx^2}\right\rangle dx\right)\{Y(t)\}\end{aligned} \tag{8.6.12}$$

where $s, m = i,\ j, k, l$, or

$$[k]^{(e)} = EIA\int_0^L \begin{bmatrix} \frac{d^2N_i}{dx^2}\,\frac{d^2N_i}{dx^2} & \frac{d^2N_i}{dx^2}\,\frac{d^2N_j}{dx^2} & \frac{d^2N_i}{dx^2}\,\frac{d^2N_k}{dx^2} & \frac{d^2N_i}{dx^2}\,\frac{d^2N_l}{dx^2} \\ \frac{d^2N_j}{dx^2}\,\frac{d^2N_i}{dx^2} & \frac{d^2N_j}{dx^2}\,\frac{d^2N_j}{dx^2} & \frac{d^2N_j}{dx^2}\,\frac{d^2N_k}{dx^2} & \frac{d^2N_j}{dx^2}\,\frac{d^2N_l}{dx^2} \\ \frac{d^2N_k}{dx^2}\,\frac{d^2N_i}{dx^2} & \frac{d^2N_k}{dx^2}\,\frac{d^2N_j}{dx^2} & \frac{d^2N_k}{dx^2}\,\frac{d^2N_k}{dx^2} & \frac{d^2N_k}{dx^2}\,\frac{d^2N_l}{dx^2} \\ \frac{d^2N_l}{dx^2}\,\frac{d^2N_i}{dx^2} & \frac{d^2N_l}{dx^2}\,\frac{d^2N_j}{dx^2} & \frac{d^2N_l}{dx^2}\,\frac{d^2N_k}{dx^2} & \frac{d^2N_l}{dx^2}\,\frac{d^2N_l}{dx^2} \end{bmatrix} dx. \tag{8.6.13}$$

Substituting for the shape functions from Eq. (8.6.11) in Eq. (8.6.13) and carrying out the integrations, yields

$$[k]^{(e)} = \frac{EIA}{L^3}\begin{bmatrix} 81 & -\frac{405}{2} & 162 & -\frac{81}{2} \\ -\frac{405}{2} & 567 & -\frac{1053}{2} & 162 \\ 162 & -\frac{1053}{2} & 567 & -\frac{405}{2} \\ -\frac{81}{2} & 162 & -\frac{405}{2} & 81 \end{bmatrix}$$

The elements of the mass matrix are

$$[m]^{(e)}\{\ddot{Y}\} = \rho A\int_0^L N_m\,\frac{\partial^2 y}{\partial t^2}dx = \left(\rho A\int_0^L \{N\}\langle N\rangle dx\right)\{\ddot{Y}\} \tag{8.6.14}$$

or

$$[m]^{(e)} = \rho A\int_0^L \begin{bmatrix} N_iN_i & N_iN_j & N_iN_k & N_iN_l \\ N_jN_i & N_jN_j & N_jN_k & N_jN_l \\ N_kN_i & N_kN_j & N_kN_k & N_kN_l \\ N_lN_i & N_lN_j & N_lN_k & N_lN_l \end{bmatrix} dx \tag{8.6.15}$$

Substituting for the shape functions from Eq. (8.6.11), the mass matrix becomes

$$[m]^{(e)} = \rho A L \begin{bmatrix} \frac{8}{105} & \frac{33}{560} & -\frac{3}{140} & \frac{19}{1680} \\ \frac{33}{560} & \frac{27}{70} & -\frac{27}{560} & -\frac{3}{140} \\ -\frac{3}{140} & -\frac{27}{560} & \frac{27}{70} & \frac{33}{560} \\ \frac{19}{1680} & -\frac{3}{140} & \frac{33}{560} & \frac{8}{105} \end{bmatrix}.$$

The force matrices are

$$\{f\}^{(e)} = \{f_q\} + \{f_s\} + \{f_b\} \tag{8.6.16}$$

where $\{f_q\}$ is the force due to the lateral applied force $q(x,t)$ on the element and is

$$\{f_q\}^{(e)} = A \int_0^L q(x,t)\{N\}dx. \tag{8.6.17}$$

The force matrix $\{f_s\}$ is the boundary force due to the applied shear force on the element boundary. This force is cancelled out between all the adjacent elements, except the first node of the first element F_{s1} and the last node of the last element of the solution domain F_{sNN} and is written for the whole solution domain (after element assembly) as

$$\langle F_s \rangle = \langle F_{s1} 0\ 0\ 0\ \cdots\ 0\ F_{sNN} \rangle \tag{8.6.18}$$

provided that the applied concentrated external shear force on the beam is zero everywhere, except at the end boundaries. In this case, if F_{s1} and F_{sNN} are the boundary shear forces, then

$$\begin{aligned} F_{s1} &= -EIA\frac{\partial^3 y}{\partial x^3} \qquad \text{at} \quad x = 0 \\ F_{sNN} &= +EIA\frac{\partial^3 y}{\partial x^3} \qquad \text{at} \quad x = L \end{aligned} \tag{8.6.19}$$

where L is the total length of the beam. Similarly, the bending force $\{f_b\}$ is the boundary force due to the applied external moment on the element boundary. These forces will also cancel each other out between any two adjacent elements, except where a local bending moment at a node is applied, where the bending moment is then

$$f_b = EIA \left. \frac{\partial^2 y}{\partial x^2} \frac{dN_m}{dx} \right|_{x=0 \text{ or } L}. \tag{8.6.20}$$

Now, consider a C^1-continuous element, where at two end nodal points the deflection and slope of the beam are unknowns, as shown in Fig. 8.7. The third order shape function is

$$y^{(e)} = a + bx + cx^2 + dx^3. \tag{8.6.21}$$

The nodal degrees of freedom are

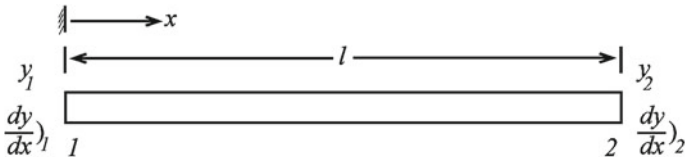


Fig. 8.7 C^1-continuous beam element

$$\begin{cases} y = y_1 \\ x = 0 \end{cases} \qquad \begin{cases} \left(\dfrac{dy}{dx} = \dfrac{dy}{dx}\right)_1 = \theta_1 \\ x = 0 \end{cases} \qquad \begin{cases} y = y_2 \\ x = L \end{cases} \qquad \begin{cases} \left(\dfrac{dy}{dx} = \dfrac{dy}{dx}\right)_2 = \theta_2 \\ x = L \end{cases} . \tag{8.6.22}$$

Substituting in Eq. (8.6.21) gives

$$y^{(e)} = N_1 Y_1 + N_2 \theta_1 + N_3 Y_2 + N_4 \theta_2 \tag{8.6.23}$$

where

$$\begin{aligned} N_1 &= 1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3} \qquad & N_2 &= x - \frac{2x^2}{L} + \frac{x^3}{L^2} \\ N_3 &= \frac{3x^2}{L^2} - \frac{2x^3}{L^3} \qquad & N_4 &= -\frac{x^2}{L} + \frac{x^3}{L^2}. \end{aligned} \tag{8.6.24}$$

Substituting for the shape functions from Eqs. (8.6.24) in Eq. (8.6.13) and integrating, the stiffness matrix becomes

$$[k]^{(e)} = \frac{EIA}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}.$$

The mass matrix from Eqs. (8.6.24) and (8.6.15) becomes

$$[m]^{(e)} = \rho AL \begin{bmatrix} \frac{13}{35} & \frac{11}{210}L & \frac{9}{70} & -\frac{43}{420}L \\ \frac{11}{210}L & \frac{1}{105}L^2 & \frac{13}{420}L & -\frac{11}{420}L^2 \\ \frac{9}{70} & \frac{13}{420}L & \frac{13}{35} & -\frac{101}{210}L \\ -\frac{43}{420}L & -\frac{11}{420}L^2 & -\frac{101}{210}L & \frac{71}{105}L^2 \end{bmatrix}.$$

The force matrices are evaluated similar to Eqs. (8.6.17), (8.6.18), and (8.6.20).

It is interesting to note that the Galerkin method provides a complete and flexible model, when proper weak formulations are considered. When, for example, the con-

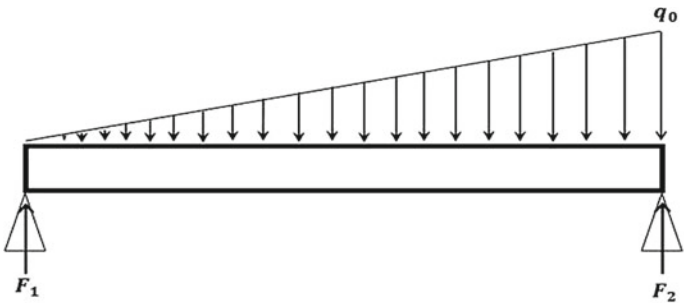


Fig. 8.8 Simply supported beam under linear load

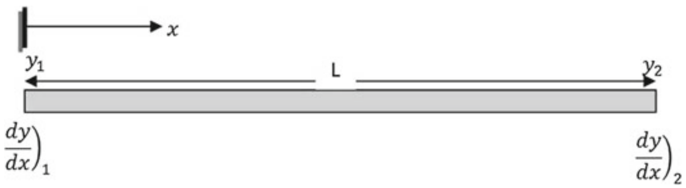


Fig. 8.9 C^1 continuous element

centrated sheer forces or bending moments are present on a beam under lateral vibration, they are directly considered through the boundary forces (8.6.19) and (8.6.22). When these boundary forces are absent, the terms obtained by the weak formulations are moved into the proper places in the stiffness matrix (Figs. 8.8 and 8.9).

Example 1 As an example, consider a simply supported beam under a linear lateral distributed load (q), as shown in Fig. 8.1. The following numerical values are assumed:

$$\begin{aligned} q_0 &= 1 \quad \frac{N}{m} \\ \text{Total length} &= 1 \quad m \\ I &= 1 \quad mm^4 \\ E &= 1 \quad (GPa) \\ \text{Number of elements (NE)} &= 4 \end{aligned}$$

Using a C^1-continuous element and the Galerkin method, divide the beam into four elements and obtain the elements of the global stiffness and force matrices.

Solution: Each of the following equations may be used to employ the Galerkin method

$$M(x) = EI\frac{d^2y}{dx^2} \tag{8.6.25}$$

$$V(x) = EI\frac{d^3y}{dx^3} \tag{8.6.26}$$

$$q(x) = EI\frac{d^4y}{dx^4} \tag{8.6.27}$$

The last equation of beam deflection under linear distributed load q is written in the form

$$\frac{d^2}{dx^2}\left(EI\frac{d^2y}{dx^2}\right) = q(x). \tag{8.6.28}$$

To solve the equation by the finite element method, the solution domain is discredited into NE elements and NN nodes. Then, in each element a shape function N_i is used to approximate the lateral deflection.

$$y(x) = \langle N_i(x)\rangle^e \{Y_i\}^e. \tag{8.6.29}$$

Equation (8.6.28) is written in the form

$$\frac{d^2}{dx^2}\left(EI\frac{d^2y}{dx^2}\right) - q(x) = R(x) \tag{8.6.30}$$

where $R(x)$ is the residue. Applying the Galerkin method, the residue is made orthogonal with respect to the approximating shape function N_i as

$$\int N_i(x)R(x)dV = 0. \tag{8.6.31}$$

We write this equation for the solution domain with NE number of elements as

$$\sum_{e=1}^{e=NE} \oint_e \langle N_i(x)\rangle^e R(x)dV = 0. \tag{8.6.32}$$

Substituting Eq. (8.6.28) into Eq. (8.6.32) gives

$$\sum_{e=1}^{e=NE} \oint_e N_i(x)\left[\frac{d^2}{dx^2}\left(EI\frac{d^2y}{dx^2}\right) - q(x)\right]dV = 0 \tag{8.6.33}$$

and

$$dV = Adx$$

where A is the element's area. To obtain the kinematical boundary conditions, weak formulation is applied:

$$\int_0^{L_e} N_i(x)^e \frac{d^4y}{dx^4}dx = \frac{d^3y}{dx^3}N_i(x)^e|_0^{L_e} - \int_0^{L_e}\left(\frac{d^3y}{dx^3}\frac{dN_i(x)^e}{dx}\right)dx$$

and finally

$$\int_0^{L_e} N_i(x)^e \frac{d^4y}{dx^4}dx = \frac{d^3y}{dx^3}N_i(x)^e|_0^{L_e} - \frac{d^2y}{dx^2}\frac{dN_i(x)^e}{dx}|_0^{L_e} + \int_0^{L_e}\left(\frac{d^2y}{dx^2}\frac{d^2N_i^e(x)}{dx^2}\right)dx \tag{8.6.34}$$

Substitution in Eq. (8.6.33) gives

$$EIA\int_0^{L_e}\frac{d^2y}{dx^2}\frac{d^2N_i^e(x)}{dx^2}dx = A\int_0^{L_e} q(x)N_i(x)^e dx - EIA\frac{d^3y}{dx^3}N_i(x)^e|_0^{L_e} + EIA\frac{d^2y}{dx^2}\frac{dN_i(x)^e}{dx}|_0^{L_e} \tag{8.6.35}$$

Dividing through by A and substituting for y from Eq. (8.6.29) yields

$$EI\int_0^{L_e}\left\{\frac{d^2N_m(x)^e}{dx^2}\right\}\langle\frac{d^2N_i(x)^e}{dx^2}\rangle dx = \int_0^{L_e} q(x)N_i(x)^e dx - EI\frac{d^3y}{dx^3}N_i(x)^e|_0^{L_e} + EI\frac{d^2y}{dx^2}\frac{dN_i(x)^e}{dx}|_0^{L_e} m, i = 1, 2, 3, \ldots., NN \tag{8.6.36}$$

Assembling this equation for all the members in the solution domain, we arrive at the final finite element equilibrium equation

$$[K]\{X\} = \{f\} \tag{8.6.37}$$

where the stiffness and force matrices of the base element (e) are

$$\begin{aligned}
[K]^e &= \int EI\{\frac{d^2N_m(x)^e}{dx^2}\}\frac{d^2N_i(x)^e}{dx^2}\rangle dx \\
\{f\}_q^e &= \int \{N_i(x)^e\}q(x)dx \\
\{f\}_{shear}^e &= \frac{d^3y}{dx^3}N_i(x)^e|_0^{L_e} \\
\{f\}_{moment}^e &= EIA\frac{d^2y}{dx^2}\frac{dN_i(x)^e}{dx}|_0^{L_e}
\end{aligned} \tag{8.6.38}$$

where $[K]^e$ and $\{f\}^e$ are the element stiffness and force matrices, respectively. The last two equations denote the natural boundary conditions, describing the shear and bending moment boundary forces. These two boundary forces appear at the very ends of the beam, where the shear and bending moment are given. They may also appear between the end edges of the beam, where shear force or bending moments are externally applied on the beam.

In this example, a cubic C^1 continuous element is considered, where nodal points of the element deflection and slope at two ends are defined as the nodal degrees of freedom. A third order polynomial is considered as

$$y(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3.$$

Evaluating the constants α_n in terms of the nodal variables give

$$Y(x) = \langle N_1^e \quad N_2^e \quad N_3^e \quad N_4^e \rangle . \begin{Bmatrix} y_1 \\ y_1' \\ y_2 \\ y_2' \end{Bmatrix} \tag{8.6.39}$$

where the shape functions are

$$\begin{aligned} N_1(x) &= 1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3} \quad N_2(x) = x - \frac{2x^2}{L} + \frac{x^3}{L^2} \\ N_3(x) &= \frac{3x^2}{L^2} - \frac{2x^3}{L^3} \quad N_4(x) = -\frac{x^2}{L} + \frac{x^3}{L^2}. \end{aligned} \tag{8.6.40}$$

The transformation from the global coordinate x to the local coordinate ξ is

$$\xi = \frac{x}{L}. \tag{8.6.41}$$

The shape functions in terms of the local coordinate become

$$\begin{aligned} N_1(\xi) &= 2\xi^3 - 3\xi^2 + 1 \quad N_2(\xi) = (\xi^3 - 2\xi^2 + \xi)L \\ N_3(\xi) &= 3\xi^2 - 2\xi^3 \quad N_4(x) = (\xi^3 - \xi^2)L. \end{aligned} \tag{8.6.42}$$

and

$$dx = L d\xi$$

Thus,

$$\frac{d^2 N_i(x)^e}{dx^2} = \frac{1}{L^2} \begin{bmatrix} (12\xi - 6) \\ L(6\xi - 2) \\ -(12\xi - 6) \\ L(6\xi - 2) \end{bmatrix}. \tag{8.6.43}$$

Considering the following integral

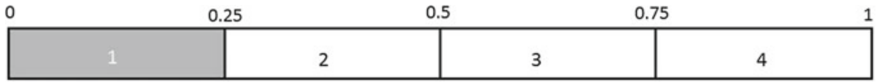


Fig. 8.10 Beam with 4 elements

$$\int_{L_e} \xi^m d\xi = \frac{L_e}{m+1} \tag{8.6.44}$$

and substituting the shape functions and their derivatives from Eqs. (8.6.42) and (8.6.43) into Eq. (8.6.38), and evaluating the integrals according to Eq. (8.6.44), the stiffness matrix becomes

$$[K]^e = \frac{EIA}{L_e^3} \begin{pmatrix} 12 & 6L_e & -12 & 6L_e \\ 6L_e & 4L_e^2 & -6L_e & 2L_e^2 \\ -12 & -6L_e & 12 & -6L_e \\ 6L_e & 2L_e^2 & -6L_e & 4L_e^2 \end{pmatrix}.$$

For the linear variation of $q(x)$ in the base element (e) from q_1^e to q_2^e, we have

$$q(x) = (1 - \frac{x}{L_e})q_1^e + \frac{x}{L_e}q_1^e = [(1-\xi) \quad \xi] \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}. \tag{8.6.45}$$

Therefore, the force matrix becomes

$$\{F_q\}^e = \int \begin{Bmatrix} 1 - 3\xi^2 + 2\xi^3 \\ L_e(\xi - 2\xi^2 + \xi^3) \\ 3\xi^2 - 2\xi^3 \\ L_e(\xi^3 - \xi^2) \end{Bmatrix} [1-\xi \quad \xi] d\xi \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

$$\{F\}_q^e = \frac{L_e}{20} \begin{Bmatrix} 7 & 3 \\ L_e & 2\frac{L_e}{3} \\ 3 & 7 \\ -2\frac{L_e}{3} & -L_e \end{Bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}.$$

Using the numerical values of the given problem, we find:
Element stiffness matrix:

$$[K]^e = \begin{bmatrix} 768 & 96 & -768 & 96 \\ 96 & 16 & -96 & 8 \\ -768 & -96 & 768 & -96 \\ 96 & 8 & -96 & 16 \end{bmatrix}$$

The global stiffness matrix for four elements, as shown in Fig. 8.10, is

$$[K]^{global} = \begin{bmatrix} 768 & 96 & -768 & 96 & 0 & 0 & 0 & 0 & 0 & 0 \\ 96 & 16 & -96 & 8 & 0 & 0 & 0 & 0 & 0 & 0 \\ -768 & -96 & 1536 & 0 & -768 & 96 & 0 & 0 & 0 & 0 \\ 96 & 8 & 0 & 32 & -96 & 8 & 0 & 0 & 0 & 0 \\ 0 & 0 & -768 & -96 & 1536 & 0 & -768 & 96 & 0 & 0 \\ 0 & 0 & 96 & 8 & 0 & 32 & -96 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 & -768 & -96 & 1536 & 0 & -768 & 96 \\ 0 & 0 & 0 & 0 & 96 & 8 & 0 & 32 & -96 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & -768 & -96 & 68 & -96 \\ 0 & 0 & 0 & 0 & 0 & 0 & 96 & 8 & -96 & 16 \end{bmatrix}.$$

The element force matrix:

$$[F]^{element} = \begin{Bmatrix} -0.0094 \\ -0.0005 \\ -0.0219 \\ 0.0008 \end{Bmatrix}.$$

The global force matrix for four elements is

$$[F]^{global} = \begin{Bmatrix} -0.0094 \\ -0.0005 \\ -0.0625 \\ -0.0010 \\ -0.125 \\ -0.0010 \\ -0.1875 \\ -0.0010 \\ -0.1156 \\ 0.0188 \end{Bmatrix}.$$

Solving the finite element matrix equation, the following results are obtained:

$$y = \begin{Bmatrix} y_1 \\ \theta_1 \\ y_2 \\ \theta_2 \\ y_3 \\ \theta_3 \\ y_4 \\ \theta_4 \\ y_5 \\ \theta_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -0.0194 \\ -0.0044 \\ -0.0144 \\ -0.0065 \\ -0.0012 \\ -0.0048 \\ 0.0142 \\ 0 \\ 0.0222 \end{Bmatrix}.$$

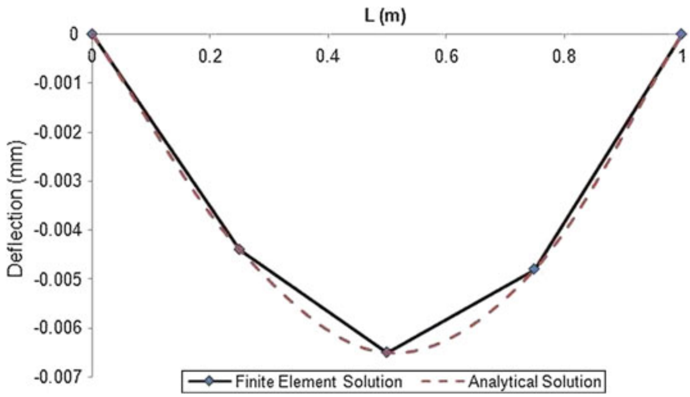


Fig. 8.11 Deflection of beam under linear load

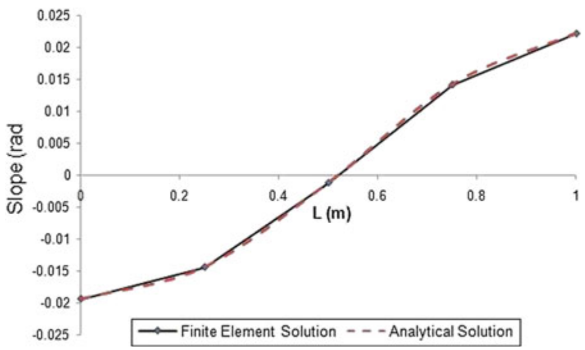


Fig. 8.12 Slop of beam under linear load

The deflection and slope of the beam under the lateral static load, using the finite element solution, are shown in Figs. 8.11 and 8.12. The results are compared with the analytical solution, in which the plots are closely related.

8.7 Timoshenko Beam

According to the Timoshenko beam theory, a beam subjected to the lateral load and bending moment distribution along the length deforms such that its plane sections remain plane after deformation, but not necessarily perpendicular to the longitudinal axis of the beam. This assumption results in a situation in which the transverse shear deformation exists, contrary to the Euler beam theory, in which this assumption was neglected.

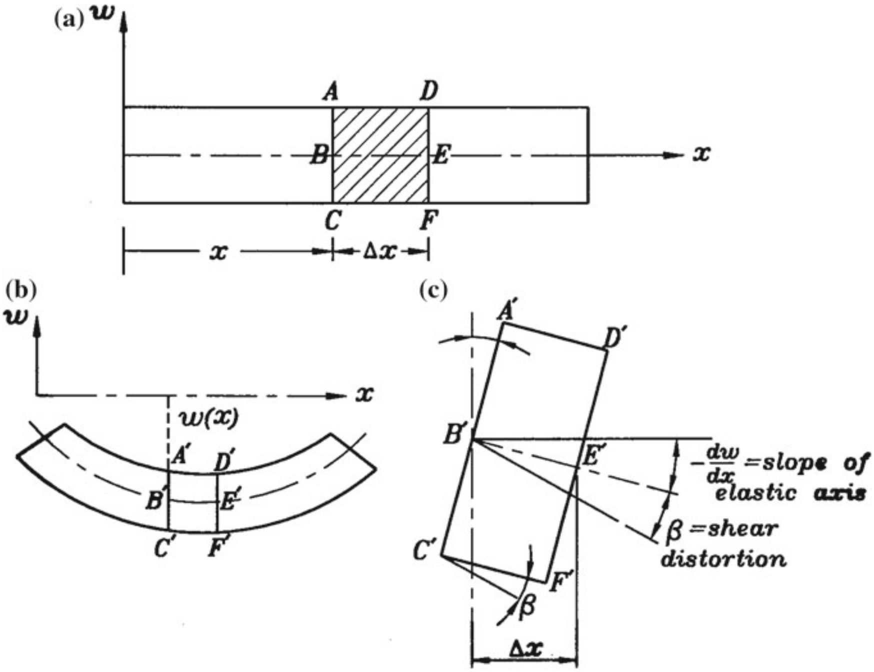


Fig. 8.13 Deformation of a Timoshenko beam element

Consider a beam as shown in Fig. 8.13a. The lateral deflection is shown by w. An element of the beam at distance x from the origin is shown by length Δx. After lateral deformation of the beam, the element is deformed downward by amount $w(x)$ to the position shown in Fig. 8.13b. The straight plane section ABC of Fig. 8.13a, which was originally perpendicular to the beam axis, is deformed into the straight line $A'B'C'$ which is not perpendicular to the beam axis and is rotated at angle $\psi(x)$. The angle $\psi(x)$ is not equal to the slope of the beam dw/dx, and thus, the line $A'B'C'$ is not perpendicular to the beam axis. The difference $\psi - (-dw/dx)$ is a measure of shear stress and is related to the shear deformation. The components of displacement across the beam height are

$$
\begin{aligned}
u &= -z\psi(x)\\
v &= 0\\
w &= w(x).
\end{aligned}
\tag{8.7.1}
$$

The nonzero stresses are

$$\sigma_{xx} = -Ez\,\frac{d\psi}{dx}$$
$$\tau_{zx} = Gk\,(-\psi + \frac{dw}{dx}) \tag{8.7.2}$$

where G is the shear modulus and k is a constant.

The moment M and shear force V are related to the normal stress σ_{xx} and shear stress τ_{zx} by

$$M = \int_A \sigma_{xx} z dA = -EI\,\frac{d\psi}{dx}$$
$$V = \int_A \tau_{zx} dA = kGA\left(-\psi + \frac{dw}{dx}\right). \tag{8.7.3}$$

From the elementary beam theory, the shear force and bending moment are given as

$$\frac{dV}{dx} = -f(x)$$
$$\frac{dM}{dx} = V \tag{8.7.4}$$

where $f(x)$ is the distribution of the lateral load per unit length. Substituting Eqs. (8.7.3) in (8.7.4) yields

$$\frac{d}{dx}\left[kGA\left(-\psi + \frac{dw}{dx}\right)\right] + f(x) = 0 \tag{8.7.5}$$
$$\frac{d}{dx}\left(EI\,\frac{d\psi}{dx}\right) + kGA\left(-\psi + \frac{dw}{dx}\right) = 0 \tag{8.7.6}$$

The system of ordinary differential equations for the functions ψ and w must be solved simultaneously to give the static distributions of these functions along the axis of beam.

When the applied lateral force is a function of time, $f = f(x, t)$, the beam vibrates and the inertia term must be included in the governing equations. The system of equations governing the lateral vibration of the Timoshenko beam are

$$\frac{\partial}{\partial x}\left[kGA\left(-\psi + \frac{\partial w}{\partial x}\right)\right] + f(x, t) = m\ddot{w} \tag{8.7.7}$$
$$\frac{\partial}{\partial x}\left(EI\frac{\partial \psi}{\partial x}\right) + kAG\left(-\psi + \frac{\partial w}{\partial x}\right) = J\ddot{\psi} \tag{8.7.8}$$

where m is the mass of the beam per unit length and J is the mass moment of inertia per unit length. When the mechanical and geometrical properties are constant along the beam, the system of Eqs. (8.7.7) and (8.7.8) become

$$-kAG\,\frac{\partial\psi}{\partial x}+kGA\,\frac{\partial^2 w}{\partial x^2}+f(x,t)-m\ddot{w}=0 \tag{8.7.9}$$

$$EI\,\frac{\partial^2\psi}{\partial x^2}-kAG\psi+kAG\,\frac{\partial w}{\partial x}-J\ddot{\psi}=0. \tag{8.7.10}$$

The finite element model of the beam is prepared by dividing the beam into a number of elements. For the base element (e), the approximating shape functions for the dependent functions ψ and w are

$$\psi^*=\sum_{i=1}^{m}\phi_i(x)\psi_i(t) \tag{8.7.11}$$

$$w^*=\sum_{i=1}^{n}N_i(x)w_i(t) \tag{8.7.12}$$

Substituting the shape functions for ψ^* and w^* in Eqs. (8.7.9) and (8.7.10), the residues are obtained. The residues are then made orthogonal with respect to the shape functions according to the standard Galerkin method as

$$\int_{V(e)}\left[kAG\,\frac{\partial^2 w^*}{\partial x^2}-kAG\,\frac{\partial\psi^*}{\partial x}+f(x,t)-m\ddot{w}^*\right]N_i dx=0 \tag{8.7.13}$$

$$\int_{V(e)}\left[EI\,\frac{\partial^2\psi^*}{\partial x^2}-kAG\psi^*+kAG\,\frac{\partial w^*}{\partial x}-J\ddot{\psi}^*\right]\phi_i dx=0. \tag{8.7.14}$$

The weak formulation of term $\frac{\partial^2 w^*}{\partial x^2}$ in Eq. (8.7.13) and term $\frac{\partial^2\psi^*}{\partial x^2}$ in Eq. (8.7.14) yield

$$kAG\,\frac{\partial w^*}{\partial x}\,N_i\Big|_0^L-kAG\int_0^L\frac{\partial w^*}{\partial x}\,\frac{dN_i}{dx}dx-kAG\psi^*N_i\big|_0^L+kAG\int_0^L\frac{dN_i}{dx}\,\psi^*dx$$
$$+\int_0^L f(x,t)N_i dx-m\,\frac{d^2}{dt^2}\int_0^L w^*N_i dx=0 \tag{8.7.15}$$

$$EI\,\frac{\partial\psi^*}{\partial x}\,\phi_i\Big|_0^L-EI\int_0^L\frac{\partial\psi^*}{\partial x}\,\frac{d\phi_i}{dx}\,dx-kAG\int_0^L\psi^*\phi_i dx$$
$$+kAG\int_0^L\frac{\partial w^*}{\partial x}\,\phi_i dx-J\,\frac{d^2}{dt^2}\int_0^L\psi^*\phi_i dx=0. \tag{8.7.16}$$

Substituting for w^* and ψ^* from Eqs. (8.7.11) and (8.7.12) yields

$$m\ddot{w}_j\int_0^L N_jN_i dx+kAGw_j\int_0^L\frac{dN_j}{dx}\,\frac{dN_i}{dx}dx-kAG\psi_j\int_0^L\phi_j\frac{dN_i}{dx}dx$$

$$= \int_0^L f(x,t) N_i dx + kAG \left. \frac{\partial \omega^*}{\partial x} N_i \right|_0^L + kAG\psi^* N_i|_0^L \tag{8.7.17}$$

$$J\ddot{\psi}_j \int_0^L \phi_j \phi_i dx + EI\psi_j \int_0^L \frac{d\phi_j}{dx}\frac{d\phi_i}{dx} dx + kAG\psi_j \int_0^L \phi_j \phi_i dx$$
$$- kAGw_j \int_0^L \frac{dN_j}{dx} \phi_i dx = EI \left. \frac{\partial \psi^*}{\partial x} \phi_i \right|_0^L . \tag{8.7.18}$$

The system of Eqs. (8.7.17) and (8.7.18) in matrix form, when all elements in the solution domain are considered and their associated matrices are assembled, is written as

$$[M]\{\ddot{w}\} + [K_1]\{w\} + [K_2]\{\psi\} = \{F_1\} + \{F_1^{BC}\} \tag{8.7.19}$$
$$[J]\{\ddot{\psi}\} + [K_3]\{\psi\} + [K_4]\{w\} = \{F_2^{BC}\} \tag{8.7.20}$$

where the submatrices for the base element (e) are

$$[m]_{ij}^{(e)} = m \int_0^L N_i N_j dx$$
$$[k_1]_{ij}^{(e)} = kAG \int_0^L \frac{dN_i}{dx}\frac{dN_j}{dx} dx$$
$$[k_2]_{ij}^{(e)} = kAG \int_0^L \frac{dN_i}{dx} \phi_j dx$$
$$\{f_1\}^{BC} = KAG(\frac{\partial \omega^*}{\partial x} N_i - \psi^* N_i)_0^L$$
$$\{f_1^{BC}\}_i^{(e)} = kAG(\frac{dw^*}{dx} N_i + \psi^* N_i)\Big|_0^L$$
$$[j]_{ij}^{(e)} = J \int_0^L \phi_i \phi_j dx$$
$$[k_3]_{ij}^{(e)} = EI \int_0^L \frac{d\phi_i}{dx}\frac{d\phi_j}{dx} dx + kAG \int_0^L \phi_i \phi_j dx$$
$$[k_4]_{ij}^{(e)} = -kAG \int_0^L \phi_i \frac{dN_j}{dx} dx$$
$$\{f_2\}^{BC} = EI \left. \frac{\partial \psi^*}{\partial x} \phi_i \right|_0^L . \tag{8.7.21}$$

Using a proper dynamic algorithm for the solution, the system of Eqs. (8.7.19) and (8.7.20) is solved in the time domain. The initial and boundary conditions must be known for input into the dynamic algorithm.

Now, by assuming linear shape functions for the unknown variables in Eqs. (8.7.11) and (8.7.12), the values of the element matrices are evaluated. The shape functions $\phi_i(x)$ and $N_i(x)$ are

$$N_i = \begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix}, \qquad N_1 = \frac{x_2 - x}{L}, \qquad N_2 = \frac{x - x_1}{L} \tag{8.7.22}$$

$$\phi_i = \begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix}, \qquad \phi_1 = \frac{x_2 - x}{L}, \qquad \phi_2 = \frac{x - x_1}{L}. \tag{8.7.23}$$

By using these shape functions, the element mass matrices in Eq. (8.7.21) become

$$\begin{aligned}
[M]_{ij}^{(e)} &= m \int_0^L \begin{bmatrix} N_1^2 & N_1 N_2 \\ N_1 N_2 & N_2^2 \end{bmatrix} dx = \frac{mL}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \\
[J]_{ij}^{(e)} &= J \int_0^L \begin{bmatrix} \phi_1^2 & \phi_1 \phi_2 \\ \phi_1 \phi_2 & \phi_2^2 \end{bmatrix} dx = \frac{JL}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.
\end{aligned} \tag{8.7.24}$$

The element stiffness matrices in the equation become

$$\begin{aligned}
[K_1]_{ij}^{(e)} &= kAG \int_0^L \begin{bmatrix} \frac{dN_1}{dx}\frac{dN_1}{dx} & \frac{dN_1}{dx}\frac{dN_2}{dx} \\ \frac{dN_2}{dx}\frac{dN_1}{dx} & \frac{dN_2}{dx}\frac{dN_2}{dx} \end{bmatrix} dx = \frac{kAG}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \\
[K_2]_{ij}^{(e)} &= kAG \int_0^L \begin{bmatrix} \frac{dN_1}{dx}\phi_1 & \frac{dN_1}{dx}\phi_2 \\ \frac{dN_2}{dx}\phi_1 & \frac{dN_2}{dx}\phi_2 \end{bmatrix} dx = \frac{kAG}{2} \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \\
[K_3]_{ij}^{(e)} &= EI \int_0^L \begin{bmatrix} \frac{d\phi_1}{dx}\frac{d\phi_1}{dx} & \frac{d\phi_1}{dx}\frac{d\phi_2}{dx} \\ \frac{d\phi_2}{dx}\frac{d\phi_1}{dx} & \frac{d\phi_2}{dx}\frac{d\phi_2}{dx} \end{bmatrix} dx - kAG \int_0^L \begin{bmatrix} \phi_1^2 & \phi_1\phi_2 \\ \phi_1\phi_2 & \phi_2^2 \end{bmatrix} dx \\
&= \frac{EI}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - \frac{kAGL}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \\
[K_4]_{ij}^{(e)} &= kAG \int_0^L \begin{bmatrix} \phi_1 \frac{dN_1}{dx} & \phi_1 \frac{dN_2}{dx} \\ \phi_2 \frac{dN_1}{dx} & \phi_2 \frac{dN_2}{dx} \end{bmatrix} dx = \frac{kAG}{2} \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}.
\end{aligned} \tag{8.7.25}$$

The system of equations for the base element (e) can be written as

$$[M']^{(e)}\{\ddot{X}\} + [K']^{(e)}\{X\} = \{F'\}^{(e)} \tag{8.7.26}$$

where

$$\{X\}^T = \{w_1,\ \psi_1,\ w_2,\ \psi_2\}^T. \tag{8.7.27}$$

In this notation, the element matrices are

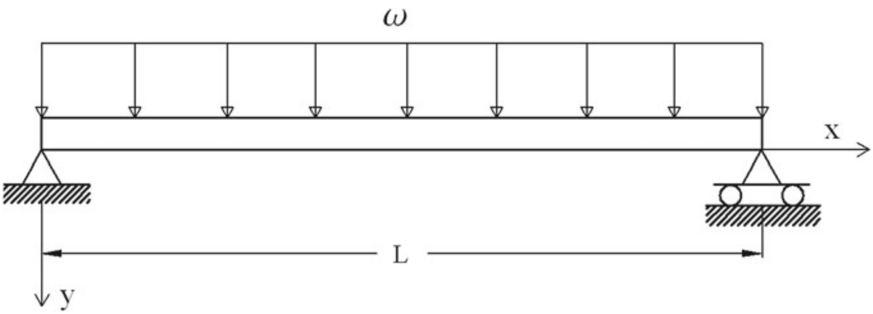


Fig. 8.14 Beam under uniform distributed load

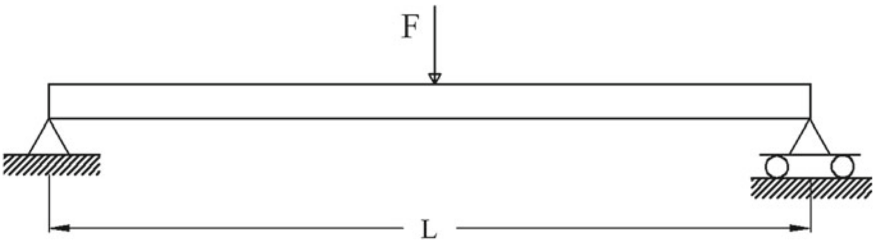


Fig. 8.15 Beam under concentrated load

$$[M']^{(e)} = \frac{L}{6}\begin{bmatrix} 2m & 0 & m & 0 \\ 0 & 2J & 0 & J \\ m & 0 & 2m & 0 \\ 0 & J & 0 & 2J \end{bmatrix}$$

$$[K']^{(e)} = \begin{bmatrix} \frac{kAG}{L} & -\frac{kAG}{2} & -\frac{kAG}{L} & -\frac{kAG}{2} \\ -\frac{kAG}{2} & \frac{EI}{L} - \frac{kAGL}{3} & \frac{kAG}{2} & -\frac{EI}{L} - \frac{kAGL}{6} \\ \frac{kAG}{L} & \frac{kAG}{2} & -\frac{kAG}{L} & \frac{kAG}{2} \\ -\frac{kAG}{2} & -\frac{EI}{L} - \frac{kAGL}{6} & \frac{kAG}{2} & \frac{EI}{L} - \frac{kAGL}{3} \end{bmatrix}. \tag{8.7.28}$$

8.8 Problems

1. Consider a beam of rectangular cross-section $b \times h$ and length L, as shown in Fig. 8.15. The beam is under the uniform lateral distributed load w and is simply supported at the ends.
   Divide the beam into three elements of equal length and write the global stiffness and force matrices $[K]$ and $\{F\}$.
2. A beam of rectangular cross-section $b \times h$ and length L under a concentrated force F acting at the mid-length of the beam, as shown in Fig. 8.15, is assumed.

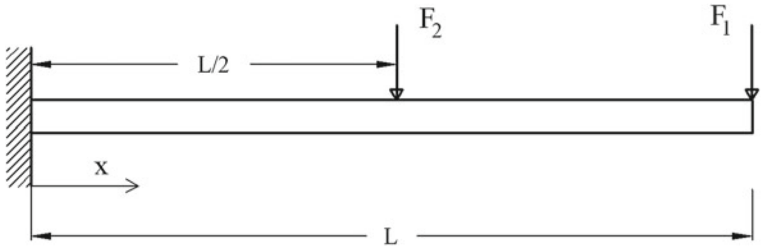


Fig. 8.16 Cantilever beam under concentrated load

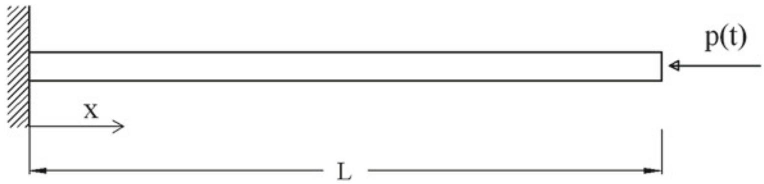


Fig. 8.17 Axial vibration of steel rod

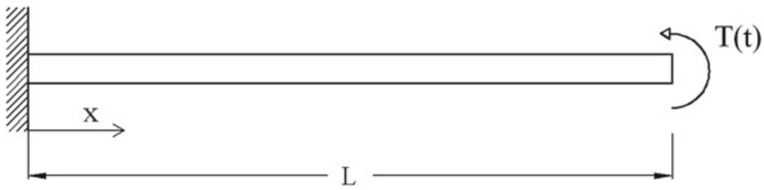


Fig. 8.18 Steel bar under torsional moment

The beam's boundary conditions are assumed to be simply supported. Divide the beam into three elements of equal length and calculate the global stiffness and force matrices.

3. Assume a rectangular cross-section cantilever beam, as shown in Fig. 8.16. The length of the beam is L. The concentrated forces F_1 and F_2 are acting at the end and mid-length of the beam, respectively. The beam is divided into four elements of equal length. Find the global stiffness and force matrices.
4. A steel rod of circular cross-section of length L and cross-sectional area A, as shown in Fig. 8.17, is considered. Dynamic load $p(t)$ is applied to the free end of the rod at $x = L$ along the axial direction. The end $x = 0$ is fixed. Divide the rod into two elements of equal length $L/2$ and calculate the stiffness, mass, and force matrices related to the axial vibration when $L = 20\,\text{cm}$ $A = 1\,\text{cm}^2$, and $p(t) = 100e^{-t}$ N.
5. Consider a steel bar with circular cross-section under torsional vibration due to a time dependent torsional moment $T(t)$ acting at the free end of the bar at $x = L$ (see Fig. 8.18). The end $x = 0$ is fixed, and the initial condition is at $t = 0$ is at

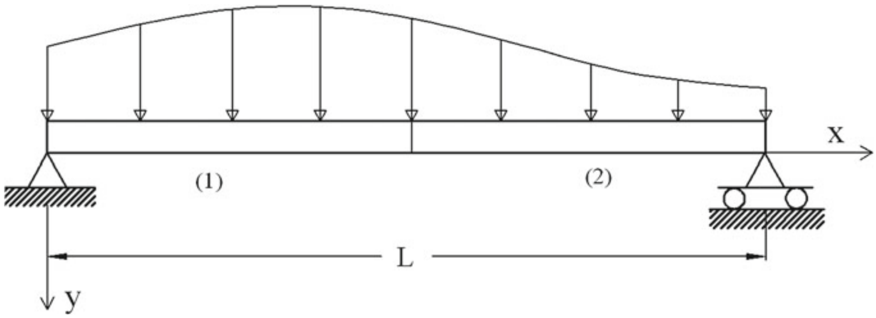


Fig. 8.19 Beam under non-uniform distributed load

$t = 0$ is $\theta(L, 0) = \theta_0$ and $\partial\theta(L, 0)/\partial t = 0$. Dividing the bar into two elements of equal length $L/2$, find the global mass, stiffness, and force matrices when $L = 20$ cm, cross-sectional diameter of 1 cm, and under dynamic torsional moment $T(t) = 100e^{-0.1t}$ N-cm.

6. Employ the C^1-continuous shape functions given by Eq. (8.6.24) and obtain the mass, stiffness, and force matrices of a simply supported steel beam under lateral force of $q(x, t) = 50 \sin \lambda x \sin \omega t$. The beam is divided into two elements, as shown in Fig. 8.19. Assume that $L = 30\ cm$ and $I = 2\ cm^4$.

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Chapter 9
Elasticity, Galerkin Formulations

Abstract The chapter begins with derivation of the equations of motion of an elastic continuum and presentation of the basic equations of theory of elasticity. Employing the Galerkin method, the equations of motion are made orthogonal with respect to the assumed element shape function. A base element is considered and the weak formulation is applied to the terms of higher order derivatives and the finite element equation for the three-dimensional elasticity is obtained. The resulting equations are reduced to the case of two-dimensional elasticity, plane stress, and plane strain conditions. Employing the two-dimensional simplex shape functions, the member of element matrices are obtained.

9.1 Introduction

In this chapter, the problems of linear elasticity are discussed. The finite element formulations are presented based on the Galerkin method. The governing equations of three-dimensional linear elasticity are given without detail treatments for the derivations. It is expected that the reader is already familiar with the basic assumptions, concepts, and equations. The three-dimensional equations are reduced to two-dimensional plane strain and plane stress problems and the related finite element formulations are presented.

9.2 Basic Equations of Elasticity

A homogeneous and isotropic elastic continuum is considered. We assume that the continuum occupies the volume V and is bounded by exterior surface S at a time t. The continuum is under the action of the external surface traction force with components t_i^n acting on a surface where its outer unit normal vector is $\vec{n}$. Also, the internal

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Solid Mechanics and Its Applications 216, DOI: 10.1007/978-3-319-08037-6_9,

body force with components X_i per unit volume is present in the continuum. It is further assumed that the acting forces, either surface tractions or body forces, are time-dependent dynamic forces resulting in elastodynamic behavior of the continuum. In the case of elastostatic problems, the mass matrix is ignored and the dynamic problem reduces to that of static elasticity. The total force on the body is

$$F_i = \int_S t_i^n dS + \int_V X_i dV. \tag{9.2.1}$$

Using Cauchy's formula

$$t_i^n = \sigma_{ij} n_j \tag{9.2.2}$$

the Gauss theorem can be used to transform the surface integral of the traction forces into a volume integral as

$$\int_S t_i^n dS = \int_S \sigma_{ji} n_j dS = \int_V \sigma_{ji,j} dV. \tag{9.2.3}$$

Thus, the total force acting on the body in components is

$$F_i = \int_V (\sigma_{ji,j} + X_i) dV. \tag{9.2.4}$$

We designate the linear momentum by

$$\mathcal{P}_i = \int_V \rho \dot{u}_i dV \tag{9.2.5}$$

where ρ is the mass density. The Newton law of motion requires that

$$F_i = \dot{\mathcal{P}}_i \tag{9.2.6}$$

Upon substitution from Eqs. (9.2.4) and (9.2.5) in (9.2.6), we obtain

$$\int_V (\sigma_{ji,j} + X_i) dV = \int_V \rho \ddot{u}_i dV. \tag{9.2.7}$$

Since the volume V is arbitrary, Eq. (9.2.7) reduces to the following equation of motion [1]:

$$\sigma_{ji,j} + X_i = \rho \ddot{u}_i. \tag{9.2.8}$$

The expanded form of the equations of motion in Cartesian coordinates is

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + X = \rho \ddot{u}$$
$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + Y = \rho \ddot{v}$$
$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + Z = \rho \ddot{w}. \tag{9.2.9}$$

For the infinitesimal theory of elasticity, the strain-displacement relations, referring to the Cartesian coordinates system, is

$$\epsilon_{ij} = \tfrac{1}{2}\,(u_{j,i} + u_{i,j}). \tag{9.2.10}$$

Here, ϵ_{ij} is the symmetric strain tensor, and u_i is the displacement tensor. In terms of the conventional Cartesian coordinates system, the six strain-displacement relations are

$$\epsilon_{xx} = \frac{\partial u}{\partial x} \qquad \epsilon_{yy} = \frac{\partial v}{\partial y} \qquad \epsilon_{zz} = \frac{\partial w}{\partial z}$$
$$\epsilon_{xy} = \frac{1}{2}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) \qquad \epsilon_{yz} = \frac{1}{2}(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}) \qquad \epsilon_{zx} = \frac{1}{2}(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}). \tag{9.2.11}$$

Eliminating the displacement components u, v, and w in Eqs. (9.2.11) provides six independent equations among the strain components called the compatibility equations. These equations in rectangular Cartesian coordinates in conventional form are

$$\frac{\partial^2 \epsilon_{xx}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \epsilon_{xy}}{\partial x \partial y}$$
$$\frac{\partial^2 \epsilon_{yy}}{\partial z^2} + \frac{\partial^2 \epsilon_{zz}}{\partial y^2} = 2\frac{\partial^2 \epsilon_{yz}}{\partial y \partial z}$$
$$\frac{\partial^2 \epsilon_{zz}}{\partial x^2} + \frac{\partial^2 \epsilon_{xx}}{\partial z^2} = 2\frac{\partial^2 \epsilon_{xz}}{\partial x \partial z}$$
$$\frac{\partial^2 \epsilon_{xx}}{\partial y \partial z} = \frac{\partial}{\partial x}(\frac{\partial \epsilon_{xz}}{\partial y} + \frac{\partial \epsilon_{xy}}{\partial z} - \frac{\partial \epsilon_{yz}}{\partial x})$$
$$\frac{\partial^2 \epsilon_{yy}}{\partial x \partial z} = \frac{\partial}{\partial y}(\frac{\partial \epsilon_{xy}}{\partial z} + \frac{\partial \epsilon_{yz}}{\partial x} - \frac{\partial \epsilon_{xz}}{\partial y})$$
$$\frac{\partial^2 \epsilon_{zz}}{\partial x \partial y} = \frac{\partial}{\partial z}(\frac{\partial \epsilon_{xz}}{\partial y} + \frac{\partial \epsilon_{yz}}{\partial x} - \frac{\partial \epsilon_{xy}}{\partial z}). \tag{9.2.12}$$

In the classical theory of linear elasticity, the components of a strain tensor are linear functions of the components of the stress tensor. The linear relations are called *Hooke's law* and are

$$\epsilon_{ij}^{e} = \frac{1}{2G}\,(\sigma_{ij} - \frac{\nu}{1+\nu}\,\sigma_{kk}\delta_{ij}) \tag{9.2.13}$$

where G is the shear modulus and ν is Poisson's ratio. Equation (9.2.13) is known as the constitutive law of linear elasticity. Solving this equation for stress tensor σ_{ij} gives

$$\sigma_{ij} = 2G[\epsilon_{ij} + \frac{\nu}{1-2\nu}\,\epsilon_{kk}\delta_{ij}] \tag{9.2.14}$$

where δ_{ij} is the Kronecker delta.

It is sometimes useful to write the stress-strain relations in terms of the Lamé constants λ and μ, where μ is the same as shear modulus G. The strain tensor in terms of the Lamé constants is related to the stress tensor by

$$\epsilon_{ij} = \frac{1}{2\mu}(\sigma_{ij} - \frac{\lambda}{3\lambda+2\mu}\sigma_{kk}\delta_{ij}). \tag{9.2.15}$$

Solving for stress tensor σ_{ij} gives

$$\sigma_{ij} = 2\mu\epsilon_{ij} + \lambda\epsilon_{kk}\delta_{ij} \tag{9.2.16}$$

where μ and λ are the Lamé constants and are related to G and ν as

$$\mu = G \qquad\qquad \lambda = \frac{2G\nu}{1-2\nu}. \tag{9.2.17}$$

The complete relationships between the elastic constants are given in reference. There are six equations between the six components of stress and six components of strain. Substituting Eq. (9.2.10) in Eq. (9.2.16) yields

$$\sigma_{ij} = \mu(u_{i,j} + u_{j,i}) + \lambda u_{k,k}\delta_{ij}. \tag{9.2.18}$$

The derivative of σ_{ij} with respect to x_j is

$$\sigma_{ij,j} = \mu u_{i,jj} + \mu u_{j,ji} + \lambda u_{j,ji}. \tag{9.2.19}$$

Substituting Eq. (9.2.19) in Eq. (9.2.8) results in an equation of motion in terms of the displacement components as

$$\mu u_{i,jj} + (\lambda+\mu)u_{j,ji} + X_i = \rho\ddot{u}_i. \tag{9.2.20}$$

This equation is called the *Navier equation* and is equivalent to the three equilibrium equations, six compatibility equations, and six stress-strain relations. The expanded form of this equation is

$$
\begin{aligned}
\mu\nabla^2 u + (\lambda+\mu)\frac{\partial}{\partial x}(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}) + X &= \rho\ddot{u}\\
\mu\nabla^2 v + (\lambda+\mu)\frac{\partial}{\partial y}(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}) + Y &= \rho\ddot{v}\\
\mu\nabla^2 w + (\lambda+\mu)\frac{\partial}{\partial z}(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}) + Z &= \rho\ddot{w}.
\end{aligned}
\tag{9.2.21}
$$

The finite element approximation follows by discretizing the solution domain into a number of elements and nodal points. The base element (e) is considered and the displacement components are approximated by proper shape functions N, as

$$
\begin{aligned}
u^{(e)}(x,y,z,t) &= \langle N(x,y,z)\rangle^{(e)}\{U(t)\}^{(e)}\\
v^{(e)}(x,y,z,t) &= \langle N(x,y,z)\rangle^{(e)}\{V(t)\}^{(e)}\\
w^{(e)}(x,y,z,t) &= \langle N(x,y,z)\rangle^{(e)}\{W(t)\}^{(e)}.
\end{aligned}
\tag{9.2.22}
$$

These approximations are of Kantrovich type, where the space and time variables are separated. The matrix of shape function in expanded form is

$$
\langle N(x,y,z)\rangle^{(e)} = \langle N_1 \quad N_2 \ldots N_r\rangle^{(e)} \tag{9.2.23}
$$

where r is the number of nodal points in the based element (e). Following the standard Galerkin method, the equations of motion (9.2.21) are made orthogonal with respect to the shape function matrices (9.2.23) over the volume of the base element (e) as

$$
\begin{aligned}
&\int_{V(e)}\left[\mu\nabla^2 u + (\lambda+\mu)\,\frac{\partial}{\partial x}\,(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}) + X - \rho\ddot{u}\right]N_s dV = 0\\
&\int_{V(e)}\left[\mu\nabla^2 v + (\lambda+\mu)\,\frac{\partial}{\partial y}\,(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}) + Y - \rho\ddot{v}\right]N_s dV = 0\\
&\int_{V(e)}\left[\mu\nabla^2 w + (\lambda+\mu)\,\frac{\partial}{\partial z}\,(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}) + Z - \rho\ddot{w}\right]N_s dV = 0\\
&\qquad s = 1, 2, \ldots r
\end{aligned}
\tag{9.2.24}
$$

These equations constitute the basic governing finite element equations for the equilibrium of the base element (e).

