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Éric Gourgoulhon

Special Relativity in General Frames

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To Valérie and Maxime

Foreword

The theory of special relativity holds a distinctive place within physics. Rather than being a specific physical theory, it is (similar to thermodynamics or analytical mechanics) a general *theoretical framework* within which various dynamical theories can be formulated. In this respect, a modern presentation of special relativity must put forward its essential structures before illustrating them by concrete applications to specific dynamical problems. Such is the challenge (so successfully met!) of the beautiful book by Éric Gourgoulhon.

Contrary to most textbooks on special relativity, which mix the presentation of this theory with that of its historical development and which sometimes write the specific form of "Lorentz transformations" before indicating that they leave a certain quadratic form invariant, the book by Éric Gourgoulhon is centred, from the very beginning, on the essential structure of the theory, i.e. the chrono-geometric structure of the four-dimensional Poincaré–Minkowski spacetime. The aim is to train the reader to formulate any relativity question in terms of four-dimensional geometry. The word *geometry* has here the meaning of "synthetic geometry" (à la Euclid) in contrast with "analytic geometry" (à la Descartes). Under the expert guidance of Éric Gourgoulhon, the reader will learn to set, and to solve, any problem of relativity by drawing spacetime diagrams, made of curves, straight lines, planes, hyperplanes, cones and vectors. He will get accustomed to visualizing the motion of a particle as a line in spacetime, to think about the twin paradox as an application of the "spacetime triangle inequality", to express the local frame of an observer as a four-dimensional generalization of the Serret–Frenet triad, to compute a spatial distance as a geometric mean of time intervals (via the hyperbolic generalization of the power of a point with respect to a circle) or to understand the Sagnac effect by considering two helices in spacetime wound in opposite directions.

Besides the pedagogical characteristic of being centred on a geometric formulation, the book by Éric Gourgoulhon is remarkable in many other ways. First of all, it is fully up to date and very complete in its coverage of the notions and results where special relativity plays an important role: from Thomas precession to the foundations of general relativity, including tensor calculus, exterior differential calculus, classical electrodynamics, the general notion of energy–momentum tensor and a noteworthy chapter on relativistic hydrodynamics. In addition, this book is sprinkled with enlightening historical notes, in which the author summarizes in a condensed, albeit very informative way the (sometimes very recent) results by historians of science. Finally, the book is richly laden with many examples of applications of special relativity to concrete physical problems. The reader will learn the role of special relativity in various domains of modern astrophysics (supernova nebulae, relativistic jets, micro-quasars) in the description of the quark-gluon plasma produced in heavy ion collisions, as well as in many high-technology experiments: from laser gyrometers to the LHC, including modern replications of the Michelson–Morley experiment, matter wave interferometers, synchrotrons and their radiation, and the comparison of atomic clocks embarked on planes, satellites or the International Space Station.

I am sure that the remarkably rich book by Éric Gourgoulhon will attract the keen interest of many readers and will enable them to understand and master one of the fundamental pillars (with general relativity and quantum theory) of modern physics.

Bures-sur-Yvette, France Thibault Damour

Preface

This book presents a geometrical introduction to special relativity. By *geometrical*, it is meant that the adopted point of view is four dimensional from the very beginning. The mathematical framework is indeed, from the first chapter, that of *Minkowski spacetime*, and the basic objects are the vectors in this space (often called *4-vectors*). Physical laws are translated in terms of geometrical operations (scalar product, orthogonal projection, etc.) on objects of Minkowski spacetime (4-vectors, worldlines, etc.).

Many relativity textbooks start rather by a three-dimensional approach, using space + time decompositions based on inertial observers. Only in the second stage they introduce 4-vectors and Minkowski spacetime. In this respect, they are faithful to the historical development of relativity. A more axiomatic approach is adopted here, setting from the very beginning the full mathematical framework as one of the postulates of the theory. From this point of view, the chosen approach is similar to that adopted in classical mechanics or quantum mechanics, where usually the exposition does not follow the history of the theory. The history of relativity is undoubtedly rich and fascinating, but the objective of this book is the learning of special relativity within a consistent and operational setting, from the bases up to advanced topics. The text is, however, enriched with historical notes, which include references to the original works and to the studies by historians of science.

Usually, the geometric approach is reserved for *general relativity*, i.e. for the incorporation of the gravitational field in relativity theory.[1] We employ it here for special relativity, taking into account a geometric structure much simpler than that of general relativity: while the latter is based on the concept of *differentiable manifold*, special relativity relies entirely on the concept of *affine space*, which can be identified with the space $\mathbb{R}^4$. Consequently, the mathematical prerequisites are relatively limited; they are mostly linear algebra at the level of the first two years of university. The mathematics used here is actually the same as those of a course

[1]Two notable exceptions are the monographs by Costa de Beauregard (1949) and Synge (1956).

of classical mechanics, provided one is ready to take into account two things: (i) vectors do not belong to a linear space of dimension three, but four, and (ii) the scalar product of two vectors is not the standard scalar product in Euclidean space but is given by a privileged symmetric bilinear form, the so-called *metric tensor*. Once this is accepted, physical results are obtained faster than by means of the "classic" three-dimensional formulation, and a more profound understanding of relativity is acquired. Moreover, learning general relativity is made much easier, starting from such an approach.

In connection with the four-dimensional approach, another characteristic of this monograph is to lay the discussion of physically measurable effects on the most general type of observer, i.e. allowing for accelerated and rotating frames. On the opposite, most of (all?) special relativity treatises are based on a privileged class of observers: the *inertial* ones. Although it is true that for these observers the perception of physical phenomena is the simplest one (for instance, for an inertial observer, light in vacuum moves along a straight line and at a constant speed), the real world is made of accelerated and rotating observers. Therefore, it seems conceptually clearer to discuss first the measures performed by a generic observer and to treat afterwards the particular case of inertial observers. Conversely, if one restricts first to inertial observers, it becomes cumbersome to extend the discussion to general observers. As a matter of fact, this is to a great extent the source of the various "paradoxes" that appeared in the course of the development of relativity. As mentioned above, the three-dimensional approach to relativity is based on inertial observers, since one may associate with each observer of this kind a global decomposition of spacetime in a "time" part and a "space" part.

One of the consequences of the "general observer" approach adopted here is the least weight attributed to the famous *Lorentz transformation* between the frames of two inertial observers. This transformation, which is usually introduced in the first chapter of a relativity course, appears here only in Chap. 6. In particular, the physical effects of time dilation or aberration of light are derived (geometrically) in Chaps. 2 and 4, without appealing explicitly to the Lorentz transformation. Similarly, the *principle of relativity*, on which special relativity has been founded at the beginning of the twentieth century (hence its name!), is mentioned here only in Chap. 9, at the occasion of a historical note.

The plan of the book is as follows. The full mathematical framework (Minkowski spacetime) is set in Chap. 1. The concepts of worldline and proper time are then introduced (Chap. 2) and are illustrated by a detailed exposition of the famous "twin paradox". Chapter 3 is entirely devoted to the definition of an observer and his (local) rest space. This is done in the most general way, taking into account acceleration as well as rotation. The notion of observer being settled, we are in position to address kinematics. This is performed in two steps: (i) by fixing the observer in Chap. 4 (introduction of the Lorentz factor, as well as relative velocity and relative acceleration) and (ii) by discussing all the effects induced by a change of observer in Chap. 5 (laws of velocity composition and acceleration composition, Doppler effect, aberration, image formation, "superluminal" motions in astrophysics). The two chapters that follow are entirely devoted to the Lorentz group, exploring its algebraic structure (Chap. 6), with the introduction of boosts and Thomas rotation, and its Lie group structure (Chap. 7). Chapter 8 focuses on the privileged class of inertial observers, with the introduction of the Poincaré group and its Lie algebra. The dynamics starts in Chap. 9, where the notion of 4-momentum is presented, as well as the principle of its conservation for any isolated system. On its side, Chap. 10 is devoted to the conservation of angular momentum and to the concepts of centre of inertia and spin. Relativistic dynamics is subsequently reformulated in Chap. 11 by means of a principle of least action. The conservation laws appear then as consequences of Noether theorem. A Hamiltonian formulation of the dynamics of relativistic particles is also presented in this chapter. Chapter 12 focuses on accelerated observers, discussing kinematical aspects (Rindler horizon, clock synchronization, Thomas precession) as well as dynamical ones (spectral shift, motion of free particles). A second type of non-inertial observers is studied in Chap. 13: the rotating ones. This chapter ends with an extensive discussion of the Sagnac effect and its application to laser gyrometers in inertial guidance systems on board airplanes.

The second part of the book opens in Chap. 14, where the physical object under focus is no longer a particle but a field. This part starts by three purely mathematical chapters to introduce the notions of tensor (Chap. 14), tensor field (Chap. 15) and integration over a subdomain of spacetime (Chap. 16). Among other things, these chapters present the p-forms and exterior calculus, which are very useful not only for electromagnetism but also for hydrodynamics. We felt necessary to devote an entire chapter to integration in order to introduce with enough details and examples the notions of submanifold of Minkowski spacetime, area and volume element; integral of a scalar or vector field; and flux integral. The chapter ends by the famous Stokes' theorem and its applications. Equipped with these mathematical tools, we proceed to electromagnetism in Chap. 17. Here again, the emphasis is put on the four-dimensional aspect: the electromagnetic field tensor $\boldsymbol{F}$ is introduced first, and the electric and magnetic field vectors $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$ appear in a second stage. The motion of charged particles and the various types of particle accelerators are discussed in this chapter. Chapter 18 presents Maxwell equations, here also in a four-dimensional form, which is intrinsically simpler than the classical set of three-dimensional equations involving $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$. The Liénard–Wiechert potentials are derived in this chapter, leading to the electromagnetic field generated by a charged particle in arbitrary motion. Chapter 19 introduces the concept of energy–momentum tensor, a fundamental tool for the dynamics of continuous media in relativity. The principles of conservation of energy–momentum and angular momentum are notably presented in a "continuous" version, as opposed to the "discrete" version considered in Chaps. 9 and 10. The energy–momentum of the electromagnetic field can then be discussed in depth in Chap. 20. In that chapter, the energy and momentum radiated away by a moving charge are computed. A particular case is constituted by synchrotron radiation, whose applications in astrophysics and in synchrotron facilities are discussed. Chapter 21 introduces relativistic hydrodynamics, first in a standard form and next making use of the exterior calculus presented in Chaps. 14–16. The latter approach facilitates greatly the derivation of relativistic generalizations of the classical theorems of fluid mechanics. Two particularly important and contemporary applications are explored in this chapter: relativistic jets in astrophysics and the quark-gluon plasma produced in heavy ion colliders. At last, the book ends by the problem of gravitation (Chap. 22): after some discussion about the unsuccessful attempts to incorporate gravitation in special relativity, the theory of general relativity is briefly introduced. Let us point out that the study of accelerated observers performed in Chap. 12 allows one, via the equivalence principle, to treat easily some relativistic effects of gravitation, such as the gravitational redshift or the bending of light rays.

The book contains six purely mathematical chapters (Chaps. 1, 6, 7, 14, 15 and 16). The aim is to introduce in a consistent and gradual way all the tools required for special relativity, up to rather advanced topics. As a monograph devoted to a theory whose foundations are more than a hundred years old, the book does not contain any truly original result. One may, however, note the general expression of the 4-acceleration of a particle in terms of its acceleration and velocity both relative to a generic observer (i.e. accelerated or rotating) [Eq. (4.60)]; the composition law of relative accelerations resulting from a change of observer and providing the relativistic generalization of centripetal and Coriolis accelerations [Eq. (5.56)]; the complete classification of restricted Lorentz transformations from a null eigenvector (Sect. 6.4); the elementary and relatively short derivation of Thomas rotation in the most general case (Sect. 6.7.2); the expressions of energy and momentum relative to an observer, taking into account the acceleration and rotation of that observer [Eqs. (9.12) and (9.13)]; the computation of the discrepancy between the rest space of an observer and his simultaneity hypersurface (Sect. 12.3); the expression of the 4-acceleration of an observer in terms of physically measurable quantities [Eq. (12.73)]; the equation of motion of a free particle in Rindler coordinates [Eqs. (12.75) and (12.82)]; and the demonstration that the nonrelativistic limit of the canonical equation of fluid dynamics is the Crocco equation (Sect. 21.5.4).

One of the book's limitations is the classical domain: no topic related to quantum mechanics is treated. In particular, spinors and representations of the Poincaré group are not discussed (see, e.g., Cartan (1966), Naber (2012), Penrose and Rindler (1984), Naïmark (1962)). Although these notions are not quantum by themselves, they are mostly used in relativistic quantum theory, notably to write Dirac equation—which we do not address here.

Notes

Notations: In order to facilitate the reading, mathematical notations and symbols introduced in the course of the text are collected in the notation index (p. 761). Throughout the text, the abbreviation *iff* stands for *if, and only if.*

Web page: The page http://relativite.obspm.fr/sperel is devoted to the book. It contains the errata, the clickable list of bibliographic references, all the links listed in Appendix B, as well as various complements. The reader is invited to use this page to report any error that he/she may find in the text.

This book has been first published in French language by EDP Sciences & CNRS Editions in 2010 (Gourgoulhon 2010). The differences with respect to that version are rather minor: they regard some improvements in the presentation and in the figures, as well as some updates in the bibliography.

Meudon, France Éric Gourgoulhon

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Contents

Chapter 1
Minkowski Spacetime

1.1 Introduction

This first chapter is purely mathematical: there is no direct mention of physical objects. The aim is to set the geometrical framework for special relativity, i.e. to introduce *Minkowski spacetime*. Later on, when dealing with physics, the outcomes of measurements will be modelled as mathematical operations in that space, such as scalar products.

Let us point out that the mathematics required for the foundations of special relativity are rather elementary. They involve linear algebra at the level of the first two years of university. For the benefit of the reader, the definitions of the basic algebraic structures are recalled in Appendix A.

1.2 The Four Dimensions

1.2.1 Spacetime as an Affine Space

Relativity has performed the fusion of *space* and *time*, two entirely distinct concepts in Galilean mechanics. Four numbers are required to determine an event in the "space-and-time continuum": three for its spatial position (for instance its Cartesian coordinates (x, y, z) or the spherical ones (r, θ, φ)) and one for its date. The general mathematical structure corresponding to such a four-dimensional continuum is a *manifold*. Without entering into technical details,[1] let us say that, given an integer $n \geq 1$, a *manifold of dimension* n is a set that "locally resembles" $\mathbb{R}^n$ (in the present case $n = 4$), but may differ from $\mathbb{R}^n$ at a global scale. Regarding the dimension

[1]The precise definition of a manifold will be given in Sect. 7.2.1.

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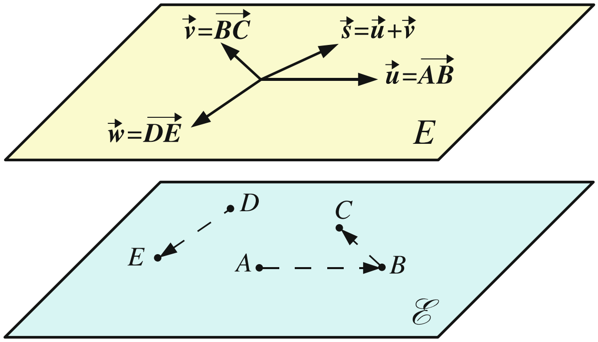

Fig. 1.1 Affine space $\mathscr{E}$ and the underlying vector space E (for graphical purposes, the dimension of $\mathscr{E}$ is reduced to 2, whereas the actual dimension of spacetime is 4)

$n = 2$, standard examples of manifolds are the plane, the cylinder, the sphere and the torus.

As far as special relativity is concerned, the chosen manifold is the simplest that one could imagine, namely, an *affine space* of dimension 4. We are familiar with the structure of affine space of dimension 3. It involves the notion of *points* that can be joined two by two by *vectors*. More precisely (cf. Fig. 1.1 and Berger (1987a)), an ***affine space*** of dimension n on $\mathbb{R}$ is a non-empty set $\mathscr{E}$ such that there exists a vector space[2] E of dimension n on $\mathbb{R}$ and a mapping[3]

$$\begin{aligned} \mathscr{V} : \mathscr{E} \times \mathscr{E} &\longrightarrow E \\ (A, B) &\longmapsto \mathscr{V}(A, B) =: \overrightarrow{AB} \end{aligned} \tag{1.1}$$

that obeys the following two properties:

- For any point $O \in \mathscr{E}$, the function

$$\begin{aligned} \mathscr{V}_O : \mathscr{E} &\longrightarrow E \\ M &\longmapsto \overrightarrow{OM} \end{aligned} \tag{1.2}$$

  is bijective.
- For any triplet (A, B, C) of elements of $\mathscr{E}$, Chasles' relation holds:

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}. \tag{1.3}$$

The elements of $\mathscr{E}$ are called ***points*** and E is called the ***vector space underlying*** $\mathscr{E}$.

[2]The definition and basic properties of a *vector space* are recalled in Appendix A.

[3]In all the text, we use the symbols $:=$ and $=:$ to denote a definition, the defined object standing on the side of the ':'. In the present case, this means that $\overrightarrow{AB}$ is defined as being $\mathscr{V}(A, B)$.

Example 1.1. An affine space of dimension 1 is a straight line and an affine space of dimension 2 is a plane. Still in dimension 2, a counterexample is a sphere.

Choosing for spacetime a structure as simple as an affine space is sufficient to treat electromagnetism, hydrodynamics and relativistic quantum field theory. On the other side, it does not allow one to incorporate gravitation into relativity in a satisfactory manner. We will see in Chap. 22 that relativistic gravity requires the general notion of a manifold, not reduced to an affine space; this is the realm of general relativity, which we shall not treat in this book (beside the brief presentation in Sect. 22.4).

Accordingly, in what follows, we shall call ***spacetime***, and denote by $\mathscr{E}$, an affine space of dimension 4 on $\mathbb{R}$. We shall note E the underlying vector space, which is isomorphic to $\mathbb{R}^4$. The elements of $\mathscr{E}$ are called ***events*** and those of E are called ***vectors***, or ***four-vectors***, abridged as ***4-vectors***.

Remark 1.1. The term *four-vector* or *4-vector* introduced by the physicist stands for nothing but a *vector* for the mathematician, that is to say the element of a vector space (E in the present case). The prefix "4-" simply recalls that such a vector belongs to a vector space of dimension 4 on $\mathbb{R}$. These vectors are hence distinguished from the vectors of three-dimensional vector spaces usually manipulated by the non-relativist physicist. Since in this book the framework is four-dimensional from the very beginning, we shall not use the word *4-vector* and shall refer to the elements of E simply as *vectors*.

1.2.2 A Few Notations

Vectors in E are denoted by boldface characters with an arrow above them, for instance: $\vec{\boldsymbol{u}}$, $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{e}}_0$. The components of a vector with respect to a basis of E are denoted with an index placed at the top right of the vector symbol, ranging from 0 to 3 (and not from 1 to 4). Hence if $(\vec{\boldsymbol{e}}_0, \vec{\boldsymbol{e}}_1, \vec{\boldsymbol{e}}_2, \vec{\boldsymbol{e}}_3)$ is a vector basis of E, the components of a vector $\vec{\boldsymbol{v}}$ with respect to it are the four real numbers (v^0, v^1, v^2, v^3) such that

$$\vec{\boldsymbol{v}} = v^0\,\vec{\boldsymbol{e}}_0 + v^1\,\vec{\boldsymbol{e}}_1 + v^2\,\vec{\boldsymbol{e}}_2 + v^3\,\vec{\boldsymbol{e}}_3 = \sum_{\alpha=0}^{3} v^\alpha\,\vec{\boldsymbol{e}}_\alpha \tag{1.4}$$

Remark 1.2. The index of a vector component lies in the upper position (e.g. v^0) and not in the lower one (e.g. v_0). This writing will be fully justified in Chap. 14.

Summations over an index ranging from 0 to 3, as in (1.4), are very frequent, therefore justifying the use of an abridged notation, named ***Einstein summation convention***: the signs $\sum$ are suppressed and each time an index appears twice in a formula, in an upper position and a lower one, the summation is implicit over all the values taken by the index. Moreover, the index range is 0 to 3 if it is a letter from the Greek alphabet ($\alpha, \beta, \ldots$) and 1 to 3 only if it is a letter from the Latin alphabet ($i, j, \ldots$). Hence formula (1.4) will be written as

$$\boxed{\vec{v} = v^\alpha \vec{e}_\alpha = v^0 \vec{e}_0 + v^i \vec{e}_i}. \tag{1.5}$$

Remark 1.3. As mentioned above, all implicit summations are to be taken over an index that appears both in an upper and a lower position, as in (1.5). If, in some rare cases, a summation must be performed over indices located at the same level, we shall make it explicit by reintroducing the symbol $\sum$.

1.2.3 Affine Coordinate System

One defines an ***affine coordinate system*** on $\mathscr{E}$, also called a ***affine frame*** of $\mathscr{E}$, as the set formed by a point $O \in \mathscr{E}$ and a basis $(\vec{e}_0, \vec{e}_1, \vec{e}_2, \vec{e}_3)$ of E. Each point $M \in \mathscr{E}$ is then characterized by its ***affine coordinates*** (x^0, x^1, x^2, x^3), which constitute the unique 4-tuple of real numbers such that

$$\overrightarrow{OM} = x^\alpha \, \vec{e}_\alpha. \tag{1.6}$$

The point O is called the ***origin*** of the considered affine coordinate system.

Remark 1.4. Affine coordinates constitute an "identification tag" of points of $\mathscr{E}$ that is purely mathematical. In Chap. 3, we shall introduce a *physical* coordinate system based on the notion of observer.

1.2.4 Constant c

Considering $\mathscr{E}$ as a four-dimensional space describing both "space" and "time" means implicitly that the same physical dimension is given to space and time, otherwise summations like (1.6) would be meaningless, since they amount to adding a duration to a length. By convention, we shall choose this common dimension to be that of a *length*, the corresponding SI unit being the metre. To recover times with the usual dimension, one shall introduce a conversion factor with the dimension of a velocity: this is the constant[4]

$$\boxed{c := 2.99792458 \times 10^8 \ \mathrm{m\,s^{-1}}}. \tag{1.7}$$

[4]Cf. footnote 3 p. 2.

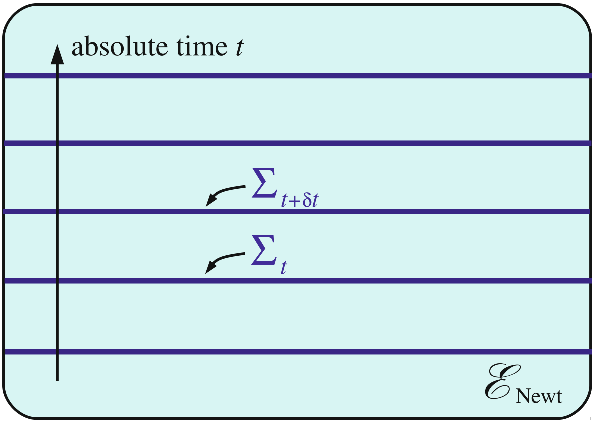

Fig. 1.2 Newtonian spacetime $\mathscr{E}_{\rm Newt}$. Two dimensions have been suppressed, so that the affine space $\mathscr{E}_{\rm Newt}$ is drawn as a plane. $\mathscr{E}_{\rm Newt}$ is foliated by the hyperplanes Σ_t (reduced here to horizontal straight lines but being actually three-dimensional) that represent the successive states of the Newtonian absolute space

Taking the risk to kill the suspense, let us tell at once that this constant corresponds to the speed of light in vacuum as measured by an inertial observer, as we shall see explicitly in Sect. 4.6.2.

1.2.5 Newtonian Spacetime

While Newtonian physics has developed without this concept, one may perfectly speak about *spacetime* regarding it. The ***Newtonian spacetime*** $\mathscr{E}_{\rm Newt}$ is then an affine space of dimension 4, as the affine space $\mathscr{E}$ of special relativity discussed above. The difference is that $\mathscr{E}_{\rm Newt}$ is equipped with a particular structure that implements Newton's concepts of *absolute time* and *absolute space*. This structure consists in the foliation by a family $(\Sigma_t)_{t\in\mathbb{R}}$ of affine subspaces of dimension 3: each Σ_t is Newton's absolute space at Newton's absolute time t (cf. Fig. 1.2). An affine subspace of dimension 3 of an affine space of dimension 4 is called a ***hyperplane***. The corresponding subspace of the vector space underlying the affine space is called a ***vector hyperplane***. Let us recall that more generally, the prefix *hyper* stands for a subspace of dimension one unit less than the ambient space.

In pictorial terms (cf. Fig. 1.2), one may say that the Newtonian spacetime is the pile of "portraits" of the absolute space at all the successive instants of absolute time:

$$\mathscr{E}_{\rm Newt} = \bigcup_{t\in\mathbb{R}} \Sigma_t . \tag{1.8}$$

The foliation $(\Sigma_t)_{t\in\mathbb{R}}$ is the mathematical translation of the concept of *Newtonian simultaneity*: the date t of a given event is the same for all observers, thereby defining the *Newtonian absolute time*.

Remark 1.5. The concept of spacetime introduced above is not very fruitful in Newtonian physics. One usually considers instead the evolution along the time t of the three-dimensional space Σ_t.

Historical note: *The mention of time as a fourth dimension can be found in texts dating back to the eighteenth century (Archibald 1914): it is present in the article* dimension *of the famous* Encyclopédie *by D. Diderot and J. Le Rond d'Alembert and in the* Traité des fonctions analytiques *by J.L. Lagrange (1797). One may also mention the representation of train timetables as spacetime diagrams in the nineteenth century (a reproduction of which can be found on p. 55 of Hartle's book (Hartle 2003)). But it is only after the advent of special relativity that a truly four-dimensional formalism has been developed to recast Newtonian physics, notably by Élie Cartan[5] in 1923–1925 (Cartan 1923, 1924, 1925) and Edward Milne[6] in 1934 (Milne 1934).*

1.3 Metric Tensor

The spacetime of special relativity, $\mathscr{E}$, and the Newtonian spacetime, $\mathscr{E}_{\rm Newt}$, are both affine spaces of dimension 4 on $\mathbb{R}$. The distinction between the two theories lies at the level of the fundamental structures introduced on these two spaces. We have seen in the preceding subsection that for $\mathscr{E}_{\rm Newt}$, the fundamental structure is the foliation by the subspaces Σ_t representing the state of the Newtonian three-dimensional space at the absolute time t. For the relativistic spacetime $\mathscr{E}$, the fundamental structure is rather different. It is provided by the metric tensor, which we shall now introduce.

1.3.1 Scalar Product on Spacetime

In nonrelativistic classical physics, where the spacetime $\mathscr{E}_{\rm Newt}$ is generally not considered, the basic framework is the Newtonian absolute space (denoted by Σ_t in Sect. 1.2.5). The latter is an affine space of dimension 3 on $\mathbb{R}$. The basic objects

[5]**Élie Cartan** (1869–1951): French mathematician, founder of the calculus on differential forms (the so-called *exterior calculus*, which we shall introduce in Sect. 15.5); Élie Cartan was very interested in special and general relativity; he was the father of Henri Cartan, a founding member of the Bourbaki group.

[6]**Edward A. Milne** (1896–1950): British astrophysicist, well known for having developed a model of an expanding universe within Newtonian gravitation.

are then the vectors[7] $\vec{v}$ of the underlying vector space (isomorphic to $\mathbb{R}^3$). On this vector space, an important structure is the *scalar product* of two vectors:

$$\vec{u} \cdot \vec{v} = u^1 v^1 + u^2 v^2 + u^3 v^3, \tag{1.9}$$

where the u^i's and v^i's are the components of vectors $\vec{u}$ and $\vec{v}$ in some orthonormal basis. The scalar product is at the basis of all the geometry. It notably allows one to define the norm of a vector, the angle between two vectors and the orthogonality between two subspaces (for instance, between a straight line and a plane). The scalar product (1.9), which involves only + signs between the $u^i v^i$ terms, is called ***Euclidean***.

The geometry of relativistic physics differs from that of Newtonian physics in two ways:

1. As discussed above, the base space is no longer of dimension 3, but of dimension 4 (it "includes time"!).
2. The employed scalar product is no longer Euclidean: there exists a vector basis of E where it takes the form $\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}} = -u^0 v^0 + u^1 v^1 + u^2 v^2 + u^3 v^3$, whereas a Euclidean scalar product would contain only + signs, as in (1.9).

More precisely, the vector space E underlying the spacetime $\mathscr{E}$ is endowed with a symmetric bilinear form $\boldsymbol{g}$ that is nondegenerate and of signature $(-, +, +, +)$. Let us recall that:

- ***Bilinear form*** means that $\boldsymbol{g}$ is a function $E \times E \longrightarrow \mathbb{R}$ (i.e. it associates with any pair of vectors $(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}})$ the real number $\boldsymbol{g}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}})$) that is linear with respect to each of its arguments: for any $\lambda \in \mathbb{R}$ and $(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}) \in E^3$,

$$\begin{aligned}
&\boldsymbol{g}(\lambda \vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}) = \lambda \boldsymbol{g}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}), \quad \boldsymbol{g}(\vec{\boldsymbol{u}}, \lambda \vec{\boldsymbol{v}}) = \lambda \boldsymbol{g}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}),\\
&\boldsymbol{g}(\vec{\boldsymbol{u}} + \vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}) = \boldsymbol{g}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{w}}) + \boldsymbol{g}(\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}),\\
&\boldsymbol{g}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}} + \vec{\boldsymbol{w}}) = \boldsymbol{g}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}) + \boldsymbol{g}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{w}}).
\end{aligned}$$

- ***Symmetric*** means that $\boldsymbol{g}(\vec{\boldsymbol{v}}, \vec{\boldsymbol{u}}) = \boldsymbol{g}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}})$ for any pair $(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}) \in E^2$.
- ***Nondegenerate*** means that there does not exist any vector $\vec{\boldsymbol{u}}$ but the zero vector satisfying $\forall \vec{\boldsymbol{v}} \in E,\ \boldsymbol{g}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}) = 0$.
- Of ***signature*** $(-, +, +, +)$ means that there exists a basis of the vector space E such that $\boldsymbol{g}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}})$ is expressed in terms of the components (u^α) and (v^α) of $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$ with respect to this basis as follows:

$$\boldsymbol{g}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}) = -u^0 v^0 + u^1 v^1 + u^2 v^2 + u^3 v^3. \tag{1.10}$$

[7] The vectors belonging to spaces of dimension 3 are denoted by a non-boldface symbol to distinguish them from vectors in E (the 4-vectors).

Thanks to a classical result of linear algebra, Sylvester's law of inertia (Berger 1987b; Deheuvels 1981), in any other basis where $\boldsymbol{g}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}})$ has a diagonal writing (i.e. without any cross term like $u^0 v^1$), $\boldsymbol{g}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}})$ is the algebraic sum of four terms, one of which has a minus sign and the remaining three of them have a plus sign, as in (1.10). This property is therefore independent of the basis where $\boldsymbol{g}$ is diagonalized: it is intrinsic to $\boldsymbol{g}$ and actually sets all the properties of $\boldsymbol{g}$, hence the term *signature*.

The signature $(-,+,+,+)$ is qualified as ***Lorentzian***, whereas the signature $(+,+,+,+)$ would have been called ***Euclidean*** or ***Riemannian***. The property of being a nondegenerate symmetric bilinear form characterizes a ***scalar product***.[8] For instance, the scalar product (1.9) of the three-dimensional Euclidean space is a nondegenerate symmetric bilinear form of signature $(+,+,+)$. $\boldsymbol{g}$ is thus a scalar product on E, which justifies the following notation:

$$\boxed{\forall\, (\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}) \in E^2, \quad \vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}} := \boldsymbol{g}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}})}. \tag{1.11}$$

We shall say that two vectors $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$ are ***orthogonal*** (without mentioning *with respect to the scalar product* $\boldsymbol{g}$) iff[9] $\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}} = 0$.

The bilinear form $\boldsymbol{g}$ defined above is called the ***metric tensor*** of the spacetime $\mathscr{E}$. It is also called the ***metric*** of $\mathscr{E}$. The metric tensor rules entirely the geometry in spacetime: whenever we shall speak about *orthogonal* vectors or about a subspace *orthogonal* to a given vector, it will always be meant orthogonality with respect to $\boldsymbol{g}$.

Remark 1.6. The justification for the word *tensor* in the name of $\boldsymbol{g}$ will be given in Chap. 14. Note that, from a purely mathematical point of view, one should qualify $\boldsymbol{g}$ as *pseudo-metric* rather than *metric*, for $\boldsymbol{g}$ does not provide $\mathscr{E}$ with the structure of *metric space*: given two points A and B in $\mathscr{E}$, the relation $d(A, B) := \sqrt{\boldsymbol{g}(\overrightarrow{AB}, \overrightarrow{AB})}$ does not define a *distance* on $\mathscr{E}$ in the usual mathematical meaning. Indeed for a genuine distance, $d(A, B) = 0$ iff A and B coincide. Here, due the signature of $\boldsymbol{g}$, we may have $d(A, B) = 0$ with A and B distinct; we may also have $d(A, B)$ be an imaginary number: this occurs whenever $\boldsymbol{g}(\overrightarrow{AB}, \overrightarrow{AB}) < 0$, which is allowed by the minus sign in (1.10).

Remark 1.7. In many textbooks, the signature $(+,-,-,-)$ is used for the metric tensor instead of $(-,+,+,+)$ (cf. Appendix C). This reflects a mere change of convention, which amounts to using $\boldsymbol{g}' = -\boldsymbol{g}$ instead of $\boldsymbol{g}$. The resulting

[8] Usually one adds the condition of Euclidean signature; we shall not do it here.

[9] Throughout the text, we use the abbreviation *iff* for *if, and only if*.

physics is identical. The attention of the reader is however drawn on the change of sign that this implies in many formulas! Each convention has its advantages and drawbacks, and of course its proponents and detractors! The reasons for which the signature $(-, +, +, +)$ has been adopted here are the following ones:

1. From a pure mathematical point of view, it is obvious that three plus signs and a single minus sign are simpler than the reverse, independently of the meaning of these signs.
2. The convention $(-, +, +, +)$ is used in almost all general relativity books and actually in all the recent ones, such as (Misner et al. 1973; Hartle 2003; Carroll 2004; Straumann 2013; Choquet-Bruhat 2009). Using it therefore facilitates the learning of general relativity from the present text.
3. With the convention $(-, +, +, +)$, the scalar product induced by $\boldsymbol{g}$ on spacelike hyperplanes (the three-dimensional "slices" of spacetime at $t = \text{const}$, where t is timelike coordinate) is Euclidean; it therefore coincides with the "usual" scalar product. This allows one to use without any ambiguity the notation (1.11), i.e. $\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}}$, without having to specify whether the dot stands for the scalar product induced by $\boldsymbol{g}$ or whether it stands for the Euclidean scalar product of the three-dimensional space to which $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$ belong. In other words, the convention $(-, +, +, +)$ allows one not to distinguish between "4-vectors" and "3-vectors". With the convention $(+, -, -, -)$, one would have had instead $\boldsymbol{g}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}) = -\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}}$, if the dot was standing for the Euclidean scalar product of the three-dimensional space.

1.3.2 *Matrix of the Metric Tensor*

Given a vector basis $(\vec{\boldsymbol{e}}_0, \vec{\boldsymbol{e}}_1, \vec{\boldsymbol{e}}_2, \vec{\boldsymbol{e}}_3)$ of E, the ***matrix of g with respect to this basis*** is the matrix $(g_{\alpha\beta})$ defined by

$$\boxed{g_{\alpha\beta} := \boldsymbol{g}(\vec{\boldsymbol{e}}_\alpha, \vec{\boldsymbol{e}}_\beta)}. \tag{1.12}$$

The matrix $(g_{\alpha\beta})$ is symmetric, since $\boldsymbol{g}$ is a symmetric bilinear form. It allows one to express the scalar product of two vectors $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$ in terms of their components (u^α) and (v^α) with respect to the basis $(\vec{\boldsymbol{e}}_\alpha)$ [cf. (1.4)]. Indeed, thanks to the bilinearity of $\boldsymbol{g}$,

$$\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}} = \boldsymbol{g}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}) = \boldsymbol{g}\left(u^\alpha \vec{\boldsymbol{e}}_\alpha, v^\beta \vec{\boldsymbol{e}}_\beta\right) = u^\alpha v^\beta \underbrace{\boldsymbol{g}(\vec{\boldsymbol{e}}_\alpha, \vec{\boldsymbol{e}}_\beta)}_{g_{\alpha\beta}}.$$

Hence

$$\boxed{\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}} = g_{\alpha\beta}\, u^\alpha\, v^\beta}. \tag{1.13}$$

Remark 1.8. In the above writing, Einstein summation convention, as introduced in Sect. 1.2.2, applies to all repeated indices (α and β), i.e. one should read

$$\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}} = \sum_{\alpha=0}^{3} \sum_{\beta=0}^{3} g_{\alpha\beta}\, u^\alpha\, v^\beta .$$

Since the bilinear form $\boldsymbol{g}$ is nondegenerate, the matrix $g := (g_{\alpha\beta})$ is invertible.

Proof. Let $(u^\alpha) \in \mathbb{R}^4$ be an element of the kernel of g: $g_{\alpha\beta} u^\beta = 0$. Then $\forall (v^\alpha) \in \mathbb{R}^4$, $g_{\alpha\beta} v^\alpha u^\beta = 0$. But from (1.13), $g_{\alpha\beta} v^\alpha u^\beta = \boldsymbol{g}(\vec{\boldsymbol{v}}, \vec{\boldsymbol{u}})$ where $\vec{\boldsymbol{v}}$ (resp. $\vec{\boldsymbol{u}}$) is the vector of E whose components in the basis $(\vec{\boldsymbol{e}}_\alpha)$ are (v^α) [resp. (u^α)]. The nondegeneracy condition of $\boldsymbol{g}$ implies then that $\vec{\boldsymbol{u}} = 0$. Hence, $u^\alpha = 0$, and we conclude that the kernel of the matrix g is reduced to $\{0\}$, which implies that g is invertible. □

By convention, the components of the inverse matrix g^{-1} are denoted by $g^{\alpha\beta}$, so that the matrix product $g^{-1} g = \mathbb{I}_4$, where $\mathbb{I}_4 := \mathrm{diag}(1, 1, 1, 1)$ is the identity matrix of size 4, is equivalent to

$$\boxed{g^{\alpha\mu} g_{\mu\beta} = \delta^\alpha{}_\beta}. \tag{1.14}$$

In this formula, use has been made of Einstein summation convention on the repeated index μ, and the components of $\mathbb{I}_4$ have been noted by the ***Kronecker symbol***: $\delta^\alpha{}_\beta := 1$ if $\alpha = \beta$ and $\delta^\alpha{}_\beta := 0$ otherwise.

1.3.3 Orthonormal Bases

A basis $(\vec{\boldsymbol{e}}_0, \vec{\boldsymbol{e}}_1, \vec{\boldsymbol{e}}_2, \vec{\boldsymbol{e}}_3)$ of the vector space E is said to be ***orthonormal*** (for the metric $\boldsymbol{g}$) iff

$$\vec{\boldsymbol{e}}_0 \cdot \vec{\boldsymbol{e}}_0 = -1 \tag{1.15a}$$

$$\vec{\boldsymbol{e}}_i \cdot \vec{\boldsymbol{e}}_i = 1 \qquad \text{for} \quad 1 \le i \le 3 \tag{1.15b}$$

$$\vec{\boldsymbol{e}}_\alpha \cdot \vec{\boldsymbol{e}}_\beta = 0 \qquad \text{for} \quad \alpha \neq \beta. \tag{1.15c}$$

Remark 1.9. There does not exist any basis of E for which $\vec{\boldsymbol{e}}_\alpha \cdot \vec{\boldsymbol{e}}_\beta = \delta^\alpha{}_\beta$ for all values of α and β between 0 and 3. One may have $|\vec{\boldsymbol{e}}_\alpha \cdot \vec{\boldsymbol{e}}_\beta| = \delta^\alpha{}_\beta$, but the signature $(-, +, +, +)$ of $\boldsymbol{g}$ imposes one of the scalar products to be negative (Sylvester's law of inertia mentioned in Sect. 1.3.1).

We read on (1.15) that the matrix of $\boldsymbol{g}$ with respect to an orthonormal basis is

$$g_{\alpha\beta} = \boldsymbol{g}(\vec{\boldsymbol{e}}_\alpha, \vec{\boldsymbol{e}}_\beta) = \eta_{\alpha\beta}, \tag{1.16}$$

when $(\eta_{\alpha\beta})$ stands for the following constant matrix

$$\boxed{\eta_{\alpha\beta} := \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}, \tag{1.17}$$

that we shall call ***Minkowski matrix***.

Thanks to (1.13), we deduce from (1.17) that, within an orthonormal basis, the scalar product of two vectors $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$ is expressed in terms of their components (u^α) and (v^α) by

$$\boxed{\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}} = \eta_{\alpha\beta}\, u^\alpha\, v^\beta = -u^0v^0 + u^1v^1 + u^2v^2 + u^3v^3.}_{\text{orthon. basis}} \tag{1.18}$$

We recover formula (1.10). Thus, the orthonormal bases are those on which one can read directly the signature $(-,+,+,+)$ of $\boldsymbol{g}$.

1.3.4 Classification of Vectors with Respect to *g*

A fundamental property of the Euclidean scalar product (1.9) of the three-dimensional Newtonian space is to be ***positive definite***, i.e. $\vec{v} \cdot \vec{v} \geq 0$ for any vector $\vec{v}$ and $\vec{v} \cdot \vec{v} = 0$ iff $\vec{v} = 0$. In the present case, the signature $(-,+,+,+)$ prevents $\boldsymbol{g}$ to be positive definite. The scalar product of a vector $\vec{\boldsymbol{v}}$ with itself can take any sign and be null without $\vec{\boldsymbol{v}}$ being zero. Accordingly, the vectors are classified in three types (apart from the zero vector); a vector $\vec{\boldsymbol{v}} \in E$ is said:

- ***timelike*** iff $\boldsymbol{g}(\vec{\boldsymbol{v}}, \vec{\boldsymbol{v}}) < 0$
- ***spacelike*** iff $\boldsymbol{g}(\vec{\boldsymbol{v}}, \vec{\boldsymbol{v}}) > 0$
- ***null*** or ***lightlike*** iff $\vec{\boldsymbol{v}} \neq 0$ and $\boldsymbol{g}(\vec{\boldsymbol{v}}, \vec{\boldsymbol{v}}) = 0$

These definitions with a strong physical connotation will be justified in Chap. 2.

Remark 1.10. Despite their name, null vectors should not be confused with the zero vector. In mathematical literature, null vectors are also called ***isotropic vectors***.

1.3.5 Norm of a Vector

Given a vector $\vec{\boldsymbol{v}} \in E$, its ***norm with respect to the metric tensor g*** is defined as the positive or null real number

$$\boxed{\|\vec{v}\|_g := \sqrt{|\vec{v}\cdot\vec{v}|} = \sqrt{|g(\vec{v},\vec{v})|}}. \tag{1.19}$$

Note that for a spacelike vector, $\|\vec{v}\|_g = \sqrt{\vec{v}\cdot\vec{v}}$, whereas for a timelike one, $\|\vec{v}\|_g = \sqrt{-\vec{v}\cdot\vec{v}}$. Besides, the following equivalence holds:

$$\forall \vec{v} \in E \setminus \{0\}, \quad \|\vec{v}\|_g = 0 \iff \vec{v} \text{ is lightlike}. \tag{1.20}$$

Remark 1.11. The function $\| \ \|_g$ is not a *norm* on E in the usual meaning of the word: a norm $\| \ \|$ on a vector space must indeed obey the following properties (Deheuvels 1981): (i) $\forall \vec{v} \in E,\ \|\vec{v}\| \geq 0$, (ii) $\|\vec{v}\| = 0 \Rightarrow \vec{v} = 0$, (iii) $\forall (\lambda, \vec{v}) \in \mathbb{R} \times E,\ \|\lambda \vec{v}\| = |\lambda| \|\vec{v}\|$ and (iv) $\forall (\vec{u}, \vec{v}) \in E^2,\ \|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$. The function $\| \ \|_g$ defined by (1.19) satisfies conditions (i) and (iii) (the latter thanks to the bilinearity of g) but neither condition (ii) ($\|\vec{v}\|_g = 0$ whenever $\vec{v}$ is lightlike) nor condition (iv) (it is easy to violate it by choosing for $\vec{u}$ and $\vec{v}$ two null vectors such that $\vec{u} + \vec{v}$ is not null). Hence the pair $(E, \| \ \|_g)$ is *not* a normed vector space.

We shall say that a vector $\vec{v} \in E$ is a ***unit vector*** iff $\|\vec{v}\|_g = 1$. There are two classes of unit vectors: the timelike ones, for which $g(\vec{v}, \vec{v}) = -1$, and the spacelike ones, for which $g(\vec{v}, \vec{v}) = 1$. We shall discuss this point further in Sect. 1.4.3.

1.3.6 Spacetime Diagrams

To draw figures representing spacetime, we shall suppress one or two dimensions, to get, respectively, a three-dimensional view in perspective or a two-dimensional plane view. A ***spacetime diagram*** is a two-dimensional plot with a timelike vector drawn in the vertical direction and a spacelike vector drawn in the horizontal direction, these two vectors being orthogonal with respect to the metric tensor g. The two arrows representing them are perpendicular on the figure (they form an angle of 90°). But not all pairs of vectors orthogonal with respect to g can be drawn in this way, because of the conflict between the Lorentzian signature of g and the Euclidean signature of the "standard" metric used to draw the figure. This aspect of spacetime diagrams is illustrated in Figs. 1.3 and 1.4, on which we call the attention of the reader before proceeding further.

Let us first discuss Fig. 1.3. We consider an orthonormal basis $(\vec{e}_\alpha)$ (with respect to the metric g, as defined in Sect. 1.3.3). To get a two-dimensional figure, we draw only the first two vectors of this basis: $\vec{e}_0$ is by definition a unit timelike vector, $\vec{e}_0 \cdot \vec{e}_0 = -1$ [cf. (1.15a)]; $\vec{e}_1$ is a unit spacelike vector, $\vec{e}_1 \cdot \vec{e}_1 = 1$ [cf. (1.15b)]; and $\vec{e}_0$ and $\vec{e}_1$ are orthogonal to each other, $\vec{e}_0 \cdot \vec{e}_1 = 0$ [cf. (1.15c)]. In Fig. 1.3, one has *arbitrarily* chosen to represent $\vec{e}_0$ and $\vec{e}_1$ by two perpendicular arrows, with $\vec{e}_0$ vertical and $\vec{e}_1$ horizontal. Besides, four other vectors have been drawn: $\vec{a}_0$, $\vec{a}_1$, $\vec{u}$ and $\vec{v}$, whose components in the basis $(\vec{e}_\alpha)$ are

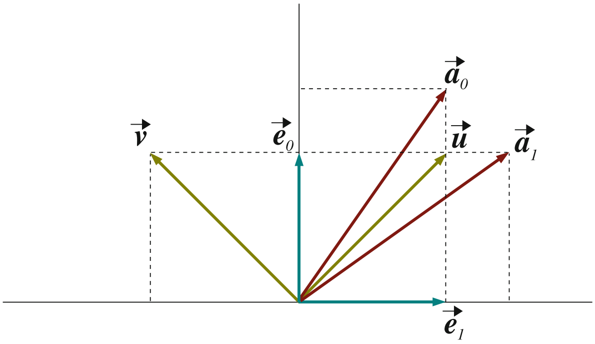

Fig. 1.3 Vectors of the vector space E underlying the spacetime $\mathscr{E}$; two dimensions have been suppressed, so that the figure is plane. $\vec{e}_0$ and $\vec{e}_1$ are the first two vectors of an orthonormal basis: $\vec{e}_0 \cdot \vec{e}_0 = -1$, $\vec{e}_0 \cdot \vec{e}_1 = 0$ and $\vec{e}_1 \cdot \vec{e}_1 = 1$. The other vectors are $\vec{a}_0 = \sqrt{2}\vec{e}_0 + \vec{e}_1$, $\vec{a}_1 = \vec{e}_0 + \sqrt{2}\vec{e}_1$, $\vec{u} = \vec{e}_0 + \vec{e}_1$ and $\vec{v} = \vec{e}_0 - \vec{e}_1$

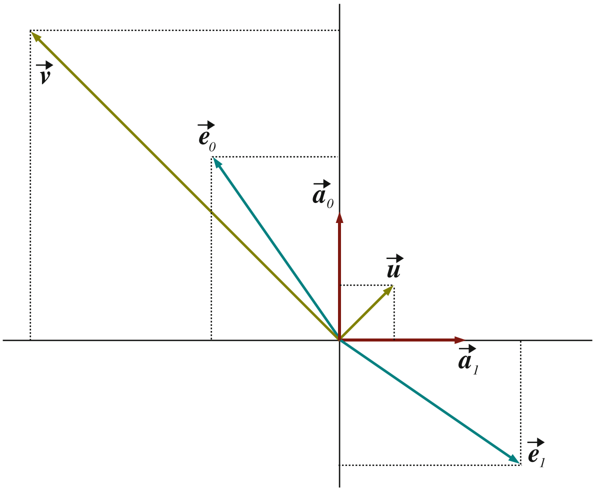

Fig. 1.4 Same vectors of E as in Fig. 1.3, but in a representation built onto the orthonormal basis $(\vec{a}_0, \vec{a}_1)$, instead of $(\vec{e}_0, \vec{e}_1)$. Even if the figure looks pretty different from Fig. 1.3, it shows exactly the same vectors. In particular, one can check that the vector equalities $\vec{u} = \vec{e}_0 + \vec{e}_1$ and $\vec{v} = \vec{e}_0 - \vec{e}_1$ still hold. Besides, the null vectors, i.e. $\vec{u}$ and $\vec{v}$, are drawn at an angle of $\pm 45°$ from the vertical, as in Fig. 1.3

$$a_0^\alpha = (\sqrt{2}, 1, 0, 0), \quad a_1^\alpha = (1, \sqrt{2}, 0, 0), \quad u^\alpha = (1, 1, 0, 0), \quad v^\alpha = (1, -1, 0, 0). \tag{1.21}$$

Note that, despite the arrows representing them are not perpendicular in Fig. 1.3, the vectors $\vec{a}_0$ and $\vec{a}_1$ are orthogonal with respect to $\boldsymbol{g}$. It is easy to check it, using the fact that in the orthonormal basis $(\vec{e}_\alpha)$, the scalar product is expressed via the Minkowski matrix according to formula (1.18):

$$\vec{a}_0 \cdot \vec{a}_1 = \eta_{\alpha\beta}\, a_0^\alpha\, a_1^\beta = -\sqrt{2} \times 1 + 1 \times \sqrt{2} + 0 \times 0 + 0 \times 0 = 0.$$

Moreover, $\vec{a}_0$ and $\vec{a}_1$ are unit vectors, $\vec{a}_0$ being timelike and $\vec{a}_1$ spacelike:

$$\vec{a}_0 \cdot \vec{a}_0 = \eta_{\alpha\beta}\, a_0^\alpha\, a_0^\beta = -\sqrt{2} \times \sqrt{2} + 1 \times 1 + 0 \times 0 + 0 \times 0 = -1,$$
$$\vec{a}_1 \cdot \vec{a}_1 = \eta_{\alpha\beta}\, a_1^\alpha\, a_1^\beta = -1 \times 1 + \sqrt{2} \times \sqrt{2} + 0 \times 0 + 0 \times 0 = 1.$$

Hence the norm of $\vec{a}_1$ with respect to $\boldsymbol{g}$ is the same as that of $\vec{e}_1$, namely, 1, while these two vectors are represented in Fig. 1.3 by arrows of different lengths.

On the opposite, the vectors $\vec{u}$ and $\vec{v}$ are drawn in Fig. 1.3 with perpendicular arrows, although they are not orthogonal with respect to $\boldsymbol{g}$:

$$\vec{u} \cdot \vec{v} = \eta_{\alpha\beta}\, u^\alpha\, v^\beta = -1 \times 1 + 1 \times (-1) + 0 \times 0 + 0 \times 0 = -2 \neq 0.$$

Both are null vectors:

$$\vec{u} \cdot \vec{u} = \eta_{\alpha\beta}\, u^\alpha\, u^\beta = -1 \times 1 + 1 \times 1 + 0 \times 0 + 0 \times 0 = 0,$$
$$\vec{v} \cdot \vec{v} = \eta_{\alpha\beta}\, v^\alpha\, v^\beta = -1 \times 1 + (-1) \times (-1) + 0 \times 0 + 0 \times 0 = 0.$$

The vectors $\vec{a}_0$ and $\vec{a}_1$ being unit vectors and orthogonal, with $\vec{a}_0 \cdot \vec{a}_0 = -1$ and $\vec{a}_1 \cdot \vec{a}_1 = 1$, can be completed by two unit spacelike vectors properly chosen, for instance, $\vec{e}_2$ and $\vec{e}_3$, to constitute a new orthonormal basis: $(\vec{a}_\alpha) := (\vec{a}_0, \vec{a}_1, \vec{e}_2, \vec{e}_3)$. One can easily obtain the components of the vectors $\vec{e}_0$, $\vec{e}_1$, $\vec{u}$ and $\vec{v}$ in the basis[10] $(\vec{a}_\alpha)$:

$$e'^\alpha_0 = (\sqrt{2}, -1, 0, 0), \quad e'^\alpha_1 = (-1, \sqrt{2}, 0, 0), \quad u'^\alpha = (\sqrt{2} - 1, \sqrt{2} - 1, 0, 0),$$
$$v'^\alpha = (\sqrt{2} + 1, -\sqrt{2} - 1, 0, 0). \tag{1.22}$$

We have stressed above that the representation of the vectors $(\vec{e}_0, \vec{e}_1)$ by two perpendicular arrows in Fig. 1.3 was an arbitrary choice. Let us then draw a new figure by favouring the orthonormal basis $(\vec{a}_0, \vec{a}_1)$ instead of $(\vec{e}_0, \vec{e}_1)$: we obtain Fig. 1.4, where $\vec{a}_0$ and $\vec{a}_1$ are drawn with two perpendicular arrows. The drawing of vectors $\vec{e}_0$, $\vec{e}_1$, $\vec{u}$ and $\vec{v}$ is performed via the components (1.22). While Fig. 1.4 looks different from Fig. 1.3, both figures are equally valid representations of the same vector space E. The drawn vectors are the same in the two figures, and none of the orthonormal basis $(\vec{e}_\alpha)$ and $(\vec{a}_\alpha)$ is privileged with respect to the metric $\boldsymbol{g}$.

There are two common features of Figs. 1.3 and 1.4 that are worth noticing:

[10]These components are denoted with a prime to distinguish them from the components in the basis $(\vec{e}_\alpha)$.

1. Two vectors that are orthogonal with respect to $\boldsymbol{g}$ define directions that are symmetric with respect to one of the main bisectors of the figure (i.e. one of the two straight lines of slope $\pm 45°$ through the origin).
2. Null vectors are always drawn as arrows with a slope equal to $\pm 45°$.

These two properties are common to all spacetime diagrams.

Proof. Property 2 is actually a special case of Property 1, for a null vector is by definition orthogonal to itself, so that the only way to be its own symmetric with respect to one of the bisectors is to lie on that bisector. Let us then demonstrate Property 1: if $(\vec{e}_0, \vec{e}_1)$ is the orthonormal basis supporting the spacetime diagram and $\vec{u} = u^0\, \vec{e}_0 + u^1\, \vec{e}_1$ and $\vec{v} = v^0\, \vec{e}_0 + v^1\, \vec{e}_1$ are generic vectors, (1.18) yields

$$\vec{u} \cdot \vec{v} = 0 \iff -u^0 v^0 + u^1 v^1 = 0 \iff \frac{u^0}{u^1} = \frac{v^1}{v^0}.$$

The last equality establishes Property 1. □

The fact that null vectors are always drawn at $\pm 45°$ shows that spacetime diagrams put forward the only directions that can be canonically associated with the metric tensor $\boldsymbol{g}$, namely, the null directions (vanishing scalar square with respect to $\boldsymbol{g}$). We shall discuss the latter in more details in the following section.

1.4 Null Cone and Time Arrow

1.4.1 Definitions

In the vector space E, the set $\mathscr{I}$ composed by the zero vector and all null vectors is called the ***null cone***[11] of the metric $\boldsymbol{g}$. Generically, the term ***cone*** means that if $\vec{v} \in \mathscr{I}$, then $\forall \lambda \in \mathbb{R},\ \lambda \vec{v} \in \mathscr{I}$.

The null cone is depicted in Fig. 1.5. It separates the timelike vectors from the spacelike ones: the former are located inside the cone, while the latter are outside it. The null vectors are by definition located on $\mathscr{I}$. The null cone is composed of the

[11] The name *light cone* is also used; in the present text, we reserve it for the affine counterpart of $\mathscr{I}$, i.e. for the subset of $\mathscr{E}$ formed by all straight lines through a given point and whose direction lies along a vector of $\mathscr{I}$, as we shall discuss in Sect. 2.5.2.

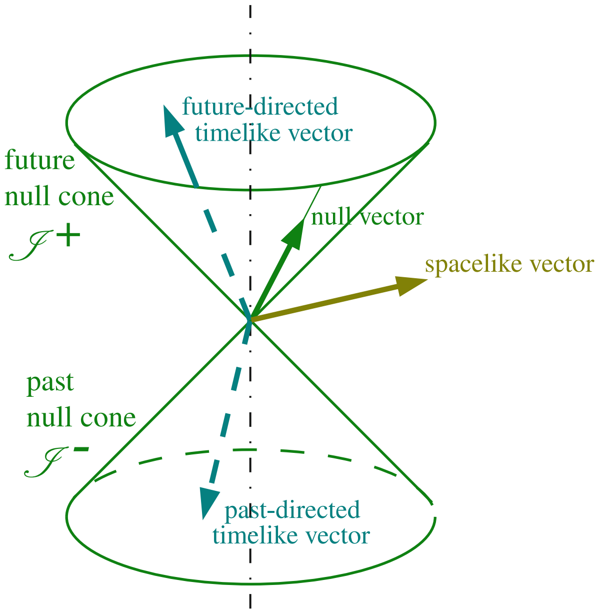

Fig. 1.5 Null cone of the metric $\boldsymbol{g}$ (a spatial dimension has been suppressed)

zero vector (its apex) and two sheets (or *nappes*). Choosing a ***time arrow*** amounts to selecting one of these two sheets and to call it the ***future null cone***; we shall denote it by $\mathscr{I}^+$. The other sheet is then called the ***past null cone*** and denoted by $\mathscr{I}^-$. Timelike and null vectors can then classified in two types:

- Vectors located inside of, or onto, $\mathscr{I}^+$ are said ***future-directed***.
- Vectors located inside of, or onto, $\mathscr{I}^-$ are said ***past-directed***.

Remark 1.12. In Figs. 1.3 and 1.4 discussed in Sect. 1.3.6, the directions of the vectors $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$ mark the trace of the null cone onto the figure plane.

1.4.2 Two Useful Lemmas

We shall often make use of the following two lemmas:

1. Two timelike vectors, $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$ say, are located inside the same sheet ($\mathscr{I}^+$ or $\mathscr{I}^-$) of $\boldsymbol{g}$'s null cone iff $\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}} < 0$.
2. Two null and noncollinear vectors, $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$ say, are located on the same sheet of $\boldsymbol{g}$'s null cone iff $\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}} < 0$.

Proof (Lemma 1). Let us introduce an orthonormal basis $(\vec{e}_\alpha)$ such that $\vec{u} = u^0 \vec{e}_0$ with $u^0 > 0$. It suffices to set $\vec{e}_0 := \|\vec{u}\|_g^{-1} \vec{u}$ and to complete the basis with three vectors $\vec{e}_i$ that are orthogonal to $\vec{e}_0$. We may expand $\vec{v}$ onto this basis: $\vec{v} = v^0 \vec{e}_0 + v^i \vec{e}_i$. The vectors $\vec{u}$ and $\vec{v}$ are then located inside the same sheet of $\mathscr{I}$ iff $v^0 > 0$. Now

$$\vec{u} \cdot \vec{v} = (u^0 \vec{e}_0) \cdot (v^0 \vec{e}_0 + v^i \vec{e}_i) = u^0 v^0 \underbrace{\vec{e}_0 \cdot \vec{e}_0}_{-1} + u^0 v^i \underbrace{\vec{e}_0 \cdot \vec{e}_i}_{0} = -u^0 v^0.$$

Since $u^0 > 0$, we have the equivalence $v^0 > 0 \iff \vec{u} \cdot \vec{v} < 0$. □

Proof (Lemma 2). We notice that an orthonormal basis $(\vec{e}_\alpha)$ can always be found so that

$$\vec{u} = u^0(\vec{e}_0 + \vec{e}_1) \quad \text{and} \quad \vec{v} = v^0(\vec{e}_0 + \cos\varphi\, \vec{e}_1 + \sin\varphi\, \vec{e}_2),$$

with $u^0 \in \mathbb{R}^*$, $v^0 \in \mathbb{R}^*$ and $\varphi \in]0, 2\pi[$ (0 is excluded since $\vec{u}$ and $\vec{v}$ are not collinear). Then

$$\vec{u} \cdot \vec{v} = u^0 v^0 (-1 + \cos\varphi).$$

$\vec{u}$ and $\vec{v}$ belong to the same sheet of the null cone iff $u^0 v^0 > 0$. The lemma results then from $\cos\varphi < 1$ (since $\varphi \neq 0$). □

1.4.3 Classification of Unit Vectors

In Sect. 1.3.5 we have defined unit vectors as the vectors whose norm with respect to $\boldsymbol{g}$ is equal to 1. Let us denote by $\mathscr{U}$ the set of timelike unit vectors and by $\mathscr{S}$ that of spacelike unit vectors:

$$\boxed{\mathscr{U} := \{\vec{v} \in E, \quad \vec{v} \cdot \vec{v} = -1\} \subset E} \tag{1.23}$$

$$\boxed{\mathscr{S} := \{\vec{v} \in E, \quad \vec{v} \cdot \vec{v} = 1\} \subset E}. \tag{1.24}$$

In addition, we shall denote $\mathscr{U}^+$ (resp. $\mathscr{U}^-$) the subset of $\mathscr{U}$ constituted by future-directed (resp. past-directed) vectors:

$$\boxed{\mathscr{U}^+ := \{\vec{v} \in E, \quad \vec{v} \cdot \vec{v} = -1 \text{ and } \vec{v} \text{ future-directed}\} \subset E} \tag{1.25}$$

$$\boxed{\mathscr{U}^- := \{\vec{v} \in E, \quad \vec{v} \cdot \vec{v} = -1 \text{ and } \vec{v} \text{ past-directed}\} \subset E}. \tag{1.26}$$

We have obviously $\mathscr{U} = \mathscr{U}^+ \cup \mathscr{U}^-$.

Given a point $O \in \mathscr{E}$, let us call $\mathscr{U}_O^+$, $\mathscr{U}_O^-$ and $\mathscr{S}_O$ the sets of points of $\mathscr{E}$ that can be connected to O by a vector belonging to, respectively, $\mathscr{U}^+$, $\mathscr{U}^-$ and $\mathscr{S}$:

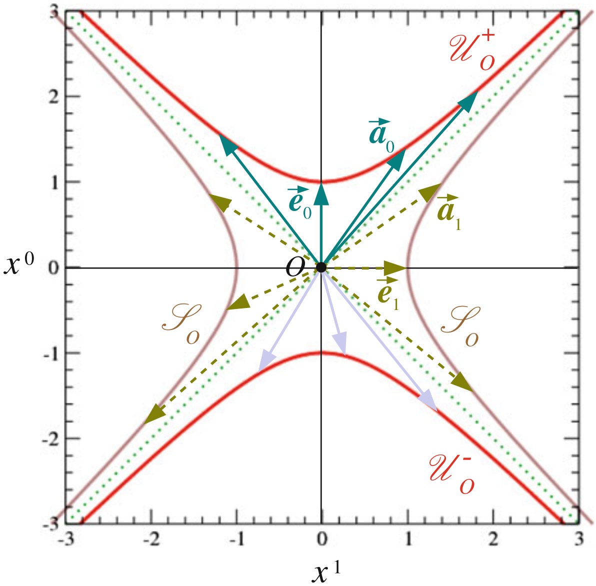

Fig. 1.6 Unit vectors in the plane $\mathrm{Span}(\vec{e}_0, \vec{e}_1)$ generated by vectors $\vec{e}_0$ and $\vec{e}_1$ and centred around a point $O \in \mathscr{E}$. *Solid-line arrows* represent timelike unit vectors ($\vec{v} \cdot \vec{v} = -1$), and *dashed-line arrows* represent spacelike unit vectors ($\vec{v} \cdot \vec{v} = 1$). $\vec{a}_0$ and $\vec{a}_1$ are the same vectors as in Figs. 1.3 and 1.4. The extremities of timelike unit vectors span the hyperbola $(x^0)^2 - (x^1)^2 = 1$, whose branches are denoted by $\mathscr{U}_O^+$ and $\mathscr{U}_O^-$. The extremities of spacelike unit vectors span the hyperbola $(x^1)^2 - (x^0)^2 = 1$, denoted $\mathscr{S}_O$; its focuses are located on the horizontal line $x^0 = 0$. The *dotted lines*, which are the asymptotes of these two hyperbolas, are also the trace of $\boldsymbol{g}$'s null cone

$$\boxed{\mathscr{U}_O^+ := \left\{ M \in \mathscr{E}, \quad \overrightarrow{OM} \in \mathscr{U}^+ \right\} \subset \mathscr{E}} \tag{1.27}$$

$$\boxed{\mathscr{U}_O^- := \left\{ M \in \mathscr{E}, \quad \overrightarrow{OM} \in \mathscr{U}^- \right\} \subset \mathscr{E}} \tag{1.28}$$

$$\boxed{\mathscr{S}_O := \left\{ M \in \mathscr{E}, \quad \overrightarrow{OM} \in \mathscr{S} \right\} = \left\{ M \in \mathscr{E}, \quad \overrightarrow{OM} \cdot \overrightarrow{OM} = 1 \right\} \subset \mathscr{E}}. \tag{1.29}$$

We shall also define $\mathscr{U}_O := \mathscr{U}_O^+ \cup \mathscr{U}_O^-$. The sets $\mathscr{U}_O^+$, $\mathscr{U}_O^-$ and $\mathscr{S}_O$ can be considered as representations in the affine space $\mathscr{E}$ of the subsets $\mathscr{U}^+$, $\mathscr{U}^-$ and $\mathscr{S}$ of the vector space E (cf. Fig. 1.6); these representations are associated with the point O.

In a Euclidean space, the sets $\mathscr{U}_O^+$ and $\mathscr{U}_O^-$ would be empty, and $\mathscr{S}_O$ would be a sphere of unit radius centred on O. Things are different in the space $(\mathscr{E}, \boldsymbol{g})$. To determine the sets $\mathscr{U}_O^+$, $\mathscr{U}_O^-$ and $\mathscr{S}_O$, let us consider an orthonormal basis $(\vec{e}_\alpha)$ of

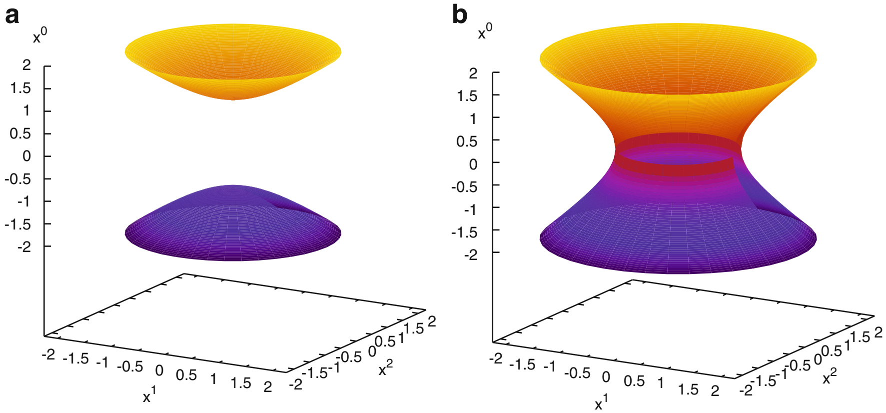

Fig. 1.7 Hyperboloids formed by the unit vectors arising from a given point $O \in \mathscr{E}$: (**a**) timelike unit vectors (the hyperboloid has two sheets, $\mathscr{U}_O^+$ and $\mathscr{U}_O^-$); (**b**) spacelike unit vectors (the hyperboloid has a single sheet, $\mathscr{S}_O$). The dimension x^3 has been suppressed for the drawing: actually these hyperboloids are "surfaces" of dimension 3 and not 2

$(E, \boldsymbol{g})$. A point $M \in \mathscr{E}$ belongs to $\mathscr{U}_O$ or $\mathscr{S}_O$ iff

$$\overrightarrow{OM} \cdot \overrightarrow{OM} = \pm 1, \tag{1.30}$$

with $+1$ for $\mathscr{S}_O$ and -1 for $\mathscr{U}_O$. Let (x^0, x^1, x^2, x^3) be the coordinates of M in the affine frame defined by O and the basis $(\vec{\boldsymbol{e}}_\alpha)$ (cf. Sect. 1.2.3). From (1.6) and (1.18), the condition (1.30) is equivalent to

$$-(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 = \pm 1. \tag{1.31}$$

Let us first consider the case of $\mathscr{U}_O$, i.e. the case where the right-hand side of (1.31) is -1. We recognize then in (1.31) the equation of a three-dimensional hyperboloid[12] of two sheets, which are $\mathscr{U}_O^+$ and $\mathscr{U}_O^-$. The trace of the hyperboloid $\mathscr{U}_O$ in the plane (x^0, x^1) is a unit hyperbola shown in Fig. 1.6. A two-dimensional view of the hyperboloid $\mathscr{U}_O$ is provided in Fig. 1.7a; it can be obtained by rotating the plane of Fig. 1.6 around the vertical axis $x^1 = 0$. $\mathscr{U}_O^+$ is the upper sheet and $\mathscr{U}_O^-$ the lower one. Let us notice that the future (resp. past) sheet of $\boldsymbol{g}$'s null cone is asymptotic to $\mathscr{U}_O^+$ (resp. $\mathscr{U}_O^-$).

Let us now consider $\mathscr{S}_O$. The right-hand side of (1.31) must then be set to $+1$. We recognize the equation of a three-dimensional hyperboloid of one sheet. Its trace in the plane (x^0, x^1) is a unit hyperbola shown in Fig. 1.6. A two-dimensional view of the hyperboloid $\mathscr{S}_O$ is provided in Fig. 1.7b.

[12]This hyperboloid has one more dimension than a standard hyperboloid, which is a two-dimensional surface.

Remark 1.13. As for Fig. 1.3, one should not be deceived by the Euclidean (and thus unphysical!) metric underlying Figs. 1.6 and 1.7. Indeed, one might think that the hyperboloid $\mathscr{U}_O$ defines privileged points with respect to O in the spacetime $\mathscr{E}$, namely, the points where the distance between the sheets $\mathscr{U}_O^+$ and $\mathscr{U}_O^-$ is minimal (points M such that $\overrightarrow{OM} = \pm\vec{\boldsymbol{e}}_0$), which one could call the two "tops" of $\mathscr{U}_0$. But one should keep in mind that the distance employed in this reasoning is not physical: it is that provided by the Euclidean metric, not by $\boldsymbol{g}$. For instance, if we would redraw Fig. 1.6 by privileging the basis $(\vec{\boldsymbol{a}}_0, \vec{\boldsymbol{a}}_1)$ rather than $(\vec{\boldsymbol{e}}_0, \vec{\boldsymbol{e}}_1)$, as we did in Fig. 1.4, then the tops of the hyperboloid $\mathscr{U}_O$ would show up as the points M such that $\overrightarrow{OM} = \pm\vec{\boldsymbol{a}}_0$ and would therefore be different from the preceding tops. There are actually as many pairs of $\mathscr{U}_O$'s "tops" as there are unit timelike vectors. In other words, all the points of $\mathscr{U}_O$ are equivalent.

1.5 Spacetime Orientation

Orienting a two-dimensional space (a plane) amounts to defining the so-called *clockwise* and *anticlockwise* directions. For a three-dimensional space, it amounts to defining *right-handed* bases. This is performed by selecting a reference vector basis and defining a basis to be *right-handed* iff its determinant with respect to the reference basis is positive. We shall extend this notion to the four-dimensional space E.

Let us recall that in dimension three, the determinant is nothing but an antisymmetric trilinear form. Since we are in dimension four, let us consider the set $\mathscr{A}_4(E)$ of antisymmetric *four*-linear forms on E, i.e. of mappings

$$\begin{aligned} \boldsymbol{A}: \ & E \times E \times E \times E \longrightarrow \mathbb{R} \\ & (\vec{\boldsymbol{u}}_1, \vec{\boldsymbol{u}}_2, \vec{\boldsymbol{u}}_3, \vec{\boldsymbol{u}}_4) \longmapsto \boldsymbol{A}(\vec{\boldsymbol{u}}_1, \vec{\boldsymbol{u}}_2, \vec{\boldsymbol{u}}_3, \vec{\boldsymbol{u}}_4), \end{aligned} \tag{1.32}$$

that are linear with respect to each of their arguments and that change sign in any permutation of two arguments. The dimension of E being four, a classical result from linear algebra states that $\mathscr{A}_4(E)$ is a vector space on $\mathbb{R}$ of dimension one. Consequently all the antisymmetric four-linear forms on E are proportional to each other. Invoking the metric tensor $\boldsymbol{g}$ allows one to single out certain elements of $\mathscr{A}_4(E)$: those that result in ± 1 when applied to an orthonormal basis with respect to $\boldsymbol{g}$. Indeed, if $(\vec{\boldsymbol{e}}_\alpha)$ is an orthonormal basis of $(E, \boldsymbol{g})$ and if $\boldsymbol{A} \in \mathscr{A}_4(E)$ is such that $|\boldsymbol{A}(\vec{\boldsymbol{e}}_0, \vec{\boldsymbol{e}}_1, \vec{\boldsymbol{e}}_2, \vec{\boldsymbol{e}}_3)| = 1$, then one can show[13] that for any other orthonormal basis of E, $(\vec{\boldsymbol{e}}'_0, \vec{\boldsymbol{e}}'_1, \vec{\boldsymbol{e}}'_2, \vec{\boldsymbol{e}}'_3)$ say, $|\boldsymbol{A}(\vec{\boldsymbol{e}}'_0, \vec{\boldsymbol{e}}'_1, \vec{\boldsymbol{e}}'_2, \vec{\boldsymbol{e}}'_3)| = 1$.

Since the dimension of $\mathscr{A}_4(E)$ is one, there are only two antisymmetric four-linear forms satisfying the above property. They are the opposite of each

[13] We will prove it in Sect. 14.4.4.

other. Defining an ***orientation*** of E amounts to choosing one of these two forms that we shall denote by $\boldsymbol{\epsilon}$ and call the ***Levi–Civita tensor***[14] associated with the metric $\boldsymbol{g}$. Then

$$\boxed{(\vec{e}_\alpha) \text{ orthonormal basis} \implies \boldsymbol{\epsilon}(\vec{e}_0, \vec{e}_1, \vec{e}_2, \vec{e}_3) = \pm 1}. \tag{1.33}$$

A basis (not necessarily orthonormal) $(\vec{e}_\alpha)$ of E is qualified of ***right-handed*** iff $\boldsymbol{\epsilon}(\vec{e}_0, \vec{e}_1, \vec{e}_2, \vec{e}_3) > 0$ and ***left-handed*** iff $\boldsymbol{\epsilon}(\vec{e}_0, \vec{e}_1, \vec{e}_2, \vec{e}_3) < 0$. When applied to four vectors of E, $\boldsymbol{\epsilon}$ provides their determinant with respect to any right-handed orthonormal basis. The Levi–Civita tensor generalizes thus the notion of *mixed product* (also called *scalar triple product*) of the Euclidean three-dimensional space. The antisymmetry of Levi–Civita tensor implies that for any permutation of four elements, i.e. any member σ of the *symmetric group* $\mathfrak{S}_4$, and for any 4-tuple $(\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4)$ of vectors:

$$\boxed{\boldsymbol{\epsilon}(\vec{u}_{\sigma(1)}, \vec{u}_{\sigma(2)}, \vec{u}_{\sigma(3)}, \vec{u}_{\sigma(4)}) = (-1)^{k(\sigma)} \boldsymbol{\epsilon}(\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4)}, \tag{1.34}$$

where $k(\sigma)$ is the number of transpositions (permutation that changes only two elements) required to decompose σ. The permutation σ is called ***even*** (resp. ***odd***) iff $k(\sigma)$ is even (resp. odd).

Remark 1.14. The reader who used to manipulate the three-dimensional Levi–Civita tensor is warned that in dimension 4, a cyclic permutation is odd and not even.

Since E is a vector space over the field $\mathbb{R}$, whose characteristic is different from 2, the property of antisymmetry of the Levi–Civita tensor $\boldsymbol{\epsilon}$ is equivalent to saying that $\boldsymbol{\epsilon}$ is an ***alternating form***, i.e. it results in zero whenever two of its arguments are equal. For instance, $\boldsymbol{\epsilon}(\vec{u}, \vec{v}, \vec{u}, \vec{w}) = 0$.

Remark 1.15. The Levi–Civita tensor does more than defining an orientation in E (any four-linear form $\alpha\boldsymbol{\epsilon}$ with $\alpha > 0$ would suffice in this role): it also provides a *volume element* on spacetime, since it generalizes the mixed product and takes the value ± 1 on orthonormal bases [property (1.33)]. We shall explore this aspect of $\boldsymbol{\epsilon}$ in much details in Chap. 16, which is devoted to the integration in $\mathscr{E}$.

[14]The term *tensor* will be justified in Chap. 14.

1.6 Vector/Linear Form Duality

1.6.1 Linear Forms and Dual Space

A fundamental concept associated with a vector space, like E, is that of a ***linear form***, i.e. a function that maps vectors to real numbers:

$$\begin{aligned} \boldsymbol{\omega} : E &\longrightarrow \mathbb{R} \\ \vec{\boldsymbol{u}} &\longmapsto \boldsymbol{\omega}(\vec{\boldsymbol{u}}), \end{aligned} \tag{1.35}$$

in a linear way:

$$\forall \lambda \in \mathbb{R},\ \forall (\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}) \in E^2, \quad \boldsymbol{\omega}(\lambda \vec{\boldsymbol{u}} + \vec{\boldsymbol{v}}) = \lambda \boldsymbol{\omega}(\vec{\boldsymbol{u}}) + \boldsymbol{\omega}(\vec{\boldsymbol{v}}). \tag{1.36}$$

Linear forms are much used in physics and, more particularly, in relativity.

Like in quantum mechanics, we shall use the *bra-ket* notation to denote the action of a linear form onto a vector:

$$\boxed{\langle \boldsymbol{\omega}, \vec{\boldsymbol{u}} \rangle := \boldsymbol{\omega}(\vec{\boldsymbol{u}})}. \tag{1.37}$$

The set of all linear forms on E is canonically equipped with the structure of a vector space on $\mathbb{R}$. It is called the ***dual vector space*** of E and denoted by E^*. As a vector space, E^* has the same dimension as E, namely, four.

Given a basis $(\vec{\boldsymbol{e}}_\alpha)$ of E, there exists a unique 4-tuple of linear forms[15] $(\boldsymbol{e}^\alpha)$ that (i) constitutes a basis of E^* and (ii) satisfies

$$\langle \boldsymbol{e}^\alpha, \vec{\boldsymbol{e}}_\beta \rangle = \delta^\alpha{}_\beta, \tag{1.38}$$

where $\delta^\alpha{}_\beta$ is the Kronecker symbol defined in Sect. 1.3.2. $(\boldsymbol{e}^\alpha)$ is called the ***dual basis*** of the basis $(\vec{\boldsymbol{e}}_\alpha)$ of E.

The components of a linear form $\boldsymbol{\omega}$ in the basis $(\boldsymbol{e}^\alpha)$ are denoted by a lower index:

$$\boxed{\boldsymbol{\omega} = \omega_\alpha\, \boldsymbol{e}^\alpha}. \tag{1.39}$$

Then, for any vector $\vec{\boldsymbol{u}}$ of components (u^α) in the basis $(\vec{\boldsymbol{e}}_\alpha)$,

$$\langle \boldsymbol{\omega}, \vec{\boldsymbol{u}} \rangle = \omega_\alpha \langle \boldsymbol{e}^\alpha, \vec{\boldsymbol{u}} \rangle = \omega_\alpha \langle \boldsymbol{e}^\alpha,\ u^\beta \vec{\boldsymbol{e}}_\beta \rangle = \omega_\alpha u^\beta \underbrace{\langle \boldsymbol{e}^\alpha, \vec{\boldsymbol{e}}_\beta \rangle}_{\delta^\alpha{}_\beta} = \omega_\alpha u^\alpha.$$

[15] Following a standard notation, we are using the same letter $\boldsymbol{e}$ for the linear form $\boldsymbol{e}^\alpha$ and the vector $\vec{\boldsymbol{e}}_\alpha$: the distinction between the two is performed by the arrow and the position of the index α.

Hence, in terms of components, the action of a linear form onto a vector is simply

$$\boxed{\langle \boldsymbol{\omega}, \vec{\boldsymbol{u}} \rangle = \omega_\alpha u^\alpha}. \tag{1.40}$$

1.6.2 Metric Duality

The metric tensor $\boldsymbol{g}$ allows one to establish an isomorphism (i.e. a bijective linear map, cf. Appendix A) between E and its dual E^*, as follows:

$$\begin{aligned} \Phi_{\boldsymbol{g}} : E &\longrightarrow E^* \\ \vec{\boldsymbol{u}} &\longmapsto \underline{\boldsymbol{u}} : E \longrightarrow \mathbb{R} \\ &\qquad\quad \vec{\boldsymbol{v}} \longmapsto \boldsymbol{g}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}). \end{aligned} \tag{1.41}$$

Hence $\Phi_{\boldsymbol{g}}$ is the mapping that sends a vector $\vec{\boldsymbol{u}}$ to the linear form $\underline{\boldsymbol{u}}$, the action of which consists in performing the scalar product of vectors with $\vec{\boldsymbol{u}}$.

Proof. Thanks to the bilinearity of $\boldsymbol{g}$, $\Phi_{\boldsymbol{g}}$ is well defined (i.e. it takes its values in E^*) and is linear. Moreover, since $\boldsymbol{g}$ is nondegenerate (cf. Sect. 1.3.1), the kernel of $\Phi_{\boldsymbol{g}}$ is reduced to the zero vector. The linear mapping $\Phi_{\boldsymbol{g}}$ is then necessarily injective (cf. Appendix A). Since E and E^* are two vector spaces of the same finite dimension, we conclude that $\Phi_{\boldsymbol{g}}$ is bijective: this is a vector space isomorphism. □

Since $\Phi_{\boldsymbol{g}}$ is bijective, for any element $\boldsymbol{\omega}$ of E^*, there exists a unique vector in E, which we shall denote by $\vec{\boldsymbol{\omega}}$, such that

$$\boldsymbol{\omega} = \Phi_{\boldsymbol{g}}(\vec{\boldsymbol{\omega}}). \tag{1.42}$$

An explicit expression for $\vec{\boldsymbol{\omega}}$ can be found as follows. Let $(\vec{\boldsymbol{e}}_\alpha)$ be a basis of E. For any vector $\vec{\boldsymbol{v}} = v^\beta \vec{\boldsymbol{e}}_\beta$, (1.40) gives $\langle \boldsymbol{\omega}, \vec{\boldsymbol{v}} \rangle = \omega_\beta v^\beta$. Let then $\vec{\boldsymbol{\omega}} \in E$ be the vector whose components ω^α within the basis $(\vec{\boldsymbol{e}}_\alpha)$ are defined by

$$\omega^\alpha := g^{\alpha\beta} \omega_\beta, \tag{1.43}$$

where $(g^{\alpha\beta})$ is the inverse of the matrix of $\boldsymbol{g}$'s components in the basis $(\vec{\boldsymbol{e}}_\alpha)$ (cf. Sect. 1.3.2). Performing the matrix multiplication of (1.43) by $(g_{\alpha\beta})$, we get $g_{\alpha\beta}\omega^\alpha = \omega_\beta$, so that the relation $\langle \boldsymbol{\omega}, \vec{\boldsymbol{v}} \rangle = \omega_\beta v^\beta$ becomes

$$\langle \boldsymbol{\omega}, \vec{\boldsymbol{v}} \rangle = g_{\alpha\beta}\omega^\alpha v^\beta = \boldsymbol{g}(\vec{\boldsymbol{\omega}}, \vec{\boldsymbol{v}}).$$

This shows that $\boldsymbol{\omega}$ is the image of $\vec{\boldsymbol{\omega}}$ by $\Phi_{\boldsymbol{g}}$. Hence we may write

$$\begin{aligned} \Phi_{\boldsymbol{g}}^{-1} : E^* &\longrightarrow E \\ \boldsymbol{\omega} &\longmapsto \vec{\boldsymbol{\omega}} \ / \ \forall \vec{\boldsymbol{v}} \in E, \quad \langle \boldsymbol{\omega}, \vec{\boldsymbol{v}} \rangle = \boldsymbol{g}(\vec{\boldsymbol{\omega}}, \vec{\boldsymbol{v}}). \end{aligned} \tag{1.44}$$

The vector spaces E and E^* being both of dimension 4 over $\mathbb{R}$, they are isomorphic (as well as being isomorphic to $\mathbb{R}^4$). The specific isomorphism $\Phi_{\boldsymbol{g}}$ discussed above is the only one that can be associated naturally with the metric tensor $\boldsymbol{g}$. We shall call ***g-duality*** or ***metric duality*** the couple $(\Phi_{\boldsymbol{g}}, \Phi_{\boldsymbol{g}}^{-1})$, i.e. (i) the association by $\Phi_{\boldsymbol{g}}$ of a linear form to any vector and (ii) the association by $\Phi_{\boldsymbol{g}}^{-1}$ of a vector to any linear form.

Given a vector $\vec{\boldsymbol{u}} \in E$, we shall denote by $\underline{\boldsymbol{u}}$ the linear form image of $\vec{\boldsymbol{u}}$ by $\boldsymbol{g}$-duality, as given explicitly by (1.41). Conversely, given a linear form $\boldsymbol{\omega}$, we shall denote by $\vec{\boldsymbol{\omega}}$ the vector image of $\boldsymbol{\omega}$ by $\boldsymbol{g}$-duality, as given explicitly by (1.44). Accordingly, the scalar product $\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}}$ of two vectors $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$ can be considered as the linear form $\underline{\boldsymbol{u}}$ acting onto the vector $\vec{\boldsymbol{v}}$ or, $\boldsymbol{g}$ being symmetric, as the linear form $\underline{\boldsymbol{v}}$ acting onto the vector $\vec{\boldsymbol{u}}$:

$$\boxed{\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}} = \boldsymbol{g}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}) = \langle \underline{\boldsymbol{u}}, \vec{\boldsymbol{v}} \rangle = \langle \underline{\boldsymbol{v}}, \vec{\boldsymbol{u}} \rangle}. \tag{1.45}$$

Similarly, the scalar $\langle \boldsymbol{\omega}, \vec{\boldsymbol{v}} \rangle$ can be considered as the linear form $\boldsymbol{\omega}$ acting onto the vector $\vec{\boldsymbol{v}}$ as well as the scalar product of the vectors $\vec{\boldsymbol{\omega}}$ and $\vec{\boldsymbol{v}}$:

$$\boxed{\langle \boldsymbol{\omega}, \vec{\boldsymbol{v}} \rangle = \vec{\boldsymbol{\omega}} \cdot \vec{\boldsymbol{v}} = \boldsymbol{g}(\vec{\boldsymbol{\omega}}, \vec{\boldsymbol{v}})}. \tag{1.46}$$

Remark 1.16. We have seen in Sect. 1.6.1 that any basis $(\vec{\boldsymbol{e}}_\alpha)$ of E is associated with a unique basis of E^*: the so-called dual basis $(\boldsymbol{e}^\alpha)$. Besides, one can associate with each basis vector $\vec{\boldsymbol{e}}_\alpha$ a linear form $\underline{\boldsymbol{e}}_\alpha$ via the metric duality introduced above. A legitimate question is then whether the 4-tuple $(\underline{\boldsymbol{e}}_\alpha)$ coincide with the basis $(\boldsymbol{e}^\alpha)$. The answer is no. Indeed, thanks to (1.38) and (1.12),

$$\langle \boldsymbol{e}^\alpha, \vec{\boldsymbol{e}}_\beta \rangle = \delta^\alpha{}_\beta \qquad \text{and} \qquad \langle \underline{\boldsymbol{e}}_\alpha, \vec{\boldsymbol{e}}_\beta \rangle = \boldsymbol{g}(\vec{\boldsymbol{e}}_\alpha, \vec{\boldsymbol{e}}_\beta) = g_{\alpha\beta}. \tag{1.47}$$

Now, the matrix $(g_{\alpha\beta})$ is necessarily different from $(\delta^\alpha{}_\beta) = \mathrm{diag}(1, 1, 1, 1)$ because of the signature $(-, +, +, +)$ of $\boldsymbol{g}$.

Remark 1.17. The mappings $\Phi_{\boldsymbol{g}}$ and $\Phi_{\boldsymbol{g}}^{-1}$, which establish the metric duality, are also called the ***musical isomorphisms***, because a flat and a sharp symbols are sometimes used instead of, respectively, the underbar and the arrow used here:

$$\boldsymbol{u}^\flat = \underline{\boldsymbol{u}} \qquad \text{and} \qquad \boldsymbol{\omega}^\sharp = \vec{\boldsymbol{\omega}}.$$

1.7 Minkowski Spacetime

At this stage, we have introduced all the mathematical tools required for the foundation of special relativity:

We shall call ***Minkowski spacetime*** the 4-tuple $(\mathscr{E}, \boldsymbol{g}, \mathscr{I}^+, \boldsymbol{\epsilon})$ where:

- $\mathscr{E}$ is an affine space of dimension four over $\mathbb{R}$, the underlying vector space being denoted by E (E is of course isomorphic to $\mathbb{R}^4$); $\mathscr{E}$ is called *spacetime* and its elements are called *events*.
- $\boldsymbol{g}$ is a bilinear form on E that is symmetric, nondegenerate and has the signature $(-,+,+,+)$; $\boldsymbol{g}$ is called the *metric tensor*.
- $\mathscr{I}^+$ is one of the two sheets of $\boldsymbol{g}$'s null cone, called the *future null cone*.
- $\boldsymbol{\epsilon}$ is a four-linear form on E that is antisymmetric and results in ± 1 when applied to any basis that is orthonormal with respect to $\boldsymbol{g}$; $\boldsymbol{\epsilon}$ is called the *Levi–Civita tensor* associated with the metric $\boldsymbol{g}$.

Note that once $\mathscr{E}$ and $\boldsymbol{g}$ have been set, there are only two possible choices for $\mathscr{I}^+$ and two possible choices for $\boldsymbol{\epsilon}$. Choosing $\mathscr{I}^+$ corresponds to choosing a *time arrow* (Sect. 1.4), while choosing $\boldsymbol{\epsilon}$ makes E an *oriented* vector space (Sect. 1.5). The Minkowski spacetime is thus a four-dimensional affine space, endowed with a scalar product of signature $(-,+,+,+)$, a time arrow and an orientation. In addition, the scalar product is employed to establish a duality between the vector space E and the space E^* of linear forms on E.

We depicted the Newtonian spacetime in Fig. 1.2, along with its fundamental structure, namely, the foliation $(\Sigma_t)_{t\in\mathbb{R}}$ by three-dimensional spaces at fixed absolute time. The fundamental structure of Minkowski spacetime, the metric tensor $\boldsymbol{g}$, cannot be depicted so simply, but one may represent it by drawing $\boldsymbol{g}$'s null cone at various points, thereby obtaining Fig. 1.8.

Having set the mathematical framework of special relativity, we shall proceed with physics. More specifically, there remains (i) to define relations between physical concepts like time, space, particle and photon and mathematical objects of Minkowski spacetime and (ii) to express physical laws in terms of mathematical operations in Minkowski spacetime.

Remark 1.18. The definitions and properties given in this chapter remain valid in spaces of dimension $n > 4$, provided that these spaces are endowed with a metric of signature $(-,+,\cdots,+)$ (1 minus sign and $n-1$ plus signs). Such spaces are encountered, for instance, in string theory ($n = 10$ or $n = 11$) (Penrose 2007).

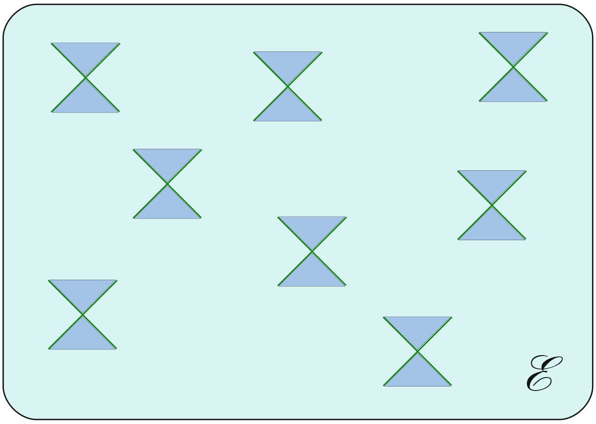

Fig. 1.8 Spacetime $\mathscr{E}$ of special relativity. Two dimensions have been suppressed, so that the affine space $\mathscr{E}$ appears like a plane. The fundamental structure of special relativity, the metric tensor $\boldsymbol{g}$, is represented by its null cone drawn at various points. This figure is to be compared with Fig. 1.2 depicting Newtonian spacetime

Historical note: *The concept of four-dimensional spacetime is not to be found in the founding article of special relativity, written by Albert Einstein*[16] *in 1905 (Einstein 1905b). It appeared first in a long article of Henri Poincaré,*[17] *known as the "Palermo memoir", written in 1905 and published the year after (Poincaré 1906). In this text, Poincaré introduces* $(\mathrm{i}t, x, y, z)$*, with* $\mathrm{i}^2 = -1$*, as the coordinates of a point in a four-dimensional space and considers the quadratic form* $-t^2 + x^2 + y^2 + z^2$ *as giving the "distance" in that space. Poincaré also uses the associated bilinear form, which we call today the metric tensor, to form scalar products. But it is only in 1908, with Hermann Minkowski,*[18] *that the concept of four-dimensional spacetime took all its extent (Minkowski 1908) (cf. also a preliminary version in (Minkowski 1907)). As a matter of fact, the title of a synthesis work by Minkowski published*

[16]**Albert Einstein** (1879–1955): German theoretical physicist (he was granted the Swiss citizenship in 1901 and the American one in 1940) who was the main founder of relativity, both special and general, and the author of major advances in quantum theory and Brownian motion; he got the 1921 Nobel Prize in Physics for his explanation of the photoelectric effect and his contribution to theoretical physics (without any explicit mention of relativity!).

[17]**Henri Poincaré** (1854–1912): French mathematician and theoretical physicist, considered as the last universal scientist, mastering most of the mathematics and physics of his time. His work ranges from algebraic topology to differential equations, going through celestial mechanics, the theory of chaos and, of course, relativity.

[18]**Hermann Minkowski** (1864–1909): Mathematician born in the Russian Empire, who became a German citizen at 8 years old. Specialist of number theory and geometry, he became interested in mathematical physics and more specifically in electrodynamics and relativity, at the University of Göttingen, under the influence of David Hilbert. He died prematurely, from appendicitis, at the age of 44.

in 1909 is "Space and Time" *(Minkowski 1909). It contains the famous sentence:* "space for itself, and time for itself shall completely reduce to a mere shadow, and only some sort of union of the two shall preserve independence". *The spacetime $\mathscr{E}$ is clearly defined in this text and called* Welt *(*world *or* universe*), with the traditional parameters (t, x, y, z) considered as coordinates of points in $\mathscr{E}$. Minkowski also defined the vectors* (Vektor) *on that space. The metric tensor* $\boldsymbol{g}$ *is not introduced explicitly, but Minkowski defines nevertheless the orthogonality of two vectors by the condition $-c^2 t_1 t_2 + x_1 x_2 + y_1 y_2 + z_1 z_2 = 0$, where (ct_1, x_1, y_1, z_1) and (ct_2, x_2, y_2, z_2) are the components of the two vectors in a basis called orthonormal in the present formulation. This fully justifies the name* Minkowski matrix *given to (1.17). The appellations* timelike vector *and* spacelike vector *have been forged by Minkowski. Moreover, Minkowski has introduced the two-dimensional spacetime diagrams, as that of Fig. 1.6. For more details about Minkowski's contribution, one could read Miller (1998) (Sect. 7.4.6), Walter (1999a) and Damour (2008), the latter putting the emphasis on the comparison with Poincaré. It is worth mentioning that Arnold Sommerfeld*[19] *contributed a lot to popularize Minkowski's four-dimensional approach, as early as 1909. In particular, Sommerfeld forged the term* four-vector *(Sommerfeld 1910a,b).*

The notation $\boldsymbol{g}$ *for the metric tensor originates from the works of the Italian mathematicians Gregorio Ricci (1853–1925) and Tullio Levi–Civita (1873–1941) around 1900. They used it as the initial of* geometry. *Within the context of relativity, it first appeared in an article by Einstein in 1913 (Einstein 1913a).*

1.8 Before Going Further...

In this chapter, the following notions have been defined. By alphabetic order:

- Affine coordinate system
- Affine space
- Alternating form
- Bilinear form
- Dual basis
- Dual vector space
- Event
- Four-linear form
- Future-directed (resp. past-directed) null vector
- Future-directed (resp. past-directed) timelike vector
- Future null cone

[19]**Arnold Sommerfeld** (1868–1951) : German theoretical physicist, pioneer of quantum mechanics. He authored many works in atomic physics and introduced the fine structure constant α in 1915. Sommerfeld supervised many theses, including those of Werner Heisenberg and Wolfgang Pauli (cf. p. 542), and wrote renowned physics textbooks.

- $\boldsymbol{g}$-duality
- Hyperplane
- Left-handed basis
- Levi–Civita tensor
- Linear form
- Lorentzian signature
- Matrix of the metric tensor with respect to a vector basis
- Minkowski matrix
- Minkowski spacetime
- Metric duality
- Metric tensor
- Nondegenerate symmetric bilinear form
- Norm of a vector with respect to the metric tensor
- Null cone
- Null vector
- Orientation of spacetime
- Orthogonal vectors
- Orthonormal vector basis
- Past null cone
- Positive definite symmetric bilinear form
- Right-handed basis
- Scalar product
- Signature of a symmetric bilinear form
- Spacelike vector
- Spacetime
- Time arrow
- Timelike vector
- Unit vector
- Vector on spacetime

The reader is invited to check whether he has assimilated each of these notions before proceeding further.

Chapter 2
Worldlines and Proper Time

2.1 Introduction

Having introduced the mathematical framework of special relativity, we may move to the basics of (non-quantum) physics, namely, the description of the motion of a particle or a physical system idealized to a pointlike particle. We shall notably see the interpretation of the metric tensor $\boldsymbol{g}$ as the operator giving the elapsed time along the trajectory of a particle.

2.2 Worldline of a Particle

Special relativity being a non-quantum theory,[1] particles are described as points, as in classical mechanics. Actually, we shall use the word ***particle*** or ***point particle*** to cover either an elementary particle or a physical system whose spatial extension can be neglected at the scale of the phenomenon under study. A "particle at a given instant" will be represented by a point in the spacetime $\mathscr{E}$ (an *event*), and the "successive positions" of the particle will draw a one-dimensional curve in the affine space $\mathscr{E}$. Let us note that, at this stage, we cannot give some meaning to the phrase "at a given instant" if we wish to preserve the mixed space/time character of $\mathscr{E}$ and not to split it into some space part and some time part. Therefore, we shall define a particle by its entirety in spacetime, namely, a curve of $\mathscr{E}$, which we shall call the ***worldline*** of the particle.

The link between physics and the mathematics introduced in Chap. 1 consists in stating that the so-called *massive* particles do not follow any kind of worldline in Minkowski spacetime, but only those that are timelike:

[1] Minkowski spacetime is however the arena for relativistic quantum field theory.

É. Gourgoulhon, *Special Relativity in General Frames*, Graduate Texts in Physics,
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Any ***massive particle*** is represented by a piecewise twice continuously differentiable curve $\mathscr{L}$ of Minkowski spacetime $(\mathscr{E}, \boldsymbol{g})$ such that any vector tangent to $\mathscr{L}$ is timelike.

Let us recall that a vector $\vec{\boldsymbol{v}} \in E$ is timelike iff $\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{v}} = \boldsymbol{g}(\vec{\boldsymbol{v}}, \vec{\boldsymbol{v}}) < 0$ (cf. Sect. 1.3.4) and that a *piecewise twice continuously differentiable* curve means that there exists some function

$$\begin{aligned} \varphi : \mathbb{R} &\longrightarrow \mathscr{E} \\ \lambda &\longmapsto A = \varphi(\lambda) \end{aligned} \tag{2.1}$$

that is (i) twice differentiable with a continuous second derivative (i.e. of class C^2) on each interval of a finite subdivision of $\mathbb{R}$ and (ii) such that $\mathscr{L}$ is the image set of φ: $\mathscr{L} = \varphi(\mathbb{R})$. If φ is injective, it is called a ***parametrization*** of $\mathscr{L}$.

Remark 2.1. Of course, for a given worldline $\mathscr{L}$, there exists an infinite number of parametrizations: if φ is one of them, any bijective function $f : \mathbb{R} \to \mathbb{R}$ of class C^2 induces a new parametrization $\tilde{\varphi} := \varphi \circ f$. A priori, a parametrization of $\mathscr{L}$ is a purely mathematical operation. We shall introduce in Sect. 2.3 a parametrization with physical grounds: that provided by the "elapsed time" (the so-called *proper time*) along $\mathscr{L}$.

Remark 2.2. We may consider the above statement as the formal definition of a *massive particle*. The notion of *mass* will be introduced in Chap. 9, and we shall see that indeed massive particles, as defined above, have a nonvanishing mass.

Remark 2.3. By demanding that the worldline be timelike, we exclude hypothetical particles called ***tachyons*** (Bilaniuk et al. 1962; Feinberg 1967; Recami 1987; Boratav and Kerner 1991; Fayngold 2002). These particles would on the contrary move on spacelike worldlines. Note that there is no consistent relativistic theory that allows a given worldline to change its type on some part of it: a worldline is either always timelike (ordinary massive particles), null (photons, Sect. 2.5) or spacelike (tachyons). We shall elaborate more on tachyons in Sect. 4.3.3.

A parametrization φ of $\mathscr{L}$ induces a one-parameter family of vectors of E—at each point of $\mathscr{L}$, we may consider the ***derivative vector*** of φ at this point:

$$\forall \lambda \in \mathbb{R}, \qquad \vec{\boldsymbol{v}}(\lambda) := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \overrightarrow{A(\lambda)A(\lambda + \varepsilon)}, \tag{2.2}$$

where we have used the notation $A(\lambda)$ for $\varphi(\lambda)$ [generic point of $\mathscr{L}$, cf. (2.1)]. $\vec{\boldsymbol{v}}$ is called the ***field of tangent vectors*** associated with the parametrization φ. One may give a more "physical" expression to $\vec{\boldsymbol{v}}$: denoting by $\mathrm{d}\lambda$ the increase ε of the parameter λ and by $\mathrm{d}\vec{\boldsymbol{x}}$ the infinitesimal vector joining the point $A(\lambda)$ to the point $A(\lambda + \mathrm{d}\lambda)$ (cf. Fig. 2.1), we get

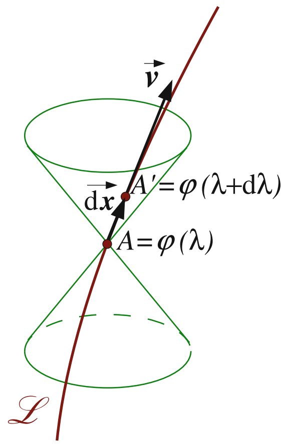

Fig. 2.1 Worldline of a massive particle, with the tangent vector $\vec{v}$ associated with the parametrization $\varphi(\lambda)$. The null cone at point A is shown. Since $\vec{v}$ is timelike, it is located inside the null cone

$$\forall \lambda \in \mathbb{R}, \qquad \boxed{\vec{v}(\lambda) = \frac{\mathrm{d}\vec{x}}{\mathrm{d}\lambda}}. \tag{2.3}$$

From the definition of a worldline, the vector $\vec{v}(\lambda)$ must be timelike for all values of the parameter λ: $\vec{v}(\lambda) \cdot \vec{v}(\lambda) < 0$.

If $(O; \vec{e}_\alpha)$ is an affine frame of $\mathscr{E}$ (cf. Sect. 1.2.3) and $(x^\alpha(\lambda))$ the affine coordinates of $A = \varphi(\lambda)$ in this frame, the components of the tangent vector $\vec{v}(\lambda)$ with respect to the basis $(\vec{e}_\alpha)$ are the derivatives of the functions $x^\alpha(\lambda)$: $v^\alpha(\lambda) = \mathrm{d}x^\alpha/\mathrm{d}\lambda$; hence,

$$\vec{v}(\lambda) = \frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\, \vec{e}_\alpha. \tag{2.4}$$

2.3 Proper Time

2.3.1 Definition

We have already noticed in Sect. 1.3.1 that the metric tensor $\boldsymbol{g}$ does not define a metric on $\mathscr{E}$ in the strict mathematical sense and that it should be called instead *pseudo-metric tensor* (cf. Remark 1.6 p. 8). As a consequence, the norm with respect to $\boldsymbol{g}$ introduced in Sect. 1.3.5, $\|\ \|_{\boldsymbol{g}}$, is not a norm in the mathematical sense. In particular, $\|\vec{v}\|_{\boldsymbol{g}} = 0$ is not equivalent to $\vec{v} = 0$. However, if the "norm" $\|\ \|_{\boldsymbol{g}}$ is taken only on timelike vectors (such as the tangent vectors to massive particle worldlines), i.e. if one considers the mapping

$$\begin{array}{rcl} E_{\text{timelike}} & \longrightarrow & \mathbb{R}^+ \\ \vec{v} & \longmapsto & \|\vec{v}\|_{\boldsymbol{g}} = \sqrt{-\boldsymbol{g}(\vec{v}, \vec{v})}, \end{array} \tag{2.5}$$

then one gets a function that vanishes only for $\vec{v} = 0$, as for any norm.[2] Accordingly, one may use $\boldsymbol{g}$ to measure "lengths" along a given worldline. The fundamental physical interpretation of the metric tensor $\boldsymbol{g}$ consists in stating that these "lengths" correspond to the elapsed time along the worldline:

Let A and A' be two infinitely close events on the worldline $\mathscr{L}$ of a given massive particle (cf. Fig. 2.1). Let $\mathrm{d}\vec{x}$ be the infinitesimal vector connecting A and A'. The vector $\mathrm{d}\vec{x}$ is tangent to $\mathscr{L}$, and from the definition of a worldline, it is timelike. We may then set

$$\boxed{\begin{cases} c\,\mathrm{d}\tau := \|\mathrm{d}\vec{x}\|_{g} = \sqrt{-\boldsymbol{g}(\mathrm{d}\vec{x},\mathrm{d}\vec{x})} & \text{if } \mathrm{d}\vec{x} \text{ is future-directed} \\ c\,\mathrm{d}\tau := -\|\mathrm{d}\vec{x}\|_{g} = -\sqrt{-\boldsymbol{g}(\mathrm{d}\vec{x},\mathrm{d}\vec{x})} & \text{if } \mathrm{d}\vec{x} \text{ is past-directed} \end{cases}}\,. \tag{2.6}$$

Let us recall that the future/past-directed properties have been defined in Sect. 1.4. Thanks to the c factor (cf. Sect. 1.2.4), the dimension of $\mathrm{d}\tau$ is time, $\boldsymbol{g}$ having no dimension and $\mathrm{d}\vec{x}$ having the dimension of length (cf. the convention adopted in Sect. 1.2.4). $\mathrm{d}\tau$ is called the ***proper time*** elapsed between the events A and A' on $\mathscr{L}$.

If the displacement $\mathrm{d}\vec{x}$ is represented by its components $(\mathrm{d}x^{\alpha})$ in some orthonormal basis of $(E,\boldsymbol{g})$, the scalar product $\boldsymbol{g}(\mathrm{d}\vec{x},\mathrm{d}\vec{x})$ can be expressed according to (1.18), so that (2.6) becomes

$$\boxed{c\,\mathrm{d}\tau = \pm\sqrt{(\mathrm{d}x^0)^2 - (\mathrm{d}x^1)^2 - (\mathrm{d}x^2)^2 - (\mathrm{d}x^3)^2},}_{\text{orthonormal basis}} \tag{2.7}$$

where the sign $\pm$ corresponds to the two cases considered in (2.6).

Given a parametrization $\varphi(\lambda)$ of $\mathscr{L}$, one may express the proper time in terms of the associated tangent vector field $\vec{v}$. Let us suppose that $\vec{v}$ is future-directed. Would this not be the case, the change of parameter $\lambda \mapsto -\lambda$ would provide a future-directed tangent vector. Then, we have

$$\mathrm{d}\vec{x} = \vec{v}\,\mathrm{d}\lambda, \tag{2.8}$$

where $\mathrm{d}\lambda$ is the difference of parameter between A' and A: $A = \varphi(\lambda)$, $A' = \varphi(\lambda + \mathrm{d}\lambda)$ (cf. Fig. 2.1). Thanks to $\boldsymbol{g}$'s bilinearity, (2.6) can be written

$$c\,\mathrm{d}\tau = \sqrt{-\boldsymbol{g}(\vec{v},\vec{v})}\,\mathrm{d}\lambda. \tag{2.9}$$

[2]It is however still not a norm in the mathematical meaning, for it does not satisfy the triangle inequality $\|\vec{v}+\vec{w}\| \le \|\vec{v}\| + \|\vec{w}\|$.

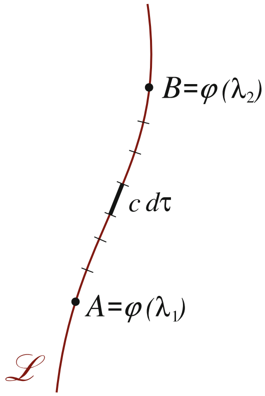

Fig. 2.2 Proper time between events A and B along a worldline $\mathscr{L}$

Remark 2.4. Choosing a parametrization such that $\vec{\boldsymbol{v}}$ is future-directed ensures that $\mathrm{d}\tau$ has the correct sign, namely, is positive (resp. negative) if $\mathrm{d}\vec{\boldsymbol{x}}$ is future-directed (resp. past-directed). Let us stress that, although Eq. (2.9) lets appear the parametrization φ of $\mathscr{L}$, the value of $\mathrm{d}\tau$ is independent of that parametrization, as it is clear on (2.6).

The definition of proper time can be extended to events with a finite separation along a worldline, by integrating (2.6) between these two events. Hence if A and B are two events of some worldline $\mathscr{L}$ (cf. Fig. 2.2) and if φ is a parametrization of $\mathscr{L}$ such that $A = \varphi(\lambda_1)$ and $B = \varphi(\lambda_2)$, we set

$$\boxed{\tau(A, B) := \int_A^B \mathrm{d}\tau = \frac{1}{c}\int_{\lambda_1}^{\lambda_2} \sqrt{-\boldsymbol{g}\left(\vec{\boldsymbol{v}}(\lambda), \vec{\boldsymbol{v}}(\lambda)\right)}\,\mathrm{d}\lambda}, \tag{2.10}$$

where $\vec{\boldsymbol{v}}(\lambda)$ is the tangent vector field associated with the parametrization φ. As for $\mathrm{d}\tau$, $\tau(A, B)$ does not depend on the choice of the parametrization φ. On the other hand, it depends on the worldline connecting A to B.

2.3.2 *Ideal Clock*

Let us call ***clock*** any physical device that (i) can be reduced to a point particle (at the scale of the phenomenon under study), (ii) follows a timelike worldline $\mathscr{L}$ and (iii) provides a sequence of "signals", i.e. a sequence of events $\ldots, E_{-1}, E_0, E_1, E_2, \ldots$ sampling $\mathscr{L}$ (Fig. 2.3). Each E_k is called a ***tick***.

An ***ideal clock*** is then defined as a clock for which the proper time $\tau(E_k, E_{k+N})$ between two ticks E_k and E_{k+N} is equal to a constant K times the number N of elapsed ticks:

$$\tau(E_k, E_{k+N}) = K\,N. \tag{2.11}$$

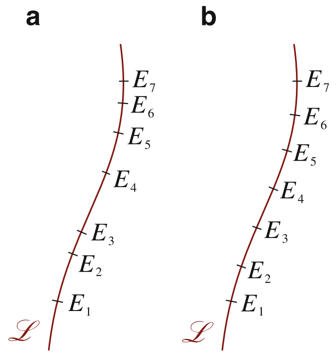

Fig. 2.3 (**a**) Generic clock; (**b**) ideal clock

Among all the clocks, ideal clocks are characterized by the fact that the proportionality; factor K is the same at each point of their worldline (Fig. 2.3). In other words, the time indicated by an ideal clock is the proper time along the clock's worldline.

Remark 2.5. Relativity has banished the concept of *absolute time* (cf. Sect. 1.2.5). It however introduces along each worldline a privileged time: that given by the metric tensor according to (2.6). An ideal clock is a clock that displays this time. Obviously it varies from one worldline to the other, i.e. the quantity $\tau(A, B)$ defined by (2.10) depends upon the worldline connecting A and B. It is in that sense that relativity has suppressed absolute time.

An ideal clock is a "theoretical" device that can be more or less well approximated by an actual device. To know whether a given experimental clock constitutes a good approximation of an ideal clock, one may check if the laws of kinematics and dynamics (which will be developed in the coming chapters and are expressed in terms of the proper time) are satisfied when experiments are described with the time given by this clock. For instance, a pendulum held fixed with respect to the Earth constitutes a relatively good approximation (at the human scale!) of an ideal clock. But this is no longer true in a strongly accelerated frame with respect to the Earth: the pendulum motion looses any periodicity if the acceleration is not constant. An atomic clock constitutes a much better approximation of an ideal clock, because it provides a time that depends very weakly on its state of acceleration, at least for accelerations smaller than the centripetal acceleration of an electron around the atomic nucleus, which is about $10^{23}\,\mathrm{m\,s^{-2}}$.

Remark 2.6. Since it is related to the fundamental object of relativity, namely, the metric tensor $\boldsymbol{g}$, the proper time is the only truly *physical time*, in the following meaning. The definition of time along a given worldline is a priori arbitrary: one can choose the time provided by any clock. The distinctive feature of proper time is that the physical laws expressed in terms of it are simpler than if expressed in terms of an arbitrary time, because the basic physical laws involve the metric tensor, to which proper time is directly related. Considering the example mentioned above, the pendulum beats are periodic functions of the proper time in an inertial frame. To paraphrase Poincaré (1898), we may say that it is for a matter of *commodity* that one uses proper time and not an arbitrary time.

Remark 2.7. When considering a human being, the proper time is also the most convenient one to describe her/his physiological evolution, given the physical nature of physiological processes. Admitting that the physiological time is indeed the one perceived by consciousness, one may think about the proper time along a worldline as the time "felt" by a human observer moving along this worldline.

Remark 2.8. The fundamental concept that appears once the metric tensor and the worldlines have been introduced is that of *time* and not of *length*. We shall discuss this further below.

2.4 Four-Velocity and Four-Acceleration

2.4.1 Four-Velocity

We have seen in Sect. 2.2 that one may associate many tangent vector fields to a given worldline $\mathscr{L}$, namely, the tangent fields linked to all possible parametrizations of $\mathscr{L}$. The introduction of proper time in Sect. 2.3 allows us to select a tangent vector field independent of any parametrization and thereby intrinsic to the worldline: the ***four-velocity***, or ***4-velocity*** for short, of a massive particle evolving along a worldline $\mathscr{L}$ is the vector of E defined at any point $A \in \mathscr{L}$ by

$$\boxed{\vec{u} := \frac{1}{c}\frac{\mathrm{d}\vec{x}}{\mathrm{d}\tau}}, \tag{2.12}$$

where $\mathrm{d}\vec{x}$ is an infinitesimal vector tangent to $\mathscr{L}$ and future-directed (cf. Sect. 1.4) and $\mathrm{d}\tau$ is the proper time interval corresponding to $\mathrm{d}\vec{x}$ via (2.6). If one wishes to give a rigorous mathematical meaning to (2.12), it suffices to parametrize the worldline $\mathscr{L}$ by c times its proper time: $\lambda = c\tau$. Such a parametrization is unique, up to the choice of some origin. The vector $\vec{u}$ is then nothing but the derivative of that parametrization, as defined in Sect. 2.2. As a derivative, it is of course independent of the origin of proper time.

If $\vec{v}$ is a future-directed tangent vector field associated with some parametrization $\varphi(\lambda)$ of $\mathscr{L}$, we may insert (2.8) and (2.9) into (2.12) and get

$$\vec{u} = \frac{\vec{v}}{\sqrt{-\boldsymbol{g}(\vec{v}, \vec{v})}} = \frac{\vec{v}}{\|\vec{v}\|_{\boldsymbol{g}}}. \tag{2.13}$$

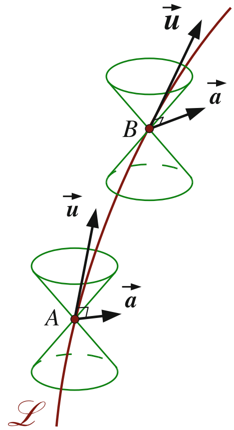

Fig. 2.4 4-velocity $\vec{u}$ and 4-acceleration $\vec{a}$ at two points A and B of a timelike worldline $\mathscr{L}$

This identity can be viewed as the definition of a *unit* tangent vector ($\vec{u}$) from an arbitrary tangent vector ($\vec{v}$). It is actually trivial to check on (2.13) that[3]

$$\boxed{\vec{u} \cdot \vec{u} = -1}. \tag{2.14}$$

We could even have introduced the 4-velocity $\vec{u}$ as the unique future-directed unit vector tangent to $\mathscr{L}$. The definition (2.12) has more the aspect of a "velocity". Note however that $\vec{u}$ is dimensionless, thanks to the factor $1/c$ in (2.12).

Remark 2.9. Many authors define the 4-velocity with the dimension of a velocity, by setting $\vec{u} := \mathrm{d}\vec{x}/\mathrm{d}\tau$ instead of (2.12). Equation (2.14) becomes then $\vec{u} \cdot \vec{u} = -c^2$. We follow here the convention of Landau and Lifshitz (1975), preferring a dimensionless 4-velocity, because many expressions are simplified when $\vec{u}$ is a unit vector. Moreover, from a pedagogical point of view, the dimensionless character of the 4-velocity is valuable in avoiding the confusion with an "ordinary" velocity, which is a different concept (in particular, it is relative to some observer, contrary to the 4-velocity, cf. Remark 2.10 below).

The property (2.14) implies that the 4-velocity belongs to the set $\mathscr{U}^+$ introduced in Sect. 1.4.3:

$$\boxed{\vec{u} \in \mathscr{U}^+}. \tag{2.15}$$

Conversely, any element of $\mathscr{U}^+$ can be considered as a 4-velocity. We conclude that $\mathscr{U}^+$ is nothing but the set of all possible 4-velocities.

The 4-velocity at two points A and B of a worldline is depicted in Fig. 2.4. The null cone of $\boldsymbol{g}$ is also drawn at these two points (cf. Sect. 1.4): as a timelike future-directed vector, $\vec{u}$ is located inside the future null cone $\mathscr{I}^+$.

[3]Let us recall that the notation $\vec{u} \cdot \vec{u}$ stands for $\boldsymbol{g}(\vec{u}, \vec{u})$.

Remark 2.10. The reader might have been surprised by the fact that, in the theory of *relativity*, the 4-velocity has not been defined *relatively* to a frame or an observer. On the contrary, it has been introduced as an *absolute* quantity, which depends only on the considered worldline, the latter being obviously independent of any observer. Actually, the 4-velocity is different from a velocity and is not a directly measurable quantity. After having introduced the concept of observer in Chap. 3, we shall define in Chap. 4 the "ordinary" velocity of a point particle with respect to an observer. It will be a function of the 4-velocities of the particle and the observer, and will be a measurable quantity, as the ratio of a length by a time.

2.4.2 Four-Acceleration

It is natural to define the ***four-acceleration***, or ***4-acceleration*** for short, as the vector of E that measures the variation of the 4-velocity field $\vec{\boldsymbol{u}}$ along the worldline $\mathscr{L}$:

$$\boxed{\vec{\boldsymbol{a}} := \frac{1}{c}\frac{\mathrm{d}\vec{\boldsymbol{u}}}{\mathrm{d}\tau}}, \tag{2.16}$$

where τ stands for the proper time along $\mathscr{L}$. The above expression takes a rigorous mathematical meaning if $\mathscr{L}$ is parametrized by $\lambda = c\tau$: the vector $\vec{\boldsymbol{a}}$ is then nothing but the second derivative of this parametrization.

$\vec{\boldsymbol{u}}$ being dimensionless and $c\tau$ having the dimension of a length [cf. Eq. (2.6)], the dimension of the 4-acceleration is that of the inverse of a length and not that of an acceleration.[4]

Two basic properties of the 4-acceleration follow easily from its definition:

- $\vec{\boldsymbol{a}}$ is orthogonal to $\vec{\boldsymbol{u}}$ (with respect to the metric $\boldsymbol{g}$):

$$\boxed{\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{u}} = 0}. \tag{2.17}$$

Proof. One has

$$\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{u}} = \frac{1}{c}\frac{\mathrm{d}\vec{\boldsymbol{u}}}{\mathrm{d}\tau} \cdot \vec{\boldsymbol{u}} = \frac{1}{2c}\frac{d}{\mathrm{d}\tau}(\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{u}}) = \frac{1}{2c}\frac{d}{\mathrm{d}\tau}(-1) = 0. \qquad \square$$

- $\vec{\boldsymbol{a}}$ is either the zero vector or a spacelike vector.

Proof. If $\vec{\boldsymbol{a}} \neq 0$, thanks to (2.17) and by means of the Gram–Schmidt process (Deheuvels 1981), we may find an orthogonal basis of E of the type $(\vec{\boldsymbol{u}}, \vec{\boldsymbol{a}}, \vec{\boldsymbol{e}}_1, \vec{\boldsymbol{e}}_2)$. In this basis, taking into account (2.14), the matrix of $\boldsymbol{g}$ is

[4] As for the 4-velocity, which has not the dimension of a velocity, cf. Remark 2.9.

$$g_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{a}} & 0 & 0 \\ 0 & 0 & \vec{\boldsymbol{e}}_1 \cdot \vec{\boldsymbol{e}}_1 & 0 \\ 0 & 0 & 0 & \vec{\boldsymbol{e}}_2 \cdot \vec{\boldsymbol{e}}_2 \end{pmatrix}.$$

Since the signature of $\boldsymbol{g}$ is $(-, +, +, +)$, the diagonal terms but -1 are necessarily strictly positive (Sylvester's law of inertia, Sect. 1.3.1); hence, in particular, $\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{a}} > 0$, which proves that $\vec{\boldsymbol{a}}$ is spacelike. □

We conclude that

$$\boxed{\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{a}} \geq 0}, \tag{2.18}$$

with $\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{a}} = 0$ iff $\vec{\boldsymbol{a}} = 0$.

It is worth to note that the above demonstration uses only the fact that $\vec{\boldsymbol{a}}$ is orthogonal to $\vec{\boldsymbol{u}}$; we have therefore established a very useful property:

Any vector orthogonal to a timelike vector is necessarily spacelike or zero.

The 4-acceleration at two points of a worldline is depicted in Fig. 2.4. Being spacelike, this vector is located outside the null cone, contrary to $\vec{\boldsymbol{u}}$. Note that the orthogonality between $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{u}}$ does not imply the orthogonality in the usual (Euclidean) sense of the arrows representing $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{u}}$ in Fig. 2.4 (cf. the discussion about the graphical representation of vectors in Sect. 1.3.6). We shall see in Sect. 2.7.3 a geometrical interpretation of the 4-acceleration involving the *curvature* of the worldline.

Remark 2.11. As for the 4-velocity (cf. Remark 2.10), the 4-acceleration is an absolute quantity, independent of any frame or observer.

Historical note: *The concepts of* worldline*,* 4-velocity *and* 4-acceleration *have been introduced by Hermann Minkowski (cf. p. 26). They appear in a publication of 1908 (Minkowski 1908) and play a central role in the famous article on spacetime published the year after (Minkowski 1909) and discussed at the end of Chap. 1. Note however that, as soon as 1905, in the "Palermo memoir" (Poincaré 1906), Henri Poincaré (cf. p. 26) let appear a four-dimensional vector that was nothing but the 4-velocity, although without any explicit mention of a worldline. The concept of* proper time*, as exposed above, namely, the length given by the metric tensor along a worldline, is also due to Minkowski: in the publications (Minkowski 1908) and (Minkowski 1909), he wrote the relations (2.6) and (2.10) (making use of the components (1.17) of* $\boldsymbol{g}$ *in an orthonormal basis). Besides, the relation (2.17) expressing the orthogonality of the 4-acceleration and the 4-velocity appears clearly in the 1909 text (Minkowski 1909).*

2.5 Photons

2.5.1 Null Geodesics

In Sect. 2.2, we have postulated that massive particles follow worldlines that are timelike. We shall now define the worldlines of massless particles, the first of them being photons. As for any point particle, a photon is represented by a one-dimensional curve in Minkowski spacetime (its worldline). Whereas the worldlines of massive particles show a great variety (all the curves with timelike tangent vectors), photons are compelled to follow quite specific curves, *straight lines*, the direction vector of which is null:

> In vacuum, a ***massless particle***, and in particular a photon, is represented by a straight line of $\mathscr{E}$ whose direction vector is a null vector of the metric $\boldsymbol{g}$, i.e. a vector $\vec{\boldsymbol{v}}$ obeying $\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{v}} = 0$. Such a line is called a ***null geodesic*** of spacetime. If the particle is a photon, it is also called a ***light ray***.

This principle justifies the choice of qualifier *lightlike* given to null vectors of $\boldsymbol{g}$ (cf. Sect. 1.3.4). When we shall treat electromagnetism (Chaps. 17–20), we shall verify that the wave solutions to Maxwell equations in vacuum propagate along null directions of the metric tensor.

Remark 2.12. A null geodesic is a special case of a ***null curve***, i.e. a curve of $\mathscr{E}$ whose tangent vectors are null vectors. There exists null curves that are not straight lines and thus not null geodesics. An example is the helix defined by the parametric equation $x^0(\lambda) = r\lambda$, $x^1(\lambda) = r\cos\lambda$, $x^2(\lambda) = r\sin\lambda$, $x^3(\lambda) = 0$, with $r > 0$, in some affine coordinate system (x^α) associated with an orthonormal basis.

Remark 2.13. The notion of *proper time* introduced for massive particles cannot be extended to photons, because (2.6) would result in $\mathrm{d}\tau = 0$ (for $\mathrm{d}\vec{\boldsymbol{x}}$ is null along a null geodesic). This would mean that an ideal clock carried by a photon is frozen. Consequently, the 4-velocity of a photon cannot be defined. In other words, there does not exist any null vector that is a unit one (since by definition, the scalar square of a null vector is zero).

2.5.2 Light Cone

Let us consider an event A in the spacetime $\mathscr{E}$. The worldlines of all the photons that encounter A (photon passing through A, or emitted at A or received at A) form a subset of $\mathscr{E}$ that is the image of the null cone of $\boldsymbol{g}$ in E (Sect. 1.4) under the identification of the pair $(\mathscr{E}, A)$ (affine space $\mathscr{E}$ with A as an origin) with the vector space E (cf. Fig. 2.4). More precisely, let $\vec{\boldsymbol{u}}$ be the 4-velocity of a massive particle passing through A and $(\vec{\boldsymbol{e}}_1, \vec{\boldsymbol{e}}_2, \vec{\boldsymbol{e}}_3)$ three vectors such that $(\vec{\boldsymbol{u}}, \vec{\boldsymbol{e}}_1, \vec{\boldsymbol{e}}_2, \vec{\boldsymbol{e}}_3)$ is an orthonormal basis of $(E, \boldsymbol{g})$. $(A; \vec{\boldsymbol{u}}, \vec{\boldsymbol{e}}_1, \vec{\boldsymbol{e}}_2, \vec{\boldsymbol{e}}_3)$ is then an (orthogonal) affine coordinate system of $\mathscr{E}$ (cf. Sect. 1.2.3). A point $M \in \mathscr{E}$ of affine coordinates (x^0, x^1, x^2, x^3) belongs to the worldline of a photon that encounters A iff $\overrightarrow{AM}$ is a null vector: $\boldsymbol{g}(\overrightarrow{AM}, \overrightarrow{AM}) = 0$. From (1.6) and (1.18), this is equivalent to

$$-(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 = 0. \tag{2.19}$$

Such an equation defines a three-dimensional cone of apex A in the affine space $\mathscr{E}$, which is called the ***light cone*** of event A. We shall denote it by $\mathscr{I}(A)$, the sheet corresponding to the future (resp. past) null cone being denoted $\mathscr{I}^+(A)$ (resp. $\mathscr{I}^-(A)$). $\mathscr{I}^+(A)$ is called the ***future light cone*** of event A and $\mathscr{I}^-(A)$ the ***past light cone*** of A.

The null cone of apex A separates the events that are related to A by a timelike vector to those that are related to A by a spacelike vector. Figure 2.4 shows the light cones of two points A and B on the worldline of a massive particle.

Remark 2.14. The light cone is entirely determined by the considered event and does not depend upon the worldline passing through this event. Note also that the light cones of different events can be deduced from each other by a mere translation (see Fig. 2.4).

2.6 Langevin's Traveller and Twin Paradox

Having introduced formally the proper time at Sect. 2.3, let us now study it in a specific case, which puts forward its dependency with respect to the considered worldline. The "experiment" to be described has been designed by Paul Langevin[5] in 1911 (Langevin 1911). It is known as *Langevin's traveller* and it illustrates the so-called *twin paradox*. Beside proper time, it provides also a nice illustration of the concepts of 4-velocity and 4-acceleration introduced in Sect. 2.4.

[5]**Paul Langevin** (1872–1946): French physicist, known for his work on the magnetic properties of materials and Brownian motion. As a friend of Einstein since 1911, he contributed a lot in the diffusion of relativity in France (Paty 1999a). He was also president of the French League of Human Rights from 1944 to 1946.

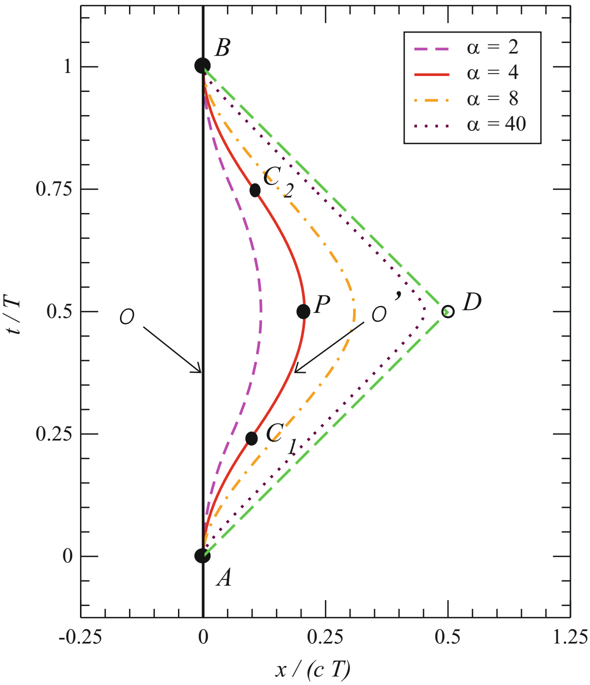

Fig. 2.5 Worldlines of the twins $\mathscr{O}$ and $\mathscr{O}'$: that of $\mathscr{O}$ is the *vertical line* $x = 0$ and that of $\mathscr{O}'$ is represented for different values of the parameter α. Between the events A ($\mathscr{O}'$ departure) and B ($\mathscr{O}'$ return), the worldline of $\mathscr{O}'$ is made of three arcs of hyperbola: AC_1, C_1C_2 and C_2B, defined by (2.20) (the points C_1, C_2 and P have been drawn only for $\alpha = 4$). *Long-dashed lines* indicate a null geodesic issued from A (segment $[AD]$) and a null geodesic arriving in B (segment $[DB]$); $[AD] \cup [DB]$ is thus the worldline of a photon emitted at A and reflected at D in order to meet observer $\mathscr{O}$ in B

2.6.1 Twins' Worldlines

Let us consider two observers $\mathscr{O}$ and $\mathscr{O}'$ that we shall model as two particles on timelike worldlines equipped with ideal clocks.[6] We take for the worldline $\mathscr{L}$ of $\mathscr{O}$ the simplest that one may think of: a straight line of $\mathscr{E}$. The worldline $\mathscr{L}'$ of $\mathscr{O}'$ is chosen to coincide with $\mathscr{L}$ until some event A; this is the very reason why $\mathscr{O}$ and $\mathscr{O}'$ may be called *twins*. At A, $\mathscr{O}'$ separates from $\mathscr{O}$ and travels until the event P. He then moves back and meets up with $\mathscr{O}$ at the event B, after which the worldlines $\mathscr{L}$ and $\mathscr{L}'$ coincide again (cf. Fig. 2.5).

Since $\mathscr{L}$ is a straight line, the 4-velocity $\vec{\boldsymbol{u}}$ of $\mathscr{O}$ is constant. This implies that the 4-acceleration of $\mathscr{O}$ vanishes. Let then $(\vec{\boldsymbol{e}}_\alpha)$ be an orthonormal basis of $(E, \boldsymbol{g})$ such

[6] We shall define more precisely the concept of *observer* in Chap. 3, the present version being sufficient for the purpose of this section.

that $\vec{u}$ is equal to the (constant) vector $\vec{e}_0$. We consider the affine coordinate system $(x^0 = ct, x^1 = x, x^2 = y, x^3 = z)$ defined by this basis and having A as origin (cf. Sect. 1.2.3). The points M of $\mathscr{L}$ obey the relation $\overrightarrow{AM} = ct\,\vec{e}_0$. Differentiating, we get $\mathrm{d}\overrightarrow{AM} = c\,\mathrm{d}t\,\vec{e}_0$, so that formula (2.6) along with the property $\boldsymbol{g}(\vec{e}_0, \vec{e}_0) = -1$ shows that the coordinate t coincides with $\mathscr{O}$'s proper time.

Let us now define precisely the worldline of $\mathscr{O}'$. For simplicity, we suppose that $\mathscr{O}'$ travels always in the same direction, which we shall select to be that of $\vec{e}_1$. The spatial trajectory of $\mathscr{O}'$ as perceived by $\mathscr{O}$ is then a line segment, travelled in one way and then in the reverse way. The corresponding worldline in Minkowski spacetime is contained in the plane through A and generated by the vectors $(\vec{e}_0, \vec{e}_1)$, i.e. the plane (t, x). The precise shape of $\mathscr{L}'$ depends on the velocity of $\mathscr{O}'$ with respect to $\mathscr{O}$. We shall choose $\mathscr{L}'$ between A and B to be made of three arcs of hyperbola, AC_1, C_1C_2 et C_2B (cf. Fig. 2.5), defined in terms of the affine coordinates (ct, x, y, z) by the following equations:

$$\text{for } t \in \left[0, \frac{T}{4}\right]: \quad x(t) = \frac{cT}{\alpha}\left[\sqrt{1+\alpha^2\,(t/T)^2} - 1\right] \tag{2.20a}$$

$$\text{for } t \in \left[\frac{T}{4}, \frac{3T}{4}\right]: \quad x(t) = \frac{cT}{\alpha}\left[-\sqrt{1+\alpha^2\,(t/T-1/2)^2} + 2\sqrt{1+\frac{\alpha^2}{16}} - 1\right] \tag{2.20b}$$

$$\text{for } t \in \left[\frac{3T}{4}, T\right]: \quad x(t) = \frac{cT}{\alpha}\left[\sqrt{1+\alpha^2\,(t/T-1)^2} - 1\right], \tag{2.20c}$$

where T is $\mathscr{O}$'s proper time elapsed between the events A and B, so that $t(A) = 0$, $t(C_1) = T/4$, $t(C_2) = 3T/4$ and $t(B) = T$. The parameter $\alpha \in \mathbb{R}$ is dimensionless and allows us to consider a whole family of worldlines for $\mathscr{O}'$, as shown in Fig. 2.5. If $\alpha = 0$, $\mathscr{L}'$ coincides with $\mathscr{L}$ and for $\alpha \neq 0$, Eq. (2.20a) leads to

$$\left(\alpha\frac{x}{cT} + 1\right)^2 - \left(\alpha\frac{t}{T}\right)^2 = 1, \tag{2.21}$$

which is the equation of a hyperbola in the plane (t, x), having the "horizontal" line $t = 0$ as the foci axis. Similarly, (2.20b) defines a hyperbola of foci axis the line $t = T/2$ and (2.20c) a hyperbola of foci axis the line $t = T$ (cf. Fig. 2.5). The choice of arcs of hyperbola will be justified in Sect. 2.6.4, where we shall see that it implies a constant norm of the 4-acceleration. This constitutes a relativistic generalization of the uniformly accelerated motion, as we shall discuss in Sect. 12.2. We shall call the observer $\mathscr{O}'$ following the worldline defined above ***Langevin's traveller***.

Let P be the mid-journey event (maximal distance from $\mathscr{O}$, cf. Fig. 2.5). Its position depends upon α and is obtained by setting $t = T/2$ in (2.20b):

$$x(P) = \frac{2cT}{\alpha}\left(\sqrt{1+\frac{\alpha^2}{16}}-1\right) = \frac{\alpha}{8}\,\frac{cT}{\sqrt{1+\alpha^2/16}+1}. \tag{2.22}$$

2.6.2 Proper Time of Each Twin

We have seen above that the proper time of $\mathscr{O}$ coincides with the coordinate t of the affine system (ct, x, y, z). To determine the proper time t' of $\mathscr{O}'$, let us parametrize the worldline $\mathscr{L}'$ by $\lambda = t$. An infinitesimal displacement $\mathrm{d}\vec{\boldsymbol{x}}'$ along $\mathscr{L}'$ has the components $\mathrm{d}x'^{\alpha} = (c\,\mathrm{d}t, \mathrm{d}x, 0, 0)$ in the orthonormal basis $(\vec{\boldsymbol{e}}_\alpha)$, where $\mathrm{d}x$ is related to $\mathrm{d}t$ by differentiating (2.20):

$$\mathrm{d}x = (-1)^k \frac{\alpha(t/T - k/2)}{\sqrt{1+\alpha^2\,(t/T-k/2)^2}}\, c\,\mathrm{d}t, \tag{2.23}$$

where the integer k takes the following values : $k = 0$ for $0 \le t \le T/4$, $k = 1$ for $T/4 \le t \le 3T/4$ and $k = 2$ for $3T/4 \le t \le T$. The basis $(\vec{\boldsymbol{e}}_\alpha)$ being orthonormal, the proper time t' along $\mathscr{L}'$ is given by formula (2.7):

$$\mathrm{d}t' = \frac{1}{c}\sqrt{(\mathrm{d}x'^0)^2 - (\mathrm{d}x'^1)^2 - (\mathrm{d}x'^2)^2 - (\mathrm{d}x'^3)^2} = \frac{1}{c}\sqrt{c^2\mathrm{d}t^2 - \mathrm{d}x^2}. \tag{2.24}$$

Substituting (2.23) for $\mathrm{d}x$ yields

$$\mathrm{d}t' = \frac{\mathrm{d}t}{\sqrt{1+\alpha^2\,(t/T-k/2)^2}}. \tag{2.25}$$

Thanks to the change of variable $\alpha(t/T - k/2) = \sinh u$, this equation is easily integrated into[7]

$$t' = \frac{T}{\alpha}\,\mathrm{arsinh}\left[\alpha\left(\frac{t}{T}-\frac{k}{2}\right)\right] + \frac{k}{2}\,T', \tag{2.26}$$

where arsinh stands for the inverse hyperbolic sine ($\mathrm{arsinh}\, x = \ln(x + \sqrt{x^2+1})$) and

$$T' := \frac{4T}{\alpha}\mathrm{arsinh}\left(\frac{\alpha}{4}\right). \tag{2.27}$$

[7] Let us recall that $\mathrm{d}(\sinh u) = \cosh u\,\mathrm{d}u$ and $\sqrt{1+\sinh^2 u} = \cosh u$.

The integration constant $kT'/2$, which appears in (2.26), has been chosen in each of the domains $k = 0$ ($t \in [0, T/4]$), $k = 1$ ($t \in [T/4, 3T/4]$) and $k = 2$ ($t \in [3T/4, T]$) in order to enforce the continuity of t', starting from $t' = 0$ at $t = 0$. The relation (2.26) between the proper times t and t' is plotted in Fig. 2.6. At the particular points A ($k = 0, t = 0$), C_1 ($k = 0, t = T/4$), P ($k = 1, t = T/2$), C_2 ($k = 1, t = 3T/4$) and B ($k = 2, t = T$), it results in

$$t'(A) = 0, \quad t'(C_1) = \frac{T'}{4}, \quad t'(P) = \frac{T'}{2}, \quad t'(C_2) = \frac{3T'}{4}, \quad t'(B) = T'. \tag{2.28}$$

We notice that the inequality $t' \leq t$ always holds (cf. Fig. 2.6). In particular, when $\mathscr{O}$ and $\mathscr{O}'$ meet again in B, the elapsed proper time from $\mathscr{O}'$ departure is $t(B) = T$ for $\mathscr{O}$, whereas the elapsed proper time for $\mathscr{O}'$, $t'(B) = T'$, is given by (2.27). Whenever $\alpha \neq 0$, we have $T' \neq T$, and the ratio of the two elapsed proper times is

$$\boxed{\frac{T'}{T} = \frac{t'(B) - t'(A)}{t(B) - t(A)} = \frac{4}{\alpha} \operatorname{arsinh}\left(\frac{\alpha}{4}\right) \leq 1}. \tag{2.29}$$

The ratio T'/T is plotted as a function of α in Fig. 2.7. For the worldlines drawn in Fig. 2.5, its value is 0.96 ($\alpha = 2$), 0.88 ($\alpha = 4$), 0.72 ($\alpha = 8$) and 0.30 ($\alpha = 40$).

2.6.3 The "Paradox"

The result (2.29) constitutes the so-called ***twin paradox***. Actually, this is not a paradox for this does not generate any contradiction in the theory of relativity, as discussed below; this is simply a surprising result for a "nonrelativistic" physicist: in Newtonian theory, the times given by the clocks of each twin would be the same when they meet in B, provided that they have been synchronized in A.

The paradoxical aspect of Langevin's traveller arises from a naive interpretation of the principle of relativity: from the point of view of twin $\mathscr{O}$, the twin $\mathscr{O}'$ is the traveller and the above computation shows that when $\mathscr{O}'$ is back, he is younger than $\mathscr{O}$. But from the point of view of $\mathscr{O}'$, it is $\mathscr{O}$ who is travelling. When the twins meet again, $\mathscr{O}$ should then be younger. Since both points of view should be equally valid according to the principle of relativity, a paradox appears: at the event B, $\mathscr{O}'$ cannot be both younger and older than $\mathscr{O}$. Actually, this argument is false because the two twins $\mathscr{O}$ and $\mathscr{O}'$ do not follow equivalent worldlines in Minkowski spacetime. The worldline of $\mathscr{O}$ is a very peculiar curve: a straight line, which implies that $\mathscr{O}$'s 4-acceleration is vanishing. On the contrary, the 4-acceleration of $\mathscr{O}'$ is nonzero, as we shall see below. Since the two twins are not equivalent, the relativity principle cannot be invoked and the paradox disappears. For a more detailed discussion, we refer the reader to Grandou and Rubin (2009).

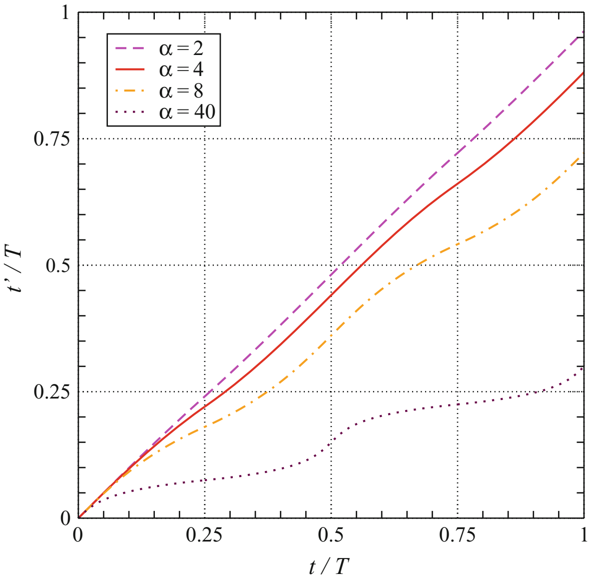

Fig. 2.6 Proper time t' of the twin $\mathscr{O}'$ (Langevin's traveller) as a function of the proper time t of the twin $\mathscr{O}$, for various values of the parameter α. Note that at the instants $t = 0$, $t = T/2$ and $t = T$, where the two worldlines are parallel (cf. Fig. 2.5), the slope of the curve is 45°, which means that t' flows at the same rate as t. On the other side, at the instants $t = T/4$ and $t = 3T/4$, where the inclination of $\mathscr{L}'$ differs the most from that of $\mathscr{L}$, the slope of the curve is the smallest

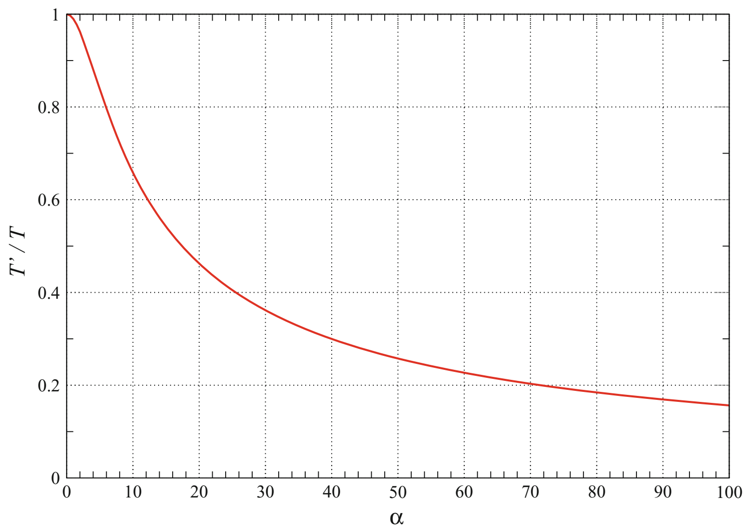

Fig. 2.7 Ratio between the proper time elapsed between A and B for $\mathscr{O}'$ and that elapsed for $\mathscr{O}$, as a function of the parameter α [cf. formula (2.29)]

Remark 2.15. From the four-dimensional point of view adopted in this book, the solution of the twin "paradox" appears rather trivial: the proper time has been defined as the length given by the metric tensor $\boldsymbol{g}$ along a worldline, and it seems obvious that the length between two points A and B depends upon the path chosen between these two points. A sceptical mind could reply: "there is nothing revolutionary in this with respect to the Newtonian time, because everything relies on the definition of proper time as the length of worldlines with respect to $\boldsymbol{g}$; this is an arbitrary definition of "time". It is therefore not surprising that it results in a strange behaviour". However, we have already mentioned in Sect. 2.3 that the time defined from $\boldsymbol{g}$ is the actual physical time, in the sense that the equations of dynamics take a simple form when expressed with it (we shall see it explicitly in Chap. 9). We shall actually see in Sect. 2.6.6 some experimental realizations of the twin paradox, showing that the time provided by atomic clocks between two events A and B do depend on the worldline between these two events. It is therefore not a mere semantic effect!

Remark 2.16. Since $\operatorname{arsinh} x = \ln(x + \sqrt{x^2 + 1})$, we deduce from formula (2.29) that when $\alpha \to +\infty$, the ratio T'/T goes to 0, behaving as $4 \ln \alpha/\alpha$. This is not surprising if one contemplates Fig. 2.5: when $\alpha \to +\infty$, the worldline $\mathscr{L}'$ approaches the worldline $[AD] \cup [DB]$ of a photon emitted in A and reflected back to B in D. Each segment $[AD]$ and $[DB]$ is a piece of null geodesic and hence as a vanishing metric length (cf. Remark 2.13 in p. 39). Therefore we understand why T', which is nothing but the metric length of $\mathscr{L}'$ between A and B, converges to zero when $\alpha \to +\infty$.

Historical note: *This is in fact Albert Einstein who, in the seminal 1905 article (Einstein 1905b), already pointed out that two clocks initially synchronized and, at the same position, would not show the same time if they are read at the same place after having travelled on different paths. Einstein gave an approximate formula (valid for small velocities) of the delay between t' and t. During a conference in Zurich in 1911, he illustrated the effect by describing a round-trip journey of a living organism locked in a moving box (cf. Damour (2006), p. 34). In order to make the effect more spectacular, Paul Langevin (cf. p. 40) imagined in 1911 a human being leaving the Earth aboard a "projectile", travelling towards some star at a velocity close to that of light and coming back on Earth after 2 years, whereas 200 years have been elapsed on our planet (Langevin 1911; Paty 1999a). Let us note that in the 1911 text (Langevin 1911), Langevin did not speak explicitly of "twins" but of a "traveller" and "the Earth". Besides, he gave clearly the explanation of the dissymmetry between the two by mentioning the acceleration felt by the traveller.*

2.6.4 4-Velocity and 4-Acceleration

Let us compute the 4-velocity $\vec{u}'$ of Langevin's traveller $\mathscr{O}'$ at each point of his worldline. From the definition (2.12), we have

$$\vec{u}' = \frac{1}{c}\frac{\mathrm{d}\vec{x}'}{\mathrm{d}t'}.$$

The components of $\vec{u}'$ in the orthonormal basis $(\vec{e}_\alpha)$ are thus $u'^\alpha = c^{-1}\mathrm{d}x'^\alpha/\mathrm{d}t'$. Since $\mathrm{d}x'^\alpha = (c\,\mathrm{d}t, \mathrm{d}x, 0, 0)$, we get $u'^2 = 0$, $u'^3 = 0$,

$$u'^0 = \frac{1}{c}\frac{dx'^0}{\mathrm{d}t'} = \frac{\mathrm{d}t}{\mathrm{d}t'} \qquad \text{and} \qquad u'^1 = \frac{1}{c}\frac{dx'^1}{\mathrm{d}t'} = \frac{1}{c}\frac{\mathrm{d}x}{\mathrm{d}t'} = \frac{1}{c}\frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}t'}.$$

By means of (2.25) and (2.23), we obtain

$$u'^0 = \sqrt{1+\alpha^2\,(t/T-k/2)^2} \tag{2.30a}$$

$$u'^1 = (-1)^k\alpha\,(t/T-k/2)\,. \tag{2.30b}$$

Given the definition of the integer k, we note that if $\alpha > 0$, then for $0 \le t \le T/2$, $u'^1 \ge 0$ ($\mathscr{O}'$ is moving away from $\mathscr{O}$ in the direction of increasing x), whereas for $T/2 \le t \le T$, $u'^1 \le 0$ ($\mathscr{O}'$ is moving towards $\mathscr{O}$). The vector $\vec{u}'$, as given by (2.30), is drawn at some selected points of $\mathscr{L}'$ in Fig. 2.8.

Let us notice that, from (2.26),

$$\alpha\left(\frac{t}{T}-\frac{k}{2}\right) = \sinh\left[\frac{\alpha}{T}\left(t'-\frac{k}{2}T'\right)\right], \tag{2.31}$$

so that the components of $\vec{u}'$ can be expressed in terms of the proper time t' according to

$$u'^0 = \cosh\left[\frac{\alpha}{T}\left(t'-\frac{k}{2}T'\right)\right] \tag{2.32a}$$

$$u'^1 = (-1)^k\sinh\left[\frac{\alpha}{T}\left(t'-\frac{k}{2}T'\right)\right]. \tag{2.32b}$$

Remark 2.17. Thanks to the identity $\cosh^2 x - \sinh^2 x = 1$, it is easily checked on these formulas that $\vec{u}'\cdot\vec{u}' = -(u'^0)^2 + (u'^1)^2 = -1$, as it should be for any 4-velocity [Eq. (2.14)].

Let us now compute the 4-acceleration $\vec{a}'$ of $\mathscr{O}'$. By definition [cf. Eq. (2.16)],

$$\vec{a}' = \frac{1}{c}\frac{\mathrm{d}\vec{u}'}{\mathrm{d}t'}.$$

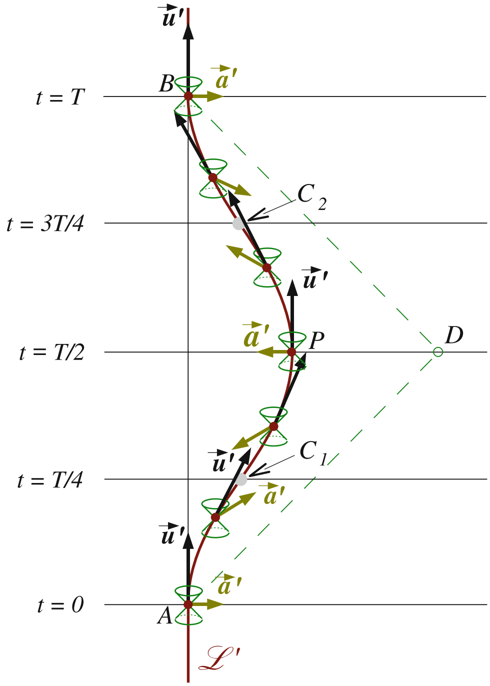

Fig. 2.8 Worldline $\mathscr{L}'$ of Langevin's traveller $\mathscr{O}'$ with the 4-velocity $\vec{u}'$ and 4-acceleration $\vec{a}'$ at some selected points. This figure corresponds to the case $\alpha = 4$ (solid line in Fig. 2.5), i.e. to the acceleration $\gamma = 4c/T$. At the event A, the 4-acceleration changes sharply from 0 to $\vec{a}' = \gamma c^{-2}\,\vec{e}_1$ (ignition of the rocket engine). Its norm stays then constant (equal to γc^{-2}), until the return event B, where $\vec{a}'$ vanishes again (rocket engine stopped). The event C_1 is the sudden change of direction of acceleration by 180° (thrust reversing). $\mathscr{O}'$ is subsequently slowed down until P and then sped up towards $\mathscr{O}$, until C_2. At this point, a new thrust reversing occurs, so that $\mathscr{O}'$ is slowed down until it reaches B

Accordingly, the components of $\vec{a}'$ in the orthonormal basis $(\vec{e}_\alpha)$ are

$$a'^0 = \frac{1}{c}\frac{du'^0}{\mathrm{d}t'}, \quad a'^1 = \frac{1}{c}\frac{du'^1}{\mathrm{d}t'}, \quad a'^2 = 0 \quad \text{and} \quad a'^3 = 0.$$

Taking the derivative of (2.32) with respect to t', we get

$$a'^0 = \frac{\alpha}{cT}\sinh\left[\frac{\alpha}{T}\left(t' - \frac{k}{2}T'\right)\right] \tag{2.33a}$$

$$a'^1 = (-1)^k\frac{\alpha}{cT}\cosh\left[\frac{\alpha}{T}\left(t' - \frac{k}{2}T'\right)\right]. \tag{2.33b}$$

As a check, the orthogonality between the 4-acceleration and the 4-velocity [Eq. (2.17)] is recovered from (2.32) and (2.33): $\vec{\boldsymbol{a}}' \cdot \vec{\boldsymbol{u}}' = -a'^0 u'^0 + a'^1 u'^1 = 0$. Thanks to (2.31), we can express the components of $\vec{\boldsymbol{a}}'$ in terms of t instead of t':

$$a'^0 = \frac{\alpha^2}{cT}\left(\frac{t}{T} - \frac{k}{2}\right) \tag{2.34a}$$

$$a'^1 = (-1)^k \frac{\alpha}{cT}\sqrt{1 + \alpha^2\left(\frac{t}{T} - \frac{k}{2}\right)^2}. \tag{2.34b}$$

We notice that a'^1 has a sudden change of sign when k goes from 0 to 1, i.e. when $t = T/4$, as well as when k goes from 1 to 2, i.e. when $t = 3T/4$. More precisely, if $\alpha > 0$, formula (2.34b) yields

$$t \in \left[0, \frac{T}{4}\right] \implies a'^1 > 0, \quad t \in \left[\frac{T}{4}, \frac{3T}{4}\right] \implies a'^1 < 0,$$
$$t \in \left[\frac{3T}{4}, T\right] \implies a'^1 > 0. \tag{2.35}$$

Physically, if we consider that $\mathscr{O}'$ is travelling in some spaceship, the sudden change of sign of a'^1 corresponds to a thrust reversing operated on the rocket engine (events C_1 and C_2 in Fig. 2.8).

Let us evaluate the scalar square of $\vec{\boldsymbol{a}}'$. The basis $(\vec{\boldsymbol{e}}_\alpha)$ being orthonormal, we have $\vec{\boldsymbol{a}}' \cdot \vec{\boldsymbol{a}}' = -(a'^0)^2 + (a'^1)^2$. From (2.33) or (2.34), we get easily

$$\vec{\boldsymbol{a}}' \cdot \vec{\boldsymbol{a}}' = \frac{\alpha^2}{c^2 T^2}. \tag{2.36}$$

The right-hand side being clearly positive, we recover the property (2.18), namely, that $\vec{\boldsymbol{a}}'$ is a spacelike vector. More remarkably, (2.36) shows that the norm of the 4-acceleration,

$$a' := \left\|\vec{\boldsymbol{a}}'\right\|_g = \sqrt{\vec{\boldsymbol{a}}' \cdot \vec{\boldsymbol{a}}'} = \frac{|\alpha|}{cT}, \tag{2.37}$$

does not depend upon t': it is therefore constant along the worldline $\mathscr{L}'$ between A and B. This property is specific to the spacetime motion along an arc of hyperbola, which we have chosen for $\mathscr{O}'$.

We have seen in Sect. 2.4.2 that the dimension of a' is the inverse of a length, in agreement with (2.37), α being dimensionless. To let appear a quantity with the dimension of an acceleration, it suffices to multiply a' by c^2. We thus introduce the parameter

$$\gamma := \alpha \frac{c}{T}, \tag{2.38}$$

instead of α. γ has the dimension of an acceleration and is related to the norm of the 4-acceleration of $\mathscr{O}'$ by

$$\boxed{a' = \frac{|\gamma|}{c^2}}. \tag{2.39}$$

We shall see in Chap. 12 that γ is actually the acceleration felt by the observer $\mathscr{O}'$ in his local frame.

Remark 2.18. Note that $\gamma \neq \mathrm{d}^2x/\mathrm{d}t^2$, i.e. γ is not the second derivative of the function $x(t)$ defining the worldline of $\mathscr{O}'$. The latter is obtained by taking the derivative of $\mathrm{d}x/\mathrm{d}t$ as given by (2.23). One gets, after substituting γ/c for α/T,

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = (-1)^k\gamma\left[1 + \frac{\gamma^2}{c^2}\left(t - \frac{k}{2}T\right)^2\right]^{-3/2}. \tag{2.40}$$

We conclude that one has $|\gamma| \simeq |\mathrm{d}^2x/\mathrm{d}t^2|$ only in the nonrelativistic limit $|\gamma|T \ll c$.

Remark 2.19. In many textbooks,[8] the twin paradox is exposed from a worldline $\mathscr{L}'$ simpler than the three arcs of hyperbola considered here, namely, a straight line segment from A to P as well as from P to B (see Fig. 2.9). The computations are then simpler than those presented above, the equation of $\mathscr{L}'$ being $x(t) = Vt$ for $t \in [0, T/2]$ and $x(t) = V(T - t)$ for $t \in [T/2, T]$, with $V := 2x(P)/T$. We have then $\mathrm{d}x = \pm V\mathrm{d}t$, so that evaluating $\mathrm{d}t'$ according to formula (2.24), we get $\mathrm{d}t' = \sqrt{1 - (V/c)^2}\,\mathrm{d}t$, which is easily integrated and leads to the proper time ratio

$$\frac{T'}{T} = \sqrt{1 - \frac{V^2}{c^2}} \leq 1. \tag{2.41}$$

However, this configuration is not physical for it corresponds to an infinite acceleration of $\mathscr{O}'$ at A (the 4-velocity jumping suddenly from $\vec{\boldsymbol{u}}'(A_-)$ to $\vec{\boldsymbol{u}}'(A_+)$, cf. Fig. 2.9) as well as in P and B. On the contrary, the "tri-hyperbolic" worldline considered here involves always a finite acceleration. It admits thus a clear physical interpretation in terms of a (rocket) engine of constant thrust, switched on at A, inverted in C_1 and C_2 and switched off at B. This demonstrates that the twin paradox is not an artefact resulting from an infinite acceleration.

[8] Two exceptions are the books by Møller (1952) and by Marder (1971).

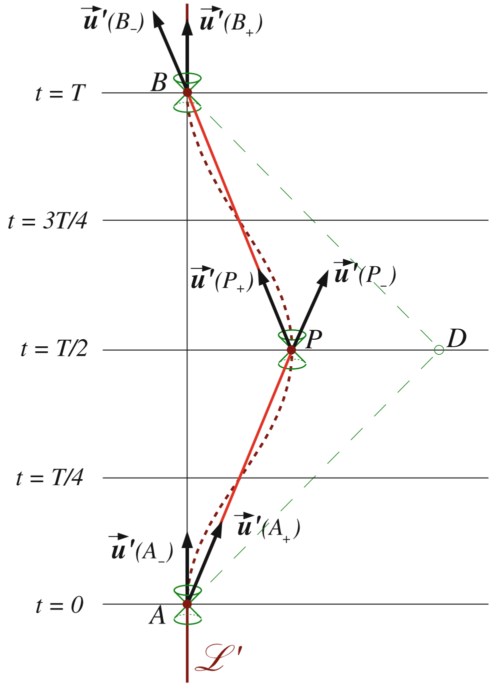

Fig. 2.9 Simplified worldline for the Langevin's traveller $\mathscr{O}'$: $\mathscr{L}'$ is reduced to a line segment between A and P, as well as between P and B. The 4-acceleration of $\mathscr{O}'$ is then infinite at A, P and B, as indicated by the jumps of the 4-velocity $\vec{u}'$ at these points. On the opposite, the "tri-hyperbolic" worldline (*dotted curve*, the same as in Figs. 2.5 and 2.8) yields always a finite 4-acceleration

2.6.5 A Round Trip to the Galactic Centre

Let $d := x(P)$ be the maximal distance of $\mathscr{O}'$ with respect to $\mathscr{O}$. We may reexpress formulas (2.22) and (2.29) in terms of the acceleration γ via (2.38):

$$d = \frac{2c^2}{\gamma}\left[\sqrt{1+\left(\frac{\gamma T}{4c}\right)^2}-1\right] \qquad \text{and} \qquad \frac{T'}{T} = \frac{4c}{\gamma T}\operatorname{arsinh}\left(\frac{\gamma T}{4c}\right). \tag{2.42}$$

The second relation allows one to express T in terms of T' as

$$\boxed{T = T_* \sinh\left(\frac{T'}{T_*}\right)}, \qquad \text{with} \quad T_* := \frac{4c}{\gamma} = \frac{4}{\alpha}T. \tag{2.43}$$

T_* is the timescale that can be built from c and the acceleration γ; in Newtonian physics, this would be 4 times the time required to reach the light velocity, starting from a zero velocity with the acceleration γ. Substituting T by the above value in the expression of d and noticing that $\sqrt{1+\sinh^2 x} = \cosh x$, we get

$$\boxed{d = \frac{cT_*}{2}\left[\cosh\left(\frac{T'}{T_*}\right) - 1\right]}. \tag{2.44}$$

When $T' \ll T_*$, the Taylor expansions of (2.43) and (2.44) lead to

$$T' \ll T_* \Longrightarrow \begin{cases} T \simeq T' \\ d \simeq \dfrac{c}{4T_*}(T')^2 = 2 \times \dfrac{1}{2}\gamma\left(\dfrac{T'}{4}\right)^2. \end{cases} \tag{2.45}$$

If $T' \ll T_*$, $\mathscr{O}'$ has not the time to reach a relativistic velocity with respect to $\mathscr{O}$ and (2.45) gives the Newtonian results, as it should be: no differential ageing and the travelled distance $d/2$ at a constant acceleration γ (phase $[AC_1]$ lasting $T'/4$) being equal to $\gamma/2$ times the square of travel time. This is indeed the expected result for a vanishing initial velocity.

Conversely, if $T' \gg T_*$, the ultra-relativistic regime is reached and relations (2.43) and (2.44) lead to

$$T' \gg T_* \Longrightarrow \begin{cases} T \simeq \dfrac{T_*}{2}\exp(T'/T_*) \\ d \simeq \dfrac{cT_*}{4}\exp(T'/T_*) = \dfrac{1}{2}cT. \end{cases} \tag{2.46}$$

Remark 2.20. Formula (2.43), which relates T to T', depends on a single parameter: the acceleration γ, via the time $T_* = 4c/\gamma$. One should however not conclude that the twin paradox is a phenomenon intrinsically linked to acceleration. It should rather be perceived as the reflect of the dissymmetry of the worldlines between A and B. It turns out that in Minkowski spacetime $(\mathscr{E}, \boldsymbol{g})$, the only way for $\mathscr{L}'$ to depart from a straight line (worldline $\mathscr{L}$) is to have some episode of nonvanishing 4-acceleration. If $\mathscr{E}$ is given a topology different from that of an affine space, then it is possible to have $T \neq T'$ with $\mathscr{L}$ and $\mathscr{L}'$ both having a vanishing 4-acceleration. It suffices that $\mathscr{E}$ has a non-simply connected topology,[9] as shown is the study (Uzan et al. 2002).

Let us apply the above formulas to the "concrete" case where $\mathscr{O}'$ is an astronaut in some spaceship. To consider an acceleration bearable for a human being, let us take the value of Earth's gravity: $\gamma = 1\ g = 9.81\ \mathrm{m\,s^{-2}}$. This has even the advantage to create an artificial gravity aboard the spaceship that simulates the terrestrial environment and makes the journey comfortable. The corresponding time parameter defined by (2.43) is $T_* = 4c/\gamma = 1.22 \times 10^8$ s $= 3.87$ yr, and formulas (2.43) and (2.44) lead to values of T and d as functions of T' listed in Table 2.1. We observe that if $\mathscr{O}'$ is travelling for more than a year, then the difference between

[9] An example of such spaces is a torus or, more generally, any compact domain with periodic boundary conditions.

Table 2.1 Properties of various trips of Langevin's traveller, when the acceleration is fixed to $\gamma = 9.81\,\mathrm{m\,s^{-2}}$: T' is the round-trip duration measured by him, T is the duration of the same trip but measured by the "sedentary" observer $\mathscr{O}$ and d is the maximal achieved distance between $\mathscr{O}'$ and $\mathscr{O}$ (1 light-year = 9.46×10^{15} m). One checks that if $T \gg T_* = 3.87$ yr, then $d \simeq cT/2$, in agreement with (2.46)

T' [yr]	T [yr]	d [light-year]
1	1.01	0.065
2	2.09	0.26
4	4.75	1.13
8	15.0	5.82
16	120	58
32	7.50×10^3	3.74×10^3
39.5	5.20×10^4	2.60×10^4
56	3.68×10^6	1.84×10^6
64	2.90×10^7	1.45×10^7
80	1.81×10^9	9.03×10^8
90	2.39×10^{10}	1.19×10^{10}
100	3.15×10^{11}	1.58×10^{11}

the "onboard" proper time T' and the "harbour" proper time T is noticeable. With a journey lasting for 8 years, $\mathscr{O}'$ can reach the closest stars from the Solar System. If he is travelling for $T' = 16$ years, when he is back on Earth, $T = 120$ years will have elapsed, implying that he will not be able to report his journey to his acquaintances but to their children. Table 2.1 shows that the centre of the Galaxy, located a roughly 26,000 light-years, can be reached within a journey of round-trip duration of only 39.5 years. In this case, it is not guaranteed that there will be anybody interested by the traveller's account at the return, for 52,000 years will have elapsed on Earth! Let us not speak about a round trip to Andromeda Galaxy, located at 2 million light-years, because while it takes only 56 years for the astronaut, his return will take place on an Earth aged by 3 million years and, at the very least, he will face some language issue…

Of course, in the above description, we have limited ourselves to pure kinematic considerations and have not taken into account the energetic cost of such travels: maintaining an acceleration of 1 g during several years requires an enormous amount of energy and forbids such travels with today technology. Nevertheless, we shall keep in mind that relativity allows one, at least theoretically, to visit the Galactic centre and even to reach the border of the observable universe within a human lifetime ($d \sim 12$ billion light-years for a round trip of 90 year, cf. Table 2.1), travelling at less than the speed of light! ($\mathscr{O}'$'s worldline is always located inside the light cone, cf. Fig. 2.8). Hence it is not correct to say that it is not possible for a person to travel further than a hundred light-year or so away from Earth because relativity forbids to travel faster than light. On the contrary, the solution is offered by relativity itself: it remains true that for any observer that he may encounter on his way, $\mathscr{O}'$ is travelling slower than light[10]; this implies that for people observing him from the Earth, $\mathscr{O}'$ will take at least 26,000 years to reach the Galactic centre. On the contrary, for $\mathscr{O}'$, only 20 years will have elapsed when he will arrive at the Galactic centre.

One lesson from the above example is that relativity allows for ***time travel to the future***: one may say that $\mathscr{O}'$ is travelling to $\mathscr{O}$'s future, since when $\mathscr{O}'$ meets again $\mathscr{O}$ at B, $\mathscr{O}$ is older than him. The numbers listed in Table 2.1 show even that this time travel to the future can be of millions of years. On the other side, special relativity does not allow for *time travel to the past*: even if we take the point of view of $\mathscr{O}$, when $\mathscr{O}$ meets again $\mathscr{O}'$ at B, the latter is younger than him but still older than when he left him at A.

Remark 2.21. It is Minkowski spacetime structure that forbids the time travel to the past: all the light cones being parallel (cf. Figs. 1.8 and 2.8), one can show that it is not possible for the worldline $\mathscr{L}'$ to meet $\mathscr{L}$ at a point B located in the past of A while staying inside the light cone of any of its points. However, if a gravitational field is present, the spacetime structure is no longer that of an affine space, as we shall see in Chap. 22, but that of a “curved” space ruled by general relativity. The light cones are then no longer parallel with respect to each other and it is possible, under certain conditions (quite extreme though…), to have $\mathscr{L}'$ such that B is anterior to A. This is the time machine of science-fiction novels! We shall not discuss this subject further and refer the interested reader (who would not be?) to Lehoucq (2004), Davies (2002), Thorne (1994).

2.6.6 Experimental Verifications

Undoubtedly, the twin paradox puts forward an effect that is not part of everyday life, namely, the dependency of time upon the motion of bodies. Actually the velocities of people and objects around us are very small with respect to the velocity of light, and we have seen that the time shift is sizeable only if T' is of the order T_*, which implies a velocity close to c [cf. (2.43) under the form $V_* := \gamma T_* \sim c$]. Nevertheless, even if the effect is too small for our senses, it can be exhibited by a sufficiently sensitive experiment. This turned out to be possible in the 1970s, thanks to atomic clocks.

2.6.6.1 Hafele–Keating Experiment (1971)

The first experimental reproduction of the twin paradox has been performed in 1971 by J.C. Hafele of Washington University at Saint Louis (Missouri) and Richard E. Keating of the US Naval Observatory (Hafele 1972b; Hafele and Keating 1972a,b)

[10]We shall make precise the notion of velocity in Chap. 4.

(cf. also Hafele (1972a)). Four caesium atomic clocks have been loaded on airline jets for a journey round the Earth; when back to their starting point, they have been compared with atomic clocks stayed on the ground. Two journeys have taken place. The first one has been performed eastward from 4 to 7 October 1971, with 12 stopovers, 3 changes of plane (Boeing 747 and 707) and a total of 41 h of flight. The second one has been performed westward from 13 to 17 October 1971, with 13 stopovers, 2 changes of plane (Boeing 707), totalizing 49 h of flight. The corresponding worldlines are rather different from that of Langevin's traveller defined in Sect. 2.6.1: the motion around the Earth being circular and not linear, the worldlines rather looks like helices. The precise trajectory of the clocks was pretty complicated because of the different stopovers, the experiment being performed on commercial airlines. Thanks to the flight data provided by the pilots, it has been possible to reconstruct the worldline $\mathscr{L}'$ followed by the onboard clocks. Another difference with Sect. 2.6.1 is that the worldline $\mathscr{L}$ of the reference clock staying on the ground is not a straight line but also a helix, due to the Earth rotation. However, even if the worldlines $\mathscr{L}$ and $\mathscr{L}'$ are more complicated than in Sect. 2.6.1, one could similarly compute, by means of (2.10), the proper times T along $\mathscr{L}$ and T' along $\mathscr{L}'$ between the plane departure (event A) and its return (event B). These theoretical values have been then compared with the actual measurements given by the clocks.

In addition to the shape of the worldlines, another complication arises from the fact that the clocks aboard the planes were travelling higher in the gravitational potential of the Earth than the clocks stayed on the ground. A general relativistic effect then takes place: the *gravitational redshift*, that we shall discuss in Chap. 22. It results in a difference between the proper times T' and T, in the direction of increasing T'. This effect has a magnitude comparable to that of the special relativistic effect that we are interested in here. The verification of the twin paradox must thus take this into account.

In Chap. 13, we shall compute the value $T' - T$ in the framework of special relativity, by means of simplified airplane trajectories. The precise computation, relying on the actual trajectories, results in $T' = T - 184 \pm 18\,\mathrm{ns}$ (nanoseconds) for the eastward journey. The 18 ns error bar is related to the uncertainties in the reconstruction of the airplane worldline (uncertainties in position and velocity). Hence, the clocks that have travelled eastward must be younger by 184 ns than those stayed on the ground. This value must be corrected from the general relativistic effect mentioned above; the latter goes in the reverse direction: it increases T' by $144 \pm 14\,\mathrm{ns}$. Accordingly, the theoretical prediction is $T' = T - 40 \pm 32\,\mathrm{ns}$. The observed value, obtained by taking the average over the four clocks, in order to reduce the experimental error, is $T' = T - 59 \pm 10\,\mathrm{ns}$.

Regarding the westward journey (counterrotating with respect to the Earth), the worldline $\mathscr{L}'$ (a helix at first approximation) deviates less from a straight line than the worldline $\mathscr{L}$. We are thus in the case where special relativity predicts $T' > T$, as we shall see explicitly in Chap. 13. The computation leads to $T' = T + 96 \pm 10\,\mathrm{ns}$, to which the gravitational redshift effect must be added (always with the result of increasing T'), to arrive at $T' = T + 275 \pm 21\,\mathrm{ns}$. The measured value is $T' = T + 273 \pm 7\,\mathrm{ns}$.

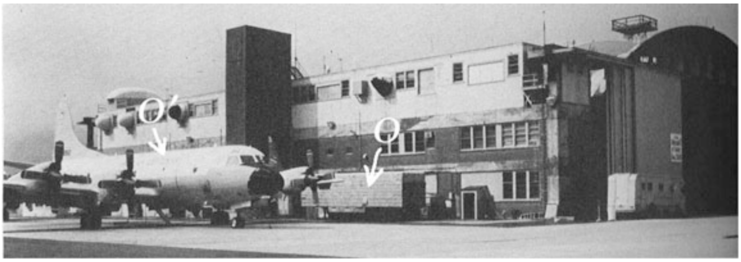

Fig. 2.10 Airplane carrying the atomic clocks (observer $\mathscr{O}'$) in Alley experiment (1975), parked near the truck containing the reference atomic clocks (observer $\mathscr{O}$), at the Naval Air Station Patuxent River (Eastern coast of United States). This picture may be considered as a view of the event A, where $\mathscr{O}$ and $\mathscr{O}'$, who were following the same worldline, are on the verge to separate [Credit: C.O. Alley (1983)]

Given the error bars, we conclude that Hafele–Keating experiment has confirmed that the proper time elapsed between two events does depend on the worldline related them. This may be seen as the experimental demonstration that the actual time is not Newton's absolute time, but relativity's time.

2.6.6.2 Alley Experiment (1975)

A more precise experiment with atomic clocks in airplane has been performed in 1975 by Carroll O. Alley of the University of Maryland (USA) (Alley 1983). This time, an aircraft entirely devoted to the experiment has been used instead of regular airline planes. It was an antisubmarine aircraft Lockheed P-3C Orion, which has the capability to fly non-stop for 16 h. On 22 November 1975 six atomic clocks (three caesium ones and three rubidium ones) have been loaded for a 15-h flight turning around Chesapeake Bay, on the Northeast coast of the United States (worldline $\mathscr{L}'$). A set of identical atomic clocks was installed in a trailer on the base from which the aircraft departed (worldline $\mathscr{L}$) (cf. Fig. 2.10). The average speed of the plane was $540\,\mathrm{km\,h^{-1}} = 150\,\mathrm{m\,s^{-1}} = 5\times 10^{-7}\,c$ and the altitude was 7,600 m during the 5 first hours, 9,100 m during the 5 next hours and 10,700 m during the 5 last hours. The computation of the proper times along $\mathscr{L}'$ and $\mathscr{L}$ leads to the following theoretical prediction:

$$T' = T \underbrace{-\,5.7\mathrm{ns}}_{\mathrm{SR}} \underbrace{+\,52.8\mathrm{ns}}_{\mathrm{GR}} = T + 47.1\mathrm{ns}, \tag{2.47}$$

where "SR" labels the contribution of special relativity (kinematic effect considered in this chapter) and "GR" the contribution of general relativity (gravitational redshift). The value measured at the return is in agreement with (2.47) within a relative accuracy of 1.5%. Since the kinematic effect is a tenth of the total effect, we conclude that Alley experiment has confirmed the twin paradox with an accuracy of the order of 15%.

Remark 2.22. Other tests about the dependency of proper time with respect to the motion will be presented in Chaps. 4 and 5. They are much more precise than the experiments described above. We have limited ourselves to the last ones because they are directly interpretable in terms of the twin paradox.

2.7 Geometrical Properties of a Worldline

2.7.1 Timelike Geodesics

In the study of Langevin's traveller, we have observed that $T > T'$ as long as the worldline $\mathscr{L}'$ departs from $\mathscr{L}$ (i.e. as long as $\alpha \neq 0$). Given the definition of proper time, we may state in an equivalent manner that between events A and B, the straight worldline $\mathscr{L}$ has a length (given by the metric tensor $\boldsymbol{g}$) *larger* than that of the curved worldline $\mathscr{L}'$. We shall show now that Langevin's traveller reflects the most general case: if two points of $\mathscr{E}$ can be joined by a timelike straight line, all the other timelike curves joining them have a smaller metric length. This result is of course the exact opposite of what holds in a Euclidean space, where the straight line is always the shortest path between two points.

Let A and B be two points of $\mathscr{E}$ such that B is located inside the future light cone of A. These two points can then be joined by timelike curves (i.e. worldlines of massive particles). A particular worldline is the straight line $\mathscr{L}_0$ through A and B. Let $(\vec{\boldsymbol{e}}_\alpha)$ be an orthonormal basis of $(E, \boldsymbol{g})$ such that $\vec{\boldsymbol{e}}_0$ coincides with the 4-velocity of $\mathscr{L}_0$. We introduce the affine coordinate system $(x^0 = ct, x^1 = x, x^2 = y, x^3 = z)$ associated with $(\vec{\boldsymbol{e}}_\alpha)$ and centred on A (cf. Fig. 2.11). Let $\mathscr{L}$ be a timelike worldline connecting A and B. As $\mathscr{L}$ must stay inside the light cone of each of its points, we can use the affine coordinate t as a regular parameter[11] along $\mathscr{L}$. Let then X, Y and Z be three functions $\mathbb{R} \to \mathbb{R}$ giving the position of $\mathscr{L}$ in terms of the affine coordinates (x^α), according to

$$x = X(t), \quad y = Y(t), \quad z = Z(t). \tag{2.48}$$

The components in the basis $(\vec{\boldsymbol{e}}_\alpha)$ of the elementary displacement vector $\mathrm{d}\vec{\boldsymbol{x}}$ along $\mathscr{L}$ are then

$$\mathrm{d}x^\alpha = (c\,\mathrm{d}t, \dot{X}\mathrm{d}t, \dot{Y}\mathrm{d}t, \dot{Z}\mathrm{d}t), \tag{2.49}$$

[11] If the timelike constraint was relaxed, then $\mathscr{L}$ could move "backward in time" and t would not be a good parameter along it.

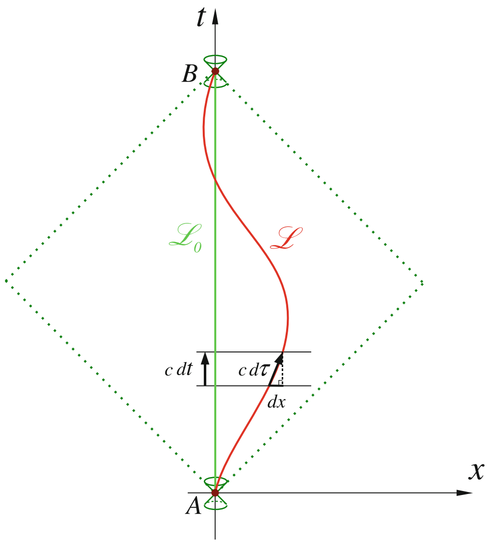

Fig. 2.11 Comparing the metric length (proper time) of two worldlines joining two events A and B: a *straight line* and a *curved line*. Since $c^2 \mathrm{d}\tau^2 = c^2 \mathrm{d}t^2 - \mathrm{d}x^2$, the *curved line* is, with respect to the metric $\boldsymbol{g}$, shorter than the *straight line*

where the derivatives of the functions X, Y and Z are indicated with a dot. The length of $\mathscr{L}$ (with respect to $\boldsymbol{g}$) between the points A and B is given by formula (2.10):

$$c\,\tau(A,B) = c\int_A^B \mathrm{d}\tau = \int_A^B \sqrt{-\boldsymbol{g}(\mathrm{d}\vec{\boldsymbol{x}},\mathrm{d}\vec{\boldsymbol{x}})}. \tag{2.50}$$

Now, since $(\vec{\boldsymbol{e}}_\alpha)$ is an orthonormal basis, $-\boldsymbol{g}(\mathrm{d}\vec{\boldsymbol{x}},\mathrm{d}\vec{\boldsymbol{x}}) = -\eta_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta = c^2\mathrm{d}t^2 - (\dot{X}\,\mathrm{d}t)^2 - (\dot{Y}\,\mathrm{d}t)^2 - (\dot{Z}\,\mathrm{d}t)^2$. Hence

$$\begin{aligned} c\,\tau(A,B) &= c\int_A^B \sqrt{1-\frac{1}{c^2}\left[(\dot{X})^2+(\dot{Y})^2+(\dot{Z})^2\right]}\,\mathrm{d}t \\ &\le c\int_A^B \mathrm{d}t = c[t(B)-t(A)]. \end{aligned} \tag{2.51}$$

Since $c[t(B)-t(A)] = c\,\tau_0(A,B)$ is the length of the straight line $\mathscr{L}_0$ between A and B, we conclude that $\mathscr{L}_0$ *maximizes* the metric length (proper time) between A and B, among all the possible worldlines.

For this reason, one calls ***timelike geodesic*** any timelike straight line of $\mathscr{E}$. Note that the term *geodesic* must be understood as a curve of extremal length, not necessarily minimal. To summarize, the null geodesics introduced in Sect. 2.5.1 correspond to minima of the metric length, whereas the timelike geodesics to maxima.

Remark 2.23. The timelike geodesic between A and B providing the upper bound on the metric length between these two points, one may ask about the lower bound on this length, taking into account that it must be positive or zero [cf. the integral (2.50)]. The answer is given by the example of Langevin's traveller: the lower bound is zero. Indeed, when the parameter α tends to infinity, the length of the worldline $\mathscr{L}'$ between A and B shrinks to zero, as shown by formula (2.29) (see also Remark 2.16 p. 46).

2.7.2 Vector Field Along a Worldline

Given a timelike worldline, $\mathscr{L}$ let us say, we have already encountered two kinds of vector fields defined along it: (i) the tangent vector fields associated with the various parametrizations of $\mathscr{L}$, among which the 4-velocity and (ii) the 4-acceleration field introduced in Sect. 2.4.2 (which is nowhere tangent to $\mathscr{L}$). More generally, let us define a ***vector field along the worldline*** $\mathscr{L}$ as a mapping

$$\begin{aligned} \vec{\boldsymbol{v}} : \mathscr{L} &\longrightarrow E \\ A &\longmapsto \vec{\boldsymbol{v}}(A). \end{aligned} \tag{2.52}$$

Since the points A of $\mathscr{L}$ are often labelled with their proper time τ, we shall also write $\vec{\boldsymbol{v}}(\tau)$ for $\vec{\boldsymbol{v}}(A(\tau))$.

One says that the vector field $\vec{\boldsymbol{v}}$ is ***differentiable*** at a point $A(\tau) \in \mathscr{L}$ iff the limit

$$\boxed{\frac{\mathrm{d}\vec{\boldsymbol{v}}}{\mathrm{d}\tau} := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[\vec{\boldsymbol{v}}(\tau + \varepsilon) - \vec{\boldsymbol{v}}(\tau) \right]} \tag{2.53}$$

exists. The vector $\mathrm{d}\vec{\boldsymbol{v}}/\mathrm{d}\tau$ is then called the ***derivative of*** $\vec{\boldsymbol{v}}$ ***along*** $\mathscr{L}$ at point $A(\tau)$. Given a basis $(\vec{\boldsymbol{e}}_\alpha)$ of E, we may write $\vec{\boldsymbol{v}}(\tau) = v^\alpha(\tau)\, \vec{\boldsymbol{e}}_\alpha$. It is then easy to see that $\vec{\boldsymbol{v}}$ is differentiable iff the components $v^\alpha(\tau)$ are differentiable functions $\mathbb{R} \to \mathbb{R}$. Moreover, the components of the derivative are nothing but the derivatives of the components:

$$\boxed{\frac{\mathrm{d}\vec{\boldsymbol{v}}}{\mathrm{d}\tau} = \frac{\mathrm{d}v^\alpha}{\mathrm{d}\tau}\, \vec{\boldsymbol{e}}_\alpha}. \tag{2.54}$$

2.7.3 Curvature and Torsions

This section can be skipped during a first reading.

Along any timelike worldline $\mathscr{L}$, one may define, from a pure geometrical viewpoint, an orthonormal basis, the ***Serret–Frenet tetrad*** $(\vec{\boldsymbol{e}}_0, \vec{\boldsymbol{e}}_1, \vec{\boldsymbol{e}}_2, \vec{\boldsymbol{e}}_3)$, which characterizes the *curvature* and *torsion* of the worldline. Usually, the Serret–Frenet tetrad is constructed in a Euclidean space, from the *arc-length parameter* s along the curve. In Minkowski spacetime, which is not Euclidean (cf. Sect. 1.3.1), the Serret–Frenet tetrad is constructed instead from the metric length $c\tau$, τ being the proper time along the curve.

The first vector of the Serret–Frenet tetrad is nothing but the 4-velocity $\vec{u}$ of $\mathscr{L}$: $\vec{e}_0 := \vec{u}$. The vector $\vec{e}_0$ is thus timelike, unit and tangent to $\mathscr{L}$. Let us assume that the 4-acceleration $\vec{a}$ of $\mathscr{L}$ is nonvanishing. If this is not the case, $\mathscr{L}$ is reduced to a straight line and the Serret–Frenet approach is useless. The second vector of the Serret–Frenet tetrad is defined by

$$\vec{e}_1 := \frac{1}{a}\vec{a} = \frac{1}{ac}\frac{\mathrm{d}\vec{e}_0}{\mathrm{d}\tau}, \qquad \text{where} \qquad a := \|\vec{a}\|_g = \sqrt{\vec{a}\cdot\vec{a}}. \tag{2.55}$$

The second equality in a's definition is meaningful for $\vec{a}$ is a spacelike vector [cf. Eq. (2.18)]. The positive number a is called the ***curvature*** of the worldline $\mathscr{L}$ at the considered point. From our conventions (cf. Sect. 2.4.2), the dimension of a is the inverse of a length. The quantity a^{-1} is called the ***curvature radius*** of $\mathscr{L}$ at the considered point. In a Euclidean space, a^{-1} would be the radius of the circle that approximates the best the curve $\mathscr{L}$ at the considered point. However, Minkowski spacetime being not a metric space, the notion of circle is not defined in the present context. A second interpretation of the curvature radius is this time transposable to Minkowski spacetime: a^{-1} is the distance to $\mathscr{L}$ at which two hyperplanes orthogonal to $\vec{u}$ at two neighbouring points of $\mathscr{L}$ intersect. We shall show it at Sect. 3.7.

Let us consider now the derivative of the vector $\vec{e}_1$ along $\mathscr{L}$, following the definition (2.53). Since $\vec{e}_1$ is a unit vector, $\mathrm{d}\vec{e}_1/\mathrm{d}\tau$ is orthogonal to $\vec{e}_1$; it is thus expressible as a linear combination of $\vec{e}_0$ and a unit vector $\vec{e}_2$ orthogonal to both $\vec{e}_0$ and $\vec{e}_1$:

$$\frac{1}{c}\frac{\mathrm{d}\vec{e}_1}{\mathrm{d}\tau} = a\,\vec{e}_0 + T_1\,\vec{e}_2. \tag{2.56}$$

The fact that the coefficient of $\vec{e}_0$ in the above formula is a can be checked by expanding the identity $d/\mathrm{d}\tau(\vec{e}_0\cdot\vec{e}_1) = 0$. If $\mathrm{d}\vec{e}_1/\mathrm{d}\tau$ is not collinear to $\vec{e}_0$, relation (2.56) constitutes the definition of both the scalar $T_1 \geq 0$ and the unit vector $\vec{e}_2$. T_1 is called the ***first torsion*** of the worldline $\mathscr{L}$. If $T_1 = 0$, $\mathscr{L}$ is contained in the plane generated by $(\vec{e}_0, \vec{e}_1)$. In the general case, let O be a point of $\mathscr{L}$ and let us set $\tau(O) = 0$. Given a point $A(\tau) \in \mathscr{L}$ close to O and of proper time τ, we may perform a Taylor expansion of the vector $\overrightarrow{OA}$ in terms of the dimensionless parameter

$$\varepsilon := a_0 c\tau, \tag{2.57}$$

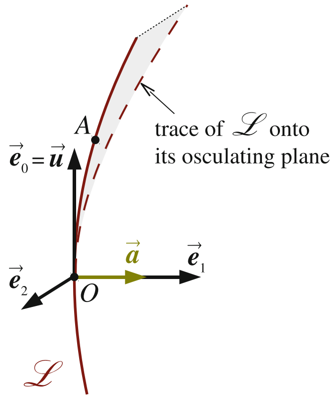

Fig. 2.12 Serret–Frenet tetrad at some point O of the worldline $\mathscr{L}$ (the vector $\vec{e}_3$ is not drawn)

where a_0 is $\mathscr{L}$'s curvature at O. Expanding up to power 3 in ε, we get

$$\overrightarrow{OA}(\tau) = \varepsilon \frac{d\,\overrightarrow{OA}}{d\varepsilon} + \frac{\varepsilon^2}{2}\frac{\mathrm{d}^2\overrightarrow{OA}}{d\varepsilon^2} + \frac{\varepsilon^3}{6}\frac{\mathrm{d}^3\overrightarrow{OA}}{d\varepsilon^3} + O(\varepsilon^4), \tag{2.58}$$

with $d^k\overrightarrow{OA}/d\varepsilon^k = (ca_0)^{-k}\, d^k\overrightarrow{OA}/\mathrm{d}\tau^k$, and from Eqs. (2.12), (2.55) and (2.56),

$$\frac{1}{c}\frac{d\,\overrightarrow{OA}}{\mathrm{d}\tau} = \vec{e}_0 \tag{2.59a}$$

$$\frac{1}{c^2}\frac{\mathrm{d}^2\overrightarrow{OA}}{\mathrm{d}\tau^2} = a\,\vec{e}_1 \tag{2.59b}$$

$$\frac{1}{c^3}\frac{\mathrm{d}^3\overrightarrow{OA}}{\mathrm{d}\tau^3} = a^2\vec{e}_0 + \frac{1}{c}\frac{da}{\mathrm{d}\tau}\vec{e}_1 + aT_1\vec{e}_2. \tag{2.59c}$$

Hence

$$\boxed{\begin{aligned}\overrightarrow{OA}(\tau) = {} & \left(1 + \frac{(ac\tau)^2}{6}\right)c\tau\,\vec{e}_0 + \left(a + \frac{da}{\mathrm{d}\tau}\frac{\tau}{3}\right)\frac{(c\tau)^2}{2}\vec{e}_1 \\ & + \frac{aT_1}{6}(c\tau)^3\,\vec{e}_2 + O((ac\tau)^4).\end{aligned}} \tag{2.60}$$

In this equality, the quantities a, $da/\mathrm{d}\tau$ and T_1, as well as the vectors $\vec{e}_0$, $\vec{e}_1$ and $\vec{e}_2$, have to be taken at the point O.

The expansion (2.60) shows that, up to the order $(ac\tau)^2$, the worldline stays in the plane $(O; \vec{e}_0, \vec{e}_1)$. This plane is called the ***osculating plane*** of $\mathscr{L}$ at O. The first torsion T_1, which appears at the order $(ac\tau)^3$ in the expansion (2.60), measures thus the departure of the worldline from its osculating plane (cf. Fig. 2.12).

Let us assume that $T_1 \neq 0$, i.e. that $\vec{e}_2$ is well defined. Since the latter is a unit vector ($\vec{e}_2 \cdot \vec{e}_2 = 1$), $\mathrm{d}\vec{e}_2/\mathrm{d}\tau$ is orthogonal to $\vec{e}_2$ and thus can be written as a linear combination of $\vec{e}_0$, $\vec{e}_1$ and a unit vector $\vec{e}_3$ orthogonal to $\vec{e}_0$, $\vec{e}_1$ and $\vec{e}_2$:

$$\frac{1}{c}\frac{\mathrm{d}\vec{e}_2}{\mathrm{d}\tau} = \alpha\,\vec{e}_0 + \beta\,\vec{e}_1 + T_2\,\vec{e}_3.$$

The coefficients α and β are determined from the scalar products $\vec{e}_0 \cdot \vec{e}_2 = 0$ and $\vec{e}_1 \cdot \vec{e}_2 = 0$; taking the derivative of the first one with respect to τ, we get $\alpha = 0$, whereas taking the derivative of the second one yields $\beta = -T_1$. Hence

$$\frac{1}{c}\frac{\mathrm{d}\vec{e}_2}{\mathrm{d}\tau} = -T_1\,\vec{e}_1 + T_2\,\vec{e}_3. \tag{2.61}$$

If $\mathrm{d}\vec{e}_2/\mathrm{d}\tau$ is not collinear to $\vec{e}_1$, this relation constitutes the definition of both the scalar $T_2 \geq 0$ and the unit vector $\vec{e}_3$. T_2 is called the ***second torsion*** of the worldline $\mathscr{L}$. If $T_2 = 0$, $\mathscr{L}$ is contained in the affine subspace of $\mathscr{E}$ of dimension 3 (hyperplane) and generated by $(\vec{e}_0, \vec{e}_1, \vec{e}_2)$. In the general case, (2.60) shows that at the order $(ac\tau)^3$, $\mathscr{L}$ is contained in the hyperplane $(O; \vec{e}_0, \vec{e}_1, \vec{e}_2)$, that we shall call the ***osculating hyperplane*** of the worldline at the point O. It is easy to see that $T_2\vec{e}_3$ is involved at the order $(ac\tau)^4$ in the expansion of $\overrightarrow{OA}(\tau)$. The second torsion measures thus the departure of $\mathscr{L}$ from its osculating hyperplane.

Let us suppose that $T_2 \neq 0$ and evaluate $\mathrm{d}\vec{e}_3/\mathrm{d}\tau$. Since $\vec{e}_3$ is a unit vector, $\mathrm{d}\vec{e}_3/\mathrm{d}\tau$ is orthogonal to $\vec{e}_3$. It is then necessarily a linear combination of the vectors $\vec{e}_0$, $\vec{e}_1$ and $\vec{e}_2$:

$$\frac{1}{c}\frac{\mathrm{d}\vec{e}_3}{\mathrm{d}\tau} = \alpha\,\vec{e}_0 + \beta\,\vec{e}_1 + \gamma\,\vec{e}_2.$$

Taking the derivative with respect to τ of the identities $\vec{e}_0 \cdot \vec{e}_3 = 0$, $\vec{e}_1 \cdot \vec{e}_3 = 0$ and $\vec{e}_2 \cdot \vec{e}_3 = 0$, we get $\alpha = 0$, $\beta = 0$ and $\gamma = -T_2$, so that we may write

$$\frac{1}{c}\frac{\mathrm{d}\vec{e}_3}{\mathrm{d}\tau} = -T_2\,\vec{e}_2. \tag{2.62}$$

Altogether, (2.55), (2.56), (2.61) and (2.62) can be written as

$$\boxed{c^{-1}\begin{pmatrix} \mathrm{d}\vec{e}_0/\mathrm{d}\tau \\ \mathrm{d}\vec{e}_1/\mathrm{d}\tau \\ \mathrm{d}\vec{e}_2/\mathrm{d}\tau \\ \mathrm{d}\vec{e}_3/\mathrm{d}\tau \end{pmatrix} = \begin{pmatrix} 0 & a & 0 & 0 \\ a & 0 & T_1 & 0 \\ 0 & -T_1 & 0 & T_2 \\ 0 & 0 & -T_2 & 0 \end{pmatrix} \begin{pmatrix} \vec{e}_0 \\ \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{pmatrix}}. \tag{2.63}$$

We shall see at Sect. 3.5.3 that the matrix appearing in the above formula can be interpreted in terms of the *4-rotation* of the Serret–Frenet tetrad.

Historical note: *The interpretation of the norm of the 4-acceleration as the curvature of the worldline appeared as early as 1908 in an article by Hermann Minkowski (cf. p. 26) (1908) and subsequently in his famous text on spacetime (Minkowski 1909).*

Chapter 3
Observers

3.1 Introduction

The notion of *affine coordinate system* or *affine frame* introduced in Sect. 1.2.3 is purely mathematical. The aim of this chapter is to define and discuss *physical observers* and *frames* in special relativity. With the material introduced at this stage, the only physical measure that can be described is that of proper time along a worldline by means of an ideal clock. To move to the full concept of observer, one shall extend the spacetime domain where measures can be performed outside the worldline. This raises the problem of simultaneity, from which we shall start the discussion.

3.2 Simultaneity and Measure of Time

3.2.1 The Problem

Let us consider a particle $\mathscr{O}$ moving on a *timelike* worldline $\mathscr{L}_0$ ($\mathscr{O}$ is a massive particle in the terminology introduced in Sect. 2.2). $\mathscr{O}$ is to be thought of not as an elementary particle but rather as some device (or even a human being) whose size is neglected (cf. the discussion in Sect. 2.2). Indeed, after having it equipped with a "frame" in Sect. 3.4, we shall define $\mathscr{O}$ as an "observer". We assume that $\mathscr{O}$ carries an ideal clock (cf. Sect. 2.3.2) so that he can measure the proper time between any two events on his worldline. He may then set a date to each event of $\mathscr{L}_0$ by choosing an event of $\mathscr{L}_0$ as the origin of proper time ($t = 0$). But how can $\mathscr{O}$ set a date to events that do not occur on his worldline?

A first answer consists in attributing the date t_A to any event simultaneous with the event A of proper time t_A along $\mathscr{O}$'s worldline. But such definition supposes that the notion of *simultaneity* is given a priori. This is fine in Newton's theory, which stipulates the existence of an absolute time, independent of any observer (cf.

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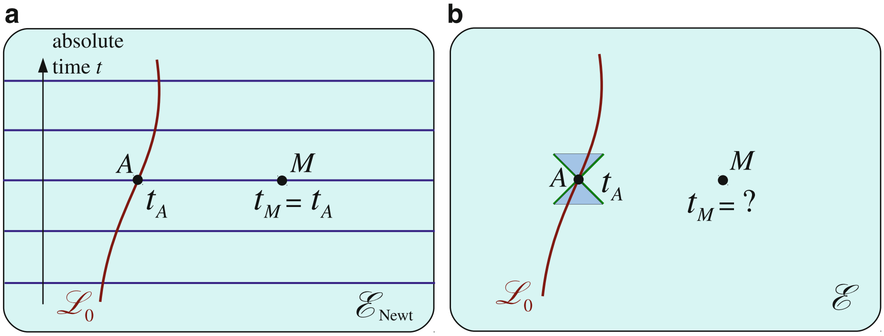


Fig. 3.1 Dating problem: (**a**) in Newtonian spacetime; (**b**) in Minkowski spacetime

Sect. 1.2.5 and Fig. 3.1a). But things are different for Minkowski spacetime, where no temporal "split" is given a priori (cf. Sect. 1.7 and Fig. 3.1b). The only privileged structures in Minkowski spacetime $\mathscr{E}$ are those related to the metric tensor, namely, the light cones (cf. Fig. 1.8 and Sect. 2.5.2). Now the light cones do not induce any slicing of $\mathscr{E}$ by spacelike hypersurfaces similar to the slicing of the Newtonian spacetime shown in Figs. 1.2 and 3.1a.

3.2.2 *Einstein–Poincaré Simultaneity*

Let us suppose that observer $\mathscr{O}$ is equipped, in addition to his ideal clock, with a device for emitting and receiving photons. Let A be an event of proper time t along $\mathscr{O}$'s worldline and M any event in $\mathscr{E}$. We shall say that M is ***simultaneous to*** A ***for observer*** $\mathscr{O}$ iff

$$\boxed{t = \frac{1}{2}(t_1 + t_2)}, \tag{3.1}$$

where t_1 is the proper time (with respect to $\mathscr{O}$) of the emission by $\mathscr{O}$ of a photon that reaches the event M and is immediately reflected to reach again observer $\mathscr{O}$ at the proper time t_2 (cf. Fig. 3.2). The ***date of the event*** M ***with respect to observer*** $\mathscr{O}$ is then defined as t.

The above definition is called ***Einstein–Poincaré criterion of simultaneity***; it is quite natural and can be interpreted naively by considering that the "time $t - t_1$" taken by light to travel from $\mathscr{O}$ to M is the same as the "time $t_2 - t$" taken to travel from M to $\mathscr{O}$. We are saying "naively" because the notion of "travel time" depends on the adopted definition of date and therefore on the concept of simultaneity.

Remark 3.1. In Einstein's view (Einstein 1905b), the definition of simultaneity given above fits nicely with his postulate of constancy of the velocity of light.

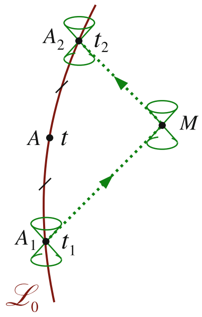


Fig. 3.2 Definition of Einstein–Poincaré simultaneity: A and M are simultaneous with respect to observer $\mathscr{O}$ iff A occurs at the mid-travelling time of the photon making a round trip from $\mathscr{O}$ to M

In the more geometrical settings adopted here, this definition is very acceptable for it relies on the light cones (via the photons' trajectories, cf. Fig. 3.2), which are the only canonical structures of Minkowski spacetime. Moreover, this definition is operational, being based on a physical criterion (measure of the round-trip time of an electromagnetic signal).

Remark 3.2. Hans Reichenbach has proposed in 1924 (Reichenbach 1924) a definition of simultaneity based on the criterion

$$t = (1 - \varepsilon)t_1 + \varepsilon t_2, \tag{3.2}$$

where ε is a constant chosen in the interval $]0, 1[$. This criterion, called ***ε-simultaneity***, reduces to the Einstein–Poincaré criterion (3.1) for $\varepsilon = 1/2$. There would be no logical inconsistency in choosing (3.2) with $\varepsilon \neq 1/2$, but this would result in an unnecessarily complicated formulation of special relativity (cf. Friedman (1983) and Vallisneri (2000) for a discussion).

Historical note: *Henri Poincaré (cf. p. 26) has been among the first ones to question the obvious character of simultaneity, as early as 1898 (Poincaré 1898). He pointed that we do not have the direct intuition of the simultaneity of two distant events nor even of their order of occurrence. He has shown that these notions are intimately linked to the definition of time itself. Poincaré concluded that simultaneity has to result from some* convention, *which must be clearly stated. A criterion for selecting between various conventions can be the simplicity in expressing the laws of physics. In Sect. 2.3, the same criterion made us prefer the use of proper time among all possible timescales along a given worldline. In 1900, Poincaré introduced the idea of synchronizing the clocks of a moving frame by exchanging light signals (Poincaré 1900). Albert Einstein used the same method in 1905 (Einstein 1905b) to give the definition (3.1) of simultaneity of two events with respect to a given observer. An important difference between Poincaré's analysis and Einstein's one is that according to Poincaré, the time t given by formula (3.1) is just an* "apparent time" *(Poincaré 1900), which differs from the* "real time" *when the observer is moving with respect to aether. On the other hand, for Einstein, there is no aether, and the time (3.1) of a "moving" observer is as legitimate as the time of an observer*

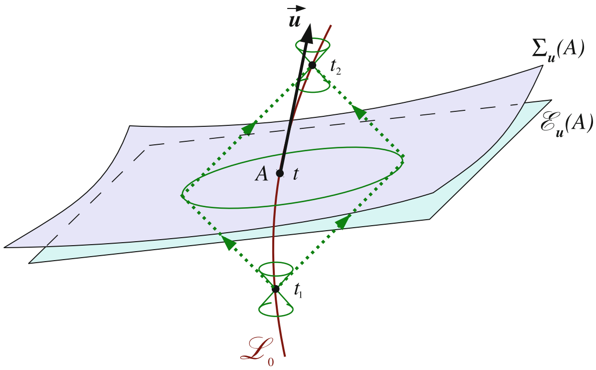


Fig. 3.3 Simultaneity hypersurface $\Sigma_{\boldsymbol{u}}(A)$ and local rest space $\mathscr{E}_{\boldsymbol{u}}(A)$ of an event A along the worldline $\mathscr{L}_0$

"at rest". For more details on the different perceptions of time between Poincaré and Einstein, we refer to Damour (2006, 2007), Darrigol (2004, 2006), Galison (2003), Reignier (2007), Rougé (2008) and Walter (2008). In particular, Darrigol's article (Darrigol 2004) is discussing in detail the influence of Poincaré on the definition (3.1) of simultaneity, and Walter's article (Walter 2008) provides a deep analysis of Poincaré's conception of spacetime.

3.2.3 Local Rest Space

The set of events that are simultaneous to an event A on $\mathscr{O}$'s worldline is a surface[1] of dimension 3 of the affine space $\mathscr{E}$, which intersects $\mathscr{L}_0$ at A (see Fig. 3.3). Being of dimension 3 in a space of dimension 4, one says that it is a ***hypersurface***.[2] We shall call it the ***simultaneity hypersurface of*** A ***for*** $\mathscr{O}$ and denote it by $\Sigma_{\boldsymbol{u}}(A)$ or $\Sigma_{\boldsymbol{u}}(t)$, $\vec{\boldsymbol{u}}$ being the 4-velocity of observer $\mathscr{O}$ and t the proper time of A with respect to $\mathscr{O}$.

An important geometrical property of the simultaneity hypersurface is its orthogonality (with respect to the metric tensor $\boldsymbol{g}$) to the worldline of the considered observer, as we are going to prove.

Let A be an event on $\mathscr{L}_0$ of proper time t and B an event not belonging to $\mathscr{L}_0$. Let us consider the emission of a photon by $\mathscr{O}$ (event $A_1 \in \mathscr{L}_0$) that is reflected at B to be received by $\mathscr{O}$ at the event A_2 (cf. Fig. 3.4). We assume that B is located *close*

[1]The proper technical word is *submanifold*, whose precise definition will be given in Chap. 16.

[2]A special type of hypersurface is of course a hyperplane, as encountered already in Sect. 1.2.5.

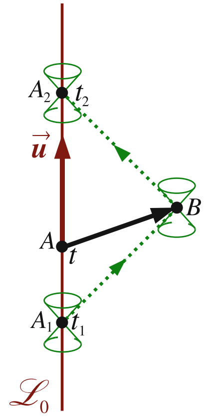


Fig. 3.4 Event B in the neighbourhood of the worldline $\mathscr{L}_0$: a light signal emitted from $\mathscr{L}_0$ in A_1 is instantaneously reflected in B to reach again $\mathscr{L}_0$ at A_2

to A, in the sense that the curvature of $\mathscr{L}_0$ can be neglected between A_1 and A_2. We may then consider that the vectors $\overrightarrow{A_1A}$ and $\overrightarrow{A_2A}$ are collinear:

$$\overrightarrow{A_1A} = c(t - t_1)\,\vec{u}(A) \quad \text{and} \quad \overrightarrow{A_2A} = c(t - t_2)\,\vec{u}(A), \tag{3.3}$$

where t_1 (resp. t_2) is the proper time of $\mathscr{O}$ at A_1 (resp. A_2). The equalities (3.3) result from the very definition of proper time and the unit character of $\vec{u}(A)$. By definition of A_1, the vector $\overrightarrow{A_1B}$ is null: $\overrightarrow{A_1B} \cdot \overrightarrow{A_1B} = 0$. Using Chasles' relation $\overrightarrow{A_1B} = \overrightarrow{A_1A} + \overrightarrow{AB}$, we get

$$\overrightarrow{A_1A} \cdot \overrightarrow{A_1A} + 2\overrightarrow{A_1A} \cdot \overrightarrow{AB} + \overrightarrow{AB} \cdot \overrightarrow{AB} = 0.$$

Now, from (3.3) and $\vec{u}(A) \cdot \vec{u}(A) = -1$, we have $\overrightarrow{A_1A} \cdot \overrightarrow{A_1A} = -c^2(t - t_1)^2$ and $\overrightarrow{A_1A} \cdot \overrightarrow{AB} = c(t - t_1)\,\vec{u}(A) \cdot \overrightarrow{AB}$. Hence,

$$-c^2(t - t_1)^2 + 2c(t - t_1)\,\vec{u}(A) \cdot \overrightarrow{AB} + \overrightarrow{AB} \cdot \overrightarrow{AB} = 0. \tag{3.4}$$

Similarly, the lightlike character of $\overrightarrow{A_2B}$ leads to

$$-c^2(t - t_2)^2 + 2c(t - t_2)\,\vec{u}(A) \cdot \overrightarrow{AB} + \overrightarrow{AB} \cdot \overrightarrow{AB} = 0. \tag{3.5}$$

If the proper times t, t_1 and t_2 are obtained from the reading of $\mathscr{O}$'s ideal clock, (3.4)–(3.5) constitute a linear system for two unknowns: the scalar products $\vec{u}(A)\cdot\overrightarrow{AB}$ and $\overrightarrow{AB}\cdot\overrightarrow{AB}$. The determinant of this system is $2c(t-t_1)-2c(t-t_2) = 2c(t_2-t_1) \neq 0$ since $B \notin \mathscr{L}_0$. There is therefore a unique solution. The latter is easily obtained by subtracting (3.5) from (3.4):

$$\vec{u}(A) \cdot \overrightarrow{AB} = c\left[t - \frac{1}{2}(t_1 + t_2)\right] \tag{3.6}$$

$$\overrightarrow{AB} \cdot \overrightarrow{AB} = c^2(t - t_1)(t_2 - t). \tag{3.7}$$

By comparing (3.6) to the Einstein–Poincaré simultaneity criterion (3.1), we deduce immediately that

$$\boxed{B \text{ is simultaneous to } A \text{ for } \mathscr{O} \iff \vec{u}(A) \cdot \overrightarrow{AB} = 0}. \tag{3.8}$$

We have thus established the following important property:

In the neighbourhood of an event A of the worldline of observer $\mathscr{O}$, the events simultaneous to A for $\mathscr{O}$ are those located in directions orthogonal—with respect to the metric $\boldsymbol{g}$—to $\mathscr{O}$'s worldline.

In the above statement, *neighbourhood* means that the curvature of $\mathscr{L}_0$ can be neglected (cf. Remark 3.3 below). The events simultaneous to A constitute thus an affine subspace of $\mathscr{E}$, namely, the affine subspace through A and orthogonal to $\vec{u}(A)$. Since the bilinear form $\boldsymbol{g}$ is nondegenerate (cf Sect. 1.3.1), this space is of dimension 3; it is thus an hyperplane of $\mathscr{E}$ (cf. Sect. 1.2.5). Moreover, this hyperplane is ***spacelike***, which means that all vectors parallel to it are spacelike. Indeed, we have seen in Sect. 2.4.2 that any nonzero vector orthogonal to a timelike vector ($\vec{u}(A)$ in the present case) is necessarily spacelike. We shall denote this hyperplane by $\mathscr{E}_{\boldsymbol{u}}(A)$ and call it the ***local rest space of observer*** $\mathscr{O}$ ***at*** A. If t is the proper time of event A, we shall also denote $\mathscr{E}_{\boldsymbol{u}}(A)$ by $\mathscr{E}_{\boldsymbol{u}}(t)$.

Remark 3.3. In this definition, the qualifier *local* reminds us that the simultaneity of events located in $\mathscr{E}_{\boldsymbol{u}}(A)$ and A has been established a priori only for events close to A so that one can neglect the curvature of $\mathscr{L}_0$. The affine space $\mathscr{E}_{\boldsymbol{u}}(A)$ is actually the space tangent in A to the simultaneity hypersurface $\Sigma_{\boldsymbol{u}}(A)$ (cf. Fig. 3.3). We shall see in Sect. 12.3 that $\Sigma_{\boldsymbol{u}}(A)$ and $\mathscr{E}_{\boldsymbol{u}}(A)$ coincide—i.e. $\mathscr{E}_{\boldsymbol{u}}(A)$ contains all events simultaneous to A, even the far ones—when the 4-acceleration $\vec{a}$ of $\mathscr{O}$ vanishes ($\mathscr{L}_0$ is then a straight line of $\mathscr{E}$, and $\mathscr{O}$ is an inertial observer) or when $\|\vec{a}\|_{\boldsymbol{g}}$ is constant and the curve $\mathscr{L}_0$ has no torsion. For all the other motions, $\mathscr{E}_{\boldsymbol{u}}(A)$ constitutes some approximation of the simultaneity hypersurface of A, the validity of which will be discussed in terms of the distance to $\mathscr{L}_0$ in Sect. 12.3.

Remark 3.4. The ε-simultaneity proposed by Reichenbach [Eq. (3.2)] would lead to a local rest space that would not be orthogonal to the worldline whenever $\varepsilon \neq 1/2$. Consequently, the simultaneity thus defined would not have a direct connection with the metric tensor $\boldsymbol{g}$.

The vector subspace of E made of all the vectors orthogonal to $\vec{u}(A)$ is denoted $E_{\boldsymbol{u}}(A)$, or $E_{\boldsymbol{u}}(t)$, t being the proper time of event A. $E_{\boldsymbol{u}}(A)$ is a vector space of dimension 3 (because $\boldsymbol{g}$ is nondegenerate) whose vectors are all spacelike. It is the vector space underlying the affine space $\mathscr{E}_{\boldsymbol{u}}(A)$. We shall call it by the same name, i.e. the ***local rest space of observer*** $\mathscr{O}$ ***at*** A.

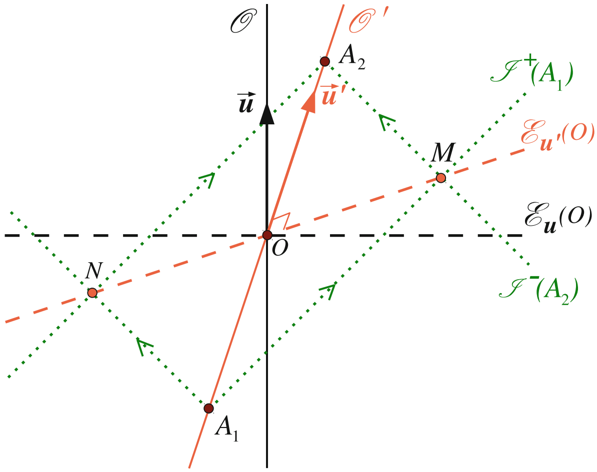
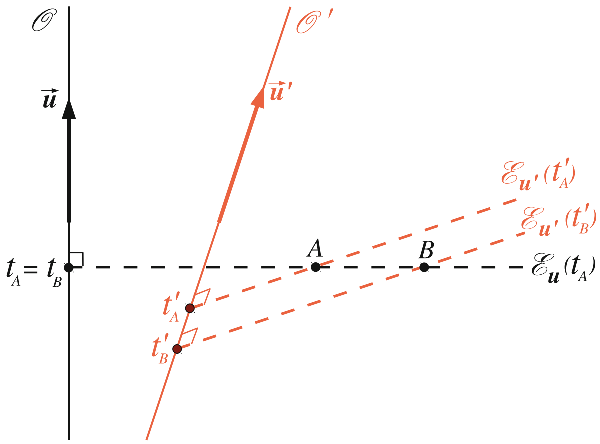


Fig. 3.5 Geometrical construction of the local rest space $\mathscr{E}_{u'}(O)$ of the event O for observer $\mathscr{O}'$: the events A_1 and A_2 are symmetric with respect to O along the worldline of $\mathscr{O}'$; the events M and N, lying on the intersection of the future light cone $\mathscr{I}^+(A_1)$ of A_1 with the past light cone $\mathscr{I}^-(A_2)$ of A_2, are simultaneous with O for $\mathscr{O}'$. The construction can be repeated by varying A_1 and A_2 in order to obtain new events in $\mathscr{E}_{u'}(O)$. The local rest space $\mathscr{E}_{u'}(O)$ hence built is reduced to a single dimension in the figure (*dashed line* through N, O and M). It is orthogonal to the 4-velocity $\vec{u}'$, despite not being drawn at 90° from $\vec{u}'$ (cf. the discussion of Sect. 1.3.6 about graphical representation in Minkowski spacetime)

3.2.4 Nonexistence of Absolute Time

The adopted definition of simultaneity allows any observer $\mathscr{O}$ to set a date to all events in $\mathscr{E}$, even to those lying outside his worldline. However, two different observers will not attribute, in general, the same date to a given event. This is so because the two observers will have different simultaneity hypersurfaces, as one can see on Fig. 3.5. This figure represents two observers, $\mathscr{O}$ and $\mathscr{O}'$, of respective 4-velocity $\vec{u}$ and $\vec{u}'$ and whose worldlines intersect at the event O. Moreover all the figure is drawn close enough to O so that the worldlines $\mathscr{L}$ and $\mathscr{L}'$ of $\mathscr{O}$ and $\mathscr{O}'$ can be approximated by straight lines. Having chosen to draw $\mathscr{L}$ as a vertical line, the local rest space of $\mathscr{O}$ at O, $\mathscr{E}_u(O)$, which is orthogonal to $\mathscr{L}$, appears as a horizontal line (cf. the discussion in Sect. 1.3.6). Light rays being lines inclined at $\pm 45°$ in this graphics, the construction of Fig. 3.5 shows that the local rest space of $\mathscr{O}'$ at O, $\mathscr{E}_{u'}(O)$, is inclined with respect to $\mathscr{E}_u(O)$ with the same angle (in absolute value) as $\mathscr{L}'$ with respect to $\mathscr{L}$. We recover actually the property of symmetry with respect to the main bisectors established in Sect. 1.3.6. Thus the two spaces do not coincide. As a consequence:

Fig. 3.6 Relativity of the notion of simultaneity: events A and B are simultaneous for observer $\mathscr{O}$, but not for observer $\mathscr{O}'$



Two events that are simultaneous for a given observer will not necessarily be so for a second observer. In other words, there does not exist a unique time associated with each event in $\mathscr{E}$, but only times defined relatively to observers.

This important point is illustrated in Fig. 3.6: the events A and B are simultaneous for observer $\mathscr{O}$, who attributes the same date $t_A = t_B$ to them, but not for observer $\mathscr{O}'$, who attributes to A a date posterior to that of B ($t'_A > t'_B$).

3.2.5 Orthogonal Projector Onto the Local Rest Space

The introduction of the local rest space $\mathscr{E}_{\boldsymbol{u}}(A)$ at any event A of the worldline of an observer $\mathscr{O}$ is naturally accompanied by the split of vectors of E into a part tangent to $\mathscr{E}_{\boldsymbol{u}}(A)$, i.e. a part belonging to the vector subspace $E_{\boldsymbol{u}}(A)$ and a part orthogonal to it, i.e. a part collinear to $\vec{\boldsymbol{u}}(A)$ (cf. Fig. 3.7):

$$\forall \vec{\boldsymbol{v}} \in E, \quad \vec{\boldsymbol{v}} = \perp_u \vec{\boldsymbol{v}} + \alpha \vec{\boldsymbol{u}}, \qquad \text{with} \quad \perp_u \vec{\boldsymbol{v}} \in E_{\boldsymbol{u}}(A) \text{ and } \alpha \in \mathbb{R}. \tag{3.9}$$

In the above writing, one should read $\vec{\boldsymbol{u}} = \vec{\boldsymbol{u}}(A)$. The decomposition (3.9) is unique: taking the scalar product with $\vec{\boldsymbol{u}}$, one gets indeed

$$\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}} = \underbrace{\vec{\boldsymbol{u}} \cdot \perp_u \vec{\boldsymbol{v}}}_{0} + \alpha \underbrace{\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{u}}}_{-1},$$

which fully determines α as $\alpha = -\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}}$. We may then rewrite (3.9) as

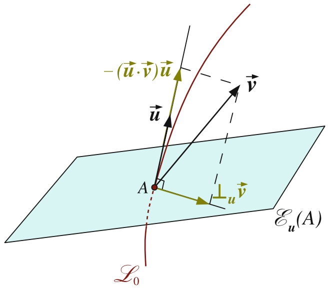


Fig. 3.7 Orthogonal decomposition of a vector $\vec{v}$ into a part $\perp_u \vec{v}$ that is orthogonal to the 4-velocity $\vec{u}$ of some observer and a part collinear to $\vec{u}$. Note that $\perp_u \vec{v} \in E_u(A)$, where $E_u(A)$ is the observer's local rest space at A

$$\boxed{\forall \vec{v} \in E, \quad \vec{v} = \perp_u \vec{v} - (\vec{u} \cdot \vec{v})\, \vec{u}}. \tag{3.10}$$

The part $\perp_u \vec{v}$ of this decomposition is called the ***orthogonal projection of*** $\vec{v}$ onto $E_u(A)$. The mapping

$$\begin{aligned} \perp_u : E &\longrightarrow E_u(A) \\ \vec{v} &\longmapsto \perp_u \vec{v} = \vec{v} + (\vec{u} \cdot \vec{v})\, \vec{u} \end{aligned} \tag{3.11}$$

is an endomorphism of E (cf. Appendix A), called the ***orthogonal projector onto*** the vector subspace $E_u(A)$. Since $\vec{u} \cdot \vec{v} = \langle \underline{u}, \vec{v} \rangle$ (metric duality between vectors and linear forms discussed in Sect. 1.6), we may write this endomorphism as

$$\boxed{\perp_u = \mathrm{Id} + \langle \underline{u}, . \rangle\, \vec{u}}, \tag{3.12}$$

where Id stands for the identity operator.

Remark 3.5. In the Euclidean space $\mathbb{R}^3$, the orthogonal projector onto the plane orthogonal to some unit vector $\vec{u}$ is written as $\perp_u = \mathrm{Id} - \langle \underline{u}, . \rangle\, \vec{u}$. The change of sign with respect to (3.12) is of course due to the signature $(-,+,+,+)$ of the metric $\boldsymbol{g}$ and to the timelike character of vector $\vec{u}$, whereas for the Euclidean scalar product, the signature is $(+,+,+)$ and all vectors are spacelike.

Three properties follow immediately from the definition of the orthogonal projector:

$$\perp_u \vec{u} = 0, \tag{3.13}$$

$$\forall \vec{v} \in E_u(A), \quad \perp_u \vec{v} = \vec{v} \tag{3.14}$$

$$\perp_u \circ \perp_u = \perp_u. \tag{3.15}$$

The first two properties, along with that of linearity, would be sufficient to entirely define $\perp_{\boldsymbol{u}}$. The third property is called ***idempotence*** and is characteristic of a projector.

In addition, the decomposition (3.10) of all vectors in E is accounted for by stating that E is the ***direct sum*** of the vector subspaces $E_{\boldsymbol{u}}(A)$ (hyperplane) and $\mathrm{Span}(\vec{\boldsymbol{u}}(A))$ (vector line generated by $\vec{\boldsymbol{u}}(A)$):

$$\boxed{E = E_{\boldsymbol{u}}(A) \overset{\perp}{\oplus} \mathrm{Span}(\vec{\boldsymbol{u}}(A))}. \tag{3.16}$$

3.2.6 Euclidean Character of the Local Rest Space

We have seen in Sect. 3.2.3 that all vectors of $E_{\boldsymbol{u}}(A)$ are spacelike. It follows that the restriction of the metric tensor $\boldsymbol{g}$ to $E_{\boldsymbol{u}}(A)$ is a *positive definite* bilinear form (cf. Sect. 1.3.4):

$$\forall \vec{\boldsymbol{v}} \in E_{\boldsymbol{u}}(A), \quad \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{v}} := \boldsymbol{g}(\vec{\boldsymbol{v}}, \vec{\boldsymbol{v}}) \geq 0 \quad \text{and} \quad \boldsymbol{g}(\vec{\boldsymbol{v}}, \vec{\boldsymbol{v}}) = 0 \iff \vec{\boldsymbol{v}} = 0. \tag{3.17}$$

This means that the local rest space $\mathscr{E}_{\boldsymbol{u}}(A)$ equipped with the metric $\boldsymbol{g}|_{E_{\boldsymbol{u}}(A)}$ ($\boldsymbol{g}$ restricted to $E_{\boldsymbol{u}}(A)$) is a *Euclidean space* (cf. Sect. 1.3.1). Being three-dimensional, it is therefore fully similar to the space $\mathbb{R}^3$ equipped with the "usual" scalar product. In particular, the restriction to $E_{\boldsymbol{u}}(A)$ of the function $\| \ \|_{\boldsymbol{g}}$ introduced in Sect. 1.3.5 ($\|\vec{\boldsymbol{v}}\|_{\boldsymbol{g}} = \sqrt{\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{v}}}$ for $\vec{\boldsymbol{v}} \in E_{\boldsymbol{u}}(A)$) does define a norm in the standard mathematical sense. This norm induces a distance d between points of $\mathscr{E}_{\boldsymbol{u}}(A)$, conferring to this space the structure of a *metric space*:

$$\forall (M, N) \in \mathscr{E}_{\boldsymbol{u}}(A) \times \mathscr{E}_{\boldsymbol{u}}(A), \quad d(M, N) := \left\| \overrightarrow{MN} \right\|_{\boldsymbol{g}}. \tag{3.18}$$

The distance d obeys *Pythagoras' theorem*. Moreover, one may define the ***angle*** θ between two vectors $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{w}}$ of $E_{\boldsymbol{u}}(A)$ by the standard formula:

$$\cos\theta = \frac{\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{w}}}{\|\vec{\boldsymbol{v}}\|_{\boldsymbol{g}} \ \|\vec{\boldsymbol{w}}\|_{\boldsymbol{g}}} = \frac{\boldsymbol{g}(\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}})}{\sqrt{\boldsymbol{g}(\vec{\boldsymbol{v}}, \vec{\boldsymbol{v}})\, \boldsymbol{g}(\vec{\boldsymbol{w}}, \vec{\boldsymbol{w}})}}. \tag{3.19}$$

The above results may be summarized as follows:

> In the local rest space $\mathscr{E}_{\boldsymbol{u}}(A)$, all of the vector calculus is identical to the calculus in the usual Euclidean three-dimensional space.

3.3 Measuring Spatial Distances

Having discussed the dating process by an observer, let us now focus on the measure of spatial distances, i.e. the measure of length of spacelike vectors with respect to the metric tensor $\boldsymbol{g}$. We shall see that this type of measurement can be achieved solely by means of clocks and a device for emitting and receiving null signals (photons) (Sect. 3.3.1). In particular, there is no need for a "ruler". On the contrary, the notion of ruler is to be defined from pure temporal measurements (Sect. 3.3.2).

3.3.1 *Synge Formula*

Let us consider the situation depicted in Fig. 3.4, namely, some observer $\mathscr{O}$, of worldline $\mathscr{L}_0$ and 4-velocity $\vec{\boldsymbol{u}}$, along with an event $A \in \mathscr{L}_0$ and an event B close to A (in the sense that the length $\|\overrightarrow{AB}\|_{\boldsymbol{g}}$ is small with respect to the curvature radius of $\mathscr{L}_0$). Let us assume further that B is such that $\overrightarrow{AB}$ is spacelike (but we shall not suppose that $\overrightarrow{AB}$ is orthogonal to $\vec{\boldsymbol{u}}$, i.e. that $B \in \mathscr{E}_{\boldsymbol{u}}(A)$). We would like to evaluate the length of $\overrightarrow{AB}$, i.e. $\|\overrightarrow{AB}\|_{\boldsymbol{g}}$. Since $\overrightarrow{AB}$ is spacelike, the event A_1 of emission of a photon by $\mathscr{O}$ towards B is necessarily located before A on the worldline $\mathscr{L}_0$. Similarly, the event A_2 of reception by $\mathscr{O}$ of the photon reflected in B is necessarily after A on $\mathscr{L}_0$ (cf. Fig. 3.4). In Sect. 3.2.3, we have computed the scalar square $\overrightarrow{AB} \cdot \overrightarrow{AB}$ in terms of $\mathscr{O}$'s proper times t, t_1 and t_2 of, respectively, A, A_1 and A_2, obtaining formula (3.7), which we have not exploited yet. This formula yields

$$\boxed{\left\|\overrightarrow{AB}\right\|_{\boldsymbol{g}} = c\sqrt{(t-t_1)(t_2-t)}}. \tag{3.20}$$

The equality (3.20) allows one to compute the spatial length $\|\overrightarrow{AB}\|_{\boldsymbol{g}}$ solely from the measure of the proper times t, t_1 and t_2 along $\mathscr{L}_0$. We shall call it ***Synge formula***. In view of it, we may say that, in relativity, time is a primary notion and length a derived one.

Let us notice that the current definition of the unit of length is in full agreement with this: in 1983, the *Conférence Générale des Poids et Mesures*[3] has defined the metre as the fraction $1/299792458$ of the distance travelled by light in vacuum during one second.

Remark 3.6. Synge formula can be seen as the "Minkowskian equivalent" of a well-known formula of Euclidean geometry: the formula expressing the *power of a point*

[3]General Conference on Weights and Measures.

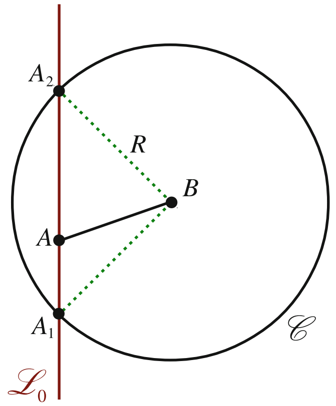


Fig. 3.8 Power of the point A with respect to the circle $\mathscr{C}$ of centre B in the Euclidean plane. This figure is to be compared with Fig. 3.4 in Minkowski spacetime. In this last case, the radius of $\mathscr{C}$, measured with respect to $\boldsymbol{g}$, is zero

with respect to a circle (Damour 2009). Indeed let us consider the circle $\mathscr{C}$ of centre B and radius R in the Euclidean plane, as well as a straight line $\mathscr{L}_0$ intersecting $\mathscr{C}$ in two points, A_1 and A_2 (cf. Fig. 3.8). Let A be a point of the line segment A_1A_2. The ***power of** A **with respect to*** $\mathscr{C}$ is defined as $P(A) := \|\overrightarrow{AB}\|^2 - R^2$. It satisfies $P(A) = \overrightarrow{AA_1} \cdot \overrightarrow{AA_2}$ independently of the choice of $\mathscr{L}_0$, hence

$$\|\overrightarrow{AB}\|^2 = \overrightarrow{AA_1} \cdot \overrightarrow{AA_2} + R^2.$$

The "Minkowskian version" of this relation is obtained by setting $R = 0$. Indeed, $R = \|\overrightarrow{BA_1}\| = \|\overrightarrow{BA_2}\|$, and since B and A_1, as well as B and A_2, are linked by a null geodesic, $\|\overrightarrow{BA_1}\|_{\boldsymbol{g}} = \|\overrightarrow{BA_2}\|_{\boldsymbol{g}} = 0$. Hence, we get

$$\left\|\overrightarrow{AB}\right\|_{\boldsymbol{g}}^2 = \overrightarrow{AA_1} \cdot \overrightarrow{AA_2}.$$

Writing $\overrightarrow{AA_1} = c(t_1 - t)\,\vec{\boldsymbol{u}}$ and $\overrightarrow{AA_2} = c(t_2 - t)\,\vec{\boldsymbol{u}}$ and using $\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{u}} = -1$, we recover the square of Synge formula (3.20).

Historical note: *It seems that formula (3.20) has been established first by Alfred A. Robb*[4] *in 1936 (Robb 1936). It has been emphasized in 1956 by John L. Synge*[5]

[4]**Alfred A. Robb** (1873–1936): British physicist, known mainly for his work in special relativity, for which he developed an axiomatic approach (Briginshaw 1979).

[5]**John L. Synge** (1897–1995): Irish mathematical physicist, who explored many domains: applied mathematics, differential geometry, hydrodynamics, optics, elasticity and relativity. He has notably written two famous relativity treatises: one on special relativity (Synge 1956), where the geometric approach is privileged, and the other one on general relativity (Synge 1960).

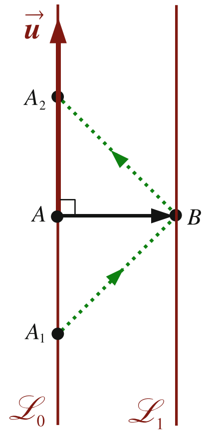


Fig. 3.9 Ruler in spacetime: $\mathscr{L}_0$ and $\mathscr{L}_1$ are the worldlines of the two extremities of the ruler

in his treatise about special relativity (Synge 1956) [Eq. (70) in Chap. I; see also Eq. (16) in Chap. III of (Synge 1960)]. We have named it in his honour.

3.3.2 Born's Rigidity Criterion

An ***infinitesimal ruler*** equipping observer $\mathscr{O}$ is defined by a timelike worldline $\mathscr{L}_1$ staying always in some infinitesimal neighbourhood of $\mathscr{O}$'s worldline, $\mathscr{L}_0$ (cf. Fig. 3.9). By *infinitesimal neighbourhood*, it must be understood that the spatial distance between $\mathscr{L}_0$ and $\mathscr{L}_1$, defined as $d := \|\vec{s}\|_{g} = \sqrt{\vec{s}\cdot\vec{s}}$ where $\vec{s}$ is a vector connecting $\mathscr{L}_0$ and $\mathscr{L}_1$ and orthogonal to $\vec{u}$, can be neglected in front of $\mathscr{L}_0$'s curvature radius. The points of the worldlines $\mathscr{L}_0$ and $\mathscr{L}_1$ constitute the two extremities of the ruler.

Let us suppose that $\mathscr{O}$ sends a photon towards the ruler's extremity $\mathscr{L}_1$ (event A_1, cf. Fig. 3.9) and that the photon is reflected once it reaches $\mathscr{L}_1$ (event B), coming back to $\mathscr{L}_0$ (event A_2). Let A be the orthogonal projection of B onto $\mathscr{L}_0$ (cf. Fig. 3.9). From what we have seen in Sect. 3.2.3, A and B are two simultaneous events for $\mathscr{O}$. In other words, B belongs to $\mathscr{E}_{\boldsymbol{u}}(A)$. From the definition of an infinitesimal ruler given above, $\|\overrightarrow{AB}\|_{g}$ is very small with respect to $\mathscr{L}_0$'s curvature radius at A (cf. Sect. 2.7.3). We may then approximate the arc A_1A_2 of $\mathscr{L}_0$ by a line segment and use Synge formula (3.20) to get the length of $\overrightarrow{AB}$. In the present case, $t_2 - t = t - t_1$ since A and B are simultaneous for $\mathscr{O}$. Synge formula reduces then to $\|\overrightarrow{AB}\|_{g} = c(t - t_1) = c(t_2 - t)$, which can be rewritten solely in terms of the photon emission time (t_1) and the reception time of the returned photon (t_2):

$$\boxed{\left\|\overrightarrow{AB}\right\|_{g} = \frac{1}{2}\,c\,(t_2 - t_1)}. \tag{3.21}$$

The left-hand side of this equality is the length, with respect to the metric tensor $\boldsymbol{g}$, of the spacelike vector $\overrightarrow{AB}$. This vector belongs to $\mathscr{O}$'s local rest space at A. Accordingly, we may call the quantity (3.21) the ***length of the infinitesimal ruler with respect to $\mathscr{O}$ at the time*** $t = (t_1 + t_2)/2$.

One naturally defines a ***rigid ruler*** as a ruler whose length does not vary, i.e. a ruler for which $\|\overrightarrow{AB}\|_{\boldsymbol{g}}$ does not depend upon the points A and B on the worldlines $\mathscr{L}_0$ and $\mathscr{L}_1$, provided that $\overrightarrow{AB}$ is orthogonal to $\vec{\boldsymbol{u}}$. The equality (3.21) provides a physical criterion to check the rigidity of a ruler: it suffices to make sure that the round-trip time of a photon between the two extremities of the ruler is constant. This criterion is called ***Born's rigidity criterion***.

Remark 3.7. The notion of rigidity defined above is one-dimensional, in so far as it concerns only pairs of point particles (the two extremities of the ruler). To generalize it to the three-dimensional case, it would be natural to define an extended object as *rigid* iff all the pairs of adjacent points constitute a rigid ruler. However, one can show that such a definition is too restrictive—much beyond the Newtonian equivalent—to be useful (cf. Chap. I of Synge's textbook (Synge 1956) or (Giulini 2006, 2009) for more details). Hence there is no natural definition of a rigid solid in relativity.

Historical note: *The rigidity criterion presented above has been proposed in the case of a three-dimensional solid by Max Born*[6] *in 1909 (Born 1909).*

3.4 Local Frame

The notion of *frame* is related to labelling the events in the spacetime $\mathscr{E}$ by a 4-tuple of real numbers, (t, x^1, x^2, x^3) where t is some "time" and (x^1, x^2, x^3) three "spatial" coordinates. The difference with a *coordinate system* on $\mathscr{E}$, like the affine coordinate system introduced in Sect. 1.2.3, is that the labelling is performed by some observer, via physical operations such as the reading of a clock or the emission and reception of light signals.

3.4.1 Local Frame of an Observer

Let us consider an observer $\mathscr{O}$ of worldline $\mathscr{L}_0$ and 4-velocity $\vec{\boldsymbol{u}}$, the frame of which is to be defined. We have seen in Sect. 3.2.2 how to attribute a time tag t at events close to $\mathscr{L}_0$, via the slicing of $\mathscr{E}$ by $\mathscr{O}$'s local rest spaces $\mathscr{E}_{\boldsymbol{u}}(t)$ (Fig. 3.10). There

[6]**Max Born** (1882–1970): German physicist, author of numerous studies in quantum mechanics, optics and solid-state physics. He received the Nobel Prize in Physics in 1954 for having introduced the probabilistic interpretation of the wave function in quantum mechanics in 1926.

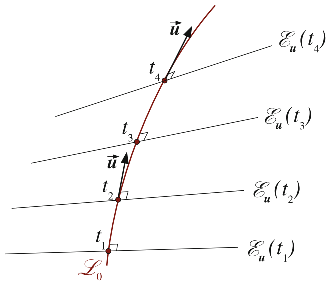


Fig. 3.10 Slicing of spacetime by the local rest spaces $\mathscr{E}_{\boldsymbol{u}}$ of a given worldline $\mathscr{L}_0$: the events in the slice $\mathscr{E}_{\boldsymbol{u}}(t)$ are given the temporal coordinate t, which is the proper time along $\mathscr{L}_0$

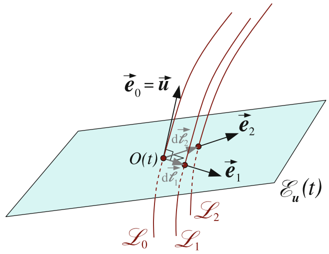


Fig. 3.11 Local frame of an observer with worldline $\mathscr{L}_0$. A dimension has been suppressed, so that neither the worldline $\mathscr{L}_3$ nor the vector $\vec{e}_3$ are represented

remains thus to attribute the three spatial coordinates (x^1, x^2, x^3). This is achieved as follows.

We define a ***local frame*** along $\mathscr{L}_0$ as a 4-tuple of vectors $(\vec{e}_\alpha(t)) = (\vec{e}_0(t), \vec{e}_1(t), \vec{e}_2(t), \vec{e}_3(t))$ defined at any point $O(t)$ of $\mathscr{L}_0$ and obeying the following properties (cf. Fig. 3.11):

1. $(\vec{e}_\alpha(t))$ is a right-handed orthonormal basis of $(E, \boldsymbol{g})$ for all $O(t) \in \mathscr{L}_0$, i.e.

$$\forall t \in \mathbb{R}, \quad \boldsymbol{g}(\vec{e}_\alpha(t), \vec{e}_\beta(t)) = \eta_{\alpha\beta} \quad \text{and} \quad \boldsymbol{\epsilon}(\vec{e}_0(t), \vec{e}_1(t), \vec{e}_2(t), \vec{e}_3(t)) = 1, \tag{3.22}$$

   where $(\eta_{\alpha\beta})$ is the Minkowski matrix (1.17) and $\boldsymbol{\epsilon}$ the Levi–Civita tensor (the second equality expressing right handedness, cf. Sect. 1.5).
2. $\vec{e}_0 = \vec{u}$ (the 4-velocity along $\mathscr{L}_0$).
3. For each $\alpha \in \{0, 1, 2, 3\}$, the field $\vec{e}_\alpha(t)$ is differentiable (cf. Sect. 2.7.2), which means that when moving from an event $O(t) \in \mathscr{L}_0$ to a neighbouring event $O(t + \mathrm{d}t)$, the orthonormal basis $(\vec{e}_\alpha)$ varies in an infinitesimal manner.

> We may then define formally an ***observer*** $\mathscr{O}$ as the set formed by a timelike worldline $\mathscr{L}_0$ and a local frame $(\vec{e}_\alpha)$ along $\mathscr{L}_0$.

The property (3.22) implies that the three vectors $(\vec{e}_1(t), \vec{e}_2(t), \vec{e}_3(t))$ are orthogonal to $\vec{e}_0(t) = \vec{u}(t)$; they therefore belong to $E_{\boldsymbol{u}}(t) := E_{\boldsymbol{u}}(O(t))$.

Physically, a local frame is realized as follows: one considers four point particles infinitely close to each other. One of them is chosen as the "origin", and the three others are placed at the tops of a rectangular trihedron around this origin. These point particles are represented by four timelike worldlines $\mathscr{L}_0$, $\mathscr{L}_1$, $\mathscr{L}_2$ and $\mathscr{L}_3$(cf. Fig. 3.11). Each of the point particle is equipped with an ideal clock as well as a device for emitting and sending light signals (photons). For $i \in \{1, 2, 3\}$, let $\mathrm{d}\vec{\boldsymbol{\ell}}_i$ be the spacelike vector orthogonal to $\vec{\boldsymbol{u}}$ and connecting $\mathscr{L}_0$ to $\mathscr{L}_i$ (cf. Fig. 3.11). One defines the three unit vectors $\vec{\boldsymbol{e}}_i$ by

$$\mathrm{d}\vec{\boldsymbol{\ell}}_i = \mathrm{d}\ell_i \, \vec{\boldsymbol{e}}_i, \tag{3.23}$$

where $\mathrm{d}\ell_i := \|\mathrm{d}\vec{\boldsymbol{\ell}}_i\|_{\boldsymbol{g}}$. The rigidity of each ruler $(\mathscr{L}_0, \mathscr{L}_i)$ is controlled via the method exposed in Sect. 3.3.2. On the other hand, the orthogonality of each of the couples $(\vec{\boldsymbol{e}}_i, \vec{\boldsymbol{e}}_j)$ $(i \neq j)$ is checked by measuring the length of the separation vector $\mathrm{d}\vec{\boldsymbol{\ell}}_{ij} := \mathrm{d}\vec{\boldsymbol{\ell}}_j - \mathrm{d}\vec{\boldsymbol{\ell}}_i$ by means of Synge formula (3.20)[7] and by checking that Pythagoras' relation holds at any instant:

$$\mathrm{d}\vec{\boldsymbol{\ell}}_{ij} \cdot \mathrm{d}\vec{\boldsymbol{\ell}}_{ij} = \mathrm{d}\vec{\boldsymbol{\ell}}_i \cdot \mathrm{d}\vec{\boldsymbol{\ell}}_i + \mathrm{d}\vec{\boldsymbol{\ell}}_j \cdot \mathrm{d}\vec{\boldsymbol{\ell}}_j. \tag{3.24}$$

This relation implies that $\mathrm{d}\vec{\boldsymbol{\ell}}_i \cdot \mathrm{d}\vec{\boldsymbol{\ell}}_j = 0$, and moreover, each quantity involved in it is chronometrically measurable according to (3.20).

3.4.2 Coordinates with Respect to an Observer

We are now in position to define the four ***coordinates*** (t, x^1, x^2, x^3) ***of an event*** M ***with respect to an observer*** $\mathscr{O}$ (cf. Fig. 3.12). We shall assume that M is located "not too far" from $\mathscr{O}$'s worldline $\mathscr{L}_0$, in the sense of being in the region of spacetime where the slicing by $\mathscr{O}$'s local rest spaces $\mathscr{E}_{\boldsymbol{u}}(t)$ is regular (i.e. the slices do not intersect, as in Fig. 3.10); this will be made precise in Sect. 3.7. There exists then a unique local rest space $\mathscr{E}_{\boldsymbol{u}}(t)$ of $\mathscr{O}$ containing M (cf. Fig. 3.12). The proper time t labelling this local rest space is chosen as the first coordinate of M with respect to observer $\mathscr{O}$. Note that if M is close to $\mathscr{L}_0$, so that $\mathscr{E}_{\boldsymbol{u}}(t)$ can be identified to the simultaneity hypersurface $\Sigma_{\boldsymbol{u}}(t)$ around M, then t is nothing but the date of M with respect to $\mathscr{O}$, as defined in Sect. 3.2. Let then $O(t)$ be the event of $\mathscr{L}_0$ of proper time t, i.e. the intersection of the hyperplane $\mathscr{E}_{\boldsymbol{u}}(t)$ with $\mathscr{L}_0$. Since both $O(t)$ and M belong to $\mathscr{E}_{\boldsymbol{u}}(t)$, we may expand the vector $\overrightarrow{O(t)M}$ onto the basis $\vec{\boldsymbol{e}}_i(t)$ of $E_{\boldsymbol{u}}(t)$ arising from the local frame of $\mathscr{O}$ at $O(t)$ (cf. Fig. 3.12):

$$\overrightarrow{O(t)M} = x^i \, \vec{\boldsymbol{e}}_i(t). \tag{3.25}$$

[7] One must employ formula (3.20) and not (3.21), since a priori $\mathrm{d}\vec{\boldsymbol{\ell}}_{ij}$ is neither orthogonal to $\mathscr{L}_i$ nor to $\mathscr{L}_j$.

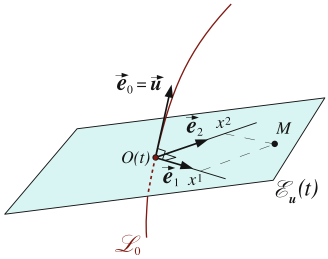


Fig. 3.12 Coordinates (t, x^1, x^2, x^3) of an event M with respect to the local frame of an observer of worldline $\mathscr{L}_0$

This defines the three spatial coordinates (x^i) of M with respect to observer $\mathscr{O}$ and completes the definition of the coordinates (t, x^1, x^2, x^3) of M with respect to $\mathscr{O}$.

Historical note: *The coordinates with respect to a generic observer (i.e. not necessarily inertial) defined above have been introduced by Charles W. Misner,*[8] *Kip S. Thorne*[9] *and John A. Wheeler*[10] *in their monumental textbook* Gravitation *(Misner et al. 1973). They can be viewed as a generalization to the rotating case (this last concept will be discussed in Sect. 3.5) of coordinates introduced previously by John L. Synge (cf. p. 74) and called by him* ***Fermi coordinates*** *(Synge 1960).*

3.4.3 Reference Space of an Observer

The local rest spaces $\mathscr{E}_{\boldsymbol{u}}(t)$ of an observer $\mathscr{O}$ form a family (parametrized by $\mathscr{O}$'s proper time t) of hyperplanes in Minkowski spacetime $\mathscr{E}$, as illustrated in Fig. 3.10. In this respect, they are abstract spaces. The introduction of the local frame and the associated coordinates allows one to define a three-dimensional vector space that is closer to the "usual" physical space "perceived" by an observer. Moreover this space will be unique, contrary to the local rest spaces, which constitute a one-parameter family.

---

[8]**Charles W. Misner**: American theoretical physicist born in 1932, expert of general relativity and cosmology; co-author with K. S. Thorne and J. A. Wheeler of the most famous treatise about general relativity: *Gravitation* (Misner et al. 1973) (1280 pages!), published in 1973 and 6 chapters of which are devoted to special relativity.

[9]**Kip S. Thorne** : American theoretical physicist born in 1940, author of many advances in general relativity and relativistic astrophysics. Beside the treatise *Gravitation* (Misner et al. 1973), he wrote the excellent popular book (Thorne 1994).

[10]**John A. Wheeler** (1911–2008) : American theoretical physicist, well known for major contributions to particle physics, nuclear fission and general relativity. He was one of the latest collaborators of Einstein in Princeton. In the late 1960s, he coined the word *black hole*.

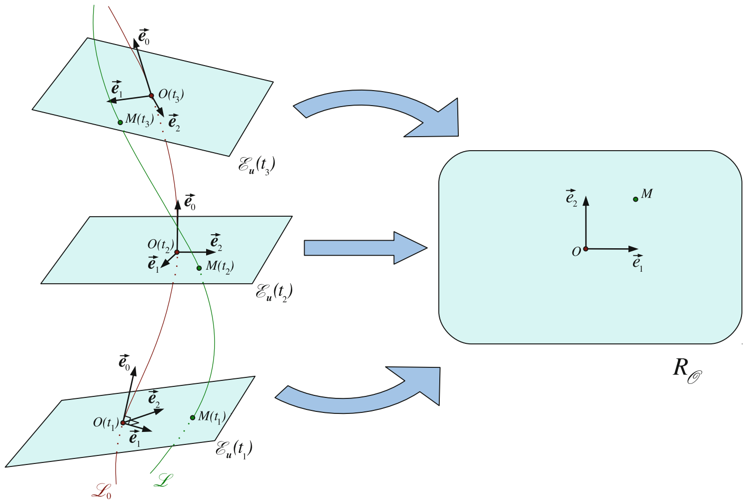


Fig. 3.13 Local frame $(\vec{e}_\alpha(t))$ and reference space $R_{\mathscr{O}}$ of an observer $\mathscr{O}$. $\mathscr{L}$ is the worldline of a point particle fixed with respect to $\mathscr{O}$

The construction is very natural, relying on the "cartography" that an observer can make of his environment when representing an event M of spatial coordinates (x^1, x^2, x^3) in his local frame by the point (x^1, x^2, x^3) in the space $\mathbb{R}^3$. More precisely, we shall call ***reference space of observer*** $\mathscr{O}$ a three-dimension Euclidean vector space $R_{\mathscr{O}}$ equipped with an orthonormal basis $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ and a mapping

$$\begin{array}{rccc} \varphi: & \mathscr{E} & \longrightarrow & R_{\mathscr{O}} \\ & M(t, x^i) & \longmapsto & \vec{x} = x^i\,\vec{e}_i \end{array}, \tag{3.26}$$

where (t, x^i) are the coordinates of M with respect to $\mathscr{O}$. Strictly speaking, φ is not a mapping from $\mathscr{E}$ to $R_{\mathscr{O}}$, but only from the domain of $\mathscr{E}$ where the local frame coordinates are well defined (domain to be discussed in Sect. 3.7) to $R_{\mathscr{O}}$.

It is worth noticing that φ induces an isomorphism $\bar{\varphi}_t$ between each local rest space of $\mathscr{O}$ and $R_{\mathscr{O}}$, according to

$$\begin{array}{rccc} \bar{\varphi}_t: & E_{\boldsymbol{u}}(t) & \longrightarrow & R_{\mathscr{O}} \\ & \vec{\boldsymbol{v}} = v^i\,\vec{\boldsymbol{e}}_i(t) & \longmapsto & \vec{v} = v^i\,\vec{e}_i \end{array}. \tag{3.27}$$

This isomorphism is sending the basis $(\vec{\boldsymbol{e}}_i(t))$ of the local frame at instant t to the Euclidean basis $(\vec{e}_i)$ of $R_{\mathscr{O}}$: $\bar{\varphi}_t(\vec{\boldsymbol{e}}_i(t)) = \vec{e}_i$. Hence, we may say that the three spacelike vectors of $\mathscr{O}$'s local fame are held fixed in $R_{\mathscr{O}}$, whereas they evolve with t in E (cf. Fig. 3.13). More generally, a vector field $\vec{\boldsymbol{v}} = \vec{\boldsymbol{v}}(t)$ along the worldline $\mathscr{L}_0$ (cf. Sect. 2.7.2) is said ***fixed with respect to*** $\mathscr{O}$ iff

$$\forall t \in \mathbb{R}, \quad \vec{\boldsymbol{v}}(t) = v^\alpha\,\vec{\boldsymbol{e}}_\alpha(t), \tag{3.28}$$

where the v^α's are four constant numbers. According to this definition, a vector field $\vec{v}$ along $\mathscr{L}_0$ obeying $\forall t \in \mathbb{R},\ \vec{v}(t) \in E_{\boldsymbol{u}}(t)$ is fixed with respect to $\mathscr{O}$ iff $\bar{\varphi}_t(\vec{v}(t))$ does not depend on t.

Similarly, a point particle is said ***fixed with respect to observer*** $\mathscr{O}$ iff the spatial coordinates (x^i) with respect to $\mathscr{O}$ are the same for all events on the particle's worldline. Such a particle is depicted in Fig. 3.13.

3.5 Four-Rotation of a Local Frame

Let us investigate the evolution of the local frame $(\vec{e}_\alpha(t))$ along the worldline of observer $\mathscr{O}$, as depicted in Fig. 3.13. In particular, we will show that the derivative of the basis vector $\vec{e}_\alpha(t)$ with respect to the proper time t involves only the 4-acceleration $\vec{a}$ of $\mathscr{O}$ and a vector of $\mathscr{O}$'s local rest space called the *4-rotation of the local frame*.

3.5.1 Variation of the Local Frame Along the Worldline

We aim at evaluating the derivative with respect to t of a given vector $\vec{e}_\alpha$ of the local frame, as defined in Sect. 2.7.2. This derivative is itself a vector, and since $(\vec{e}_\alpha)$ is a basis of E, there exists a unique 4-tuple of real functions of t, $(\Omega^\beta{}_\alpha)_{0\leq\beta\leq 3}$, such that

$$\frac{\mathrm{d}\vec{e}_\alpha}{\mathrm{d}t} = \Omega^\beta{}_\alpha\, \vec{e}_\beta . \tag{3.29}$$

The order of indices α and β is chosen for future convenience.

More generally, let us express the temporal variation of a vector field $\vec{v}(t)$ *fixed* with respect to $\mathscr{O}$, in the sense defined in Sect. 3.4.3. We have, from (3.28) and (3.29),

$$\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = v^\alpha \frac{\mathrm{d}\vec{e}_\alpha}{\mathrm{d}t} = \Omega^\beta{}_\alpha\, v^\alpha\, \vec{e}_\beta = (\Omega^\alpha{}_\beta\, v^\beta)\, \vec{e}_\alpha ,$$

hence

$$\vec{v} \text{ fixed with respect to } \mathscr{O} \implies \boxed{\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \boldsymbol{\Omega}(\vec{v})}, \tag{3.30}$$

where $\boldsymbol{\Omega}$ is the endomorphism of E having $(\Omega^\alpha{}_\beta)$ as matrix in the basis $(\vec{e}_\alpha)$. Note that $\boldsymbol{\Omega}$ is a function of t since $\Omega^\alpha{}_\beta = \Omega^\alpha{}_\beta(t)$. In particular, applying (3.30) to $\vec{v} = \vec{u}$ and invoking the definition of the 4-acceleration $\vec{a}$ of $\mathscr{O}$, we get

$$\boldsymbol{\Omega}(\vec{u}) = c\, \vec{a} . \tag{3.31}$$

Thanks to the $\boldsymbol{g}$-duality introduced in Sect. 1.6.2, we may associate with $\boldsymbol{\Omega}$ a unique bilinear form $\underline{\boldsymbol{\Omega}}$ on E. Indeed, for any vector $\vec{\boldsymbol{w}}$, $\boldsymbol{\Omega}(\vec{\boldsymbol{w}})$ is a vector in E. If $\vec{\boldsymbol{v}}$ is another vector in E, $\vec{\boldsymbol{v}} \cdot \boldsymbol{\Omega}(\vec{\boldsymbol{w}})$ is a scalar, which we shall define as the bilinear form $\underline{\boldsymbol{\Omega}}$ acting on the pair $(\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}})$:

$$\underline{\boldsymbol{\Omega}}(\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}) := \vec{\boldsymbol{v}} \cdot \boldsymbol{\Omega}(\vec{\boldsymbol{w}}). \tag{3.32}$$

Remark 3.8. The attention of the reader is drawn on the order of the arguments of $\underline{\boldsymbol{\Omega}}$: the second one is that on which acts the endomorphism $\boldsymbol{\Omega}$.