9.3 Galerkin Finite Element Formulation

Consider a three-dimensional elastic continuum under the action of the surface traction forces t_i^n and the body forces X_i. The solution domain of the problem is divided into an arbitrary number of elements. The shape function N_l is selected to describe the displacement components in the base element (e). According to the Galerkin method, the following orthogonal form is considered [2, 3]:

$$\int_{V(e)} (\sigma_{ij,j} + X_i - \rho \ddot{u}_i) N_l dV = 0 \qquad l = 1, 2, \ldots r \tag{9.3.1}$$

where r is the total number of the nodal points of the base element (e). Applying the weak formulation to the first term yields

$$\int_{V(e)} (\sigma_{ij,j}) N_l dV = \int_{S(e)} \sigma_{ij} n_j N_l dS - \int_{V(e)} \frac{\partial N_l}{\partial x_j} \sigma_{ij} dV \tag{9.3.2}$$

where n_j is the component of the unit outer normal vector to the boundary. Substituting Eq. (9.3.2) in Eq. (9.3.1) gives

$$\int_{S(e)} \sigma_{ij} n_j N_l dS - \int_{V(e)} \frac{\partial N_l}{\partial x_j} \sigma_{ij} dV + \int_{V(e)} X_i N_l dV - \int_{V(e)} \rho \ddot{u}_i N_l dV = 0. \tag{9.3.3}$$

According to Cauchy's formula, the traction force components acting on the boundary are related to the stress tensor as

$$t_i^n = \sigma_{ij} n_j \tag{9.3.4}$$

Thus, the first term of Eq. (9.3.3) is

$$\int_{S(e)} \sigma_{ij} n_j N_l dS = \int_{S(e)} t_i N_l dS. \tag{9.3.5}$$

The stress tensor is related to the displacement components, using Eqs. (9.2.10) and (9.2.14), as

$$\sigma_{ij} = G(u_{i,j} + u_{j,i} + 2A u_{k,k} \delta_{ij}) \tag{9.3.6}$$

where $A = \nu/(1 - 2\nu)$. Substituting in the second term of Eq. (9.3.3) yields

$$\int_{V(e)} \frac{\partial N_l}{\partial x_j} \sigma_{ij} dV = G \int_{V(e)} \frac{\partial N_l}{\partial x_j} (u_{i,j} + u_{j,i} + 2A u_{k,k} \delta_{ij}) dV. \tag{9.3.7}$$

Substituting in Eq. (9.3.3) yields

$$\int_{V(e)} \rho \ddot{u}_i N_l dV + G \int_{V(e)} \frac{\partial N_l}{\partial x_j} (u_{i,j} + u_{j,i} + 2A u_{k,k} \delta_{ij}) dV$$
$$= \int_{V(e)} X_i N_l dV + \int_{S(e)} t_i N_l dS. \tag{9.3.8}$$

Now, the base element (e) is considered, and the displacement components in the element (e) are approximated as

$$u_i^{(e)}(x, y, z, t) = \langle N(x, y, z) \rangle^{(e)} \{U_i(t)\}^{(e)} = N_m U_{mi}. \tag{9.3.9}$$

Using this approximation, Eq. (9.3.8) is

$$\begin{aligned}&\int_{V(e)} \rho N_l N_m \ddot{U}_{mi} dV + G(\int_{V(e)} \frac{\partial N_l}{\partial x_j}\frac{\partial N_m}{\partial x_j} dV)U_{mi}\\&+G(\int_{V(e)} \frac{\partial N_l}{\partial x_j}\frac{\partial N_m}{\partial x_i} dV)U_{mj} + 2AG(\int_{V(e)} \frac{\partial N_l}{\partial x_i}\frac{\partial N_m}{\partial x_j} dV)U_{mj}\\&=\int_{V(e)} X_i N_l dV + \int_{S(e)} t_i N_l dS \quad l,m = 1,2,\ldots r \quad i,j = 1,2,3.\end{aligned} \tag{9.3.10}$$

Equation (9.3.10), after assembling all the element matrices and substituting for the shape function, results in the finite element equation

$$[M]\{\ddot{\Delta}\} + [K]\{\Delta\} = \{F\}. \tag{9.3.11}$$

For a two-dimensional problem, i and j take the values 1 and 2. In this case, two equations are obtained in the x and y-directions as

$$\begin{aligned}&(\int_{V(e)} \rho N_l N_m dV)\ddot{U}_m + \Big[2G(A+1)\int_{V(e)} \frac{\partial N_l}{\partial x}\frac{\partial N_m}{\partial x} dV\\&+G\int_{V(e)} \frac{\partial N_l}{\partial y}\frac{\partial N_m}{\partial y} dV\Big]U_m + \Big[G\int_{V(e)} \frac{\partial N_l}{\partial y}\frac{\partial N_m}{\partial x} dV\\&+2AG\int_{V(e)} \frac{\partial N_l}{\partial x}\frac{\partial N_m}{\partial y} dV\Big]V_m = \int_{V(e)} X N_l dV + \int_{S(e)} t_x N_l dS.\end{aligned} \tag{9.3.12}$$

$$\begin{aligned}&(\int_{V(e)} \rho N_l N_m dV)\ddot{V}_m + \Big[2G(A+1)\int_{V(e)} \frac{\partial N_l}{\partial y}\frac{\partial N_m}{\partial y} dV\\&+G\int_{V(e)} \frac{\partial N_l}{\partial x}\frac{\partial N_m}{\partial x} dV\Big]V_m + \quad [G\int_{V(e)} \frac{\partial N_l}{\partial x}\frac{\partial N_m}{\partial y} dV\\&+2AG\int_{V(e)} \frac{\partial N_l}{\partial y}\frac{\partial N_m}{\partial x} dV\Big]U_m = \int_{V(e)} Y N_l dV + \int_{S(e)} t_y N_l dS.\end{aligned} \tag{9.3.13}$$

The definition of the mass, stiffness, and force matrices of the element (e) are

$$[m]^{(e)}_{ml} = \begin{bmatrix} \int_{V(e)} \rho N_l N_m dV & 0 \\ 0 & \int_{V(e)} \rho N_l N_m dV \end{bmatrix} \tag{9.3.14}$$

$$[k]^{(e)}_{ml} = \left[\begin{matrix} 2G(A+1)\int_{V(e)} \frac{\partial N_l}{\partial x}\frac{\partial N_m}{\partial x} dV + G\int_{V(e)} \frac{\partial N_l}{\partial y}\frac{\partial N_m}{\partial y} dV & . \\ & . \\ G\int_{V(e)} \frac{\partial N_l}{\partial x}\frac{\partial N_m}{\partial y} dV + 2AG\int_{V(e)} \frac{\partial N_l}{\partial y}\frac{\partial N_m}{\partial x} dV & . \end{matrix}\right.$$

$$
\begin{aligned}
. \quad & G \int_{V(e)} \frac{\partial N_l}{\partial y} \frac{\partial N_m}{\partial x} dV + 2AG \int_{V(e)} \frac{\partial N_l}{\partial x} \frac{\partial N_m}{\partial y} dV \\
. \\
. \; & 2G(A+1) \int_{V(e)} \frac{\partial N_l}{\partial y} \frac{\partial N_m}{\partial y} dV + G \int_{V(e)} \frac{\partial N_l}{\partial x} \frac{\partial N_m}{\partial x} dV \Bigg]
\end{aligned}
\tag{9.3.15}
$$

$$
\{f\}_l^{(e)} = \begin{Bmatrix} \int_{V(e)} X N_l dV + \int_{S(e)} t_x N_l dS \\ \int_{V(e)} Y N_l dV + \int_{S(e)} t_y N_l dS \end{Bmatrix} \tag{9.3.16}
$$

and the unknown matrix is

$$
\{\delta\} = \begin{Bmatrix} U \\ V \end{Bmatrix}. \tag{9.3.17}
$$

9.4 Two-Dimensional Elasticity

The classical theory of elasticity distinguishes two conditions of plane stress and plane strain problems. The conditions correspond to the real practical problems of elasticity. The two conditions are discussed and the proper finite element formulations are presented in the following.

Plane stress condition

According to this assumption, the normal and shear stresses along the $z-$axis are assumed to be zero,

$$
\sigma_{zz} = \tau_{xz} = \tau_{yz} = 0 \tag{9.4.1}
$$

From Hooke's law

$$
\begin{aligned}
\epsilon_{xz} &= \epsilon_{yz} = 0 \\
\epsilon_{zz} &= -\frac{\nu}{E} (\sigma_{xx} + \sigma_{yy}).
\end{aligned}
\tag{9.4.2}
$$

This condition corresponds to the case where w, the deflection component along the $z-$direction, is a function of x and y ($w = w(x, y)$). By this assumption, w disappears from the first and second equilibrium equations, and Eqs. (9.2.21) reduce to

$$
\begin{aligned}
\mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \mu \left(\frac{1+\nu}{1-\nu}\right) \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + X = \rho \ddot{u} \\
\mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \mu \left(\frac{1+\nu}{1-\nu}\right) \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + Y = \rho \ddot{v}.
\end{aligned}
\tag{9.4.3}
$$

Consider the base element (e) with the shape functions given as

$$u^{(e)}(x, y, t) = \langle N(x, y)\rangle^{(e)}\{U(t)\}^{(e)}$$
$$v^{(e)}(x, y, t) = \langle N(x, y)\rangle^{(e)}\{V(t)\}^{(e)} \tag{9.4.4}$$

where $\langle N(x, y)\rangle^{(e)} = \langle N1 \;\; N2 \ldots N_r\rangle^{(e)}$ and r is the number of nodes of the base element (e). The Galerkin approximations for the element (e) are

$$\int_{V(e)} \left[\mu\,(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) + \mu\,(\frac{1+\nu}{1-\nu})\,\frac{\partial}{\partial x}\,(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) + X - \rho\ddot{u}\right] N_l dV = 0$$
$$\int_{V(e)} \left[\mu\,(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}) + \mu\,(\frac{1+\nu}{1-\nu})\,\frac{\partial}{\partial y}\,(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) + Y - \rho\ddot{v}\right] N_l dV = 0$$
$$l = 1, 2, \ldots r. \tag{9.4.5}$$

Now, the weak formulation may be applied to the terms with second order derivatives.

Reason for weak formulation

The mathematical explanation of a physical problem is in the form of a set of partial differential equations associated with a number of boundary conditions. The boundary conditions related to the problems of solid mechanics are described in terms of kinematical form, in which the displacement components are defined on the boundary, or of forced type where the forces (or stresses) are defined on the boundary. When the displacement-based finite element formulation is employed, satisfaction of the kinematical boundary conditions is easy to handle, but the force type of boundary conditions is complicated. The same is true when force-formulations are employed, in which the force type boundary conditions are easy to handle and the kinematical boundary conditions become a complicated issue. An ideal finite element formulation is the one which provides a model in which both kinematical and forced types of boundary conditions become available in the final numerical model. This is an essential tool in the finite element modelling of plate and shell problems, in which the displacement type of boundary conditions appear on the edges of the plates or shells in the form of clamped or simply supported boundaries. The force boundary conditions on the plates or shells appear in the form of applied bending moments or in-plane forces. An ideal finite element model for these types of structures is the one which can handle both types of kinematical and forced boundary conditions.

The Galerkin finite element formulation has the advantage of preparing a more general, detailed, and complete numerical model to handle more different and various types of boundary conditions, either in the form of kinematical or forced. The process of weak formulation is to provide a means to prepare such a general and complete model. It is not that we should always apply the weak formulation to the terms of higher order derivatives. A *proper and ideal* finite element model is the one which provides a menu that can handle all possible types of boundary conditions associated with that problem. Here is why the potential to create all possible types of boundary conditions becomes important. The Galerkin method, if properly handled, is capable of providing such an ideal numerical model.

To show the details of weak formulations and justifications for weak formulations, we consider the following example.

Example 1 Consider a rectangular solution domain in the Rectangular Cartesian Coordinates. In this special example, the unit outer normal vectors in the x and y-directions coincide with unit vectors along the coordinate axes, where $n_x = n_y = 1$. The weak formulation of the first term of Eq. (9.4.5) is

$$\mu \int_{V(e)} \frac{\partial^2 u}{\partial x^2}\ N_l dV = \mu \int_{\Gamma(e)} \frac{\partial u}{\partial x}\ N_l d\Gamma - \mu \int_{V(e)} \frac{\partial u}{\partial x}\ \frac{\partial N_l}{\partial x}\ dV$$
$$= \mu \int_{\Gamma(e)} \epsilon_x N_l d\Gamma - \mu \int_{V(e)} \langle \frac{\partial N_1}{\partial x} \dots \frac{\partial N_r}{\partial x} \rangle \{U(t)\}\ \frac{\partial N_l}{\partial x} dV.$$

It is noticed that the weak formulation of the first term results in the x-component of strain on the boundary. This term may then be related to the stress along the x-direction, providing a means to satisfy the forced type of boundary condition, in addition to the possibility of satisfying the kinematical boundary condition along the same axis (which is u).

The weak formulation of the second term results in $\frac{\partial u}{\partial y}$, which does not have any physical meaning by itself. We may keep this term from the weak formulation and add it to the weak formulation of the first term of the second equation of (9.4.5) $\frac{\partial^2 v}{\partial x^2}$ with respects to x, to obtain an expression for the shear strain $\epsilon_{xy} = \frac{1}{2}\ (\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})$. This is if the decision is made to bring such a boundary condition into the finite element formulations. If not, the weak formulation of the second term is done as

$$\mu \int_{V(e)} \frac{\partial^2 u}{\partial y^2}\ N_l dV = \mu \int_{\Gamma(e)} \langle \frac{\partial N_1}{\partial y} \dots \frac{\partial N_r}{\partial y} \rangle \{U(t)\}\ N_l d\Gamma$$
$$- \mu \int_{V(e)} \langle \frac{\partial N_1}{\partial y} \dots \frac{\partial N_r}{\partial y} \rangle \{U(t)\}\ \frac{\partial N_l}{\partial y} dV \tag{9.4.6}$$

where the integral over the boundary of the element will be transformed into the stiffness matrix as coefficients of the unknown matrix U.

With similar reasoning, we may proceed into the weak formulation process. The third term is

$$\mu (\frac{1+\nu}{1-\nu}) \int_{V(e)} \frac{\partial^2 u}{\partial x^2}\ N_l dV = \mu (\frac{1+\nu}{1-\nu}) \int_{\Gamma(e)} \epsilon_x N_l d\Gamma$$
$$- \mu (\frac{1+\nu}{1-\nu}) \int_{V(e)} \langle \frac{\partial N_1}{\partial x} \cdots \frac{\partial N_r}{\partial x} \rangle \{U(t)\} \frac{\partial N_l}{\partial x} dV. \tag{9.4.7}$$

The fourth term is

$$\mu(\frac{1+\nu}{1-\nu}) \int_{V(e)} \frac{\partial^2 v}{\partial x \partial y} N_l dV = \mu(\frac{1+\nu}{1-\nu}) \int_{\Gamma(e)} \frac{\partial v}{\partial x} N_l d\Gamma - \mu(\frac{1+\nu}{1-\nu}) \int_{V(e)} \frac{\partial v}{\partial x} \frac{\partial N_l}{\partial y} dV$$
$$= \mu(\frac{1+\nu}{1-\nu}) \int_{\Gamma(e)} \langle \frac{\partial N_l}{\partial x} \cdots \frac{\partial N_r}{\partial x} \rangle \{V(t)\} N_l d\Gamma - \mu(\frac{1+\nu}{1-\nu}) \int_{V(e)} \langle \frac{\partial N_1}{\partial x} \cdots \frac{\partial N_l}{\partial x} \rangle \{V(t)\} \frac{\partial N_l}{\partial y} dV. \quad (9.4.8)$$

The fifth term is

$$\int_{V(e)} X N_l dV. \quad (9.4.9)$$

The sixth term is

$$\int_{V(e)} \rho \langle N_1 \ldots N_r \rangle \{\ddot{U}(t)\} N_l dV \quad (9.4.10)$$

Substituting into the first of Eqs. (9.4.5) gives

$$\left[\int_{V(e)} \rho \langle N \rangle N_l dV\right] \{\ddot{U}\} + \left[\frac{2\mu}{1-\nu} \int_{V(e)} \langle \frac{\partial N}{\partial x} \rangle \frac{\partial N_l}{\partial x} dV + \mu \int_{V(e)} \langle \frac{\partial N}{\partial y} \rangle \frac{\partial N_l}{\partial y} dV - \mu \int_{\Gamma(e)} \langle \frac{\partial N}{\partial y} \rangle N_l d\Gamma\right] \{U(t)\}$$
$$+ \left[\mu(\frac{1+\nu}{1-\nu}) \int_{V(e)} \langle \frac{\partial N}{\partial x} \rangle \frac{\partial N_l}{\partial y} dV - \mu(\frac{1+\nu}{1-\nu}) \int_{\Gamma(e)} \langle \frac{\partial N}{\partial x} \rangle N_l d\Gamma\right] \{V(t)\}$$
$$= \int_{V(e)} X N_l dV + \frac{2\mu}{1-\nu} \int_{\Gamma(e)} \epsilon_x N_l d\Gamma \quad l = 1, 2, \ldots r \quad (9.4.11)$$

With a similar procedure, the weak formulation of the second of Eqs. (9.4.5) gives

$$\left[\int_{V(e)} \rho \langle N \rangle N_l dV\right] \{\ddot{V}\} + \left[\mu(\frac{1+\nu}{1-\nu}) \int_{V(e)} \langle \frac{\partial N}{\partial y} \rangle \frac{\partial N_l}{\partial x} dV + \mu(\frac{1+\nu}{1-\nu}) \int_{\Gamma(e)} \langle \frac{\partial N}{\partial y} \rangle N_l d\Gamma\right] \{V(t)\} + \left[\mu \int_{V(e)} \langle \frac{\partial N}{\partial x} \rangle \frac{\partial N_l}{\partial x} dV\right.$$
$$\left. + \frac{2\mu}{1-\nu} \int_{V(e)} \langle \frac{\partial N}{\partial y} \rangle \frac{\partial N_l}{\partial y} dV - \mu \int_{\Gamma(e)} \langle \frac{\partial N}{\partial x} \rangle N_l d\Gamma\right] \{U(t)\}$$

$$= \int_{V(e)} Y N_l dV + \frac{2\mu}{1-\nu} \int_{\Gamma(e)} \epsilon_y N_l d\Gamma \qquad l = 1, 2, \ldots r \qquad (9.4.12)$$

The system of Eqs. (9.4.11) and (9.4.12) provides a linear set of $2r \times 2r$ equations describing the dynamic equations of the base element (e). A detailed review of these two equations reveals that both types of kinematical and forced boundary conditions are considered in the resulting equations. When the constraint is applied on the displacement components U and V on the boundary, then the force matrix related to the strain is transformed into the displacement components on the boundary employing the shape functions and substituted into the stiffness matrix. On the other hand, when force is applied on the boundary, then the force matrix containing the strain is properly changed to include the forced boundary condition, and the displacement components on the same boundary are unknown and should be obtained solving the finite element equation.

Equations (9.4.11) and (9.4.12), once written in the global coordinate system, are added for all elements in the solution domain, and ultimately provide the finite element equation of motion as

$$[M]\{\ddot{\Delta}\} + [K]\{\Delta\} = \{F(t)\} \qquad (9.4.13)$$

in which the elements of each matrix for the base element (e) are

$$\langle \delta \rangle^{(e)} = \langle \langle \delta_1 \rangle \quad \langle \delta_2 \rangle \cdots \langle \delta_r \rangle \rangle^{(e)} \qquad (9.4.14)$$

where $\langle \delta_i \rangle = \langle U_i \quad V_i \rangle$. The members of the other matrices for the base element (e) of thickness h are given in the following. The members of the force matrix are

$$\begin{aligned} f_{2i-1} &= \frac{2\mu h}{1-\nu} \int_{\Gamma(e)} \epsilon_x N_i dl + h \int_{A(e)} X N_i dA \\ f_{2i} &= \frac{2\mu h}{1-\nu} \int_{\Gamma(e)} \epsilon_y N_i dl + h \int_{A(e)} Y N_i dA \qquad i = 1, 2, 3. \end{aligned} \qquad (9.4.15)$$

The members of the mass matrix are

$$\begin{aligned} m_{2i-1,2j-1} &= h \int_{A(e)} \rho N_i N_j dA \\ m_{2i-1,2j} &= 0 \\ m_{2i,2j-1} &= 0 \\ m_{2i,2j} &= h \int_{A(e)} \rho N_i N_j dA \qquad i = 1, 2, 3 \quad j = 1, 2, 3. \end{aligned} \qquad (9.4.16)$$

The members of the stiffness matrix are

$$
\begin{aligned}
k_{2i-1,2j-1} &= \frac{2\mu h}{1-\nu} \int_{A(e)} \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x}\, dA + \mu h \int_{A(e)} \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y}\, dA \\
&\quad - \mu h \int_{\Gamma(e)} N_i \frac{\partial N_j}{\partial y}\, dl \\
k_{2i-1,2j} &= \mu h \frac{1+\nu}{1-\nu} \int_{A(e)} \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial x}\, dA - \mu h \frac{1+\nu}{1-\nu} \int_{\Gamma(e)} N_i \frac{\partial N_j}{\partial x} dl \\
k_{2i,2j-1} &= \mu h \frac{1+\nu}{1-\nu} \int_{A(e)} \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial y}\, dA - \mu h \frac{1+\nu}{1-\nu} \int_{\Gamma(e)} N_i \frac{\partial N_j}{\partial y} dl \\
k_{2i,2j} &= \mu h \int_{A(e)} \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x}\, dA + \frac{2\mu h}{1-\nu} \int_{A(e)} \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} dA \\
&\quad - \mu h \int_{\Gamma(e)} N_i \frac{\partial N_j}{\partial x} dl \qquad i = 1, 2, 3 \quad j = 1, 2, 3.
\end{aligned} \tag{9.4.17}
$$

Plane strain condition

According to this assumption, the normal and shear strains along the $z-$axis are assumed to be zero as

$$
\epsilon_{zz} = \epsilon_{xz} = \epsilon_{yz} = 0. \tag{9.4.18}
$$

Using these assumptions in Hooke's law gives

$$
\begin{aligned}
\tau_{xz} &= \tau_{yz} = 0 \\
\sigma_{zz} &= \frac{E\nu}{(1+\nu)(1-2\nu)} (\sigma_{xx} + \sigma_{yy}).
\end{aligned} \tag{9.4.19}
$$

Substituting Eq. (9.2.10) in (9.2.16), with the assumption of plane strain condition (9.4.19), and substituting the results in Eq. (9.2.8), the equations of motion in terms of the displacement components for the plane strain condition become

$$
\begin{aligned}
\mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + \frac{\mu}{1-2\nu} \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + X = \rho \ddot{u} \\
\mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + \frac{\mu}{1-2\nu} \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + Y = \rho \ddot{v}.
\end{aligned} \tag{9.4.20}
$$

Following a similar procedure as described for the plane stress condition, beginning from Eq. (9.4.5), the final form of the finite element equation becomes

$$
[M]\{\ddot{\Delta}\} + [K]\{\Delta\} = \{F\} \tag{9.4.21}
$$

where the definition of matrix $\{\delta\}$ is the same as Eq. (9.4.14) and the members of the force matrix are

$$f_{2i-1} = \mu h \,(\frac{1-\nu}{1-2\nu}) \int_{\Gamma(e)} \epsilon_x N_i dl + h \int_{A(e)} X N_i dA$$
$$f_{2i} = \mu h \,(\frac{1-\nu}{1-2\nu}) \int_{\Gamma(e)} \epsilon_y N_i dl + h \int_{A(e)} Y N_i dA. \tag{9.4.22}$$

The elements of the mass matrix are

$$\begin{aligned} m_{2i-1,2j-1} &= h \int_{A(e)} \rho N_i N_j dA \\ m_{2i-1,2j} &= m_{2i,2j-1} = 0 \\ m_{2i,2j} &= h \int_{A(e)} \rho N_i N_j dA \qquad i = 1, 2, \ldots r \qquad j = 1, 2, \ldots r. \end{aligned} \tag{9.4.23}$$

The members of the stiffness matrix are

$$\begin{aligned} k_{2i-1,2j-1} &= \mu h \,(\frac{1-\nu}{1-2\nu}) \int_{A(e)} \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} dA + \mu h \int_{A(e)} \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} dA \\ &\quad - \mu h \int_{\Gamma(e)} N_i \frac{\partial N_j}{\partial y} dl \\ k_{2i-1,2j} &= \frac{\mu h}{1-2\nu} \int_{A(e)} \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial x} dA - \frac{\mu h}{1-2\nu} \int_{\Gamma(e)} N_i \frac{\partial N_j}{\partial x} dl \\ k_{2i,2j-1} &= \frac{\mu h}{1-2\nu} \int_{A(e)} \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial y} dA - \frac{\mu h}{1-2\nu} \int_{\Gamma(e)} N_i \frac{\partial N_j}{\partial y} dl \\ k_{2i,2j} &= \mu h \int_{A(e)} \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} dA + \mu h \,(\frac{1-\nu}{1-2\nu}) \int_{A(e)} \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} dA \\ &\quad - \mu h \int_{\Gamma(e)} N_i \frac{\partial N_j}{\partial x} dl \qquad i = 1, 2, \ldots r \qquad j = 1, 2, \ldots r \end{aligned} \tag{9.4.24}$$

Once the element shape function is defined, all the matrices for the element (e) are determined and assembled to give the final finite element equilibrium equation. In the next section, a triangular simplex element in two-dimensions is considered, and the members of the mass, stiffness, and force matrices are derived.

9.5 Two-Dimensional Simplex Element

Consider a two-dimensional triangular element with linear assumption for the shape functions for u and v as

$$u^{(e)} = \langle N_i \quad N_j \quad N_k \rangle^{(e)} \begin{Bmatrix} U_i \\ U_j \\ U_k \end{Bmatrix}^{(e)}$$

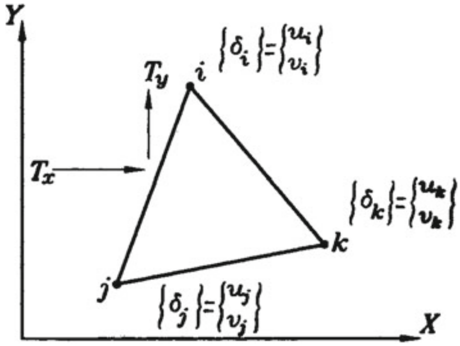

Fig. 9.1 Triangular simplex element

$$v^{(e)} = \langle N_i \quad N_j \quad N_k \rangle^{(e)} \begin{Bmatrix} V_i \\ V_j \\ V_k \end{Bmatrix}^{(e)}. \tag{9.5.1}$$

The thickness of the element is assumed to be h. The side ij is assumed to be under the external traction forces T_x and T_y. The elements of the matrices for the two-dimensional plane stress problem are (Fig. 9.1)

$$\begin{aligned} f_1 = f_3 &= \frac{\mu \epsilon_x A_{ij}}{1-\nu} + \frac{XV}{3} \\ f_5 &= \frac{XV}{3} \\ f_2 = f_4 &= \frac{\mu \epsilon_y A_{ij}}{1-\nu} + \frac{YV}{3} \\ f_6 &= \frac{YV}{3}. \end{aligned} \tag{9.5.2}$$

The elements of the stiffness matrix are
where

$$A = \frac{\mu h(1+\nu)}{4A(1-\nu)} \qquad B = \frac{\mu h}{4A} \qquad C = \frac{\mu A_{ij}}{4A}$$
$$D = \frac{\mu h}{2A(1-\nu)} \qquad E = \frac{\mu A_{ij}(1+\nu)}{4A(1-\nu)} \qquad A_{ij} = h l_{ij}. \tag{9.5.3}$$

Here, l_{ij} is the side length ij of the element (e). Note that the stiffness matrix in this case is not symmetric. The members of the mass matrix are

$$m_{11} = \frac{\rho h A^{(e)}}{6} = \frac{\rho V^{(e)}}{6} = \frac{m^{(e)}}{6} = m_{22} = m_{33} = m_{44} = m_{55} = m_{66}$$

$$
\begin{array}{ll}
k_{11} = Db_i^2 + Bc_i^2 - Cc_i & k_{12} = Ab_ic_i - Eb_i \\
k_{13} = Db_ib_j + Bc_ic_j - Cc_j & k_{14} = Ab_jc_i - Eb_j \\
k_{15} = Db_ib_k + Bc_ic_k - Cc_k & k_{16} = Ab_kc_i - Eb_k \\
k_{21} = Db_ib_j + Bc_ic_j - Cc_i & k_{22} = Ab_ic_j - Eb_i \\
k_{23} = Db_j^2 + Bc_j^2 - Cc_j & k_{24} = Ab_jc_j - Eb_j \\
k_{25} = Db_jb_k + Bc_jc_k - Cc_k & k_{26} = Ab_kc_j - Eb_k \\
k_{31} = Db_ib_k + Bc_ic_k & k_{32} = Ab_ic_k \\
k_{33} = Db_jb_k + Bc_jc_k & k_{34} = Ab_jc_k \\
k_{35} = Db_k^2 - Bc_k^2 & k_{36} = Ab_kc_k \\
k_{41} = Ab_ic_i - Ec_i & k_{42} = Bb_i^2 + Dc_i^2 - Cb_i \\
k_{43} = Ab_ic_j - Ec_j & k_{44} = Bb_ib_j + Dc_ic_j - Cb_j \\
k_{45} = Ab_ic_k - Ec_k & k_{46} = Bb_ib_k + Dc_ic_k - Cb_k \\
k_{51} = Ab_jc_i - Ec_i & k_{52} = Bb_ib_j + Dc_ic_j - Cb_i \\
k_{53} = Ab_jc_j - Ec_j & k_{54} = Bb_j^2 + Dc_j^2 - Cb_j \\
k_{55} = Ab_jc_k - Ec_k & k_{56} = Bb_jb_k + Dc_jc_k - Cb_k \\
k_{61} = Ab_kc_i & k_{62} = Bb_ib_k + Dc_ic_k \\
k_{63} = Ab_kc_j & k_{64} = Bb_jb_k + Dc_jc_k \\
k_{65} = Ab_kc_k & k_{66} = Bb_k^2 + Dc_k^2
\end{array}
$$

$$
\begin{aligned}
m_{13} &= m_{31} = M_{15} = m_{51} = m_{24} = m_{42} = m_{26} = m_{62} = m_{35} = m_{53} \\
&= m_{46} = m_{64} = \frac{\rho h A^{(e)}}{12} = \frac{\rho V^{(e)}}{12} = \frac{m^{(e)}}{12} \quad \text{all other } m_{ij} = 0. \qquad (9.5.4)
\end{aligned}
$$

The mass matrix is symmetric.

The elements of the matrices for the two-dimensional plane strain condition are obtained in a similar manner and are as given. The elements of the force matrix are

$$
\begin{aligned}
f_{2i-1} &= \frac{\mu h \epsilon_x (1-\nu)}{1-2\nu} \int_{\Gamma(e)} N_i dl + h \int_{A(e)} X N_i dA \\
f_{2i} &= \frac{\mu h \epsilon_y (1-\nu)}{1-2\nu} \int_{\Gamma(e)} N_i dl + h \int_{A(e)} Y N_i dA. \qquad (9.5.5)
\end{aligned}
$$

Substituting for N_i and integrating gives

$$
\begin{aligned}
f_1 &= \frac{\mu \epsilon_x A_{ij} (1-\nu)}{2(1-2\nu)} + \frac{XV}{3} \\
f_3 &= \frac{\mu \epsilon_x A_{ij} (1-\nu)}{2(1-2\nu)} + \frac{XV}{e} \\
f_5 &= \frac{XV}{3} \\
f_2 &= \frac{\mu \epsilon_y A_{ij} (1-\nu)}{2(1-2\nu)} + \frac{YV}{3} \\
f_4 &= \frac{\mu \epsilon_y A_{ij} (1-\nu)}{2(1-2\nu)} + \frac{YV}{3}
\end{aligned}
$$

$$f_6 = \frac{YV}{3} \tag{9.5.6}$$

where it was assumed that the traction force acts on the $(ij)-$side.

The elements of the stiffness matrix are

$$\begin{array}{ll}
k_{11} = A'b_i^2 + B'c_i^2 - C'c_i & k_{12} = D'b_ic_i - E'b_i \\
k_{13} = A'b_ib_j + B'c_ic_j - C'c_j & k_{14} = D'b_jc_i - E'b_j \\
k_{15} = A'b_ib_k + B'c_ic_k - C'c_k & k_{16} = D'b_kc_i - E'b_k \\
k_{21} = A'b_ib_j + B'c_ic_j - C'c_i & k_{22} = D'b_ic_j - E'b_i \\
k_{23} = A'b_j^2 + B'c_j^2 - C'c_j & k_{24} = D'c_jb_j - E'b_j \\
k_{25} = A'b_jb_k + B'c_jc_k - C'c_k & k_{26} = D'c_jb_k - E'b_k \\
k_{31} = A'b_kb_i + B'c_kc_i & k_{32} = D'c_kb_i \\
k_{33} = A'b_kb_j + B'c_kc_j & k_{34} = D'c_kb_j \\
k_{35} = A'b_k^2 + B'c_k^2 & k_{36} = D'b_kb_k \\
k_{41} = D'b_ic_i - E'c_i & k_{42} = A'c_i^2 + B'b_i^2 - C'b_i \\
k_{43} = D'b_ic_j - E'c_j & k_{44} = A'c_ic_j + B'b_ib_j - C'b_j \\
k_{45} = D'b_ic_k - E'c_k & k_{46} = A'c_ic_k + B'b_ib_k - C'b_k \\
k_{51} = D'b_jc_i - E'c_i & k_{52} = A'c_ic_j + B'b_ib_j - C'b_i \\
k_{53} = D'b_jc_j - E'c_j & k_{54} = A'c_j^2 + B'b_j^2 - C'b_j \\
k_{55} = D'b_jc_k - E'c_k & k_{56} = A'c_jc_k + B'b_jb_k - C'b_k \\
k_{61} = D'b_kc_i & k_{62} = A'c_kc_i + B'b_kb_i \\
k_{63} = D'b_kc_j & k_{64} = A'c_kc_j + C'b_kb_j \\
k_{65} = D'b_kc_k & k_{66} = A'c_k^2 + B'b_k^2
\end{array}$$

where

$$\begin{aligned}
A' &= \frac{\mu h(1-\nu)}{4A(1-2\nu)} \qquad B' = \frac{\mu h}{4A} \qquad C' = \frac{\mu A_{ij}}{4A} \\
D' &= \frac{\mu h}{4A(1-2\nu)} \qquad E' = \frac{\mu A_{ij}}{4A(1-2\nu)}.
\end{aligned} \tag{9.5.7}$$

The elements of the mass matrix are

$$\begin{aligned}
m_{ii} &= \frac{m}{6} \\
m_{13} &= m_{31} = M_{15} = m_{51} = m_{24} = m_{42} = m_{26} = m_{62} = m_{35} = m_{53} \\
&= m_{46} = m_{64} = \frac{m^{(e)}}{12} \qquad \text{all other } m_{ij} = 0.
\end{aligned} \tag{9.5.8}$$

Once the elements of the mass, stiffness, and force matrices are calculated for one element, an algorithm is used to generate and assemble the global mass, stiffness, and force matrices. The finite element solution of such an assembled system provides the general displacement matrix at the required time. The strain and stress matrices are then calculated using the displacement matrix.

Example 2 Consider a two-dimensional elasticity problem.

(a) Reduce Eq. (9.3.10) to the two-dimensional plane strain condition.

(b) Do part (a) for the plane stress condition.

(c) Consider a triangular simplex element and derive the members of the stiffness and mass matrices using Eqs. (9.3.14), (9.3.15), and (9.3.16).

(a) For a two-dimensional plane strain condition, the displacement component along the z direction is zero and the other components of the displacement are independent of the z coordinate. The procedure in deriving Eq. (9.3.10) remains unchanged, except that the term u_z is eliminated from the trial and test function spaces. Thus, the final equations have index range $l, m = 1, 2$ instead of $l, m = 1, 2, 3$. The result is Eqs. (9.3.12) and (9.3.13) in the x and y directions, respectively.

(b) For the two-dimensional plane stress condition, we have $\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$ and the other components Of the stress tensor are independent of the z coordinate. The stress-displacement relation, Eq. (9.3.6), which is used in the procedure for deriving Eq. (9.3.10), should be modified accordingly. For the plane stress condition, Eq. (9.3.6) should be modified by reducing the index range to $i, j, k = 1, 2$ and using $A = \nu/(1-\nu)$ instead of $\nu/(1-2\nu)$. The final equations (9.3.12) and (9.3.13) remain unchanged, except that the new value of A should be used.

(c) From Eq. (4.8.9), integrals of the element mass matrix could be obtained. There are two forms of integrals in the element mass matrix:

$$\int_{V(e)} \rho N_i N_i dV = \int_{V(e)} \rho N_i^2 dV = \frac{2!0!0!}{(2+0+0+2)!} 2S = \frac{S}{6}$$

$$\int_{V(e)} \rho N_i N_j dV = \frac{1!1!0!}{(1+1+0+2)!} 2S = \frac{S}{12}$$

where $dV = tdS$, t is the thickness (which is assumed to be unity), and dS is the area element. The term $\int_{V(e)} \rho N_l N_m dV$ $(l, m = 1, 2, 3)$ becomes

$$\int_{V(e)} \rho N_l N_m dV = \frac{\rho S}{12} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

Therefore, the element mass matrix for the simplex element becomes

$$[m]^{(e)} = \frac{\rho S}{12} \begin{bmatrix} 2 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 \end{bmatrix}.$$

Now, the element stiffness matrix is calculated. The integrals of the element stiffness matrix are

$$\int_{V(e)} \frac{\partial N_l}{\partial x} \frac{\partial N_m}{\partial x} dV = \int_{S(e)} \begin{Bmatrix} b_i \\ b_j \\ b_k \end{Bmatrix} \begin{Bmatrix} b_i & b_j & b_k \end{Bmatrix} \frac{1}{2S} \frac{1}{2S} dS$$
$$= \frac{1}{4S} \begin{bmatrix} b_i b_i & b_i b_j & b_i b_k \\ b_j b_i & b_j b_j & b_j b_k \\ b_k b_i & b_k b_j & b_k b_k \end{bmatrix}$$

and

$$\int_{V(e)} \frac{\partial N_l}{\partial y} \frac{\partial N_m}{\partial y} dV = \int_{S(e)} \begin{Bmatrix} c_i \\ c_j \\ c_k \end{Bmatrix} \begin{Bmatrix} c_i & c_j & c_k \end{Bmatrix} \frac{1}{2S} \frac{1}{2S} dS$$
$$= \frac{1}{4S} \begin{bmatrix} c_i c_i & c_i c_j & c_i c_k \\ c_j c_i & c_j c_j & c_j c_k \\ c_k c_i & c_k c_j & c_k c_k \end{bmatrix}$$

and

$$\int_{V(e)} \frac{\partial N_l}{\partial x} \frac{\partial N_m}{\partial y} dV = \int_{S(e)} \begin{Bmatrix} b_i \\ b_j \\ b_k \end{Bmatrix} \begin{Bmatrix} c_i & c_j & c_k \end{Bmatrix} \frac{1}{2S} \frac{1}{2S} dS$$
$$= \frac{1}{4S} \begin{bmatrix} b_i c_i & b_i c_j & b_i c_k \\ b_j c_i & b_j c_j & b_j c_k \\ b_k c_i & b_k c_j & b_k c_k \end{bmatrix}.$$

The terms involved in the element stiffness matrix become

$$2G(A+1) \int_{V(e)} \frac{\partial N_l}{\partial x} \frac{\partial N_m}{\partial x} dV + G \int_{V(e)} \frac{\partial N_l}{\partial y} \frac{\partial N_m}{\partial y} dV =$$
$$\frac{G}{4S} \begin{bmatrix} Bb_i b_i + c_i c_i & Bb_i b_j + c_i c_j & Bb_i b_k + c_i c_k \\ Bb_j b_i + c_j c_i & Bb_j b_j + c_j c_j & Bb_j b_k + c_j c_k \\ Bb_k b_i + c_k c_i & Bb_k b_j + c_k c_j & Bb_k b_k + c_k c_k \end{bmatrix}$$

and

$$2G(A+1) \int_{V(e)} \frac{\partial N_l}{\partial y} \frac{\partial N_m}{\partial y} dV + G \int_{V(e)} \frac{\partial N_l}{\partial x} \frac{\partial N_m}{\partial x} dV =$$
$$\frac{G}{4S} \begin{bmatrix} Bc_i c_i + b_i b_i & Bc_i c_j + b_i b_j & Bc_i c_k + b_i b_k \\ Bc_j c_i + b_j b_i & Bc_j c_j + b_j b_j & Bc_j c_k + b_j b_k \\ Bc_k c_i + b_k b_i & Bc_k c_j + b_k b_j & Bc_k c_k + b_k b_k \end{bmatrix}$$

and

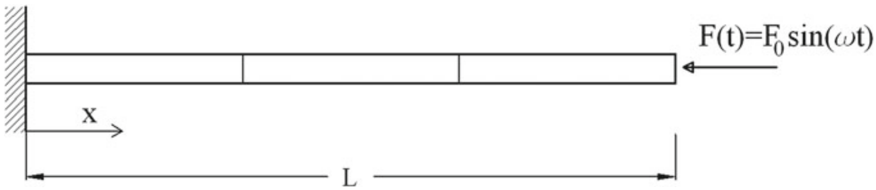

Fig. 9.2 An elastic rod under dynamic load

$$G\int_{V(e)}\frac{\partial N_l}{\partial y}\,\frac{\partial N_m}{\partial x}dV+2AG\int_{V(e)}\frac{\partial N_l}{\partial x}\,\frac{\partial N_m}{\partial y}dV=$$
$$\frac{G}{4S}\begin{bmatrix} c_ib_i+2Ab_ic_i & c_ib_j+2Ab_ic_j & c_ib_k+2Ab_ic_k\\ c_jb_i+2Ab_jc_i & c_jb_j+2Ab_jc_j & c_jb_k+2Ab_jc_k\\ c_kb_i+2Ab_kc_i & c_kb_j+2Ab_kc_j & c_kb_k+2Ab_kc_k \end{bmatrix}$$

and

$$G\int_{V(e)}\frac{\partial N_l}{\partial x}\,\frac{\partial N_m}{\partial y}dV+2AG\int_{V(e)}\frac{\partial N_l}{\partial y}\,\frac{\partial N_m}{\partial x}dV=$$
$$\left(G\int_{V(e)}\frac{\partial N_l}{\partial y}\,\frac{\partial N_m}{\partial x}dV+2A^*G\int_{V(e)}\frac{\partial N_l}{\partial x}\,\frac{\partial N_m}{\partial y}dV\right)^T$$

where $B=2(A+1)$. Therefore, the element stiffness matrix for the simplex element becomes

$$[k]^{(e)}=\frac{G}{4S}\times$$
$$\begin{bmatrix} Bb_ib_i+c_ic_i & Bb_ib_j+c_ic_j & Bb_ib_k+c_ic_k & c_ib_i+2Ab_ic_i & c_ib_j+2Ab_ic_j & c_ib_k+2Ab_ic_k\\ Bb_jb_i+c_jc_i & Bb_jb_j+c_jc_j & Bb_jb_k+c_jc_k & c_jb_i+2Ab_jc_i & c_jb_j+2Ab_jc_j & c_jb_k+2Ab_jc_k\\ Bb_kb_i+c_kc_i & Bb_kb_j+c_kc_j & Bb_kb_k+c_kc_k & c_kb_i+2Ab_kc_i & c_kb_j+2Ab_kc_j & c_kb_k+2Ab_kc_k\\ c_ib_i+2Ab_ic_i & c_jb_i+2Ab_jc_i & c_kb_i+2Ab_kc_i & Bc_ic_i+b_ib_i & Bc_ic_j+b_ib_j & Bc_ic_k+b_ib_k\\ c_ib_j+2Ab_ic_j & c_jb_j+2Ab_jc_j & c_kb_j+2Ab_kc_j & Bc_jc_i+b_jb_i & Bc_jc_j+b_jb_j & Bc_jc_k+b_jb_k\\ c_ib_k+2Ab_ic_k & c_jb_k+2Ab_jc_k & c_kb_k+2Ab_kc_k & Bc_kc_i+b_kb_i & Bc_kc_j+b_kb_j & Bc_kc_k+b_kb_k \end{bmatrix}$$

9.6 Problems

1. Reduce the general finite element equations (9.3.12) and (9.3.13) for one-dimensional elasticity and obtain the members of the stiffness and mass matrices.
2. Consider the one-dimensional elasticity of an elastic rod. Divide the rod into a number of elements, and for the base element (e), write the members of the stiffness and mass matrices associated with a one-dimensional simplex element.

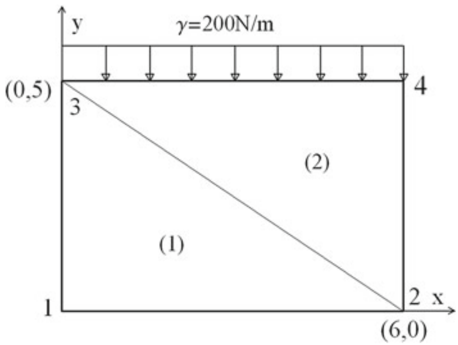

Fig. 9.3 A two-dimensional elastic domain

3. A rod of constant cross-section A, length L, mass density ρ, and the modules of elasticity E is considered. The rod is under a dynamic load $F = F_0 \sin \omega t$ at its free end at $x = L$ and fixed at the other end $x = 0$ (Fig. 9.2).

   Divide the rod into three elements of equal length and derive the elements of the global mass, stiffness, and force matrices of the rod.
4. Use Eqs. (9.3.12) and (9.3.13), and reduce them to the two-dimensional plane strain condition.
5. Do Problem 4 for the plane stress condition.
6. Consider a triangular simplex element and derive the members of the stiffness and mass matrices of Problem 4.
7. Assume a two-dimensional solution domain with two triangular simplex elements, as shown in Fig. 9.3.

   The coordinates of nodes 2 and 3 are (6,0) and (0,5) centimeters, respectively. The problem is assumed to be in a plane stress condition. The side (3-4) is under a static uniform distributed load of 200 N/cm. Obtain the members of the global stiffness and force matrices.
8. Compare the results of Problem 6 with those given in Sect. 9.5.

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Chapter 10
Elasticity, Variational Formulations

Abstract The derivation of finite element equation of motion based on the variational formulation is presented in this chapter. The Hamilton principle for elastic continuum is derived and basic relations for the linear elasticity, the constitutive law, and the kinematical relations, are presented. Employing the linear shape functions for three displacement components, the elements of the mass, stiffness, and force matrices are derived. The two-dimensional plane stress and plane strain elasticity are discussed and the axisymmetric elasticity formulation follows.

10.1 Introduction

In this Chapter, finite element analysis of the problems of elasticity based on variational formulations and the Ritz method are discussed. The field variables are considered to be the displacement components. The formulations start with the general form of Hamilton's principle for non-conservative external forces, and the general form of the three-dimensional elastodynamic finite element equation is derived. The problem is then reduced to the two-dimensional plane stress and plane strain problems. Finally, the axisymmetric elasticity problems are discussed and the related finite element equations are given. Through out the chapter, the material under consideration is assumed to be isotropic and homogeneous. The detail element matrices are derived using the first order isoparametric elements. The formulations may be extended to derive the element matrices based on the higher order elements.

10.2 Hamilton's Principle

Consider an elastic continuum under isothermal conditions, but loaded on its boundary surface by the traction forces. At an arbitrary point of the continuum, the traction force vector T_i and the body force vector X_i act. The virtual work of these forces

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due to the virtual displacement δu_i is [1]

$$\delta W = \int_S t_i^n \delta u_i dS + \int_V X_i \delta u_i dV \tag{10.2.1}$$

where T_i is the component of the traction vector at a point on a plane the outer unit normal vector of which is $\vec{n}$. From Cauchy's formula [2]

$$t_i^n = \sigma_{ij} n_j \tag{10.2.2}$$

where σ_{ij} is the stress tensor at the point. Substituting Eq. (10.2.2) in the first term of Eq. (10.2.1) and using the Gauss integral theorem to convert the area integral into the volume integral gives

$$\int_A t_i^n \delta u_i dA = \int_A \sigma_{ij} n_j \delta u_i dA = \int_V (\sigma_{ij} \delta u_i)_{,j} dV. \tag{10.2.3}$$

Carrying out the differentiation, gives

$$\int_A t_i^n \delta u_i dA = \int_V \sigma_{ij,j} \delta u_i dV + \int_V \sigma_{ij} \delta u_{i,j} dV. \tag{10.2.4}$$

The strain tensor is

$$\epsilon_{ij} = \tfrac{1}{2} \left(u_{i,j} + u_{j,i}\right). \tag{10.2.5}$$

Considering the above relation, and also the symmetry of the stress tensor, the last term in Eq. (10.2.4) can be written as

$$\int_V \sigma_{ij} \delta u_{i,j} dV = \int_V \frac{1}{2}(\sigma_{ij} \delta u_{i,j} + \sigma_{ji} \delta u_{j,i}) dV =$$
$$\int_V \sigma_{ij} \, \frac{1}{2}(\delta u_{i,j} + \delta u_{j,i}) dV = \int_V \sigma_{ij} \delta \epsilon_{ij} dV. \tag{10.2.6}$$

Substituting Eq. (10.2.6) in Eq. (10.2.4) and using the equilibrium equation

$$\sigma_{ij,j} + X_i = \rho \ddot{u}_i \tag{10.2.7}$$

where ρ is the mass density yields

$$\int_A t_i^n \delta u_i dA = -\int_V (X_i - \rho \ddot{u}_i) \delta u_i dV + \int_V \sigma_{ij} \delta \epsilon_{ij} dV \tag{10.2.8}$$

or

$$\int_V \sigma_{ij} \delta \epsilon_{ij} dV + \int_V \rho \ddot{u}_i \delta u_i dV = \int_A T_i^n \delta u_i dA + \int_V X_i \delta u_i dV. \tag{10.2.9}$$

The first term on the left-hand side of Eq. (10.2.9) is

$$\int_V \sigma_{ij}\delta\epsilon_{ij}dV = \delta\int_V U_0 dV = \delta U \tag{10.2.10}$$

where U_0 is the strain energy function per unit volume, and U is the total strain energy function. The time integration of Eq. (10.2.9) in the time interval t_1 to t_2 leads to

$$\begin{aligned}\int_{t_1}^{t_2}\int_V \delta U_0 dV dt + \int_{t_1}^{t_2}\int_V \rho\ddot{u}_i\delta u_i dV dt &= \int_{t_1}^{t_2}\int_A T_i^n\delta u_i dA dt\\ &+ \int_{t_1}^{t_2}\int_V X_i\delta u_i dV dt.\end{aligned} \tag{10.2.11}$$

The inertia term may be written as

$$\begin{aligned}\int_{t_1}^{t_2}\int_V \rho\ddot{u}_i\delta u_i dV dt &= \int_V (\int_{t_1}^{t_2}\rho\ddot{u}_i\delta u_i dt)dV\\ &= \int_V \rho\dot{u}_i\delta u_i|_{t_1}^{t_2} dV - \int_{t_1}^{t_2}\int_V \rho\dot{u}_i\delta\dot{u}_i dV dt\\ &= -\int_{t_1}^{t_2}(\delta\int_V \tfrac{1}{2}\rho\dot{u}_i\dot{u}_i dV)dt = -\int_{t_1}^{t_2}\delta K dt\end{aligned} \tag{10.2.12}$$

where the following conditions are used:

$$\delta u_i(t_1) = 0 \qquad \delta u_i(t_2) = 0. \tag{10.2.13}$$

The conditions (10.2.13) are used, since the values of u_i at $t = t_1$ and $t = t_2$ are known. The kinetic energy is defined as

$$K = \tfrac{1}{2}\int_V \rho\dot{u}_i\dot{u}_i dV. \tag{10.2.14}$$

Calling the virtual work of the traction force T_i and the body force X_i by δA as

$$\delta A = \int_A t_i^n\delta u_i dA + \int_V X_i\delta u_i dV \tag{10.2.15}$$

and substituting Eqs. (10.2.15) and (10.2.12) in (10.2.11) yields

$$\int_{t_1}^{t_2}\delta(U - K - A)dt = 0. \tag{10.2.16}$$

Calling the Lagrangian by

$$L = K - U, \tag{10.2.17}$$

Equation (10.2.16) becomes

$$\delta \int_{t_1}^{t_2} L dt + \delta \int_{t_1}^{t_2} A dt = 0 \tag{10.2.18}$$

where the variation symbol is placed before the time integration, as the virtual operator applies only to the space coordinates. Equation (10.2.18) is called *Hamilton's principle*, and expresses the law of dynamic equilibrium of a system. For elastodynamics problems, the expression

$$F = L + A \tag{10.2.19}$$

is the functional, in which its stationary value is associated with the dynamic equilibrium of a system. This variational principle is reduced to the principle of minimum potential energy by setting the kinetic energy expression equal to zero. The resulting expression is the functional of the elasto-static problems and may be used to derive the equilibrium equations of the elasticity problems.

10.3 Basic Relations of Linear Elasticity

The infinitesimal theory of linear elasticity is based on the linear strain-displacement relations and Hooke's law. The strain-displacement relations of the infinitesimal theory of elasticity in the rectangular Cartesian coordinates are

$$\begin{aligned}
\epsilon_{xx} &= \frac{\partial u}{\partial x} & \epsilon_{xy} &= \tfrac{1}{2}\,(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) \\
\epsilon_{yy} &= \frac{\partial v}{\partial y} & \epsilon_{yz} &= \tfrac{1}{2}\,(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}) \\
\epsilon_{zz} &= \frac{\partial w}{\partial z} & \epsilon_{zx} &= \tfrac{1}{2}\,(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z})
\end{aligned} \tag{10.3.1}$$

where u, v, and w are the displacement components along the x, y, and z axes, respectively. Equation (10.3.1), in matrix form, are

$$\begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \epsilon_{xy} \\ \epsilon_{yz} \\ \epsilon_{zx} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{1}{2}\frac{\partial}{\partial y} & \frac{1}{2}\frac{\partial}{\partial x} & 0 \\ 0 & \frac{1}{2}\frac{\partial}{\partial z} & \frac{1}{2}\frac{\partial}{\partial y} \\ \frac{1}{2}\frac{\partial}{\partial z} & 0 & \frac{1}{2}\frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}. \tag{10.3.2}$$

Calling

$$\begin{aligned} \{\epsilon\}^T &= \langle\ \epsilon_{xx}\ \epsilon_{yy}\ \epsilon_{zz}\ \epsilon_{xy}\ \epsilon_{yz}\ \epsilon_{zx}\ \rangle \\ \{f\}^T &= \langle\ u\ v\ w\ \rangle \end{aligned} \tag{10.3.3}$$

and the mathematical operator $[d]$ by

$$[d] = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{1}{2}\frac{\partial}{\partial y} & \frac{1}{2}\frac{\partial}{\partial x} & 0 \\ 0 & \frac{1}{2}\frac{\partial}{\partial z} & \frac{1}{2}\frac{\partial}{\partial y} \\ \frac{1}{2}\frac{\partial}{\partial z} & 0 & \frac{1}{2}\frac{\partial}{\partial x} \end{bmatrix} \tag{10.3.4}$$

the strain-displacement relations (10.3.2) are written as

$$\{\epsilon\} = [d]\{f\}. \tag{10.3.5}$$

The stress-strain relations of linear elasticity from Hooke's law in terms of the modulus of elasticity E and Poisson's ratio ν are

$$\begin{aligned} \epsilon_{xx} &= \frac{1}{E}\,[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})] \\ \epsilon_{yy} &= \frac{1}{E}\,[\sigma_{yy} - \nu(\sigma_{zz} + \sigma_{xx})] \\ \epsilon_{zz} &= \frac{1}{E}\,[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})] \end{aligned}$$

$$\begin{aligned}\epsilon_{xy} &= \frac{\tau_{xy}}{2G}\\ \epsilon_{yz} &= \frac{\tau_{yz}}{2G}\\ \epsilon_{zx} &= \frac{\tau_{zx}}{2G}\end{aligned} \tag{10.3.6}$$

where σ denotes normal stress, τ stands for the shear stress, E is the modulus of elasticity, ν is Poisson's ratio, and G is the shear modulus, which is related to E and ν as

$$G = \frac{E}{2(1+\nu)}. \tag{10.3.7}$$

Solving Eq. (10.3.6) for the stresses and writing them in matrix form gives

$$\{\sigma\} = [D]\{\epsilon\} \tag{10.3.8}$$

where

$$\{\sigma\}^T = \langle\ \sigma_{xx}\ \sigma_{yy}\ \sigma_{zz}\ \tau_{xy}\ \tau_{yz}\ \tau_{zx}\ \rangle. \tag{10.3.9}$$

The matrix of elastic constants $[D]$ is obtained as

$$[D] = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\ & 1 & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & sym. & & \frac{1-2\nu}{(1-\nu)} & 0 & 0 \\ & & & & \frac{1-2\nu}{(1-\nu)} & 0 \\ & & & & & \frac{1-2\nu}{(1-\nu)} \end{bmatrix}. \tag{10.3.10}$$

10.4 Finite Element Approximation

Consider a three-dimensional simplex base element (e) in the rectangular Cartesian coordinates, as shown in Fig. 10.1. Nodal coordinates are known in the global $xyz-$coordinates. We apply the finite element approximation to the displacement field. The resulting equations are then called the displacement based finite element model. In this model, the displacement fields for u, v, and w are compatible and their continuity in the solution domain is insured. In addition, the kinematical boundary conditions are easily applied. On the other hand, to satisfy the traction boundary conditions, they must be transformed into the displacement components, providing a complicated formulation. That is, while the solution domain is described in terms of the displacement components, the boundary conditions are given in terms of the forces. To handle the forced boundary conditions, the relationship between the traction and displacement components must be used to provide a proper formulation to

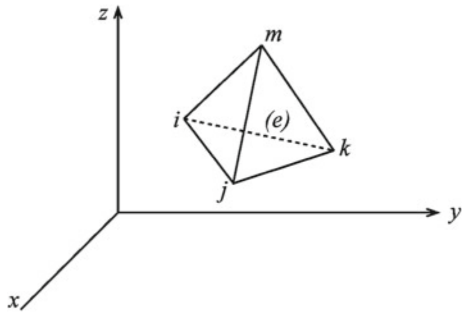

Fig. 10.1 Three-dimensional simplex element

satisfy the forced boundary conditions. For a general discussion of the displacement and stress-based finite element modeling, the reader is referred to the discussion given in [3, 4].

Considering the displacement formulation, the displacement components u, v, and w may be approximated by linear shape functions as

$$\begin{aligned} u^{(e)}(x, y, z, t) &= a_1 + a_2 x + a_3 y + a_4 z \\ v^{(e)}(x, y, z, t) &= a_5 + a_6 x + a_7 y + a_8 z \\ w^{(e)}(x, y, z, t) &= a_9 + a_{10} x + a_{11} y + a_{12} z \end{aligned} \tag{10.4.1}$$

where a_1 through a_{12} are functions of time and are found in terms of the nodal displacements as

$$\begin{cases} u = U_i \\ x = x_i \\ y = y_i \\ z = z_i \end{cases} \quad \begin{cases} u = U_j \\ x = x_j \\ y = y_j \\ z = z_j \end{cases} \quad \begin{cases} u = U_k \\ x = x_k \\ y = y_k \\ z = z_k \end{cases} \quad \begin{cases} u = U_m \\ x = x_m \\ y = y_m \\ z = z_m \end{cases} . \tag{10.4.2}$$

The same conditions are considered for v and w. Conditions (10.4.2) are substituted in the first of Eq. (10.4.1). Four equations are obtained for the four unknowns a_1 through a_4, which are then solved and obtained in terms of U_i, U_j, U_k, and U_m. Similar conditions are used for v and w and are substituted in the second and third of Eq. (10.4.1), through which the constants a_5 through a_{12} are obtained in terms of the nodal values of v and w. The results are

$$\begin{aligned} u^{(e)}(x, y, z, t) &= \langle\, N(x, y, z)\,\rangle^{(e)}\{U(t)\} \\ v^{(e)}(x, y, z, t) &= \langle\, N(x, y, z)\,\rangle^{(e)}\{V(t)\} \\ w^{(e)}(x, y, z, t) &= \langle\, N(x, y, z)\,\rangle^{(e)}\{W(t)\} \end{aligned} \tag{10.4.3}$$

where

$$\langle\, N \,\rangle^{(e)} = \langle\, N_i \; N_j \; N_k \; N_m \,\rangle^{(e)} \tag{10.4.4}$$

and the shape functions are

$$
\begin{aligned}
N_i &= \frac{a_i + b_i x + c_i y + d_i z}{6V} \\
N_j &= \frac{a_j + b_j x + c_j y + d_j z}{6V} \\
N_k &= \frac{a_k + b_k x + c_k y + d_k z}{6V} \\
N_m &= \frac{a_m + b_m x + c_m y + d_m z}{6V}
\end{aligned}
\tag{10.4.5}
$$

where the constants a_r, b_r, c_r, and d_r, $r = i, j, k, m$ are functions of the nodal coordinates, (see Chap. 4). The approximations given by Eq. (10.4.3) are of Kantrovich type, in which space and time are separated in the given form. We transfer Eq. (10.4.3) into the matrix form as

$$
\{f\}^{(e)} = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}^{(e)} = [N]^{(e)}\{\zeta\}^{(e)} \tag{10.4.6}
$$

where

$$
[N]^{(e)} = \begin{bmatrix} N_i & 0 & 0 & N_j & 0 & 0 & N_k & 0 & 0 & N_m & 0 & 0 \\ 0 & N_i & 0 & 0 & N_j & 0 & 0 & N_k & 0 & 0 & N_m & 0 \\ 0 & 0 & N_i & 0 & 0 & N_j & 0 & 0 & N_k & 0 & 0 & N_m \end{bmatrix} \tag{10.4.7}
$$

and

$$
\{\zeta\}^{(e)T} = \langle\langle\zeta_i\rangle\ \langle\zeta_j\rangle\ \langle\zeta_k\rangle\ \langle\zeta_m\rangle\rangle. \tag{10.4.8}
$$

Here, the matrix $\{\zeta(t)\}$ is the nodal displacement matrix which is a function of time, i.e.,

$$
\{\zeta_i(t)\}^{(e)} = \begin{Bmatrix} U_i(t) \\ V_i(t) \\ W_i(t) \end{Bmatrix}^{(e)}. \tag{10.4.9}
$$

From Eq. (10.3.5), the strain matrix is

$$
\{\epsilon\} = [d]\{f\}.
$$

Substituting from Eq. (10.4.6) gives

$$
\{\epsilon\}^{(e)} = [d][N]^{(e)}\{\zeta\}^{(e)}.
$$

Calling

$$
[B]^{(e)} = [d][N]^{(e)} \tag{10.4.10}
$$

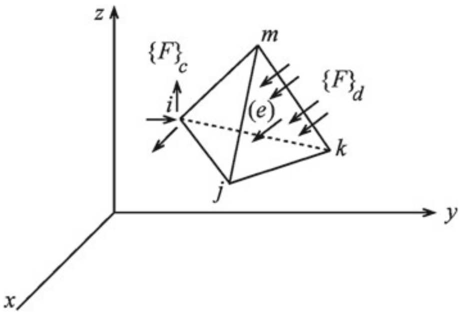

Fig. 10.2 Distributed and concentrated nodal forces

the strain matrix for the element (e) becomes

$$\{\epsilon\}^{(e)} = [B]^{(e)}\{\zeta\}^{(e)}. \tag{10.4.11}$$

For the simplex element shown in Fig. 10.1, the elements of matrix $[B]$ are

$$[B]^{(e)} = \frac{1}{6V}\begin{bmatrix} b_i & 0 & 0 & b_j & 0 & 0 & b_k & 0 & 0 & b_m & 0 & 0 \\ 0 & c_i & 0 & 0 & c_j & 0 & 0 & c_k & 0 & 0 & c_m & 0 \\ 0 & 0 & d_i & 0 & 0 & d_j & 0 & 0 & d_k & 0 & 0 & d_m \\ c_i/2 & b_i/2 & 0 & c_j/2 & b_j/2 & 0 & c_k/2 & b_k/2 & 0 & c_m/2 & b_m/2 & 0 \\ 0 & d_i/2 & c_i/2 & 0 & d_j/2 & c_j/2 & 0 & d_k/2 & c_k/2 & 0 & d_m/2 & c_m/2 \\ d_i/2 & 0 & b_i/2 & d_j/2 & 0 & b_j/2 & d_k/2 & 0 & b_k/2 & d_m/2 & 0 & b_m/2 \end{bmatrix}. \tag{10.4.12}$$

This matrix is constant for the simplex elements in the rectangular Cartesian coordinates.