The matrix $(\Omega_{\alpha\beta})$ of the bilinear form $\underline{\boldsymbol{\Omega}}$ in the basis $(\vec{\boldsymbol{e}}_\alpha)$ is obtained by writing

$$\Omega_{\alpha\beta} v^\alpha w^\beta = \underline{\boldsymbol{\Omega}}(\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}) = \boldsymbol{g}(\vec{\boldsymbol{v}}, \boldsymbol{\Omega}(\vec{\boldsymbol{w}})) = g_{\alpha\mu}\, v^\alpha\, \Omega^\mu{}_\beta\, w^\beta = (g_{\alpha\mu}\, \Omega^\mu{}_\beta)\, v^\alpha\ w^\beta,$$

where $(g_{\alpha\beta})$ is the matrix of $\boldsymbol{g}$ in the basis $(\vec{\boldsymbol{e}}_\alpha)$ (cf. Eq. (1.12)). We find thus

$$\Omega_{\alpha\beta} = g_{\alpha\mu}\, \Omega^\mu{}_\beta. \tag{3.33}$$

The orthonormal character of the basis $(\vec{\boldsymbol{e}}_\alpha)$ implies that the bilinear form $\underline{\boldsymbol{\Omega}}$ is antisymmetric. Indeed, taking the time derivative of the identity $\vec{\boldsymbol{e}}_\alpha \cdot \vec{\boldsymbol{e}}_\beta = \eta_{\alpha\beta}$, where $(\eta_{\alpha\beta})$ is Minkowski matrix (1.17), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(\vec{\boldsymbol{e}}_\alpha \cdot \vec{\boldsymbol{e}}_\beta) = 0 = \frac{\mathrm{d}\vec{\boldsymbol{e}}_\alpha}{\mathrm{d}t} \cdot \vec{\boldsymbol{e}}_\beta + \vec{\boldsymbol{e}}_\alpha \cdot \frac{\mathrm{d}\vec{\boldsymbol{e}}_\beta}{\mathrm{d}t},$$

which can be rewritten as $\boldsymbol{\Omega}(\vec{\boldsymbol{e}}_\alpha) \cdot \vec{\boldsymbol{e}}_\beta = -\vec{\boldsymbol{e}}_\alpha \cdot \boldsymbol{\Omega}(\vec{\boldsymbol{e}}_\beta)$. The definition (3.32) of $\underline{\boldsymbol{\Omega}}$ then leads to

$$\underline{\boldsymbol{\Omega}}(\vec{\boldsymbol{e}}_\beta, \vec{\boldsymbol{e}}_\alpha) = -\underline{\boldsymbol{\Omega}}(\vec{\boldsymbol{e}}_\alpha, \vec{\boldsymbol{e}}_\beta), \tag{3.34}$$

displaying the antisymmetry of $\underline{\boldsymbol{\Omega}}$.

Remark 3.9. The result (3.34) is equivalent to $\Omega_{\beta\alpha} = -\Omega_{\alpha\beta}$, but does not imply $\Omega^\beta{}_\alpha = -\Omega^\alpha{}_\beta$. Indeed, given the signature $(-,+,+,+)$ of $\boldsymbol{g}$, it is easy to see, via (3.33), that (3.34) is equivalent to

$$\Omega^0{}_0 = 0, \quad \Omega^i{}_0 = \Omega^0{}_i \quad \text{and} \quad \Omega^i{}_j = -\Omega^j{}_i. \tag{3.35}$$

(Let us recall that the Latin indices i, j take their values in $\{1, 2, 3\}$).

We shall decompose $\underline{\boldsymbol{\Omega}}$ with respect to $\mathscr{O}$'s 4-velocity $\vec{\boldsymbol{u}}$ by means of a method that can be applied to any antisymmetric bilinear form. This type of decomposition will be useful at various occurrences in the book. We therefore devote a full section to it.

3.5.2 *Orthogonal Decomposition of Antisymmetric Bilinear Forms*

Let $\boldsymbol{A}$ be an antisymmetric bilinear form on E:

$$\forall(\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}) \in E^2, \quad \boldsymbol{A}(\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}) = -\boldsymbol{A}(\vec{\boldsymbol{w}}, \vec{\boldsymbol{v}}). \tag{3.36}$$

Given a unit timelike vector $\vec{\boldsymbol{u}}$ (in practice, it will be the 4-velocity of some observer), there exists a unique linear form $\boldsymbol{q} \in E^*$ and a unique vector $\vec{\boldsymbol{b}} \in E$ such that

$$\boxed{\boldsymbol{A} = \underline{\boldsymbol{u}} \otimes \boldsymbol{q} - \boldsymbol{q} \otimes \underline{\boldsymbol{u}} + \boldsymbol{\epsilon}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{b}}, ., .)}, \quad \boxed{\langle \boldsymbol{q}, \vec{\boldsymbol{u}} \rangle = 0} \text{ and } \boxed{\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{b}} = 0}. \tag{3.37}$$

In (3.37), $\boldsymbol{\epsilon}$ is the Levi–Civita tensor introduced in Sect. 1.5, and $\otimes$ is the tensor product: given two linear forms $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ on E, the ***tensor product*** of $\boldsymbol{q}_1$ by $\boldsymbol{q}_2$ is the bilinear form defined by

$$\boxed{\begin{aligned} \boldsymbol{q}_1 \otimes \boldsymbol{q}_2 : E \times E &\longrightarrow \mathbb{R} \\ (\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}) &\longmapsto \langle \boldsymbol{q}_1, \vec{\boldsymbol{v}} \rangle \langle \boldsymbol{q}_2, \vec{\boldsymbol{w}} \rangle \end{aligned}}, \tag{3.38}$$

where the right-hand side is nothing but the product of the two real numbers $\langle \boldsymbol{q}_1, \vec{\boldsymbol{v}} \rangle$ and $\langle \boldsymbol{q}_2, \vec{\boldsymbol{w}} \rangle$. Accordingly, formula (3.37) can be made explicit as

$$\forall(\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}) \in E^2, \quad \boldsymbol{A}(\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}) = \langle \underline{\boldsymbol{u}}, \vec{\boldsymbol{v}} \rangle \langle \boldsymbol{q}, \vec{\boldsymbol{w}} \rangle - \langle \boldsymbol{q}, \vec{\boldsymbol{v}} \rangle \langle \underline{\boldsymbol{u}}, \vec{\boldsymbol{w}} \rangle + \boldsymbol{\epsilon}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{b}}, \vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}), \tag{3.39}$$

with, by definition of the linear form $\underline{\boldsymbol{u}}$, $\langle \underline{\boldsymbol{u}}, \vec{\boldsymbol{v}} \rangle = \vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}}$ [Eq. (1.45)].

Let us establish (3.37). First of all, we note that its right hand does define an antisymmetric bilinear form, in particular thanks to the fully antisymmetric character of the Levi–Civita tensor. Then, we set

$$\boxed{\boldsymbol{q} := \boldsymbol{A}(., \vec{\boldsymbol{u}})}. \tag{3.40}$$

$\boldsymbol{q}$ is thus the linear form defined by $\forall \vec{\boldsymbol{v}} \in E$, $\langle \boldsymbol{q}, \vec{\boldsymbol{v}} \rangle = \boldsymbol{A}(\vec{\boldsymbol{v}}, \vec{\boldsymbol{u}})$. Given the antisymmetry of $\boldsymbol{A}$, it is clear that $\langle \boldsymbol{q}, \vec{\boldsymbol{u}} \rangle = 0$. The second equation in (3.37) is therefore fulfilled. Next, let us define

$$\boldsymbol{B} := \boldsymbol{A} - \underline{\boldsymbol{u}} \otimes \boldsymbol{q} + \boldsymbol{q} \otimes \underline{\boldsymbol{u}}. \tag{3.41}$$

$\boldsymbol{B}$ is clearly an antisymmetric bilinear form. It vanishes if one of its argument is $\vec{\boldsymbol{u}}$:

$$\boldsymbol{B}(., \vec{\boldsymbol{u}}) = \underbrace{\boldsymbol{A}(., \vec{\boldsymbol{u}})}_{\boldsymbol{q}} - \underline{\boldsymbol{u}}\ \underbrace{\langle \boldsymbol{q}, \vec{\boldsymbol{u}} \rangle}_{0} + \boldsymbol{q}\ \underbrace{\langle \underline{\boldsymbol{u}}, \vec{\boldsymbol{u}} \rangle}_{-1} = 0.$$

Let us determine the action of $\boldsymbol{B}$ in the hyperplane $E_{\boldsymbol{u}}$ normal to $\vec{\boldsymbol{u}}$. As we have seen in Sect. 3.2.6, $(E_{\boldsymbol{u}}, \boldsymbol{g})$ is a Euclidean space. Let then $(\vec{\boldsymbol{e}}_i) = (\vec{\boldsymbol{e}}_1, \vec{\boldsymbol{e}}_2, \vec{\boldsymbol{e}}_3)$ be an orthonormal basis of $(E_{\boldsymbol{u}}, \boldsymbol{g})$. If $\vec{\boldsymbol{u}}$ is the 4-velocity of an observer, one may of course select the $\vec{\boldsymbol{e}}_i$'s to be the three spatial vectors of the observer's local frame. Let us define the following three real numbers:

$$b^1 := \boldsymbol{B}(\vec{\boldsymbol{e}}_2, \vec{\boldsymbol{e}}_3), \quad b^2 := \boldsymbol{B}(\vec{\boldsymbol{e}}_3, \vec{\boldsymbol{e}}_1), \quad b^3 := \boldsymbol{B}(\vec{\boldsymbol{e}}_1, \vec{\boldsymbol{e}}_2). \tag{3.42}$$

and introduce the vector

$$\vec{\boldsymbol{b}} := b^i\, \vec{\boldsymbol{e}}_i \ \in E_{\boldsymbol{u}}. \tag{3.43}$$

It is clear that $\vec{\boldsymbol{b}}$ satisfies the third equation of (3.37): $\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{b}} = 0$. Moreover, for any pair $(\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}})$ of vectors of $E_{\boldsymbol{u}}$, the antisymmetry of $\boldsymbol{B}$ yields

$$\begin{aligned}\boldsymbol{B}(\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}) &= \boldsymbol{B}(v^i \vec{\boldsymbol{e}}_i, w^j \vec{\boldsymbol{e}}_j) = v^i w^j\, \boldsymbol{B}(\vec{\boldsymbol{e}}_i, \vec{\boldsymbol{e}}_j) \\ &= v^1 w^2 b^3 - v^2 w^1 b^3 - v^1 w^3 b^2 + v^3 w^1 b^2 + v^2 w^3 b^1 - v^3 w^2 b^1 \\ &= \begin{vmatrix} b^1 & v^1 & w^1 \\ b^2 & v^2 & w^2 \\ b^3 & v^3 & w^3 \end{vmatrix}.\end{aligned} \tag{3.44}$$

This shows that $\boldsymbol{B}(\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}})$ is the mixed product (scalar triple product) of vectors $(\vec{\boldsymbol{b}}, \vec{\boldsymbol{v}}, \vec{\boldsymbol{w}})$ in the Euclidean space $(E_{\boldsymbol{u}}, \boldsymbol{g})$, provided that one has chosen an orientation making $(\vec{\boldsymbol{e}}_i)$ be a right-handed basis. Let us recall that the choice of an orientation of a vector space of dimension n amounts to the choice of a fully antisymmetric n-linear form. We did so in Sect. 1.5 in the case $n = 4$ by selecting the Levi–Civita tensor $\boldsymbol{\epsilon}$. In the present context, $n = 3$ and it is natural to select the antisymmetric trilinear form $\boldsymbol{\epsilon}_{\boldsymbol{u}}$ defined from $\boldsymbol{\epsilon}$ by

$$\forall\, \vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_2, \vec{\boldsymbol{v}}_3 \in E_{\boldsymbol{u}}, \quad \boldsymbol{\epsilon}_{\boldsymbol{u}}(\vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_2, \vec{\boldsymbol{v}}_3) := \boldsymbol{\epsilon}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_2, \vec{\boldsymbol{v}}_3). \tag{3.45}$$

Since $\boldsymbol{\epsilon}$ is an antisymmetric four-linear form, it is clear that $\boldsymbol{\epsilon}_{\boldsymbol{u}}$ is an antisymmetric trilinear form on $E_{\boldsymbol{u}}$. In addition, it satisfies $\boldsymbol{\epsilon}_{\boldsymbol{u}}(\vec{\boldsymbol{e}}_1, \vec{\boldsymbol{e}}_2, \vec{\boldsymbol{e}}_3) = 1$ if $(\vec{\boldsymbol{u}}, \vec{\boldsymbol{e}}_1, \vec{\boldsymbol{e}}_2, \vec{\boldsymbol{e}}_3)$ is a right-handed orthonormal basis of $(E, \boldsymbol{g}, \boldsymbol{\epsilon})$, which we shall assume from now on. By metric duality, $\boldsymbol{\epsilon}_{\boldsymbol{u}}$ induces the ***cross product*** of two vectors of $E_{\boldsymbol{u}}$ by

$$\forall (\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}) \in E_{\boldsymbol{u}}^2, \quad \boxed{\vec{\boldsymbol{v}} \times_{\boldsymbol{u}} \vec{\boldsymbol{w}} := \vec{\boldsymbol{\epsilon}}_{\boldsymbol{u}}(\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}, .) = \vec{\boldsymbol{\epsilon}}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}, .)}, \tag{3.46}$$

where the notation $\vec{\boldsymbol{\epsilon}}_{\boldsymbol{u}}(\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}, .\,)$ stands for the vector of $E_{\boldsymbol{u}}$ associated by $\boldsymbol{g}$-duality (cf. Sect. 1.6) to the linear form $E_{\boldsymbol{u}} \longrightarrow \mathbb{R}$, $\vec{\boldsymbol{z}} \longmapsto \boldsymbol{\epsilon}_{\boldsymbol{u}}(\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}, \vec{\boldsymbol{z}})$. Similarly $\vec{\boldsymbol{\epsilon}}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}, .\,)$ stands for the vector in E that is $\boldsymbol{g}$-dual of the linear form $E \longrightarrow \mathbb{R}$, $\vec{\boldsymbol{z}} \longmapsto \boldsymbol{\epsilon}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}, \vec{\boldsymbol{z}})$. Hence, we may write the ***mixed product*** of three vectors $(\vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_2, \vec{\boldsymbol{v}}_3)$ in $E_{\boldsymbol{u}}$ as

$$\boldsymbol{\epsilon}_{\boldsymbol{u}}(\vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_2, \vec{\boldsymbol{v}}_3) = (\vec{\boldsymbol{v}}_1 \times_{\boldsymbol{u}} \vec{\boldsymbol{v}}_2) \cdot \vec{\boldsymbol{v}}_3 = (\vec{\boldsymbol{v}}_2 \times_{\boldsymbol{u}} \vec{\boldsymbol{v}}_3) \cdot \vec{\boldsymbol{v}}_1 = (\vec{\boldsymbol{v}}_3 \times_{\boldsymbol{u}} \vec{\boldsymbol{v}}_1) \cdot \vec{\boldsymbol{v}}_2. \tag{3.47}$$

Since $\boldsymbol{\epsilon}_{\boldsymbol{u}}(\vec{\boldsymbol{e}}_1, \vec{\boldsymbol{e}}_2, \vec{\boldsymbol{e}}_3) = 1$, (3.44) can be rewritten as

$$\forall (\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}) \in E_{\boldsymbol{u}}^2, \quad \boldsymbol{B}(\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}) = \boldsymbol{\epsilon}_{\boldsymbol{u}}(\vec{\boldsymbol{b}}, \vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}) = \boldsymbol{\epsilon}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{b}}, \vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}). \tag{3.48}$$

Given the definition (3.41) of $\boldsymbol{B}$, this establishes the decomposition (3.37) of $\boldsymbol{A}$. There remains to prove the uniqueness of the linear form $\boldsymbol{q}$ and of the vector $\vec{\boldsymbol{b}}$. For $\boldsymbol{q}$, this is easy: if the first two identities of (3.37) are fulfilled, then necessarily

$$\forall \vec{\boldsymbol{v}} \in E, \quad \boldsymbol{A}(\vec{\boldsymbol{v}}, \vec{\boldsymbol{u}}) = \langle \underline{\boldsymbol{u}}, \vec{\boldsymbol{v}} \rangle \underbrace{\langle \boldsymbol{q}, \vec{\boldsymbol{u}} \rangle}_{0} - \langle \boldsymbol{q}, \vec{\boldsymbol{v}} \rangle \underbrace{\langle \underline{\boldsymbol{u}}, \vec{\boldsymbol{u}} \rangle}_{-1} + \underbrace{\boldsymbol{\epsilon}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{b}}, \vec{\boldsymbol{v}}, \vec{\boldsymbol{u}})}_{0} = \langle \boldsymbol{q}, \vec{\boldsymbol{v}} \rangle,$$

which shows that $\boldsymbol{q} = \boldsymbol{A}(., \vec{\boldsymbol{u}})$. The choice (3.40) was thus the only one possible. Regarding $\vec{\boldsymbol{b}}$, we notice from (3.39) that if the action of $\boldsymbol{A}$ is restricted to $E_{\boldsymbol{u}}$, we get

$$\forall (\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}) \in E_{\boldsymbol{u}}^2, \quad \boldsymbol{A}(\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}) = \boldsymbol{\epsilon}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{b}}, \vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}) = \boldsymbol{\epsilon}_{\boldsymbol{u}}(\vec{\boldsymbol{b}}, \vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}). \tag{3.49}$$

If $\vec{\boldsymbol{b}}' \in E_{\boldsymbol{u}}$ is such that the decomposition (3.37) holds with $\vec{\boldsymbol{b}}$ replaced by $\vec{\boldsymbol{b}}'$, we deduce from (3.49) that

$$\forall (\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}) \in E_{\boldsymbol{u}}^2, \quad \boldsymbol{\epsilon}_{\boldsymbol{u}}(\vec{\boldsymbol{b}}' - \vec{\boldsymbol{b}}, \vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}) = 0.$$

Since the trilinear form $\boldsymbol{\epsilon}_{\boldsymbol{u}}$ is nondegenerate on $E_{\boldsymbol{u}}$, we conclude that $\vec{\boldsymbol{b}}' - \vec{\boldsymbol{b}} = 0$, which establishes the uniqueness of $\vec{\boldsymbol{b}}$ and completes the demonstration of (3.37). □

Remark 3.10. The representation of an antisymmetric bilinear form by a linear form and a vector, both orthogonal to $\vec{\boldsymbol{u}}$, is understandable by counting the degrees of freedom: the matrix of $\boldsymbol{A}$ in the basis $(\vec{\boldsymbol{e}}_\alpha)$ is an antisymmetric 4×4 matrix; it has therefore only 6 independent components. The linear form $\boldsymbol{q}$, which must satisfy $\langle \boldsymbol{q}, \vec{\boldsymbol{u}} \rangle = 0$, has 3 independent components. The same thing holds for the vector $\vec{\boldsymbol{b}}$, which obeys the constraint $\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{b}} = 0$. We have thus $3 + 3 = 6$, as it should be.

Remark 3.11. The linear form $\boldsymbol{q}$ and the vector $\vec{\boldsymbol{b}}$ are sometimes called, respectively, the ***electric part*** and the ***magnetic part*** of the bilinear form $\boldsymbol{A}$ with respect to $\vec{\boldsymbol{u}}$. We shall see in Chap. 17 that such denominations arise from the decomposition of the electromagnetic field tensor, which is an antisymmetric bilinear form.

Remark 3.12. In Chap. 14, we will perform a rewriting of the decomposition (3.37), in terms of operations specific to antisymmetric multilinear forms (exterior product and Hodge star); this will result in Eq. (14.80).

3.5.3 Application to the Variation of the Local Frame

Let us apply the decomposition (3.37) to the antisymmetric bilinear form $\underline{\boldsymbol{\Omega}}$ introduced above. From (3.40), $\boldsymbol{q} = \underline{\boldsymbol{\Omega}}(.,\vec{\boldsymbol{u}})$. Now, by means of (3.32) and (3.31), we get

$$\forall \vec{\boldsymbol{v}} \in E, \quad \underline{\boldsymbol{\Omega}}(\vec{\boldsymbol{v}},\vec{\boldsymbol{u}}) = \vec{\boldsymbol{v}} \cdot \boldsymbol{\Omega}(\vec{\boldsymbol{u}}) = c\,\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{v}},$$

which shows that $\underline{\boldsymbol{\Omega}}(.,\vec{\boldsymbol{u}}) = c\underline{\boldsymbol{a}}$, hence $\boldsymbol{q} = c\underline{\boldsymbol{a}}$. Regarding the vector $\vec{\boldsymbol{b}}$ in the decomposition (3.37), we shall rather consider $-\vec{\boldsymbol{b}}$, which we shall denote by $\vec{\boldsymbol{\omega}}$. The decomposition (3.37) is then written as

$$\boxed{\underline{\boldsymbol{\Omega}} = c\,\underline{\boldsymbol{u}} \otimes \underline{\boldsymbol{a}} - c\,\underline{\boldsymbol{a}} \otimes \underline{\boldsymbol{u}} - \boldsymbol{\epsilon}(\vec{\boldsymbol{u}},\vec{\boldsymbol{\omega}},.,.)}. \tag{3.50}$$

Therefore, from (3.32), for any pair of vectors $(\vec{\boldsymbol{v}},\vec{\boldsymbol{w}})$,

$$\begin{aligned}\boldsymbol{\Omega}(\vec{\boldsymbol{v}}) \cdot \vec{\boldsymbol{w}} &= \underline{\boldsymbol{\Omega}}(\vec{\boldsymbol{w}},\vec{\boldsymbol{v}}) \\ &= c\,(\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{w}})(\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{v}}) - c\,(\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{w}})(\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}}) \underbrace{-\boldsymbol{\epsilon}(\vec{\boldsymbol{u}},\vec{\boldsymbol{\omega}},\vec{\boldsymbol{w}},\vec{\boldsymbol{v}})}_{+\boldsymbol{\epsilon}(\vec{\boldsymbol{u}},\vec{\boldsymbol{\omega}},\vec{\boldsymbol{v}},\vec{\boldsymbol{w}})}.\end{aligned}$$

By extending the definition (3.46) of the cross product $\times_u$ to the entire E via $\boldsymbol{\epsilon}(\vec{\boldsymbol{u}},\vec{\boldsymbol{\omega}},\vec{\boldsymbol{v}},\vec{\boldsymbol{w}}) =: (\vec{\boldsymbol{\omega}} \times_u \vec{\boldsymbol{v}}) \cdot \vec{\boldsymbol{w}}$, we get the following expression for the endomorphism $\boldsymbol{\Omega}$:

$$\forall \vec{\boldsymbol{v}} \in E, \quad \boxed{\boldsymbol{\Omega}(\vec{\boldsymbol{v}}) = c(\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{v}})\,\vec{\boldsymbol{u}} - c(\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}})\,\vec{\boldsymbol{a}} + \vec{\boldsymbol{\omega}} \times_u \vec{\boldsymbol{v}}}. \tag{3.51}$$

Rewriting this relation for each of the vectors $\vec{\boldsymbol{e}}_\alpha$ of $\mathscr{O}$'s local frame, we get

$$\boxed{\frac{\mathrm{d}\vec{\boldsymbol{e}}_\alpha}{\mathrm{d}t} = \underbrace{c(\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{e}}_\alpha)\,\vec{\boldsymbol{u}} - c(\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{e}}_\alpha)\,\vec{\boldsymbol{a}}}_{\text{Fermi–Walker part}} + \underbrace{\vec{\boldsymbol{\omega}} \times_u \vec{\boldsymbol{e}}_\alpha}_{\substack{\text{spatial} \\ \text{rotation part}}}}, \tag{3.52}$$

where the denomination *Fermi–Walker* and *spatial rotation* will be justified hereafter.

Remark 3.13. Since $\vec{\boldsymbol{e}}_0 = \vec{\boldsymbol{u}}$, $\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{u}} = 0$, $\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{u}} = -1$ and $\vec{\boldsymbol{\omega}} \times_u \vec{\boldsymbol{u}} = \vec{\boldsymbol{\epsilon}}(\vec{\boldsymbol{u}},\vec{\boldsymbol{\omega}},\vec{\boldsymbol{u}},.) = 0$, we check that (3.52) gives $\mathrm{d}\vec{\boldsymbol{e}}_0/\mathrm{d}t = \mathrm{d}\vec{\boldsymbol{u}}/\mathrm{d}t = c\,\vec{\boldsymbol{a}}$.

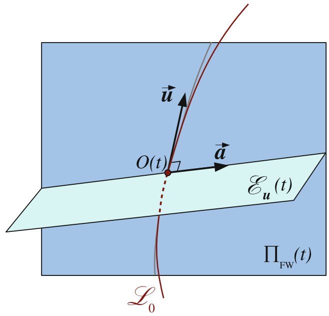


Fig. 3.14 Evolution of the 4-velocity along a worldline $\mathscr{L}_0$: the vertical plane $\Pi_{FW}(t)$ is the plane defined by $(\vec{u}(t), \vec{a}(t))$ (osculating plane of $\mathscr{L}_0$), and the hyperplane $\mathscr{E}_{\boldsymbol{u}}(t)$ (graphically reduced to 2 dimensions) is the hyperplane orthogonal to $\vec{u}(t)$; it contains $\vec{a}(t)$

Regarding the classic three-dimensional space, the time variation of a moving frame of three orthogonal unit vectors $(\vec{e}_i)$ is entirely due to the rotation of this frame. In the present case, the moving frame contains a fourth vector, $\vec{e}_0 = \vec{u}$, which may vary even in the absence of any spatial rotation, when the observer is accelerated (cf. Sect. 2.4). It is therefore natural to decompose $\boldsymbol{\Omega}$ in two parts:

$$\boxed{\boldsymbol{\Omega} = \boldsymbol{\Omega}_{\mathrm{FW}} + \boldsymbol{\Omega}_{\mathrm{rot}}}, \tag{3.53}$$

with

$$\forall \vec{v} \in E, \quad \boldsymbol{\Omega}_{\mathrm{FW}}(\vec{v}) := c(\vec{a} \cdot \vec{v})\, \vec{u} - c(\vec{u} \cdot \vec{v})\, \vec{a} \tag{3.54}$$

$$\boldsymbol{\Omega}_{\mathrm{rot}}(\vec{v}) := \vec{\omega} \times_{\boldsymbol{u}} \vec{v}. \tag{3.55}$$

$\boldsymbol{\Omega}_{\mathrm{FW}}$ is related solely to the 4-acceleration $\vec{a}$ of observer $\mathscr{O}$, which makes $\vec{e}_0 = \vec{u}$ change its direction when t varies. $\boldsymbol{\Omega}_{\mathrm{FW}}$ is named ***Fermi–Walker tensor***. $\boldsymbol{\Omega}_{\mathrm{FW}}$ is acting only on the components of the vectors in the plane Span$(\vec{u}, \vec{a})$, which is the osculating plane of $\mathscr{O}$'s worldline, as we have seen in Sect. 2.7.3 (cf. Fig. 3.14).

The remaining part of $\boldsymbol{\Omega}$, namely, $\boldsymbol{\Omega}_{\mathrm{rot}}$, represents the *spatial rotation* of the triad $(\vec{e}_i)$; it is called ***spatial rotation tensor***. The vector $\vec{\omega} \in E_{\boldsymbol{u}}$ is named the ***four-rotation***, or ***4-rotation*** for short, of the local frame $(\vec{e}_\alpha)$ and, by extension, of observer $\mathscr{O}$. The physical dimension of $\boldsymbol{\Omega}_{\mathrm{rot}}$ can be read on (3.30): it is the inverse of a time. Since $\vec{u}$ and $\boldsymbol{\epsilon}$ are dimensionless, we deduce that the dimension of $\vec{\omega}$ is the inverse of a time as well, i.e. the dimension of an angular velocity. $\vec{\omega}$ is a spacelike vector orthogonal to $\vec{u}$ (from the third property in (3.37) with $\vec{b} = -\vec{\omega}$):

$$\boxed{\vec{u} \cdot \vec{\omega} = 0}. \tag{3.56}$$

Moreover, we have $\boldsymbol{\Omega}_{\mathrm{rot}}(\vec{\omega}) = \vec{\omega} \times_{\boldsymbol{u}} \vec{\omega} = 0$, which shows that $\boldsymbol{\Omega}_{\mathrm{rot}}$ acts only in changing the basis $(\vec{e}_\alpha)$ in the two-dimensional subspace of $E_{\boldsymbol{u}}$ orthogonal to $\vec{\omega}$. In other words, the action of $\boldsymbol{\Omega}_{\mathrm{rot}}$ consists in a rotation in the plane orthogonal to both $\vec{u}$ and $\vec{\omega}$.

We shall say that $\mathscr{O}$ is an ***accelerated observer*** iff $\vec{\boldsymbol{a}} \neq 0$ and that $\mathscr{O}$ is a ***rotating observer***, iff $\vec{\boldsymbol{\omega}} \neq 0$. We shall see in Chaps. 12 and 13 that $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{\omega}}$ are quantities measurable by $\mathscr{O}$, contrary to the 4-velocity $\vec{\boldsymbol{u}}$: by means of physical experiments, $\mathscr{O}$ can determine his 4-acceleration (Sect. 12.4.5) and his 4-rotation (Sect. 13.2.2).

Example 3.1. The Serret–Frenet tetrad $(\vec{\boldsymbol{e}}_\alpha)$ introduced in Sect. 2.7.3 for the geometrical description of a worldline obeys to the criteria defining a local frame, as stated in Sect. 3.4.1. Given a worldline $\mathscr{L}_0$, one may, at least theoretically, consider the observer (i) whose worldline is $\mathscr{L}_0$ and (ii) whose local frame in the Serret–Frenet tetrad of $\mathscr{L}_0$. We shall not discuss here about the physical realization of such an observer, i.e. about the physical processes by which an observer ensures himself that his local frame coincides with the Serret–Frenet tetrad of his worldline. The evolution of the Serret–Frenet tetrad along the worldline is given by (2.63). The matrix appearing in this equation is nothing but the matrix $\Omega^\beta{}_\alpha$ (row index: α; column index: β) considered at Sect. 3.5.1 [Eq. (3.29)]. Incidentally, we may check that the matrix (2.63) obeys the properties (3.35). By comparing formulas (3.52) and (2.63), we obtain the Fermi–Walker tensor in terms of the curvature a of the worldline:

$$\underline{\boldsymbol{\Omega}}_{\rm FW} = ca(\underline{\boldsymbol{e}}_1 \otimes \underline{\boldsymbol{e}}_0 - \underline{\boldsymbol{e}}_0 \otimes \underline{\boldsymbol{e}}_1) \tag{3.57}$$

as well as the 4-rotation of the Serret–Frenet tetrad in terms of the first and second torsions, T_1 and T_2, of the worldline:

$$\vec{\boldsymbol{\omega}} = cT_2\,\vec{\boldsymbol{e}}_1 + cT_1\,\vec{\boldsymbol{e}}_3. \tag{3.58}$$

Historical note: *The denomination* Fermi–Walker *for the change in the tetrad* $(\vec{\boldsymbol{e}}_\alpha)$ *that is not related to any spatial rotation arises from the "nonrotating" coordinates introduced in the vicinity of a worldline by Enrico Fermi*[11] *in 1922 (Fermi 1922) and Arthur G. Walker*[12] *in 1932 (Walker 1932).*

3.5.4 Inertial Observers

Among all observers, it is natural to distinguish the ones whose local frame is constant along their worldline, i.e. satisfies

[11]**Enrico Fermi** (1901–1954): Italian physicist, author of fundamental works in quantum mechanics, statistical physics and nuclear physics; he was awarded the 1938 Nobel Prize in Physics. He wrote the article (Fermi 1922) about the coordinates at the vicinity of a worldline at the age of 21, while he was still a student at the Scuola Normale Superiore in Pisa.

[12]**Arthur G. Walker** (1909–2001): British mathematician, expert in differential geometry and well known for his work in cosmology; he was also a talented dancer.

$$\forall \alpha \in \{0, 1, 2, 3\}, \quad \boxed{\frac{\mathrm{d}\vec{e}_\alpha}{\mathrm{d}t} = 0}. \qquad (3.59)$$

Such observers are called ***inertial observers***.

In view of the evolution law (3.52), an observer whose 4-acceleration $\vec{a}$ and 4-rotation $\vec{\omega}$ both vanish is an inertial observer. Conversely, if an observer obeys (3.59), then, by definition of the 4-acceleration [Eq. (2.16)], $\vec{a} = c^{-1}\mathrm{d}\vec{u}/\mathrm{d}t = c^{-1}\mathrm{d}\vec{e}_0/\mathrm{d}t = 0$. For $i \in \{1, 2, 3\}$, Eq. (3.52) reduces then to $\mathrm{d}\vec{e}_i/\mathrm{d}t = \vec{\omega} \times_u \vec{e}_i$. Since $\mathrm{d}\vec{e}_i/\mathrm{d}t = 0$, this implies $\vec{\omega} = 0$. We therefore conclude that

An observer is inertial iff

$$\forall t \in \mathbb{R}, \quad \boxed{\vec{a}(t) = 0} \quad \text{and} \quad \boxed{\vec{\omega}(t) = 0}, \qquad (3.60)$$

where t, $\vec{a}$ and $\vec{\omega}$ are, respectively, the proper time, the 4-acceleration and the 4-rotation of the observer.

Inertial observers are the simplest observers in Minkowski spacetime, and historically, special relativity was first formulated only in terms of them. We shall study these observers in detail in Chap. 8.

Remark 3.14. Inertial observers are sometimes called *Galilean observers*. We shall not use such a terminology here, partly to avoid any confusion with pre-relativistic mechanics.

3.6 Derivative of a Vector Field Along a Worldline

3.6.1 Absolute Derivative

Let $\vec{v} = \vec{v}(t)$ be a vector field along the worldline $\mathscr{L}_0$ of some observer $\mathscr{O}$ of proper time t (cf. Sect. 2.7.2). The derivative of $\vec{v}$ along $\mathscr{L}_0$, $\mathrm{d}\vec{v}/\mathrm{d}t$, has been defined in Sect. 2.7.2 [Eq. (2.53)]. We shall call it ***absolute derivative of the field $\vec{v}$ along $\mathscr{L}_0$*** to distinguish it from the other types of derivatives introduced hereafter.

Denoting by $(v^\alpha(t))$ the components of $\vec{v}(t)$ in $\mathscr{O}$'s local frame $(\vec{e}_\alpha(t))$, we have $\vec{v}(t) = v^\alpha(t)\, \vec{e}_\alpha(t)$, so that

$$\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \frac{\mathrm{d}v^\alpha}{\mathrm{d}t}\, \vec{e}_\alpha + v^\alpha \frac{\mathrm{d}\vec{e}_\alpha}{\mathrm{d}t}.$$

Substituting expression (3.52) for $\mathrm{d}\vec{e}_\alpha/\mathrm{d}t$, we get immediately

$$\boxed{\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \frac{\mathrm{d}v^\alpha}{\mathrm{d}t}\,\vec{e}_\alpha + c(\vec{a}\cdot\vec{v})\,\vec{u} - c(\vec{u}\cdot\vec{v})\,\vec{a} + \vec{\omega}\times_u\vec{v}}. \tag{3.61}$$

3.6.2 Derivative with Respect to an Observer

Using the same notations as above, we call ***derivative of*** $\vec{v}$ ***with respect to observer*** $\mathscr{O}$ the vector field defined along $\mathscr{L}_0$ by

$$\boxed{\boldsymbol{D}_{\mathscr{O}}\vec{v} := \frac{\mathrm{d}v^\alpha}{\mathrm{d}t}\,\vec{e}_\alpha}. \tag{3.62}$$

Hence $\boldsymbol{D}_{\mathscr{O}}\vec{v}$ measures the variation of $\vec{v}$ along the worldline $\mathscr{L}_0$ that is solely due to the variation of $\vec{v}$'s components in $\mathscr{O}$'s local frame. In view of the definition of a fixed vector field with respect to $\mathscr{O}$ given in Sect. 3.4.3, we have

$$\boxed{\vec{v}\ \text{fixed w.r.t.}\ \mathscr{O} \iff \boldsymbol{D}_{\mathscr{O}}\vec{v} = 0}. \tag{3.63}$$

In particular, the derivative of each of the vectors of $\mathscr{O}$'s local frame is zero:

$$\forall\alpha\in\{0,1,2,3\},\quad \boldsymbol{D}_{\mathscr{O}}\vec{e}_\alpha = 0. \tag{3.64}$$

For $\alpha = 0$, one may of course replace $\vec{e}_0$ by $\vec{u}$:

$$\boldsymbol{D}_{\mathscr{O}}\vec{u} = 0. \tag{3.65}$$

An important property of $\boldsymbol{D}_{\mathscr{O}}$ is that the derivative of a vector lying in $\mathscr{O}$'s local rest space stays in this space:

$$\forall\vec{v}\in E_{\boldsymbol{u}},\quad \boldsymbol{D}_{\mathscr{O}}\vec{v}\in E_{\boldsymbol{u}}. \tag{3.66}$$

Indeed, if $\vec{v}\in E_{\boldsymbol{u}}$, then $\vec{v} = v^i\,\vec{e}_i$ and $\vec{u}\cdot\boldsymbol{D}_{\mathscr{O}}\vec{v} = (\mathrm{d}v^i/\mathrm{d}t)\,\vec{u}\cdot\vec{e}_i = 0$, since $\vec{u}\cdot\vec{e}_i = 0$.

Remark 3.15. The property (3.66) is not satisfied by the absolute derivative. Indeed, if $\vec{v}\in E_{\boldsymbol{u}}$, then $\vec{u}\cdot\mathrm{d}\vec{v}/\mathrm{d}t = -\mathrm{d}\vec{u}/\mathrm{d}t\cdot\vec{v} = -c\,\vec{a}\cdot\vec{v}\neq 0$ in general, because $\vec{a}$ and $\vec{v}$ are two vectors in the hyperplane $E_{\boldsymbol{u}}$ and have a priori no reason to be orthogonal.

By means of (3.61), we may express the derivative with respect to an observer in terms of the absolute derivative and the observer's 4-acceleration and 4-rotation:

$$\boxed{\boldsymbol{D}_{\mathscr{O}}\vec{\boldsymbol{v}} = \frac{\mathrm{d}\vec{\boldsymbol{v}}}{\mathrm{d}t} - c(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{v}})\,\vec{\boldsymbol{u}} + c(\vec{\boldsymbol{u}}\cdot\vec{\boldsymbol{v}})\,\vec{\boldsymbol{a}} - \vec{\boldsymbol{\omega}} \times_u \vec{\boldsymbol{v}}}. \tag{3.67}$$

In particular, if $\mathscr{O}$ is an inertial observer [$\vec{\boldsymbol{a}} = 0$ and $\vec{\boldsymbol{\omega}} = 0$, cf. (3.60)], the two derivatives coincide:

$$\boldsymbol{D}_{\mathscr{O}}\vec{\boldsymbol{v}} = \frac{\mathrm{d}\vec{\boldsymbol{v}}}{\mathrm{d}t} \qquad (\mathscr{O}\text{ inertial}). \tag{3.68}$$

3.6.3 Fermi–Walker Derivative

The ***Fermi–Walker derivative*** of a vector field $\vec{\boldsymbol{v}}$ along a worldline $\mathscr{L}_0$ is the vector field along $\mathscr{L}_0$ defined by [cf. (3.54)]

$$\boxed{\boldsymbol{D}^{\mathrm{FW}}_{\boldsymbol{u}}\vec{\boldsymbol{v}} := \frac{\mathrm{d}\vec{\boldsymbol{v}}}{\mathrm{d}t} - \boldsymbol{\Omega}_{\mathrm{FW}}(\vec{\boldsymbol{v}}) = \frac{\mathrm{d}\vec{\boldsymbol{v}}}{\mathrm{d}t} - c(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{v}})\,\vec{\boldsymbol{u}} + c(\vec{\boldsymbol{u}}\cdot\vec{\boldsymbol{v}})\,\vec{\boldsymbol{a}}}. \tag{3.69}$$

Comparing with (3.67), we notice that Fermi–Walker derivative is the derivative with respect to a nonrotating observer ($\vec{\boldsymbol{\omega}} = 0$) having $\mathscr{L}_0$ as worldline. For a rotating observer, the two derivatives are related by

$$\boxed{\boldsymbol{D}_{\mathscr{O}}\vec{\boldsymbol{v}} = \boldsymbol{D}^{\mathrm{FW}}_{\boldsymbol{u}}\vec{\boldsymbol{v}} - \vec{\boldsymbol{\omega}} \times_u \vec{\boldsymbol{v}}}. \tag{3.70}$$

One says that a vector field $\vec{\boldsymbol{v}}$ is ***Fermi–Walker transported*** along the worldline $\mathscr{L}_0$ iff $\boldsymbol{D}^{\mathrm{FW}}_{\boldsymbol{u}}\vec{\boldsymbol{v}} = 0$. Then, the equivalence (3.63) means that for a nonrotating observer, the notions of fixed vector field and Fermi–Walker transported vector field are identical:

$$\boxed{\vec{\boldsymbol{v}}\text{ fixed w.r.t. }\mathscr{O} \iff \vec{\boldsymbol{v}}\text{ Fermi–Walker transported along }\mathscr{L}_0.}_{\;\vec{\boldsymbol{\omega}}=0} \tag{3.71}$$

Remark 3.16. As it is clear on the definition (3.69), the concept of Fermi–Walker derivative relies only on the worldline $\mathscr{L}_0$ and not on any observer that may have $\mathscr{L}_0$ as a worldline. Let us recall that there exists an infinite number of such observers: they differ by their 4-rotation $\vec{\boldsymbol{\omega}}$.

Remark 3.17. We have already noticed that the Fermi–Walker derivative can be considered as a particular case of the derivative with respect to an observer (a non-rotating one). It therefore obeys properties (3.65) and (3.66) :

$$\boldsymbol{D}^{\mathrm{FW}}_{\boldsymbol{u}}\vec{\boldsymbol{u}} = 0 \qquad \text{and} \qquad \forall\vec{\boldsymbol{v}} \in E_{\boldsymbol{u}}, \quad \boldsymbol{D}^{\mathrm{FW}}_{\boldsymbol{u}}\vec{\boldsymbol{v}} \in E_{\boldsymbol{u}}. \tag{3.72}$$

Remark 3.18. In some textbooks (Hawking and Ellis 1973; Straumann 2013), the terms *Fermi derivative* and *Fermi transport* are used instead of *Fermi–Walker derivative* and *Fermi–Walker transport*. The *Fermi–Walker* terminology employed here is used, among others, in the textbooks (Ferraro 2007; Misner et al. 1973; Synge 1960).

Let us assume that the vector $\vec{\boldsymbol{v}}$ is orthogonal to the 4-velocity $\vec{\boldsymbol{u}}$ at any point of $\mathscr{L}_0$, i.e. that $\vec{\boldsymbol{v}} \in E_{\boldsymbol{u}}$. The last term in (3.69) then vanishes, yielding

$$\boldsymbol{D}^{\mathrm{FW}}_{\boldsymbol{u}} \vec{\boldsymbol{v}} = \frac{\mathrm{d}\vec{\boldsymbol{v}}}{\mathrm{d}t} - c(\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{v}})\, \vec{\boldsymbol{u}} = \frac{\mathrm{d}\vec{\boldsymbol{v}}}{\mathrm{d}t} - \left(\frac{\mathrm{d}\vec{\boldsymbol{u}}}{\mathrm{d}t} \cdot \vec{\boldsymbol{v}}\right) \vec{\boldsymbol{u}} = \frac{\mathrm{d}\vec{\boldsymbol{v}}}{\mathrm{d}t} + \left(\vec{\boldsymbol{u}} \cdot \frac{\mathrm{d}\vec{\boldsymbol{v}}}{\mathrm{d}t}\right) \vec{\boldsymbol{u}}.$$

We recognize the orthogonal projection onto $E_{\boldsymbol{u}}$ [cf. (3.11)]:

$$\boxed{\forall \vec{\boldsymbol{v}} \in E_{\boldsymbol{u}}, \quad \boldsymbol{D}^{\mathrm{FW}}_{\boldsymbol{u}} \vec{\boldsymbol{v}} = \perp_{\boldsymbol{u}} \frac{\mathrm{d}\vec{\boldsymbol{v}}}{\mathrm{d}t}}. \tag{3.73}$$

Hence, for the vectors in the local rest space $E_{\boldsymbol{u}}$, the Fermi–Walker derivative is nothing but the orthogonal projection onto $E_{\boldsymbol{u}}$ of the absolute derivative. In particular, we recover the fact that the Fermi–Walker derivative of a vector in $E_{\boldsymbol{u}}$ is a vector in $E_{\boldsymbol{u}}$ [Property (3.72)].

3.7 Locality of an Observer's Frame

We investigate here by which extent the coordinates with respect to a given observer $\mathscr{O}$, as defined in Sect. 3.4.2, are local.

Let O be an event of proper t on the worldline $\mathscr{L}_0$ where the 4-acceleration of $\mathscr{O}$ does not vanish: $\vec{\boldsymbol{a}} \neq 0$ (cf. Fig. 3.15). Let (x^α) be the affine coordinate system on $\mathscr{E}$ (cf. Sect. 1.2.3) defined by the point O and the vectors $(\vec{\boldsymbol{u}}(t), \vec{\boldsymbol{\varepsilon}}_1, \vec{\boldsymbol{\varepsilon}}_2, \vec{\boldsymbol{\varepsilon}}_3)$, where

$$\vec{\boldsymbol{\varepsilon}}_1 := a^{-1}\, \vec{\boldsymbol{a}}(t), \qquad a := \|\vec{\boldsymbol{a}}(t)\|_{\boldsymbol{g}} = \sqrt{\vec{\boldsymbol{a}}(t) \cdot \vec{\boldsymbol{a}}(t)} \tag{3.74}$$

and $\vec{\boldsymbol{\varepsilon}}_2$ and $\vec{\boldsymbol{\varepsilon}}_3$ are two unit vectors orthogonal to each other, as well as to $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{\varepsilon}}_1$. In particular, $\vec{\boldsymbol{\varepsilon}}_1$ is the second vector of the Serret–Frenet tetrad introduced in Sect. 2.7.3. The basis $(\vec{\boldsymbol{u}}(t), \vec{\boldsymbol{\varepsilon}}_i)$ hence constructed is orthonormal. Each event $M \in \mathscr{E}$ is labelled by its affine coordinates (x^α) so that [cf. (1.6)]

$$\overrightarrow{OM} = x^0\, \vec{\boldsymbol{u}} + x^i\, \vec{\boldsymbol{\varepsilon}}_i. \tag{3.75}$$

The equation of the hyperplane $\mathscr{E}_{\boldsymbol{u}}(t)$ in terms of the affine coordinates (x^α) is

$$\mathscr{E}_{\boldsymbol{u}}(t): \qquad x^0 = 0. \tag{3.76}$$

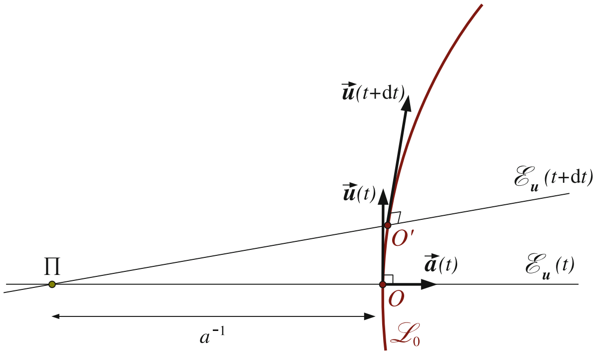


Fig. 3.15 Non-globality of the frame of an accelerated observer: the local rest spaces $\mathscr{E}_{\boldsymbol{u}}(t)$ and $\mathscr{E}_{\boldsymbol{u}}(t + \mathrm{d}t)$ intersect in a plane Π located at the distance $a^{-1} = (\vec{a} \cdot \vec{a})^{-1/2}$ from O. On this two-dimensional figure, Π is reduced to a point and $\mathscr{E}_{\boldsymbol{u}}(t)$ and $\mathscr{E}_{\boldsymbol{u}}(t + \mathrm{d}t)$ to lines

The equation of the neighbouring hyperplane $\mathscr{E}_{\boldsymbol{u}}(t + \mathrm{d}t)$ is determined as follows. The point $O' := O + c\,\mathrm{d}t\,\vec{u}$ belongs to $\mathscr{E}_{\boldsymbol{u}}(t + \mathrm{d}t)$ (cf. Fig. 3.15). The normal to $\mathscr{E}_{\boldsymbol{u}}(t + \mathrm{d}t)$ being $\vec{u}(t + \mathrm{d}t)$, a point $M \in \mathscr{E}$ belongs to $\mathscr{E}_{\boldsymbol{u}}(t + \mathrm{d}t)$ iff $\vec{u}(t + \mathrm{d}t) \cdot \overrightarrow{O'M} = 0$, i.e. iff $\vec{u}(t + \mathrm{d}t) \cdot \overrightarrow{OM} = \vec{u}(t + \mathrm{d}t) \cdot \overrightarrow{OO'}$. Now, at first order in $\mathrm{d}t$, $\vec{u}(t + \mathrm{d}t) \cdot \overrightarrow{OO'} = -c\,\mathrm{d}t$. Hence

$$M \in \mathscr{E}_{\boldsymbol{u}}(t + \mathrm{d}t) \iff \vec{u}(t + \mathrm{d}t) \cdot \overrightarrow{OM} = -c\,\mathrm{d}t.$$

By definition of the 4-acceleration, $\vec{u}(t + \mathrm{d}t) = \vec{u}(t) + c\,\mathrm{d}t\,\vec{a} = \vec{u}(t) + ac\,\mathrm{d}t\,\vec{\varepsilon}_1$. Using this relation as well as (3.75), we get

$$M \in \mathscr{E}_{\boldsymbol{u}}(t + \mathrm{d}t) \iff (\vec{u}(t) + ac\,\mathrm{d}t\,\vec{\varepsilon}_1) \cdot (x^0\,\vec{u} + x^i\,\vec{\varepsilon}_i) = -c\,\mathrm{d}t.$$

Expanding and using the orthogonality of the basis $(\vec{u}, \vec{\varepsilon}_i)$, we obtain the equation of the hyperplane $\mathscr{E}_{\boldsymbol{u}}(t + \mathrm{d}t)$ within the affine coordinate system (x^α):

$$\mathscr{E}_{\boldsymbol{u}}(t + \mathrm{d}t): \qquad -x^0 + ac\,\mathrm{d}t\,x^1 = -c\,\mathrm{d}t. \tag{3.77}$$

From (3.76) and (3.77), the equation of the intersection of the hyperplanes $\mathscr{E}_{\boldsymbol{u}}(t)$ and $\mathscr{E}_{\boldsymbol{u}}(t + \mathrm{d}t)$ is $x^0 = 0$ and $0 + ac\,\mathrm{d}t\,x^1 = -c\,\mathrm{d}t$, i.e.

$$\begin{cases} x^0 = 0 \\ x^1 = -a^{-1}. \end{cases} \tag{3.78}$$

We conclude that if $\vec{a}(t) \neq 0$, the two hyperplanes $\mathscr{E}_{\boldsymbol{u}}(t)$ and $\mathscr{E}_{\boldsymbol{u}}(t + \mathrm{d}t)$ intersect in a (two-dimensional) plane, Π, whose equation in the affine coordinates (x^α) is (3.78) (cf. Fig. 3.15). This plane marks the limit of applicability of the local frame of observer $\mathscr{O}$. Indeed, let us consider an event $M \in \Pi$. The affine coordinates of M

are $x^\alpha = (0, -a^{-1}, x^2, x^3)$. They are uniquely defined but are purely mathematical. Regarding the physical coordinates with respect to observer $\mathscr{O}$, as defined in Sect. 3.4.2, the point M can be labelled twice. First of all, M belongs to the local rest space $\mathscr{E}_{\boldsymbol{u}}(t)$. The observer $\mathscr{O}$ attributes then the coordinates (t, y^i) to M, where, according to (3.25), the (y^i)'s are three real numbers such that $\overrightarrow{OM} = y^i \, \vec{\boldsymbol{e}}_i(t)$. But M also belongs to the local rest space $\mathscr{E}_{\boldsymbol{u}}(t + \mathrm{d}t)$. Still from (3.25), the observer $\mathscr{O}$ attributes to it the coordinates $(t + \mathrm{d}t, z^i)$ where the z^i's are three real numbers such that $\overrightarrow{O'M} = z^i \, \vec{\boldsymbol{e}}_i(t + \mathrm{d}t)$. Therefore, there is not a unique set of coordinates for points of Π within the local frame of $\mathscr{O}$. The same thing holds for events located beyond Π, i.e. events with $x^1 < -a^{-1}$.

We note that $\vec{\boldsymbol{\varepsilon}}_1$ being a unit vector, $|x^1|$ is the distance with respect to the metric tensor $\boldsymbol{g}$ between the point O and the plane Π. In conclusion, the local frame of an observer with 4-acceleration $\vec{\boldsymbol{a}}$ can be safely used to set coordinates to events located at a distance r from the observer's worldline such that

$$\boxed{r \ll a^{-1} = \|\vec{\boldsymbol{a}}\|_{\boldsymbol{g}}^{-1} = (\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{a}})^{-1/2}}. \tag{3.79}$$

We shall see in Chap. 4 that $a = \gamma/c^2$, where γ is the amplitude of the acceleration of $\mathscr{O}$ measured by an inertial observer whose worldline is tangent to that of $\mathscr{O}$ at O [Eq. (4.64)]. For modest accelerations, the criterion (3.79) is not very constraining at a laboratory scale. Indeed, for $\gamma = 10\,\mathrm{m\,s^{-2}}$, $a^{-1} = c^2/\gamma \simeq 9 \times 10^{15}\mathrm{m} \simeq 1$ light-year !

Chapter 4
Kinematics 1: Motion with Respect to an Observer

4.1 Introduction

Having introduced the notion of observer in the preceding chapter, we are in position to discuss *kinematics*, i.e. the description of the motion of a particle with respect to a given observer. We shall distinguish the case of a massive particle (Sects. 4.2–4.5) from that of a massless one (Sect. 4.6). The latter corresponds to a photon and is thus relevant for describing the propagation of light with respect to an observer.

In the next chapter, we shall no longer consider a single observer but two of them and shall derive the transformation laws of kinematic quantities when moving from one observer to the other.

4.2 Lorentz Factor

4.2.1 Definition

Let us consider an observer $\mathscr{O}$, of worldline $\mathscr{L}$ and 4-velocity $\vec{u}$, as well as a massive particle $\mathscr{P}$, of worldline $\mathscr{L}'$ and 4-velocity $\vec{u}'$ (cf. Fig. 4.1). We assume that $\mathscr{L}'$ is located in the vicinity of $\mathscr{L}$, in the sense that $\mathscr{L}'$ can be described in $\mathscr{O}$'s local frame. From the analysis of Sect. 3.7, this means that the spatial distance between $\mathscr{L}$ and $\mathscr{L}'$ is always smaller than $\|\vec{a}\|_g^{-1}$, where $\vec{a}$ stands for $\mathscr{O}$'s 4-acceleration.

At the instant t of $\mathscr{O}$'s proper time, the position of $\mathscr{P}$ "perceived" by $\mathscr{O}$ is the intersection $M(t)$ of $\mathscr{P}$'s worldline with $\mathscr{O}$'s local rest space at t, $\mathscr{E}_{\boldsymbol{u}}(t)$ (cf. Fig. 4.1). Given some infinitesimal increment $\mathrm{d}t$ of $\mathscr{O}$'s proper time, let $\mathrm{d}t'$ be the elapsed proper time of $\mathscr{P}$ when it moves from $M(t)$ to $M(t + \mathrm{d}t)$ along its worldline. Contrary to what Newtonian physics would state, $\mathrm{d}t'$ is a priori not equal to $\mathrm{d}t$. The ratio of these two increments of proper time (one for $\mathscr{O}$: $\mathrm{d}t$, the other one for $\mathscr{P}$: $\mathrm{d}t'$)

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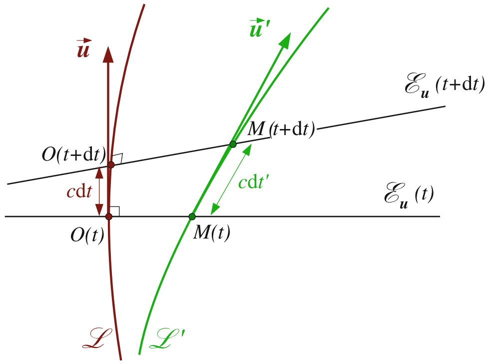

Fig. 4.1 Motion of a massive particle (worldline $\mathscr{L}'$ and 4-velocity $\vec{u}'$) with respect of an observer $\mathscr{O}$ (worldline $\mathscr{L}$, 4-velocity $\vec{u}$ and local rest space $\mathscr{E}_{u}(t)$)

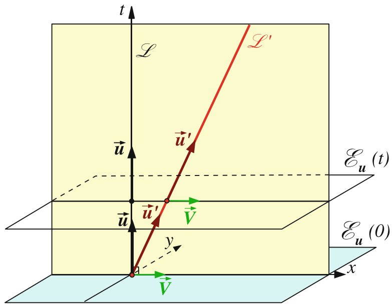

Fig. 4.2 Example 4.1: uniform linear motion

is called the ***Lorentz factor*** of particle $\mathscr{P}$ with respect to observer $\mathscr{O}$ and is denoted by Γ:

$$\boxed{\mathrm{d}t = \Gamma\,\mathrm{d}t'}. \tag{4.1}$$

Example 4.1. The simplest example of motion that one may think of is that represented in Fig. 4.2: both worldlines $\mathscr{L}$ and $\mathscr{L}'$ are straight lines of $\mathscr{E}$. This implies that $\mathscr{O}$'s 4-velocity $\vec{u}$ is constant. We shall suppose that the three spacelike vectors $\vec{e}_i$ of $\mathscr{O}$'s local frame $(\vec{e}_\alpha)$ are constant as well, in addition to $\vec{e}_0 = \vec{u}$; $\mathscr{O}$ is then an inertial observer (cf. Sect. 3.5.4). Let us denote by $(x^0 = ct,\ x^1 = x,\ x^2 = y,\ x^3 = z)$ the coordinates associated with $\mathscr{O}$ (cf. Sect. 3.4.2) and let us consider the case where $\mathscr{L}'$ is the line of parametric equation

$$x(t) = vt, \quad y(t) = 0, \quad z(t) = 0, \tag{4.2}$$

where v is a constant such that $|v| < c$. $\mathscr{P}$ has a uniform linear motion with respect to $\mathscr{O}$ of velocity v in the direction of the x-axis. The vector $\overrightarrow{M(t)M(t+\mathrm{d}t)}$ along $\mathscr{L}'$ has the components $\mathrm{d}x^\alpha = (c\,\mathrm{d}t, v\,\mathrm{d}t, 0, 0)$ in the basis $(\vec{e}_\alpha)$. From formula (2.7) with $\tau = t'$, the corresponding increase $\mathrm{d}t'$ of $\mathscr{P}$'s proper time is

$$\mathrm{d}t' = \frac{1}{c}\sqrt{(c\,\mathrm{d}t)^2 - (v\,\mathrm{d}t)^2} = \mathrm{d}t\,\sqrt{1-(v/c)^2}.$$

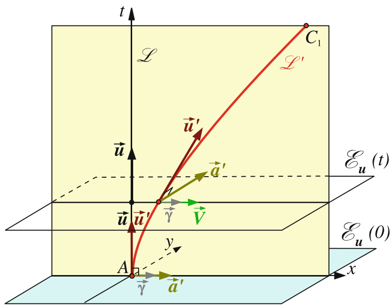

Fig. 4.3 Example 4.2: uniformly accelerated motion (Langevin's traveller between events A and C_1)

Comparing with (4.1), one deduces the value of the Lorentz factor of $\mathscr{P}$ with respect to $\mathscr{O}$:

$$\Gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}. \tag{4.3}$$

Notice that Γ is constant, since v is.

Example 4.2. A second example is provided by Langevin's traveller discussed in Sect. 2.6 and whose worldline between events A (rocket engine ignition) and C_1 (thrust reversing) is depicted in Fig. 4.3. The "sedentary" twin $\mathscr{O}$ is then an inertial observer, as in Example 4.1. His proper time t coincides with the affine coordinate t introduced in Sect. 2.6.1, and $\mathscr{O}$'s local rest spaces are the hyperplanes $\mathscr{E}_{\boldsymbol{u}}(t)$ defined by $t = \text{const}$. In these circumstances, the comparison of (2.25) with (4.1) leads to the Lorentz factor of Langevin's traveller $\mathscr{O}'$ with respect to $\mathscr{O}$:

$$\Gamma = \sqrt{1 + \frac{\gamma^2}{c^2}\left(t - \frac{k}{2}T\right)^2} \tag{4.4}$$

(use has been made of (2.38) to let appear the acceleration γ instead of the parameter α). We notice that $\Gamma = 1$ at A ($t = 0, k = 0$), P ($t = T/2, k = 1$) and B ($t = T, k = 2$) and that Γ takes its maximum value at C_1 ($t = T/4, k = 0$) and C_2 ($t = 3T/4, k = 1$), which is

$$\Gamma_{\max} = \sqrt{1 + (T/T_*)^2}, \tag{4.5}$$

where $T_* := 4c/\gamma$ is the time introduced in Sect. 2.6.5.

Example 4.3. As a last example, let us consider that $\mathscr{P}$ is a point particle in ***uniform circular motion*** in the plane $z = 0$ of $\mathscr{O}$'s reference space, $\mathscr{O}$ being still an inertial observer. This means that $\mathscr{P}$'s worldline obeys the following equations:

$$\begin{cases} x(t) = R\cos\Omega t \\ y(t) = R\sin\Omega t \\ z(t) = 0, \end{cases} \tag{4.6}$$

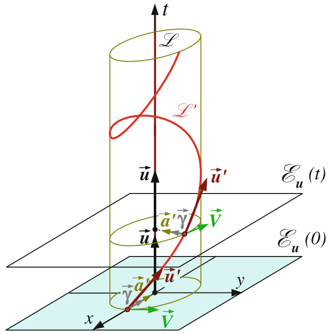

Fig. 4.4 Example 4.3: uniform circular motion

where R and Ω are two positive constants such that $R\Omega < c$. The worldline $\mathscr{L}'$ is a helix when represented in a spacetime diagram based on $\mathscr{O}$'s coordinates, as in Fig. 4.4. In particular, $\mathscr{L}'$ is not confined to a plane, contrary to Examples 4.1 and 4.2. Along $\mathscr{L}'$, the elementary displacement vector $\overrightarrow{M(t)M(t+\mathrm{d}t)}$ has the components $\mathrm{d}x^\alpha = (c\,\mathrm{d}t, -R\Omega\sin\Omega t\,\mathrm{d}t, R\Omega\cos\Omega t\,\mathrm{d}t, 0)$ in the basis $(\vec{e}_\alpha)$. From formula (2.7), the corresponding increase $\mathrm{d}t'$ of $\mathscr{P}$'s proper time is

$$\mathrm{d}t' = \frac{1}{c}\sqrt{(c\,\mathrm{d}t)^2 - (R\Omega\sin\Omega t\,\mathrm{d}t)^2 - (R\Omega\cos\Omega t\,\mathrm{d}t)^2} = \mathrm{d}t\,\sqrt{1-(R\Omega/c)^2}.$$

Comparing with (4.1), we deduce the Lorentz factor of $\mathscr{P}$ with respect to $\mathscr{O}$:

$$\Gamma = \left[1 - \left(\frac{R\Omega}{c}\right)^2\right]^{-1/2}. \tag{4.7}$$

Notice that Γ is constant, since R and Ω are.

4.2.2 *Expression in Terms of the 4-Velocity and the 4-Acceleration*

Let $O(t)$ be the event of proper time t along the worldline of observer $\mathscr{O}$. We are going to see that the Lorentz factor Γ can be expressed in terms of $\vec{u}$ ($\mathscr{O}$'s 4-velocity), $\vec{u}'$ ($\mathscr{P}$'s 4-velocity), $\vec{a}$ ($\mathscr{O}$'s 4-acceleration) and $\overrightarrow{OM}$ ($\mathscr{P}$'s position vector in $\mathscr{O}$'s local frame). Indeed, let us express that $M(t+\mathrm{d}t)$ belongs to the local rest space $\mathscr{E}_u(t+\mathrm{d}t)$, i.e. that the vectors $\vec{u}$ and $\overrightarrow{OM}$ are orthogonal at the proper time $t + \mathrm{d}t$ [Eq. (3.8)]:

$$\vec{u}(t+\mathrm{d}t)\cdot\overrightarrow{O(t+\mathrm{d}t)M(t+\mathrm{d}t)} = 0. \tag{4.8}$$

Now, from the very definition of the 4-acceleration,

$$\vec{u}(t+\mathrm{d}t) = \vec{u}(t) + c\,\mathrm{d}t\,\vec{a}(t)$$

and, by Chasles' relation,

$$\overrightarrow{O(t+\mathrm{d}t)M(t+\mathrm{d}t)} = \overrightarrow{O(t+\mathrm{d}t)O(t)} + \overrightarrow{O(t)M(t)} + \overrightarrow{M(t)M(t+\mathrm{d}t)}.$$

In this last relation, the definition of the 4-velocity enables one to write $\overrightarrow{O(t+\mathrm{d}t)O(t)} = -c\,\mathrm{d}t\,\vec{u}(t)$ and $\overrightarrow{M(t)M(t+\mathrm{d}t)} = c\,\mathrm{d}t'\,\vec{u}'(t)$ (cf. Fig. 4.1). Taking into account the above relations, (4.8) becomes

$$-c\,\mathrm{d}t\,\vec{u}(t)\cdot\vec{u}(t) + \vec{u}(t)\cdot\overrightarrow{O(t)M(t)} + c\,\mathrm{d}t'\,\vec{u}(t)\cdot\vec{u}'(t) - c^2\,\mathrm{d}t^2\,\vec{a}(t)\cdot\vec{u}(t)$$
$$+c\,\mathrm{d}t\,\vec{a}(t)\cdot\overrightarrow{O(t)M(t)} + c^2\,\mathrm{d}t\,\mathrm{d}t'\,\vec{a}(t)\cdot\vec{u}'(t) = 0.$$

Now $\vec{u}(t)\cdot\vec{u}(t) = -1$ [Eq. (2.14)], $\vec{u}(t)\cdot\overrightarrow{O(t)M(t)} = 0$ [$M(t)\in\mathscr{E}_{\boldsymbol{u}}(t)$; cf. Sect. 3.2.3], $\vec{a}(t)\cdot\vec{u}(t) = 0$ [property (2.17) of the 4-acceleration]. Moreover, the term $c^2\,\mathrm{d}t\,\mathrm{d}t'\vec{a}(t)\cdot\vec{u}'(t)$ is of second order in $\mathrm{d}t$. At first order in $\mathrm{d}t$, there remains then

$$c\,\mathrm{d}t + c\,\mathrm{d}t'\,\vec{u}\cdot\vec{u}' + c\,\mathrm{d}t\,\vec{a}\cdot\overrightarrow{OM} = 0.$$

In the above writing, all vectors have to be considered at time t so that the explicit mention of t has been omitted. Substituting expression (4.1) for $\mathrm{d}t$, we get finally

$$\boxed{\Gamma = -\frac{\vec{u}\cdot\vec{u}'}{1+\vec{a}\cdot\overrightarrow{OM}}}. \tag{4.9}$$

Notice that in the cases where (i) $\mathscr{P}$ intersects $\mathscr{O}$'s worldline ($\overrightarrow{OM} = 0$) or (ii) $\mathscr{O}$ has a vanishing 4-acceleration (for instance, $\mathscr{O}$ is an inertial observer), the above expression reduces to

$$\boxed{\Gamma = -\vec{u}\cdot\vec{u}'}_{M\in\mathscr{L}\ \text{or}\ \vec{a}=0}. \tag{4.10}$$

Thus, from a geometrical point of view, the Lorentz factor between two observers whose worldlines are crossing each other is nothing but minus the scalar product of their 4-velocities.

Remark 4.1. The expression (4.10) of Γ is symmetric with respect to $\mathscr{O}$ and $\mathscr{P}$, but not (4.9). In other words, if the worldlines of $\mathscr{O}$ and $\mathscr{P}$ intersect themselves, the Lorentz factor of $\mathscr{P}$ with respect to $\mathscr{O}$ is the same as the Lorentz factor of $\mathscr{O}$ with respect to $\mathscr{P}$.

Example 4.4. Let us consider Example 4.2 again, namely, Langevin's traveller. Since $\mathscr{O}$'s 4-acceleration vanishes, formula (4.10) holds. Moreover, the 4-velocity $\vec{u}$ coincides with the vector $\vec{e}_0$ of the orthonormal basis introduced in Sect. 2.6. Formula (4.10) reduces then to $\Gamma = -\vec{e}_0\cdot\vec{u}' = -\vec{e}_0\cdot(u'^{\alpha}\vec{e}_{\alpha}) = -u'^{\alpha}\eta_{0\alpha} = -u'^{0}(-1)$, i.e.

$$\Gamma = u'^0. \tag{4.11}$$

As a check, using expression (2.30a) of u'^0, we recover (4.4). Moreover (2.32a) leads to an alternative expression of the Lorentz factor, in terms of $\mathscr{O}'$'s proper time:

$$\Gamma = \cosh\left[\frac{4}{T_*}\left(t' - \frac{k}{2}T'\right)\right], \tag{4.12}$$

where use has been made of (2.43). Γ reaches its maximal value at C_1 ($t' = T'/4, k = 0$) and C_2 ($t' = 3T'/4, k = 1$), which is

$$\Gamma_{\max} = \cosh(T'/T_*). \tag{4.13}$$

This perfectly agrees with the result (4.5), given the relations $T/T_* = \sinh(T'/T_*)$ [Eq. (2.43)] and $\cosh x = \sqrt{1 + \sinh^2 x}$.

4.2.3 Time Dilation

A lower bound on the Lorentz factor can be easily inferred from relation (4.10). Indeed, let us introduce the components of $\mathscr{P}$'s 4-velocity with respect to $\mathscr{O}$'s local frame $(\vec{\boldsymbol{e}}_\alpha) = (\vec{\boldsymbol{e}}_0 = \vec{\boldsymbol{u}}, \vec{\boldsymbol{e}}_i)$ (cf. Sect. 3.4.1): $\vec{\boldsymbol{u}}' = u'^0 \vec{\boldsymbol{u}} + u'^i \vec{\boldsymbol{e}}_i$. Since $(\vec{\boldsymbol{e}}_\alpha)$ is an orthonormal basis, (4.10) yields

$$\Gamma = -\vec{\boldsymbol{u}} \cdot (u'^0 \vec{\boldsymbol{u}} + u'^i \vec{\boldsymbol{e}}_i) = -u'^0 \underbrace{\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{u}}}_{-1} - u'^i \underbrace{\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{e}}_i}_{0} = u'^0. \tag{4.14}$$

On the other side, the constraint $\vec{\boldsymbol{u}}' \cdot \vec{\boldsymbol{u}}' = -1$, which any 4-velocity must obey, can be written (thanks to the orthonormality of $(\vec{\boldsymbol{e}}_\alpha)$):

$$\vec{\boldsymbol{u}}' \cdot \vec{\boldsymbol{u}}' = -(u'^0)^2 + \sum_{i=1}^{3} (u'^i)^2 = -1.$$

Since $u'^0 > 0$ (for $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{u}}'$ are both future-directed), we get

$$u'^0 = \sqrt{1 + \sum_{i=1}^{3} (u'^i)^2}.$$

The right-hand side is manifestly larger than or equal to 1. Therefore (4.14) allows one to conclude

$$\boxed{\Gamma \geq 1}_{M \in \mathscr{L} \text{ or } \vec{\boldsymbol{a}} = 0}, \tag{4.15}$$

the equality being achieved iff $u'^i = 0$, i.e. iff $\vec{\boldsymbol{u}}' = \vec{\boldsymbol{u}}$, i.e. iff $\mathscr{P}$ coincides with $\mathscr{O}$.

Given the definition (4.1) of the Lorentz factor as a ratio between two proper times, the property $\Gamma \geq 1$ is named ***time dilation***:

$$\boxed{\mathrm{d}t \geq \mathrm{d}t'}_{M \in \mathscr{L} \text{ or } \vec{a}=0} \quad . \tag{4.16}$$

In other words, when $\mathscr{P}$'s worldline crosses that of $\mathscr{O}$ or when $\mathscr{O}$ has a vanishing 4-acceleration, the proper time measured by $\mathscr{O}$ between the events $M(t)$ and $M(t + \mathrm{d}t)$ on $\mathscr{P}$'s worldline is larger or equal to the proper time measured by $\mathscr{P}$ himself between these two events.

We are facing again the relativity of time, already discussed in Chap. 2 with the twin paradox. Note however a difference: in Chap. 2, only times of events occurring along the worldline of an observer were involved. Here, thanks to the definition of simultaneity (Chap. 3), we are dealing with time intervals between events away from observer $\mathscr{O}$'s worldline.

Remark 4.2. Contrary to what Fig. 4.1 might suggest, we do have $\mathrm{d}t \geq \mathrm{d}t'$, and not $\mathrm{d}t \leq \mathrm{d}t'$. Indeed, one shall not forget that $\mathrm{d}t$ and $\mathrm{d}t'$ are the lengths $O(t)O(t + \mathrm{d}t)$ and $M(t)M(t + \mathrm{d}t)$ with respect to the metric tensor; they therefore do not correspond to the Euclidean lengths of the segments drawn in Fig. 4.1.

Example 4.5. The Lorentz factor of Example 4.1 (uniform linear motion) satisfies clearly $\Gamma \geq 1$ [cf. Eq. (4.3)]. We have even $\Gamma > 1$ if $v \neq 0$.

Example 4.6. On each of the expressions (4.4) and (4.12) obtained above for the Lorentz factor of Langevin's traveller, it is obvious that $\Gamma \geq 1$, in full agreement with (4.15). Moreover, it is worth noticing that formula (2.29) obtained in Sect. 2.6.2 implies $T \geq T'$, which is an integrated form of (4.16).

Example 4.7. Similarly, in Example 4.3 (uniform circular motion), we have clearly $\Gamma \geq 1$ [cf. Eq. (4.7)] and even $\Gamma > 1$ if $R \neq 0$ and $\Omega \neq 0$.

4.3 Velocity Relative to an Observer

4.3.1 Definition

For each instant of $\mathscr{O}$'s proper time t, the position of particle $\mathscr{P}$ is marked in $\mathscr{O}$'s local frame $(\vec{e}_\alpha(t))$ by three real numbers $(x^1(t), x^2(t), x^3(t))$ such that (cf. Sect. 3.4.2)

$$\overrightarrow{O(t)M(t)} = x^i(t)\, \vec{e}_i(t). \tag{4.17}$$

The motion of $\mathscr{P}$ with respect to $\mathscr{O}$ is thus defined by the "position vector" $\vec{x}(t) = x^i(t)\,\vec{e}_i$ in $\mathscr{O}$'s reference space $R_{\mathscr{O}}$ (cf. Sect. 3.4.3). It is then natural to define the ***velocity of particle $\mathscr{P}$ relative to observer $\mathscr{O}$*** as the derivative of the vector $\vec{x}(t)$ with respect to the proper time t:

$$\vec{V} := \frac{\mathrm{d}\vec{x}}{\mathrm{d}t}, \tag{4.18}$$

The vector $\vec{V}$ belongs to the reference space $R_{\mathscr{O}}$. Via the correspondence (3.27), $\vec{V}$ can be identified to the following vector in the local rest space $E_{\boldsymbol{u}}(t)$:

$$\boxed{\vec{\boldsymbol{V}}(t) := \frac{\mathrm{d}x^i}{\mathrm{d}t}\,\vec{\boldsymbol{e}}_i(t)}. \tag{4.19}$$

Given that the components of the vector $\overrightarrow{O(t)M(t)}$ in $\mathscr{O}$'s local frame are $\underrightarrow{(0, x^i(t))}$ [Eq. (4.17)], we may say that $\vec{\boldsymbol{V}}(t)$ is nothing but the derivative of $\overrightarrow{O(t)M(t)}$ with respect to observer $\mathscr{O}$, as defined in Sect. 3.6.2:

$$\boxed{\vec{\boldsymbol{V}}(t) = \boldsymbol{D}_{\mathscr{O}}\,\overrightarrow{O(t)M(t)}}. \tag{4.20}$$

Example 4.8. For Example 4.1, the equation of motion (4.2) leads immediately to

$$\vec{\boldsymbol{V}}(t) = v\,\vec{\boldsymbol{e}}_1, \tag{4.21}$$

that is to say, to a constant velocity along the x-axis (cf. Fig. 4.2).

Example 4.9. Let us proceed with Langevin's traveller (Example 4.2). The local frame of the "sedentary" twin $\mathscr{O}$ coincides with the constant orthonormal basis $(\vec{\boldsymbol{e}}_0, \vec{\boldsymbol{e}}_1, \vec{\boldsymbol{e}}_2, \vec{\boldsymbol{e}}_3)$ introduced in Sect. 2.6.1. The motion of Langevin's traveller is governed by

$$\overrightarrow{OM(t)} = ct\,\vec{\boldsymbol{e}}_0 + x(t)\,\vec{\boldsymbol{e}}_1,$$

where $O := A$ and the function $x(t)$ is defined by (2.20). The definition (4.19) leads then to the velocity $\vec{\boldsymbol{V}}(t) = \mathrm{d}x/\mathrm{d}t\;\vec{\boldsymbol{e}}_1$, with $\mathrm{d}x/\mathrm{d}t$ given by (2.23); hence,

$$\vec{\boldsymbol{V}}(t) = (-1)^k \frac{\gamma\,(t - kT/2)}{\sqrt{1 + \gamma^2\,(t - kT/2)^2/c^2}}\,\vec{\boldsymbol{e}}_1. \tag{4.22}$$

The velocity of $\mathscr{O}'$ relative to $\mathscr{O}$ is therefore collinear to $\vec{\boldsymbol{e}}_1$ (cf. Fig. 4.3), in agreement with the unidirectional motion of Langevin's traveller.

Example 4.10. In the case of Example 4.3, the equation of motion (4.6) yields

$$\vec{\boldsymbol{V}}(t) = -R\Omega\sin\Omega t\;\vec{\boldsymbol{e}}_1 + R\Omega\cos\Omega t\;\vec{\boldsymbol{e}}_2. \tag{4.23}$$

Note that in the general case $\vec{V} \neq \mathrm{d}\overrightarrow{OM}/\mathrm{d}t$, where $\mathrm{d}\overrightarrow{OM}/\mathrm{d}t$ is the derivative of the vector field $\overrightarrow{O(t)M(t)}$ along $\mathscr{O}$'s worldline, in the sense defined in Sects. 2.7.2 and 3.6.1. Indeed, $\vec{V}$ is the derivative of $\overrightarrow{OM}$ *with respect to observer* $\mathscr{O}$ [Eq. (4.20)], and we have seen in Sect. 3.6.2 that the derivatives $\boldsymbol{D}_{\mathscr{O}}$ and $\mathrm{d}/\mathrm{d}t$ are related by (3.67). Since in the present case $\vec{\boldsymbol{u}} \cdot \overrightarrow{OM} = 0$, this formula leads to

$$\boxed{\vec{V} = \frac{\mathrm{d}\overrightarrow{OM}}{\mathrm{d}t} - c(\vec{\boldsymbol{a}} \cdot \overrightarrow{OM})\,\vec{\boldsymbol{u}} - \vec{\boldsymbol{\omega}} \times_u \overrightarrow{OM}}. \tag{4.24}$$

This expression can be rewritten to let appear the Fermi–Walker derivative of $\overrightarrow{OM}$ along $\mathscr{O}$'s worldline (cf. Sect. 3.6.3 and Eq. (3.69) with $\vec{\boldsymbol{u}} \cdot \overrightarrow{OM} = 0$):

$$\boxed{\boldsymbol{D}^{\mathrm{FW}}_{\boldsymbol{u}}\,\overrightarrow{OM} = \vec{V} + \vec{\boldsymbol{\omega}} \times_u \overrightarrow{OM}}. \tag{4.25}$$

4.3.2 4-Velocity and Lorentz Factor in Terms of the Velocity

Let us write

$$\begin{aligned}
\mathrm{d}\overrightarrow{OM} &= \overrightarrow{O(t+\mathrm{d}t)M(t+\mathrm{d}t)} - \overrightarrow{O(t)M(t)} \\
&= \overrightarrow{O(t+\mathrm{d}t)O(t)} + \overrightarrow{O(t)M(t)} + \overrightarrow{M(t)M(t+\mathrm{d}t)} - \overrightarrow{O(t)M(t)} \\
&= \overrightarrow{O(t+\mathrm{d}t)O(t)} + \overrightarrow{M(t)M(t+\mathrm{d}t)}.
\end{aligned} \tag{4.26}$$

Since $\overrightarrow{O(t+\mathrm{d}t)O(t)} = -c\,\mathrm{d}t\,\vec{\boldsymbol{u}}$ and $\overrightarrow{M(t)M(t+\mathrm{d}t)} = c\,\mathrm{d}t'\,\vec{\boldsymbol{u}}' = c\Gamma^{-1}\mathrm{d}t\,\vec{\boldsymbol{u}}'$, we get

$$\frac{\mathrm{d}\overrightarrow{OM}}{\mathrm{d}t} = c\left(\Gamma^{-1}\vec{\boldsymbol{u}}' - \vec{\boldsymbol{u}}\right).$$

Combining this relation with (4.24), we obtain an expression of $\mathscr{P}$'s 4-velocity $\vec{\boldsymbol{u}}'$ in terms of quantities relative to $\mathscr{O}$:

$$\boxed{\vec{\boldsymbol{u}}' = \Gamma\left[(1 + \vec{\boldsymbol{a}} \cdot \overrightarrow{OM})\,\vec{\boldsymbol{u}} + \frac{1}{c}\left(\vec{V} + \vec{\boldsymbol{\omega}} \times_u \overrightarrow{OM}\right)\right]}. \tag{4.27}$$

From a geometrical point of view, this relation constitutes the orthogonal decomposition of the 4-velocity $\vec{\boldsymbol{u}}'$ in a part along $\vec{\boldsymbol{u}}$ [the term $\Gamma(1 + \vec{\boldsymbol{a}} \cdot \overrightarrow{OM})\,\vec{\boldsymbol{u}}$] and a part in the vector hyperplane E_u normal to $\vec{\boldsymbol{u}}$ [the term $\Gamma/c\,(\vec{V} + \vec{\boldsymbol{\omega}} \times_u \overrightarrow{OM})$].

Substituting (4.9) for Γ in (4.27), we get

$$\vec{\boldsymbol{u}}' = -(\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{u}}')\,\vec{\boldsymbol{u}} + \frac{\Gamma}{c}\left(\vec{V} + \vec{\boldsymbol{\omega}} \times_u \overrightarrow{OM}\right).$$

We deduce from this an expression of the relative velocity $\vec{V}$ in terms of the 4-velocity $\vec{u}'$, the Lorentz factor Γ and the 4-rotation $\vec{\omega}$ of observer $\mathscr{O}$:

$$\boxed{\vec{V} = \frac{c}{\Gamma}\perp_u \vec{u}' - \vec{\omega} \times_u \overrightarrow{OM}}, \tag{4.28}$$

where $\perp_u$ stands for the orthogonal projector onto the vector hyperplane E_u normal to $\vec{u}$: $\perp_u := \mathrm{Id} + \langle \underline{u}, . \rangle\, \vec{u}$ (cf. Sect. 3.2.5).

Let us now insert expression (4.27) of $\vec{u}'$ into the 4-velocity normalization relation $\vec{u}' \cdot \vec{u}' = -1$. Expanding and using the orthogonality of vectors $\vec{V}$ and $\vec{\omega} \times_u \overrightarrow{OM}$ with $\vec{u}$, we get

$$-1 = \Gamma^2 \left[-(1 + \vec{a} \cdot \overrightarrow{OM})^2 + \frac{1}{c^2} \left(\vec{V} + \vec{\omega} \times_u \overrightarrow{OM} \right) \cdot \left(\vec{V} + \vec{\omega} \times_u \overrightarrow{OM} \right) \right], \tag{4.29}$$

hence

$$\boxed{\Gamma = \left[(1 + \vec{a} \cdot \overrightarrow{OM})^2 - \frac{1}{c^2} \left(\vec{V} + \vec{\omega} \times_u \overrightarrow{OM} \right) \cdot \left(\vec{V} + \vec{\omega} \times_u \overrightarrow{OM} \right) \right]^{-1/2}}. \tag{4.30}$$

Remark 4.3. Even if $\vec{V} = 0$ (particle $\mathscr{P}$ fixed with respect to $\mathscr{O}$), (4.30) shows that one may have $\Gamma \neq 1$, provided that $\vec{a} \neq 0$ (accelerated observer) or $\vec{\omega} \neq 0$ (rotating observer). We shall elaborate on this point in Chaps. 12 and 13.

In the cases where (i) $\mathscr{P}$ intersects $\mathscr{O}$'s worldline ($\overrightarrow{OM} = 0$) or (ii) $\mathscr{O}$ is an inertial observer (which implies $\vec{a} = 0$ and $\vec{\omega} = 0$; cf. Sect. 3.5.4), expressions (4.27), (4.28) and (4.30) reduce to

$$\boxed{\vec{u}' = \Gamma \left(\vec{u} + \frac{1}{c} \vec{V} \right)}_{M \in \mathscr{L} \text{ or } \mathscr{O} \text{ inertial}} \tag{4.31}$$

$$\boxed{\vec{V} = \frac{c}{\Gamma} \perp_u \vec{u}'}_{M \in \mathscr{L} \text{ or } \mathscr{O} \text{ inertial}} \tag{4.32}$$

$$\boxed{\Gamma = \left(1 - \frac{1}{c^2} \vec{V} \cdot \vec{V} \right)^{-1/2}.}_{M \in \mathscr{L} \text{ or } \mathscr{O} \text{ inertial}} \tag{4.33}$$

The vectors $\vec{u}$, $\vec{u}'$ and $\vec{V}$ are depicted in Fig. 4.5.

Remark 4.4. We recover on (4.33) that $\Gamma \geq 1$ [Eq. (4.15)], whatever the relative velocity $\vec{V}$, since $\vec{V} \cdot \vec{V} \geq 0$, $\vec{V}$ being spacelike.

Example 4.11. For the uniform linear motion of Example 4.1, (4.31) combined with the expression (4.21) of the relative velocity leads to

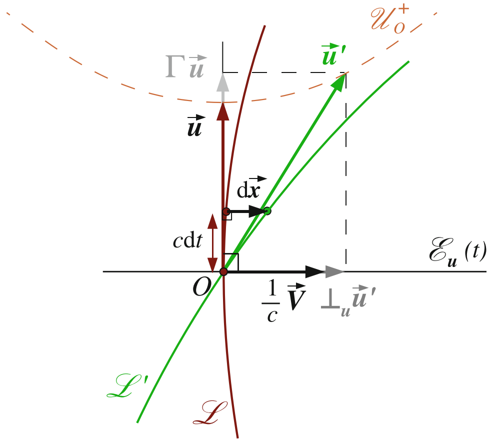

Fig. 4.5 Motion of particle $\mathscr{P}$ (worldline $\mathscr{L}'$ and 4-velocity $\vec{u}'$) with respect to observer $\mathscr{O}$ (worldline $\mathscr{L}$ and 4-velocity $\vec{u}$): case where the two worldlines cross (event O). $\mathscr{U}_O^+$ is the hyperboloid introduced in Sect. 1.4.3. The velocity $\vec{V}$ of $\mathscr{P}$ relative to $\mathscr{O}$ is, up to a Lorentz factor, the orthogonal projection of $\vec{u}'$ onto $\mathscr{O}$'s local rest space $\mathscr{E}_u$: $\vec{V} = c\Gamma^{-1}\perp_u \vec{u}'$

$$\vec{u}' = \Gamma\left(\vec{e}_0 + \frac{v}{c}\vec{e}_1\right), \tag{4.34}$$

where Γ is the function of v given by (4.3). Since v is constant, the vector $\vec{u}'$ is constant along $\mathscr{L}'$; it is depicted in Fig. 4.2.

Example 4.12. The 4-velocity of Langevin's traveller considered in Sect. 2.6 is given by (2.30) :

$$\vec{u}' = \sqrt{1 + \frac{\gamma^2}{c^2}\left(t - \frac{k}{2}T\right)^2}\,\vec{e}_0 + (-1)^k\frac{\gamma}{c}\left(t - \frac{k}{2}T\right)\vec{e}_1. \tag{4.35}$$

Since $\mathscr{O}$'s 4-velocity is $\vec{u} = \vec{e}_0$, it follows that the orthogonal projection of $\vec{u}'$ onto E_u is

$$\perp_u \vec{u}' = (-1)^k\frac{\gamma}{c}\left(t - \frac{k}{2}T\right)\vec{e}_1.$$

In view of expressions (4.4) and (4.22) for, respectively, Γ and $\vec{V}$, we observe that $\perp_u \vec{u}' = (\Gamma/c)\vec{V}$. In other words, (4.32) is fulfilled. Equation (4.35), once combined with (4.4), gives then (4.31). We check similarly relation (4.33) from (4.4) and (4.22).

Example 4.13. For the uniform circular motion considered in Example 4.3, Eq. (4.31), combined with the expression (4.23) of the relative velocity, yields

$$\vec{u}' = \Gamma\left(\vec{e}_0 - \frac{R\Omega}{c}\sin\Omega t\;\vec{e}_1 + \frac{R\Omega}{c}\cos\Omega t\;\vec{e}_2\right), \tag{4.36}$$

where Γ is the function of R and Ω given by (4.7). The vector $\vec{u}'$ is depicted in Fig. 4.4.

4.3.3 Maximum Relative Velocity

Let us consider one of the two cases mentioned above:

(i) The worldline of $\mathscr{P}$ crosses that of $\mathscr{O}$ at proper time t ($\overrightarrow{OM} = 0$).
(ii) $\mathscr{O}$ is an inertial observer ($\vec{\boldsymbol{a}} = 0$ and $\vec{\boldsymbol{\omega}} = 0$).

Formula (4.29), which reflects the normalization of $\mathscr{P}$'s 4-velocity $\vec{\boldsymbol{u}}'$, reduces then to

$$-1 = \Gamma^2 \left(-1 + \frac{1}{c^2}\vec{\boldsymbol{V}} \cdot \vec{\boldsymbol{V}}\right).$$

One deduces immediately that $-1 + c^{-2}\vec{\boldsymbol{V}} \cdot \vec{\boldsymbol{V}} < 0$, i.e.

$$\boxed{\left\|\vec{\boldsymbol{V}}\right\|_{\boldsymbol{g}} < c}, \tag{4.37}$$

where $\|\vec{\boldsymbol{V}}\|_{\boldsymbol{g}} := \sqrt{\vec{\boldsymbol{V}} \cdot \vec{\boldsymbol{V}}}$ is the norm of the velocity vector with respect to the metric tensor $\boldsymbol{g}$ (cf. Sect. 1.3.5). We may thus state:

> The constant c introduced in Sect. 1.2.4 is a strict upper bound for the relative velocity of any massive particle observed locally ($\overrightarrow{OM} = 0$) by a given observer. Moreover, for inertial observers, this result can be extended to distant observations ($\overrightarrow{OM} \neq 0$).

Example 4.14. For the uniform linear motion of Example 4.1, (4.21) yields $\|\vec{\boldsymbol{V}}\|_{\boldsymbol{g}} = v$, so that (4.37) is fulfilled, in view of the assumption on the parameter v.

Example 4.15. Let us check (4.37) for the Langevin's traveller. The norm of $\vec{\boldsymbol{V}}$ can be read directly on (4.22), since $\vec{\boldsymbol{e}}_1$ is a unit vector. $\|\vec{\boldsymbol{V}}\|_{\boldsymbol{g}}$ reaches its maximum at the points of thrust reversing C_1 ($t = T/4$, $k = 0$) and C_2 ($t = 3T/4$, $k = 1$):

$$\max \left\|\vec{\boldsymbol{V}}\right\|_{\boldsymbol{g}} = \frac{\gamma T/4}{\sqrt{1 + (\gamma T/4c)^2}} = c\,\frac{T/T_*}{\sqrt{1 + (T/T_*)^2}} = c\,\frac{\alpha/4}{\sqrt{1 + (\alpha/4)^2}}.$$

It is clear on that expression that $\max \|\vec{\boldsymbol{V}}\|_{\boldsymbol{g}} < c$ and that

$$\lim_{\gamma T \to \infty} \max \left\|\vec{\boldsymbol{V}}\right\|_{\boldsymbol{g}} = \lim_{T/T_* \to \infty} \max \left\|\vec{\boldsymbol{V}}\right\|_{\boldsymbol{g}} = \lim_{\alpha \to \infty} \max \left\|\vec{\boldsymbol{V}}\right\|_{\boldsymbol{g}} = c.$$

This last result is in agreement with Fig. 2.5, which shows that for $\alpha \to \infty$, the worldline of Langevin's traveller approaches that of a photon.

Example 4.16. For the uniform circular motion of Example 4.3, Eq. (4.23) results in $\|\vec{V}\|_g = R\Omega$, so that (4.37) is fulfilled, given the assumptions on the parameters R and Ω.

Remark 4.5. For the hypothetical particles of the tachyon class, which move on spacelike worldlines instead of timelike ones (cf. Remark 2.3 p. 30), we would find $\|\vec{V}\|_g > c$ instead of (4.37). The velocity of light is thus the *minimum* velocity for a tachyon with respect to an "ordinary" (i.e. timelike) observer. However, one can show that tachyons cannot be used to transmit information faster than light between two observers (Feinberg 1967; Recami 1987). Moreover, the presence of tachyons is a source of instabilities in quantum field theory. Let us stress that, to date, there is no experimental evidence of the existence of tachyons.

4.3.4 Component Expressions

$(\vec{e}_\alpha(t))$ being observer $\mathscr{O}$'s local frame, as defined in Sect. 3.4.1, we have $\vec{e}_0 = \vec{u}$. The components of $\vec{u}$ with respect to the basis $(\vec{e}_\alpha(t))$ are thus simply

$$u^\alpha = (1, 0, 0, 0). \tag{4.38}$$

The components of the velocity $\vec{V}$ of particle $\mathscr{P}$ relative to $\mathscr{O}$ in the same frame are by definition [cf. Eq. (4.19)]

$$V^\alpha = (0, V^1, V^2, V^3) \quad \text{with} \quad V^i = \frac{\mathrm{d}x^i}{\mathrm{d}t}, \tag{4.39}$$

where $x^i = x^i(t)$ is the equation of $\mathscr{P}$'s trajectory in $\mathscr{O}$'s reference frame. Since the basis $(\vec{e}_\alpha)$ is orthonormal, we have $\vec{V} \cdot \vec{V} = (V^i \vec{e}_i) \cdot (V^j \vec{e}_j) = V^i V^j \, \vec{e}_i \cdot \vec{e}_j = V^i V^j \eta_{ij} = V^i V^j \delta_{ij} = \sum_{i=1}^3 (V^i)^2$. In particular, the norm of $\vec{V}$ introduced above can be expressed as

$$\left\|\vec{V}\right\|_g = \sqrt{(V^1)^2 + (V^2)^2 + (V^3)^2}. \tag{4.40}$$

Let us focus on one of the cases (i) or (ii) of Sect. 4.3.3, namely, the cases for which (4.31)–(4.33) hold. The expression (4.33) for the Lorentz factor becomes then

$$\boxed{\Gamma = \left[1 - \frac{1}{c^2} \sum_{i=1}^3 (V^i)^2\right]^{-1/2}.}_{M\in\mathscr{L} \text{ or } \mathscr{O} \text{ inertial}} \tag{4.41}$$

The components of $\mathscr{P}$'s 4-velocity are immediately deduced from (4.31), (4.38) and (4.39) :

$$\boxed{u'^{\alpha} = \left(\Gamma,\ \Gamma\frac{V^1}{c},\ \Gamma\frac{V^2}{c},\ \Gamma\frac{V^3}{c}\right).}_{M\in\mathscr{L} \text{ or } \mathscr{O} \text{ inertial}} \tag{4.42}$$

Example 4.17. For Langevin's traveller ($\mathscr{O}$ is then inertial), the above formulas are easy to check. Indeed, the components of $\vec{u}'$ are $u'^{\alpha} = (u'^0, u'^1, 0, 0)$, with u'^0 and u'^1 read on (4.35). The components of $\vec{V}$ are $V^i = (V^1, 0, 0)$, with V^1 given by (4.22). We have already observed that $u'^0 = \Gamma$ [Eq. (4.11)], and we verify easily that $u'^1 = \Gamma V^1/c$.

Historical note: *The Lorentz factor, as given by (4.41) and appearing as a proportionality factor between two times, has been introduced in 1904 by Hendrik A. Lorentz*[1] *(1904). This factor actually appears in many anterior works by Lorentz, dating back to 1895 (Lorentz 1895), but only as the ratio between two lengths. Let us stress that in Lorentz's 1904 interpretation, the two times whose ratio is Γ do not have the same status: one is the "real" time, and the other is named* "local time" *by Lorentz. It is only with the 1905 breakthrough of Albert Einstein (cf. p. 26) (Einstein 1905b) that the Lorentz factor has acquired its current signification, as the ratio of two times, one as physical as the other. The phenomenon of time dilation (Sect. 4.2.3) has been clearly described in Einstein's article (Einstein 1905b) as an effect truly measurable by means of clocks.*

The fact that the velocity of light is a limit speed for material bodies has been enounced by Henri Poincaré (cf. p. 26) in 1904 (Poincaré 1904), as a consequence of the increase of the inertia of a body with its velocity (whereas the result obtained in Sect. 4.3.3 is purely kinematical).

4.4 Experimental Verifications of Time Dilation

We present below a few key experiments that confirmed time dilation. For a much more complete review, the reader is referred to Chap. 9 of Zhang's book (Zhang 1997) or to Will's review articles (Will 2006a,b).

4.4.1 Atmospheric Muons

The ***muon*** (symbol μ^-) is, with the electron and the tau, one of the three charged particles in the *lepton* family—those particles that are not subject to the strong interaction. The muon has the same electric charge as the electron but is 207 times

[1]**Hendrik A. Lorentz** (1853–1928): Dutch theoretical physicist, author of many works on electromagnetism, the electron theory and relativity; he received the 1902 Nobel Prize in Physics for the explanation of the Zeeman effect (cf. p. 146).

more massive. Contrary to the electron, it is unstable, and its mean lifetime is $\tau_0 = 2.2 \times 10^{-6}$s. The muon decays into an electron (e^-), a muon neutrino (ν_μ) and an electron antineutrino ($\bar{\nu}_e$):

$$\mu^- \longrightarrow e^- + \nu_\mu + \bar{\nu}_e.$$

Muons are constantly produced in the upper atmosphere by the interaction of cosmic rays (cf. p. 277) with the nitrogen and oxygen atoms.[2] They are created with a speed close to c, for the cosmic rays are ultra-relativistic, as we shall see in Sect. 9.2.3. In nonrelativistic physics, muons should not reach the ground because they travel only the distance $d = c\tau_0 \simeq 660$ m during their mean lifetime. However an appreciable muon flux is detected on the ground. The explanation is that the actual travelled distance is $d = c\tau$, with $\tau = \Gamma\tau_0$, where $\Gamma \gg 1$ is the Lorentz factor of muons with respect to the terrestrial observer.

In 1941, Bruno Rossi[3] and David B. Hall have compared the muon fluxes in detectors located at two different points of Colorado State: Echo Lake (altitude $z_1 = 3240$ m) and Denver (altitude $z_2 = 1616$ m) (Rossi and Hall 1941). The flux at $z = z_2$, corrected by the atmospheric absorption, turned out to be lower than that measured at $z = z_1$. Rossi and Hall deduced that muons are decaying between z_1 and z_2 and have been able to estimate the muon mean lifetime (up to 10% of its modern value). Moreover, by selecting the muons in two distinct linear momentum ranges, by means of lead plates of various thicknesses, Rossi and Hall have noted that the muons of lowest momentum [and hence of lowest velocity; cf. Eq. (9.17)] have, between z_1 and z_2, a larger decay rate than those of higher momentum. This is in qualitative agreement with the relation $\tau = \Gamma\tau_0$, with Γ being an increasing function of momentum (and thus of velocity). Unfortunately, Rossi and Hall could not perform a quantitative test because they had no precise measure of the momentum of the most rapid muons (only a lower bound).

The quantitative test was performed in 1963 by the American physicists David H. Frisch (1918–1991) and James H. Smith (1963). They measured the muon flux at the top of Mount Washington in the New Hampshire (altitude $z_1 = 1910$ m) and at Cambridge in Massachusetts (altitude $z_2 = 3$ m), taking care of selecting muons with a velocity[4] in the range $0.9950\,c \le V \le 0.9954\,c$ at $z = z_1$. The observed

[2]Incidentally, the muon has been discovered in this manner in 1937; cf. the historical note below.

[3]**Bruno Rossi** (1905–1993): Italian physicist, who moved to United States in 1939 to escape from the fascist regime. Specialist of cosmic rays, he was also a pioneer of X-ray astronomy in the 1960s. The *Rossi X-ray Timing Explorer (RXTE)* satellite (1995–2012) was named after him.

[4]Actually, the muon selection has been performed from their penetration depth in iron plates, which provides their energy E and, in a second stage, their velocity, by assuming the relation $E = mc^2/\sqrt{1 - V^2/c^2}$; cf. Eq. (9.16).

decay time of muons was $\tau = (8.8 \pm 0.8)\tau_0$, which is in good agreement with the Lorentz factor deduced from the velocity according to formula (4.33) : $\Gamma = (1 - V^2/c^2)^{-1/2} = 8.4 \pm 2.0$.

Historical note: *The first muons have been discovered in cosmic rays in 1937 by Carl D. Anderson*[5] *and his student Seth H. Neddermeyer (1907–1988) (Neddermeyer and Anderson 1937), as well as by Jabez C. Street (1906–1989) and Edward C. Stevenson (1937). But at that time, the new particle was mistaken for the meson predicted in 1935 by Hideki Yukawa,*[6] *as the vector of the interaction between protons and neutrons in the atomic nucleus. Ten years later, in 1947, three Italian physicists, Marcello Conversi (1917–1988), Ettore Pancini (1915–1981) and Oreste Piccioni (1915–2002) (Conversi et al. 1947), have shown that the particle observed in 1937 could not be the meson, because it was interacting too weakly with nuclei. The first true meson was discovered the same year, as a* ***pi meson*** *or* ***pion****, by the Brazilian Cesare Lattes (1924–2005), the Italian Giuseppe Occhialini*[7] *(1907–1993) and the British Cecil F. Powell (1903–1969) (Lattes et al. 1947).*

4.4.2 Other Tests

Muons can be produced in particle accelerators by the decay of pions (π^+ or π^-). The pions, which are created by bombarding a target with high-energy protons, have a mean lifetime of 2.6×10^{-8}s. They decay into muons (or antimuons μ^+) according to $\pi^- \to \mu^- + \bar{\nu}_\mu$ and $\pi^+ \to \mu^+ + \nu_\mu$. The muons hence produced are then directed to a storage ring (cf Sect. 17.5.5). By studying the muon decay in such a ring at CERN, the time dilation corresponding to a Lorentz factor $\Gamma = 29.3$ ($V = 0.9994\,c$) has been checked with a relative accuracy of 10^{-3} (Bailey et al. 1979).

Another famous experiment has been put forward as a test of time dilation: that of Ives and Stilwell (1938). Since it relies on the Doppler effect, we shall rather discuss it in Sect. 5.5. Finally, let us mention that the experiments by Hafele and Keating (1971) and by Alley (1975) described in Sect. 2.6.6 provide some experimental support to time dilation as well.

[5]**Carl D. Anderson** (1905–1991): American physicist working at Caltech, who discovered two elementary particles: the positron (the electron antiparticle) in 1932 (Anderson 1932, 1933), for which he received the Nobel Prize in Physics in 1936, and the muon in 1937.

[6]**Hideki Yukawa** (1907–1981): Japanese theoretical physicist, pioneer of particle physics; he was awarded the Nobel Prize in Physics in 1949 (the first Japanese one!) for his prediction of the meson.

[7]The satellite *BeppoSAX* (1996–2003), dedicated to X-ray astronomy, has been named to honour him, *Beppo* being the nickname of Giuseppe.

4.5 Acceleration Relative to an Observer

4.5.1 Definition

In Sect. 4.3.1, we have defined the velocity of the point particle $\mathscr{P}$ relative to an observer $\mathscr{O}$ as the first derivative of the position vector $\vec{x}(t)$ of $\mathscr{P}$ in the reference space $R_{\mathscr{O}}$. It is then natural to define the ***acceleration of $\mathscr{P}$ relative to $\mathscr{O}$*** as the second derivative of the position vector:

$$\vec{\gamma} := \frac{\mathrm{d}^2\vec{x}}{\mathrm{d}t^2}. \tag{4.43}$$

Let us recall that, in this expression, t is $\mathscr{O}$'s proper time. Thanks to the correspondence (3.27), $\vec{\gamma}$ is identified to the following vector of the local rest space $E_{\boldsymbol{u}}(t)$:

$$\boxed{\vec{\boldsymbol{\gamma}}(t) := \frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2}\,\vec{\boldsymbol{e}}_i(t)}. \tag{4.44}$$

Example 4.18. Considering Example 4.1 of Sect. 4.2.1, the equation of motion (4.2) leads immediately to $\vec{\boldsymbol{\gamma}}(t) = 0$, in agreement with the concept of uniform linear motion.

Example 4.19. Regarding Example 4.2 (Langevin's traveller), the acceleration of the travelling twin relative to the "sedentary" twin $\mathscr{O}$ obeys $\vec{\boldsymbol{\gamma}} = \mathrm{d}^2x/\mathrm{d}t^2\,\vec{\boldsymbol{e}}_1$, with $\mathrm{d}^2x/\mathrm{d}t^2$ given by (2.40); hence,

$$\vec{\boldsymbol{\gamma}}(t) = (-1)^k\gamma\left[1 + \frac{\gamma^2}{c^2}\left(t - \frac{k}{2}T\right)^2\right]^{-3/2}\vec{\boldsymbol{e}}_1. \tag{4.45}$$

Note that the norm of $\vec{\boldsymbol{\gamma}}$ is not equal to the parameter γ appearing in the right-hand side of the above formula, except for $t = 0$, $T/2$ or T (cf. Remark 2.18 p. 50).

Example 4.20. Regarding Example 4.3 of Sect. 4.2.1, the equation of motion (4.6) leads to

$$\vec{\boldsymbol{\gamma}}(t) = -R\Omega^2\cos\Omega t\;\vec{\boldsymbol{e}}_1 - R\Omega^2\sin\Omega t\;\vec{\boldsymbol{e}}_2 = -\Omega^2\,\overrightarrow{O(t)M(t)}. \tag{4.46}$$

This relative acceleration vector is thus purely centripetal.

4.5.2 Relation to the Secondw Derivative of the Position Vector

According to definitions (4.19) and (4.44), $\vec{\boldsymbol{\gamma}}$ is the derivative of the vector $\vec{\boldsymbol{V}}$ with respect to observer $\mathscr{O}$ (cf. Sect. 3.6.2):

$$\boxed{\vec{\gamma} = \boldsymbol{D}_{\mathscr{O}}\vec{V} = \boldsymbol{D}_{\mathscr{O}}\boldsymbol{D}_{\mathscr{O}}\overrightarrow{OM}}, \tag{4.47}$$

where use has been made of (4.20) to write the second equality, having abridged the vector $\overrightarrow{O(t)M(t)}$ to $\overrightarrow{OM}$. Hence the relative acceleration $\vec{\gamma}$ is the second derivative of the vector $\overrightarrow{OM}$ *with respect to observer* $\mathscr{O}$.

Let us now relate $\vec{\gamma}$ to the *absolute* second derivative of the position vector, i.e. $\mathrm{d}^2\overrightarrow{OM}/\mathrm{d}t^2$. To this aim, let us start to express $\vec{\gamma}$ in terms of the Fermi–Walker derivative of $\vec{V}$, according to (3.70):

$$\vec{\gamma} = \boldsymbol{D}^{\mathrm{FW}}_{\boldsymbol{u}}\vec{V} - \vec{\omega} \times_u \vec{V}. \tag{4.48}$$

The vector $\vec{V}$ itself is related to the Fermi–Walker derivative of the position vector via (4.25). Taking another derivative of this formula, we get

$$\boldsymbol{D}^{\mathrm{FW}}_{\boldsymbol{u}}\boldsymbol{D}^{\mathrm{FW}}_{\boldsymbol{u}}\overrightarrow{OM} = \boldsymbol{D}^{\mathrm{FW}}_{\boldsymbol{u}}\vec{V} + \boldsymbol{D}^{\mathrm{FW}}_{\boldsymbol{u}}\left(\vec{\omega} \times_u \overrightarrow{OM}\right). \tag{4.49}$$

Since $\vec{\omega} \times_u \overrightarrow{OM}$ is by definition a vector belonging to $E_{\boldsymbol{u}}$, formula (3.73) holds:

$$\boldsymbol{D}^{\mathrm{FW}}_{\boldsymbol{u}}\left(\vec{\omega} \times_u \overrightarrow{OM}\right) = \perp_u\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\vec{\omega} \times_u \overrightarrow{OM}\right)\right] = \perp_u\left[\frac{\mathrm{d}}{\mathrm{d}t}\vec{\boldsymbol{\epsilon}}(\vec{u}, \vec{\omega}, \overrightarrow{OM}, .)\right], \tag{4.50}$$

where the second equality results from the definition (3.46) of the vector product. Since $\boldsymbol{\epsilon}$ is a constant multilinear form, the derivative of $\vec{\boldsymbol{\epsilon}}(\vec{u}, \vec{\omega}, \overrightarrow{OM}, .)$ is computed via Leibniz rule:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\vec{\boldsymbol{\epsilon}}(\vec{u}, \vec{\omega}, \overrightarrow{OM}, .) &= \vec{\boldsymbol{\epsilon}}\left(\frac{\mathrm{d}\vec{u}}{\mathrm{d}t}, \vec{\omega}, \overrightarrow{OM}, .\right) + \vec{\boldsymbol{\epsilon}}\left(\vec{u}, \frac{\mathrm{d}\vec{\omega}}{\mathrm{d}t}, \overrightarrow{OM}, .\right) \\ &\quad + \vec{\boldsymbol{\epsilon}}\left(\vec{u}, \vec{\omega}, \frac{\mathrm{d}\overrightarrow{OM}}{\mathrm{d}t}, .\right) \\ &= c\,\vec{\boldsymbol{\epsilon}}(\vec{a}, \vec{\omega}, \overrightarrow{OM}, .) + \frac{\mathrm{d}\vec{\omega}}{\mathrm{d}t} \times_u \overrightarrow{OM} + \vec{\omega} \times_u \frac{\mathrm{d}\overrightarrow{OM}}{\mathrm{d}t} \\ &= c\,\vec{\boldsymbol{\epsilon}}(\vec{a}, \vec{\omega}, \overrightarrow{OM}, .) + \frac{\mathrm{d}\vec{\omega}}{\mathrm{d}t} \times_u \overrightarrow{OM} \\ &\quad + \vec{\omega} \times_u \left(\vec{V} + \vec{\omega} \times_u \overrightarrow{OM}\right),\end{aligned} \tag{4.51}$$

where, to get the last line, we have used (4.24) and the fact that $\vec{\omega} \times_u \vec{u} = 0$. Let us evaluate the components of the vector $\vec{b} := \vec{\boldsymbol{\epsilon}}(\vec{a}, \vec{\omega}, \overrightarrow{OM}, .)$ onto the basis $(\vec{e}_\alpha)$. Since the latter is orthonormal, one has $b^0 = -\vec{e}_0 \cdot \vec{b} = -\vec{u} \cdot \vec{b} = -\boldsymbol{\epsilon}(\vec{a}, \vec{\omega}, \overrightarrow{OM}, \vec{u}) = \boldsymbol{\epsilon}(\vec{u}, \vec{\omega}, \overrightarrow{OM}, \vec{a}) = (\vec{\omega} \times_u \overrightarrow{OM}) \cdot \vec{a}$. On the other side, $b^i = \vec{e}_i \cdot \vec{b} = \boldsymbol{\epsilon}(\vec{a}, \vec{\omega}, \overrightarrow{OM}, \vec{e}_i)$. Now $\vec{a}$, $\vec{\omega}$, $\overrightarrow{OM}$ and $\vec{e}_i$ are four vectors in the vector space $E_{\boldsymbol{u}}$, which is three-dimensional. They are thus not linearly independent; $\boldsymbol{\epsilon}$ being a totally antisymmetric four-linear form, we get $b^i = 0$. Hence $\vec{\boldsymbol{b}} = b^0\,\vec{\boldsymbol{u}} = [\vec{\boldsymbol{a}}\cdot(\vec{\boldsymbol{\omega}} \times_u \overrightarrow{OM})]\,\vec{\boldsymbol{u}}$, i.e.

$$\vec{\boldsymbol{\epsilon}}(\vec{\boldsymbol{a}}, \vec{\boldsymbol{\omega}}, \overrightarrow{OM}, .) = [\vec{\boldsymbol{a}}\cdot(\vec{\boldsymbol{\omega}} \times_u \overrightarrow{OM})]\,\vec{\boldsymbol{u}}. \tag{4.52}$$

Substituting this expression in (4.51), we obtain the useful relation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\vec{\boldsymbol{\omega}} \times_u \overrightarrow{OM}\right) = \frac{\mathrm{d}\vec{\boldsymbol{\omega}}}{\mathrm{d}t} \times_u \overrightarrow{OM} + \vec{\boldsymbol{\omega}} \times_u \left(\vec{\boldsymbol{V}} + \vec{\boldsymbol{\omega}} \times_u \overrightarrow{OM}\right) + c[\vec{\boldsymbol{a}}\cdot(\vec{\boldsymbol{\omega}} \times_u \overrightarrow{OM})]\,\vec{\boldsymbol{u}}. \tag{4.53}$$

To use this result in (4.50), it must be projected onto $E_{\boldsymbol{u}}$. The last two terms, being vector products $\times_u$, are already in $E_{\boldsymbol{u}}$. The projection of the last term, which is collinear to $\vec{\boldsymbol{u}}$, results in zero. We obtain thus

$$\boldsymbol{D}^{\mathrm{FW}}_{\boldsymbol{u}}\left(\vec{\boldsymbol{\omega}} \times_u \overrightarrow{OM}\right) = \frac{\mathrm{d}\vec{\boldsymbol{\omega}}}{\mathrm{d}t} \times_u \overrightarrow{OM} + \vec{\boldsymbol{\omega}} \times_u \vec{\boldsymbol{V}} + \vec{\boldsymbol{\omega}} \times_u \left(\vec{\boldsymbol{\omega}} \times_u \overrightarrow{OM}\right). \tag{4.54}$$

Finally, the combination of (4.49), (4.48) and (4.54) yields

$$\boxed{\boldsymbol{D}^{\mathrm{FW}}_{\boldsymbol{u}}\boldsymbol{D}^{\mathrm{FW}}_{\boldsymbol{u}}\,\overrightarrow{OM} = \vec{\boldsymbol{\gamma}} + \vec{\boldsymbol{\omega}} \times_u \left(\vec{\boldsymbol{\omega}} \times_u \overrightarrow{OM}\right) + 2\vec{\boldsymbol{\omega}} \times_u \vec{\boldsymbol{V}} + \frac{\mathrm{d}\vec{\boldsymbol{\omega}}}{\mathrm{d}t} \times_u \overrightarrow{OM}}. \tag{4.55}$$

The term $\vec{\boldsymbol{\omega}} \times_u \left(\vec{\boldsymbol{\omega}} \times_u \overrightarrow{OM}\right)$ is called the ***centripetal acceleration*** and the term $2\vec{\boldsymbol{\omega}} \times_u \vec{\boldsymbol{V}}$ the ***Coriolis acceleration***.

Let us relate now the Fermi–Walker second derivative $\boldsymbol{D}^{\mathrm{FW}}_{\boldsymbol{u}}\boldsymbol{D}^{\mathrm{FW}}_{\boldsymbol{u}}\,\overrightarrow{OM}$ to the absolute second derivative of the position vector along $\mathscr{L}$, i.e. $\mathrm{d}^2\overrightarrow{OM}/\mathrm{d}t^2$. From the definition (3.69) of Fermi–Walker derivative, we get (using $\vec{\boldsymbol{u}}\cdot\overrightarrow{OM} = 0$)

$$\boldsymbol{D}^{\mathrm{FW}}_{\boldsymbol{u}}\overrightarrow{OM} = \frac{\mathrm{d}\overrightarrow{OM}}{\mathrm{d}t} - c(\vec{\boldsymbol{a}}\cdot\overrightarrow{OM})\,\vec{\boldsymbol{u}}$$

Using (3.69) once again, as well as the property $\boldsymbol{D}^{\mathrm{FW}}_{\boldsymbol{u}}\vec{\boldsymbol{u}} = 0$ [Eq. (3.72)], we obtain

$$\boldsymbol{D}^{\mathrm{FW}}_{\boldsymbol{u}}\boldsymbol{D}^{\mathrm{FW}}_{\boldsymbol{u}}\,\overrightarrow{OM} = \frac{\mathrm{d}^2\overrightarrow{OM}}{\mathrm{d}t^2} - c\left(\vec{\boldsymbol{a}}\cdot\frac{\mathrm{d}\overrightarrow{OM}}{\mathrm{d}t}\right)\vec{\boldsymbol{u}} + c\left(\vec{\boldsymbol{u}}\cdot\frac{\mathrm{d}\overrightarrow{OM}}{\mathrm{d}t}\right)\vec{\boldsymbol{a}}$$
$$-c\left[\frac{\mathrm{d}}{\mathrm{d}t}(\vec{\boldsymbol{a}}\cdot\overrightarrow{OM})\right]\vec{\boldsymbol{u}}.$$

Now, from (4.24),

$$\vec{\boldsymbol{a}}\cdot\frac{\mathrm{d}\overrightarrow{OM}}{\mathrm{d}t} = \vec{\boldsymbol{a}}\cdot\left(\vec{\boldsymbol{V}} + \vec{\boldsymbol{\omega}} \times_u \overrightarrow{OM}\right) \quad \text{and} \quad \vec{\boldsymbol{u}}\cdot\frac{\mathrm{d}\overrightarrow{OM}}{\mathrm{d}t} = -c(\vec{\boldsymbol{a}}\cdot\overrightarrow{OM}),$$

hence

$$\boldsymbol{D}^{\mathrm{FW}}_{\boldsymbol{u}}\boldsymbol{D}^{\mathrm{FW}}_{\boldsymbol{u}}\,\overrightarrow{OM} = \frac{\mathrm{d}^2\overrightarrow{OM}}{\mathrm{d}t^2} - c\left[2\vec{\boldsymbol{a}}\cdot\left(\vec{\boldsymbol{V}} + \vec{\boldsymbol{\omega}} \times_u \overrightarrow{OM}\right) + \frac{\mathrm{d}\vec{\boldsymbol{a}}}{\mathrm{d}t}\cdot\overrightarrow{OM}\right]\vec{\boldsymbol{u}}$$
$$-c^2(\vec{\boldsymbol{a}}\cdot\overrightarrow{OM})\,\vec{\boldsymbol{a}}.$$

Inserting this relation in (4.55), we get

$$\boxed{\begin{aligned}\frac{\mathrm{d}^2\overrightarrow{OM}}{\mathrm{d}t^2} = {} & \vec{\boldsymbol{\gamma}} + \vec{\boldsymbol{\omega}} \times_u \left(\vec{\boldsymbol{\omega}} \times_u \overrightarrow{OM}\right) + 2\vec{\boldsymbol{\omega}} \times_u \vec{\boldsymbol{V}} + \frac{\mathrm{d}\vec{\boldsymbol{\omega}}}{\mathrm{d}t} \times_u \overrightarrow{OM} \\ & + c^2(\vec{\boldsymbol{a}}\cdot\overrightarrow{OM})\vec{\boldsymbol{a}} + c\left[2\vec{\boldsymbol{a}}\cdot\left(\vec{\boldsymbol{V}} + \vec{\boldsymbol{\omega}} \times_u \overrightarrow{OM}\right) + \frac{\mathrm{d}\vec{\boldsymbol{a}}}{\mathrm{d}t}\cdot\overrightarrow{OM}\right]\vec{\boldsymbol{u}}.\end{aligned}} \tag{4.56}$$

Remark 4.6. The terms in the first line, which involve the centripetal acceleration and the Coriolis acceleration, are "Newtonian" (they remain at the nonrelativistic limit), whereas those of the second line are relativistic. This can be seen by writing, thanks to Eq. (4.64) below, $\vec{\boldsymbol{a}} = \vec{\boldsymbol{\gamma}}_0/c^2$, where $\vec{\boldsymbol{\gamma}}_0$ is the acceleration of $\mathscr{O}$ relative to an inertial observer whose 4-velocity coincides momentarily with that of $\mathscr{O}$. The nonrelativistic limit is then obtained via $\vec{\boldsymbol{\gamma}}_0\cdot\overrightarrow{OM}/c^2 \to 0$ and $\vec{\boldsymbol{V}}/c \to 0$.

4.5.3 Expression of the 4-Acceleration

Let us derive now the analogue of (4.27), namely, the expression of the 4-acceleration $\vec{\boldsymbol{a}}'$ of particle $\mathscr{P}$ in terms of its acceleration $\vec{\boldsymbol{\gamma}}$ relative to observer $\mathscr{O}$ and $\mathscr{O}$'s 4-velocity $\vec{\boldsymbol{u}}$, 4-acceleration $\vec{\boldsymbol{a}}$ and 4-rotation $\vec{\boldsymbol{\omega}}$. We have, from the definition of 4-acceleration [Eq. (2.16)],

$$\vec{\boldsymbol{a}}' = \frac{1}{c}\frac{\mathrm{d}\vec{\boldsymbol{u}}'}{\mathrm{d}t'} = \frac{1}{c}\frac{\mathrm{d}\vec{\boldsymbol{u}}'}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}t'} = \frac{\Gamma}{c}\frac{\mathrm{d}\vec{\boldsymbol{u}}'}{\mathrm{d}t},$$

where we have used the definition (4.1) of the Lorentz factor. Taking the derivative with respect to t of expression (4.27) for $\vec{\boldsymbol{u}}'$, we thus get

$$\vec{\boldsymbol{a}}' = \frac{\Gamma}{c}\left\{\frac{\mathrm{d}\Gamma}{\mathrm{d}t}\left[(1 + \vec{\boldsymbol{a}}\cdot\overrightarrow{OM})\,\vec{\boldsymbol{u}} + \frac{1}{c}\left(\vec{\boldsymbol{V}} + \vec{\boldsymbol{\omega}} \times_u \overrightarrow{OM}\right)\right]\right.$$
$$\left. + \Gamma\left[\frac{\mathrm{d}}{\mathrm{d}t}(\vec{\boldsymbol{a}}\cdot\overrightarrow{OM})\vec{\boldsymbol{u}} + c(1 + \vec{\boldsymbol{a}}\cdot\overrightarrow{OM})\,\vec{\boldsymbol{a}} + \frac{1}{c}\frac{\mathrm{d}}{\mathrm{d}t}\left(\vec{\boldsymbol{V}} + \vec{\boldsymbol{\omega}} \times_u \overrightarrow{OM}\right)\right]\right\}. \tag{4.57}$$

Now, by means of (4.24),

$$\frac{\mathrm{d}}{\mathrm{d}t}(\vec{\boldsymbol{a}} \cdot \overrightarrow{OM}) = \frac{\mathrm{d}\vec{\boldsymbol{a}}}{\mathrm{d}t} \cdot \overrightarrow{OM} + \vec{\boldsymbol{a}} \cdot \left(\vec{\boldsymbol{V}} + \vec{\boldsymbol{\omega}} \times_u \overrightarrow{OM}\right). \tag{4.58}$$

On the other side, from (3.69) and (4.48),

$$\frac{\mathrm{d}\vec{\boldsymbol{V}}}{\mathrm{d}t} = \boldsymbol{D}_u^{\mathrm{FW}} \vec{\boldsymbol{V}} + c(\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{V}})\, \vec{\boldsymbol{u}} = \vec{\boldsymbol{\gamma}} + \vec{\boldsymbol{\omega}} \times_u \vec{\boldsymbol{V}} + c(\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{V}})\, \vec{\boldsymbol{u}},$$

so that combining with (4.53) leads to

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\vec{\boldsymbol{V}} + \vec{\boldsymbol{\omega}} \times_u \overrightarrow{OM}\right) = \vec{\boldsymbol{\gamma}} + \vec{\boldsymbol{\omega}} \times_u \left(\vec{\boldsymbol{\omega}} \times_u \overrightarrow{OM}\right) + 2\vec{\boldsymbol{\omega}} \times_u \vec{\boldsymbol{V}} + \frac{\mathrm{d}\vec{\boldsymbol{\omega}}}{\mathrm{d}t} \times_u \overrightarrow{OM} \\ + c\left[\vec{\boldsymbol{a}} \cdot \left(\vec{\boldsymbol{V}} + \vec{\boldsymbol{\omega}} \times_u \overrightarrow{OM}\right)\right] \vec{\boldsymbol{u}}.\end{aligned} \tag{4.59}$$

Substituting (4.58) and (4.59) into (4.57), we find

$$\boxed{\begin{aligned}\vec{\boldsymbol{a}}' = \frac{\Gamma^2}{c^2}\Bigg\{ & \vec{\boldsymbol{\gamma}} + \vec{\boldsymbol{\omega}} \times_u \left(\vec{\boldsymbol{\omega}} \times_u \overrightarrow{OM}\right) + 2\vec{\boldsymbol{\omega}} \times_u \vec{\boldsymbol{V}} + \frac{\mathrm{d}\vec{\boldsymbol{\omega}}}{\mathrm{d}t} \times_u \overrightarrow{OM} \\ & + c^2(1 + \vec{\boldsymbol{a}} \cdot \overrightarrow{OM})\, \vec{\boldsymbol{a}} + \frac{1}{\Gamma}\frac{\mathrm{d}\Gamma}{\mathrm{d}t}\left(\vec{\boldsymbol{V}} + \vec{\boldsymbol{\omega}} \times_u \overrightarrow{OM}\right) \\ & + c\left[2\vec{\boldsymbol{a}} \cdot \left(\vec{\boldsymbol{V}} + \vec{\boldsymbol{\omega}} \times_u \overrightarrow{OM}\right) + \frac{\mathrm{d}\vec{\boldsymbol{a}}}{\mathrm{d}t} \cdot \overrightarrow{OM} + \frac{1}{\Gamma}\frac{\mathrm{d}\Gamma}{\mathrm{d}t}(1 + \vec{\boldsymbol{a}} \cdot \overrightarrow{OM})\right] \vec{\boldsymbol{u}} \Bigg\}.\end{aligned}} \tag{4.60}$$

The derivative $\mathrm{d}\Gamma/\mathrm{d}t$ appearing in this formula can be evaluated from expression (4.30); using (4.58) and (4.59), we get

$$\boxed{\begin{aligned}\frac{1}{\Gamma}\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = \frac{\Gamma^2}{c^2}\Bigg\{ & (\vec{\boldsymbol{V}} + \vec{\boldsymbol{\omega}} \times_u \overrightarrow{OM}) \cdot \Bigg[\vec{\boldsymbol{\gamma}} + \vec{\boldsymbol{\omega}} \times_u (\vec{\boldsymbol{\omega}} \times_u \overrightarrow{OM}) + 2\vec{\boldsymbol{\omega}} \times_u \vec{\boldsymbol{V}} \\ & + \frac{\mathrm{d}\vec{\boldsymbol{\omega}}}{\mathrm{d}t} \times_u \overrightarrow{OM}\Bigg] - c^2(1 + \vec{\boldsymbol{a}} \cdot \overrightarrow{OM})\Big[\vec{\boldsymbol{a}} \cdot (\vec{\boldsymbol{V}} + \vec{\boldsymbol{\omega}} \times_u \overrightarrow{OM}) \\ & + \frac{\mathrm{d}\vec{\boldsymbol{a}}}{\mathrm{d}t} \cdot \overrightarrow{OM}\Big]\Bigg\}.\end{aligned}} \tag{4.61}$$

If $\mathscr{O}$'s 4-rotation vanishes, the above expressions simplify somewhat. By means of (4.30), we may write

$$\boxed{\begin{aligned}\vec{\boldsymbol{a}}' = \frac{\Gamma^2}{c^2}\Bigg\{ & \vec{\boldsymbol{\gamma}} + \frac{\Gamma^2}{c^2}\left[\vec{\boldsymbol{\gamma}}\cdot\vec{\boldsymbol{V}} - c^2(1+\vec{\boldsymbol{a}}\cdot\overrightarrow{OM})\left(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{V}} + \frac{\mathrm{d}\vec{\boldsymbol{a}}}{\mathrm{d}t}\cdot\overrightarrow{OM}\right)\right]\vec{\boldsymbol{V}} \\ & + c^2(1+\vec{\boldsymbol{a}}\cdot\overrightarrow{OM})\vec{\boldsymbol{a}} + \frac{\Gamma^2}{c}\Bigg[(1+\vec{\boldsymbol{a}}\cdot\overrightarrow{OM})\vec{\boldsymbol{\gamma}}\cdot\vec{\boldsymbol{V}} \\ & - \vec{\boldsymbol{V}}\cdot\vec{\boldsymbol{V}}\left(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{V}} + \frac{\mathrm{d}\vec{\boldsymbol{a}}}{\mathrm{d}t}\cdot\overrightarrow{OM}\right) + \frac{c^2}{\Gamma^2}\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{V}}\Bigg]\vec{\boldsymbol{u}}\Bigg\}.\end{aligned}}_{(\vec{\boldsymbol{\omega}}=0)} \tag{4.62}$$

If, in addition to $\vec{\boldsymbol{\omega}} = 0$, one has $\vec{\boldsymbol{a}} = 0$, that is to say, if $\mathscr{O}$ is an inertial observer, the simplification is even greater:

$$\boxed{\vec{\boldsymbol{a}}' = \frac{\Gamma^2}{c^2}\left[\vec{\boldsymbol{\gamma}} + \frac{\Gamma^2}{c^2}(\vec{\boldsymbol{\gamma}}\cdot\vec{\boldsymbol{V}})\left(\vec{\boldsymbol{V}} + c\,\vec{\boldsymbol{u}}\right)\right].}_{\mathscr{O}\ \text{inertial}} \tag{4.63}$$

Moreover, if at the considered instant t, $\mathscr{P}$ is momentarily at rest with respect to $\mathscr{O}$: $\vec{\boldsymbol{V}} = 0$ (which implies $\vec{\boldsymbol{u}}' = \vec{\boldsymbol{u}}$ and $\Gamma = 1$), the above formula reduces to

$$\boxed{\vec{\boldsymbol{a}}' = \frac{1}{c^2}\,\vec{\boldsymbol{\gamma}}.}_{\mathscr{O}\ \text{inertial and}\ \vec{\boldsymbol{V}}=0} \tag{4.64}$$

We conclude that

> The vector 4-acceleration of particle $\mathscr{P}$ can be interpreted (up to a factor c^2) as the acceleration relative to an inertial observer whose worldline is tangent to that of $\mathscr{P}$ at the considered event.

Remark 4.7. In Newtonian physics, the acceleration $\vec{\boldsymbol{\gamma}}$ of a point particle relative to an *inertial* observer is independent of that observer. Things are different in relativity: formula (4.63) shows that $\vec{\boldsymbol{\gamma}}$ depends upon the velocity $\vec{\boldsymbol{V}}$ relative to the inertial observer. This last quantity being obviously not invariant by a change of inertial observer, $\vec{\boldsymbol{\gamma}}$, will vary in a change of observer, even if the 4-acceleration $\vec{\boldsymbol{a}}'$ is held fixed.

Example 4.21. For Example 4.1 of Sect. 4.2.1, we have seen above that $\vec{\boldsymbol{\gamma}} = 0$ (Example 4.18), so that (4.63) yields $\vec{\boldsymbol{a}}' = 0$, in agreement with the worldline $\mathscr{L}'$ being a straight line of $\mathscr{E}$.

Example 4.22. For Langevin's traveller, relation (4.63), once combined with expressions (4.4), (4.22) and (4.45) of, respectively, Γ, $\vec{\boldsymbol{V}}$ and $\vec{\boldsymbol{\gamma}}$, yields

$$\vec{a}' = \frac{\gamma}{c^2}\left[\frac{\gamma}{c}\left(t - \frac{k}{2}T\right)\,\vec{e}_0 + (-1)^k\sqrt{1 + \frac{\gamma^2}{c^2}\left(t - \frac{k}{2}T\right)^2}\,\vec{e}_1\right]. \tag{4.65}$$

This result is in full agreement with formulas (2.34a), (2.34b) and (2.38) obtained in Chap. 2. We have already noticed in Chap. 2 that the norm of $\vec{a}'$ is constant [cf. Eq. (2.39)]:

$$\left\|\vec{a}'\right\|_g = \frac{|\gamma|}{c^2}. \tag{4.66}$$

On the other side, the norm of the relative acceleration $\vec{\gamma}$ is not constant, as seen on (4.45). The vector $\vec{a}'$ is depicted at two different events in Fig. 4.3.

Example 4.23. In the case of Example 4.3 introduced in Sect. 4.2.1, formulas (4.23) and (4.46) show that $\vec{\gamma}\cdot\vec{V} = 0$, so that (4.63) reduces to

$$\vec{a}' = \frac{\Gamma^2}{c^2}\vec{\gamma}. \tag{4.67}$$

The vectors 4-acceleration and acceleration relative to $\mathscr{O}$ are therefore collinear (contrary, for instance, to the case of Example 4.22), as one can see in Fig. 4.4. Evaluating the norm of $\vec{a}'$ from the above relation, (4.7) and (4.46), we get

$$\left\|\vec{a}'\right\|_g = \frac{1}{R}\left(\frac{c^2}{R^2\Omega^2} - 1\right)^{-1}. \tag{4.68}$$

Hence the norm of the 4-acceleration is constant, as in Example 4.22.

A useful formula is that relating the norm of the 4-acceleration $\vec{a}'$ to the norm of the relative acceleration $\vec{\gamma}$. Taking the scalar square of (4.63), we get

$$\begin{aligned}
\vec{a}'\cdot\vec{a}' &= \frac{\Gamma^4}{c^4}\left[\vec{\gamma}\cdot\vec{\gamma} + 2\frac{\Gamma^2}{c^2}(\vec{\gamma}\cdot\vec{V})(\vec{\gamma}\cdot\vec{V} + c\,\underbrace{\vec{\gamma}\cdot\vec{u}}_{0}) + \frac{\Gamma^4}{c^4}(\vec{\gamma}\cdot\vec{V})^2\left(\vec{V}\cdot\vec{V} - c^2\right)\right] \\
&= \frac{\Gamma^4}{c^4}\left\{\vec{\gamma}\cdot\vec{\gamma} + \frac{\Gamma^2}{c^2}(\vec{\gamma}\cdot\vec{V})^2\left[2 + \Gamma^2\underbrace{\left(\frac{1}{c^2}\vec{V}\cdot\vec{V} - 1\right)}_{-\Gamma^{-2}}\right]\right\},
\end{aligned}$$

hence the relatively simple formula:

$$\boxed{\vec{a}'\cdot\vec{a}' = \frac{\Gamma^4}{c^4}\left[\vec{\gamma}\cdot\vec{\gamma} + \frac{\Gamma^2}{c^2}\left(\vec{\gamma}\cdot\vec{V}\right)^2\right].}_{\mathscr{O}\text{ inertial}} \tag{4.69}$$

Thanks to the identity $(\vec{\gamma}\times_u\vec{V})^2 = \gamma^2V^2 - (\vec{\gamma}\cdot\vec{V})^2$, we can write

$$\boxed{\vec{a}' \cdot \vec{a}' = \frac{\Gamma^6}{c^4}\left[\vec{\gamma}\cdot\vec{\gamma} - \frac{1}{c^2}(\vec{\gamma} \times_u \vec{V})^2\right].}_{\mathscr{O}\ \text{inertiel}} \qquad (4.70)$$

Besides, in the case where $\vec{V} \neq 0$, $\vec{\gamma}$ can be split into a part along $\vec{V}$ and a part orthogonal to $\vec{V}$, according to $\vec{\gamma} =: \gamma_\parallel \vec{n} + \vec{\gamma}_\perp$, with $\vec{V} = V\vec{n}$, $\vec{n}\cdot\vec{n} = 1$ and $\vec{n}\cdot\vec{\gamma}_\perp = 0$. Then $\vec{\gamma}\cdot\vec{\gamma} = \gamma_\parallel^2 + \gamma_\perp^2$ and $(\vec{\gamma}\cdot\vec{V})^2 = \gamma_\parallel^2 V^2$, so that (4.69) can be written as

$$\boxed{\vec{a}' \cdot \vec{a}' = \frac{\Gamma^4}{c^4}\left(\Gamma^2\gamma_\parallel^2 + \gamma_\perp^2\right).}_{\mathscr{O}\ \text{inertial}} \qquad (4.71)$$

It is easy to invert formula (4.63) to express $\vec{\gamma}$ in terms of $\vec{a}'$. Indeed the scalar product of (4.63) by $\vec{V}$ leads to

$$\vec{a}'\cdot\vec{V} = \frac{\Gamma^2}{c^2}\left[\vec{\gamma}\cdot\vec{V} + \frac{\Gamma^2}{c^2}(\vec{\gamma}\cdot\vec{V})(\vec{V}\cdot\vec{V} + 0)\right] = \frac{\Gamma^2}{c^2}\Bigg(\underbrace{1 + \frac{\Gamma^2}{c^2}\vec{V}\cdot\vec{V}}_{\Gamma^2}\Bigg)\vec{\gamma}\cdot\vec{V}$$
$$= \frac{\Gamma^4}{c^2}\vec{\gamma}\cdot\vec{V}.$$

Substituting this relation for $\vec{\gamma}\cdot\vec{V}$ in (4.63), we get

$$\boxed{\vec{\gamma} = \Gamma^{-2}\left[c^2\,\vec{a}' - (\vec{a}'\cdot\vec{V})\left(\vec{V} + c\,\vec{u}\right)\right].}_{\mathscr{O}\ \text{inertial}} \qquad (4.72)$$

4.6 Photon Motion

Let us now turn to the description of the motion of photons (or more generally of massless particles) with respect to a given observer. Let us recall that a characteristic of these particles is to have worldlines that are straight lines of $\mathscr{E}$ and oriented along null vectors (the so-called null geodesics; cf. Sect. 2.5.1).

4.6.1 Propagation Direction of a Photon

As in Sect. 4.2.1, let us consider some observer $\mathscr{O}$ of worldline $\mathscr{L}$ and 4-velocity $\vec{u}$. Let $\mathscr{P}$ be a photon moving on the null geodesic Δ, in the vicinity of $\mathscr{L}$ (cf. Fig. 4.6). Let $M(t)$ be the position of the photon at the proper time t (with respect to $\mathscr{O}$): $M(t)$ is the intersection of Δ with the local rest space $\mathscr{E}_{\boldsymbol{u}}(t)$ of $\mathscr{O}$ at proper time t.

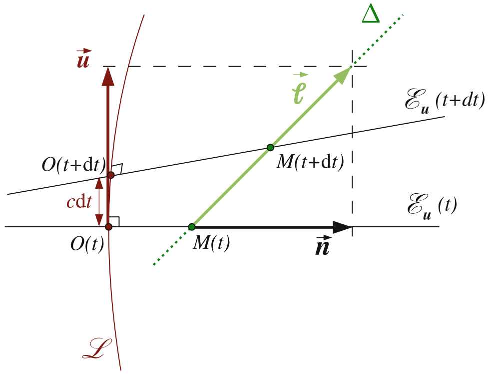

Fig. 4.6 Motion of a photon (null geodesic Δ) with respect to an observer (worldline $\mathscr{L}$, 4-velocity $\vec{u}$ and local rest space $\mathscr{E}_u(t)$)

The main difference with the motion of a massive particle, as described in Sect. 4.2.1, is the lack of a unit vector tangent to the photon's worldline: vectors tangent to Δ cannot be normalized because their scalar square vanishes (since they are null vectors). All that one can do is to select a null vector adapted to observer $\mathscr{O}$ as follows: at the point $M(t) \in \Delta$, let us define $\vec{\ell}(t)$ as the unique null vector parallel to Δ such that

$$\vec{\ell}(t) \cdot \vec{u}(t) = -1. \tag{4.73}$$

This amounts to demanding that the orthogonal decomposition of $\vec{\ell}$ with respect to $\vec{u}$ is $\vec{\ell} = \alpha\vec{u} + \perp_u \vec{\ell}$, with $\alpha = 1$ [cf. Eq. (3.10)]. Let us set $\vec{n} := \perp_u \vec{\ell}$, so that (cf. Fig. 4.6)

$$\boxed{\vec{\ell} = \vec{u} + \vec{n}} \qquad \text{with} \quad \vec{u} \cdot \vec{n} = 0. \tag{4.74}$$

By definition the vector $\vec{n}$ lies in $\mathscr{O}$'s local rest space: $\vec{n} \in E_u$. Moreover, it is a unit vector. Indeed, the property $\vec{\ell} \cdot \vec{\ell} = 0$ implies

$$\underbrace{\vec{u} \cdot \vec{u}}_{-1} + 2\,\underbrace{\vec{u} \cdot \vec{n}}_{0} + \vec{n} \cdot \vec{n} = 0,$$

hence

$$\boxed{\vec{n} \cdot \vec{n} = 1}. \tag{4.75}$$

We shall call the unit vector $\vec{n}$ the ***propagation direction of photon $\mathscr{P}$ with respect to observer $\mathscr{O}$***.

4.6.2 Velocity of Light

The ***velocity of the photon relative to observer*** $\mathscr{O}$ is defined in the same manner as the velocity of a massive particle at Sect. 4.3.1, i.e. as the derivative of the position vector of the photon with respect to $\mathscr{O}$:

$$\boxed{\vec{V}(t) := \frac{\mathrm{d}x^i}{\mathrm{d}t}\,\vec{e}_i(t)}, \tag{4.76}$$

where the $\vec{e}_i(t)$'s are the three spatial vectors of $\mathscr{O}$'s local frame at time t and the $x^i(t)$'s are the coordinates of the point $M(t)$ in that frame:

$$\overrightarrow{O(t)M(t)} = x^i(t)\,\vec{e}_i(t). \tag{4.77}$$

The computation of $\vec{V}$ performed in Sect. 4.3.1 for a massive particle remains valid in the present case, so that we get the same equation as (4.24). The difference with Sect. 4.3 arises from the expression of $\mathrm{d}\overrightarrow{OM}/\mathrm{d}t$. To evaluate it, we may still start from (4.26). There appears the infinitesimal vector $\overrightarrow{M(t)M(t+\mathrm{d}t)}$. It is by definition parallel to Δ and thus collinear to the vector $\vec{\ell}(t)$ introduced above (cf. Fig. 4.6). Moreover, this is a first-order quantity in $\mathrm{d}t$, so that we can write

$$\overrightarrow{M(t)M(t+\mathrm{d}t)} = \lambda\, c\,\mathrm{d}t\,\vec{\ell}(t),$$

where $\lambda \in \mathbb{R}$. The value of λ is found by expressing that the event $M(t+\mathrm{d}t)$ belongs to the local rest space $\mathscr{E}_{\boldsymbol{u}}(t+\mathrm{d}t)$, through the orthogonality of the vectors $\vec{u}(t+\mathrm{d}t)$ and $\overrightarrow{O(t+\mathrm{d}t)M(t+\mathrm{d}t)}$. We obtain successively

$$\begin{aligned}
&\vec{u}(t+\mathrm{d}t)\cdot\overrightarrow{O(t+\mathrm{d}t)M(t+\mathrm{d}t)} = 0,\\
&[\vec{u}(t) + c\,\mathrm{d}t\,\vec{a}(t)]\cdot\left[\overrightarrow{O(t+\mathrm{d}t)O(t)} + \overrightarrow{O(t)M(t)} + \overrightarrow{M(t)M(t+\mathrm{d}t)}\right] = 0,\\
&[\vec{u}(t) + c\,\mathrm{d}t\,\vec{a}(t)]\cdot\left[-c\,\mathrm{d}t\,\vec{u}(t) + \overrightarrow{O(t)M(t)} + \lambda\,c\,\mathrm{d}t\,\vec{\ell}(t)\right] = 0,\\
&-c\,\mathrm{d}t(-1) + 0 + \lambda\,c\,\mathrm{d}t(-1) - c^2\,\mathrm{d}t^2\times 0 + c\,\mathrm{d}t\,\vec{a}(t)\cdot\overrightarrow{O(t)M(t)}\\
&\qquad + c^2\mathrm{d}t^2\,\lambda\,\vec{a}(t)\cdot\vec{\ell}(t) = 0,\\
&c\,\mathrm{d}t - \lambda\,c\,\mathrm{d}t + c\,\mathrm{d}t\,\vec{a}(t)\cdot\overrightarrow{OM}(t) = 0,\\
&\lambda = 1 + \vec{a}(t)\cdot\overrightarrow{OM}(t).
\end{aligned}$$

Note that we have used the property (4.73) and have neglected second-order terms in $\mathrm{d}t$. Hence

$$\overrightarrow{M(t)M(t+\mathrm{d}t)} = (1 + \vec{\boldsymbol{a}} \cdot \overrightarrow{OM})\, c\, \mathrm{d}t\, \vec{\boldsymbol{\ell}}.$$

Substituting this value for $\overrightarrow{M(t)M(t+\mathrm{d}t)}$ in (4.26) and writing $\overrightarrow{O(t+\mathrm{d}t)O(t)} = -c\,\mathrm{d}t\,\vec{\boldsymbol{u}}$, we get

$$\frac{\mathrm{d}\overrightarrow{OM}}{\mathrm{d}t} = -c\,\vec{\boldsymbol{u}} + c(1 + \vec{\boldsymbol{a}} \cdot \overrightarrow{OM})\,\vec{\boldsymbol{\ell}} = c(\vec{\boldsymbol{a}} \cdot \overrightarrow{OM})\,\vec{\boldsymbol{u}} + c(1 + \vec{\boldsymbol{a}} \cdot \overrightarrow{OM})\,\vec{\boldsymbol{n}},$$

where the second equality stems from the replacement of $\vec{\boldsymbol{\ell}}$ by $\vec{\boldsymbol{u}} + \vec{\boldsymbol{n}}$ [Eq. (4.74)]. There remains to insert this result into (4.24) to get the final expression of the photon velocity with respect to observer $\mathscr{O}$:

$$\boxed{\vec{V} = c(1 + \vec{\boldsymbol{a}} \cdot \overrightarrow{OM})\,\vec{\boldsymbol{n}} - \vec{\boldsymbol{\omega}} \times_u \overrightarrow{OM}}. \tag{4.78}$$

In the cases where (i) the photon's worldline crosses the worldline $\mathscr{L}$ of $\mathscr{O}$ at the proper time t ($\overrightarrow{OM} = 0$) or (ii) $\mathscr{O}$ is an inertial observer ($\vec{\boldsymbol{a}} = 0$ and $\vec{\boldsymbol{\omega}} = 0$), the above expression simplifies to

$$\boxed{\vec{V} = c\,\vec{\boldsymbol{n}}}_{\;M\in\mathscr{L}\text{ or }\mathscr{O}\text{ inertial}}. \tag{4.79}$$

Since $\vec{\boldsymbol{n}}$ is a unit vector, we deduce immediately that the norm of $\vec{V}$ is equal to c:

$$\boxed{\left\|\vec{V}\right\|_{g} = c}_{\;M\in\mathscr{L}\text{ or }\mathscr{O}\text{ inertial}}. \tag{4.80}$$

Let us stress that this result is valid for any observer $\mathscr{O}$ as long as the photon crosses his worldline ($M \in \mathscr{L}$):

The velocity of light as measured by any observer at a point of his worldline has a norm always equal to the constant c.

We recover thus one of the two historical postulates of Einstein (1905b), the second one being the relativity principle, to be discussed in Sect. 9.3.4. Note however that Einstein's postulate regarded only inertial observers. The result obtained here holds for any kind of observer, provided that the measurement is performed at the observer's position ($\overrightarrow{OM} = 0$). Would this be not the case, (4.78) might lead to $\|\vec{V}\|_g \neq c$, if $\mathscr{O}$ is not inertial, i.e. is accelerated ($\vec{\boldsymbol{a}} \neq 0$) or rotating ($\vec{\boldsymbol{\omega}} \neq 0$).

Remark 4.8. The constancy of the velocity of light, which is a postulate in Einstein's original formulation (Einstein 1905b), appears here as a derived result. It is actually a consequence of the principle stated in Sect. 2.5.1, namely, that photons follow null geodesics. Thus this principle can be seen as a geometrical version of Einstein's postulate.

Historical note: *The finiteness of the velocity of light was first shown by Jean-Dominique Cassini*[8] *and Ole C. Rømer.*[9] *In August 1676, Cassini announced at the French Academy of Sciences that light requires 10 to 11 minutes to cross the radius of the Earth orbit (Bobis and Lequeux 2008). However, Cassini became soon sceptical about this result and left Rømer to publish the discovery alone at the end of 1676 (Rømer 1676). The measure of Cassini and Rømer was based on observations of the eclipses of one of Jupiter's satellites, Io. Indeed, due to the variation of Jupiter–Earth distance and the finite time of propagation of light, Io eclipses observed from Earth occur with a delay or an advance of up to 10 minutes with respect to mean ephemerides. The inferred value of the velocity of light is* $c \simeq$ *radius of the Earth orbit* / $11\mathrm{min} \simeq 2 \times 10^8\,\mathrm{m\,s^{-1}}$ *in modern units. Even if the value of* c *obtained by Cassini and Rømer was only* $2/3$ *of the correct value, their measurement has established the finite character of* c*. A more precise measure was performed in 1728 by James Bradley,*[10] *from stellar aberration (to be discussed in Sect. 5.6) (Bradley 1728). He obtained* $c = 10210\, V_\oplus$*, where* $V_\oplus$ *is the orbital velocity of Earth with respect to an inertial frame centred on the Sun, and he concluded that light takes* 8 min 12 s *to travel the Sun–Earth distance (the modern value is* 8 min 19 s*). Since at that time, the value of* $V_\oplus$ *was poorly known, Bradley has not expressed his measure in standard velocity units. The first precise determination of* c *in metre per second has been achieved only in the nineteenth century, by Hippolyte Fizeau.*[11] *In 1849, he obtained* $c = 3.15 \times 10^8\,\mathrm{m\,s^{-1}}$ *with a device based on rotating cogwheel (Fizeau 1849). This value will be refined to* $c = 2.98 \times 10^8\,\mathrm{m\,s^{-1}}$ *by Léon Foucault*[12] *in 1862 (Foucault 1862), with a device involving a rotating mirror. Let us recall that since 1983, the value of* c *is fixed by convention to* $2.99792458 \times 10^8\,\mathrm{m\,s^{-1}}$ *[Eq. (1.7)] and provides the definition of the metre.*

[8]**Jean-Dominique Cassini** (1625–1712): Italian–French astronomer, first director of the newly built Paris Observatory in 1671, under the reign of Louis XIV. Apart from the study of Io's eclipses, he discovered four Saturn's moons and the famous "Cassini division" in Saturn's rings. He also supervised the first measure of the size of the Solar System by means of the parallax of Mars.

[9]**Ole C. Rømer** (1644–1710): Danish astronomer who worked at Paris Observatory from 1672 to 1679, at the invitation of Jean Picard (cf. p. 157) and under the supervision of Cassini.

[10]**James Bradley** (1693–1762): British astronomer, famous for this explanation of stellar aberration (Sect. 5.6.3) and the discovery of Earth nutation.

[11]**Hippolyte Fizeau** (1819–1896): French physicist who authored many studies on light; in addition to the measurement of c, he notably discovered the Doppler effect on light waves (Sect. 5.5).

[12]**Léon Foucault** (1819–1868): French physicist and astronomer, famous for this works in optics (measure of c, Foucault test for telescope mirrors), electromagnetism (Foucault currents) and mechanics (Foucault pendulum).

4.6.3 Experimental Tests of the Invariance of the Velocity of Light

In all this section, $\mathscr{O}$ is an observer attached to the Earth, and we shall denote by $\vec{V}_{\rm light}$ the velocity of light relative to $\mathscr{O}$, as defined by (4.76), and by $V_{\rm light}$ its norm (with respect to $\boldsymbol{g}$). From (4.80), relativity predicts that, for any local measure, $V_{\rm light} = c$, whatever the state of motion of observer $\mathscr{O}$.

When one wants to check a prediction from a theory, such as the constancy of the velocity of light (4.80), one should, strictly speaking, do it in the framework of a ***test theory***, i.e. a wider theory that contains free parameters and that reduces to the theory to be checked for well-defined values of the parameters. The advantage of such an approach is to quantify easily the possible violation of the theory by the experimental determination of the free parameters. Usually one obtains upper bounds on the absolute values of the parameters when the theory to be tested corresponds to the value zero of the parameters. In the case of special relativity, a test theory frequently used is that developed by Robertson (1949), Tourrenc et al. (1996) and R. Mansouri and R.U. Sexl (1977); it contains three free parameters. This is a *kinematical* test theory, for it concerns only relations between observers. More recent works are based instead on a *dynamical* test theory, i.e. a theory based on some generalization of the equations of motion; it consists in an extension of the standard model of particle physics, introduced by D. Colladay and V.A. Kostelecký in 1998 (Colladay and Kostelecký 1998; Kostelecký and Russel 2011) and called ***SME*** (for *Standard Model Extension*). Such a test theory contains $19 + 48n$ free parameters, n being the number of types of elementary particles involved in the model. In the present book however, the experimental tests will be presented quite succinctly and not within the framework of a test theory. For such a presentation, we refer to the book by Zhang (1997) or to the review articles by Lämmerzahl (2006), Wolf et al. (2006), Mattingly (2005) or Kostelecký and Russel (2011).

4.6.3.1 Arago Experiment (1810)

In 1810, François Arago[13] showed that the light from stars is propagating, with respect to a terrestrial observer, at the same velocity whatever the direction of the star with respect to the direction of motion of the Earth around the Sun (Arago 1853).[14] In view of the Galilean law of velocity composition, one would expect that the velocity of light with respect to the terrestrial observer would be, in absolute

[13]**François Arago** (1786–1853): French astronomer, well known for his works in optics. He was director of Paris Observatory and minister of the French Second Republic, where he acted for the abolition of slavery in French colonies (1848).

[14]Arago's results have been presented to the French Academy of Sciences in 1810, but the corresponding article has been published only in 1853 because the original manuscript had been lost.

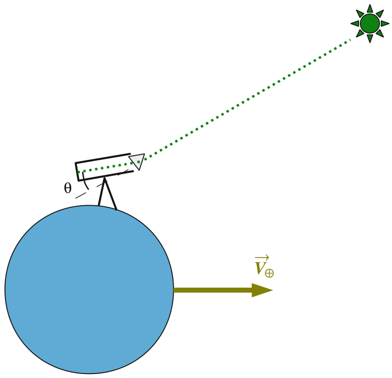

Fig. 4.7 Arago experiment (1810): measure of the velocity of light with respect to the Earth, for the light coming from various stars, by means of the refraction caused by a prism at the telescope entry

value, $V_{\text{light}} = c_\odot - V_\oplus$ when the star is located in the direction of the Earth motion around the Sun and $V_{\text{light}} = c_\odot + V_\oplus$ in the opposite direction. In these formulas, $V_\oplus = 30\ \text{km s}^{-1}$ is the Earth velocity with respect to an inertial frame centred on the Sun, and $c_\odot \simeq c$ is the velocity of light in that frame.

To perform his experiment, Arago has placed a prism in front of the objective of a refracting telescope and has measured the deviation angle for stars located in different directions (cf. Fig. 4.7). The underlying assumption is that this angle is given by the Snell–Descartes law, $n_1 \sin\theta_1 = n_2 \sin\theta_2$, applied to each interface of the prism. In this law, the refraction index n_1 is inversely proportional to the velocity of light in the medium. One should therefore find different deviation angles according the direction of the star with respect to the direction of motion of the Earth around the Sun. Given the refraction index of the prism and the value $V_\oplus = 30\ \text{km s}^{-1}$, the amplitude of the differences should be $28''$. The outcome of the experiment clearly contradicts this: all the deviation angles are equal within $\pm 5''$, and the small discrepancies are not correlated with the direction of the star with respect to the Earth motion. To confirm the result, Arago performed the experiment with two different prisms and at two different epoch of the year (March and October). The conclusion is thus that the velocity of light with respect to the Earth is constant and depends neither from the source star nor from the motion of the Earth with respect to that star.

Remark 4.9. Arago experiment constitutes the very first evidence of a relativistic effect, almost a century before the formulation of special relativity! For a more detailed discussion of this experiment, see Eisenstaedt (2007) and Ferraro and Sforza (2005).

Historical note: *The result of the experiment left Arago quite perplex. At this time, he was a proponent of the corpuscular theory of light (soon after, he became in favour of the wave theory). To explain the lack of deviation, he put forward the following hypotheses: (i) a star emits "light rays" within a full range of velocities, these velocities combining with the Earth's one according to the Galilean law, and (ii) the human eye is sensitive only to rays having a well-defined velocity, hence the*

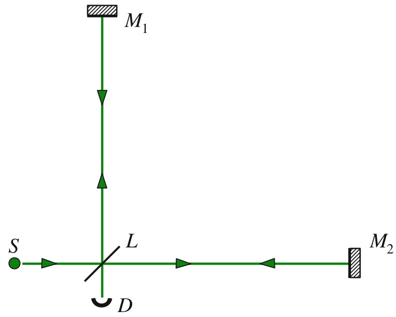

Fig. 4.8 Sketch of a Michelson interferometer : S is the source, L the semi-transparent mirror, M_1 and M_2 the two end mirrors and D the detector

result. Hypothesis (i) fits well with the corpuscular theory of light (each corpuscle may have a velocity different from the others) (Eisenstaedt 2007). On the other hand, it accommodates badly with the wave theory of light (Young's double-slit experiment dates from 1801!), because in that theory, the velocity of light is constant with respect to aether, which is the "medium" supporting the light waves. To reconcile Arago's result with the wave theory, Augustin Fresnel[15] introduced the hypothesis of partial aether dragging by transparent materials (Fresnel 1818) (cf. Sect. 2.6 of the textbook (Ferraro 2007)).

4.6.3.2 Michelson–Morley Experiment (1887)

In 1887 Albert A. Michelson[16] and Edward W. Morley[17] performed a measurement of the difference of velocity of light between two orthogonal directions (Michelson and Morley 1887). The apparatus used by Michelson and Morley, known nowadays as a ***Michelson interferometer***, is depicted in Fig. 4.8: a source S emits a light beam that is split by a semi-transparent mirror L. Each half-beam makes a round trip to the mirrors M_1 and M_2. The two half-beams are then recombined at the level of L, which generates interference fringes, recorded by the detector D. Michelson and Morley apparatus was installed on a marble table floating onto mercury, thereby allowing for an easy rotation of the whole device.

Michelson and Morley have not observed any displacement of the interference fringes after rotating the apparatus by 90°. This implies that the velocity of light is the same in the two directions. In particular, this result contradicts the prediction based on the aether model and the Galilean addition of velocities: due to the motion

[15]**Augustin Fresnel** (1788–1827): French physicist, cofounder of the wave theory of light; he invented the lens bearing his name and that equips lighthouses.

[16]**Albert A. Michelson** (1852–1931): American physicist who devoted his life to precision optics and, in particular, to the measure of V_{light}; he was awarded the Nobel Prize in Physics in 1907 (the first American!).

[17]**Edward W. Morley** (1838–1923): American chemist, mostly known for his work with Michelson.

of Earth with respect to aether, the velocity of light with respect to the laboratory would not be the same in two different directions.

Historical note: *In order to explain the negative result of Michelson and Morley, i.e. the lack of imprint of the Earth motion with respect to aether, George F. FitzGerald*[18] *forged in 1889 the hypothesis of length contraction of material bodies in the direction of their motion with respect to aether (FitzGerald 1889). The contraction factor proposed by FitzGerald is nothing but the Lorentz factor* $\Gamma = (1 - V^2/c^2)^{-1/2}$*, where* V *is Earth's velocity relative to aether. The contraction of the interferometer arm in the direction of Earth motion explains then Michelson–Morley result. Hendrik A. Lorentz (cf. p. 108) made the same remark three years after (Lorentz 1892) and has shown that, in the aether theory, the contraction factor* Γ *can be derived by considering that the cohesion forces of the interferometer arms are of electromagnetic origin: the contraction results then from the modification of the forces induced by the motion of Earth across aether.*

4.6.3.3 Kennedy–Thorndike Experiment (1932)

The Michelson–Morley experiment is sensitive to the variation of V_{light} in two different directions. In other words, it tests the isotropy of $\vec{V}_{\text{light}}$. More precisely, it shows that the velocity of light measured by an observer $\mathscr{O}$ does not depend upon the *direction* of $\mathscr{O}$'s motion with respect to a somewhat privileged frame (e.g. aether). But this experiment does not prove that the velocity of light is independent from the *norm* of the velocity of $\mathscr{O}$ with respect to aether. This last property has been established in 1932 by Roy J. Kennedy and Edward M. Thorndike (1932), on the basis of an experiment that differs from that of Michelson–Morley by three aspects:

- The interferometer arms have different lengths, L_1 and L_2 say.
- The interferometer is held fixed in the laboratory (no rotation).
- The fringes are monitored over a long period: several months.

Due to the Earth revolution around the Sun, the norm of the observer velocity with respect to aether should change in an appreciable manner during several months. A simple computation shows that the resulting phase difference of the two interferometer beams is proportional to $L_1 - L_2$ (cf. Sect. 5.4 de Giulini (2005) or Sect. 3.IV.2 de Simon (2004)). But Kennedy and Thorndike did not observed any displacement of the fringes. This demonstrates the invariance of V_{light} with respect of the amplitude of the observer velocity relative to some hypothetical aether.

[18]**George F. FitzGerald** (1851–1901): Irish physicist, who worked mostly on Maxwell theory and electromagnetism.

4.6.3.4 Independence with Respect to the Source Motion

One may wonder about the dependency of $\vec{V}_{\text{light}}$ with respect to the motion of the light source. Let us write

$$\vec{V}_{\text{light}} = \vec{V}_0 + k\vec{V}_{\text{s}}, \tag{4.81}$$

where $\vec{V}_{\text{s}}$ is the velocity of the light source relative to observer $\mathscr{O}$, $\vec{V}_0$ is the velocity of light if the source was at rest with respect to $\mathscr{O}$ and k is some constant. The relativity prediction (4.79) amounts to $k = 0$.

A confirmation of this prediction has been provided by the observation of binary stars. As noticed by de Sitter[19] in 1913 (de Sitter 1913a,b,c), if the velocity of light added with that of the source, this would imply some irregularity in the apparent motion of binary stars. This is easily understood by considering a small star in circular orbit around a massive companion and setting the observer in the orbital plane. It is then a simple exercise to compute the travel time of light and to conclude that the image of the star would take less time to complete the half-orbit on the observer side than that on the opposite side. At de Sitter's time, the observations led to the upper bound $|k| < 0.002$ (de Sitter 1913b).

A weak point of de Sitter's demonstration is that the light from stars is not received directly by the observer but is scattered by the interstellar medium: the photons are absorbed and reemitted many times before reaching the Earth. Consequently, in (4.81), $\vec{V}_{\text{s}}$ should rather be the mean velocity of the interstellar medium with respect to the Earth. This mean velocity being constant, no effect would be observed even if $k \neq 0$. Fortunately, the scattering by the interstellar medium is important only for optical wavelengths or larger ones. It is almost nonexistent for X-rays. By observing three X-ray pulsars, i.e. three binary systems where one of the stars is a neutron star accreting matter from its companion, Kenneth Brecher has shown in 1977 (Brecher 1977) that

$$|k| < 2 \times 10^{-9}, \tag{4.82}$$

which constitutes an excellent test for relativity.

Another test of the independence of the velocity of light from the source is provided by particle physics. In an experiment performed at CERN in 1964, T. Alväger et al. (1964) determined the velocity of gamma photons emitted during the decay of neutral pions. A π^0 pion (cf. p. 110) has indeed a mean lifetime of 8×10^{-17} s, and its main decay mode is a double photon creation (in the gamma ray regime):

$$\pi^0 \longrightarrow \gamma + \gamma.$$

[19] **Willem de Sitter** (1872–1934): Dutch physicist and astronomer, famous for having introduced a cosmological model within general relativity, called *de Sitter universe*.

The pions produced during collisions in the PS synchrotron at CERN being ultra-relativistic ($\Gamma \sim 45$), we are in the case where $\|\vec{V}_s\|_g \simeq c$ in (4.81). The velocity of the gamma photons was determined by measuring their time of flight between two detectors separated by 31 m; this led to (Alväger et al. 1964)

$$k = (-3 \pm 13) \times 10^{-5}, \tag{4.83}$$

which is a value fully compatible with the $k = 0$ of special relativity.

4.6.3.5 Modern Experiments

Modern experiments regarding the constancy of V_{light} are based on the precise determination of resonant frequencies of a microwave or optical cavity, either of Fabry–Perot type or circular. Indeed the frequency of a cavity eigenmode is

$$\nu = \frac{N V_{\text{light}}}{2nL}, \tag{4.84}$$

where L is the cavity length, n the index of the medium filling the cavity and $N \in \mathbb{N}$ the eigenmode number. If V_{light} was varying, by some amount δV_{light}, this formula would imply that ν would vary as well. In 1979, by means of a Fabry–Perot cavity and a helium-neon laser of wavelength $\lambda = 3.39\ \mu$m, Alain Brillet and J.L. Hall have obtained

$$\frac{|\delta V_{\text{light}}|}{c} < 10^{-14}$$

for a round trip in the cavity (Brillet and Hall 1979). Nowadays, circular cryogenic cavities (Wolf et al. 2003, 2004, 2006; Müller et al. 2007) or a system of two perpendicular optical cavities (Eisele et al. 2009; Herrmann et al. 2009) is used. This has allowed to reach upper bounds of the order of 10^{-15} (Wolf et al. 2006, 2004) or even 10^{-17} (Eisele et al. 2009; Herrmann et al. 2009) on some parameters of the theoretical framework SME mentioned above, those parameters being zero for special relativity.

On another side, without performing any experiment but simply using public data from the Global Positioning System (GPS, to be discussed in Sect. 22.3.3.5), which involves atomic clocks in satellites and ground-based ones, Peter Wolf and Gérard Petit have obtained $|\delta V_{\text{light}}|/c < 5 \times 10^{-9}$ for the one-way travel of an electromagnetic signal (Wolf and Petit 1997).

Finally, some recent astrophysical observations have provided noticeable proofs on the independence of V_{light} from the energy E of the photons. In special relativity, all photons, whatever their energy, must travel on null geodesics. So V_{light} must be independent of the photon energy. On 28 July 2006, astronomers using the Cherenkov telescope HESS observed high-energy gamma photons ($E \sim 800$ GeV) from an explosion in the active galaxy PKS 2155-304 (Aharonian et al. 2008). They compared the arrival time of photons in two energy bands and set an upper bound on the difference. From the knowledge of the distance of the galaxy, they deduced that

$$\frac{V_{\text{light}}}{c} = 1 + \alpha \frac{E}{1\ \text{GeV}} + \beta \left(\frac{E}{1\ \text{GeV}}\right)^2, \text{ with } |\alpha| < 2 \times 10^{-18} \text{ and } |\beta| < 5 \times 10^{-19},$$

which is a pretty tight constraint on the dependence of V_{light} on the photon energy.

For a more detailed review of experiments, we refer to Will (2006a), Wolf et al. (2006), Ehlers and Lämmerzahl (2006) and Mattingly (2005). The historical experiments of the nineteenth century and beginning of the twentieth century are discussed, among others, in Darrigol (2000), Miller (1998) and Tonnelat (1959). It is also worth reading the article (Ellis and Uzan 2005) devoted to the different aspects of c (velocity of light, time/length conversion constant, velocity of propagation of gravitation, etc.) and their relations with experiment.

Chapter 5
Kinematics 2: Change of Observer

5.1 Introduction

The preceding chapter was devoted to the motion of a particle as perceived by a given observer and introduced the notions of velocity and acceleration. We move now to the manner by which two distinct observers perceive the same particle. We shall notably establish the various laws of transformation of relative quantities as one moves from one observer to the other: transformation of lengths (FitzGerald–Lorentz contraction, Sect. 5.2), velocities (Sect. 5.3), accelerations (Sect. 5.4), frequencies (Doppler effect, Sect. 5.5), observation angles (aberration, Sect. 5.6) and images (Sect. 5.7).

5.2 Relations Between Two Observers

In all this chapter, we consider two observers, $\mathscr{O}$ and $\mathscr{O}'$, of respective worldlines $\mathscr{L}$ and $\mathscr{L}'$ and 4-velocities $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{u}}'$.

5.2.1 Reciprocity of the Relative Velocity

Observer $\mathscr{O}'$ has a certain velocity relative to $\mathscr{O}$, and similarly, $\mathscr{O}$ has a velocity relative to $\mathscr{O}'$. How are these two velocities related to one another? In Galilean physics, they would simply be the opposite of each other. We are going to see that this is no longer the case in special relativity.

For simplicity, let us consider the case where the worldlines $\mathscr{L}$ and $\mathscr{L}'$ of the two observers cross at the same event O. One may then use the reduced formulas (4.31)–(4.33). To put observers $\mathscr{O}$ and $\mathscr{O}'$ on the same footing, let us perform a slight change of notation: we shall denote by $\vec{\boldsymbol{U}}$ (and no longer by $\vec{\boldsymbol{V}}$) the velocity of $\mathscr{O}'$ relative

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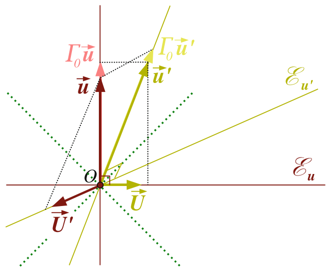

Fig. 5.1 Relative velocities of two observers: $\mathscr{O}$ (4-velocity $\vec{u}$ and local rest space $\mathscr{E}_u$) and $\mathscr{O}'$ (4-velocity $\vec{u}'$ and local rest space $\mathscr{E}_{u'}$); $\vec{U}$ is the velocity of $\mathscr{O}'$ relative to $\mathscr{O}$ and $\vec{U}'$ the velocity of $\mathscr{O}$ relative to $\mathscr{O}'$

to $\mathscr{O}$. We shall then denote by $\vec{U}'$ the velocity of $\mathscr{O}$ relative to $\mathscr{O}'$. Accordingly, formula (4.31) leads to the following relations between the 4-velocities of the two observers (cf. Fig. 5.1):

$$\vec{u}' = \Gamma_0 \left(\vec{u} + \frac{1}{c} \vec{U} \right) \quad \text{with} \quad \vec{u} \cdot \vec{U} = 0, \tag{5.1}$$

$$\vec{u} = \Gamma_0 \left(\vec{u}' + \frac{1}{c} \vec{U}' \right) \quad \text{with} \quad \vec{u}' \cdot \vec{U}' = 0, \tag{5.2}$$

where $\Gamma_0 = -\vec{u} \cdot \vec{u}'$ is the Lorentz factor of $\mathscr{O}'$ with respect to $\mathscr{O}$. It is the same in (5.1) and (5.2), thanks to the symmetry of the scalar product $\vec{u} \cdot \vec{u}'$ (cf. Remark 4.1 p. 99). Formula (4.33) leads to the following relations:

$$\Gamma_0 = \left(1 - \frac{1}{c^2} \vec{U} \cdot \vec{U} \right)^{-1/2} = \left(1 - \frac{1}{c^2} \vec{U}' \cdot \vec{U}' \right)^{-1/2}. \tag{5.3}$$

We deduce from them that the scalar squares of the relative velocities of the two observers are identical:

$$\vec{U} \cdot \vec{U} = \vec{U}' \cdot \vec{U}', \tag{5.4}$$

or, equivalently, that $\|\vec{U}\|_g = \|\vec{U}'\|_g$. In Galilean physics, this property would be immediate since the velocity of $\mathscr{O}$ relative to $\mathscr{O}'$ is the exact opposite of that of $\mathscr{O}'$ relative to $\mathscr{O}$:

$$\vec{U}' = -\vec{U} \qquad \text{(nonrelativistic)}. \tag{5.5}$$

The relation (5.5) is not possible in the relativistic framework since the vectors $\vec{U}$ and $\vec{U}'$ belong to distinct vector spaces (except if $\mathscr{O}$ and $\mathscr{O}'$ coincide): $\vec{U} \in E_u$ and $\vec{U}' \in E_{u'}$ (cf. Fig. 5.1). The vector spaces E_u and $E_{u'}$ have a non-trivial intersection,[1] but $\vec{U}$ and $\vec{U}'$ do not belong to it. From (5.1) and (5.2), it is easy to

[1]It is the two-dimensional vector subspace (a vector plane) formed by all vectors orthogonal to both $\vec{u}$ and $\vec{u}'$.

establish the relation between $\vec{U}$ and $\vec{U}'$. Substituting (5.1) for $\vec{u}'$ in (5.2), we get indeed

$$\vec{u} = \Gamma_0 \left[\Gamma_0 \left(\vec{u} + \frac{1}{c}\vec{U} \right) + \frac{1}{c}\vec{U}' \right].$$

Thanks to the identity $1 - \Gamma_0^{-2} = c^{-2}\vec{U} \cdot \vec{U}$ [cf. (5.3)], this equation may be rewritten as

$$\boxed{\vec{U}' = -\Gamma_0 \left[\vec{U} + \frac{1}{c}(\vec{U} \cdot \vec{U})\, \vec{u} \right]}. \tag{5.6}$$

At the nonrelativistic limit, $\|\vec{U}\|_g/c \to 0$, $\Gamma_0 \to 1$ and we recover (5.5). Besides, we check that (5.6) implies property (5.4), thanks to $\vec{u} \cdot \vec{U} = 0$, $\vec{u} \cdot \vec{u} = -1$ and (5.3).

Equation (5.6) can be recast in a more geometrical form. Let us indeed compute the orthogonal projection of $\vec{U}$ onto the hyperplane $E_{u'}$ normal to the 4-velocity of $\mathscr{O}'$: from (3.12) and (5.1),

$$\begin{aligned} \perp_{u'}\vec{U} &= \vec{U} + (\vec{u}' \cdot \vec{U})\, \vec{u}' = \vec{U} + \left[\Gamma_0 \left(\vec{u} + \frac{1}{c}\vec{U} \right) \cdot \vec{U} \right] \Gamma_0 \left(\vec{u} + \frac{1}{c}\vec{U} \right) \\ &= \vec{U} + \frac{\Gamma_0^2}{c^2}(\vec{U} \cdot \vec{U}) \left(c\,\vec{u} + \vec{U} \right) = \Gamma_0^2 \left[\vec{U} + \frac{1}{c}(\vec{U} \cdot \vec{U})\, \vec{u} \right], \end{aligned}$$

where use has been made of the identity $1 + \Gamma_0^2(\vec{U} \cdot \vec{U})/c^2 = \Gamma_0^2$, which can be deduced from (5.3). By comparing the above expression with (5.6), we observe that, up to some Lorentz factor, $\vec{U}'$ is nothing but minus the orthogonal projection of $\vec{U}$ onto the local rest space $E_{u'}$ of observer $\mathscr{O}'$:

$$\boxed{\vec{U}' = -\frac{1}{\Gamma_0} \perp_{u'}\vec{U}}. \tag{5.7}$$

The Galilean limit is clear from this formula, because in this limit, all the local rest spaces converge to a single affine subspace of $\mathscr{E}$: the set of events of same Newtonian absolute time, denoted Σ_t in Sect. 1.2.5. Therefore, at the Galilean limit, $\perp_{u'}\vec{U} = \perp_{u}\vec{U} = \vec{U}$. Since in addition, $\Gamma_0 \to 1$, we deduce that $\vec{U}' = -\vec{U}$, i.e. we recover (5.5).

The symmetry $\mathscr{O} \leftrightarrow \mathscr{O}'$ implies that, analogously to (5.6) and (5.7),

$$\boxed{\vec{U} = -\Gamma_0 \left[\vec{U}' + \frac{1}{c}(\vec{U}' \cdot \vec{U}')\, \vec{u}' \right] = -\frac{1}{\Gamma_0} \perp_{u}\vec{U}'}. \tag{5.8}$$

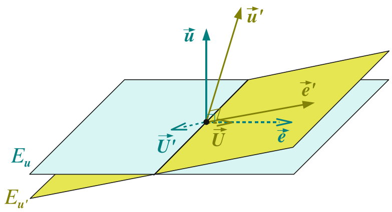

Fig. 5.2 4-velocities $\vec{u}$ and $\vec{u}'$ of observers $\mathscr{O}$ and $\mathscr{O}'$ and the unit spacelike vectors $\vec{e}$ and $\vec{e}'$

5.2.2 Length Contraction

Let us introduce the unit vector in the direction of the velocity of $\mathscr{O}'$ relative to $\mathscr{O}$:

$$\boxed{\vec{e} := \frac{1}{U}\vec{U}}, \qquad \text{where} \quad U := \left\|\vec{U}\right\|_{g}. \tag{5.9}$$

Let us recall that the norm with respect to g is defined by (1.19). $\vec{e}$ is a spacelike unit vector: $\vec{e}\cdot\vec{e} = 1$ ($\vec{e} \in \mathscr{S}$; cf. Sect. 1.4.3). In addition, it belongs to $\mathscr{O}$'s local rest space: $\vec{e} \in E_u$ (cf. Fig. 5.2). If $\vec{U} = 0$, formula (5.9) cannot be used to define $\vec{e}$; the latter can be then chosen to be any unit vector of E_u.

By combining (5.7) and (5.9), $\mathscr{O}$'s velocity relative to $\mathscr{O}'$ can be written as $\vec{U}' = -(U/\Gamma_0)\perp_{u'}\vec{e}$, which can be recast as

$$\boxed{\vec{U}' = U'\,\vec{e}\,'}, \qquad \text{with} \quad \boxed{U' := -U} \tag{5.10}$$

and

$$\boxed{\vec{e}\,' := \frac{1}{\Gamma_0}\perp_{u'}\vec{e}}. \tag{5.11}$$

Thanks to the property $\vec{U}'\cdot\vec{U}' = \vec{U}\cdot\vec{U} = U^2$ [Eq. (5.4)], (5.10) implies $U^2 = U^2\vec{e}\,'\cdot\vec{e}\,'$. It follows that $\vec{e}\,'$ must be a unit vector: $\vec{e}\,'\cdot\vec{e}\,' = 1$ ($\vec{e}\,' \in \mathscr{S}$). By definition $\vec{e}\,' \in E_{u'}$ (cf. Fig. 5.2), and an explicit expression is obtained by comparing the writings (5.6) (with $\vec{U} = U\,\vec{e}$) and (5.10) of $\vec{U}'$:

$$\boxed{\vec{e}\,' = \Gamma_0\left(\vec{e} + \frac{U}{c}\,\vec{u}\right)}. \tag{5.12}$$

This formula can be easily inverted by switching the roles of $\mathscr{O}$ and $\mathscr{O}'$:

$$\vec{e} = \Gamma_0\left(\vec{e}\,' + \frac{U'}{c}\vec{u}'\right) = \Gamma_0\left(\vec{e}\,' - \frac{U}{c}\vec{u}'\right). \tag{5.13}$$

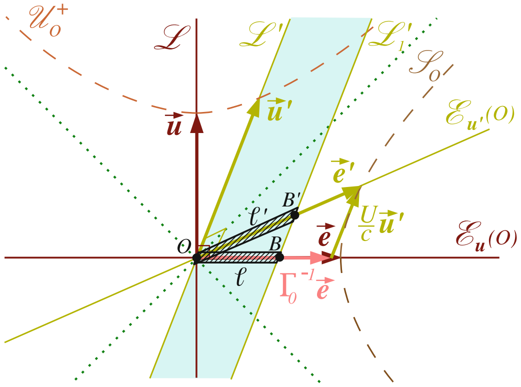

Fig. 5.3 Motion of a ruler attached to observer $\mathscr{O}'$: the ruler's extremities follow the worldlines $\mathscr{L}'$ and $\mathscr{L}_1'$. The coloured domain is the part of spacetime covered by the ruler. $\mathscr{U}_O^+$ (resp. $\mathscr{S}_O$) is the hyperboloid formed by unit timelike (resp. spacelike) vectors originating from O (cf. Sect. 1.4.3 and Fig. 1.6). At the event O, the ruler perceived by $\mathscr{O}'$ is the segment $[OB']$ of the local rest space $\mathscr{E}_{u'}(O)$. At the same event, observer $\mathscr{O}$ perceives the ruler as the segment $[OB]$ of the local rest space $\mathscr{E}_u(O)$

The above relation between a unit vector from the local rest space of $\mathscr{O}'$, $\vec{e}\,'$, and a unit vector from local rest space of $\mathscr{O}$, $\vec{e}$, is at the heart of the so-called "length contraction" phenomenon. Indeed, let us suppose that $\mathscr{O}'$ is travelling with a ruler aligned in the direction $\vec{e}\,'$ (cf. Sect. 3.3.2). The worldline of one of the ruler's extremities is then the worldline of $\mathscr{O}'$, i.e. $\mathscr{L}'$; let us denote by $\mathscr{L}_1'$ the worldline of the second extremity (cf. Fig. 5.3). Let B' be the event of $\mathscr{L}_1'$ simultaneous, with respect to $\mathscr{O}'$, of the event O defined by the crossing of the worldlines of $\mathscr{O}$ and $\mathscr{O}'$: $B' = \mathscr{L}_1' \cap \mathscr{E}_{u'}(O)$. The ruler length ℓ' measured by $\mathscr{O}'$ is then the norm of the vector $\overrightarrow{OB'}$ with respect to $\boldsymbol{g}$ (cf. Sect. 3.3.2). Since $\vec{e}\,'$ is a unit vector and the ruler is aligned along $\vec{e}\,'$, we have

$$\overrightarrow{OB'} = \ell'\,\vec{e}\,'. \tag{5.14}$$

From the point of view of observer $\mathscr{O}$, the ruler extremity at the instant of event O is the point B of $\mathscr{L}_1'$ that is simultaneous to O with respect to $\mathscr{O}$: $B = \mathscr{L}_1' \cap \mathscr{E}_u(O)$ (cf. Fig. 5.3). The ruler length ℓ measured by $\mathscr{O}$ is then the norm of $\overrightarrow{OB}$. This vector is necessarily in the direction of the orthogonal projection of $\vec{e}\,'$ onto E_u; this direction being parallel to $\vec{e}$ [cf. Eq. (5.12)], and $\vec{e}$ being a unit vector, we have

$$\overrightarrow{OB} = \ell\,\vec{e}. \tag{5.15}$$

Moreover, assuming that the ruler size is negligible with respect to the curvature radius of $\mathscr{L}_1'$, we have $\overrightarrow{BB'} = \alpha\,\vec{u}'$, with $\alpha \in \mathbb{R}$ (cf. Fig. 5.3). Accordingly

$$\overrightarrow{OB'} = \overrightarrow{OB} + \overrightarrow{BB'} = \ell\,\vec{e} + \alpha\,\vec{u}'.$$

On the other hand, by combining (5.14) and (5.13), we get

$$\overrightarrow{OB'} = \frac{\ell'}{\Gamma_0}\,\vec{e} + \frac{U\ell'}{c}\,\vec{u}'. \tag{5.16}$$

Since the vectors $\vec{e}$ and $\vec{u}'$ are never collinear, the coefficients in front of them in the above two expressions of $\overrightarrow{OB'}$ must be equal. Hence

$$\boxed{\ell = \frac{\ell'}{\Gamma_0}} \tag{5.17}$$

and $\alpha = U\ell'/c$.

The result (5.17) is known under the name ***length contraction***, or ***FitzGerald–Lorentz contraction***.[2] Since $\Gamma_0 \geq 1$ [cf. Eq. (4.15)], (5.17) implies indeed that $\ell \leq \ell'$. In other words, the ruler is shorter for the observer with respect to whom it is moving.

It is worth noticing that the length contraction occurs only in the direction of the motion of $\mathscr{O}'$ with respect to $\mathscr{O}$. Indeed, in directions orthogonal to it, the local rest spaces of $\mathscr{O}$ and $\mathscr{O}'$ coincide (cf. Fig. 5.2), which implies $\vec{e} = \vec{e}\,'$ and thus the equality of the lengths along these directions. Besides, we shall see in Sect. 5.7 that the length contraction is not directly observable on physical images of a moving object.

Remark 5.1. The derivation of (5.17) shows clearly that the phenomenon of length contraction is a direct consequence of the relativity of simultaneity. If the local rest spaces $\mathscr{E}_{\boldsymbol{u}}(O)$ and $\mathscr{E}_{\boldsymbol{u}'}(O)$ were coinciding, there would be no effect.

5.3 Law of Velocity Composition

5.3.1 General Form

Let us consider a massive particle $\mathscr{P}$ and two observers, $\mathscr{O}$ and $\mathscr{O}'$. We are looking for a relation between the velocity of $\mathscr{P}$ relative to $\mathscr{O}$, $\vec{V}$ say, and that relative to $\mathscr{O}'$, $\vec{V}'$ say (cf. Table 5.1). We shall restrict ourselves to the case where the worldlines of $\mathscr{P}$, $\mathscr{O}$ and $\mathscr{O}'$ intersect at the same event O (cf. Fig. 5.4). The formulas to be used are then (4.31)–(4.33). Let us denote by $\vec{v}$ the 4-velocity of $\mathscr{P}$, leaving $\vec{u}$ and $\vec{u}'$ for the 4-velocities of $\mathscr{O}$ and $\mathscr{O}'$, as in the previous section. Equation (4.31) results in the following two decompositions of $\vec{v}$:

[2] Cf. the historical note p. 126.

Table 5.1 Notations used in this chapter; in Sects. 5.2 and 5.3, one has $\Gamma'_{\mathscr{O}} = \Gamma_{\mathscr{O}'} = \Gamma_0$

	4-velocity	4-accel.	Lor. fact. / $\mathscr{O}$	Lor. fact. / $\mathscr{O}'$	Velocity / $\mathscr{O}$	Velocity / $\mathscr{O}'$	Acceler. / $\mathscr{O}$	Acceler. / $\mathscr{O}'$
$\mathscr{O}$	$\vec{u}$	$\vec{a}_{\mathscr{O}}$	1	$\Gamma'_{\mathscr{O}}$	0	$\vec{U}'$	0	$\vec{\gamma}'_{\mathscr{O}}$
$\mathscr{O}'$	$\vec{u}'$	$\vec{a}_{\mathscr{O}'}$	$\Gamma_{\mathscr{O}'}$	1	$\vec{U}$	0	$\vec{\gamma}_{\mathscr{O}'}$	0
$\mathscr{P}$	$\vec{v}$	$\vec{a}_{\mathscr{P}}$	Γ	Γ'	$\vec{V}$	$\vec{V}'$	$\vec{\gamma}$	$\vec{\gamma}'$

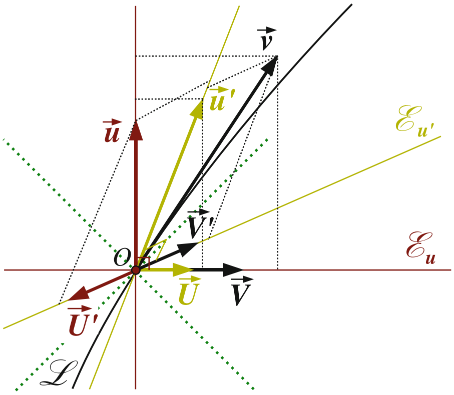

Fig. 5.4 Motion of particle $\mathscr{P}$ with respect to observers $\mathscr{O}$ and $\mathscr{O}'$, at the same event O. $\mathscr{L}$ is the worldline of $\mathscr{P}$, $\vec{v}$ its 4-velocity, $\vec{V}$ its velocity relative to $\mathscr{O}$ and $\vec{V}'$ its velocity relative to $\mathscr{O}'$. $\vec{u}$ is the 4-velocity of $\mathscr{O}$ and $\vec{U}'$ his velocity relative to $\mathscr{O}'$. $\vec{u}'$ is the 4-velocity of $\mathscr{O}'$ and $\vec{U}$ his velocity relative to $\mathscr{O}$. Note that the vectors $\vec{U}$ and $\vec{V}$ belong to the local rest space of $\mathscr{O}$, $E_{\boldsymbol{u}}$, whereas $\vec{U}'$ and $\vec{V}'$ belong to the local rest space of $\mathscr{O}'$, $E_{\boldsymbol{u}'}$. The figure corresponds to the case where $\vec{u}$, $\vec{u}'$ and $\vec{v}$ are coplanar

$$\vec{v} = \Gamma\left(\vec{u} + \frac{1}{c}\vec{V}\right) = \Gamma'\left(\vec{u}' + \frac{1}{c}\vec{V}'\right), \quad \text{with} \quad \vec{u}\cdot\vec{V} = \vec{u}'\cdot\vec{V}' = 0, \tag{5.18}$$

where Γ (resp. Γ') is the Lorentz factor of $\mathscr{P}$ with respect to observer $\mathscr{O}$ (resp. $\mathscr{O}'$): from (4.10) and (4.33),

$$\Gamma = -\vec{u}\cdot\vec{v} = \left(1 - \frac{1}{c^2}\vec{V}\cdot\vec{V}\right)^{-1/2}, \tag{5.19}$$

$$\Gamma' = -\vec{u}'\cdot\vec{v} = \left(1 - \frac{1}{c^2}\vec{V}'\cdot\vec{V}'\right)^{-1/2}. \tag{5.20}$$

We deduce from (5.18) that

$$\vec{V}' = \frac{\Gamma}{\Gamma'}\left(c\,\vec{u} + \vec{V}\right) - c\,\vec{u}'. \tag{5.21}$$

Let us project this relation onto $E_{\boldsymbol{u}'}$, via the operator $\perp_{\boldsymbol{u}'}$ (cf. Sect. 3.2.5); since $\perp_{\boldsymbol{u}'}\vec{\boldsymbol{V}}' = \vec{\boldsymbol{V}'}$, $\perp_{\boldsymbol{u}'}\vec{\boldsymbol{u}} = \Gamma_0 c^{-1}\vec{\boldsymbol{U}'}$ [cf. Eq. (5.2)] and $\perp_{\boldsymbol{u}'}\vec{\boldsymbol{u}}' = 0$, we get

$$\vec{\boldsymbol{V}}' = \frac{\Gamma}{\Gamma'}\left[\Gamma_0\,\vec{\boldsymbol{U}'} + \perp_{\boldsymbol{u}'}\vec{\boldsymbol{V}}\right]. \tag{5.22}$$

To evaluate the term Γ/Γ', let us proceed as follows. The relative velocity $\vec{\boldsymbol{V}}'$ is by definition orthogonal to the 4-velocity of $\mathscr{O}'$: $\vec{\boldsymbol{u}}'\cdot\vec{\boldsymbol{V}}' = 0$. Substituting (5.21) for $\vec{\boldsymbol{V}}'$ and expanding yields

$$\frac{\Gamma}{\Gamma'}\big(c\,\underbrace{\vec{\boldsymbol{u}}'\cdot\vec{\boldsymbol{u}}}_{-\Gamma_0} + \vec{\boldsymbol{u}}'\cdot\vec{\boldsymbol{V}}\big) - c\,\underbrace{\vec{\boldsymbol{u}}'\cdot\vec{\boldsymbol{u}}'}_{-1} = 0. \tag{5.23}$$

Now, from (5.1) and the property $\vec{\boldsymbol{u}}\cdot\vec{\boldsymbol{V}} = 0$,

$$\vec{\boldsymbol{u}}'\cdot\vec{\boldsymbol{V}} = \Gamma_0\left(\vec{\boldsymbol{u}} + \frac{1}{c}\vec{\boldsymbol{U}}\right)\cdot\vec{\boldsymbol{V}} = \frac{\Gamma_0}{c}\,\vec{\boldsymbol{U}}\cdot\vec{\boldsymbol{V}}. \tag{5.24}$$

Substituting this value in (5.23), we obtain the expression of $\mathscr{P}$'s Lorentz factor with respect to $\mathscr{O}'$:

$$\boxed{\Gamma' = \Gamma\,\Gamma_0\left(1 - \frac{1}{c^2}\vec{\boldsymbol{U}}\cdot\vec{\boldsymbol{V}}\right)}. \tag{5.25}$$

Let us eliminate $\vec{\boldsymbol{U}}$ from this formula: from (5.6) and $\vec{\boldsymbol{u}}\cdot\vec{\boldsymbol{V}} = 0$, we get immediately

$$\vec{\boldsymbol{U}}'\cdot\vec{\boldsymbol{V}} = -\Gamma_0\,\vec{\boldsymbol{U}}\cdot\vec{\boldsymbol{V}}. \tag{5.26}$$

Moreover, since $\vec{\boldsymbol{U}}' \in E_{\boldsymbol{u}'}$, one may write $\vec{\boldsymbol{U}}'\cdot\vec{\boldsymbol{V}} = \vec{\boldsymbol{U}}'\cdot\perp_{\boldsymbol{u}'}\vec{\boldsymbol{V}}$ and recast (5.25) as

$$\boxed{\Gamma' = \Gamma\left[\Gamma_0 + \frac{1}{c^2}\vec{\boldsymbol{U}}'\cdot\perp_{\boldsymbol{u}'}\vec{\boldsymbol{V}}\right]}. \tag{5.27}$$

Substituting this value in (5.22), we get

$$\boxed{\vec{\boldsymbol{V}}' = \frac{1}{\Gamma_0 + \frac{1}{c^2}\vec{\boldsymbol{U}}'\cdot\perp_{\boldsymbol{u}'}\vec{\boldsymbol{V}}}\left[\perp_{\boldsymbol{u}'}\vec{\boldsymbol{V}} + \Gamma_0\,\vec{\boldsymbol{U}}'\right]}, \tag{5.28}$$

with, from (3.12), (5.24) and (5.26),

$$\perp_{\boldsymbol{u}'}\vec{\boldsymbol{V}} := \vec{\boldsymbol{V}} + (\vec{\boldsymbol{u}}'\cdot\vec{\boldsymbol{V}})\,\vec{\boldsymbol{u}}' = \vec{\boldsymbol{V}} + \frac{\Gamma_0}{c}(\vec{\boldsymbol{U}}\cdot\vec{\boldsymbol{V}})\,\vec{\boldsymbol{u}}' = \vec{\boldsymbol{V}} - \frac{1}{c}(\vec{\boldsymbol{U}}'\cdot\vec{\boldsymbol{V}})\,\vec{\boldsymbol{u}}'. \tag{5.29}$$

Formula (5.28) is the law of velocity composition that we sought for: it expresses the velocity $\vec{V}'$ of $\mathscr{P}$ with respect to the "new" observer $\mathscr{O}'$, in terms of the velocity $\vec{V}$ of $\mathscr{P}$ with respect to the "old' observer $\mathscr{O}$ and the velocity $\vec{U}'$ of $\mathscr{O}$ relative to $\mathscr{O}'$. The quantity Γ_0 must be seen as the function of $\vec{U}'$ given by (5.3). At the nonrelativistic limit, $U'/c \to 0$ and $\Gamma_0 \to 1$, so that (5.29) leads to $\perp_{u'}\vec{V} \simeq \vec{V}$ (cf. the discussion p. 133); formula (5.28) reduces then to the well-known velocity addition law

$$\vec{V}' = \vec{V} + \vec{U}' \qquad \text{(nonrelativistic).} \tag{5.30}$$

Remark 5.2. It is clear on (5.28) that the vectors $\vec{V}'$ and $\vec{V}$ do not belong to the same hyperplane of E (except if $\vec{U}' = 0$; cf. Remark 5.3): $\vec{V}$ has first to be projected onto $E_{u'}$, via the operator $\perp_{u'}$, before being added to observer $\mathscr{O}$'s velocity $\vec{U}'$ (corrected by the Lorentz factor Γ_0).

Remark 5.3. If $\vec{U}' = 0$, observers $\mathscr{O}$ and $\mathscr{O}'$ coincide in O, formula (5.29) reduces to $\perp_{u'}\vec{V} = \vec{V}$ and the law (5.28) yields $\vec{V}' = \vec{V}$, as it should.

5.3.2 Decomposition in Parallel and Transverse Parts

Observer $\mathscr{O}$ is measuring two velocities: the velocity $\vec{V}$ of particle $\mathscr{P}$ and the velocity $\vec{U}$ of observer $\mathscr{O}'$. It is then instructive to decompose $\vec{V}$ into a part parallel to $\vec{U}$ and a part transverse to $\vec{U}$ (more precisely orthogonal to $\vec{U}$ with respect to $\boldsymbol{g}$). Here we assume that $\vec{U} \neq 0$. Otherwise observers $\mathscr{O}$ and $\mathscr{O}'$ coincide at O, and we are in the trivial situation where $\vec{V}' = \vec{V}$.

The parallel and transverse parts of $\vec{V}$ with respect to $\vec{U}$ are defined by

$$\vec{V} =: V_{\parallel}\,\vec{e} + \vec{V}_{\perp}, \qquad \text{with} \quad \vec{e}\cdot\vec{V}_{\perp} = 0, \tag{5.31}$$

where $\vec{e}$ is the unit vector in the direction of $\vec{U}$ introduced in Sect. 5.2.2 [Eq. (5.9)]. It is worth noticing that $\vec{V}_{\perp}$ is both orthogonal to $\vec{u}$ and $\vec{u}'$:

$$\vec{V}_{\perp} \in E_{u} \cap E_{u'}. \tag{5.32}$$

Proof. $\vec{V}_{\perp} \in E_u$ since $\vec{V} \in E_u$ and $\vec{e} \in E_u$. Moreover, thanks to (5.1),

$$\vec{u}'\cdot\vec{V}_{\perp} = \Gamma_0\left(\vec{u} + \frac{1}{c}U\,\vec{e}\right)\cdot\vec{V}_{\perp} = \Gamma_0\Big(\underbrace{\vec{u}\cdot\vec{V}_{\perp}}_{0} + \frac{1}{c}U\,\underbrace{\vec{e}\cdot\vec{V}_{\perp}}_{0}\Big) = 0,$$

which shows that $\vec{V}_{\perp} \in E_{u'}$. □

Similarly, the parallel and transverse parts of $\vec{V}'$ with respect to $\vec{U}'$ are defined by

$$\vec{V}' =: V'_{\parallel}\,\vec{e}\,' + \vec{V}'_{\perp}, \qquad \text{with} \quad \vec{e}\,'\cdot\vec{V}'_{\perp} = 0, \tag{5.33}$$

where $\vec{e}\,'$ is the unit vector in the direction of $\vec{U}'$ introduced in Sect. 5.2.2 [Eq. (5.10)]. As for $\vec{V}_\perp$, we have

$$\vec{V}'_\perp \in E_u \cap E_{u'}. \tag{5.34}$$

Let us express the projection of $\vec{V}$ onto $E_{u'}$, which appears in the velocity composition law (5.28). The operator $\perp_{u'}$ being linear, we get

$$\perp_{u'}\vec{V} = \perp_{u'}(V_\| \,\vec{e} + \vec{V}_\perp) = V_\| \perp_{u'}\vec{e} + \perp_{u'}\vec{V}_\perp.$$

Now, from (5.11), $\perp_{u'}\vec{e} = \Gamma_0\,\vec{e}\,'$ and from (5.32), $\perp_{u'}\vec{V}_\perp = \vec{V}_\perp$. Hence

$$\perp_{u'}\vec{V} = \Gamma_0 V_\| \,\vec{e}\,' + \vec{V}_\perp. \tag{5.35}$$

In particular,

$$\vec{U}'\cdot\perp_{u'}\vec{V} = U'\Gamma_0 V_\| \underbrace{\vec{e}\,'\cdot\vec{e}\,'}_{1} + U'\,\vec{e}\,'\cdot\vec{V}_\perp.$$

Now $\vec{e}\,'\cdot\vec{V}_\perp = \Gamma_0[\vec{e} + (U/c)\vec{u}]\cdot\vec{V}_\perp = \Gamma_0[0 + (U/c)\times 0] = 0$; hence,

$$\vec{U}'\cdot\perp_{u'}\vec{V} = \Gamma_0 U' V_\|. \tag{5.36}$$

In view of (5.35) and (5.36), formula (5.28) can be recast as

$$\vec{V}' = \frac{1}{1 + U'V_\|/c^2}\left[(V_\| + U')\vec{e}\,' + \frac{1}{\Gamma_0}\vec{V}_\perp\right]. \tag{5.37}$$

Comparing with (5.33), we obtain

$$\boxed{V'_\| = \frac{V_\| + U'}{1 + U'V_\|/c^2}} \tag{5.38a}$$

$$\boxed{\vec{V}'_\perp = \frac{1}{\Gamma_0\left(1 + U'V_\|/c^2\right)}\,\vec{V}_\perp}. \tag{5.38b}$$

Besides, from (5.36) and (5.27), we get the expression of the Lorentz factor of $\mathscr{P}$ with respect to $\mathscr{O}'$:

$$\boxed{\Gamma' = \Gamma\Gamma_0\left(1 + \frac{U'V_\|}{c^2}\right)}. \tag{5.39}$$

Since $U' = -U$ [Eq. (5.10)], the above formulas are obviously equivalent to

$$\boxed{V'_{\parallel} = \frac{V_{\parallel} - U}{1 - U V_{\parallel}/c^2}} \tag{5.40a}$$

$$\boxed{\vec{\boldsymbol{V}}'_{\perp} = \frac{1}{\Gamma_0\left(1 - U V_{\parallel}/c^2\right)}\,\vec{\boldsymbol{V}}_{\perp}} \tag{5.40b}$$

$$\boxed{\Gamma' = \Gamma\,\Gamma_0\left(1 - \frac{U V_{\parallel}}{c^2}\right)}. \tag{5.40c}$$

Remark 5.4. Many authors present a version of the law of velocity composition in which all velocity vectors belong to the same three-dimensional vector space (cf., for instance, the textbooks by Møller (1952), Fock (1955) or, at the limit of low velocities, Landau and Lifshitz (1975)). This amounts to introducing the following "representing" vector of $\vec{\boldsymbol{V}}$ in the space $E_{\boldsymbol{u}'}$:

$$\vec{\boldsymbol{V}}_{*} := V_{\parallel}\,\vec{\boldsymbol{e}}\,' + \vec{\boldsymbol{V}}_{\perp}. \tag{5.41}$$

We have $\vec{\boldsymbol{V}}_{*} \in E_{\boldsymbol{u}'}$ since $\vec{\boldsymbol{e}}\,' \in E_{\boldsymbol{u}'}$ and $\vec{\boldsymbol{V}}_{\perp} \in E_{\boldsymbol{u}'}$, so that the three vectors $\vec{\boldsymbol{V}}_{*}$, $\vec{\boldsymbol{V}}'$ and $\vec{\boldsymbol{U}}'$ do belong to the same three-dimensional vector space, namely, $E_{\boldsymbol{u}'}$. The identification $\vec{\boldsymbol{V}} \leftrightarrow \vec{\boldsymbol{V}}_{*}$ amounts to considering an orthonormal basis $(\vec{\boldsymbol{e}}_i)$ in the local rest space of $\mathscr{O}$ and an orthonormal basis $(\vec{\boldsymbol{e}}'_i)$ in the local rest space of $\mathscr{O}'$, such that $\vec{\boldsymbol{e}}_1 = \vec{\boldsymbol{e}} = U^{-1}\vec{\boldsymbol{U}}$ [cf. Eq. (5.9)] and $\vec{\boldsymbol{e}}'_1 = \vec{\boldsymbol{e}}' = (U')^{-1}\vec{\boldsymbol{U}}'$ [cf. Eq. (5.10)]. If (V^i) denote the components of $\vec{\boldsymbol{V}}$ within the basis $(\vec{\boldsymbol{e}}_i)$, the vector $\vec{\boldsymbol{V}}_{*}$ is then defined by $\vec{\boldsymbol{V}}_{*} := V^i\,\vec{\boldsymbol{e}}'_i$. Comparing with (5.35) and (5.41), we get

$$\Gamma_0^{-1}\perp_{\boldsymbol{u}'}\vec{\boldsymbol{V}} = \vec{\boldsymbol{V}}_{*} + (\Gamma_0^{-1} - 1)\vec{\boldsymbol{V}}_{\perp}.$$

In particular,

$$\vec{\boldsymbol{U}}'\cdot\Gamma_0^{-1}\perp_{\boldsymbol{u}'}\vec{\boldsymbol{V}} = \vec{\boldsymbol{U}}'\cdot\vec{\boldsymbol{V}}_{*} + (\Gamma_0^{-1} - 1)\underbrace{\vec{\boldsymbol{U}}'\cdot\vec{\boldsymbol{V}}_{\perp}}_{0} = \vec{\boldsymbol{U}}'\cdot\vec{\boldsymbol{V}}_{*}.$$

The law of velocity composition (5.28) can be then written as

$$\vec{\boldsymbol{V}}' = \frac{1}{1 + \frac{1}{c^2}\vec{\boldsymbol{U}}'\cdot\vec{\boldsymbol{V}}_{*}}\left[\vec{\boldsymbol{V}}_{*} + (\Gamma_0^{-1} - 1)\vec{\boldsymbol{V}}_{\perp} + \vec{\boldsymbol{U}}'\right]. \tag{5.42}$$

Now, from (5.41), $\vec{\boldsymbol{V}}_{\perp} = \vec{\boldsymbol{V}}_{*} - V_{\parallel}\,\vec{\boldsymbol{e}}\,' = \vec{\boldsymbol{V}}_{*} - (\vec{\boldsymbol{e}}\,'\cdot\vec{\boldsymbol{V}}_{*})\,\vec{\boldsymbol{e}}\,'$, which, thanks to (5.10), can be recast as

$$\vec{\boldsymbol{V}}_{\perp} = \vec{\boldsymbol{V}}_{*} - \frac{1}{U'^2}(\vec{\boldsymbol{U}}'\cdot\vec{\boldsymbol{V}}_{*})\,\vec{\boldsymbol{U}}'.$$

Substituting this value in (5.42) and using the identity $(\Gamma_0^{-1} - 1)/U'^2 = -c^{-2}\Gamma_0/(1+\Gamma_0)$, which is easily deduced from $\Gamma_0 = (1-U'^2/c^2)^{-1/2}$ [Eq. (5.3)], we obtain

$$\vec{V}' = \frac{1}{1+\frac{1}{c^2}\vec{U}'\cdot\vec{V}_*}\left\{\vec{V}_* + \vec{U}' + \frac{\Gamma_0}{c^2(1+\Gamma_0)}\left[(\vec{U}'\cdot\vec{V}_*)\,\vec{U}' - (\vec{U}'\cdot\vec{U}')\,\vec{V}_*\right]\right\}.$$

It is this relation between $\vec{V}'$, $\vec{V}_*$ and $\vec{U}'$ [Γ_0 being the function of $\vec{U}'$ given by (5.3)] that is sometimes presented as the law of velocity composition.[3] The more complicated structure of this equation, as compared with (5.28), shows clearly the advantage of adopting a four-dimensional point of view, instead of a three-dimensional one.

5.3.3 *Collinear Velocities*

When the velocities $\vec{V}$ and $\vec{U}$ of, respectively, $\mathscr{P}$ and $\mathscr{O}'$ relative to $\mathscr{O}$ are collinear, the above formulas simplify significantly. Let us first note that in this particular case, the three 4-velocities $\vec{u}$, $\vec{u}'$ and $\vec{v}$ are coplanar and the relative velocities $\vec{V}$ and $\vec{V}'$ have a vanishing transverse part:

$$\vec{V}_\perp = 0 \qquad \text{and} \qquad \vec{V}'_\perp = 0. \tag{5.43}$$

This is the case depicted in Fig. 5.4. Setting $V := V_\parallel$ and $V' := V'_\parallel$, we can write

$$\vec{V} = V\,\vec{e} \qquad \text{and} \qquad \vec{V}' = V'\,\vec{e}\,'. \tag{5.44}$$

The law of velocity composition (5.38a) yields immediately

$$\boxed{V' = \frac{V+U'}{1+U'V/c^2}}. \tag{5.45}$$

whereas (5.38b) is reduced to $0 = 0$. Moreover, the law (5.39) for the transformation of the Lorentz factor becomes, in the present case,

$$\boxed{\Gamma' = \Gamma\Gamma_0\left(1+\frac{U'V}{c^2}\right)}. \tag{5.46}$$

[3] Cf., for instance, Eqs. (55') in Chap. II of Møller (1952), (16.08) in Fock (1955), (4) in Takahashi (1982), (4) in Ungar (1991) and (25) in Costella et al. (2001).

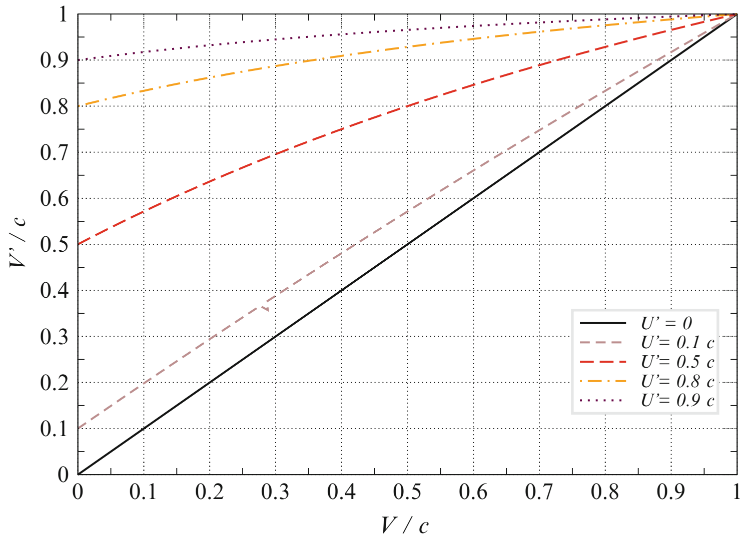

Fig. 5.5 Velocity V' of particle $\mathscr{P}$ relative to $\mathscr{O}'$, as a function of the velocity V of $\mathscr{P}$ relative to $\mathscr{O}$ and of the velocity U' of $\mathscr{O}$ relative to $\mathscr{O}'$, as given by (5.45) (case of collinear velocities)

Remark 5.5. Since $U' = -U$ [Eq. (5.10)], the above formulas can also be written as

$$V' = \frac{V - U}{1 - UV/c^2} \qquad \text{and} \qquad \Gamma' = \Gamma\Gamma_0\left(1 - \frac{UV}{c^2}\right). \tag{5.47}$$

The norm of the relative velocity $\vec{V}$ is $\|\vec{V}\|_g := \sqrt{\vec{V} \cdot \vec{V}} = |V|$ and that of $\vec{V}'$ is $\|\vec{V}'\|_g := \sqrt{\vec{V}' \cdot \vec{V}'} = |V'|$. We have seen in Sect. 4.3.3 that one has always $\|\vec{V}\|_g < c$. Formula (5.45) ensures that $\|\vec{V}'\|_g < c$ for any value of U' such that $|U'| < c$, as it can be seen in Fig. 5.5. This would of course not be the case for the Galilean formula $V' = V + U'$.

5.3.4 Alternative Formula

We can derive a formula for $\vec{V}'$ that generalizes the nonrelativistic law

$$\vec{V}' = \vec{V} - \vec{U} \qquad \text{(nonrelativistic)}, \tag{5.48}$$

$\vec{U}$ being the velocity of $\mathscr{O}'$ relative to $\mathscr{O}$. Formula (5.48) is of course equivalent to (5.30) since in the nonrelativistic case, $\vec{U}' = -\vec{U}$.

The starting point is (5.21). Using (5.25) to express Γ/Γ' and (5.1) to replace $\vec{u}'$, we get

$$\vec{V}' = \frac{\Gamma_0}{1 - \frac{1}{c^2}\vec{U}\cdot\vec{V}}\left[\left(\Gamma_0^{-2} - 1 + \frac{1}{c^2}\vec{U}\cdot\vec{V}\right)c\,\vec{u} + \Gamma_0^{-2}\vec{V} - \left(1 - \frac{1}{c^2}\vec{U}\cdot\vec{V}\right)\vec{U}\right].$$

Now, from (5.3), $\Gamma_0^{-2} = 1 - c^{-2}\vec{U}\cdot\vec{U}$. Hence

$$\boxed{\begin{aligned}\vec{V}' = \frac{\Gamma_0}{1 - \frac{1}{c^2}\vec{U}\cdot\vec{V}}\Bigg\{&\vec{V} - \vec{U} + \frac{1}{c^2}\left[(\vec{U}\cdot\vec{V})\,\vec{U} - (\vec{U}\cdot\vec{U})\,\vec{V}\right]\\ &+\frac{1}{c}\left[\vec{U}\cdot(\vec{V}-\vec{U})\right]\vec{u}\Bigg\}.\end{aligned}} \tag{5.49}$$

This formula expresses the velocity $\vec{V}'$ of particle $\mathscr{P}$ relative to the "new" observer $\mathscr{O}'$, in terms of the velocity $\vec{V}$ of $\mathscr{P}$ relative to the "old" observer $\mathscr{O}$ and the velocity $\vec{U}$ of $\mathscr{O}'$ relative to $\mathscr{O}$ (in contrast with (5.28), which involves the velocity $\vec{U}'$ of $\mathscr{O}$ relative to $\mathscr{O}'$). At the nonrelativistic limit, $\|\vec{U}\|_g/c \to 0$, $\Gamma_0 \to 1$, and (5.49) reduces to the law (5.48).

Remark 5.6. The vectors $\vec{V}$, $\vec{U}$ and $\vec{V}'$ do not belong to the same vector subspace of E: $\vec{V} \in E_{\boldsymbol{u}}$, $\vec{U} \in E_{\boldsymbol{u}}$ and $\vec{V}' \in E_{\boldsymbol{u}'}$ (cf. Fig. 5.4). In formula (5.49), the nonrelativistic term $\vec{V} - \vec{U}$, as well as the relativistic term $c^{-2}[(\vec{U}\cdot\vec{V})\,\vec{U} - (\vec{U}\cdot\vec{U})\,\vec{V}]$, are in $E_{\boldsymbol{u}}$. It is the last term, namely, the one along $\vec{u}$, that makes the result leave the space $E_{\boldsymbol{u}}$ and send it to $E_{\boldsymbol{u}'}$.

Historical note: *The law of velocity composition, in a form equivalent to (5.38), has been obtained in 1905 by Albert Einstein (cf. p. 26) (1905b) and Henri Poincaré (cf. p. 26) (1906).*

5.3.5 Experimental Verification: Fizeau Experiment

In 1850, Hippolyte Fizeau (cf. p. 122) has performed an experiment that is interpreted today as a test of the law of velocity composition (5.45) (Fizeau 1851). The experimental setup is depicted in Fig. 5.6: the light emitted by a source S reaches a beam splitter L. Each subsequent beam crosses a U-shaped tube in which water is circulating. The upper beam in Fig. 5.6, which we shall call number 1 (full arrow), moves in the sense opposite to water, before reaching the mirror M and propagating in the lower branch of the tube, still in countermotion with respect to water. On the contrary, the beam number 2, which starts in the lower branch (empty arrow in Fig. 5.6), is always travelling in the same sense as water. Thanks to the lens l_1, the two beams are recombined and interfere at the level of the detector D.

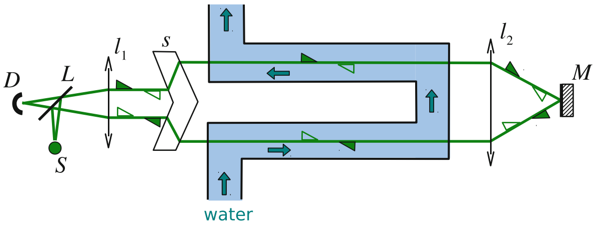

Fig. 5.6 Fizeau experiment

The phase difference reflects the dissymmetry between the two beams, which is interpreted as a difference of propagation speed.

With respect to water, the velocity of light in each beam is c/n, where $n \simeq 1.33$ is the refraction index of water. The velocities of beams 1 and 2 with respect to the laboratory are then

$$c_1 = \frac{c}{n} - \alpha V \qquad \text{and} \qquad c_2 = \frac{c}{n} + \alpha V, \tag{5.50}$$

where V is the velocity of water with respect to the laboratory and α is a coefficient that takes the value 1 for the Galilean law of velocity addition. For the relativistic law, α is determined from (5.47):

$$c_1 = \frac{c/n - V}{1 - (c/n)V/c^2} \simeq \frac{c}{n} - \left(1 - \frac{1}{n^2}\right) V,$$

where, in the second equality, only first-order terms in V/c have been kept. By comparing with (5.50), we get the relativity prediction:

$$\alpha = 1 - \frac{1}{n^2}. \tag{5.51}$$

The first-order expansion of c_2 would lead to the same result. If ℓ is the length of one branch of the U-shaped tube, the difference of travel time between the two beams is $\Delta t = 2\ell/c_2 - 2\ell/c_1$. It results in the phase shift $\Delta\phi = 2\pi \Delta t/T$, T being the radiation period, related to the wavelength by $T = \lambda/c$. From (5.50) and still at the first order in V/c, we obtain

$$\Delta\phi = 8\pi n^2 \frac{\ell}{\lambda} \frac{V}{c} \alpha. \tag{5.52}$$

The measure of this phase shift, via the position of the interference fringes, provides α. The obtained value is in agreement with the prediction (5.51) arising from the relativistic law of velocity composition. In particular, the value $\alpha = 1$ of the Galilean law is not recovered.

Historical note: *The experiment has been performed by Fizeau more than half a century before the advent of special relativity! At that time, Fizeau interpreted it as the confirmation of the hypothesis of partial aether dragging by moving bodies.*

This hypothesis had been emitted by Augustin Fresnel (cf. p. 125) in 1818 to explain the result of Arago experiment (cf. historical note p. 124). Within Fresnel hypothesis, α is the coefficient of aether dragging and takes the value $1-1/n^2$, as the relativistic result (5.51). Fizeau experiment has been repeated with an enhanced precision by Michelson (cf. p. 125) and Morley (cf. p. 125) in (1886), and in the years 1914–1919 by Pieter Zeeman,[4] who substituted water by solid bodies (glass or quartz), in order to increase the refraction index n. On the theoretical ground, the prediction (5.51) of special relativity has been established by Max Laue[5] in (1907). For Albert Einstein, Fizeau experiment constituted one of the main supports of special relativity, at the same level, if not higher, than Michelson–Morley experiment described in Sect. 4.6.3 (Einstein 1956). It must however be noticed that none of these two experiments is mentioned in Einstein's famous 1905 article (Einstein 1905b).

5.4 Law of Acceleration Composition

After the velocities, let us focus now on the composition of the relative accelerations introduced in Sect. 4.5. We would like to express the acceleration $\vec{\boldsymbol{\gamma}}'$ of particle $\mathscr{P}$ relative to observer $\mathscr{O}'$ in terms of the acceleration $\vec{\boldsymbol{\gamma}}$ of $\mathscr{P}$ relative to observer $\mathscr{O}$ and, among others, the acceleration $\vec{\boldsymbol{\gamma}}'_{\mathscr{O}}$ of $\mathscr{O}$ relative to $\mathscr{O}'$ (cf. Table 5.1 for the notations). Contrary to what we did for velocities, we shall no longer suppose that $\mathscr{O}$, $\mathscr{O}'$ and $\mathscr{P}$ meet at the same event O. However, in order to simplify the problem, we shall suppose that $\mathscr{O}'$ is an inertial observer: his 4-acceleration and 4-rotation then vanish.

The 4-acceleration of $\mathscr{P}$, $\vec{\boldsymbol{a}}_{\mathscr{P}}$, is related to the acceleration $\vec{\boldsymbol{\gamma}}$ and velocity $\vec{\boldsymbol{V}}$ of $\mathscr{P}$ relative to $\mathscr{O}$ by formula (4.60) (cf. Table 5.1 for some change in notations):

$$\begin{aligned}\vec{\boldsymbol{a}}_{\mathscr{P}} = \frac{\Gamma^2}{c^2}\Bigg\{&\vec{\boldsymbol{\gamma}} + \vec{\boldsymbol{\omega}} \times_u \left(\vec{\boldsymbol{\omega}} \times_u \overrightarrow{OM}\right) + 2\vec{\boldsymbol{\omega}} \times_u \vec{\boldsymbol{V}} + \frac{\mathrm{d}\vec{\boldsymbol{\omega}}}{\mathrm{d}t} \times_u \overrightarrow{OM} \\ &+c^2(1 + \vec{\boldsymbol{a}}_{\mathscr{O}} \cdot \overrightarrow{OM})\, \vec{\boldsymbol{a}}_{\mathscr{O}} + \frac{1}{\Gamma}\frac{\mathrm{d}\Gamma}{\mathrm{d}t}\left(\vec{\boldsymbol{V}} + \vec{\boldsymbol{\omega}} \times_u \overrightarrow{OM}\right) + c\Bigg[\frac{\mathrm{d}\vec{\boldsymbol{a}}_{\mathscr{O}}}{\mathrm{d}t} \cdot \overrightarrow{OM} \\ &+2\vec{\boldsymbol{a}}_{\mathscr{O}} \cdot \left(\vec{\boldsymbol{V}} + \vec{\boldsymbol{\omega}} \times_u \overrightarrow{OM}\right) + \frac{1}{\Gamma}\frac{\mathrm{d}\Gamma}{\mathrm{d}t}(1 + \vec{\boldsymbol{a}}_{\mathscr{O}} \cdot \overrightarrow{OM})\Bigg]\vec{\boldsymbol{u}}\Bigg\}. \end{aligned} \tag{5.53}$$

[4]**Pieter Zeeman** (1865–1943): Dutch physicist, former assistant of Lorentz, he discovered that a spectral line splits in several lines in presence of a magnetic field (the so-called Zeeman effect, for which he received the 1902 Nobel Prize in Physics).

[5]**Max Laue** (1879–1960): German physicist, who got the 1914 Nobel Prize in Physics for the discovery of the diffraction of X-rays in crystals. Former student and assistant of Planck (cf. p. 279), he contributed significantly to the development of special relativity, of which he wrote the very first textbook in 1911 (Laue 1911c). His father having been ennobled in 1913, he changed his name to **Max von Laue**.

In this formula, $\vec{\boldsymbol{\omega}}$ is the 4-rotation of observer $\mathscr{O}$ and the time derivative of the Lorentz factor Γ is given by (4.61), with $\vec{\boldsymbol{a}}$ replaced by $\vec{\boldsymbol{a}}_{\mathscr{O}}$. If, on the other side, the same 4-acceleration $\vec{\boldsymbol{a}}_{\mathscr{P}}$ is expressed in terms of quantities relative to $\mathscr{O}'$, and no longer $\mathscr{O}$, we may use the simplified formula (4.63), since $\mathscr{O}'$ is an inertial observer. Adapting it to the notations defined in Table 5.1, we get

$$\vec{\boldsymbol{a}}_{\mathscr{P}} = \frac{\Gamma'^2}{c^2}\left[\vec{\boldsymbol{\gamma}}' + \frac{\Gamma'^2}{c^2}(\vec{\boldsymbol{\gamma}}'\cdot\vec{\boldsymbol{V}}')\left(\vec{\boldsymbol{V}}' + c\,\vec{\boldsymbol{u}}'\right)\right]. \tag{5.54}$$

Similarly, applying (4.63) to $\mathscr{O}$'s 4-acceleration, we get (still with Table 5.1 notations)

$$\vec{\boldsymbol{a}}_{\mathscr{O}} = \frac{\Gamma_{\mathscr{O}}'^{\,2}}{c^2}\left[\vec{\boldsymbol{\gamma}}'_{\mathscr{O}} + \frac{\Gamma_{\mathscr{O}}'^{\,2}}{c^2}(\vec{\boldsymbol{\gamma}}'_{\mathscr{O}}\cdot\vec{\boldsymbol{U}}')\left(\vec{\boldsymbol{U}}' + c\,\vec{\boldsymbol{u}}'\right)\right]. \tag{5.55}$$

From (5.54), we deduce that

$$\vec{\boldsymbol{\gamma}}' + \frac{\Gamma'^2}{c^2}(\vec{\boldsymbol{\gamma}}'\cdot\vec{\boldsymbol{V}}')\vec{\boldsymbol{V}}' = \frac{c^2}{\Gamma'^2}\perp_{u'}\vec{\boldsymbol{a}}_{\mathscr{P}},$$

where $\perp_{\boldsymbol{u}'}$ stands for the orthogonal projector onto the vector space $E_{\boldsymbol{u}'}$ (cf. Sect. 3.2.5). Let us substitute (5.53) for $\vec{\boldsymbol{a}}_{\mathscr{P}}$ in the right-hand side of this expression. Using $\perp_{\boldsymbol{u}'}\vec{\boldsymbol{u}} = (\Gamma'_{\mathscr{O}}/c)\,\vec{\boldsymbol{U}}'$ [Eq. (4.32)] and the following relation deduced from (5.55),

$$\perp_{u'}\vec{\boldsymbol{a}}_{\mathscr{O}} = \frac{\Gamma_{\mathscr{O}}'^{\,2}}{c^2}\left[\vec{\boldsymbol{\gamma}}'_{\mathscr{O}} + \frac{\Gamma_{\mathscr{O}}'^{\,2}}{c^2}(\vec{\boldsymbol{\gamma}}'_{\mathscr{O}}\cdot\vec{\boldsymbol{U}}')\vec{\boldsymbol{U}}'\right],$$

we obtain

$$\boxed{\begin{aligned}\vec{\boldsymbol{\gamma}}' + \frac{\Gamma'^2}{c^2}(\vec{\boldsymbol{\gamma}}'\cdot\vec{\boldsymbol{V}}')\vec{\boldsymbol{V}}' = \frac{\Gamma^2}{\Gamma'^2}\Bigg\{&\perp_{u'}\Bigg[\vec{\boldsymbol{\gamma}} + \vec{\boldsymbol{\omega}}\times_u\left(\vec{\boldsymbol{\omega}}\times_u\overrightarrow{OM}\right) + 2\vec{\boldsymbol{\omega}}\times_u\vec{\boldsymbol{V}}\\ &+\frac{\mathrm{d}\vec{\boldsymbol{\omega}}}{\mathrm{d}t}\times_u\overrightarrow{OM}\Bigg] + \Gamma_{\mathscr{O}}'^{\,2}(1+\vec{\boldsymbol{a}}_{\mathscr{O}}\cdot\overrightarrow{OM})\left[\vec{\boldsymbol{\gamma}}'_{\mathscr{O}} + \frac{\Gamma_{\mathscr{O}}'^{\,2}}{c^2}(\vec{\boldsymbol{\gamma}}'_{\mathscr{O}}\cdot\vec{\boldsymbol{U}}')\vec{\boldsymbol{U}}'\right]\\ &+\frac{1}{\Gamma}\frac{\mathrm{d}\Gamma}{\mathrm{d}t}\perp_{u'}\left(\vec{\boldsymbol{V}} + \vec{\boldsymbol{\omega}}\times_u\overrightarrow{OM}\right) + \Gamma'_{\mathscr{O}}\Bigg[2\vec{\boldsymbol{a}}_{\mathscr{O}}\cdot\left(\vec{\boldsymbol{V}} + \vec{\boldsymbol{\omega}}\times_u\overrightarrow{OM}\right)\\ &+\frac{\mathrm{d}\vec{\boldsymbol{a}}_{\mathscr{O}}}{\mathrm{d}t}\cdot\overrightarrow{OM} + \frac{1}{\Gamma}\frac{\mathrm{d}\Gamma}{\mathrm{d}t}(1+\vec{\boldsymbol{a}}_{\mathscr{O}}\cdot\overrightarrow{OM})\Bigg]\vec{\boldsymbol{U}}'\Bigg\}.\end{aligned}} \tag{5.56}$$

At the Galilean limit, $\|\vec{\boldsymbol{V}}'\|_g/c \to 0$, $\|\vec{\boldsymbol{U}}'\|_g/c \to 0$, all the local rest spaces coincide, which implies $\perp_{u'} = \mathrm{Id}$ and $\times_u = \times$. In addition, $\Gamma = \Gamma' = \Gamma'_{\mathscr{O}} = 1$

and, from (5.55), all the terms with $\vec{a}_{\mathscr{O}}$ have a c^{-2} factor, hence converge to zero. From (4.61), the same thing holds for $d\Gamma/\mathrm{d}t$. Finally, (5.56) reduces to

$$\vec{\gamma}' = \vec{\gamma} + \vec{\gamma}'_{\mathscr{O}} + \vec{\omega} \times \left(\vec{\omega} \times \overrightarrow{OM}\right) + 2\vec{\omega} \times \vec{V} + \frac{\mathrm{d}\vec{\omega}}{\mathrm{d}t} \times \overrightarrow{OM} \qquad \text{(nonrelativistic).} \tag{5.57}$$

We recognize the Galilean law of acceleration composition, with the centripetal term, $\vec{\omega} \times (\vec{\omega} \times \overrightarrow{OM})$ (causing the centrifugal force) and the Coriolis term $2\vec{\omega} \times \vec{V}$.

5.5 Doppler Effect

The Doppler effect consists in the change of frequency of a periodic phenomenon induced by the motion of the emitter with respect to the receiver. The effect is well known for acoustic waves: from everyday experience, a sound emitted by a car is more acute when the car is approaching than when it is receding. We shall study here the Doppler effect within the framework of relativity and apply it mostly to electromagnetic waves.

5.5.1 Derivation

Let us consider an observer $\mathscr{O}'$ emitting light signals at regular intervals $\Delta t'_{\mathrm{em}}$ of his proper time t'. The signals are received by a second observer, $\mathscr{O}$, who found them separated by the interval Δt_{rec} of his proper time t. Let us search for the link between Δt_{rec} and $\Delta t'_{\mathrm{em}}$, by supposing that (i) $\mathscr{O}$ and $\mathscr{O}'$ are sufficiently close so that the curvatures of their worldlines can be neglected or (ii) $\mathscr{O}$ is an inertial observer. In both cases, $\mathscr{O}$'s worldline can be treated as a straight line (Fig. 5.7).

Let P_1 and P_2 be the events of emission of two successive light signals by $\mathscr{O}'$ and M_1 and M_2 the events of reception of these signals by $\mathscr{O}$ (cf. Fig. 5.7). The proper time of $\mathscr{O}'$ elapsed between P_1 and P_2 is then $\Delta t'_{\mathrm{em}}$ and that of $\mathscr{O}$ between M_1 and M_2 is Δt_{rec}. Let us denote by t_1^{rec}, t_2^{rec}, t_1^{em} and t_2^{em} the dates, *all relative to observer* $\mathscr{O}$, of the events P_1, P_2, M_1 and M_2, respectively. They are related by

$$t_1^{\mathrm{rec}} = t_1^{\mathrm{em}} + \frac{r_1}{c} \quad \text{and} \quad t_2^{\mathrm{rec}} = t_2^{\mathrm{em}} + \frac{r_2}{c},$$

where r_1 (resp. r_2) is the distance between $\mathscr{O}'$ and $\mathscr{O}$ at the date t_1^{em} (resp. t_2^{em}), distance measured in $\mathscr{O}$'s local rest space. In other words, $r_1 = \|\overrightarrow{O_1 P_1}\|_{\boldsymbol{g}}$ and $r_2 = \|\overrightarrow{O_2 P_2}\|_{\boldsymbol{g}}$, O_1 (resp. O_2) being the event of $\mathscr{O}$'s worldline that is simultaneous to P_1 (resp. P_2) (cf. Fig. 5.7). We deduce from the above relations the expression of the interval $\Delta t_{\mathrm{rec}} := t_2^{\mathrm{rec}} - t_1^{\mathrm{rec}}$:

$$\Delta t_{\mathrm{rec}} = \Delta t_{\mathrm{em}} + \frac{1}{c}(r_2 - r_1), \tag{5.58}$$

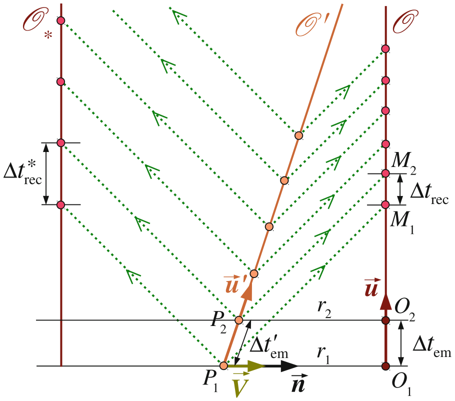

Fig. 5.7 Doppler effect: the period $\Delta t_{\rm rec}$ measured by $\mathscr{O}$ of light signals emitted by $\mathscr{O}'$ is different from the period $\Delta t'_{\rm em}$ measured by the emitter. In the figure $\vec{n}\cdot\vec{V} > 0$, so that $\Delta t_{\rm rec} < \Delta t'_{\rm em}$. A second receiver, $\mathscr{O}_*$, has the same 4-velocity as $\mathscr{O}$, but is such that $\vec{n}\cdot\vec{V} < 0$, so that $\Delta t^*_{\rm rec} > \Delta t'_{\rm em}$

where $\Delta t_{\rm em} := t_2^{\rm em} - t_1^{\rm em}$ is the emission period in terms of $\mathscr{O}$'s proper time. To evaluate $r_2 - r_1$, we note that, from the very definition of the velocity $\vec{V}$ of $\mathscr{O}'$ relative to $\mathscr{O}$,

$$\overrightarrow{O_2P_2} = \overrightarrow{O_1P_1} + \Delta t_{\rm em}\,\vec{V}.$$

Let $\vec{n}$ be the unit vector in $\mathscr{O}$'s local rest space at time $t_1^{\rm em}$ that is directed from P_1 to O_1. We have $\overrightarrow{O_1P_1} = -r_1\,\vec{n}$, and the scalar square of the above relation yields

$$r_2^2 = \overrightarrow{O_2P_2}\cdot\overrightarrow{O_2P_2} = r_1^2 - 2r_1\Delta t_{\rm em}\,\vec{n}\cdot\vec{V} + (\Delta t_{\rm em})^2V^2,$$

hence

$$r_2 - r_1 = \frac{r_2^2 - r_1^2}{r_2 + r_1} = -\Delta t_{\rm em}\frac{2}{1 + r_2/r_1}\left(\vec{n}\cdot\vec{V} - \frac{\Delta t_{\rm em}V^2}{2r_1}\right).$$

Substituting this formula in (5.58), we get

$$\Delta t_{\rm rec} = \Delta t_{\rm em}\left[1 - \frac{2}{1 + r_2/r_1}\left(\frac{\vec{n}\cdot\vec{V}}{c} - \frac{\Delta t_{\rm em}V^2}{2cr_1}\right)\right]. \tag{5.59}$$

Let us now suppose that the emission period $\Delta t_{\rm em}$ is very small, in the sense that $V\Delta t_{\rm em}/r_1 \ll 1$. We may then neglect the last term in the above expression and write $r_2/r_1 \simeq 1$, so that

$$\Delta t_{\rm rec} \simeq \left(1 - \frac{\vec{n}\cdot\vec{V}}{c}\right)\Delta t_{\rm em}. \tag{5.60}$$

Besides, $\Delta t_{\rm em}$ is related to the proper emission period $\Delta t'_{\rm em}$ by the Lorentz factor between $\mathscr{O}$ and $\mathscr{O}'$, $\Gamma = (1 - V^2/c^2)^{-1/2}$, according to (4.1): $\Delta t_{\rm em} = \Gamma\Delta t'_{\rm em}$. We thus obtain

$$\boxed{\Delta t_{\rm rec} = \Gamma\left(1 - \frac{\vec{n}\cdot\vec{V}}{c}\right)\Delta t'_{\rm em}}. \tag{5.61}$$

If one considers a whole series of signals, as illustrated in Fig. 5.7, and if the period between two signals is constant, it is natural to introduce the emission frequency $f'_{\rm em} = (\Delta t'_{\rm em})^{-1}$ and the reception one: $f_{\rm rec} = (\Delta t_{\rm rec})^{-1}$. Relation (5.61) becomes then

$$\boxed{f_{\rm rec} = \frac{f'_{\rm em}}{\Gamma(1 - \vec{\boldsymbol{n}} \cdot \vec{\boldsymbol{V}}/c)}}. \tag{5.62}$$

The fact that $f_{\rm rec} \neq f'_{\rm em}$ (except if $\vec{\boldsymbol{V}} = 0$) constitutes the ***Doppler effect*** (also called sometimes ***Doppler–Fizeau effect***). The proportionality coefficient $[\Gamma(1 - \vec{\boldsymbol{n}} \cdot \vec{\boldsymbol{V}}/c)]^{-1}$ between $f'_{\rm em}$ and $f_{\rm rec}$ is called the ***Doppler factor***.

At the nonrelativistic limit $\|\vec{\boldsymbol{V}}\|_{\boldsymbol{g}} \ll c$, (5.62) reduces to

$$f_{\rm rec} = \left(1 + \frac{\vec{\boldsymbol{n}} \cdot \vec{\boldsymbol{V}}}{c}\right) f'_{\rm em} \qquad \text{(nonrelativistic).} \tag{5.63}$$

The frequency change given by this formula is called the ***first-order Doppler effect***, for it is first order in V/c. At this order, the Doppler effect vanishes if the emitter velocity is orthogonal to the direction of observation ($\vec{\boldsymbol{n}} \cdot \vec{\boldsymbol{V}} = 0$). At the second order in V/c, the so-called ***transverse Doppler effect*** appears. It remains even if $\vec{\boldsymbol{n}} \cdot \vec{\boldsymbol{V}} = 0$ because of the Lorentz factor in (5.62).

If the emitter velocity $\vec{\boldsymbol{V}}$ relative to the receiver is the direction of observation $\vec{\boldsymbol{n}}$, we can write $\vec{\boldsymbol{V}} = V\,\vec{\boldsymbol{n}}$, with $V = \pm\|\vec{\boldsymbol{V}}\|_{\boldsymbol{g}}$ (+ sign if $\mathscr{O}'$ is moving towards $\mathscr{O}$, − sign otherwise). Then $\vec{\boldsymbol{n}} \cdot \vec{\boldsymbol{V}} = V$ and $\Gamma = (1 - V^2/c^2)^{-1/2}$, so that formula (5.62) becomes

$$\boxed{f_{\rm rec} = \sqrt{\frac{1 + V/c}{1 - V/c}}\; f'_{\rm em}.}_{\;\vec{\boldsymbol{V}} = V\,\vec{\boldsymbol{n}}} \tag{5.64}$$

Historical note: *The Doppler effect has been predicted by Christian Doppler*[6] *in 1842 (Doppler 1842), both for sound waves and light, on the basis of propagation within aether. Doppler gave the specific example of binary stars, where the effect could be detected, arising from the motion of each star around the centre of mass of the system. The effect has also been predicted independently by Hippolyte Fizeau (cf. p. 122) in 1848. The first observational evidence was obtained on stars in 1868 by the English astronomer William Huggins (1824–1910). The first observation in laboratory dates from 1895. Doppler and Fizeau obtained the nonrelativistic formula (5.63). It is Albert Einstein who, in the seminal 1905 article (Einstein 1905b), derived the relativistic formula (5.62) (actually an equivalent form). In 1907, he suggested to search for the effect by observing atomic spectral lines (Einstein 1907), which was achieved by Ives and by Stilwell in 1938 (see below).*

[6]**Christian A. Doppler** (1804–1853): Austrian mathematician and physicist, mostly known for the prediction of the effect bearing his name.

5.5.2 *Experimental Verifications*

The observations of the Doppler effect mentioned above were checking only the first-order effect [Eq. (5.63)]. The measure of the transverse Doppler effect, which is specific to special relativity, is much more delicate, because the effect scales as V^2/c^2 and is thereby much smaller than the first-order effect, which scales as V/c.

5.5.2.1 Ives–Stilwell Experiment

The first detection of the transverse Doppler effect has been performed in 1938, by Herbert Ives[7] and G.R. Stilwell (1938). To get rid of the first-order Doppler effect, Ives and Stilwell measured the radiation emitted by atoms moving in two opposite directions and aligned with the observer. For atoms moving towards the observer, the measured frequency, f_1, is given by (5.64). For those moving in the opposite direction, the frequency f_2 is obtained by replacing V by $-V$. The arithmetic mean is then

$$\frac{f_1 + f_2}{2} = \frac{1}{2}\left(\sqrt{\frac{1+V/c}{1-V/c}} + \sqrt{\frac{1-V/c}{1+V/c}}\right) f_0, \quad \text{i.e.} \quad \frac{f_1 + f_2}{2} = \Gamma f_0, \tag{5.65}$$

where f_0 is the emission frequency of the atoms at rest [$f_0 = f'_{\mathrm{em}}$ with notations of Eq. (5.64)]. A nonrelativistic theory, based on (5.63), would predict $(f_1 + f_2)/2 = f_0$. The presence of the Lorentz factor Γ in (5.65) is thus the thing to test. Ives and Stilwell used hydrogen atoms. To get sizeable velocities, of the order of $V \sim 4 \times 10^{-3}c$, H_2^+ and H_3^+ ions were produced by means of an electric arc in a hydrogen tube and accelerated in an electric potential difference of $\sim 10^4$ volts. These ions decomposed then into hydrogen atoms, keeping their initial velocity. The hydrogen atoms are formed in an excited state and decay by emitting lines in the Balmer series. Using a grating spectrograph, the second line of this series, the H_β line at the wavelength $\lambda_0 = 486$ nm, could be detected. Ives and Stilwell measured a shift by a few picometres of the mean line $(f_1 + f_2)/2$ with respect to f_0, in agreement with formula (5.65) (with Γ expanded at first order in V^2/c^2), with an accuracy of the order of 1%.

Remark 5.7. In the computation of Sect. 5.5.1, the factor Γ in (5.65) appears as a time-dilation term when passing from (5.60) to (5.61). We may thus consider that Ives–Stilwell experiment constitutes an experimental verification of the time dilation discussed in Chap. 4. Historically speaking, this is even the first one, since

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[7] **Herbert E. Ives** (1882–1953): American physicist and engineer, pioneer of television and fax. Amazingly, he was an opponent to the theory of relativity during his entire life! He notably interpreted the result of his famous experiment with Stilwell as an evidence in favour of the electromagnetic theory of Larmor and Lorentz, which was based on the aether.

it has been performed three years before the measurement on atmospheric muons presented in Sect. 4.4.1.

5.5.2.2 Modern Experiments

A very precise test of the relativistic Doppler effect has been performed in 2007 by a team of the Max Planck Institute in Heidelberg (Reinhardt et al. 2007). As in the Ives–Stilwell experiment, two groups of emitting particles are set up to propagate in opposite directions with the same velocity modulus V. The idea is now to measure not the mean frequency as in (5.65) but the product of frequencies:

$$f_1 f_2 = \sqrt{\frac{1+V/c}{1-V/c}}\sqrt{\frac{1-V/c}{1+V/c}}\, f_0^2, \quad \text{i.e.} \quad f_1 f_2 = f_0^2. \tag{5.66}$$

The product is thus independent of the particle velocity V. This is interesting because it is difficult to measure V with a high precision. It should be noted that a nonrelativistic theory, based on (5.63), would predict $f_1 f_2 = (1 - V^2) f_0^2$, instead of (5.66). By observing the lines emitted by $^7\mathrm{Li}^+$ ions accelerated at $V = 0.03c$ and $V = 0.064c$ in a storage ring (cf. Sect. 17.5.5), the Heidelberg team has confirmed formula (5.66), and thereby special relativity, with a relative deviation smaller than 10^{-9} (Reinhardt et al. 2007).

5.6 Aberration

Aberration is the difference between the incidence angles of a same light ray perceived by two observers in relative motion. As for the Doppler effect, it is not a purely relativistic effect, for it reflects essentially the finiteness of the propagation velocity of the signal. For instance, aberration is well known to the pedestrian walking in the rain: although the rain drops are falling vertically with respect to the ground, the pedestrian must incline his umbrella in order not to be wet.

5.6.1 Theoretical Expression

The aberration phenomenon appears when relating the velocity $\vec{V}$ of a photon relative to some observer $\mathscr{O}$ to the velocity $\vec{V}'$ of the same photon relative to a second observer $\mathscr{O}'$. Let us suppose that the worldlines of the photon and the two observers cross at the same event O. Then, from (4.79),

$$\vec{V} = c\,\vec{n} \quad \text{and} \quad \vec{V}' = c\,\vec{n}', \tag{5.67}$$

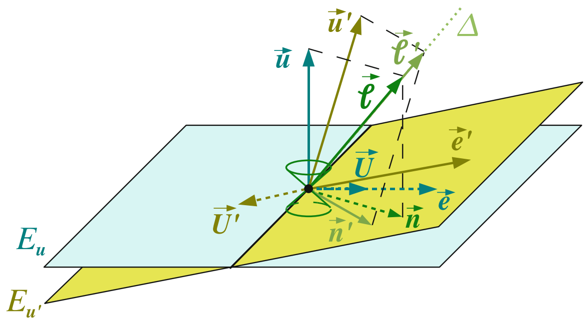

Fig. 5.8 Motion of a photon (null geodesic Δ) with respect to two observers, $\mathscr{O}$ and $\mathscr{O}'$, of respective 4-velocities $\vec{u}$ and $\vec{u}'$. The unit vector $\vec{n} \in E_u$ (resp. $\vec{n}' \in E_{u'}$) gives the propagation direction of the photon with respect to $\mathscr{O}$ (resp. $\mathscr{O}'$)

where the propagation direction vectors $\vec{n}$ and $\vec{n}'$ with respect to $\mathscr{O}$ and $\mathscr{O}'$ are defined by

$$\vec{\ell} = \vec{u} + \vec{n}, \qquad \vec{\ell} \parallel \Delta, \qquad \vec{u} \cdot \vec{n} = 0, \tag{5.68}$$

$$\vec{\ell}' = \vec{u}' + \vec{n}', \qquad \vec{\ell}' \parallel \Delta, \qquad \vec{u}' \cdot \vec{n}' = 0, \tag{5.69}$$

Δ being the photon's null geodesic and $\vec{u}$ (resp. $\vec{u}'$) the 4-velocity of observer $\mathscr{O}$ (resp. $\mathscr{O}'$) (cf. Fig. 5.8). The vector $\vec{\ell}$ (resp. $\vec{\ell}'$) is the null vector tangent to Δ and adapted to observer $\mathscr{O}$ (resp. $\mathscr{O}'$) [cf. (4.74)]. These two vectors are of course collinear:

$$\vec{\ell} = \lambda \vec{\ell}', \tag{5.70}$$

where λ is a strictly positive number, to be determined.

As in Sect. 5.3, let us denote by Γ_0 the Lorentz factor relating $\mathscr{O}$ and $\mathscr{O}'$ and by $\vec{U}$ (resp. $\vec{U}'$) the velocity of $\mathscr{O}'$ (resp. $\mathscr{O}$) relative to $\mathscr{O}$ (resp. $\mathscr{O}'$). All these quantities obey (5.1)–(5.3). Let us introduce the same unit vectors $\vec{e} \in E_u$ and $\vec{e}\,' \in E_{u'}$ as in Sect. 5.3.2 (cf. Fig. 5.2):

$$\vec{U} = U\vec{e} \quad \text{and} \quad \vec{U}' = U'\vec{e}\,', \quad \text{with} \quad U' = -U. \tag{5.71}$$

Let $\theta \in [0, \pi]$ be the incidence angle of the photon with respect to the direction of motion of $\mathscr{O}'$, as measured by $\mathscr{O}$: θ is the supplement of the angle between vectors $\vec{e}$ and $\vec{n}$ [cf. Fig. 5.9 and Eq. (3.19)]; hence,

$$\vec{e} \cdot \vec{n} = -\cos\theta. \tag{5.72}$$

The parallel/transverse decomposition of $\vec{n}$ is thus

$$\vec{n} =: -\cos\theta\, \vec{e} + \vec{n}_\perp, \qquad \text{with} \quad \vec{e} \cdot \vec{n}_\perp = 0 \quad \text{and} \quad \vec{n}_\perp \cdot \vec{n}_\perp = \sin^2\theta. \tag{5.73}$$

On the other side, combining (5.2) and (5.71) leads to $\vec{u} = \Gamma_0[\vec{u}' - (U/c)\vec{e}\,']$. Thanks to this identity, as well as (5.73) and (5.13), the relation $\vec{\ell} = \vec{u} + \vec{n}$ [Eq. (5.68)] becomes

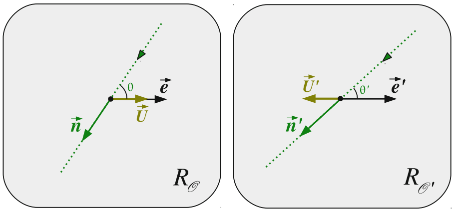

Fig. 5.9 Aberration phenomenon. *Left:* photon trajectory in the reference space of observer $\mathscr{O}$. *Right:* trajectory of the same photon in the reference space of observer $\mathscr{O}'$. The vector $\vec{U} = U\,\vec{e}$ (resp. $\vec{U}' = -U\,\vec{e}\,'$) is the velocity of $\mathscr{O}'$ with respect to $\mathscr{O}$ (resp. $\mathscr{O}$ with respect to $\mathscr{O}'$). The angles θ and θ' are related by formula (5.77) or (5.79)

$$\vec{\boldsymbol{\ell}} = \Gamma_0\left(\vec{\boldsymbol{u}}' - \frac{U}{c}\vec{\boldsymbol{e}}\,'\right) - \cos\theta\,\Gamma_0\left(\vec{\boldsymbol{e}}\,' - \frac{U}{c}\vec{\boldsymbol{u}}'\right) + \vec{\boldsymbol{n}}_\perp,$$

which can be written as

$$\vec{\boldsymbol{\ell}} = \Gamma_0\left(1 + \frac{U}{c}\cos\theta\right)\left[\vec{\boldsymbol{u}}' - \frac{\cos\theta + \frac{U}{c}}{1 + \frac{U}{c}\cos\theta}\,\vec{\boldsymbol{e}}\,' + \frac{1}{\Gamma_0\left(1 + \frac{U}{c}\cos\theta\right)}\,\vec{\boldsymbol{n}}_\perp\right]. \tag{5.74}$$

By comparing with (5.69) and (5.70), one can infer the proportionality coefficient λ between $\vec{\boldsymbol{\ell}}$ and $\vec{\boldsymbol{\ell}}\,'$: $\lambda = \Gamma_0[1 + (U/c)\cos\theta]$, as well as the propagation direction vector with respect to $\mathscr{O}'$:

$$\boxed{\vec{\boldsymbol{n}}' = -\frac{\cos\theta + \frac{U}{c}}{1 + \frac{U}{c}\cos\theta}\,\vec{\boldsymbol{e}}\,' + \frac{1}{\Gamma_0\left(1 + \frac{U}{c}\cos\theta\right)}\,\vec{\boldsymbol{n}}_\perp}. \tag{5.75}$$

The photon's incident angle θ' measured by $\mathscr{O}'$ with respect to the direction $\vec{\boldsymbol{e}}\,'$ is given by (cf. Fig. 5.9 and compare with (5.72))

$$\cos\theta' = -\vec{\boldsymbol{e}}\,' \cdot \vec{\boldsymbol{n}}'. \tag{5.76}$$

Since $\vec{\boldsymbol{e}}\,' \cdot \vec{\boldsymbol{n}}_\perp = 0$ (for $\vec{\boldsymbol{n}}_\perp \in E_{\boldsymbol{u}} \cap E_{\boldsymbol{u}'}$), (5.75) yields

$$\boxed{\cos\theta' = \frac{\cos\theta + \frac{U}{c}}{1 + \frac{U}{c}\cos\theta}}. \tag{5.77}$$

The transverse part of $\vec{\boldsymbol{n}}'$ being defined by a formula similar to (5.73): $\vec{\boldsymbol{n}}' =: -\cos\theta'\,\vec{\boldsymbol{e}}\,' + \vec{\boldsymbol{n}}'_\perp$, with $\vec{\boldsymbol{e}}\,' \cdot \vec{\boldsymbol{n}}'_\perp = 0$, we deduce form (5.75) that

$$\boxed{\vec{n}'_{\perp} = \frac{1}{\Gamma_0 \left(1 + \frac{U}{c}\cos\theta\right)}\, \vec{n}_{\perp}}. \tag{5.78}$$

Remark 5.8. Since the photon's velocity relative to $\mathscr{O}$ is $\vec{V} = c\,\vec{n}$ [Eq. (4.79)], and that relative to $\mathscr{O}'$ is $\vec{V}' = c\,\vec{n}'$, we note, after setting $V_{\parallel} = -c\cos\theta$ [cf. Eq. (5.72)] and $V'_{\parallel} = -c\cos\theta'$, that (5.77) and (5.78) are in full agreement with formulas (5.40a)–(5.40b) established for a massive particle.

Thanks to the trigonometric identity $\tan^2(\theta/2) = (1-\cos\theta)/(1+\cos\theta)$, Eq. (5.77) can be written as

$$\boxed{\tan\frac{\theta'}{2} = \sqrt{\frac{1-\frac{U}{c}}{1+\frac{U}{c}}}\,\tan\frac{\theta}{2}}. \tag{5.79}$$

This formula admits a simple geometrical interpretation, which we shall discuss in Sect. 5.7.3.

The fact that the angle θ' is different from θ is called ***aberration*** (or ***aberration of light***). We observe from (5.79) that the following inequality always holds:

$$\boxed{\theta' \leq \theta}. \tag{5.80}$$

Two particular cases are worth mentioning:

- If the photon and $\mathscr{O}'$ are moving in the same direction ($\theta = \pi$) [resp. in opposite directions ($\theta = 0$)] for $\mathscr{O}$, (5.77) leads to $\cos\theta' = -1$ (resp. $\cos\theta = 1$), i.e. $\theta' = \pi$ (resp. $\theta' = 0$): there is no aberration effect.
- If the direction of light propagation with respect to $\mathscr{O}$ is perpendicular to the velocity of $\mathscr{O}'$, $\theta = \pi/2$ and $\vec{n} = \vec{n}_{\perp}$ (this is the classical case of the "pedestrian in the rain"), then (5.77) reduces to

$$\cos\theta' = \frac{U}{c}, \tag{5.81}$$

which implies $\theta' < \pi/2$ for $U > 0$: observer $\mathscr{O}'$ does not perceive the light in the direction perpendicular to $\mathscr{O}$'s motion (the pedestrian must incline his umbrella).

Historical note: *The aberration formula (5.77) has been obtained as early as 1905 by Albert Einstein in the famous article (Einstein 1905b).*

5.6.2 Distortion of the Celestial Sphere

The aberration phenomenon is easily visualized by considering a uniform grid on the celestial sphere of observer $\mathscr{O}$. By ***celestial sphere***, it is meant the sphere $\mathscr{S}$ in

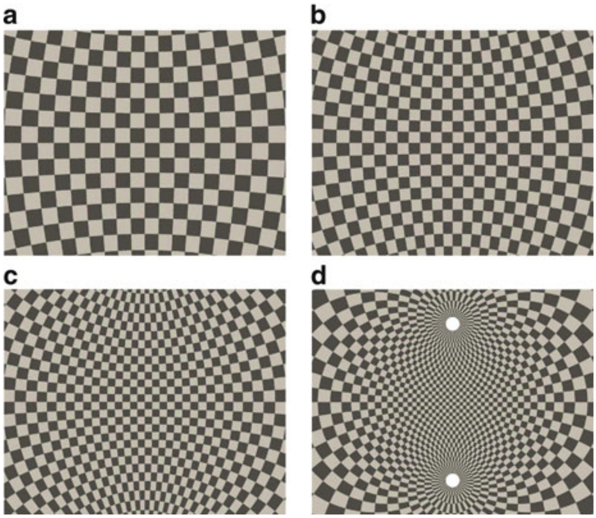

Fig. 5.10 Images perceived by observer $\mathscr{O}'$ of the spherical coordinate grid set on the celestial sphere of observer $\mathscr{O}$. The axis of each view is the direction of the motion of $\mathscr{O}'$ with respect to $\mathscr{O}$, with an opening angle of 90°. The four views correspond to different values of the velocity U of $\mathscr{O}'$ relative to $\mathscr{O}$: (**a**) $U = 0$, (**b**) $U = 0.3\,c$, (**c**) $U = 0.6\,c$ and (**d**) $U = 0.9\,c$. These images have been computed by Alain Riazuelo (2009), by means of the aberration formula (5.79)

$\mathscr{O}$'s reference frame centred on O (the position of $\mathscr{O}$) and of unit radius, the latter choice being arbitrary. Each light ray that arrives to O cuts $\mathscr{S}$ in a unique point. One may thus identify $\mathscr{S}$ to the set of directions centred on O and establish a bijective map between $\mathscr{S}$ and the set of null geodesics that constitute the past light cone $\mathscr{I}^-(O(t))$, $O(t)$ being the spacetime position of $\mathscr{O}$ at the instant t of his proper time. An alternative definition of $\mathscr{S}$ is being the intersection of $\mathscr{I}^-(O(t))$ with the hyperplane $\mathscr{E}_{\boldsymbol{u}}(t_0)$ for some $t_0 < t$ (cf. Fig. 6.1 in the next chapter).