To obtain the finite element solution, Hamilton's principle is used. From Eq. (10.2.9), we have

$$\int_V \sigma_{ij}\delta\epsilon_{ij}dV + \int_V \rho\ddot{u}_i\delta u_i dV = \int_A t_i^n \delta u_i dA + \int_V X_i \delta u_i dV. \tag{10.4.13}$$

The matrix form of this equation for the base element (e) is

$$\int_{V(e)} \{\delta\epsilon\}^T\{\sigma\}dV + \int_{V(e)} \rho\{\delta f\}^T\{\ddot{f}\}dV = \int_{A(e)} \{\delta f\}^T\{F_d\}dA$$
$$+ \sum_{i=1}^{n} \delta\{\zeta\}^T\{F_c\} + \int_{V(e)} \{\delta f\}^T\{X\}dV \tag{10.4.14}$$

where the surface traction forces are divided into two types, the distributed traction force over the boundary of the element $\{F_d\}$, and the concentrated force acting on a concentrated nodal point on the element boundary $\{F_c\}$, as shown in Fig. 10.2.

The virtual work of the distributed force is obtained by its product with the distributed virtual displacement $\delta\{f\}$, while the virtual work of the concentrated boundary force is obtained by its product with the nodal point displacement matrix $\delta\{\zeta\}$. In the latter case, it is convenient to choose a nodal point under the point of effect of the concentrated force.

Substituting from Eqs. (10.3.8), (10.4.6), and (10.4.11) in Eq. (10.4.14) gives

$$\delta\{\zeta\}^T\left(\int_{V(e)}[B]^T[D][B]dV\right)\{\zeta\}+\delta\{\zeta\}^T\left(\int_{V(e)}\rho[N]^T[N]dV\right)\{\ddot{\zeta}\}$$
$$=\delta\{\zeta\}^T\int_{A(e)}[N]^T\{F_d\}dA+\delta\{\zeta\}^T\int_{V(e)}[N]^T\{X\}dV+\delta\{\zeta\}^T\sum\{F_c\} \tag{10.4.15}$$

It is noted that the matrix $\{\zeta\}$ is the nodal point displacement matrix and is independent of coordinate variables, and is thus taken out of the integral sign. Since $\delta\{\zeta\}$ is the virtual displacement of the nodal points and is arbitrary, it is omitted from both sides of Eq. (10.4.15). Adding up the element equilibrium equations for all the elements in the solution domain, Eq. (10.4.15) provides

$$[M]\{\ddot{\Delta}\}+[K]\{\Delta\}=\{F_{BF}\}+\{F_d\}+\{F_c\} \tag{10.4.16}$$

where $[M]$, $[K]$, and $\{F\}$ are the total global mass, stiffness, and force matrices and $\{\Delta\}$ is the total unknown nodal displacement matrix. The mass, stiffness, and force matrices of the base element (e) are

$$\begin{aligned}
[m]^{(e)} &= \int_{V(e)}\rho[N]^T[N]dV\\
[k]^{(e)} &= \int_{V(e)}[B]^T[D][B]dV\\
\{f_{BF}\}^{(e)} &= \int_{V(e)}[N]^T\{X\}dV\\
\{f_d\}^{(e)} &= \int_{A(e)}[N]^T\{f_d\}dA\\
\{f_c\}^{(e)} &= \{f_c\}^{(e)}.
\end{aligned} \tag{10.4.17}$$

Equation (10.4.16) is a system of second order, coupled with an ordinary differential equation in time. It is called the finite element semi-discretized equilibrium equation, since although the nodal displacement matrix $\{\zeta\}$ is discretized in space, it is a continuous function of time. The dynamic equilibrium Eq. (10.4.16) may be solved by means of modal methods or direct integration techniques in the time domain. The first method uncouples the equations for the natural frequencies and their associated mode vectors. The second method establishes a time integration technique, and with a proper time increment, marches through time, as described in Chap. 7.

10.5 Two-Dimensional Elasticity

In the classical theory of elasticity, reduction from the three-dimensional theory to the two-dimensional theory is through the plane stress or plane strain assumptions. Each of these assumptions has valid physical applications. The finite element formulation based on the displacement components is discussed in this section.

10.5.1 Plane Strain Condition

According to the plane strain assumption, the normal and shear strains in the $z-$direction are zero, i.e.,

$$\epsilon_{zz} = \epsilon_{xz} = \epsilon_{yz} = 0. \tag{10.5.1}$$

Setting $\epsilon_{zz} = 0$, the normal stress in the $z-$direction from Hooke's law becomes

$$\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy}). \tag{10.5.2}$$

The strain-stress relations, using Eq. (10.5.2), are

$$\begin{aligned}
\epsilon_{xx} &= \frac{1+\nu}{E}\left[(1-\nu)\sigma_{xx} - \nu\sigma_{yy}\right]\\
\epsilon_{yy} &= \frac{1+\nu}{E}\left[(1-\nu)\sigma_{yy} - \nu\sigma_{xx}\right]\\
\epsilon_{xy} &= \frac{\tau_{xy}}{G} = \frac{2(1+\nu)}{E}\,\tau_{xy}.
\end{aligned} \tag{10.5.3}$$

Solving for the stresses gives

$$\{\sigma\} = [D]\{\epsilon\} \tag{10.5.4}$$

in which the elastic constants matrix is

$$[D] = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1 & \frac{\nu}{1-\nu} & 0\\ \frac{\nu}{1-\nu} & 1 & 0\\ 0 & 0 & \frac{1-2\nu}{2(1-\nu)}\end{bmatrix}. \tag{10.5.5}$$

The strain-displacement relations for the two-dimensional plane strain condition reduce to

$$\epsilon_{xx} = \frac{\partial u}{\partial x} \quad \epsilon_{yy} = \frac{\partial v}{\partial y} \quad \epsilon_{xy} = \tfrac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right). \tag{10.5.6}$$

In matrix form,

$$\{\epsilon\} = [d]\{f\} \tag{10.5.7}$$

where

$$[d] = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \frac{1}{2}\dfrac{\partial}{\partial y} & \frac{1}{2}\dfrac{\partial}{\partial x} \end{bmatrix} \tag{10.5.8}$$

and

$$\{f\}^T = \langle u \ v\rangle \quad \{\epsilon\}^T = \langle \epsilon_{xx} \ \epsilon_{yy} \ \epsilon_{xy}\rangle. \tag{10.5.9}$$

Now, the finite element solution requires the discretization of the solution domain into a number of elements. Considering the base element (e), the displacements u and v are approximated as

$$\begin{aligned} u^{(e)}(x, y, t) &= \langle N(x, y)\rangle^{(e)}\{U(t)\} \\ v^{(e)}(x, y, t) &= \langle N(x, y)\rangle^{(e)}\{V(t)\} \end{aligned} \tag{10.5.10}$$

where

$$\langle N(x, y)\rangle^{(e)} = \langle N_1 \ N_2 \ \ldots \ N_r\rangle^{(e)} \tag{10.5.11}$$

and r is the number of nodal points in the element (e). Equation (10.5.10) in matrix form is written as

$$\{f\}^{(e)} = [N]\{\zeta\}^{(e)} \tag{10.5.12}$$

where $\langle \zeta_i\rangle = \langle U_i \quad V_i\rangle$. Substituting Eq. (10.5.12) in Eq. (10.5.7) gives

$$\{\epsilon\}^{(e)} = [d]\{f\}^{(e)} = [d][N]^{(e)}\{\zeta\}^{(e)}. \tag{10.5.13}$$

Calling

$$[B]^{(e)} = [d][N]^{(e)} \tag{10.5.14}$$

the strain matrix for the element (e) becomes

$$\{\epsilon\}^{(e)} = [B]^{(e)}\{\zeta\}^{(e)}. \tag{10.5.15}$$

For example, matrix $[B]$ for a simplex two-dimensional triangular element, using the linear shape function, is

$$[B]^{(e)} = \frac{1}{2A}\begin{bmatrix} b_i & 0 & b_j & 0 & b_k & 0 \\ 0 & c_i & 0 & c_j & 0 & c_k \\ c_i/2 & b_i/2 & c_j/2 & b_j/2 & c_k/2 & b_k/2 \end{bmatrix}^{(e)}. \tag{10.5.16}$$

This matrix is used to evaluate the stiffness matrix.

10.5.2 Plane Stress Condition

The Plane stress condition is assumed when

$$\sigma_{zz} = \tau_{xz} = \tau_{yz} = 0. \tag{10.5.17}$$

This condition results in non-zero strain in the $z-$ direction as

$$\epsilon_{zz} = \frac{-\nu}{E}\,(\sigma_{xx} + \sigma_{yy}). \tag{10.5.18}$$

The stress-strain relations from Hooke's law are

$$\begin{aligned} \epsilon_{xx} &= \frac{1}{E}\,(\sigma_{xx} - \nu\sigma_{yy}) \\ \epsilon_{yy} &= \frac{1}{E}\,(\sigma_{yy} - \nu\sigma_{xx}) \\ \epsilon_{xy} &= \frac{1}{G}\,\tau_{xy} = \frac{2(1+\nu)}{E}\,\tau_{xy}. \end{aligned} \tag{10.5.19}$$

Solving for the stresses gives the stress-strain relations in matrix form as

$$\{\sigma\} = [D]\{\epsilon\} \tag{10.5.20}$$

in which the matrix of elastic constants for the plane stress condition becomes

$$[D] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}. \tag{10.5.21}$$

The strain-displacement relations remain the same as Eq. (10.5.7). Defining the shape function matrix $[N]$ for the displacement matrix $\{f\}$, as given by Eq. (10.5.12), the strain-displacement relations in terms of the shape function matrix become similar to Eq. (10.5.15). The matrix $[B]$ for the base element (e), assuming linear shape function, becomes identical with Eq. (10.5.16).

The stiffness matrix $[k]$ for the base element (e) is defined by Eq. (10.4.17) as

$$[k]^{(e)} = \int_{V(e)} [B]^T [D][B]dV \tag{10.5.22}$$

For a two-dimensional problem $dV = hdA$ in which h is the thickness of the element. The elasticity constant matrix $[D]$ for the plane strain condition is given by Eq. (10.5.5), and for the plane stress condition is given by Eq. (10.5.21). The final dimension of the stiffness matrix $[k]$ for a linear two-dimensional triangular element is 6×6.

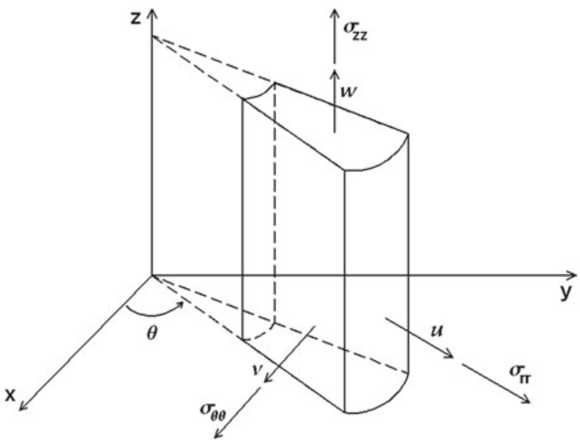

Fig. 10.3 An element of axisymmetric body under axisymmetric load ($v = 0$)

10.6 Axisymmetric Elasticity

Consider an elastic body of revolution under axisymmetric loads. We describe the geometry of the body in the cylindrical-coordinates with r, θ, and z indicating the radial, circumferential, and axial directions, respectively. The displacement components along the coordinates r, θ, and z are called u, v, and w, respectively. Since the body has three-dimensional configuration, both the axial stress and strain are non-zero, while the shear stresses and strains in $r\theta$ and θz-directions are zero due to the axisymmetric geometry and load distribution. Therefore, the stress and strain matrices are

$$\begin{aligned} \langle \sigma \rangle &= \langle \sigma_{rr} \ \sigma_{\theta\theta} \ \sigma_{zz} \ \tau_{rz} \rangle \\ \langle \epsilon \rangle &= \langle \epsilon_{rr} \ \epsilon_{\theta\theta} \ \epsilon_{zz} \ \epsilon_{rz} \rangle . \end{aligned} \tag{10.6.1}$$

The strain-displacement relations for this type of loading are

$$\begin{aligned} \epsilon_{rr} &= \frac{\partial u}{\partial r} \\ \epsilon_{\theta\theta} &= \frac{u}{r} \\ \epsilon_{zz} &= \frac{\partial w}{\partial z} \\ \epsilon_{rz} &= \tfrac{1}{2} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right) . \end{aligned} \tag{10.6.2}$$

Figure 10.3 shows an element of the body in the cylindrical coordinates with the direction of the stresses and displacements. Equation (10.6.2) in matrix form is

$$\{\epsilon\} = [d]\{f\} \tag{10.6.3}$$

where

$$\{f\} = \begin{Bmatrix} u \\ w \end{Bmatrix}$$

$$[d] = \begin{bmatrix} \dfrac{\partial}{\partial r} & 0 \\ \frac{1}{r} & 0 \\ 0 & \dfrac{\partial}{\partial z} \\ \frac{1}{2}\dfrac{\partial}{\partial z} & \frac{1}{2}\dfrac{\partial}{\partial r} \end{bmatrix}. \tag{10.6.4}$$

The strain-stress relations from Hooke's law are

$$\begin{aligned} \epsilon_{rr} &= \frac{1}{E}\left[\sigma_{rr} - \nu(\sigma_{\theta\theta} + \sigma_{zz})\right] \\ \epsilon_{\theta\theta} &= \frac{1}{E}\left[\sigma_{\theta\theta} - \nu(\sigma_{rr} + \sigma_{zz})\right] \\ \epsilon_{zz} &= \frac{1}{E}\left[\sigma_{zz} - \nu(\sigma_{rr} + \sigma_{\theta\theta})\right] \\ \epsilon_{rz} &= \frac{1}{G}\tau_{rz} = \frac{2(1+\nu)}{E}\tau_{rz}. \end{aligned} \tag{10.6.5}$$

Solving Eq. (10.6.5) for the stresses yields the relations in matrix form as

$$\{\sigma\} = [D]\{\epsilon\} \tag{10.6.6}$$

where the elasticity constant matrix is

$$[D] = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 0 \\ \frac{\nu}{1-\nu} & 1 & \frac{\nu}{1-\nu} & 0 \\ \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 1 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix}. \tag{10.6.7}$$

The finite element approximation of the body is in the rz-plane, which is the plane of revolution. The stresses and strains are all independent of the angle θ. Thus, an axisymmetric plane of the body in the rz-plane is sufficient for the analysis. Now, the section of the body in the rz-plane may be considered and drawn in the global rz-coordinates and divided into a number of elements. A base element (e) in the rz-plane is shown in Fig. 10.4.

Assuming the shape function N for the displacement vector, the displacement matrix in terms of the nodal displacement matrix $\{\zeta\}$ and shape function $[N]$ is

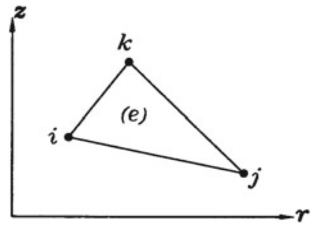

Fig. 10.4 An element in rz−plane

$$\{f\}^{(e)} = [N]^{(e)}\{\zeta\}^{(e)}. \tag{10.6.8}$$

Substituting Eq. (10.6.8) in (10.6.3) gives the strain matrix of element (e) as

$$\{\epsilon\}^{(e)} = [d][N]^{(e)}\{\zeta\}^{(e)} \tag{10.6.9}$$

or

$$\{\epsilon\}^{(e)} = [B]^{(e)}\{\zeta\}^{(e)} \tag{10.6.10}$$

where $[B]^{(e)} = [d][N]^{(e)}$. Due to the nature of the operator matrix $[d]$ in cylindrical coordinates, as given by Eq. (10.6.4), matrix $[B]$ is in general a function of the variable r. For the simplex linear triangular element, matrix $[N]$ is

$$[N]^{(e)} = \begin{bmatrix} N_i & 0 & N_j & 0 & N_k & 0 \\ 0 & N_i & 0 & N_j & 0 & N_k \end{bmatrix} \tag{10.6.11}$$

where

$$\begin{aligned} N_i &= \frac{a_i + b_i r + c_i z}{2A} \\ N_j &= \frac{a_j + b_j r + c_j z}{2A} \\ N_k &= \frac{a_k + b_k r + c_k z}{2A}. \end{aligned} \tag{10.6.12}$$

Substituting Eq. (10.6.12) in Eqs. (10.6.8) and (10.6.9) gives

$$[B]^{(e)} = \frac{1}{2A}\begin{bmatrix} b_i & 0 & b_j & 0 & b_k & 0 \\ \frac{2AN_i}{r} & 0 & \frac{2AN_j}{r} & 0 & \frac{2AN_k}{r} & 0 \\ 0 & c_i & 0 & c_j & 0 & c_k \\ \frac{c_i}{2} & \frac{b_i}{2} & \frac{c_j}{2} & \frac{b_j}{2} & \frac{c_k}{2} & \frac{b_k}{2} \end{bmatrix}. \tag{10.6.13}$$

Matrix $[B]$ for the axisymmetric elements, even when a triangular linear element is considered, is a function of the variable r and is not constant. The element integrations, as required for the stiffness and force matrices, are more complicated and need special mathematical techniques. One technique is to approximate the variables r and z at the centroid of the element. For example, the stiffness matrix for the element (e) is approximated as

$$[k]^{(e)} = \int_{V(e)} [B]^T [D][B] dV = [\bar{B}]^T [D][\bar{B}] \int_{V(e)} dV \tag{10.6.14}$$

where $[\bar{B}]$ is evaluated at the centroid of the triangular element (e). The integral of volume is

$$V = 2\pi \bar{R} A. \tag{10.6.15}$$

Here, A is the cross-sectional area of the element (e), and $\bar{R} = (r_i + r_j + r_k)/3$. The final approximate value for the element matrix $[k]$ is

$$[k]^{(e)} = [\bar{B}]^T [D][\bar{B}] 2\pi \bar{R} A. \tag{10.6.16}$$

The body force matrix can be evaluated using area coordinates. By the definition

$$\{f_{BF}\}^{(e)} = \int_{V(e)} [N]^T \{X\} dV. \tag{10.6.17}$$

For the axisymmetric condition

$$\{f_{BF}\}^{(e)} = \int_{V(e)} \begin{bmatrix} N_i & 0 \\ 0 & N_i \\ N_j & 0 \\ 0 & N_j \\ N_k & 0 \\ 0 & N_k \end{bmatrix} \begin{Bmatrix} \mathbf{R} \\ \mathbf{Z} \end{Bmatrix} 2\pi r dA \tag{10.6.18}$$

where **R** and **Z** are the components of the body force vector in the r and z-directions at the centroid of the element. In terms of the area coordinates

$$\{f_{BF}\}^{(e)} = \int_{A(e)} \begin{bmatrix} rL_1 & 0 \\ 0 & rL_1 \\ rL_2 & 0 \\ 0 & rL_2 \\ rL_3 & 0 \\ 0 & rL_3 \end{bmatrix} \begin{Bmatrix} \mathbf{R} \\ \mathbf{Z} \end{Bmatrix} 2\pi dA. \tag{10.6.19}$$

Substituting for r its expression in terms of the area coordinates as

$$r = R_i L_1 + R_j L_2 + R_k L_3 \tag{10.6.20}$$

in Eq. (10.6.19) results in integrations of terms of the type L_i^2 and L_iL_j. In Eq. (10.6.19), R_i, R_j, and R_k are the nodal coordinates of nodes i, j, and k, respectively. Using the rule of the area integration yields

$$\{f_{BF}\}^{(e)} = \frac{2\pi A}{12}\begin{Bmatrix} (2R_i + R_j + R_k)\mathbf{R} \\ (2R_i + R_j + R_k)\mathbf{Z} \\ (R_i + 2R_j + R_k)\mathbf{R} \\ (R_i + 2R_j + R_k)\mathbf{Z} \\ (R_i + R_j + 2R_k)\mathbf{R} \\ (R_i + R_j + 2R_k)\mathbf{Z} \end{Bmatrix}. \tag{10.6.21}$$

Expression (10.6.21) is the exact integration of the body force matrix when a linear shape function in a triangular element is used.

The force matrix related to the distributed traction force is

$$\{f_d\}^{(e)} = \int_{A(e)} [N]^T\{f_d\}dA. \tag{10.6.22}$$

In terms of the area coordinates, we have

$$\{f_d\}^{(e)} = \int_{A(e)} \begin{bmatrix} L_1 & 0 \\ 0 & L_1 \\ L_2 & 0 \\ 0 & L_2 \\ L_3 & 0 \\ 0 & L_3 \end{bmatrix} \begin{Bmatrix} f_r \\ f_z \end{Bmatrix} dA. \tag{10.6.23}$$

Assuming a triangular element with external constant distributed traction forces acting on its $ij-$side, Eq. (10.6.23) becomes ($N_k = 0$ on $ij-$side)

$$\{f_d\}^{(e)} = \int_{L_{ij}} \begin{bmatrix} rL_1 & 0 \\ 0 & rL_1 \\ rL_2 & 0 \\ 0 & rL_2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} f_r \\ f_z \end{Bmatrix} 2\pi dL_{ij} \tag{10.6.24}$$

where $dA = 2\pi r dL$. Using the rule of length integral, we get

$$\{f_d\}^{(e)} = \frac{2\pi L_{ij}}{6} \begin{Bmatrix} (2R_i + R_j)f_r \\ (2R_i + R_j)f_z \\ (R_i + 2R_j)f_r \\ (R_i + 2R_j)f_z \\ 0 \\ 0 \end{Bmatrix} \tag{10.6.25}$$

where L_{ij} is the length of the $ij-$side of the element.

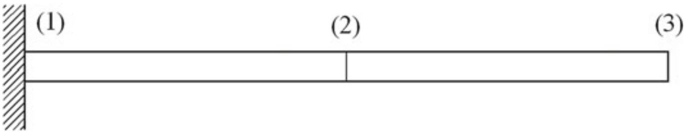

Fig. 10.5 An elastic rod under static load

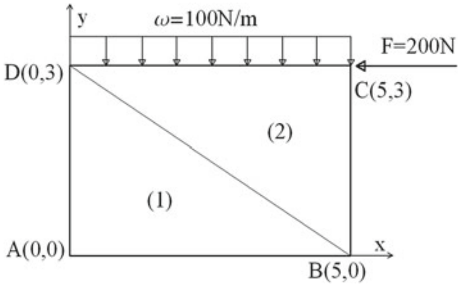

Fig. 10.6 A plane stress elastic domain

10.7 Problems

1. Reduce the general form of the finite element Eq. (10.4.15) to the one-dimensional case. Obtain the members of the stiffness and mass matrices for the base element (e) with a simplex linear shape function.
2. An elastic rod of cross-sectional area A, length L, mass density ρ, and the modules of elasticity E is assumed. The rod is divided into two elements of equal length, as shown in Fig. 10.5.

   The rod is under static load. Compute the members of the global stiffness matrix for the simplex linear element.
3. Reconsider Problem 2, where, at the free end node 3, the dynamic load $F = F_0 \sin \omega t$ is acting. Derive the members of the global mass, stiffness, and force matrices of the bar.
4. Use Eqs. (10.5.5) and (10.5.16) and obtain the elements of the stiffness matrix of a plane strain condition.
5. For a triangular simplex element, use Eq. (10.4.15) to derive the members of the element mass matrix.
6. For a triangular element of axisymmetric elasticity, derive the elements of the stiffness matrix using Eq. (10.6.14).
7. Consider a rectangular domain in the plane stress condition, as shown in Fig. 10.6.

   The coordinates of the node are shown in centimeters on the figure. The distributed force $w = 100\ N/cm$ is applied on the top surface of element (2) and a horizontal force $F = 200\ N$ is acting on node C. Nodes A and B are fixed. The domain is divided into two elements. Write the global stiffness and force matrices, and using the algorithm of wiping rows and columns, find the horizontal and vertical displacements at nodes C and D.

References

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Chapter 11
Torsion of Prismatic Bars

Abstract This chapter deals with the torsion of prismatic bars. Bars of general cross-sectional area are considered and the equilibrium equation of the bars under torsional couple is derived, using the concept of stress function. The final form of the equilibrium equation is that of the Poisson equation, where its functional is similar to that of the elastic membrane. Employing a base element (e), the Ritz method is used to derive the stiffness and force matrices of the finite element equation.

11.1 Introduction

The bars of the general cross-sectional area may be under the application of a torsional moment. The stresses, as the result of the torsional moment, are pure shear stresses acting on the cross-section of the bar. For the bars of the circular cross-sectional area, the shear stress is obtained using the classical theory of strength of materials. This subject is usually studied under the topic of power transmitting shafts. For the bars of triangular, elliptical, and rectangular cross-sections, the analytical methods of analysis are available in the theory of elasticity. However, there are no analytical solutions for shear stresses produced by the applied torsional moment for bars having other geometrical types of cross-sections.

The governing equation of the bars under the torsional moment is Poisson's equation in terms of the stress function. The analytical solution of Poisson's equation is simple, as the standard method of separation of variables may be used. To obtain the constants of integration, however, the boundary conditions should be used. The constants of integration, using boundary conditions, may be obtained only for the classical types of cross-sectional geometries, such as circle, ellipse, triangle, and rectangle.

For bars of general cross-section under torsion, the finite element method may be used for stress analysis. The Galerkin or Ritz method may be applied, as both a

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boundary value problem and the variational formulations for the torsion problems exist.

11.2 Equilibrium Equation for Torsion of Bars

Consider a prismatic bar of general cross-sectional area and length L. The cross-section is constant along the bar. The bar is fixed at one end in the xy—plane, as shown in Fig. 11.1. At distance z away from the fixed end, the torsional couple T is applied. Due to the applied couple T, the bar twists at an angle of α. Calling the angle of rotation per unit length by θ,

$$\alpha = \theta z, \tag{11.2.1}$$

we assume that both θ and α are small. The section of the bar at distance z from the fixed end is shown in Fig. 11.2.

The point P on the cross-section with coordinates (x, y, z) moves to point $P'(x + u,\ y + v,\ z + w)$ after deformation. The projection of P' on xy—plane is P_1', as shown in Fig. 11.2. The angle of rotation between lines OP and OP_1' is α. If α is small, then $\cos\alpha \approx 1$ and $\sin\alpha = \alpha$, and the components of displacements are

$$\begin{aligned} u &= r\cos(\beta+\alpha) - r\cos\beta = r\cos\alpha\cos\beta - r\sin\alpha\sin\beta - r\cos\beta \\ &\approx -y\alpha \\ v &= r\sin(\beta+\alpha) - r\sin\beta = r\sin\alpha\cos\beta + r\cos\alpha\sin\beta - r\sin\beta \\ &\approx x\alpha. \end{aligned} \tag{11.2.2}$$

Combining with Eq. (11.2.1), gives us

$$\begin{aligned} u &= -\theta yz \\ v &= \theta xz. \end{aligned} \tag{11.2.3}$$

The deflection component along the z—axis is denoted by w and is called the *warping function*, and is assumed to be a function of x and y, as $w = w(x, y)$. The strain components are thus

$$\begin{aligned} \epsilon_{xx} &= \epsilon_{yy} = \epsilon_{zz} = \epsilon_{xy} = 0 \\ \epsilon_{xz} &= \tfrac{1}{2}\left(\frac{\partial w}{\partial x} - y\theta\right) \\ \epsilon_{yz} &= \tfrac{1}{2}\left(\frac{\partial w}{\partial y} + x\theta\right). \end{aligned} \tag{11.2.4}$$

Using Hooke's law, the stresses are

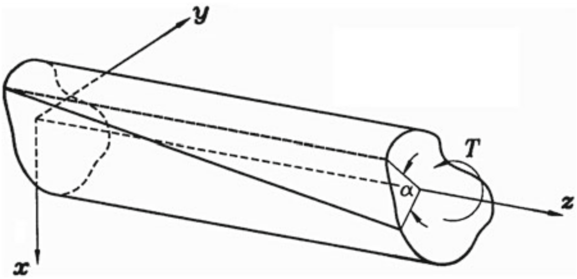
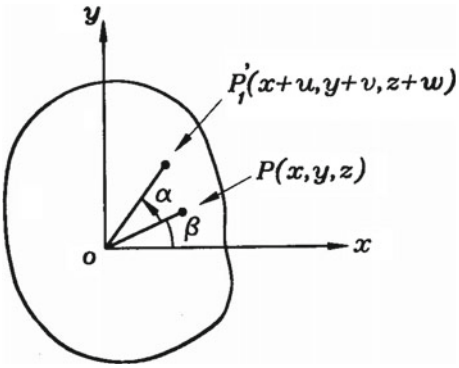


Fig. 11.1 A prismatic bar under torsional couple

Fig. 11.2 Cross-section of the bar at point z from the fixed end



$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \tau_{xy} = 0$$
$$\tau_{xz} = G\left(\frac{\partial w}{\partial x} - y\theta\right)$$
$$\tau_{yz} = G\left(\frac{\partial w}{\partial y} + x\theta\right). \tag{11.2.5}$$

The equilibrium equations of the three-dimensional theory of elasticity, in the absence of body force, are

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0$$
$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 0$$
$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = 0. \tag{11.2.6}$$

Substituting the stresses from Eq. (11.2.5) into (11.2.6), the first two equilibrium Eq. (11.2.6) are identically satisfied and the third equilibrium equation yields

$$\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0. \tag{11.2.7}$$

The solution to Eq. (11.2.7) results in an expression for the warping function.

The compatibility equations for linear elasticity from Eq. (9.2.12) are now used to relate the strain components. Four compatibility equations are identically satisfied using the strain components (11.2.4), and the last two are

$$\begin{aligned} \frac{\partial}{\partial x}\left(\frac{\partial \epsilon_{yz}}{\partial x} - \frac{\partial \epsilon_{xz}}{\partial y}\right) &= 0 \\ -\frac{\partial}{\partial y}\left(-\frac{\partial \epsilon_{yz}}{\partial x} + \frac{\partial \epsilon_{xz}}{\partial y}\right) &= 0. \end{aligned} \tag{11.2.8}$$

Integration yields

$$\frac{\partial \epsilon_{yz}}{\partial x} - \frac{\partial \epsilon_{xz}}{\partial y} = \text{const} = \theta. \tag{11.2.9}$$

Substituting from Hooke's law

$$\begin{aligned} \epsilon_{xz} &= \frac{1}{2G}\,\tau_{xz} \\ \epsilon_{yz} &= \frac{1}{2G}\,\tau_{yz}, \end{aligned} \tag{11.2.10}$$

the compatibility Eq. (11.2.9) becomes

$$\frac{\partial \tau_{yz}}{\partial x} - \frac{\partial \tau_{xz}}{\partial y} = 2G\theta. \tag{11.2.11}$$

Now, introducing the stress function ϕ such that

$$\tau_{xz} = \frac{\partial \phi}{\partial y} \quad \tau_{yz} = -\frac{\partial \phi}{\partial x}, \tag{11.2.12}$$

the compatibility Eq. (11.2.11), in terms of the stress function, becomes

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -2G\theta. \tag{11.2.13}$$

The solution to Eq. (11.2.13) for the stress function ϕ provides the expressions for the shear stresses from Eq. (11.2.12). However, the solution of Eq. (11.2.13) for the stress function ϕ needs boundary conditions to evaluate the constants of integration.

The general form of the expressions for boundary conditions in three-dimensional elasticity for a body under surface tractions is

$$\begin{aligned}\sigma_{xx}l + \tau_{xy}m + \tau_{xz}n &= T_x\\ \tau_{xy}l + \sigma_{yy}m + \tau_{yz}n &= T_y\\ \tau_{xz}l + \tau_{yz}m + \sigma_{zz}n &= T_z\end{aligned} \tag{11.2.14}$$

where $l,\ m$, and n are the cosine directions of a unit outer normal vector to the boundary, and $T_x,\ T_y$, and T_z are the components of the traction force acting on the boundary. For the case of torsion, the first two of Eq. (11.2.14) are identically satisfied, and the third equation is reduced to

$$\tau_{xz}l + \tau_{yz}m = 0. \tag{11.2.15}$$

For the torsion problems, the traction force on the boundary is zero. From Fig. 11.3, the cosine directors of unit outer normal vector $\vec{n}$ to the boundary of the bar cross-section in xy—plane are

$$\begin{aligned}l &= \cos(n, x) = \frac{dy}{ds}\\ m &= \cos(n, y) = \frac{dx}{ds}.\end{aligned} \tag{11.2.16}$$

Substituting Eqs. (11.2.16) and (11.2.12) into Eq. (11.2.15) gives

$$\frac{\partial\phi}{\partial y}\frac{dy}{ds} + \frac{\partial\phi}{\partial x}\frac{dx}{ds} = \frac{d\phi}{ds} = 0. \tag{11.2.17}$$

Integrating with respect to s yields

$$\phi = cte \quad \text{on the boundary}\ . \tag{11.2.18}$$

Since the derivative of ϕ with respect to x and y has physical meaning (sheer stresses), the value of ϕ itself on the boundary may be selected as any constant, such as zero. Thus, the value of the constant in Eq. (11.2.18) is selected as zero.

The torque acting on the section of the bar is computed from the expression

$$\begin{aligned}T &= \int\int(\tau_{yz}x - \tau_{xz}y)dxdy\\ &= -\int\int\left(\frac{\partial\phi}{\partial x}x + \frac{\partial\phi}{\partial y}y\right)dxdy.\end{aligned} \tag{11.2.19}$$

Integrating by parts and making use of the boundary conditions gives us

$$T = 2\int\int\phi dxdy. \tag{11.2.20}$$

This equation relates the applied torque T to the stress function ϕ.

11.3 Finite Element Solution

The equilibrium equation of torsion of prismatic bars in terms of the stress function ϕ is identical to the membrane equation. All the finite element techniques that were discussed for the solution of elastic shallow membrane problems also apply to torsion problems.

The equilibrium equation for the torsion of the bar from Eq. (11.2.13) is

$$\nabla^2\phi = -2G\theta. \tag{11.3.1}$$

From the membrane analogy, the expression for the functional V is

$$V = \frac{1}{2}\int_D \left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + G\theta\phi\right] dxdy. \tag{11.3.2}$$

The boundary condition for ϕ is

$$\phi = 0 \quad \text{on} \quad \partial D \tag{11.3.3}$$

where D is the solution domain, being the cross-sectional area of the prismatic bar, and ∂D is its boundary.

To solve the torsion problem by the finite element method, the cross-section of the prismatic bar is divided into an arbitrary number of elements with a proper approximating shape function for the base element (e). The finite element model is obtained through the Galerkin method, if Eq. (11.3.1) is used. On the other hand, the variational formulation based on the Ritz method may be used, if the finite element method is based on the functional expression given by Eq. (11.3.2). These techniques were described in Chap. 6 for conducting heat transfer problems.

To obtain the finite element model, the solution domain is divided into a number of elements and the shape function for the stress function ϕ in the base element (e) is approximated as

$$\phi^{(e)}(x, y) = \langle N(x, y)\rangle\{\Phi\}^{(e)}, \tag{11.3.4}$$

where $N(x, y)$ is the approximating function for the stress function in element (e). Using the variation formulation and the Ritz method, Eq. (11.3.4) is substituted in the expression for the functional in Eq. (11.3.2) and the Ritz method is applied as

$$\begin{aligned}\frac{\partial V}{\partial\phi_i} &= \sum_{e=1}^{NE}\frac{\partial V^{(e)}}{\partial\phi_i}\\ &= \sum_{e=1}^{NE}\int_{D(e)}\left[\frac{\partial\phi}{\partial x}\,\frac{\partial}{\partial\phi_i}\left(\frac{\partial\phi}{\partial x}\right) + \frac{\partial\phi}{\partial y}\,\frac{\partial}{\partial\phi_i}\left(\frac{\partial\phi}{\partial y}\right) + G\theta\,\frac{\partial\phi}{\partial\phi_i}\right] dxdy = 0.\end{aligned} \tag{11.3.5}$$

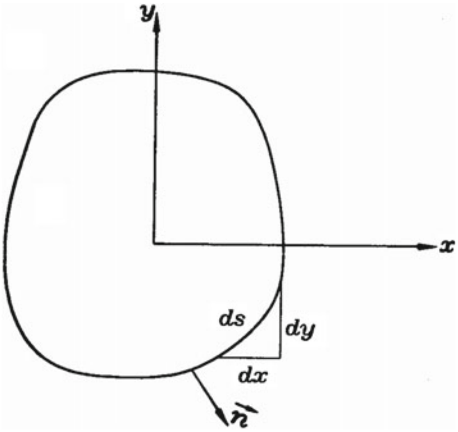


Fig. 11.3 Cross-section of the prismatic bar in xy—plane

This leads to the final form of the finite element equilibrium equation

$$[K]\{\Phi\} = \{F\} \tag{11.3.6}$$

where the element stiffness and force matrices are

$$[k]^{(e)} = \int_{D(e)} [B]^T [D][B] dV$$
$$\{f\}^{(e)} = G\theta \int_{D(e)} \{N\} dxdy \tag{11.3.7}$$

where $[D] = [I]$ is the unit matrix. The equilibrium Eq. (11.3.6) is subjected to the boundary conditions $\phi = 0$.

Once the stress function matrix $\{\phi\}$ is determined from the solution of Eq. (11.3.6), the shear stresses τ_{xz} and τ_{yz} can be determined from Eq. (11.2.12) as

$$\tau_{xz} = \frac{\partial \phi}{\partial y} \qquad \tau_{yz} = -\frac{\partial \phi}{\partial x}. \tag{11.3.8}$$

For element (e), the shear stresses are

$$\tau_{xz}^{(e)} = \frac{\partial}{\partial y} \left(\langle N \rangle \{\phi\}\right)^{(e)}$$
$$\tau_{xz}^{(e)} = \langle \frac{dN}{dy} \rangle^{(e)} \{\phi\}^{(e)}. \tag{11.3.9}$$

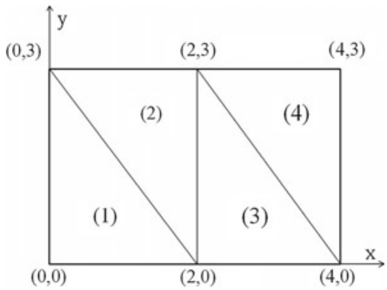


Fig. 11.4 A prismatic bar of rectangular cross-section

Similarly

$$\tau_{yz}^{(e)} = \langle -\frac{dN}{dx} \rangle^{(e)} \{\phi\}^{(e)}. \tag{11.3.10}$$

Since the element shape function and the nodal values for ϕ in element (e) are known, the distribution of the shear stresses τ_{xz} and τ_{yz} in the element (e) are found from Eqs. (11.3.9) and (11.3.10).

11.4 Problems

1. Consider a prismatic bar under torsional couple T. The Cross-section of the bar is divided into a number of triangular elements. Obtain the members of the stiffness and force matrices of the base element (e).

2. A prismatic bar of rectangular cross-section under the torsional couple T is considered. The cross-section of the bar is divided into four elements, as shown in Fig. 11.4. Determine the global stiffness and force matrices.

Further Readings

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Chapter 12
Thermoelasticity

Abstract The chapter begins with an explanation of the basic governing equations of linear thermoelasticity, including the equations of motion, the linear thermoelastic constitutive law, and the kinematical relations. Using the governing equations, the Navier equations of motion in terms of the displacement components are derived. The condition for the case of zero thermal stresses is then discussed. The displacement-based finite element equations in conjunction with the heat conduction equation are derived employing the Galerkin method. The final of the finite element equation of motion is reduced to the case of two-dimensional thermoelasticity problems, where the element matrices for the base element (e) are derived. The dynamic finite element equation is further reduced to the one-dimensional case, where for a one-dimensional simplex element the detail of element matrices are calculated.

12.1 Introduction

In general, the variation of the temperature field within an elastic continuum results in thermal stresses. The influence of the temperature field in the governing equations of thermoelasticity is through the constitutive law. The theory of linear thermoelasticity is based on the linear addition of thermal strains to mechanical strains. While the equilibrium and compatibility equations remain the same for elasticity problems, the main difference rests in the constitutive law. On this basis, many techniques that have been developed for solving the elasticity problems are also applicable to the thermoelasticity problems. There are, however, special classes of thermoelasticity problem, such as the coupled thermoelasticity problem, which require entirely different mathematical approaches and means of analysis. Even for these classes of problem, some of the basic equations remain the same. It is, therefore, necessary to define the basic laws of thermoelasticity and to derive the governing equations. In this chapter, the thermoelastic material is assumed to be homogeneous and isotropic, with constant material properties.

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12.2 Governing Equations

The general governing equations of the classical theory of thermoelasticity are the equations of motion, the compatibility equations, and the constitutive law of linear thermoelasticity. The equation of motion was derived and is given by Eq. (9.2.8) as

$$\sigma_{ij,j} + X_i = \rho \ddot{u}_i \qquad \text{in V.} \tag{12.2.1}$$

For the infinitesimal theory of thermoelasticity, the displacement gradient is small, so that the strain tensor ϵ_{ij} becomes linear as

$$\epsilon_{ij} = \tfrac{1}{2}\,(u_{j,i} + u_{i,j}). \tag{12.2.2}$$

In the classical theory of linear thermoelasticity, the components of the strain tensor are linear functions of the components of the elastic strain tensor and thermal strain; that is [1],

$$\epsilon_{ij} = \epsilon_{ij}^{e} + \epsilon_{ij}^{T} \tag{12.2.3}$$

where ϵ_{ij}^{e} denotes the elastic strain and ϵ_{ij}^{T} stands for the thermal strain. Consider a cubic element the temperature of which is raised from the reference temperature T_0, at which strains and stresses are zero, to the temperature T. The sides of the element are free from traction. The thermal strain of the element due to the temperature change is

$$\epsilon_{ij}^{T} = \alpha(T - T_0)\delta_{ij} \tag{12.2.4}$$

where $T - T_0$ is the temperature change, and α is the coefficient of linear thermal expansion. The relation (12.2.4) represents a property of an isotropic body, in which a temperature change $T - T_0$ results in no change of shear angles, the only result being a change of volume in the element.

The elastic strain tensor is linearly proportional to the stress tensor as

$$\epsilon_{ij}^{e} = \frac{1}{2G}\,(\sigma_{ij} - \frac{\nu}{1+\nu}\,\sigma_{kk}\delta_{ij}) \tag{12.2.5}$$

where G is the shear modulus and ν is Poisson's ratio. Equation (12.2.5) is known as the constitutive law of linear elasticity, or Hooke's law. From Eqs. (12.2.4) and (12.2.5), the total strain tensor is

$$\epsilon_{ij} = \frac{1}{2G}\,(\sigma_{ij} - \frac{\nu}{1+\nu}\,\sigma_{kk}\delta_{ij}) + \alpha(T - T_0)\delta_{ij}. \tag{12.2.6}$$

Equation (12.2.6) is called the constitutive law of linear thermoelasticity. Solving this equation for the stress tensor σ_{ij} gives

$$\sigma_{ij} = 2G\left[\epsilon_{ij} + \frac{\nu}{1-2\nu}\,(\epsilon_{kk} - \frac{1+\nu}{\nu}\,\alpha(T-T_0))\delta_{ij}\right]. \tag{12.2.7}$$

It is sometimes useful to write the stress-strain relations in terms of Lamé constants λ and μ, where μ is the same as shear modulus G. In terms of Lamé constants, $\lambda = 2G\nu/(1-2\nu)$ and $\mu = G$, the strain tensor is related to the stress tensor by

$$\epsilon_{ij} = \frac{1}{2\mu}(\sigma_{ij} - \frac{\lambda}{3\lambda+2\mu}\sigma_{kk}\delta_{ij}) + \alpha(T-T_0)\delta_{ij}. \tag{12.2.8}$$

Solving for the stress tensor σ_{ij}, gives

$$\sigma_{ij} = 2\mu\epsilon_{ij} + [\lambda\epsilon_{kk} - \alpha(3\lambda+2\mu)(T-T_0)]\delta_{ij}. \tag{12.2.9}$$

From Eq. (12.2.6), denoting the first invariant of the strain tensor by $e = I_1^{'} = \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}$ and the first invariant of the stress tensor by $I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$, we obtain

$$e - 3\alpha(T-T_0) = \frac{1}{2G}\,\frac{1-2\nu}{1+\nu}\,I_1. \tag{12.2.10}$$

For an isotropic elastic material in a state of uniform temperature, Eq. (12.2.10) reduces to

$$I_1 = 2G\,\frac{1+\nu}{1-2\nu}\,e = 3Ke. \tag{12.2.11}$$

The constant K is called the *bulk modulus*. The quantity e, for the infinitesimal displacement field, is the change of volume per unit volume of the material. Thus, Eq. (12.2.11) relates the first invariant of the stress tensor to the first invariant of the strain tensor. For the special case of hydrostatic compression

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -p, \qquad \sigma_{xy} = \sigma_{yz} = \sigma_{zx} = 0, \tag{12.2.12}$$

the first invariant of the stress tensor is $I_1 = -3p$. Substituting for e using Eq. (12.2.2) into (12.2.11), we obtain

$$\frac{\Delta V}{V} = -\frac{p}{K} \tag{12.2.13}$$

where V is the volume and ΔV is the change of volume. Equation (12.2.13) states that a material under hydrostatic compression is compressible, provided that the bulk modulus of the material is of a finite value. For incompressible materials, the bulk modulus K approaches infinity. For this special case, from Eqs. (12.2.13) and (12.2.11), $\nu = 1/2$, and since $G = E/2(1+\nu)$, $G = E/3$. The value of $\nu = 1/2$ is an upper limit of Poisson's ratio.

The stress-strain relations in terms of Young's modulus and Poisson's ratio are frequently used, and they are

$$\begin{aligned}
\epsilon_{xx} &= \frac{1}{E}\left[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\right] + \alpha(T - T_0) \\
\epsilon_{yy} &= \frac{1}{E}\left[\sigma_{yy} - \nu(\sigma_{zz} + \sigma_{xx})\right] + \alpha(T - T_0) \\
\epsilon_{zz} &= \frac{1}{E}\left[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})\right] + \alpha(T - T_0) \\
\epsilon_{xy} &= \frac{\sigma_{xy}}{2G} \\
\epsilon_{yz} &= \frac{\sigma_{yz}}{2G} \\
\epsilon_{zx} &= \frac{\sigma_{zx}}{2G}.
\end{aligned} \tag{12.2.14}$$

The relationship between Lamé constants λ and μ, Young's modulus E, Poisson's ratio ν, and the bulk modulus K are [2] (see also the complete table in Hetnarski and Ignaczak [3], p. 144)

$$\begin{aligned}
\lambda &= \frac{E\nu}{(1+\nu)(1-2\nu)} = \frac{2G\nu}{1-2\nu} = \frac{G(E-2G)}{3G-E} = \frac{3K\nu}{1+\nu} = \frac{3K(3K-E)}{9K-E} \\
&= K - \frac{2G}{3} \\
\mu &= G = \frac{E}{2(1+\nu)} = \frac{\lambda(1-2\nu)}{2\nu} = \frac{3}{2}(K-\lambda) = \frac{3KE}{9K-E} = \frac{3K(1-2\nu)}{2(1+\nu)} \\
E &= 2G(1+\nu) = \frac{\lambda(1+\nu)(1-2\nu)}{\nu} = \frac{G(3\lambda+2G)}{\lambda+G} = 3K(1-2\nu) \\
&= \frac{9KG}{3K+G} = \frac{9K(K-\lambda)}{3K-\lambda} \\
\nu &= \frac{E}{2G} - 1 = \frac{\lambda}{2(\lambda+G)} = \frac{3K-E}{6K} = \frac{\lambda}{3K-\lambda} = \frac{3K-2G}{2(3K+G)} \\
K &= \frac{E}{3(1-2\nu)} = \lambda + \frac{2G}{3} = \frac{2G(1+\nu)}{3(1-2\nu)} = \frac{GE}{3(3G-E)} = \frac{\lambda(1+\nu)}{3\nu}.
\end{aligned} \tag{12.2.15}$$

12.3 Displacement Formulation

The equation of motion in terms of the stress tensor is

$$\sigma_{ij,j} + X_i = \rho \ddot{u}_i. \tag{12.3.1}$$

The stresses can be expressed in terms of the strains and then in terms of the displacements. Substituting for the strain tensor ϵ_{ij} in terms of the displacement u_i, Eq. (12.2.7) gives

$$\sigma_{ij} = \mu(u_{i,j} + u_{j,i}) + [\lambda u_{k,k} - \alpha(3\lambda + 2\mu)(T - T_0)]\delta_{ij}. \tag{12.3.2}$$

Taking the partial derivative of Eq. (12.3.2) and substituting into the equation of motion (12.3.1) yields [4–6]

$$\mu u_{i,kk} + (\lambda + \mu)u_{k,ki} - (3\lambda + 2\mu)\alpha T_{,i} + X_i = \rho \ddot{u}_i. \tag{12.3.3}$$

Equation (12.3.3) is called the *Navier equation*. It is expressed in terms of the displacement components along the three coordinate axes.

The boundary conditions must be satisfied on the surface boundary of the body. If the traction components on the boundary are given as t_i^n, Cauchy's formula, see Eq. (9.3.4),

$$t_i^n = \sigma_{ij} n_j \tag{12.3.4}$$

states the boundary conditions. Since, however, the problem formulation is in terms of displacements, the prescribed traction on the boundary can be related to the displacement components by Eqs. (12.3.2) and (12.3.4) as

$$t_i^n = \{\mu(u_{i,j} + u_{j,i}) + [\lambda u_{k,k} - \alpha(3\lambda + 2\mu)(T - T_0)]\delta_{ij}\}n_j. \tag{12.3.5}$$

Writing Eq. (12.3.5) in components yields

$$\begin{aligned}
t_x^n &= \lambda e n_x + G(n_x \frac{\partial u}{\partial x} + n_y \frac{\partial u}{\partial y} + n_z \frac{\partial u}{\partial z}) + G(n_x \frac{\partial u}{\partial x} + n_y \frac{\partial v}{\partial x} + n_z \frac{\partial w}{\partial x}) \\
&\quad - \frac{E\alpha(T - T_0)}{1 - 2\nu} n_x \\
t_y^n &= \lambda e n_y + G(n_x \frac{\partial v}{\partial x} + n_y \frac{\partial v}{\partial y} + n_z \frac{\partial v}{\partial z}) + G(n_x \frac{\partial u}{\partial y} + n_y \frac{\partial v}{\partial y} + n_z \frac{\partial w}{\partial y}) \\
&\quad - \frac{E\alpha(T - T_0)}{1 - 2\nu} n_y \\
t_z^n &= \lambda e n_z + G(n_x \frac{\partial w}{\partial x} + n_y \frac{\partial w}{\partial y} + n_z \frac{\partial w}{\partial z}) + G(n_x \frac{\partial u}{\partial z} + n_y \frac{\partial v}{\partial z} + n_z \frac{\partial w}{\partial z}) \\
&\quad - \frac{E\alpha(T - T_0)}{1 - 2\nu} n_z
\end{aligned} \tag{12.3.6}$$

where n_x, n_y, and n_z are the cosine directions of the unit outer normal vector to the boundary, and $e = \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}$ is the first invariant of the strain tensor.

The traction boundary conditions along with the thermal boundary conditions will fully define the displacement and temperature fields. It should be noted that the solution to Navier equation (12.3.3), simultaneously satisfies the compatibility condition and the constitutive law, and, therefore, is an acceptable solution for a problem of thermoelasticity.

12.4 Temperature Distribution for Zero Thermal Stress

Generally, when a body is exposed to a thermal gradient, it is expected that thermal stresses will be developed within the body. The question arises as to whether any type of thermal gradient results in thermal stresses [7, 8].

Consider a freely supported body, so that no constraint prevents its thermal expansion. We further assume that the boundary traction $\bar{t}_i^n$ and the body force X_i are zero. Setting all the stress components equal to zero, $\sigma_{ij} = 0$, the surface boundary condition

$$\bar{t}_i^n = n_j \bar{\sigma}_{ij} \tag{12.4.1}$$

is identically satisfied, and the governing equation in terms of the stresses reduces to [8]

$$T_{,ij} + \frac{3\lambda + 2\mu}{\lambda + 2\mu} T_{,kk} \delta_{ij} = 0. \tag{12.4.2}$$

In the expanded form, this equation reads as

$$\begin{aligned}
&(\frac{1+\nu}{1-\nu})\nabla^2 T + \frac{\partial^2 T}{\partial x^2} = 0 \\
&(\frac{1+\nu}{1-\nu})\nabla^2 T + \frac{\partial^2 T}{\partial y^2} = 0 \\
&(\frac{1+\nu}{1-\nu})\nabla^2 T + \frac{\partial^2 T}{\partial z^2} = 0 \\
&\frac{\partial^2 T}{\partial x \partial y} = 0, \qquad \frac{\partial^2 T}{\partial y \partial z} = 0, \qquad \frac{\partial^2 T}{\partial z \partial x} = 0.
\end{aligned} \tag{12.4.3}$$

Adding up the first three equations results in $\nabla^2 T = 0$. This means that the only possible temperature distribution which produces zero thermal stresses in a body of a simply connected region is when

$$\begin{aligned}
&\nabla^2 T = 0 \\
&\frac{\partial^2 T}{\partial x \partial y} = \frac{\partial^2 T}{\partial y \partial z} = \frac{\partial^2 T}{\partial z \partial x} = 0.
\end{aligned} \tag{12.4.4}$$

The unique solution for the temperature distribution satisfying Eq. (12.4.4) is

$$T - T_0 = a + bx + cy + dz = B_0 + B_i x_i \tag{12.4.5}$$

where $a, \; b, \; c$, and d are some arbitrary constants of integration, and $B_0 = a, \; B_1 = b$, $B_2 = c$, and $B_3 = d$. A temperature distribution of this form will not produce any thermal stresses in a body of a simply connected region provided that the body has not been constrained by its boundary in any direction.

In general, a linear distribution of temperature in a body, of either a simply or multiply connected region, results in zero thermal stresses, provided that the boundaries are free of traction [8].

12.5 Finite Element Formulation

For a quasi-static or quasi-stationary thermoelasticity, the temperature must be known in order to evaluate the resulting thermal stresses. The temperature distribution is obtained by solving the heat conduction equation. Let us assume that the heat conduction equation is solved and the temperature distribution is known. To obtain the resulting thermal stresses, a finite element formulation may be developed based on the Galerkin method. The finite element model of the problem is obtained by discretizing the solution domain into a number of arbitrary elements. In each base element (e), the components of displacement and temperature change are approximated by the shape functions [8]

$$\begin{aligned} u_i^{(e)}(x_1, x_2, x_3, t) &= U_{mi}(t) N_m(x_1, x_2, x_3) \\ \theta^{(e)}(x_1, x_2, x_3, t) &= \theta_m(t) N_m(x_1, x_2, x_3) \qquad m = 1, 2, ..., r \end{aligned} \tag{12.5.1}$$

where r is the total number of nodal points in the base element (e). The summation convention is used for the dummy index m. This is a Kantrovitch type of approximation, in which the time and space functions are separated into distinct functions. Here, $U_{mi}(t)$ is the component of displacement at each nodal point, and $\theta_m(t)$ is the temperature change at each nodal point, all being functions of time. The shape function $N_m(x_1, x_2, x_3)$ is a function of space variables.