Let us mark the points of $\mathscr{S}$ by their spherical coordinates (ϑ, φ) based on the triad $(\vec{\boldsymbol{e}}_i)$ of $\mathscr{O}$'s local frame: $\vartheta \in [0, \pi]$ is the colatitude, with $\vartheta = 0$ along the axis $\vec{\boldsymbol{e}}_3$, and $\varphi \in [0, 2\pi[$ is the azimuth in the plane $(\vec{\boldsymbol{e}}_1, \vec{\boldsymbol{e}}_2)$, with $\varphi = 0$ along the axis $\vec{\boldsymbol{e}}_1$. Figure 5.10a shows a part of the spherical coordinate grid, view from the position of $\mathscr{O}$ with an opening angle of 90° in the direction of the motion of $\mathscr{O}'$. Each square of the draughtboard corresponds to an increment of 5° in ϑ and φ. The aberration effect is visualized by drawing the same draughtboard, but viewed by observer $\mathscr{O}'$, with the same opening angle (90°) in the direction of this motion, for different values of the velocities of $\mathscr{O}'$ relative to $\mathscr{O}$ (Fig. 5.10). This amounts to drawing the spherical coordinate grid of $\mathscr{O}$ onto the celestial sphere of $\mathscr{O}'$. For $U = 0$, the view is of course identical to that of $\mathscr{O}$ (Fig. 5.10a). For $U > 0$, due to the property $\theta' \leq \theta$ [Eq. (5.80)], directions that were outside the field of view of $\mathscr{O}$ are appearing in the field of view of $\mathscr{O}'$. Remarkably, for $U = 0.9c$, the two poles of the spherical coordinates (ϑ, φ) are part of the field of view of $\mathscr{O}'$ (Fig. 5.10d)!

5.6.3 Experimental Verifications

The main experimental verification of aberration is provided by the observation of stars. $\mathscr{O}'$ is then an observer on Earth. As a first approximation, one may consider that the Earth follows a circular orbit around the Sun, at the velocity $\vec{U}$ with respect to an inertial observer $\mathscr{O}$ at rest with respect to the Sun. The orbital plane is called the ***ecliptic plane*** and the perpendicular direction the ***ecliptic pole axis***. The light from a star located in the direction of one of the ecliptic poles is perceived by a terrestrial observer with an aberration angle given by formula (5.81). The angle with respect to the axis of Earth orbit is $\alpha = \pi/2 - \theta'$ (cf. Fig. 5.9, where the ecliptic pole axis would appear as the vertical direction), so that (5.81) yields

$$\sin\alpha = \frac{U}{c}. \tag{5.82}$$

In the case of the Earth, $U = 30\ \mathrm{km\,s^{-1}}$, which results in $\alpha = 10^{-4}\ \mathrm{rad} = 20''$. If the Earth would have a uniform linear motion, this angle would not be detectable, but thanks to the orbital motion, the direction of motion of the Earth is changing perpetually. Consequently, the image of a star located at the ecliptic North pole $(\theta = \pi/2)$ draws within one year a circle of radius $20''$ around the pole. For a star located at an arbitrary ecliptic latitude, the figure is an ellipse.

One should mention a second effect due to the orbital motion of the Earth: ***parallax***, which is the variation of the viewing angle of a star located at a finite distance when the Earth runs on its orbit. The aberration can be distinguished from parallax by two properties:

- A much larger amplitude: the parallax is at most $0.77''$, a value achieved by the closest star (Proxima Centauri), and decays with the distance to the star, whereas the aberration angle is the same for all stars, taking the fixed value of $20''$.
- The phase in the ellipse drawn in the sky is not the same function of the position of the Earth on its orbit for the two effects (cf. Sect. 2.3 of Ferraro (2007) for details).

Since the eighteenth century, aberration is routinely measured in astronomy.

Historical note: *The stellar aberration has been observed for the first time in 1680 by Jean Picard,*[8] *on the Northern Star, by means of one of the first telescope with reticle (Picard 1680) (Article VIII). Picard was, however, not capable to interpret his observations, neither were his successors, among them John Flamsteed,*[9] *during almost half a century (cf. Liebscher and Brosche 1998). This explains the name*

[8] **Jean Picard** (1620–1682): French astronomer, who performed the first precise measurement of the Earth radius, by determining the length of one degree of latitude by triangulation. He made the observations leading to the discovery of aberration at Uraniborg—Tycho Brahe's observatory.

[9] **John Flamsteed** (1646–1719): first British Royal Astronomer, founder of Greenwich Observatory and author of a catalogue of about 3,000 stars.

aberration *given to this phenomenon. Only in 1728 James Bradley (cf. p. 122) provided the correct explanation (Bradley 1728) (within a nonrelativistic framework, of course!). He measured by himself the aberration of the star γ Draconis, getting the correct value of* $20''$. *According to formula (5.82), he obtained incidentally the value of c, in units of U (orbital Earth velocity), as we have already mentioned in the historical note p. 122.*

5.7 Images of Moving Objects

5.7.1 Image and Instantaneous Position

In view of the length contraction discussed in Sect. 5.2.2, one may naively think that the image of an object in fast motion is deduced from its image at rest by a mere FitzGerald–Lorentz contraction in the direction of motion. A spherical object would therefore appear as an axisymmetric ellipsoid with the small axis in the direction of motion. We are going to see that this is not the case. Indeed, one should not confuse the *position* of an object at some fixed instant t of an observer's proper time and the *image* perceived by the observer at the instant t.

More precisely, let us consider an event O of proper time t on the worldline of an observer $\mathscr{O}$. A three-dimensional object describes a ***worldtube*** in $\mathscr{E}$, which is the domain of $\mathscr{E}$ filled by the worldlines of all the particles constituting the object (cf. Fig. 5.11). For observer $\mathscr{O}$, the ***position*** of the object at the instant t is the intersection $\mathscr{T}_1$ between the worldtube and $\mathscr{O}$'s local rest space at the instant t, $\mathscr{E}_{\boldsymbol{u}}(O)$. On the other hand, the ***image*** or ***photography*** of the object perceived by $\mathscr{O}$ at the instant t is generated by all the photons that arrive at O, having been emitted by the object. Geometrically, this means that the image of the object is determined by the intersection $\mathscr{T}_2$ of the worldtube with the past light cone of vertex O, $\mathscr{I}^-(O)$ (cf. Fig. 5.11). $\mathscr{T}_1$ is the set of events of the worldtube that are simultaneous to O with respect to $\mathscr{O}$, whereas the events in $\mathscr{T}_2$ are not simultaneous with respect to $\mathscr{O}$. Since $\mathscr{T}_1 \neq \mathscr{T}_2$, it is then conceivable that the relation between the image and the instantaneous position of an object can be complicated. In particular, the FitzGerald–Lorentz contraction described in Sect. 5.2.2 regards only the position of the object and not its image. This last one can actually be elongated rather than shortened, as shown in Fig. 5.12: a ruler moving towards the observer appears elongated (upper figure), whereas when moving perpendicularly to the observer, it appears shortened (lower figure).

5.7.2 Apparent Rotation

An important effect regarding the visual aspect of moving objects is some apparent rotation. It is easy to evaluate this effect in the special case of a cube moving perpendicularly to the line of sight of an inertial observer $\mathscr{O}$. This observer

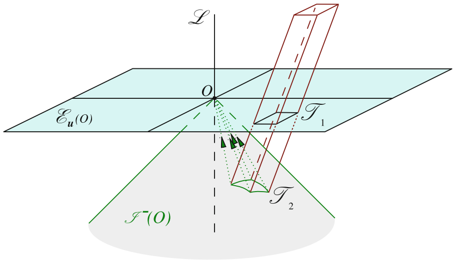

Fig. 5.11 Difference between the position $\mathscr{T}_1$ of an object at a given instant in the local rest space $\mathscr{E}_{\boldsymbol{u}}(O)$ of an observer and its image perceived by that observer

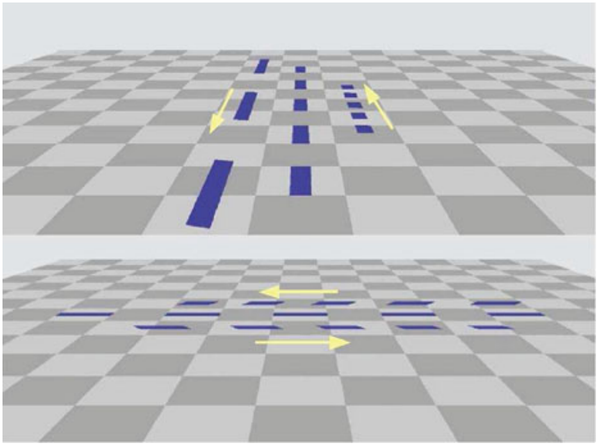

Fig. 5.12 Moving rulers. *Upper figure:* motion in the direction of the observer—the left rulers is approaching at the velocity $0.7\,c$, the middle one is at rest and the right one is receding at the velocity $0.7\,c$. *Lower figure:* motion perpendicular to the line of sight—the background ruler moves leftward at the velocity $0.7\,c$, the middle one is at rest and the foreground one moves rightward at the velocity $0.7\,c$ [Image computed by Ute Kraus (2005) and reproduced with permission]

perceives, in addition to the face of the cube oriented towards him, the face opposite to the direction of motion, as if the cube was rotated by some angle $\theta \neq 0$ (cf. Fig. 5.13). Let us suppose that $\mathscr{O}$ is far away from the cube, so that the light rays arriving to him can be considered as parallel. Let a be the proper length (i.e. the length measured by an observer at rest with respect to the cube) of a cube's edge; let $\vec{V} = V\,\vec{e}_x$ be the cube's velocity relative to $\mathscr{O}$ and $\Gamma = (1 - V^2/c^2)^{-1/2}$ the corresponding Lorentz factor. In $\mathscr{O}$'s local rest space, the cube is contracted in the direction of motion by the factor Γ (FitzGerald–Lorentz contraction; cf. Sect. 5.2.2). Let us consider a photon emitted at the instant $t = 0$ of $\mathscr{O}$'s proper time from the edge of the back face with respect to the direction of motion and opposite to the line of sight (cf. Fig. 5.13). At $\mathscr{O}$'s proper time $t = a/c$, this photon is at the same level as those just emitted by the cube's face in front of the observer. All these photons will therefore reach the observer at the same time, resulting in the image shown in

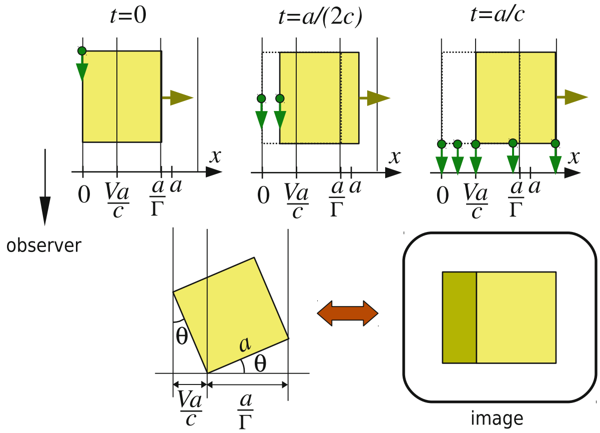

Fig. 5.13 Image of a cube moving perpendicularly to the direction of observation

the box of Fig. 5.13. On this image, the width of the back face is $\ell_1 = Va/c$, whereas that of the face oriented towards the observer is $\ell_2 = a/\Gamma = a\sqrt{1 - V^2/c^2}$. Setting

$$\theta := \arcsin(V/c), \tag{5.83}$$

we have then $\ell_1 = a \sin\theta$ and $\ell_2 = a \cos\theta$. We conclude that the image is identical to that of a cube that would be at rest with respect to $\mathscr{O}$ and would have been rotated by the angle θ given by (5.83) (cf. Fig. 5.13).

Remark 5.9. The phenomenon of apparent rotation is not intrinsically relativistic but reflects mostly the finite time of propagation of light. However, relativity enters via the FitzGerald–Lorentz contraction of the face oriented towards the observer. If there would be no contraction, the image of the cube would be more elongated and would not be similar to that resulting from a pure rotation, since one would have $\ell_2 = a \neq a \cos\theta$.

Remark 5.10. The apparent rotation is visible on the images of the moving rulers in the lower panel of Fig. 5.12.

5.7.3 Image of a Sphere

After the image of a cube, let us now determine that of a sphere. The result is somewhat surprising:

> The image of a moving sphere is a perfect disk, as if the sphere were at rest.

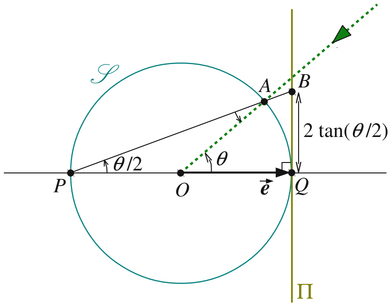

Fig. 5.14 Celestial sphere and stereographic projection

In particular, there is no visible contraction in the direction of motion. We are going to demonstrate this from the aberration formula established in Sect. 5.6.1.

Let us consider indeed two observers $\mathscr{O}$ and $\mathscr{O}'$ whose worldlines intersect at some event O. Let $\mathscr{S}$ be the celestial sphere of $\mathscr{O}$ (cf. Sect. 5.6.2). Each light ray that arrives at O corresponds to a unique point of $\mathscr{S}$ (cf. Fig. 5.14). Let Q be the point of $\mathscr{S}$ located in the direction of the motion of $\mathscr{O}'$ with respect to $\mathscr{O}$, i.e. such that[10] $\overrightarrow{OQ} = \vec{e}$, the velocity $\vec{U}$ of $\mathscr{O}'$ relative to $\mathscr{O}$ being $\vec{U} = U\vec{e}$. Let Π be the plane tangent to $\mathscr{S}$ at Q and P the point of $\mathscr{S}$ opposite to Q. The ***stereographic projection*** from the pole P maps every point $A \in \mathscr{S} \setminus \{P\}$ to a point of Π, defined as the intersection B of the line PA with Π (cf. Fig 5.14). Except for those passing through P, each light ray arriving at O can be labelled by the polar coordinates (ρ, φ) of the point B in the plane Π, choosing Q as the origin: ρ is the distance from Q to B, and φ is the rotation angle about the axis PQ. It is easy to see that if the light ray forms the angle θ with the direction PQ, then[11]

$$\rho = 2\tan\frac{\theta}{2}. \tag{5.84}$$

The aberration formula (5.79) can be then interpreted as the contraction by the constant factor $\sqrt{(1-U/c)/(1+U/c)}$ when moving from the plane of stereographic projection of observer $\mathscr{O}$ to those of $\mathscr{O}'$. Now the stereographic projection has the property to transform any circle on the celestial sphere $\mathscr{S}$ not containing P into a circle of Π (see, e.g. Berger 1987b). A circle through P is transformed into a straight line of Π. The converse is true: any circle of Π is transformed into a circle on $\mathscr{S}$ by the inverse stereographic projection. We deduce from this that if an object looks spherical to observer $\mathscr{O}$, it also looks spherical for $\mathscr{O}'$. Indeed, if the object's contour is a circle (not through P) on the celestial sphere of $\mathscr{O}$, its stereographic projection is a circle of Π. The transformation (5.79), along with $\varphi' = \varphi$, maps this

[10] Let us recall that, by convention, $\mathscr{S}$ has a unit radius.

[11] *Proof.* The triangle POA being isosceles, the angle OPA is necessarily equal to $\theta/2$; QB is the opposite edge to this angle in the right triangle PQB with $PQ = 2$, hence (5.84) □

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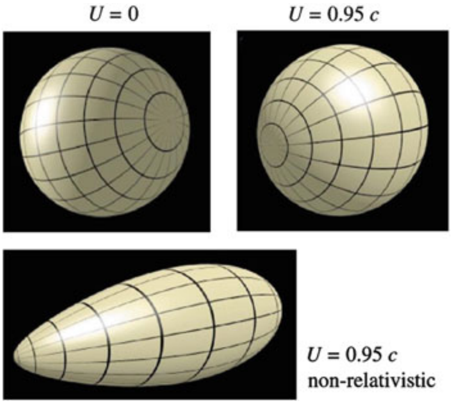

Fig. 5.15 Images of a moving sphere. The bottom image is obtained in Galilean theory, i.e. by taking into account only the finiteness of the velocity of light [Images computed by Daniel Weiskopf (2002) and reproduced with permission]

circle into a circle of the stereographic plane Π' associated with $\mathscr{O}'$. By the inverse stereographic projection, one obtains a circle on the celestial sphere $\mathscr{S}'$ of $\mathscr{O}'$. The only thing that may change is the angular size of the circle.

Remark 5.11. If one would take into account only the finiteness of the velocity of light in a nonrelativistic theory (Galilean theory), a moving sphere would appear elongated in the direction of motion, as illustrated in Fig. 5.15. Thanks to the FitzGerald–Lorentz contraction, it appears exactly spherical. It is also clear on Fig. 5.15 that the moving sphere seems to have been rotated, showing a part of its face opposite to the direction of motion. This is of course the apparent rotation discussed in Sect. 5.7.2.

Historical note: *It seems that the first person to wonder about the image of an object in relativistic motion was Anton Lampa*[12] *in 1924 (Lampa 1924). He computed explicitly the aspect of a moving ruler. But his work has been ignored, most authors supposing more or less implicitly that the image of a moving object is simply distorted by the FitzGerald–Lorentz contraction in the direction of motion. For instance, the illustrations of the popular book by George Gamow,*[13] Mr Tompkins in Wonderland, *published in 1939 (Gamow 1939), show the wheels of a "relativistic bicycle" as ellipses flattened in the direction of motion, whereas the exact shape is more complicated, as one can check p. 30 of the book (Nollert and Ruder 2008). It is only in 1959 that Roger Penrose*[14] *(1959) demonstrated*

[12]**Anton Lampa** (1868–1938): Austrian physicist; interested very early by relativity, he helped Einstein to get a professor position at Prague University in 1911.

[13]**George Gamow** (1904–1968): Russian physicist (naturalized American in 1940), friend of Landau (cf. p. 445) with whom he studied in Leningrad; he is one of the fathers of the Big Bang model.

[14]**Roger Penrose** : British mathematician born in 1931, who performed major advances in general relativity and invented non-periodic tilings of the plane; he also devised an impossible triangular figure, known as *Penrose triangle*.

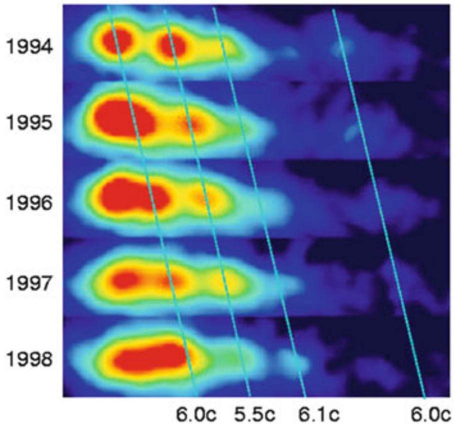

Fig. 5.16 Images from the Hubble Space Telescope of knots in the jet of the galaxy M87, over several years. These images involve only the central part of the jet; a global view of the latter provided in Fig. 21.4 [Source: Biretta et al. (1999)]

that a sphere appears exactly circular to any observer, whatever his state of motion with respect to it. The same year, the American physicist James Terrell conducted a systematic study of the appearance of moving objects (Terrel 1959), putting forward the rotation effect described in Sect. 5.7.2. After the outcome of computers, numerous images and movies of objects or landscapes in relativistic motion have been produced; see, for instance, Nollert and Ruder (2008), Kraus (2005), Kraus et al. (2002), Müller and Weiskopf (2011) and [W4] to [W7] (Appendix *B*).

5.7.4 Superluminal Motions

The preceding examples of relativistic images are academic ones: nobody has ever seen a ruler, a cube or a sphere moving with a velocity close to c in a laboratory. On the other hand, macroscopic relativistic motions are frequently observed in astrophysics, notably in the form of jets. We shall discuss astrophysical jets in more details in Sect. 21.7.1; here we focus on a remarkable kinematical property of certain relativistic jets: they show an *apparent* velocity larger than c! Such a motion is called ***superluminal***. An example is provided in Fig. 5.16, which shows the progression over several years of features (knots) in the jet emitted by the nucleus of the galaxy M87. On the plane of the sky, the corresponding angular velocity is $\omega = 0.024''$ per year. Knowing the distance of M87 to the Earth, $D = 52$ million light-years, one deduces the jet velocity: $V_{\rm app} = D\omega \simeq 6c$! There is, however, no contradiction with the result of Sect. 4.3.3, according to which the velocity of any material body relative to an inertial observer must be lower than c. Indeed $V_{\rm app}$ is not the velocity relative to an observer, as we have defined it in Sect. 4.3.1. This last one is obtained from the *position* of the object in the observer's local frame at two successive instants of the observer's proper time. On the contrary, the apparent velocity $V_{\rm app}$ is determined from two successive *images* of the object. We recover here the difference between image and instantaneous position underlined in Sect. 5.7.1.

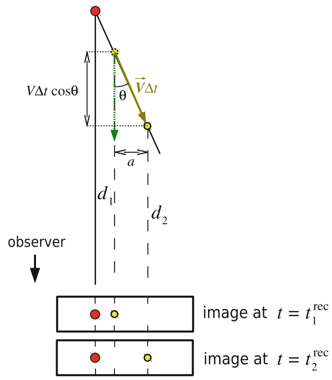

Fig. 5.17 Apparent superluminal motion. Figure in the reference space of observer $\mathscr{O}$. $\Delta t := t_2^{\mathrm{em}} - t_1^{\mathrm{em}}$

It is easy to see that V_{app} can be larger than c by means of the following simple model, illustrated by Fig. 5.17. Let us consider a source that emits some blob of matter (knot) at the constant velocity $\vec{V}$ relative to the inertial observer $\mathscr{O}$, who is supposed to be far away (astronomical situation). At the instant t_1^{em} (of $\mathscr{O}$'s proper time), the knot emits some luminous signal, which reaches $\mathscr{O}$ at $t_1^{\mathrm{rec}} = t_1^{\mathrm{em}} + d_1/c$, where d_1 is the distance between the emission point and $\mathscr{O}$ in the reference space of the latter. At the instant $t_2^{\mathrm{em}} > t_1^{\mathrm{em}}$, the knot emits a second luminous signal, which reaches $\mathscr{O}$ at $t_2^{\mathrm{rec}} = t_2^{\mathrm{em}} + d_2/c$, with (cf. Fig. 5.17),

$$d_2 = d_1 - V(t_2^{\mathrm{em}} - t_1^{\mathrm{em}})\cos\theta,$$

where $V := \|\vec{V}\|_g$ and θ is the angle between $\vec{V}$ and the line of sight. The distance perpendicular to the line of sight and travelled between the two emissions is (cf. Fig. 5.17)

$$a = V(t_2^{\mathrm{em}} - t_1^{\mathrm{em}})\sin\theta.$$

The apparent velocity is nothing but this distance divided by the elapsed time between the receptions of the two signals:

$$V_{\mathrm{app}} = \frac{a}{t_2^{\mathrm{rec}} - t_1^{\mathrm{rec}}} = \frac{V(t_2^{\mathrm{em}} - t_1^{\mathrm{em}})\sin\theta}{t_2^{\mathrm{em}} + \frac{1}{c}\left[d_1 - V(t_2^{\mathrm{em}} - t_1^{\mathrm{em}})\cos\theta\right] - t_1^{\mathrm{em}} - \frac{1}{c}d_1},$$

hence

$$V_{\mathrm{app}} = \frac{V\sin\theta}{1 - \frac{V}{c}\cos\theta}. \tag{5.85}$$

It is clear that if V is sufficiently large (but still lower than c!) and θ is sufficiently small, the quantity V_{app} can be arbitrarily large. The apparent superluminal velocities are thus generated by relativistic motions along a direction close to the line of sight. In the case of the jet of M87 shown in Fig. 5.16, $V \geq 0.986\,c$ (which corresponds to a Lorentz factor $\Gamma \geq 6$) and $\theta \leq 19°$.

Historical note: *Apparent superluminal motions in astrophysical sources have been predicted in 1966 by Martin Rees*[15] *(1966). The first observations of this phenomenon occurred in 1970 on the quasar 3C 279, from radio images at high angular resolution (obtained via very long baseline interferometry) (Whitney et al. 1971) (cf. Suzy Collin-Zahn's book (Collin-Zahn 2009) for more details). Since then, superluminal motions are frequently observed in quasars and active galactic nuclei, as, for instance, M87 (Fig. 5.16). In 1994, a superluminal motion has been observed in our galaxy, in the micro-quasar GRS* $1915 + 105$ *(a black hole accreting matter from a companion star; cf. Sect. 21.7.1) (Mirabel and Rodríguez 1994, 1999); the apparent velocity is* $V_{\rm app} = 1.25\, c$*, which corresponds to some matter ejection at the velocity* $V = 0.92\, c$ *and with the angle* $\theta \simeq 70°$.

[15]**Martin Rees**: British astrophysicist born in 1942, author of numerous works in cosmology and in the physics of galaxies and quasars

Chapter 6
Lorentz Group

6.1 Introduction

As we have introduced it in Chap. 3, an observer is characterized by his worldline $\mathscr{L}$ and his local frame $(\vec{e}_\alpha(t))$. The latter constitutes, at any point of $\mathscr{L}$, an orthonormal basis of $(E, \boldsymbol{g})$. The study of the transition from one observer to the other is thus equivalent to the study of the transformations of E that map an orthonormal basis to another one. These mappings are the famous Lorentz transformations, to which this chapter is devoted. Like Chap. 1, this is a purely mathematical chapter. The definitions of basic algebraic concepts used here are recalled in Appendix A.

6.2 Lorentz Transformations

6.2.1 Definition and Characterization

One calls ***Lorentz transformation*** any linear map

$$\begin{aligned} \boldsymbol{\Lambda} : E &\longrightarrow E \\ \vec{v} &\longmapsto \boldsymbol{\Lambda}(\vec{v}) \end{aligned} \tag{6.1}$$

such that[1]

$$\boxed{\forall (\vec{v}, \vec{w}) \in E \times E, \quad \boldsymbol{g}(\boldsymbol{\Lambda}(\vec{v}), \boldsymbol{\Lambda}(\vec{w})) = \boldsymbol{g}(\vec{v}, \vec{w})}\,. \tag{6.2}$$

[1]We have made explicit the metric tensor $\boldsymbol{g}$ in the expression of the scalar product, but we could of course have written $\forall (\vec{v}, \vec{w}) \in E \times E,\ \boldsymbol{\Lambda}(\vec{v}) \cdot \boldsymbol{\Lambda}(\vec{w}) = \vec{v} \cdot \vec{w}$ instead of (6.2).

É. Gourgoulhon, *Special Relativity in General Frames*, Graduate Texts in Physics, DOI 10.1007/978-3-642-37276-6_6, © Springer-Verlag Berlin Heidelberg 2013

$\boldsymbol{\Lambda}$ being a linear map from the vector space E to itself, it is called an ***endomorphism*** of E (cf. Appendix A). The property (6.2) is often expressed by stating that $\boldsymbol{\Lambda}$ ***preserves the scalar product*** $\boldsymbol{g}$. In particular, the scalar square of a vector is preserved by $\boldsymbol{\Lambda}$, which implies that the norm with respect to $\boldsymbol{g}$, as introduced in Sect. 1.3.5, is conserved:

$$\forall \vec{\boldsymbol{v}} \in E, \quad \|\boldsymbol{\Lambda}(\vec{\boldsymbol{v}})\|_{\boldsymbol{g}} = \|\vec{\boldsymbol{v}}\|_{\boldsymbol{g}} . \tag{6.3}$$

For this reason, one says that a Lorentz transformation is an ***isometry*** of the space $(E, \boldsymbol{g})$.

An endomorphism of E is entirely characterized by its matrix in a given basis. One therefore calls ***Lorentz matrix*** any real 4×4 matrix that represents a Lorentz transformation in an *orthonormal* basis of $(E, \boldsymbol{g})$, i.e. any matrix $\Lambda = (\Lambda^{\alpha}{}_{\beta})$ such that there exists a Lorentz transformation $\boldsymbol{\Lambda}$ and an orthonormal basis $(\vec{\boldsymbol{e}}_{\alpha})$ obeying

$$\boxed{\boldsymbol{\Lambda}(\vec{\boldsymbol{e}}_{\beta}) = \Lambda^{\alpha}{}_{\beta}\, \vec{\boldsymbol{e}}_{\alpha}}. \tag{6.4}$$

Remark 6.1. The first index, α, labels the rows of the matrix Λ; it is in upper position and gives the components of the vector $\boldsymbol{\Lambda}(\vec{\boldsymbol{e}}_{\beta})$ onto the basis $(\vec{\boldsymbol{e}}_{\alpha})$. The second index, β, labels the columns of the matrix Λ; it is in lower position and refers to the basis vector whose image is taken by $\boldsymbol{\Lambda}$. The matrix Λ is hence formed by writing, column by column, the images of the vectors of the basis $(\vec{\boldsymbol{e}}_{\alpha})$.

Thanks to this convention, the components (w^{α}) of the image $\vec{\boldsymbol{w}} = \boldsymbol{\Lambda}(\vec{\boldsymbol{v}})$ are expressed in terms of the components (v^{α}) of $\vec{\boldsymbol{v}}$ according to

$$\boxed{w^{\alpha} = \Lambda^{\alpha}{}_{\beta}\, v^{\beta}}. \tag{6.5}$$

Proof. Using the linearity of $\boldsymbol{\Lambda}$ and formula (6.4), one has $\vec{\boldsymbol{w}} = w^{\alpha}\vec{\boldsymbol{e}}_{\alpha} = \boldsymbol{\Lambda}(\vec{\boldsymbol{v}}) = \boldsymbol{\Lambda}(v^{\beta}\vec{\boldsymbol{e}}_{\beta}) = v^{\beta}\boldsymbol{\Lambda}(\vec{\boldsymbol{e}}_{\beta}) = v^{\beta}\Lambda^{\alpha}{}_{\beta}\, \vec{\boldsymbol{e}}_{\alpha}$. □

The Lorentz transformations can be characterized as follows:

> An endomorphism of E is a Lorentz transformation iff it maps any orthonormal basis of $(E, \boldsymbol{g})$ into another such basis.

Proof. Let $\boldsymbol{\Lambda}$ be a Lorentz transformation and $(\vec{\boldsymbol{e}}_{\alpha})$ an orthonormal basis of $(E, \boldsymbol{g})$. By definition $\vec{\boldsymbol{e}}_{\alpha} \cdot \vec{\boldsymbol{e}}_{\beta} = \eta_{\alpha\beta}$, where η is the Minkowski matrix (1.17). Since $\boldsymbol{\Lambda}$ preserves the scalar product, we have $\boldsymbol{\Lambda}(\vec{\boldsymbol{e}}_{\alpha}) \cdot \boldsymbol{\Lambda}(\vec{\boldsymbol{e}}_{\beta}) = \eta_{\alpha\beta}$, i.e. $(\boldsymbol{\Lambda}(\vec{\boldsymbol{e}}_{\alpha}))$ is an orthonormal basis of $(E, \boldsymbol{g})$. The converse is easy to establish: let $\boldsymbol{\Lambda}$ be an endomorphism of E that maps any orthonormal basis into an orthonormal basis. Let us consider two vectors $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{w}}$ in E and expand them onto an orthonormal basis

$(\vec{e}_\alpha)$: $\vec{v} = v^\alpha\, \vec{e}_\alpha$ and $\vec{w} = w^\alpha\, \vec{e}_\alpha$. We have then, by linearity of $\boldsymbol{\Lambda}$, $\boldsymbol{\Lambda}(\vec{v})\cdot\boldsymbol{\Lambda}(\vec{w}) = v^\alpha w^\beta \boldsymbol{\Lambda}(\vec{e}_\alpha)\cdot\boldsymbol{\Lambda}(\vec{e}_\beta)$. But since $\boldsymbol{\Lambda}$ maps an orthonormal basis into another one, $\boldsymbol{\Lambda}(\vec{e}_\alpha)\cdot\boldsymbol{\Lambda}(\vec{e}_\beta) = \eta_{\alpha\beta} = \vec{e}_\alpha\cdot\vec{e}_\beta$, so that $\boldsymbol{\Lambda}(\vec{v})\cdot\boldsymbol{\Lambda}(\vec{w}) = v^\alpha w^\beta\, \vec{e}_\alpha\cdot\vec{e}_\beta = \vec{v}\cdot\vec{w}$, which shows that $\boldsymbol{\Lambda}$ is a Lorentz transformation. □

Remark 6.2. Let us recall that the order of vectors is important in the definition of an orthonormal basis (cf. Sect. 1.3.3): if $(\vec{e}_\alpha)$ is an orthonormal basis of $(E, \boldsymbol{g})$, then $\vec{e}_0$ is the vector of scalar square -1 and $\vec{e}_1$, $\vec{e}_2$ and $\vec{e}_3$ are those of scalar square $+1$. Consequently, an endomorphism $\boldsymbol{\Lambda}$ that transforms an orthonormal basis $(\vec{e}_0, \vec{e}_1, \vec{e}_2, \vec{e}_3)$ in such a way that $(\boldsymbol{\Lambda}(\vec{e}_1), \boldsymbol{\Lambda}(\vec{e}_0), \boldsymbol{\Lambda}(\vec{e}_2), \boldsymbol{\Lambda}(\vec{e}_3))$ is an orthonormal basis is not a Lorentz transformation.

6.2.2 Lorentz Group

The set of all Lorentz transformations, equipped with the composition law $\circ$, constitutes a group. It is called ***Lorentz group*** and denoted by the symbol $\mathrm{O}(3,1)$.

Proof. All the axioms defining a group (cf. Appendix A) are satisfied:

- The law $\circ$ is an internal law: if $\boldsymbol{\Lambda}_1$ and $\boldsymbol{\Lambda}_2$ are two Lorentz transformations, then for any pair $(\vec{v}, \vec{w})$ of vectors of E,

$$\begin{aligned}(\boldsymbol{\Lambda}_1\circ\boldsymbol{\Lambda}_2)(\vec{v})\cdot(\boldsymbol{\Lambda}_1\circ\boldsymbol{\Lambda}_2)(\vec{w}) &= \boldsymbol{\Lambda}_1(\boldsymbol{\Lambda}_2(\vec{v}))\cdot\boldsymbol{\Lambda}_1(\boldsymbol{\Lambda}_2(\vec{w}))\\ &= \boldsymbol{\Lambda}_2(\vec{v})\cdot\boldsymbol{\Lambda}_2(\vec{w}) = \vec{v}\cdot\vec{w},\end{aligned}$$

  which shows that the composite function $\boldsymbol{\Lambda}_1\circ\boldsymbol{\Lambda}_2$ is a Lorentz transformation.
- The law $\circ$ is obviously associative: $\boldsymbol{\Lambda}_1\circ(\boldsymbol{\Lambda}_2\circ\boldsymbol{\Lambda}_3) = (\boldsymbol{\Lambda}_1\circ\boldsymbol{\Lambda}_2)\circ\boldsymbol{\Lambda}_3$; it is so for any function $E \to E$ and thus, in particular, for Lorentz transformations.
- There exists an identity element: the identity function $\mathrm{Id} : E \to E,\ \vec{v} \mapsto \vec{v}$, which is obviously a Lorentz transformation.
- Any element has an inverse: let $\boldsymbol{\Lambda}$ be a Lorentz transformation and $(\vec{e}_\alpha)$ an orthonormal basis of $(E, \boldsymbol{g})$. Then, by virtue of (6.2), $(\boldsymbol{\Lambda}(\vec{e}_\alpha))$ is an orthonormal basis. $\boldsymbol{\Lambda}$ is thus an invertible endomorphism. One has to show that its inverse, $\boldsymbol{\Lambda}^{-1}$, is also a Lorentz transformation: since $\boldsymbol{\Lambda}\circ\boldsymbol{\Lambda}^{-1} = \mathrm{Id}$, we have $\forall(\vec{v}, \vec{w}) \in E\times E$, $[\boldsymbol{\Lambda}\circ\boldsymbol{\Lambda}^{-1}(\vec{v})]\cdot[\boldsymbol{\Lambda}\circ\boldsymbol{\Lambda}^{-1}(\vec{w})] = \vec{v}\cdot\vec{w}$. On the other side, since $\boldsymbol{\Lambda}$ is a Lorentz transformation, we have $[\boldsymbol{\Lambda}\circ\boldsymbol{\Lambda}^{-1}(\vec{v})]\cdot[\boldsymbol{\Lambda}\circ\boldsymbol{\Lambda}^{-1}(\vec{w})] = \boldsymbol{\Lambda}^{-1}(\vec{v})\cdot\boldsymbol{\Lambda}^{-1}(\vec{w})$. Hence $\boldsymbol{\Lambda}^{-1}(\vec{v})\cdot\boldsymbol{\Lambda}^{-1}(\vec{w}) = \vec{v}\cdot\vec{w}$, which shows that $\boldsymbol{\Lambda}^{-1}$ is a Lorentz transformation. □

O(3, 1) is actually a subgroup of the ***general linear group of*** E, GL(E), which is the group of all the automorphisms of E. Let us recall that ***automorphism*** means invertible endomorphism, i.e. a bijective linear map from the vector space E to itself (cf. Appendix A). In the vocabulary of linear algebra, the Lorentz group is the ***orthogonal group*** associated with the scalar product $\boldsymbol{g}$. This explains the notation O(3, 1): O stands for "orthogonal", and (3, 1) refers to the signature $(-,+,+,+)$ of $\boldsymbol{g}$ (three $+$ and one $-$).

Remark 6.3. The notation O(3, 1) constitutes an extension of the well-known notations O(2) and O(3) for the isometry groups of, respectively, the Euclidean plane and Euclidean three-dimensional space. Indeed, the signature of the Euclidean scalar product is $(+,+)$ (plane) or $(+,+,+)$ (3-space), so that the corresponding orthogonal groups are O(2, 0) and O(3, 0), which is abridged in O(2) and O(3).

An immediate corollary of the group structure of O(3, 1) is that the set of all Lorentz matrices, equipped with the law of matrix multiplication, constitutes a subgroup of the group of 4×4 invertible real matrices.

6.2.3 Properties of Lorentz Transformations

Given $\boldsymbol{\Lambda} \in \mathrm{O}(3,1)$ and a subset F of E, one says that F is ***invariant under*** $\boldsymbol{\Lambda}$, or ***stable under*** $\boldsymbol{\Lambda}$, iff

$$\forall \vec{v} \in F, \quad \boldsymbol{\Lambda}(\vec{v}) \in F. \tag{6.6}$$

This property is equivalent to $\boldsymbol{\Lambda}(F) \subset F$. Moreover, one says that F is ***strictly invariant under*** $\boldsymbol{\Lambda}$ iff $\boldsymbol{\Lambda}$ restricted to F is the identity map:

$$\forall \vec{v} \in F, \quad \boldsymbol{\Lambda}(\vec{v}) = \vec{v}. \tag{6.7}$$

If F is strictly invariant under $\boldsymbol{\Lambda}$, it is of course stable under $\boldsymbol{\Lambda}$.

An immediate property of Lorentz transformations is to leave invariant the null cone of $\boldsymbol{g}$, $\mathscr{I}$ (cf. Sect. 1.4):

$$\boxed{\forall \boldsymbol{\Lambda} \in \mathrm{O}(3,1), \quad \boldsymbol{\Lambda}(\mathscr{I}) = \mathscr{I}}. \tag{6.8}$$

Proof. One has

$$\forall \vec{v} \in E, \quad \vec{v} \in \mathscr{I} \iff \vec{v} \cdot \vec{v} = 0 \iff \boldsymbol{\Lambda}(\vec{v}) \cdot \boldsymbol{\Lambda}(\vec{v}) = 0 \iff \boldsymbol{\Lambda}(\vec{v}) \in \mathscr{I},$$

where the part $\Leftarrow$ of the second equivalence is justified by the fact that if $\boldsymbol{\Lambda}$ is a Lorentz transformation, its inverse is also a Lorentz transformation. Thus we have $\boldsymbol{\Lambda}(\mathscr{I}) \subset \mathscr{I}$. Now every vector $\vec{v} \in \mathscr{I}$ has an inverse image in $\mathscr{I}$ by $\boldsymbol{\Lambda}$, namely, the vector $\boldsymbol{\Lambda}^{-1}(\vec{v})$. Therefore the $\subset$ sign can be replaced by an $=$ sign, leading to (6.8). □

Another important property of Lorentz transformations is to have a determinant equal to $+1$ or -1:

$$\boxed{\forall \boldsymbol{\Lambda} \in \mathrm{O}(3,1), \quad \det \boldsymbol{\Lambda} = \pm 1}. \tag{6.9}$$

Let us recall that the ***determinant of an endomorphism*** is the determinant Δ of the matrix of this endomorphism in any vector basis of E, Δ being independent of the choice of the basis. Using notations of (6.4), we have thus $\det \boldsymbol{\Lambda} = \det(\Lambda^{\alpha}{}_{\beta})$. To show the property (6.9), let us consider an orthonormal basis $(\vec{\boldsymbol{e}}_{\alpha})$ of $(E, \boldsymbol{g})$ and let us denote by Λ the matrix of $\boldsymbol{\Lambda}$ in this basis. In terms of components relative to the basis $(\vec{\boldsymbol{e}}_{\alpha})$, the defining property (6.2) of Lorentz transformations is written as

$$\forall (\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}) \in E \times E, \quad \eta_{\alpha\beta}\, [\boldsymbol{\Lambda}(\vec{\boldsymbol{v}})]^{\alpha} [\boldsymbol{\Lambda}(\vec{\boldsymbol{w}})]^{\beta} = \eta_{\mu\nu}\, v^{\mu} w^{\nu},$$

where we have taken into account the orthonormal feature of $(\vec{\boldsymbol{e}}_{\alpha})$ by replacing the components $g_{\alpha\beta}$ of $\boldsymbol{g}$ by the components $\eta_{\alpha\beta}$ of Minkowski matrix (1.17): $\eta_{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1)$. By means of (6.5), we get

$$\forall (\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}) \in E \times E, \quad \underbrace{\eta_{\alpha\beta}\, \Lambda^{\alpha}{}_{\mu}\, v^{\mu}\, \Lambda^{\beta}{}_{\nu}\, w^{\nu}}_{\eta_{\alpha\beta}\, \Lambda^{\alpha}{}_{\mu}\, \Lambda^{\beta}{}_{\nu}\, v^{\mu} w^{\nu}} = \eta_{\mu\nu}\, v^{\mu} w^{\nu}.$$

This identity being valid for any couple $(\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}})$ of vectors in E, we deduce that

$$\boxed{\eta_{\alpha\beta}\, \Lambda^{\alpha}{}_{\mu}\, \Lambda^{\beta}{}_{\nu} = \eta_{\mu\nu}}. \tag{6.10}$$

Let us express this formula in terms of matrix products. We recognize in $\eta_{\alpha\beta}\, \Lambda^{\beta}{}_{\nu}$ the matrix product $\eta\, \Lambda$. On the other hand, for $\eta_{\alpha\beta}\, \Lambda^{\alpha}{}_{\mu} = \Lambda^{\alpha}{}_{\mu}\, \eta_{\alpha\beta}$, the summation index α is ill placed to read directly a matrix product; one shall in fact consider the transpose of Λ, i.e. the matrix ${}^{\mathrm{t}}\Lambda$ defined by

$$\boxed{({}^{\mathrm{t}}\Lambda)_{\alpha}{}^{\beta} := \Lambda^{\beta}{}_{\alpha}}. \tag{6.11}$$

Then, $\Lambda^{\alpha}{}_{\mu}\, \eta_{\alpha\beta} = ({}^{\mathrm{t}}\Lambda)_{\mu}{}^{\alpha}\, \eta_{\alpha\beta} = ({}^{\mathrm{t}}\Lambda\, \eta)_{\mu\beta}$. Thus (6.10) is equivalent to

$$\boxed{{}^{\mathrm{t}}\Lambda\, \eta\, \Lambda = \eta}. \tag{6.12}$$

Since the determinant of a matrix product is the product of the determinants, we obtain immediately

$$(\det {}^{\mathrm{t}}\Lambda)(\det \eta)(\det \Lambda) = \det \eta.$$

$\det \eta$ being nonvanishing ($\det \eta = -1$), one may simplify this expression and use the identity $\det {}^{\mathrm{t}}\Lambda = \det \Lambda$ to get $(\det \Lambda)^2 = 1$. This demonstrates (6.9).

6.3 Subgroups of O(3,1)

6.3.1 Proper Lorentz Group SO(3,1)

In view of (6.9), Lorentz transformations can be classified in two categories: those of determinant $+1$ and those of determinant -1. A Lorentz transformation of determinant $+1$ is called a ***proper Lorentz transformation***. If instead its determinant is -1, it is called an ***improper Lorentz transformation***. A proper Lorentz transformation preserves the orientation of the vector bases of E: if $(\vec{e}_\alpha)$ is a right-handed basis, as defined in Sect. 1.5, then $(\boldsymbol{\Lambda}(\vec{e}_\alpha))$ is a right-handed basis too. This property follows from the following formula:

$$\boldsymbol{\varepsilon}(\boldsymbol{\Lambda}(\vec{e}_0), \boldsymbol{\Lambda}(\vec{e}_1), \boldsymbol{\Lambda}(\vec{e}_2), \boldsymbol{\Lambda}(\vec{e}_3)) = (\det \boldsymbol{\Lambda})\, \boldsymbol{\varepsilon}(\vec{e}_0, \vec{e}_1, \vec{e}_2, \vec{e}_3), \tag{6.13}$$

where $\boldsymbol{\varepsilon}$ is the Levi–Civita tensor introduced in Sect. 1.5 to define the orientation of E. Actually, formula (6.13) is very general: it is valid for any endomorphism $\boldsymbol{\Lambda}$ and any antisymmetric 4-linear form $\boldsymbol{\varepsilon}$. It could even be considered as the definition of the determinant of an endomorphism.

The identity is clearly a proper Lorentz transformation. Moreover, the standard formula $\det(\boldsymbol{\Lambda}_1 \circ \boldsymbol{\Lambda}_2) = (\det \boldsymbol{\Lambda}_1)(\det \boldsymbol{\Lambda}_2)$ implies that the composition of two proper Lorentz transformations is still a proper Lorentz transformation. It also implies that the inverse of a proper Lorentz transformation is proper (take $\boldsymbol{\Lambda}_2 = \boldsymbol{\Lambda}_1^{-1}$ and use $\det \mathrm{Id} = 1$). We conclude that the set of all proper Lorentz transformations constitutes a subgroup of the Lorentz group $\mathrm{O}(3,1)$. This subgroup is called the ***proper Lorentz group*** and is denoted by $\mathrm{SO}(3,1)$.

Remark 6.4. Here again, the notation $\mathrm{SO}(3,1)$ constitutes a generalization of the notations $\mathrm{SO}(2)$ and $\mathrm{SO}(3)$ for the rotation groups of, respectively, the Euclidean plane and the Euclidean three-dimensional space. Let us recall that rotations are nothing but the isometries of the Euclidean space whose determinant is $+1$.

Remark 6.5. The set of all improper Lorentz transformations is not a group, for the identity does not belong to it. Moreover, the composition of two improper Lorentz transformations is a proper Lorentz transformation.

We get immediately from (6.13) that if $\boldsymbol{\Lambda}$ is a proper Lorentz transformation, then $\boldsymbol{\varepsilon}(\boldsymbol{\Lambda}(\vec{e}_0), \boldsymbol{\Lambda}(\vec{e}_1), \boldsymbol{\Lambda}(\vec{e}_2), \boldsymbol{\Lambda}(\vec{e}_3)) = \boldsymbol{\varepsilon}(\vec{e}_0, \vec{e}_1, \vec{e}_2, \vec{e}_3)$. Thanks to the 4-linearity of $\boldsymbol{\varepsilon}$, we conclude that

$$\forall \boldsymbol{\Lambda} \in \mathrm{SO}(3,1),\ \forall (\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4) \in E^4,$$
$$\boxed{\boldsymbol{\varepsilon}(\boldsymbol{\Lambda}(\vec{v}_1), \boldsymbol{\Lambda}(\vec{v}_2), \boldsymbol{\Lambda}(\vec{v}_3), \boldsymbol{\Lambda}(\vec{v}_4)) = \boldsymbol{\varepsilon}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4)}\,. \tag{6.14}$$

6.3.2 Orthochronous Lorentz Group

Let $\vec{\boldsymbol{u}}$ be a future-directed timelike vector and $\boldsymbol{\Lambda} \in \mathrm{O}(3,1)$; then $\boldsymbol{\Lambda}(\vec{\boldsymbol{u}})$ is also timelike. Let $\vec{\boldsymbol{v}}$ be another future-directed timelike vector. By virtue of Lemma 1 of Sect. 1.4.2, we have $\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}} < 0$. Since $\boldsymbol{\Lambda}(\vec{\boldsymbol{u}}) \cdot \boldsymbol{\Lambda}(\vec{\boldsymbol{v}}) = \vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}}$ (for $\boldsymbol{\Lambda}$ is a Lorentz transformation), we get $\boldsymbol{\Lambda}(\vec{\boldsymbol{u}}) \cdot \boldsymbol{\Lambda}(\vec{\boldsymbol{u}}) < 0$. Invoking again Lemma 1 of Sect. 1.4.2, we conclude that the vectors $\boldsymbol{\Lambda}(\vec{\boldsymbol{u}})$ and $\boldsymbol{\Lambda}(\vec{\boldsymbol{v}})$ are either both future-directed or both past-directed. Consequently, we may classify Lorentz transformations in two distinct categories:

1. Those that map any future-directed timelike vector into a future-directed timelike vector; they are called ***orthochronous Lorentz transformations***.
2. Those that map any future-directed timelike vector into a past-directed timelike vector; they are called ***antichronous Lorentz transformations***.

Thanks to Lemma 1 of Sect. 1.4.2, $\boldsymbol{\Lambda}$ is orthochronous iff

$$\forall \vec{\boldsymbol{u}} \in E, \ \vec{\boldsymbol{u}} \text{ timelike} \implies \boldsymbol{\Lambda}(\vec{\boldsymbol{u}}) \cdot \vec{\boldsymbol{u}} < 0. \tag{6.15}$$

It is clear that (i) the identity is orthochronous, (ii) the composition of two orthochronous Lorentz transformations is still orthochronous and (iii) the inverse of an orthochronous Lorentz transformations is orthochronous. We conclude that the set of all orthochronous Lorentz transformations is thus a subgroup of the Lorentz group, called the ***orthochronous Lorentz group***; we shall denote it by $\mathrm{O}_\mathrm{o}(3,1)$.

Remark 6.6. The set of antichronous Lorentz transformations is not a group, for the identity does not belong to it.

The orthochronous condition is easily expressed in terms of the matrix Λ of the Lorentz transformation $\boldsymbol{\Lambda}$ in an orthonormal basis $(\vec{\boldsymbol{e}}_\alpha)$. Indeed, from (6.4),

$$\boldsymbol{\Lambda}(\vec{\boldsymbol{e}}_0) \cdot \vec{\boldsymbol{e}}_0 = \Lambda^\alpha{}_0 \underbrace{\vec{\boldsymbol{e}}_\alpha \cdot \vec{\boldsymbol{e}}_0}_{-\delta_{0\alpha}} = -\Lambda^0{}_0. \tag{6.16}$$

Equation (6.15) shows that $\boldsymbol{\Lambda}$ is orthochronous iff $\boldsymbol{\Lambda}(\vec{\boldsymbol{e}}_0) \cdot \vec{\boldsymbol{e}}_0 < 0$. This last condition is thus equivalent to $\Lambda^0{}_0 > 0$. Actually, in this case, one has necessarily $\Lambda^0{}_0 \geq 1$. Indeed, picking $\mu = 0$ and $\nu = 0$ in (6.10), we get successively

$$\Lambda^\alpha{}_0 \, \eta_{\alpha\beta} \, \Lambda^\beta{}_0 = \eta_{00} = -1,$$

$$-(\Lambda^0{}_0)^2 + \sum_{i=1}^{3} (\Lambda^i{}_0)^2 = -1,$$

$$(\Lambda^0{}_0)^2 = 1 + \sum_{i=1}^{3} (\Lambda^i{}_0)^2.$$

We have thus $(\Lambda^0{}_0)^2 \geq 1$, so that $\Lambda^0{}_0 > 0$ (resp. $\Lambda^0{}_0 < 0$) is actually equivalent to $\Lambda^0{}_0 \geq 1$ (resp. $\Lambda^0{}_0 \leq -1$). We conclude that a Lorentz transformation is orthochronous iff its matrix Λ in some orthonormal basis obeys

$$\boxed{\Lambda^0{}_0 \geq 1}. \tag{6.17}$$

In the opposite case, one has necessarily $\Lambda^0{}_0 \leq -1$.

6.3.3 Restricted Lorentz Group

A Lorentz transformation that is both proper and orthochronous is called a ***restricted Lorentz transformation***. In terms of its matrix Λ in a given orthonormal basis of $(E, \boldsymbol{g})$, a restricted Lorentz transformation is characterized by

$$\det \Lambda = 1 \qquad \text{and} \qquad \Lambda^0_0 \geq 1. \tag{6.18}$$

The local frame $(\vec{\boldsymbol{e}}_\alpha)$ of an observer being a right-handed basis of E with $\vec{\boldsymbol{e}}_0$ future-directed (being a 4-velocity!), we note that the restricted Lorentz transformations are those that relate the local frames of two observers, hence their importance.

It is clear that the set of all restricted Lorentz transformations constitutes a subgroup of both SO(3, 1) and $\mathrm{O_o}(3, 1)$. It is naturally called the ***restricted Lorentz group*** and denoted by $\mathrm{SO_o}(3, 1)$.

6.3.4 Reduction of the Lorentz Group to $SO_o(3, 1)$

Summarizing the preceding results, one may write the Lorentz group as the union of four nonintersecting components:

$$\mathrm{O}(3,1) = \underbrace{\mathrm{SO_o}(3,1)\ \cup\ \mathrm{SO_a}(3,1)}_{\mathrm{SO}(3,1)} \cup \mathrm{O_o^-}(3,1)\ \cup\ \mathrm{O_a^-}(3,1), \tag{6.19}$$

where $\mathrm{SO_a}(3, 1)$ stands for the set of antichronous proper Lorentz transformations, $\mathrm{O_o^-}(3, 1)$ for the set of orthochronous improper Lorentz transformations and $\mathrm{O_a^-}(3, 1)$ for the set of antichronous improper Lorentz transformations. Note that among the above four components, only $\mathrm{SO_o}(3, 1)$ is a group.

It is easy to reduce a generic Lorentz transformation to a restricted one (i.e. to an element of $\mathrm{SO_o}(3, 1)$). Indeed let us define the ***spacetime inversion operator*** as the opposite of the identity operator on E: $\boldsymbol{I} := -\,\mathrm{Id}$. Moreover, given a right-handed orthonormal basis of $(E, \boldsymbol{g})$, $(\vec{\boldsymbol{e}}_\alpha)$ say, let us introduce the following endomorphisms $E \to E$:

$$\forall \vec{v} = v^\alpha \vec{e}_\alpha \in E, \quad \boldsymbol{T}(\vec{v}) := -v^0 \vec{e}_0 + v^i \vec{e}_i, \quad \boldsymbol{P}(\vec{v}) := v^0 \vec{e}_0 - v^i \vec{e}_i. \tag{6.20}$$

We shall call $\boldsymbol{T}$ the ***time reversal operator associated with the basis*** $(\vec{e}_\alpha)$ and $\boldsymbol{P}$ the ***space inversion operator associated with the basis*** $(\vec{e}_\alpha)$. $\boldsymbol{P}$ is also called the ***parity inversion operator associated with the basis*** $(\vec{e}_\alpha)$ (hence the letter P). The matrices of each of these operators with respect to the basis $(\vec{e}_\alpha)$ are

$$I^\alpha_{\ \beta} = \mathrm{diag}(-1, -1, -1, -1),$$

$$T^\alpha_{\ \beta} = \mathrm{diag}(-1, 1, 1, 1) \qquad \text{and} \qquad P^\alpha_{\ \beta} = \mathrm{diag}(1, -1, -1, -1).$$

As they map an orthonormal basis to an orthonormal basis, these three operators are Lorentz transformations. More precisely, since $\det \boldsymbol{I} = 1$, $\boldsymbol{I}(\vec{e}_0) = -\vec{e}_0$, $\det \boldsymbol{T} = -1$, $\boldsymbol{T}(\vec{e}_0) = -\vec{e}_0$, $\det \boldsymbol{P} = -1$ and $\boldsymbol{P}(\vec{e}_0) = \vec{e}_0$, we have

$$\boldsymbol{I} \in \mathrm{SO_a}(3,1), \quad \boldsymbol{T} \in \mathrm{O_a^-}(3,1), \quad \boldsymbol{P} \in \mathrm{O_o^-}(3,1). \tag{6.21}$$

In addition, each of these operators is an involution: $\boldsymbol{I}^{-1} = \boldsymbol{I}$, $\boldsymbol{T}^{-1} = \boldsymbol{T}$ and $\boldsymbol{P}^{-1} = \boldsymbol{P}$. Let us consider now a Lorentz transformation $\boldsymbol{\Lambda}$ that is not a restricted one. There are three possibilities:

1. $\boldsymbol{\Lambda} \in \mathrm{SO_a}(3,1)$: then $\boldsymbol{I} \circ \boldsymbol{\Lambda} \in \mathrm{SO_o}(3,1)$, because $\det(\boldsymbol{I} \circ \boldsymbol{\Lambda}) = (+1)(+1) = +1$ and $\boldsymbol{I} \circ \boldsymbol{\Lambda}$ is orthochronous, for both $\boldsymbol{I}$ and $\boldsymbol{\Lambda}$ are antichronous. Since $\boldsymbol{I}$ is its own inverse, we get

$$\boldsymbol{\Lambda} = \boldsymbol{I} \circ \boldsymbol{\Lambda}_0, \quad \text{with} \quad \boldsymbol{\Lambda}_0 \in \mathrm{SO_o}(3,1). \tag{6.22}$$

2. $\boldsymbol{\Lambda} \in \mathrm{O_o^-}(3,1)$: then $\boldsymbol{P} \circ \boldsymbol{\Lambda} \in \mathrm{SO_o}(3,1)$, because $\det(\boldsymbol{P} \circ \boldsymbol{\Lambda}) = (-1)(-1) = +1$ and $\boldsymbol{P} \circ \boldsymbol{\Lambda}$ is orthochronous, for both $\boldsymbol{P}$ and $\boldsymbol{\Lambda}$ are orthochronous. Since $\boldsymbol{P}$ is its own inverse, we get

$$\boldsymbol{\Lambda} = \boldsymbol{P} \circ \boldsymbol{\Lambda}_0, \quad \text{with} \quad \boldsymbol{\Lambda}_0 \in \mathrm{SO_o}(3,1). \tag{6.23}$$

3. $\boldsymbol{\Lambda} \in \mathrm{O_a^-}(3,1)$: then $\boldsymbol{T} \circ \boldsymbol{\Lambda} \in \mathrm{SO_o}(3,1)$, because $\det(\boldsymbol{T} \circ \boldsymbol{\Lambda}) = (-1)(-1) = +1$ and $\boldsymbol{T} \circ \boldsymbol{\Lambda}$ is orthochronous, for both $\boldsymbol{T}$ and $\boldsymbol{\Lambda}$ are antichronous. Since $\boldsymbol{T}$ is its own inverse, we get

$$\boldsymbol{\Lambda} = \boldsymbol{T} \circ \boldsymbol{\Lambda}_0, \quad \text{with} \quad \boldsymbol{\Lambda}_0 \in \mathrm{SO_o}(3,1). \tag{6.24}$$

The results (6.22), (6.23) and (6.24) show that thanks to the inversion operators $\boldsymbol{I}$, $\boldsymbol{P}$ and $\boldsymbol{T}$, one can reduce any Lorentz transformation to a restricted one.

Remark 6.7. The set of the four operators $\{\mathrm{Id}, \boldsymbol{I}, \boldsymbol{T}, \boldsymbol{P}\}$, equipped with the composition law $\circ$, is a finite subgroup of the Lorentz group $\mathrm{O}(3,1)$. Indeed, we have $\boldsymbol{I} \circ \boldsymbol{T} = \boldsymbol{P}$, $\boldsymbol{I} \circ \boldsymbol{P} = \boldsymbol{T}$, etc. $\{\mathrm{Id}, \boldsymbol{I}, \boldsymbol{T}, \boldsymbol{P}\}$ is thus stable with respect to the law $\circ$. Moreover, the identity belongs to this set, and each element is its own inverse. This 4-element group is isomorphic to the ***Klein group*** $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Let us recall that, up to some isomorphism, there exists only two groups of order 4: the Klein group and the cyclic group $\mathbb{Z}/4\mathbb{Z}$; they are both abelian.

Remark 6.8. Reasoning as above on the determinants and the images of $\vec{e}_0$, it is easy to see that

$$\forall \boldsymbol{\Lambda}_0 \in \mathrm{SO_o}(3,1), \quad \forall \boldsymbol{\Lambda} \in \mathrm{O}(3,1), \quad \boldsymbol{\Lambda} \circ \boldsymbol{\Lambda}_0 \circ \boldsymbol{\Lambda}^{-1} \in \mathrm{SO_o}(3,1). \tag{6.25}$$

This property means that $\mathrm{SO_o}(3,1)$ is a normal subgroup of $\mathrm{O}(3,1)$ [cf. Eq. (A.3) in Appendix A]. One can then consider the quotient group $\mathrm{O}(3,1)/\mathrm{SO_o}(3,1)$. It is isomorphic to the group $\{\mathrm{Id}, \boldsymbol{I}, \boldsymbol{T}, \boldsymbol{P}\}$ considered above:

$$\mathrm{O}(3,1)/\mathrm{SO_o}(3,1) \simeq \{\mathrm{Id}, \boldsymbol{I}, \boldsymbol{T}, \boldsymbol{P}\} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}. \tag{6.26}$$

6.4 Classification of Restricted Lorentz Transformations

In view of the result of Sect. 6.3.4, we shall restrict ourselves from now on to the study of $\mathrm{SO_o}(3,1)$, i.e. to restricted Lorentz transformations. We shall exhibit their general form and classify them in different types.

6.4.1 Invariant Null Direction

Let us call ***null direction*** any vector line $\Delta \subset E$ formed by null vectors: $\Delta = \mathrm{Span}(\vec{\ell})$, with $\vec{\ell} \cdot \vec{\ell} = 0$. The straight lines of the affine space $\mathscr{E}$ corresponding to null directions are nothing but the null geodesics defined in Sect. 2.5.1 (photon worldlines).

The starting point of our study is the following property[2]:

> Any restricted Lorentz transformation admits at least one invariant null direction.

In other words, given $\boldsymbol{\Lambda} \in \mathrm{SO_o}(3,1)$, there exists a null direction $\Delta = \mathrm{Span}(\vec{\ell})$ such that[3] $\boldsymbol{\Lambda}(\Delta) = \Delta$. Equivalently, there exists a nonvanishing null vector $\vec{\ell}$ such

[2]Cf. Sect. 6.2.3 for the definition of invariance under a Lorentz transformation.

[3]A priori, the definition of invariance stipulates only that $\boldsymbol{\Lambda}(\Delta) \subset \Delta$, but every vector $\vec{v} \in \Delta$ has an inverse image in Δ by $\boldsymbol{\Lambda}$, namely, the vector $\boldsymbol{\Lambda}^{-1}(\vec{v})$. Therefore the $\subset$ sign can be replaced by an $=$ sign.

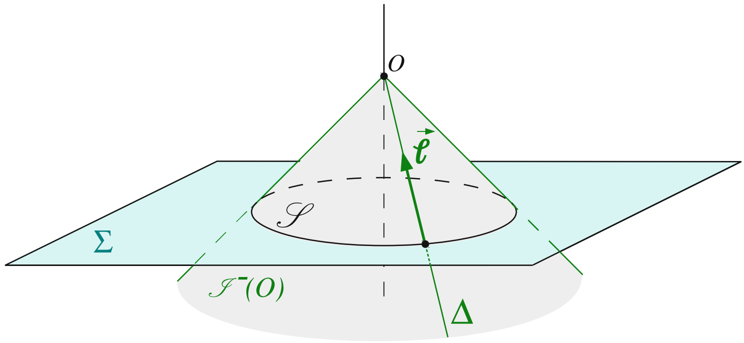

Fig. 6.1 Sphere $\mathscr{S}$, intersection of the past light cone of O, $\mathscr{I}^-(O)$, with a spacelike hyperplane Σ. For the drawing, one dimension has been suppressed, so that $\mathscr{S}$ appears as a circle

that $\boldsymbol{\Lambda}(\vec{\ell}) = \lambda\vec{\ell}$, with $\lambda \in \mathbb{R} \setminus \{0\}$. $\vec{\ell}$ is then a null eigenvector of $\boldsymbol{\Lambda}$. If $\lambda = 1$, the null direction Δ is strictly invariant under $\boldsymbol{\Lambda}$.

Proof. We shall demonstrate the above property by an algebraic method in Sect. 7.5.5. Here we provide instead a demonstration based on a topological argument. Let us consider the intersection of the past light cone of some event $O \in \mathscr{E}$, $\mathscr{I}^-(O)$, with some spacelike hyperplane Σ, i.e. some three-dimensional affine subspace of $\mathscr{E}$ such that any vector parallel to it is spacelike (cf. Sect. 3.2.3). $\mathscr{S} := \mathscr{I}^-(O) \cap \Sigma$ has the topology of a sphere (cf. Fig. 6.1). It can be seen as the celestial sphere of an observer that would have Σ as a local rest space (cf. Sect. 5.6.2 and Fig. 5.11). Each null direction Δ can be identified with a point of $\mathscr{S}$: this "point" defines, up to a scaling factor, a direction vector of Δ. The property (6.8) of invariance of the null cone $\mathscr{I}$ under $\boldsymbol{\Lambda}$ means that $\boldsymbol{\Lambda}$ maps each null direction in another null direction. This implies that $\boldsymbol{\Lambda}$ induces a mapping from the sphere $\mathscr{S}$ to itself. This mapping $\mathscr{S} \to \mathscr{S}$ is continuous, because $\boldsymbol{\Lambda}$ is so (as any linear map). Moreover, it preserves the orientation since $\boldsymbol{\Lambda}$ is a restricted Lorentz transformation. We may then invoke a topology theorem[4] which stipulates that any continuous and orientation-preserving mapping from the sphere to itself has a fixed point. There exists thus a "point" Δ of $\mathscr{S}$ such that $\boldsymbol{\Lambda}(\Delta) = \Delta$. □

[4]Cf., for instance, corollary 18.2.5.6 in Berger's book (Berger 1987b), according to which any continuous mapping $\mathscr{S} \to \mathscr{S}$ of degree different from -1 admits at least a fixed point; this applies to an orientation-preserving mapping for its degree is positive (by the very definition of the degree). One can also obtain this theorem as a consequence of a more general result: the so-called *Lefschetz fixed-point theorem*.

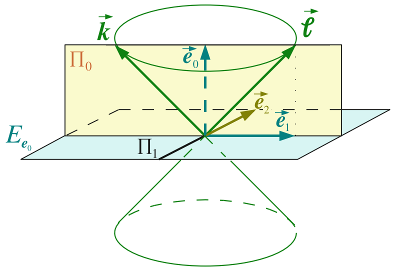

Fig. 6.2 Definition of vectors $\vec{e}_0$ and $\vec{e}_1$ from two null vectors $\vec{\ell}$ and $\vec{k}$, the former being an eigenvector of $\boldsymbol{\Lambda}$. The plane $\Pi_0 = \mathrm{Span}(\vec{\ell}, \vec{k}) = \mathrm{Span}(\vec{e}_0, \vec{e}_1)$ is timelike. For the drawing, its orthogonal complementary, Π_1, has been reduced to a single dimension

6.4.2 Decomposition with Respect to an Invariant Null Direction

The above property greatly simplifies the study of Lorentz transformations: it guarantees the existence of a null eigenvector $\vec{\ell}$ for any $\boldsymbol{\Lambda} \in \mathrm{SO_o}(3,1)$. We shall start from this eigenvector to decompose $\boldsymbol{\Lambda}$. Let us choose $\vec{\ell}$ to be future-directed and let us denote by λ the associated eigenvalue: $\boldsymbol{\Lambda}(\vec{\ell}) = \lambda \vec{\ell}$. Since $\boldsymbol{\Lambda}$ is an automorphism of E, we have $\lambda \neq 0$. Moreover, $\boldsymbol{\Lambda}$ being orthochronous, we have $\lambda > 0$. We are thus allowed to set $\psi := \ln \lambda$ and to write

$$\boxed{\boldsymbol{\Lambda}(\vec{\ell}) = \mathrm{e}^{\psi}\, \vec{\ell}}, \qquad \psi \in \mathbb{R}. \tag{6.27}$$

The second stage in the decomposition consists in picking a future-directed null vector, $\vec{k}$, that is not collinear to $\vec{\ell}$. By virtue of Lemma 2 of Sect. 1.4.2, we have $\vec{\ell} \cdot \vec{k} < 0$. At the price of rescaling it by a positive factor, we can always choose $\vec{k}$ so that

$$\vec{\ell} \cdot \vec{k} = -2. \tag{6.28}$$

Let us then define

$$\vec{e}_0 := \frac{1}{2}(\vec{\ell} + \vec{k}) \qquad \text{and} \qquad \vec{e}_1 := \frac{1}{2}(\vec{\ell} - \vec{k}). \tag{6.29}$$

These formulas can be inverted in (cf. Fig. 6.2):

$$\boxed{\vec{\ell} = \vec{e}_0 + \vec{e}_1} \qquad \text{and} \qquad \boxed{\vec{k} = \vec{e}_0 - \vec{e}_1}. \tag{6.30}$$

Taking into account that $\vec{\boldsymbol{\ell}}\cdot\vec{\boldsymbol{\ell}} = 0$, $\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{k}} = 0$ and $\vec{\boldsymbol{\ell}}\cdot\vec{\boldsymbol{k}} = -2$, we observe that $\vec{\boldsymbol{e}}_0\cdot\vec{\boldsymbol{e}}_0 = -1$, $\vec{\boldsymbol{e}}_1\cdot\vec{\boldsymbol{e}}_1 = 1$ and $\vec{\boldsymbol{e}}_0\cdot\vec{\boldsymbol{e}}_1 = 0$. $(\vec{\boldsymbol{e}}_0, \vec{\boldsymbol{e}}_1)$ is thus an orthonormal basis of the vector plane

$$\Pi_0 := \mathrm{Span}(\vec{\boldsymbol{e}}_0, \vec{\boldsymbol{e}}_1) = \mathrm{Span}(\vec{\boldsymbol{\ell}}, \vec{\boldsymbol{k}}). \tag{6.31}$$

A plane such as Π_0, i.e. spanned by a timelike vector ($\vec{\boldsymbol{e}}_0$) and a spacelike one ($\vec{\boldsymbol{e}}_1$), is called a ***timelike plane***. The metric induced by $\boldsymbol{g}$ in Π_0 is Lorentzian: $\mathrm{sign}\, \boldsymbol{g}|_{\Pi_0} = (-,+)$. The orthogonal complementary (with respect to $\boldsymbol{g}$) of Π_0, $\Pi_0^{\perp}$, is a plane too, which we shall denote by Π_1. We have the decomposition

$$E = \Pi_0 \overset{\perp}{\oplus} \Pi_1. \tag{6.32}$$

This means that every vector of E can be written in a unique way as the sum of two orthogonal vectors, one in Π_0 and the other in Π_1. The signature of the restriction of $\boldsymbol{g}$ to Π_0 being $(-,+)$, one must have $\mathrm{sign}\, \boldsymbol{g}|_{\Pi_1} = (+,+)$ to recover the global signature $(-,+,+,+)$ of $\boldsymbol{g}$. Consequently all the vectors of Π_1 are spacelike. Π_1 is accordingly called a ***spacelike plane***.

Remark 6.9. The orthogonal decomposition (6.32) can be qualified of *2 + 2 decomposition* of E (since the vector subspaces Π_0 and Π_1 are both two-dimensional). On the other hand, the orthogonal decomposition (3.16) encountered in Chap. 3 is a *3 + 1 decomposition*.

Remark 6.10. The signature of the restriction of the metric $\boldsymbol{g}$ to Π_0 is $(-,+)$, whereas that of the restriction of $\boldsymbol{g}$ to Π_1 is $(+,+)$. One says that $(\Pi_1, \boldsymbol{g})$ is a ***Euclidean plane*** and $(\Pi_0, \boldsymbol{g})$ a ***Minkowskian plane***. The latter is a two-dimensional analogue of the vector space $(E, \boldsymbol{g})$ underlying Minkowski spacetime. In particular, it has a null cone, made of two vector lines, $\mathrm{Span}(\vec{\boldsymbol{\ell}})$ and $\mathrm{Span}(\vec{\boldsymbol{k}})$, which constitute the intersection of the null cone of $(E, \boldsymbol{g})$ with Π_0 (cf. Fig. 6.2). A Minkowskian plane is sometimes called an ***Artinian plane*** (Berger 1987b).

Thanks to (6.32), $\boldsymbol{\Lambda}(\vec{\boldsymbol{k}})$ can be orthogonally split into a part in Π_0, $a\,\vec{\boldsymbol{\ell}} + b\,\vec{\boldsymbol{k}}$ say, and a part (possibly zero) in Π_1, $\vec{\boldsymbol{m}}$ say: $\boldsymbol{\Lambda}(\vec{\boldsymbol{k}}) = a\,\vec{\boldsymbol{\ell}} + b\,\vec{\boldsymbol{k}} + \vec{\boldsymbol{m}}$. Now, since $\boldsymbol{\Lambda}$ is a Lorentz transformation and $\vec{\boldsymbol{\ell}}\cdot\vec{\boldsymbol{k}} = -2$, we must have [cf. (6.2)]

$$\underbrace{\boldsymbol{\Lambda}(\vec{\boldsymbol{\ell}})}_{\mathrm{e}^{\psi}\,\vec{\boldsymbol{\ell}}}\cdot\underbrace{\boldsymbol{\Lambda}(\vec{\boldsymbol{k}})}_{a\,\vec{\boldsymbol{\ell}}+b\,\vec{\boldsymbol{k}}+\vec{\boldsymbol{m}}} = -2.$$

Given that $\vec{\boldsymbol{\ell}}\cdot\vec{\boldsymbol{\ell}} = 0$, $\vec{\boldsymbol{\ell}}\cdot\vec{\boldsymbol{k}} = -2$ and $\vec{\boldsymbol{\ell}}\cdot\vec{\boldsymbol{m}} = 0$ (for $\vec{\boldsymbol{m}} \in \Pi_1 = \Pi_0^{\perp}$), we get $\mathrm{e}^{\psi} b(-2) = -2$, i.e. $b = \mathrm{e}^{-\psi}$. On the other hand, still invoking the fact that $\boldsymbol{\Lambda}$ is a Lorentz transformation, the property $\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{k}} = 0$ implies $\boldsymbol{\Lambda}(\vec{\boldsymbol{k}})\cdot\boldsymbol{\Lambda}(\vec{\boldsymbol{k}}) = 0$, i.e.

$$(a\,\vec{\boldsymbol{\ell}} + \mathrm{e}^{-\psi}\,\vec{\boldsymbol{k}} + \vec{\boldsymbol{m}})\cdot(a\,\vec{\boldsymbol{\ell}} + \mathrm{e}^{-\psi}\,\vec{\boldsymbol{k}} + \vec{\boldsymbol{m}}) = 0.$$

Expanding and simplifying, we get

$$\vec{\boldsymbol{m}} \cdot \vec{\boldsymbol{m}} = 4\mathrm{e}^{-\psi} a.$$

Now, as a vector in Π_1, $\vec{\boldsymbol{m}}$ is either zero or spacelike, so that $\vec{\boldsymbol{m}} \cdot \vec{\boldsymbol{m}} \geq 0$, which implies $a \geq 0$. We may then introduce $\alpha := \sqrt{a}\mathrm{e}^{\psi/2}/2$, so that the above equation is equivalent to $\|\vec{\boldsymbol{m}}\|_{\boldsymbol{g}} = 4\alpha\mathrm{e}^{-\psi}$. If $\alpha \neq 0$, $\|\vec{\boldsymbol{m}}\|_{\boldsymbol{g}} \neq 0$ and we may define the unit vector $\vec{\boldsymbol{e}}_2 := \|\vec{\boldsymbol{m}}\|_{\boldsymbol{g}}^{-1}\vec{\boldsymbol{m}} = \mathrm{e}^{\psi}/(4\alpha)\,\vec{\boldsymbol{m}}$ in Π_1. If $\alpha = 0$, we consider whatever unit vector $\vec{\boldsymbol{e}}_2 \in \Pi_1$. Gathering all the above results, we write

$$\boxed{\boldsymbol{\Lambda}(\vec{\boldsymbol{k}}) = \mathrm{e}^{-\psi}\left(4\alpha^2\,\vec{\boldsymbol{\ell}} + \vec{\boldsymbol{k}} + 4\alpha\,\vec{\boldsymbol{e}}_2\right)}. \tag{6.33}$$

Let us associate with $\vec{\boldsymbol{e}}_2$ a unit vector $\vec{\boldsymbol{e}}_3 \in \Pi_1$ in order to form an orthonormal basis of $(\Pi_1, \boldsymbol{g})$, in such a way that $(\vec{\boldsymbol{e}}_0, \vec{\boldsymbol{e}}_1, \vec{\boldsymbol{e}}_2, \vec{\boldsymbol{e}}_3)$ is a *right-handed* orthonormal basis of $(E, \boldsymbol{g})$. Since $\vec{\boldsymbol{\ell}}$ and $\vec{\boldsymbol{k}}$ are two noncollinear vectors in Π_0, another basis of E is

$$(\vec{\boldsymbol{e}}^{\,*}_{\alpha}) := (\vec{\boldsymbol{\ell}}, \vec{\boldsymbol{k}}, \vec{\boldsymbol{e}}_2, \vec{\boldsymbol{e}}_3). \tag{6.34}$$

Of course, this is not an orthonormal basis. Let us expand $\boldsymbol{\Lambda}(\vec{\boldsymbol{e}}_2)$ onto that basis:

$$\boldsymbol{\Lambda}(\vec{\boldsymbol{e}}_2) = u\,\vec{\boldsymbol{\ell}} + v\,\vec{\boldsymbol{k}} + x\,\vec{\boldsymbol{e}}_2 + y\,\vec{\boldsymbol{e}}_3.$$

The coefficients (u, v, x, y) are determined from $\boldsymbol{\Lambda}$'s property of preserving scalar products. For instance, the condition $\boldsymbol{\Lambda}(\vec{\boldsymbol{\ell}})\cdot\boldsymbol{\Lambda}(\vec{\boldsymbol{e}}_2) = \vec{\boldsymbol{\ell}}\cdot\vec{\boldsymbol{e}}_2 = 0$, along with (6.27) and (6.28), leads to $-2\mathrm{e}^{\psi}v = 0$; hence, $v = 0$. Next, the condition $\boldsymbol{\Lambda}(\vec{\boldsymbol{e}}_2)\cdot\boldsymbol{\Lambda}(\vec{\boldsymbol{e}}_2) = \vec{\boldsymbol{e}}_2\cdot\vec{\boldsymbol{e}}_2 = 1$ leads to $x^2 + y^2 = 1$, which allows us to introduce $\varphi \in [0, 2\pi[$ so that $x = \cos\varphi$ and $y = \sin\varphi$. Finally, the condition $\boldsymbol{\Lambda}(\vec{\boldsymbol{k}})\cdot\boldsymbol{\Lambda}(\vec{\boldsymbol{e}}_2) = \vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{e}}_2 = 0$, along with (6.33), gives

$$\mathrm{e}^{-\psi}\left(4\alpha^2\,\vec{\boldsymbol{\ell}} + \vec{\boldsymbol{k}} + 4\alpha\,\vec{\boldsymbol{e}}_2\right)\cdot\left(u\,\vec{\boldsymbol{\ell}} + \cos\varphi\,\vec{\boldsymbol{e}}_2 + \sin\varphi\,\vec{\boldsymbol{e}}_3\right) = 0.$$

Expanding, we obtain $-2u + 4\alpha\cos\varphi = 0$; hence, $u = 2\alpha\cos\varphi$ and finally

$$\boxed{\boldsymbol{\Lambda}(\vec{\boldsymbol{e}}_2) = 2\alpha\cos\varphi\,\vec{\boldsymbol{\ell}} + \cos\varphi\,\vec{\boldsymbol{e}}_2 + \sin\varphi\,\vec{\boldsymbol{e}}_3}. \tag{6.35}$$

Similarly let us expand $\boldsymbol{\Lambda}(\vec{\boldsymbol{e}}_3)$ onto the basis $(\vec{\boldsymbol{e}}^{\,*}_{\alpha})$:

$$\boldsymbol{\Lambda}(\vec{\boldsymbol{e}}_3) = u'\,\vec{\boldsymbol{\ell}} + v'\,\vec{\boldsymbol{k}} + x'\,\vec{\boldsymbol{e}}_2 + y'\,\vec{\boldsymbol{e}}_3.$$

The conservation by $\boldsymbol{\Lambda}$ of the scalar products $\vec{\boldsymbol{\ell}}\cdot\vec{\boldsymbol{e}}_3 = 0$, $\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{e}}_3 = 0$, $\vec{\boldsymbol{e}}_2\cdot\vec{\boldsymbol{e}}_3 = 0$ and $\vec{\boldsymbol{e}}_3\cdot\vec{\boldsymbol{e}}_3 = 1$ leads to $u' = -2\alpha\sin\varphi$, $v' = 0$, $x' = -\sin\varphi$ and $y' = \cos\varphi$. Hence

$$\boxed{\boldsymbol{\Lambda}(\vec{\boldsymbol{e}}_3) = -2\alpha\sin\varphi\,\vec{\boldsymbol{\ell}} - \sin\varphi\,\vec{\boldsymbol{e}}_2 + \cos\varphi\,\vec{\boldsymbol{e}}_3}. \tag{6.36}$$

Let us collect the results (6.27), (6.33), (6.35) and (6.36) by writing the matrix of $\boldsymbol{\Lambda}$ in the basis $(\vec{e}^{\,*}_\alpha) := (\vec{\ell}, \vec{k}, \vec{e}_2, \vec{e}_3)$:

$$\boxed{(\Lambda^*)^\alpha{}_\beta = \begin{pmatrix} \mathrm{e}^\psi & 4\alpha^2\mathrm{e}^{-\psi} & 2\alpha\cos\varphi & -2\alpha\sin\varphi \\ 0 & \mathrm{e}^{-\psi} & 0 & 0 \\ 0 & 4\alpha\mathrm{e}^{-\psi} & \cos\varphi & -\sin\varphi \\ 0 & 0 & \sin\varphi & \cos\varphi \end{pmatrix}}. \tag{6.37}$$

Thanks to (6.29), it is easy to deduce the matrix of $\boldsymbol{\Lambda}$ in the orthonormal basis $(\vec{e}_\alpha) := (\vec{e}_0, \vec{e}_1, \vec{e}_2, \vec{e}_3)$:

$$\boxed{\Lambda^\alpha{}_\beta = \begin{pmatrix} \cosh\psi + 2\alpha^2\mathrm{e}^{-\psi} & \sinh\psi - 2\alpha^2\mathrm{e}^{-\psi} & 2\alpha\cos\varphi & -2\alpha\sin\varphi \\ \sinh\psi + 2\alpha^2\mathrm{e}^{-\psi} & \cosh\psi - 2\alpha^2\mathrm{e}^{-\psi} & 2\alpha\cos\varphi & -2\alpha\sin\varphi \\ 2\alpha\mathrm{e}^{-\psi} & -2\alpha\mathrm{e}^{-\psi} & \cos\varphi & -\sin\varphi \\ 0 & 0 & \sin\varphi & \cos\varphi \end{pmatrix}}, \tag{6.38}$$

where use has been made of the identities $\mathrm{e}^\psi + \mathrm{e}^{-\psi} = 2\cosh\psi$ and $\mathrm{e}^\psi - \mathrm{e}^{-\psi} = 2\sinh\psi$. The matrix of $\boldsymbol{\Lambda}$ in any of the bases $(\vec{e}^{\,*}_\alpha)$ and $(\vec{e}_\alpha)$ depends on three parameters: $\psi \in \mathbb{R}$, $\alpha \in \mathbb{R}^+$ and $\varphi \in [0, 2\pi[$. Formulas (6.37) and (6.38) constitute the most general expression of a restricted Lorentz transformation. Let us now investigate special cases, where some of the parameters ψ, α and φ vanish.

6.4.3 *Spatial Rotations*

If $\psi = 0$ and $\alpha = 0$, the matrix of $\boldsymbol{\Lambda}$ in the orthonormal basis $(\vec{e}_\alpha)$ is identical to that in the basis $(\vec{e}^{\,*}_\alpha)$ and takes the form

$$\boxed{\Lambda^\alpha{}_\beta = (\Lambda^*)^\alpha{}_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\varphi & -\sin\varphi \\ 0 & 0 & \sin\varphi & \cos\varphi \end{pmatrix}}. \tag{6.39}$$

We observe that the plane $\Pi_0 = \mathrm{Span}(\vec{e}_0, \vec{e}_1) = \mathrm{Span}(\vec{\ell}, \vec{k})$ is strictly invariant under such a transformation and that the action of $\boldsymbol{\Lambda}$ in the plane $\Pi_1 = \Pi_0^\perp = \mathrm{Span}(\vec{e}_2, \vec{e}_3)$, where the metric $\boldsymbol{g}$ is Euclidean, is nothing but an "ordinary" rotation of angle φ. This justifies the following definition:

One calls ***spatial rotation*** any restricted Lorentz transformation that leaves a timelike plane strictly invariant. The orthogonal complementary of that plane is called the ***plane of the spatial rotation***. It is spacelike and stable under the spatial rotation.

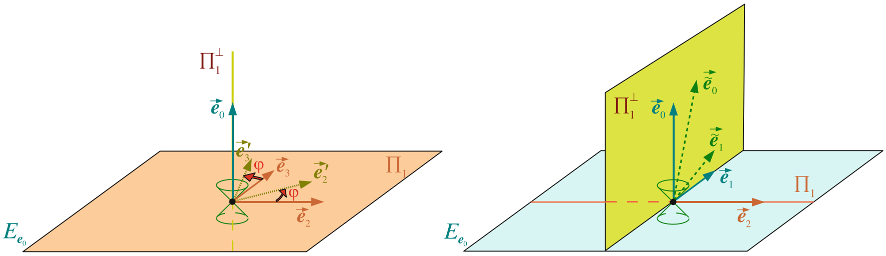

Fig. 6.3 Two representations of a spatial rotation of plane Π_1 and angle φ. *Left:* the dimension along $\vec{e}_1$ has been suppressed, so that $\Pi_1^\perp = \mathrm{Span}(\vec{e}_0, \vec{e}_1) = \Pi_0$ is drawn as a line, whereas it is actually a vector plane. Moreover, Π_1 seems to coincide with the hyperplane E_{e_0}, whereas it is only a two-dimensional subspace of E_{e_0}. *Right:* the dimension along $\vec{e}_3$ has been suppressed, so that the rotation plane $\Pi_1 = \mathrm{Span}(\vec{e}_2, \vec{e}_3)$ is now drawn as a line. For the spatial rotation, all the orthonormal bases of $\Pi_1^\perp$, such as $(\vec{e}_0, \vec{e}_1)$ and $(\vec{e}\,'_0, \vec{e}\,'_1)$ shown here, are equivalent and lead to the matrix (6.39)

It is clear that $\boldsymbol{\Lambda}$ defined by (6.39) is a spatial rotation, of plane Π_1 (cf. Fig. 6.3). Conversely, if $\boldsymbol{\Lambda} \in \mathrm{SO}_o(3,1)$ leaves a timelike plane Π_0 strictly invariant, it is easy to see, by repeating the reasoning of Sect. 6.4.2 with $\psi = 0$ and $\alpha = 0$, that its matrix in a right-handed orthonormal basis having the first two vectors in Π_0 is necessarily of the form (6.39). The parameter $\varphi \in [0, 2\pi[$ appearing in (6.39) is then called the ***spatial rotation angle***. A spatial rotation is thus entirely defined by its plane and its angle. The angle cosine is related to the trace of the transformation by

$$\boxed{\cos\varphi = \frac{1}{2}\mathrm{tr}\boldsymbol{\Lambda} - 1}. \tag{6.40}$$

This formula follows immediately from the matrix (6.39) (let us recall that the trace of an endomorphism is independent of the basis of E chosen for expressing its matrix).

Remark 6.11. In the three-dimensional Euclidean space, a rotation is defined by an angle φ and one direction named the *rotation axis*. This axis is the orthogonal complementary of the plane Π in which the rotation acts. In the four-dimensional case considered here, the notion of rotation axis has no longer any meaning since the orthogonal complementary to Π is not a line but a plane, as illustrated in Fig. 6.3.

The action of a spatial rotation on a vector $\vec{v} = v^\alpha\, \vec{e}_\alpha \in E$ is deduced from the matrix (6.39) via (6.5):

$$\begin{aligned}\boldsymbol{\Lambda}(\vec{v}) = \Lambda^\alpha{}_\beta v^\beta\, \vec{e}_\alpha &= v^0\, \vec{e}_0 + v^1\, \vec{e}_1 + (v^2\cos\varphi - v^3\sin\varphi)\vec{e}_2 \\ &\quad + (v^2\sin\varphi + v^3\cos\varphi)\vec{e}_3.\end{aligned}$$

Setting $\vec{n} := \vec{e}_1$ for denoting the unit vector of the rotation axis in the Euclidean hyperplane E_{e_0} (cf. the above remark), we may rewrite this formula as

$$\boldsymbol{\Lambda}(\vec{\boldsymbol{v}}) = \underbrace{v^0}_{-\vec{\boldsymbol{e}}_0\cdot\vec{\boldsymbol{v}}}\ \vec{\boldsymbol{e}}_0 + \underbrace{v^1}_{\vec{\boldsymbol{n}}\cdot\vec{\boldsymbol{v}}}\ \vec{\boldsymbol{n}} + \cos\varphi(\underbrace{v^2\vec{\boldsymbol{e}}_2 + v^3\vec{\boldsymbol{e}}_3}_{\vec{\boldsymbol{v}}-v^0\vec{\boldsymbol{e}}_0-v^1\vec{\boldsymbol{n}}}) + \sin\varphi(\underbrace{v^2\vec{\boldsymbol{e}}_3 - v^3\vec{\boldsymbol{e}}_2}_{\vec{\boldsymbol{n}}\times_{\boldsymbol{e_0}}\vec{\boldsymbol{v}}}).$$

We thus obtain ***Rodrigues formula***[5]:

$$\boxed{\boldsymbol{\Lambda}(\vec{\boldsymbol{v}}) = \cos\varphi\ \vec{\boldsymbol{v}} + \sin\varphi\ \vec{\boldsymbol{n}} \times_{\boldsymbol{e_0}} \vec{\boldsymbol{v}} + (1-\cos\varphi)\left[(\vec{\boldsymbol{n}}\cdot\vec{\boldsymbol{v}})\vec{\boldsymbol{n}} - (\vec{\boldsymbol{e}}_0\cdot\vec{\boldsymbol{v}})\vec{\boldsymbol{e}}_0\right]}. \tag{6.41}$$

This formula expresses the spatial rotation $\boldsymbol{\Lambda}$ in terms of its angle φ and an orthonormal basis $(\vec{\boldsymbol{e}}_0, \vec{\boldsymbol{n}})$ of the plane Π_0 that is left strictly invariant by $\boldsymbol{\Lambda}$.

6.4.4 Lorentz Boosts

If $\alpha = 0$ and $\varphi = 0$, the matrix (6.37) of $\boldsymbol{\Lambda}$ in the basis $(\vec{\boldsymbol{e}}^{\,*}_\alpha)$ reduces to

$$\boxed{(\Lambda^*)^\alpha{}_\beta = \begin{pmatrix} e^\psi & 0 & 0 & 0 \\ 0 & e^{-\psi} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}, \tag{6.42}$$

whereas the matrix (6.38) in the orthonormal basis $(\vec{\boldsymbol{e}}_\alpha)$ becomes

$$\boxed{\Lambda^\alpha{}_\beta = \begin{pmatrix} \cosh\psi & \sinh\psi & 0 & 0 \\ \sinh\psi & \cosh\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}. \tag{6.43}$$

We notice that $\boldsymbol{\Lambda}$ leaves the plane $\Pi_1 = \mathrm{Span}(\vec{\boldsymbol{e}}_2, \vec{\boldsymbol{e}}_3)$ strictly invariant and that this plane is spacelike, contrary to a spatial rotation, for which the strictly invariant plane is timelike. One introduces then the following definition:

One calls ***Lorentz boost***, or simply ***boost***, any restricted Lorentz transformation that leaves a spacelike plane strictly invariant. The orthogonal complementary of this plane is called the ***plane of the Lorentz boost***. It is timelike and stable under the Lorentz boost.

[5] **Olinde Rodrigues** (1795–1851): French mathematician, disciple of the utopian philosopher Saint-Simon. He derived formula (6.41) in 1840 (without the $\vec{\boldsymbol{e}}_0$ term since he considered only a three-dimensional space) and used it to study the composition of two rotations.

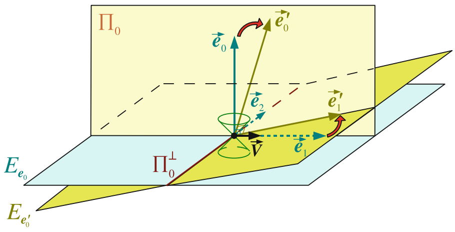

Fig. 6.4 Lorentz boost of plane Π_0, with $\vec{e}'_0 := \boldsymbol{\Lambda}(\vec{e}_0)$ and $\vec{e}'_1 := \boldsymbol{\Lambda}(\vec{e}_1)$

It is clear that the transformation $\boldsymbol{\Lambda}$ defined by (6.42) or (6.43) is a boost of plane Π_0 (cf. Fig. 6.4). Conversely, if $\boldsymbol{\Lambda} \in \mathrm{SO_o}(3,1)$ leaves a spacelike plane Π_1 strictly invariant, one may introduce a right-handed orthonormal basis, $(\vec{e}_\alpha)$, such that $\Pi_1 = \mathrm{Span}(\vec{e}_2, \vec{e}_3)$ and $\Pi_1^\perp = \mathrm{Span}(\vec{e}_0, \vec{e}_1)$. Let us then expand $\boldsymbol{\Lambda}(\vec{e}_0)$ and $\boldsymbol{\Lambda}(\vec{e}_1)$ on this basis:

$$\boldsymbol{\Lambda}(\vec{e}_0) = a^\alpha\, \vec{e}_\alpha \qquad \text{and} \qquad \boldsymbol{\Lambda}(\vec{e}_1) = b^\alpha\, \vec{e}_\alpha.$$

The properties $\boldsymbol{\Lambda}(\vec{e}_0)\cdot\boldsymbol{\Lambda}(\vec{e}_2) = \vec{e}_0\cdot\vec{e}_2 = 0$ and $\boldsymbol{\Lambda}(\vec{e}_2) = \vec{e}_2$ imply $a^2 = 0$. One shows similarly that $a^3 = 0$, $b^2 = 0$ and $b^3 = 0$; hence, $\boldsymbol{\Lambda}(\vec{e}_0) = a^0\,\vec{e}_0 + a^1\,\vec{e}_1$ and $\boldsymbol{\Lambda}(\vec{e}_1) = b^0\,\vec{e}_0 + b^1\,\vec{e}_1$. Since $\boldsymbol{\Lambda}$ preserves the scalar products, we have then

$$\boldsymbol{\Lambda}(\vec{e}_0)\cdot\boldsymbol{\Lambda}(\vec{e}_0) = -1 = -(a^0)^2 + (a^1)^2 \tag{6.44a}$$

$$\boldsymbol{\Lambda}(\vec{e}_0)\cdot\boldsymbol{\Lambda}(\vec{e}_1) = 0 = -a^0b^0 + a^1b^1 \tag{6.44b}$$

$$\boldsymbol{\Lambda}(\vec{e}_1)\cdot\boldsymbol{\Lambda}(\vec{e}_1) = 1 = -(b^0)^2 + (b^1)^2. \tag{6.44c}$$

Moreover, the orthochronous character of $\boldsymbol{\Lambda}$ implies $\vec{e}_0\cdot\boldsymbol{\Lambda}(\vec{e}_0) = -a^0 < 0$ [cf. Eq. (6.15)], i.e. $a^0 > 0$. Equation (6.44a) shows then that $a^0 \geq 1$. One may therefore introduce $\psi \in \mathbb{R}$ such that $a^0 =: \cosh\psi$ and solve (6.44a) by $a^1 = \sinh\psi$. On its side, Eq. (6.44c) implies $|b^1| \geq 1$. The case $b^1 < 0$ is excluded for it would lead to a change of orientation between the bases $(\vec{e}_0, \vec{e}_1)$ and $(\boldsymbol{\Lambda}(\vec{e}_0), \boldsymbol{\Lambda}(\vec{e}_1))$ in $\Pi_1^\perp$. Now, as a restricted Lorentz transformation, $\boldsymbol{\Lambda}$ must preserve the orientation of the basis $(\vec{e}_\alpha)$, which amounts to preserving the orientation of $(\vec{e}_0, \vec{e}_1)$, since $\boldsymbol{\Lambda}(\vec{e}_2) = \vec{e}_2$ and $\boldsymbol{\Lambda}(\vec{e}_3) = \vec{e}_3$. We may thus set $b^1 =: \cosh\psi'$ with $\psi' \in \mathbb{R}$ and solve (6.44c) by $b^0 = \sinh\psi'$. Equation (6.44b) becomes then

$$-\cosh\psi\,\sinh\psi' + \sinh\psi\,\cosh\psi' = 0,$$

i.e. $\sinh(\psi - \psi') = 0$. We deduce immediately $\psi' = \psi$, which shows that the matrix of $\boldsymbol{\Lambda}$ is indeed of the form (6.43).

From the matrix (6.43), one can relate the parameter ψ to the trace of $\boldsymbol{\Lambda}$:

$$\boxed{\cosh\psi = \frac{1}{2}\mathrm{tr}\boldsymbol{\Lambda} - 1}. \tag{6.45}$$

Since the trace of an endomorphism is independent of the basis chosen to express its matrix, we conclude that ψ is independent of the choice of the basis $(\vec{e}_\alpha)$ adapted to the plane of $\boldsymbol{\Lambda}$ (in the sense that $\Pi_0 = \mathrm{Span}(\vec{e}_0, \vec{e}_1)$). The parameter $\psi \in \mathbb{R}$ is thus intrinsic to $\boldsymbol{\Lambda}$ and is called the ***rapidity*** of the boost $\boldsymbol{\Lambda}$. Moreover, the above demonstration has shown that

A boost is entirely determined by its plane and its rapidity.

Remark 6.12. Formula (6.45) is similar to (6.40), with the cosine replaced by the hyperbolic cosine. Moreover, (6.45) can be obtained from (6.42) as well as (6.43), thereby illustrating the independence of the trace from the basis used to express the matrix of $\boldsymbol{\Lambda}$.

An alternative parametrization of Lorentz boosts makes use of the quantity

$$\boxed{V := c \tanh\psi} \tag{6.46}$$

instead of ψ. Thanks to the c factor, V has the dimension of a velocity. From the properties of a hyperbolic tangent, $|V| < c$. We therefore shall call V the ***velocity parameter*** of the boost $\boldsymbol{\Lambda}$. The identity $1 - \tanh^2\psi = \cosh^{-2}\psi$ yields

$$\boxed{\cosh\psi = \left(1 - \frac{V^2}{c^2}\right)^{-1/2} =: \Gamma}. \tag{6.47}$$

The parameter Γ is called the ***Lorentz factor*** of the boost $\boldsymbol{\Lambda}$. Given the definitions of V and Γ, we can rewrite the matrix (6.43) as

$$\boxed{\Lambda^\alpha{}_\beta = \begin{pmatrix} \Gamma & \Gamma V/c & 0 & 0 \\ \Gamma V/c & \Gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}. \tag{6.48}$$

We shall discuss about the kinematical interpretation of Γ and V as the Lorentz factor and the relative velocity between two observers in Sect. 6.6.1.

6.4.5 Null Rotations

The last case where only one of the parameters (ψ, α, φ) defined in Sect. 6.4.2 is nonvanishing is the case $\psi = 0$ and $\varphi = 0$. The matrix (6.37) of $\boldsymbol{\Lambda}$ in the basis $(\vec{e}^{\,*}_\alpha)$ reduces then to

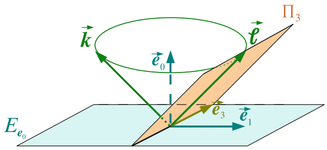

Fig. 6.5 Null plane $\Pi_3 = \mathrm{Span}(\vec{\ell}, \vec{e}_3)$. The dimension along $\vec{e}_2$ has been suppressed, so that the plane orthogonal to Π_3, $\Pi_3^{\perp} = \mathrm{Span}(\vec{\ell}, \vec{e}_2)$, is reduced to the line along $\vec{\ell}$

$$\boxed{(\Lambda^*)^{\alpha}{}_{\beta} = \begin{pmatrix} 1 & 4\alpha^2 & 2\alpha & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 4\alpha & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}, \tag{6.49}$$

whereas the matrix (6.38) in the orthonormal basis $(\vec{e}_\alpha)$ becomes

$$\boxed{\Lambda^{\alpha}{}_{\beta} = \begin{pmatrix} 1+2\alpha^2 & -2\alpha^2 & 2\alpha & 0 \\ 2\alpha^2 & 1-2\alpha^2 & 2\alpha & 0 \\ 2\alpha & -2\alpha & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}. \tag{6.50}$$

We observe on (6.49) that $\boldsymbol{\Lambda}$ leaves the vectors $\vec{e}_0^{\,*} = \vec{\ell}$ and $\vec{e}_3^{\,*} = \vec{e}_3$ invariant. We deduce then that the vector plane

$$\Pi_3 := \mathrm{Span}(\vec{\ell}, \vec{e}_3) \tag{6.51}$$

is strictly invariant by $\boldsymbol{\Lambda}$ (cf. Fig. 6.5). Contrary to the planes Π_0 and Π_1 considered above, Π_3 is neither timelike nor spacelike. Indeed, the metric induced by $\boldsymbol{g}$ in Π_3 is degenerate[6]: the nonzero vector $\vec{\ell}$ belongs to Π_3 and is orthogonal to all vectors in Π_3. Indeed $\vec{\ell}$ is by construction orthogonal to $\vec{e}_3$ and, being null, is also orthogonal to itself. On the contrary, the metric induced in a timelike or spacelike plane is never degenerate. One says that Π_3 is a ***null plane***.

Remark 6.13. The degeneracy of $\boldsymbol{g}$ in Π_3 is usually expressed by the following signature statement: sign $\boldsymbol{g}|_{\Pi_3} = (0, +)$.

All vectors of Π_3 are either collinear to $\vec{\ell}$ (and thus null) or spacelike.

Proof. If $\vec{v} \in \Pi_3$, then $\vec{v} = a\,\vec{\ell} + b\,\vec{e}_3$ and the properties $\vec{\ell}\cdot\vec{\ell} = 0$, $\vec{\ell}\cdot\vec{e}_3 = 0$ and $\vec{e}_3 \cdot \vec{e}_3 = 1$ lead to $\vec{v}\cdot\vec{v} = b^2 \geq 0$. The equality holds iff $b = 0$, i.e. if $\vec{v}$ is collinear to $\vec{\ell}$, otherwise, $\vec{v}\cdot\vec{v} > 0$ and $\vec{v}$ is spacelike. ☐

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[6]Cf. the definition of nondegeneracy given in Sect. 1.3.1.

All null planes share this property, namely, they contain only one null direction, the others being spacelike.

Proof. Let us suppose that a plane Π contains two distinct null directions, $\vec{\boldsymbol{\ell}}$ and $\vec{\boldsymbol{k}}$ say. Then $\Pi = \mathrm{Span}(\vec{\boldsymbol{\ell}}, \vec{\boldsymbol{k}})$, and we are back to the situation considered in Sect. 6.4.2 for Π_0: the plane Π is necessarily timelike [cf. (6.31)]. A null plane can therefore have only one null direction. Now, by virtue of the property established in Sect. 2.4.2, a timelike vector cannot be orthogonal to a null vector. All the non-null vectors of Π are therefore necessarily spacelike. □

Since $\mathrm{Span}(\vec{\boldsymbol{\ell}})$ is the only null direction contained in Π_3, the intersection of Π_3 with the null cone of $\boldsymbol{g}$, $\mathscr{I}$, is reduced to that direction. Consequently, a null plane is tangent to the null cone, as shown in Fig. 6.5. On the opposite, a timelike plane intersects $\mathscr{I}$ in two distinct null directions (cf. Fig. 6.2) and a spacelike plane intersects $\mathscr{I}$ only in $\{0\}$.

A property that characterizes a null plane is that one cannot decompose the vector space E as a direct sum of Π_3 and $\Pi_3^{\perp}$, as in (6.32). Indeed, in the present case,

$$\Pi_3^{\perp} = \mathrm{Span}(\vec{\boldsymbol{\ell}}, \vec{\boldsymbol{e}}_2) \tag{6.52}$$

and $\Pi_3 \cap \Pi_3^{\perp} = \mathrm{Span}(\vec{\boldsymbol{\ell}}) \neq \{0\}$. Moreover, $\Pi_3 \cup \Pi_3^{\perp}$ is not generating E but only the hyperplane $\mathrm{Span}(\vec{\boldsymbol{\ell}}, \vec{\boldsymbol{e}}_2, \vec{\boldsymbol{e}}_3)$.

Remark 6.14. $\Pi_3^{\perp}$ is itself a null plane, $\vec{\boldsymbol{e}}_2$ playing the role of $\vec{\boldsymbol{e}}_3$.

In view of what precedes:

One calls ***null rotation*** any restricted Lorentz transformation that leaves a null plane strictly invariant. This plane is called the ***plane of the null rotation***.

Remark 6.15. The reader may notice a certain dissymmetry in our definitions of the planes of the various transformations: the plane of a spatial rotation or a boost is the plane where the transformation acts. It is therefore not strictly invariant, contrary to the plane of a null rotation.

The map $\boldsymbol{\Lambda}$ whose matrix is (6.49) is clearly a null rotation of plane Π_3. Conversely, if $\boldsymbol{\Lambda} \in \mathrm{SO}_o(3,1)$ leaves a null plane Π_3 strictly invariant, let us denote by $\vec{\boldsymbol{\ell}}$ a generator of the null direction in Π_3 and by $\vec{\boldsymbol{e}}_3$ a (spacelike) unit vector in Π_3. One can complete $\vec{\boldsymbol{e}}_3$ to form an orthonormal basis $(\vec{\boldsymbol{e}}_\alpha)$ such that $\vec{\boldsymbol{\ell}} = \vec{\boldsymbol{e}}_0 + \vec{\boldsymbol{e}}_1$. Setting $\vec{\boldsymbol{k}} := \vec{\boldsymbol{e}}_0 - \vec{\boldsymbol{e}}_1$, we may then follow the reasoning of Sect. 6.4.2 with the supplementary conditions $\boldsymbol{\Lambda}(\vec{\boldsymbol{\ell}}) = \vec{\boldsymbol{\ell}}$ and $\boldsymbol{\Lambda}(\vec{\boldsymbol{e}}_3) = \vec{\boldsymbol{e}}_3$, which implies $\psi = 0$ and $\varphi = 0$ [cf. Eqs. (6.27) and (6.36)]. We arrive then necessarily to the matrix (6.49). We conclude that a null rotation $\boldsymbol{\Lambda}$ is entirely determined by its plane and a parameter $\alpha \in \mathbb{R}^+$, the latter being related to $\boldsymbol{\Lambda}$'s matrix by (6.49) or (6.50), depending on the considered basis.

Remark 6.16. Null rotations are sometimes called *lightlike rotations* (Sexl and Urbantke 2001), *singular Lorentz transformations* (Synge 1956; Parizet 2008) or *parabolic Lorentz transformations*. A web page dedicated to null rotations is [W20].

6.4.6 Four-Screws

Given a timelike vector plane $\Pi \subset E$, one calls ***four-screw***, or ***4-screw*** for short, ***of plane*** Π any restricted Lorentz transformation that is the composition of a boost of plane Π and a spatial rotation of plane $\Pi^{\perp}$. A 4-screw corresponds to the case $\alpha = 0$ is the decomposition presented in Sect. 6.4.2. Its matrix in the basis $(\vec{e}^{\,*}_{\alpha})$ is

$$\boxed{(\Lambda^*)^{\alpha}{}_{\beta} = \begin{pmatrix} e^{\psi} & 0 & 0 & 0 \\ 0 & e^{-\psi} & 0 & 0 \\ 0 & 0 & \cos\varphi & -\sin\varphi \\ 0 & 0 & \sin\varphi & \cos\varphi \end{pmatrix}.} \tag{6.53}$$

whereas its matrix in the orthonormal basis $(\vec{e}_{\alpha})$ is

$$\boxed{\Lambda^{\alpha}{}_{\beta} = \begin{pmatrix} \cosh\psi & \sinh\psi & 0 & 0 \\ \sinh\psi & \cosh\psi & 0 & 0 \\ 0 & 0 & \cos\varphi & -\sin\varphi \\ 0 & 0 & \sin\varphi & \cos\varphi \end{pmatrix}.} \tag{6.54}$$

It is then clear that

$$\boldsymbol{\Lambda} = \boldsymbol{S} \circ \boldsymbol{R} = \boldsymbol{R} \circ \boldsymbol{S}, \tag{6.55}$$

where $\boldsymbol{S}$ is the boost of plane $\Pi_0 = \mathrm{Span}(\vec{\boldsymbol{\ell}}, \vec{\boldsymbol{k}}) = \mathrm{Span}(\vec{e}_0, \vec{e}_1)$ and rapidity ψ and $\boldsymbol{R}$ is the spatial rotation of plane $\Pi_0^{\perp} = \Pi_1 = \mathrm{Span}(\vec{e}_2, \vec{e}_3)$ and angle φ. Let us stress that the decomposition (6.55) is commutative. This follows directly from the bloc structure of matrices (6.53) and (6.54).

Remark 6.17. The 4-screws are sometimes called *loxodromic Lorentz transformations* (Sexl and Urbantke 2001). Besides, it is clear that boosts and spatial rotations are special cases of 4-screws (corresponding, respectively, to $\varphi = 0$ and $\psi = 0$).

One has the following characterization:

A restricted Lorentz transformation is a 4-screw iff it leaves invariant two distinct null directions.

Proof. It is clear on (6.53) that a 4-screw has such a property, $\vec{\boldsymbol{\ell}}$ and $\vec{\boldsymbol{k}}$ being two null eigenvectors of $\boldsymbol{\Lambda}$. Conversely, if $\boldsymbol{\Lambda} \in \mathrm{SO_o}(3,1)$ leaves invariant two distinct null directions, one may choose a vector $\vec{\boldsymbol{\ell}}$ along the first one, a vector $\vec{\boldsymbol{k}}$ along the second one and impose $\vec{\boldsymbol{\ell}} \cdot \vec{\boldsymbol{k}} = -2$. We may then repeat the argument of Sect. 6.4.2, with the additional property that $\vec{\boldsymbol{k}}$ is an eigenvector of $\boldsymbol{\Lambda}$. From (6.33), this implies $\alpha = 0$, showing that $\boldsymbol{\Lambda}$ is a 4-screw. □

6.4.7 Eigenvectors of a Restricted Lorentz Transformation

An advantage of the general decomposition (6.37) of a restricted Lorentz transformation $\boldsymbol{\Lambda}$ is to make possible an easy computation of its eigenvalues and eigenvectors. Indeed, the characteristic polynomial $P(\lambda) := \det(\Lambda^* - \lambda \mathbb{I}_4)$ of the matrix (6.37) is easily computable:

$$P(\lambda) = (\lambda - \mathrm{e}^{\psi})(\lambda - \mathrm{e}^{-\psi})(\lambda^2 - 2\lambda\cos\varphi + 1).$$

If $\varphi \neq 0$ and $\varphi \neq \pi$, the polynomial $\lambda^2 - 2\lambda\cos\varphi + 1$ has no real root. In this case, $\boldsymbol{\Lambda}$ has only two real eigenvalues: $\lambda_1 = \mathrm{e}^{\psi}$ and $\lambda_2 = \mathrm{e}^{-\psi}$. If $\varphi = 0$ (resp. π), the eigenvalue $\lambda_3 = 1$ (resp. $\lambda_3 = -1$) must be added, with a multiplicity 2. The corresponding eigenvectors are

$$\lambda_1 = \mathrm{e}^{\psi}: \quad \vec{\boldsymbol{v}}_1 = \vec{\boldsymbol{\ell}} \tag{6.56a}$$

$$\begin{aligned}\lambda_2 = \mathrm{e}^{-\psi}: \quad \vec{\boldsymbol{v}}_2 = \; & \alpha^2 \mathrm{e}^{-\psi}\, \vec{\boldsymbol{\ell}} + \tfrac{1}{2}(\cosh\psi - \cos\varphi)\, \vec{\boldsymbol{k}} \\ & + \alpha(\mathrm{e}^{-\psi} - \cos\varphi)\, \vec{\boldsymbol{e}}_2 + \alpha\sin\varphi\, \vec{\boldsymbol{e}}_3\end{aligned} \tag{6.56b}$$

$$\lambda_3 = 1 \text{ (case } \varphi = 0) \;:\; \vec{\boldsymbol{v}}_3 = \vec{\boldsymbol{e}}_3 \;\text{ and }\; \vec{\boldsymbol{v}}'_3 = -2\alpha\, \vec{\boldsymbol{\ell}} + (\mathrm{e}^{\psi} - 1)\, \vec{\boldsymbol{e}}_2 \tag{6.56c}$$

$$\lambda_3 = -1 \text{ (case } \varphi = \pi) \;:\; \vec{\boldsymbol{v}}_3 = \vec{\boldsymbol{e}}_3 \;\text{ and }\; \vec{\boldsymbol{v}}'_3 = 2\alpha\, \vec{\boldsymbol{\ell}} + (\mathrm{e}^{\psi} + 1)\, \vec{\boldsymbol{e}}_2. \tag{6.56d}$$

Proof. The eigenvector $\vec{\boldsymbol{v}}_1$ follows from (6.27). Using (6.37), it is easy to see that $\boldsymbol{\Lambda}(\vec{\boldsymbol{v}}_2) = \mathrm{e}^{-\psi}\, \vec{\boldsymbol{v}}_2$, $\boldsymbol{\Lambda}(\vec{\boldsymbol{v}}'_3) = \vec{\boldsymbol{v}}'_3$ for $\varphi = 0$ and $\boldsymbol{\Lambda}(\vec{\boldsymbol{v}}'_3) = -\vec{\boldsymbol{v}}'_3$ for $\varphi = \pi$. □

The eigenvectors $\vec{\boldsymbol{v}}_1$ and $\vec{\boldsymbol{v}}_2$ are always null. This is obvious for $\vec{\boldsymbol{v}}_1$, and the direct computation of $\vec{\boldsymbol{v}}_2 \cdot \vec{\boldsymbol{v}}_2$ shows it for $\vec{\boldsymbol{v}}_2$. Note that if $\boldsymbol{\Lambda}$ is a 4-screw ($\alpha = 0$), $\vec{\boldsymbol{v}}_2 \propto \vec{\boldsymbol{k}}$. If $\boldsymbol{\Lambda}$ is a null rotation ($\psi = \varphi = 0$), then $\lambda_1 = \lambda_2 = 1$ and $\vec{\boldsymbol{v}}_2 \propto \vec{\boldsymbol{\ell}}$. In the case $\varphi = 0$ or π, $\vec{\boldsymbol{v}}_3$ is always spacelike. Regarding $\vec{\boldsymbol{v}}'_3$, we have $\vec{\boldsymbol{v}}'_3 \cdot \vec{\boldsymbol{v}}'_3 = (\mathrm{e}^{\psi} \pm 1)^2$ with $-$ for $\varphi = 0$ and $+$ for $\varphi = \pi$. We conclude that $\vec{\boldsymbol{v}}'_3$ is spacelike, except if $\varphi = 0$ and $\psi = 0$ ($\boldsymbol{\Lambda}$ is then a null rotation), in which case $\vec{\boldsymbol{v}}'_3$ is null and collinear to $\vec{\boldsymbol{\ell}}$.

6.4.8 Summary

We have seen above that a restricted Lorentz transformation $\boldsymbol{\Lambda}$ admits one or two invariant null directions: those generated by the null eigenvectors[7] $\vec{\boldsymbol{v}}_1$ and $\vec{\boldsymbol{v}}_2$. These two null directions coincide iff $\vec{\boldsymbol{v}}_2$ is collinear to $\vec{\boldsymbol{v}}_1 = \vec{\boldsymbol{\ell}}$. Now, from (6.56b),

$$\vec{\boldsymbol{v}}_2 \propto \vec{\boldsymbol{\ell}} \iff \begin{cases} \cosh\psi - \cos\varphi = 0 \\ \alpha(\mathrm{e}^{-\psi} - \cos\varphi) = 0 \\ \alpha \sin\varphi = 0 \end{cases} \iff \begin{cases} \psi = 0 \\ \varphi = 0. \end{cases}$$

Thus $\boldsymbol{\Lambda}$ admits a unique invariant null direction iff $\boldsymbol{\Lambda}$ is a null rotation. We have seen in Sect. 6.4.6 that if $\boldsymbol{\Lambda}$ admits two distinct invariant null directions, it is necessarily a 4-screw.

Collecting the previous results, we may state:

Any element $\boldsymbol{\Lambda}$ of the restricted Lorentz group $\mathrm{SO}_\mathrm{o}(3,1)$ leaves invariant at least one null direction; moreover:

- If such a direction is unique, $\boldsymbol{\Lambda}$ is a null rotation (with $\alpha \neq 0$).
- If there exist exactly two invariant null directions, $\boldsymbol{\Lambda}$ is a 4-screw of nonvanishing rapidity or nonvanishing rotation angle.
- If there exist three or more invariant null directions, $\boldsymbol{\Lambda}$ is the identity.

In particular,

There does not exist any other restricted Lorentz transformation than null rotations and 4-screws (the identity being considered as the case $\alpha = 0$ of a null rotation or the case $(\psi, \varphi) = (0, 0)$ of a 4-screw).

In other words, given $\boldsymbol{\Lambda} \in \mathrm{SO}_\mathrm{o}(3,1)$, there exists necessarily a basis $(\vec{\boldsymbol{e}}^{\,*}_\alpha) = (\vec{\boldsymbol{\ell}}, \vec{\boldsymbol{k}}, \vec{\boldsymbol{e}}_2, \vec{\boldsymbol{e}}_3)$ of E such that $\vec{\boldsymbol{\ell}}$ and $\vec{\boldsymbol{k}}$ are timelike, $\vec{\boldsymbol{e}}_2$ and $\vec{\boldsymbol{e}}_3$ are unit and spacelike, the planes $\Pi_0 = \mathrm{Span}(\vec{\boldsymbol{\ell}}, \vec{\boldsymbol{k}})$ and $\Pi_1 = \mathrm{Span}(\vec{\boldsymbol{e}}_2, \vec{\boldsymbol{e}}_3)$ are orthogonal and the matrix of $\boldsymbol{\Lambda}$ in this basis is either (6.49) (null rotation) or (6.53) (4-screw).

[7] If $\varphi = 0$ and $\psi = 0$, $\vec{\boldsymbol{v}}'_3$ is also a null eigenvector, but there is no need to distinguish this case since $\vec{\boldsymbol{v}}'_3$ is then collinear to $\vec{\boldsymbol{v}}_1$.

Historical note: *Lorentz boosts have actually been discovered quite early, in 1887, by Woldemar Voigt*[8] *(1887), as changes of coordinates*[9] $(ct, x, y, z) \mapsto (ct', x', y,', z')$ *that leave invariant the wave equation* $-c^{-2}\partial^2\Phi/\partial t^2 + \partial^2\Phi/\partial x^2 + \partial^2\Phi/\partial y^2 + \partial^2\Phi/\partial z^2 = 0$. *Lorentz boosts have been subsequently rediscovered by Joseph Larmor*[10] *in 1900 (Larmor 1900) and by Hendrik A. Lorentz (cf. p. 108) in 1904 (Lorentz 1904), as changes of coordinates that leave invariant the equations of Maxwell electrodynamics. The name* Lorentz transformations *was given to them in 1905 by Henri Poincaré (cf. p. 26) (1905b). The fact that boosts sharing the same plane form a group*[11] *has been established independently by Einstein (1905b) and Poincaré (1906) in 1905. This is Poincaré (1906) who added spatial rotations to form the Lorentz group, in the sense defined here (more precisely, Poincaré considered only the restricted Lorentz group* $\mathrm{SO}_o(3,1)$*). The appellation* Lorentz group *is moreover due to him (Poincaré 1906).*

6.5 Polar Decomposition

In Sect. 6.4, we have performed a decomposition of restricted Lorentz transformations from a null eigenvector. Here we present another useful decomposition, based on a unit timelike vector.

6.5.1 Statement and Demonstration

Given a unit timelike vector $\vec{e}_0 \in E$, any restricted Lorentz transformation $\boldsymbol{\Lambda} \in \mathrm{SO}_o(3,1)$ can be written in a unique way as the product

$$\boxed{\boldsymbol{\Lambda} = \boldsymbol{S} \circ \boldsymbol{R}}, \tag{6.57}$$

[8] **Woldemar Voigt** (1850–1919): German physicist who studied crystals, thermodynamics and electro-optics. He notably discovered some birefringence in gas induced by a magnetic field (the so-called *Voigt effect*).

[9] In our language, we should say *change of affine coordinates on $\mathscr{E}$ that are associated with an orthonormal basis of E*. In addition, the transformation found by Voigt was actually $\Gamma^{-1}\boldsymbol{\Lambda}$, where $\boldsymbol{\Lambda}$ is a boost and Γ its Lorentz factor.

[10] **Joseph Larmor** (1857–1942): British physicist from Northern Ireland, who worked in electromagnetism and thermodynamics; he authored a treatise on "Aether and Matter" (Larmor 1900) and left his name to *Larmor precession* (precession of a body carrying a magnetic moment in an external magnetic field).

[11] This is trivial from the definition of boosts given in Sect. 6.4.4, but this was not for Einstein and Poincaré, who were defining boosts from expression (6.48).

where $\boldsymbol{S}$ is a boost whose plane contains $\vec{\boldsymbol{e}}_0$ and $\boldsymbol{R}$ is a spatial rotation whose plane is orthogonal to $\vec{\boldsymbol{e}}_0$. The writing (6.57) is called ***polar decomposition of $\boldsymbol{\Lambda}$ relative to $\vec{\boldsymbol{e}}_0$***.

Remark 6.18. The right-hand side of (6.57), although looking similar to (6.55), is not necessarily a 4-screw, since a priori the planes of $\boldsymbol{S}$ and $\boldsymbol{R}$ are not orthogonal. Indeed, we shall see in Sect. 6.5.2 that if $\boldsymbol{\Lambda}$ is null rotation, the planes of $\boldsymbol{S}$ and $\boldsymbol{R}$ are not orthogonal.

Proof. Let us set $\vec{\boldsymbol{e}}'_0 := \boldsymbol{\Lambda}(\vec{\boldsymbol{e}}_0)$ and distinguish two cases: $\vec{\boldsymbol{e}}'_0 = \vec{\boldsymbol{e}}_0$ and $\vec{\boldsymbol{e}}'_0 \neq \vec{\boldsymbol{e}}_0$. In the first case, $\boldsymbol{\Lambda}$ is acting only in the hyperplane normal to $\vec{\boldsymbol{e}}_0$, $E_{\boldsymbol{e}_0}$. Indeed, from the very definition of a Lorentz transformation [Eq. (6.2)],

$$\forall \vec{\boldsymbol{v}} \in E_{\boldsymbol{e}_0}, \quad \vec{\boldsymbol{e}}_0 \cdot \boldsymbol{\Lambda}(\vec{\boldsymbol{v}}) = \boldsymbol{\Lambda}(\vec{\boldsymbol{e}}_0) \cdot \boldsymbol{\Lambda}(\vec{\boldsymbol{v}}) = \vec{\boldsymbol{e}}_0 \cdot \vec{\boldsymbol{v}} = 0.$$

This shows that $\boldsymbol{\Lambda}(\vec{\boldsymbol{v}}) \in E_{\boldsymbol{e}_0}$ for any $\vec{\boldsymbol{v}} \in E_{\boldsymbol{e}_0}$. The hyperplane $E_{\boldsymbol{e}_0}$ is thus invariant under $\boldsymbol{\Lambda}$. Since $(E_{\boldsymbol{e}_0}, \boldsymbol{g})$ is a Euclidean three-dimensional space, we deduce that $\boldsymbol{\Lambda}$ is necessarily a spatial rotation, as defined in Sect. 6.4.3. We have thus established (6.57) with $\boldsymbol{S} = \mathrm{Id}$ and $\boldsymbol{R} = \boldsymbol{\Lambda}$.

Let us now consider the case where $\vec{\boldsymbol{e}}'_0 \neq \vec{\boldsymbol{e}}_0$. The vector $\vec{\boldsymbol{e}}'_0$ is then necessarily not collinear to $\vec{\boldsymbol{e}}_0$. Indeed, since $\vec{\boldsymbol{e}}_0$ and $\vec{\boldsymbol{e}}'_0$ are two unit vectors, the only possibility of collinearity compatible with $\vec{\boldsymbol{e}}'_0 \neq \vec{\boldsymbol{e}}_0$ would be $\vec{\boldsymbol{e}}'_0 = -\vec{\boldsymbol{e}}_0$. But then $\vec{\boldsymbol{e}}_0$ and $\vec{\boldsymbol{e}}'_0$ would not have the same time orientation, which is impossible since $\boldsymbol{\Lambda}$ is orthochronous. Therefore, the subspace $\Pi := \mathrm{Span}(\vec{\boldsymbol{e}}_0, \vec{\boldsymbol{e}}'_0)$ is two-dimensional (vector plane). Moreover, Π is a timelike plane, as defined in Sect. 6.4.2, for it contains timelike directions (those of $\vec{\boldsymbol{e}}_0$ and $\vec{\boldsymbol{e}}'_0$). Let then $\boldsymbol{S}$ be the boost of plane Π and Lorentz factor $\Gamma := -\vec{\boldsymbol{e}}_0 \cdot \vec{\boldsymbol{e}}'_0$. These two conditions define entirely $\boldsymbol{S}$ (as shown in Sect. 6.4.4, a boost is fully characterized by its plane and its rapidity $\psi = \operatorname{arcosh} \Gamma$). By construction, $\boldsymbol{S}$ maps $\vec{\boldsymbol{e}}_0$ to $\vec{\boldsymbol{e}}'_0$:

$$\boldsymbol{S}(\vec{\boldsymbol{e}}_0) = \vec{\boldsymbol{e}}'_0.$$

Let us define

$$\boldsymbol{R} := \boldsymbol{S}^{-1} \circ \boldsymbol{\Lambda}. \tag{6.58}$$

As the composition of two such transformations, $\boldsymbol{R}$ is a restricted Lorentz transformation. Moreover it satisfies

$$\boldsymbol{R}(\vec{\boldsymbol{e}}_0) = \boldsymbol{S}^{-1}(\boldsymbol{\Lambda}(\vec{\boldsymbol{e}}_0)) = \boldsymbol{S}^{-1}(\vec{\boldsymbol{e}}'_0) = \vec{\boldsymbol{e}}_0.$$

The above discussion of the case $\vec{\boldsymbol{\Lambda}}(\vec{\boldsymbol{e}}_0) = \vec{\boldsymbol{e}}_0$, once applied to $\boldsymbol{R}$, shows that $\boldsymbol{R}$ is a spatial rotation whose plane is orthogonal to $\vec{\boldsymbol{e}}_0$. We deduce then from (6.58) that $\boldsymbol{\Lambda}$ is indeed expressible as (6.57).

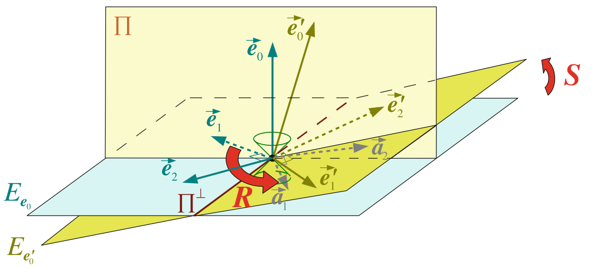

Fig. 6.6 Polar decomposition of a restricted Lorentz transformation $\boldsymbol{\Lambda}$; the transformation is fully defined by its action on the orthonormal basis $(\vec{e}_\alpha)$: $\vec{e}^{\,\prime}_\alpha = \boldsymbol{\Lambda}(\vec{e}_\alpha)$. $\boldsymbol{\Lambda}$ is decomposed into a rotation $\boldsymbol{R}$ within the hyperplane E_{e_0}, leading to $\vec{a}_i := \boldsymbol{R}(\vec{e}_i)$, followed by a boost $\boldsymbol{S}$ of plane $\Pi = \mathrm{Span}(\vec{e}_0, \vec{e}^{\,\prime}_0)$

To show the uniqueness of the decomposition (6.57), let us suppose that $\boldsymbol{\Lambda} = \boldsymbol{S}' \circ \boldsymbol{R}'$ with $\boldsymbol{S}'$ and $\boldsymbol{R}'$ having the same properties with respect to $\vec{e}_0$ as $\boldsymbol{S}$ and $\boldsymbol{R}$. We have then $\boldsymbol{S}' = \boldsymbol{S} \circ \boldsymbol{R} \circ \boldsymbol{R}'^{-1}$, so that $\boldsymbol{S}'(\vec{e}_0) = \boldsymbol{S} \circ \boldsymbol{R} \circ \boldsymbol{R}'^{-1}(\vec{e}_0)$. Now $\vec{e}_0$ is invariant under the rotations $\boldsymbol{R}'^{-1}$ and $\boldsymbol{R}$; hence, $\boldsymbol{S}'(\vec{e}_0) = \boldsymbol{S}(\vec{e}_0)$. We conclude that the boosts $\boldsymbol{S}$ and $\boldsymbol{S}'$ have the same plane and same Lorentz factor $\Gamma = -\vec{e}_0 \cdot \boldsymbol{S}(\vec{e}_0) = -\vec{e}_0 \cdot \boldsymbol{S}'(\vec{e}_0)$. They thus coincide (cf. Sect. 6.4.4): $\boldsymbol{S}' = \boldsymbol{S}$. It follows immediately that $\boldsymbol{R}' = \boldsymbol{R}$. □

Remark 6.19. The polar decomposition (6.57) is a particular case of a theorem of linear algebra called the ***polar decomposition theorem***. This theorem stipulates that any invertible real matrix Λ is expressible in a unique way as the product of a positive definite symmetric matrix[12] S and an orthogonal matrix[13] R: $\Lambda = SR$ (cf., for instance, Mneimné and Testard (1986)). This theorem can be applied to the present case, because the matrix of a spatial rotation is orthogonal [cf. (6.39)] and that of a boost is symmetric (we have seen it with (6.43), which is valid in an adapted basis and shall see it in a general basis in Sect. 6.6.2) and positive definite: from the results of Sect. 6.4.7, the eigenvalues of $\boldsymbol{S}$ are e^{ψ}, $\mathrm{e}^{-\psi}$ and 1 (ψ being $\boldsymbol{S}$'s rapidity) and are all strictly positive.

The polar decomposition is illustrated in Fig. 6.6, where $\boldsymbol{\Lambda}$ is represented by its action on an orthonormal basis $(\vec{e}_\alpha)$, by drawing the two bases $(\vec{e}_\alpha)$ and $(\vec{e}^{\,\prime}_\alpha) := \boldsymbol{\Lambda}(\vec{e}_\alpha)$.

[12] That is to say a symmetric matrix whose eigenvalues are all strictly positive.

[13] Here *orthogonal* is used in the usual sense, i.e. to qualify a matrix whose transpose is its inverse: ${}^{\mathrm{t}}RR = \mathbb{I}_4$, and not to mark any orthogonality with respect to the metric tensor $\boldsymbol{g}$.

6.5.2 Explicit Forms

We have seen in Sect. 6.4.8 that a restricted Lorentz transformation $\boldsymbol{\Lambda}$ is either a 4-screw or a null rotation. In the case of 4-screw, the polar decomposition with respect to a vector $\vec{e}_0$ belonging to the plane of $\boldsymbol{\Lambda}$ is immediate: it is given by (6.55). Moreover, in this case, the planes of $\boldsymbol{S}$ and $\boldsymbol{R}$ are orthogonal and the product $\boldsymbol{S} \circ \boldsymbol{R}$ commutes.

In the case where $\boldsymbol{\Lambda}$ is a null rotation of parameter α (cf. Sect. 6.4.5), one reads on (6.50) that

$$\vec{e}'_0 = \boldsymbol{\Lambda}(\vec{e}_0) = (1 + 2\alpha^2)\,\vec{e}_0 + 2\alpha^2\,\vec{e}_1 + 2\alpha\,\vec{e}_2.$$

The boost $\boldsymbol{S}$ of the polar decomposition is the boost of plane $\Pi := \mathrm{Span}(\vec{e}_0, \vec{e}'_0)$ and Lorentz factor $\Gamma = -\vec{e}_0 \cdot \vec{e}'_0$. In the present case,

$$\Gamma = 1 + 2\alpha^2. \tag{6.59}$$

It is easily checked that an orthonormal basis of Π is $(\vec{e}_0, \vec{\varepsilon}_1)$, where $\vec{\varepsilon}_1$ is the unit spacelike vector defined by

$$\vec{\varepsilon}_1 = \frac{1}{\sqrt{1+\alpha^2}}\,(\alpha\,\vec{e}_1 + \vec{e}_2)\,. \tag{6.60}$$

In addition, one may choose an orthonormal basis $(\vec{\varepsilon}_2, \vec{\varepsilon}_3)$ of $\Pi^{\perp}$ as follows:

$$\vec{\varepsilon}_2 := -\frac{1}{\sqrt{1+\alpha^2}}\,(\vec{e}_1 - \alpha\,\vec{e}_2) \qquad \text{and} \qquad \vec{\varepsilon}_3 := \vec{e}_3.$$

$(\vec{e}_0, \vec{\varepsilon}_1, \vec{\varepsilon}_2, \vec{\varepsilon}_3)$ is then an orthonormal basis of $(E, \boldsymbol{g})$. The explicit form of $\boldsymbol{R}$ is obtained from (6.58), which, in terms of matrices in the basis $(\vec{e}_\alpha)$, gives

$$R = S^{-1}\,\Lambda.$$

The matrix S is obtained from (i) the matrix of $\boldsymbol{S}$ in the basis $(\vec{e}_0, \vec{\varepsilon}_1, \vec{\varepsilon}_2, \vec{\varepsilon}_3)$, which is of the type (6.48), and (ii) the change of basis matrix, from $(\vec{e}_0, \vec{\varepsilon}_1, \vec{\varepsilon}_2, \vec{\varepsilon}_3)$ to $(\vec{e}_\alpha)$. It is then a simple exercise to get

$$R^\alpha{}_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1-\alpha^2}{1+\alpha^2} & \frac{2\alpha}{1+\alpha^2} & 0 \\ 0 & -\frac{2\alpha}{1+\alpha^2} & \frac{1-\alpha^2}{1+\alpha^2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

By comparing with (6.39), we deduce that $\boldsymbol{R}$ is a spatial rotation of plane $\Pi_R = \mathrm{Span}(\vec{e}_1, \vec{e}_2)$ and angle φ given by

$$\cos\varphi = \frac{1-\alpha^2}{1+\alpha^2} \qquad \text{and} \qquad \sin\varphi = -\frac{2\alpha}{1+\alpha^2}. \tag{6.61}$$

Remark 6.20. If $\alpha = 0$, formulas (6.59) and (6.61) lead to $\Gamma = 1$ and $\varphi = 0$, i.e. $\boldsymbol{S} = \mathrm{Id}$ and $\boldsymbol{R} = \mathrm{Id}$, as it should be.

Remark 6.21. Since the plane of $\boldsymbol{S}$ is $\Pi = \mathrm{Span}(\vec{e}_0, \vec{\boldsymbol{\varepsilon}}_1)$, with $\vec{\boldsymbol{\varepsilon}}_1$ linked to $\vec{e}_1$ and $\vec{e}_2$ by (6.60), and the plane of $\boldsymbol{R}$ is $\Pi_R = \mathrm{Span}(\vec{e}_1, \vec{e}_2)$, we notice that, for a null rotation, the factors $\boldsymbol{S}$ and $\boldsymbol{R}$ of the polar decomposition are not acting in orthogonal planes, contrary to what happens for a 4-screw.

6.6 Properties of Lorentz Boosts

Lorentz boosts have been introduced in Sect. 6.4.4. We detail here some of their properties.

6.6.1 *Kinematical Interpretation*

Let us consider a Lorentz boost $\boldsymbol{\Lambda}$ of plane Π. A right-handed orthonormal basis $(\vec{e}_\alpha)$ of $(E, \boldsymbol{g})$ is said to be ***adapted to*** $\boldsymbol{\Lambda}$ iff $\vec{e}_0$ is future-directed and (cf. Fig. 6.4)

$$\Pi = \mathrm{Span}(\vec{e}_0, \vec{e}_1) \quad \text{and} \quad \Pi^\perp = \mathrm{Span}(\vec{e}_2, \vec{e}_3). \tag{6.62}$$

An example of adapted basis is that considered in Sect. 6.4.4. It follows that the matrix of a boost in an adapted basis is of the type (6.43) when expressed in terms of the rapidity ψ or (6.48) when expressed in terms of $\Gamma := \cosh\psi$ and $V := c\tanh\psi$.

Remark 6.22. Values $V < 0$ are allowed: they correspond to $\psi < 0$ and simply mean that the vector $\vec{e}_1$ is in the direction opposite to the orthogonal projection of $\boldsymbol{\Lambda}(\vec{e}_0)$ on the hyperplane E_{e_0}.

We set

$$\vec{e}'_0 := \boldsymbol{\Lambda}(\vec{e}_0), \tag{6.63}$$

and observe on (6.48) that

$$\boxed{\Gamma = -\vec{e}_0 \cdot \vec{e}'_0}. \tag{6.64}$$

Since $\boldsymbol{\Lambda}$ is orthochronous, we have $\Gamma \geq 1$ [cf. Eqs. (6.16) and (6.17)]. Physically Γ can be interpreted as the Lorentz factor between two observers whose worldlines cross at the same event [cf. Eq. (4.10)]. Indeed, $\vec{e}_0$ and $\vec{e}'_0$ are future-directed unit timelike vectors and are thereby eligible for being 4-velocities. Let then $\mathscr{O}$ be an observer whose 4-velocity $\vec{u}$ at some event $O \in \mathscr{E}$ obeys

$$\vec{u}(O) = \vec{e}_0 \tag{6.65}$$

and whose local frame at O coincides with $(\vec{e}_\alpha)$. Let $\mathscr{O}'$ be an observer at the same event O, whose 4-velocity satisfies $\vec{u}'(O) = \vec{e}'_0$ and whose local frame at O coincides with $(\vec{e}'_0, \vec{e}'_1 := \boldsymbol{\Lambda}(\vec{e}_1), \vec{e}_2, \vec{e}_3)$. The orthogonal decomposition of $\vec{e}'_0$ with respect to $\mathscr{O}$ is given by (4.31) where we make $\vec{u}' = \vec{e}'_0$ and $\vec{u} = \vec{e}_0$ (cf. Fig. 6.4):

$$\vec{e}'_0 = \Gamma\left(\vec{e}_0 + \frac{1}{c}\vec{V}\right). \tag{6.66}$$

In this formula, $\vec{V} \in E_{\boldsymbol{e}_0}$ is the velocity of observer $\mathscr{O}'$ relative to $\mathscr{O}$ at O. Besides, from the matrix (6.48), $\vec{e}'_0 = \Gamma[\vec{e}_0 + (V/c)\vec{e}_1]$. Comparing with (6.66) yields

$$\vec{V} = V\,\vec{e}_1. \tag{6.67}$$

The relation (6.47) between Γ and V appears thus as the standard expression (4.33) of the Lorentz factor in terms of the relative velocity.

Let us now investigate the action of the boost $\boldsymbol{\Lambda}$ on a generic vector $\vec{v} \in E$. Denoting by (v^α) the components of $\vec{v}$ in the basis $(\vec{e}_\alpha)$ adapted to $\boldsymbol{\Lambda}$, we have $\vec{v} = v^\alpha\,\vec{e}_\alpha$ and, given the matrix (6.48) of $\boldsymbol{\Lambda}$ in this basis,

$$\boldsymbol{\Lambda}(\vec{v}) = \Gamma\left(v^0 + \frac{V}{c}v^1\right)\vec{e}_0 + \Gamma\left(\frac{V}{c}v^0 + v^1\right)\vec{e}_1 + v^2\vec{e}_2 + v^3\vec{e}_3.$$

Let us express the term $v^2\vec{e}_2 + v^3\vec{e}_3$ in terms of the orthogonal projection $\perp_{\boldsymbol{e}_0}\vec{v}$ of $\vec{v}$ onto the hyperplane $E_{\boldsymbol{e}_0}$ (cf. Sect. 3.2.5); we have $\perp_{\boldsymbol{e}_0}\vec{v} = v^i\,\vec{e}_i$; Hence

$$v^2\vec{e}_2 + v^3\vec{e}_3 = \perp_{\boldsymbol{e}_0}\vec{v} - v^1\vec{e}_1$$

and

$$\boldsymbol{\Lambda}(\vec{v}) = \Gamma v^0\vec{e}_0 + \Gamma\frac{V}{c}\left(v^1\vec{e}_0 + v^0\vec{e}_1\right) + \perp_{\boldsymbol{e}_0}\vec{v} + (\Gamma - 1)v^1\vec{e}_1.$$

Now, since $(\vec{e}_\alpha)$ is an orthonormal basis, $v^0 = -\vec{e}_0 \cdot \vec{v}$ and $v^1 = \vec{e}_1 \cdot \vec{v}$. Using (6.67) as $\vec{e}_1 = V^{-1}\vec{V}$ and substituting $\vec{u}(O)$ for $\vec{e}_0$ [Eq. (6.65)], we get

$$\boxed{\boldsymbol{\Lambda}(\vec{\boldsymbol{v}}) = -\Gamma(\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}})\,\vec{\boldsymbol{u}} + \frac{\Gamma}{c}\left[(\vec{\boldsymbol{V}} \cdot \vec{\boldsymbol{v}})\,\vec{\boldsymbol{u}} - (\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}})\,\vec{\boldsymbol{V}}\right] + \perp_u \vec{\boldsymbol{v}} + \frac{\Gamma - 1}{V^2}(\vec{\boldsymbol{V}} \cdot \vec{\boldsymbol{v}})\,\vec{\boldsymbol{V}}}, \tag{6.68}$$

where the 4-velocity $\vec{\boldsymbol{u}}$ and the relative velocity $\vec{\boldsymbol{V}}$ are to be taken at the event O. From (6.47), one has $\Gamma - 1 = (\Gamma^2 - 1)/(1+\Gamma) = \Gamma^2 V^2/[c^2(1+\Gamma)]$, so that an expression equivalent to (6.68) is

$$\boxed{\boldsymbol{\Lambda}(\vec{\boldsymbol{v}}) = -\Gamma(\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}})\,\vec{\boldsymbol{u}} + \frac{\Gamma}{c}\left[(\vec{\boldsymbol{V}} \cdot \vec{\boldsymbol{v}})\,\vec{\boldsymbol{u}} - (\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}})\,\vec{\boldsymbol{V}}\right] + \perp_u \vec{\boldsymbol{v}} + \frac{\Gamma^2}{c^2(1+\Gamma)}(\vec{\boldsymbol{V}} \cdot \vec{\boldsymbol{v}})\,\vec{\boldsymbol{V}}}. \tag{6.69}$$

The relation (6.68) or (6.69) shows that a Lorentz boost of plane Π can be expressed entirely in terms of a unit timelike vector $\vec{\boldsymbol{u}}$ (a 4-velocity) belonging to Π and a spacelike vector $\vec{\boldsymbol{V}}$ obeying (i) $\vec{\boldsymbol{V}}$ is in the plane Π, (ii) $\vec{\boldsymbol{V}}$ is orthogonal to $\vec{\boldsymbol{u}}$ and (iii) $\vec{\boldsymbol{V}} \cdot \vec{\boldsymbol{V}} < c^2$. The factor Γ that appears in (6.68) and (6.69) is fully determined by $\vec{\boldsymbol{V}}$ via $\Gamma = (1 - \vec{\boldsymbol{V}} \cdot \vec{\boldsymbol{V}}/c^2)^{-1/2}$. We shall call $\vec{\boldsymbol{V}}$ the ***velocity of the boost $\boldsymbol{\Lambda}$ with respect to $\vec{\boldsymbol{u}}$***.

Remark 6.23. Formula (6.68) [or (6.69)] is analogous to Rodrigues formula (6.41) for spatial rotations.

Remark 6.24. For a given boost, the velocity parameter V, defined by (6.46), is unique, but the vector $\vec{\boldsymbol{V}}$ depends on the choice of the 4-velocity $\vec{\boldsymbol{u}} \in \Pi$. However one has always $\|\vec{\boldsymbol{V}}\|_g = |V|$.

Collecting the above results, we may state:

Given two observers $\mathscr{O}$ and $\mathscr{O}'$ and two values $\vec{\boldsymbol{u}}(A)$ and $\vec{\boldsymbol{u}}\,'(A')$ of their 4-velocities at two points A and A' of their respective worldlines, there exists a unique Lorentz boost $\boldsymbol{\Lambda}$ such that

$$\boldsymbol{\Lambda}(\vec{\boldsymbol{u}}(A)) = \vec{\boldsymbol{u}}\,'(A').$$

If $\vec{\boldsymbol{u}}(A) = \vec{\boldsymbol{u}}\,'(A')$, $\boldsymbol{\Lambda}$ is the identity. If $\vec{\boldsymbol{u}}(A) \neq \vec{\boldsymbol{u}}\,'(A')$, $\boldsymbol{\Lambda}$ is the boost of plane $\mathrm{Span}(\vec{\boldsymbol{u}}(A), \vec{\boldsymbol{u}}\,'(A'))$ and Lorentz factor $\Gamma = -\vec{\boldsymbol{u}}(A) \cdot \vec{\boldsymbol{u}}\,'(A')$. Moreover, if $A = A'$ (the worldlines cross each other), the vector $\vec{\boldsymbol{V}}$ appearing in expressions (6.68) or (6.69) of $\boldsymbol{\Lambda}$ is nothing but the velocity of $\mathscr{O}'$ relative to $\mathscr{O}$ at A.

6.6.2 Expression in a General Basis

Given a Lorentz boost $\boldsymbol{\Lambda}$ of plane Π, let us consider a right-handed orthonormal basis $(\vec{e}_\alpha)$ such that

$$\vec{e}_0 \in \Pi. \tag{6.70}$$

This condition is weaker than (6.62), which defines an adapted basis, since it is no longer demanded that $\vec{e}_1 \in \Pi$. We shall call a basis satisfying (6.70) a ***basis semi-adapted to*** $\boldsymbol{\Lambda}$.

Let $\vec{V}$ be the velocity of $\boldsymbol{\Lambda}$ with respect to $\vec{e}_0$ [cf. Eq. (6.68) with $\vec{u} = \vec{e}_0$]. Since $\vec{V} \in E_{\boldsymbol{e}_0}$, we can write $\vec{V} = V^i\,\vec{e}_i$. In particular

$$\forall \vec{v} \in E, \quad \vec{V} \cdot \vec{v} = \eta_{\alpha\beta} V^\alpha v^\beta = \delta_{ij} V^i v^j = \sum_{i=1}^{3} V^i v^i.$$

On the other hand,

$$\forall \vec{v} \in E, \quad \vec{u} \cdot \vec{v} = \vec{e}_0 \cdot \vec{v} = -v^0 \quad \text{and} \quad \perp_{\boldsymbol{u}} \vec{v} = \perp_{\boldsymbol{e}_0} \vec{v} = v^i\,\vec{e}_i = \delta^i_{\ j} v^j\,\vec{e}_i.$$

In view of these relations, one reads directly on (6.68) the expression of the matrix of $\boldsymbol{\Lambda}$ with respect to the semi-adapted basis $(\vec{e}_\alpha)$:

$$\boxed{\Lambda^\alpha_{\ \beta} = \left(\begin{array}{c|c} \Lambda^0_{\ 0} & \Lambda^0_{\ j} \\ \hline \Lambda^i_{\ 0} & \Lambda^i_{\ j} \end{array}\right) = \left(\begin{array}{c|c} \Gamma & \Gamma \dfrac{V^j}{c} \\ \hline \Gamma \dfrac{V^i}{c} & \delta^i_{\ j} + \dfrac{\Gamma - 1}{V^2} V^i V^j \end{array}\right)}. \tag{6.71}$$

Using instead (6.69) leads to the alternative expression

$$\boxed{\Lambda^\alpha_{\ \beta} = \left(\begin{array}{c|c} \Lambda^0_{\ 0} & \Lambda^0_{\ j} \\ \hline \Lambda^i_{\ 0} & \Lambda^i_{\ j} \end{array}\right) = \left(\begin{array}{c|c} \Gamma & \Gamma \dfrac{V^j}{c} \\ \hline \Gamma \dfrac{V^i}{c} & \delta^i_{\ j} + \dfrac{\Gamma^2}{c^2(1+\Gamma)} V^i V^j \end{array}\right)}. \tag{6.72}$$

We check that in the particular case $(V^1, V^2, V^3) = (V, 0, 0)$, (6.71) reduces to (6.48).

An interesting property appears immediately on expressions (6.71) and (6.72): the matrix of a Lorentz boost in a semi-adapted basis is symmetric:

$$\boxed{\Lambda^\alpha_{\ \beta} = \Lambda^\beta_{\ \alpha}}. \tag{6.73}$$

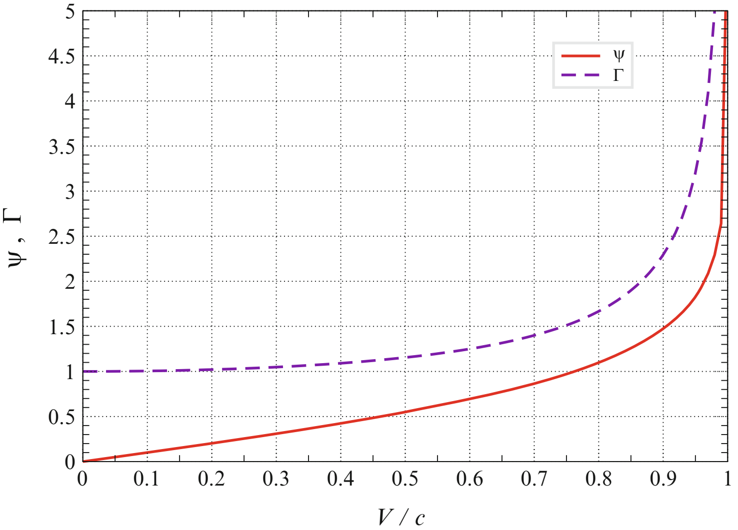

Fig. 6.7 Rapidity ψ and Lorentz factor Γ as functions of the velocity V of a Lorentz boost

6.6.3 Rapidity

The rapidity ψ of a Lorentz boost $\boldsymbol{\Lambda}$ has been defined in Sect. 6.4.4. It is related to the velocity V of $\boldsymbol{\Lambda}$ by formula (6.46), $V = c \tanh \psi$, and to the Lorentz factor Γ of $\boldsymbol{\Lambda}$ by formula (6.47): $\Gamma = \cosh \psi$. These two formulas can be inverted, yielding, respectively, to

$$\boxed{\psi = \operatorname{artanh} \frac{V}{c} = \frac{1}{2} \ln \left(\frac{1 + V/c}{1 - V/c} \right)}, \tag{6.74}$$

$$\boxed{\psi = \operatorname{arcosh} \Gamma = \ln(\Gamma + \sqrt{\Gamma^2 - 1})}. \tag{6.75}$$

In these equations, the second equality stems from the standard logarithmic expressions of inverse hyperbolic functions. Note that at the nonrelativistic limit, $V \ll c$, Eq. (6.74) reduces to

$$\psi \simeq \frac{V}{c} \qquad (V \ll c). \tag{6.76}$$

In this limit, the rapidity coincides thus with the velocity normalized by c. In the general case, the rapidity is plotted as a function of V in Fig. 6.7. The rapidity admits a nice geometrical interpretation, as the area (with respect to the Euclidean metric) of the grey surface in Fig. 6.8. We leave the demonstration as an exercise for the reader.

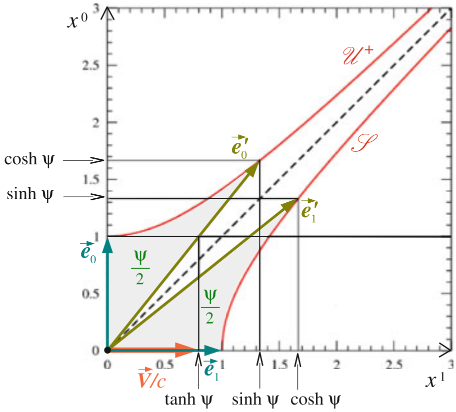

Fig. 6.8 Graphical interpretation of the rapidity ψ of a Lorentz boost $\boldsymbol{\Lambda}$. $(\vec{e}_0, \vec{e}_1)$ is an orthonormal basis of $\boldsymbol{\Lambda}$'s plane, $\vec{e}\,'_0 := \boldsymbol{\Lambda}(\vec{e}_0)$ and $\vec{e}\,'_1 := \boldsymbol{\Lambda}(\vec{e}_1)$. Denoting by (x^0, x^1) the affine coordinates associated with $(\vec{e}_0, \vec{e}_1)$, the curve $\mathscr{U}^+$ is a branch of the hyperbola $(x^0)^2 - (x^1)^2 = 1$ and the curve $\mathscr{S}$ a branch of the hyperbola $(x^1)^2 - (x^0)^2 = 1$. They are the same as in Fig. 1.6. The rapidity ψ is nothing but the total area of the grey surfaces. In this figure, $\psi = \ln 3 \simeq 1.0986$, which corresponds to $\Gamma = \cosh\psi = 5/3$, $\sinh\psi = 4/3$ and $V = 4c/5$

Let us recall that the matrix of a Lorentz boost in an adapted basis is expressed in terms of the rapidity according to (6.43).

Remark 6.25. Expression (6.43) resembles the matrix of a rotation in which the sines and cosines would have been replaced by their hyperbolic versions. We can indeed compare (6.43) to the matrix (6.39) of a spatial rotation of angle φ in the plane $\mathrm{Span}(\vec{e}_2, \vec{e}_3)$. Note that the sign $-$ in factor of $\sin\varphi$ in the third row of (6.39) does not appear in (6.43). To deepen this analogy, let us associate with the rapidity ψ of a Lorentz boost the complex imaginary "angle"

$$\varphi^* := \mathrm{i}\psi. \tag{6.77}$$

Then, by Euler formula,

$$\cosh\psi = \frac{e^{\psi} + e^{-\psi}}{2} = \frac{e^{-i\varphi^*} + e^{i\varphi^*}}{2} = \cos\varphi^*,$$

$$\sinh\psi = \frac{e^{\psi} - e^{-\psi}}{2} = \frac{e^{-i\varphi^*} - e^{i\varphi^*}}{2} = -\mathrm{i}\sin\varphi^*.$$

In addition, let us introduce the complex imaginary "vector" $\vec{e}^{\,*}_0 := \mathrm{i}\,\vec{e}_0$. This amounts to considering complex values for the first component of any vector $\vec{v} \in E$, since we can write $\vec{v} = v^\alpha \vec{e}_\alpha = v^0_* \vec{e}^{\,*}_0 + v^i \vec{e}_i$, with

$$v^0 = \mathrm{i}v^0_*.$$

Such a procedure is called a ***Wick rotation***, the multiplication by i being equivalent to a rotation of $\pi/2$ in the complex plane. Wick rotation is widely used in quantum field theory (cf., e.g. Maggiore (2005)). From (6.43), we get

$$\begin{aligned}
\boldsymbol{\Lambda}(\vec{\boldsymbol{e}}^{\,*}_0) &= \mathrm{i}\,\boldsymbol{\Lambda}(\vec{\boldsymbol{e}}_0) = \mathrm{i}\,(\cosh\psi\,\vec{\boldsymbol{e}}_0 + \sinh\psi\,\vec{\boldsymbol{e}}_1) = \mathrm{i}\,\big(\cos\varphi^*\,\vec{\boldsymbol{e}}_0 - \mathrm{i}\sin\varphi^*\,\vec{\boldsymbol{e}}_1\big) \\
&= \cos\varphi^*\,\vec{\boldsymbol{e}}^{\,*}_0 + \sin\varphi^*\,\vec{\boldsymbol{e}}_1 \\
\boldsymbol{\Lambda}(\vec{\boldsymbol{e}}_1) &= \sinh\psi\,\vec{\boldsymbol{e}}_0 + \cosh\psi\,\vec{\boldsymbol{e}}_1 = -\mathrm{i}\sin\varphi^*\,\vec{\boldsymbol{e}}_0 + \cos\varphi^*\,\vec{\boldsymbol{e}}_1 \\
&= -\sin\varphi^*\,\vec{\boldsymbol{e}}^{\,*}_0 + \cos\varphi^*\,\vec{\boldsymbol{e}}_1,
\end{aligned}$$

so that the matrix of $\boldsymbol{\Lambda}$ in the complex "basis" $(\vec{\boldsymbol{e}}^{\,*}_0, \vec{\boldsymbol{e}}_1, \vec{\boldsymbol{e}}_2, \vec{\boldsymbol{e}}_3)$ is

$$\Lambda^{*\alpha}{}_{\beta} = \begin{pmatrix} \cos\varphi^* & -\sin\varphi^* & 0 & 0 \\ \sin\varphi^* & \cos\varphi^* & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Comparing with (6.39), we recognize the matrix of the rotation of angle φ^* in the plane $\Pi = \mathrm{Span}(\vec{\boldsymbol{e}}^{\,*}_0, \vec{\boldsymbol{e}}_1)$. The fact that introducing complex numbers makes a Lorentz boost of plane Π resemble a rotation in this plan is not surprising if one expresses the components of the metric tensor in the complex "basis" $(\vec{\boldsymbol{e}}^{\,*}_0, \vec{\boldsymbol{e}}_1, \vec{\boldsymbol{e}}_2, \vec{\boldsymbol{e}}_3)$; since $\vec{\boldsymbol{e}}^{\,*}_0 \cdot \vec{\boldsymbol{e}}^{\,*}_0 = \mathrm{i}^2\,\vec{\boldsymbol{e}}_0 \cdot \vec{\boldsymbol{e}}_0 = (-1)(-1) = 1$, these components are indeed

$$g^*_{\alpha\beta} = \mathrm{diag}(1, 1, 1, 1).$$

Hence $\boldsymbol{g}$ appears as a "Euclidean" metric (cf. Sect. 1.3.1). This is the virtue of Wick rotation: to transform a Minkowskian problem into a Euclidean one. Lorentz transformations being defined as the transformations that preserve $\boldsymbol{g}$, they can then be seen as the isometries of a four-dimensional Euclidean space. Consequently, all restricted Lorentz can be decomposed into rotations in vector planes. If the plane Π of the rotation is spacelike, the transformation is a spatial rotation, if Π is timelike, the transformation is a boost and if Π is null, it is a null rotation (cf. Sect. 6.4.5).

Historical note: *Rapidity has been introduced in 1908 (Minkowski 1908) by Hermann Minkowski (cf. p. 26), who preferred to handle* iψ *rather than* ψ *(cf. (6.77) and the discussion in Walter (1999b)). Rapidity has also been employed under this form by Arnold Sommerfeld (cf. p. 27) in 1909 (Sommerfeld 1909) to reduce the calculus on boosts to ordinary trigonometry, via some imaginary angles. The word "rapidity" has been coined by Alfred A. Robb (cf. p. 74) in 1911 (Robb 1911). However, as early as 1905 (Poincaré 1906), Henri Poincaré (cf. p. 26) showed that boosts can be considered as "complex rotations" in spacetime (cf. Remark 6.25 p. 200). Wick rotation owes its name to the Italian theoretical physicist Gian-Carlo*

Wick (1909–1992). From what precedes, it would be more appropriate to call it Poincaré–Wick rotation.

6.6.4 Eigenvalues

We have seen in Sect. 6.4 that any Lorentz boost $\boldsymbol{\Lambda}$ is diagonalizable [cf. Eq. (6.42)] and that its eigenvalues and eigenvectors are[14]

$$\boxed{\lambda_+ = \mathrm{e}^{\psi}} : \qquad \vec{\boldsymbol{\ell}}_+ = \vec{\boldsymbol{e}}_0 + \vec{\boldsymbol{e}}_1 \tag{6.78a}$$

$$\boxed{\lambda_- = \mathrm{e}^{-\psi}} : \qquad \vec{\boldsymbol{\ell}}_- = \vec{\boldsymbol{e}}_0 - \vec{\boldsymbol{e}}_1 \tag{6.78b}$$

$$\boxed{\lambda_0 = 1} : \qquad \vec{\boldsymbol{e}}_2 \ \text{ and } \ \vec{\boldsymbol{e}}_3, \tag{6.78c}$$

where ψ is $\boldsymbol{\Lambda}$'s rapidity and $(\vec{\boldsymbol{e}}_\alpha)$ an orthonormal basis adapted to $\boldsymbol{\Lambda}$. The eigenvectors $\vec{\boldsymbol{\ell}}_+$ and $\vec{\boldsymbol{\ell}}_-$ are null. They are depicted in Fig. 6.9. Thanks to (6.74), one can reexpress the eigenvalues λ_+ and λ_- in terms of $\boldsymbol{\Lambda}$'s velocity parameter V:

$$\boxed{\lambda_+ = \sqrt{\frac{1 + V/c}{1 - V/c}}} \quad \text{and} \quad \boxed{\lambda_- = \sqrt{\frac{1 - V/c}{1 + V/c}}}. \tag{6.79}$$

The variation of λ_+ and λ_- with V is represented in Fig. 6.10.

Remark 6.26. One can recover that a Lorentz boost has two null eigenvectors without appealing to the results of Sect. 6.4. It suffices to note that $\boldsymbol{\Lambda}$ leaves invariant both its plane Π and the null cone $\mathscr{I}$ [cf. (6.8)]. The image of a vector in $\mathscr{I} \cap \Pi$, such as $\vec{\boldsymbol{\ell}}_+$ or $\vec{\boldsymbol{\ell}}_-$, is thus both in $\mathscr{I}$ and in Π, hence in $\mathscr{I} \cap \Pi$. This set is made of only two vector lines (those drawn with dashed lines in Fig. 6.9), we deduce that $\boldsymbol{\Lambda}(\vec{\boldsymbol{\ell}}_+)$ must be collinear either to $\vec{\boldsymbol{\ell}}_+$ or to $\vec{\boldsymbol{\ell}}_-$. But the latter case is excluded by continuity (limit $V \to 0$, where $\boldsymbol{\Lambda}$ reduces to the identity).

6.7 Composition of Boosts and Thomas Rotation

Let $\boldsymbol{\Lambda}_1$ and $\boldsymbol{\Lambda}_2$ be two Lorentz boosts, of respective planes Π_1 and Π_2 and Lorentz factors Γ_1 and Γ_2. We would like to determine the composed transformation

[14]This is immediate on the diagonal matrix (6.42); this can also be found by setting $\alpha = 0$ and $\varphi = 0$ in (6.56), with the change of notion $\vec{\boldsymbol{\ell}} \to \vec{\boldsymbol{\ell}}_+$ and $\vec{\boldsymbol{k}} \to \vec{\boldsymbol{\ell}}_-$.

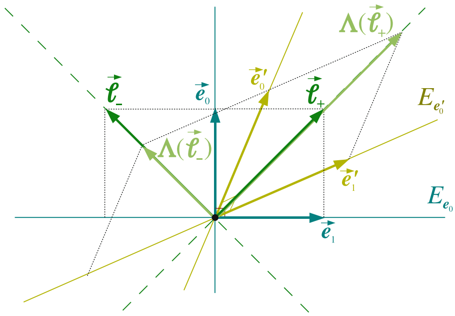

Fig. 6.9 Eigenvectors $\vec{\ell}_+$ and $\vec{\ell}_-$ of a Lorentz boost $\boldsymbol{\Lambda}$, corresponding, respectively, to the eigenvalues $\lambda_+ = \mathrm{e}^{\psi}$ and $\lambda_- = \mathrm{e}^{-\psi}$. Since $\vec{\ell}_+ = \vec{e}_0 + \vec{e}_1$ and $\vec{\ell}_- = \vec{e}_0 - \vec{e}_1$, one has $\boldsymbol{\Lambda}(\vec{\ell}_+) = \lambda_+ \vec{\ell}_+ = \vec{e}\,'_0 + \vec{e}\,'_1$ and $\boldsymbol{\Lambda}(\vec{\ell}_-) = \lambda_- \vec{\ell}_- = \vec{e}\,'_0 - \vec{e}\,'_1$, with $\vec{e}\,'_0 := \boldsymbol{\Lambda}(\vec{e}_0)$ and $\vec{e}\,'_1 := \boldsymbol{\Lambda}(\vec{e}_1)$. It is clear on the figure that $\lambda_+ \geq 1$ and $\lambda_- \leq 1$

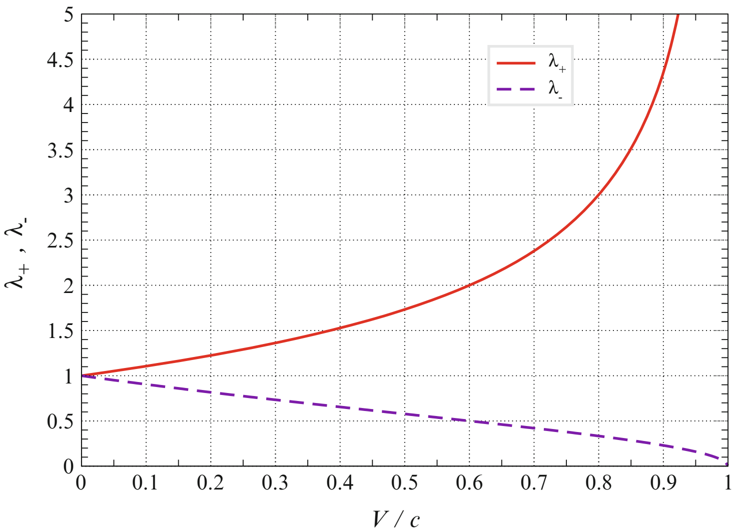

Fig. 6.10 Eigenvalues different from 1 of a Lorentz boost, as functions of the velocity parameter V

$$\boldsymbol{\Lambda} := \boldsymbol{\Lambda}_2 \circ \boldsymbol{\Lambda}_1. \tag{6.80}$$

We know already that it is a restricted Lorentz transformation, for $\mathrm{SO}_0(3,1)$ is a group. But is it a boost or a more general transformation, such as the product of a boost by a spatial rotation, as shown in Sect. 6.5? We are going to investigate first the simplest case, where the planes Π_1 and Π_2 coincide (Sect. 6.7.1), before moving to the general case (Sect. 6.7.2), which will lead us to Thomas rotation.

6.7.1 Coplanar Boosts

We assume here that $\Pi_1 = \Pi_2$. In this case, $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}_2 \circ \boldsymbol{\Lambda}_1$ is a Lorentz boost of plane

$$\Pi := \Pi_1 = \Pi_2. \tag{6.81}$$

Proof. Π being a timelike plane, $\Pi^\perp$ is a spacelike plane, which is moreover strictly invariant under $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}_2 \circ \boldsymbol{\Lambda}_1$, since it is strictly invariant under both $\boldsymbol{\Lambda}_1$ and $\boldsymbol{\Lambda}_2$. From the very definition of a Lorentz boost given in Sect. 6.4.4, it is then clear that $\boldsymbol{\Lambda}$ is a Lorentz boost of plane Π. □

Since $\Pi_1 = \Pi_2$, any basis adapted to $\boldsymbol{\Lambda}_1$ is also adapted to $\boldsymbol{\Lambda}_2$ [cf. condition (6.62)]. Let then $(\vec{e}_\alpha)$ be such a basis. The matrices of $\boldsymbol{\Lambda}_1$ and $\boldsymbol{\Lambda}_2$ in this basis are given by (6.48) in which (Γ, V) must be replaced by, respectively, (Γ_1, V_1) and (Γ_2, V_2), with

$$\Gamma_1 := \left(1 - V_1^2/c^2\right)^{-1/2} \quad \text{and} \quad \Gamma_2 := \left(1 - V_2^2/c^2\right)^{-1/2}.$$

The matrix of $\boldsymbol{\Lambda} := \boldsymbol{\Lambda}_2 \circ \boldsymbol{\Lambda}_1$ in the basis $(\vec{e}_\alpha)$ is obtained as the product of the matrices of $\boldsymbol{\Lambda}_2$ and $\boldsymbol{\Lambda}_1$:

$$\Lambda^\alpha{}_\beta = \begin{pmatrix} \Gamma_1\Gamma_2(1 + V_1V_2/c^2) & \Gamma_1\Gamma_2(V_1 + V_2)/c & 0 & 0 \\ \Gamma_1\Gamma_2(V_1 + V_2)/c & \Gamma_1\Gamma_2(1 + V_1V_2/c^2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{6.82}$$

Comparing with (6.48), we conclude that $\boldsymbol{\Lambda}_2 \circ \boldsymbol{\Lambda}_1$ is a Lorentz boost of plane Π and of Lorentz factor Γ and velocity V given by

$$\boxed{\Gamma = \Gamma_1\Gamma_2\left(1 + \frac{V_1V_2}{c^2}\right)} \quad \text{and} \quad \boxed{V = \frac{V_1 + V_2}{1 + V_1V_2/c^2}}. \tag{6.83}$$

We recognize in (6.83) the law of velocity composition established in Chap. 4 in the case of collinear velocities [formula (5.45)]. Indeed the following kinematical interpretation holds (cf. Fig. 6.11): let $\mathscr{O}'$ be an observer of 4-velocity $\vec{\boldsymbol{u}}' = \vec{\boldsymbol{e}}_0$, $\mathscr{O}$ an observer of 4-velocity $\vec{\boldsymbol{u}} = \boldsymbol{\Lambda}_1(\vec{\boldsymbol{e}}_0)$ and $\mathscr{P}$ a point particle of 4-velocity $\vec{\boldsymbol{v}} = \boldsymbol{\Lambda}_2 \circ \boldsymbol{\Lambda}_1(\vec{\boldsymbol{e}}_0)$. The composite function $\boldsymbol{\Lambda}_2 \circ \boldsymbol{\Lambda}_1$ is then the Lorentz transformation to move from $\mathscr{O}'$ to $\mathscr{P}$, so that the velocity obtained in (6.83) is the velocity denoted by V' in Sect. 5.3.3. More generally, the correspondence with the notations used in Chap. 4 is given in Table 6.1. We are indeed in the case treated in Sect. 5.3.3, namely, that where the 4-velocities $\vec{\boldsymbol{u}}$, $\vec{\boldsymbol{u}}'$ and $\vec{\boldsymbol{v}}$ are coplanar (they all belong to the plane $\Pi = \Pi_1 = \Pi_2$; cf. Fig. 6.11). The velocities $\vec{\boldsymbol{U}} = -V_1\,\vec{\boldsymbol{e}}'_1$ ($\vec{\boldsymbol{e}}'_1 := \boldsymbol{\Lambda}_1(\vec{\boldsymbol{e}}_1)$) and $\vec{\boldsymbol{V}} = V_2\,\vec{\boldsymbol{e}}'_1$ of, respectively, $\mathscr{O}'$ and $\mathscr{P}$ relative to $\mathscr{O}$ are then collinear and

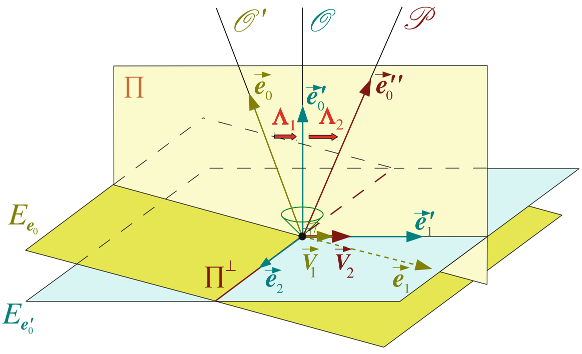

Fig. 6.11 Composition of two boosts, $\boldsymbol{\Lambda}_1$ and $\boldsymbol{\Lambda}_2$, having the same plane Π. Observers $\mathscr{O}'$ and $\mathscr{O}$, of respective 4-velocities $\vec{\boldsymbol{u}}\,' = \vec{\boldsymbol{e}}_0$ and $\vec{\boldsymbol{u}} = \vec{\boldsymbol{e}}\,'_0 := \boldsymbol{\Lambda}_1(\vec{\boldsymbol{e}}_0)$, as well as the point particle $\mathscr{P}$, of 4-velocity $\vec{\boldsymbol{v}} = \vec{\boldsymbol{e}}\,''_0 := \boldsymbol{\Lambda}_2 \circ \boldsymbol{\Lambda}_1(\vec{\boldsymbol{e}}_0)$, are the same than those considered in Chap. 4

Table 6.1 Kinematical interpretation of the composition of two boosts. This table makes the link between notations of Sect. 5.3 (left-hand side of the equalities) and those of this chapter (right-hand side of the equalities). $(\vec{\boldsymbol{e}}_\alpha)$ is an orthonormal basis adapted to $\boldsymbol{\Lambda}_1$ and whose first vector coincides with the 4-velocity of $\mathscr{O}'$. Note that $\vec{\boldsymbol{e}}\,'_1 := \boldsymbol{\Lambda}_1(\vec{\boldsymbol{e}}_1)$ and $\vec{\boldsymbol{\varepsilon}}$ is the unit vector of $\Pi_2 \cap E_{\boldsymbol{u}}$ that coincides with $\vec{\boldsymbol{e}}\,'_1$ for coplanar boosts

Observer	$\mathscr{O}'$	$\Longrightarrow$	$\mathscr{O}$	$\longrightarrow$	$\mathscr{P}$
4-velocity	$\vec{\boldsymbol{u}}' = \vec{\boldsymbol{e}}_0$		$\vec{\boldsymbol{u}} = \boldsymbol{\Lambda}_1(\vec{\boldsymbol{e}}_0)$		$\vec{\boldsymbol{v}} = \boldsymbol{\Lambda}_2 \circ \boldsymbol{\Lambda}_1(\vec{\boldsymbol{e}}_0)$
Boost		$\boldsymbol{\Lambda}_1$		$\boldsymbol{\Lambda}_2$	
Lorentz fact.		$\Gamma_0 = \Gamma_1$		$\Gamma = \Gamma_2$	
Velocity/$\mathscr{O}'$		$\vec{\boldsymbol{U}}' = V_1 \vec{\boldsymbol{e}}_1$			
Velocity/$\mathscr{O}$		$-\vec{\boldsymbol{U}} = \vec{\boldsymbol{V}}_1$		$\vec{\boldsymbol{V}} = \vec{\boldsymbol{V}}_2$	
		$= V_1 \vec{\boldsymbol{e}}\,'_1$		$= V_2\, \vec{\boldsymbol{\varepsilon}}$	
			$\boldsymbol{\Lambda}_2 \circ \boldsymbol{\Lambda}_1$		
			$\Gamma' = \Gamma$		
			$\vec{\boldsymbol{V}}' = \vec{\boldsymbol{V}}$		

formula (5.45) holds. Taking into account the notation changes listed in Table 6.1, this formula is exactly Eq. (6.83) obtained here. Similarly, one can check that Eq. (6.83), which gives the Lorentz factor of $\boldsymbol{\Lambda}_2 \circ \boldsymbol{\Lambda}_1$ coincides with Eq. (5.46), obtained in Chap. 4.

The symmetry of formulas (6.83) in $(\Gamma_1, V_1) \leftrightarrow (\Gamma_2, V_2)$ shows that the composition of boosts sharing the same plane is commutative:

$$\boldsymbol{\Lambda}_2 \circ \boldsymbol{\Lambda}_1 = \boldsymbol{\Lambda}_1 \circ \boldsymbol{\Lambda}_2. \tag{6.84}$$

Let us now express $\boldsymbol{\Lambda}_2 \circ \boldsymbol{\Lambda}_1$ in terms of the rapidities ψ_1 and ψ_2 of $\boldsymbol{\Lambda}_1$ and $\boldsymbol{\Lambda}_2$. In view of (6.47) and (6.46), the rapidity ψ of $\boldsymbol{\Lambda}_2 \circ \boldsymbol{\Lambda}_1$ can be inferred from the result (6.83):

$$\cosh\psi = \cosh\psi_1 \cosh\psi_2 + \sinh\psi_1 \sinh\psi_2 .$$

We recognize in the right-hand side the hyperbolic cosine of the sum $\psi_1 + \psi_2$. Hence

$$\boxed{\psi = \psi_1 + \psi_2}. \tag{6.85}$$

The composition of coplanar Lorentz boosts is thus very simple in terms of rapidity: the rapidity of the result is nothing but the sum of the rapidities of the two components.

Remark 6.27. If, following Remark 6.25 (p. 200), one considers boosts as "rotations" of imaginary angle $\varphi^* = \mathrm{i}\psi$, the result (6.85) simply means that the composite of two rotations having the same plane is a rotation whose angle is the sum of the individual angles of each rotation. Moreover, we shall see in Chap. 7 a deeper interpretation of this result: there exists necessarily a parametrization of coplanar boosts such that the composition law is reduced to the addition of the parameters (cf. Sect. 7.2.3). It turns out that rapidity is the parameter that makes this actual.

6.7.2 Thomas Rotation

Let us now investigate the case where the planes Π_1 and Π_2 of the boosts $\boldsymbol{\Lambda}_1$ and $\boldsymbol{\Lambda}_2$ are different. We shall treat only the case that corresponds physically to a change of observer, namely, we shall assume that the intersection between Π_1 and Π_2 is timelike. We can then introduce the 4-velocity $\vec{\boldsymbol{u}} = \vec{\boldsymbol{e}}'_0$ of the "intermediate" observer $\mathscr{O}$ (cf. Fig. 6.12):

$$\Pi_1 \cap \Pi_2 = \mathrm{Span}(\vec{\boldsymbol{e}}'_0), \qquad \vec{\boldsymbol{e}}'_0 \cdot \vec{\boldsymbol{e}}'_0 = -1.$$

The 4-velocity of the first observer[15], $\mathscr{O}'$, is then

$$\vec{\boldsymbol{e}}_0 := \boldsymbol{\Lambda}_1^{-1}(\vec{\boldsymbol{e}}'_0) \in \Pi_1 \tag{6.86}$$

and that of the point particle $\mathscr{P}$ is

$$\vec{\boldsymbol{e}}''_0 := \boldsymbol{\Lambda}_2(\vec{\boldsymbol{e}}'_0) = \boldsymbol{\Lambda}_2 \circ \boldsymbol{\Lambda}_1(\vec{\boldsymbol{e}}_0) \in \Pi_2. \tag{6.87}$$

Two vectors that naturally appear in the problem are the following unit vectors (cf. Fig. 6.12):

$$\vec{\boldsymbol{e}}_1 \in \Pi_1 \cap E_{\boldsymbol{e}_0}, \qquad \vec{\boldsymbol{e}}_1 \cdot \vec{\boldsymbol{e}}_1 = 1 \tag{6.88}$$

$$\vec{\boldsymbol{\varepsilon}} \in \Pi_2 \cap E_{\boldsymbol{e}'_0}, \qquad \vec{\boldsymbol{\varepsilon}} \cdot \vec{\boldsymbol{\varepsilon}} = 1. \tag{6.89}$$

[15] As in Sect. 6.7.1, we are using the notations of Table 6.1 to make the link with Chap. 4.

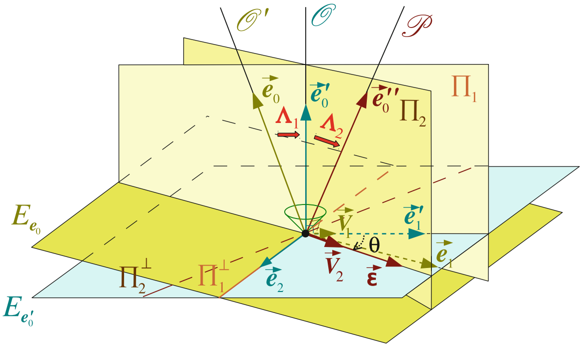

Fig. 6.12 Composition of two Lorentz boosts, $\boldsymbol{\Lambda}_1$ and $\boldsymbol{\Lambda}_2$, of different planes Π_1 and Π_2. The orthonormal basis $(\vec{e}_\alpha)$ is adapted to $\boldsymbol{\Lambda}_1$ and such that $\vec{e}_3 \in \Pi_1^\perp \cap \Pi_2^\perp$. The figure is a view of the three-dimensional space orthogonal to $\vec{e}_3$. We have noted $\vec{e}\,'_0 := \boldsymbol{\Lambda}_1(\vec{e}_0)$, $\vec{e}\,'_1 := \boldsymbol{\Lambda}_1(\vec{e}_1)$ and $\vec{e}\,''_0 := \boldsymbol{\Lambda}_2 \circ \boldsymbol{\Lambda}_1(\vec{e}_0)$. The unit vector $\vec{\varepsilon} \in \Pi_2 \cap E_{e'_0}$ is such that $\vec{\varepsilon} = \cos\theta\, \vec{e}\,'_1 + \sin\theta\, \vec{e}_2$

Conditions (6.88) and (6.89) define $\vec{e}_1$ and $\vec{\varepsilon}$ up to some sign; we shall choose this sign to ensure that $\vec{e}_1$ has the same direction as the orthogonal projection of $\vec{e}\,'_0$ onto the hyperplane E_{e_0} and $\vec{\varepsilon}$ has the same direction as the orthogonal projection of $\vec{e}\,''_0$ onto the hyperplane $E_{e'_0}$. We define

$$\vec{e}\,'_1 := \boldsymbol{\Lambda}_1(\vec{e}_1) \in \Pi_1 \cap E_{e'_0}. \tag{6.90}$$

$\vec{e}\,'_1$ and $\vec{\varepsilon}$ are two vectors of $E_{e'_0}$ related to the velocities $\vec{V}_1$ and $\vec{V}_2$ of transformations $\boldsymbol{\Lambda}_1$ and $\boldsymbol{\Lambda}_2$ with respect to $\vec{e}\,'_0$ via

$$\vec{V}_1 = V_1\, \vec{e}\,'_1 \qquad \text{and} \qquad \vec{V}_2 = V_2\, \vec{\varepsilon}. \tag{6.91}$$

Note that $\vec{V}_1$ and $\vec{V}_2$ can be interpreted as physical velocities: $-\vec{V}_1$ is the velocity of $\mathscr{O}'$ relative to $\mathscr{O}$ and $\vec{V}_2$ is the velocity of $\mathscr{P}$ relative to $\mathscr{O}$ (cf. Fig. 6.12 and Table 6.1).

The fundamental parameter in the problem is the angle θ between the velocities $\vec{V}_1$ and $\vec{V}_2$, i.e. the angle θ defined by

$$\cos\theta := \vec{e}\,'_1 \cdot \vec{\varepsilon}, \quad \theta \in [0, \pi]. \tag{6.92}$$

If $\theta = 0$ or $\theta = \pi$, the planes Π_1 and Π_2 coincide and we are back to the case treated in Sect. 6.7.1.

Let us consider the two vector planes $\Pi_1^\perp$ and $\Pi_2^\perp$. They are both entirely contained in the spacelike hyperplane $E_{e'_0}$ (cf. Fig. 6.12). If Π_1 and Π_2 do not coincide, $\Pi_1^\perp$ and $\Pi_2^\perp$ do not coincide either and their intersection is necessarily a vector line[16] of $E_{e'_0}$. Let us then introduce the unit vector $\vec{e}_3 \in E_{e'_0}$ such that

$$\Pi_1^\perp \cap \Pi_2^\perp = \mathrm{Span}(\vec{e}_3), \qquad \vec{e}_3 \cdot \vec{e}_3 = 1. \tag{6.93}$$

This requirement defines $\vec{e}_3$ up to some sign. The vector $\vec{e}_3$ plays an important role because it defines a direction invariant by the composite transformation $\boldsymbol{\Lambda}_2 \circ \boldsymbol{\Lambda}_1$. Indeed, since $\vec{e}_3 \in \Pi_1^\perp$, $\boldsymbol{\Lambda}_1(\vec{e}_3) = \vec{e}_3$ and since $\vec{e}_3 \in \Pi_2^\perp$, $\boldsymbol{\Lambda}_2(\vec{e}_3) = \vec{e}_3$.

Given that $\vec{e}\,'_1 \in \Pi_1$ [cf. (6.90)], $\vec{e}\,'_1$ and $\vec{e}_3$ are two orthogonal unit vectors of $E_{e'_0}$. Let then $\vec{e}_2$ be the unique vector of $E_{e'_0}$ such that $(\vec{e}\,'_1, \vec{e}_2, \vec{e}_3)$ is a right-handed orthonormal basis of $E_{e'_0}$. We have thus

$$\Pi_1^\perp = \mathrm{Span}(\vec{e}_2, \vec{e}_3).$$

Since $\Pi_1 = \mathrm{Span}(\vec{e}_0, \vec{e}_1)$, the 4-tuple $(\vec{e}_\alpha) = (\vec{e}_0, \vec{e}_1, \vec{e}_2, \vec{e}_3)$ is an orthonormal basis of $E = \Pi_1 \oplus \Pi_1^\perp$. Moreover, it is adapted to $\boldsymbol{\Lambda}_1$. Let us compute the matrix of $\boldsymbol{\Lambda}_2 \circ \boldsymbol{\Lambda}_1$ in this basis. We start by writing the action of $\boldsymbol{\Lambda}_1$ on the basis $(\vec{e}_\alpha)$. Since the latter is an adapted basis, it suffices to use the matrix (6.43):

$$\boldsymbol{\Lambda}_1(\vec{e}_0) = \vec{e}\,'_0 = c_1\,\vec{e}_0 + s_1\,\vec{e}_1 \tag{6.94a}$$

$$\boldsymbol{\Lambda}_1(\vec{e}_1) = \vec{e}\,'_1 = s_1\,\vec{e}_0 + c_1\,\vec{e}_1 \tag{6.94b}$$

$$\boldsymbol{\Lambda}_1(\vec{e}_2) = \vec{e}_2, \qquad \boldsymbol{\Lambda}_1(\vec{e}_3) = \vec{e}_3, \tag{6.94c}$$

with the following abbreviations:

$$c_1 := \cosh\psi_1 = \Gamma_1 \quad \text{and} \quad s_1 := \sinh\psi_1 = \Gamma_1 V_1/c. \tag{6.95}$$

To evaluate the action of $\boldsymbol{\Lambda}_2$, we remark first that the image of $(\vec{e}_\alpha)$ by $\boldsymbol{\Lambda}_1$, namely, $(\vec{e}\,'_0, \vec{e}\,'_1, \vec{e}_2, \vec{e}_3)$, is a semi-adapted basis to $\boldsymbol{\Lambda}_2$, since $\vec{e}\,'_0 \in \Pi_2$ (cf. Sect. 6.6.2). In the hyperplane normal to $\vec{e}\,'_0$, $E_{e'_0}$, the direction of the boost $\boldsymbol{\Lambda}_2$ is that of $\vec{\varepsilon}$. Let us expand this vector on the basis $(\vec{e}\,'_1, \vec{e}_2, \vec{e}_3)$ of $E_{e'_0}$. Since $\vec{\varepsilon} \in \Pi_2$ and $\vec{e}_3 \in \Pi_2^\perp$, $\vec{\varepsilon}$ has no component along $\vec{e}_3$ and we may write, in view of (6.92), $\vec{\varepsilon} = \cos\theta\,\vec{e}\,'_1 \pm \sin\theta\,\vec{e}_2$. We are free to choose $\vec{e}_2$ in order to have a $+$ sign in front of $\sin\theta$, i.e. to ensure that the component of $\vec{\varepsilon}$ along $\vec{e}_2$ is positive (because $\sin\theta \geq 0$, since $\theta \in [0, \pi]$). Indeed we have seen above that $\vec{e}_3$ is defined up to some sign, and we take this opportunity to choose $(\vec{e}_2, \vec{e}_3)$ to fulfil the two conditions: (i) $(\vec{e}\,'_1, \vec{e}_2, \vec{e}_3)$ is a right-handed orthonormal basis of $(E_{e'_0}, \boldsymbol{g})$, and (ii) $\vec{e}_2 \cdot \vec{\varepsilon} \geq 0$. We have then

$$\vec{\varepsilon} = \cos\theta\,\vec{e}\,'_1 + \sin\theta\,\vec{e}_2. \tag{6.96}$$

[16]This line is reduced to a point in Fig. 6.12.

The components of the velocity $\vec{V}_2 = V_2\,\vec{\varepsilon}$ [Eq. (6.91)] of $\boldsymbol{\Lambda}_2$ with respect to $\vec{e}\,'_0$ are then $V_2^i = (V_2\cos\theta, V_2\sin\theta, 0)$. The generic expression (6.71) of a boost in a semi-adapted basis yields then

$$\boldsymbol{\Lambda}_2(\vec{e}\,'_0) = c_2\,\vec{e}\,'_0 + s_2\cos\theta\,\vec{e}\,'_1 + s_2\sin\theta\,\vec{e}_2 \tag{6.97a}$$

$$\boldsymbol{\Lambda}_2(\vec{e}\,'_1) = s_2\cos\theta\,\vec{e}\,'_0 + \left[1+(c_2-1)\cos^2\theta\right]\vec{e}\,'_1 + (c_2-1)\sin\theta\cos\theta\,\vec{e}_2 \tag{6.97b}$$

$$\boldsymbol{\Lambda}_2(\vec{e}_2) = s_2\sin\theta\,\vec{e}\,'_0 + (c_2-1)\sin\theta\cos\theta\,\vec{e}\,'_1 + \left[1+(c_2-1)\sin^2\theta\right]\vec{e}_2 \tag{6.97c}$$

$$\boldsymbol{\Lambda}_2(\vec{e}_3) = \vec{e}_3, \tag{6.97d}$$

with abbreviations analogous to (6.95):

$$c_2 := \cosh\psi_2 = \Gamma_2 \quad \text{and} \quad s_2 := \sinh\psi_2 = \Gamma_2 V_2/c. \tag{6.98}$$

Substituting (6.94a) and (6.94b) for, respectively, $\vec{e}\,'_0$ and $\vec{e}\,'_1$ in the system (6.97), we get

$$\begin{aligned}\boldsymbol{\Lambda}_2\circ\boldsymbol{\Lambda}_1(\vec{e}_0) &= (c_1c_2 + s_1s_2\cos\theta)\,\vec{e}_0 + (s_1c_2 + c_1s_2\cos\theta)\,\vec{e}_1\\ &\quad + s_2\sin\theta\,\vec{e}_2\end{aligned} \tag{6.99a}$$

$$\begin{aligned}\boldsymbol{\Lambda}_2\circ\boldsymbol{\Lambda}_1(\vec{e}_1) &= \left[s_1 + c_1s_2\cos\theta + s_1(c_2-1)\cos^2\theta\right]\vec{e}_0\\ &\quad + \left[c_1 + s_1s_2\cos\theta + c_1(c_2-1)\cos^2\theta\right]\vec{e}_1\\ &\quad + (c_2-1)\sin\theta\cos\theta\,\vec{e}_2\end{aligned} \tag{6.99b}$$

$$\begin{aligned}\boldsymbol{\Lambda}_2\circ\boldsymbol{\Lambda}_1(\vec{e}_2) &= \sin\theta\,[c_1s_2 + s_1(c_2-1)\cos\theta]\,\vec{e}_0\\ &\quad + \sin\theta\,[s_1s_2 + c_1(c_2-1)\cos\theta]\,\vec{e}_1\\ &\quad + \left[1+(c_2-1)\sin^2\theta\right]\vec{e}_2\end{aligned} \tag{6.99c}$$

$$\boldsymbol{\Lambda}_2\circ\boldsymbol{\Lambda}_1(\vec{e}_3) = \vec{e}_3. \tag{6.99d}$$

As previously noticed, the vector $\vec{e}_3$ is invariant under $\boldsymbol{\Lambda}_2\circ\boldsymbol{\Lambda}_1$. The matrix $\Lambda^\alpha{}_\beta$ of $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}_2\circ\boldsymbol{\Lambda}_1$ in the basis $(\vec{e}_\alpha)$ is obtained by storing the four vectors (6.99) in four columns. Let us first observe that if $\theta = 0$, this matrix reduces to (6.82), as it should (case of coplanar boosts). If $\theta \neq 0$ and $\theta \neq \pi$, we notice that the matrix $\Lambda^\alpha{}_\beta$ is not symmetric. Therefore it cannot be the matrix of a Lorentz boost in a semi-adapted basis (cf. Sect. 6.6.2). According to the polar decomposition theorem (Sect. 6.5), $\boldsymbol{\Lambda}$ can be written as the product of a spatial rotation $\boldsymbol{R}$ by a boost $\boldsymbol{S}$ [Eq. (6.57)]:

$$\boxed{\boldsymbol{\Lambda} = \boldsymbol{\Lambda}_2\circ\boldsymbol{\Lambda}_1 = \boldsymbol{S}\circ\boldsymbol{R}}, \tag{6.100}$$

with the plane of $\boldsymbol{R}$ contained in $E_{\boldsymbol{e}_0}$ and the plane of $\boldsymbol{S}$ containing $\vec{\boldsymbol{e}}_0$. The couple $(\boldsymbol{S}, \boldsymbol{R})$ is unique and is determined as follows.

Since the plane of the spatial rotation $\boldsymbol{R}$ is orthogonal to $\vec{\boldsymbol{e}}_0$, we have necessarily $\boldsymbol{R}(\vec{\boldsymbol{e}}_0) = \vec{\boldsymbol{e}}_0$, so that the image of $\vec{\boldsymbol{e}}_0$ by $\boldsymbol{S}$ is $\boldsymbol{\Lambda}_2 \circ \boldsymbol{\Lambda}_1(\vec{\boldsymbol{e}}_0)$. It is given directly by (6.99a):

$$\vec{\boldsymbol{e}}''_0 = \boldsymbol{S}(\vec{\boldsymbol{e}}_0) = (c_1 c_2 + s_1 s_2 \cos\theta)\, \vec{\boldsymbol{e}}_0 + (s_1 c_2 + c_1 s_2 \cos\theta)\, \vec{\boldsymbol{e}}_1 + s_2 \sin\theta\, \vec{\boldsymbol{e}}_2. \tag{6.101}$$

The plane Π of the Lorentz boost $\boldsymbol{S}$ is entirely determined by $\vec{\boldsymbol{e}}_0$ and $\vec{\boldsymbol{e}}''_0$:

$$\Pi = \mathrm{Span}(\vec{\boldsymbol{e}}_0, \vec{\boldsymbol{e}}''_0).$$

The Lorentz factor of $\boldsymbol{S}$ is $\Gamma = -\vec{\boldsymbol{e}}_0 \cdot \vec{\boldsymbol{e}}''_0$ [cf. Eq. (6.64)]. Using (6.101), we get

$$\Gamma = c_1 c_2 + s_1 s_2 \cos\theta, \tag{6.102}$$

that is, in view of (6.95) and (6.98),

$$\boxed{\Gamma = \Gamma_1 \Gamma_2 \left(1 + \frac{V_1 V_2}{c^2}\cos\theta\right)}. \tag{6.103}$$

As a check, if $\theta = 0$, we recover (6.83). The velocity $\vec{\boldsymbol{V}}$ of $\boldsymbol{S}$ with respect to $\vec{\boldsymbol{e}}_0$ is given by (6.66), $\Gamma/c\, \vec{\boldsymbol{V}}$ being the orthogonal projection of $\boldsymbol{S}(\vec{\boldsymbol{e}}_0)$ onto $E_{\boldsymbol{e}_0}$. We read on (6.101) that

$$\frac{\Gamma}{c}\vec{\boldsymbol{V}} = (s_1 c_2 + c_1 s_2 \cos\theta)\, \vec{\boldsymbol{e}}_1 + s_2 \sin\theta\, \vec{\boldsymbol{e}}_2. \tag{6.104}$$

Substituting (6.103) for Γ, we get

$$\boxed{\vec{\boldsymbol{V}} = \frac{1}{1 + \frac{V_1 V_2 \cos\theta}{c^2}} \left[(V_1 + V_2\cos\theta)\, \vec{\boldsymbol{e}}_1 + \frac{V_2}{\Gamma_1} \sin\theta\, \vec{\boldsymbol{e}}_2 \right]}. \tag{6.105}$$

If $\theta = 0$, we obtain $\vec{\boldsymbol{V}} = V \vec{\boldsymbol{e}}_1$ with $V = (V_1 + V_2)/(1 + V_1 V_2/c^2)$; i.e. we recover (6.83). Besides, recalling that $\boldsymbol{S}$ describes the transition from observer $\mathscr{O}'$ to the point particle $\mathscr{P}$ (cf. Table 6.1), $\vec{\boldsymbol{V}}$ is nothing but the velocity of $\mathscr{P}$ relative to $\mathscr{O}'$, denoted by $\vec{\boldsymbol{V}}'$ in Chap. 4. Similarly $-\vec{\boldsymbol{V}}_1$ is the velocity of $\mathscr{O}'$ relative to $\mathscr{O}$, denoted by $\vec{\boldsymbol{U}}$ in Chap. 4, and $\vec{\boldsymbol{V}}_2$ the velocity of $\mathscr{P}$ relative to $\mathscr{O}$, denoted by $\vec{\boldsymbol{V}}$ in Chap. 4. Moreover, $-\vec{\boldsymbol{e}}_1$ is the vector denoted by $\vec{\boldsymbol{e}}'$ in Sect. 5.3.2, and $-\vec{\boldsymbol{e}}'_1$ the vector denoted by $\vec{\boldsymbol{e}}$, so that $U' = V_1$ and the decomposition (5.31) becomes $\vec{\boldsymbol{V}}_2 = -V_\parallel\, \vec{\boldsymbol{e}}'_1 + \vec{\boldsymbol{V}}_\perp$ with, from (6.91) and (6.96), $V_\parallel = -V_2\cos\theta$ and $\vec{\boldsymbol{V}}_\perp = -V_2 \sin\theta\, \vec{\boldsymbol{e}}_2$. On its side, the decomposition (5.33) is written $\vec{\boldsymbol{V}}' = -V'_\parallel\, \vec{\boldsymbol{e}}_1 + \vec{\boldsymbol{V}}'_\perp$.

Equation (6.105) is then in perfect agreement with formulas (5.40a)–(5.40b). Of course (6.103) agrees with (5.40c).

In view of (6.72) and (6.104), we are in position to write the matrix of the boost $\boldsymbol{S}$ in the basis $(\vec{e}_\alpha)$:

$$S^\alpha{}_\beta = \begin{pmatrix} c_1c_2 + s_1s_2\cos\theta & s_1c_2 + c_1s_2\cos\theta & s_2\sin\theta & 0 \\ s_1c_2 + c_1s_2\cos\theta & 1 + \frac{(s_1c_2+c_1s_2\cos\theta)^2}{1+\Gamma} & \frac{s_2\sin\theta(s_1c_2+c_1s_2\cos\theta)}{1+\Gamma} & 0 \\ s_2\sin\theta & \frac{s_2\sin\theta(s_1c_2+c_1s_2\cos\theta)}{1+\Gamma} & 1+\frac{s_2^2\sin^2\theta}{1+\Gamma} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{6.106}$$

Let us determine now the second element of the polar decomposition (6.100) of $\boldsymbol{\Lambda}_2 \circ \boldsymbol{\Lambda}_1$, namely, the spatial rotation $\boldsymbol{R}$. Since $\boldsymbol{\Lambda}_2 \circ \boldsymbol{\Lambda}_1(\vec{e}_3) = \vec{e}_3$ and $\boldsymbol{S}(\vec{e}_3) = \vec{e}_3$, we have $\boldsymbol{R}(\vec{e}_3) = \boldsymbol{S}^{-1} \circ \boldsymbol{\Lambda}_2 \circ \boldsymbol{\Lambda}_1(\vec{e}_3) = \vec{e}_3$. In addition $\boldsymbol{R}(\vec{e}_0) = \vec{e}_0$. Accordingly, the plane of the spatial rotation $\boldsymbol{R}$ is (cf. Sect. 6.4.3)

$$\Pi_{\boldsymbol{R}} = \mathrm{Span}(\vec{e}_1, \vec{e}_2) \tag{6.107}$$

and the matrix of $\boldsymbol{R}$ in the basis $(\vec{e}_\alpha)$ is of the type (6.39), with the permutation $(\vec{e}_1, \vec{e}_2, \vec{e}_3) \to (\vec{e}_3, \vec{e}_1, \vec{e}_2)$:

$$R^\alpha{}_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi_{\mathrm{T}} & -\sin\varphi_{\mathrm{T}} & 0 \\ 0 & \sin\varphi_{\mathrm{T}} & \cos\varphi_{\mathrm{T}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{6.108}$$

with $\varphi_{\mathrm{T}} \in [0, 2\pi[$. The spatial rotation $\boldsymbol{R}$ is called ***Thomas rotation***.

Remark 6.28. Thomas rotation is sometimes called *Wigner rotation* (see, e.g. Ferraro (2007)) or *Thomas–Wigner rotation* (see, e.g. Rhodes and Semon (2004)). φ_{T} is sometimes called *Wigner angle* (Aravind 1997). Thomas rotation is at the root of *Thomas precession*, which we shall study in Sect. 12.5.

Remark 6.29. We have not encountered Thomas rotation in Chap. 5 because we were in fine relating the 4-velocity of particle $\mathscr{P}$, $\vec{\boldsymbol{v}} = \boldsymbol{\Lambda}_2 \circ \boldsymbol{\Lambda}_1(\vec{e}_0)$, to the 4-velocity of observer $\mathscr{O}'$, $\vec{\boldsymbol{u}}' = \vec{e}_0$ (cf. Table 6.1), and the relation between the two 4-velocities is entirely described by the Lorentz boost $\boldsymbol{S}$, as for any two 4-velocities (cf. the grey box at the end of Sect. 6.6.1). The supplementary rotation that appears here is due to the fact that we are actually considering the transformation of entire orthonormal bases and not only of the first element of these bases (the 4-velocity).

6.7.3 Thomas Rotation Angle

We read on (6.108) that

$$\cos\varphi_{\mathrm{T}} = \boldsymbol{R}(\vec{e}_1)\cdot\vec{e}_1 = [\boldsymbol{S}^{-1}\circ\boldsymbol{\Lambda}_2\circ\boldsymbol{\Lambda}_1(\vec{e}_1)]\cdot\vec{e}_1 = [\boldsymbol{\Lambda}_2\circ\boldsymbol{\Lambda}_1(\vec{e}_1)]\cdot\boldsymbol{S}(\vec{e}_1),$$

where the last equality stems from the fact that $\boldsymbol{S}$ is a Lorentz transformation. Substituting (6.99) for $\boldsymbol{\Lambda}_2\circ\boldsymbol{\Lambda}_1(\vec{e}_1)$ and reading $\boldsymbol{S}(\vec{e}_1)$ on (6.106), we get

$$\begin{aligned}\cos\varphi_{\mathrm{T}} = &-\left[s_1 + c_1 s_2\cos\theta + s_1(c_2-1)\cos^2\theta\right](s_1c_2 + c_1s_2\cos\theta)\\ &+\left[c_1 + s_1s_2\cos\theta + c_1(c_2-1)\cos^2\theta\right]\left[1 + \frac{(s_1c_2+c_1s_2\cos\theta)^2}{1+\Gamma}\right]\\ &+(c_2-1)\sin\theta\cos\theta\,\frac{s_2\sin\theta(s_1c_2+c_1s_2\cos\theta)}{1+\Gamma}.\end{aligned}$$

Expanding and taking into account simplifications due to the identities $c_1^2 - s_1^2 = 1$, $c_2^2 - s_2^2 = 1$ and $\sin^2\theta = 1-\cos^2\theta$, we obtain

$$\cos\varphi_{\mathrm{T}} = \frac{1}{1+\Gamma}\left[c_1 + c_2 + s_1s_2\cos\theta + (c_1-1)(c_2-1)\cos^2\theta\right].$$

Thanks to (6.102), we may rewrite this expression as

$$\cos\varphi_{\mathrm{T}} = 1 - \frac{(c_1-1)(c_2-1)}{1+\Gamma}\sin^2\theta. \tag{6.109}$$

Similarly, we read on (6.108) that

$$\sin\varphi_{\mathrm{T}} = \boldsymbol{R}(\vec{e}_1)\cdot\vec{e}_2 = [\boldsymbol{S}^{-1}\circ\boldsymbol{\Lambda}_2\circ\boldsymbol{\Lambda}_1(\vec{e}_1)]\cdot\vec{e}_2 = [\boldsymbol{\Lambda}_2\circ\boldsymbol{\Lambda}_1(\vec{e}_1)]\cdot\boldsymbol{S}(\vec{e}_2).$$

Substituting (6.99) for $\boldsymbol{\Lambda}_2\circ\boldsymbol{\Lambda}_1(\vec{e}_1)$ and reading $\boldsymbol{S}(\vec{e}_2)$ on (6.106), we get

$$\begin{aligned}\sin\varphi_{\mathrm{T}} = &-\left[s_1 + c_1s_2\cos\theta + s_1(c_2-1)\cos^2\theta\right]s_2\sin\theta\\ &+\left[c_1 + s_1s_2\cos\theta + c_1(c_2-1)\cos^2\theta\right]\frac{s_2\sin\theta(s_1c_2+c_1s_2\cos\theta)}{1+\Gamma}\\ &+(c_2-1)\sin\theta\cos\theta\left(1 + \frac{s_2^2\sin^2\theta}{1+\Gamma}\right).\end{aligned}$$

After some expansion and simplifications, we obtain

$$\sin\varphi_{\mathrm{T}} = -\frac{\sin\theta}{1+\Gamma}\left[s_1s_2 + (c_1-1)(c_2-1)\cos\theta\right]. \tag{6.110}$$

Let us reexpress the results (6.109) and (6.110) by making explicit c_1, s_1, c_2 and s_2 via (6.95) and (6.98):

$$\boxed{\cos\varphi_{\mathrm{T}} = 1 - \frac{(\Gamma_1 - 1)(\Gamma_2 - 1)}{1 + \Gamma}\sin^2\theta} \tag{6.111a}$$

$$\boxed{\sin\varphi_{\mathrm{T}} = -\sin\theta\,\frac{\Gamma_1\Gamma_2}{1+\Gamma}\frac{V_1V_2}{c^2}\left(1 + \frac{\Gamma_1}{1+\Gamma_1}\frac{\Gamma_2}{1+\Gamma_2}\frac{V_1V_2}{c^2}\cos\theta\right)}. \tag{6.111b}$$

A few comments are appropriate. First of all, if $\theta = 0$, (6.111) leads to $\varphi_{\mathrm{T}} = 0$, which implies $\boldsymbol{R} = \mathrm{Id}$. $\boldsymbol{\Lambda}_2 \circ \boldsymbol{\Lambda}_1$ is then a pure Lorentz boost, in agreement with the analysis of Sect. 6.7.1. Next, we notice that the property $\sin\theta \geq 0$ (arising from the definition of θ as lying in the interval $[0, \pi]$) implies $\sin\varphi_{\mathrm{T}} \leq 0$, hence

$$\boxed{-\pi \leq \varphi_{\mathrm{T}} \leq 0}. \tag{6.112}$$

Thomas rotation is thus taking place in the clockwise direction in the plane $\Pi_{\boldsymbol{R}} = \mathrm{Span}(\vec{\boldsymbol{e}}_1, \vec{\boldsymbol{e}}_2)$. Let us recall that the vector $\vec{\boldsymbol{e}}_2$ has been chosen so that $\vec{\boldsymbol{e}}\,'_1 \times_{\boldsymbol{e}'_0} \vec{\boldsymbol{e}}_2$ has the same direction as $\vec{\boldsymbol{V}}_1 \times_{\boldsymbol{e}'_0} \vec{\boldsymbol{V}}_2$.

In the particular case where $\theta = \pi/2$ (Π_1 and Π_2 are orthogonal planes), (6.103) reduces to $\Gamma = \Gamma_1\Gamma_2$ and formulas (6.111) simplify to

$$\cos\varphi_{\mathrm{T}} = \frac{\Gamma_1 + \Gamma_2}{1 + \Gamma_1\Gamma_2} \qquad \left(\theta = \frac{\pi}{2}\right) \tag{6.113a}$$

$$\sin\varphi_{\mathrm{T}} = -\frac{\Gamma_1\Gamma_2}{1 + \Gamma_1\Gamma_2}\frac{V_1V_2}{c^2} \qquad \left(\theta = \frac{\pi}{2}\right). \tag{6.113b}$$

By means of (6.103), we can eliminate the angle θ from formula (6.111a). Indeed (6.103) leads to

$$\sin^2\theta = 1 - \cos^2\theta = 1 - c^4\frac{(\Gamma - \Gamma_1\Gamma_2)^2}{(\Gamma_1V_1)^2(\Gamma_2V_2)^2} = 1 - \frac{(\Gamma - \Gamma_1\Gamma_2)^2}{(\Gamma_1^2 - 1)(\Gamma_2^2 - 1)}. \tag{6.114}$$

Substituting this expression for $\sin^2\theta$ in (6.111a), we get

$$\cos\varphi_{\mathrm{T}} = 1 - \frac{1}{1+\Gamma}\left[(\Gamma_1 - 1)(\Gamma_2 - 1) - \frac{(\Gamma - \Gamma_1\Gamma_2)^2}{(\Gamma_1 + 1)(\Gamma_2 + 1)}\right],$$

or equivalently,

$$\boxed{\cos\varphi_{\mathrm{T}} = \frac{(1 + \Gamma + \Gamma_1 + \Gamma_2)^2}{(1+\Gamma)(1+\Gamma_1)(1+\Gamma_2)} - 1.} \tag{6.115}$$

This formula is remarkably symmetric with respect to the three Lorentz factors. In view of the identity $\cos\varphi_{\mathrm{T}} = 2\cos^2(\varphi_{\mathrm{T}}/2) - 1$, we can rewrite it as

$$\cos\frac{\varphi_{\mathrm{T}}}{2} = \frac{1 + \Gamma + \Gamma_1 + \Gamma_2}{\sqrt{2(1+\Gamma)(1+\Gamma_1)(1+\Gamma_2)}}. \tag{6.116}$$

Moreover, we may let appear the rapidities ψ, ψ_1 and ψ_2 of the boosts $\boldsymbol{S}$, $\boldsymbol{\Lambda}_1$ and $\boldsymbol{\Lambda}_2$, which are related to the Lorentz factors via $\Gamma = \cosh\psi$. Thanks to the identity $1 + \cosh\psi = \cosh^2(\psi/2)$, (6.116) becomes then ***Macfarlane formula***:

$$\boxed{\cos\frac{\varphi_{\mathrm{T}}}{2} = \frac{1 + \cosh\psi + \cosh\psi_1 + \cosh\psi_2}{4\cosh(\psi/2)\cosh(\psi_1/2)\cosh(\psi_2/2)}}. \tag{6.117}$$

Similarly, we can rewrite formula (6.111b) for $\sin\varphi_{\mathrm{T}}$ by expressing $V_1 V_2 \cos\theta$ in terms of Γ, Γ_1 and Γ_2, thanks to (6.103). We obtain in this way ***Stapp formula***:

$$\boxed{\sin\varphi_{\mathrm{T}} = -\sin\theta\ \Gamma_1\Gamma_2\frac{V_1V_2}{c^2}\frac{1+\Gamma+\Gamma_1+\Gamma_2}{(1+\Gamma)(1+\Gamma_1)(1+\Gamma_2)}}. \tag{6.118}$$

In view of the identity $\sin\varphi_{\mathrm{T}} = 2\sin(\varphi_{\mathrm{T}}/2)\cos(\varphi_{\mathrm{T}}/2)$ and of (6.116), we get then

$$\boxed{\sin\frac{\varphi_{\mathrm{T}}}{2} = -\sin\theta\ \frac{\Gamma_1\Gamma_2\,V_1V_2/c^2}{\sqrt{2(1+\Gamma)(1+\Gamma_1)(1+\Gamma_2)}}}. \tag{6.119}$$

We can also use the identity $\Gamma_1 V_1 = c\sqrt{\Gamma_1^2 - 1}$ to let appear only Lorentz factors in this formula and obtain the simple expression:

$$\boxed{\sin\frac{\varphi_{\mathrm{T}}}{2} = -\sin\theta\ \sqrt{\frac{(\Gamma_1-1)(\Gamma_2-1)}{2(1+\Gamma)}}}. \tag{6.120}$$

Taking into account the identities $\cosh\psi_1 - 1 = 2\sinh^2(\psi_1/2)$ and $\cosh\psi + 1 = 2\cosh^2(\psi/2)$, (6.120) can be rewritten as

$$\boxed{\sin\frac{\varphi_{\mathrm{T}}}{2} = -\sin\theta\ \frac{\sinh(\psi_1/2)\sinh(\psi_2/2)}{\cosh(\psi/2)}}. \tag{6.121}$$

Remark 6.30. If necessary, we may eliminate $\sin\theta$ from the above formulas and let appear only the Lorentz factors Γ, Γ_1 and Γ_2. It suffices to use (6.114) in the form

$$\sin\theta = \left[\frac{1 + 2\Gamma\Gamma_1\Gamma_2 - \Gamma^2 - \Gamma_1^2 - \Gamma_1^2}{(\Gamma_1^2 - 1)(\Gamma_2^2 - 1)}\right]^{1/2}. \qquad (6.122)$$

Historical note: *The problem of the composition of two noncoplanar Lorentz boosts has been investigated by Arnold Sommerfeld (cf. p. 27) in 1909 (Sommerfeld 1909) and by Émile Borel*[17] *in 1913 (Borel 1913, 1914). These two authors have notably obtained formula (6.103), giving the Lorentz factor of the boost* $\boldsymbol{S}$*. Let us note that they have first written formula (6.102) involving rapidities and hyperbolic trigonometry.*

Regarding Thomas rotation per se, it seems that it has been mentioned first by Émile Borel in 1913 (Borel 1913). In the particular case $\theta = \pi/2$*, it had been noticed as early as 1909 by Arnold Sommerfeld (1909) (cf. the discussion in Belloni and Reina (1986)). Still in this case, formula (6.113b) has been obtained by Paul Langevin (cf. p. 40) in 1926 (Langevin 1926). The expression of the rotation angle has been obtained by Llewellyn Thomas*[18] *in 1926 in the particular case where the velocity* V_2 *is infinitely small (Thomas 1926, 1927). This is the case that intervenes in the study of* Thomas precession*, to be discussed in Sect. 12.5. In a series of works started in 1939 (Wigner 1939, 1957), Eugene P. Wigner*[19] *investigated Lorentz transformations that leave invariant the 4-momentum vector* $\vec{\boldsymbol{p}}$ *of a particle (this vector will be defined in Chap. 9; it is parallel to the 4-velocity of the particle). Wigner stressed that the product of three Lorentz boosts of different planes that leaves* $\vec{\boldsymbol{p}}$ *invariant is not the identity but a spatial rotation in the hyperplane normal to* $\vec{\boldsymbol{p}}$*: this is Thomas rotation, as one can easily see by rewriting (6.100) as* $\boldsymbol{S}^{-1} \circ \boldsymbol{\Lambda}_2 \circ \boldsymbol{\Lambda}_1 = \boldsymbol{R}$*. Since* $\boldsymbol{R}(\vec{\boldsymbol{e}}_0) = \vec{\boldsymbol{e}}_0$*, it is then clear that the combination of the three boosts* $\boldsymbol{S}^{-1}$*,* $\boldsymbol{\Lambda}_2$ *and* $\boldsymbol{\Lambda}_1$ *leaves* $\vec{\boldsymbol{e}}_0$ *invariant (one may always consider* $\vec{\boldsymbol{e}}_0$ *as being related to the 4-momentum of a particle by* $\vec{\boldsymbol{e}}_0 = \vec{\boldsymbol{p}}/\|\vec{\boldsymbol{p}}\|_{\boldsymbol{g}}$*). It seems that the name* Wigner rotation *was given to* $\boldsymbol{R}$ *by the French physicist Amitabha Chakrabarti in 1964 (Chakrabarti 1964). Let us note that Wigner has not given the expression of the angle* φ_{T} *in any of his works, except in the particular case* $\theta = \pi/2$ *(Wigner 1957), where he recovered Langevin formula (6.113b) (Langevin 1926). It is only in 1956, i.e. 30 years after Thomas' study, that an explicit and general formula was given for* φ_{T}*. It has been obtained by the American physicist Henry P. Stapp (1956), in the form (6.118). The same formula has also been derived by the Russian physicist Vladimir Ivanovich Ritus in 1961 (Ritus 1961). Formula (6.117), which involves* $\cos\varphi_{\mathrm{T}}$*, has been obtained by Alan J. Macfarlane in 1962 (Macfarlane 1962).*

[17]**Émile Borel** (1871–1956): French mathematician, pioneer of measure theory and of the study of probabilities, founder of the Henri Poincaré Institute in Paris and cofounder of CNRS.

[18]**Llewellyn H. Thomas** (1903–1992): British physicist, who moved to the USA in 1929; known for his works in atomic physics.

[19]**Eugene P. Wigner** (1902–1995): Hungarian mathematician and theoretical physicist, naturalized American in 1937; he performed fundamentals studies on symmetries in quantum mechanics, as well as in nuclear physics and particle physics; he received the Nobel Prize in Physics in 1963.

6.7.4 Conclusion

Lorentz boosts sharing the same plane form a subgroup of the restricted Lorentz group $\mathrm{SO_o}(3,1)$. Moreover, this subgroup is abelian (i.e. commutative; cf. (6.84)). On the other hand, the set of all Lorentz boosts does not constitute a subgroup of $\mathrm{SO_o}(3,1)$, since the composition of two boosts of different planes does not yield a boost, but the product of a boost by a spatial rotation, the latter being known as *Thomas rotation*. As a consequence, the polar decomposition (6.57) does not generate a "factorization" of the restricted Lorentz group $\mathrm{SO_o}(3,1)$ in two groups. One can even show that such a factorization does not exist, in the sense that $\mathrm{SO_o}(3,1)$ is a simple group (cf. Appendix A). We shall not establish here the simplicity of $\mathrm{SO_o}(3,1)$. The demonstration can be found p. 146 of Sexl and Urbantke (2001).

Historical note: *The simplicity of the restricted Lorentz group* $\mathrm{SO_o}(3,1)$ *has been shown by Eugene P. Wigner (cf. p. 215) in 1939 (Wigner 1939).*

Chapter 7
Lorentz Group as a Lie Group

7.1 Introduction

As the preceding one, this chapter is purely mathematical. Moreover, it can be skipped during a first lecture. It contains mathematics (in particular topology) of a level slightly higher than those involved up to now, which were mostly linear algebra. However, no a priori knowledge about Lie groups is required; all the necessary notions are introduced in the text, on the specific example of the Lorentz group. As for Chap. 6, the definitions of basic algebraic concepts used in this chapter are recalled in Appendix A.

7.2 Lie Group Structure

7.2.1 Definitions

The Lorentz group $\mathrm{O}(3,1)$ is a "continuous" group, in the sense that its elements depend on continuous parameters (for instance, rapidity or rotation angle). More precisely, $\mathrm{O}(3,1)$ is a ***Lie group*** (Choquet-Bruhat et al. 1977; Godement 2004; Eschrig 2011):

- It is a group (for the composition law $\circ$).
- It is also a differentiable manifold.
- The operations $(\boldsymbol{\Lambda}_1, \boldsymbol{\Lambda}_2) \mapsto \boldsymbol{\Lambda}_1 \circ \boldsymbol{\Lambda}_2$ and $\boldsymbol{\Lambda} \mapsto \boldsymbol{\Lambda}^{-1}$ are continuous.[1]

[1]One can show that they are then necessarily differentiable.

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In this definition, there appears the notion of *differentiable manifold*. A (real) ***manifold*** is a Hausdorff[2] second countable[3] topological space $\mathscr{M}$ such that each point has a neighbourhood homeomorphic[4] to an open set of $\mathbb{R}^n$. The integer n, which must be the same for all the neighbourhoods, is called the ***dimension of the manifold***. Broadly speaking, a manifold is a set $\mathscr{M}$ such that on any part that is not too large, one can label the points by n real numbers and consider that $\mathscr{M}$ locally "resembles" $\mathbb{R}^n$. On a larger part of $\mathscr{M}$, the resemblance might be lost; typical examples are the sphere and the torus: both locally resemble $\mathbb{R}^2$, but obviously not globally.

Given an open subset $\mathscr{U} \subset \mathscr{M}$, a ***coordinate system*** or ***chart*** on $\mathscr{U}$ is a homeomorphism:

$$\begin{aligned} \Phi : \mathscr{U} \subset \mathscr{M} &\longrightarrow \Phi(\mathscr{U}) \subset \mathbb{R}^n \\ p \quad &\longmapsto (x^1, \ldots, x^n). \end{aligned}$$

Usually, one needs more than one coordinate system to cover $\mathscr{M}$. An ***atlas*** on $\mathscr{M}$ is a finite set of pairs $(\mathscr{U}_k, \Phi_k)_{1 \le k \le K}$, where K is a nonzero integer, $\mathscr{U}_k$ an open set of $\mathscr{M}$ and Φ_k a chart on $\mathscr{U}_k$, such that the union of all $\mathscr{U}_k$ covers $\mathscr{M}$. A ***differentiable manifold*** (as considered in the above definition of a Lie group) is a manifold equipped with an atlas such that when two charts overlap, the mapping converting the coordinates of one chart to those of the other chart is a differentiable function $\mathbb{R}^n \to \mathbb{R}^n$.

In the case of a Lie group, the dimension n of the group considered as a manifold is called the ***dimension of the Lie group***, and the coordinates are rather called the parameters of the group.

Example 7.1. The group of rotations in the Euclidean plane, SO(2), is a Lie group of dimension 1, since it has only one parameter: the rotation angle. In the three-dimensional Euclidean space, the group of rotations, SO(3), is a Lie group of dimension 3, the 3 parameters being, for instance, the 3 Euler angles defining a rotation [cf. (7.79) below] or the 3 angles (θ, ϕ, φ), where (θ, ϕ) give the direction of the rotation axis and φ is the rotation angle.

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[2] ***Hausdorff*** means *separated*: any two distinct points admit disjoint open neighbourhoods.

[3] A topological space is ***second-countable*** iff there exists a countable family $(\mathscr{U}_k)_{k \in \mathbb{N}}$ of open sets such that any open set of can be written as the union (possibly infinite) of some members of this family. This property excludes "unreasonably large" manifolds. In particular, it allows for a differentiable manifold of dimension n to be embedded smoothly in the Euclidean space $\mathbb{R}^{2n}$ (Whitney theorem).

[4] A ***homeomorphism*** between two topological spaces is a bijective continuous map, whose inverse is continuous as well.

7.2.2 Dimension of the Lorentz group

The Lorentz group O(3, 1) is a Lie group of dimension 6.

Proof. Given an orthonormal basis $(\vec{e}_\alpha)$ of $(E, \boldsymbol{g})$, each element $\boldsymbol{\Lambda}$ of O(3, 1) can be uniquely represented by its matrix $\Lambda = (\Lambda^\alpha{}_\beta)$ with respect to $(\vec{e}_\alpha)$. The set of all real-valued 4×4 matrices is obviously a manifold of dimension 16. The necessary and sufficient condition for $\boldsymbol{\Lambda}$ to belong to O(3, 1) is that its matrix obeys (6.12): ${}^{\mathrm{t}}\Lambda\, \eta\, \Lambda = \eta$. We observe that this relation is an equality between symmetric matrices, whatever the value of Λ: this is obvious for the right-hand side, which is Minkowski matrix $\eta = \mathrm{diag}(-1, 1, 1, 1)$. For the left-hand side, it suffices to consider the transpose:

$$
{}^{\mathrm{t}}\left({}^{\mathrm{t}}\Lambda\, \eta\, \Lambda\right) = {}^{\mathrm{t}}\Lambda \underbrace{{}^{\mathrm{t}}\eta}_{\eta} \underbrace{{}^{\mathrm{t}}\left({}^{\mathrm{t}}\Lambda\right)}_{\Lambda} = {}^{\mathrm{t}}\Lambda\, \eta\, \Lambda,
$$

which shows that it is indeed symmetric. Equation (6.12) is thus a set of 10 independent conditions—the 10 independent components of a symmetric 4×4 matrix. From the initial 16 degrees of freedom of a real 4×4 matrix, there remains then $16 - 10 = 6$ of them. Hence Lorentz matrices can be parametrized by 6 real numbers. Consequently the dimension of the Lie group O(3, 1) is 6. □

It is instructive to give a second proof, by means of the polar decomposition established in Sect. 6.5:

Proof. The polar decomposition theorem states that given a unit timelike vector $\vec{e}_0$, any element $\boldsymbol{\Lambda}$ of the restricted Lorentz group $\mathrm{SO}_o(3, 1)$ can be written in a unique way as the product of a spatial rotation $\boldsymbol{R}$ whose plane is normal to $\vec{e}_0$ by a Lorentz boost $\boldsymbol{S}$ whose plane contains $\vec{e}_0$ [cf. (6.57)]. Three parameters are required to specify $\boldsymbol{R}$ since it is a rotation in the three-dimensional Euclidean space $(E_{\boldsymbol{e}_0}, \boldsymbol{g})$ (cf. Example 7.1). In addition, we have seen in Sect. 6.6.1 that the boost $\boldsymbol{S}$ is entirely defined by a vector $\vec{V}$ in $E_{\boldsymbol{e}_0}$, called the velocity of $\boldsymbol{S}$ with respect to $\vec{e}_0$. Since $\vec{V}$ has 3 independent components, we deduce that 3 parameters are necessary to represent $\boldsymbol{S}$. Therefore, from the polar decomposition, we see that $3+3 = 6$ real parameters are involved to represent $\boldsymbol{\Lambda}$. In other words, the dimension of the Lie group $\mathrm{SO}_o(3, 1)$ is 6. Moreover, we have seen in Sect. 6.3.4 that the full Lorentz group O(3, 1) has four components [cf. (6.19)], one of which being $\mathrm{SO}_o(3, 1)$ and the three others being deduced from $\mathrm{SO}_o(3, 1)$ by the product with one of the inversion operators $\boldsymbol{I}$, $\boldsymbol{P}$ and $\boldsymbol{T}$. We conclude that O(3, 1) has the same dimension than $\mathrm{SO}_o(3, 1)$, namely, 6. □

The subgroup formed by the Lorentz boosts having the same plane (cf. Sect. 6.7.1) is a Lie group of dimension 1, the only parameter being the Lorentz factor Γ, or equivalently the rapidity ψ.

O(3, 1) is actually a subgroup of the Lie group GL(E) formed by all invertible endomorphisms of E (cf. Appendix A). The dimension of GL(E) is 16. Similarly, SO(3, 1) and $\mathrm{SO_o}(3,1)$ are subgroups of the Lie group SL(E) formed by all the endomorphisms of E whose determinant is 1. The dimension of SL(E) is 15.

7.2.3 *Topology of the Lorentz Group*

According to the polar decomposition with respect to a unit timelike vector $\vec{e}_0$ (cf. Sect. 6.5), any element of the restricted Lorentz group $\mathrm{SO_o}(3,1)$ can be written in a unique way $\boldsymbol{\Lambda} = \boldsymbol{S} \circ \boldsymbol{R}$, where $\boldsymbol{R}$ is a spatial rotation whose plane is normal to $\vec{e}_0$ and $\boldsymbol{S}$ is a boost whose plane contains $\vec{e}_0$. We deduce immediately that the parameters of the restricted Lorentz group $\mathrm{SO_o}(3,1)$ can be chosen as follows:

$$(\theta_1, \phi_1, \varphi, \theta_2, \phi_2, \psi) \in [0,\pi] \times [0, 2\pi[\times [0,\pi] \times [0,\pi] \times [0, 2\pi[\times [0, \infty[, \qquad (7.1)$$

where (θ_1, ϕ_1) define the axis of the rotation $\boldsymbol{R}$ in $E_{\boldsymbol{e}_0}$ ($\theta_1 \in [0,\pi]$ being the colatitude and $\phi_1 \in [0, 2\pi[$ the azimuth), φ is the angle of $\boldsymbol{R}$: $\varphi \in [0,\pi]$ since a rotation of angle $\varphi \in]\pi, 2\pi[$ is equal to a rotation of angle $2\pi - \varphi$ in the opposite direction $(\pi - \theta_1, 2\pi - \phi_1)$. Moreover, if $\varphi = \pi$, the points of parameters (θ_1, ϕ_1) and $(\pi - \theta_1, 2\pi - \varphi_1)$ must be identified, for they correspond to the same rotation. Besides, (θ_2, ϕ_2) define the axis of the boost $\boldsymbol{S}$ (intersection of its plane with $E_{\boldsymbol{e}_0}$, $\theta_2 \in [0,\pi]$ being the colatitude $\phi_2 \in [0, 2\pi[$ the azimuth) and $\psi \in [0, \infty[$ is the rapidity of $\boldsymbol{S}$. Since $\psi \in [0,\infty[$, we deduce from the parametrization (7.1) that $\mathrm{SO_o}(3,1)$ is not compact, which implies that O(3, 1) is not compact as well:

> The Lorentz group O(3, 1) and the restricted Lorentz group $\mathrm{SO_o}(3,1)$ are non-compact spaces.

Remark 7.1. The conclusion would of course have been the same if we had chosen as parameter the Lorentz factor $\Gamma \in [1, \infty[$ instead of ψ, or the velocity $V \in [0, c[$, the interval $[0, c[$ being non-compact for it is not closed. On the opposite, the rotation group of the three-dimensional Euclidean space, SO(3), is a compact space.

Remark 7.2. We have seen in Sect. 6.7.1 that the set of all boosts having the same plane is a subgroup of $\mathrm{SO_o}(3,1)$. It is a Lie group of dimension one and non-compact, since it can be parametrized by the rapidity $\psi \in [0, +\infty[$. Now a theorem about Lie groups stipulates that any non-compact Lie group of dimension one admits a parametrization that makes it diffeomorphic to $(\mathbb{R}, +)$ (group formed by the set of real numbers equipped with the addition) (Lévy-Leblond and Provost 1979). In the present case, the parametrization that exhibits this diffeomorphism is rapidity, thanks to the addition law (6.85).

Beside the non-compactness, another consequence of (7.1) is

> The restricted Lorentz group $\mathrm{SO_o}(3, 1)$ is connected.

This follows immediately from the connectedness of the space $[0, \pi] \times [0, 2\pi[\times[0, \pi]\times[0, \pi]\times[0, 2\pi[\times[0, \infty[$. On the other hand, $\mathrm{O}(3, 1)$ is not connected: along with (6.26), the decomposition (6.19) shows that

> The Lorentz group $\mathrm{O}(3, 1)$ has four connected components:
>
> $$\mathrm{SO_o}(3,1), \quad \boldsymbol{I}\,\mathrm{SO_o}(3,1) = \mathrm{SO_a}(3,1),$$
>
> $$\boldsymbol{T}\,\mathrm{SO_o}(3,1) = \mathrm{O}^-_{\mathrm{a}}(3,1) \quad \text{and} \quad \boldsymbol{P}\,\mathrm{SO_o}(3,1) = \mathrm{O}^-_{\mathrm{o}}(3,1),$$
>
> where the notations are those of Sect. 6.3.4. Note that only $\mathrm{SO_o}(3, 1)$ is a Lie subgroup. Furthermore, $\mathrm{SO_o}(3, 1)$ is the connected component that contains the identity.

7.3 Generators and Lie Algebra

7.3.1 Infinitesimal Lorentz Transformations

Let us focus on *infinitesimal* Lorentz transformations, i.e. transformations infinitely close to the identity. They are necessary in the connected component of $\mathrm{O}(3, 1)$ that contains the identity, namely, $\mathrm{SO_o}(3, 1)$. They are thus restricted Lorentz transformations. Let us write such a transformation as

$$\boxed{\boldsymbol{\Lambda} = \mathrm{Id} + \varepsilon \boldsymbol{L}}, \tag{7.2}$$

where $\varepsilon \in \mathbb{R}$ is a small parameter and $\boldsymbol{L} \in \mathscr{L}(E)$, $\mathscr{L}(E)$ being the vector space of all endomorphisms on E (i.e. linear maps $E \to E$). From the definition (6.2) of a Lorentz transformation, one has the successive equivalences:

$$
\begin{aligned}
\boldsymbol{\Lambda} \in \mathrm{O}(3,1) &\iff \forall(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}) \in E^2, \quad \boldsymbol{\Lambda}(\vec{\boldsymbol{u}}) \cdot \boldsymbol{\Lambda}(\vec{\boldsymbol{v}}) = \vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}} \\
&\iff \forall(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}) \in E^2, \quad [\vec{\boldsymbol{u}} + \varepsilon \boldsymbol{L}(\vec{\boldsymbol{u}})] \cdot [\vec{\boldsymbol{v}} + \varepsilon \boldsymbol{L}(\vec{\boldsymbol{v}})] = \vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}} \\
&\iff \forall(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}) \in E^2, \quad \vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}} + \varepsilon[\vec{\boldsymbol{u}} \cdot \boldsymbol{L}(\vec{\boldsymbol{v}}) + \vec{\boldsymbol{v}} \cdot \boldsymbol{L}(\vec{\boldsymbol{u}})] = \vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}} \\
\boldsymbol{\Lambda} \in \mathrm{O}(3,1) &\iff \forall(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}) \in E^2, \quad \vec{\boldsymbol{u}} \cdot \boldsymbol{L}(\vec{\boldsymbol{v}}) = -\vec{\boldsymbol{v}} \cdot \boldsymbol{L}(\vec{\boldsymbol{u}}), \qquad (7.3)
\end{aligned}
$$

where to get the last but one line, we have to drop the second-order term in ε. Introducing the set

$$
\boxed{\mathrm{so}(3,1) := \left\{ \boldsymbol{L} \in \mathscr{L}(E) \ / \ \forall(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}) \in E^2, \ \ \vec{\boldsymbol{u}} \cdot \boldsymbol{L}(\vec{\boldsymbol{v}}) = -\vec{\boldsymbol{v}} \cdot \boldsymbol{L}(\vec{\boldsymbol{u}}) \right\}}, \qquad (7.4)
$$

we conclude that for any infinitesimal ε,

$$
\boxed{\mathrm{Id} + \varepsilon \boldsymbol{L} \in \mathrm{O}(3,1) \iff \boldsymbol{L} \in \mathrm{so}(3,1)}. \qquad (7.5)
$$

Note the lower-case letters in the symbol so(3, 1), not to be confused with SO(3, 1) (the proper Lorentz group introduced in Sect. 6.3.1). By virtue of (7.5), one may associate with any element of so(3, 1) an infinitesimal Lorentz transformation. We shall see below that one can even associate a finite Lorentz transformation, via the so-called exponential map.

7.3.2 Structure of Lie Algebra

It is clear that so(3, 1) is a vector subspace of $\mathscr{L}(E)$. Indeed, if $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ both satisfy (7.3), then for any $\alpha \in \mathbb{R}$, $\alpha \boldsymbol{L}_1 + \boldsymbol{L}_2$ satisfies (7.3) too. Let us determine the dimension of this vector space. The dimension of $\mathscr{L}(E)$ is 16 (for one can identify each element of $\mathscr{L}(E)$ by its matrix in a given basis and the dimension of the vector space formed by all real 4×4 matrices is 16). Let us express the condition $\boldsymbol{L} \in \mathrm{so}(3,1)$ in terms of the matrix $L = (L^{\alpha}{}_{\beta})$ of $\boldsymbol{L}$ with respect to an orthonormal basis $(\vec{\boldsymbol{e}}_\alpha)$. The matrix of $\boldsymbol{g}$ in $(\vec{\boldsymbol{e}}_\alpha)$ being Minkowski matrix $\eta = (\eta_{\alpha\beta})$, the condition (7.3) becomes $\eta_{\alpha\beta} u^\alpha L^{\beta}{}_{\gamma} v^\gamma = -\eta_{\alpha\beta} v^\alpha L^{\beta}{}_{\gamma} u^\gamma$, i.e. $\eta_{\alpha\beta} L^{\beta}{}_{\gamma} u^\alpha v^\gamma = -\eta_{\gamma\beta} L^{\beta}{}_{\alpha} u^\alpha v^\gamma$; hence, $\eta_{\alpha\beta} L^{\beta}{}_{\gamma} = -\eta_{\gamma\beta} L^{\beta}{}_{\alpha}$. We recognize in $\eta_{\alpha\beta} L^{\beta}{}_{\gamma}$ the matrix product of η by L; the condition $\boldsymbol{L} \in \mathrm{so}(3,1)$ is thus equivalent to

$$
\boxed{\eta\, L = -{}^{\mathrm{t}}(\eta\, L)}. \qquad (7.6)
$$

In other words, $\boldsymbol{L} \in \mathrm{so}(3,1)$ iff the matrix $\eta\, L$ is antisymmetric. This gives 10 constraints on the components of $\eta\, L$, and thus 10 constraints on the components of $L = \eta\,(\eta\, L)$ (recall that $\eta^{-1} = \eta$). We conclude that the dimension of the vector space so(3, 1) is $16 - 10 = 6$.

Besides, we know that the vector space $\mathscr{L}(E)$ equipped with the composition law $\circ$ is an algebra over $\mathbb{R}$ (cf. Appendix A). It is then natural to ask whether so(3, 1) is a subalgebra of $(\mathscr{L}(E), \circ)$, in addition of being a vector subspace of $\mathscr{L}(E)$. The answer is no, for so(3, 1) is not stable under $\circ$: if $\boldsymbol{L}_1 \in \mathrm{so}(3,1)$ and $\boldsymbol{L}_2 \in \mathrm{so}(3,1)$, in general $\boldsymbol{L}_1 \circ \boldsymbol{L}_2 \notin \mathrm{so}(3,1)$. Indeed, applying the property (7.3) to successively $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$, we get

$$\forall(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}) \in E^2, \quad \vec{\boldsymbol{u}} \cdot \boldsymbol{L}_1 \circ \boldsymbol{L}_2(\vec{\boldsymbol{v}}) = -\boldsymbol{L}_2(\vec{\boldsymbol{v}}) \cdot \boldsymbol{L}_1(\vec{\boldsymbol{u}}) = \vec{\boldsymbol{v}} \cdot \boldsymbol{L}_2 \circ \boldsymbol{L}_1(\vec{\boldsymbol{u}}), \tag{7.7}$$

which shows that a priori $\boldsymbol{L}_1 \circ \boldsymbol{L}_2$ does not satisfy (7.3). On the other hand, switching the roles of $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ in (7.7) and subtracting the result from (7.7), we get

$$\forall(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}) \in E^2, \quad \vec{\boldsymbol{u}} \cdot [\boldsymbol{L}_1, \boldsymbol{L}_2](\vec{\boldsymbol{v}}) = -\vec{\boldsymbol{v}} \cdot [\boldsymbol{L}_1, \boldsymbol{L}_2](\vec{\boldsymbol{u}}), \tag{7.8}$$

where we have introduced the ***commutator*** of $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$:

$$\boxed{[\boldsymbol{L}_1, \boldsymbol{L}_2] := \boldsymbol{L}_1 \circ \boldsymbol{L}_2 - \boldsymbol{L}_2 \circ \boldsymbol{L}_1}. \tag{7.9}$$

$[\boldsymbol{L}_1, \boldsymbol{L}_2]$ is an endomorphism of E and (7.8) shows that

$$\forall(\boldsymbol{L}_1, \boldsymbol{L}_2) \in \mathrm{so}(3,1) \times \mathrm{so}(3,1), \quad [\boldsymbol{L}_1, \boldsymbol{L}_2] \in \mathrm{so}(3,1). \tag{7.10}$$

so(3, 1) is thus stable under the commutator. Moreover, the commutator obeys the following three properties, where $\boldsymbol{L}_1$, $\boldsymbol{L}_2$ and $\boldsymbol{L}_3$ are generic elements of so(3, 1) and $\alpha \in \mathbb{R}$:

- $[\,,]$ is antisymmetric:
$$[\boldsymbol{L}_1, \boldsymbol{L}_2] = -[\boldsymbol{L}_2, \boldsymbol{L}_1]; \tag{7.11}$$
- $[\,,]$ is bilinear, i.e. linear with respect to each of its arguments:
$$[\alpha \boldsymbol{L}_1 + \boldsymbol{L}_2, \boldsymbol{L}_3] = \alpha[\boldsymbol{L}_1, \boldsymbol{L}_3] + [\boldsymbol{L}_2, \boldsymbol{L}_3]. \tag{7.12}$$
- $[\,,]$ satisfies ***Jacobi identity***:
$$[\boldsymbol{L}_1, [\boldsymbol{L}_2, \boldsymbol{L}_3]] + [\boldsymbol{L}_2, [\boldsymbol{L}_3, \boldsymbol{L}_1]] + [\boldsymbol{L}_3, [\boldsymbol{L}_1, \boldsymbol{L}_2]] = 0. \tag{7.13}$$

  This last identity can be checked easily from the definition (7.9) of the commutator.

Any vector space endowed with an internal composition law $[\,,]$ that satisfies properties (7.11), (7.12) and (7.13) is called a ***Lie algebra***. The internal composition law $[\,,]$ is called ***Lie bracket***. The space so(3, 1), endowed with

$[\,,\,]$ as defined by (7.9), is thus a Lie algebra. It is called the ***Lie algebra of the Lorentz group***, or simply ***Lorentz algebra***. Note that the dimension of $\mathrm{so}(3,1)$, as a vector space over $\mathbb{R}$, is the same as that of the Lorentz group $O(3,1)$, as a manifold over $\mathbb{R}$, namely, 6. Actually this property holds for any Lie algebra associated with a Lie group.

Remark 7.3. Thanks to the bilinearity of the Lie bracket, a Lie algebra is an algebra over $\mathbb{R}$ for the "product" $[\,,\,]$ (cf. Appendix A), but this algebra is not associative; for in general, $[\boldsymbol{L}_1,[\boldsymbol{L}_2,\boldsymbol{L}_3]] \neq [[\boldsymbol{L}_1,\boldsymbol{L}_2],\boldsymbol{L}_3]$.

Remark 7.4. The reader familiar with differentiable manifolds will have noticed that $\mathrm{so}(3,1)$ is the tangent space of the manifold $O(3,1)$ at the point Id [cf. Eq. (7.2)]. It is thus not surprising that the dimension of $\mathrm{so}(3,1)$ is the same as that of the basis manifold.

Remark 7.5. In group theory, the ***commutator*** of two elements a and b is the element $aba^{-1}b^{-1}$. It is equal to the identity element iff a and b commute. In the group $(\mathrm{SO}_o(3,1),\circ)$, the commutator of two Lorentz transformations $\boldsymbol{\Lambda}_1$ and $\boldsymbol{\Lambda}_2$ is thus $\boldsymbol{\Lambda}_1 \circ \boldsymbol{\Lambda}_2 \circ \boldsymbol{\Lambda}_1^{-1} \circ \boldsymbol{\Lambda}_2^{-1}$. If $\boldsymbol{\Lambda}_1$ and $\boldsymbol{\Lambda}_2$ are two infinitesimal transformations of the type (7.2), $\boldsymbol{\Lambda}_1 = \mathrm{Id} + \varepsilon \boldsymbol{L}_1$, $\boldsymbol{\Lambda}_2 = \mathrm{Id} + \varepsilon \boldsymbol{L}_2$, the commutator of $\boldsymbol{\Lambda}_1$ and $\boldsymbol{\Lambda}_2$, in the group-theoretical sense, is related to the commutator of $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$, in the Lie algebra sense [i.e. defined by (7.9)], by

$$\boldsymbol{\Lambda}_1 \circ \boldsymbol{\Lambda}_2 \circ \boldsymbol{\Lambda}_1^{-1} \circ \boldsymbol{\Lambda}_2^{-1} = \mathrm{Id} + \varepsilon^2 [\boldsymbol{L}_1,\boldsymbol{L}_2] + O(\varepsilon^3). \tag{7.14}$$

To show it, it suffices to observe that at the second order in ε, $\boldsymbol{\Lambda}_1^{-1} = \mathrm{Id} - \varepsilon \boldsymbol{L}_1 + \varepsilon^2 \boldsymbol{L}_1 \circ \boldsymbol{L}_1$ (idem for $\boldsymbol{\Lambda}_2^{-1}$) and to compute $\boldsymbol{\Lambda}_1 \circ \boldsymbol{\Lambda}_2 \circ \boldsymbol{\Lambda}_1^{-1} \circ \boldsymbol{\Lambda}_2^{-1}$, still at the second order in ε. In particular, if $\boldsymbol{\Lambda}_1$ and $\boldsymbol{\Lambda}_2$ commute, $[\boldsymbol{L}_1,\boldsymbol{L}_2] = 0$.

7.3.3 Generators

Let us look for a basis of the vector space $\mathrm{so}(3,1)$. To this aim, let us employ a matrix representation of the elements of $\mathrm{so}(3,1)$. Let $(\vec{\boldsymbol{e}}_\alpha)$ be an orthonormal basis of $(E,\boldsymbol{g})$. We have seen above that an endomorphism $\boldsymbol{L}$ belongs to $\mathrm{so}(3,1)$ iff its matrix $L = (L^\alpha{}_\beta)$ with respect to $(\vec{\boldsymbol{e}}_\alpha)$ is such that $\eta\, L$ is an antisymmetric matrix [Eq. (7.6)], i.e. iff there exist 6 real numbers k_1, k_2, k_3, j_1, j_2 and j_3 such that

$$(\eta\, L)^\alpha{}_\beta = \begin{pmatrix} 0 & -k_1 & -k_2 & -k_3 \\ k_1 & 0 & -j_3 & j_2 \\ k_2 & j_3 & 0 & -j_1 \\ k_3 & -j_2 & j_1 & 0 \end{pmatrix}.$$

The matrix L is then obtained via $L = \eta^{-1}\,(\eta\, L)$. Since $\eta^{-1} = \eta = \mathrm{diag}(-1, 1, 1, 1)$, we get

$$L^{\alpha}{}_{\beta} = \begin{pmatrix} 0 & k_1 & k_2 & k_3 \\ k_1 & 0 & -j_3 & j_2 \\ k_2 & j_3 & 0 & -j_1 \\ k_3 & -j_2 & j_1 & 0 \end{pmatrix}. \tag{7.15}$$

Note the change in the signs of the k_i terms in the first row with respect to ηL; as a result, the matrix L is neither antisymmetric nor symmetric. In view of (7.15), a basis of so(3, 1) is formed by the 6 endomorphisms $\boldsymbol{K}_1$, $\boldsymbol{K}_2$, $\boldsymbol{K}_3$, $\boldsymbol{J}_1$, $\boldsymbol{J}_2$ and $\boldsymbol{J}_3$ whose matrices in the orthonormal basis $(\vec{\boldsymbol{e}}_\alpha)$ are

$$(K_1)^{\alpha}{}_{\beta} = \begin{pmatrix} 0&1&0&0\\ 1&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}, \quad (K_2)^{\alpha}{}_{\beta} = \begin{pmatrix} 0&0&1&0\\ 0&0&0&0\\ 1&0&0&0\\ 0&0&0&0 \end{pmatrix}, \tag{7.16a}$$

$$(K_3)^{\alpha}{}_{\beta} = \begin{pmatrix} 0&0&0&1\\ 0&0&0&0\\ 0&0&0&0\\ 1&0&0&0 \end{pmatrix}, \quad (J_1)^{\alpha}{}_{\beta} = \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&-1\\ 0&0&1&0 \end{pmatrix}, \tag{7.16b}$$

$$(J_2)^{\alpha}{}_{\beta} = \begin{pmatrix} 0&0&0&0\\ 0&0&0&1\\ 0&0&0&0\\ 0&-1&0&0 \end{pmatrix}, \quad (J_3)^{\alpha}{}_{\beta} = \begin{pmatrix} 0&0&0&0\\ 0&0&-1&0\\ 0&1&0&0\\ 0&0&0&0 \end{pmatrix}. \tag{7.16c}$$

Accordingly, any element $\boldsymbol{L}$ of so(3, 1) can be written in a unique way as

$$\boldsymbol{L} = k_1\boldsymbol{K}_1 + k_2\boldsymbol{K}_2 + k_3\boldsymbol{K}_3 + j_1\boldsymbol{J}_1 + j_2\boldsymbol{J}_2 + j_3\boldsymbol{J}_3, \tag{7.17}$$

with $(k_1, k_2, k_3, j_1, j_2, j_3) \in \mathbb{R}^6$. The endomorphisms $\boldsymbol{K}_1$, $\boldsymbol{K}_2$, $\boldsymbol{K}_3$, $\boldsymbol{J}_1$, $\boldsymbol{J}_2$ and $\boldsymbol{J}_3$ are called the ***generators of the Lorentz group*** associated with the orthonormal basis $(\vec{\boldsymbol{e}}_\alpha)$. Note that the matrices of the $\boldsymbol{K}_i$'s are symmetric and those of the $\boldsymbol{J}_i$'s are antisymmetric. Moreover, the action of $\boldsymbol{J}_i$ is nothing but the cross product by $\vec{\boldsymbol{e}}_i$ in $E_{\boldsymbol{e}_0}$. Indeed, from the definition (3.46) of the cross product, one has, for any vector $\vec{\boldsymbol{v}} = v^\alpha \vec{\boldsymbol{e}}_\alpha \in E$,

$$\begin{aligned} \vec{\boldsymbol{e}}_1 \times_{\boldsymbol{e}_0} \vec{\boldsymbol{v}} &= \vec{\boldsymbol{\epsilon}}(\vec{\boldsymbol{e}}_0, \vec{\boldsymbol{e}}_1, \vec{\boldsymbol{v}}, .\,) = v^\alpha \vec{\boldsymbol{\epsilon}}(\vec{\boldsymbol{e}}_0, \vec{\boldsymbol{e}}_1, \vec{\boldsymbol{e}}_\alpha, .\,) \\ &= v^2 \vec{\boldsymbol{\epsilon}}(\vec{\boldsymbol{e}}_0, \vec{\boldsymbol{e}}_1, \vec{\boldsymbol{e}}_2, .\,) + v^3 \vec{\boldsymbol{\epsilon}}(\vec{\boldsymbol{e}}_0, \vec{\boldsymbol{e}}_1, \vec{\boldsymbol{e}}_3, .\,) = v^2\, \vec{\boldsymbol{e}}_3 - v^3 \vec{\boldsymbol{e}}_2 = \boldsymbol{J}_1(\vec{\boldsymbol{v}}). \end{aligned}$$

We establish analogous formulas for $\boldsymbol{J}_2$ and $\boldsymbol{J}_3$, so that we can conclude

$$\forall \vec{v} \in E, \quad \boxed{\boldsymbol{J}_i(\vec{v}) = \vec{e}_i \times_{\boldsymbol{e}_0} \vec{v}}, \qquad 1 \leq i \leq 3. \tag{7.18}$$

A condensed writing of the generators is obtained by introducing the 6 endomorphisms defined by

$$\boxed{\mathscr{J}_{\alpha\beta} := \langle \underline{e}_\alpha, \cdot \rangle \, \vec{e}_\beta - \langle \underline{e}_\beta, \cdot \rangle \, \vec{e}_\alpha}, \quad \alpha, \beta \in \{0, 1, 2, 3\} \, / \, \alpha < \beta. \tag{7.19}$$

Let us recall that $\underline{e}_\alpha$ is the linear form associated with the vector $\vec{e}_\alpha$ by metric duality (cf. Sect. 1.6) and the above notation means that for any vector $\vec{v} \in E$, $\mathscr{J}_{\alpha\beta}(\vec{v})$ is the vector of E defined by $\mathscr{J}_{\alpha\beta}(\vec{v}) = \langle \underline{e}_\alpha, \vec{v} \rangle \, \vec{e}_\beta - \langle \underline{e}_\beta, \vec{v} \rangle \, \vec{e}_\alpha = (\vec{e}_\alpha \cdot \vec{v}) \, \vec{e}_\beta - (\vec{e}_\beta \cdot \vec{v}) \, \vec{e}_\alpha$. Since $\langle \underline{e}_0, \vec{e}_0 \rangle = -1$, $\langle \underline{e}_0, \vec{e}_i \rangle = 0$ and $\langle \underline{e}_i, \vec{e}_j \rangle = \delta_{ij}$, one checks easily that

$$\boldsymbol{K}_i = -\mathscr{J}_{0i}, \quad \boldsymbol{J}_1 = \mathscr{J}_{23}, \quad \boldsymbol{J}_2 = -\mathscr{J}_{13}, \quad \boldsymbol{J}_3 = \mathscr{J}_{12}. \tag{7.20}$$

Let us pick as infinitesimal Lorentz transformation a spatial rotation $\boldsymbol{R}$ of angle $\delta\varphi \ll 1$ and axis $\vec{n} = n^i \vec{e}_i$ in $E_{\boldsymbol{e}_0}$. Expanding Rodrigues formula (6.41) at first order in $\delta\varphi$, we get, for any $\vec{v} \in E$,

$$\boldsymbol{R}(\vec{v}) = \vec{v} + \delta\varphi \, n^i \, \vec{e}_i \times_{\boldsymbol{e}_0} \vec{v}.$$

Now (7.18) gives $\vec{e}_i \times_{\boldsymbol{e}_0} \vec{v} = \boldsymbol{J}_i(\vec{v})$; hence,

$$\boxed{\boldsymbol{R} = \mathrm{Id} + \delta\varphi \, n^i \, \boldsymbol{J}_i}. \tag{7.21}$$

This expression is of the type (7.2), $\delta\varphi$ playing the role of the small parameter ε. We conclude thus that

> For $i \in \{1, 2, 3\}$, $\boldsymbol{J}_i$ is the generator of spatial rotations in the plane orthogonal to $\vec{e}_0$ and $\vec{e}_i$.

Let us now choose as infinitesimal Lorentz transformation a boost $\boldsymbol{S}$ of rapidity $\delta\psi \ll 1$ and plane $\mathrm{Span}(\vec{e}_0, \vec{n})$, with $\vec{n} = n^i \vec{e}_i$ being a unit vector. Its matrix is given by (6.72), with $\Gamma = \cosh(\delta\psi)$, $V^i = V n^i$, $V = c \tanh \delta\psi$. At first order in $\delta\psi$, we get

$$S^\alpha{}_\beta = \left(\begin{array}{c|c} 1 & \delta\psi \, n^j \\ \hline \delta\psi \, n^i & \delta^i{}_j \end{array} \right).$$

Comparing with the matrices K_1, K_2 and K_3 given by (7.16), we conclude that

$$\boxed{\boldsymbol{S} = \mathrm{Id} + \delta\psi\, n^i \boldsymbol{K}_i}. \tag{7.22}$$

This expression is of the type (7.2), $\delta\psi$ playing the role of the small parameter ε. Hence

> For $i \in \{1, 2, 3\}$, $\boldsymbol{K}_i$ is the generator of Lorentz boosts of plane $\mathrm{Span}(\vec{\boldsymbol{e}}_0, \vec{\boldsymbol{e}}_i)$.