Substituting Eq. (12.5.1) into Eq. (12.2.1) and applying the weighted residual integral with respect to the weighting functions $N_m(x_1, x_2, x_3)$, the formal Galerkin approximation reduces to

$$\int_{V(e)} (\sigma_{ij,j} + X_i - \rho \ddot{u}_i) N_l dV = 0 \qquad l = 1, 2, ..., r. \tag{12.5.2}$$

Applying the weak formulation to the first term, yields

$$\int_{V(e)} (\sigma_{ij,j}) N_l dv = \int_{A(e)} \sigma_{ij} n_j N_l dA - \int_{V(e)} \frac{\partial N_l}{\partial x_j} \sigma_{ij} dV \tag{12.5.3}$$

where n_j is the component of the unit outer normal vector to the boundary. Substituting Eq. (12.5.3) in Eq. (12.5.2) gives

$$\int_{A(e)} \sigma_{ij} n_j N_l dA - \int_{V(e)} \frac{\partial N_l}{\partial x_j} \sigma_{ij} dV + \int_{V(e)} X_i N_l dV - \int_{V(e)} \rho \ddot{u}_i N_l dV = 0. \tag{12.5.4}$$

According to Cauchy's formula, the traction force components acting on the boundary are related to the stress tensor as

$$t_i = \sigma_{ij} n_j. \tag{12.5.5}$$

Thus, the first term of Eq. (12.5.4) is

$$\int_{A(e)} \sigma_{ij} n_j N_l dA = \int_{A(e)} t_i N_l dA. \tag{12.5.6}$$

From Hooke's law, the stress tensor is related to the strain tensor, or the displacement components, and temperature change $\theta = T - T_0$, T_0 being the reference temperature, as

$$\sigma_{ij} = G(u_{i,j} + u_{j,i}) + \lambda u_{k,k}\delta_{ij} - \beta\theta\delta_{ij}. \tag{12.5.7}$$

Substituting for σ_{ij} in the second term of Eq. (12.5.4) yields

$$\int_{V(e)} \frac{\partial N_l}{\partial x_j} \sigma_{ij} dV = \int_{V(e)} \frac{\partial N_l}{\partial x_j} [G(u_{i,j} + u_{j,i}) + \lambda u_{k,k}\delta_{ij} - \beta\theta\delta_{ij}] dV. \tag{12.5.8}$$

Substituting this expression in Eq. (12.5.4) gives

$$\begin{aligned} &\int_{V(e)} \rho \ddot{u}_i N_l dV + \int_{V(e)} \frac{\partial N_l}{\partial x_j} [G(u_{i,j} + u_{j,i}) + \lambda u_{k,k}\delta_{ij}] dV \\ &- \int_{V(e)} \beta\theta \frac{\partial N_l}{\partial x_i} dV = \int_{V(e)} X_i N_l dV + \int_{A(e)} t_i N_l dA. \end{aligned} \tag{12.5.9}$$

Now, the base element (e) with r nodal points is considered, and the displacement components and temperature change in the element (e) are approximated by Eq. (12.5.1). Using these approximations, Eq. (12.5.9) becomes

$$\begin{aligned} &(\int_{V(e)} \rho N_l N_m dV)\ddot{U}_{mi} + (\int_{V(e)} G \frac{\partial N_l}{\partial x_j} \frac{\partial N_m}{\partial x_j} dV) U_{mi} \\ &+(\int_{V(e)} G \frac{\partial N_l}{\partial x_j} \frac{\partial N_m}{\partial x_i} dV) U_{mj} + (\int_{V(e)} \lambda \frac{\partial N_l}{\partial x_i} \frac{\partial N_m}{\partial x_j} dV) U_{mj} \\ &-(\int_{V(e)} \beta \frac{\partial N_l}{\partial x_i} N_m dV)\theta_m = \int_{V(e)} X_i N_l dV + \int_{A(e)} t_i N_l dA \\ &\qquad l, m = 1, 2, ..., r \quad i, j = 1, 2, 3. \end{aligned} \tag{12.5.10}$$

Equation (12.5.10) is the finite element approximation of the equation of motion.

The Galerkin approximation may also be applied to the energy equation given by Eq. (6.2.1) as

$$\int_{V(e)} (q_{i,i} + \rho c \frac{\partial \theta}{\partial t} - Q) N_l dV = 0 \quad l = 1, 2, \cdots, r. \tag{12.5.11}$$

The weak formulation of the heat flux gradient $q_{i,i}$ gives

$$\int_{V(e)} q_{i,i} N_l dV = \int_{V(e)} (\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}) N_l dV = \int_{A(e)} (\vec{q} \cdot \vec{n}) N_l dA - \int_{V(e)} q_i \frac{\partial N_l}{\partial x_i} dV \tag{12.5.12}$$

where $A(e)$ is the boundary surface of the base element (e). Substituting Eq. (12.5.12) in Eq. (12.5.11) and rearranging the terms gives

$$\int_{V(e)} \rho c \frac{\partial \theta}{\partial t} N_l dV - \int_{V(e)} q_i \frac{\partial N_l}{\partial x_i} dV = \int_{V(e)} Q N_l dV - \int_{A(e)} (\vec{q} \cdot \vec{n}) N_l dA \quad l = 1, 2, \cdots, r. \tag{12.5.13}$$

Substituting for the temperature change θ its approximate values in the base element (e) from Eq. (12.5.1) gives

$$(\int_{V(e)} k \frac{\partial N_m}{\partial x_i} \frac{\partial N_l}{\partial x_i} dV) \theta_m + (\int_{V(e)} \rho c N_m N_l dV) \dot{\theta}_m = \int_{V(e)} Q N_l dV - \int_{A(e)} (\vec{q} \cdot \vec{n}) N_l dA. \tag{12.5.14}$$

Equation (12.5.14) is the finite element approximation of the heat conduction equation.

Equations (12.5.10) and (12.5.14) may be assembled into a matrix form, resulting in the general finite element equations given by

$$[M]\{\ddot{\Delta}\} + [C]\{\dot{\Delta}\} + [K]\{\Delta\} = \{F\} \tag{12.5.15}$$

where $[M]$, $[C]$, and $[K]$ are the mass, damping, and stiffness matrices, respectively. Matrix $\{\Delta\}^T = \langle U_i, \theta \rangle$ is the matrix of unknowns, and $\{F\}$ is the known mechanical and thermal force matrix.

For a two-dimensional problem, l and m take the values $1, 2, ...r$. In this case, Eq. (12.5.10) reduces into two equations in the x and y-directions, as

$$\begin{aligned}&(\int_{V(e)} \rho N_l N_m dV)\ddot{U}_m + \left[\int_{V(e)} (2G+\lambda)\frac{\partial N_l}{\partial x}\ \frac{\partial N_m}{\partial x} dV\right.\\&\quad \left.+ \int_{V(e)} G\frac{\partial N_l}{\partial y}\ \frac{\partial N_m}{\partial y} dV\right] U_m + \left[\int_{V(e)} G\frac{\partial N_l}{\partial y}\ \frac{\partial N_m}{\partial x} dV\right.\\&\quad \left.+ \int_{V(e)} \lambda\frac{\partial N_l}{\partial x}\ \frac{\partial N_m}{\partial y} dV\right] V_m - \left[\int_{V(e)} \beta N_m \frac{\partial N_l}{\partial x} dV\right]\theta_m\\&= \int_{V(e)} X N_l dV + \int_{A(e)} t_x N_l dA \end{aligned} \tag{12.5.16}$$

$$\begin{aligned}&(\int_{V(e)} \rho N_l N_m dV)\ddot{V}_m + \left[(2G+\lambda)\int_{V(e)} \frac{\partial N_l}{\partial y}\ \frac{\partial N_m}{\partial y} dV\right.\\&\quad \left.+ \int_{V(e)} G\frac{\partial N_l}{\partial x}\ \frac{\partial N_m}{\partial x} dV\right] V_m + \left[\int_{V(e)} G\frac{\partial N_l}{\partial x}\ \frac{\partial N_m}{\partial y} dV\right.\\&\quad \left.+ \int_{V(e)} \lambda\frac{\partial N_l}{\partial y}\ \frac{\partial N_m}{\partial x} dV\right] U_m - \left[\int_{V(e)} \beta N_m \frac{\partial N_l}{\partial y} dV\right]\theta_m\\&= \int_{V(e)} Y N_l dV + \int_{A(e)} t_y N_l dA. \end{aligned} \tag{12.5.17}$$

The energy equation (12.5.14) for a two-dimensional problem becomes

$$\begin{aligned}&(\int_{V(e)} \rho c N_m N_l dV)\dot{\theta}_m + (\int_{V(e)} k\ \frac{\partial N_m}{\partial x}\ \frac{\partial N_l}{\partial x} dV\\&\quad + \int_{V(e)} k\ \frac{\partial N_m}{\partial y}\ \frac{\partial N_l}{\partial y} dV)\theta_m = \int_{V(e)} Q N_l dV - \int_{V(e)} (\vec{q}\cdot\vec{n}) N_l dA. \end{aligned} \tag{12.5.18}$$

The elements of the mass, damping, stiffness, and force matrices of the base element (e) are

$$[M]^{(e)} = \begin{bmatrix} \left[\int_{V(e)} \rho N_l N_m dV\right] & 0 & 0\\ 0 & \left[\int_{V(e)} \rho N_l N_m dV\right] & 0\\ 0 & 0 & 0 \end{bmatrix}. \tag{12.5.19}$$

The damping matrix is

$$[C]^{(e)} = \begin{bmatrix} 0\ 0 & 0\\ 0\ 0 & 0\\ 0\ 0 & \left[\int_{v(e)} \rho c N_m N_l dV\right] \end{bmatrix} \tag{12.5.20}$$

and the stiffness matrix is

$$[k]^{(e)} = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} \tag{12.5.21}$$

where

$$\begin{aligned}
[k_{11}^{lm}] &= \left[\int_{V(e)} (2G+\lambda)\frac{\partial N_l}{\partial x}\ \frac{\partial N_m}{\partial x} dV + \int_{V(e)} G\frac{\partial N_l}{\partial y}\ \frac{\partial N_m}{\partial y} dV\right] \\
[k_{12}^{lm}] &= \left[\int_{V(e)} G\frac{\partial N_l}{\partial y}\ \frac{\partial N_m}{\partial x} dV + \int_{V(e)} \lambda\frac{\partial N_l}{\partial x}\ \frac{\partial N_m}{\partial y} dV\right] \\
[k_{13}^{lm}] &= -\left[\int_{V(e)} \beta N_m \frac{\partial N_l}{\partial x} dV\right] \\
[k_{21}^{lm}] &= \left[\int_{V(e)} G\frac{\partial N_l}{\partial x}\ \frac{\partial N_m}{\partial y} dV + \int_{V(e)} \lambda\frac{\partial N_l}{\partial y}\ \frac{\partial N_m}{\partial x} dV\right] \\
[k_{22}^{lm}] &= \left[\int_{V(e)} (2G+\lambda)\frac{\partial N_l}{\partial y}\ \frac{\partial N_m}{\partial y} dV + \int_{V(e)} G\frac{\partial N_l}{\partial x}\ \frac{\partial N_m}{\partial x} dV\right] \\
[k_{23}^{lm}] &= -\left[\int_{V(e)} \beta N_m \frac{\partial N_l}{\partial y} dV\right] \\
[k_{31}^{lm}] &= [k_{32}]^{ml} = 0 \\
[k_{33}^{lm}] &= \left[\int_{V(e)} k\ \frac{\partial N_m}{\partial x}\ \frac{\partial N_l}{\partial x} dV + \int_{V(e)} k\ \frac{\partial N_m}{\partial y}\ \frac{\partial N_l}{\partial y} dV\right].
\end{aligned} \tag{12.5.22}$$

The force matrix is

$$\{f\}_l^{(e)} = \left\{ \begin{array}{c} \left\{\int_{V(e)} XN_l dV + \int_{A(e)} t_x N_l dA\right\} \\ \left\{\int_{V(e)} YN_l dV + \int_{A(e)} t_y N_l dA\right\} \\ \left\{\int_{V(e)} QN_l dV - \int_{V(e)} (\vec{q}\cdot\vec{n}) N_l dA\right\} \end{array} \right\} \tag{12.5.23}$$

and the unknown matrix is

$$\{\delta\}^{(e)} = \left\{ \begin{array}{c} \{U\} \\ \{V\} \\ \{\theta\} \end{array} \right\}. \tag{12.5.24}$$

The initial and general forms of the thermal boundary conditions are one, or combinations, of the following:

$$
\begin{aligned}
\theta(x, y, z, 0) &= 0(x, y, z) \text{ at } t = 0\\
\theta(x, y, z, t) &= \theta_s \text{ on } A_1 \text{ and } t > 0\\
q_x l + q_y m + q_z n &= -q^{''} \text{ on } A_2 \text{ and } t > 0\\
q_x l + q_y m + q_z n &= h(\theta + T_0 - T_\infty) \text{ on } A_3 \text{ and } t > 0\\
q_x l + q_y m + q_z n &= \sigma\epsilon(\theta + T_0)^4 - \alpha_{ab} q_r \text{ on } A_4 \text{ and } t > 0
\end{aligned}
\tag{12.5.25}
$$

where $T_0(x, y, z)$ is the known initial temperature, θ_s is the known specified temperature change on a part of the boundary surface A_1, $q^{''}$ is the known heat flux on the boundary A_2, h and T_∞ are the convection coefficient and ambient temperature specified on a part of the boundary surface A_3, respectively, σ is the Stefan-Boltzmann constant, ϵ is the radiation coefficient of the boundary surface, α_{ab} is the boundary surface absorption coefficient, and q_r is the rate of thermal flux reaching the boundary surface per unit area, all specified on boundary surface A_4. The cosine directors of the unit outer normal vector to the boundary in the x, y, and z-directions are shown by l, m, and n, respectively. According to the boundary conditions given by Eq. (12.5.25), the last surface integral of the energy equation (12.5.14) may be decomposed into four integrals over A_1 through A_4 as

$$
\begin{aligned}
\int_{A(e)} (\vec{q} \cdot \vec{n}) N_l dA = \int_{A_2} q^{''} N_l dA - \int_{A_3} h(\theta + T_0 - T_\infty) N_l dA\\
- \int_{A_4} (\sigma\epsilon(\theta + T_0)^4 - \alpha_{ab} q_r) N_l dA \quad l = 1, 2, \cdots r.
\end{aligned}
\tag{12.5.26}
$$

Note that the signs of the integrals in Eq. (12.5.26) depend upon the direction of the heat input. The positive sign is defined when the heat is given to the body, and the negative when the heat is removed from the body. That is, $q^{''}$ is defined positive in Eq. (12.5.26), since we have assumed that the heat flux is given to the body. On the other hand, we have assumed negative convection on the surface area A_3, which means the heat is removed from the A_3 boundary by convection. Similarly, the boundary A_4 is assumed to radiate to the ambient, as the sign of this integral is considered negative.

In order to discuss the method in more detail, a one-dimensional problem is considered [8, 9]. The equation of motion in terms of displacement is

$$
(\lambda + 2G)\frac{\partial^2 u}{\partial x^2} - \beta\frac{\partial \theta}{\partial x} = \rho\frac{\partial^2 u}{\partial t^2} \tag{12.5.27}
$$

and the first law of thermodynamics reduces to

$$
k\frac{\partial^2 \theta}{\partial x^2} - \rho c\frac{\partial \theta}{\partial t} = 0. \tag{12.5.28}
$$

Taking a line element of length L, the approximating function for the axial displacement in base element (e) is assumed to be linear in x as

$$u(x,t)^{(e)} = N_i U_i + N_j U_j = \langle N\rangle^{(e)} \{U\}^{(e)} \tag{12.5.29}$$

where the piecewise linear shape function $\langle N\rangle$ is $N_i = (L-\eta)/L$, $N_j = \eta/L$, and $\eta = x - x_i$. Similarly, the temperature change is approximated by

$$\theta(x,t)^{(e)} = N_i\theta_i + N_j\theta_j = \langle N\rangle^{(e)} \{\theta\}^{(e)}. \tag{12.5.30}$$

Employing the formal Galerkin method and applying the weak form to the first and second terms of Eq. (12.5.27) and the first term of Eq. (12.5.28) results in the following system of equations:

$$\begin{aligned}&(\int_0^L \rho N_l N_m d\eta)\ddot{U}_m + (\int_0^L (2G+\lambda)\frac{\partial N_l}{\partial\eta}\ \frac{\partial N_m}{\partial\eta} d\eta)U_m\\&-(\int_0^L \beta N_m \frac{\partial N_l}{\partial \eta} d\eta)\theta_m = t_x N_l\Big|_i^j + \int_0^L X N_l d\eta\end{aligned} \tag{12.5.31}$$

$$\begin{aligned}&(\int_0^L \rho c N_m N_l d\eta)\dot{\theta}_m + (\int_0^L k\ \frac{\partial N_m}{\partial\eta}\ \frac{\partial N_l}{\partial\eta} d\eta)\theta_m\\&= -\ (\vec{q}\cdot\vec{n})N_l|_i^j + \int_0^L Q N_l d\eta\end{aligned} \tag{12.5.32}$$

This system of equations may be written in matrix form as

$$[M]\{\ddot{\Delta}\} + [C]\{\dot{\Delta}\} + [K]\{\Delta\} = \{F\} \tag{12.5.33}$$

where the mass, damping, stiffness and force matrices for the first order element are 4×4 matrices and are defined as

$$[M]^{(e)} = \int_0^L \begin{bmatrix} \rho N_i N_i & 0 & \rho N_i N_j & 0\\ 0 & 0 & 0 & 0\\ \rho N_j N_i & 0 & \rho N_j N_j & 0\\ 0 & 0 & 0 & 0\end{bmatrix} d\eta \tag{12.5.34}$$

$$[C]^{(e)} = \int_0^L \begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & \rho c N_i N_i & 0 & \rho c N_i N_j\\ 0 & 0 & 0 & 0\\ 0 & \rho c N_j N_i & 0 & \rho c N_j N_j\end{bmatrix} d\eta \tag{12.5.35}$$

$$[K]^{(e)} = \int_0^L \begin{bmatrix} (2G+\lambda)(\frac{\partial N_i}{\partial \eta})^2 & -\beta N_i \frac{\partial N_i}{\partial \eta} & (2G+\lambda)\frac{\partial N_i}{\partial \eta}\frac{\partial N_j}{\partial \eta} & -\beta N_j \frac{\partial N_i}{\partial \eta} \\ 0 & k(\frac{\partial N_i}{\partial \eta})^2 & 0 & k\frac{\partial N_j}{\partial \eta}\frac{\partial N_i}{\partial \eta} \\ (2G+\lambda)\frac{\partial N_j}{\partial \eta}\frac{\partial N_i}{\partial \eta} & -\beta N_i \frac{\partial N_j}{\partial \eta} & (2G+\lambda)(\frac{\partial N_j}{\partial \eta})^2 & -\beta N_j \frac{\partial N_j}{\partial \eta} \\ 0 & k\frac{\partial N_i}{\partial \eta}\frac{\partial N_j}{\partial \eta} & 0 & k(\frac{\partial N_j}{\partial \eta})^2 \end{bmatrix} d\eta \tag{12.5.36}$$

$$\{F\}^{(e)} = \begin{Bmatrix} t_x N_i|_0^L + \int_0^L X N_i d\eta \\ -q_x N_i|_0^L + \int_0^L R N_i d\eta \\ t_x N_j\big|_L + \int_0^L X N_j d\eta \\ -q_x N_j\big|_0^L + \int_0^L R N_j d\eta \end{Bmatrix} \tag{12.5.37}$$

Upon substitution of the shape functions in the foregoing equations, the submatrices for the base element (e) are

$$[M]^{(e)} = \begin{bmatrix} \frac{\rho L}{3} & 0 & \frac{\rho L}{6} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\rho L}{6} & 0 & \frac{\rho L}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{12.5.38}$$

$$[C]^{(e)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{\rho c L}{3} & 0 & \frac{\rho c L}{6} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{\rho c L}{6} & 0 & \frac{\rho c L}{3} \end{bmatrix} \tag{12.5.39}$$

$$[K]^{(e)} = \begin{bmatrix} \frac{(2G+\lambda)}{L} & \frac{\beta}{2} & -\frac{(2G+\lambda)}{L} & \frac{\beta}{2} \\ 0 & \frac{k}{L} & 0 & -\frac{k}{L} \\ -\frac{(2G+\lambda)}{L} & -\frac{\beta}{2} & \frac{(2G+\lambda)}{L} & -\frac{\beta}{2} \\ 0 & -\frac{k}{L} & 0 & \frac{k}{L} \end{bmatrix} \tag{12.5.40}$$

$$\{F\}^{(e)} = \begin{Bmatrix} -t_x|_0 + \frac{XL}{2} \\ q_x|_0 + \frac{RL}{2} \\ t_x|_L + \frac{XL}{2} \\ -q_x|_L + \frac{RL}{2} \end{Bmatrix} \tag{12.5.41}$$

and the matrix of unknown nodal value is

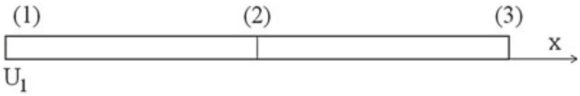

Fig. 12.1 A one-dimensional rod with known nodal temperatures

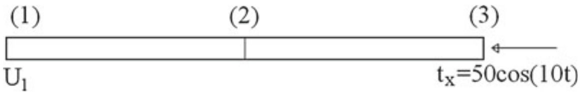

Fig. 12.2 A one-dimensional rod with known nodal temperatures under axial dynamic load

$$\{\Delta\}^{(e)} = \begin{Bmatrix} U_i \\ \theta_i \\ U_j \\ \theta_j \end{Bmatrix}. \tag{12.5.42}$$

Once the nodal temperature changes to θ and the forces are known, the axial thermal displacements can be obtained.

The general rule for the solution of thermal stress problems is that the temperature distribution must be known in the solution domain prior to thermal displacement analysis. That is, the heat conduction equation must be considered and solved independently to obtain the temperature distribution. Temperature distribution may be in the steady state condition, or a function of time in transient or dynamic forms. Still, the heat conduction equation is uncoupled from the thermoelasticity equations, even with dynamic temperature distributions. In industrial applications, this situation occurs almost all of the time.

For the coupled assumption, the duration of the period of time of thermal shock application must be much smaller than the magnitude of the first time period of the body's natural frequency to cause thermal coupling. In many thermal stress problems, this situation does not occur, and thus, thermal stress problems may be solved with sufficient accuracy under uncoupled conditions.

12.6 Problems

1. A one-dimensional rod of constant cross-section A and length L is considered. The rod is made into two elements of equal length $L/2$ and nodal points 1,2, and 3, as shown in Fig. 12.1.

   The steady-state temperature changes at nodes 1,2, and 3 are θ_1, θ_2, and θ_3 and are known. The axial displacement at node 1 is fixed ($U_1 = 0$). Find the elements of the stiffness and force matrices and obtain the thermal axial displacement of nodes 2 and 3.

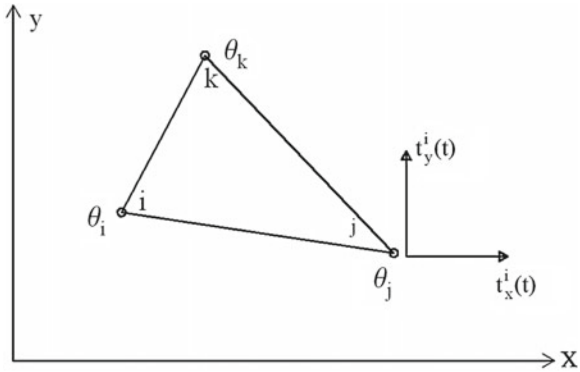

Fig. 12.3 A linear triangular element with known nodal temperatures under axial dynamic loads at node *j*

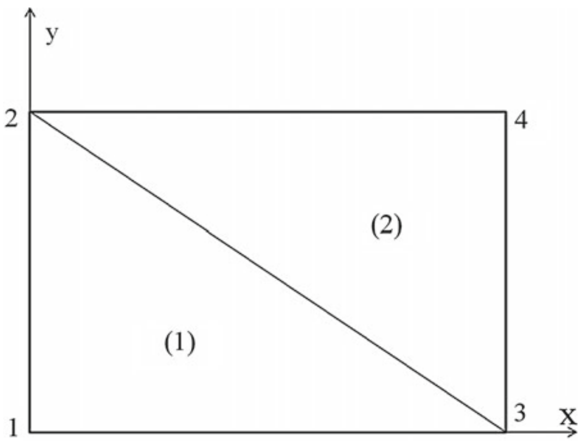

Fig. 12.4 A two-dimensional simple plane strain rectangular domain with known nodal temperatures

2. Reconsider Problem 1, where an axial dynamic force $t_x = 50 \cos 10t$ is applied at node 3. Node 1 is fixed, and the nodal temperature changes are constant with time.

   Find the elements of the mass, stiffness, and force matrices, and obtain the axial displacements at nodes 2 and 3. The mass density is assumed to be ρ (Fig. 12.2).
3. Check Eqs. (12.5.16) and (12.5.17) to show that these equations are associated with the simple plane strain thermoelasticity condition.
4. Reduce Eq. (12.5.10) for a two-dimensional thermoelasticity plane stress condition.
5. Employ a linear triangular element and use Eqs. (12.5.16) and (12.5.17) to obtain the elements of the mass, stiffness, and force matrices.

The element load condition is shown in Fig. 12.3. Temperature changes at nodes i, j, and k are θ_i, θ_j, and θ_k, respectively, and node j is under two dynamic concentrated forces $t_x^j(t)$ and $t_y^j(t)$. The mass density of the element is assumed to be ρ.

6. Consider a two-dimensional rectangular domain with simple plane strain condition. The domain is divided into two triangular elements, as shown in Fig. 12.4.

   Temperature changes at nodes are known to be θ_1 to θ_4, and are constant with time. The kinematic boundary conditions are $u_1 = v_1 = 0$ and $u_2 = 0$. Find the elements of the stiffness and force matrices and obtain the thermal displacements at nodes 2,3, and 4.

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Chapter 13
Incompressible Viscous Fluid Flow

Abstract The flow of incompressible and viscous fluid is considered in this chapter. The continuity condition and the equations of motion, the Navier-Stokes equations for the Newtonian fluid, are derived. The Stokes equation for the low Reynolds number flow is discussed and its associated functional expression is presented. The governing equations are then made dimensionless and employing the Galerkin method, the finite element equation is derived, using two different shape functions for the velocity components and pressure. The general finite element equations are then reduced to the case of two-dimensional fluid flow and the elements of mass, stiffness, and force matrices are calculated. The general form of boundary conditions are discussed and the selection/limitation of the shape functions for the velocity components and pressure is explained. As another method of fluid flow problems, the vorticity transport technique is presented and the finite element modeling of the fluid flow in terms of the vorticity and stream function is presented and the element matrices are derived. Since the governing finite element equations are nonlinear, the method of linearization technique is discussed. The finite element matrices for two-dimensional flow of an incompressible viscous fluid employing two-dimensional simplex elements are derived.

13.1 Introduction

Newton's law of motion, which is used to derive the equations of motion of a solid elastic continuum, is employed to derive the equations of motion of viscous fluid flow. The derivations for the Newtonian fluids, in which the stress tensor is linearly related to the rate of the strain tensor, is directly follows from Newton's equation of motion for a continuum by proper substitution for the stress tensor. The resulting equilibrium equations are called the Navier-Stokes equations.

The laws governing the motion of fluid are essentially those of classical mechanics. The energy principle and the balance of force and momentum equations all hold for

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fluid flow too. The exception is that the general laws and principles that are based on energy equations, are not well-defined and formulated for the fluid flow problems. In other words, the variational formulation of fluid flow problems does not exist. Therefore, the finite element analysis of fluid dynamic problems of viscous flow based on the Ritz method is not yet developed, as the functional of such problems does not exist.

The Galerkin method is a well-accepted technique for the finite element formulation of fluid dynamic problems. In particular, since the Navier-Stokes equations are nonlinear partial differential equations, the Galerkin method, which has a strong rate of convergence, is an appropriate tool for handling such nonlinear partial differential equations.

In this chapter, the basic governing equations of fluid dynamics are derived. Using the Galerkin method, two-dimensional Navier-Stokes equations are employed with continuity equations to derive the finite element equation of motion. As an alternate method, the Navier-Stokes equations are transfered into the vorticity transport equations, for which the Galerkin method is employed to derive the associated finite element equilibrium equations. A triangular simplex element is used to derive the element mass and stiffness matrices. Finally, an iterative method is proposed to solve the resulting nonlinear finite element equations.

13.2 Continuity Equation

The continuity equation is derived from the principle of conservation of mass. Consider an element of fluid volume dv. The mass of the volume element is ρdv, where ρ is the mass density of fluid. This element of mass, during the motion of fluid, is not changed with time, and thus [1, 2]

$$\frac{D}{Dt}(\rho dv) = 0 \tag{13.2.1}$$

where D/Dt is called the *substantial derivative*, and is defined as

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_i \frac{\partial}{\partial x_i} \tag{13.2.2}$$

where u_i is the component of the fluid velocity vector. The physical concept of Eq. (13.2.1) is that the mass is conserved. Expanding the derivative, Eq. (13.2.1) becomes

$$\rho \frac{D(dv)}{Dt} + dv \frac{D\rho}{Dt} = 0 \tag{13.2.3}$$

since

$$\frac{D(dv)}{Dt} = dv \frac{\partial u_i}{\partial x_i}. \tag{13.2.4}$$

Thus, substituting in Eq. (13.2.3), gives

$$\rho \frac{\partial u_i}{\partial x_i} + \frac{D\rho}{Dt} = 0 \tag{13.2.5}$$

This equation may be written as

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u_i)}{\partial x_i} = 0. \tag{13.2.6}$$

Equation (13.2.6) is called the continuity equation and describes the conservation of mass. For the incompressible fluids, $\rho = cte$, and thus, Eq. (13.2.6) reduces to

$$\frac{\partial u_i}{\partial x_i} = 0. \tag{13.2.7}$$

In conventional notation, the continuity equation is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{13.2.8}$$

This equation must always be satisfied for any flow of incompressible fluid in the absence of fluid shock.

13.3 Equation of Motion

Newton's equation of motion for a continuum is

$$\frac{\partial \tau_{ij}}{\partial x_j} + \rho B_i = \rho \alpha_i \tag{13.3.1}$$

in which τ_{ij} is the stress tensor, ρ is the mass density, B_i is the body force tensor per unit volume, and a_i is the acceleration tensor. Fluid is a continuum that does not stand shear stress, and flows under the applied shear stress.

For a Newtonian fluid, the stress tensor is linearly dependent on the pressure and rate of the strain tensor, as [1, 2]

$$\tau_{ij} = -p\delta_{ij} + \tau'_{ij} \tag{13.3.2}$$

where p is the pressure, τ'_{ij} is the viscous stress tensor and is a function of the derivatives of the velocity components, and δ_{ij} is the Kronecker delta. Calling

$$D_{ij} = \tfrac{1}{2}\,(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}) \tag{13.3.3}$$

the shear stress tensor τ'_{ij} for an isotropic fluid is defined as

$$\tau'_{ij} = \lambda\Delta\delta_{ij} + 2\mu D_{ij} \tag{13.3.4}$$

where

$$\Delta = D_{11} + D_{22} + D_{33} = D_{kk}. \tag{13.3.5}$$

The coefficients λ and μ are the fluid constants and have the dimension of (force) $\times$ (time) / (length)^2. Substituting Eq. (13.3.4) in (13.3.2), gives

$$\tau_{ij} = (-p + \lambda\Delta)\delta_{ij} + 2\mu D_{ij}. \tag{13.3.6}$$

Expanding the equation, we get

$$\begin{aligned}
\tau_{11} &= -p + \lambda\Delta + 2\mu D_{11}\\
\tau_{22} &= -p + \lambda\Delta + 2\mu D_{22}\\
\tau_{33} &= -p + \lambda\Delta + 2\mu D_{33}\\
\tau_{12} &= 2\mu D_{12}\\
\tau_{23} &= 2\mu D_{23}\\
\tau_{13} &= 2\mu D_{13}.
\end{aligned} \tag{13.3.7}$$

To find the constants λ and μ, consider a fluid flow field in which $u_1 = f(x_2),\ u_2 = u_3 = 0$. For this flow

$$\begin{aligned}
D_{11} &= D_{22} = D_{33} = D_{13} = D_{23} = 0\\
D_{12} &= \tfrac{1}{2}\,\frac{df}{dx_2}.
\end{aligned} \tag{13.3.8}$$

Thus, from Eq. (13.3.6), the shear stress is

$$\begin{aligned}
\tau_{11} &= \tau_{22} = \tau_{33} = -p\\
\tau_{13} &= \tau_{23} = 0\\
\tau_{12} &= \mu\,\frac{df}{dx_2}.
\end{aligned} \tag{13.3.9}$$

Equation (13.3.9) shows that the constant μ is the proportionality coefficient between the shear stress and the velocity gradient. This proportionality constant is called *viscosity*. From Eq. (13.3.6)

$$\frac{1}{3}\,\tau_{kk} = \left(\lambda + \frac{2}{3}\,\mu\right)\Delta. \tag{13.3.10}$$

This equation relates the mean stress ($\tau_{kk}/3$) to the rate of volume change (Δ). The sum of the constant coefficients ($\lambda + 2/3\mu$) is called the *bulk modulus*, as

$$k = \lambda + \frac{2}{3}\,\mu. \tag{13.3.11}$$

For incompressible fluids, k tends to infinity.

13.4 Incompressible Newtonian Fluid Flow

For the incompressible fluids, the rate of volume change is zero ($\Delta = 0$), and thus, the relation between the stress and velocity gradients from Eq. (13.3.6) is [1, 2]

$$\tau_{ij} = -p\delta_{ij} + 2\mu D_{ij}. \tag{13.4.1}$$

In this equation, the pressure is substituted by the mean normal stress ($p = -\tau_{kk}/3$). The velocity gradient is

$$D_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{13.4.2}$$

Substituting in Eq. (13.4.1), gives

$$\tau_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{13.4.3}$$

The equation of motion (13.3.1) for fluid flow is written as

$$\frac{\partial \tau_{ij}}{\partial x_j} + \rho B_i = \rho\,\frac{Du_i}{Dt} = \rho\left(\frac{\partial u_i}{\partial t} + u_j\,\frac{\partial u_i}{\partial x_j}\right). \tag{13.4.4}$$

Using Eq. (13.4.3), the left side of Eq. (13.4.4) is

$$\begin{aligned}\tau_{ij,j} &= -\,p_{,j}\delta_{ij} + \mu(u_{i,j})_{,j} + \mu(u_{j,j})_{,i}\\ &= -\,p_{,j}\delta_{ij} + \mu(u_{i,j})_{,j}\end{aligned} \tag{13.4.5}$$

in which, due to the continuity equation for incompressible fluid, $(u_{j,i})_{,j} = (u_{j,j})_{,i} = 0$. Substituting Eq. (13.4.5) in (13.4.4) gives

$$\rho(\frac{\partial u_i}{\partial t} + u_j u_{i,j}) = \rho B_i - p_{,i} + \mu(u_{i,j})_{,j}. \tag{13.4.6}$$

This equation is called the Navier-Stokes equation for the motion of incompressible fluid flow. Expanding Eq. (13.4.6) and using the conventional notation for the velocity components as u, v, and w, gives

$$\begin{aligned}
\rho(\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} + w\,\frac{\partial u}{\partial z}) &= \rho B_x - \frac{\partial p}{\partial x} + \mu(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}) \\
\rho(\frac{\partial v}{\partial t} + u\,\frac{\partial v}{\partial x} + v\,\frac{\partial v}{\partial y} + w\,\frac{\partial v}{\partial z}) &= \rho B_y - \frac{\partial p}{\partial y} + \mu(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}) \\
\rho(\frac{\partial w}{\partial t} + u\,\frac{\partial w}{\partial x} + v\,\frac{\partial w}{\partial y} + w\,\frac{\partial w}{\partial z}) &= \rho B_z - \frac{\partial p}{\partial z} + \mu(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}).
\end{aligned} \tag{13.4.7}$$

Along with the Navier-Stokes Eq. (13.4.7), the continuity equation must be considered. For incompressible fluids, the continuity equation is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{13.4.8}$$

The system of partial differential Eqs. (13.4.7) and (13.4.8) must be solved simultaneously for the dependent functions u, v, w, and p.

13.5 Stokes Equation

For low Reynolds number fluid flow problems, called *creeping flow*, the convective term $u_j\,\frac{\partial u_i}{\partial x_j}$ in the Navier-Stokes equations may be ignored. The resulting equations are called Stokes equations. For steady-state incompressible fluid flow, the Stokes equations for two-dimensional flow are

$$\begin{aligned}
\mu\nabla^2 u - \frac{\partial p}{\partial x} &= 0 \\
\mu\nabla^2 v - \frac{\partial p}{\partial y} &= 0
\end{aligned} \tag{13.5.1}$$

which should be solved simultaneously with the continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{13.5.2}$$

In Eq. (13.5.1), the body force components are ignored. Unlike the Navier-Stokes equations, in which the expression for the functional is not known, the Stokes equations have a functional. The expression for the functional is

$$I(u, v, p) = \frac{\mu}{2} \int_{\Omega} \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} \, d\Omega - \int_{\Omega} p \frac{\partial u_i}{\partial x_i} \, d\Omega. \tag{13.5.3}$$

Using the technique of calculus of variation described in Chap. 2, the minimum of functional (13.5.3) may be obtained. The class of admissible functions is selected as

$$\begin{aligned} u^* &= u + \epsilon_1 \eta_1 \\ v^* &= v + \epsilon_2 \eta_2 \\ p^* &= p + \epsilon_3 \eta_3 \end{aligned} \tag{13.5.4}$$

in which $u, \ v$, and p are the functions minimizing the functional (13.5.3), $\eta_1, \ \eta_2$, and η_3 are the variational functions satisfying the homogeneous boundary conditions, and $\epsilon_1, \ \epsilon_2$, and ϵ_3 are the variational parameters, respectively. It is verified that when Eq. (13.5.4) are substituted in the expression of functional (13.5.3) and the method of calculus of variation is applied, the set of boundary value Eqs. (13.5.1) and (13.5.2) are obtained.

13.6 Dimensionless Form of Equations

The most basic and important rule in any computational method is to write and use the dimensionless form of the governing equations. An even more important point is that, when the finite element method is used, the governing equations must be dimensionless in terms of the characteristic length of the element. Otherwise, if the equations are dimensionless in terms of the problem's characteristic length, such as the total length of a plate, the order of magnitude of the members of the total assembled mass and stiffness matrices are subjected to high variations from very small to very high numbers. This high variation of the magnitudes of the members of the mass and stiffness matrices results in a very unstable solution to the problem. An optimum method for preparing the dimensionless form of the governing equations, is to make them dimensionless with respect to the order of magnitude of the element's characteristic length. This should be done in such a way that the resulting assembled stiffness and mass matrices have the lowest possible variations of the order of magnitude of their members.

This law should also be observed between the mass and stiffness matrices in comparison with each other. That is, the order of magnitude of the members of the assembled mass and stiffness matrices must be as close as possible. With this law in mind, the Navier-Stokes equations are now made dimensionless to prepare the proper finite element model.

The dimensionless quantities for the Navier-Stokes and continuity equations are introduced as

$$\begin{aligned} x_i^* &= x_i/l \\ u_i^* &= u_i/u_0 \\ t^* &= tu_0/l \\ p^* &= p/(\rho u_0^2) \end{aligned} \tag{13.6.1}$$

in which l is a characteristic length (of the order of element size) and u_0 is the base velocity (or free stream velocity). With the introduction of the dimensionless quantities, the continuity and the Navier-Stokes equations in dimensionless form become

$$\frac{\partial u_i^*}{\partial x_i^*} = 0 \tag{13.6.2}$$

$$\frac{\partial u_i^*}{\partial t^*} + u_j^* \frac{\partial u_i^*}{\partial x_j^*} = \frac{l_i}{Fr^2} - \frac{\partial p^*}{\partial x_i^*} + \frac{1}{Re} \frac{\partial^2 u_i^*}{\partial x_j^* \partial x_j^*} \tag{13.6.3}$$

in which Fr is the Froud number and Re is the Reynolds number defined as

$$\begin{aligned} Fr &= u_0/\sqrt{lB} \\ Re &= \rho u_0 l/\mu. \end{aligned} \tag{13.6.4}$$

13.7 Galerkin Finite Element Formulations

To prepare the finite element model of the viscous fluid flow problems, the Navier-Stokes equations must be modified for the weak formulations of the terms of higher order derivatives. The dimensionless form of the equation is

$$\dot{u}_i + u_j u_{i,j} = \frac{l_i}{Fr^2} - p_{,i} + \frac{1}{Re} (u_{i,j})_{,j}. \tag{13.7.1}$$

This equation may be written as

$$\dot{u}_i + u_j u_{i,j} = \frac{l_i}{Fr^2} + \left[-p\delta_{ij} + \frac{1}{Re} (u_{i,j}) \right]_{,j}. \tag{13.7.2}$$

Now, the solution domain is divided into a number of elements, and the base element (e) is considered. The velocity components u_i are approximated with identical shape functions ϕ and the pressure is approximated with a shape function called ψ, as [3]

$$u_i^{(e)} = \phi_j u_{ij}^{(e)} \qquad j = 1, 2, \ldots s$$
$$p^{(e)} = \psi_n p_n^{(e)} \qquad n = 1, 2, \ldots r \tag{13.7.3}$$

where s and r are the number of nodal points of velocity and pressure shape functions, respectively. Applying the Galerkin method, the Navier-Stokes equations are made orthogonal with respect to the shape function of velocity and the continuity equation is made orthogonal with respect to the shape function of pressure as

$$\int_\Omega \phi_m \left\{ \dot{u}_i + u_j u_{i,j} - \frac{l_i}{Fr^2} - \left[-p\delta_{ij} + \frac{1}{Re}\,(u_{i,j}) \right]_{,j} \right\} d\Omega = 0$$
$$i = 1, 2, 3 \qquad m = 1, 2, \ldots s \tag{13.7.4}$$

$$\int_\Omega \psi_n u_{i,i} d\Omega = 0 \qquad n = 1, 2, \ldots r \tag{13.7.5}$$

From Eq. (13.4.5),

$$\frac{\partial \tau_{ij}}{\partial x_j} = -\frac{\partial p}{\partial x_j}\,\delta_{ij} + \frac{1}{Re}\,\frac{\partial^2 u_i}{\partial x_j \partial x_j} \tag{13.7.6}$$

or

$$\tau_{ij,j} = \left(-p\delta_{ij} + \frac{1}{Re}\,u_{i,j} \right)_{,j}. \tag{13.7.7}$$

Substituting Eq. (13.7.7) in (13.7.4), gives

$$\int_\Omega \phi_m \left[(\dot{u}_i + u_j u_{i,j}) - \frac{l_i}{Fr^2} - \tau_{ij,j} \right] d\Omega = 0$$
$$i = 1, 2, 3 \qquad m = 1, 2, \ldots s. \tag{13.7.8}$$

Applying the weak formulation to the last term of Eq. (13.7.8) gives

$$\int_\Omega \phi_m \left[(\dot{u}_i + u_j u_{i,j}) \right] d\Omega + \int_\Omega \phi_{m,j} \tau_{ij} d\Omega$$
$$= \int_\Omega \phi_m \frac{l_i}{Fr^2} d\Omega + \int_\Gamma \phi_m \tau_{ij} n_j d\Gamma. \tag{13.7.9}$$

In Eq. (13.7.9), $\tau_{ij} n_j$ is the component of the traction vector T_i acting on the boundary. Also, since

$$\tau_{ij} = -p\delta_{ij} + \frac{1}{Re}\,(u_{i,j} + u_{j,i}). \tag{13.7.10}$$

Thus,

$$\int_\Omega \phi_m(\dot{u}_i + u_j u_{i,j})d\Omega - \int_\Omega p\phi_{m,i}d\Omega + \int_\Omega \phi_{m,j}\frac{1}{Re}(u_{i,j} + u_{j,i})d\Omega$$
$$= \int_\Omega \phi_m \frac{l_i}{Fr^2}d\Omega + \int_\Gamma \phi_m T_i d\Gamma. \tag{13.7.11}$$

The continuity equation is

$$-\int_\Omega \psi_n u_{i,i}d\Omega = 0. \tag{13.7.12}$$

The system of Eqs. (13.7.11) and (13.7.12) provides the basic finite element equation to be used for the three-dimensional viscous fluid flow problems.

13.8 Two-Dimensional Fluid Flow

Consider viscous incompressible flow in two-dimensional coordinate system xy. The conventional notation for the velocity components in two-dimensions are u and v in the x and y-directions, respectively. Dividing the fluid flow domain into a number of elements, the velocity components u v, and the pressure p in the base element are approximated as

$$u^{(e)} = \langle\phi\rangle\{u\}^{(e)} = u_i\phi_i^{(e)}$$
$$v^{(e)} = \langle\phi\rangle\{v\}^{(e)} = v_i\phi_i^{(e)} \qquad i = 1, 2, \ldots s \tag{13.8.1}$$
$$p^{(e)} = \langle\psi\rangle\{p\}^{(e)} = p_j\psi_j^{(e)} \qquad j = 1, 2, \ldots r \tag{13.8.2}$$

in which $\langle\phi\rangle$ is the approximating shape function for velocity and $\langle\psi\rangle$ is the approximating shape function for pressure. The number of nodal points of element (e) to describe the velocity is s and for pressure is r. The Galerkin approximation for the two-dimensional fluid flow from Eqs. (13.7.11) and (13.7.12), in terms of the conventional notation, is [3]

$$\int_\Omega \phi_m\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right)d\Omega - \int_\Omega p\frac{\partial \phi_m}{\partial x}d\Omega$$
$$+\frac{1}{Re}\int_\Omega \frac{\partial \phi_m}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}\right)d\Omega + \frac{1}{Re}\int_\Omega \frac{\partial \phi_m}{\partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)d\Omega$$
$$= \int_\Omega \phi_m\frac{l_x}{Fr^2}d\Omega + \int_\Gamma \phi_m T_x d\Gamma \qquad m = 1, 2, \ldots s \tag{13.8.3}$$

$$\int_\Omega \phi_m \left(\frac{\partial v}{\partial t} + u\ \frac{\partial v}{\partial x} + v\ \frac{\partial v}{\partial y}\right) d\Omega - \int_\Omega p\ \frac{\partial \phi_m}{\partial y} d\Omega$$
$$+ \frac{1}{Re}\int_\Omega \frac{\partial \phi_m}{\partial x}\ \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) d\Omega + \frac{1}{Re}\int_\Omega \frac{\partial \phi_m}{\partial y}\ \left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial y}\right) d\Omega$$
$$= \int_\Omega \phi_m\ \frac{l_y}{Fr^2} d\Omega + \int_\Gamma \phi_m T_y d\Gamma \qquad m = 1, 2, \ldots s \tag{13.8.4}$$
$$- \int_\Omega \psi_n \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) d\Omega \qquad n = 1, 2, \ldots r. \tag{13.8.5}$$

The negative sign in Eq. (13.8.5) is used to make the part of the stiffness matrix symmetric. Substituting from Eqs. (13.8.1) and (13.8.2) in Eqs. (13.8.3), (13.8.4) and (13.8.5) gives

$$\int_\Omega \phi_i \left(\dot{u}_j \phi_j + u_k \phi_k u_j\ \frac{\partial \phi_j}{\partial x} + v_k \phi_k u_j\ \frac{\partial \phi_j}{\partial y}\right) d\Omega - \int_\Omega \frac{\partial \phi_i}{\partial x}\ p_m \psi_m d\Omega$$
$$+ \frac{1}{Re}\int_\Omega \left[u_j \left(\frac{\partial \phi_i}{\partial x}\ \frac{\partial \phi_j}{\partial x} + \frac{\partial \phi_i}{\partial y}\ \frac{\partial \phi_j}{\partial y}\right)\right] d\Omega + \frac{1}{Re}\int_\Omega \left[u_j\ \frac{\partial \phi_j}{\partial x}\ \frac{\partial \phi_i}{\partial x}\right.$$
$$\left. + v_j\ \frac{\partial \phi_j}{\partial x}\ \frac{\partial \phi_i}{\partial y}\right] d\Omega = \int_\Omega \phi_i\ \frac{l_x}{Fr^2} d\Omega + \int_\Gamma \phi_i T_x d\Gamma \tag{13.8.6}$$
$$\int_\Omega \phi_i \left(\dot{v}_j \phi_j + u_k \phi_k v_j\ \frac{\partial \phi_j}{\partial x} + v_k \phi_k v_j\ \frac{\partial \phi_j}{\partial y}\right) d\Omega - \int_\Omega \frac{\partial \phi_i}{\partial y}\ p_m \psi_m d\Omega$$
$$+ \frac{1}{Re}\int_\Omega \left[v_j \left(\frac{\partial \phi_i}{\partial x}\ \frac{\partial \phi_j}{\partial x} + \frac{\partial \phi_i}{\partial y}\ \frac{\partial \phi_j}{\partial y}\right)\right] d\Omega + \frac{1}{Re}\int_\Omega \left[v_j\ \frac{\partial \phi_j}{\partial y}\ \frac{\partial \phi_i}{\partial y}\right.$$
$$\left. + u_j\ \frac{\partial \phi_j}{\partial y}\ \frac{\partial \phi_i}{\partial x}\right] d\Omega = \int_\Omega \phi_i\ \frac{l_y}{Fr^2} d\Omega + \int_\Gamma \phi_i T_y d\Gamma \tag{13.8.7}$$
$$- \left[\int_\Omega \left(u_j \psi_i\ \frac{\partial \phi_j}{\partial x} + v_j \psi_i\ \frac{\partial \phi_j}{\partial y}\right) d\Omega\right] = 0. \tag{13.8.8}$$

These equations in matrix form are rewritten as

$$\left[\int_\Omega \phi_i \phi_j d\Omega\right] \dot{u}_j + \left[\frac{1}{Re}\int_\Omega \left(2\ \frac{\partial \phi_i}{\partial x}\ \frac{\partial \phi_j}{\partial x} + \frac{\partial \phi_i}{\partial y}\ \frac{\partial \phi_j}{\partial y}\right) d\Omega\right] u_j$$
$$+ \left[\frac{1}{Re}\int_\Omega \frac{\partial \phi_j}{\partial x}\ \frac{\partial \phi_i}{\partial y} d\Omega\right] v_j + \left[u_k \int_\Omega \phi_i \phi_k\ \frac{\partial \phi_j}{\partial x} d\Omega + v_k \int_\Omega \phi_i \phi_k\ \frac{\partial \phi_j}{\partial y} d\Omega\right] u_j$$
$$- \left[\int_\Omega \frac{\partial \phi_i}{\partial x}\ \psi_m d\Omega\right] p_m = \int_\Omega \phi_i\ \frac{l_x}{Fr^2}\ d\Omega + \int_\Gamma \phi_i T_x d\Gamma \tag{13.8.9}$$
$$\left[\int_\Omega \phi_i \phi_j d\Omega\right] \dot{v}_j + \left[\frac{1}{Re}\int_\Omega \left(2\ \frac{\partial \phi_i}{\partial y}\ \frac{\partial \phi_j}{\partial y} + \frac{\partial \phi_i}{\partial x}\ \frac{\partial \phi_j}{\partial x}\right) d\Omega\right] v_j$$
$$+ \left[\frac{1}{Re}\int_\Omega \frac{\partial \phi_j}{\partial y}\ \frac{\partial \phi_i}{\partial x} d\Omega\right] u_j + \left[u_k \int_\Omega \phi_i \phi_k\ \frac{\partial \phi_j}{\partial x} d\Omega + v_k \int_\Omega \phi_i \phi_k\ \frac{\partial \phi_j}{\partial y} d\Omega\right] v_j$$

$$-\left[\int_{\Omega} \frac{\partial \phi_i}{\partial y}\ \psi_m d\Omega\right] p_m = \int_{\Omega} \phi_i\ \frac{l_y}{Fr^2}\ d\Omega + \int_{\Gamma} \phi_i T_y d\Gamma \tag{13.8.10}$$

$$-\left[\int_{\Omega} \psi_i\ \frac{\partial \phi_j}{\partial x} d\Omega\right] u_j - \left[\int_{\Omega} \psi_i\ \frac{\partial \phi_j}{\partial y}\ d\Omega\right] v_j = 0. \tag{13.8.11}$$

Defining the element matrices as

$$[m_{ij}]^{(e)} = \begin{bmatrix} \int_{\Omega} \phi_i \phi_j d\Omega & 0 \\ 0 & \int_{\Omega} \phi_i \phi_j d\Omega \end{bmatrix} \tag{13.8.12}$$

$$[k_{ij}]^{(e)} = \frac{1}{Re} \begin{bmatrix} \int_{\Omega} \left(2\ \frac{\partial \phi_i}{\partial x}\ \frac{\partial \phi_j}{\partial x} + \frac{\partial \phi_i}{\partial y}\ \frac{\partial \phi_j}{\partial y}\right) d\Omega & \int_{\Omega} \frac{\partial \phi_i}{\partial y}\ \frac{\partial \phi_j}{\partial x} \\ \int_{\Omega} \frac{\partial \phi_i}{\partial x}\ \frac{\partial \phi_j}{\partial y} & \int_{\Omega} \left(\frac{\partial \phi_i}{\partial x}\ \frac{\partial \phi_j}{\partial x} + 2\ \frac{\partial \phi_i}{\partial y}\ \frac{\partial \phi_j}{\partial y}\right) d\Omega \end{bmatrix} \tag{13.8.13}$$

$$[N(u)_{ij}]^{(e)} = \begin{bmatrix} u_k \int_{\Omega} \phi_k \phi_i\ \frac{\partial \phi_j}{\partial x}\ d\Omega + v_k \int_{\Omega} \phi_k \phi_i\ \frac{\partial \phi_j}{\partial y}\ d\Omega & 0 \\ 0 & u_k \int_{\Omega} \phi_k \phi_i\ \frac{\partial \phi_j}{\partial x}\ d\Omega + v_k \int_{\Omega} \phi_k \phi_i\ \frac{\partial \phi_j}{\partial y}\ d\Omega \end{bmatrix} \tag{13.8.14}$$

$$[c_{ij}]^{(e)} = \begin{bmatrix} -\int_{\Omega} \psi_j\ \frac{\partial \phi_i}{\partial x}\ d\Omega \\ -\int_{\Omega} \psi_j\ \frac{\partial \phi_i}{\partial y}\ d\Omega \end{bmatrix} \tag{13.8.15}$$

$$\{f_{ij}\}^{(e)} = \begin{Bmatrix} -\int_{\Gamma} \phi_i T_x d\Gamma + \int_{\Omega} \phi_i\ \frac{l_x}{Fr^2}\ d\Omega \\ -\int_{\Gamma} \phi_i T_y d\Gamma + \int_{\Omega} \phi_i\ \frac{l_y}{Fr^2}\ d\Omega \end{Bmatrix}. \tag{13.8.16}$$

The finite element equation describing two-dimensional viscous fluid motion becomes

$$[M]\{\dot{U}\} + [K + N(U)]\{U\} + [C]\{P\} = \{F\} \tag{13.8.17}$$

$$[C]^T \{U\} = 0. \tag{13.8.18}$$

The set of Eqs. (13.8.17) and (13.8.18) for steady state fluid flow is reduced to

$$\begin{bmatrix} [K] + [N(u)] & \{C\} \\ \{C\}^T & 0 \end{bmatrix} \begin{Bmatrix} \{U\} \\ P \end{Bmatrix} = \begin{Bmatrix} \{F\} \\ 0 \end{Bmatrix}. \tag{13.8.19}$$

The system of Eqs. (13.8.17) and (13.8.18), or Eq. (13.8.19), is nonlinear, due to the existence of matrix $[N(u)]$. This matrix is a function of the velocity matrix, which itself is unknown. Therefore, the finite element modeling of the inviscid fluid flow through the Navier-Stokes equations is nonlinear in velocity, as are the original basic equations. There are a number of techniques available to apply and solve the resulting nonlinear finite element equations. One of these techniques is an iterative method of solution with increment placed on the Reynolds number. This technique is described later on in this chapter.

13.9 Boundary Conditions

The finite element method described in the preceding section was based on velocity and pressure as dependent functions. This method is called the *conventional formulation*, and as with the boundary conditions, the velocity components and pressure values must be known on the boundary.

Calling the normal traction and normal velocity T_n and u_n, and the tangential traction and velocity T_t and u_t, the normal and sheer stresses on the boundary Γ of the fluid flow field are [3, 4]

$$\begin{aligned} T_n &= -p + 2\mu \frac{\partial u_n}{\partial n} \\ T_t &= \mu \left(\frac{\partial u_n}{\partial t} + \frac{\partial u_t}{\partial n} \right) \end{aligned} \tag{13.9.1}$$

in which the subscripts n and t represent the normal and tangential directions on the boundary Γ. The boundary condition on the outlet flow of a flow field is $T_n = T_t = 0$. This condition is equivalent to the traction free boundary condition. For high Reynold's numbers, this condition results in a reliable solution, while that may not be the case for the low Reynolds numbers. For a symmetric flow field, the condition of $u_n = T_t = 0$ along the symmetry axes may be used. The boundary conditions on a rigid boundary are satisfied by $u_n = u_t = 0$.

It is usually not possible directly to define the pressure on a fluid flow boundary. For an incompressible fluid flow, the pressure distribution at each point adjusts itself such that the continuity condition $\nabla \cdot \vec{U} = 0$ is satisfied. Now, if the pressure on a fluid flow boundary is defined through the boundary conditions, the continuity condition $\nabla \cdot \vec{U} = 0$ along the given boundary may not be truly satisfied [3]. This may cause complications in a reliable solution field and result in a physically incompatible fluid field, as the law of conservation of mass is violated along the boundary where the pressure is defined. It is thus recommended to use the condition of normal traction

$$T_n = -p + \frac{du_n}{dn} \tag{13.9.2}$$

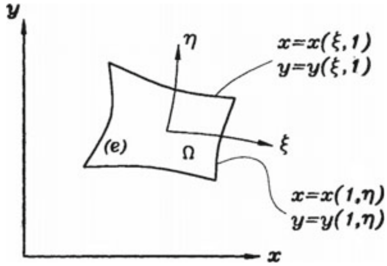

Fig. 13.1 A general four-sides element

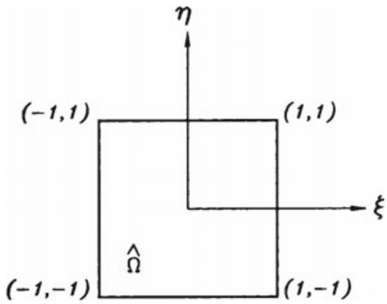

Fig. 13.2 Element in local coordinates $\xi\eta$

on the boundary where pressure is being specified, especially at high Reynold's numbers. When condition (13.9.2) is used on a boundary, the pressure is specified on the boundary and the continuity condition is also satisfied.