Finally, let us consider the null rotation of plane $\mathrm{Span}(\vec{\boldsymbol{e}}_0 + \vec{\boldsymbol{e}}_1, \vec{\boldsymbol{e}}_3)$ and infinitesimal parameter $\delta\alpha$ ($|\delta\alpha| \ll 1$). Its matrix is given by (6.50); at first order in $\delta\alpha$, it reduces to

$$\Lambda^{\alpha}{}_{\beta} = \begin{pmatrix} 1 & 0 & 2\delta\alpha & 0 \\ 0 & 1 & 2\delta\alpha & 0 \\ 2\delta\alpha & -2\delta\alpha & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Comparing with matrices (7.16), we obtain

$$\boxed{\boldsymbol{\Lambda} = \mathrm{Id} + 2\delta\alpha(\boldsymbol{K}_2 - \boldsymbol{J}_3)}. \tag{7.23}$$

Remark 7.6. We may check that the null vector $\vec{\boldsymbol{\ell}} := \vec{\boldsymbol{e}}_0 + \vec{\boldsymbol{e}}_1$ is invariant under the transformation (7.23), as it should:

$$\begin{aligned} \boldsymbol{\Lambda}(\vec{\boldsymbol{\ell}}) &= \vec{\boldsymbol{\ell}} + 2\delta\alpha[\boldsymbol{K}_2(\vec{\boldsymbol{\ell}}) - \boldsymbol{J}_3(\vec{\boldsymbol{\ell}})] \\ &= \vec{\boldsymbol{\ell}} + 2\delta\alpha\big[\underbrace{\boldsymbol{K}_2(\vec{\boldsymbol{e}}_0)}_{\vec{\boldsymbol{e}}_2} + \underbrace{\boldsymbol{K}_2(\vec{\boldsymbol{e}}_1)}_{0} - \underbrace{\boldsymbol{J}_3(\vec{\boldsymbol{e}}_0)}_{0} - \underbrace{\boldsymbol{J}_3(\vec{\boldsymbol{e}}_1)}_{\vec{\boldsymbol{e}}_2}\big] = \vec{\boldsymbol{\ell}}. \end{aligned}$$

7.3.4 Link with the Variation of a Local Frame

It is instructive to make the link between what precedes and the law established in Chap. 3 for the variation of the local frame $(\vec{\boldsymbol{e}}_\alpha)$ of an observer $\mathscr{O}$. Let $\mathrm{d}t$ be an infinitesimal increment of $\mathscr{O}$'s proper time t. The transition from $(\vec{\boldsymbol{e}}_\alpha(t))$ to $(\vec{\boldsymbol{e}}_\alpha(t + \mathrm{d}t))$ along $\mathscr{O}$'s worldline is a change of orthonormal basis (cf. Fig. 3.13). It corresponds thus to a unique Lorentz transformation; moreover, this transformation is infinitesimal. From the results of Sect. 7.3.1, we may write

$$\vec{\boldsymbol{e}}_\alpha(t + \mathrm{d}t) = [\mathrm{Id} + \mathrm{d}t\, \boldsymbol{L}]\, \vec{\boldsymbol{e}}_\alpha(t), \tag{7.24}$$

with $\boldsymbol{L} \in \mathrm{so}(3,1)$ and $\mathrm{d}t$ playing the role of the small parameter ε. Note that a priori $\boldsymbol{L}$ depends upon t. We deduce immediately from (7.24) that $\boldsymbol{L}(\vec{\boldsymbol{e}}_\alpha(t))$ is nothing but the derivative of $\vec{\boldsymbol{e}}_\alpha(t)$ with respect to t: $\boldsymbol{L}\,(\vec{\boldsymbol{e}}_\alpha(t)) = \mathrm{d}\vec{\boldsymbol{e}}_\alpha/\mathrm{d}t$. The endomorphism $\boldsymbol{L}$ is thus fully determined by formula (3.52). Substituting $\vec{\boldsymbol{e}}_0$ for $\vec{\boldsymbol{u}}$ in it yields

$$\boldsymbol{L}\,(\vec{\boldsymbol{e}}_\alpha) = c(\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{e}}_\alpha)\,\vec{\boldsymbol{e}}_0 - c(\vec{\boldsymbol{e}}_0 \cdot \vec{\boldsymbol{e}}_\alpha)\,\vec{\boldsymbol{a}} + \vec{\boldsymbol{\omega}} \times_{\boldsymbol{e}_0} \vec{\boldsymbol{e}}_\alpha, \tag{7.25}$$

where $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{\omega}}$ are, respectively, the 4-acceleration and the 4-rotation of observer $\mathscr{O}$. Let us check that the operator $\boldsymbol{L}$ defined by (7.25) has the required form for a member of the Lie algebra $\mathrm{so}(3,1)$, namely, can be expanded onto the basis $(\boldsymbol{K}_i, \boldsymbol{J}_i)$ according to (7.17). Since the vectors $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{\omega}}$ are both orthogonal to $\vec{\boldsymbol{e}}_0$, we can expand them as[5] $\vec{\boldsymbol{a}} = a^i\,\vec{\boldsymbol{e}}_i$ and $\vec{\boldsymbol{\omega}} = \omega^i\,\vec{\boldsymbol{e}}_i$. In view of (7.18) and (7.19), we may rewrite (7.25) as

$$\boldsymbol{L}(\vec{\boldsymbol{e}}_\alpha) = -ca^i\ \mathscr{J}_{0i}(\vec{\boldsymbol{e}}_\alpha) + \omega^i\,\boldsymbol{J}_i(\vec{\boldsymbol{e}}_\alpha).$$

Now, from (7.20), $-\mathscr{J}_{0i} = \boldsymbol{K}_i$. Hence

$$\boldsymbol{L} = ca^i\,\boldsymbol{K}_i + \omega^i\,\boldsymbol{J}_i. \tag{7.26}$$

We conclude that $\boldsymbol{L}$ does take the form (7.17), as it should. Combining (7.26) and (7.24), we obtain

$$\boxed{\vec{\boldsymbol{e}}_\alpha(t+\mathrm{d}t) = \left[\mathrm{Id} + \mathrm{d}t\ \left(ca^i\,\boldsymbol{K}_i + \omega^i\,\boldsymbol{J}_i\right)\right]\ \vec{\boldsymbol{e}}_\alpha(t)}. \tag{7.27}$$

We may then reinterpret the 4-acceleration and 4-rotation of observer $\mathscr{O}$ as follows: the components (a^i) of the 4-acceleration (multiplied by $c\,\mathrm{d}t$) are the coefficients of the three generators $\boldsymbol{K}_i$ of Lorentz boosts in the transition from $(\vec{\boldsymbol{e}}_\alpha(t))$ to $(\vec{\boldsymbol{e}}_\alpha(t+\mathrm{d}t))$, and the components (ω^i) of the 4-rotation (multiplied by $\mathrm{d}t$) are the coefficients of the three generators $\boldsymbol{J}_i$ of spatial rotations in the same transition.

7.4 Reduction of O(3,1) to Its Lie Algebra

7.4.1 Exponential Map

On the space $\mathscr{L}(E)$ of all the endomorphisms of E, one defines the ***exponential*** as the map that associates with any $\boldsymbol{A} \in \mathscr{L}(E)$ an invertible endomorphism $\exp \boldsymbol{A} \in \mathrm{GL}(E)$, according to[6]

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[5]Let us recall that the index i ranges from 1 to 3.

[6]Let us recall that $\mathrm{GL}(E)$ stands for the general linear group of E, formed by all the invertible endomorphisms (automorphisms) of E (cf. p. 170).

$$\begin{aligned} \exp : \mathscr{L}(E) &\longrightarrow \mathrm{GL}(E) \\ \boldsymbol{A} \longmapsto \exp \boldsymbol{A} \ &:= \mathrm{Id} + \boldsymbol{A} + \frac{1}{2}\boldsymbol{A} \circ \boldsymbol{A} + \frac{1}{6}\boldsymbol{A} \circ \boldsymbol{A} \circ \boldsymbol{A} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}\boldsymbol{A}^n, \end{aligned} \tag{7.28}$$

where the notation $\boldsymbol{A}^n := \boldsymbol{A} \circ \cdots \circ \boldsymbol{A}$ (n times) has been used, with the convention $\boldsymbol{A}^0 = \mathrm{Id}$. The fact that the target space in (7.28) is $\mathrm{GL}(E)$ results from the following property (see, e.g. Mneimné and Testard (1986)):

$$\forall \boldsymbol{A} \in \mathscr{L}(E), \quad \det(\exp \boldsymbol{A}) = \mathrm{e}^{\mathrm{tr}\,\boldsymbol{A}}. \tag{7.29}$$

We have then necessarily $\det(\exp \boldsymbol{A}) \neq 0$, so that $\exp \boldsymbol{A}$ is always invertible.

The definition of the exponential can be extended to matrices of endomorphisms by replacing in (7.28) the composition operator $\circ$ by the matrix multiplication. In other words, the ***exponential*** of any real 4×4 matrix A is the matrix

$$\exp A := \mathbb{I}_4 + A + \frac{1}{2}A\,A + \frac{1}{6}A\,A\,A + \cdots = \sum_{n=0}^{\infty}\frac{1}{n!}A^n, \tag{7.30}$$

with $A^n := A \cdots A$ (n times) and $A^0 = \mathbb{I}_4$ (identity matrix of size 4). One can show easily that the series (7.30) is convergent for any standard matrix norm. Incidentally, this shows the convergence of the series (7.28) for endomorphisms. If A is the matrix of an endomorphism $\boldsymbol{A}$ in some basis of E, $\exp A$ is the matrix of $\exp \boldsymbol{A}$ in the same basis. In particular, $\exp A$ obeys the change-of-basis formula:

$$\exp(PAP^{-1}) = P\,(\exp A)\,P^{-1}, \tag{7.31}$$

where P is any invertible matrix (thus, representing a change of basis). Formula (7.31) follows from the trivial identity $(PAP^{-1})^n = PA^nP^{-1}$ and the definition (7.30). Another useful property of the matrix exponentiation is that it commutes with transposition:

$$\exp({}^{\mathrm{t}}A) = {}^{\mathrm{t}}(\exp A). \tag{7.32}$$

Again, this results from the trivial identity $({}^{\mathrm{t}}A)^n = {}^{\mathrm{t}}(A^n)$ and the definition (7.30). Moreover, if two matrices commute, the exponential of their sum is the product of their exponentials:

$$AB = BA \implies \exp(A + B) = \exp A\,\exp B. \tag{7.33}$$

Indeed, if the matrices commute, they behave formally like real numbers with respect to the addition and multiplication laws, so that one can expand the terms $(A+B)^n$ in the series (7.30) defining $\exp(A+B)$ according to the binomial theorem. We obtain then the same properties as for the exponential of real numbers, namely,

$\exp(A+B) = \exp A\ \exp B$. An immediate consequence of (7.33) with $B = -A$ is that for any matrix A,

$$(\exp A)^{-1} = \exp(-A). \tag{7.34}$$

The major interest of the exponential map for our purpose is to establish a connection between the Lie algebra of the Lorentz group and the restricted Lorentz group:

$$\boxed{\begin{aligned} \exp : \mathrm{so}(3,1) &\longrightarrow \mathrm{SO_o}(3,1) \\ \boldsymbol{L} &\longmapsto \exp \boldsymbol{L} \end{aligned}}. \tag{7.35}$$

Proof. We have to show that for any $\boldsymbol{L} \in \mathrm{so}(3,1)$, $\exp \boldsymbol{L} \in \mathrm{SO_o}(3,1)$. From (7.6), the matrix L of $\boldsymbol{L}$ in an orthonormal basis satisfies $\eta\, L = -{}^{\mathrm{t}}(\eta\, L) = -{}^{\mathrm{t}}L\, {}^{\mathrm{t}}\eta = -{}^{\mathrm{t}}L\, \eta$, for the Minkowski matrix η is symmetric. Since it is moreover its own inverse, we get $\eta\, L\, \eta = -{}^{\mathrm{t}}L$. Taking the exponential, there comes $\exp(\eta\, L\, \eta) = \exp(-{}^{\mathrm{t}}L)$. By expressing the left-hand side via (7.31) (for $\eta^{-1} = \eta$) and the right-hand side via (7.34), we get $\eta\,(\exp L)\,\eta = (\exp {}^{\mathrm{t}}L)^{-1}$; hence,

$$(\exp {}^{\mathrm{t}}L)\,\eta\,(\exp L) = \eta^{-1}.$$

Using $\eta^{-1} = \eta$ and (7.32), we obtain finally

$${}^{\mathrm{t}}(\exp L)\,\eta\,\exp L = \eta.$$

We recognize the criterion (6.12) for belonging to the Lorentz group; hence, $\exp \boldsymbol{L} \in \mathrm{O}(3,1)$. Besides, formula (7.29) leads to $\det(\exp \boldsymbol{L}) = 1$, for $\mathrm{tr}\, \boldsymbol{L} = 0$ for any $\boldsymbol{L} \in \mathrm{so}(3,1)$ [cf. Eq. (7.15)]. We have thus $\exp \boldsymbol{L} \in \mathrm{SO}(3,1)$. Now $\mathrm{SO}(3,1)$ has two connected components: $\mathrm{SO_o}(3,1)$ and $\mathrm{SO_a}(3,1)$ (cf. Sect. 7.2.3). But as $\mathrm{so}(3,1)$ is connected (it is a vector space over $\mathbb{R}$) and exp is a continuous map, the image of $\mathrm{so}(3,1)$ by exp must be connected and thus entirely contained in one of the connected components of $\mathrm{SO}(3,1)$. Since $\exp(0) = \mathrm{Id} \in \mathrm{SO_o}(3,1)$, it is necessarily the component $\mathrm{SO_o}(3,1)$. We have thus

$$\forall \boldsymbol{L} \in \mathrm{so}(3,1), \quad \exp \boldsymbol{L} \in \mathrm{SO_o}(3,1), \tag{7.36}$$

which proves that the mapping (7.35) is well defined. □

Remark 7.7. Formula (7.2) appears as a particular case of (7.35), namely, that for which the expansion of $\exp(\varepsilon \boldsymbol{L})$ is limited to the first order in ε, resulting in an infinitesimal Lorentz transformation.

One can show, but it is difficult (cf. Gallier (2011)), that

> The exponential map (7.35) is surjective: any restricted Lorentz transformation is the exponential of some element of the Lie algebra of the Lorentz group.

Remark 7.8. The surjectivity of the exponential map is well known for connected compact Lie groups, as, for instance, SO(3). The difficulty here arises from the non-compactness of $\mathrm{SO_o}(3,1)$ (cf. Sect. 7.2.3). For non-compact Lie groups, a classical result is that the group elements can be written as products of a finite number of exponentials. The remarkable fact is that for $\mathrm{SO_o}(3,1)$, this number can be reduced to one. On the other hand, the exponential map (7.35) is not injective, as we shall see below.

The different algebraic and topological structures introduced up to now are depicted in Fig. 7.1.

7.4.2 Generation of Lorentz Boosts

Let us consider a Lorentz boost $\boldsymbol{\Lambda}$ of rapidity ψ and plane Π. Let $(\vec{\boldsymbol{e}}_\alpha)$ be an orthonormal basis semi-adapted to $\boldsymbol{\Lambda}$: $\vec{\boldsymbol{e}}_0 \in \Pi$ (cf. Sect. 6.6.2). The velocity of $\boldsymbol{\Lambda}$ with respect to $\vec{\boldsymbol{e}}_0$ is $\vec{\boldsymbol{V}} = V\,\vec{\boldsymbol{n}} = c\tanh\psi\,\vec{\boldsymbol{n}}$, where $\vec{\boldsymbol{n}} = n^i\,\vec{\boldsymbol{e}}_i$ is a unit vector in $E_{\boldsymbol{e}_0}$. Let $N \in \mathbb{N}\setminus\{0\}$. Since the composition of two boosts having the same plane Π leads to a boost of plane Π and of rapidity the sum of the individual rapidities (cf. Sect. 6.7.1), we can write

$$\boldsymbol{\Lambda} = \prod_{p=1}^{N} \boldsymbol{\Lambda}_{(\delta\psi)}, \tag{7.37}$$

where $\delta\psi := \psi/N$ and $\boldsymbol{\Lambda}_{(\delta\psi)}$ is the boost of plane Π and rapidity $\delta\psi$. When $N \to +\infty$, $\boldsymbol{\Lambda}_{(\delta\psi)}$ is an infinitesimal boost; it is therefore of the type (7.22):

$$\boldsymbol{\Lambda}_{(\delta\psi)} = \mathrm{Id} + \delta\psi\, n^i\,\boldsymbol{K}_i = \exp(\delta\psi\, n^i\,\boldsymbol{K}_i).$$

The second equality, involving the exponential, is valid to the first order in $\delta\psi$. Equation (7.37) becomes then

$$\boldsymbol{\Lambda} = \prod_{p=1}^{N} \exp(\delta\psi\, n^i\,\boldsymbol{K}_i).$$

Now, since the $\delta\psi\, n^i\,\boldsymbol{K}_i$ commutes with itself, one can apply formula (7.33) and write the product of exponentials as the exponential of the sum. Given that $\sum_{p=1}^{N}\delta\psi\, n^i\,\boldsymbol{K}_i = N\delta\psi\, n^i\,\boldsymbol{K}_i = \psi\, n^i\,\boldsymbol{K}_i$, we obtain thus

$$\boxed{\boldsymbol{\Lambda} = \exp(\psi\, n^i\,\boldsymbol{K}_i)}. \tag{7.38}$$

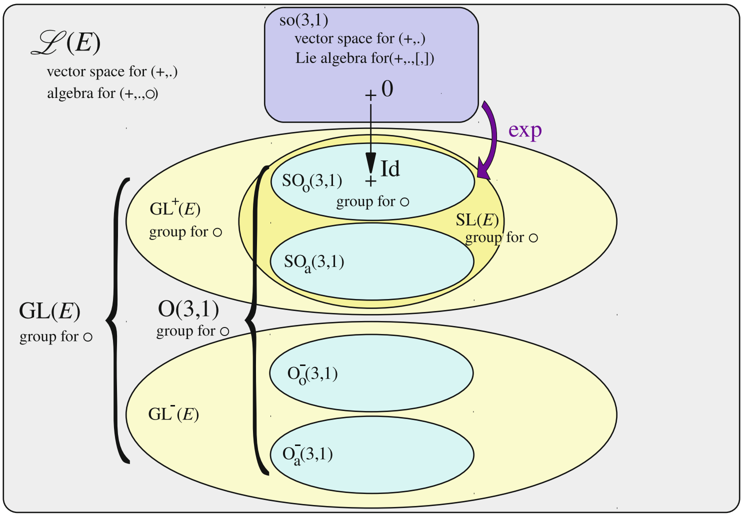

Fig. 7.1 Lorentz group O(3, 1) and its Lie algebra so(3, 1), as subsets of $\mathscr{L}(E)$, the space of all the endomorphisms of the vector space E underlying Minkowski spacetime $\mathscr{E}$. For each set, the algebraic structure relative to the composition laws $+$ (addition), . (multiplication by a real number), $\circ$ (composition) and $[,]$ (commutator) is indicated. $\mathrm{GL}(E)$ is the *general linear group* of E, formed by all invertible endomorphisms. $\mathrm{GL}^+(E)$ is the subgroup of $\mathrm{GL}(E)$ formed by endomorphisms of positive determinant, and $\mathrm{GL}^-(E)$ is the part of $\mathrm{GL}(E)$ formed by endomorphisms of negative determinant. $\mathrm{SL}(E)$ is the *special linear group* of E, i.e. the subgroup of $\mathrm{GL}^+(E)$ formed by all the endomorphisms whose determinant is one. O(3, 1) is the *Lorentz group* and $\mathrm{SO_o}(3,1)$ the *restricted Lorentz group*, formed by proper orthochronous Lorentz transformations; $\mathrm{SO_o}(3,1)$ is a subgroup of $\mathrm{SL}(E)$. $\mathrm{SO_a}(3,1)$ is the part of O(3, 1) formed by antichronous proper Lorentz transformations, $\mathrm{O_o^-}(3,1)$ that formed by orthochronous improper Lorentz transformations and $\mathrm{O_a^-}(3,1)$ that formed by antichronous improper Lorentz transformations. so(3, 1) is the *Lie algebra of the Lorentz group*; it is a vector subspace of $\mathscr{L}(E)$, whose image by the exponential map is $\mathrm{SO_o}(3,1)$. As a vector space, so(3, 1) contains the element 0 (vanishing linear map), while as a group for $\circ$, $\mathrm{SO_o}(3,1)$ contains the element Id (identity map). The latter is actually the image of 0 by the exponential map. Regarding topology, the figure respects connectedness: $\mathrm{GL}(E)$ appears with its two connected components: $\mathrm{GL}^+(E)$ and $\mathrm{GL}^-(E)$, and O(3, 1) appears with its four connected components: $\mathrm{SO_o}(3,1)$, $\mathrm{SO_a}(3,1)$, $\mathrm{O_o^-}(3,1)$ and $\mathrm{O_a^-}(3,1)$. On the other hand, compactness is not respected: none of the above spaces is compact, despite being depicted with compact figures (ellipses and rectangles). Similarly, the dimensions are not respected: $\mathscr{L}(E)$ and so(3, 1) are vector spaces of dimension 16 and 6, respectively (hard to depict!); $\mathrm{GL}(E)$, $\mathrm{SL}(E)$, O(3, 1) and $\mathrm{SO_o}(3,1)$ are Lie groups of dimension 16, 15, 6 and 6, respectively

The equality shows that the endomorphisms $\boldsymbol{K}_i$ of so(3, 1) generate not only the infinitesimal boosts, as we have seen in Sect. 7.3.3, but also, thanks to the exponential map, *all* Lorentz boosts. They thus deserve the name of *generator of the Lorentz group* (without any mention of infinitesimal character) given to them in Sect. 7.3.3.

Remark 7.9. If $\vec{n}$ coincides with one of the vectors $\vec{e}_i$, for instance, $\vec{n} = \vec{e}_1$, relation (7.38) can be established by a direct computation of the exponential of the matrix ψK_1, following the definition (7.30), without invoking infinitesimal transformations. It suffices to observe that, for $n > 0$, $(K_1)^n = (K_1)^2$ if n is even and $(K_1)^n = K_1$ if n is odd, with $(K_1)^2 = \mathrm{diag}(1, 1, 0, 0)$. This simplifies greatly the series (7.30), which becomes

$$\exp(\psi K_1) = \mathbb{I}_4 + \sum_{p=0}^{\infty} \left[\frac{\psi^{2p+1}}{(2p+1)!} K_1 + \frac{\psi^{2p+2}}{(2p+2)!} (K_1)^2 \right]$$

$$= \begin{pmatrix} \sum\limits_{p=0}^{\infty} \frac{\psi^{2p}}{(2p)!} & \sum\limits_{p=0}^{\infty} \frac{\psi^{2p+1}}{(2p+1)!} & 0 & 0 \\ \sum\limits_{p=0}^{\infty} \frac{\psi^{2p+1}}{(2p+1)!} & \sum\limits_{p=0}^{\infty} \frac{\psi^{2p}}{(2p)!} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

We recognize in this expression the Taylor expansions of the functions $\cosh \psi$ and $\sinh \psi$. Comparing with (6.43), we conclude that $\exp(\psi \boldsymbol{K}_1)$ is the Lorentz boost of rapidity ψ and plane $\Pi = \mathrm{Span}(\vec{e}_0, \vec{e}_1)$, thereby demonstrating (7.38) in the particular case $(n^i) = (1, 0, 0)$.

7.4.3 Generation of Spatial Rotations

Let us consider now a spatial rotation $\boldsymbol{R}$ such that $\boldsymbol{R}(\vec{e}_0) = \vec{e}_0$. Let $\varphi \in [0, 2\pi[$ be $\boldsymbol{R}$'s angle and $\vec{n}$ the unit vector defining $\boldsymbol{R}$'s axis in $E_{\boldsymbol{e}_0}$. The composition of two rotations that leave $\vec{e}_0$ invariant and have the same axis $\vec{n}$ being a rotation of the same type, we may write, for any $N \in \mathbb{N} \setminus \{0\}$,

$$\boldsymbol{R} = \prod_{p=1}^{N} \boldsymbol{R}_{(\delta\varphi)},$$

where $\delta\varphi := \varphi/N$ and $\boldsymbol{R}_{(\delta\varphi)}$ is the rotation of angle $\delta\varphi$ around $\vec{n}$ in $E_{\boldsymbol{e}_0}$. The argument is then similar to that of Sect. 7.4.2, starting from expression (7.21) of an infinitesimal rotation. We thus obtain

$$\boxed{\boldsymbol{R} = \exp(\varphi\, n^i \boldsymbol{J}_i)} . \tag{7.39}$$

Hence, the endomorphisms $\boldsymbol{J}_i$ of so(3, 1) generate not only the infinitesimal rotations, as we have seen in Sect. 7.3.3, but also, thanks to the exponential map, *all* the spatial rotations. They thus deserve the name of *generator of the Lorentz group* (without any mention of infinitesimal character) given to them in Sect. 7.3.3.

Remark 7.10. The writing (7.39), which can be made explicit as

$$\boldsymbol{R} = \exp(\varphi_1 \boldsymbol{J}_1 + \varphi_2 \boldsymbol{J}_2 + \varphi_3 \boldsymbol{J}_3),$$

with $\varphi_i := \varphi n^i$, does not mean that $\boldsymbol{R}$ can be decomposed into a rotation of angle φ_3 in the plane $\mathrm{Span}(\vec{e}_1, \vec{e}_2)$, followed by a rotation of angle φ_2 in the plane $\mathrm{Span}(\vec{e}_3, \vec{e}_1)$ and a rotation of angle φ_1 in the plane $\mathrm{Span}(\vec{e}_2, \vec{e}_3)$. Indeed, in general

$$\exp(\varphi_1 \boldsymbol{J}_1 + \varphi_2 \boldsymbol{J}_2 + \varphi_3 \boldsymbol{J}_3) \neq \exp(\varphi_1 \boldsymbol{J}_1) \circ \exp(\varphi_2 \boldsymbol{J}_2) \circ \exp(\varphi_3 \boldsymbol{J}_3),$$

for the endomorphisms $\boldsymbol{J}_1$, $\boldsymbol{J}_2$ and $\boldsymbol{J}_3$ do not commute, so that formula (7.33) is not applicable.

Remark 7.11. As in Sect. 7.4.2 (cf. Remark 7.9), one can establish (7.39) directly from the definition (7.30) of the exponential map, in the particular case where, for instance, $\vec{n} = \vec{e}_1$. It suffices to observe that $(J_1)^2 = -(J_1)^4$ and $(J_1)^3 = -J_1$, with $(J_1)^4 = \mathrm{diag}(0, 0, 1, 1)$. We obtain then in the expression of the matrix $\exp(\varphi J_1)$ the Taylor expansions of $\cos\varphi$ and $\sin\varphi$. Comparing with (6.39) leads to (7.39).

7.4.4 Structure Constants

We have mentioned in Sect. 7.4.1 that the restricted Lorentz group $\mathrm{SO_o}(3,1)$ is entirely generated by the Lie algebra $\mathrm{so}(3,1)$, via the exponential map. We have proved it explicitly for Lorentz boosts (Sect. 7.4.2) and spatial rotations (Sect. 7.4.3). What about the group law of $\mathrm{SO_o}(3,1)$? In other words, does there exist any simple connection between the composition of two elements $\boldsymbol{\Lambda}_1$ of $\boldsymbol{\Lambda}_2$ of $\mathrm{SO_o}(3,1)$ and some operation in the Lie algebra $\mathrm{so}(3,1)$? Thanks to the surjectivity of the exponential map (7.35), it is always possible to find two elements $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ in $\mathrm{so}(3,1)$ such that $\boldsymbol{\Lambda}_1 = \exp \boldsymbol{L}_1$ and $\boldsymbol{\Lambda}_2 = \exp \boldsymbol{L}_2$. If $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ commute, the answer to the above question is simple: from (7.33) we have

$$[\boldsymbol{L}_1, \boldsymbol{L}_2] = 0 \implies \boldsymbol{\Lambda}_1 \circ \boldsymbol{\Lambda}_2 = \exp(\boldsymbol{L}_1 + \boldsymbol{L}_2). \tag{7.40}$$

Thus in this case, the composition law $\circ$ corresponds to a mere addition in the Lie algebra $\mathrm{so}(3,1)$.

In the general case, the answer is provided by ***Baker–Campbell–Hausdorff formula***:

$$\boxed{\boldsymbol{\Lambda}_1 \circ \boldsymbol{\Lambda}_2 = \exp\left(\boldsymbol{L}_1 + \boldsymbol{L}_2 + \frac{1}{2}[\boldsymbol{L}_1, \boldsymbol{L}_2] + \frac{1}{12}[\boldsymbol{L}_1, [\boldsymbol{L}_1, \boldsymbol{L}_2]] - \frac{1}{12}[\boldsymbol{L}_2, [\boldsymbol{L}_1, \boldsymbol{L}_2]] + \cdots\right)}. \tag{7.41}$$

The following terms are complicated but are all formed by nested commutators, of the type $[\boldsymbol{L}_a, [\boldsymbol{L}_b, \ldots [\boldsymbol{L}_1, \boldsymbol{L}_2] \ldots]]$ $(a, b = 1, 2)$. The demonstration of this formula can be found in the textbooks Godement (2004) and Mneimné and Testard (1986).

Baker–Campbell–Hausdorff formula shows that the product of two elements of the restricted Lorentz group $\mathrm{SO_o}(3, 1)$ can be expressed entirely by means of the commutator and the addition law in the Lie algebra $\mathrm{so}(3, 1)$. In other words, all the information on the group law of $\mathrm{SO_o}(3, 1)$ is encoded in the Lie algebra $\mathrm{so}(3, 1)$, whose structure (vector space + Lie bracket) is simpler than that of the non-abelian group $\mathrm{SO_o}(3, 1)$.

Remark 7.12. We have already noticed in Sect. 7.3.2 (Remark 7.5) that for infinitesimal Lorentz transformations, the commutator of two elements of the group $\mathrm{SO_o}(3, 1)$ was entirely defined by the commutator of the Lie algebra $\mathrm{so}(3, 1)$ [Eq. (7.14)].

Let us consider the generators of the Lorentz group associated with an orthonormal basis $(\vec{\boldsymbol{e}}_\alpha)$ of $(E, \boldsymbol{g})$ (cf. Sect. 7.3.3), and let us denote them by $\boldsymbol{G}_a$ with $a \in \{1, 2, 3, 4, 5, 6\}$, according to

$$\boldsymbol{G}_1 := \boldsymbol{K}_1, \quad \boldsymbol{G}_2 := \boldsymbol{K}_2, \quad \boldsymbol{G}_3 := \boldsymbol{K}_3, \quad \boldsymbol{G}_4 := \boldsymbol{J}_1, \quad \boldsymbol{G}_5 := \boldsymbol{J}_2, \quad \boldsymbol{G}_6 := \boldsymbol{J}_3. \tag{7.42}$$

The 6-tuple $(\boldsymbol{G}_a)$ constitutes then a basis of the vector space $\mathrm{so}(3, 1)$, and thanks to the bilinearity of the Lie bracket, the computation of $[\boldsymbol{L}_1, \boldsymbol{L}_2]$ is reduced to the computation of the Lie brackets of the generators, namely, $[\boldsymbol{G}_a, \boldsymbol{G}_b]$. The latter can be expanded on the basis $(\boldsymbol{G}_a)$, thereby defining $6^3 = 216$ real coefficients $C_{ab}{}^{c}$:

$$[\boldsymbol{G}_a, \boldsymbol{G}_b] = \sum_{c=1}^{6} C_{ab}{}^{c}\, \boldsymbol{G}_c. \tag{7.43}$$

The coefficients $C_{ab}{}^{c}$ are called ***structure constants*** of the Lorentz group. Thanks to the Baker–Campbell–Hausdorff formula (7.41), all the information about the group law $\mathrm{SO_o}(3, 1)$ is contained in these numbers, hence their name.

By means of the explicit forms (7.16) of the generators, one obtains the following values for the Lie brackets of the generators of the Lorentz group:

$$\boxed{\begin{aligned} [\boldsymbol{K}_i, \boldsymbol{K}_j] &= -\sum_{k=1}^{3} \epsilon_{ijk}\, \boldsymbol{J}_k \\ [\boldsymbol{K}_i, \boldsymbol{J}_j] &= -\sum_{k=1}^{3} \epsilon_{ijk}\, \boldsymbol{K}_k \\ [\boldsymbol{J}_i, \boldsymbol{J}_j] &= \sum_{k=1}^{3} \epsilon_{ijk}\, \boldsymbol{J}_k, \end{aligned}} \tag{7.44}$$

where ϵ_{ijk} stands for the fully antisymmetric symbol of 3 indices: $\epsilon_{ijk} = 0$ if any two of the indices i, j and k are equal, $\epsilon_{ijk} = 1$ if (i, j, k) is an even permutation of $(1, 2, 3)$ and $\epsilon_{ijk} = -1$ otherwise. We read on (7.44) that the structure constants of the Lorentz group are very simple: $C_{ab}^{\ \ c} = 0$, 1 or -1.

Remark 7.13. While the right-hand sides of (7.44) are written as sums over the index k, they are actually limited to a single term, thanks to the antisymmetry of ϵ_{ijk}. Explicitly

$$\begin{aligned}
&[\boldsymbol{K}_1, \boldsymbol{K}_2] = -\boldsymbol{J}_3, \qquad &&[\boldsymbol{K}_2, \boldsymbol{K}_3] = -\boldsymbol{J}_1, \qquad &&[\boldsymbol{K}_3, \boldsymbol{K}_1] = -\boldsymbol{J}_2,\\
&[\boldsymbol{K}_1, \boldsymbol{J}_2] = -\boldsymbol{K}_3, &&[\boldsymbol{K}_2, \boldsymbol{J}_3] = -\boldsymbol{K}_1, &&[\boldsymbol{K}_3, \boldsymbol{J}_1] = -\boldsymbol{K}_2,\\
&[\boldsymbol{J}_1, \boldsymbol{K}_2] = -\boldsymbol{K}_3, &&[\boldsymbol{J}_2, \boldsymbol{K}_3] = -\boldsymbol{K}_1, &&[\boldsymbol{J}_3, \boldsymbol{K}_1] = -\boldsymbol{K}_2,\\
&[\boldsymbol{J}_1, \boldsymbol{J}_2] = \boldsymbol{J}_3, &&[\boldsymbol{J}_2, \boldsymbol{J}_3] = \boldsymbol{J}_1, &&[\boldsymbol{J}_3, \boldsymbol{J}_1] = \boldsymbol{J}_2.
\end{aligned} \tag{7.45}$$

The first equation of (7.44) shows that the Lie bracket of two generators of Lorentz boosts is a generator of spatial rotations. This feature is intimately related to Thomas rotation, discussed in Sect. 6.7. Indeed, let us consider two boosts $\boldsymbol{\Lambda}_1$ and $\boldsymbol{\Lambda}_2$, of rapidities ψ_1 and ψ_2. If $\boldsymbol{\Lambda}_1$ and $\boldsymbol{\Lambda}_2$ have the same plane, for instance, $\mathrm{Span}(\vec{e}_0, \vec{e}_1)$, (7.38) leads to $\boldsymbol{\Lambda}_1 = \exp(\psi_1 \boldsymbol{K}_1)$ and $\boldsymbol{\Lambda}_2 = \exp(\psi_2 \boldsymbol{K}_1)$. Since obviously $[\boldsymbol{K}_1, \boldsymbol{K}_1] = 0$, formula (7.40) is applicable and we conclude that $\boldsymbol{\Lambda}_1 \circ \boldsymbol{\Lambda}_2 = \exp((\psi_1 + \psi_2)\boldsymbol{K}_1)$. Hence $\boldsymbol{\Lambda}_1 \circ \boldsymbol{\Lambda}_2$ is a boost, of the same plane as $\boldsymbol{\Lambda}_1$ and $\boldsymbol{\Lambda}_2$ and of rapidity $\psi_1 + \psi_2$. We recover the result of Sect. 6.7.1. On the contrary, if $\boldsymbol{\Lambda}_1$ and $\boldsymbol{\Lambda}_2$ do not have the same plane, for instance, if $\boldsymbol{\Lambda}_1 = \exp(\psi_1 \boldsymbol{K}_1)$ and $\boldsymbol{\Lambda}_2 = \exp(\psi_2 \boldsymbol{K}_2)$, one must use the Baker–Campbell–Hausdorff formula (7.41) to evaluate the product $\boldsymbol{\Lambda}_1 \circ \boldsymbol{\Lambda}_2$, setting $\boldsymbol{L}_1 := \psi_1 \boldsymbol{K}_1$ and $\boldsymbol{L}_2 := \psi_2 \boldsymbol{K}_2$. Since from (7.45), $[\boldsymbol{K}_1, \boldsymbol{K}_2] = -\boldsymbol{J}_3$, $[\boldsymbol{K}_1, [\boldsymbol{K}_1, \boldsymbol{K}_2]] = -[\boldsymbol{K}_1, \boldsymbol{J}_3] = -\boldsymbol{K}_2$ and $[\boldsymbol{K}_2, [\boldsymbol{K}_1, \boldsymbol{K}_2]] = -[\boldsymbol{K}_2, \boldsymbol{J}_3] = \boldsymbol{K}_1$, (7.41) leads to

$$\boldsymbol{\Lambda}_1 \circ \boldsymbol{\Lambda}_2 = \exp\left(\psi_1 \boldsymbol{K}_1 + \psi_2 \boldsymbol{K}_2 - \frac{1}{2}\psi_1\psi_2 \boldsymbol{J}_3 - \frac{1}{12}\psi_1^2\psi_2 \boldsymbol{K}_2 - \frac{1}{12}\psi_1\psi_2^2 \boldsymbol{K}_1 + \cdots\right). \tag{7.46}$$

The appearance of $\boldsymbol{J}_3$ in this expression means that the product $\boldsymbol{\Lambda}_1 \circ \boldsymbol{\Lambda}_2$ contains a spatial rotation: this is Thomas rotation.

Historical note: *It is Sophus Lie*[7] *who showed how to reduce the study of continuous groups of transformations (today called* Lie groups*) to that of their*

[7]**Sophus Lie** (1842–1899): Norwegian mathematician, essentially known for the foundation of the theory of Lie groups. As an adolescent, he contemplated some military career, but his strong myopia forced him to choose an academic career instead! During a stay in Paris in 1870, at the contact of Camille Jordan, he started the study of continuous groups of transformations. After the declaration of war of France to Prussia in July 1870, he was arrested near Paris, being suspected to be a German spy: his mathematical notes had been mistaken for coded messages! He was released thanks to the intervention of the mathematician Gaston Darboux.

Lie algebras. In the case of the Lorentz group, the generators have been exhibited in 1905 by Henri Poincaré (cf. p. 26), in the famous "Palermo memoir" (Poincaré 1906).

7.5 Relations Between the Lorentz Group and SL(2,C)

There exists an intimate link between the restricted Lorentz group $\mathrm{SO_o}(3,1)$ and the ***special linear group*** $\mathrm{SL}(2,\mathbb{C})$, the latter being the set of all 2×2 complex matrices of unit determinant, equipped with the law of matrix multiplication[8]:

$$\boxed{\mathrm{SL}(2,\mathbb{C}) := \{A \in \mathrm{Mat}(2,\mathbb{C}),\ \det A = 1\}}.$$

It is clear that $\mathrm{SL}(2,\mathbb{C})$ is a group for matrix multiplication. Moreover, it is a Lie group, of the same dimension on $\mathbb{R}$ as the Lorentz group, namely, 6. Indeed $\mathrm{Mat}(2,\mathbb{C})$ is a manifold of dimension 8 on $\mathbb{R}$ (4 complex numbers are required to form a 2×2 matrix, and each complex number is composed of 2 real numbers) and the condition $\det A = 1$, which is an equality between two complex numbers, fixes 2 real degrees of freedom. The link between $\mathrm{SL}(2,\mathbb{C})$ and $\mathrm{SO_o}(3,1)$ is performed by the *spinor map*, which we introduce below.

7.5.1 Spinor Map

In order to construct the spinor map, let us consider the set of all 2×2 complex matrices that are ***Hermitian*** (one also says ***self-adjoint***):

$$\mathrm{Herm}(2,\mathbb{C}) := \{H \in \mathrm{Mat}(2,\mathbb{C}),\ H^\dagger = H\}, \tag{7.47}$$

where $H^\dagger := {}^{\mathrm{t}}\bar{H}$, i.e. the transpose of the matrix obtained by taking the complex conjugate (denoted here by an overbar) of each coefficient of H. Expressing H in terms of its coefficients, $H = (H_{ab})_{1\le a,b\le 2}$, it is easy to see that

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \in \mathrm{Herm}(2,\mathbb{C}) \iff \begin{cases} H_{11} \in \mathbb{R}, \quad H_{22} \in \mathbb{R} \\ H_{21} = \bar{H}_{12}. \end{cases} \tag{7.48}$$

[8] $\mathrm{Mat}(2,\mathbb{C})$ stands for the set of 2×2 matrices with complex coefficients.

It follows immediately that for any $\lambda \in \mathbb{R}$ and any pair (H, H') of Hermitian matrices, $\lambda H + H' \in \mathrm{Herm}(2,\mathbb{C})$. $\mathrm{Herm}(2,\mathbb{C})$ is thus a vector space over[9] $\mathbb{R}$. Its dimension is 4: (7.48) provides 4 independent constraints on the 8 real numbers required to describe a 2×2 complex matrix. More precisely, in view of (7.48), one can represent an element H of $\mathrm{Herm}(2,\mathbb{C})$ in a unique way by a 4-tuple $(v^0, v^1, v^2, v^3) \in \mathbb{R}^4$ defined by $v^0 := (H_{11} + H_{22})/2$, $v^1 := \mathrm{Re}\, H_{12} = \mathrm{Re}\, H_{21}$, $v^2 := -\mathrm{Im}\, H_{12} = \mathrm{Im}\, H_{21}$ and $v^3 := (H_{11} - H_{22})/2$. Accordingly

$$\boxed{H = \begin{pmatrix} v^0 + v^3 & v^1 - \mathrm{i}v^2 \\ v^1 + \mathrm{i}v^2 & v^0 - v^3 \end{pmatrix}}, \quad (v^0, v^1, v^2, v^3) \in \mathbb{R}^4. \tag{7.49}$$

This writing is equivalent to

$$H = v^0\sigma_0 + v^1\sigma_1 + v^2\sigma_2 + v^3\sigma_3, \tag{7.50}$$

with[10]

$$\boxed{\sigma_0 := \mathbb{I}_2,\ \sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_2 := \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix},\ \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}, \tag{7.51}$$

σ_1, σ_2 and σ_3 are called ***Pauli matrices*** (cf. p. 542). The expansion (7.50) shows that $(\sigma_0, \sigma_1, \sigma_2, \sigma_3)$ is a basis of the four-dimensional vector space $\mathrm{Herm}(2,\mathbb{C})$. Moreover,

For a given orthonormal basis of $(E, \boldsymbol{g})$, $(\vec{\boldsymbol{e}}_\alpha)$, the mapping

$$\boxed{\begin{aligned} \mathscr{H} : \quad & E && \longrightarrow \mathrm{Herm}(2,\mathbb{C}) \\ & \vec{\boldsymbol{v}} = v^\alpha \vec{\boldsymbol{e}}_\alpha && \longmapsto H = v^\alpha \sigma_\alpha. \end{aligned}} \tag{7.52}$$

is an isomorphism between the vector space E underlying Minkowski spacetime and $\mathrm{Herm}(2,\mathbb{C})$. In addition,

$$\forall \vec{\boldsymbol{v}} \in E, \quad \boxed{\det \mathscr{H}(\vec{\boldsymbol{v}}) = -\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{v}}}. \tag{7.53}$$

[9] But not over $\mathbb{C}$, since multiplying H_{11} by $\lambda = \mathrm{i}$ would lead to a violation of the first condition in (7.48).

[10] $\mathbb{I}_2 := \mathrm{diag}(1, 1)$ is the 2×2 identity matrix.

Proof. Since $(\vec{e}_\alpha)$ is a basis of E and (σ_α) a basis of $\mathrm{Herm}(2,\mathbb{C})$, it is clear that $\mathscr{H}$ is an isomorphism between the vector spaces E and $\mathrm{Herm}(2,\mathbb{C})$. From (7.49), $\det H = (v^0)^2-(v^1)^2-(v^2)^2-(v^3)^2$. On the other hand, $\vec{v}\cdot\vec{v} = -(v^0)^2+(v^1)^2+(v^2)^2+(v^3)^2$, since the basis $(\vec{e}_\alpha)$ is orthonormal. Hence (7.53) holds. □

The group $\mathrm{SL}(2,\mathbb{C})$ acts onto the vector space $\mathrm{Herm}(2,\mathbb{C})$ via the operation

$$\boxed{\begin{array}{llll}\forall A\in \mathrm{SL}(2,\mathbb{C}), & \Phi_A : \mathrm{Herm}(2,\mathbb{C}) & \longrightarrow & \mathrm{Herm}(2,\mathbb{C}) \\ & \qquad\qquad H & \longmapsto & A\,H\,A^\dagger.\end{array}} \tag{7.54}$$

For each $A\in\mathrm{SL}(2,\mathbb{C})$, the mapping Φ_A is well defined, i.e. it takes its values in $\mathrm{Herm}(2,\mathbb{C})$. Indeed, thanks to the property $(A\,B)^\dagger = B^\dagger\,A^\dagger$,

$$(A\,H\,A^\dagger)^\dagger = (H\,A^\dagger)^\dagger\,A^\dagger = (\underbrace{A^{\dagger\dagger}}_{A}\ \underbrace{H^\dagger}_{H})\,A^\dagger = A\,H\,A^\dagger.$$

Besides, for any $A\in\mathrm{SL}(2,\mathbb{C})$, Φ_A is an automorphism (i.e. a bijective linear map) of the vector space $\mathrm{Herm}(2,\mathbb{C})$: Φ_A is clearly linear and it is bijective:

$$\forall (H,H')\in \mathrm{Herm}(2,\mathbb{C})^2,\quad A\,H\,A^\dagger = H' \iff H = A^{-1}\,H'\,(A^\dagger)^{-1}.$$

Thanks to the isomorphism $\mathscr{H}$ between the vector spaces E and $\mathrm{Herm}(2,\mathbb{C})$ defined by (7.52), one may associate with each Φ_A an automorphism $\boldsymbol{\Lambda}_A$ of E. This amounts to setting

$$\boldsymbol{\Lambda}_A := \mathscr{H}^{-1}\circ\Phi_A\circ\mathscr{H}. \tag{7.55}$$

Explicitly, if $\vec{v}\in E$ and $H := \mathscr{H}(\vec{v})$, $\boldsymbol{\Lambda}_A(\vec{v})$ is the vector of E such that $H' := \mathscr{H}(\boldsymbol{\Lambda}_A(\vec{v})) = A\,H\,A^\dagger$. We have then

$$\det H' = (\underbrace{\det A}_{1})(\det H)(\underbrace{\det A^\dagger}_{1}) = \det H.$$

From the property (7.53), we deduce that $\boldsymbol{\Lambda}_A(\vec{v})\cdot\boldsymbol{\Lambda}_A(\vec{v}) = \vec{v}\cdot\vec{v}$, which shows that $\boldsymbol{\Lambda}_A$ is a Lorentz transformation. Let us show that it is actually a restricted Lorentz transformation, i.e. that it is proper and orthochronous [cf. (6.18)]. To this aim, let us express the matrix of $\boldsymbol{\Lambda}_A$ in terms of the coefficients of the matrix A:

$$A = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix},\quad \text{with } (\alpha,\beta,\gamma,\delta)\in\mathbb{C}^4 \quad\text{and}\quad \alpha\delta-\beta\gamma = 1, \tag{7.56}$$

the last condition reflecting the fact that A belongs to $\mathrm{SL}(2,\mathbb{C})$: $\det A = 1$. By virtue of the isomorphism $\mathscr{H}$, the matrix $(\Lambda_A)^\alpha{}_\beta$ of $\boldsymbol{\Lambda}_A$ in the basis $(\vec{e}_\alpha)$ of E is identical to the matrix of Φ_A in the basis (σ_α) of $\mathrm{Herm}(2,\mathbb{C})$. The latter matrix is obtained by performing the two matrix multiplications that appear in (7.54), using expressions (7.49) for H and (7.56) for A. Using again (7.49) to determine the components (v'^{α}) of the result in the basis (σ_α), we get

$$(\Lambda_A)^{\alpha}{}_{\beta} = \frac{1}{2}\begin{pmatrix} \alpha\bar{\alpha}+\beta\bar{\beta}+\gamma\bar{\gamma}+\delta\bar{\delta} & \alpha\bar{\beta}+\bar{\alpha}\beta+\gamma\bar{\delta}+\bar{\gamma}\delta & \mathrm{i}(\bar{\alpha}\beta-\alpha\bar{\beta}+\bar{\gamma}\delta-\gamma\bar{\delta}) & \alpha\bar{\alpha}-\beta\bar{\beta}+\gamma\bar{\gamma}-\delta\bar{\delta} \\ \alpha\bar{\gamma}+\bar{\alpha}\gamma+\beta\bar{\delta}+\bar{\beta}\delta & \alpha\bar{\delta}+\bar{\alpha}\delta+\beta\bar{\gamma}+\bar{\beta}\gamma & \mathrm{i}(\beta\bar{\gamma}-\bar{\beta}\gamma-\alpha\bar{\delta}+\bar{\alpha}\delta) & \alpha\bar{\gamma}+\bar{\alpha}\gamma-\beta\bar{\delta}-\bar{\beta}\delta \\ \mathrm{i}(\alpha\bar{\gamma}-\bar{\alpha}\gamma+\beta\bar{\delta}-\bar{\beta}\delta) & \mathrm{i}(\beta\bar{\gamma}-\bar{\beta}\gamma+\alpha\bar{\delta}-\bar{\alpha}\delta) & \alpha\bar{\delta}+\bar{\alpha}\delta-\beta\bar{\gamma}-\bar{\beta}\gamma & \mathrm{i}(\alpha\bar{\gamma}-\bar{\alpha}\gamma+\bar{\beta}\delta-\beta\bar{\delta}) \\ \alpha\bar{\alpha}+\beta\bar{\beta}-\gamma\bar{\gamma}-\delta\bar{\delta} & \alpha\bar{\beta}+\bar{\alpha}\beta-\gamma\bar{\delta}-\bar{\gamma}\delta & \mathrm{i}(\bar{\alpha}\beta-\alpha\bar{\beta}-\bar{\gamma}\delta+\gamma\bar{\delta}) & \alpha\bar{\alpha}-\beta\bar{\beta}-\gamma\bar{\gamma}+\delta\bar{\delta} \end{pmatrix}. \tag{7.57}$$

We can check that for any pair (α,β), $(\Lambda_A)^{\alpha}{}_{\beta} \in \mathbb{R}$, as it should. For instance, $(\Lambda_A)^{0}{}_{1} = \mathrm{Re}(\alpha\bar{\beta} + \gamma\bar{\delta})$ and $(\Lambda_A)^{0}{}_{2} = \mathrm{Im}(\alpha\bar{\beta} + \gamma\bar{\delta})$. Moreover, we have

$$(\Lambda_A)^{0}{}_{0} = \frac{1}{2}\left(\alpha\bar{\alpha} + \beta\bar{\beta} + \gamma\bar{\gamma} + \delta\bar{\delta}\right) = \frac{1}{2}\left(|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2\right) > 0,$$

which shows that $\boldsymbol{\Lambda}_A$ is an orthochronous Lorentz transformation (cf. Sect. 6.3.2). There remains to show that $\boldsymbol{\Lambda}_A$ is a proper Lorentz transformation, i.e. that $\det \boldsymbol{\Lambda}_A = 1$. We could compute directly the determinant of matrix $(\Lambda_A)^{\alpha}{}_{\beta}$ displayed in (7.57), but this is not so tempting... Another way consists in rewriting (7.49) as $h = T\,v$, where $h = (H_{11}, H_{12}, H_{21}, H_{22})$, $v = (v^0, v^1, v^2, v^3)$ and T is the matrix

$$T^{\alpha}{}_{\beta} = (\sigma_\beta)_{ab} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -\mathrm{i} & 0 \\ 0 & 1 & \mathrm{i} & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}, \tag{7.58}$$

where $(a,b) = (1,1), (1,2), (2,1)$ and $(2,2)$ for, respectively, $\alpha = 0$, 1, 2 and 3. Writing $H = v^\mu \sigma_\mu$ and $H' = v'^\mu \sigma_\mu = (\Lambda_A)^{\mu}{}_{\nu} v^\nu \sigma_\mu$ in the relation $H' = A\,H\,A^\dagger$ and identifying the coefficients of v^α, one obtains four 2×2 matrix identities:

$$(\Lambda_A)^{\mu}{}_{\alpha}\sigma_\mu = A\,\sigma_\alpha\,A^\dagger, \qquad 0 \le \alpha \le 3.$$

Thanks to (7.58) and $A^\dagger = {}^{\mathrm{t}}\bar{A}$, these four identities are easily transformed into a unique 4×4 matrix relation:

$$T\,\Lambda_A = (A \otimes \bar{A})\,T, \tag{7.59}$$

where $A \otimes \bar{A}$ stands for the ***Kronecker product*** of the matrix A by the matrix $\bar{A}$, i.e. the 4×4 matrix whose writing in terms of 2×2 blocs is

$$A \otimes \bar{A} := \left(\begin{array}{c|c} A_{11}\bar{A} & A_{12}\bar{A} \\ \hline A_{21}\bar{A} & A_{22}\bar{A} \end{array}\right) = \left(\begin{array}{c|c} \alpha\bar{A} & \beta\bar{A} \\ \hline \gamma\bar{A} & \delta\bar{A} \end{array}\right). \tag{7.60}$$

Equation (7.59) leads to the following expression for the matrix of $\boldsymbol{\Lambda}_A$:

$$\boxed{\Lambda_A = T^{-1}\,(A \otimes \bar{A})\,T}. \tag{7.61}$$

Using the property $\det(A \otimes B) = (\det A)^2 (\det B)^2$, valid for any Kronecker product of 2×2 matrices, we obtain the desired result:

$$\det \Lambda_A = (\det T)^{-1} \det(A \otimes \bar{A}) \det T = \det(A \otimes \bar{A}) = (\underbrace{\det A}_{1})^2 (\underbrace{\det \bar{A}}_{1})^2 = 1.$$

$\boldsymbol{\Lambda}_A$ is thus a proper Lorentz transformation.

Remark 7.14. From the matrices T and A given by (7.58) and (7.56), it is an easy exercise to show that the matrix product (7.61) results in (7.57).

Having shown that for any $A \in \mathrm{SL}(2,\mathbb{C})$, $\boldsymbol{\Lambda}_A$ is a restricted Lorentz transformation, i.e. an element of $\mathrm{SO_o}(3,1)$, we call ***spinor map*** and denote by $\mathscr{S}$ the map from $\mathrm{SL}(2,\mathbb{C})$ to $\mathrm{SO_o}(3,1)$ that associates $\boldsymbol{\Lambda}_A$ to A:

$$\boxed{\begin{array}{llll} \mathscr{S}: \mathrm{SL}(2,\mathbb{C}) & \longrightarrow \mathrm{SO_o}(3,1) & & \\ \qquad A & \longmapsto \boldsymbol{\Lambda}_A : E \longrightarrow E & & \\ & \qquad\qquad \vec{\boldsymbol{v}} \longmapsto \vec{\boldsymbol{v}}' \ / \ \mathscr{H}(\vec{\boldsymbol{v}}') = A\, \mathscr{H}(\vec{\boldsymbol{v}})\, A^{\dagger}, & & \end{array}} \tag{7.62}$$

where $\mathscr{H}$ is the isomorphism (7.52) between the vector spaces E and $\mathrm{Herm}(2,\mathbb{C})$. The explicit form of the spinor map, in terms of the matrix A and the matrix of $\boldsymbol{\Lambda}_A$ in the basis $(\vec{\boldsymbol{e}}_\alpha)$ of E, is provided by (7.56)–(7.57) or, alternatively, by (7.61).

Example 7.2. Let us consider the matrix

$$A = \begin{pmatrix} \cos(\varphi/2) & -\mathrm{i}\sin(\varphi/2) \\ -\mathrm{i}\sin(\varphi/2) & \cos(\varphi/2) \end{pmatrix}, \qquad \varphi \in [0, 2\pi[.$$

It satisfies $\det A = \cos^2(\varphi/2) - (-1)\sin^2(\varphi/2) = 1$ and thus belongs to $\mathrm{SL}(2,\mathbb{C})$. Plugging the coefficients of A into (7.57) and using standard trigonometric identities, we obtain that Λ_A is identical to the matrix (6.39); $\mathscr{S}(A)$ is thus the spatial rotation of plane $\mathrm{Span}(\vec{\boldsymbol{e}}_2, \vec{\boldsymbol{e}}_3)$ and angle φ.

Example 7.3. The matrix

$$A = \begin{pmatrix} \cosh(\psi/2) & \sinh(\psi/2) \\ \sinh(\psi/2) & \cosh(\psi/2) \end{pmatrix}, \qquad \psi \in \mathbb{R}$$

clearly belongs to $\mathrm{SL}(2,\mathbb{C})$: $\det A = \cosh^2(\psi/2) - \sinh^2(\psi/2) = 1$. Plugging the coefficients of A into (7.57), we realize that the matrix Λ_A coincides with (6.43); $\mathscr{S}(A)$ is thus the Lorentz boost of plane $\mathrm{Span}(\vec{e}_0, \vec{e}_1)$ and rapidity ψ.

Example 7.4. The matrix

$$A = \begin{pmatrix} 1 + \mathrm{i}\alpha & -\mathrm{i}\alpha \\ \mathrm{i}\alpha & 1 - \mathrm{i}\alpha \end{pmatrix}, \quad \alpha \in \mathbb{R}$$

is in $\mathrm{SL}(2,\mathbb{C})$: $\det A = 1 + \alpha^2 - \alpha^2 = 1$. Plugging its coefficients into (7.57) leads to the matrix (6.50): $\mathscr{S}(A)$ is thus the null rotation of plane $\mathrm{Span}(\vec{e}_0 + \vec{e}_1, \vec{e}_3)$ and parameter α.

Remark 7.15. We have proved that the spinor map takes its values in $\mathrm{SO_o}(3,1)$ by an algebraic method, namely, by computations showing that $(\Lambda_A)^0{}_0 > 0$ and $\det \Lambda_A = 1$. An alternative proof relies on a topological argument: the Lie group $\mathrm{SL}(2,\mathbb{C})$ is connected (proof below) and the map $\mathscr{S}$ is continuous [it is clear on (7.57) or on (7.61)]. The image $\mathscr{S}(\mathrm{SL}(2,\mathbb{C}))$ must therefore be connected. Moreover, it must contain the identity, since $\mathscr{S}(\mathbb{I}_2) = \mathrm{Id}$. Now, the identity is contained in the connected component $\mathrm{SO_o}(3,1)$ of $\mathrm{O}(3,1)$ (cf. Fig. 7.1); we conclude that $\mathscr{S}(\mathrm{SL}(2,\mathbb{C})) \subset \mathrm{SO_o}(3,1)$. The starting point of this demonstration, i.e. the connectedness of $\mathrm{SL}(2,\mathbb{C})$, is easy to obtain: given an element $A \in \mathrm{SL}(2,\mathbb{C})$, let us construct a path from $\mathbb{I}_2$ to A within $\mathrm{Mat}(2,\mathbb{C})$ by setting

$$\begin{aligned} B : [0,1] &\longrightarrow \mathrm{Mat}(2,\mathbb{C}) \\ t &\longmapsto B(t) := [1 - \lambda(t)]\,\mathbb{I}_2 + \lambda(t)\,A, \end{aligned}$$

where $t \mapsto \lambda(t)$ is a path in $\mathbb{C}$ such that $\lambda(0) = 0$ and $\lambda(1) = 1$. We have then $B(0) = \mathbb{I}_2$ and $B(1) = A$. By making A explicit as in (7.56), we observe that $\det B(t)$ is a second-order polynomial in $\lambda(t)$. It has thus at most two zeros in $\mathbb{C}$ and we can always choose a path $\lambda(t)$ that does not encounter these two zeros. Let then $\mu(t) \in \mathbb{C} \setminus \{0\}$ be one of the two square roots of $\det B(t)$: $\mu(t)^2 = \det B(t)$. The mapping $t \mapsto \tilde{A}(t) := \mu(t)^{-1} B(t)$ defines a path from $\mathbb{I}_2$ to A within $\mathrm{SL}(2,\mathbb{C})$, since by construction $\det \tilde{A}(t) = 1$. We conclude that $\mathrm{SL}(2,\mathbb{C})$ is path-connected and thus connected.

The starting and the arrival sets of $\mathscr{S}$ being groups, it is natural to ask whether $\mathscr{S}$ is a group homomorphism, i.e. a map that preserves the group structure (cf. Appendix A). The answer is positive:

> The spinor map $\mathscr{S}$ is a homomorphism between the special linear group $\mathrm{SL}(2,\mathbb{C})$ and the restricted Lorentz group $\mathrm{SO_o}(3,1)$.

Proof. If $(A, B) \in \mathrm{SL}(2,\mathbb{C})^2$ and $H \in \mathrm{Herm}(2,\mathbb{C})$, then

$$\Phi_A \circ \Phi_B(H) = A(B\ H\ B^\dagger)A^\dagger = AB\ H\ B^\dagger A^\dagger = AB\ H\ (AB)^\dagger = \Phi_{AB}(H),$$

which shows that $\Phi_A \circ \Phi_B = \Phi_{AB}$ and, via (7.55), that $\boldsymbol{\Lambda}_A \circ \boldsymbol{\Lambda}_B = \boldsymbol{\Lambda}_{AB}$. □

7.5.2 The Spinor Map from SU(2) to SO(3)

A particular subgroup of $\mathrm{SL}(2,\mathbb{C})$ is the ***special unitary group*** SU(2), defined by

$$\mathrm{SU}(2) := \{A \in \mathrm{SL}(2,\mathbb{C}), \quad A^{-1} = A^\dagger\}. \tag{7.63}$$

It is clear that it is a subgroup of $\mathrm{SL}(2,\mathbb{C})$. If the coefficients of A are denoted by $(\alpha, \beta, \gamma, \delta)$ as in (7.56), then, thanks to the property $\det A = 1$,

$$A^{-1} = \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}.$$

We deduce immediately that

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SU}(2) \iff \begin{cases} \gamma = -\bar{\beta} \\ \delta = \bar{\alpha} \\ |\alpha|^2 + |\beta|^2 = 1. \end{cases} \tag{7.64}$$

Example 7.5. The identity matrix and the three Pauli matrices (7.51) multiplied by i are all elements of SU(2):

$$\sigma_0 = \mathbb{I}_2,\ \mathrm{i}\sigma_1 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix},\ \mathrm{i}\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ \mathrm{i}\sigma_3 = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}. \tag{7.65}$$

Similarly, the matrix A of Example 7.2 p. 241 is in SU(2).

Using $\gamma = -\bar{\beta}$, $\delta = \bar{\alpha}$ and setting $\alpha =: x_1 + \mathrm{i}x_2$ and $\beta =: x_3 + \mathrm{i}x_4$, with $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$, we get $|\alpha|^2 = x_1^2 + x_2^2$ and $|\beta|^2 = x_3^2 + x_4^2$, so that (7.64) can be expressed as

$$A \in \mathrm{SU}(2) \iff x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1. \tag{7.66}$$

The right-hand side being the equation of the ***hypersphere*** $\mathbb{S}^3$ in the Euclidean space $\mathbb{R}^4$, we deduce that, as a manifold over $\mathbb{R}$, SU(2) can be identified to $\mathbb{S}^3$:

$$\boxed{\mathrm{SU}(2) \sim \mathbb{S}^3}. \tag{7.67}$$

Since $\mathbb{S}^3$ is a connected compact manifold, we conclude that SU(2) is a connected compact Lie group, of dimension 3 on $\mathbb{R}$. For $A \in \mathrm{SU}(2)$, the image of the vector $\vec{\boldsymbol{e}}_0$ by $\boldsymbol{\Lambda}_A$ is given by the first column of the matrix (7.57); thanks to the properties (7.64), we note that $(\Lambda_A)^\alpha{}_0 = \delta^\alpha{}_0$; hence,

$$\boldsymbol{\Lambda}_A(\vec{\boldsymbol{e}}_0) = \vec{\boldsymbol{e}}_0.$$

This shows that $\boldsymbol{\Lambda}_A$ is a spatial rotation, in the hyperplane $E_{\boldsymbol{e}_0}$. By identifying the set of all spatial rotations in $(E_{\boldsymbol{e}_0}, \boldsymbol{g})$ with the group of rotations in the three-dimensional Euclidean space, SO(3), we may state that the spinor map sends SU(2) to SO(3):

$$\mathscr{S} : \ \mathrm{SU}(2) \longrightarrow \mathrm{SO}(3). \tag{7.68}$$

Remark 7.16. Via the isomorphism $\mathscr{H}$ defined by (7.52), the hyperplane $E_{\boldsymbol{e}_0}$ corresponds to the vector space of traceless 2×2 Hermitian matrices [it suffices to set $v^0 = 0$ in (7.49)].

From (7.66), it is clear that $|x_1| \leq 1$. Let us then introduce $\varphi \in [0, 2\pi]$ such that $x_1 =: \cos(\varphi/2)$. Similarly, let us introduce $(n^1, n^2, n^3) \in \mathbb{R}^3$ such that $x_2 =: -n^3 \sin(\varphi/2)$, $x_3 =: -n^2 \sin(\varphi/2)$ and $x_4 =: -n^1 \sin(\varphi/2)$. Then any element of SU(2) can be written as

$$A = \begin{pmatrix} \cos\frac{\varphi}{2} - \mathrm{i}\, n^3 \sin\frac{\varphi}{2} & -\sin\frac{\varphi}{2}(n^2 + \mathrm{i}\, n^1) \\ \sin\frac{\varphi}{2}(n^2 - \mathrm{i}\, n^1) & \cos\frac{\varphi}{2} + \mathrm{i}\, n^3 \sin\frac{\varphi}{2} \end{pmatrix}, \tag{7.69}$$

$$\text{with} \quad (n^1)^2 + (n^2)^2 + (n^3)^2 = 1. \tag{7.70}$$

One can rewrite this relation in terms of the matrices (7.65) as

$$\boxed{A = \cos\frac{\varphi}{2}\, \mathbb{I}_2 - \sin\frac{\varphi}{2}\left(n^1\, \mathrm{i}\sigma_1 + n^2\, \mathrm{i}\sigma_2 + n^3\, \mathrm{i}\sigma_3\right)}. \tag{7.71}$$

We read on (7.69) that $\alpha = \cos(\varphi/2) - \mathrm{i}\, n^3 \sin(\varphi/2)$ and $\beta = -\sin(\varphi/2)(n^2 - \mathrm{i}\, n^1)$. Inserting these values, as well as $\gamma = -\bar{\beta}$ and $\delta = \bar{\alpha}$ into the matrix (7.57), we get the expression of the image of a generic element of SU(2) by the spinor map:

$$(\Lambda_A)^\alpha{}_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi + (n^1)^2(1-\cos\varphi) & n^1 n^2(1-\cos\varphi) - n^3 \sin\varphi & n^1 n^3(1-\cos\varphi) + n^2 \sin\varphi \\ 0 & n^1 n^2(1-\cos\varphi) + n^3 \sin\varphi & \cos\varphi + (n^2)^2(1-\cos\varphi) & n^2 n^3(1-\cos\varphi) - n^1 \sin\varphi \\ 0 & n^1 n^3(1-\cos\varphi) - n^2 \sin\varphi & n^2 n^3(1-\cos\varphi) + n^1 \sin\varphi & \cos\varphi + (n^3)^2(1-\cos\varphi) \end{pmatrix}.$$

By comparing with Rodrigues formula (6.41), we note that this is the matrix of a rotation in $E_{\boldsymbol{e}_0}$, of angle φ and whose axis is defined by the vector $\vec{\boldsymbol{n}} := n^i\, \vec{\boldsymbol{e}}_i$. Condition (7.70) ensures that $\vec{\boldsymbol{n}}$ is a unit vector. φ and $\vec{\boldsymbol{n}}$ being arbitrary, we have shown that any rotation in E_{e_0} is the image of an element of SU(2) by the spinor map. In other words, the mapping $\mathscr{S}:\ \mathrm{SU}(2) \longrightarrow \mathrm{SO}(3)$ is surjective. On the other side, $\mathscr{S}$ is not injective; for instance, the identity of SO(3) has two inverse images: $\mathbb{I}_2$ [set $\varphi = 0$ in (7.69)] and $-\mathbb{I}_2$ [set $\varphi = 2\pi$ in (7.69)]. We shall elaborate more on this below.

Remark 7.17. The ***field of quaternions*** $\mathbb{H}$ can be defined as the subalgebra of $\mathrm{Mat}(2,\mathbb{C})$, considered as an algebra over $\mathbb{R}$ (cf. Appendix A), generated by the matrices (7.65):

$$\mathbb{H} := \mathrm{Span}_{\mathbb{R}}(\mathbf{1}, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}), \tag{7.72}$$

where

$$\mathbf{1} := \mathbb{I}_2, \quad \boldsymbol{i} := -\mathrm{i}\sigma_1, \quad \boldsymbol{j} := -\mathrm{i}\sigma_2, \quad \text{and} \quad \boldsymbol{k} := -\mathrm{i}\sigma_3 \tag{7.73}$$

and the index $\mathbb{R}$ on Span reminds one that only linear combinations with real coefficients of matrices $\mathbf{1}$, $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are allowed. $\mathbb{H}$ is an algebra of dimension 4 over $\mathbb{R}$, such that any nonzero element admits an inverse for the multiplication. It is therefore a (noncommutative) field, which extends the field of real numbers (dimension 1 over $\mathbb{R}$) and the field of complex numbers (dimension 2 over $\mathbb{R}$). The matrices (7.73) obey ***Hamilton relations***[11]

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -\mathbf{1}, \tag{7.74}$$

as it can easily be checked from (7.65). The conjugate and the norm of a quaternion $q = t\mathbf{1} + u\boldsymbol{i} + v\boldsymbol{j} + w\boldsymbol{k}$ are defined by, respectively, $q^* := t\mathbf{1} - u\boldsymbol{i} - v\boldsymbol{j} - w\boldsymbol{k}$ and $\|q\| := \sqrt{q\,q^*} = \sqrt{t^2 + u^2 + v^2 + w^2}$. In view of (7.73), the writing (7.71) of a generic element of SU(2) becomes

$$A = \cos\frac{\varphi}{2}\,\mathbf{1} + \sin\frac{\varphi}{2}\left(n^1\,\boldsymbol{i} + n^2\,\boldsymbol{j} + n^3\,\boldsymbol{k}\right). \tag{7.75}$$

Given the constraint (7.70), SU(2) appears then as the set of quaternions of unit norm:

$$\mathrm{SU}(2) = \{q \in \mathbb{H}, \quad \|q\| = 1\}. \tag{7.76}$$

Since (i) the spinor map $\mathscr{S}$ is a group homomorphism and (ii) every rotation in SO(3) admits an inverse image by $\mathscr{S}$, the composition of two rotations is reduced to a multiplication in $\mathbb{H}$. Accordingly, quaternions are used today to compute products of rotations in computer graphics, cybernetics and celestial mechanics (satellite motions).

[11] **William R. Hamilton** (1805–1865): Irish mathematician and physicist. In 1927, he founded what is called today the *Hamiltonian mechanics* (cf. Chap. 11). He introduced the quaternions in 1843 and imagined the relations (7.74) while walking on a Dublin bridge on 16 October of that year.

Remark 7.18. The surjectivity of the spinor map $\mathrm{SU}(2) \longrightarrow \mathrm{SO}(3)$ can also be established by considering the parametrization of SO(3) by the three Euler angles $(\hat{\varphi}, \hat{\theta}, \hat{\psi})$, instead of φ and $\vec{\boldsymbol{n}}$. Let us recall that the ***Euler angles*** of a spatial rotation $\boldsymbol{R}$ are defined as the three angles linking the basis $(\vec{\boldsymbol{e}}_i)$ of E_{e_0} to its image $(\vec{\boldsymbol{\varepsilon}}_i) := (\boldsymbol{R}(\vec{\boldsymbol{e}}_i))$ as follows. In E_{e_0}, one calls ***line of nodes*** the intersection of planes $\mathrm{Span}(\vec{\boldsymbol{e}}_1, \vec{\boldsymbol{e}}_2)$ and $\mathrm{Span}(\vec{\boldsymbol{\varepsilon}}_1, \vec{\boldsymbol{\varepsilon}}_2)$. The Euler angle $\hat{\varphi}$ is then the angle between $\vec{\boldsymbol{e}}_1$ and the line of nodes, $\hat{\theta}$ is the angle between $\vec{\boldsymbol{e}}_3$ and $\vec{\boldsymbol{\varepsilon}}_3$ and $\hat{\psi}$ is the angle between the line of nodes and $\vec{\boldsymbol{\varepsilon}}_1$. It results from these definitions that $\boldsymbol{R}$ can be written as the product of three rotations:

$$\boldsymbol{R} = \boldsymbol{R}_3 \circ \boldsymbol{R}_2 \circ \boldsymbol{R}_1. \tag{7.77}$$

$\boldsymbol{R}_1$ is the rotation of plane $\mathrm{Span}(\vec{\boldsymbol{e}}_1, \vec{\boldsymbol{e}}_2)$ and angle $\hat{\varphi}$; it maps $\vec{\boldsymbol{e}}_1$ onto the line of nodes. Setting $\vec{\boldsymbol{e}}'_\alpha := \boldsymbol{R}_1(\vec{\boldsymbol{e}}_\alpha)$, $\boldsymbol{R}_2$ is the rotation of plane $\mathrm{Span}(\vec{\boldsymbol{e}}'_2, \vec{\boldsymbol{e}}'_3)$ and angle $\hat{\theta}$; it maps $\vec{\boldsymbol{e}}'_3 = \vec{\boldsymbol{e}}_3$ to $\vec{\boldsymbol{\varepsilon}}_3$. Setting $\vec{\boldsymbol{e}}''_\alpha := \boldsymbol{R}_2(\vec{\boldsymbol{e}}'_\alpha)$, $\boldsymbol{R}_3$ is the rotation of plane $\mathrm{Span}(\vec{\boldsymbol{e}}''_1, \vec{\boldsymbol{e}}''_2)$ and angle $\hat{\psi}$; it maps $\vec{\boldsymbol{e}}''_1$ and $\vec{\boldsymbol{e}}''_2$ to, respectively, $\vec{\boldsymbol{\varepsilon}}_1$ and $\vec{\boldsymbol{\varepsilon}}_2$.

Denoting by R_1 the matrix of $\boldsymbol{R}_1$ in the basis $(\vec{\boldsymbol{e}}_\alpha)$, by R'_2 that of $\boldsymbol{R}_2$ in the basis $(\vec{\boldsymbol{e}}'_\alpha)$ and by R''_3 that of $\boldsymbol{R}_3$ in the basis $(\vec{\boldsymbol{e}}''_\alpha)$, it is easy to show that the matrix of $\boldsymbol{R}$ in the basis $(\vec{\boldsymbol{e}}_\alpha)$ is

$$R = R_1\, R'_2\, R''_3. \tag{7.78}$$

The matrices in the right-hand side having a very simple form, of the type (6.39), the matrix R is easily computed:

$$R^\alpha{}_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\hat{\varphi}\cos\hat{\psi} - \cos\hat{\theta}\sin\hat{\varphi}\sin\hat{\psi} & -\cos\hat{\varphi}\sin\hat{\psi} - \cos\hat{\theta}\sin\hat{\varphi}\cos\hat{\psi} & \sin\hat{\theta}\sin\hat{\varphi} \\ 0 & \sin\hat{\varphi}\cos\hat{\psi} + \cos\hat{\theta}\cos\hat{\varphi}\sin\hat{\psi} & -\sin\hat{\varphi}\sin\hat{\psi} + \cos\hat{\theta}\cos\hat{\varphi}\cos\hat{\psi} & -\sin\hat{\theta}\cos\hat{\varphi} \\ 0 & \sin\hat{\theta}\sin\hat{\psi} & \sin\hat{\theta}\cos\hat{\psi} & \cos\hat{\theta} \end{pmatrix}. \tag{7.79}$$

Incidentally, note that the relation (7.78) is a matrix product in the reverse order with respect of the product $R = R_3\, R_2\, R_1$ that one would infer from the composition law (7.77). But in this last case, the R_i's would be the matrices of the $\boldsymbol{R}_i$'s all taken in the same basis $(\vec{\boldsymbol{e}}_\alpha)$, contrary to R'_2 and R''_3, which are relative to the bases $(\vec{\boldsymbol{e}}'_\alpha)$ and $(\vec{\boldsymbol{e}}''_\alpha)$ and have a simpler form.

One calls ***Cayley–Klein parameters*** of the rotation $\boldsymbol{R}$ the two complex numbers defined by

$$\begin{cases} \alpha := -\cos\dfrac{\hat{\theta}}{2}\, \mathrm{e}^{-\mathrm{i}(\hat{\varphi}+\hat{\psi})/2} \\ \beta := \mathrm{i}\sin\dfrac{\hat{\theta}}{2}\, \mathrm{e}^{\mathrm{i}(\hat{\psi}-\hat{\varphi})/2}. \end{cases} \tag{7.80}$$

They satisfy $|\alpha|^2 + |\beta|^2 = 1$. As a result, the matrix A constructed from α and β according to (7.64) belongs to SU(2). By plugging these values of α and β, as well as $\gamma = -\bar{\beta}$ and $\delta = \bar{\alpha}$ in (7.57), we obtain the matrix (7.79). We conclude that $\boldsymbol{R} = \mathscr{S}(A)$ and that the map $\mathscr{S} :\ \mathrm{SU}(2) \longrightarrow \mathrm{SO}(3)$ is surjective.

7.5.3 The Spinor Map and Lorentz Boosts

Let us consider an element of $\mathrm{SL}(2,\mathbb{C})$ that is Hermitian: $A \in \mathrm{SL}(2,\mathbb{C}) \cap \mathrm{Herm}(2,\mathbb{C})$. A can be written as in (7.50): $A = v^\mu \sigma_\mu = \mathscr{H}(v^\mu \vec{\boldsymbol{e}}_\mu)$, with $(v^\mu) \in \mathbb{R}^4$. In view of (7.53), the condition $\det A = 1$ (belonging to $\mathrm{SL}(2,\mathbb{C})$) is then equivalent to

$$(v^0)^2 - (v^1)^2 - (v^2)^2 - (v^3)^2 = 1. \tag{7.81}$$

This implies $v^0 \geq 1$ or $v^0 \leq -1$. Let us consider the first case. We can then introduce $\psi \in \mathbb{R}$ so that $v^0 =: \cosh(\psi/2)$ and define $n^i := v^i/\sinh(\psi/2)$ if $\psi \neq 0$ and $n^i := (1,0,0)$ if $\psi = 0$. The condition (7.81) reduces then to

$$(n^1)^2 + (n^2)^3 + (n^3)^2 = 1.$$

We conclude that any Hermitian matrix of $\mathrm{SL}(2,\mathbb{C})$ can be written as

$$\boxed{\pm A = \cosh\frac{\psi}{2}\,\mathbb{I}_2 + \sinh\frac{\psi}{2}\left(n^1\sigma_1 + n^2\sigma_2 + n^3\sigma_3\right)}, \tag{7.82}$$

where (n^i) are the components of a unit vector of $(E_{\boldsymbol{e}_0}, \boldsymbol{g})$ and the sign $+$ (resp. $-$) corresponds to the case $v^0 \geq 1$ (resp. $v^0 \leq -1$). This expression is the "hyperbolic" counterpart of (7.71).

The components $(\alpha, \beta, \gamma, \delta)$ of A, as defined by (7.56), are

$$\alpha = \cosh\frac{\psi}{2} + n^3 \sinh\frac{\psi}{2}, \quad \beta = \bar{\gamma} = \sinh\frac{\psi}{2}(n^1 - in^2), \quad \delta = \cosh\frac{\psi}{2} - n^3 \sinh\frac{\psi}{2}.$$

Plugging these values in (7.57), we obtain the matrix of the image of A by the spinor map:

$$(\Lambda_A)^\alpha{}_\beta = \begin{pmatrix} \cosh\psi & n^1\sinh\psi & n^2\sinh\psi & n^3\sinh\psi \\ n^1\sinh\psi & 1+(\cosh\psi-1)(n^1)^2 & (\cosh\psi-1)n^1n^2 & (\cosh\psi-1)n^1n^3 \\ n^2\sinh\psi & (\cosh\psi-1)n^1n^2 & 1+(\cosh\psi-1)(n^2)^2 & (\cosh\psi-1)n^2n^3 \\ n^3\sinh\psi & (\cosh\psi-1)n^1n^3 & (\cosh\psi-1)n^2n^3 & 1+(\cosh\psi-1)(n^3)^2 \end{pmatrix}. \tag{7.83}$$

By comparing with (6.71), we recognize, via (6.46) and (6.47), the matrix of the Lorentz boost of rapidity ψ and plane $\Pi = \mathrm{Span}(\vec{\boldsymbol{e}}_0, \vec{\boldsymbol{n}})$ with $\vec{\boldsymbol{n}} = n^i \vec{\boldsymbol{e}}_i$. We have thus shown that any Lorentz boost whose plane contains $\vec{\boldsymbol{e}}_0$ admits an inverse image by the spinor map $\mathscr{S}$.

Example 7.6. For $n^i = (1,0,0)$, we recognize in (7.82) the matrix A of Example 7.3 p. 241 and (7.83) reduces to the matrix (6.43) of a Lorentz boost in an adapted basis.

7.5.4 Covering of the Restricted Lorentz Group by SL(2,C)

We have shown in Sect. 7.5.2 that any rotation whose plane is normal to $\vec{e}_0$ admits an inverse image by $\mathscr{S}$. In Sect. 7.5.3, we have shown that the same property holds for any Lorentz boost whose plane contains $\vec{e}_0$. Since (i) any element of the restricted Lorentz group $\mathrm{SO}_o(3,1)$ results from the composition of a spatial rotation and a Lorentz boost, both having the above properties with respect to $\vec{e}_0$ [polar decomposition (6.57)] and (ii) $\mathscr{S}$ is a homomorphism between the groups $\mathrm{SL}(2,\mathbb{C})$ and $\mathrm{SO}_o(3,1)$, we conclude that any element of $\mathrm{SO}_o(3,1)$ has an inverse image by $\mathscr{S}$. In other words, the spinor map (7.62) is surjective.

We have noticed above that $\mathscr{S}$ is not injective. Let us show that actually any element of $\mathrm{SO}_o(3,1)$ has exactly two inverse images by $\mathscr{S}$. Let A and B be two elements of $\mathrm{SL}(2,\mathbb{C})$ such that $\mathscr{S}(A) = \mathscr{S}(B)$. This relation is equivalent to $\boldsymbol{\Lambda}_A \circ \boldsymbol{\Lambda}_B^{-1} = \mathrm{Id}$. Now, since $\mathscr{S} : A \mapsto \boldsymbol{\Lambda}_A$ is a homomorphism, $\boldsymbol{\Lambda}_B^{-1} = \boldsymbol{\Lambda}_{B^{-1}}$ and $\boldsymbol{\Lambda}_A \circ \boldsymbol{\Lambda}_{B^{-1}} = \boldsymbol{\Lambda}_{AB^{-1}}$. Hence,

$$\mathscr{S}(A) = \mathscr{S}(B) \iff \mathscr{S}(AB^{-1}) = \mathrm{Id}. \tag{7.84}$$

The problem is then reduced to determine the inverse images of the identity. From (7.61),

$$\mathscr{S}(A) = \mathrm{Id} \iff T^{-1}(A \otimes \bar{A})\, T = \mathbb{I}_4 \iff (A \otimes \bar{A})\, T = T \iff A \otimes \bar{A} = \mathbb{I}_4.$$

Expressing A in terms of its coefficients, according to (7.56), and using the definition (7.60) of the Kronecker product, the relation $A \otimes \bar{A} = \mathbb{I}_4$ is equivalent to

$$\alpha\bar{\alpha} = \alpha\bar{\delta} = \delta\bar{\alpha} = \delta\bar{\delta} = 1 \quad \text{and} \quad \beta = \gamma = 0.$$

If one adds the condition $\det A = 1$ [A belongs to $\mathrm{SL}(2,\mathbb{C})$], one obtains moreover $\alpha\delta - \beta\gamma = 1$. There are then only two solutions: $(\alpha,\delta) = (1,1)$ or $(\alpha,\delta) = (-1,-1)$. This shows that the only inverse images of the identity by $\mathscr{S}$ are the matrices $\mathbb{I}_2$ and $-\mathbb{I}_2$. The equivalence (7.84) becomes then

$$\forall (A,B) \in \mathrm{SL}(2,\mathbb{C})^2, \quad \mathscr{S}(A) = \mathscr{S}(B) \iff (A = B \ \text{ or } \ A = -B)\,.$$

Hence, each element of $\mathrm{SO}_o(3,1)$ has exactly two inverse images by the spinor map, which are opposite from each other. Since $\{\mathbb{I}_2, -\mathbb{I}_2\}$ is a normal subgroup of $\mathrm{SL}(2,\mathbb{C})$ (cf. Appendix A), we may summarize the preceding results as follows:

The restricted Lorentz group $\mathrm{SO}_o(3,1)$ is isomorphic to the quotient of the special linear group $\mathrm{SL}(2,\mathbb{C})$ by $\{\mathbb{I}_2, -\mathbb{I}_2\}$:

$$\boxed{\mathrm{SO}_o(3,1) \simeq \mathrm{SL}(2,\mathbb{C})/\{\mathbb{I}_2, -\mathbb{I}_2\}}. \tag{7.85}$$

The subgroup of $\mathrm{SO}_o(3,1)$ formed by the rotations in a fixed spacelike hyperplane is isomorphic to the quotient of the special unitary group of index 2 by $\{\mathbb{I}_2, -\mathbb{I}_2\}$:

$$\boxed{\mathrm{SO}(3) \simeq \mathrm{SU}(2)/\{\mathbb{I}_2, -\mathbb{I}_2\}}. \tag{7.86}$$

Note that SU(2) is a subgroup of $\mathrm{SL}(2,\mathbb{C})$. The above isomorphisms are implemented by the spinor map $\mathscr{S}$ defined by (7.62). The result (7.85) is expressed by stating that $\mathrm{SL}(2,\mathbb{C})$ is a ***double covering group*** of $\mathrm{SO}_o(3,1)$.

Remark 7.19. In the language of Lie group theory, $\mathrm{SL}(2,\mathbb{C})$ is the ***universal covering group*** of $\mathrm{SO}_o(3,1)$ (Bacry 1967; Deheuvels 1981; Gallier 2011; Godement 2004; Mneimné and Testard 1986). Any connected Lie group admits indeed a universal covering group, the latter being simply connected. It is easy to see that $\mathrm{SO}_o(3,1)$ is not simply connected. For instance, a path in $\mathrm{SO}_o(3,1)$ made of spatial rotations of fixed plane and whose angles vary from 0 to 2π starts from the identity and ends on it. It is thus a loop. This loop cannot by continuously transformed into a point (the identity). Hence, $\mathrm{SO}_o(3,1)$ is not simply connected. On the contrary, it can be shown that $\mathrm{SL}(2,\mathbb{C})$ is simply connected (cf., e.g. Sect. 2.7 of Godement (2004)).

7.5.5 Existence of Null Eigenvectors

An interesting application of the covering of $\mathrm{SO}_o(3,1)$ by $\mathrm{SL}(2,\mathbb{C})$ is the following result, which we have used as starting point for the classification of Lorentz transformations in Sect. 6.4:

Any restricted Lorentz transformation admits a null eigenvector or, equivalently, an invariant null direction. Moreover, the corresponding eigenvalue is strictly positive.

Proof. Let us consider $\boldsymbol{\Lambda} \in \mathrm{SO}_o(3,1)$. From the surjectivity of the spinor map, there exists $A \in \mathrm{SL}(2,\mathbb{C})$ such that $\boldsymbol{\Lambda} = \mathscr{S}(A)$. Since A is a matrix over $\mathbb{C}$, its characteristic polynomial admits at least one (complex) zero, which means that A

admits at least one eigenvalue $\mu \in \mathbb{C}$. One has necessarily $\mu \neq 0$ since $\det A = 1 \neq 0$. Let $U = (u, v) \in \mathbb{C}^2$ be the corresponding eigenvector:

$$A\,U = \mu\,U.$$

From the components of U, let us construct the matrix

$$H := \begin{pmatrix} \bar{u}u & \bar{v}u \\ \bar{u}v & \bar{v}v \end{pmatrix}. \tag{7.87}$$

We notice that $H \in \mathrm{Herm}(2, \mathbb{C})$ [cf. Eq. (7.48)]. By the isomorphism (7.52), there exists then a vector $\vec{\boldsymbol{v}} \in E$ such that $H = \mathscr{H}(\vec{\boldsymbol{v}})$. Note that $\vec{\boldsymbol{v}} \neq 0$ since $H \neq 0$. From the definition (7.62) of the spinor map, $\boldsymbol{\Lambda}(\vec{\boldsymbol{v}})$ is the image of the matrix $A\,H\,A^\dagger$ by $\mathscr{H}^{-1}$. Now, denoting the coefficients of A as in (7.56), we have

$$A\,H = \begin{pmatrix} \bar{u}(\alpha u + \beta v) & \bar{v}(\alpha u + \beta v) \\ \bar{u}(\gamma u + \delta v) & \bar{v}(\gamma u + \delta v) \end{pmatrix}.$$

Since $(\alpha u + \beta v,\ \gamma u + \delta v) = A\,U$ and U is an eigenvector of A, we have $\alpha u + \beta v = \mu u$ and $\gamma u + \delta v = \mu v$, so that the above equation becomes

$$A\,H = \begin{pmatrix} \bar{u}\mu u & \bar{v}\mu u \\ \bar{u}\mu v & \bar{v}\mu v \end{pmatrix} = \mu H.$$

We deduce immediately that (recall that $H^\dagger = H$)

$$A\,H\,A^\dagger = \mu H\,A^\dagger = \mu(A\,H^\dagger)^\dagger = \mu(A\,H)^\dagger = \mu(\mu H)^\dagger = \mu\bar{\mu}H = |\mu|^2 H.$$

This result is equivalent to

$$\boldsymbol{\Lambda}(\vec{\boldsymbol{v}}) = |\mu|^2\,\vec{\boldsymbol{v}}.$$

$\vec{\boldsymbol{v}}$ is thus an eigenvector of $\boldsymbol{\Lambda}$, of eigenvalue $|\mu|^2 > 0$. Moreover, we read on (7.87) that $\det H = 0$, which, given (7.53), implies $\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{v}} = 0$. The vector $\vec{\boldsymbol{v}}$ is thus null. □

7.5.6 Lie Algebra of SL(2,C)

Let us determine the Lie algebra of the group $\mathrm{SL}(2, \mathbb{C})$, as we did for the Lorentz group in Sect. 7.3, namely, by studying infinitesimal transformations. An element of $\mathrm{SL}(2, \mathbb{C})$ close to the identity can be written as

$$A = \mathbb{I}_2 + \varepsilon B, \tag{7.88}$$

where $\varepsilon \in \mathbb{R}$ is a small parameter and $B \in \mathrm{Mat}(2, \mathbb{C})$. At first order in ε, we have

$$\det A = 1 + \varepsilon \operatorname{tr} B. \tag{7.89}$$

Proof. The general formula governing the variation of the determinant of an invertible matrix is

$$\delta \ln \det A = \operatorname{tr}\left(A^{-1}\, \delta A\right), \tag{7.90}$$

where δ stands for any variation that obeys Leibniz rule and tr is the trace operator. Applying this formula to $\delta = d/d\varepsilon$, we get (7.89). □

The condition $A \in \mathrm{SL}(2, \mathbb{C})$, i.e. $\det A = 1$, is thus equivalent to $\operatorname{tr} B = 0$. We conclude that the Lie algebra of $\mathrm{SL}(2, \mathbb{C})$ is formed by the 2×2 complex matrices of vanishing trace[12]:

$$\boxed{\mathrm{sl}(2, \mathbb{C}) = \{B \in \mathrm{Mat}(2, \mathbb{C}),\ \operatorname{tr} B = 0\}}. \tag{7.91}$$

It is clear that $\mathrm{sl}(2, \mathbb{C})$ is a vector space. Its dimension over $\mathbb{R}$ is $8 - 2 = 6$ (the constraint $\operatorname{tr} B = 0$ in $\mathbb{C}$ being equivalent to 2 equalities in $\mathbb{R}$); this is the same dimension as $\mathrm{SL}(2, \mathbb{C})$ as a Lie group over $\mathbb{R}$, as it should be. The Lie bracket associated with $\mathrm{sl}(2, \mathbb{C})$ is nothing but the commutator of matrices:

$$[B_1, B_2] := B_1\, B_2 - B_2\, B_1.$$

This operator is internal to $\mathrm{sl}(2, \mathbb{C})$ for it preserves the vanishing of the trace: $\operatorname{tr}[B_1, B_2] = \operatorname{tr}(B_1\, B_2) - \operatorname{tr}(B_2\, B_1) = 0$, thanks to the general property $\operatorname{tr}(B_1\, B_2) = \operatorname{tr}(B_2\, B_1)$. Moreover, the commutator satisfies the three properties (7.11)–(7.13).

A basis of the vector space $\mathrm{sl}(2, \mathbb{C})$ is formed by the Pauli matrices (7.51), augmented by their products by i:

$$\boxed{\mathrm{sl}(2, \mathbb{C}) = \mathrm{Span}_{\mathbb{R}}(\sigma_1,\ \sigma_2,\ \sigma_3,\ \mathrm{i}\sigma_1,\ \mathrm{i}\sigma_2,\ \mathrm{i}\sigma_3)}. \tag{7.92}$$

Let us compute the image under the spinor map of an element $A \in \mathrm{SL}(2, \mathbb{C})$ close to the identity. Let $\vec{\boldsymbol{v}} \in E$ and $H := \mathscr{H}(\vec{\boldsymbol{v}}) \in \mathrm{Herm}(2, \mathbb{C})$. Setting $H' := \Phi_A(H) = A\, H\, A^\dagger$ and substituting (7.88) for A, we get, at first order in ε,

$$H' = (\mathbb{I}_2 + \varepsilon B)\, H\, \underbrace{(\mathbb{I}_2 + \varepsilon B)^\dagger}_{\mathbb{I}_2 + \varepsilon B^\dagger} \simeq H + \varepsilon(B\, H + H\, B^\dagger).$$

We deduce that

$$\mathscr{S}(A) = \boldsymbol{\Lambda}_A = \mathrm{Id} + \varepsilon \boldsymbol{L},$$

[12] As for that of the Lorentz group, the Lie algebra of $\mathrm{SL}(2, \mathbb{C})$ is denoted by the same letters than the group but in lower case.

where $\boldsymbol{L}$ is the endomorphism of E that corresponds, via $\mathscr{H}$, to the following endomorphism of $\mathrm{Herm}(2,\mathbb{C})$:

$$\begin{aligned} \Phi'_B : \mathrm{Herm}(2,\mathbb{C}) &\longrightarrow \mathrm{Herm}(2,\mathbb{C}) \\ H \quad &\longmapsto B\,H + H\,B^\dagger. \end{aligned} \tag{7.93}$$

The map Φ'_B is well defined because if $H' := \Phi'_B(H)$, then $H'^\dagger = (B\,H)^\dagger + (H\,B^\dagger)^\dagger = H\,B^\dagger + B\,H = H'$, which shows that $H' \in \mathrm{Herm}(2,\mathbb{C})$. Since $\boldsymbol{\Lambda}_A \in \mathrm{SO_o}(3,1)$, $\boldsymbol{L}$ is necessarily in the Lie algebra of the Lorentz group [cf. Eq. (7.2)]. Thus, the spinor map induces a mapping[13] between the Lie algebra of $\mathrm{SL}(2,\mathbb{C})$ (where B takes its values) to the Lie algebra of $\mathrm{SO_o}(3,1)$:

$$\boxed{\begin{aligned} \mathscr{S}' : \mathrm{sl}(2,\mathbb{C}) &\longrightarrow \mathrm{so}(3,1) \\ B \quad &\longmapsto \boldsymbol{L} : E \longrightarrow E \\ &\qquad\quad \vec{\boldsymbol{v}} \longmapsto \vec{\boldsymbol{v}}'/\mathscr{H}(\vec{\boldsymbol{v}}') = B\,\mathscr{H}(\vec{\boldsymbol{v}}) + \mathscr{H}(\vec{\boldsymbol{v}})B^\dagger. \end{aligned}} \tag{7.94}$$

In a manner analogous to (7.55), we can write

$$\mathscr{S}'(B) := \mathscr{H}^{-1} \circ \Phi'_B \circ \mathscr{H}. \tag{7.95}$$

It is clear that $\mathscr{S}'$ is a linear map between the vector spaces $\mathrm{sl}(2,\mathbb{C})$ and $\mathrm{so}(3,1)$. Moreover, $\mathscr{S}'$ preserves the Lie bracket:

$$\forall (B_1, B_2) \in \mathrm{sl}(2,\mathbb{C})^2, \quad \mathscr{S}'([B_1, B_2]) = [\mathscr{S}'(B_1), \mathscr{S}'(B_2)]. \tag{7.96}$$

Proof. If $\boldsymbol{L}_1 := \mathscr{S}'(B_1)$ and $\boldsymbol{L}_2 := \mathscr{S}'(B_2)$, we have, from (7.9), $[\boldsymbol{L}_1, \boldsymbol{L}_2] = \boldsymbol{L}_1 \circ \boldsymbol{L}_2 - \boldsymbol{L}_2 \circ \boldsymbol{L}_1$, so that, via $H = \mathscr{H}(\vec{\boldsymbol{v}})$ and $H' = \mathscr{H}([\boldsymbol{L}_1, \boldsymbol{L}_2](\vec{\boldsymbol{v}}))$,

$$\begin{aligned} H' &= \Phi'_{B_1} \circ \Phi'_{B_2}(H) - \Phi'_{B_2} \circ \Phi'_{B_1}(H) \\ &= B_1(B_2\,H + H\,B_2^\dagger) + (B_2\,H + H\,B_2^\dagger)B_1^\dagger - B_2(B_1\,H + H\,B_1^\dagger) \\ &\quad -(B_1\,H + H\,B_1^\dagger)B_2^\dagger \\ &= (B_1\,B_2 - B_2\,B_1)\,H + H(B_1\,B_2 - B_2\,B_1)^\dagger = \Phi'_{[B_1,B_2]}(H). \end{aligned}$$

This shows that $[\boldsymbol{L}_1, \boldsymbol{L}_2] = \mathscr{S}'([B_1, B_2])$ and establishes (7.96). □

Since it preserves the Lie bracket, the map $\mathscr{S}'$ is called a ***Lie algebra morphism***.

---

[13] Considering $\mathrm{SL}(2,\mathbb{C})$ and $\mathrm{SO_o}(3,1)$ as differentiable manifolds (of dimension 6) over $\mathbb{R}$, this mapping is actually the *differential* of the spinor map taken at the point $\mathbb{I}_2$; accordingly, it goes from the vector space tangent to $\mathrm{SL}(2,\mathbb{C})$ at $\mathbb{I}_2$, i.e. $\mathrm{sl}(2,\mathbb{C})$, to the vector space tangent to $\mathrm{SO_o}(3,1)$ at $\mathscr{S}(\mathbb{I}_2) = \mathrm{Id}$, i.e. $\mathrm{so}(3,1)$ (cf. Remark 7.4 p. 224).

Let us compute the image by $\mathscr{S}'$ of the first element of the basis (7.92), namely, the Pauli matrix σ_1. Let $\vec{\boldsymbol{v}} = v^\alpha \vec{\boldsymbol{e}}_\alpha \in E$ and $\vec{\boldsymbol{v}}\,' = v'^\alpha \vec{\boldsymbol{e}}_\alpha$ be the image of $\vec{\boldsymbol{v}}$ by $\mathscr{S}'(\sigma_1)$. The matrix representing $\vec{\boldsymbol{v}}$ (resp. $\vec{\boldsymbol{v}}\,'$) in $\mathrm{Herm}(2,\mathbb{C})$ being $H = v^\alpha \sigma_\alpha$ (resp. $H' = v'^\alpha \sigma_\alpha$), we have, since $\sigma_1^\dagger = \sigma_1$,

$$H' = \sigma_1 H + H \sigma_1 = v^\alpha (\sigma_1 \sigma_\alpha + \sigma_\alpha \sigma_1).$$

Now, as one can check easily on (7.51),

$$\sigma_1 \sigma_0 = \sigma_0 \sigma_1 = \sigma_1, \; \sigma_1 \sigma_1 = \sigma_0, \; \sigma_1 \sigma_2 = -\sigma_2 \sigma_1 = \mathrm{i}\sigma_3, \; \sigma_1 \sigma_3 = -\sigma_3 \sigma_1 = -\mathrm{i}\sigma_2.$$

We deduce that $H' = 2v^1 \sigma_0 + 2v^0 \sigma_1$ and thus that

$$v'^0 = 2v^1, \quad v'^1 = 2v^0, \quad v'^2 = 0 \quad \text{and} \quad v'^3 = 0.$$

Comparing with the matrix (7.16a) of the endomorphism $\boldsymbol{K}_1$ in the basis $(\vec{\boldsymbol{e}}_\alpha)$, we deduce immediately that $\vec{\boldsymbol{v}}\,' = 2\boldsymbol{K}_1(\vec{\boldsymbol{v}})$; hence,

$$\mathscr{S}'(\sigma_1) = 2\boldsymbol{K}_1. \tag{7.97}$$

One shows similarly that

$$\mathscr{S}'(\sigma_2) = 2\boldsymbol{K}_2, \quad \mathscr{S}'(\sigma_3) = 2\boldsymbol{K}_3 \tag{7.98a}$$

$$\mathscr{S}'(\mathrm{i}\sigma_1) = -2\boldsymbol{J}_1, \quad \mathscr{S}'(\mathrm{i}\sigma_2) = -2\boldsymbol{J}_2, \quad \mathscr{S}'(\mathrm{i}\sigma_3) = -2\boldsymbol{J}_3. \tag{7.98b}$$

We observe that the image of the basis (7.92) of $\mathrm{sl}(2,\mathbb{C})$ is

$$(2\boldsymbol{K}_1, \; 2\boldsymbol{K}_2, \; 2\boldsymbol{K}_3, \; -2\boldsymbol{J}_1, \; -2\boldsymbol{J}_2 \; -2\boldsymbol{J}_3),$$

which is a basis of so(3, 1) (cf. Sect. 7.3.3). This implies that the linear map $\mathscr{S}'$ is a vector space isomorphism. Having already shown that it is a Lie algebra morphism, we thus conclude:

The Lie algebras so(3, 1) and $\mathrm{sl}(2,\mathbb{C})$ are isomorphic:

$$\boxed{\mathrm{so}(3,1) \simeq \mathrm{sl}(2,\mathbb{C})}\,. \tag{7.99}$$

A realization of this isomorphism is provided by the map $\mathscr{S}'$ defined by (7.94).

Remark 7.20. Despite their Lie algebras are isomorphic, the Lie groups $\mathrm{SO}_o(3,1)$ and $\mathrm{SL}(2,\mathbb{C})$ are not isomorphic, $\mathrm{SL}(2,\mathbb{C})$ being "twice as large as" $\mathrm{SO}_o(3,1)$ [cf. Eq. (7.85)].

7.5.7 Exponential Map on sl(2,C)

As for the Lie algebra of the Lorentz group, one can define the exponential map from the Lie algebra $\mathrm{sl}(2,\mathbb{C})$ to the group $\mathrm{SL}(2,\mathbb{C})$. The definition is identical to that given by the series (7.30). A general result from Lie group theory (cf., e.g. Sect. 6.2 of Godement's book (Godement 2004)) states that since $\mathscr{S}$ is a Lie group homomorphism and $\mathscr{S}'$ is its differential at the point $\mathbb{I}_2$ (cf. footnote 13), the following diagram is commutative:

$$\begin{array}{ccccc} & \mathrm{sl}(2,\mathbb{C}) & \xrightarrow{\mathscr{S}'} & \mathrm{so}(3,1) & \\ \exp & \downarrow & & \downarrow & \exp \\ & \mathrm{SL}(2,\mathbb{C}) & \xrightarrow{\mathscr{S}} & \mathrm{SO}_o(3,1) & \end{array}$$

This means that

$$\forall B \in \mathrm{sl}(2,\mathbb{C}), \quad \boxed{\mathscr{S}(\exp B) = \exp(\mathscr{S}'(B))}. \tag{7.100}$$

Remark 7.21. We have already mentioned in Sect. 7.4.1 that the exponential map from $\mathrm{so}(3,1)$ to $\mathrm{SO}_o(3,1)$ is surjective. On the other side, it is not surjective from $\mathrm{sl}(2,\mathbb{C})$ to $\mathrm{SL}(2,\mathbb{C})$: the elements of $\mathrm{SL}(2,\mathbb{C})$ of the type $A = -\mathbb{I}_2 + N$ with $N \neq 0$ and nilpotent ($N^2 = 0$) do not have any inverse image by exp (cf., e.g. Gallier (2011)). In all cases, either A or $-A$ has an inverse image by exp, which, in view of (7.85), explains why $\exp :\ \mathrm{so}(3,1) \to \mathrm{SO}_o(3,1)$ is surjective.

It is instructive to study the inverse images by the exponential of the elements of the subgroup $\mathrm{SU}(2)$ of $\mathrm{SL}(2,\mathbb{C})$. We have seen in Sect. 7.5.2 that any element $A \in \mathrm{SU}(2)$ can be written as (7.71) with $\varphi \in [0, 2\pi]$ and $(n^1, n^2, n^3) \in \mathbb{R}^3$ with $\sum_{i=1}^{3}(n^i)^2 = 1$, which can be interpreted as the components of a unit vector in E_{e_0}. Let us then set $B := -(\varphi/2)n^j \mathrm{i}\sigma_j$ and evaluate

$$\exp B = \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{\varphi}{2}\right)^n \left(n^j \mathrm{i}\sigma_j\right)^n.$$

We check easily from (7.51) and (7.70) that

$$\left(n^j \mathrm{i}\sigma_j\right)^2 = -(n^j\,\sigma_j)^2 = -\left[(n^1)^2 + (n^2)^2 + (n^3)^2\right]\mathbb{I}_2 = -\mathbb{I}_2.$$

Consequently, the above series is simplified to

$$\exp B = \underbrace{\left[\sum_{p=0}^{\infty} \frac{1}{(2p)!}\left(\frac{\varphi}{2}\right)^{2p}(-1)^p\right]}_{\cos(\varphi/2)} \mathbb{I}_2 + \underbrace{\left[\sum_{p=0}^{\infty} \frac{1}{(2p+1)!}(-1)\left(\frac{\varphi}{2}\right)^{2p+1}(-1)^p\right]}_{-\sin(\varphi/2)} n^j \mathrm{i}\sigma_j .$$

We recognize the expression (7.71) of A. We conclude that any element of SU(2) can be written as an exponential, according to the simple formula

$$\boxed{A = \cos\frac{\varphi}{2}\,\mathbb{I}_2 - \sin\frac{\varphi}{2}\,n^j \mathrm{i}\sigma_j = \exp\left(-\frac{\varphi}{2}\,n^j \mathrm{i}\sigma_j\right)}. \tag{7.101}$$

An application of this formula and of the general result (7.100) consists in recovering the exponential expression (7.39) of a given spatial rotation. Indeed, if $\boldsymbol{R}$ is a rotation of angle φ and axis $\vec{\boldsymbol{n}} = n^j\,\vec{\boldsymbol{e}}_j$ in $E_{\boldsymbol{e}_0}$, then $\boldsymbol{R} = \mathscr{S}(A)$ with A of the form (7.71). The above result gives $\boldsymbol{R} = \mathscr{S}(\exp B)$. The property (7.100) allows one to write $\boldsymbol{R} = \exp(\mathscr{S}'(B))$. Now, by linearity of $\mathscr{S}'$ and using (7.98b),

$$\mathscr{S}'(B) = -\frac{\varphi}{2} n^j \underbrace{\mathscr{S}'(\mathrm{i}\sigma_j)}_{-2\boldsymbol{J}_j} = \varphi\, n^j\, \boldsymbol{J}_j .$$

We have thus $\boldsymbol{R} = \exp(\varphi\, n^j\, \boldsymbol{J}_j)$, which is nothing but the result (7.39).

Similarly, we can express the Hermitian elements of $\mathrm{SL}(2,\mathbb{C})$ considered in Sect. 7.5.3 by taking the exponential of $B := \psi/2\, n^j \sigma_j$ for $\psi \in \mathbb{R}$. A computation similar to the above one leads to

$$\boxed{A = \cosh\frac{\psi}{2}\,\mathbb{I}_2 + \sinh\frac{\psi}{2}\,n^j \sigma_j = \exp\left(\frac{\psi}{2}\,n^j \sigma_j\right)}. \tag{7.102}$$

By combining with (7.100) and (7.97)–(7.98a), we get the exponential expression $\boldsymbol{\Lambda} = \exp(\psi\, n^j\, \boldsymbol{K}_j)$ for any Lorentz boost $\boldsymbol{\Lambda}$, i.e. we recover (7.38).

Historical note: *The link between the Lorentz group and* $\mathrm{SL}(2,\mathbb{C})$ *was known to Felix Klein*[14] *in 1910 (Klein 1910) and to Élie Cartan (cf. p. 6) in 1914 (Cartan 1914).*

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[14]**Felix Klein** (1849–1925): German mathematician, who authored numerous works in group theory and non-Euclidean geometry; in 1872, he proposed the famous *Erlangen programme*, whose aim was to classify the various geometries in terms of their symmetry groups and their invariants. He founded the mathematical centre at Göttingen. The Klein group mentioned in Sect. 6.3.4 is named after him.

Chapter 8
Inertial Observers and Poincaré Group

8.1 Introduction

In Chaps. 3–5, we dealt with any kind of observers. Here we focus on the simplest observers in Minkowski spacetime: the inertial ones. On historical grounds, it should be noticed that special relativity has been forged only in terms of this kind of observers. Special relativity is still presented in this way in many textbooks. As advocated in the Preface, the present exposition of special relativity takes a different route. Inertial observers are then simply considered as a special class among all possible observers in Minkowski spacetime.

8.2 Characterization of Inertial Observers

8.2.1 *Definition*

An ***inertial observer*** has been defined in Sect. 3.5.4 as an observer $\mathscr{O}$ whose local frame $(\vec{e}_\alpha(t))$ (t being $\mathscr{O}$'s proper time) fulfils

$$\boxed{\frac{\mathrm{d}\vec{e}_\alpha}{\mathrm{d}t} = 0}, \tag{8.1}$$

i.e. each vector $\vec{e}_\alpha$ is constant along $\mathscr{O}$'s worldline (cf. Fig. 8.1). We have seen that this condition is equivalent to vanishing 4-acceleration and 4-rotation along the worldline [Eq. (3.60)]:

$$\forall t \in \mathbb{R}, \quad \boxed{\vec{a}(t) = 0} \quad \text{and} \quad \boxed{\vec{\omega}(t) = 0}. \tag{8.2}$$

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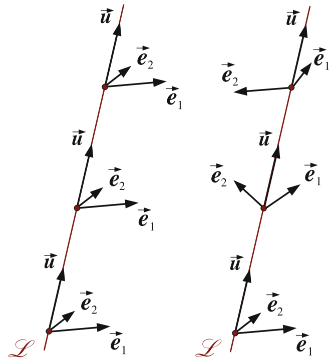

Fig. 8.1 Worldline and local frame of an inertial observer (*left*) Worldline and local frame of an observer without any 4-acceleration but with a nonvanishing 4-rotation (*right*). The local frame of the inertial observer obeys (8.1)

Let us recall that the *4-acceleration* $\vec{a}$ and the *4-rotation* $\vec{\omega}$ have been introduced in Sects. 2.4.2 and 3.5.3, respectively. These two quantities can be defined as the vectors orthogonal to $\mathscr{O}$'s 4-velocity $\vec{u}$ that rule the evolution of the local frame according to (3.52):

$$\frac{\mathrm{d}\vec{e}_\alpha}{\mathrm{d}t} = c(\vec{a} \cdot \vec{e}_\alpha)\,\vec{u} - c(\vec{u} \cdot \vec{e}_\alpha)\,\vec{a} + \vec{\omega} \times_u \vec{e}_\alpha.$$

As we have already noticed in Chap. 3 (cf. (3.68)), an immediate consequence of (8.2) is that the derivative of a vector field with respect to observer $\mathscr{O}$ (cf. Sect. 3.6.2) coincides with the absolute derivative : $\boldsymbol{D}_{\mathscr{O}}\vec{v} = \mathrm{d}\vec{v}/\mathrm{d}t$.

8.2.2 Worldline

Since the vector $\vec{e}_0$ of $\mathscr{O}$'s local frame is nothing but the 4-velocity $\vec{u}$, an immediate consequence of (8.1) is

$$\vec{u}(t) = \text{const.} \tag{8.3}$$

In other words, $\vec{u}(t)$ is the same vector of E at any point of the worldline $\mathscr{L}$ of $\mathscr{O}$. Let then $(O; \vec{\varepsilon}_0, \vec{\varepsilon}_1, \vec{\varepsilon}_2, \vec{\varepsilon}_3)$ be an affine frame[1] of $\mathscr{E}$ and $x^\alpha = x^\alpha(t)$ the equation of the worldline $\mathscr{L}$ in this frame. Since $\vec{u}$ is the derivative vector associated with the parametrization of $\mathscr{L}$ by ct, we have $\vec{u} = c^{-1}(\mathrm{d}x^\alpha/\mathrm{d}t)\,\vec{\varepsilon}_\alpha$; hence

$$\frac{\mathrm{d}x^\alpha}{\mathrm{d}t} = c u_0^\alpha, \tag{8.4}$$

[1]The vectors of the affine frame, as defined in Sect. 1.2.3, are denoted by $\vec{\varepsilon}_\alpha$ to distinguish them from those of $\mathscr{O}$'s local frame.

where the u_0^α's are the four components of $\vec{u}$ in the basis $(\vec{\varepsilon}_\alpha)$, which are constant according to (8.3). Equation (8.4) is then readily integrated, yielding

$$x^\alpha(t) = c u_0^\alpha t + x_0^\alpha, \tag{8.5}$$

the quantities (u_0^α, x_0^α) being eight constants. We recognize the equation of a straight line, parametrized by t, so that we conclude:

The worldline of any inertial observer is a straight line of Minkowski spacetime $\mathscr{E}$.

Remark 8.1. The converse is not true: an observer whose worldline is a straight line is only an observer with a vanishing 4-acceleration. To be an inertial observer, he must in addition have a vanishing 4-rotation (cf. Fig. 8.1).

8.2.3 Globality of the Local Rest Space

In Sect. 3.2.3, we have made the distinction between the *simultaneity hypersurface* $\Sigma_{\boldsymbol{u}}(t)$ of an observer $\mathscr{O}$ and his *local rest space* $\mathscr{E}_{\boldsymbol{u}}(t)$: $\Sigma_{\boldsymbol{u}}(t)$ is defined as the set of events in $\mathscr{E}$ that are simultaneous[2] to the event of proper time t on the worldline $\mathscr{L}$ of $\mathscr{O}$, namely, $O(t)$, whereas $\mathscr{E}_{\boldsymbol{u}}(t)$ is defined in a pure geometric way as the hyperplane of $\mathscr{E}$ orthogonal to $\mathscr{L}$ at $O(t)$. It is tangent to $\Sigma_{\boldsymbol{u}}(t)$ at $O(t)$ and we have seen in Sect. 3.2.3 that the difference between the two subspaces is induced by the curvature of $\mathscr{L}$. If the latter is a straight line, this difference disappears. Thus the equalities (3.3), which are the starting points of the computation yielding to $B \in \mathscr{E}_{\boldsymbol{u}}(t)$, are valid for any point $B \in \Sigma_{\boldsymbol{u}}(t)$, even if it is very far from $\mathscr{L}$. We conclude that, for an inertial observer, the simultaneity hypersurface and the local rest space coincide:

$$\forall t \in \mathbb{R}, \quad \boxed{\Sigma_{\boldsymbol{u}}(t) = \mathscr{E}_{\boldsymbol{u}}(t)}. \tag{8.6}$$

Moreover, since $\vec{u}(t) = \text{const}$ [property (8.3)], the hyperplanes $\mathscr{E}_{\boldsymbol{u}}(t)$ are parallel (cf. Fig. 8.2); thus, they never intersect, contrary to those of an accelerated observer, as discussed in Sect. 3.7 (see Fig. 3.15). Hence, there is no obstruction to the labelling of all events of $\mathscr{E}$ by the coordinates (ct, x^1, x^2, x^3) with respect to observer $\mathscr{O}$ (as defined in Sect. 3.4.2). In addition, we have $\overrightarrow{O(0)M} = \overrightarrow{O(0)O(t)} + \overrightarrow{O(t)M} = ct\,\vec{u} + \overrightarrow{O(t)M}$, so that (3.25) leads to

[2] In the Einstein–Poincaré sense, cf. Sect. 3.2.2.

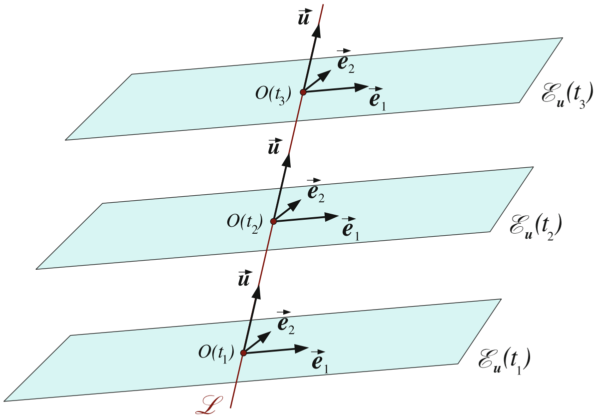

Fig. 8.2 Worldline and local rest spaces of an inertial observer

$$\overrightarrow{O(0)M} = ct\,\vec{u} + x^i\,\vec{e}_i. \tag{8.7}$$

Comparing with (1.6), we notice that the coordinates (ct, x^1, x^2, x^3) with respect to the observer $\mathscr{O}$ form an affine coordinate system of $\mathscr{E}$ centred in $O(0)$. We shall call ***inertial coordinates*** any affine coordinates of this type. Since such a coordinate system is clearly global, we shall drop the qualifier *local* in the denomination of $(\vec{e}_\alpha)$ and $\mathscr{E}_{\boldsymbol{u}}(t)$, calling them respectively the ***frame*** and the ***rest space*** of the inertial observer $\mathscr{O}$.

Remark 8.2. Inertial coordinates are sometimes called ***Minkowskian coordinates*** or ***Galilean coordinates***.

8.2.4 Rigid Array of Inertial Observers

Let $\mathscr{O}$ be an inertial observer of worldline $\mathscr{L}$, 4-velocity $\vec{u}$, rest space $\mathscr{E}_{\boldsymbol{u}}(t)$ and frame $(\vec{e}_0 = \vec{u}, \vec{e}_1, \vec{e}_2, \vec{e}_3)$. Let us consider an observer $\mathscr{O}'$ *fixed* with respect to $\mathscr{O}$, i.e. an observer whose coordinates (x^1, x^2, x^3) with respect to $\mathscr{O}$ are constant (cf. Sect. 3.4.3). The worldline $\mathscr{L}'$ of $\mathscr{O}'$ is a straight line of $\mathscr{E}$ parallel to $\mathscr{L}$.

Proof. If $O(t) := \mathscr{E}_{\boldsymbol{u}}(t) \cap \mathscr{L}$ and $O'(t) := \mathscr{E}_{\boldsymbol{u}}(t) \cap \mathscr{L}'$, then

$$\overrightarrow{O(0)O(t)} = ct\,\vec{u} \qquad \text{and} \qquad \overrightarrow{O(0)O'(t)} = ct\,\vec{u} + x^i\,\vec{e}_i.$$

The first equation gives the spacetime position of the generic point $O(t)$ of $\mathscr{L}$ and the second one the spacetime position of the generic point $O'(t)$ of $\mathscr{L}'$. Since the x^i's are constant (for $\mathscr{O}'$ is fixed with respect to $\mathscr{O}$) and the vectors $\vec{e}_i$'s are constant

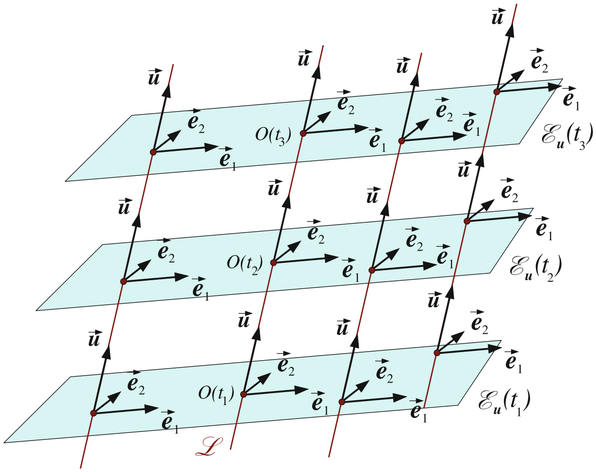

Fig. 8.3 Rigid array of inertial observers

(for $\mathscr{O}$ is an inertial observer), the vector $x^i \vec{e}_i$ is constant. We deduce that the worldline $\mathscr{L}'$ is a line parallel to $\mathscr{L}$. □

It follows that the proper time of $\mathscr{O}'$ coincides (up to the choice of some origin) with the proper time t of $\mathscr{O}$ and that the 4-velocity $\vec{u}'$ of $\mathscr{O}'$ is equal to that of $\mathscr{O}$. We can then equip $\mathscr{O}'$ with the same spatial basis $(\vec{e}_i)$ as $\mathscr{O}$ to make it an inertial observer. The rest spaces of $\mathscr{O}'$ are then the same as those of $\mathscr{O}$:

$$\mathscr{E}_{\boldsymbol{u}'}(t) = \mathscr{E}_{\boldsymbol{u}}(t). \tag{8.8}$$

This construction can obviously be extended to any set of observers fixed with respect to $\mathscr{O}$ (cf. Fig. 8.3), leading to what we shall call a ***rigid array of inertial observers***. By choosing the same origin for all the proper times, the ideal clocks carried by each observer indicate the same value: one says that they are ***synchronized***.

8.3 Poincaré Group

8.3.1 Change of Inertial Coordinates

Let us consider two inertial observers, $\mathscr{O}$ and $\mathscr{O}'$, of respective frames $(\vec{e}_\alpha)$ and $(\vec{e}\,'_\alpha)$. The coordinates (x^α) and (x'^α) of an event $M \in \mathscr{E}$ with respect to $\mathscr{O}$ and $\mathscr{O}'$ are defined by (8.7):

$$\overrightarrow{OM} = x^\alpha \vec{e}_\alpha \quad \text{and} \quad \overrightarrow{O'M} = x'^\alpha \vec{e}\,'_\alpha, \tag{8.9}$$

where O (resp. O') is the event of the worldline of $\mathscr{O}$ (resp. $\mathscr{O}'$) of proper time $t = x^0/c = 0$ (resp. $t' = x'^0/c = 0$). From Chasles' relation,

$$\overrightarrow{O'M} = \overrightarrow{O'O} + \overrightarrow{OM} = x_0'^{\alpha}\vec{\boldsymbol{e}}'_{\alpha} + x^{\alpha}\vec{\boldsymbol{e}}_{\alpha}, \tag{8.10}$$

where $(x_0'^{\alpha})$ are the coordinates of O with respect to $\mathscr{O}'$. The orthonormal bases $(\vec{\boldsymbol{e}}_{\alpha})$ and $(\vec{\boldsymbol{e}}'_{\alpha})$ are related by a unique restricted Lorentz transformation, $\boldsymbol{\Lambda}$, such that

$$\vec{\boldsymbol{e}}_{\alpha} = \boldsymbol{\Lambda}(\vec{\boldsymbol{e}}'_{\alpha}) = \Lambda^{\beta}{}_{\alpha}\vec{\boldsymbol{e}}'_{\beta}, \tag{8.11}$$

where $\Lambda^{\beta}{}_{\alpha}$ stands for the matrix of $\boldsymbol{\Lambda}$ in the basis $(\vec{\boldsymbol{e}}'_{\alpha})$ [cf. Eq. (6.5)]. Equations (8.9) and (8.10) lead then to

$$\overrightarrow{O'M} = x'^{\alpha}\vec{\boldsymbol{e}}'_{\alpha} = x_0'^{\alpha}\vec{\boldsymbol{e}}'_{\alpha} + x^{\alpha}\Lambda^{\beta}{}_{\alpha}\vec{\boldsymbol{e}}'_{\beta} = x_0'^{\alpha}\vec{\boldsymbol{e}}'_{\alpha} + x^{\beta}\Lambda^{\alpha}{}_{\beta}\vec{\boldsymbol{e}}'_{\alpha},$$

hence the relation between the two coordinate systems:

$$\boxed{x'^{\alpha} = \Lambda^{\alpha}{}_{\beta}x^{\beta} + x_0'^{\alpha}}. \tag{8.12}$$

One calls ***passive Poincaré transformation*** any change of coordinates of this type, i.e. any map of the form

$$\begin{aligned} f : \mathbb{R}^4 &\longrightarrow \mathbb{R}^4 \\ (x^{\alpha}) &\longmapsto \left(\Lambda^{\alpha}{}_{\beta}x^{\beta} + x_0'^{\alpha}\right), \end{aligned} \tag{8.13}$$

where $\Lambda^{\alpha}{}_{\beta}$ is a Lorentz matrix (cf. Sect. 6.2.1) and $x_0'^{\alpha}$ are four real numbers.

Example 8.1. A particularly important example (historically the first one!) is that where the axes of observers $\mathscr{O}$ and $\mathscr{O}'$ are ***quasiparallel***, i.e. obey $\vec{\boldsymbol{e}}'_2 = \vec{\boldsymbol{e}}_2$, $\vec{\boldsymbol{e}}'_3 = \vec{\boldsymbol{e}}_3$, the velocity $\vec{\boldsymbol{V}}$ of $\mathscr{O}'$ relative to $\mathscr{O}$ being along $\vec{\boldsymbol{e}}_1$: $\vec{\boldsymbol{V}} = V\vec{\boldsymbol{e}}_1$, and the velocity $\vec{\boldsymbol{V}}'$ of $\mathscr{O}$ relative to $\mathscr{O}'$ being along $\vec{\boldsymbol{e}}'_1$: $\vec{\boldsymbol{V}}' = V'\vec{\boldsymbol{e}}'_1$. We have then necessarily $V' = -V$ [cf. Eqs. (5.9) and (5.10)]. In this case, $\boldsymbol{\Lambda}$ is a Lorentz boost, whose matrix is given by (6.48), where V is replaced by $V' = -V$. Using this matrix in (8.12) and denoting by (ct, x, y, z) (resp. (ct', x', y', z')) the inertial coordinates (x^{α}) (resp. (x'^{α})), we get

$$\boxed{\begin{cases} ct' = \Gamma\left(ct - \frac{V}{c}x\right) \\ x' = \Gamma\,(x - Vt) \\ y' = y \\ z' = z. \end{cases}} \tag{8.14}$$

Note that the constants $x_0'^\alpha$ have been set to zero, which amounts to considering that the worldlines of $\mathscr{O}$ and $\mathscr{O}'$ intersect at $t = 0$ and $t' = 0$. The above transformation can be inverted by replacing V by $-V$:

$$\boxed{\begin{cases} ct = \Gamma\left(ct' + \frac{V}{c}x'\right) \\ x = \Gamma\left(x' + Vt'\right) \\ y = y' \\ z = z', \end{cases}} \tag{8.15}$$

Remark 8.3. We do not limit the definition (8.13) to restricted Lorentz matrices, but it is only in this case that a Poincaré transformation describes the change of coordinates induced by a change of inertial observer.

8.3.2 Active Poincaré Transformations

One calls ***Poincaré transformation*** any function

$$\begin{aligned} f : \mathscr{E} &\longrightarrow \mathscr{E} \\ M &\longmapsto f(M) \end{aligned} \tag{8.16}$$

for which there exists a Lorentz transformation $\boldsymbol{\Lambda} : E \to E$ such that

$$\forall (M, N) \in \mathscr{E}^2, \quad \overrightarrow{f(M)f(N)} = \boldsymbol{\Lambda}(\overrightarrow{MN}). \tag{8.17}$$

It is clear that the Lorentz transformation $\boldsymbol{\Lambda}$ is unique. Since $\boldsymbol{\Lambda}$ is a linear map, f is, by definition, an ***affine map***. In particular, f is entirely defined by (i) the Lorentz transformation $\boldsymbol{\Lambda}$ and (ii) the image of a single point of $\mathscr{E}$, O, say. Indeed one deduces from (8.17) that

$$\boxed{\forall M \in \mathscr{E}, \quad \overrightarrow{Of(M)} = \boldsymbol{\Lambda}(\overrightarrow{OM}) + \overrightarrow{Of(O)}}. \tag{8.18}$$

Remark 8.4. A Poincaré transformation, as defined above, sends a point of $\mathscr{E}$ to another point of $\mathscr{E}$; this is an *active* transformation, by contrast with the passive transformation considered in Sect. 8.3.1. We shall omit the qualifier *active* and use simply the term *Poincaré transformation* for these transformations. On the other hand, we shall use explicitly the qualifier *passive* for passive transformations.

Let $\mathscr{O}$ be an inertial observer, of frame $(\vec{\boldsymbol{e}}_\alpha)$, and let (x^α) be the associated system of inertial coordinates. O being the origin of the latter, let us denote by $(x_0'^\alpha)$ the coordinates of the point $f(O)$ and by (x'^α) the coordinates of the image of a generic point M, of coordinates (x^α). Equation (8.18) leads then to

$$\boxed{x'^{\alpha} = \Lambda^{\alpha}{}_{\beta} x^{\beta} + x_0'^{\alpha}}, \tag{8.19}$$

where $\Lambda^{\alpha}{}_{\beta}$ is the matrix of $\boldsymbol{\Lambda}$ in the basis $(\vec{e}_{\alpha})$, defined according to (6.4). This relation has the same structure as (8.12), the difference being that here x'^{α} stands for the coordinates of the image of a point by f in the coordinate system (x^{α}), whereas in (8.12), (x'^{α}) and (x^{α}) are two different coordinate systems.