13.10 Element Selection

A proper element for the conventional finite element formulation of viscous fluid flow in two-dimensions is a general four-sided element, as shown in Fig. 13.1. The global coordinates system xy and the local system $\xi\eta$ are shown in Fig. 13.1. With proper coordinate transformation [3]

$$\begin{aligned} x &= x(\xi, \eta) \\ y &= y(\xi, \eta) \end{aligned} \tag{13.10.1}$$

the element (e) of Fig. 13.1 is transformed into the square element (e) in domain $\widehat{\Gamma}$, as shown in Fig. 13.2, with domain variables specified as

$$\begin{aligned} -1 \le \xi \le +1 \\ -1 \le \eta \le +1. \end{aligned} \tag{13.10.2}$$

The law of transformation between the two systems is

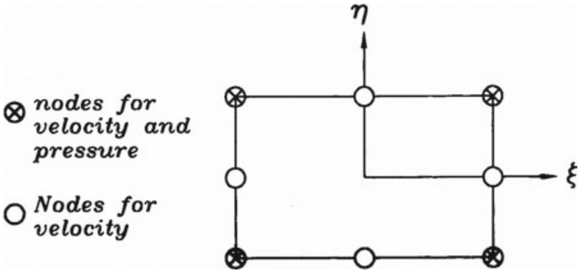

Fig. 13.3 A four-sides element for velocity and pressure

$$\begin{Bmatrix} dx \\ dy \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial x}{\partial \eta} \\ \dfrac{\partial y}{\partial \xi} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} d\xi \\ d\eta \end{Bmatrix}. \tag{13.10.3}$$

Thus, the Jacobian is

$$|J| = J = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi}. \tag{13.10.4}$$

The Jacobian is used when the derivatives of the shape functions are involved in the element matrices. Consider the relations

$$\begin{Bmatrix} \dfrac{\partial N}{\partial x} \\ \dfrac{\partial N}{\partial y} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial \xi}{\partial x} & \dfrac{\partial \eta}{\partial x} \\ \dfrac{\partial \xi}{\partial y} & \dfrac{\partial \eta}{\partial y} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial N}{\partial \xi} \\ \dfrac{\partial N}{\partial \eta} \end{Bmatrix} = \frac{1}{|J|} \begin{bmatrix} \dfrac{\partial y}{\partial \eta} & -\dfrac{\partial y}{\partial \xi} \\ -\dfrac{\partial x}{\partial \eta} & \dfrac{\partial x}{\partial \xi} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial N}{\partial \xi} \\ \dfrac{\partial N}{\partial \eta} \end{Bmatrix} \tag{13.10.5}$$

or

$$\begin{Bmatrix} \dfrac{\partial N}{\partial x} \\ \dfrac{\partial N}{\partial y} \end{Bmatrix} = [J^{-1}]^T \begin{Bmatrix} \dfrac{\partial N}{\partial \xi} \\ \dfrac{\partial N}{\partial \eta} \end{Bmatrix}. \tag{13.10.6}$$

Since the shape functions are described in forms of the natural coordinates ξ and η, their derivatives in the local coordinates are carried out, and then, from Eq. (13.10.6), are related to the variables x and y in the global coordinates.

For the conventional formulations, in which the field variables are selected to be the velocity vector and pressure, the shape function associated with velocity components must always have higher degrees compared to the shape function describing the pressure distribution in the element. For the four-sided element in two-dimensions, the shape function for velocity components is selected as a third order polynomial, and for pressure, a second order polynomial. Figure 13.3 shows an element with eight nodal points for velocity and four nodal points for pressure. As noted, the corner nodes are common for both velocity and pressure, where the middle side nodes are for velocity alone. Thus, eight nodes describe the velocity components u and v, while four nodes describe the pressure. The shape functions describing the velocity components is of order three and are.

$$
\begin{aligned}
\phi_1 &= \frac{1}{4}\,(-1+\xi\eta+\xi^2+\eta^2-\xi^2\eta-\xi\eta^2)\\
\phi_2 &= \tfrac{1}{2}\,(1-\eta-\xi^2+\xi^2\eta)\\
\phi_3 &= \frac{1}{4}\,(-1-\xi\eta+\xi^2+\eta^2-\xi^2\eta+\xi\eta^2)\\
\phi_4 &= \tfrac{1}{2}\,(1+\xi-\eta^2-\xi\eta^2)\\
\phi_5 &= \frac{1}{4}\,(-1+\xi\eta+\xi^2+\eta^2+\xi^2\eta+\xi\eta^2)\\
\phi_6 &= \tfrac{1}{2}\,(1+\eta-\xi^2-\xi^2\eta)\\
\phi_7 &= \frac{1}{4}\,(-1-\xi\eta+\xi^2+\eta^2+\xi^2\eta-\xi\eta^2)\\
\phi_8 &= \tfrac{1}{2}\,(1-\xi-\eta^2+\xi\eta^2).
\end{aligned}
\tag{13.10.7}
$$

The shape functions describing the pressure are of order two and are

$$
\begin{aligned}
\psi_1 &= \frac{1}{4}\,(1-\xi-\eta+\xi\eta)\\
\psi_2 &= \frac{1}{4}\,(1+\xi-\eta-\xi\eta)\\
\psi_3 &= \frac{1}{4}\,(1+\xi+\eta+\xi\eta)\\
\psi_4 &= \frac{1}{4}\,(1-\xi+\eta-\xi\eta).
\end{aligned}
\tag{13.10.8}
$$

13.11 Vorticity Transport

The Navier-Stokes equations in terms of the velocity components and the pressure were used to construct the finite element model of fluid dynamics problems. A different approach for describing the field equations are the governing equations of fluid dynamics in terms of the vorticity and stream function. The Navier-Stokes equations are modified into a system of equations written in terms of the vorticity and stream function and are called the *vorticity transport* [5].

Let us consider a two-dimensional incompressible viscous fluid flow. By definition, the stream function ψ is related to the velocity components as

$$
u = \frac{\partial\psi}{\partial y} \qquad\qquad v = -\frac{\partial\psi}{\partial x}.
\tag{13.11.1}
$$

The vorticity ω is defined as

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \tag{13.11.2}$$

The continuity and the Navier-Stokes equations for a two-dimensional steady-state incompressible viscous fluid flow in a dimensionless form are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{13.11.3}$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{13.11.4}$$

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{13.11.5}$$

Adding and subtracting terms $\partial v/\partial x$ and $\partial u/\partial y$ to Eq. (13.11.3) yields

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial v}{\partial x} - \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} - \frac{\partial u}{\partial y} = 0.$$

Using the definition of vorticity and stream function, this equation is written as

$$\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x \partial y} + \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \omega = 0$$

which simplifies to

$$\nabla^2 \psi + \omega = 0. \tag{13.11.6}$$

To obtain the second equation, Eq. (13.11.4) is differentiated with respect to y and Eq. (13.11.5) is differentiated with respect to x and is subtracted from the first equation to give

$$\frac{\partial u}{\partial y}\frac{\partial u}{\partial x} + u\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} + v\frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} - u\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} - v\frac{\partial^2 v}{\partial x \partial y}$$
$$= \frac{1}{Re}\left(\frac{\partial^3 u}{\partial y \partial x^2} + \frac{\partial^3 u}{\partial y^3} - \frac{\partial^3 v}{\partial x^3} - \frac{\partial^3 v}{\partial x \partial y^2}\right).$$

This equation may be rearranged as

$$\left[u\left(\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 v}{\partial x^2}\right) + v\left(\frac{\partial^2 u}{\partial y^2} - .\frac{\partial^2 v}{\partial x \partial y}\right) + \frac{\partial u}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right). - \frac{\partial v}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right] Re$$
$$= \frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right).$$

Using the continuity equation, we have

$$\left[u \frac{\partial}{\partial x} (\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}) + v \frac{\partial}{\partial y} (\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x})\right] Re$$
$$= \frac{\partial^2}{\partial x^2} (\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}) + \frac{\partial^2}{\partial y^2} (\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}).$$

Substituting from Eqs. (13.11.1) and (13.11.2), we obtain

$$\nabla^2\omega - Re\ (\frac{\partial \psi}{\partial y} \frac{\partial \omega}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial \omega}{\partial y}) = 0. \tag{13.11.7}$$

This equation relates the vorticity ω to the stream function ψ. The system of Eqs. (13.11.6) and (13.11.7) are a pair of partial differential equations relating two independent functions ω and ψ. This system of equations is substituted for the system of Eqs. (13.11.3) to (13.11.5), which relates p, u, and v. Therefore, we may either solve the system of Eqs. (13.11.3) to (13.11.5) for the independent functions p, u, and v, or solve the system of Eqs. (13.11.6) and (13.11.7) for the independent functions ω and ψ. The latter system of equations is called the *vorticity transport* equations.

13.12 Finite Element Modelling

The system of equations in terms of the vorticity transport is written as

$$\nabla^2\psi + \omega = 0 \tag{13.12.1}$$
$$\nabla^2\omega - Re\ (\frac{\partial \psi}{\partial y} \frac{\partial \omega}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial \omega}{\partial y}) = 0. \tag{13.12.2}$$

Dividing the solution domain into a number of arbitrary elements, the residues of the governing equations over the base element (e) on the weighting function w_k, using the Galerkin method, are

$$\int_{D(e)} (\nabla^2\psi^* + \omega^*) w_k dxdy = 0 \tag{13.12.3}$$
$$\int_{D(e)} \left[\nabla^2\omega^* - Re\ (\frac{\partial \psi^*}{\partial y} \frac{\partial \omega^*}{\partial x} - \frac{\partial \psi^*}{\partial x} \frac{\partial \omega^*}{\partial y})\right] w_k dxdy = 0 \tag{13.12.4}$$

where ψ^* and ω^* are the approximate functions describing the stream function and the vorticity in the element (e).

To evaluate the integrations and weak formulations, consider the integral

$$I = \int_{D(e)} A \nabla^2 B dx dy \tag{13.12.5}$$

where A and B are two arbitrary functions of x and y. This equation is written as

$$I = \int_{D(e)} A \frac{\partial^2 B}{\partial x^2} dx dy + \int_{D(e)} A \frac{\partial^2 B}{\partial y^2} dx dy \tag{13.12.6}$$

or

$$I = \int_{y-dir} \left(\int_{x-dir} A \frac{\partial^2 B}{\partial x^2} dx \right) dy + \int_{x-dir} \left(\int_{y-dir} A \frac{\partial^2 B}{\partial y^2} dy \right) dx. \tag{13.12.7}$$

The weak formulation of Eq. (13.12.7) gives

$$I = \int_{y-dir} A \frac{\partial B}{\partial x} dy - \int_{D(e)} \frac{\partial A}{\partial x} \frac{\partial B}{\partial x} dx dy + \int_{x-dir} A \frac{\partial B}{\partial y} dx - \int_{D(e)} \frac{\partial A}{\partial y} \frac{\partial B}{\partial y} dx dy. \tag{13.12.8}$$

The sum of the first and third terms in Eq. (13.12.8) is the integration of function B along the normal to the boundary curve, i.e., $A\ \partial B / \partial n$ on $\Gamma(e)$, where $\vec{n}$ is the unit outer normal vector to the boundary of the element. Thus,

$$\int_{D(e)} A \nabla^2 B dx dy = \int_{\Gamma(e)} A \frac{\partial B}{\partial n} dS - \int_{D(e)} \left(\frac{\partial A}{\partial x} \frac{\partial B}{\partial x} + \frac{\partial A}{\partial y} \frac{\partial B}{\partial y} \right) dx dy. \tag{13.12.9}$$

Using the rule of weak formulation given by Eq. (13.12.9), the weak formulation of Eq. (13.12.3) is

$$\int_{D(e)} w_k \nabla^2 \psi^* dx dy + \int_{D(e)} w_k \omega^* dx dy = 0 \tag{13.12.10}$$

or

$$\int_{\Gamma(e)} w_k \frac{\partial \psi^*}{\partial n} dS - \int_{D(e)} \left(\frac{\partial w_k}{\partial x} \frac{\partial \psi^*}{\partial x} + \frac{\partial w_k}{\partial y} \frac{\partial \psi^*}{\partial y} \right) dx dy + \int_{\Gamma(e)} w_k \omega^* dS = 0. \tag{13.12.11}$$

This equation is rearranged as

$$\int_{D(e)} \left(\frac{\partial w_k}{\partial x} \frac{\partial \psi^*}{\partial x} + \frac{\partial w_k}{\partial y} \frac{\partial \psi^*}{\partial y} \right) dx dy - \int_{D(e)} w_k \omega^* dx dy = \int_{\Gamma(e)} w_k \frac{\partial \psi^*}{\partial n} dS. \tag{13.12.12}$$

Similarly, using the rule of Eq. (13.12.9), the weak formulation of Eq. (13.12.4) yields

$$\int_{D(e)} (\frac{\partial w_k}{\partial x}\frac{\partial \omega^*}{\partial x} + \frac{\partial w_k}{\partial y}\frac{\partial \omega^*}{\partial y}) dxdy + Re\int_{D(e)} (\frac{\partial \psi^*}{\partial y}\frac{\partial \omega^*}{\partial x} - \frac{\partial \psi^*}{\partial x}\frac{\partial \omega^*}{\partial y}) w_k dxdy$$
$$= \int_{\Gamma(e)} w_k \frac{\partial \omega^*}{\partial n}\, dS. \tag{13.12.13}$$

The system of Eqs. (13.12.12) and (13.12.13) establishes the element equilibrium equations.

Consider the base element (e). The stream function and vorticity are approximated in (e) as

$$\psi^{*(e)} = \sum_{i=1}^{r} N_i \psi_i \tag{13.12.14}$$

$$\omega^{*(e)} = \sum_{i=1}^{r} N_i \omega_i \tag{13.12.15}$$

where N_i is the approximating shape function. Here, we have assumed identical approximating shape functions for ψ and ω. Replacing the weighting function w by N, the finite element equilibrium Eqs. (13.12.12) and (13.12.13) become [6]

$$\int_{D(e)} \left(\frac{\partial N_k}{\partial x}\frac{\partial \sum N_i \psi_i}{\partial x} + \frac{\partial N_k}{\partial y}\frac{\partial \sum N_i \psi_i}{\partial y} - N_k \sum N_i \omega_i\right) dxdy$$
$$= \int_{\Gamma(e)} N_k \frac{\partial \psi^*}{\partial n}\, dS$$

$$\int_{D(e)} \left(\frac{\partial N_k}{\partial x}\frac{\partial \sum N_i \omega_i}{\partial x} + \frac{\partial N_k}{\partial y}\frac{\partial \sum N_i \omega_i}{\partial y}\right) dxdy$$
$$Re\int_{D(e)} N_k \left(\frac{\partial \sum N_i \psi_i}{\partial y}\frac{\partial \sum N_j \omega_j}{\partial x} - \frac{\partial \sum N_i \psi_i}{\partial x}\frac{\partial \sum N_j \omega_j}{\partial y}\right) dxdy$$
$$= \int_{\Gamma(e)} N_k \frac{\partial \omega^*}{\partial n}\, dS.$$

These equations are rearranged as

$$\sum_i \left[\int_{D(e)} (\frac{\partial N_k}{\partial x} \frac{\partial N_i}{\partial x} + \frac{\partial N_k}{\partial y} \frac{\partial N_i}{\partial y}) dxdy \right] \psi_i - \sum_i \left[\int_{D(e)} N_k N_i dxdy \right] \omega_i$$
$$= \int_{\Gamma(e)} N_k \frac{\partial \psi^*}{\partial n} dS \qquad (13.12.16)$$

$$\sum_i \left[\int_{D(e)} (\frac{\partial N_k}{\partial x} \frac{\partial N_i}{\partial x} + \frac{\partial N_k}{\partial y} \frac{\partial N_i}{\partial y}) dxdy \right] \omega_i$$
$$+ Re \sum_{i,j} \left[\int_{D(e)} \left(N_k (\frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial x} - \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial y}) dxdy \right) \psi_i \right] \omega_j$$
$$= \int_{\Gamma(e)} N_k \frac{\partial \omega^*}{\partial n} dS. \qquad (13.12.17)$$

Note that ψ_i and ω_i are the stream function and vorticity at the nodes of the element (e), and thus, are constants with respect to the domain integrations. Considering the constant matrices of coefficients as

$$A_{ki} = \int_{D(e)} \left(\frac{\partial N_k}{\partial x} \frac{\partial N_i}{\partial x} + \frac{\partial N_k}{\partial y} \frac{\partial N_i}{\partial y} \right) dxdy$$
$$B_{ki} = \int_{D(e)} N_k N_i dxdy$$
$$C_{kij} = \int_{D(e)} N_k \left(\frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial x} - \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial y} \right) dxdy$$
$$f_{1k} = \int_{\Gamma(e)} N_k \frac{\partial \psi^*}{\partial n} dS$$
$$f_{2k} = \int_{\Gamma(e)} N_k \frac{\partial \omega^*}{\partial n} dS, \qquad (13.12.18)$$

the system of finite element Eqs. (13.12.16) and (13.12.17) are written as

$$-\sum_i B_{ki} \omega_i + \sum_i A_{ki} \psi_i \Rightarrow f_{1k} \qquad (13.12.19)$$
$$\sum_i A_{ki} \omega_i + Re \sum_{i,j} C_{kij} \psi_i \omega_j \Rightarrow f_{2k}. \qquad (13.12.20)$$

Assembling the element equilibrium equations for all the elements in the solution domain results in the final equation of motion of finite elements as

$$[A]\{\Psi\} - [B]\{\Omega\} = \{F_1\} \qquad (13.12.21)$$
$$[A]\{\Omega\} + Re[C(\Omega)]\{\Psi\} = \{F_2\}. \qquad (13.12.22)$$

It should be noted that Eq. (13.12.22) is nonlinear, as the products of ω and ψ appear in the equation.

13.13 Linearization Technique

The set of Eqs. (13.12.19) and (13.12.20) establishes a system of nonlinear equations with constant coefficients to be solved for the dependent functions Ψ and Ω on the nodal points. A method for solving this system of equations, is the incremental technique and linearization method. The increment is placed on the Reynolds number Re.

Let us, for the time being, assume that the boundary conditions are set such that $\{f_1\}$ and $\{f_2\}$ are both zero. This point is discussed later in Sect. 13.9 of this chapter. The system of Eqs. (13.12.19) and (13.12.20) is now written as [6]

$$-\sum_i B_{ki}\omega_i + \sum_i A_{ki}\psi_i = 0 \tag{13.13.1}$$

$$\sum_i A_{ki}\omega_i + Re\sum_{i,j} C_{kij}\psi_i\omega_j = 0. \tag{13.13.2}$$

Now, let us assume that at Reynolds number R, the values of ψ_i and ω_i are known and satisfy Eqs. (13.13.1) and (13.13.2). At an additional increment of Reynolds number $R + \delta R$, the stream function and the vorticity are $\psi_i + \delta\psi_i$ and $\omega_i + \delta\omega_i$. Since ψ_i and ω_i are assumed to be known at R, their incremental values $\delta\psi_i$ and $\delta\omega_i$ must be calculated at $R + \delta R$. Substituting in Eqs. (13.13.1) and (13.13.2) yields

$$-\sum_i B_{ki}(\omega_i + \delta\omega_i) + \sum_i A_{ki}(\psi_i + \delta\psi_i) = 0 \tag{13.13.3}$$

$$\sum_i A_{ki}(\omega_i + \delta\omega_i) + (R + \delta R)\sum_{i,j} C_{kij}(\psi_i + \delta\psi_i)(\omega_j + \delta\omega_j) = 0. \tag{13.13.4}$$

Expanding Eq. (13.13.3)

$$-\sum_i B_{ki}\omega_i + \sum_i A_{ki}\psi_i - \sum_i B_{ki}\delta\omega_i + \sum_i A_{ki}\delta\psi_i = 0.$$

According to Eq. (13.13.1), the first two terms are zero, and thus,

$$-\sum_i B_{ki}\delta\omega_i + \sum_i A_{ki}\delta\psi_i = 0. \tag{13.13.5}$$

Expanding Eq. (13.13.4) yields

$$\sum_i A_{ki}\omega_i + \sum_i A_{ki}\delta\omega_i + R\sum_{i,j} C_{kij}\psi_i\omega_j + R\sum_{i,j} C_{kij}\psi_i\delta\omega_j$$
$$+ R\sum_{i,j} C_{kij}\delta\psi_i\omega_j + R\sum_{i,j} C_{kij}\delta\psi_i\delta\omega_j + \delta R\sum_{i,j} C_{kij}\psi_i\omega_j$$
$$+ \delta R\sum_{i,j} C_{kij}\psi_i\delta\omega_j + \delta R\sum_{i,j} C_{kij}\delta\psi_i\omega_j + \delta R\sum_{i,j} C_{kij}\delta\psi_i\delta\omega_j. \quad (13.13.6)$$

If δR is selected small enough, then terms with $\delta\delta$ and $\delta\delta\delta$ in Eq. (13.13.6) are neglected and the equation reduces to

$$\sum_i A_{ki}\omega_i + R\sum_{i,j} C_{kij}\psi_i\omega_j + \sum_i (A_{ki} + R\sum_j C_{kij}\psi_i)\delta\omega_j$$
$$+R\sum_{i,j} C_{kij}\delta\psi_i\omega_j + \delta R\sum_{i,j} C_{kij}\psi_i\omega_j. \quad (13.13.7)$$

The first two terms, using Eq. (13.13.2), are zero, and thus,

$$\sum_i \left(A_{ki} + R\sum_j C_{kij}\psi_i\right)\delta\omega_j + R\sum_{i,j} C_{kij}\delta\psi_i\omega_j$$
$$= -\delta R\sum_{i,j} C_{kij}\psi_i\omega_j. \quad (13.13.8)$$

Equations (13.13.5) and (13.13.8) are a set of linearized equations to be solved for $\delta\psi_i$ and $\delta\omega_i$.

13.14 Triangular Simplex Element

We try to use the simplex triangular element to model the two-dimensional fluid flow problems based on vorticity transport equations. This is an interesting point to compare with the conventional formulations in which essentially higher order polynomials were used to approximate the velocity components. The efficiency and accuracy of simplex triangular element for modeling the flow problems of incompressible fluids based on vorticity transport equations are shown in references [6, 7].

Consider a two-dimensional flow of an incompressible fluid. A simplex triangular element (e) with nodes i, j, and k is considered. The stream function and the vorticity are approximated with linear shape functions as

$$\psi^{(e)} = \langle N_i \quad N_j \quad N_k \rangle \begin{Bmatrix} \psi_i \\ \psi_j \\ \psi_k \end{Bmatrix}$$

$$\omega^{(e)} = \langle N_i \quad N_j \quad N_k \rangle \begin{Bmatrix} \omega_i \\ \omega_j \\ \omega_k \end{Bmatrix} \tag{13.14.1}$$

where

$$N_i = \frac{a_i + b_i x + c_i y}{2A} \qquad N_j = \frac{a_j + b_j x + c_j y}{2A} \qquad N_k = \frac{a_k + b_k x + c_k y}{2A} \tag{13.14.2}$$

in which a, b, and c are constant coefficients and A is the area of triangle (e) (see Chap. 4). Substituting for the values of shape functions in Eq. (13.12.18), the members of the coefficient matrices are obtained through proper integrations. From the rule of area coordinates,

$$\int_{A(e)} N_i^a N_j^b N_k^c dxdy = \frac{a!b!c!}{(a+b+c+2)!}\, 2A. \tag{13.14.3}$$

Thus,

$$\begin{aligned}
A_{ii} &= \int_{A(e)} \frac{1}{4A^2}\, (b_i^2 + c_i^2) dxdy = \frac{b_i^2 + c_i^2}{4A} \\
A_{ij} &= \int_{A(e)} \frac{1}{4A^2}\, (b_i b_j + c_i c_j) dxdy = \frac{b_i b_j + c_i c_j}{4A} \\
B_{ii} &= \int_{A(e)} N_i N_i dxdy = \frac{A}{6} \\
B_{ij} &= \int_{A(e)} N_i N_j dxdy = \frac{A}{12}.
\end{aligned} \tag{13.14.4}$$

Substituting in Eq. (13.13.5) gives

$$\begin{bmatrix} cc[-k\Delta] & [k] \\ [k] & [0] \end{bmatrix} \begin{Bmatrix} c\{\delta\omega\} \\ \{\delta\psi\} \end{Bmatrix} \to \begin{Bmatrix} c\{0\} \\ \{f_1\} \end{Bmatrix} \tag{13.14.5}$$

where

$$[k\Delta]^{(e)} = \frac{A}{12}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$[k]^{(e)} = \frac{1}{4A}\begin{bmatrix} b_i^2 + c_i^2 & b_i b_j + c_i c_j & b_i b_k + c_i c_k \\ b_j b_i + c_j c_i & b_j^2 + c_j^2 & b_j b_k + c_j c_k \\ b_k b_i + c_k c_i & b_k b_j + c_k c_j & b_k^2 + c_k^2 \end{bmatrix}$$

$$\{\delta\omega\}^T = \langle \delta\omega_i \quad \delta\omega_j \quad \delta\omega_k \rangle^{(e)}$$
$$\{\delta\psi\}^T = \langle \delta\psi_i \quad \delta\psi_j \quad \delta\psi_k \rangle^{(e)}. \tag{13.14.6}$$

Equation (13.14.5), derived from Eq. (13.13.5), is linear.

To derive the element members of the matrices related to the nonlinear Eq. (13.12.20), or its equivalent linearized Eq. (13.13.8), the coefficients C_{kij} must first be evaluated. From Eq. (13.12.18),

$$C_{kij} = \int_{A(e)} N_k (\frac{\partial N_i}{\partial y}\ \frac{\partial N_j}{\partial x} - \frac{\partial N_i}{\partial y}\ \frac{\partial N_j}{\partial x}) dxdy.$$

Substituting from Eq. (13.14.2), gives

$$C_{kij} = \int_{A(e)} \frac{N_k}{4A^2}(c_i b_j - c_j b_i) dxdy.$$

Using the rule of area coordinates, the integration yields

$$C_{kij} = \frac{c_i b_j - c_j b_i}{4A^2}\int_{A(e)} N_k^1 N_j^0 N_k^0 dxdy$$

or

$$C_{kij} = \frac{c_i b_j - c_j b_i}{12A}.$$

Substituting C_{kij} in Eq. (13.13.8), with known A_{ij}, Results in

$$\begin{bmatrix} cc[-k\Delta] & [k] \\ [k + k\psi] & [k\omega] \end{bmatrix}\begin{Bmatrix} c\{\delta\omega\} \\ \{\delta\psi\} \end{Bmatrix} \rightarrow \begin{Bmatrix} c\{0\} \\ \{f\} \end{Bmatrix} \tag{13.14.7}$$

in which the definition of submatrices is

$$[k\psi]^{(e)} = R \begin{bmatrix} \sum_l C_{ili}\psi_l & \sum_l C_{ilj}\psi_l & \sum_l C_{ilk}\psi_l \\ \sum_l C_{jli}\psi_l & \sum_l C_{jlj}\psi_l & \sum_l C_{jlk}\psi_l \\ \sum_l C_{kli}\psi_l & \sum_l C_{klj}\psi_l & \sum_l C_{klk}\psi_l \end{bmatrix}$$

$$[k\omega]^{(e)} = R \begin{bmatrix} \sum_l C_{iil}\omega_l & \sum_l C_{ijl}\omega_l & \sum_l C_{ikl}\omega_l \\ \sum_l C_{jil}\omega_l & \sum_l C_{jjl}\omega_l & \sum_l C_{jkl}\omega_l \\ \sum_l C_{kil}\omega_l & \sum_l C_{kjl}\omega_l & \sum_l C_{kkl}\omega_l \end{bmatrix}$$

$$\{f\}^{(e)} = -\delta R \begin{Bmatrix} \sum_{l,m} C_{ilm}\psi_l\omega_m \\ \sum_{l,m} C_{jlm}\psi_l\omega_m \\ \sum_{l,m} C_{klm}\psi_l\omega_m \end{Bmatrix}. \tag{13.14.8}$$

The definitions of $[\Delta k]$ and $[k]$ are given in Eq. (13.14.6). The system of Eqs. (13.14.5) and (13.14.8) is linear in terms of the nodal unknowns $\{\delta\psi\}$ and $\{\delta\omega\}$. The problem must be solved incrementally, i.e. marched through the Reynolds number δR. The numerical solution must start from the initial state where, at Reynolds number R, the values of ψ and ω at all nodal points are known. With a proper small value of δR, the solution is incrementally solved through the advancement of the Reynolds number until the required Reynolds number, by which the problem is defined, is reached.

13.15 Boundary Conditions

The general form of the finite element equations of two-dimensional incompressible steady-state fluid flow problems for the base element (e) was derived as

$$-\sum_i B_{ki}\omega_i + \sum_i A_{ki}\psi_i \Rightarrow f_{1k} \tag{13.15.1}$$

$$\sum_i A_{ki}\omega_i + R\sum_{i,j} C_{kij}\psi_i\omega_j \Rightarrow f_{2k}. \tag{13.15.2}$$

The definitions of the force matrices f_{1k} and f_{2k} are given by Eq. (13.12.18). These forces are evaluated on the boundary of the solution domain, and are directly calculated knowing the boundary conditions.

Consider a solution domain with boundary C. We divide the boundary curve C into two parts,; namely, C_1 and C_2. The boundary C_1 is assumed to be where the stream function ψ and $\partial\psi/\partial n$ are defined, and C_2 is the part of the boundary where the inflow and outflow are defined, and thus, ψ and ω are known, as shown in Fig. 13.4.

The boundary C_1 is fixed and is the rigid boundary of the fluid field. The fluid arrives from the left side through boundary C_2 and flows out from the right boundary C_2. The value of ψ is known on the boundary C_1 and the values of ψ and ω are known on the boundary C_2.

According to definition, the element force matrices f_{1k} and f_{2k} are

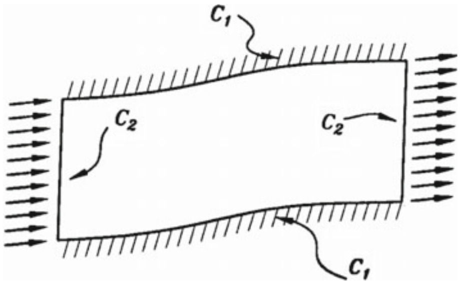

Fig. 13.4 The solution domain of a steady-state two-dimensional fluid flow

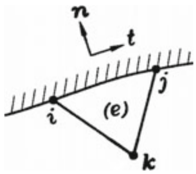

Fig. 13.5 Element (*e*) near the solid boundary

$$f_{1k} = \int_{\Gamma(e)} N_k \frac{\partial \psi^*}{\partial n} \, dS$$
$$f_{2k} = \int_{\Gamma(e)} N_k \frac{\partial \omega^*}{\partial n} \, dS. \tag{13.15.3}$$

These forces cancel each other out between any two adjacent elements inside the solution domain, as the integrations over any common side of two adjacent elements become positive (for one element) and negative with equal magnitude (for the adjacent element), so that the sums of two integrations over the common side of two adjacent elements cancel each other out.

Now, the value of forces f_{1k} and f_{2k} on the solid boundary of the solution domain, such as C_1, will be considered. We take the element (e) with one side ij on the solid boundary, as shown in Fig. 13.5. We also designate the directions n and t as normal and tangent to the solid boundary. According to the definition of stream function, from Eq. (13.11.1),

$$v_t = \frac{\partial \psi}{\partial n} \quad v_n = -\frac{\partial \psi}{\partial t} \tag{13.15.4}$$

where v_t and v_n are the tangential and normal velocities on the solid boundary. It immediately follows that, due to the fluid viscosity, $v_t = 0$, and due to the solid boundary, $v_n = 0$. Therefore, on solid boundaries, the force f_{1k} from Eq. (13.15.3) is zero.

From Eq. (13.11.2), the vorticity on the solid boundary is

$$\omega = \frac{\partial v_n}{\partial t} - \frac{\partial v_t}{\partial n}. \tag{13.15.5}$$

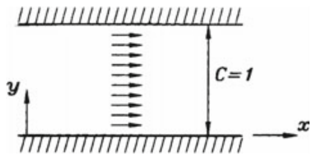

Fig. 13.6 Poiseuille flow in a straight pipe

Moving along the solid wall (t-direction), v_n remains zero, and thus, $\partial v_n/\partial t = 0$. On the other hand, $\partial v_t/\partial n$ is not zero and is proportional to the shear stress. Since the shear stress along the n-direction varies, $\partial w/\partial n \neq 0$. This suggests that the force f_{2k} from Eq. (13.15.3) must be evaluated on the solid wall. In this case, one may use the shape function for ω from Eq. (13.14.1), substituting it in the second of Eq. (13.15.3) for f_{2k}, and evaluate the integral.

Now, consider the part of boundary C_2, where the inflow and outflow are defined. These boundaries are usually defined in such a way that the flow conditions are either given or estimated, such as free stream conditions or undisturbed flow patterns defined far from the solution domain. Since the velocity profiles on this type of boundary is known, both ψ and ω functions are readily calculated from Eqs. (13.11.1) and (13.11.2), and are set on C_2.

To explain the foregoing discussion, consider the Poiseuille flow in a pipe, as shown in Fig. 13.6. From Eqs. (13.12.1) and (13.12.2), since the variations of ψ and ω along the x-direction is zero and the velocity component along the y-direction is also zero, we have

$$\frac{\partial^2 \psi}{\partial y^2} + \omega = 0 \tag{13.15.6}$$

$$\frac{\partial^2 w}{\partial y^2} = 0. \tag{13.15.7}$$

Integrating Eq. (13.15.6) gives

$$\omega = Ay + B \tag{13.15.8}$$

in which A and B are the constants of integration. Substituting Eq. (13.15.8) in (13.15.6) and integrating, gives

$$\frac{\partial \psi}{\partial y} = -\frac{A}{2}\, y^2 - By + C \tag{13.15.9}$$

$$\psi = -\frac{A}{6}\, y^3 - \frac{B}{2}\, y^2 + Cy + D. \tag{13.15.10}$$

The boundary conditions are

$$y = 0 \quad \psi = 0 \qquad \frac{\partial \psi}{\partial y} = 0 \tag{13.15.11}$$

$$y = C = 1 \quad \psi = 1 \qquad \frac{\partial \psi}{\partial y} = 0. \tag{13.15.12}$$

In Eq. (13.15.12), the value of ψ is arbitrarily set to 1, as the difference of ψ with respect to y is important, not its numerical value. From conditions (13.15.11) and (13.15.12),

$$C = D = 0 \quad A = 12 \qquad B = -6 \tag{13.15.13}$$

and thus,

$$\omega = 12y - 6 \tag{13.15.14}$$

$$\psi = 3y^2 - 2y^3. \tag{13.15.15}$$

This example demonstrates how the boundary conditions on a solid wall are imposed. Since the functional relationship of ω and ψ with respect to y are known, their numerical values at any arbitrary point along the y-direction is calculated from Eqs. (13.15.14) and (13.15.15). This boundary condition is of C_2-type.

13.16 Problems

1. Consider a two-dimensional fluid flow of an incompressible viscous fluid in a general flow field. While for this type of fluid flow, the linear triangular element is not a proper choice for describing the velocity components u and v and the pressure p, the procedure for employing such an element to obtain the mass, stiffness, and force matrices remains the same as those of higher order elements. For this reason, try to use Eqs. (13.8.9) to (13.8.11) to obtain the elements of the mass, stiffness, and force matrices employing the linear triangular element.
2. For a two-dimensional fluid flow employing the formulations based on the vorticity transport, derive the matrix elements of Eq. (13.12.18) for a linear triangular element and verify those given by Eqs. (13.14.4) to (13.14.6).
3. For the linear triangular element, derive the members of the matrix C_{kij} and the final form of the matrices given by Eq. (13.14.8).

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Chapter 14
One-Dimensional Higher Order Elements

Abstract The chapter begins with the definition of straight one-dimensional quadratic element, where the natural coordinate and the Jacobian matrix is then calculated. As an application, the field problem is selected and the stiffness and force matrices are calculated. The cubic element in the general and local coordinates are discussed the Jacobian of transformation is calculated. The layer-wise theory is then described and it is applied to a composite beam under static and dynamic loading conditions.

14.1 Introduction

In Chap. 4, different types of elements and shape functions were described. To increase the accuracy of the finite element solution, the approximating shape function may be selected of higher order polynomials. It was discussed that the polynomials of the second and third orders are usually employed to describe the field approximation within the elements. The polynomials of orders higher than three are, in general, not recommended due to the possibility of having roots within the element domain.

In this chapter, the one-dimensional higher order polynomials are used to approximate the function in the base element. The application of such elements is discussed in the beam's layer-wise approach to the finite element.

14.2 One-Dimensional Quadratic Element

Consider a one-dimensional straight element. To describe a quadratic shape function, three nodal points must be considered on the element. Figure 14.1 shows the element (e) with three nodes i, j, and k. Node j is taken in the middle of the element. If

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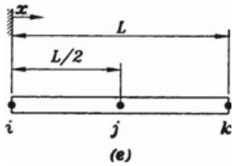


Fig. 14.1 A quadratic straight element

we assume the local coordinate x at origin i, the second order approximation for the dependent function ϕ within the element (e) is

$$\phi^{(e)} = a_1 + a_2 x + a_3 x^2. \tag{14.2.1}$$

To evaluate the constants $a_1,\ a_2$, and a_3, we write

$$\begin{cases} \phi = \phi_i \\ x = 0 \end{cases} \qquad \begin{cases} \phi = \phi_j \\ x = L/2 \end{cases} \qquad \begin{cases} \phi = \phi_k \\ x = L \end{cases}. \tag{14.2.2}$$

Substituting Eq. (14.2.2) in (14.2.1), gives

$$\begin{aligned} a_1 &= \phi_i \\ a_2 &= \frac{4\phi_j - 3\phi_i - \phi_k}{L} \\ a_3 &= \frac{2}{L^2}(\phi_i - 2\phi_j + \phi_k). \end{aligned} \tag{14.2.3}$$

With the known values for $a_1,\ a_2$, and a_3, the shape function (14.2.1) becomes

$$\phi^{(e)} = \langle N_i \quad N_j \quad N_k \rangle \begin{Bmatrix} \phi_i \\ \phi_j \\ \phi_k \end{Bmatrix}^{(e)} \tag{14.2.4}$$

in which the shape functions are

$$\begin{aligned} N_i &= (1 - \frac{2x}{L})(1 - \frac{x}{L}) \\ N_j &= \frac{4x}{L}(1 - \frac{x}{L}) \\ N_k &= -\frac{x}{L}(1 - \frac{2x}{L}). \end{aligned} \tag{14.2.5}$$

The shape functions N have two important properties. The first property is that their sum is 1,

$$N_i + N_j + N_K = 1. \tag{14.2.6}$$

The second property is that their value is 1 when evaluated at the associated nodal point and 0 when evaluated at the other nodes, i.e.,

$$N_\alpha = \begin{cases} 1 & \text{on node } \alpha \\ 0 & \text{on other nodes} \end{cases} \tag{14.2.7}$$

14.3 Natural Coordinates, Jacobian Matrix

The natural coordinate system ξ may be considered with a coordinate transformation from x. Referring to Fig. 14.2, the transformation law is

$$\xi = \frac{2x}{L} - 1. \tag{14.3.1}$$

With this coordinate transformation, $\xi = 0$ at $x = L/2$ (at node j) and

$$\begin{aligned} \xi &= -1 \quad \text{at node } i \\ \xi &= +1 \quad \text{at node } k. \end{aligned} \tag{14.3.2}$$

The new variable ξ varies from -1 to $+1$ in the element,

$$-1 \leq \xi \leq 1. \tag{14.3.3}$$

The shape functions N, in terms of the natural coordinate ξ, are obtained using Eqs. (14.2.5) and (14.3.1) as

$$\begin{aligned} N_i &= -\frac{\xi}{2}\,(1-\xi) \\ N_j &= (1+\xi)(1-\xi) \\ N_k &= \frac{\xi}{2}\,(1+\xi). \end{aligned} \tag{14.3.4}$$

Since the variable is changed from x to ξ, the element integrations involving the shape functions and their derivatives require the Jacobian matrix. To evaluate the Jacobian matrix, we need the law of coordinate transformation.

$$x = N_i X_i + N_j X_j + N_k X_k \tag{14.3.5}$$

in which where X_i, X_j, and X_k are the coordinates of nodes i, j, and k in the global coordinates system, and N_i, N_j, and N_k are the shape functions in terms of the variable ξ, as given by Eq. (14.3.4).

The derivative of the shape function N_α with respect to ξ is related to the derivative with respect to x as

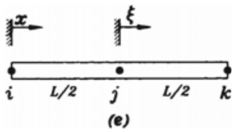


Fig. 14.2 One-dimensional straight element with natural coordinates system

$$\frac{dN_\alpha}{d\xi} = \frac{dN_\alpha}{dx}\frac{dx}{d\xi} \tag{14.3.6}$$

or

$$\frac{dN_\alpha}{dx} = \frac{1}{dx/d\xi}\frac{dN_\alpha}{d\xi}. \tag{14.3.7}$$

The term $dx/d\xi$ is called the Jacobian matrix of coordinate transformation from the variable x to ξ. Using Eq. (14.3.5), the Jacobian matrix, which, for one-dimensional problems, is a 1×1 matrix, is

$$[J] = \frac{dx}{d\xi} = \frac{dN_i}{d\xi} X_i + \frac{dN_j}{d\xi} X_j + \frac{dN_k}{d\xi} X_k \tag{14.3.8}$$

From Eq. (14.3.4),

$$\begin{aligned} \frac{dN_i}{d\xi} &= -\tfrac{1}{2}(1-\xi) + \frac{\xi}{2} = -\tfrac{1}{2} + \frac{\xi}{2} + \frac{\xi}{2} = \xi - \tfrac{1}{2} \\ \frac{dN_j}{d\xi} &= 1 - \xi - (1+\xi) = 1 - \xi - 1 - \xi = -2\xi \\ \frac{dN_k}{d\xi} &= \tfrac{1}{2}(1+\xi) + \frac{\xi}{2} = \tfrac{1}{2} + \xi. \end{aligned} \tag{14.3.9}$$

Thus, the Jacobian matrix for the straight quadratic element is

$$[J] = X_i\left(\xi - \tfrac{1}{2}\right) - 2X_j\xi + X_k\left(\xi + \tfrac{1}{2}\right); \tag{14.3.10}$$

when the coordinates of nodes i, j, and k are known, Eq. (14.3.10) is fully determined. For equi-distanced nodes, the Jacobian matrix of this element becomes constant independent of ξ

$$[J] = 0 \times \left(\xi - \tfrac{1}{2}\right) - 2 \times \frac{L}{2} \times \xi + L \times \left(\xi + \tfrac{1}{2}\right) = \frac{L}{2}. \tag{14.3.11}$$

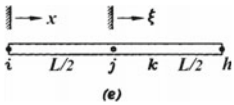


Fig. 14.3 One-dimensional quadratic element

14.4 Application to the Field Problems

The application of the quadratic element in the field problems is discussed in this section. The stiffness and force matrices were obtained in Chap. 5, as

$$[k]^{(e)} = \int_{V(e)} [B]^T [k][B] dV + \int_{S_2(e)} h\{N\}\langle N\rangle dS$$
$$\{f\}^{(e)} = \int_{V(e)} Q\{N\} dV + \int_{S_1(e)} q^{''}\{N\} dS - \int_{S_2(e)} h\phi_\infty \{N\} dS. \tag{14.4.1}$$

Considering the quadratic element, the shape function for the dependent function ϕ for the base element (e) is (Fig. 14.3)

$$\phi^{(e)} = \langle N_i \quad N_j \quad N_k\rangle \begin{Bmatrix} \phi_i \\ \phi_j \\ \phi_k \end{Bmatrix}^{(e)}. \tag{14.4.2}$$

The gradient matrix for this case is

$$\{g\}^{(e)} = \frac{d\phi}{dx} =< \frac{dN_i}{dx} \; \frac{dN_j}{dx} \; \frac{dN_k}{dx} > \{ \begin{matrix} \phi_i \\ \phi_j \\ \phi_k \end{matrix} \}^{(e)}. \tag{14.4.3}$$

Substituting for N from Eq. (14.2.5) in (14.4.3) and recalling the definition of matrix $\langle B\rangle^{(e)}$ gives

$$\langle B\rangle^{(e)} =< (\frac{4x}{L^2} - \frac{3}{L})(\frac{4}{L} - \frac{8x}{L^2})(\frac{4x}{L^2} - \frac{1}{L}) > \tag{14.4.4}$$

Since $[k] = k_{xx}$ and $dV = Adx$, A being the element cross-sectional area, the first term of the stiffness matrix becomes

$$[k_1]^{(e)} = \int \{B\}[k]\langle B\rangle A dx$$
$$= k_{xx} A \int_0^L \begin{bmatrix} \left(\frac{4x}{L^2}-\frac{3}{L}\right)^2 & \left(\frac{4x}{L^2}-\frac{3}{L}\right)\left(\frac{4}{L}-\frac{8x}{L^2}\right) & \left(\frac{4x}{L^2}-\frac{3}{L}\right)\left(\frac{4x}{L^2}-\frac{1}{L}\right) \\ & \left(\frac{4}{L}-\frac{8x}{L^2}\right)^2 & \left(\frac{4}{L}-\frac{8x}{L^2}\right)\left(\frac{4x}{L^2}-\frac{1}{L}\right) \\ \text{sym} & & \left(\frac{4x}{L^2}-\frac{1}{L}\right)^2 \end{bmatrix} dx. \tag{14.4.5}$$

After integration we get

$$[k_1]^{(e)} = \frac{Ak_{xx}}{6L} \begin{bmatrix} 14 & -16 & 2 \\ -16 & 32 & -16 \\ 2 & -16 & 14 \end{bmatrix}. \tag{14.4.6}$$

The above integration can also be evaluated using natural coordinates. The matrix $\langle B\rangle^{(e)}$ is

$$\langle B\rangle^{(e)} = \frac{d\xi}{dx} < \frac{dN_i}{d\xi}\frac{dN_j}{d\xi}\frac{dN_k}{d\xi} >= [J]^{-1} < \left(\xi - \tfrac{1}{2}\right)(-2\xi)\left(\tfrac{1}{2}+\xi\right) > . \tag{14.4.7}$$

The first term of the stiffness matrix becomes

$$[k_1]^{(e)} = s\int \{B\}[k]\langle B\rangle A[J]d\xi$$
$$= \frac{2k_{xx}A}{L}\int_{-1}^{+1} \begin{bmatrix} \left(\xi-\frac{1}{2}\right)^2 & \left(\xi-\frac{1}{2}\right)(-2\xi) & \left(\xi^2-\frac{1}{4}\right) \\ & 4\xi^2 & (-2\xi)\left(\xi+\frac{1}{2}\right) \\ \text{sym} & & \left(\xi+\frac{1}{2}\right)^2 \end{bmatrix} d\xi, \tag{14.4.8}$$

which after integration yields the same result as that obtained by using the x coordinate given in Eq. (14.4.6). The second part of the stiffness matrix is

$$[k_2]^{(e)} = \int_S h\{N\}\langle N\rangle dS = h\int_A \begin{bmatrix} N_iN_i & N_iN_j & N_iN_k \\ N_jN_i & N_jN_j & N_jN_k \\ N_kN_i & N_kN_j & N_kN_k \end{bmatrix} dA. \tag{14.4.9}$$

When convection from the peripheral surface is considered, the integration yields

$$[k_2]^{(e)} = \frac{PhL}{30}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix} \tag{14.4.10}$$

in which P is the periphery of the element.

The force matrix is the sum of terms of Eq. (14.4.1), in which the first term is

$$\int_{V(e)} Q\{N\}dV = AQ\int_0^L \begin{Bmatrix} N_i \\ N_j \\ N_k \end{Bmatrix} dx = \frac{AQL}{6}\begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix}. \tag{14.4.11}$$

The force related to the heat flux q'' applied to the peripheral area of the element is

$$\int_{S_1(e)} q\{N\}dS = q''P\int_0^L \begin{Bmatrix} N_i \\ N_j \\ N_k \end{Bmatrix} dx = \frac{q''PL}{6}\begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix}. \tag{14.4.12}$$

The last term of the force matrix is

$$\int_{S_2(e)} h\phi_\infty\{N\}dS = hP\phi_\infty\int_0^L \begin{Bmatrix} N_i \\ N_j \\ N_k \end{Bmatrix} dx. \tag{14.4.13}$$

If convection occurs from the cross-section of node i, then

$$-\int_{S_2(e)} h\phi_\infty \begin{Bmatrix} N_i \\ 0 \\ 0 \end{Bmatrix} dS = -h\phi_\infty A_i \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}. \tag{14.4.14}$$

Thus, the total force for the given condition is

$$\{f\}^{(e)} = \frac{AQL}{6}\begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix} + \frac{q''pL}{6}\begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix} - h\phi_\infty A_i \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}. \tag{14.4.15}$$

14.5 Straight Cubic Element

A cubic element may be used to improve the accuracy of the finite element solution. For this purpose, a straight element with four nodes is considered, as shown in Fig. 14.4. The nodal points j and k are considered at distances $L/3$ from nodes i and m and from each other, so that the element is equally divided into three distances. The element shape function in terms of a third-order polynomial is

$$\phi^{(e)} = a_1 + a_2x + a_3x^2 + a_4x^3. \tag{14.5.1}$$

The constants a_1 through a_4 are obtained using the following conditions:

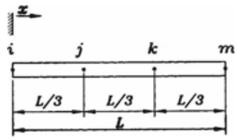


Fig. 14.4 A one-dimensional cubic element

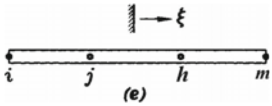


Fig. 14.5 A one-dimensional cubic element in terms of natural coordinate

$$\begin{cases} x = x_i \\ \phi = \phi_i \end{cases} \begin{cases} x = x_j \\ \phi = \phi_j \end{cases} \begin{cases} x = x_k \\ \phi = \phi_k \end{cases} \begin{cases} x = x_m \\ \phi = \phi_m \end{cases}. \tag{14.5.2}$$

Substituting the condition (14.5.2) in Eq. (14.5.1), the constants a_1 through a_4 are obtained, which, upon substitution, gives

$$\phi^{(e)} = N_i\phi_i + N_j\phi_j + N_k\phi_k + N_m\phi_m = \langle N\rangle^{(e)}\{\phi\}^{(e)} \tag{14.5.3}$$

in which the shape functions N are

$$\begin{aligned} N_i &= (1 - \frac{3x}{L})(1 - \frac{3x}{2L})(1 - \frac{x}{L}) \\ N_j &= \frac{9x}{L}(1 - \frac{3x}{2L})(1 - \frac{x}{L}) \\ N_k &= -\frac{9x}{2L}(1 - \frac{3x}{L})(1 - \frac{x}{L}) \\ N_m &= \frac{x}{L}(1 - \frac{3x}{L})(1 - \frac{3x}{2L}). \end{aligned} \tag{14.5.4}$$

The shape functions may be written in terms of the natural coordinates ξ. Considering the definition for ξ, and from Fig. 14.5, we have

$$\begin{aligned} N_i &= -\frac{9}{16}(\xi - 1)(\xi + \frac{1}{3})(\xi - \frac{1}{3}) \\ N_j &= \frac{27}{16}(\xi + 1)(\xi - 1)(\xi - \frac{1}{3}) \\ N_k &= -\frac{27}{16}(\xi + 1)(\xi - 1)(\xi + \frac{1}{3}) \\ N_m &= \frac{9}{16}(\xi + 1)(\xi + \frac{1}{3})(\xi - \frac{1}{3}) \end{aligned} \tag{14.5.5}$$

in which the natural coordinate ξ is defined as the length ratio and is

$$-1 \le \xi \le +1 \tag{14.5.6}$$

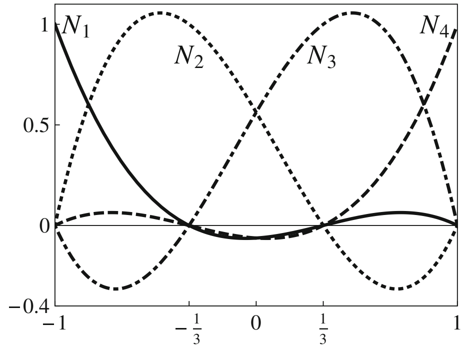


Fig. 14.6 C^0-continuous shape functions for a one-dimensional cubic element

and its value in terms of the x-coordinate is given by Eq. (14.3.1).

The coordinate transformation from the x-coordinate to the ξ-coordinate is

$$x = N_i X_i + N_j X_j + N_k X_k + N_m X_m \tag{14.5.7}$$

in which X_i, X_j, X_k, and X_m are the coordinates of nodes i, j, k, and m in the global system. The Jacobian matrix in this case is 1×1 and is given as

$$[J] = \frac{dx}{d\xi} = X_i \frac{dN_i}{d\xi} + X_j \frac{dN_j}{d\xi} + X_k \frac{dN_k}{d\xi} + X_m \frac{dN_m}{d\xi}. \tag{14.5.8}$$

Substituting for the derivative of the shape functions with respect to ξ, provides the expression for the Jacobian matrix as $[J] = L/2$.

Example 1 Consider a one-dimensional cubic element with nodes placed at successive equal distances:

(a) Find the relation between x and ξ for this element.
(b) Calculate the Jacobian matrix.
(c) Find the elements of the stiffness matrix $[k_1]^{(e)} + [k_2]^{(e)}$ and the force vector $\{f_1\}^{(e)} + \{f_2\}^{(e)} + \{f_3\}^{(e)}$ for this element.
(d) Do steps (a) to (c) for a C^1-continuous two-node element.

Solution: The coordinate transformation from the x-coordinate to the ξ-coordinate is given by Eq. (14.5.7), in which the shape functions N_i are given in terms of the ξ-coordinate in Eq. (14.5.5). These shape functions are plotted in Fig. 14.6.

By substituting the expressions for the shape functions N_i in terms of the ξ-coordinate, Eq. (14.5.5) into Eq. (14.5.7), the relation between x and ξ can be obtained. For an element with equi-distanced nodes, the nodes coordinates can be written as

$$
\begin{aligned}
X_i &= 0 \\
X_j &= L/3 \\
X_k &= 2L/3 \\
X_m &= L
\end{aligned}
\tag{14.5.9}
$$

where L is the element length. By substituting Eq. (14.5.5) and Eq. (14.5.9) into Eq. (14.5.8), the relation between x and ξ is obtained. The result after algebraic simplification is

$$
x = \frac{L}{2}(1+\xi). \tag{14.5.10}
$$

The Jacobian matrix for this element is 1×1 and is given by Eq. (14.5.8) as

$$
[J] = \frac{dx}{d\xi} = X_i \frac{dN_i}{d\xi} + X_j \frac{dN_j}{d\xi} + X_k \frac{dN_k}{d\xi} + X_m \frac{dN_m}{d\xi}.
$$

By substituting Eq. (14.5.5) and Eq. (14.5.9) into the above equation and using Eq. (14.5.10), the Jacobian matrix is obtained as

$$
[J] = \frac{dx}{d\xi} = \frac{L}{2}.
$$

The element stiffness matrix $[k_1]^{(e)}$ is

$$
\begin{aligned}
[k_1]^{(e)} &= \int_{V(e)} [B]^T [k][B] dV = Ak_{xx} \int_0^L \{\frac{dN}{dx}\} < \frac{dN}{dx} > dx \\
&= Ak_{xx} \int_{-1}^{+1} \{\frac{dN}{d\xi}\frac{d\xi}{dx}\} < \frac{dN}{d\xi}\frac{d\xi}{dx} > \frac{dx}{d\xi} d\xi \\
&= Ak_{xx} \int_{-1}^{+1} \{\frac{dN}{d\xi}\} < \frac{dN}{d\xi} > [J] d\xi.
\end{aligned}
\tag{14.5.11}
$$

The derivatives of the shape functions with respect to the ξ-coordinate, $\frac{dN_i}{d\xi}$, are obtained from Eq. (14.5.5) as

$$
\begin{aligned}
\frac{dN_i}{d\xi} &= \frac{1}{16}\,(1 + 18\xi - 27\xi^2) \\
\frac{dN_j}{d\xi} &= \frac{9}{16}\,(-3 - 2\xi + 9\xi^2) \\
\frac{dN_k}{d\xi} &= -\frac{9}{16}\,(-3 + 2\xi + 9\xi^2) \\
\frac{dN_m}{d\xi} &= \frac{1}{16}\,(-1 + 18\xi + 27\xi^2).
\end{aligned}
$$

By substituting the above equations into the expression for the stiffness matrix $[k_1]^{(e)}$, Eq. (14.5.11), and after performing the integrations, we have

$$[k_1]^{(e)} = \frac{Ak_{xx}}{40L}\begin{bmatrix} 148 & -189 & 54 & -13 \\ -189 & 432 & -297 & 54 \\ 54 & -297 & 432 & -189 \\ -13 & 54 & -189 & 148 \end{bmatrix}.$$

The element stiffness matrix $[k_2]^{(e)}$ is

$$\begin{aligned}[k_2]^{(e)} &= \int_{S_2(e)} h\{N\}\langle N\rangle dS = hp\int_0^L \{N\}\langle N\rangle dx \\ &= hp\int_{-1}^{+1}\{N\}\langle N\rangle\frac{dx}{d\xi}d\xi = hp\int_{-1}^{+1}\{N\}\langle N\rangle[J]d\xi\end{aligned}$$

where p is the periphery of the element ($dS = pdx$). After performing the integrations, the stiffness matrix $[k_2]^{(e)}$ is obtained as

$$[k_2]^{(e)} = \frac{phL}{1680}\begin{bmatrix} 128 & 99 & -36 & 19 \\ 99 & 648 & -81 & -36 \\ -36 & -81 & 648 & 99 \\ 19 & -36 & 99 & 128 \end{bmatrix}.$$

The element force vector $\{f_1\}^{(e)}$ is

$$\begin{aligned}\{f_1\}^{(e)} &= \int_{V(e)} Q\{N\}dV = AQ\int_0^L \{N\}dx \\ &= AQ\int_{-1}^{+1}\{N\}\frac{dx}{d\xi}d\xi = AQ\int_{-1}^{+1}\{N\}[J]d\xi.\end{aligned}$$

After performing the integrations, the force vector $\{f_1\}^{(e)}$ becomes

$$\{f_1\}^{(e)} = \frac{AQL}{8}\begin{Bmatrix} 1 \\ 3 \\ 3 \\ 1 \end{Bmatrix}.$$

The other terms of the force vector, $\{f_2\}^{(e)}$ and $\{f_3\}^{(e)}$, are obtained in a similar manner:

$$\{f_2\}^{(e)} = \int_{S_1(e)} q'' \{N\} dS = q'' p \int_0^L \{N\} dx$$
$$= q'' p \int_{-1}^{+1} \{N\}[J] d\xi = \frac{q'' pL}{8} \begin{Bmatrix} 1 \\ 3 \\ 3 \\ 1 \end{Bmatrix}$$

and

$$\{f_3\}^{(e)} = -\int_{S_2(e)} h\phi_\infty \{N\} dS = -ph\phi_\infty \int_0^L \{N\} dx$$
$$= -ph\phi_\infty \int_{-1}^{+1} \{N\}[J] d\xi = -\frac{ph\phi_\infty L}{8} \begin{Bmatrix} 1 \\ 3 \\ 3 \\ 1 \end{Bmatrix}.$$

If convection occurs from the cross-section of node i, then

$$\{f_3\}^{(e)} = -h\phi_\infty \int_{S_2(e)} \begin{Bmatrix} N_i \\ 0 \\ 0 \\ 0 \end{Bmatrix} dS = -h\phi_\infty A_i \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

where A_i is the cross-section area at node i.

Now, the previous calculations will be done for a C^1-continuous two-node element. First, the element shape functions are obtained in terms of the natural coordinate ξ. Two end nodal points are defined per element, and on each of these two nodes, the primary variable ϕ and its derivative $d\phi/dx$ are the corresponding unknowns. Having four unknowns for each element, a third-order polynomial may be used to define the element shape function

$$\phi^{(e)} = a_1 + a_2\xi + a_3\xi^2 + a_4\xi^3 \tag{14.5.12}$$

in which the variable ξ varies from -1 at node i to $+1$ at node j through the element. The constants a_1 through a_4 are obtained using the following conditions:

$$\begin{cases} \xi = -1 \\ \phi = \phi_1 \end{cases} \begin{cases} \xi = -1 \\ \dfrac{d\phi}{d\xi} = \left.\dfrac{d\phi}{d\xi}\right|_1 \end{cases} \begin{cases} \xi = +1 \\ \phi = \phi_2 \end{cases} \begin{cases} \xi = +1 \\ \dfrac{d\phi}{d\xi} = \left.\dfrac{d\phi}{d\xi}\right|_2 \end{cases}.$$

By substituting these conditions into Eq. (14.5.12), the constants a_1 through a_4 are obtained. Equation (14.5.12), after substitution of the values for a_1 through a_4, is written in the form

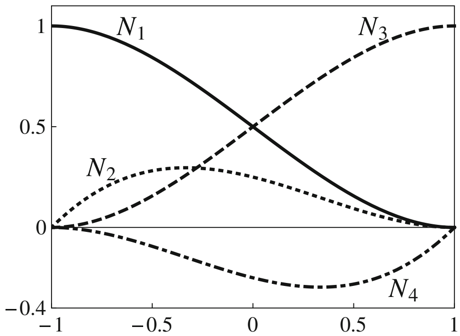


Fig. 14.7 C^1-continuous shape functions for a one-dimensional two-node element

$$\phi^{(e)} = N_1\phi_1 + N_2 \left.\frac{d\phi}{d\xi}\right|_1 + N_3\phi_2 + N_4 \left.\frac{d\phi}{d\xi}\right|_2$$

in which the shape functions N_i are

$$\begin{aligned} N_1 &= \frac{1}{4}(\xi^3 - 3\xi + 2) \\ N_2 &= \frac{1}{4}(\xi^3 - \xi^2 - \xi + 1) \\ N_3 &= -\frac{1}{4}(\xi^3 - 3\xi - 2) \\ N_4 &= \frac{1}{4}(\xi^3 + \xi^2 - \xi - 1). \end{aligned} \tag{14.5.13}$$

These shape functions are plotted in Fig. 14.7.

Now, the relation between x and ξ may be found. The coordinate transformation from the x-coordinate to the ξ-coordinate is

$$x = N_1 X_1 + N_2 \left.\frac{dx}{d\xi}\right|_1 + N_3 X_2 + N_4 \left.\frac{dx}{d\xi}\right|_2 .$$

By using the condition

$$\begin{aligned} X_1 = 0 \quad \left.\frac{dx}{d\xi}\right|_1 &= \frac{L}{2} \\ X_2 = L \quad \left.\frac{dx}{d\xi}\right|_2 &= \frac{L}{2} \end{aligned}$$

the relation between x and ξ is obtained as

$$x = \frac{L}{2}(1+\xi)$$

which is identical to the one for the previous case with a straight cubic element. Similar to the previous case, the Jacobian matrix for this element is 1×1 and is given by

$$[J] = \frac{dx}{d\xi} = \frac{L}{2}$$

The element stiffness matrix $[k_1]^{(e)}$ is

$$[k_1]^{(e)} = \int_{V(e)} [B]^T [k][B] dV = Ak_{xx} \int_0^L \{\frac{dN}{dx}\} < \frac{dN}{dx} > dx$$
$$= Ak_{xx} \int_{-1}^{+1} \{\frac{dN}{d\xi}\} < \frac{dN}{d\xi} > [J] d\xi.$$

By using the expressions for the shape functions given by Eq. (14.5.13), the integrations in expression for the stiffness matrix $[k_1]^{(e)}$ are evaluated. The result is

$$[k_1]^{(e)} = \frac{Ak_{xx}}{30L} \begin{bmatrix} 36 & 3L & -36 & 3L \\ 3L & 4L^2 & -3L & -L^2 \\ -36 & -3L & 36 & -3L \\ 3L & -L^2 & -3L & 4L^2 \end{bmatrix}.$$

The terms of the stiffness matrix $[k_2]^{(e)}$ are obtained in a similar manner

$$[k_2]^{(e)} = \int_{S_2(e)} h\{N\}\langle N\rangle dS =$$
$$= hp \int_0^L \{N\}\langle N\rangle dx = hp \int_{-1}^{+1} \{N\}\langle N\rangle [J] d\xi$$
$$= phL \begin{bmatrix} 156 & 22L & 53 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 53 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix}.$$

The force vector $\{f_1\}^{(e)}$ is

$$\{f_1\}^{(e)} = \int_{V(e)} Q\{N\} dV = AQ \int_0^L \{N\} dx$$
$$= AQ \int_{-1}^{+1} \{N\}[J] d\xi = \frac{AQL}{12} \begin{Bmatrix} 6 \\ L \\ 6 \\ -L \end{Bmatrix}.$$

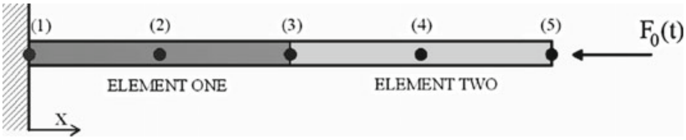


Fig. 14.8 Axial vibration of a cantilever beam

Similarly, the other terms of the force vector $\{f_2\}^{(e)}$ and $\{f_3\}^{(e)}$ are

$$\{f_2\}^{(e)} = \int_{S_1(e)} q^{''}\{N\}dS = q^{''}p\int_0^L \{N\}dx$$
$$= q^{''}p\int_{-1}^{+1}\{N\}[J]d\xi = \frac{q^{''}pL}{12}\begin{Bmatrix} 6 \\ L \\ 6 \\ -L \end{Bmatrix}$$

and

$$\{f_3\}^{(e)} = -\int_{S_2(e)} h\phi_\infty\{N\}dS = -ph\phi_\infty\int_0^L \{N\}dx$$
$$= -ph\phi_\infty\int_{-1}^{+1}\{N\}[J]d\xi = -\frac{ph\phi_\infty L}{12}\begin{Bmatrix} 6 \\ L \\ 6 \\ -L \end{Bmatrix}.$$

If convection occurs from the cross-section of node 1, then

$$\{f_3\}^{(e)} = -h\phi_\infty\int_{S_2(e)}\begin{Bmatrix} N_1 \\ N_2 \\ 0 \\ 0 \end{Bmatrix}dS = -h\phi_\infty A_1\begin{Bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{Bmatrix}$$

where A_1 is the cross- section area at node 1.

Example 2 Consider the axial vibration of a cantilever beam under force $F_0(t)$, shown in Fig. 14.8. The beam is descritized using two quadratic elements. Derive the local and global system matrices.

Solution: The axial vibration of a beam was formulated in Sect. 4 of Chap. 8. The weak formulation gives

$$\left(\int_0^L \rho AN_lN_m\,dx\right)\ddot{U}_m + \left(\int_0^L EA\frac{\partial N_l}{\partial x}\frac{\partial N_m}{\partial x}\,dx\right)U_m = F(x)N_m|_0^L + \int_0^L pAN_m\,dx \tag{14.5.14}$$

From which the elements of the mass, stiffness, and force matrix are

$$[m_{ij}]^{(e)} = \rho A \int_0^L \begin{bmatrix} N_1N_1 & N_1N_2 & N_1N_3 \\ N_2N_1 & N_2N_2 & N_2N_3 \\ N_3N_1 & N_3N_2 & N_3N_3 \end{bmatrix} dx \tag{14.5.15}$$

and

$$[k_{ij}]^{(e)} = EA \int_0^L \begin{bmatrix} \frac{dN_1}{dx}\frac{dN_1}{dx} & \frac{dN_1}{dx}\frac{dN_2}{dx} & \frac{dN_1}{dx}\frac{dN_3}{dx} \\ \frac{dN_2}{dx}\frac{dN_1}{dx} & \frac{dN_2}{dx}\frac{dN_2}{dx} & \frac{dN_2}{dx}\frac{dN_3}{dx} \\ \frac{dN_3}{dx}\frac{dN_1}{dx} & \frac{dN_3}{dx}\frac{dN_2}{dx} & \frac{dN_3}{dx}\frac{dN_3}{dx} \end{bmatrix} dx \tag{14.5.16}$$

and

$$\{f_i\}^{(e)} = F(x) \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} \Big|_0^L + pA \int_0^L \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} dx \tag{14.5.17}$$

respectively. Substituting for the values of the shape Functions, the mass, stiffness, and force matrices for a quadratic element are obtained as

$$[m_{ij}]^{(e)} = \frac{\rho AL}{15} \begin{bmatrix} 2 & 1 & -1 \\ 1 & 8 & 1 \\ -1 & 1 & 2 \end{bmatrix} \tag{14.5.18}$$

and

$$[k_{ij}]^{(e)} = \frac{EA}{3L} \begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} \tag{14.5.19}$$

and

$$\{f_i\}^{(e)} = \begin{Bmatrix} -F(0) \\ 0 \\ F(L) \end{Bmatrix} \tag{14.5.20}$$

respectively. Assembling the local mass matrices of elements one and two gives the global mass matrix as

$$[M_{ij}] = \frac{\rho AL}{15}\begin{bmatrix} 2 & 1 & -1 & 0 & 0 \\ 1 & 8 & 1 & 0 & 0 \\ -1 & 1 & 4 & 1 & -1 \\ 0 & 0 & 1 & 8 & 1 \\ 0 & 0 & -1 & 1 & 2 \end{bmatrix}. \tag{14.5.21}$$

In a similar way, the global stiffness and force matrices are obtained as

$$[K_{ij}] = \frac{EA}{3L}\begin{bmatrix} 7 & -8 & 1 & 0 & 0 \\ -8 & 16 & -8 & 0 & 0 \\ 1 & -8 & 14 & -8 & 1 \\ 0 & 0 & -8 & 16 & -8 \\ 0 & 0 & 1 & -8 & 7 \end{bmatrix} \tag{14.5.22}$$

and

$$\{F_i\} = \begin{Bmatrix} -F_1 \\ 0 \\ 0 \\ 0 \\ -F_0(t) \end{Bmatrix} \tag{14.5.23}$$

respectively.

14.6 Layer-Wise Theory of Composite Beams

As an example of higher order one-dimensional elements, the composite beams may be considered. There are two finite element approaches for the analysis of composite structures: the equivalent layer theory and the layer-wise theory. The first theory considers composite layer stacking as one single layer and assumes a continuous distribution of the displacement components across the thickness of the structure. The second theory, on the other hand, takes into account the layers, and applies the finite element approximation both along the layers and across the layers (through the thickness). Each layer of composite may be divided into an arbitrary number of elements. The shape functions along and across the layers may be considered linear or of higher order polynomials. In this section, as a practical example, a composite beam is considered, and with the layer-wise theory, shape functions in both directions of the beam (along the length of the beam and across the beam's cross-section) are considered of higher order polynomials.

In designing composite multi-layer structures, a critical issue is the inter-laminar stress situation. Many composite defects arise from the lack of knowledge of stress distribution across the layers and the resulting delamination or crack propagations along the laminate faces. Among different parameters influencing inter-laminar debounding, temperature distribution and the resulting thermal stresses are important.

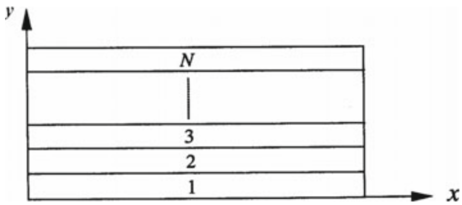


Fig. 14.9 A model of composite multilayer beam

Since the coefficient of thermal expansion along the fiber is substantially lower than in the transverse direction, the thermal gradient within an angle-ply layered composite structure causes significant normal and transverse shear stresses.

The first-order and higher order shear theories are developed to investigate the stress distribution caused by mechanical and thermal loads in composite structures. The works of Stavsky [1], Nemirovski [2], Shaldyrvan [3], Rao [4], Wu and Tauchert [5, 6], Khadeir and Reddy [7], and Huang and Tauchert [8] are examples of these types of analysis. In these theories, the displacement field is described in terms of infinitely differentiable functions with respect to thickness variable, but the proposed model cannot properly describe the distribution of the transverse shear stresses across the thickness due to the induced thermal bendings.

The number of publications dealing with the calculation of thermal stresses in layered composites is small. Huang and Tauchert [9], Thanjitham and Choi [10], Noor and Burton [11], and Tanigawa and Murakami [12, 13] are some examples. Chen et al. [14], Cho et al. [15], and Eslami et al. [16, 17] have considered the transverse stresses in their models and investigated the static and dynamic behavior of thick composite layered beams under thermal stress conditions.

Consider a thick beam consisting of k orthotropic layers. The x-axis is taken along the beam length and the z-axis is along the thickness, as shown in Fig. 14.9.

The displacement components along the x and z axes are u and w, respectively. Using the layer-wise approach of the finite element and the Kantrovich approximation along the thickness, the displacement components are approximated by

$$\begin{aligned} u &= \phi_j(z)U^j(x,t) \\ w &= \phi_j(z)W^j(x,t) \qquad j = 1, 2, \ldots N \end{aligned} \tag{14.6.1}$$

in which U^j and W^j are the displacements at node j, and ϕ_j is the continuous piecewise Lagrangian functions of which the second-order form is

$$\begin{aligned}
\phi_1(z) &= \psi_2^{(1)}(z) && z_1 \le z \le z_3 \\
\phi_{2i}(z) &= \psi_2^{(i)}(z) && z_{2i-1} \le z \le z_{2i+1} \\
\phi_{2i+1}(z) &= \psi_3^{(i)}(z) && z_{2i-1} \le z \le z_{2i+1} \\
\phi_{2i+i}(z) &= \psi_1^{(i+1)}(z)\ z_{2i+1} \le z \le z_{2i+3} \\
\phi_N(z) &= \psi_3^{Ne}(z) && z_{N-2} \le z \le z_N
\end{aligned} \tag{14.6.2}$$

where

$$\begin{aligned}
\psi_1^{(k)}(z) &= (1 - \frac{\bar{z}}{h_k})(1 - \frac{2\bar{z}}{h_k}) \\
\psi_2^{(k)}(z) &= \frac{4\bar{z}}{h_k}(1 - \frac{\bar{z}}{h_k}) \\
\psi_3^{(k)}(z) &= -\frac{\bar{z}}{h_k}(1 - \frac{2\bar{z}}{h_k}) \quad 0 \le \bar{z} \le h_k
\end{aligned} \tag{14.6.3}$$

with $N = 2N_e + 1$, in which N is the number of nodes, and N_e is the number of elements across the thickness. The number N_e may be equal to or different from the true number of physical laminate layers; with the increase of the number N_e, the accuracy of the solution increases. In Eq. (14.6.3), $\bar{z}$ denotes the local coordinate between two adjacent nodes, z_j is the coordinate of node j, $\bar{z} = z - z_b^k$, in which z_b^k is the lower coordinate of the k-th layer, and h_k is the thickness of the k-th layer.

For small deflection analysis, the strain-displacement relations are

$$\epsilon_{ij} = \frac{1}{2}\,(u_{i,j} + u_{j,i}) \tag{14.6.4}$$

Substituting the first of Eq. (14.6.1) in (14.6.4) gives

$$\begin{aligned}
\epsilon_{xx} &= \epsilon_1 = \phi_j\ \frac{\partial U^j}{\partial x} \\
\epsilon_{zz} &= \epsilon_3 = \frac{d\phi_j}{dz}\ W^j \\
2\epsilon_{xz} &= \epsilon_5 = \frac{d\phi_j}{dz}\ U^j + \phi_j\ \frac{\partial W^j}{\partial x}.
\end{aligned} \tag{14.6.5}$$

The stress–strain relations for the two-dimensional state of the k-th lamina made of orthotropic material [18] is

$$\begin{Bmatrix} \sigma_1 \\ \sigma_3 \\ \sigma_5 \end{Bmatrix}^{(k)} = \begin{bmatrix} \bar{C}_{11} & \bar{C}_{13} & 0 \\ \bar{C}_{13} & \bar{C}_{33} & 0 \\ 0 & 0 & \bar{C}_{55} \end{bmatrix}^{(k)} \begin{Bmatrix} \epsilon_1 - \alpha_{xx}\Delta T \\ \epsilon_3 - \alpha_{zz}\Delta T \\ \epsilon_5 \end{Bmatrix}^{(k)} \tag{14.6.6}$$

in which $\sigma_1 = \sigma_{xx}$, $\sigma_3 = \sigma_{zz}$, $\sigma_5 = \tau_{xz}$, and α_{xx} and α_{zz} are the coefficients of thermal expansion along the x and z directions. The coefficients $\bar{C}_{ij}^{(k)}$ are the transformed elastic constants $C_{ij}^{(k)}$ from the (xyz) system to the orthotropic laminate's coordinate with

$$C_{11} = \frac{E_1}{1 - \nu_{13}\nu_{31}} \qquad C_{33} = \frac{E_3}{1 - \nu_{13}\nu_{31}}$$
$$C_{13} = \frac{\nu_{13}E_1}{1 - \nu_{13}\nu_{31}} \qquad C_{55} = G_{13} \tag{14.6.7}$$

in which E_1 and E_3 are the elastic moduli along the x and z directions of lamina, ν_{ij} is the Poisson ratio, and G_{13} is the shear modulus. According to the principle of virtual work,

$$\int_V \sigma_{ij}\delta\epsilon_{ij}dV = \int_V B_i\delta u_i dV + \int_S t_i^n \delta u_i dS - \int_V \rho\ddot{u}_i\delta u_i dV \tag{14.6.8}$$

in which B_i is the body force. For elastic analysis in the absence of body force in Eq. (14.6.8), using Eqs. (14.6.1), (14.6.5), and (14.6.6), and substituting in Eq. (14.6.8) yields

$$\int_A N_1^j \frac{\partial\delta U_j}{\partial x}dA + \int_A \hat{Q}_3^j\delta W_j dA + \int_A Q_5^j\delta U_j dA + \int_A \hat{Q}_5^j\frac{\partial\delta W_j}{\partial x}dA$$
$$+ \int_A I^{jm}\ddot{U}_j\delta U_m dA + \int_A \hat{I}^{jm}\ddot{W}_j\delta W_m dA$$
$$= \int_\Gamma N_1^j\delta U_j dL + \int_\Gamma \hat{Q}_5^j\delta W_j dL + \int_A (f_1\delta W_1 + f_n\delta W_n)dA \tag{14.6.9}$$

in which

$$N_1^j = \sum_{k=1}^{N-1}\int_{z_k}^{z_{k+1}} \sigma_1\phi^j dz$$
$$Q_5^j = \sum_{k=1}^{N-1}\int_{z_k}^{z_{k+1}} \sigma_5\frac{d\phi^j}{dz}dz$$
$$\hat{Q}_5^j = \sum_{k=1}^{N-1}\int_{z_k}^{z_{k+1}} \sigma_5\hat{\phi}^j dz$$
$$\hat{Q}_3^j = \sum_{k=1}^{N-1}\int_{z_k}^{z_{k+1}} \sigma_3\frac{d\hat{\phi}^j}{dz}dz$$
$$(I^{jm}, \ \hat{I}^{jm}) = \sum_{k=1}^{N-1}\int_{z_k}^{z_{k+1}} \rho(\phi^j\phi^m, \ \hat{\phi}^j\hat{\phi}^m)dz \tag{14.6.10}$$

In Eq. (14.6.9), f_1 and f_n are the distributed forces applied to the lower and upper surfaces of the beam and I is the cross-section perpendicular to the z-axis and Γ is the perimeter of the cross-section.

To solve Eq. (14.6.9), the finite element method is used along the x-axis. Thus, considering a shape function of the form

$$\begin{aligned} U_j(x,t) &= \langle N(x)\rangle\{U_j(t)\} \\ W_j(x,t) &= \langle N(x)\rangle\{W_j(t)\} \end{aligned} \tag{14.6.11}$$

and substituting Eqs. (14.6.10) and (14.6.11) in (14.6.9) yields

$$\sum_{m=1}^{n}[K_{uu}]^{jm}\{U_m\}+\sum_{m=1}^{n}[K_{uw}]^{jm}\{W_m\}+\sum_{m=1}^{n}[M_u]^{jm}\{\ddot{U}_m\} = b\{N_1^j\}+b\int_{x_e}^{x_{e+1}}T_1^j\{\frac{dN}{dx}\}dx \tag{14.6.12}$$

$$\sum_{m=1}^{n}[K_{wu}]^{jm}\{U_m\}+\sum_{m=1}^{n}[K_{ww}]^{jm}\{W_m\}+\sum_{m=1}^{n}[M_w]^{jm}\{\ddot{W}_m\} = b\{\hat{Q}_x\}+b\int_{x_e}^{x_{e+1}}T_3^j\{N\}dx+\int_{x_e}^{x_{e+1}}f_j\{N\}dx. \tag{14.6.13}$$

Equations (14.6.12) and (14.6.13) are the finite element equilibrium equations of the beam. The element stiffness and mass matrices for a base element (e) are

$$\begin{aligned}
[K_{uu}]^{jm} &= C_{11}^{jm}b\int_{x_e}^{x_{e+1}}\{\frac{dN}{dx}\}<\frac{dN}{dx}>dx + C_{55}^{jm}b\int_{x_e}^{x_{e+1}}\{N\}\langle N\rangle dx \\
[K_{uw}]^{jm} &= \hat{C}_{13}^{jm}b\int_{x_e}^{x_{e+1}}\{\frac{dN}{dx}\}<N>dx + \hat{B}_{55}^{jm}b\int_{x_e}^{x_{e+1}}\{N\}<\frac{dN}{dx}>dx \\
[K_{wu}]^{jm} &= \hat{B}_{13}^{jm}b\int_{x_e}^{x_{e+1}}\{N\}<\frac{dN}{dx}>dx + \hat{C}_{55}^{jm}b\int_{x_e}^{x_{e+1}}\{\frac{dN}{dx}\}<N>dx \\
[K_{ww}]^{jm} &= \hat{C}_{33}^{jm}b\int_{x_e}^{x_{e+1}}\{N\}<N>dx + \hat{D}_{55}^{jm}b\int_{x_e}^{x_{e+1}}\{\frac{dN}{dx}\}<\frac{dN}{dx}>dx \\
[M_u]^{jm} &= I^{jm}b\int_{x_e}^{x_{e+1}}\{N\}<N>dx \\
[M_w]^{jm} &= \hat{I}^{jm}b\int_{x_e}^{x_{e+1}}\{N\}<N>dx
\end{aligned} \tag{14.6.14}$$

and the constants are

$$C_{11}^{jm} = \sum_{k=1}^{N-1}\int_{z_k}^{z_{k+1}}\bar{C}_{11}\phi^j\phi^m dz$$

$$
\begin{aligned}
C_{55}^{jm} &= \sum_{k=1}^{N-1} \int_{z_k}^{z_{k+1}} \bar{C}_{55} \frac{d\phi^j}{dz} \frac{d\phi^m}{dz} dz \\
\hat{C}_{13}^{jm} &= \sum_{k=1}^{N-1} \int_{z_k}^{z_{k+1}} \bar{C}_{13} \frac{d\hat{\phi}^j}{dz} \phi^m dz \\
\hat{B}_{55}^{jm} &= \sum_{k=1}^{N-1} \int_{z_k}^{z_{k+1}} \bar{C}_{55} \hat{\phi}^j \frac{d\phi^m}{dz} dz \\
\hat{B}_{13}^{jm} &= \sum_{k=1}^{N-1} \int_{z_k}^{z_{k+1}} \bar{C}_{13} \hat{\phi}^j \frac{d\phi^m}{dz} dz \\
\hat{C}_{55}^{jm} &= \sum_{k=1}^{N-1} \int_{z_k}^{z_{k+1}} \bar{C}_{55} \frac{d\hat{\phi}^j}{dz} \phi^m dz \\
\hat{C}_{33}^{jm} &= \sum_{k=1}^{N-1} \int_{z_k}^{z_{k+1}} \bar{C}_{33} \frac{d\hat{\phi}^j}{dz} \frac{d\hat{\phi}^m}{dz} dz \\
\hat{D}_{55}^{jm} &= \sum_{k=1}^{N-1} \int_{z_k}^{z_{k+1}} \bar{C}_{55} \hat{\phi}^j \hat{\phi}^m dz \\
T_1^j &= \sum_{k=1}^{N-1} \int_{z_k}^{z_{k+1}} (\alpha_{xx} \bar{C}_{11} + \alpha_{zz} \bar{C}_{13}) \Delta T \phi^j dz \\
\hat{T}_3^j &= \sum_{k=1}^{N-1} \int_{z_k}^{z_{k+1}} (\alpha_{xx} \bar{C}_{13} + \alpha_{zz} \bar{C}_{33}) \Delta T \frac{d\phi^j}{dz} dz
\end{aligned}
\tag{14.6.15}
$$

Static Thermal Stresses

Consider a cantilever beam of three layers, for which the bottom layer is number (1), the middle layer is number (2), and the top layer is number (3). The layers (1) and (3) are divided into 3 numerical layers and layer (2) is divided into 2 numerical layers. Since a second order approximation polynomial is used, a total of 17 nodes are considered along the thickness. The beam length is divided into 80 elements, where the element sizes are proportionally smaller close to the clamped edge. A uniform temperature rise of $\Delta T = 10^3\,^{\circ}$F is applied to the beam. The selection of this temperature rise is for the purpose of comparing the results with references. Since the governing equations are linear, the results can be applied to a lower temperature rise using the proper scale.

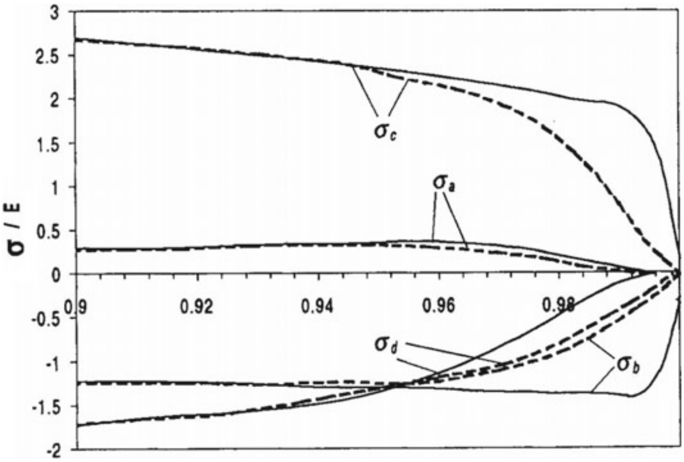


Fig. 14.10 Distribution of axial thermal stress based on layer-wise theory

The physical properties of the layers are:
Layer 1:
$E_1^{(1)} = E(1)_3 = 6.89$ GPa, $\alpha_x^{(1)} + \alpha_z^{(1)} = 13 \times 10^{-6}/°\text{F}$, $\nu_{13}^{(1)} = \nu_{31}^{(1)} = 0.33$, $h^{(1)} = 5$ cm.
Layer 2:
$E_1^{(2)} = E(2)_3 = 3.0$ GPa, $\alpha_x^{(2)} + \alpha_z^{(2)} = 2.5 \times 10^{-6}/°\text{F}$, $\nu_{13}^{(2)} = \nu_{31}^{(2)} = 0.33$, $h^{(2)} = 0.1$ cm.
Layer 3:
$E_1^{(3)} = E(3)_3 = 20.69$ GPa, $\alpha_x^{(3)} + \alpha_z^{(3)} = 6.5 \times 10^{-6}/°\text{F}$, $\nu_{13}^{(3)} = \nu_{31}^{(3)} = 0.25$, $h^{(3)} = 5$ cm.

Figure 14.10 shows the distribution of axial stress along the length of the beam. The stresses σ_a, σ_b, σ_c, and σ_d are axial stresses at the free bottom surface, between the bottom layer and the inner glued surface, between the upper layer and the inner glued surface, and at the free upper surface, respectively. The solid lines are the results of the layer-wise theory, and the dotted lines are the results given by Chen et al. [14]. The axial stress at the interface between the lower plate and the middle plate, and between the upper surface and the middle plate, are shown in Fig. 14.11. The solid lines are the results of the layer-wise theory and the dotted lines are the results of Cho et al. [15]. The results of Cho are obtained using the higher order equivalent layer and are compared with the SAP IV computer program (dashed dotted line). In Fig. 14.12, the normal stress versus the length of the beam are plotted and compared. The solid line is the result of the layer-wise theory, the dashed line is due to Chen, the dotted line is the result of SAP IV, and the dashed-dotted line is the result of Cho. As shown, the maximum lateral normal stress is located at the end of the beam.

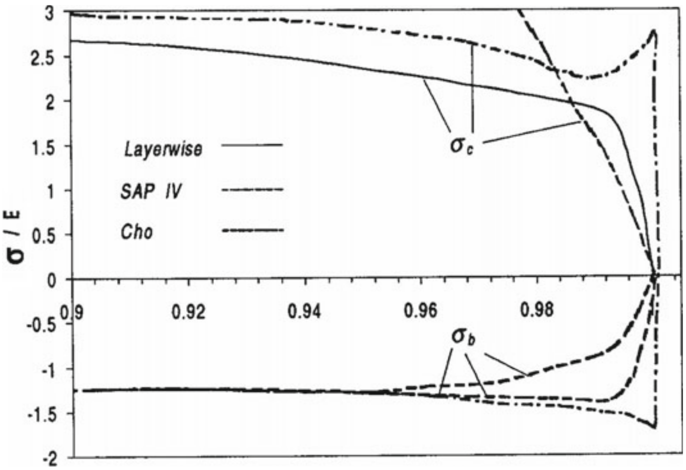


Fig. 14.11 Comparison of results of axial thermal stresses

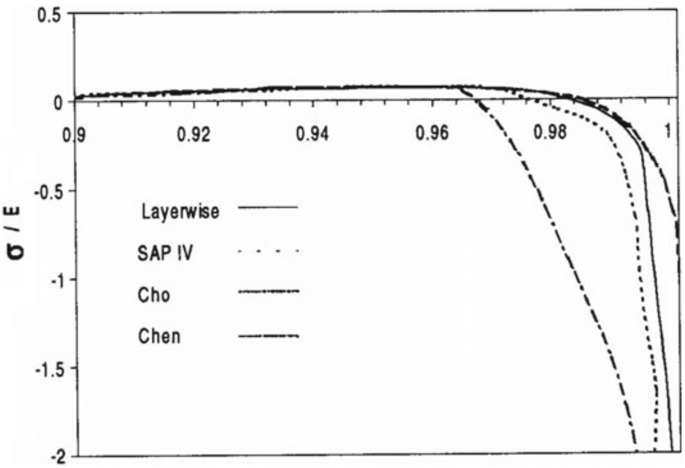


Fig. 14.12 Comparison of lateral normal stress

Dynamic Thermal Stresses

As an example, consider a composite beam made of four unsymmetric layers of graphite epoxy arranged as (0/90/0/90). Both ends of the beam are clamped. The beam is divided into 80 elements along the length and 8 quadratic elements (17 nodes) along the thickness. The physical properties of the beam are assumed to be $E_{11} = 180$ GPa, $E_{22} = E_{33} = 10$ GPa, $G_{13} = 7.17$ GPa, $G_{23} = 2.87$ GPa,

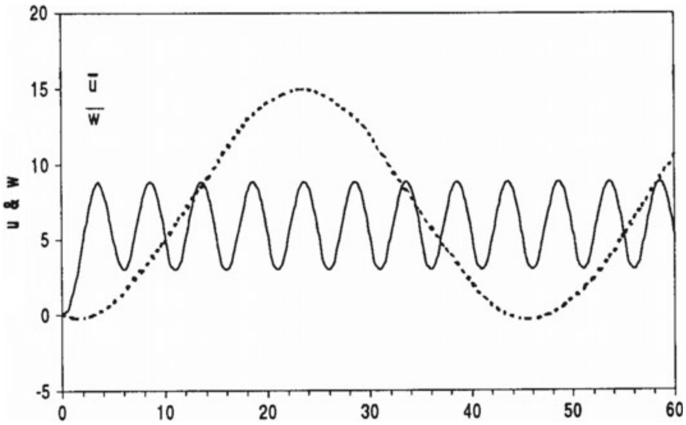


Fig. 14.13 Lateral deflection and axial displacement at middle length

$\nu_{12} = \nu_{13} = 0.28$, $\nu_{23} = 0.33$, $\rho = 2 \times 10^3$ kg/m^3, $\alpha_{11} = 0.02 \times 10^{-6} 1/°C$, $\alpha_{22} = \alpha_{33} = 22.5 \times 10^{-6}$ 1/°C.

The dimensionless parameters are defined as

$$\bar{t} = \frac{t}{\sqrt{\rho l^2/E_{11}}} \qquad \bar{u} = \frac{u}{\alpha_{11} T_0 L}$$
$$\bar{w} = \frac{w}{\alpha_{11} T_0 L} \qquad \bar{z} = \frac{z}{h} \qquad \bar{x} = \frac{x}{L} \tag{14.6.16}$$

for which L is the length of the beam, h the thickness, T_0 the ambient temperature, and ρ the mass density of the layers. The temperature rise is assumed to be applied uniformly to the beam as

$$\bar{T} = 0.1 \times (1 - e^{-10t}) \tag{14.6.17}$$

Figure 14.13 shows the lateral deflection and axial displacement at the middle length of beam (at $\bar{x} = 1/2$). The period of lateral vibration is 40 and axial vibration is 5 nondimensional units. In Figs. 14.14 and 14.15, the normal and shear transverse stresses between layers 1 and 2 (*a*), layers 2 and 3 (*b*) and layers 3 and 4 (*c*) versus time are shown. It should be noted that the period of vibration of transverse shear stress is about 40 times that of the transverse normal stress. Figure 14.16 shows the time history of the axial thermal stress at the top of layers 1 and 2 at the middle length of the beam. The curve (*a*) shows the axial thermal stress at the top of layer 1, and curve (*b*) at the top of layer 2.

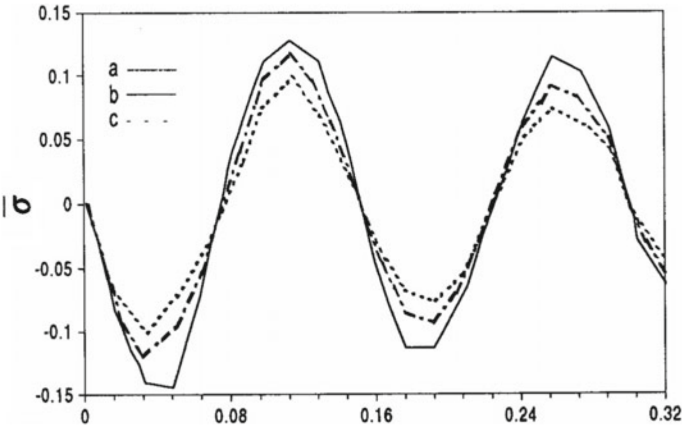


Fig. 14.14 Lateral normal stress

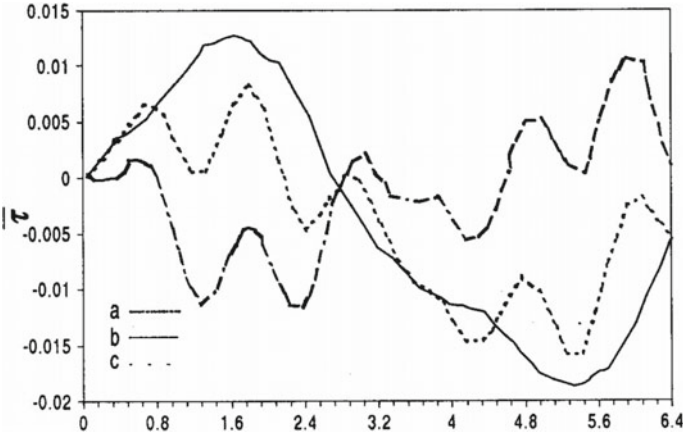


Fig. 14.15 Transverse shear stress

14.7 Problems

1. Consider a straight one-dimensional quadratic element, as shown in Fig. 14.17. Verify that

$$\int_{(e)} \{N(x)\}dx = \int_{(e)} \{N(\xi)\}|J|d\xi.$$

2. The first part of the stiffness matrix of Eq. (14.4.1) is

$$[k_1]^{(e)} = \int_{V(e)} [B]^T [k][B]dv.$$

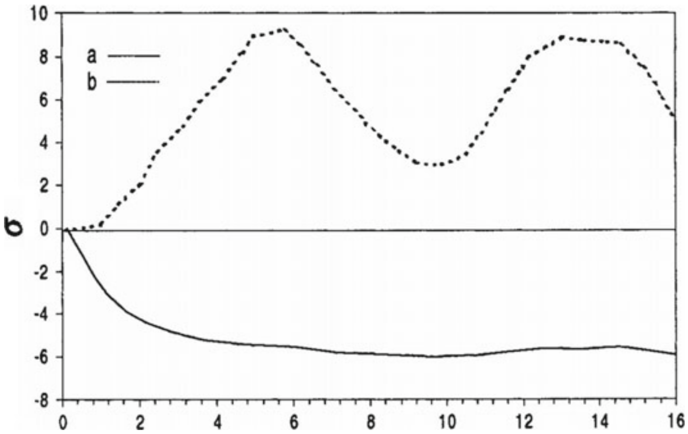
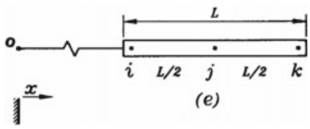


Fig. 14.16 Time history of the axial thermal stress at the top of layer 1 and 2 at middle length

Fig. 14.17 One-dimensional quadratic element



In terms of the global coordinate system variable x, $[k_1]^{(e)}$ is calculated and given in Eq. (14.4.6). Use the natural coordinates (14.3.4) and the Jacobian (14.3.10) to arrive at the same matrix (14.4.6).

3. For the straight cubic element, the shape functions in the global coordinates are given in Eq. (14.5.4). Derive these equations in terms of the natural coordinates (14.5.5) and calculate the Jacobian matrix.
4. Find the elements of the stiffness matrix $[k_1]^{(e)}$ for a straight cubic element using the natural coordinates and the Jacobian matrix.
5. The shape functions given in Eq. (14.5.4) are for a C^0-continuous third order element. Find the associated shape function for a C^1-continuous element in terms of the natural coordinate ξ.

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Chapter 15
Two-Dimensional Higher Order Elements

Abstract The triangular elements with quadratic and cubic shape functions in terms of the local area coordinates are presented and coordinate transformation law with the associated Jacobian matrix calculations are given in the chapter. As an application, the Field problem in two dimensions is considered and employing the quadratic triangular element, the stiffness and force matrices are calculated. The quadrilateral element is discussed in the following and the shape functions in the global and local coordinates are obtained. The field problem is reconsidered and the element matrices are calculated employing the bilinear quadrilateral element.

15.1 Introduction

For a higher accuracy of solution in two-dimensional domains, we may select two-dimensional higher-order elements. In this case, the approximating polynomial for describing the dependent function is selected of second or third order degrees. As described in Chap. 4, these elements may be of triangular or four-sided shapes with straight or curved sides. A second or third order isoparametric element can be selected to fit the curved boundary geometry of the solution domain, in which the solution accuracy is well-considered.

When triangular elements are considered, the area coordinates may be selected to describe the approximating shape functions. This selection will properly provide a formulation which is easy for the element integrations. The triangular elements of second and third order polynomials can be described in terms of the area coordinates, as will be discussed in this chapter.

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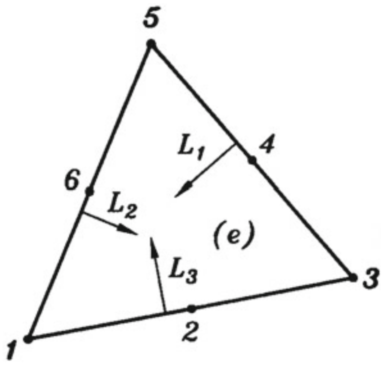


Fig. 15.1 Quadratic element

15.2 Triangular Element

Consider a triangular element with straight sides. A quadratic element will have a total of six nodes, three at the apex and the other three at the mid-sides, as shown in Fig. 15.1. The area coordinates L_1, L_2, and L_3 are shown in the figure, and their definitions are given by Eq. (4.8.6). The shape functions in terms of the area coordinates are

$$
\begin{aligned}
N_1 &= L_1(2L_1 - 1) \qquad N_2 = 4L_1L_2 \\
N_3 &= L_2(2L_2 - 1) \qquad N_4 = 4L_2L_3 \\
N_5 &= L_3(2L_3 - 1) \qquad N_6 = 4L_3L_1
\end{aligned} \tag{15.2.1}
$$

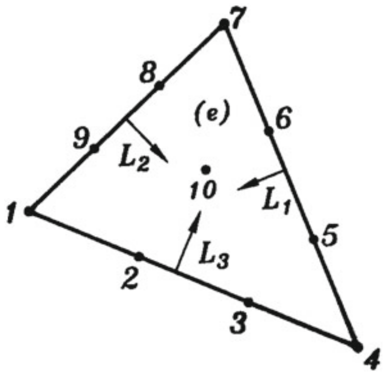
in which the following relation relates L_1, L_2, and L_3.

$$L_1 + L_2 + L_3 = 1 \tag{15.2.2}$$

It is noted that the three variables L_1, L_2, and L_3 are not independent. When carrying out the element differentiations, the dependency relation (15.2.2) must be considered. The mathematical relations for the derivatives of the shape functions will be discussed in the next section.

A triangular element with cubic shape function in terms of the area coordinates is shown in Fig. 15.2. The nodal points are defined in the element, three at the triangular apex, six equally spaced on the sides, and one at the element centroid. Ten cubic shape functions defined in terms of the area coordinates are

Fig. 15.2 Cubic element



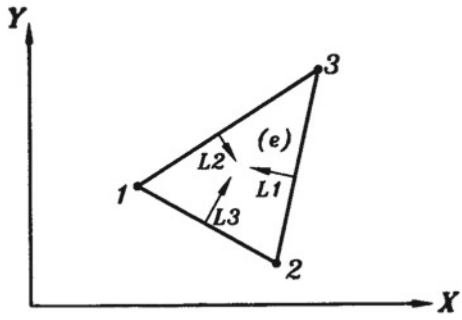
$$
\begin{aligned}
N_1 &= \tfrac{1}{2}\, L_1(3L_1 - 1)(3L_1 - 2)\\
N_2 &= \tfrac{9}{2}\, L_1L_2(3L_1 - 1)\\
N_3 &= \tfrac{9}{2}\, L_1L_2(3L_2 - 1)\\
N_4 &= \tfrac{1}{2}\, L_2(3L_2 - 1)(3L_2 - 2)\\
N_5 &= \tfrac{9}{2}\, L_2L_3(3L_2 - 1)\\
N_6 &= \tfrac{9}{2}\, L_2L_3(3L_3 - 1)\\
N_7 &= \tfrac{1}{2}\, L_3(3L_3 - 1)(3L_3 - 2)\\
N_8 &= \tfrac{9}{2}\, L_3L_1(3L_3 - 1)\\
N_9 &= \tfrac{9}{2}\, L_1L_3(3L_1 - 1)\\
N_{10} &= 27L_1L_2L_3
\end{aligned}
\qquad (15.2.3)
$$

The area coordinates L_1, L_2, and L_3 are related to each other through Eq. (15.2.2). Consideration of Eq. (15.2.2) leaves two independent variables among the three variables L_1, L_2, and L_3.

15.3 Jacobian Matrix

The element matrices must be calculated using the coordinates. The final form of the element matrices, after the necessary differentiations and integrations, must be obtained in the global coordinate system. This requirement provides the means to evaluate the Jacobian matrix. When the area coordinates are used, the coordinate transformation from the two-dimensional x and y coordinates to the area coordinates L_1, L_2, and L_3 must be known to evaluate the Jacobian matrix. The Law of coordinate transformation is

Fig. 15.3 Area coordinates



$$
\begin{aligned}
x &= L_1 X_1 + L_2 X_2 + L_3 X_3 \\
y &= L_1 Y_1 + L_2 Y_2 + L_3 Y_3
\end{aligned}
\tag{15.3.1}
$$

in which (X_1, Y_1), (X_2, Y_2), and (X_3, Y_3) are the x and y coordinates of the apex of the triangular element, as shown in Fig. 15.3. In the two-dimensional (x, y) coordinates, two independent variables x and y are considered as the problem variables, while in terms of the area coordinates, three variables L_1, L_2, and L_3 are introduced. Since the area coordinates are always related through the following equation,

$$
L_1 + L_2 + L_3 = 1 \tag{15.3.2}
$$

only two of the three area coordinates are independent. Let us assume that L_1 and L_2 are two independent variables. Thus, from Eq. (15.3.2),

$$
L_3 = 1 - L_1 - L_2. \tag{15.3.3}
$$

Now, according to the definition, the Jacobian matrix for the coordinate transformation from x and y to L_1 and L_2 is

$$
[J] = \begin{bmatrix} \dfrac{\partial x}{\partial L_1} & \dfrac{\partial y}{\partial L_1} \\ \dfrac{\partial x}{\partial L_2} & \dfrac{\partial y}{\partial L_2} \end{bmatrix}. \tag{15.3.4}
$$

The differential of the shape functions with respect to the new variables L_1 and L_2, when $N_s = N_s(x, y)$, are

$$
\begin{aligned}
\frac{\partial N_s}{\partial L_1} &= \frac{\partial N_s}{\partial x}\frac{\partial x}{\partial L_1} + \frac{\partial N_s}{\partial y}\frac{\partial y}{\partial L_1} \\
\frac{\partial N_s}{\partial L_2} &= \frac{\partial N_s}{\partial x}\frac{\partial x}{\partial L_2} + \frac{\partial N_s}{\partial y}\frac{\partial y}{\partial L_2}
\end{aligned}
\tag{15.3.5}
$$

or, in terms of the Jacobian matrix,

$$\begin{Bmatrix} \dfrac{\partial N_s}{\partial L_1} \\ \dfrac{\partial N_s}{\partial L_2} \end{Bmatrix} = [J] \begin{Bmatrix} \dfrac{\partial N_s}{\partial x} \\ \dfrac{\partial N_s}{\partial y} \end{Bmatrix}. \tag{15.3.6}$$

The inverse relation is

$$\begin{Bmatrix} \dfrac{\partial N_s}{\partial x} \\ \dfrac{\partial N_s}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \dfrac{\partial N_s}{\partial L_1} \\ \dfrac{\partial N_s}{\partial L_2} \end{Bmatrix}. \tag{15.3.7}$$

Since L_1 and L_2 depend on L_3, the derivative of N_s with respect to L_1 is related to L_3 as

$$\frac{\partial N_s}{\partial L_1} = \frac{\partial N_s}{\partial L_1}\frac{\partial L_1}{\partial L_1} + \frac{\partial N_s}{\partial L_3}\frac{\partial L_3}{\partial L_1} \tag{15.3.8}$$

From Eq. (15.3.3)

$$\frac{\partial N_s}{\partial L_1} = \frac{\partial N_s}{\partial L_1} - \frac{\partial N_s}{\partial L_3}. \tag{15.3.9}$$

Similarly,

$$\frac{\partial N_s}{\partial L_2} = \frac{\partial N_s}{\partial L_2} - \frac{\partial N_s}{\partial L_3}. \tag{15.3.10}$$

That is, when the derivative of shape functions with respect to the area coordinates are evaluated, the dependency relation (15.3.3) must be taken into account. This is also valid for the evaluation of the Jacobian matrix. Considering the law of coordinate transformation from Eq. (15.3.1), the members of the Jacobian matrix from Eq. (15.3.4) are evaluated as

$$\begin{aligned} \frac{\partial x}{\partial L_1} &= \frac{\partial x}{\partial L_1} - \frac{\partial x}{\partial L_3} = X_1 - X_3 \\ \frac{\partial x}{\partial L_2} &= \frac{\partial x}{\partial L_2} - \frac{\partial x}{\partial L_3} = X_2 - X_3 \\ \frac{\partial y}{\partial L_1} &= \frac{\partial y}{\partial L_1} - \frac{\partial y}{\partial L_3} = Y_1 - Y_3 \\ \frac{\partial y}{\partial L_2} &= \frac{\partial y}{\partial L_2} - \frac{\partial y}{\partial L_3} = Y_2 - Y_3. \end{aligned} \tag{15.3.11}$$

Thus, the Jacobian matrix is

$$[J] = \begin{bmatrix} X_1 - X_3 & Y_1 - Y_3 \\ X_2 - X_3 & Y_2 - Y_3 \end{bmatrix}. \tag{15.3.12}$$

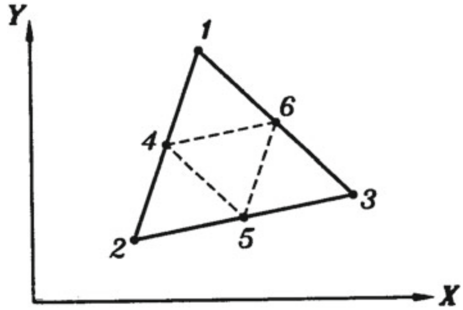


Fig. 15.4 A quadratic triangular element

15.4 Quadratic Element

As for the application of the quadratic triangular elements, we may consider the two-dimensional field problems. A triangular element with six nodes is shown in Fig. 15.4. The shape functions in terms of the variables x and y have a general form given by

$$\phi^{(e)} = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 y^2 + a_6 xy. \tag{15.4.1}$$

Using the nodal point coordinates, the constants a_1 through a_6 are calculated in terms of the nodal values of ϕ, and Eq. (15.4.1) is written in matrix form as

$$\phi^{(e)}(x, y) = \langle N(x, y)\rangle \{\Phi\}^{(e)} \tag{15.4.2}$$

where

$$\begin{aligned} \langle N(x, y)\rangle^{(e)} &= \langle N_1 \quad N_2 \quad N_3 \quad N_4 \quad N_5 \quad N_6\rangle^{(e)} \\ \langle \Phi\rangle^{(e)} &= \langle \Phi_1 \quad \Phi_2 \quad \Phi_3 \quad \Phi_4 \quad \Phi_5 \quad \Phi_6\rangle^{(e)}. \end{aligned} \tag{15.4.3}$$

The gradient matrix of element (e) is

$$\{g\}^{(e)} = \begin{Bmatrix} \dfrac{\partial \phi}{\partial x} \\ \dfrac{\partial \phi}{\partial y} \end{Bmatrix} = [B]^{(e)}\{\Phi\} \tag{15.4.4}$$

in which

$$[B]^{(e)} = \begin{bmatrix} \dfrac{\partial N_1}{\partial x} & \dfrac{\partial N_2}{\partial x} & \dfrac{\partial N_3}{\partial x} & \dfrac{\partial N_4}{\partial x} & \dfrac{\partial N_5}{\partial x} & \dfrac{\partial N_6}{\partial x} \\ \dfrac{\partial N_1}{\partial y} & \dfrac{\partial N_2}{\partial y} & \dfrac{\partial N_3}{\partial y} & \dfrac{\partial N_4}{\partial y} & \dfrac{\partial N_5}{\partial y} & \dfrac{\partial N_6}{\partial y} \end{bmatrix}. \tag{15.4.5}$$

Using the area coordinates, the transformation law from the global to local area coordinates is

$$\left\{\begin{array}{c}\dfrac{\partial N_i}{\partial L_1}\\ \dfrac{\partial N_i}{\partial L_2}\end{array}\right\} = [J]\left\{\begin{array}{c}\dfrac{\partial N_i}{\partial x}\\ \dfrac{\partial N_i}{\partial y}\end{array}\right\}. \tag{15.4.6}$$

Similarly, the transformation law for matrix $[B]^{(e)}$ from the local to the global coordinate is

$$[B_g] = [J]^{-1}[B_l] \tag{15.4.7}$$

where

$$[B_l]^{(e)} = \begin{bmatrix} \dfrac{\partial N_1}{\partial L_1} & \dfrac{\partial N_2}{\partial L_1} & \dfrac{\partial N_3}{\partial L_1} & \dfrac{\partial N_4}{\partial L_1} & \dfrac{\partial N_5}{\partial L_1} & \dfrac{\partial N_6}{\partial L_1} \\ \dfrac{\partial N_1}{\partial L_2} & \dfrac{\partial N_2}{\partial L_2} & \dfrac{\partial N_3}{\partial L_2} & \dfrac{\partial N_4}{\partial L_2} & \dfrac{\partial N_5}{\partial L_2} & \dfrac{\partial N_6}{\partial L_2} \end{bmatrix}. \tag{15.4.8}$$

For quadratic elements, the relations between the shape function N and the area coordinates L are

$$\begin{aligned} N_1 &= L_1(2L_1 - 1) \\ N_2 &= L_2(2L_2 - 1) \\ N_3 &= L_3(2L_3 - 1) \\ N_4 &= 4L_1L_2 \\ N_5 &= 4L_2L_3 \\ N_6 &= 4L_3L_1. \end{aligned} \tag{15.4.9}$$

Considering the following constraint equation among the area coordinates,

$$L_1 + L_2 + L_3 = 1, \tag{15.4.10}$$

and selecting L_1 and L_2 as the independent variables, Eq. (15.4.8) becomes,

$$[B_l]^{(e)} = \begin{bmatrix} 4L_1 - 1 & 0 & 4L_1 + 4L_2 - 3 & 4L_2 & -4L_2 & 4 - 4L_2 - 8L_1 \\ 0 & 4L_2 - 1 & 4L_1 + 4L_2 - 3 & 4L_1 & 4 - 4L_1 - 8L_2 & -4L_1 \end{bmatrix}. \tag{15.4.11}$$

For field problems, the stiffness matrix for the base element (e) was derived and given by Eq. (5.2.18) as

$$[k]^{(e)} = \int [B_g]^T [D][B_g] dA. \tag{15.4.12}$$

Using Eq. (15.4.7)

$$[B_g]^T = \left([J]^{-1}[B_l]\right)^T = [B_l]^T \left([J]^{-1}\right)^T \tag{15.4.13}$$

Thus, substituting in Eq. (15.4.12) gives

$$[k]^{(e)} = \int [B_g]^T [D][B_g] dx dy = \int [B_L]^T [J^{-1}]^T [D][J]^{-1} [B_L] |J| dL_1 dL_2 \tag{15.4.14}$$

in which the indices g and L are referred to the global and local coordinate systems and $[J]$ is the Jacobian matrix. To evaluate the Jacobian matrix, the law of coordinates transformation between the local and global coordinate systems are considered as

$$\begin{aligned} x &= L_1 X_1 + L_2 X_2 + L_3 X_3 \\ y &= L_1 Y_1 + L_2 Y_2 + L_3 Y_3. \end{aligned} \tag{15.4.15}$$

Selecting L_1 and L_2 as the independent variables, the Jacobian matrix is

$$[J] = \begin{bmatrix} \dfrac{\partial x}{\partial L_1} & \dfrac{\partial y}{\partial L_1} \\ \dfrac{\partial x}{\partial L_2} & \dfrac{\partial y}{\partial L_2} \end{bmatrix} = \begin{bmatrix} X_1 - X_3 & Y_1 - Y_3 \\ X_2 - X_3 & Y_2 - Y_3 \end{bmatrix}. \tag{15.4.16}$$

The inverse of the Jacobian matrix is

$$[J]^{-1} = \frac{1}{|J|} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} = \frac{1}{2A^{(e)}} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \tag{15.4.17}$$

in which

$$\begin{aligned} J_{11} &= X_1 - X_3 \qquad J_{21} = X_2 - X_3 \\ J_{12} &= Y_1 - Y_3 \qquad J_{22} = Y_2 - Y_3. \end{aligned} \tag{15.4.18}$$

For the case in which

$$[D] = \begin{bmatrix} k_{xx} & 0 \\ 0 & k_{yy} \end{bmatrix} \tag{15.4.19}$$

we have

$$[J^{-1}]^T [D][J^{-1}] = \frac{1}{4A^2} \begin{bmatrix} k_{xx} J_{22}^2 + k_{yy} J_{21}^2 & -k_{xx} J_{22} J_{12} - k_{yy} J_{21} J_{11} \\ -k_{xx} J_{12} J_{22} - k_{yy} J_{11} J_{21} & k_{xx} J_{12}^2 + k_{yy} J_{11}^2 \end{bmatrix} \tag{15.4.20}$$

or

$$[J^{-1}]^T [D][J^{-1}] = \frac{1}{4A^2} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{15.4.21}$$

in which

$$
\begin{aligned}
a_{11} &= \left(k_{xx}J_{22}^2 + k_{yy}J_{21}^2\right)\\
a_{12} &= (-k_{xx}J_{22}J_{12} - k_{yy}J_{21}J_{11})\\
a_{21} &= a_{12}\\
a_{22} &= (k_{xx}J_{12}^2 + k_{yy}J_{11}^2).
\end{aligned}
\tag{15.4.22}
$$

Substituting the matrix product (15.4.20) in Eq. (15.4.14) and carrying out the integration, gives the element stiffness matrix as

$$
[k]^{(e)} = \frac{1}{2A^{(e)}}\begin{bmatrix}
\frac{1}{2}a_{11} & -\frac{1}{6}a_{12} & \frac{1}{6}\,(a_{11}+a_{12}) & \frac{2}{3}a_{12} & 0 & -\frac{2}{3}\,(a_{11}+a_{12})\\
 & \frac{1}{2}a_{22} & \frac{1}{6}\,(a_{21}+a_{23}) & \frac{2}{3}a_{21} & -\frac{2}{3}\,(a_{21}+a_{22}) & 0\\
\text{symmetric} & & \frac{1}{2}(a_{11}+2a_{12}+a_{22}) & 0 & -\frac{2}{3}\,(a_{12}+a_{22}) & -\frac{2}{3}\,(a_{11}+a_{21})\\
 & & & \frac{4}{3}\,(a_{11}+a_{21}+a_{22}) & -\frac{4}{3}\,(a_{11}+a_{21}) & -\frac{4}{3}\,(a_{12}+a_{22})\\
 & & & & \frac{4}{3}\,(a_{11}+a_{12}+a_{22}) & \frac{4}{3}\,a_{12}\\
 & & & & & \frac{4}{3}\,(a_{11}+a_{12}+a_{22})
\end{bmatrix}.
\tag{15.4.23}
$$

The force matrix related to the pressure is

$$
\{F\}^{(e)} = \int_{A(e)} p\{N\}dA = \int_{A(e)} p\begin{Bmatrix} 1\\ N_2\\ N_3\\ N_4\\ N_5\\ N_6 \end{Bmatrix} dA
\tag{15.4.24}
$$

or

$$
\{F\}^{(e)} = \int_{A(e)} p\begin{Bmatrix} L_1(2L_1-1)\\ L_2(2L_2-1)\\ L_3(2L_3-1)\\ 4L_1L_2\\ 4L_2L_3\\ 4L_3L_1 \end{Bmatrix} [J]dL_1dL_2.
\tag{15.4.25}
$$

15.5 The Quadrilateral Elements

Consider a two-dimensional quadrilateral element with four straight sides, as shown in Fig. 15.5. The interpolating shape function in the element for a dependent function ϕ is

$$\phi^{(e)} = a_1 + a_2 x + a_3 y + a_4 xy. \tag{15.5.1}$$

The constants a_1 through a_4 are obtained using the nodal point coordinates. With reference to Fig. 15.5, the nodal coordinates are

$$\begin{cases} x = a \\ y = b \\ \phi = \phi_i \end{cases} \quad \begin{cases} x = -a \\ y = b \\ \phi = \phi_j \end{cases} \quad \begin{cases} x = -a \\ y = -b \\ \phi = \phi_k \end{cases} \quad \begin{cases} x = a \\ y = -b \\ \phi = \phi_m. \end{cases} \tag{15.5.2}$$

Substituting conditions (15.5.2) in Eq. (15.5.1) and solving for the constant coefficients a_1 to a_4, gives

$$\begin{aligned} a_1 &= \tfrac{1}{4} \left(\phi_i + \phi_j + \phi_k + \phi_m\right) \\ a_2 &= \tfrac{1}{4b} \left(\phi_i - \phi_j - \phi_k + \phi_m\right) \\ a_3 &= \tfrac{1}{4a} \left(\phi_i + \phi_j - \phi_k - \phi_m\right) \\ a_4 &= \tfrac{1}{4ab} \left(\phi_i - \phi_j + \phi_k - \phi_m\right). \end{aligned} \tag{15.5.3}$$

Substituting the values of a_1 to a_4 in Eq. (15.5.1), yields

$$\phi^{(e)} = N_i \phi_i + N_j \phi_j + N_k \phi_k + N_m \phi_m \tag{15.5.4}$$

where

$$\begin{aligned} N_i &= \tfrac{1}{4ab} \,(a + x)(b + y) \\ N_j &= \tfrac{1}{4ab} \,(a - x)(b + y) \\ N_k &= \tfrac{1}{4ab} \,(a - x)(b - y) \\ N_m &= \tfrac{1}{4ab} \,(a + x)(b - y). \end{aligned} \tag{15.5.5}$$

The main advantage of using the quadrilateral element with four nodes over the triangular simplex element is that, while the number of total elements in the solution domain is half of that of triangular elements with the same number of nodal points, the approximating shape function is quadratic. Therefore, the selection of the quadrilateral elements in the finite element modeling results in lower computational time, due to the lower number of elements in the solution domain, and higher solution accuracy, compared to the simplex triangular element. This element is thus very efficient in two-dimensional modeling. Due to the quadratic nature of the shape function, the gradients of the dependent function in the x and y-directions are not

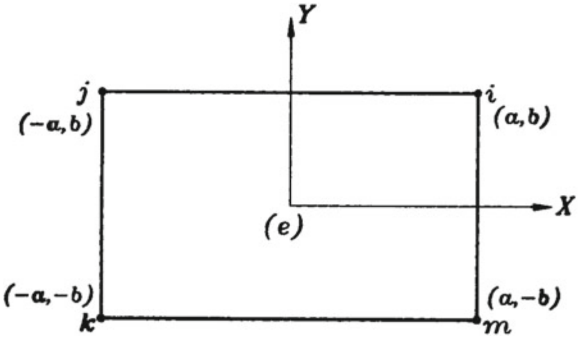
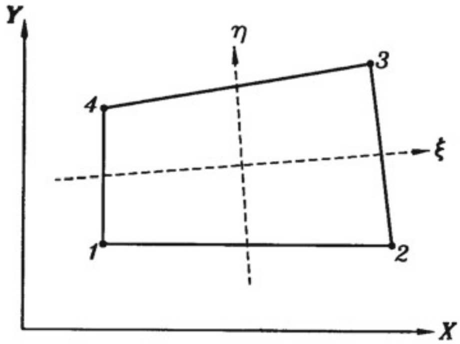


Fig. 15.5 A quadrilateral element with four nodes

Fig. 15.6 A general quadrilateral element with four nodes and natural coordinates ξ and η



constant. The gradient of the function in the x-direction, for example, is constant with respect to x, but is a linear function of y; that is,

$$\frac{\partial \phi}{\partial x} = a_2 + a_4 y$$
$$\frac{\partial \phi}{\partial y} = a_3 + a_4 x. \tag{15.5.6}$$

In Fig. 15.5, the quadrilateral element is shown by a rectangle with perpendicular sides. This element is generalized into an arbitrary four-sided element represented by a pair of natural coordinates ξ and η, as shown in Fig. 15.6. To write the shape functions in terms of the natural coordinates ξ and η, we refer to Fig. 15.6 and write the shape function N_1 as

$$N_1 = \frac{1}{4ab}(b - x)(a - y) = \frac{1}{4}\left(1 - \frac{x}{b}\right)\left(1 - \frac{y}{a}\right). \tag{15.5.7}$$

The range of the variables x/b and y/a in the element (e) are

$$-1 \leq \frac{x}{b} \leq +1 \quad -1 \leq \frac{y}{a} \leq +1. \tag{15.5.8}$$

The variables x/b and y/b may now be redefined as

$$\xi = \frac{x}{b} \quad \eta = \frac{y}{a} \tag{15.5.9}$$

The variables ξ and η are the natural coordinates, as defined in Fig. 15.6, and their range of variations in the element (e) is between -1 and $+1$. The shape function in element (e), as shown in Fig. 15.6, in terms of the natural coordinates is written as

$$\phi^{(e)} = N_1\phi_1 + N_2\phi_2 + N_3\phi_3 + N_4\phi_4 \tag{15.5.10}$$

in which

$$\begin{aligned} N_1 &= \tfrac{1}{4}\,(1-\xi)(1-\eta) \\ N_2 &= \tfrac{1}{4}\,(1+\xi)(1-\eta) \\ N_3 &= \tfrac{1}{4}\,(1+\xi)(1+\eta) \\ N_4 &= \tfrac{1}{4}\,(1-\xi)(1+\eta). \end{aligned} \tag{15.5.11}$$

The ξ and η axes in Fig. 15.6 are along the lines connecting the middle points of the sides of the quadrilateral. In the next section, Eq. (15.5.11) will be proven for a more general bilinear quadrilateral element with a general shape.

15.6 Bilinear Quadrilateral Element

Consider a bilinear quadrilateral element with nodal coordinates (x_i, y_i), $i = 1, 2, 3, 4$ in the x–y coordinates system, as shown in Fig. 15.7a. This quadrilateral element of general shape can be mapped into a square with nodal coordinates $(\pm 1, \pm 1)$, as shown in Fig. 15.7b. The coordinates of any point in the mapped element are $< \xi \quad \eta >$, while in the original, coordinates are $< x \quad y >$. The coordinate transformation law between two coordinate systems can be described in terms of the shape functions as

$$\begin{aligned} x(\xi,\eta) &= \sum_{i=1}^{4} N_i(\xi,\eta)x_i^{(e)} \\ y(\xi,\eta) &= \sum_{i=1}^{4} N_i(\xi,\eta)y_i^{(e)}. \end{aligned} \tag{15.6.1}$$

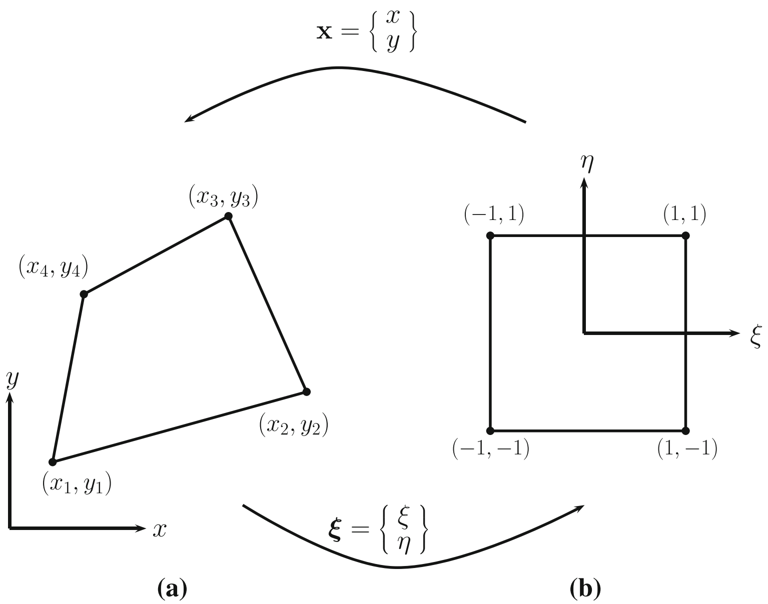


Fig. 15.7 A bilinear quadrilateral element in the global and local coordinates

in which ξ and η are called the natural coordinates. To obtain the shape functions of the coordinate transformation law, we assume the bilinear expansion of the forms

$$\begin{aligned} x(\xi,\eta) &= a_0 + a_1\xi + a_2\eta + a_3\xi\eta \\ y(\xi,\eta) &= b_0 + b_1\xi + b_2\eta + b_3\xi\eta. \end{aligned} \tag{15.6.2}$$

Using the conditions

$$\begin{aligned} x(\xi_i,\eta_i) &= x_i^{(e)} \\ y(\xi_i,\eta_i) &= y_i^{(e)} \end{aligned} \tag{15.6.3}$$

we obtain

$$\begin{aligned} x(-1,-1) &= x_1^{(e)} & y(-1,-1) &= y_1^{(e)} \\ x(1,-1) &= x_2^{(e)} & y(1,-1) &= y_2^{(e)} \\ x(1,1) &= x_3^{(e)} & y(1,1) &= y_3^{(e)} \\ x(-1,1) &= x_4^{(e)} & y(-1,1) &= y_4^{(e)}. \end{aligned} \tag{15.6.4}$$

Conditions (15.6.3) imply that

$$N_i(\xi_j) = \delta_{ij}.$$

That is, from Eq. (15.6.1),

$$x_j^{(e)} = x(\xi_j, \eta_j) = \sum_{i=1}^{4} N_i(\xi_j, \eta_j) x_i^{(e)}. \tag{15.6.5}$$

This equation is valid when $N_i(\xi_j, \eta_j) = \delta_{ij}$. Using Eqs. (15.6.2) and (15.6.4), the following matrix equations are obtained:

$$\begin{Bmatrix} x_1^e \\ x_2^e \\ x_3^e \\ x_4^e \end{Bmatrix} = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix} \tag{15.6.6}$$

$$\begin{Bmatrix} y_1^e \\ y_2^e \\ y_3^e \\ y_4^e \end{Bmatrix} = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{Bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{Bmatrix}. \tag{15.6.7}$$

Solving Eqs. (15.6.6) and (15.6.7) for $a's$ and $b's$, we obtain

$$\begin{aligned} N_1(\xi, \eta) &= \tfrac{1}{4}\,(1-\xi)(1-\eta) \\ N_2(\xi, \eta) &= \tfrac{1}{4}\,(1+\xi)(1-\eta) \\ N_3(\xi, \eta) &= \tfrac{1}{4}\,(1+\xi)(1+\eta) \\ N_4(\xi, \eta) &= \tfrac{1}{4}\,(1-\xi)(1+\eta). \end{aligned} \tag{15.6.8}$$

15.7 Application to the Field Problems

Consider a two-dimensional field problem with $\phi = \phi(x, y)$. Dividing the solution domain into a number of arbitrary quadrilateral elements, the one degree of freedom at each node is ϕ. The base element (e) with four nodes is shown in Fig. 15.6. The function ϕ in the base element (e) is approximated in terms of its nodal values $< \phi_1, \phi_2, \phi_3, \phi_4 >$ as

$$\phi^{(e)}(x, y) = < N_1 \;\; N_2 \;\; N_3 \;\; N_4 > \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix}. \tag{15.7.1}$$

The gradient matrix in element (e) is

$$\{g\}^{(e)} = \begin{Bmatrix} \dfrac{\partial \phi}{\partial x} \\ \dfrac{\partial \phi}{\partial y} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial N_1}{\partial x} & \dfrac{\partial N_2}{\partial x} & \dfrac{\partial N_3}{\partial x} & \dfrac{\partial N_4}{\partial x} \\ \dfrac{\partial N_1}{\partial y} & \dfrac{\partial N_2}{\partial y} & \dfrac{\partial N_3}{\partial y} & \dfrac{\partial N_4}{\partial y} \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix} = [B]\{\phi\}. \tag{15.7.2}$$

The gradient matrix is transformed and written in terms of the natural coordinates ξ and η as

$$\{g\}^{(e)} = \begin{Bmatrix} \dfrac{\partial \phi}{\partial x} \\ \dfrac{\partial \phi}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \dfrac{\partial \phi}{\partial \xi} \\ \dfrac{\partial \phi}{\partial \eta} \end{Bmatrix} \tag{15.7.3}$$

where $[J]$ is the Jacobian matrix of transformation and

$$\begin{Bmatrix} \dfrac{\partial \phi}{\partial \xi} \\ \dfrac{\partial \phi}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial N_1}{\partial \xi} & \dfrac{\partial N_2}{\partial \xi} & \dfrac{\partial N_3}{\partial \xi} & \dfrac{\partial N_4}{\partial \xi} \\ \dfrac{\partial N_1}{\partial \eta} & \dfrac{\partial N_2}{\partial \eta} & \dfrac{\partial N_3}{\partial \eta} & \dfrac{\partial N_4}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix} = [\bar{B}]\{\phi\}. \tag{15.7.4}$$

Thus, matrix $[B]$ is related to matrix $[\bar{B}]$ through the relation

$$[B] = [J]^{-1}[\bar{B}]. \tag{15.7.5}$$

To obtain the Jacobian matrix, we have

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= \frac{\partial \phi}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial \phi}{\partial \eta}\frac{\partial \eta}{\partial x} \\ \frac{\partial \phi}{\partial y} &= \frac{\partial \phi}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial \phi}{\partial \eta}\frac{\partial \eta}{\partial y}. \end{aligned} \tag{15.7.6}$$

The inverse transformation is

$$\begin{aligned} \frac{\partial \phi}{\partial \xi} &= \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial \xi} \\ \frac{\partial \phi}{\partial \eta} &= \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial \eta}. \end{aligned} \tag{15.7.7}$$

From Eq. (15.7.3),

$$\begin{Bmatrix} \dfrac{\partial \phi}{\partial \xi} \\ \dfrac{\partial \phi}{\partial \eta} \end{Bmatrix} = [J] \begin{Bmatrix} \dfrac{\partial \phi}{\partial x} \\ \dfrac{\partial \phi}{\partial y} \end{Bmatrix} \tag{15.7.8}$$

in which the Jacobian matrix is

$$[J] = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix}. \tag{15.7.9}$$

The coordinate transformation laws between the global (x, y) coordinates to the local (ξ, η) coordinates are

$$\begin{aligned} x_{,\,\xi} &= \textstyle\sum_{i=1}^{4} N_{i,\,\xi} x_i \quad x_{,\,\eta} = \sum_{i=1}^{4} N_{i,\,\eta} x_i \\ y_{,\,\xi} &= \textstyle\sum_{i=1}^{4} N_{i,\,\xi} y_i \quad y_{,\,\eta} = \sum_{i=1}^{4} N_{i,\,\eta} y_i. \end{aligned} \tag{15.7.10}$$

Thus, the Jacobian matrix is

$$[J] = [\bar{B}]^{(e)} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix}. \tag{15.7.11}$$

Using Eq. (15.6.8), we have

$$[\bar{B}]^{(e)} = \frac{1}{4} \begin{bmatrix} -(1-\eta) & (1-\eta) & (1+\eta) & -(1+\eta) \\ -(1-\xi) & -(1+\xi) & (1+\xi) & (1-\xi) \end{bmatrix}. \tag{15.7.12}$$

The inverse of the Jacobian matrix is

$$[J]^{-1} = \frac{1}{J} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \tag{15.7.13}$$

in which J is the determinant of the Jacobian matrix

$$J = \det[J] = J_{11} J_{22} - J_{12} J_{21}. \tag{15.7.14}$$

Now, the stiffness matrix of a field problem may be obtained. In terms of the variables (x, y) in the global coordinate system, the stiffness matrix is (see Eq. (5.2.18)),

$$[k]^{(e)} = \int\int [B]^T [k][B] dx dy. \tag{15.7.15}$$

The stiffness matrix in terms of the natural coordinates ξ and η is calculated with $[B]$ written in terms of the natural coordinates as

$$[k]^{(e)} = \int_{-1}^{1} \int_{-1}^{1} [B]^T [k][B][J] d\xi d\eta. \tag{15.7.16}$$

Substituting for $[B]$ from Eq. (15.7.5) yields

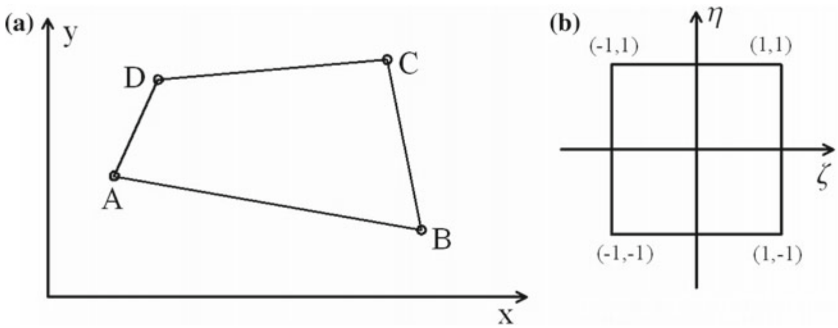


Fig. 15.8 Linear quadrilateral element

$$[k]^{(e)} = \int_{-1}^{1}\int_{-1}^{1} \left([J]^{-1}[\bar{B}]\right)^T [k]\left[[J][\bar{B}]\right][J]d\xi d\eta \tag{15.7.17}$$

or

$$[k]^{(e)} = \int_{-1}^{1}\int_{-1}^{1} [\bar{B}]^T[\bar{k}][\bar{B}][J]d\xi d\eta \tag{15.7.18}$$

in which

$$[\bar{k}] = \left([J]^{-1}\right)^T [k][J]. \tag{15.7.19}$$

The Jacobian J is, in general, a function of ξ and η. For the special cases of rectangle and parallelograms, it becomes a constant.

15.8 Problems

1. Use Eq. (15.4.25) to integrate and obtain the elements of the force matrix associated with a two-dimensional quadratic triangular element.
2. Consider the linear quadrilateral element of Fig. 15.8a with the nodal coordinates $A(2, 2)$, $B(5, 1)$, $C(6, 5)$, and $D(3, 4)$. Find the coordinate transformation laws which transform the element into the local coordinates ξ and η, as shown in Fig. 15.8b.
3. Find the Jacobian matrix of the element of Problem 1.
4. Find the determinant of the Jacobian matrix of Problem 1.
5. Use Eq. (15.7.19) to find the members of matrix $[\bar{k}]$, when $[k] = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$.
6. Use Eq. (15.7.18) to calculate the stiffness matrix of the element of Problem 1.
7. The force matrix of a two-dimensional field problem is

$$\{f\}^{(e)} = \int_A p\{N\}dA$$

Using the linear quadrilateral element, find the general form of the related force matrix.

8. Find the force matrix of the element of Problem 1, when p is a constant.

Further Readings

1. Nejad-Malayeri AR, Mansour K (1999) Finite element solution of two-dimensional potential flow using linear, second, and third order shape functions, B.Sc. project, submitted to the Aeronautical and Aerospace Department, Amirkabir University of Technology, Spring
2. Segerlind LJ (1984) Applied finite element analysis. Wiley, New York
3. Bathe KJ, Wilson EL (1976) Numerical methods in finite element analysis. Prentice Hall Inc., New Jersey

Chapter 16
Coupled Thermoelasticity

Abstract The problems of coupled linear thermoelasticity are among those classes of mechanics which seldom have analytical solution even for simple structures such as beams and rods. Therefore, finite element method is one of the most reliable numerical methods to handle the solution of structural members. The chapter begins with the Galerkin method to obtain the finite element equations of the coupled problems for general three-dimensional case. The members of each related matrice in the resulting finite element equations are calculated and given. The method is then applied to a number of problems. The function- ally graded layer under thermal shock load is analyzed in the next section. Thick spherical vessels under radially symmetric thermal shock load applied to its inside surface is dis- cussed in the next section. The coupled thermoelastic equations for an axisymmetrically loaded disk with different approximation orders is presented in the last section. Elements with various orders are employed to investigate the effects of the number of nodes in an element

16.1 Introduction

Due to the mathematical complexities encountered in the analytical treatment of coupled thermoelasticity problems, the finite element method is often preferred. The finite element method itself is based on two entirely different approaches: the variational approach based on the Ritz method, and the weighted residual methods. The variational approach, which for elastic continuum is based on the extremum of the total potential and kinetic energies, has deficiencies in handling coupled thermoelasticity problems due to the controversial functional relation of the first law of thermodynamics. On the other hand, the weighted residual method based on the Galerkin technique, which is directly applied to the governing equations, is quite efficient and has a very high rate of convergence [1–4].

M. R. Eslami, *Finite Elements Methods in Mechanics*,
Solid Mechanics and Its Applications 216, DOI: 10.1007/978-3-319-08037-6_16,

16.2 Galerkin Finite Element

The general governing equations of classical coupled thermoelasticity are the equation of motion (9.2.8) and the first law of thermodynamics as [1]

$$\sigma_{ij,j} + X_i = \rho \ddot{u}_i \quad \text{in V} \tag{16.2.1}$$

$$q_{i,i} + \rho c \dot{\theta} + \beta T_0 \dot{\epsilon}_{ii} = R \quad \text{in V.} \tag{16.2.2}$$

These equations must be solved simultaneously for the displacement components u_i and temperature change θ. The thermal boundary conditions are satisfied by either of the equations

$$\theta = \theta_s \quad \text{on } A \quad \text{for } t > t_0 \tag{16.2.3}$$

$$\theta_{,n} + a\theta = b \quad \text{on } A \quad \text{for } t > t_0 \tag{16.2.4}$$

in which $\theta_{,n}$ is the gradient of temperature change along the normal to the surface boundary A, and a and b are either constants or given functions of temperature on the boundary. The first condition is related to the specified temperature and the second condition describes the convection and radiation on the boundary.

The mechanical boundary conditions are specified through the traction vector on the boundary. The traction components are related to the stress tensor through Cauchy's formula given by

$$t_i^n = \sigma_{ij} n_j \quad \text{on } A \quad \text{for } t > t_0 \tag{16.2.5}$$

in which t_i^n is the prescribed traction component on the boundary surface the outer unit normal vector of which is $\vec{n}$. For displacement formulations, using the constitutive laws of linear thermoelasticity along with the strain-displacement relations, the traction components can be related to the displacements as

$$t_i^n = \mu(u_{i,j} + u_{j,i})n_j + \lambda u_{k,k} n_i - (3\lambda + 2\mu)\alpha\theta n_i \tag{16.2.6}$$

in which $\theta = T - T_0$ is the temperature change above the reference temperature T_0. It is further possible to have kinematical boundary conditions where the displacements are specified on the boundary as

$$u_i = \bar{u}_i(s) \quad \text{on } A \quad \text{for } t > t_0. \tag{16.2.7}$$

The system of coupled equations (16.2.1) and (16.2.2) does not have a general analytical solution. A finite element formulation may be developed based on the Galerkin method. The finite element model of the problem is obtained by discretizing the solution domain into a number of arbitrary elements. In each base element (e), the components of displacement and temperature change are approximated by the shape functions

$$u_i^{(e)}(x_1, x_2, x_3, t) = U_{mi}(t)N_m(x_1, x_2, x_3) \tag{16.2.8}$$

$$\theta^{(e)}(x_1, x_2, x_3, t) = \theta_m(t)N_m(x_1, x_2, x_3) \qquad m = 1, 2, ..., r \tag{16.2.9}$$

in which r is the total number of nodal points in the base element (e). The summation convention is used for the dummy index m. This is a Kantrovitch type of approximation, in which the time and space functions are separated into distinct functions. Here, $U_{mi}(t)$ is the component of displacement at each nodal point, and $\theta_m(t)$ is the temperature change at each nodal point, all being functions of time. The shape function $N_m(x_1, x_2, x_3)$ is a function of space variables.

Substituting Eqs. (16.2.8) and (16.2.9) into (16.2.1) and applying the weighted residual integral with respect to the weighting functions $N_m(x_1, x_2, x_3)$, the formal Galerkin approximation reduces to

$$\int_{V(e)} (\sigma_{ij,j} + X_i - \rho\ddot{u}_i)N_l dV = 0 \qquad l = 1, 2, ..., r. \tag{16.2.10}$$

Applying the weak formulation to the first term yields

$$\int_{V(e)} (\sigma_{ij,j})N_l dv = \int_{A(e)} \sigma_{ij}n_j N_l dA - \int_{V(e)} \frac{\partial N_l}{\partial x_j}\sigma_{ij} dV \tag{16.2.11}$$

in which n_j is the component of the unit outer normal vector to the boundary. Substituting Eq. (16.2.11) in (16.2.10) gives

$$\int_{A(e)} \sigma_{ij}n_j N_l dA - \int_{V(e)} \frac{\partial N_l}{\partial x_j}\sigma_{ij} dV + \int_{V(e)} X_i N_l dV - \int_{V(e)} \rho\ddot{u}_i N_l dV = 0. \tag{16.2.12}$$

According to Cauchy's formula, the traction force components acting on the boundary are related to the stress tensor as

$$t_i = \sigma_{ij}n_j. \tag{16.2.13}$$

Thus, the first term of Eq. (16.2.12) is

$$\int_{A(e)} \sigma_{ij}n_j N_l dA = \int_{A(e)} t_i N_l dA. \tag{16.2.14}$$

From Hooke's law, the stress tensor is related to the strain tensor, or the displacement components, and temperature change θ as

$$\sigma_{ij} = G(u_{i,j} + u_{j,i}) + \lambda u_{k,k}\delta_{ij} - \beta\theta\delta_{ij}. \tag{16.2.15}$$

Substituting for σ_{ij} in the second term of Eq. (16.2.12) yields

$$\int_{V(e)} \frac{\partial N_l}{\partial x_j} \sigma_{ij} dV = \int_{V(e)} \frac{\partial N_l}{\partial x_j} [G(u_{i,j} + u_{j,i}) + \lambda u_{k,k} \delta_{ij} - \beta \theta \delta_{ij}] dV. \quad (16.2.16)$$

Substituting this expression in Eq. (16.2.12) gives

$$\int_{V(e)} \rho \ddot{u}_i N_l dV + \int_{V(e)} \frac{\partial N_l}{\partial x_j} [G(u_{i,j} + u_{j,i}) + \lambda u_{k,k} \delta_{ij}] dV \\ - \int_{V(e)} \beta \theta \frac{\partial N_l}{\partial x_i} dV = \int_{V(e)} X_i N_l dV + \int_{A(e)} t_i N_l dA. \quad (16.2.17)$$

Now, the base element (e) with r nodal points is considered, and the displacement components and temperature change in the element (e) are approximated by Eqs. (16.2.8) and (16.2.9). Using these approximations, Eq. (16.2.17) becomes

$$\left(\int_{V(e)} \rho N_l N_m dV \right) \ddot{U}_{mi} + \left(\int_{V(e)} G \frac{\partial N_l}{\partial x_j} \frac{\partial N_m}{\partial x_j} dV \right) U_{mi} \\ + \left(\int_{V(e)} G \frac{\partial N_l}{\partial x_j} \frac{\partial N_m}{\partial x_i} dV \right) U_{mj} + \left(\int_{V(e)} \lambda \frac{\partial N_l}{\partial x_i} \frac{\partial N_m}{\partial x_j} dV \right) U_{mj} \\ - \left(\int_{V(e)} \beta \frac{\partial N_l}{\partial x_i} N_m dV \right) \theta_m = \int_{V(e)} X_i N_l dV + \int_{A(e)} t_i N_l dA \\ l, m = 1, 2, ..., r \; i, j = 1, 2, 3. \quad (16.2.18)$$

Equation (16.2.18) is the finite element approximation of the equation of motion. The Galerkin approximation of the energy equation given by Eq. (16.2.1) becomes

$$\int_{V(e)} \left(q_{i,i} + \rho c \frac{\partial \theta}{\partial t} + T_0 \beta \dot{u}_{i,i} - R \right) N_l dV = 0 \quad l = 1, 2, \cdots, r. \quad (16.2.19)$$

The weak formulation of the heat flux gradient $q_{i,i}$ gives

$$\int_{V(e)} q_{i,i} N_l dV = \int_{V(e)} \left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} \right) N_l dV \\ = \int_{A(e)} (\vec{q} \cdot \vec{n}) N_l dA - \int_{V(e)} q_i \frac{\partial N_l}{\partial x_i} dV \quad (16.2.20)$$

in which $A(e)$ is the boundary surface of the base element (e). Substituting Eq. (16.2.20) in (16.2.19) and rearranging the terms gives

$$\int_{V(e)} \rho c \frac{\partial \theta}{\partial t} N_l dV - \int_{V(e)} q_i \frac{\partial N_l}{\partial x_i} dV + \int_{V(e)} T_0 \beta \dot{u}_{i,i} N_l dV \\ = \int_{V(e)} R N_l dV - \int_{A(e)} (\vec{q} \cdot \vec{n}) N_l dA \quad l = 1, 2, \ldots, r. \quad (16.2.21)$$

Substituting for the displacement components u_i and temperature change θ their approximate values in the base element (e) from Eqs. (16.2.8) and (16.2.9) gives

$$\left(\int_{V(e)} k\frac{\partial N_m}{\partial x_i}\frac{\partial N_l}{\partial x_i}dV\right)\theta_m + \left(\int_{V(e)} T_0\beta\frac{\partial N_m}{\partial x_i}N_l dV\right)\dot{U}_{mi} + \left(\int_{V(e)} \rho c N_m N_l dV\right)\dot{\theta}_m = \int_{V(e)} RN_l dV - \int_{A(e)} (\vec{q}\cdot\vec{n})N_l dA. \tag{16.2.22}$$

Equation (16.2.22) is the finite element approximation of the coupled energy equation.

Equations (16.2.18) and (16.2.22) are assembled into a matrix form resulting in the general finite element coupled equation given by

$$[M]\{\ddot{\Delta}\} + [C]\{\dot{\Delta}\} + [K]\{\Delta\} = \{F\} \tag{16.2.23}$$

in which $[M]$, $[C]$, and $[K]$ are the mass, damping, and stiffness matrices, respectively. Matrix $\{\Delta\}^T = \langle U_i, \theta\rangle$ is the matrix of unknowns and $\{F\}$ is the known mechanical and thermal force matrix.

For a two-dimensional problem, l and m take the values $1, 2, \ldots r$. In this case, Eq. (16.2.18) reduces into two equations in the x and y-directions as

$$\begin{aligned}&\left(\int_{V(e)} \rho N_l N_m dV\right)\ddot{U}_m + \left[\int_{V(e)} (2G+\lambda)\frac{\partial N_l}{\partial x}\frac{\partial N_m}{\partial x}dV + \int_{V(e)} G\frac{\partial N_l}{\partial y}\frac{\partial N_m}{\partial y}dV\right]U_m + \left[\int_{V(e)} G\frac{\partial N_l}{\partial y}\frac{\partial N_m}{\partial x}dV \right.\\ &\left. + \int_{V(e)} \lambda\frac{\partial N_l}{\partial x}\frac{\partial N_m}{\partial y}dV\right]V_m - \left[\int_{V(e)} \beta N_m\frac{\partial N_l}{\partial x}dV\right]\theta_m = \int_{V(e)} XN_l dV + \int_{A(e)} t_x N_l dA\end{aligned} \tag{16.2.24}$$

$$\begin{aligned}&\left(\int_{V(e)} \rho N_l N_m dV\right)\ddot{V}_m + \left[(2G+\lambda)\int_{V(e)} \frac{\partial N_l}{\partial y}\frac{\partial N_m}{\partial y}dV + \int_{V(e)} G\frac{\partial N_l}{\partial x}\frac{\partial N_m}{\partial x}dV\right]V_m + \left[\int_{V(e)} G\frac{\partial N_l}{\partial x}\frac{\partial N_m}{\partial y}dV \right.\\ &\left. + \int_{V(e)} \lambda\frac{\partial N_l}{\partial y}\frac{\partial N_m}{\partial x}dV\right]U_m - \left[\int_{V(e)} \beta N_m\frac{\partial N_l}{\partial y}dV\right]\theta_m = \int_{V(e)} YN_l dV + \int_{A(e)} t_y N_l dA.\end{aligned} \tag{16.2.25}$$

The energy equation (16.2.22) for a two-dimensional problem becomes

$$\left(\int_{V(e)} T_0\beta \frac{\partial N_m}{\partial x} N_l dV\right)\dot{U}_m + \left(\int_{V(e)} T_0\beta \frac{\partial N_m}{\partial y} N_l dV\right)\dot{V}_m$$
$$+\left(\int_{V(e)} \rho c N_m N_l dV\right)\dot{\theta}_m + \left(\int_{V(e)} k\frac{\partial N_m}{\partial x}\frac{\partial N_l}{\partial x} dV\right.$$
$$\left.+\int_{V(e)} k\frac{\partial N_m}{\partial y}\frac{\partial N_l}{\partial y} dV\right)\theta_m = \int_{V(e)} RN_l dV - \int_{V(e)} (\vec{q}\cdot\vec{n})N_l dA. \tag{16.2.26}$$

The elements of the mass, damping, stiffness, and force matrices of the base element (e) are

$$[M]^{(e)} = \begin{bmatrix} [\int_{V(e)} \rho N_l N_m dV] & 0 & 0 \\ 0 & [\int_{V(e)} \rho N_l N_m dV] & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{16.2.27}$$

The damping matrix is

$$[C]^{(e)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ [\int_{v(e)} T_0\beta\frac{\partial N_m}{\partial x} N_l dV] & [\int_{v(e)} T_0\beta\frac{\partial N_m}{\partial y} N_l dV] & [\int_{v(e)} \rho c N_m N_l dV] \end{bmatrix} \tag{16.2.28}$$

and the stiffness matrix is

$$[k]^{(e)} = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} \tag{16.2.29}$$

in which

$$[k_{11}^{lm}] = \left[\int_{V(e)} (2G+\lambda)\frac{\partial N_l}{\partial x}\frac{\partial N_m}{\partial x} dV + \int_{V(e)} G\frac{\partial N_l}{\partial y}\frac{\partial N_m}{\partial y} dV\right]$$
$$[k_{12}^{lm}] = \left[\int_{V(e)} G\frac{\partial N_l}{\partial y}\frac{\partial N_m}{\partial x} dV + \int_{V(e)} \lambda\frac{\partial N_l}{\partial x}\frac{\partial N_m}{\partial y} dV\right]$$
$$[k_{13}^{lm}] = -\left[\int_{V(e)} \beta N_m\frac{\partial N_l}{\partial x} dV\right]$$
$$[k_{21}^{lm}] = \left[\int_{V(e)} G\frac{\partial N_l}{\partial x}\frac{\partial N_m}{\partial y} dV + \int_{V(e)} \lambda\frac{\partial N_l}{\partial y}\frac{\partial N_m}{\partial x} dV\right]$$
$$[k_{22}^{lm}] = \left[\int_{V(e)} (2G+\lambda)\frac{\partial N_l}{\partial y}\frac{\partial N_m}{\partial y} dV + \int_{V(e)} G\frac{\partial N_l}{\partial x}\frac{\partial N_m}{\partial x} dV\right]$$

$$[k_{23}^{lm}] = -\left[\int_{V(e)} \beta N_m \frac{\partial N_l}{\partial y} dV\right]$$
$$[k_{31}^{lm}] = [k_{32}]^{ml} = 0$$
$$[k_{33}^{lm}] = \left[\int_{V(e)} k \frac{\partial N_m}{\partial x}\frac{\partial N_l}{\partial x} dV + \int_{V(e)} k \frac{\partial N_m}{\partial y}\frac{\partial N_l}{\partial y} dV\right]. \tag{16.2.30}$$

The force matrix is

$$\{f\}_l^{(e)} = \left\{\begin{array}{c} \{\int_{V(e)} XN_l dV + \int_{A(e)} t_x N_l dA\} \\ \{\int_{V(e)} YN_l dV + \int_{A(e)} t_y N_l dA\} \\ \{\int_{V(e)} RN_l dV - \int_{V(e)} (\vec{q}\cdot\vec{n}) N_l dA\} \end{array}\right\} \tag{16.2.31}$$

and the unknown matrix is

$$\{\delta\}^{(e)} = \left\{\begin{array}{c} \{U\} \\ \{V\} \\ \{\theta\} \end{array}\right\}. \tag{16.2.32}$$

The initial and general form of the thermal boundary conditions are one, or a combination, of the following:

$$\begin{aligned} \theta(x, y, z, 0) &= 0(x, y, z) \text{ at } t = 0 \\ \theta(x, y, z, t) &= \theta_s \text{ on } A_1 \text{ and } t > 0 \\ q_x l + q_y m + q_z n &= -q^{''} \text{ on } A_2 \text{ and } t > 0 \\ q_x l + q_y m + q_z n &= h(\theta + T_0 - T_\infty) \text{ on } A_3 \text{ and } t > 0 \\ q_x l + q_y m + q_z n &= \sigma\epsilon(\theta + T_0)^4 - \alpha_{ab} q_r \text{ on } A_4 \text{ and } t > 0 \end{aligned} \tag{16.2.33}$$

where $T_0(x, y, z)$ is the known initial temperature, θ_s is the known specified temperature change on a part of the boundary surface A_1, $q^{''}$ is the known heat flux on the boundary A_2, h and T_∞ are the convection coefficient and ambient temperature specified on a part of the boundary surface A_3, respectively, σ is the Stefan-Boltzmann constant, ϵ is the radiation coefficient of the boundary surface, α_{ab} is the boundary surface absorption coefficient, and q_r is the rate of thermal flux reaching the boundary surface per unit area, all specified on boundary surface A_4. The cosine directors of the unit outer normal vector to the boundary in the x, y, and z-directions are shown by l, m, and n, respectively. According to the boundary conditions given by Eq. (16.2.33), the last surface integral of the energy Eq. (16.2.22) may be decomposed into four integrals over A_1 through A_4 as

$$\int_{A(e)} (\vec{q} \cdot \vec{n}) N_l dA = \int_{A_2} q^{''} N_l dA - \int_{A_3} h(\theta + T_0 - T_\infty) N_l dA$$
$$- \int_{A_4} (\sigma\epsilon(\theta + T_0)^4 - \alpha_{ab} q_r) N_l dA \quad l = 1, 2, \cdots r. \tag{16.2.34}$$

Note that the signs of the integrals in Eq. (16.2.34) depend upon the direction of the heat input. The positive sign is defined when the heat is given to the body, and is negative when the heat is removed from the body. That is, $q^{''}$ is defined positive in Eq. (16.2.34), since we have assumed that the heat flux is given to the body. On the other hand, we have assumed negative convection on the surface area A_3, which means the heat is removed from the A_3 boundary by convection. Similarly, boundary A_4 is assumed to radiate to the ambient, as the sign of this integral is considered negative.

In order to discuss the method in more detail, a one-dimensional problem is considered [5–7]. The equation of motion in terms of displacement is

$$(\lambda + 2G)\frac{\partial^2 u}{\partial x^2} - \beta\frac{\partial \theta}{\partial x} = \rho\frac{\partial^2 u}{\partial t^2} \tag{16.2.35}$$

and the first law of thermodynamics reduces to

$$k\frac{\partial^2 \theta}{\partial x^2} - \rho c\frac{\partial \theta}{\partial t} - \beta T_0\frac{\partial^2 u}{\partial x \partial t} = 0. \tag{16.2.36}$$

Taking a line element of length L, the approximating function for axial displacement for the base element (e) is assumed to be linear in x as

$$u(x, t)^{(e)} = N_i U_i + N_j U_j = \langle N\rangle^{(e)} \{U\}^{(e)} \tag{16.2.37}$$

in which the piecewise linear shape function $\langle N\rangle$ is $N_i = (L - \eta)/L$, $N_j = \eta/L$, and $\eta = x - x_i$. Similarly, the temperature change is approximated by

$$\theta(x, t)^{(e)} = N_i\theta_i + N_j\theta_j = \langle N\rangle^{(e)} \{\theta\}^{(e)}. \tag{16.2.38}$$

Employing the formal Galerkin method and applying the weak form to the first and second terms of Eq. (16.2.35) and first term of Eq. (16.2.36) results in the following system of equations:

$$\left(\int_0^L \rho N_l N_m d\eta\right) \ddot{U}_m + \left(\int_0^L (2G + \lambda)\frac{\partial N_l}{\partial \eta}\,\frac{\partial N_m}{\partial \eta} d\eta\right) U_m$$
$$- \left(\int_0^L \beta N_m \frac{\partial N_l}{\partial \eta} d\eta\right) \theta_m = \mathbf{t}_x N_l|_i^j + \int_0^L X N_l d\eta \tag{16.2.39}$$

$$\left(\int_0^L T_0\beta \frac{\partial N_m}{\partial \eta}\ N_l d\eta\right)\dot{U}_m + \left(\int_0^L \rho c N_m N_l d\eta\right)\dot{\theta}_m + \left(\int_0^L k\ \frac{\partial N_m}{\partial \eta}\ \frac{\partial N_l}{\partial \eta} d\eta\right)\theta_m$$
$$= -\ (\vec{q}\cdot\vec{n})N_l|_i^j + \int_0^L RN_l d\eta. \qquad (16.2.40)$$

This system of equations may be written in matrix form as

$$[M]\{\ddot{\Delta}\} + [C]\{\dot{\Delta}\} + [K]\{\Delta\} = \{F\} \qquad (16.2.41)$$

in which the mass, damping, stiffness and force matrices for the first order element are 4×4 matrices and are defined as

$$[M]^{(e)} = \int_0^L \begin{bmatrix} \rho N_i N_i & 0 & \rho N_i N_j & 0 \\ 0 & 0 & 0 & 0 \\ \rho N_j N_i & 0 & \rho N_j N_j & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} d\eta \qquad (16.2.42)$$

$$[C]^{(e)} = \int_0^L \begin{bmatrix} 0 & 0 & 0 & 0 \\ T_0\beta\ N_i \frac{\partial N_i}{\partial \eta} & \rho c N_i N_i & T_0\beta\ N_i \frac{\partial N_j}{\partial \eta} & \rho c N_i N_j \\ 0 & 0 & 0 & 0 \\ T_0\beta\ N_j \frac{\partial N_i}{\partial \eta} & \rho c N_j N_i & T_0\beta\ N_j \frac{\partial N_j}{\partial \eta} & \rho c N_j N_j \end{bmatrix} d\eta \qquad (16.2.43)$$

$$[K]^{(e)} = \int_0^L \begin{bmatrix} (2G+\lambda)\left(\frac{\partial N_i}{\partial \eta}\right)^2 & -\beta N_i \frac{\partial N_i}{\partial \eta} & (2G+\lambda)\frac{\partial N_i}{\partial \eta}\ \frac{\partial N_j}{\partial \eta} & -\beta N_j \frac{\partial N_i}{\partial \eta} \\ 0 & k\left(\frac{\partial N_i}{\partial \eta}\right)^2 & 0 & k\ \frac{\partial N_j}{\partial \eta}\ \frac{\partial N_i}{\partial \eta} \\ (2G+\lambda)\frac{\partial N_j}{\partial \eta}\ \frac{\partial N_i}{\partial \eta} & -\beta N_i \frac{\partial N_j}{\partial \eta} & (2G+\lambda)\left(\frac{\partial N_j}{\partial \eta}\right)^2 & -\beta N_j \frac{\partial N_j}{\partial \eta} \\ 0 & k\ \frac{\partial N_i}{\partial \eta}\ \frac{\partial N_j}{\partial \eta} & 0 & k\left(\frac{\partial N_j}{\partial \eta}\right)^2 \end{bmatrix} d\eta \qquad (16.2.44)$$

$$\{F\}^{(e)} = \begin{Bmatrix} \mathbf{t}_x N_i|_0^L + \int_0^L XN_i d\eta \\ -\ q_x N_i|_0^L + \int_0^L RN_i d\eta \\ \mathbf{t}_x N_j\big|_L + \int_0^L XN_j d\eta \\ -\ q_x N_j\big|_0^L + \int_0^L RN_j d\eta \end{Bmatrix}. \qquad (16.2.45)$$

Upon substitution of the shape functions in the foregoing equations, the submatrices for the base element (e) are

$$[M]^{(e)} = \begin{bmatrix} \frac{\rho L}{3} & 0 & \frac{\rho L}{6} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\rho L}{6} & 0 & \frac{\rho L}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad (16.2.46)$$

$$[C]^{(e)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\frac{T_0\beta}{2} & \frac{\rho c L}{3} & \frac{T_0\beta}{2} & \frac{\rho c L}{6} \\ 0 & 0 & 0 & 0 \\ -\frac{T_0\beta}{2} & \frac{\rho c L}{6} & \frac{T_0\beta}{2} & \frac{\rho c L}{3} \end{bmatrix} \tag{16.2.47}$$

$$[K]^{(e)} = \begin{bmatrix} \frac{(2G+\lambda)}{L} & \frac{\beta}{2} & -\frac{(2G+\lambda)}{L} & \frac{\beta}{2} \\ 0 & \frac{k}{L} & 0 & -\frac{k}{L} \\ -\frac{(2G+\lambda)}{L} & -\frac{\beta}{2} & \frac{(2G+\lambda)}{L} & -\frac{\beta}{2} \\ 0 & -\frac{k}{L} & 0 & \frac{k}{L} \end{bmatrix} \tag{16.2.48}$$

$$\{F\}^{(e)} = \begin{bmatrix} -\,t_x|_0 + \frac{XL}{2} \\ q_x|_0 + \frac{RL}{2} \\ t_x|_L + \frac{XL}{2} \\ -\,q_x|_L + \frac{RL}{2} \end{bmatrix} \tag{16.2.49}$$

and the matrix of unknown nodal value is

$$\{\Delta\}^{(e)} = \begin{bmatrix} U_i \\ \theta_i \\ U_j \\ \theta_j \end{bmatrix}. \tag{16.2.50}$$

16.3 Functionally Graded Layers

Functionally Graded Materials (FGMs) are high-performance, heat-resistant materials able to withstand the ultra-high temperatures and extremely large thermal gradients used in the aerospace industries. FGMs are microscopically inhomogeneous and their mechanical properties vary smoothly and continuously from one surface to the other [8]. Typically, these materials are made from a mixture of ceramic and metal.

The coupled thermoelasticity of a layer with isotropic material is discussed in the literature. Bagri et al. [9] proposed a new system of coupled equations that contains the LS, GL, and GN models [10–14] in a unified form. They employed the suggested formulation and then analytically solved the coupled system of equations for a layer using the Laplace transform. The effect of the thermomechanical coupling coefficient in the problems of coupled thermoelasticity is pointed out in Refs. [15, 16].

Except for a few cases of coupled problems, the general closed form solution for the coupled thermoelasticity problems is not available in the literature, and most of the relevant problems are solved numerically. Among the numerical procedures, the boundary and finite element method, as well as the finite difference method, are most considered for these types of problem. Tamma and Namburu [17] have an overview of these numerical methods. Chen and Lin [18] proposed a hybrid numerical method based on the Laplace transform and control volume method for analyzing the transient coupled thermoelastic problems with relaxation times involving a nonlinear radiation boundary condition. Hosseini and Eslami [19, 20]considered the boundary element formulation for the analysis of coupled thermoelastic problems in a two-dimensional finite domain and studied the coupling coefficient effects on thermal and elastic wave propagation.

The response of functionally-graded materials under dynamic thermal loads and using the coupled form equations of thermoelasticity theories are found in just a few articles. Zhang et al. [21] modeled an isotropic ceramic-metal laminated beam subjected to an abrupt heating condition, and demonstrated the influence of thermo-mechanical coupling on the thermal shock response. Praveen and Reddy [22] studied the static and dynamic response of the functionally-graded plates and showed that the response of FG plates is not intermediate to the response of the ceramic and metal plates. Bagri et al.[23] considered the classical coupled thermoelasticity theory to study the behavior of an FG layer under thermal shock load. The effect of the material composition profile on the distribution of temperature, displacement, and stress through the thickness of the layer is studied.

In this section, the dynamic response of a layer made of FGMs based on the LS theory is investigated Bagri et al. [24]. The power law form function is assumed for the material properties distribution. A suitable transfinite element method via the Laplace transform is employed to find the temperature and displacement field solution in the space domain. Finally, the temperature, displacement and stress in the physical time domain are obtained using a numerical inversion of the Laplace transform proposed by Honig and Hirdes [25]. The temperature, displacement and stress waves propagation and reflection from the boundaries of layer are studied. Also, the relaxation time and material volume fraction effects on temperature, displacement and stress variations are investigated.

Consider a ceramic-metal FG layer with thickness of L and assume that the properties of the FG layer obey a power law function as

$$P = \left(\frac{x}{L}\right)^n (P_m - P_c) + P_c \tag{16.3.1}$$

in which x is the position from the ceramic rich side of the layer, P is the effective property of FGM, n is the power law index that governs the distribution of the constituent materials through the thickness of the layer, and P_m and P_c are the properties of metal and ceramic, respectively. Meanwhile, the subscripts m and c indicate the metal and ceramic features, respectively.

For the LS theory, in the absence of body forces and heat supply, when the derivative of the relaxation time with respect to the position variable is neglected, the governing equations of an FG layer in terms of displacement and temperature are as follows [24]

$$(\lambda + 2\mu)\frac{\partial^2 u}{\partial x^2} + \frac{\partial(\lambda + 2\mu)}{\partial x}\frac{\partial u}{\partial x} - \frac{\partial[\beta(T - T_0)]}{\partial x} = \rho\frac{\partial^2 u}{\partial t^2} \tag{16.3.2}$$

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) - \rho c\frac{\partial T}{\partial t} - \rho c t_0\frac{\partial^2 T}{\partial t^2} - \beta T_0\left(t_0\frac{\partial^2}{\partial t^2} + \frac{\partial}{\partial t}\right)\frac{\partial u}{\partial x} = 0 \tag{16.3.3}$$

in which t_0 is the relaxation time proposed by Lord and Shulman. The preceding equations may be introduced in dimensionless form for convenience. The nondimensional parameters are defined as

$$\begin{aligned}
\bar{x} &= \frac{x c_m\sqrt{\rho_m(\lambda_m + 2\mu_m)}}{k_m}; \quad \bar{T} = \frac{T - T_0}{T_d} \\
\bar{t} &= \frac{t(\lambda_m + 2\mu_m)c_m}{k_m}; \quad \bar{t}_0 = \frac{t_0(\lambda_m + 2\mu_m)c_m}{k_m} \\
\bar{q}_x &= \frac{q_x}{c_m T_d\sqrt{\rho_m(\lambda_m + 2\mu_m)}}; \quad \bar{\sigma}_{xx} = \frac{\sigma_{xx}}{\beta_m T_d} \\
\bar{u} &= \frac{(\lambda_m + 2\mu_m)^{3/2}\rho_m^{1/2}\ c_m\ u}{k_m\beta_m T_d}
\end{aligned} \tag{16.3.4}$$

in which the subscript m denotes the metal properties and term T_d is a characteristic temperature used for normalizing the temperature. Using the dimensionless parameters, the governing equations (16.3.2) and (16.3.3) appear in the form

$$\begin{aligned}
&\left[\frac{(\lambda + 2\mu)}{(\lambda_m + 2\mu_m)}\frac{\partial^2}{\partial\bar{x}^2} + \frac{1}{(\lambda_m + 2\mu_m)}\frac{\partial(\lambda + 2\mu)}{\partial\bar{x}}\frac{\partial}{\partial\bar{x}} - \frac{\rho}{\rho_m}\frac{\partial^2}{\partial\bar{t}^2}\right]\bar{u} \\
&\quad - \frac{1}{\beta_m}\left(\frac{\partial\beta}{\partial\bar{x}} + \beta\frac{\partial}{\partial\bar{x}}\right)\bar{T} = 0
\end{aligned} \tag{16.3.5}$$

$$\begin{aligned}
&\left[\frac{k}{k_m}\frac{\partial^2}{\partial\bar{x}^2} + \frac{1}{k_m}\frac{\partial k}{\partial\bar{x}}\frac{\partial}{\partial\bar{x}} - \frac{\rho c}{\rho_m c_m}\left(\frac{\partial}{\partial\bar{t}} + \bar{t}_0\frac{\partial^2}{\partial\bar{t}^2}\right)\right]\bar{T} \\
&\quad - \frac{\beta_m T_0}{\rho_m c_m(\lambda_m + 2\mu_m)}\beta\left(\bar{t}_0\frac{\partial^2}{\partial\bar{t}^2} + \frac{\partial}{\partial\bar{t}}\right)\frac{\partial\bar{u}}{\partial\bar{x}} = 0.
\end{aligned} \tag{16.3.6}$$

Also, the dimensionless stress-displacement-temperature relation and the heat conduction equation for the functionally graded layer based on the LS theory are

$$\bar{\sigma}_{xx} = \frac{(\lambda + 2\mu)}{(\lambda_m + 2\mu_m)}\frac{\partial\bar{u}}{\partial\bar{x}} - \frac{\beta}{\beta_m}\bar{T} \tag{16.3.7}$$

$$\bar{q}_x + \bar{t}_0\frac{\partial\bar{q}_x}{\partial\bar{t}} = -\frac{k}{k_m}\frac{\partial\bar{T}}{\partial\bar{x}}. \tag{16.3.8}$$

The layer is occupied in the region $0 \le \bar{x} \le 1$ and the corresponding dimensionless boundary conditions are

$$\bar{q}_x = 1 - (1 + 100\bar{t})e^{-100\bar{t}}\ ; \quad \bar{\sigma}_{xx} = 0 \quad at\ \bar{x} = 0$$
$$\bar{T} = 0\ ; \quad \bar{u} = 0 \quad at\ \bar{x} = 1. \tag{16.3.9}$$

To solve the coupled system of equations, the transfinite element method may be employed. To this end, the equations may be transformed to the space domain using the Laplace transformation. Assume that the layer is initially at rest and the initial displacement, velocity, temperature, and temperature rate are zero. Applying the Laplace transformation to Eqs. (16.3.5) and (16.3.6) gives

$$\left[\frac{(\lambda + 2\mu)}{(\lambda_m + 2\mu_m)}\frac{\partial^2}{\partial \bar{x}^2} + \frac{1}{(\lambda_m + 2\mu_m)}\frac{\partial(\lambda + 2\mu)}{\partial \bar{x}}\frac{\partial}{\partial \bar{x}} - \frac{\rho}{\rho_m}s^2\right]\bar{u}^*$$
$$- \frac{1}{\beta_m}\left(\frac{\partial \beta}{\partial \bar{x}} + \beta\frac{\partial}{\partial \bar{x}}\right)\bar{T}^* = 0 \tag{16.3.10}$$
$$\left[\frac{k}{k_m}\frac{\partial^2}{\partial \bar{x}^2} + \frac{1}{k_m}\frac{\partial k}{\partial \bar{x}}\frac{\partial}{\partial \bar{x}} - \frac{\rho c}{\rho_m c_m}s(1 + \bar{t}_0 s)\right]\bar{T}^*$$
$$- \frac{\beta_m T_0}{\rho_m c_m(\lambda_m + 2\mu_m)}\beta s(1 + \bar{t}_0 s)\frac{\partial \bar{u}^*}{\partial \bar{x}} = 0. \tag{16.3.11}$$

To find the solution of the equations using the transfinite element method, the geometry of the layer may be divided into a number of discretized elements through the thickness of the layer. In the base element, the Kantorovich approximation for the displacement u and temperature T with identical shape functions is assumed as

$$\bar{u}^{*(e)} = \sum_{i=1}^{\ell} N_i \bar{U}_i^* \qquad \bar{T}^{*(e)} = \sum_{i=1}^{\ell} N_i \bar{T}_i^* \tag{16.3.12}$$

in which N_i is the shape function and terms $\bar{U}_i^*$ and $\bar{T}_i^*$ are the unknown nodal values of displacement and temperature, respectively. Substituting Eq. (16.3.12) into (16.3.10) and (16.3.11) and then employing the Galerkin finite element method, the following system of equations, applying the weak form to the terms of second order of derivatives of the space variable, is obtained:

$$\begin{bmatrix} [K_{11}]\,[K_{12}] \\ [K_{21}]\,[K_{22}] \end{bmatrix} \begin{Bmatrix} \bar{U}^* \\ \bar{T}^* \end{Bmatrix} = \begin{Bmatrix} F^* \\ Q^* \end{Bmatrix} \tag{16.3.13}$$

The submatrices $[K_{11}],\ [K_{12}],\ [K_{21}],\ [K_{22}],\ F^*$ and Q^* are

$$[K_{11}^{ij}] = \int_{\bar{x}_f}^{\bar{x}_e} \left\{\frac{(\lambda + 2\mu)}{(\lambda_m + 2\mu_m)}\frac{\partial N_i}{\partial \bar{x}}\frac{\partial N_j}{\partial \bar{x}} + \frac{\rho s^2}{\rho_m}N_i N_j\right\} d\bar{x} \tag{16.3.14}$$

$$[K_{12}^{ij}] = \frac{1}{\beta_m}\int_{\bar{x}_f}^{\bar{x}_e} \left\{\beta N_j \frac{\partial N_j}{\partial \bar{x}} + \frac{\partial \beta}{\partial \bar{x}}N_i N_j\right\} d\bar{x} \tag{16.3.15}$$

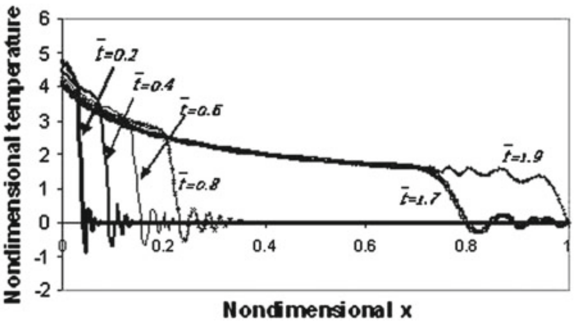


Fig. 16.1 Distribution of the nondimensional temperature through the thickness of the layer for $n = 1$

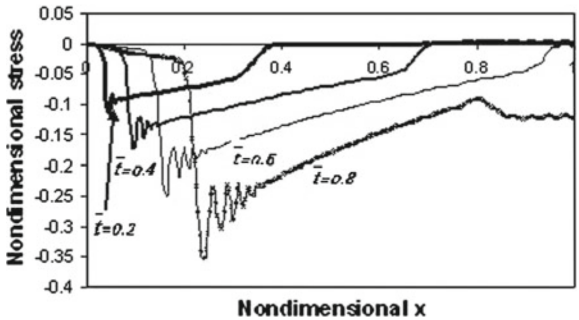


Fig. 16.2 Distribution of the nondimensional stress through the thickness of the layer for $n = 1$

$$[K_{21}^{ij}] = \frac{s\beta_m T_0}{\rho_m c_m(\lambda_m + 2\mu_m)} \int_{\bar{x}_f}^{\bar{x}_e} \beta(1 + \bar{t}_0 s) N_i \frac{\partial N_j}{\partial \bar{x}} d\bar{x} \tag{16.3.16}$$

$$[K_{22}^{ij}] = \int_{\bar{x}_f}^{\bar{x}_e} \left\{ \frac{k}{k_m} \frac{\partial N_i}{\partial \bar{x}} \frac{\partial N_j}{\partial \bar{x}} + \frac{\rho c}{\rho_m c_m} s(1 + \bar{t}_0 s) N_i N_j \right\} d\bar{x} \tag{16.3.17}$$

$$\{F^*\} = \left\{ \begin{array}{c} \frac{\beta_c}{\beta_m} \bar{T}_1^* \\ 0 \\ . \\ . \\ 0 \end{array} \right\}; \; \{Q^*\} = \left\{ \begin{array}{c} (1 + \bar{t}_0 s)\bar{q}_x^* \\ 0 \\ . \\ . \\ 0 \end{array} \right\}. \tag{16.3.18}$$

In these equations, $\bar{x}_f$ and $\bar{x}_e$ are the first and last nodes of the solution domain, respectively.

The system of Eq. (16.3.13) is solved in the space domain. To transform the results from the Laplace transform domain into the real time domain, the numerical inverse Laplace transform technique given in Ref. [25] is used.

Consider an FG layer composed of aluminum and alumina as metal and ceramic constituents, respectively. The reference temperature is assumed to be $T_0 = 298K$.

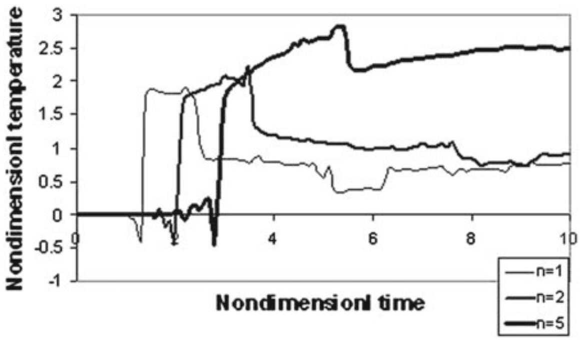


Fig. 16.3 Variations of the nondimensional temperature at the middle point of the thickness of the layer for different values of power index

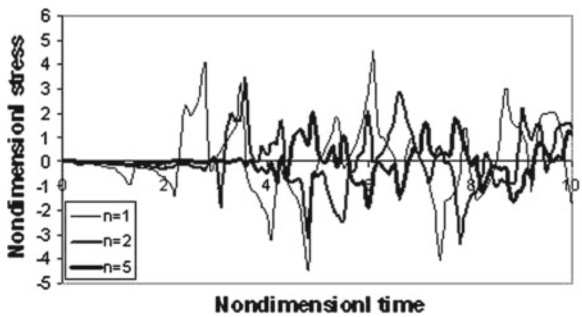


Fig. 16.4 Variations of the nondimensional stress at the middle point of the thickness of the layer for different values of power index

The relaxation time of aluminum and alumina are assumed to be $\bar{t}_{0m} = 0.64$, $\bar{t}_{0c} = 1.5625$, respectively. The linear Lagrangian polynomials are used for the shape functions in the base element. Figures 16.1 and 16.2 show the temperature and stress wave propagation and reflection from the boundaries of the layer for $n = 1$. In Fig. 16.1, the times $\bar{t} = 0.2, 0.4, 0.6, 0.8, 1.7$ show the temperature wave propagation through the thickness of the layer, while the reflection of the temperature wave occurred at time $\bar{t} = 1.9$. Figure 16.2 shows that the maximum of stress occurs at the temperature wave front. In Figs. 16.1 and 16.2, it can be seen that a conversion between the mechanical and thermal energies occurs at the temperature wave front. It may be found from the figures that the propagation velocity of waves varies through the thickness of the layer.

The effect of power law index, n, on variation of the temperature and stress at a point located at the middle point of the layer thickness is shown in Figs. 16.3 and 16.4. It is seen from Fig. 16.3 that when n increases, the speed of the temperature wave decreases. In Fig. 16.4, it is shown that the amplitude of stress variation is decreased with the increase of n.

The relaxation time effect on variation of the temperature and stress at the middle point of the thickness of the layer is investigated and is shown in Figs. 16.5 and

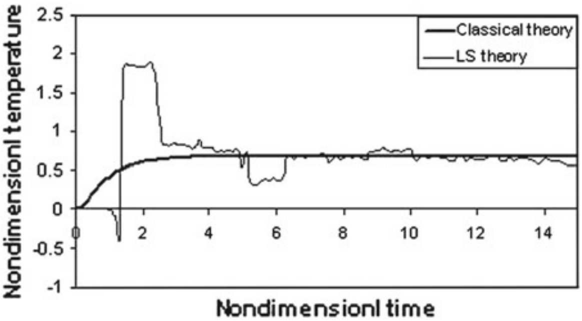


Fig. 16.5 Variations of the nondimensional temperature at the middle point of the thickness of the layer for the classical and LS theories

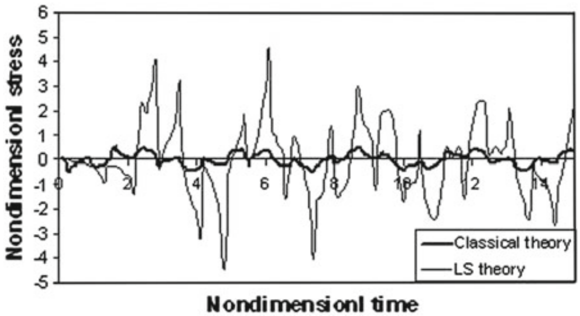


Fig. 16.6 Variations of the nondimensional stress at the middle point of the thickness of the layer for the classical and LS theories

16.6. The value of $n = 1$ is considered for the power law index. It is seen that, for the classical theory of thermoelasticity, the case when $t_0 = 0$, smaller values for amplitude of temperature and resulting stress variations are obtained. Since with the increase of relaxation time the propagation velocity of the temperature wave decreases, these maximum values of variation occur at the later times.

16.4 Coupled Thermoelasticity of Thick Spheres

The finite element analysis of coupled thermoelasticity of thick spheres and cylinders was studied by Li et al. [26], by Ghoneim [27], and by Eslami and Vahedi [28, 29]. The Galerkin method is basically used to obtain the finite element formulations. The analysis is also based on displacement formulation.

Consider a thick-walled sphere of inside and outside radii r_{in} and r_{out}, respectively. For symmetric loading condition

$$\sigma_{\theta\theta} = \sigma_{\phi\phi} \quad \epsilon_{\theta\theta} = \epsilon_{\phi\phi} \tag{16.4.1}$$

the equation of motion is

$$\frac{\partial \sigma_{rr}}{\partial r} + 2\frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = \rho \ddot{u} \tag{16.4.2}$$

and the strain-displacement relations are

$$\epsilon_{rr} = \frac{\partial u}{\partial r} \quad \epsilon_{\theta\theta} = \frac{u}{r} = \epsilon_{\phi\phi}. \tag{16.4.3}$$

From Hooke's law

$$\begin{aligned}
\sigma_{rr} &= \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\epsilon_{rr} + 2\nu\epsilon_{\theta\theta} - (1+\nu)\alpha(T - T_0)] \\
\sigma_{\theta\theta} &= \frac{E}{(1+\nu)(1-2\nu)}[\epsilon_{\theta\theta} + \nu\epsilon_{rr} - (1+\nu)\alpha(T - T_0)].
\end{aligned} \tag{16.4.4}$$

The first law of thermodynamics for a coupled condition is

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} - \frac{\rho c}{k}\dot{T} = \gamma_1(\dot{\epsilon}_{rr} + 2\dot{\epsilon}_{\theta\theta}) - R \tag{16.4.5}$$

where $\gamma_1 = (3\lambda + 2\mu)\alpha T_0/k$. Elimination of the stresses from Eqs. (16.4.2), (16.4.3), and (16.4.4) results in the equation of motion in terms of the displacement

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} - \frac{2u}{r^2} - \alpha\frac{(1+\nu)}{(1-\nu)}\frac{\partial T}{\partial r} = \frac{(1+\nu)(1-2\nu)\rho}{(1-\nu)E}\ddot{u}. \tag{16.4.6}$$

The energy equation, after substitution for strains, becomes

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} - \frac{\rho c}{k}\dot{T} = \frac{\gamma_1}{r^2}\frac{\partial}{\partial r}(r^2\dot{u}) - R. \tag{16.4.7}$$

The boundary conditions are in general given as

$$\sigma_{rr} \times n_r = \mathbf{t}_r - k\frac{\partial T}{\partial r} = q_r \tag{16.4.8}$$

where n_r is the unit vector in the radial direction. In terms of displacement, the boundary conditions at the inside and outside radii are:

At $r = r_{in}$

$$\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\left[\frac{\partial u}{\partial r} + \frac{\nu}{(1-\nu)}\frac{2u}{r} - \frac{(1+\nu)}{(1-\nu)}\alpha(T - T_0)\right] = -P_a(t)$$

$$T(t) = T_0 \left\{ 2 + \left[\left(\frac{\rho\ c\ c_1^2\ t}{k} \right)^2 - \frac{\rho\ c\ c_1^2\ t}{k} - 1 \right] e^{-\frac{\rho\ c\ c_1^2\ t}{k}} \right\} \tag{16.4.9}$$

At $r = r_{out}$

$$\begin{aligned} u &= 0 \\ k\frac{\partial T}{\partial r} &= -h_o[T - T_0] \end{aligned} \tag{16.4.10}$$

in which h_o is the convection coefficient at the inside and outside surfaces of the sphere, respectively, $T_i(t)$ is the inside surface temperature, which is assumed to vary in time and is applied as a thermal shock to the inside surface, T_0 is the constant outside ambient temperature, and $P_a(t)$ is the applied pressure shock at the inside surface, which may be considered zero.

The governing equations are changed into dimensionless form through the following formulas:

$$\begin{aligned} \bar{T} &= \frac{(T - T_0)}{T_0} \\ \bar{u} &= (\frac{1-\nu}{1+\nu})\frac{\rho\ c\ c_1\ u}{k\ \alpha\ T_0} \\ \bar{\sigma}_{rr} &= \frac{(1-2\nu)\sigma_{rr}}{E\alpha T_0} \\ \bar{r} &= \frac{\rho\ c\ c_1\ r}{k} \\ \bar{t} &= \frac{\rho\ c\ c_1^2\ t}{k} \\ c_1 &= \sqrt{\frac{(1-\nu)E}{(1+\nu)(1-2\nu)\rho}}. \end{aligned} \tag{16.4.11}$$

Using these quantities, and in the absence of heat generation, the governing equations are expressed in dimensionless form (bar is dropped for convenience)

$$\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r} + \frac{2u}{r}\right) - \frac{\partial T}{\partial r} = \ddot{u} \tag{16.4.12}$$

$$\frac{\partial}{\partial r}\left(\frac{\partial T}{\partial r}\right) + \frac{2}{r}\frac{\partial T}{\partial r} - \dot{T} = C\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\dot{u}). \tag{16.4.13}$$

The boundary conditions are

$$\frac{\partial u}{\partial r} + \gamma_2\frac{u}{r} - T = -\frac{1-2\nu}{E\alpha T_0}P_a(t)$$

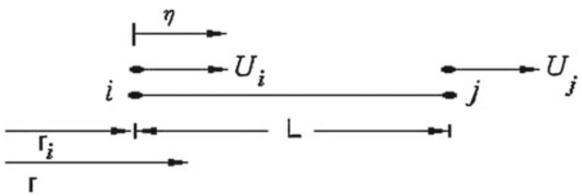


Fig. 16.7 The element (e) along the radius

$$T = 1 + (t^2 - t - 1)\, e^{-t} \quad \text{at} \quad r = a \tag{16.4.14}$$

and

$$\begin{aligned} u &= 0 \\ \frac{\partial T}{\partial r} &= -\eta_o T \quad \text{at} \quad r = b \end{aligned} \tag{16.4.15}$$

in which a and b are the dimensionless inner and outer radii of the sphere, respectively, and the parameters used in these equations are defined as

$$C = \frac{T_0(1+\nu)E\alpha^2}{\rho c(1-\nu)(1-2\nu)}; \quad \gamma_2 = \frac{2\nu}{(1-\nu)}; \quad \eta_o = \frac{h_o}{\rho c c_1}. \tag{16.4.16}$$

Due to the radial symmetry of loading conditions, the variations of the dependent functions are along the radius of the sphere. Thus, the radius of the sphere is divided into a number of line elements (NE) with nodes i and j for the base element (e), as shown in Fig. 16.7.

The displacement of element (e) is described by the linear shape function

$$u(r,t) = \alpha_1(t) + \alpha_2(t)r \tag{16.4.17}$$

in terms of unknown nodal variables, the unknown coefficients $\alpha_1(t)$ and $\alpha_2(t)$ are found from Eq. (16.4.17) as

$$\begin{aligned} U_i(t) &= \alpha_1 + \alpha_2 r_i \\ U_j(t) &= \alpha_1 + \alpha_2 r_j. \end{aligned} \tag{16.4.18}$$

Solving for α_1 and α_2 and substituting in Eq. (16.4.17) yields

$$u(\eta,t) = \frac{L-\eta}{L}\, U_i + \frac{\eta}{L} U_j \tag{16.4.19}$$

in which η is the variable in local coordinates, $\eta = r - r_i$, and L is the element length $L = r_j - r_i$. Defining the linear shape functions as

$$N_i = \frac{L-\eta}{L} \quad N_j = \frac{\eta}{L} \tag{16.4.20}$$

the displacement u is allowed to vary linearly in the base element (e) as

$$u^{(e)}(\eta, t) = N_i U_i + N_j U_j = \langle N_i \quad N_j \rangle \begin{Bmatrix} U_i \\ U_j \end{Bmatrix} = \langle N \rangle^{(e)} \, \{U\}^{(e)}. \tag{16.4.21}$$

Similarly, the temperature variation in the element (e) is assumed to vary linearly.

$$T^{(e)}(\eta, t) = N_i T_i + N_j T_j = \langle N \rangle^{(e)} \, \{T\}^{(e)} \tag{16.4.22}$$

Using Eqs. (16.4.21) and (16.4.22) and applying the formal Galerkin method to the governing equations (16.4.12) and (16.4.13) for the base element (e) yields

$$\int_{r_i}^{r_j} \left[\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r} + \frac{2u}{r}\right) - \frac{\partial T}{\partial r} - \frac{\partial^2 u}{\partial t^2} \right] r^2 N_m dr = 0 \tag{16.4.23}$$

$$\int_{r_i}^{r_j} \left[\frac{\partial}{\partial r}\left(\frac{\partial T}{\partial r}\right) + \frac{2}{r}\frac{\partial T}{\partial r} - \frac{\partial T}{\partial t} - C\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial t}\right) \right] r^2 N_m dr = 0$$
$$m = i, j. \tag{16.4.24}$$

Considering the change of variable $\eta = r - r_i$ and applying the weak formulation to the terms of second-order derivatives gives

$$\int_0^L \frac{\partial[(\eta + r_i)^2 N_m]}{\partial \eta} \frac{\partial u}{\partial \eta} d\eta - \int_0^L (\eta + r_i)^2 N_m \left(\frac{2}{(\eta + r_i)} \frac{\partial u}{\partial \eta} - \frac{2u}{(\eta + r_i)^2} \right) d\eta$$
$$+ \int_0^L (\eta + r_i)^2 N_m \frac{\partial T}{\partial \eta} d\eta + \int_0^L (\eta + r_i)^2 N_m \ddot{u} d\eta = (\eta + r_i)^2 N_m \frac{\partial u}{\partial \eta} |_0^L \tag{16.4.25}$$

$$\int_0^L \frac{\partial[(\eta + r_i)^2 N_m]}{\partial \eta} \frac{\partial T}{\partial \eta} d\eta - \int_0^L 2(\eta + r_i) N_m \frac{\partial T}{\partial \eta} d\eta + \int_0^L (\eta + r_i)^2 N_m \dot{T} d\eta$$
$$+ C \int_0^L N_m \frac{\partial}{\partial \eta}[(\eta + r_i)^2 \dot{u}] d\eta = (\eta + r_i)^2 N_m \frac{\partial T}{\partial \eta} |_0^L$$
$$m = i, j. \tag{16.4.26}$$

Substituting the shape functions for u and T from Eqs. (16.4.21) and (16.4.22) yields

$$\int_0^L \left(\frac{d[(\eta + r_i)^2 N_m]}{d\eta} \left\langle \frac{dN}{d\eta} \right\rangle - 2N_m[(\eta + r_i)\left\langle \frac{dN}{d\eta} \right\rangle - \langle N \rangle] \right) d\eta \{U\}$$
$$+ \int_0^L (\eta + r_i)^2 N_m \left\langle \frac{dN}{d\eta} \right\rangle d\eta \{T\} + \int_0^L (\eta + r_i)^2 N_m \, \langle N \rangle \, d\eta \{\ddot{U}\}$$
$$= (\eta + r_i)^2 N_m \frac{\partial u}{\partial \eta} |_0^L \tag{16.4.27}$$

$$\int_0^L \left(\frac{d[(\eta + r_i)^2 N_m]}{d\eta} \left\langle \frac{dN}{d\eta} \right\rangle - 2(\eta + r_i) N_m \left\langle \frac{dN}{d\eta} \right\rangle \right) d\eta \{T\}$$

$$+\int_0^L (\eta + r_i)^2 N_m \langle N \rangle \, d\eta \{\dot{T}\} + C \int_0^L N_m \left\langle \frac{d[(\eta + r_i)^2 N]}{d\eta} \right\rangle d\eta \{\dot{U}\}$$
$$= (\eta + r_i)^2 N_m \frac{dT}{d\eta}|_0^L.$$
$$m = i, j \qquad (16.4.28)$$

The terms on the right-hand side of Eqs. (16.4.27) and (16.4.28) are derived through the weak formulation and coincide with the natural boundary conditions. They cancel each other out between any two adjacent elements except the first node of the first element and the last node of the last element, which coincide with the given boundary conditions on the inside and outside surfaces of the sphere. These boundary conditions are

$$\begin{aligned}
-a^2 \frac{\partial u}{\partial \eta}|_1 &= 2a\gamma_2 U_1 - a^2 T_1 + a^2 \frac{1-2\nu}{E\alpha T_0} P_a(t) \\
U_M &= 0 \\
T_1 &= 1 + (t^2 - t - 1\,)\, e^{-t} \\
b^2 \frac{\partial T}{\partial \eta}|_M &= -b^2 \eta_o T_M
\end{aligned} \qquad (16.4.29)$$

in which the index 1 denotes the first note of the first element of the solution domain at $r = a$ and the index M denotes the last node of the last element of the solution domain at $r = b$. It is to be noted that due to the assumed boundary conditions at $r = a$ and $r = b$, terms $-a^2 \frac{\partial T}{\partial \eta}|_1$ and $b^2 \frac{\partial u}{\partial \eta}|_M$ vanish. Equations (16.4.27) and (16.4.28) are solved for the nodal unknown of the element (e) and are finally arranged in the form of the following matrix equations:

$$[M]\{\ddot{\Delta}\} + [C]\{\dot{\Delta}\} + [K]\{\Delta\} = \{F\}. \qquad (16.4.30)$$

The definitions of the mass, damping, stiffness, and force matrices of Eq. (16.4.30) for the base element (e) are

Mass matrix

$$[M]^{(e)} = \begin{bmatrix} m_{11} & 0 & m_{13} & 0 \\ 0 & 0 & 0 & 0 \\ m_{31} & 0 & m_{33} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad (16.4.31)$$

in which the components of the mass matrix are

$$\begin{aligned}
m_{11} &= \int_0^L (\eta + r_i)^2 N_i N_i d\eta \\
m_{13} &= \int_0^L (\eta + r_i)^2 N_i N_j d\eta \\
m_{31} &= \int_0^L (\eta + r_i)^2 N_j N_i d\eta \\
m_{33} &= \int_0^L (\eta + r_i)^2 N_j N_j d\eta
\end{aligned} \qquad (16.4.32)$$

Damping matrix

$$[C]^{(e)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ C_{21} & C_{22} & C_{23} & C_{24} \\ 0 & 0 & 0 & 0 \\ C_{41} & C_{42} & C_{43} & C_{44} \end{bmatrix} \qquad (16.4.33)$$

in which the components of the damping matrix are

$$\begin{aligned}
C_{21} &= C\int_0^L N_i \frac{d[(\eta+r_i)^2 N_i]}{d\eta} d\eta \\
C_{23} &= C\int_0^L N_i \frac{d[(\eta+r_i)^2 N_j]}{d\eta} d\eta \\
C_{41} &= C\int_0^L N_j \frac{d[(\eta+r_i)^2 N_i]}{d\eta} d\eta \\
C_{43} &= C\int_0^L N_j \frac{d[(\eta+r_i)^2 N_j]}{d\eta} d\eta \\
C_{22} &= \int_0^L (\eta+r_i)^2 N_i N_i d\eta \\
C_{24} &= \int_0^L (\eta+r_i)^2 N_i N_j d\eta \\
C_{42} &= \int_0^L (\eta+r_i)^2 N_j N_i d\eta \\
C_{44} &= \int_0^L (\eta+r_i)^2 N_j N_j d\eta
\end{aligned} \tag{16.4.34}$$

Stiffness matrix

$$[K]^{(e)} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ 0 & K_{22} & 0 & K_{24} \\ K_{31} & K_{32} & K_{33} & K_{34} \\ 0 & K_{42} & 0 & K_{44} \end{bmatrix} d\eta \tag{16.4.35}$$

in which the components of the stiffness matrix are

$$\begin{aligned}
K_{11} &= \int_0^L \left(\frac{d[(\eta+r_i)^2 N_i]}{d\eta}\frac{dN_i}{d\eta} - 2(\eta+r_i)N_i\frac{dN_i}{d\eta} + 2N_i N_i)\right) d\eta \\
K_{13} &= \int_0^L \left(\frac{d[(\eta+r_i)^2 N_i]}{d\eta}\frac{dN_j}{d\eta} - 2(\eta+r_i)N_i\frac{dN_j}{d\eta} + 2N_i N_j)\right) d\eta \\
K_{31} &= \int_0^L \left(\frac{d[(\eta+r_i)^2 N_j]}{d\eta}\frac{dN_i}{d\eta} - 2(\eta+r_i)N_j\frac{dN_i}{d\eta} + 2N_j N_i)\right) d\eta \\
K_{33} &= \int_0^L \left(\frac{d[(\eta+r_i)^2 N_j]}{d\eta}\frac{dN_j}{d\eta} - 2(\eta+r_i)N_j\frac{dN_j}{d\eta} + 2N_j N_j)\right) d\eta \\
K_{12} &= \int_0^L (\eta+r_i)^2 N_i \frac{dN_i}{d\eta} d\eta \\
K_{14} &= \int_0^L (\eta+r_i)^2 N_i \frac{dN_j}{d\eta} d\eta \\
K_{32} &= \int_0^L (\eta+r_i)^2 N_j \frac{dN_i}{d\eta} d\eta \\
K_{34} &= \int_0^L (\eta+r_i)^2 N_j \frac{dN_j}{d\eta} d\eta
\end{aligned}$$

$$K_{22} = \int_0^L \left(\frac{d[(\eta + r_i)^2 N_i]}{d\eta} \frac{dN_i}{d\eta} - 2(\eta + r_i) N_i \frac{dN_i}{d\eta} \right) d\eta$$
$$K_{24} = \int_0^L \left(\frac{d[(\eta + r_i)^2 N_i]}{d\eta} \frac{dN_j}{d\eta} - 2(\eta + r_i) N_i \frac{dN_j}{d\eta} \right) d\eta$$
$$K_{42} = \int_0^L \left(\frac{d[(\eta + r_i)^2 N_j]}{d\eta} \frac{dN_i}{d\eta} - 2(\eta + r_i) N_j \frac{dN_i}{d\eta} \right) d\eta$$
$$K_{44} = \int_0^L \left(\frac{d[(\eta + r_i)^2 N_j]}{d\eta} \frac{dN_j}{d\eta} - 2(\eta + r_i) N_j \frac{dN_j}{d\eta} \right) d\eta \tag{16.4.36}$$

Also, matrix $\{F\}$, for this special case, is related to the boundary conditions. With the inside temperature and pressure shocks and outside surface insulated, as given by Eq. (16.4.29), the final assembled form of this matrix becomes

$$\{F\} = \begin{Bmatrix} 2a\gamma_2 U_1 - a^2 T_1 + a^2 \frac{1-2\nu}{E\alpha T_0} P_a(t) \\ 1 + (t^2 - t - 1)\, e^{-t} \\ 0 \\ 0 \\ . \\ . \\ 0 \\ 0 \\ -b^2 \eta_o T_M \end{Bmatrix}. \tag{16.4.37}$$

Moreover, the matrix of unknown nodal values is

$$\{\Delta\}^{(e)} = \begin{Bmatrix} U_i \\ T_i \\ U_j \\ T_j \end{Bmatrix} \tag{16.4.38}$$

Using the linear shape functions (16.4.21) and (16.4.22) for $\{U\}$ and $\{T\}$, the component of the matrices are simplified as

Components of mass matrix

$$m_{11} = \frac{L(L^2 + 5r_i L + 10r_i^2)}{30}$$
$$m_{13} = m_{31} = \frac{L(3L^2 + 10r_i L + 10ri^2)}{60}$$
$$m_{33} = \frac{L(6L^2 + 15r_i L + 10r_i^2)}{30} \tag{16.4.39}$$

Components of damping matrix

$$C_{21} = C\frac{(L^2 + 4r_i L - 6r_i^2)}{12}$$
$$C_{23} = C\frac{(3L^2 + 8r_i L + 6r_i^2)}{12}$$
$$C_{41} = -C\frac{(L^2 + 4r_i L + 6r_i^2)}{12}$$

$$
\begin{aligned}
C_{43} &= C\frac{(9L^2 + 16r_iL + 6r_i^2)}{12} \\
C_{22} &= L\frac{(L^2 + 5r_iL + 10r_i^2)}{30} \\
C_{24} = C_{42} &= L\frac{(3L^2 + 10r_iL + 10r_i^2)}{60} \\
C_{44} &= L\frac{(6L^2 + 15r_iL + 10r_i^2)}{30}
\end{aligned}
\tag{16.4.40}
$$

Components of stiffness matrix

$$
\begin{aligned}
K_{11} = K_{33} &= \frac{L^2 + r_iL + r_i^2}{L} \\
K_{13} = K_{31} &= -\frac{r_i(L + r_i)}{L} \\
K_{12} = -K_{14} &= -\frac{1}{12}L^2 - \frac{1}{3}r_iL - \frac{1}{2}r_i^2 \\
K_{32} = -K_{34} &= -\frac{1}{4}L^2 - \frac{2}{3}r_iL - \frac{1}{2}r_i^2 \\
K_{22} = K_{44} = -K_{24} = -K_{42} &= \frac{L^2 + 3r_iL + 3r_i^2}{3L}
\end{aligned}
\tag{16.4.41}
$$

The element matrices given by Eqs. (16.4.34)–(16.4.36) are generated within a loop to construct the general matrices of Eq. (16.4.30), where after assembly of all the elements in the solution domain, they are solved using one of the numerical techniques of either the time marching or modal analysis methods.

As a numerical example, a thick sphere is considered with the following properties: $E = 70 \times 10^9$ N/m^2, $\nu = 0.3$, $\rho = 2707$ Kg/m^3, $k = 204$ W/m-K, $\alpha = 23 \times 10^{-6}$ 1/K, $c = 903$ J/Kg-K, $T_0 = 298$ K. The pressure at the inner surface of the sphere is assumed to be zero (traction free condition) and the outer surface of the sphere is insulated (with $h_o = 0$). The plot of the internal thermal shock is shown in Fig. 16.8. The distribution of temperature, radial displacement, radial stress, and hoop stress at different times are plotted in Figs. 16.9, 16.10, 16.11 and 16.12. Figures 16.12 and 16.13 show the variations of radial and hoop stresses versus the radius at different times. Figure 16.13 shows the time variation of the radial and hoop stresses and temperature at the mid-point of the thickness of the sphere.

16.5 Higher Order Elements

Chen and Lin [18] proposed a hybrid numerical method based on the Laplace transform and control volume method for analyzing the transient coupled thermoelastic problems with relaxation times involving a nonlinear radiation boundary condition. Hosseini and Eslami [19] considered the boundary element formulation for the analysis of coupled thermoelastic problems in a finite domain and studied the coupling coefficient and relaxation times effects on thermal and elastic wave propagations.

In this section, a transfinite element method using the Laplace transform is used to solve the coupled equations for an axisymmetrically loaded disk in the transformed domain. Elements with various orders are employed to investigate the effects of the number of nodes in an element. Finally, the temperature and displacement are inverted to obtain the actual physical quantities, using the numerical inversion of the Laplace transform method proposed by Honig and Hirdes [25].

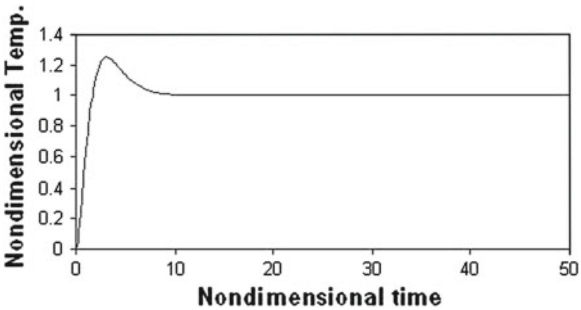


Fig. 16.8 The temperature shock applied to the inner surface of the sphere

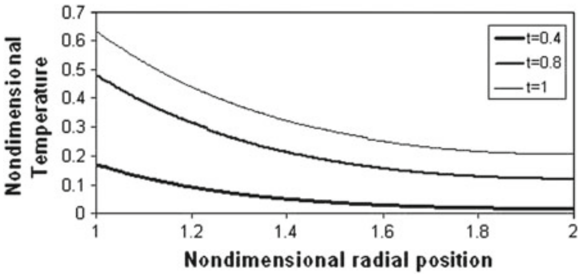


Fig. 16.9 The temperature distribution at different times

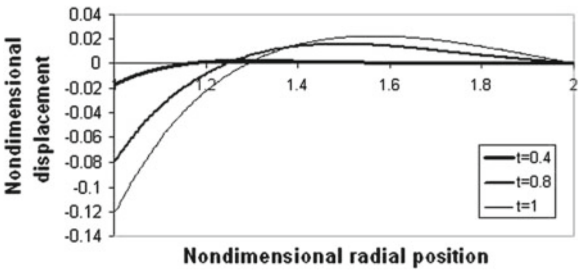


Fig. 16.10 The displacement distribution at different times

In the absence of the heat source and body forces and for isotropic materials, the nondimensionalized form of the generalized coupled thermoelastic equations of the axisymmetrically loaded circular disk based on the Lord-Shulman theory in terms of the displacement and temperature may be written as [30]

$$\left\{\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}-\frac{\partial^2}{\partial t^2}\right\}u-\frac{\partial T}{\partial r}=0 \tag{16.5.1}$$

$$\left\{\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{\partial}{\partial t}\left(1+t_0\frac{\partial}{\partial t}\right)\right\}T-C\left\{t_0\left[\frac{\partial^3}{\partial r\partial t^2}+\frac{1}{r}\frac{\partial^2}{\partial t^2}\right]+\frac{\partial^2}{\partial r\partial t}+\frac{1}{r}\frac{\partial}{\partial t}\right\}u=0. \tag{16.5.2}$$

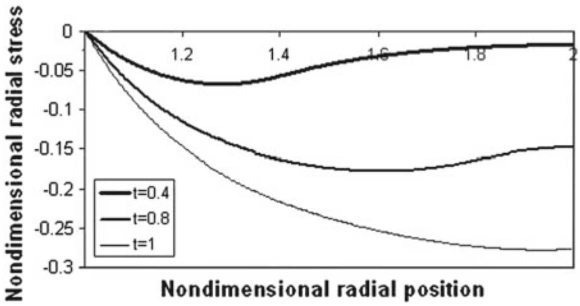


Fig. 16.11 The radial stress distribution at different times

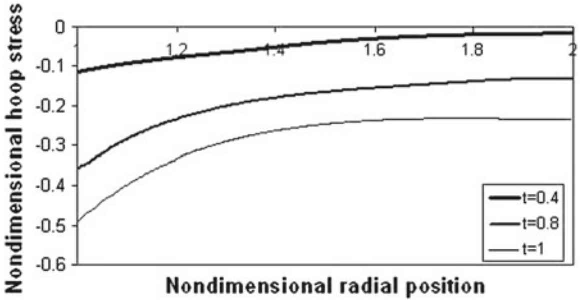


Fig. 16.12 The hoop stress distribution at different times

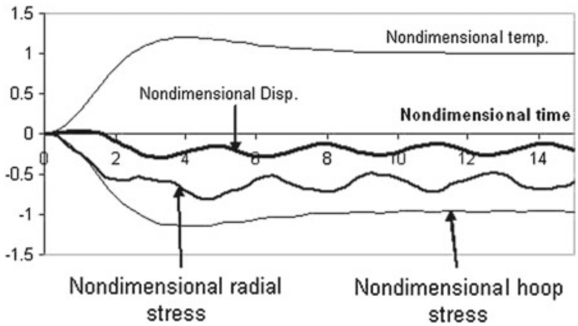


Fig. 16.13 Time history of temperature, displacement and stresses at the mid-radius point of the thickness of the sphere

Here, $C = T_0\bar{\beta}^2/\left[\rho c_e\left(\bar{\lambda} + 2\mu\right)\right]$ is the coupling coefficient. For the plane stress condition, $\bar{\lambda} = \frac{2\mu}{\lambda+2\mu}\lambda$ and $\bar{\beta} = \frac{2\mu}{\lambda+2\mu}\beta$. In the preceding Equations, $\rho,\ u,\ T_0,\ T,\ \bar{\beta},\ c_e$, and t_0 are the density, radial displacement, reference temperature, temperature change, stress-temperature moduli, thermal conductivity, specific heat and the relaxation time (proposed by Lord and Shulman), respectively, while λ and μ are Lamé constants. The dimensionless thermal and mechanical boundary conditions are

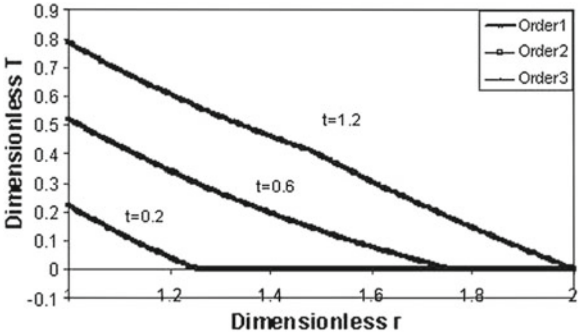


Fig. 16.14 Distribution of the dimensionless temperature along the radius of the disk at three values of the time for three types of elements

$$q_{in} = -\frac{\partial T}{\partial r}; \quad u = 0 \qquad\qquad at \quad r = a$$
$$T = 0; \qquad \sigma_{rr} = \frac{\partial u}{\partial r} + \frac{\bar{\lambda}}{\bar{\lambda} + 2\mu}\frac{u}{r} - T = 0 \qquad at \quad r = b \tag{16.5.3}$$

in which σ_{rr} and a and b are the radial stress and dimensionless inner and outer radii, respectively.

In order to derive the transfinite element formulation, the Laplace transformation is used to transform the equations into the Laplace transform domain. Applying the Galerkin finite element method to the governing equations (16.5.1) and (16.5.2) for the base element (e), yields

$$\int_0^L \left\{ -\left[\left\{ \frac{1}{(\eta + r_i)}\frac{\partial}{\partial \eta} - \frac{1}{(\eta + r_i)^2} - s^2 \right\} u - \frac{\partial T}{\partial \eta} \right] N_m(\eta + r_i) + \frac{\partial\left(N_m(\eta + r_i)\right)}{\partial \eta}\frac{\partial u}{\partial \eta} \right\} d\eta$$
$$= N_m(\eta + r_i)\frac{\partial u}{\partial \eta}\bigg|_0^L \tag{16.5.4}$$

$$\int_0^L \left\{ -\left[\left\{ \frac{1}{(\eta + r_i)}\frac{\partial}{\partial \eta} - s(1 + t_0 s) \right\} T - C\left(t_0 s^2 + s\right)\left(\frac{\partial}{\partial \eta} + \frac{1}{\eta + r_i}\right) u \right]\right.$$
$$\left. \times N_m(\eta + r_i) + \frac{\partial\left(N_m(\eta + r_i)\right)}{\partial \eta}\frac{\partial T}{\partial \eta} \right\} d\eta = N_m(\eta + r_i)\frac{\partial T}{\partial \eta}\bigg|_0^L \tag{16.5.5}$$

in which $u = \sum_{m=1}^{n} N_m U_m$ and $T = \sum_{m=1}^{n} N_m T_m$. In the preceding equations, s, N_m, $\eta = r - r_i$, r_i, L, U_m, and T_m are the Laplace parameter, shape function, local coordinates, the radius of the i-th node of the base element, the length of element in the radial direction, nodal displacement, and the nodal temperature, respectively. The terms on the right-hand sides of Eqs. (16.5.4) and (12.5.5) cancel each other out between any two adjacent elements, except the nodes located on the boundaries of the solution domain. These boundary conditions are

$$-a\frac{\partial T}{\partial \eta}\bigg|_1 = a q_{in}; \quad U_1 = 0$$
$$T_M = 0; \qquad b\frac{\partial u}{\partial \eta}\bigg|_M = -\frac{\bar{\lambda}}{\bar{\lambda} + 2\mu} U_M + b T_M \tag{16.5.6}$$

The subscripts 1 and M refer to the first and last nodes of the solution domain, respectively.

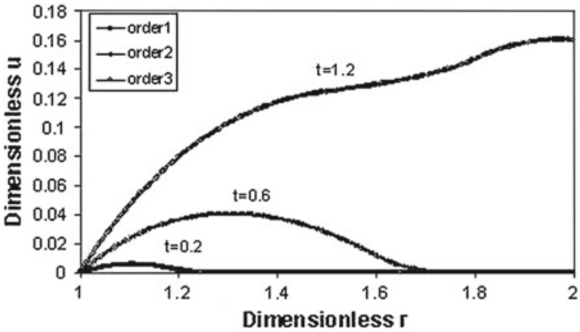
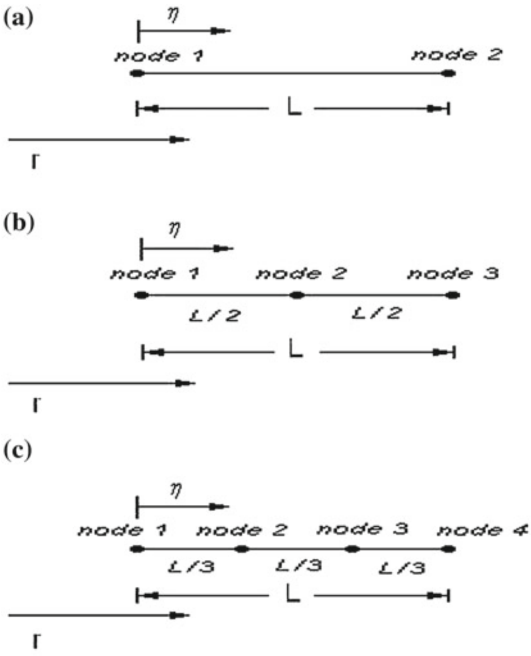


Fig. 16.15 Distribution of the dimensionless displacement along the radius of the disk at three values of the time for three types of elements



$$N_1(\eta) = 1 - \frac{\eta}{L}$$

$$N_2(\eta) = \frac{\eta}{L}$$

$$N_1(\eta) = (1 - 2\frac{\eta}{L})(1 - \frac{\eta}{L})$$

$$N_2(\eta) = 4\frac{\eta}{L}(1 - \frac{\eta}{L})$$

$$N_3(\eta) = -\frac{\eta}{L}(1 - 2\frac{\eta}{L})$$

$$N_1(\eta) = (1 - 3\frac{\eta}{L})(1 - \frac{3}{2}\frac{\eta}{L})(1 - \frac{\eta}{L})$$

$$N_2(\eta) = 9\frac{\eta}{L}(1 - \frac{3}{2}\frac{\eta}{L})(1 - \frac{\eta}{L})$$

$$N_3(\eta) = -\frac{9}{2}\frac{\eta}{L}(1 - 3\frac{\eta}{L})(1 - \frac{\eta}{L})$$

$$N_4(\eta) = \frac{\eta}{L}(1 - 3\frac{\eta}{L})(1 - \frac{3}{2}\frac{\eta}{L})$$

Fig. 16.16 Elements with linear, second, and third order shape functions

To investigate the accuracy of the method, a numerical example is considered. The material of the disk is assumed to be aluminum. The dimensionless inside and outside radii are $a = 1$ and $b = 2$. The dimensionless input heat flux is defined as the Heaviside unit step function. Since the applied boundary conditions are assumed to be axisymmetric, the radius of the disk is divided into 100 elements. Three types of shape functions, linear, second order, and third order polynomials are used for the finite element model of the problem. Results for each of these orders are plotted and compared.

Figures 16.14 and 16.15 show the wave propagation of the temperature and radial displacement along the radial direction [26]. The numerical values of the coupling parameter and the dimensionless relaxation time are assumed to be 0.01 and 0.64, respectively. The wave propagation is shown at several times. Two wave fronts for elastic and temperature waves are detected from the figures, as expected from the LS model. It is seen from the figures that the results of the three types of shape functions (Fig. 16.16) for the assumed number of elements coincide. For a smaller number of elements, the difference between the results obtained for different shape functions increases noticeably. For the assumed number of elements, the curves for radial displacement and temperature distribution are checked against the known data in the literature, in which very close agreement is observed. Figure 16.14 clearly shows the temperature wave front (the second sound effect), which is propagating along the radius of the disk.

16.6 Problems

1. Verify Eqs. (16.2.46) to (16.2.49) for the coupled thermoelasticity of a one-dimensional rod with a simplex linear element.

2. Consider a one-dimensional element, for which the axial displacement u and the temperature change θ are approximated in the element by a quadratic polynomial. Obtain the mass, damping, stiffness, and force matrices.

3. Use a C^1-continuous straight element in terms of the natural coordinates, and rework Problem 2.

4. Use Eqs. (16.2.27) to (16.2.31) and employ a linear simplex triangular element to derive the members of the mass, damping, stiffness, and force matrices. To derive these matrices, use the area coordinates. What is the determinant of the Jacobian matrix?.

5. Consider a rod of length L thermally insulated along its length. The initial temperature at $x = L$ is suddenly raised by
$$T(L, t) = T_0 e^{(-t/t_0)}.$$
Divide the rod into two elements with linear shape functions for the displacement and temperature change. Find the temperature and displacement at the nodal points for the fixed boundary conditions at $x = 0$ and at the free end $x = L$.

6. A rod of length L is considered. The heat is generated along the rod by the following equation:
$$Q(x, t) = Q_1(t) \cos (x/L).$$
The end $x = 0$ is fixed, but the end $x = L$ is free. Divide the rod into two elements with linear shape functions for the displacement and temperature change. Find the temperature and displacement at the nodal points.

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Chapter 17
Computer Programs

Abstract In this chapter, three computer codes are presented. These codes are written to solve the elevation of elastic membrane under static load, the static elasticity, and the three-dimensional transient heat conduction problems. The descriptions and de- tails of the processor and postprocessor for each computer program is presented in this chapter.

17.1 Description of the Membrane Computer Program

In the following, the different parts of the membrane program, are divided into three main categories (preprocessing, processing, and postprocessing) and discussed separately. The program is written in a C++ environment.

17.1.1 Preprocessor

In the preprocessor unit, five different options are considered for use:

1. The subprogram *mesh_rd_3nte.cpp*, which generates the mesh for a rectangular domain by three node triangular elements. The inputs to this program are the length and width of the rectangular domain and the number of divisions in each direction.
2. The subprogram *mesh_rd_6nte.cpp*, which generates the mesh for a rectangular domain by six node triangular elements. The inputs to this program are the length and width of the rectangular domain and the number of divisions in each direction.
3. The subprogram *mesh_td_3nte.cpp*, which generates the mesh for a right-angled triangular domain by three node triangular elements. The inputs to this program are the dimensions of the triangle and the number of divisions, which is assumed to be equal in both directions.

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Solid Mechanics and Its Applications 216, DOI: 10.1007/978-3-319-08037-6_17,

4. The subprogram *mesh_td_6nte.cpp*, which generates the mesh for a right-angled triangular domain by six node triangular elements. The inputs to this program are the dimensions of the triangle and the number of divisions, which is assumed to be equal in both directions.
5. The subprogram *mesh_ed_3nte.cpp*, which generates the mesh for an elliptic domain by three node triangular elements. The inputs to this program are dimensions of the ellipse and the number of divisions in the radial direction, and also the number of divisions in the angular direction for the first layer of elements around the ellipse center.

In implementation of the skyline method, it is necessary to compute the height of each column separately and then use these heights to find the address of different terms in the stiffness matrix. The individual heights of columns are computed in the subprogram *bandwidth.cpp*.

17.1.2 Processor

The processor unit is programmed for two different situations. In the first situation, the stiffness matrix is assumed in banded symmetric form. The boundary conditions are then imposed on this banded form of the stiffness matrix, and then the Gauss elimination procedure is invoked to solve it. In the second situation, the concept of skyline is invoked and the nonzero elements of the stiffness matrix are stored in an array which is more compacted in dimension. The boundary conditions are then imposed on this array, and then the LU decomposition procedure is implemented to obtain the unknown primary variables. The processor unit, where the largest amount of computing time is spent, consists of three parts:

1. A subprogram for calculating element matrices and assembling them in the global matrix. This is done in the subprogram *core-3nte.cpp* for the three node triangular element and in the subprogram *core_6nte.cpp* for the six node triangular element. The assembly of element matrices is carried out as soon as they are computed, rather than waiting until element matrices of all the elements are computed.
2. Imposition of boundary conditions on primary variables. This is done in the subprogram *boundary_sym_banded.cpp* for a banded symmetric form and in the subprogram *boundary_sym_skyline.cpp* for a skyline symmetric form.
3. Solving the system of linear equations to obtain the unknown primary variables. Similar to the previous step, two different subprograms are considered for this purpose. The first subprogram *solve_gauss_sym_banded.cpp* uses the Gauss elimination procedure for a banded symmetric form. The second subprogram *solve_LUdecom_sym_skyline.cpp* uses the LU decomposition procedure for a skyline symmetric form.

An important aspect of the computer implementation of the Gauss elimination solution procedure is that a minimum solution time should be used. In addition, the

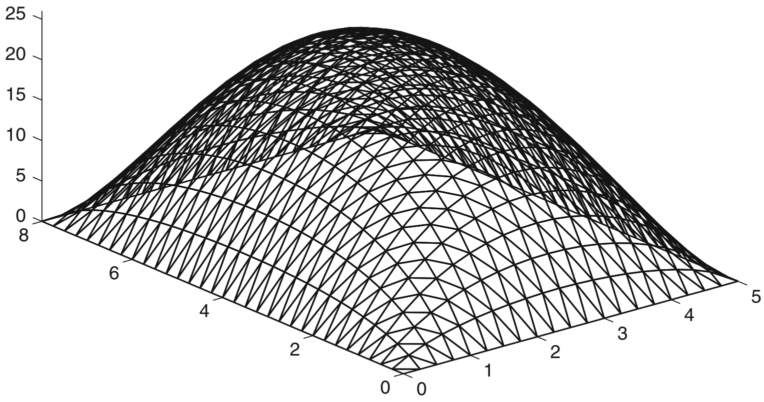

Fig. 17.1 Deformed shape of the membrane for the three node triangular mesh of a rectangular domain

high-speed storage requirements should be as small as possible to avoid the use of backup storage. The use of the skyline method, in spite of its greater complexity, could be justified in the cases in which not only out-of-band elements of the stiffness matrix are zero, but many elements inside the band are zero too.

17.1.3 Postprocessor

After solving the linear system of algebraic equations in the processor unit, it follows that one should print or plot the results in a convenient format. In the postprocessor unit, the results are written on certain files to be read by convenient software (e.g., MATLAB or MATHEMATICA) and plotted in two and three-dimensional form.The short subprogram *file_out.cpp* does this job in a C++ environment and writes the results on some files. A sample of the results is shown in Fig. 17.1.

17.2 Description of the Static Elasticity Computer Program

Like the first computer program, the different parts of the static elasticity program are divided into three categories (preprocessing, processing, and postprocessing) and discussed separately. The program is written in a C++ environment.

17.2.1 Preprocessor

In the preprocessor unit, two different options are considered for use:

1. The subprogram *mesh_LRE.cpp*, which generates the mesh for a rectangular domain by four node rectangular elements. The inputs to this program are the length and width of the rectangular domain and the number of divisions in each direction.
2. The subprogram *mesh_serendipity.cpp*, which generates the mesh for a rectangular domain by eight node rectangular elements (called serendipity elements in the literature). The inputs to this program are the length and width of the rectangular domain and the number of divisions in each direction.

17.2.2 Processor

The stiffness matrix is assumed in banded symmetric form. The boundary conditions are then imposed on this banded form of the stiffness matrix, and then the Gauss elimination procedure is invoked to solve it. The processor unit, where the largest amount of computing time is spent, consists of three parts:

1. A subprogram for calculating element matrices and assembling them in the global matrix. This is done in the subprogram *core_LRE.cpp* for the four node rectangular element and in the subprogram *core_serendipity.cpp* for the eight node rectangular element. The assembly of element matrices is carried out as soon as they are computed, rather than waiting until element matrices of all the elements are computed.
2. Imposition of boundary conditions on primary variables. This is done in the subprogram *boundary_sym_banded.cpp* for the banded symmetric form of the stiffness matrix.
3. Solving the system of linear equations to obtain the unknown primary variables. The subprogram *solve_gauss_sym_banded.cpp* uses the Gauss elimination procedure for the banded symmetric form of the stiffness matrix.

17.2.3 Postprocessor

After solving the linear system of algebraic equations in the processor unit, it follows that one should print or plot the results in a convenient format. In the postprocessor unit, the results are written on certain files to be read by convenient software (e.g., MATLAB or MATHEMATICA) and plotted in two and three-dimensional form. The short subprogram *file_out.cpp* does this job in a C++ environment and writes the results on some files. A sample of the results is shown in Fig. 17.2.

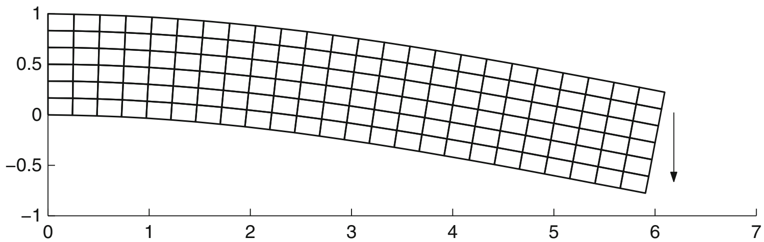

Fig. 17.2 Deformed shape of a cantilever beam under uniform vertical shear load at the right edge

17.3 Description of the 3D Transient Heat Conduction Computer Program

This computer program solves three-dimensional transient heat conduction by using sparse matrices and iterative solvers. The C++ standard template library (STL) and the concept of classes are used in this computer code.

For three-dimensional problems, the region inside the bands of the system matrices is generally more sparse than in two-dimensional problems. Also, the number of nodes is generally far greater than in two-dimensional cases ($n \times n \times n$ vs. $n \times n$). These two reasons motivate the use of sparse matrices instead of banded or skyline matrices for three-dimensional cases. In sparse matrices, only those entries of the system matrices that are nonzero are stored, and the way of numbering the DOFs and arranging them in the vector of unknowns, makes *no* difference in the size of data (in contrast to banded and skyline formats). In general, the data storage volume for sparse matrices is considerably smaller than that for banded matrices (even in the case in which the DOFs are numbered in a way so as to produce a very narrow band in the system matrices). This makes sparse formats very suitable for models involving a very large number of degrees of freedom.

There are different formats for storing a sparse matrix (coordinate format, compressed row and compressed column). In this study, the coordinate format is that in which three different vectors are used for representing a sparse matrix. One vector for storing the row index of nonzero entries, one vector for the column index and one vector for the values of nonzero entries. Before constructing these vectors, the number of nonzero entries should be known. The *map* class available in a C++ standard template library is used for this purpose to map the pair of indexes of a nonzero entry on its location in the three vectors mentioned earlier. This mapping is also used to search for nonzero entries in the assembling process from local matrices onto global matrices. After performing assemblies, the mapping has no job and is deleted, and only the three vectors mentioned earlier are representative of the sparse matrix structure. Since the dimensions of these vectors is specified in the run time of the code, two vector classes are defined in the code the dimensions of which are given by non-constant integers calculated during the run time of the code. These vector classes use the dynamic memory (heap segment) to store vectors of types *int* and *double*. It is possible to use mapping directly in all sections of the code without developing such vector representation of sparse matrices, but the problem is that the computational time increases drastically by doing so.

In the next stage, these sparse matrices should be given to a solver subprogram to obtain the responses. Since the Gauss elimination solver deteriorates the sparse structure of the system matrices in the region inside the bands, it would not be a good choice for solving equations with sparse matrices. There is another class of solvers called iterative solvers, in contrast to the Gauss elimination method, which is a direct solver. Iterative solvers have the advantage that they do *not* alter the sparse structure of the matrices. In other words, when using iterative solvers, only nonzero entries are manipulated and zero entries remain zero through the solution process. There are different iterative algorithms available for solving symmetric or non-symmetric matrices. In this study, we chose the Conjugate Gradient Method, which can be used for solving a system of equations with a symmetric positive definite coefficient matrix. Most finite element models give matrices of this type. To improve the convergence rate of an iterative solver, it is common to use a matrix called a preconditioner to transform the coefficient matrix into a form that has more convenient spectral properties. In this study, the Jacobi preconditioner is used, the simplest preconditioner for iterative solvers. More sophisticated ones are more efficient.

The pseudo-code for the Preconditioned Conjugate Gradient Method is given in the following box. It uses a preconditioner M. Further description of the method can be found in Ref. [10].

Compute $r^{(0)} = b - Ax^{(0)}$ for some initial guess $x^{(0)}$
for $i = 1, 2, \ldots$
  solve $Mz^{(i-1)} = r^{(i-1)}$
  $\rho_{i-1} = r^{(i-1)^T} z^{(i-1)}$
  if $i = 1$
    $p^{(1)} = z^{(0)}$
  else
    $\beta_{i-1} = \rho_{i-1}/\rho_{i-2}$
    $p^{(i)} = z^{(i-1)} + \beta_{i-1} p^{(i-1)}$
  endif
  $q^{(i)} = Ap^{(i)}$
  $\alpha_i = \rho_{i-1}/p^{(i)^T} q^{(i)}$
  $x^{(i)} = x^{(i-1)} + \alpha_i p^{(i)}$
  $r^{(i)} = r^{(i-1)} - \alpha_i q^{(i)}$
  check convergence; continue, if necessary
end

For the meshing process, two options are considered in this code. One is to use the subprogram *Mesh_Cube.cpp* to produce a tetrahedronal mesh for a cube, and another is to use gmesh software to produce the mesh for a 3D arbitrary shape of the solution domain. Since meshing a three-dimensional body is not a straightforward task (except

Fig. 17.3 A dodcahedronal solid that is warmed by sunshine

for very simple geometries like a cube), a convenient mesh generator software can be used to mesh a 3D body and write the mesh information on a file. This file can then be used by a finite element code to do the subsequent jobs. In this study, the gmesh software is chosen for this purpose. It is a very efficient software which is also capable of reading the results obtained from the finite element code and plotting some 2D and 3D figures of them in combination with the corresponding mesh information. In this code, by choosing the option for use of gmesh, the mesh information is first read from a file generated by gmesh. Then, different stages of calculations are performed by the code and the results are obtained. These results are then appended to the file that first contained the mesh information. After augmentation of this file by the results obtained from the finite element code, it is read by gmesh software and the results are plotted in certain figures. A sample of the results is shown in Fig. 17.3.

The different subprograms used in this code are as follows:

1. The subprogram *Det.cpp* is used for calculating the determinant of a 3×3 matrix, which is encountered frequently in this code.
2. The subprogram *Mesh_Cube.cpp* generates a tetrahedronal mesh for a cube if the gmesh software is not chosen for the mesh generation.
3. The subprogram *Core_LTE.cpp* calculates element damping and stiffness matrices for a tetrahedronal element (3D simplex element) and assembles these local matrices into the global ones.
4. The subprogram *Flux_LTE.cpp* calculates the right-hand side vector (force vector) in the matrix equation. Calculation of the force vector is separated from calculation of the element matrices because the surface heat flux and also the internal heat

source may change from one time step to another one and implementation of the subprogram *Flux_LTE.cpp* may need to be repeated in each time step of calculations.
5. The subprogram *Adjust.cpp* imposes essential boundary conditions.
6. The subprogram *Preconditioner* calculates the Jacobi preconditioner.
7. The subprogram *Conjugate_Gradient.cpp* solves the linear system of equations by the Conjugate Gradient Method.

It should be noted that the type of preconditioner used for an iterative solver has a noticeable effect on the computational time and the number of iterations required to reach a prescribed accuracy. As mentioned earlier, the Jacobi preconditioner, which is the simplest one, is used in this study. Calculating this preconditioner requires a very small computational time, but after that, the number of iterations in the solution process is large. In contrast, a more sophisticated preconditioner, like incomplete Cholesky factorization, needs more computational time to be Calculated, but then the number of the subsequent iterations are reduced. The sum of these two effects reduces the computational difference between different preconditioners for static problems (assuming the system matrix is well conditioned). But for transient and dynamic problems, the difference is large, since the preconditioner can be calculated once and then be used for all time steps. Therefore, a preconditioner like incomplete Cholesky factorization can be more efficient than the Jacobi preconditioner for transient linear problems. Improving the incomplete Cholesky factorization by using the filling technique can further reduce the computational time by trading between data storage volume and the computational time.

Some of the references that helped the author to prepare these three computer codes [1–11].

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