One calls ***translation*** any Poincaré transformation f whose associated Lorentz transformation is the identity. The vector $\overrightarrow{Of(O)}$ is then independent of the point O. Indeed, relation (8.17) with $\boldsymbol{\Lambda} = \mathrm{Id}$ leads to

$$\overrightarrow{f(O)O} + \overrightarrow{OO'} + \overrightarrow{O'f(O')} = \overrightarrow{f(O)f(O')} = \overrightarrow{OO'}, \quad \text{hence } \overrightarrow{O'f(O')} = \overrightarrow{Of(O)}.$$

The vector $\vec{v} = \overrightarrow{Of(O)}$ is called the ***translation vector***. A translation is entirely defined by a given point and its image.

For any $O \in \mathscr{E}$, one calls ***Lorentz transformation pointed at O*** any Poincaré transformation f such that $f(O) = O$. We have then the following decomposition:

Given a point $O \in \mathscr{E}$, any Poincaré transformation f can be uniquely decomposed into a Lorentz transformation pointed at O, Λ_O, say, followed by a translation T, of vector $\vec{v} = \overrightarrow{Of(O)}$:

$$\boxed{f = T \circ \Lambda_O}. \tag{8.20}$$

We shall write $f = (\vec{v}, \boldsymbol{\Lambda})$ where $\vec{v}$ is the vector of the translation T and $\boldsymbol{\Lambda}$ the Lorentz transformation associated with Λ_O (or equivalently with f).

Proof. It suffices to write (8.18) in the form

$$\forall M \in \mathscr{E}, \quad \overrightarrow{Of(M)} = \boldsymbol{\Lambda}(\overrightarrow{OM}) + \vec{v} \tag{8.21}$$

and to define Λ_O as the map such that $\overrightarrow{O\Lambda_O(M)} = \boldsymbol{\Lambda}(\overrightarrow{OM})$. □

8.3.3 Group Structure

The set of all Lorentz transformations being a group, it is easy to check that the same property holds for the set of all Poincaré transformations. In particular, if f_1 and f_2 are two Poincaré transformations, of associated Lorentz transformations $\boldsymbol{\Lambda}_1$ and $\boldsymbol{\Lambda}_2$, then the repeated application of (8.21) yields

$$\forall M \in \mathscr{E}, \quad \overrightarrow{Of_1 \circ f_2(M)} = \boldsymbol{\Lambda}_1[\overrightarrow{Of_2(M)}] + \vec{\boldsymbol{v}}_1$$
$$= \boldsymbol{\Lambda}_1[\boldsymbol{\Lambda}_2(\overrightarrow{OM}) + \vec{\boldsymbol{v}}_2] + \vec{\boldsymbol{v}}_1$$
$$= \boldsymbol{\Lambda}_1 \circ \boldsymbol{\Lambda}_2(\overrightarrow{OM}) + \boldsymbol{\Lambda}_1(\vec{\boldsymbol{v}}_2) + \vec{\boldsymbol{v}}_1.$$

Comparing with (8.21), we deduce that $f_1 \circ f_2$ is a Poincaré transformation, of associated Lorentz transformation $\boldsymbol{\Lambda}_1 \circ \boldsymbol{\Lambda}_2$. Moreover, the translation associated with $f_1 \circ f_2$ in the decomposition (8.20) with respect to O is the translation of vector

$$\vec{\boldsymbol{v}} = \vec{\boldsymbol{v}}_1 + \boldsymbol{\Lambda}_1(\vec{\boldsymbol{v}}_2). \tag{8.22}$$

The group formed by the set of all Poincaré transformations is naturally called ***Poincaré group*** and is denoted by IO(3, 1).

Remark 8.5. The Poincaré group is sometimes called the *inhomogeneous Lorentz group*, which explains the notation IO(3, 1).

For any $O \in \mathscr{E}$, the set of all Lorentz transformations pointed at O constitutes a subgroup of IO(3, 1). Moreover, this subgroup is isomorphic to the Lorentz group O(3, 1). Besides, the set of all translations constitutes also a subgroup of IO(3, 1). This subgroup is isomorphic to $(E, +)$ or to $(\mathbb{R}^4, +)$.

The decomposition (8.20) might let one think that the Poincaré group is isomorphic to the group product $\mathbb{R}^4 \times \mathrm{O}(3,1)$. This would be the case if the relation $\vec{\boldsymbol{v}} = \vec{\boldsymbol{v}}_1 + \vec{\boldsymbol{v}}_2$ held instead of (8.22). Indeed, let us recall that the ***group product*** (or ***group direct product***) of two groups, $(G_1, *_1)$ and $(G_2, *_2)$, is the Cartesian product $G_1 \times G_2$ endowed with the internal law $*$ defined by $(a_1, a_2) * (b_1, b_2) = (a_1 *_1 b_1,\ a_2 *_2 b_2)$. In the present case, $(G_1, *_1) = (\mathbb{R}^4, +)$, $(G_2, *_2) = (\mathrm{O}(3,1), \circ)$, $* = \circ$ and one should have $(\vec{\boldsymbol{v}}_1, \boldsymbol{\Lambda}_1) \circ (\vec{\boldsymbol{v}}_2, \boldsymbol{\Lambda}_2) = (\vec{\boldsymbol{v}}_1 + \vec{\boldsymbol{v}}_2,\ \boldsymbol{\Lambda}_1 \circ \boldsymbol{\Lambda}_2)$. The result (8.22) shows that we have instead

$$\boxed{(\vec{\boldsymbol{v}}_1, \boldsymbol{\Lambda}_1) \circ (\vec{\boldsymbol{v}}_2, \boldsymbol{\Lambda}_2) = (\vec{\boldsymbol{v}}_1 + \boldsymbol{\Lambda}_1(\vec{\boldsymbol{v}}_2),\ \boldsymbol{\Lambda}_1 \circ \boldsymbol{\Lambda}_2)}. \tag{8.23}$$

One then says that the Poincaré group is the ***semidirect product*** of the translation group and the Lorentz group, and this operation is denoted by the symbol $\rtimes$:

$$\boxed{\mathrm{IO}(3,1) \simeq \mathbb{R}^4 \rtimes \mathrm{O}(3,1)}, \tag{8.24}$$

the symbol $\simeq$ standing for "isomorphic to" (cf. Appendix A).

The set of all Poincaré transformations whose associated Lorentz transformation $\boldsymbol{\Lambda}$ is restricted ($\boldsymbol{\Lambda} \in \mathrm{SO_o}(3,1)$) constitutes a subgroup of IO(3, 1), naturally called the ***restricted Poincaré group*** and denoted by $\mathrm{ISO_o}(3,1)$. The elements of $\mathrm{ISO_o}(3,1)$ are those that govern the changes of inertial observers. We have of course

$$\boxed{\mathrm{ISO_o}(3,1) \simeq \mathbb{R}^4 \rtimes \mathrm{SO_o}(3,1)}. \tag{8.25}$$

A consequence of the semidirect product structure (8.25) is that the translation group $(\mathbb{R}^4, +)$ is a normal subgroup (cf. Annexe A) of the restricted Poincaré group $\mathrm{ISO_o}(3,1)$. It is indeed easy to check that the condition (A.3) is fulfilled by the product (8.23). Consequently, $\mathrm{ISO_o}(3,1)$ is not a simple group, contrary to $\mathrm{SO_o}(3,1)$ (cf. Sect. 6.7.4).

Historical note: *The name* Poincaré group *is due to Eugene P. Wigner (cf. p. 215) and appears for the first time in an article issued in 1952 (Inönü and Wigner 1952) (cf. Rougé (2008), p. 152). As we have seen in Chap. 6 (cf. the historical note p. 191), Henri Poincaré has defined the Lorentz group, rather than the group bearing his name.*

8.3.4 The Poincaré Group as a Lie Group

Since $(\mathbb{R}^4, +)$ is a Lie group of dimension 4 and $\mathrm{O}(3,1)$ a Lie group of dimension 6 (cf. Chap. 7), we deduce immediately from (8.24) that

The Poincaré group $\mathrm{IO}(3,1)$ is a Lie group of dimension 10.

Let us determine the generators of $\mathrm{IO}(3,1)$. An infinitesimal Poincaré transformation f must take the form (8.21) with $\boldsymbol{\Lambda}$ an infinitesimal Lorentz transformation and $\vec{\boldsymbol{v}}$ an infinitesimal vector. By (7.2), $\boldsymbol{\Lambda}$ is of the type $\boldsymbol{\Lambda} = \mathrm{Id} + \varepsilon \boldsymbol{L}$, with $\boldsymbol{L} \in \mathrm{so}(3,1)$—the Lie algebra of $\mathrm{SO_o}(3,1)$. Writing the infinitesimal translation vector as $\varepsilon \vec{\boldsymbol{v}}$, we then obtain

$$\forall M \in \mathscr{E}, \quad \overrightarrow{Of(M)} = \overrightarrow{OM} + \varepsilon \left[\vec{\boldsymbol{v}} + \boldsymbol{L}(\overrightarrow{OM}) \right]. \tag{8.26}$$

The term in factor of ε spans the set

$$\boxed{\mathrm{iso}(3,1) := E \times \mathrm{so}(3,1)}. \tag{8.27}$$

This set, endowed with the addition law defined by

$$\begin{aligned} &\forall (\vec{\boldsymbol{v}}_1, \boldsymbol{L}_1) \in \mathrm{iso}(3,1),\ \forall (\vec{\boldsymbol{v}}_2, \boldsymbol{L}_2) \in \mathrm{iso}(3,1), \\ &\qquad (\vec{\boldsymbol{v}}_1, \boldsymbol{L}_1) + (\vec{\boldsymbol{v}}_2, \boldsymbol{L}_2) := (\vec{\boldsymbol{v}}_1 + \vec{\boldsymbol{v}}_2, \boldsymbol{L}_1 + \boldsymbol{L}_2), \end{aligned} \tag{8.28}$$

is a vector space over $\mathbb{R}$ (recall that $\mathrm{so}(3,1)$ is a vector space over $\mathbb{R}$, so that the addition $\boldsymbol{L}_1 + \boldsymbol{L}_2$ is well defined; cf. Sect. 7.3.2). Since $\mathrm{so}(3,1)$ is a vector space of dimension 6, $\mathrm{iso}(3,1)$ is a vector space of dimension 10.

Contrary to what was done for the Lie algebra of the Lorentz group, so(3, 1), one cannot define the Lie bracket in iso(3, 1) from the commutator because one has not a priori any composition law $\circ$ in iso(3, 1). In the case of so(3, 1), the law $\circ$ stemmed from so(3, 1) being a subset of the algebra $\mathscr{L}(E)$ of endomorphisms on E, and this allowed us to define the Lie bracket via (7.9). We shall define the Lie bracket in iso(3, 1) from the law (8.23) ruling the composition of Poincaré transformations. Indeed let us consider two infinitesimal Poincaré transformations, f_1 and f_2. Thanks to (8.26), we can write their decompositions in terms of translation vectors and Lorentz transformations as

$$f_1 = (\varepsilon\vec{\boldsymbol{v}}_1, \mathrm{Id} + \varepsilon\boldsymbol{L}_1) \qquad \text{and} \qquad f_2 = (\varepsilon\vec{\boldsymbol{v}}_2, \mathrm{Id} + \varepsilon\boldsymbol{L}_2), \tag{8.29}$$

where $\vec{\boldsymbol{v}}_1 \in E$, $\vec{\boldsymbol{v}}_2 \in E$, $\boldsymbol{L}_1 \in \mathrm{so}(3,1)$, $\boldsymbol{L}_2 \in \mathrm{so}(3,1)$ and the same parameter ε has been chosen for f_1 and f_2 (this is possible via a redefinition of $\vec{\boldsymbol{v}}_2$ and $\boldsymbol{L}_2$). By the composition law (8.23), we have

$$f_1 \circ f_2 = \left(\varepsilon(\vec{\boldsymbol{v}}_1 + \vec{\boldsymbol{v}}_2) + \varepsilon^2\boldsymbol{L}_1(\vec{\boldsymbol{v}}_2),\ \mathrm{Id} + \varepsilon(\boldsymbol{L}_1 + \boldsymbol{L}_2) + \varepsilon^2\,\boldsymbol{L}_1 \circ \boldsymbol{L}_2\right).$$

Switching the indices 1 and 2 and subtracting, we get

$$f_1 \circ f_2 - f_2 \circ f_1 = \varepsilon^2\left(\boldsymbol{L}_1(\vec{\boldsymbol{v}}_2) - \boldsymbol{L}_2(\vec{\boldsymbol{v}}_1),\ \boldsymbol{L}_1 \circ \boldsymbol{L}_2 - \boldsymbol{L}_2 \circ \boldsymbol{L}_1\right).$$

We deduce that if f_1 and f_2 commute, we must have $\boldsymbol{L}_1(\vec{\boldsymbol{v}}_2) = \boldsymbol{L}_2(\vec{\boldsymbol{v}}_1)$ and $\boldsymbol{L}_1 \circ \boldsymbol{L}_2 = \boldsymbol{L}_2 \circ \boldsymbol{L}_1$. This suggests to define the Lie bracket in iso(3, 1) by

$$\begin{aligned} &\forall(\vec{\boldsymbol{v}}_1, \boldsymbol{L}_1) \in \mathrm{iso}(3,1),\ \forall(\vec{\boldsymbol{v}}_2, \boldsymbol{L}_2) \in \mathrm{iso}(3,1), \\ &\boxed{[(\vec{\boldsymbol{v}}_1, \boldsymbol{L}_1),\ (\vec{\boldsymbol{v}}_2, \boldsymbol{L}_2)] := (\boldsymbol{L}_1(\vec{\boldsymbol{v}}_2) - \boldsymbol{L}_2(\vec{\boldsymbol{v}}_1),\ [\boldsymbol{L}_1, \boldsymbol{L}_2])}. \end{aligned} \tag{8.30}$$

In this formula, $[\boldsymbol{L}_1, \boldsymbol{L}_2]$ stands of course for the Lie bracket in so(3, 1): $[\boldsymbol{L}_1, \boldsymbol{L}_2] := \boldsymbol{L}_1 \circ \boldsymbol{L}_2 - \boldsymbol{L}_2 \circ \boldsymbol{L}_1$. The function $[,]$ defined by (8.30) is internal to iso(3, 1) since $\boldsymbol{L}_1(\vec{\boldsymbol{v}}_2) - \boldsymbol{L}_2(\vec{\boldsymbol{v}}_1) \in E$ and $[\boldsymbol{L}_1, \boldsymbol{L}_2] \in \mathrm{so}(3,1)$. Moreover, it is clearly bilinear and antisymmetric. It can be shown that it obeys Jacobi identity [cf. (7.13)]. It thus satisfies all the properties defining a Lie bracket (cf. Sect. 7.3.2). We conclude that iso(3, 1), endowed with the internal law $[,]$ defined by (8.30), is a Lie algebra. It is called the ***Lie algebra of the Poincaré group*** or simply ***Poincaré algebra***.

Remark 8.6. The definition (8.30) of the Lie bracket in iso(3, 1) has been introduced above as a measure of the noncommutativity of two infinitesimal Poincaré transformations. There exists actually a fully general procedure to construct a unique Lie algebra from a given Lie group (cf., e.g. Godement (2004)). It can be shown that, once applied to the Poincaré group, this procedure results in the Lie bracket (8.30).

Given an orthonormal basis $(\vec{\boldsymbol{e}}_\alpha)$ of $(E, \boldsymbol{g})$ and the associated generators of the Lorentz group, $\boldsymbol{K}_1$, $\boldsymbol{K}_2$, $\boldsymbol{K}_3$, $\boldsymbol{J}_1$, $\boldsymbol{J}_2$ and $\boldsymbol{J}_3$ (cf. Sect. 7.3.3), the ten following elements of iso(3, 1)

$$\boxed{P_\alpha := (\vec{e}_\alpha, 0)}, \quad \alpha \in \{0, 1, 2, 3\} \tag{8.31}$$

$$\boxed{K_i := (0, \boldsymbol{K}_i)}, \quad i \in \{1, 2, 3\} \tag{8.32}$$

$$\boxed{J_i := (0, \boldsymbol{J}_i)}, \quad i \in \{1, 2, 3\} \tag{8.33}$$

constitute a vector basis of iso(3, 1). We shall call them ***generators of the Poincaré group*** associated with the orthonormal basis $(\vec{e}_\alpha)$. P_α is obviously the generator of translations along the vector $\vec{e}_\alpha$, K_i the generator of Poincaré transformations associated with Lorentz boosts of plane $\mathrm{Vect}(\vec{e}_0, \vec{e}_i)$ and J_i the generator of spatial rotations of plane $\mathrm{Vect}(\vec{e}_0, \vec{e}_i)^\perp$.

Let us evaluate the structure constants of the Poincaré group, by computing the Lie bracket of the various generators (cf. Sect. 7.4.4). From formula (8.30), we get

$$[P_\alpha, P_\beta] = \left[(\vec{e}_\alpha, 0),\ (\vec{e}_\beta, 0)\right] = (0 - 0, [0, 0]) = 0. \tag{8.34}$$

Similarly,

$$[K_i, P_\alpha] = [(0, \boldsymbol{K}_i),\ (\vec{e}_\alpha, 0)] = (\boldsymbol{K}_i(\vec{e}_\alpha) - 0,\ [\boldsymbol{K}_i, 0]) = (\boldsymbol{K}_i(\vec{e}_\alpha),\ 0)\,.$$

Now we read on the matrices (7.16) that

$$\boldsymbol{K}_i(\vec{e}_0) = \vec{e}_i \quad \text{and} \quad \boldsymbol{K}_i(\vec{e}_j) = \delta_{ij}\ \vec{e}_0.$$

We get then

$$[K_i, P_0] = (\vec{e}_i, 0) = P_i \quad \text{and} \quad [K_i, P_j] = (\delta_{ij}\ \vec{e}_0,\ 0) = \delta_{ij}\ P_0. \tag{8.35}$$

Besides, still from (8.30),

$$[J_i, P_\alpha] = [(0, \boldsymbol{J}_i),\ (\vec{e}_\alpha, 0)] = (\boldsymbol{J}_i(\vec{e}_\alpha) - 0,\ [\boldsymbol{J}_i, 0]) = (\boldsymbol{J}_i(\vec{e}_\alpha),\ 0)\,,$$

with, in view of the matrices (7.16),

$$\boldsymbol{J}_i(\vec{e}_0) = 0 \quad \text{and} \quad \boldsymbol{J}_i(\vec{e}_j) = \sum_{k=1}^{3} \epsilon_{ijk}\, \vec{e}_k.$$

We deduce that

$$[J_i, P_0] = 0 \quad \text{and} \quad [J_i, P_j] = \left(\sum_{k=1}^{3} \epsilon_{ijk}\, \vec{e}_k,\ 0\right) = \sum_{k=1}^{3} \epsilon_{ijk}\, P_k. \tag{8.36}$$

Finally, we check easily that

$$[K_i, K_j] = (0,\ [\boldsymbol{K}_i, \boldsymbol{K}_j]), \quad [K_i, J_j] = (0,\ [\boldsymbol{K}_i, \boldsymbol{J}_j]) \quad \text{and} \quad [J_i, J_j] = (0,\ [\boldsymbol{J}_i, \boldsymbol{J}_j]). \tag{8.37}$$

Collecting formulas (8.34), (8.35), (8.36), (8.37) and (7.44), we obtain the structure constants of the Poincaré group:

$$\boxed{\begin{aligned}
&[P_\alpha, P_\beta] = 0\\
&[K_i, P_0] = P_i\\
&[K_i, P_j] = \delta_{ij}\ P_0\\
&[J_i, P_0] = 0\\
&[J_i, P_j] = \sum_{k=1}^{3} \epsilon_{ijk} P_k\\
&[K_i, K_j] = -\sum_{k=1}^{3} \epsilon_{ijk}\ J_k\\
&[K_i, J_j] = -\sum_{k=1}^{3} \epsilon_{ijk}\ K_k\\
&[J_i, J_j] = \sum_{k=1}^{3} \epsilon_{ijk}\ J_k .
\end{aligned}} \tag{8.38}$$

Remark 8.7. As already noticed when discussing the structure constants of the Lorentz group (Sect. 7.4.4), the sums over the index k in the above formulas are actually limited to a single term, thanks to the antisymmetry of ϵ_{ijk}.

Chapter 9
Energy and Momentum

9.1 Introduction

After chapters devoted to the mathematical framework of special relativity and to relativistic kinematics, we address here *relativistic dynamics*. After introducing the concepts of four-momentum, mass, energy and linear momentum of a single particle (Sect. 9.2), we shall extend the definitions to particle systems in Sect. 9.3, in order to state with full generality the first of the two fundamental principles of relativistic dynamics: the conservation of four-momentum. The second principle, the conservation of angular momentum, will be discussed in the next chapter. Various applications of four-momentum conservation are presented in Sects. 9.3.7 and 9.4: the Doppler effect, collisions of particles and the Compton effect. Finally, Sect. 9.5 deals with non-isolated particles and introduces the concept of four-force.

9.2 Four-Momentum, Mass and Energy

9.2.1 Four-Momentum and Mass of a Particle

Let us consider a particle $\mathscr{P}$ of worldline $\mathscr{L}$ in Minkowski spacetime $\mathscr{E}$. This particle can either be massive, in which case $\mathscr{L}$ is a timelike curve (cf. Sect. 2.2), or be massless (for instance, a photon), in which case $\mathscr{L}$ is a null geodesic (cf. Sect. 2.5). If $\mathscr{P}$ has no internal structure, its dynamics is entirely described by a field of linear forms $\boldsymbol{p}$ defined along $\mathscr{L}$ and such that at any point $M \in \mathscr{L}$, the vector[1] $\vec{\boldsymbol{p}}(M)$ is tangent to $\mathscr{L}$ and future-directed (cf. Fig. 9.1). Moreover, it is demanded that $\boldsymbol{p}$ has the dimension of a linear momentum, i.e. a mass times a velocity.

[1]Let us recall that $\vec{\boldsymbol{p}}(M)$ is the vector associated with the linear form $\boldsymbol{p}(M)$ by metric duality; cf. Sect. 1.6 and Eq. (1.46).

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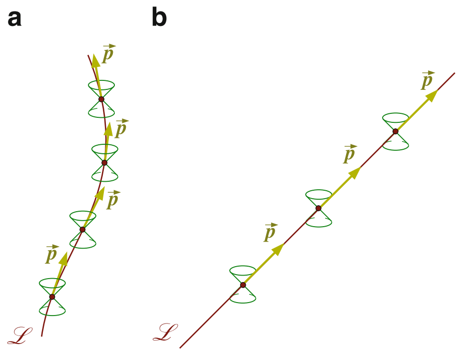

Fig. 9.1 Four-momentum vector $\vec{p}$ at various points of the worldline $\mathscr{L}$ of a particle: (**a**) case of a massive particle; (**b**) case of a massless particle (photon)

The vector $\vec{p}$ has the same dimension since $\boldsymbol{g}$ is dimensionless [cf. Eq. (1.46)]. The linear form $\boldsymbol{p}(M)$ is called ***four-momentum***, or ***4-momentum*** for short, of $\mathscr{P}$ at M. The metric dual $\vec{p}(M)$ is then called ***four-momentum vector***, or ***4-momentum vector***, of $\mathscr{P}$ at M.

We shall call ***simple particle*** the model of particle whose 4-momentum obeys to the above property and rules entirely the dynamics, in order to distinguish it from more sophisticated models, such as a *particle with spin* (to be discussed in Sect. 10.7). In some versions of these models, the 4-momentum vector $\vec{p}$ is not necessarily tangent to the worldline (Corben 1968).

Remark 9.1. In many textbooks, the 4-momentum is presented as a vector of E (a "4-vector"), on the same footing as, e.g. the 4-velocity. However, as we shall see below and in Chap. 11, the 4-momentum is fundamentally a *linear form* and not a vector. The vector $\vec{p}$ is thus a secondary quantity, deduced from the linear form $\boldsymbol{p}$ by metric duality.

The norm of $\vec{p}$ (as defined in Sect. 1.3.5) divided by c,

$$\boxed{m := \frac{1}{c}\,\|\vec{p}\|_{\boldsymbol{g}} = \frac{1}{c}\sqrt{-\langle \boldsymbol{p}, \vec{p}\rangle}}, \tag{9.1}$$

is called the ***mass of particle*** $\mathscr{P}$. The sign $-$ in (9.1) takes into account the fact that $\vec{p}\cdot\vec{p} \le 0$, $\vec{p}$ being either a timelike vector ($\mathscr{P}$ = massive particle, Fig. 9.1a) or a null one ($\mathscr{P}$ = massless particle, Fig. 9.1b). For a massive particle, $m > 0$, whereas for a massless one $m = 0$. This justifies a posteriori the denomination *massless particle* introduced in Chap. 2 for particles whose worldlines are null geodesics. A writing equivalent to (9.1) is

$$\boxed{\vec{p}\cdot\vec{p} = -m^2c^2}. \tag{9.2}$$

Remark 9.2. The quantity m defined by (9.1) or (9.2) is sometimes called the *rest mass* or *proper mass* of particle $\mathscr{P}$; we shall not use this terminology here and shall design m simply by *mass*.

Remark 9.3. We have not assumed that m is constant along $\mathscr{L}$. This will occur if $\mathscr{P}$ is an elementary particle (e.g. an electron). But if $\mathscr{P}$ is a composite particle (e.g. an atomic nucleus or an atom), m contains some binding energy and can vary. For instance, an atom in an excited state that returns to its fundamental state (by emitting a photon) has a mass that decreases slightly.

In the case where $\mathscr{P}$ is a massive particle, a second field of vectors tangent to the worldline $\mathscr{L}$ has been introduced in Chap. 2: the 4-velocity $\vec{\boldsymbol{u}}$. At each point of $\mathscr{L}$, the vectors $\vec{\boldsymbol{p}}$ and $\vec{\boldsymbol{u}}$ are thus collinear: $\vec{\boldsymbol{p}} = \alpha \vec{\boldsymbol{u}}$, with $\alpha \in \mathbb{R}$. Since $\vec{\boldsymbol{p}}$ and $\vec{\boldsymbol{u}}$ are both future-directed, one must have $\alpha \geq 0$. Furthermore, $\vec{\boldsymbol{u}}$ being unitary and the norm of $\vec{\boldsymbol{p}}$ being mc [Eq. (9.1)], we have necessarily $\alpha = mc$, whence $\vec{\boldsymbol{p}} = mc\,\vec{\boldsymbol{u}}$. Introducing the linear form $\underline{\boldsymbol{u}}$ associated with $\vec{\boldsymbol{u}}$ by metric duality (cf. Sect. 1.6), this relation can be rewritten as

$$\boxed{\boldsymbol{p} = mc\,\underline{\boldsymbol{u}}}. \tag{9.3}$$

Historical note: *The notion of 4-momentum has been introduced in 1909 by Hermann Minkowski (cf. p. 26), in his famous text on spacetime (Minkowski 1909).*

9.2.2 *Energy and Momentum Relative to an Observer*

Given an observer $\mathscr{O}$, of worldline $\mathscr{L}_0$, 4-velocity $\vec{\boldsymbol{u}}_0$ and proper time t, one calls ***energy of the particle measured by $\mathscr{O}$ at the instant*** t the quantity

$$\boxed{E := -c\,\langle \boldsymbol{p}, \vec{\boldsymbol{u}}_0 \rangle}, \tag{9.4}$$

where $\vec{\boldsymbol{u}}_0 = \vec{\boldsymbol{u}}_0(t)$ is the 4-velocity of $\mathscr{O}$ at the instant t and $\boldsymbol{p} = \boldsymbol{p}(M(t))$ the 4-momentum of $\mathscr{P}$ at the point $M(t)$ where the worldline $\mathscr{L}$ intersects $\mathscr{O}$'s local rest space at time t, $\mathscr{E}_{\boldsymbol{u}_0}(t)$ (cf. Fig. 9.2).

Remark 9.4. The *linear form* character of the 4-momentum appears clearly in the definition (9.4): $\boldsymbol{p}$ is the linear form that maps the 4-velocity $\vec{\boldsymbol{u}}_0$ of any observer $\mathscr{O}$ to the real number E quantifying the particle energy measured by $\mathscr{O}$. It is true that the equality (9.4) can be written in terms of the vector $\vec{\boldsymbol{p}}$, via a scalar product with $\vec{\boldsymbol{u}}_0$:

$$E = -c\,\vec{\boldsymbol{p}} \cdot \vec{\boldsymbol{u}}_0 = -c\,\boldsymbol{g}(\vec{\boldsymbol{p}}, \vec{\boldsymbol{u}}_0).$$

However, this writing is conceptually more complicated than (9.4), for it involves three objects in Minkowski spacetime: the vectors $\vec{\boldsymbol{p}}$ and $\vec{\boldsymbol{u}}_0$, and the metric tensor $\boldsymbol{g}$, whereas (9.4) involves only two objects: the linear form $\boldsymbol{p}$ and the vector $\vec{\boldsymbol{u}}_0$.

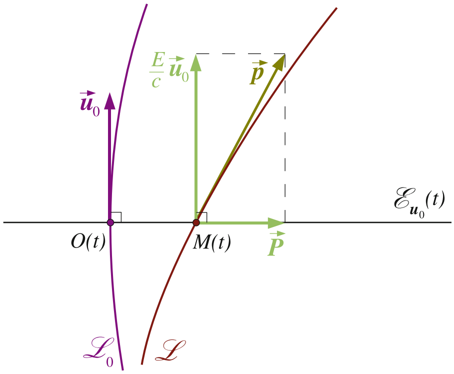

Fig. 9.2 Energy E and linear-momentum vector $\vec{P}$ of a particle with respect to an observer. $\mathscr{L}$ is the particle's worldline and $\vec{p}$ its 4-momentum vector. $\mathscr{L}_0$ is the observer's worldline, $\vec{u}_0$ its 4-velocity and $\mathscr{E}_{u_0}(t)$ its local rest space

The energy represents one component of the 4-momentum with respect to $\mathscr{O}$. The other component is the ***linear momentum of the particle measured by $\mathscr{O}$ at the instant*** t, defined by

$$\boxed{\boldsymbol{P} := \boldsymbol{p} \circ \perp_{\boldsymbol{u}_0}}, \tag{9.5}$$

where $\perp_{\boldsymbol{u}_0}$ stands for the orthogonal projector on the vector hyperplane $E_{\boldsymbol{u}_0}(t)$, local rest space of $\mathscr{O}$ (cf. Sect. 3.2.5). As in (9.4), the 4-momentum $\boldsymbol{p}$ in (9.5) is to be taken at the point $M(t)$ where $\mathscr{L}$ intersects $\mathscr{E}_{\boldsymbol{u}_0}(t)$. Let us make explicit the action of the linear form (9.5): $\boldsymbol{P} : E \longrightarrow \mathbb{R}$, $\vec{v} \mapsto \langle \boldsymbol{p}, \perp_{\boldsymbol{u}_0} \vec{v} \rangle$. Again, we should have written $\boldsymbol{P}(M(t))$, $\boldsymbol{p}(M(t))$ and $\perp_{\boldsymbol{u}_0(t)}$. The vector $\vec{P}$ associated with the linear form $\boldsymbol{P}$ by metric duality is such that (cf. Sect. 1.6)

$$\forall \vec{v} \in E, \quad \vec{P} \cdot \vec{v} = \langle \boldsymbol{P}, \vec{v} \rangle = \langle \boldsymbol{p}, \perp_{\boldsymbol{u}_0} \vec{v} \rangle = \vec{p} \cdot \perp_{\boldsymbol{u}_0} \vec{v} = \perp_{\boldsymbol{u}_0} \vec{p} \cdot \perp_{\boldsymbol{u}_0} \vec{v} = \perp_{\boldsymbol{u}_0} \vec{p} \cdot \vec{v}.$$

We conclude that

$$\boxed{\vec{P} = \perp_{\boldsymbol{u}_0} \vec{p}}. \tag{9.6}$$

Hence the linear-momentum vector measured by $\mathscr{O}$ is nothing but the orthogonal projection of the 4-momentum vector onto $\mathscr{O}$'s local rest space.

By means of the explicit form (3.11) of $\perp_{\boldsymbol{u}_0}$, we can write (9.6) as $\vec{P} = \vec{p} + (\vec{u}_0 \cdot \vec{p})\, \vec{u}_0$. Via (9.4), there comes then

$$\boxed{\vec{p} = \frac{E}{c}\, \vec{u}_0 + \vec{P}} \qquad \text{with} \quad \vec{u}_0 \cdot \vec{P} = 0. \tag{9.7}$$

Hence, $\mathscr{P}$'s energy and linear momentum measured by $\mathscr{O}$ appear as the components of the orthogonal decomposition with respect to $\mathscr{O}$ of $\mathscr{P}$'s 4-momentum vector (cf. Fig. 9.2).

Remark 9.5. For this reason, some authors call $\vec{\boldsymbol{p}}$ the *energy–momentum 4-vector*. Edwin F. Taylor and John A. Wheeler (cf. p. 79) have even coined the term *momenergy* for it, contraction of *momentum* and *energy* (Taylor and Wheeler 1992).

By metric duality, (9.7) is equivalent to

$$\boxed{\underline{\boldsymbol{p}} = \frac{E}{c}\,\underline{\boldsymbol{u}}_0 + \boldsymbol{P}} \qquad \text{with} \quad \langle \boldsymbol{P}, \vec{\boldsymbol{u}}_0 \rangle = 0. \tag{9.8}$$

The scalar square (with respect to the metric tensor $\boldsymbol{g}$) of (9.7) leads to

$$\vec{\boldsymbol{p}} \cdot \vec{\boldsymbol{p}} = \frac{E^2}{c^2} \underbrace{\vec{\boldsymbol{u}}_0 \cdot \vec{\boldsymbol{u}}_0}_{-1} + 2\frac{E}{c} \underbrace{\vec{\boldsymbol{u}}_0 \cdot \vec{\boldsymbol{P}}}_{0} + \vec{\boldsymbol{P}} \cdot \vec{\boldsymbol{P}}$$

Since according to (9.2), $\vec{\boldsymbol{p}} \cdot \vec{\boldsymbol{p}} = -m^2 c^2$, we get the ***Einstein relation***:

$$\boxed{E^2 = m^2 c^4 + \vec{\boldsymbol{P}} \cdot \vec{\boldsymbol{P}}\, c^2}. \tag{9.9}$$

In the case of a photon (or more generally a massless particle), this relation simplifies to

$$\boxed{E = \|\vec{\boldsymbol{P}}\|_{\boldsymbol{g}}\, c}_{\,m=0}. \tag{9.10}$$

Remark 9.6. It is clear from the preceding definitions that the *4-momentum* and the *mass* of a particle are "absolute" quantities, i.e. they depend only on the state of the considered particle; they are thus on the same footing as the 4-velocity and the 4-acceleration. On the opposite, the particle's energy and linear momentum are defined relatively to an observer, similarly to the velocity and the acceleration introduced in Chap. 4.

In the SI system, the unit of energy is the ***joule***: $1\ \mathrm{J} = 1\ \mathrm{kg\,m^2\,s^{-2}}$. In particle physics, it is more convenient to use the ***electronvolt*** (symbol: eV), which is the energy acquired by an electron when crossing an electric potential difference of one volt. As we shall see in Chap. 17,

$$1\ \mathrm{eV} = 1.602\,176\,487(40) \times 10^{-19}\ \mathrm{J}. \tag{9.11}$$

Multiples of the electronvolt are often used: $1\ \mathrm{keV} = 10^3\ \mathrm{eV}$, $1\ \mathrm{MeV} = 10^6\ \mathrm{eV}$, $1\ \mathrm{GeV} = 10^9\ \mathrm{eV}$, $1\ \mathrm{TeV} = 10^{12}\ \mathrm{eV}$, $1\ \mathrm{PeV} = 10^{15}\ \mathrm{eV}$ (petaelectronvolt) and $1\ \mathrm{EeV} = 10^{18}\ \mathrm{eV}$ (exaelectronvolt). Similarly, the unit of linear momentum used in particle physics is the electronvolt divided by c [cf. (9.10)]: $1\ \mathrm{eV}/c = 5.344285502 \times 10^{-28}\ \mathrm{kg\,m\,s^{-1}}$.

9.2.3 Case of a Massive Particle

If $\mathscr{P}$ is a massive particle, formula (9.3) relates its 4-momentum vector to its 4-velocity $\vec{\boldsymbol{u}}$. The latter is expressible in terms of $\mathscr{P}$'s velocity $\vec{V}$ relative to $\mathscr{O}$ and $\mathscr{P}$'s Lorentz factor Γ with respect to $\mathscr{O}$, via (4.27):

$$\vec{\boldsymbol{u}} = \Gamma \left[(1 + \vec{\boldsymbol{a}}_0 \cdot \overrightarrow{OM})\, \vec{\boldsymbol{u}}_0 + \frac{1}{c} \left(\vec{V} + \vec{\boldsymbol{\omega}} \times_{\boldsymbol{u}_0} \overrightarrow{OM} \right) \right],$$

where $\vec{\boldsymbol{a}}_0$ and $\vec{\boldsymbol{\omega}}$ are, respectively, the 4-acceleration and the 4-rotation of observer $\mathscr{O}$. Inserting this relation into (9.3), we get

$$\vec{\boldsymbol{p}} = \Gamma mc (1 + \vec{\boldsymbol{a}}_0 \cdot \overrightarrow{OM})\, \vec{\boldsymbol{u}}_0 + \Gamma m \left(\vec{V} + \vec{\boldsymbol{\omega}} \times_{\boldsymbol{u}_0} \overrightarrow{OM} \right).$$

The comparison with (9.7) leads to

$$\boxed{E = \Gamma (1 + \vec{\boldsymbol{a}}_0 \cdot \overrightarrow{OM}) mc^2}, \tag{9.12}$$

and

$$\boxed{\vec{\boldsymbol{P}} = \Gamma m \left(\vec{V} + \vec{\boldsymbol{\omega}} \times_{\boldsymbol{u}_0} \overrightarrow{OM} \right)}. \tag{9.13}$$

Note that the Lorentz factor that appears in these formulas is given by (4.30).

Let us call ***quantity of motion*** of particle $\mathscr{P}$ with respect to observer $\mathscr{O}$ the linear form

$$\boxed{\boldsymbol{q} := \Gamma m \underline{V}}, \tag{9.14}$$

where $\underline{V}$ is the linear form associated with the velocity vector $\vec{V}$ by metric duality (cf. Sect. 1.6). The vector $\vec{\boldsymbol{q}}$, metric dual of $\boldsymbol{q}$, belongs to $\mathscr{O}$'s local rest space. Equation (9.13) leads to the following relation between the linear momentum and the quantity of motion (both relative to $\mathscr{O}$):

$$\vec{\boldsymbol{P}} = \vec{\boldsymbol{q}} + \Gamma m\, \vec{\boldsymbol{\omega}} \times_{\boldsymbol{u}_0} \overrightarrow{OM}. \tag{9.15}$$

Two important particular cases are the following: (i) $\mathscr{O}$ is an inertial observer ($\vec{\boldsymbol{a}}_0 = 0$ and $\vec{\boldsymbol{\omega}} = 0$) and (ii) the worldlines of the particle and the observer cross each other ($M(t) \in \mathscr{L}_0$). The above formulas and (4.30) simplify then to

$$\boxed{E = \Gamma mc^2}_{\, M \in \mathscr{L}_0 \text{ or } \mathscr{O} \text{ inertial}} \tag{9.16}$$

$$\boxed{\vec{\boldsymbol{P}} = \vec{\boldsymbol{q}} = \Gamma m \vec{V}}_{\, M \in \mathscr{L}_0 \text{ or } \mathscr{O} \text{ inertial}} \tag{9.17}$$

$$\Gamma = \left(1 - \frac{1}{c^2}\vec{V}\cdot\vec{V}\right)^{-1/2} \qquad (M \in \mathscr{L}_0 \text{ or } \mathscr{O} \text{ inertial}). \tag{9.18}$$

Equation (9.16) is probably the best-known equation in whole physics!

When $\mathscr{O}$ is inertial, one calls ***kinetic energy*** of $\mathscr{P}$ with respect to observer $\mathscr{O}$ the quantity

$$\boxed{E_{\rm kin} := (\Gamma - 1)mc^2}. \tag{9.19}$$

Formula (9.16) becomes then

$$E = mc^2 + E_{\rm kin}. \tag{9.20}$$

The quantity mc^2 is called ***mass energy*** of particle $\mathscr{P}$. At the nonrelativistic limit, $\|\vec{V}\|_g \ll c$ and the expansion of Γ in (9.18) shows that (9.17) and (9.19) reduce to

$$\vec{P} \simeq m\vec{V} \quad \text{and} \quad E_{\rm kin} \simeq \frac{1}{2}m\vec{V}\cdot\vec{V} \qquad \text{(nonrelativistic)}. \tag{9.21}$$

We recognize the standard expressions for linear momentum and kinetic energy.

Example 9.1. For an electron $mc^2 = 511$ keV, whereas for a proton, $mc^2 = 0.938$ GeV. The former accelerator LEP of CERN (cf. Sect. 17.5.4 and Table 17.1) brought electrons up to the energy $E = 104$ GeV, which, according to (9.16), corresponds to the Lorentz factor $\Gamma = 2 \times 10^5$. The LHC, to be presented in Sect. 17.5.4, accelerates protons up to $E = 7$ TeV, which corresponds to the Lorentz factor $\Gamma = 7.5 \times 10^3$.

Example 9.2. The Earth is continuously hit by high-energy particles from the universe, called ***cosmic rays*** (Crozon 2005). They are mostly protons and their energy distribution is plotted in Fig. 9.3. One notes the existence of very high-energy particles, up to $E \sim 10^{20}$ eV. They are called ***ultrahigh-energy*** cosmic rays. The record is $E = 3 \times 10^{20}$ eV, measured in 1993 (Bird et al. 1993). Expressed in joules, this energy is $E = 52$ J, which equals the kinetic energy of a tennis ball launched at 150 km/h! The precise nature of the particle is not known. If it is a proton, its Lorentz factor, deduced from (9.16), is $\Gamma = 3 \times 10^{11}$. This implies a velocity extremely close to c: $V = (1 - 5 \times 10^{-24})c$. It is the fastest massive particle (with respect to a terrestrial observer) observed to date. Such a particle is however not directly detected: as soon as it enters into the Earth atmosphere, it interacts with atomic nuclei and gives birth to thousands of secondary particles, forming a so-called ***cosmic air shower***, illustrated in Fig. 9.4. The energy of the primary particle is inferred from the secondary particles detected on the ground. The largest detector for ultrahigh-energy cosmic rays is the ***Pierre Auger Observatory***, which started operations in 2008; it is shown in Fig. 9.4.

With (9.19), we have formally defined the kinetic energy as $(\Gamma - 1)mc^2$, where Γ is the function (9.18) of the particle velocity. In addition to the correct nonrelativistic limit (9.21), a justification of this definition is that it corresponds to the energy

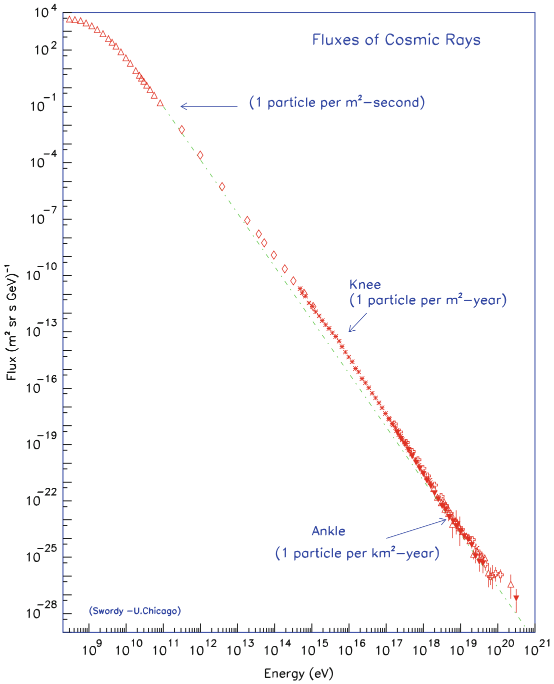

Fig. 9.3 Distribution of cosmic rays arriving on Earth as a function of their energy: the abscissa is the energy E of cosmic rays with respect to a terrestrial observer, while the ordinate is the number F of cosmic rays per unit surface, per unit solid angle and per unit time within an energy band of 1 GeV. The dashed line corresponds to a distribution obeying the power law $F \propto E^{-2.8}$. Up to $\sim 10^{10}$ eV, most cosmic rays are coming from the Sun. Between $\sim 10^{10}$ eV and $\sim 10^{15}$ eV, they originate from sources located within our galaxy, whereas above $\sim 10^{15}$ eV, they have an extragalactic origin [Source: Cronin, Gaisser and Swordy (1997)]

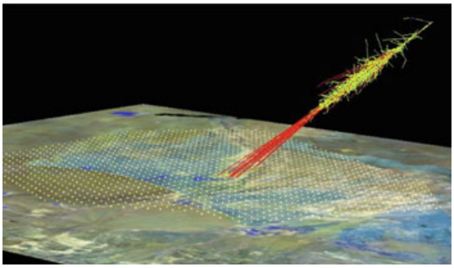

Fig. 9.4 Numerical simulation of a cosmic air shower created during the atmospheric entry of a proton of energy 10^{19} eV. The small dots on the ground are the 1,600 tanks (each filled with 12,000 L of water) of the Pierre Auger Observatory, covering an area of $3,000$ km^2 in Argentina. The particles propagating along a straight line and reaching the ground are muons. The other particles are mostly photons, electrons and positrons [Source: R. Landsberg, D. Surendran & M. SubbaRao (Cosmus group, Univ. Chicago)]

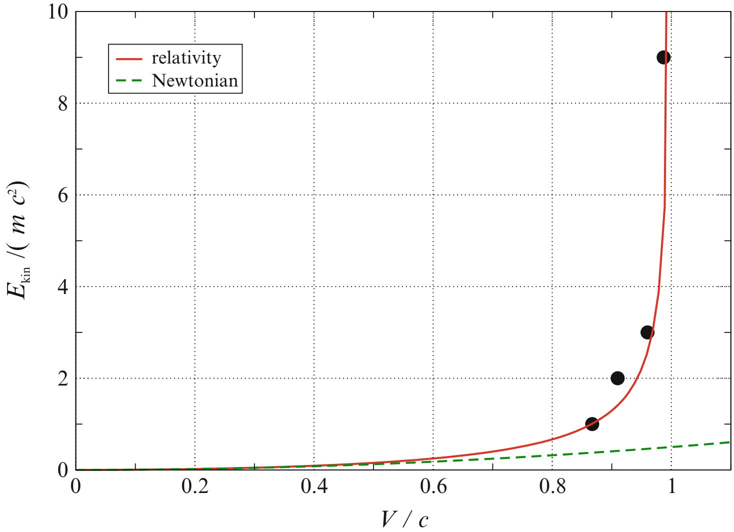

Fig. 9.5 Kinetic energy $E_{\rm kin}$ as a function of the norm V the velocity: the solid curve corresponds to the relativistic formula (9.19), the dashed curve to the Newtonian formula $\frac{1}{2}mV^2$ [Eq. (9.21)] and the dots are the experimental results obtained by W. Bertozzi in 1964 (Bertozzi 1964)

deposited as heat when relativistic particles are absorbed in matter. This has been checked experimentally by William Bertozzi in 1964 (Bertozzi 1964): he measured by calorimetry the energy laid by relativistic electrons into an aluminium disk. He also determined directly the velocity V of the electrons by measuring the time to cover the 8.4 m separating the output of the electrostatic accelerator (the electron source) to the aluminium disk. Bertozzi measurements are depicted in Fig. 9.5: they agree well with the relativistic definition (9.19) of kinetic energy and disagree strongly with the Newtonian counterpart (9.21).

Historical note: *The equivalence between mass and energy, expressed here by (9.16), has been established by Albert Einstein (cf. p. 26) in 1905 (Einstein 1905c), a few months after having published the article funding special relativity (Einstein 1905b). Let us stress that the concept of 4-momentum vector did not exist at this time (cf. historical note in Sect. 9.2.1), so that Einstein's reasoning was very different from that presented here. Regarding the relation* $\vec{\boldsymbol{P}} = \Gamma m \vec{\boldsymbol{V}}$ *[Eq. (9.17)], it appears in the specific case of a electromagnetic model for the electron in an article by Hendrik Lorentz (cf. p. 108) published in 1904 (Lorentz 1904) (cf. Darrigol (2006)). In the case of special relativity, it seems that it has been written for the first time by Max Planck*[2] *in 1906 (Planck 1906).*

[2]**Max Planck** (1858–1947): German physicist, 1918 Nobel Prize in Physics, famous for having introduced the concept of energy quantum, via the constant bearing his name [cf. Eq. (9.25)], which was the prelude to quantum mechanics. Planck supported Albert Einstein as soon as 1905, recognizing immediately the importance of relativity and contributing to its diffusion in Germany.

9.2.4 Energy and Momentum of a Photon

Let us assume that the particle $\mathscr{P}$ is a photon. In Sect. 4.6, we have introduced its propagation direction with respect to the observer $\mathscr{O}$ as the unit vector $\vec{\boldsymbol{n}}$ belonging to $\mathscr{O}$'s local rest space $E_{\boldsymbol{u}_0}$ such that the null vector $\vec{\boldsymbol{\ell}} := \vec{\boldsymbol{u}}_0 + \vec{\boldsymbol{n}}$ is tangent to $\mathscr{P}$'s worldline and future-directed [cf. Eq. (4.74)]. $\vec{\boldsymbol{\ell}}$ and the 4-momentum $\vec{\boldsymbol{p}}$ are then two tangent vectors to $\mathscr{P}$'s worldline; they must be collinear: $\vec{\boldsymbol{p}} = \alpha \vec{\boldsymbol{\ell}} = \alpha(\vec{\boldsymbol{u}}_0 + \vec{\boldsymbol{n}})$, with $\alpha \in \mathbb{R}$. Comparing with expression (9.7) for $\vec{\boldsymbol{p}}$, we deduce that $\alpha = E/c$,

$$\boxed{\vec{\boldsymbol{p}} = \frac{E}{c}\,(\vec{\boldsymbol{u}}_0 + \vec{\boldsymbol{n}})} \tag{9.22}$$

and

$$\boxed{\vec{\boldsymbol{P}} = \frac{E}{c}\,\vec{\boldsymbol{n}}}\,. \tag{9.23}$$

As a check, the scalar square of (9.23) yields (9.10), since $\vec{\boldsymbol{n}} \cdot \vec{\boldsymbol{n}} = 1$.

The photon's ***frequency relative to observer*** $\mathscr{O}$, f, is related to its energy E with respect to $\mathscr{O}$, as defined in Sect. 9.2.2, by the ***Planck–Einstein formula***

$$\boxed{E = hf}\,, \tag{9.24}$$

where h is Planck constant:

$$h = 6.6260693(11) \times 10^{-34}\ \mathrm{J\ s}. \tag{9.25}$$

Substituting (9.24) for E in (9.23), we get

$$\boxed{\vec{\boldsymbol{P}} = \frac{h}{\lambda}\,\vec{\boldsymbol{n}}} \quad \text{with} \quad \boxed{\lambda := \frac{c}{f}}\,. \tag{9.26}$$

The quantity λ is called ***wavelength*** of the photon measured by observer $\mathscr{O}$.

Remark 9.7. The frequency and the wavelength of a photon are quantities that are *relative to an observer* (since they are related to the photon energy measured by the observer). In particular, one can always find an observer with respect to which the frequency of a given photon is arbitrary small or arbitrary large. The only dynamical quantity intrinsic to a photon[3] is its 4-momentum $\boldsymbol{p}$.

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[3]If quantum effects are to be taken into account, a second intrinsic dynamical quantity must be considered: the photon's spin (cf. Sect. 10.7).

9.2.5 Relation Between P, E and the Relative Velocity

For any particle $\mathscr{P}$, massive or not, the linear momentum $\vec{\boldsymbol{P}}$, energy E and velocity $\vec{\boldsymbol{V}}$, all the three relative to observer $\mathscr{O}$, are linked by the equation

$$\boxed{\vec{\boldsymbol{P}} = \frac{E}{c^2(1 + \vec{\boldsymbol{a}}_0 \cdot \overrightarrow{OM})} \left(\vec{\boldsymbol{V}} + \vec{\boldsymbol{\omega}} \times_{u_0} \overrightarrow{OM} \right)}, \tag{9.27}$$

where $\vec{\boldsymbol{a}}_0$ is the 4-acceleration of $\mathscr{O}$ and $\vec{\boldsymbol{\omega}}$ his 4-rotation.

Proof. If $\mathscr{P}$ is a massive particle, we can get rid of Γm in (9.13) by means of (9.12); this yields exactly (9.27). If $\mathscr{P}$ is a massless particle, we may extract $\vec{\boldsymbol{n}}$ from (4.78) and insert the result in (9.23); this leads to (9.27). □

If $\mathscr{O}$ is an inertial observer or if particle $\mathscr{P}$ encounters $\mathscr{O}$ ($\overrightarrow{OM} = 0$), (9.27) simplifies to

$$\boxed{\vec{\boldsymbol{P}} = \frac{E}{c^2} \vec{\boldsymbol{V}}.}_{\; M \in \mathscr{L}_0 \text{ or } \mathscr{O} \text{ inertial}} \tag{9.28}$$

9.2.6 Components of the 4-Momentum

The components (p_α) of $\mathscr{P}$'s 4-momentum with respect to the local frame $(\vec{\boldsymbol{e}}_\alpha)$ of observer $\mathscr{O}$ are

$$p_\alpha := \langle \boldsymbol{p}, \vec{\boldsymbol{e}}_\alpha \rangle. \tag{9.29}$$

Of course, the dependency in t is implicit in this equation: with all rigour, one should write $p_\alpha(t) := \langle \boldsymbol{p}(M(t)), \vec{\boldsymbol{e}}_\alpha(t) \rangle$ [cf. Fig. 9.2]. Thanks to the linearity of $\boldsymbol{p}$, (9.29) leads to

$$\forall \vec{\boldsymbol{v}} \in E, \quad \boxed{\langle \boldsymbol{p}, \vec{\boldsymbol{v}} \rangle = p_\alpha \, v^\alpha}, \tag{9.30}$$

where the v^α's are the components of $\vec{\boldsymbol{v}}$ in the basis $(\vec{\boldsymbol{e}}_\alpha)$: $\vec{\boldsymbol{v}} = v^\alpha \vec{\boldsymbol{e}}_\alpha$.

Since $\vec{\boldsymbol{e}}_0 = \vec{\boldsymbol{u}}_0$, relation (9.8) allows one to express the 4-momentum's components in terms of $\mathscr{P}$'s energy measured by $\mathscr{O}$ and the components P_i of $\mathscr{P}$'s linear momentum measured by $\mathscr{O}$:

$$\begin{aligned} p_\alpha v^\alpha &= \frac{E}{c} \langle \underline{\boldsymbol{u}}_0, \vec{\boldsymbol{v}} \rangle + \langle \boldsymbol{P}, \vec{\boldsymbol{v}} \rangle = \frac{E}{c} v^\alpha \langle \underline{\boldsymbol{u}}_0, \vec{\boldsymbol{e}}_\alpha \rangle + v^\alpha \langle \boldsymbol{P}, \vec{\boldsymbol{e}}_\alpha \rangle \\ &= \frac{E}{c} v^0 \underbrace{\langle \underline{\boldsymbol{u}}_0, \vec{\boldsymbol{u}}_0 \rangle}_{-1} + v^i \underbrace{\langle \boldsymbol{P}, \vec{\boldsymbol{e}}_i \rangle}_{P_i} = -\frac{E}{c} v^0 + P_i v^i. \end{aligned}$$

Hence,

$$\boxed{p_\alpha = \left(-\frac{E}{c},\ P_1,\ P_2,\ P_3\right)}. \tag{9.31}$$

The components (p^α) of the 4-momentum vector $\vec{p} = p^\alpha \vec{e}_\alpha$ are deduced directly from (9.7):

$$\boxed{p^\alpha = \left(\frac{E}{c},\ P^1,\ P^2,\ P^3\right)}, \tag{9.32}$$

where the P^i's are the components of the $\vec{P}$ in the basis $(\vec{e}_i)$ of E_{u_0}. Since this basis is orthonormal, we have $P^i = P_i$. More generally, the components of $\vec{p}$ and $\boldsymbol{p}$ are related by (1.43): $p^\alpha = g^{\alpha\beta} p_\beta$. Since for an orthonormal basis $g^{\alpha\beta} = \eta^{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1)$, we have $p^0 = -p_0$ and $p^i = p_i$, which we observe on (9.31)–(9.32).

If $\mathscr{O}$ is an inertial observer or if $M(t) \in \mathscr{L}_0$, the relations (9.16) and (9.17) for a massive particle lead to

$$p_\alpha = \Gamma m(-c,\ V_1,\ V_2,\ V_3) \quad \text{and} \quad p^\alpha = \Gamma m(c,\ V^1,\ V^2,\ V^3), \tag{9.33}$$

m being the mass of $\mathscr{P}$ and the V^i's the components of $\mathscr{P}$'s velocity relative to $\mathscr{O}$, with $V_i = V^i$.

9.3 Conservation of 4-Momentum

Relativistic dynamics is based on two principles: conservation of 4-momentum and conservation of angular momentum. We present here the former, the latter being the topic of the next chapter. Before stating the principle, one must first define the total 4-momentum of a system of particles.

9.3.1 4-Momentum of a Particle System

Let us consider a system formed by a finite number of particles:

$$\mathscr{S} = \{\mathscr{P}_1, \mathscr{P}_2, \ldots, \mathscr{P}_N\}.$$

For each particle $\mathscr{P}_a$ $(a \in \{1, \ldots, N\})$, the 4-momentum $\boldsymbol{p}_a$ has been defined in Sect. 9.2.1 as a *field* of linear forms along the particle's worldline $\mathscr{L}_a$. To define

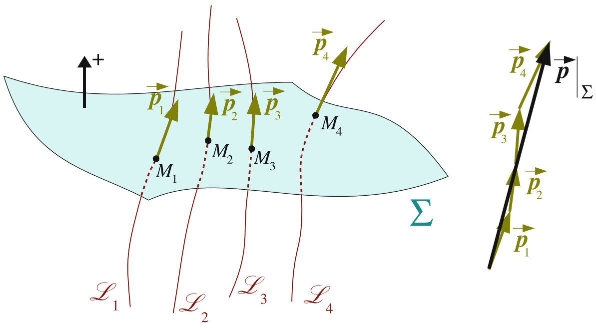

Fig. 9.6 Total 4-momentum vector $\vec{p}|_\Sigma$ of a particle system defined as the vectorial sum of the 4-momenta of each particles at the intersection point M_a of their worldlines with an oriented hypersurface Σ

the total 4-momentum of the system, it is a priori not trivial to "add" the individual 4-momenta $\boldsymbol{p}_a$, because they are defined along different worldlines. To perform the addition, one should select an event M_a on each worldline $\mathscr{L}_a$ and consider the linear form $\boldsymbol{p}_a(M_a)$, which is an element of the vector space E^* (the dual of E; cf. Sect. 1.6.1). Then one can form the sum $\sum_{a=1}^{N} \boldsymbol{p}_a(M_a)$, which is well defined within E^*. Now, there does not exist in Minkowski spacetime any canonical structure that would yield a natural choice for the points M_a. In particular, contrary to the Newtonian case, there does not exist any absolute time, which would have enabled us to define M_a as the event of $\mathscr{L}_a$ for a prescribed value of this time. If an observer $\mathscr{O}$ is given, in addition to the system $\mathscr{S}$, a solution to the above problem appears quite naturally: the events M_a can be selected as those of fixed time t with respect to $\mathscr{O}$, i.e. as the intersections of the worldlines $\mathscr{L}_a$ with $\mathscr{O}$'s local rest space at proper time t, $\mathscr{E}_{\boldsymbol{u}_0}(t)$. More generally, it suffices to provide a hypersurface[4] of $\mathscr{E}$, Σ, that cuts all the worldlines $\mathscr{L}_a$ and such that no worldline is tangent to Σ (cf. Fig. 9.6). The case of an observer appears then as the particular case where Σ is the observer's local rest space at some given instant. We shall assume that Σ is ***orientable***, i.e. that it is possible to split the normal vectors to Σ in two categories (positive and negative orientations) and this in a continuous way. After an orientation is chosen, one says that Σ is an ***oriented hypersurface***.

Example 9.3. The local rest space of an observer at a given proper time t, $\mathscr{E}_{\boldsymbol{u}_0}(t)$, is an oriented hypersurface, the positive orientation being naturally that of future-directed normal vectors. A counterexample (in a three-dimensional space) is provided by the *Moebius strip*, which is a non-orientable surface: moving a normal vector continuously along a path parallel to a strip's edge results in a vector of opposite direction after returning to the initial position.

[4]Let us recall that *hypersurface* stands for a surface of dimension $4 - 1 = 3$. It is actually a three-dimensional volume; hypersurfaces will be discussed in details in Chap. 16.

Remark 9.8. The above notion of orientation is different from that defined in Sect. 1.5: the latter can be named ***internal orientation***, whereas the former is an ***external orientation*** since it makes sense only because Σ is embedded in a larger space, namely, $\mathscr{E}$.

Given an oriented hypersurface Σ, one defines the ***total 4-momentum of the system $\mathscr{S}$ on Σ*** as the linear form on E resulting from the sum of the individual 4-momenta taken at the points of intersection of Σ with the different worldlines (cf. Fig. 9.6):

$$\boxed{\boldsymbol{p}|_{\Sigma} := \sum_{a=1}^{N} \sum_{M \in \mathscr{L}_a \cap \Sigma} \varepsilon \, \boldsymbol{p}_a(M)}, \tag{9.34}$$

where $\boldsymbol{p}_a(M)$ is 4-momentum of particle no. a at point M and $\varepsilon = +1$ if the 4-momentum vector $\vec{\boldsymbol{p}}_a(M)$ associated with $\boldsymbol{p}_a(M)$ has the direction given by the positive orientation of Σ and $\varepsilon = -1$ otherwise. The vector associated with $\boldsymbol{p}|_{\Sigma}$ by metric duality is of course denoted by $\vec{\boldsymbol{p}}|_{\Sigma}$.

Remark 9.9. In Fig. 9.6, the intersection $\mathscr{L}_a \cap \Sigma$ is limited to a single point, but this is not necessarily so in general, hence the second summation sign of (9.34). An example is provided by Fig. 9.8 below.

If Σ is the local rest space of an observer $\mathscr{O}$ at the proper time t, $\Sigma = \mathscr{E}_{\boldsymbol{u}_0}(t)$, we shall call $\boldsymbol{p}|_{\Sigma}$ the ***total 4-momentum of the system $\mathscr{S}$ at the instant t of observer $\mathscr{O}$***. In this case, the natural orientation of Σ is set by the time arrow of Minkowski spacetime (cf. Sect 1.4), and since all the vectors $\vec{\boldsymbol{p}}_a$ are future-directed (by the very definition of 4-momentum; cf. Sect. 9.2.1), one has always $\varepsilon = 1$ in (9.34). Moreover, $\mathscr{E}_{\boldsymbol{u}_0}(t)$ being a spacelike hyperplane, its intersection with a worldline $\mathscr{L}_a$ (which either timelike or null) is necessarily reduced to a single point, M_a, say. Equation (9.34) then becomes

$$\boxed{\boldsymbol{p}|_{\mathscr{E}_{\boldsymbol{u}_0}(t)} := \sum_{a=1}^{N} \boldsymbol{p}_a(M_a)} \quad \text{with} \quad \{M_a\} := \mathscr{L}_a \cap \mathscr{E}_{\boldsymbol{u}_0}(t). \tag{9.35}$$

Historical note: *The definition (9.34) for the total 4-momentum of a system on a hypersurface has been introduced in 1935 by John L. Synge (cf. p. 74) (1935).*

9.3.2 *Isolated System and Particle Collisions*

One says that a particle system is ***isolated*** iff it is not subject to any external interaction. There may be some interaction between the particles. We shall assume them to be pointlike, i.e. to occur in a domain of negligible extension with respect

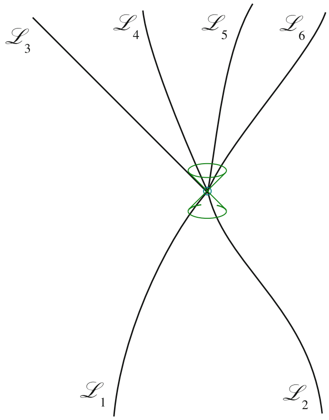

Fig. 9.7 Collision of two particles (worldlines $\mathscr{L}_1$ and $\mathscr{L}_2$) giving birth to four particles (worldlines $\mathscr{L}_3$, $\mathscr{L}_4$, $\mathscr{L}_5$ and $\mathscr{L}_6$). In this example, $\mathscr{L}_3$ corresponds to a photon, whereas the other worldlines are those of massive particles

to the problem under consideration. One calls them ***localized interactions*** or ***collisions***. If the particles are elementary ones, the detailed description of a collision involves quantum mechanics, as well as the treatment of the weak, strong and electromagnetic interactions. It is therefore out of the scope of the present book. During a collision, some particles may disappear or be created. The number of worldlines is thus not necessarily constant, as illustrated in Fig. 9.7.

9.3.3 Principle of 4-Momentum Conservation

We are now in position to state the first fundamental principle of relativistic dynamics:

If a particle system is isolated, its total 4-momentum on any closed hypersurface Σ vanishes:

$$\boxed{\mathscr{S} \text{ isolated and } \Sigma \text{ closed} \implies \boldsymbol{p}|_{\Sigma} = 0}. \tag{9.36}$$

Let us recall that ***closed*** means compact and without boundary.

Various comments are relevant:

- A closed hypersurface separates the spacetime $\mathscr{E}$ in two distinct domains: the interior and the exterior (cf. Fig. 9.8). This provides a natural orientation, and in the summation (9.34), one can choose $\varepsilon = +1$ if $\vec{\boldsymbol{p}}_a$ is directed towards the interior of Σ and $\varepsilon = -1$ otherwise.

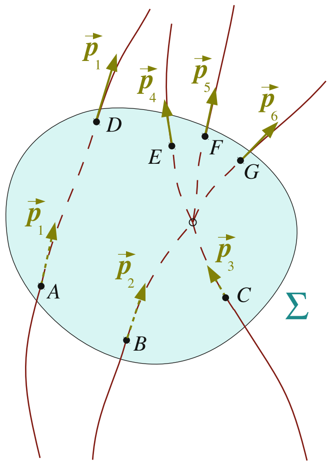

Fig. 9.8 Total 4-momentum of a particle system on a closed hypersurface: $\boldsymbol{p}|_\Sigma = \boldsymbol{p}_1(A) - \boldsymbol{p}_1(D) + \boldsymbol{p}_2(B) + \boldsymbol{p}_3(C) - \boldsymbol{p}_4(E) - \boldsymbol{p}_5(F) - \boldsymbol{p}_6(G)$

- Collisions between particles may occur inside Σ, as illustrated in Fig. 9.8. The number of ingoing 4-momenta is therefore not necessarily equal to the number of outgoing ones.
- The property (9.36) is genuinely a *conservation law* in so far as it means that "nothing is going out of Σ but things that entered in it". We shall detail this in what follows and shall show that (9.36) implies the conservation of energy and linear momentum of the system with respect to any inertial observer.
- The property (9.36) could be used, not as a principle, but as the *definition* of an isolated system.
- For a full description of the dynamics a system of relativistic particles, the above principle must be completed by the principle of angular momentum conservation, to be stated in Chap. 10.
- In the present formulation, the conservation of 4-momentum appears as a first principle (an axiom). We shall see in Chap. 11 that it is no longer the case in a Lagrangian formulation of relativistic dynamics: the conservation of 4-momentum results from the symmetries of the system, via the Noether theorem.

9.3.4 Application to an Isolated Particle: Law of Inertia

Applied to a system reduced to a single particle, the principle (9.36) implies that the 4-momentum of an isolated particle $\mathscr{P}$ is a field of *constant* linear forms along $\mathscr{P}$'s worldline $\mathscr{L}$:

$$\forall M \in \mathscr{L}, \quad \boldsymbol{p}(M) = \text{const.} \tag{9.37}$$

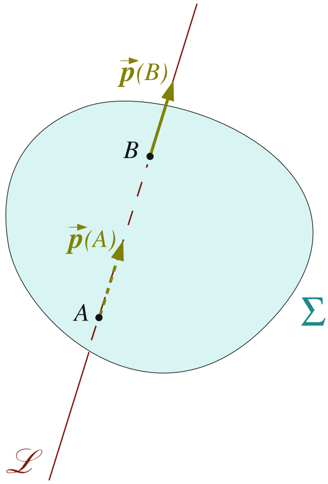

Fig. 9.9 Application of the principle of 4-momentum conservation to an isolated particle: $\vec{p}(A) = \vec{p}(B)$

Proof. If we choose a closed hypersurface Σ that cuts $\mathscr{L}$ in two points A and B (cf. Fig. 9.9), then (9.36) reduces to $\boldsymbol{p}(A) - \boldsymbol{p}(B) = 0$, i.e. $\boldsymbol{p}(A) = \boldsymbol{p}(B)$. By varying Σ and hence the points A and B, we get (9.37). □

Equation (9.37) means that the vector $\vec{p}(M)$ associated with $\boldsymbol{p}(M)$ by metric duality is constant along the worldline $\mathscr{L}$. Since this vector must be tangent to $\mathscr{L}$ at any point, we deduce from (9.37) that $\mathscr{L}$ has to be a straight line. Moreover, the mass m of the particle $\mathscr{P}$ being nothing but the metric norm of $\vec{p}$ [Eq. (9.1)], we also deduce from (9.37) that m is constant. If $\mathscr{P}$ is a massive particle, its 4-velocity, $\vec{u} = (mc)^{-1}\vec{p}$, is also a constant vector along $\mathscr{L}$. For any inertial observer $\mathscr{O}$, the velocity $\vec{V}$ of $\mathscr{P}$ relative to $\mathscr{O}$ is then constant: $\vec{V}$ is given by (4.31): $\vec{V} = c\left(\Gamma^{-1}\vec{u} - \vec{u}_0\right)$, where $\vec{u}_0$ is $\mathscr{O}$'s 4-velocity and $\Gamma = -\vec{u}\cdot\vec{u}_0$ [Eq. (4.10)]. Since $\mathscr{O}$ is an inertial observer, $\vec{u}_0$ is constant [Eq. (8.3)], and the above expression shows indeed that $\vec{V} = \text{const}$. If $\mathscr{P}$ is massless, formula (4.79) applies: $\vec{V} = c\,\vec{n}$ with $\vec{n}$ related to $\vec{p}$ and $\vec{u}_0$ by (9.22), which implies that $\vec{n}$ is constant. To summarize:

If a particle is isolated:

- Its worldline is a straight line in Minkowski spacetime.
- Its 4-momentum is constant.
- Its mass is constant.
- Its velocity relative to any inertial observer is constant.

The last result is known as the ***law of inertia***: with respect to any inertial observer, an isolated particle has a uniform rectilinear motion.

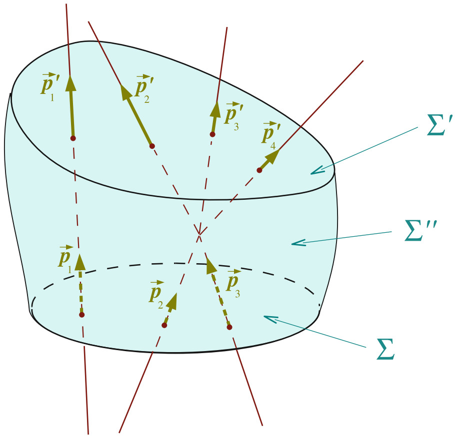

Fig. 9.10 Conservation of the 4-momentum of an isolated system: $\vec{p}|_{\Sigma'} = \vec{p}|_{\Sigma}$ with $\vec{p}|_{\Sigma'} := \vec{p}'_1 + \vec{p}'_2 + \vec{p}'_3 + \vec{p}'_4$ and $\vec{p}|_{\Sigma} = \vec{p}_1 + \vec{p}_2 + \vec{p}_3$

Historical note: *The law of inertia is one of the many facets of the* ***relativity principle****, which stipulates that the laws of physics are the same for all inertial observers. This principle has been formulated for the laws of mechanics by Galileo in 1632 (cf. Paty (1999b) for a detailed discussion). It was Henri Poincaré (cf. p. 26) who, in 1904, extended the principle to electromagnetism and erected it as a fundamental principle for all the laws of physics and named it* relativity principle *(Poincaré 1904, Paty 1996). In the 1905 historical article (Einstein 1905b), Albert Einstein (cf. p. 26) has elaborated special relativity from two postulates: (i) the relativity principle and (ii) the principle of constancy of the velocity of light in vacuum. We have already stressed in Sect. 4.6.2 that in the present formulation of special relativity, the postulate (ii) is no longer a first principle and results from the Minkowski spacetime structure (cf. Remark 4.8 p. 121). The law of inertia that we have just derived shows that the relativity principle has the same status in the present formulation.*

9.3.5 4-Momentum of an Isolated System

Let us consider an isolated system of particles $\mathscr{S}$ and two compact spacelike hypersurfaces Σ and Σ'. By ***spacelike hypersurface*** it is meant that any vector tangent to Σ or Σ' must be spacelike. Examples of such hypersurfaces are of course the hyperplanes formed by the local rest spaces of an observer. Moreover, we assume that Σ and Σ' have no intersection. We may then consider that Σ' is located entirely in the future of Σ (cf. Fig. 9.10). We shall also assume that Σ and Σ' are such that if they are completed by a third hypersurface Σ'' to form a closed hypersurface (cf. Fig. 9.10), then no worldline of a particle in $\mathscr{S}$ crosses Σ''. Note that Σ'' is not spacelike, contrary to Σ and Σ'.

Σ and Σ' being spacelike hypersurfaces, their natural orientation is provided by the time arrow of Minkowski spacetime (cf. Sect 1.4). The 4-momenta of the system are then given by formula (9.34) with $\varepsilon = +1$:

$$\boldsymbol{p}|_{\Sigma} = \sum_{a=1}^{N} \sum_{M \in \mathscr{L}_a \cap \Sigma} \boldsymbol{p}_a(M) \quad \text{and} \quad \boldsymbol{p}|_{\Sigma'} = \sum_{a=1}^{N'} \sum_{M \in \mathscr{L}_a \cap \Sigma'} \boldsymbol{p}_a(M). \tag{9.38}$$

Note that $N' \neq N$ is allowed in these formulas, in order to take into account the disintegration and the creation of particles between Σ and Σ', as illustrated in Fig. 9.10. Besides, the hypersurfaces Σ and Σ' being spacelike, it is not possible that a given worldline has more than one intersection with them. Indeed, if $M \in \mathscr{L}_a \cap \Sigma$, the worldline $\mathscr{L}_a$ is entirely contained within the light cone at M, whereas Σ is necessarily at the exterior of this light cone: if Σ crossed the interior of the light cone, it would "bend" and have portions with timelike tangents, which is not permitted for a spacelike hypersurface. We have thus $\mathscr{L}_a \cap \Sigma = \{M\}$. Consequently (9.38) simplifies into

$$\boldsymbol{p}|_{\Sigma} = \sum_{a=1}^{N} \boldsymbol{p}_a(M_a) \quad \text{and} \quad \boldsymbol{p}|_{\Sigma'} = \sum_{a=1}^{N'} \boldsymbol{p}_a(M'_a),$$

with $\{M_a\} := \mathscr{L}_a \cap \Sigma$ and $\{M'_a\} := \mathscr{L}_a \cap \Sigma'$.

Let us then apply the principle of 4-momentum conservation (9.36) to the closed hypersurface $\Sigma \cup \Sigma' \cup \Sigma''$ and the isolated system $\mathscr{S}$:

$$\boldsymbol{p}|_{\Sigma} - \boldsymbol{p}|_{\Sigma'} + \underbrace{\boldsymbol{p}|_{\Sigma''}}_{0} = 0,$$

where (i) the sign $-$ in front of $\boldsymbol{p}|_{\Sigma'}$ arises from the change of orientation between the hypersurface Σ' (positive orientation towards the future) and the hypersurface $\Sigma \cup \Sigma' \cup \Sigma''$ (positive orientation towards its interior) and (ii) the property $\boldsymbol{p}|_{\Sigma''} = 0$ stems from the fact that no worldline crosses Σ''. We have thus

$$\boxed{\boldsymbol{p}|_{\Sigma'} = \boldsymbol{p}|_{\Sigma}}. \tag{9.39}$$

This means that

The 4-momentum of an isolated system is independent of the choice of the spacelike hypersurface Σ, in so far as the latter crosses all the worldlines of the particles in the system. We can thus omit the index Σ and denote

$$\boxed{\boldsymbol{p} := \sum_{a=1}^{N} \boldsymbol{p}_a(M_a)} \quad \text{with} \quad \{M_a\} := \mathscr{L}_a \cap \Sigma. \tag{9.40}$$

The linear form $\boldsymbol{p} \in E^*$ is called the ***4-momentum of the isolated system*** $\mathscr{S}$.

Remark 9.10. In order to define the total 4-momentum of a particle system, we had to introduce an auxiliary hypersurface as a "slice of time" across the particle worldlines. For an isolated system, the above result shows that the total 4-momentum hence defined is independent of the choice of the hypersurface. It is therefore a quantity intrinsic to the system, on the same footing as the 4-momentum of a single particle. But one should keep in mind that this is valid a priori only for an *isolated* system.

The 4-momentum of an isolated system is similar to that of a single particle regarding the following property:

The 4-momentum vector $\vec{\boldsymbol{p}}$ of an isolated system is either a future-directed timelike vector or a future-directed null vector, the latter case occurring only for a system made of massless particles whose 4-momenta are all collinear.

Proof. In view of (9.40), $\vec{\boldsymbol{p}}$ is the sum of timelike or null vectors, all future-directed. It suffices then to show that the sum $\vec{\boldsymbol{p}} = \vec{\boldsymbol{p}}_1 + \vec{\boldsymbol{p}}_2$ of two such vectors is a future-directed vector, either timelike or null, the latter case occurring only if $\vec{\boldsymbol{p}}_1$ and $\vec{\boldsymbol{p}}_2$ are both null and collinear. We have

$$\vec{\boldsymbol{p}} \cdot \vec{\boldsymbol{p}} = \underbrace{\vec{\boldsymbol{p}}_1 \cdot \vec{\boldsymbol{p}}_1}_{\leq 0} + 2\vec{\boldsymbol{p}}_1 \cdot \vec{\boldsymbol{p}}_2 + \underbrace{\vec{\boldsymbol{p}}_2 \cdot \vec{\boldsymbol{p}}_2}_{\leq 0}.$$

Now, from Lemmas 1 and 2 of Sect. 1.4.2, $\vec{\boldsymbol{p}}_1 \cdot \vec{\boldsymbol{p}}_2 < 0$ if $\vec{\boldsymbol{p}}_1$ and $\vec{\boldsymbol{p}}_2$ are both timelike or are both null and noncollinear. By continuity, we also have $\vec{\boldsymbol{p}}_1 \cdot \vec{\boldsymbol{p}}_2 < 0$ if $\vec{\boldsymbol{p}}_1$ is timelike and $\vec{\boldsymbol{p}}_2$ is null. In all these cases, we have thus $\vec{\boldsymbol{p}} \cdot \vec{\boldsymbol{p}} < 0$, from which we conclude that $\vec{\boldsymbol{p}}$ is timelike. In the remaining case, namely, when $\vec{\boldsymbol{p}}_1$ and $\vec{\boldsymbol{p}}_2$ are null and collinear, we have $\vec{\boldsymbol{p}}_1 \cdot \vec{\boldsymbol{p}}_2 = 0$, which yields $\vec{\boldsymbol{p}} \cdot \vec{\boldsymbol{p}} = 0$, i.e. $\vec{\boldsymbol{p}}$ null. Finally, it is obvious that $\vec{\boldsymbol{p}}$ is future-directed, for $\vec{\boldsymbol{p}}_1$ and $\vec{\boldsymbol{p}}_2$ are. □

As for an individual particle, we define the ***mass of the isolated system*** $\mathscr{S}$ as the metric norm of the vector $\vec{p}$ (up to a c factor):

$$\boxed{m := \frac{1}{c}\,\|\vec{p}\|_g = \frac{1}{c}\sqrt{-\langle \boldsymbol{p}, \vec{p}\rangle}}. \tag{9.41}$$

This formula is well defined since $\vec{p}$ is timelike or null.

Remark 9.11. The total mass of an isolated system is not equal to the sum of the masses of the particles making the system. For example, for a system made by two particles of masses $m_1 > 0$ and $m_2 > 0$, and 4-velocities $\vec{u}_1$ and $\vec{u}_2$, formulas (9.3) and (9.41) yield to

$$\begin{aligned} m^2 &= -(m_1\vec{u}_1 + m_2\vec{u}_2)\cdot(m_1\vec{u}_1 + m_2\vec{u}_2) = m_1^2 - 2m_1m_2\vec{u}_1\cdot\vec{u}_2 + m_2^2 \\ &= (m_1+m_2)^2 + 2(\Gamma - 1)m_1m_2, \end{aligned}$$

with $\Gamma := -\vec{u}_1\cdot\vec{u}_2$. If the 4-velocities $\vec{u}_1$ and $\vec{u}_2$ are not collinear, $\Gamma > 1$ and $m > m_1 + m_2$.

If $m \neq 0$ (i.e. if $\mathscr{S}$ is not made solely of massless particles with collinear 4-momenta), the vector

$$\vec{u} := \frac{1}{mc}\,\vec{p} \tag{9.42}$$

is a future-directed timelike unit vector. By analogy with (9.3), we shall call it ***4-velocity of the isolated system*** $\mathscr{S}$. As for $\vec{p}$, $\vec{u}$ is a constant vector in E. We shall then call ***comoving inertial observer with the isolated system*** $\mathscr{S}$ any inertial observer whose 4-velocity is equal to $\vec{u}$.

9.3.6 Energy and Linear Momentum of a System

Given a particle system $\mathscr{S}$ and an observer $\mathscr{O}$, of proper time t, one calls, respectively, ***energy of the system*** $\mathscr{S}$ ***measured by*** $\mathscr{O}$ ***at time*** t and ***linear momentum of the system*** $\mathscr{S}$ ***measured by*** $\mathscr{O}$ ***at time*** t the quantities

$$\boxed{E := -c\,\langle\, \boldsymbol{p}|_{\mathscr{E}_{u_0}(t)}\,,\ \vec{u}_0(t)\rangle}, \tag{9.43}$$

$$\boxed{\boldsymbol{P} := \boldsymbol{p}|_{\mathscr{E}_{u_0}(t)} \circ \perp_{u_0}}, \tag{9.44}$$

where $\vec{u}_0(t)$ is the 4-velocity of $\mathscr{O}$ at time t and $\boldsymbol{p}|_{\mathscr{E}_{u_0}(t)}$ the 4-momentum of $\mathscr{S}$ at the instant t of observer $\mathscr{O}$, as given by (9.35).

We have of course formulas analogous to (9.6) and (9.7):

$$\boxed{\vec{P} = \perp_{u_0} \vec{p}|_{\mathscr{E}_{u_0}(t)}}, \tag{9.45}$$

$$\boxed{\vec{p}|_{\mathscr{E}_{u_0}(t)} = \frac{E}{c}\,\vec{u}_0 + \vec{P}} \qquad \text{with} \quad \vec{u}_0 \cdot \vec{P} = 0, \tag{9.46}$$

as well as the analogue of Einstein relation (9.9):

$$\boxed{E^2 = m^2c^4 + \vec{P} \cdot \vec{P}\, c^2}. \tag{9.47}$$

Substituting $\boldsymbol{p}|_{\mathscr{E}_{u_0}(t)}$ by expression (9.35) in (9.43) and (9.44) and comparing with (9.4) and (9.5), we deduce that

$$\boxed{E = \sum_{a=1}^{N} E_a} \qquad \text{and} \qquad \boxed{\boldsymbol{P} = \sum_{a=1}^{N} \boldsymbol{P}_a}, \tag{9.48}$$

where E_a and $\boldsymbol{P}_a$ are, respectively, the energy and linear momentum of particle a, both measured by $\mathscr{O}$ at time t.

Example 9.4. Let us suppose that the system $\mathscr{S}$ is isolated and choose for $\mathscr{O}$ a comoving observer. Equation (9.42) leads then to $\vec{p} = mc\,\vec{u}_0$. Comparing with (9.46), we get $E = mc^2$ and $\boldsymbol{P} = 0$. Hence, the linear momentum of a system measured by a comoving observer is vanishing.

If the system $\mathscr{S}$ is isolated, we have seen in Sect. 9.3.5 that its 4-momentum is constant: $\boldsymbol{p}|_{\mathscr{E}_{u_0}(t)} = \boldsymbol{p}$. Besides, if $\mathscr{O}$ is inertial, the 4-velocity $\vec{u}_0$ is a constant vector. We deduce then from (9.43) and (9.44) the following properties:

The energy and linear momentum of an *isolated* system measured by an *inertial* observer are constant:

$$\boxed{\frac{\mathrm{d}E}{\mathrm{d}t} = 0} \qquad \text{and} \qquad \boxed{\frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t} = 0}. \tag{9.49}$$

9.3.7 Application: Doppler Effect

Let us consider the emission of a photon $\mathscr{P}$ of 4-momentum $\boldsymbol{p}$ by an observer $\mathscr{O}'$ (4-velocity $\vec{\boldsymbol{u}}\,'$) and its reception by an inertial observer $\mathscr{O}$ (4-velocity $\vec{\boldsymbol{u}}$). The 4-momentum $\boldsymbol{p}$ is expressible in terms of $\mathscr{P}$'s energy measured by $\mathscr{O}$, $E_{\rm rec}$, and $\mathscr{P}$'s propagation direction with respect to $\mathscr{O}$, $\vec{\boldsymbol{n}}$, according to (9.22): $\boldsymbol{p} = (E_{\rm rec}/c)(\underline{\boldsymbol{u}} + \underline{\boldsymbol{n}})$. Moreover, the 4-velocity of $\mathscr{O}'$ is expressible in terms of his velocity $\vec{\boldsymbol{V}}$ and his Lorentz factor Γ relative to $\mathscr{O}$ according to (4.31): $\vec{\boldsymbol{u}}\,' = \Gamma(\vec{\boldsymbol{u}} + \vec{\boldsymbol{V}}/c)$. If the photon is not submitted to any interaction between $\mathscr{O}'$ and $\mathscr{O}$, its 4-momentum $\boldsymbol{p}$ at the reception is the same as that at the emission [cf. (9.37)]. We may then express the photon's energy relative to the emitter $\mathscr{O}'$ according to (9.4):

$$
\begin{aligned}
E'_{\rm em} &= -c\,\langle \boldsymbol{p}, \vec{\boldsymbol{u}}\,' \rangle = -c\left\langle \frac{E_{\rm rec}}{c}(\underline{\boldsymbol{u}} + \underline{\boldsymbol{n}}),\ \Gamma\left(\vec{\boldsymbol{u}} + \frac{1}{c}\vec{\boldsymbol{V}}\right)\right\rangle \\
&= \Gamma\left(1 - \frac{\vec{\boldsymbol{n}}\cdot\vec{\boldsymbol{V}}}{c}\right) E_{\rm rec},
\end{aligned}
$$

where the properties $\langle \underline{\boldsymbol{u}}, \vec{\boldsymbol{u}} \rangle = -1$, $\langle \underline{\boldsymbol{u}}, \vec{\boldsymbol{V}} \rangle = 0$ and $\langle \underline{\boldsymbol{n}}, \vec{\boldsymbol{u}} \rangle = 0$ have been used. The energy of a photon being proportional to its frequency, via the Planck–Einstein relation (9.24), $E = h\,f$, we get the following relation between the frequency at the reception event measured by $\mathscr{O}$, $f_{\rm rec}$, say, and the frequency at the emission event measured by $\mathscr{O}'$, $f'_{\rm em}$, say:

$$
\boxed{f_{\rm rec} = \frac{f'_{\rm em}}{\Gamma\left(1 - \vec{\boldsymbol{n}}\cdot\vec{\boldsymbol{V}}/c\right)}}. \qquad (9.50)
$$

We thus recover the formula for the Doppler effect obtained in Sect. 5.5 [Eq. (5.62)].

Remark 9.12. In Sect. 5.5, the Doppler formula relating $f_{\rm rec}$ and $f'_{\rm em}$ has been established by reasoning on the measure of time intervals related to periodic signals, whereas it has been obtained here from energetic considerations based on the photon's 4-momentum. The fact that the two results coincide may be seen as a validation of the proportionality relation $E = h\,f$ postulated by Einstein between the energy of a photon and the frequency of the corresponding electromagnetic wave.

9.4 Particle Collisions

9.4.1 *Localized Interactions*

The general form of a localized interaction[5] between some particles is

$$\mathscr{P}_1 + \mathscr{P}_2 + \cdots \longrightarrow \mathscr{P}'_1 + \mathscr{P}'_2 + \cdots ,$$

where $\mathscr{P}_1$, $\mathscr{P}_2$, etc. stand for the particles before the interaction and $\mathscr{P}'_1$, $\mathscr{P}'_2$, etc. for those after the interaction. Various subcases can be distinguished:

- ***Deexcitation***: $\mathscr{P}_1 \longrightarrow \mathscr{P}_1 + \mathscr{P}'_2$; a typical example is the deexcitation of an atom $\mathscr{P}_1$ by the emission of a photon $\mathscr{P}'_2$.
- ***Decay*** or ***disintegration***: $\mathscr{P}_1 \longrightarrow \mathscr{P}'_1 + \mathscr{P}'_2 + \cdots$ with $\mathscr{P}'_a \neq \mathscr{P}_1$. an example is the neutron decay (beta decay), $\mathrm{n} \longrightarrow \mathrm{p} + \mathrm{e}^- + \bar{\nu}_\mathrm{e}$; another example is the muon decay considered in Sect. 4.4.1, $\mu^- \longrightarrow \mathrm{e}^- + \nu_\mu + \bar{\nu}_\mathrm{e}$.
- ***Elastic collision***: $\mathscr{P}_1 + \mathscr{P}_2 \longrightarrow \mathscr{P}_1 + \mathscr{P}_2$, where the masses m_1 and m_2 of particles $\mathscr{P}_1$ and $\mathscr{P}_2$ are kept constant ($m_2 = 0$ if $\mathscr{P}_2$ is a photon); an example is the Compton scattering, to be discussed in Sect. 9.4.4;
- ***Inelastic collision***: $\mathscr{P}_1 + \mathscr{P}_2 \longrightarrow \mathscr{P}'_1 + \mathscr{P}'_2 + \cdots$, with $\mathscr{P}'_1 \neq \mathscr{P}_1$ or $\mathscr{P}'_2 \neq \mathscr{P}_2$, or $\mathscr{P}'_1 = \mathscr{P}_1$ and $\mathscr{P}'_2 = \mathscr{P}_2$ but $m'_1 \neq m_1$ or $m'_2 \neq m_2$
- ***Annihilation***: $\mathscr{P}_1 + \bar{\mathscr{P}}_1 \longrightarrow \mathscr{P}'_1 + \mathscr{P}'_2 + \cdots$, where $\bar{\mathscr{P}}_1$ stands for the antiparticle of $\mathscr{P}_1$; the annihilation is a particular case of an inelastic collision; often, $\mathscr{P}'_1$ and $\mathscr{P}'_2$ are photons, as, for example, in the electron-positron annihilation: $\mathrm{e}^- + \mathrm{e}^+ \longrightarrow \gamma + \gamma$.

Remark 9.13. From the above definition, a collision is *elastic* iff both the nature and the masses of the particles are unchanged. Let us recall that if a particle is not elementary, its mass may vary, after some reorganization of its constituents (variation of internal energy, cf. Remark 9.3 p. 273).

9.4.2 *Collision Between Two Particles*

Let us consider a localized interaction between two particles, $\mathscr{P}_1$ and $\mathscr{P}_2$, whose output is also two particles, $\mathscr{P}'_1$ and $\mathscr{P}'_2$, say, with possibly $\mathscr{P}'_1 = \mathscr{P}_1$ or $\mathscr{P}'_2 = \mathscr{P}_2$:

$$\mathscr{P}_1 + \mathscr{P}_2 \longrightarrow \mathscr{P}'_1 + \mathscr{P}'_2.$$

[5] Cf. Sect. 9.3.2.

The principle of 4-momentum conservation, in the form (9.39), yields

$$\boxed{\vec{p}_1 + \vec{p}_2 = \vec{p}\,'_1 + \vec{p}\,'_2}, \tag{9.51}$$

where $\vec{p}_a$ (resp. $\vec{p}\,'_a$) is the 4-momentum vector of $\mathscr{P}_a$ (resp. $\mathscr{P}'_a$).

It is customary to introduce ***Mandelstam variables***[6] as the following scalar squares:

$$s := -(\vec{p}_1 + \vec{p}_2) \cdot (\vec{p}_1 + \vec{p}_2) = -(\vec{p}\,'_1 + \vec{p}\,'_2) \cdot (\vec{p}\,'_1 + \vec{p}\,'_2) \tag{9.52a}$$

$$t := -(\vec{p}_1 - \vec{p}\,'_1) \cdot (\vec{p}_1 - \vec{p}\,'_1) = -(\vec{p}_2 - \vec{p}\,'_2) \cdot (\vec{p}_2 - \vec{p}\,'_2) \tag{9.52b}$$

$$u := -(\vec{p}_1 - \vec{p}\,'_2) \cdot (\vec{p}_1 - \vec{p}\,'_2) = -(\vec{p}_2 - \vec{p}\,'_1) \cdot (\vec{p}_2 - \vec{p}\,'_1). \tag{9.52c}$$

In each equation, the second equality results from (9.51). Since they are defined from the 4-momenta, Mandelstam variables are independent of any observer. According to the definition (9.41), s is related to the total mass m of the system by

$$s = m^2 c^2. \tag{9.53}$$

Besides, the sum of the three Mandelstam variables is related to the particles' individual masses by

$$s + t + u = (m_1^2 + m_2^2 + m_1'^2 + m_2'^2)c^2. \tag{9.54}$$

Proof. Expanding the scalar products (9.52), we get

$$s+t+u = -\vec{p}_1\cdot\vec{p}_1 - \vec{p}_2\cdot\vec{p}_2 - \vec{p}\,'_1\cdot\vec{p}\,'_1 - \vec{p}\,'_2\cdot\vec{p}\,'_2 - 2\vec{p}_1\cdot\underbrace{(\vec{p}_1 + \vec{p}_2 - \vec{p}\,'_1 - \vec{p}\,'_2)}_{0},$$

where the vanishing of the term inside the parenthesis results from (9.51). The definition (9.2) of the mass of each particle leads then immediately to (9.54). □

9.4.3 *Elastic Collision*

For an elastic collision, $\mathscr{P}'_1 = \mathscr{P}_1$ and $\mathscr{P}'_2 = \mathscr{P}_2$. Moreover, $m'_1 = m_1$ and $m'_2 = m_2$. Expanding (9.52a) and using (9.2) to let appear the particle masses, we get

---

[6] **Stanley Mandelstam**: Theoretician physicist born in South Africa in 1928; he made his career in Birmingham (United Kingdom) and then of the University of California in Berkeley. He introduced the variables bearing his name in 1958 (Mandelstam 1958), in order to study the interaction of pions with atomic nuclei.

$$s = m_1^2 c^2 + m_2^2 c^2 - 2\vec{\boldsymbol{p}}_1 \cdot \vec{\boldsymbol{p}}_2. \tag{9.55}$$

Similarly, by expanding the last term in (9.52a), we get $s = {m'_1}^2 c^2 + {m'_2}^2 c^2 - 2\vec{\boldsymbol{p}}'_1 \cdot \vec{\boldsymbol{p}}'_2$. Since $m'_1 = m_1$ and $m'_2 = m_2$, we obtain

$$\boxed{\vec{\boldsymbol{p}}'_1 \cdot \vec{\boldsymbol{p}}'_2 = \vec{\boldsymbol{p}}_1 \cdot \vec{\boldsymbol{p}}_2}. \tag{9.56}$$

In the case of two massive particles, the 4-momenta are related to the 4-velocities by (9.3), so that the above relation is equivalent to

$$\vec{\boldsymbol{u}}'_1 \cdot \vec{\boldsymbol{u}}'_2 = \vec{\boldsymbol{u}}_1 \cdot \vec{\boldsymbol{u}}_2. \tag{9.57}$$

Now, from (4.10), the scalar product $-\vec{\boldsymbol{u}}_1 \cdot \vec{\boldsymbol{u}}_2$ is nothing but the Lorentz factor between $\mathscr{P}_1$ and $\mathscr{P}_2$. Since the latter is related to the norm of the velocity $\vec{\boldsymbol{V}}_{12}$ of $\mathscr{P}_1$ relative to $\mathscr{P}_2$ via (4.33) (there is the same relation for the velocity $\vec{\boldsymbol{V}}_{21}$ of $\mathscr{P}_2$ relative to $\mathscr{P}_1$), we deduce that for an elastic collision, the norm of the relative velocity between the two particles is constant:

$$\left\| \vec{\boldsymbol{V}}'_{12} \right\|_{\boldsymbol{g}} = \left\| \vec{\boldsymbol{V}}_{12} \right\|_{\boldsymbol{g}}. \tag{9.58}$$

A custom problem in particle physics is that of an elastic collision on a ***fixed target***. This means that the collision is studied from the point of view of an inertial observer $\mathscr{O}$ (the "laboratory observer") for whom one of the particles, $\mathscr{P}_2$, say, is at rest before the collision. The 4-velocity of $\mathscr{P}_2$ before the collision is then equal to that of $\mathscr{O}$, $\vec{\boldsymbol{u}}_0$; this implies $m_2 \neq 0$ and $\vec{\boldsymbol{p}}_2 = m_2 c\, \vec{\boldsymbol{u}}_0$. The linear momentum $\boldsymbol{P}_2$ of $\mathscr{P}_2$ measured by $\mathscr{O}$ is zero before the collision. Regarding the linear momentum $\boldsymbol{P}_1$ of $\mathscr{P}_1$ measured by $\mathscr{O}$, we shall assume that it is such that the vector $\vec{\boldsymbol{P}}_1$ is collinear to the vector $\vec{\boldsymbol{e}}_1 =: \vec{\boldsymbol{e}}_x$ of $\mathscr{O}$'s frame: $\vec{\boldsymbol{P}}_1 = P_1\, \vec{\boldsymbol{e}}_x$. The input data of the problem are then the masses m_1 and m_2 of the two particles (with of course $m_1 = 0$ if $\mathscr{P}_1$ is a photon) and the energy E_1 of $\mathscr{P}_1$ measured by $\mathscr{O}$. P_1 is then deduced from E_1 and m_1 by means of the Einstein relation (9.9): $P_1^2 = E_1^2/c^2 - m_1^2 c^2$. The particles' 4-momenta before the collision are related to these data by

$$\boldsymbol{p}_1 = \frac{E_1}{c}\, \underline{\boldsymbol{u}}_0 + P_1\, \vec{\boldsymbol{e}}_x \qquad \text{and} \qquad \boldsymbol{p}_2 = m_2 c\, \underline{\boldsymbol{u}}_0. \tag{9.59}$$

The law of linear-momentum conservation (9.49) implies $\vec{\boldsymbol{P}}_1 = \vec{\boldsymbol{P}}'_1 + \vec{\boldsymbol{P}}'_2$. Consequently the three vectors $\vec{\boldsymbol{P}}_1$, $\vec{\boldsymbol{P}}'_1$ and $\vec{\boldsymbol{P}}'_2$ are coplanar. With the collision point, they define a plane in $\mathscr{O}$'s rest space, called the ***collision plane***. After the collision, the particles $\mathscr{P}_1$ and $\mathscr{P}_2$ move in directions defined by the angles θ_1 and θ_2 with $\vec{\boldsymbol{e}}_x$ (cf. Fig. 9.11a).

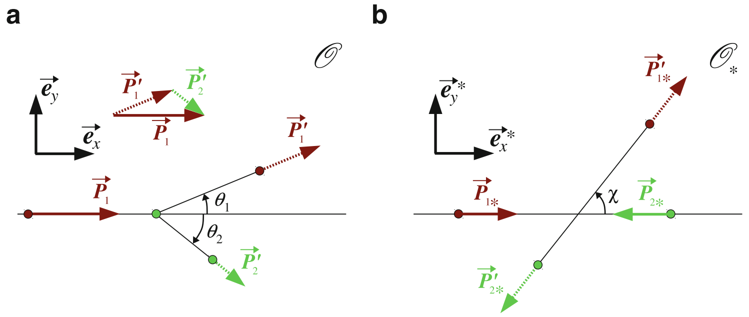

Fig. 9.11 Elastic collision: (**a**) seen in the reference space of observer $\mathscr{O}$ with respect to which the particle $\mathscr{P}_2$ is initially at rest; (**b**) seen in the reference space of an observer $\mathscr{O}_*$ comoving with the system

The problem is actually simpler when considered from the point of view of an observer *comoving with the system* $\{\mathscr{P}_1, \mathscr{P}_2\}$ (cf. Sect. 9.3.5), i.e. an observer $\mathscr{O}_*$ whose 4-velocity is given by (9.42) : $\vec{u} = (mc)^{-1}\vec{p}$, $\vec{p} = \vec{p}_1 + \vec{p}_2$ being the 4-momentum vector of the system and $m := c^{-1}\|\vec{p}\|_g$ the mass of the system,[7] which is related to Mandelstam variable s by (9.53). Substituting (9.59) for $\vec{p}_1$ and $\vec{p}_2$ in $m = c^{-1}\|\vec{p}_1 + \vec{p}_2\|_g$, we obtain m in terms of the data of the problem:

$$m = \sqrt{\left(m_2 + \frac{E_1}{c^2}\right)^2 - \frac{P_1^2}{c^2}} = \sqrt{m_1^2 + m_2^2 + 2m_2\frac{E_1}{c^2}}. \tag{9.60}$$

Let us denote by $\vec{V}$ the velocity of $\mathscr{O}_*$ relative to $\mathscr{O}$ and by Γ the corresponding Lorentz factor:

$$\vec{u} = \Gamma\left(\vec{u}_0 + \frac{1}{c}\vec{V}\right) \qquad \text{with} \qquad \Gamma = \left(1 - \vec{V}\cdot\vec{V}/c^2\right)^{-1/2}.$$

Since $\vec{p} = mc\,\vec{u}$, the decomposition (9.46) shows that the energy and the linear momentum of the system measured by $\mathscr{O}$ are $E = \Gamma mc^2$ and $\vec{P} = \Gamma m\vec{V}$. In addition, $E = E_1 + E_2 = E_1 + m_2c^2$ and $\vec{P} = \vec{P}_1 + \vec{P}_2 = P_1\vec{e}_x$. We deduce then that $\vec{V} = V\vec{e}_x$ with

$$V = \frac{P_1}{m_2 + E_1/c^2} \tag{9.61}$$

and, by forming $\Gamma = E/(mc^2)$,

[7] In Chap. 10, we will introduce a particular comoving observer: the *barycentric observer* or *centre-of-mass observer*. However, in the present problem, a generic comoving observer is sufficient.

$$\Gamma = \frac{m_2 + E_1/c^2}{\sqrt{m_1^2 + m_2^2 + 2m_2 E_1/c^2}}. \tag{9.62}$$

For $\mathscr{O}_*$, the linear momentum of $\mathscr{P}_2$ before the collision is $\vec{\boldsymbol{P}}_{2*} = \Gamma_{2*} m_2 \vec{\boldsymbol{V}}_{2*}$. Now the particle $\mathscr{P}_2$ being at rest with respect to $\mathscr{O}$, the reciprocity of the relative velocity between the observers $\mathscr{O}$ and $\mathscr{O}_*$ leads to $\Gamma_{2*} = \Gamma$ and $\vec{\boldsymbol{V}}_{2*} = -V \boldsymbol{\Lambda}(\vec{\boldsymbol{e}}_x)$ where $\boldsymbol{\Lambda}$ is the Lorentz boost from $\mathscr{O}$ to $\mathscr{O}_*$ [cf. Sect. 5.2.1 and Eqs. (5.10)–(5.11)]. We have then, using the notation $\vec{\boldsymbol{e}}^{\,*}_x := \boldsymbol{\Lambda}(\vec{\boldsymbol{e}}_x)$,

$$\vec{\boldsymbol{P}}_{2*} = -\Gamma m_2 V \, \vec{\boldsymbol{e}}^{\,*}_x.$$

Since $\mathscr{O}_*$ is a comoving observer, the total linear momentum measured by him vanishes: $\vec{\boldsymbol{P}}_* = 0$ (cf. Sect. 9.3.6). We get then

$$\vec{\boldsymbol{P}}_{1*} = -\vec{\boldsymbol{P}}_{2*} = \Gamma m_2 V \, \vec{\boldsymbol{e}}^{\,*}_x.$$

Similarly, after the collision, $\vec{\boldsymbol{P}}'_* = 0$ leads to $\vec{\boldsymbol{P}}'_{1*} = -\vec{\boldsymbol{P}}'_{2*}$. Hence, in $\mathscr{O}_*$'s reference space, the linear momenta of the two particles are the opposite of each other. Before the collision, $\vec{\boldsymbol{P}}_{1*}$ and $\vec{\boldsymbol{P}}_{2*}$ are along $\vec{\boldsymbol{e}}^{\,*}_x$. After the collision, the common direction of $\vec{\boldsymbol{P}}'_{1*}$ and $\vec{\boldsymbol{P}}'_{2*}$ does not in general coincide with $\vec{\boldsymbol{e}}^{\,*}_x$ (cf. Fig. 9.11b). Let us then call χ the angle between $\vec{\boldsymbol{e}}^{\,*}_x$ and $\vec{\boldsymbol{P}}'_{1*}$. The actual value of χ depends on the microscopic detail of the collision, which we shall not describe here. It is a supplementary parameter of the problem.

Besides, the equation of energy conservation (9.49) with respect to the inertial observer $\mathscr{O}_*$, namely, $E_{1*} + E_{2*} = E'_{1*} + E'_{2*}$, is written, given Einstein relation (9.9), $\|\vec{\boldsymbol{P}}_{2*}\|_{\boldsymbol{g}} = \|\vec{\boldsymbol{P}}_{1*}\|_{\boldsymbol{g}}$ and $\|\vec{\boldsymbol{P}}'_{2*}\|_{\boldsymbol{g}} = \|\vec{\boldsymbol{P}}'_{1*}\|_{\boldsymbol{g}}$:

$$\sqrt{m_1^2 c^2 + \left\|\vec{\boldsymbol{P}}_{1*}\right\|_{\boldsymbol{g}}^2} + \sqrt{m_2^2 c^2 + \left\|\vec{\boldsymbol{P}}_{1*}\right\|_{\boldsymbol{g}}^2} = \sqrt{m_1^2 c^2 + \left\|\vec{\boldsymbol{P}}'_{1*}\right\|_{\boldsymbol{g}}^2} + \sqrt{m_2^2 c^2 + \left\|\vec{\boldsymbol{P}}'_{1*}\right\|_{\boldsymbol{g}}^2}.$$

Note that the hypothesis of elastic collision has been used under the form $m'_1 = m_1$ and $m'_2 = m_2$. We deduce from the above relation that $\|\vec{\boldsymbol{P}}'_{1*}\|_{\boldsymbol{g}} = \|\vec{\boldsymbol{P}}_{1*}\|_{\boldsymbol{g}}$. The norms of the linear momenta with respect to $\mathscr{O}_*$ are thus all equal:

$$\left\|\vec{\boldsymbol{P}}_{1*}\right\|_{\boldsymbol{g}} = \left\|\vec{\boldsymbol{P}}_{2*}\right\|_{\boldsymbol{g}} = \left\|\vec{\boldsymbol{P}}'_{1*}\right\|_{\boldsymbol{g}} = \left\|\vec{\boldsymbol{P}}'_{2*}\right\|_{\boldsymbol{g}} = \Gamma m_2 V. \tag{9.63}$$

The Einstein relation (9.9) can be then used once again to show that the energy of each particle with respect to $\mathscr{O}_*$ is conserved:

$$E'_{1*} = E_{1*} = \sqrt{m_1^2 c^4 + (\Gamma m_2 V c)^2} = \alpha \Gamma m_2 c^2 \tag{9.64a}$$

$$E'_{2*} = E_{2*} = \Gamma m_2 c^2, \tag{9.64b}$$

where the expression of E_{2*} follows directly from $\Gamma_{2*} = \Gamma$ and we have set

$$\alpha := \sqrt{\frac{1}{\Gamma^2}\left(\frac{m_1}{m_2}\right)^2 + \frac{V^2}{c^2}} = \frac{(m_1/m_2)^2 + E_1/(m_2c^2)}{1 + E_1/(m_2c^2)}. \tag{9.65}$$

Note that if $m_1 = m_2$, $\alpha = 1$ and $m_1 < m_2 \iff \alpha < 1$ (in particular, if $m_1 = 0$, $\alpha = V/c$). Hence,

In an elastic collision, for a comoving observer, the energy and the norm of the linear momentum of each particle is conserved. Moreover, the linear momenta of the two particles are always the opposite of each other.

The 4-momentum vectors of $\mathscr{P}_1$ and $\mathscr{P}_2$ after the collision are $\vec{\boldsymbol{p}}'_a = (E'_{a*}/c)\,\vec{\boldsymbol{u}} + \vec{\boldsymbol{P}}'_{a*}$ $(a = 1, 2)$, i.e. from the definition of χ and Eqs. (9.63)–(9.64b),

$$\vec{\boldsymbol{p}}'_1 = \Gamma m_2 \left[\alpha c\,\vec{\boldsymbol{u}} + V\left(\cos\chi\,\vec{\boldsymbol{e}}^{\,*}_x + \sin\chi\,\vec{\boldsymbol{e}}^{\,*}_y\right)\right]$$
$$\vec{\boldsymbol{p}}'_2 = \Gamma m_2 \left[c\,\vec{\boldsymbol{u}} - V\left(\cos\chi\,\vec{\boldsymbol{e}}^{\,*}_x + \sin\chi\,\vec{\boldsymbol{e}}^{\,*}_y\right)\right],$$

These equations give the components $p'^{*\alpha}_a$ of each 4-momentum vector $\vec{\boldsymbol{p}}'_a$ in $\mathscr{O}_*$'s frame $(\vec{\boldsymbol{e}}^{\,*}_\alpha) = (\vec{\boldsymbol{u}}, \vec{\boldsymbol{e}}^{\,*}_x, \vec{\boldsymbol{e}}^{\,*}_y, \vec{\boldsymbol{e}}^{\,*}_z)$. Since the latter is deduced from $\mathscr{O}$'s frame $(\vec{\boldsymbol{e}}_\alpha) = (\vec{\boldsymbol{u}}_0, \vec{\boldsymbol{e}}_x, \vec{\boldsymbol{e}}_y, \vec{\boldsymbol{e}}_z)$ by the Lorentz boost $\boldsymbol{\Lambda}$, we have $\vec{\boldsymbol{p}}'_a = p'^{*\beta}_a\,\vec{\boldsymbol{e}}^{\,*}_\beta = p'^{*\beta}_a\,\boldsymbol{\Lambda}(\vec{\boldsymbol{e}}_\beta) = p'^{*\beta}_a\,\Lambda^\alpha{}_\beta\,\vec{\boldsymbol{e}}_\alpha$ [cf. Eq. (6.4)]. We deduce that the components of $\vec{\boldsymbol{p}}'_a$ in $\mathscr{O}$'s frame are $p'^\alpha_a = \Lambda^\alpha{}_\beta p'^{*\beta}_a$. The matrix $\Lambda^\alpha{}_\beta$ having the form (6.48), we get

$$(p'_1)^\alpha = \Gamma m_2\left(\Gamma c\left(\alpha + \frac{V^2}{c^2}\cos\chi\right),\ \Gamma V\left(\alpha + \cos\chi\right),\ V\sin\chi,\ 0\right) \tag{9.66}$$

$$(p'_2)^\alpha = \Gamma m_2\left(\Gamma c\left(1 - \frac{V^2}{c^2}\cos\chi\right),\ \Gamma V\left(1 - \cos\chi\right),\ -V\sin\chi,\ 0\right). \tag{9.67}$$

By virtue of the relation $\vec{\boldsymbol{p}}'_a = (E'_a/c)\,\vec{\boldsymbol{u}}_0 + \vec{\boldsymbol{P}}'_a$, the first component gives the particle's energy measured by $\mathscr{O}$ after the collision and the last three components the linear momentum measured by $\mathscr{O}$:

$$E'_1 = \Gamma^2 m_2 c^2\left(\alpha + \frac{V^2}{c^2}\cos\chi\right) \quad \text{and} \quad E'_2 = \Gamma^2 m_2 c^2\left(1 - \frac{V^2}{c^2}\cos\chi\right), \tag{9.68}$$

$$\vec{P}'_1 = \Gamma m_2 V \left[\Gamma\,(\alpha + \cos\chi)\;\vec{e}_x + \sin\chi\,\vec{e}_y\right], \tag{9.69}$$

$$\vec{P}'_2 = \Gamma m_2 V \left[\Gamma\,(1 - \cos\chi)\;\vec{e}_x - \sin\chi\,\vec{e}_y\right]. \tag{9.70}$$

Remark 9.14. We have $(P'_2)^x \geq 0$: after the collision, the target particle moves always towards the right in Fig. 9.11a. On the other hand, if $m_1 < m_2$, we have seen above that $\alpha < 1$. For $-1 \leq \cos\chi < -\alpha$, we have then $(P'_1)^x < 0$, so that the incident particle moves towards the left: it "bounces" onto the target particle.

We deduce from (9.68) (and from $E_2 = m_2 c^2$) that the energy received by $\mathscr{P}_2$ during the collision is

$$\Delta E_2 := E'_2 - E_2 = \Gamma^2 m_2 V^2\,(1 - \cos\chi)\,,$$

where the relation $\Gamma^2 - 1 = \Gamma^2 V^2/c^2$ has been used. Replacing Γ and V by the expressions (9.62) and (9.61), we obtain ΔE_2 in terms of the data of the problem:

$$\boxed{\Delta E_2 = \frac{m_2 P_1^2}{m_1^2 + m_2^2 + 2 m_2 E_1/c^2}(1 - \cos\chi)}. \tag{9.71}$$

We notice that $\Delta E_2 \geq 0$: an elastic collision always increases the energy of the target particle. The observer $\mathscr{O}$ being inertial, the law of energy conservation (9.49) implies that the incident particle must lose some energy: $\Delta E_1 = -\Delta E_2$.

The angles θ_1 and θ_2 formed by the momenta $\mathscr{P}_1$ and $\mathscr{P}_2$ with the axis $\vec{e}_x$ after the collision are such that $\tan\theta_a = (P'_a)^y/(P'_a)^x$; from (9.69) and (9.70), we get

$$\tan\theta_1 = \frac{\sin\chi}{\Gamma(\alpha + \cos\chi)} \qquad \text{and} \qquad \tan\theta_2 = -\frac{\sin\chi}{\Gamma(1 - \cos\chi)}.$$

Using (9.62) and (9.65), we obtain

$$\boxed{\tan\theta_1 = \frac{\sin\chi\sqrt{m_1^2 + m_2^2 + 2 m_2 E_1/c^2}}{m_1^2/m_2 + E_1/c^2 + (m_2 + E_1/c^2)\cos\chi}} \tag{9.72a}$$

$$\boxed{\tan\theta_2 = -\frac{\sin\chi\sqrt{m_1^2 + m_2^2 + 2 m_2 E_1/c^2}}{(m_2 + E_1/c^2)\,(1 - \cos\chi)}}. \tag{9.72b}$$

Case of Identical Particles

If $\mathscr{P}_1$ and $\mathscr{P}_2$ are two identical particles, $m_1 = m_2$ and the above formulas simplify to

$$\left.\boxed{\Delta E_2 = \frac{\Gamma_1 - 1}{2} m_1 c^2 (1 - \cos\chi)}\right._{m_1=m_2}, \tag{9.73}$$

$$\left.\boxed{\tan\theta_1 = \sqrt{\frac{2}{1+\Gamma_1}}\tan\frac{\chi}{2}}\right._{m_1=m_2}, \quad \left.\boxed{\tan\theta_2 = -\sqrt{\frac{2}{1+\Gamma_1}}\frac{1}{\tan\frac{\chi}{2}}}\right._{m_1=m_2}, \tag{9.74}$$

where $\Gamma_1 = E_1/(m_1c^2)$ is the Lorentz factor of $\mathscr{P}_1$ with respect to $\mathscr{O}$ before the collision. We notice that the product of $\tan\theta_1$ and $\tan\theta_2$ is independent of χ:

$$\left.\boxed{\tan\theta_1\ \tan\theta_2 = -\frac{2}{1+\Gamma_1}}\right._{m_1=m_2}. \tag{9.75}$$

The angle between the trajectories of $\mathscr{P}_1$ and $\mathscr{P}_2$ after the collision is $\theta = \theta_1 - \theta_2$. Thanks to the formula $\tan(\theta_1 - \theta_2) = (\tan\theta_1 - \tan\theta_2)/(1 + \tan\theta_1\ \tan\theta_2)$, we get

$$\left.\boxed{\tan\theta = \frac{2\sqrt{2(\Gamma_1+1)}}{(\Gamma_1 - 1)\sin\chi}}\right._{m_1=m_2}. \tag{9.76}$$

In the nonrelativistic limit, $\Gamma_1 = 1$ and we get $\tan\theta = +\infty$, which implies $\theta = \pi/2$. We recover here a well-known result regarding elastic shocks in Newtonian mechanics: a billiard player knows well that after the hit, the two balls leave each other at right angle (at least in the absence of any other effect on the balls). In the relativistic case, $\Gamma_1 > 1$ and the above formula shows that $\tan\theta > 0$, which implies $\theta < \pi/2$: the two particles recede from each other by making an acute angle (cf. Fig. 9.11a).

9.4.4 Compton Effect

The ***Compton scattering*** is the interaction of a photon ($\mathscr{P}_1 = \gamma$) with an electron ($\mathscr{P}_2 = \mathrm{e}^-$), the latter being at rest with respect to observer $\mathscr{O}$:

$$\gamma + \mathrm{e}^- \longrightarrow \gamma + \mathrm{e}^-. \tag{9.77}$$

Since (at the classical level) particles are conserved and $m'_1 = m_1$ ($= 0$) and $m'_2 = m_2$ ($= m_e$), it is an elastic collision. We may then apply the results of the preceding section. Setting $m_1 = 0$ in (9.71) and (9.72a), we get

$$\Delta E_1 = -\frac{E_1^2}{E_2 + 2E_1}(1 - \cos\chi) \quad \text{and} \quad \tan\theta_1 = \frac{\sin\chi\,\sqrt{E_2^2 + 2E_1E_2}}{E_1 + (E_1 + E_2)\cos\chi}, \tag{9.78}$$

where we have used $\Delta E_1 = -\Delta E_2$, $P_1 = E_1/c$ and $E_2 = m_2c^2$. Thanks to the formula $\cos^2\theta_1 = 1/(1 + \tan^2\theta_1)$, we obtain

$$\cos\theta_1 = \frac{E_1 + (E_1 + E_2)\cos\chi}{E_1 + E_2 + E_1\cos\chi}. \tag{9.79}$$

Remark 9.15. In the present case, the velocity of the comoving observer $\mathscr{O}_*$ relative to $\mathscr{O}$ is $V = c\,E_1/(E_1 + E_2)$ [set $P_1 = E_1/c$ in (9.61)]. Hence, formula (9.79) can be written as

$$\cos\theta_1 = \frac{\cos\chi + V/c}{1 + (V/c)\cos\chi}.$$

We recognize the aberration formula derived in Sect. 5.6, namely, Eq. (5.77), in which U is replaced by $-V$, θ by $\pi - \chi$ and θ' by $\pi - \theta_1$. This is not surprising since χ is the angle formed by the photon's trajectory with $\vec{e}^{\,*}_x$ in $\mathscr{O}_*$'s reference space, θ_1 is the angle formed by the photon's trajectory with $\vec{e}_x$ in $\mathscr{O}$'s reference space (cf. Fig. 9.11) and $-V\,\vec{e}^{\,*}_x$ is the velocity of $\mathscr{O}$ relative to $\mathscr{O}_*$.

The photon's energy after the collision is $E'_1 = E_1 + \Delta E_1$. The expression of $1/E'_1$ in terms of $\cos\theta_1$ is particularly simple: from (9.78), we get

$$\frac{1}{E'_1} - \frac{1}{E_1} = \frac{1 - \cos\chi}{E_2 + E_1(1 + \cos\chi)},$$

i.e. by means of (9.79),

$$\boxed{\frac{1}{E'_1} - \frac{1}{E_1} = \frac{1}{E_2}(1 - \cos\theta_1)}. \tag{9.80}$$

It is then natural to let the photon's wavelength appear, since it is proportional to the inverse of the energy: $E'_1 = hc/\lambda'$ and $E_1 = hc/\lambda$ [cf. Eqs. (9.24) and (9.26)]. Replacing E_2 by $m_2c^2 = m_ec^2$, we thus obtain

$$\boxed{\lambda' - \lambda = \frac{h}{m_e c}(1 - \cos\theta_1)}. \tag{9.81}$$

The quantity $h/(m_e c)$ is called the ***electron Compton wavelength***. Its numerical value is

$$\lambda_C := \frac{h}{m_e c} = 2.4263102 \times 10^{-12}\ \text{m}. \tag{9.82}$$

The decrease of the energy of the photon after its encounter with the electron is called the ***Compton effect***. It is noticeable only for λ of the order of, or smaller than, λ_C. The corresponding energy is $E_C := hc/\lambda_C = m_e c^2 \simeq 511$ keV. Hence, the Compton effect concerns essentially X-ray and gamma-ray photons. We note that the effect vanishes in the direction of the incident photon ($\theta_1 = 0$) and is maximal in the opposite direction ($\theta_1 = \pi$).

Direct Derivation

Formula (9.81) giving the Compton effect can be derived directly from the principle of 4-momentum conservation (9.51), without relying on the results of Sect. 9.4.3. Indeed, let us rewrite (9.51) as $\vec{\boldsymbol{p}}_1 + \vec{\boldsymbol{p}}_2 - \vec{\boldsymbol{p}}'_1 = \vec{\boldsymbol{p}}'_2$ and take the scalar square of it. Since $\vec{\boldsymbol{p}}_1 \cdot \vec{\boldsymbol{p}}_1 = 0$, $\vec{\boldsymbol{p}}'_1 \cdot \vec{\boldsymbol{p}}'_1 = 0$ and $\vec{\boldsymbol{p}}_2 \cdot \vec{\boldsymbol{p}}_2 = \vec{\boldsymbol{p}}'_2 \cdot \vec{\boldsymbol{p}}'_2 = -m_e^2 c^2$, we obtain

$$\vec{\boldsymbol{p}}_2 \cdot (\vec{\boldsymbol{p}}_1 - \vec{\boldsymbol{p}}'_1) = \vec{\boldsymbol{p}}_1 \cdot \vec{\boldsymbol{p}}'_1. \tag{9.83}$$

Let us then consider the orthogonal decompositions with respect to observer $\mathscr{O}$: $\vec{\boldsymbol{p}}_2 = m_2 c\, \vec{\boldsymbol{u}}_0 = (E_2/c)\, \vec{\boldsymbol{u}}_0$,

$$\vec{\boldsymbol{p}}_1 = \frac{E_1}{c}(\vec{\boldsymbol{u}}_0 + \vec{\boldsymbol{n}}) \qquad \text{and} \qquad \vec{\boldsymbol{p}}'_1 = \frac{E'_1}{c}(\vec{\boldsymbol{u}}_0 + \vec{\boldsymbol{n}}'), \tag{9.84}$$

where, according to (9.22), $\vec{\boldsymbol{n}}$ and $\vec{\boldsymbol{n}}'$ are the propagation directions of the photon with respect to $\mathscr{O}$ before and after the collision, respectively. By hypothesis, $\vec{\boldsymbol{n}} = \vec{\boldsymbol{e}}_x$. Equation (9.83) becomes then

$$E_2\, \vec{\boldsymbol{u}}_0 \cdot \left[E_1(\vec{\boldsymbol{u}}_0 + \vec{\boldsymbol{e}}_x) - E'_1(\vec{\boldsymbol{u}}_0 + \vec{\boldsymbol{n}}')\right] = E_1 E'_1 (\vec{\boldsymbol{u}}_0 + \vec{\boldsymbol{e}}_x) \cdot (\vec{\boldsymbol{u}}_0 + \vec{\boldsymbol{n}}').$$

Since $\vec{\boldsymbol{u}}_0 \cdot \vec{\boldsymbol{u}}_0 = -1$, $\vec{\boldsymbol{u}}_0 \cdot \vec{\boldsymbol{e}}_x = 0$, $\vec{\boldsymbol{u}}_0 \cdot \vec{\boldsymbol{n}}' = 0$ and $\vec{\boldsymbol{e}}_x \cdot \vec{\boldsymbol{n}}' = \cos\theta_1$ (from the very definition of θ_1), expanding and dividing by $E_1 E'_1 E_2$, we obtain (9.80) and, from there, the Compton effect equation (9.81).

Historical note: *At the beginning of the 1920s, various experiments had shown that X-rays scattered by matter had a smaller energy than the incident X-rays. Arthur H. Compton*[8] *measured then the angular dependence of the effect by sending X-rays*

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[8] **Arthur H. Compton** (1892–1962): American physicist, 1927 Nobel Prize in Physics for the discovery of the effect bearing his name. During the Second World War, he was responsible of the

resulting from the K_α transition of molybdenum ($\lambda = 70.8$ pm) onto graphite. He obtained the law (9.81) and, in a paper published in 1923 (Compton 1923), he gave its theoretical derivation from the conservation of energy and linear momentum in relativistic dynamics. The Compton effect provided the final proof of the corpuscular nature of light, convincing those who had been sceptical after the introduction of light quanta by Albert Einstein to explain the photoelectric effect in 1905 (Einstein 1905a).

9.4.5 Inverse Compton Scattering

Let us consider now the case where the electron is no longer initially at rest with respect to observer $\mathscr{O}$, but has a constant velocity $\vec{V}_2 =: V_e \vec{e}$, where $\vec{e}$ is a unit vector. The electron's 4-momentum before the collision is then

$$\vec{p}_2 = \frac{E_2}{c}\left(\vec{u}_0 + \frac{V_e}{c}\,\vec{e}\right). \tag{9.85}$$

The photon's 4-momentum, before and after the collision, is still given by (9.84). Equation (9.83), which does not depend on the state of motion of the electron with respect to $\mathscr{O}$, is still valid. In view of (9.84) and (9.85), it becomes

$$E_2\left(\vec{u}_0 + \frac{V_e}{c}\,\vec{e}\right)\cdot\left[E_1(\vec{u}_0 + \vec{n}) - E_1'(\vec{u}_0 + \vec{n}')\right] = E_1 E_1'(\vec{u}_0 + \vec{n})\cdot(\vec{u}_0 + \vec{n}').$$

Now $\vec{u}_0 \cdot \vec{u}_0 = -1$, $\vec{u}_0 \cdot \vec{n} = 0$, $\vec{u}_0 \cdot \vec{n}' = 0$, $\vec{u}_0 \cdot \vec{e} = 0$ and

$$\vec{e}\cdot\vec{n} = \cos\varphi, \quad \vec{e}\cdot\vec{n}' = \cos\varphi' \quad \text{and} \quad \vec{n}\cdot\vec{n}' = \cos\theta_1,$$

where φ (resp. φ') is the angle between the direction of the electron and that of the photon before (resp. after) the collision and θ_1 is the photon deviation angle. Thus, we obtain

$$\boxed{\frac{E_1'}{E_1} = \frac{1 - (V_e/c)\cos\varphi}{1 - (V_e/c)\cos\varphi' + (E_1/E_2)(1 - \cos\theta_1)}}. \tag{9.86}$$

If $V_e = 0$, this formula reduces to (9.80), as it should. We have then $E_1'/E_1 \leq 1$: the photon loses some energy during the collision: this is the Compton effect. On the other hand, if $V_e \neq 0$, one may have $E_1'/E_1 > 1$, i.e. the photon is gaining some energy. One names this the ***inverse Compton effect*** and calls the collision an ***inverse***

"Metallurgical Laboratory" in Chicago—the cover name for facilities producing the uranium and the plutonium of the first atomic bombs.

Compton scattering. In particular, when the electron is much more energetic than the photon, $E_1/E_2 \ll 1$ and (9.86) simplifies in

$$\frac{E_1'}{E_1} \simeq \frac{1 - (V_e/c)\cos\varphi}{1 - (V_e/c)\cos\varphi'}.$$

We observe that the photon energy increase is maximum when $\varphi = \pi$ and $\varphi' = 0$, i.e. when the photon propagates initially in the direction opposite to that of the electron and is scattered in the same direction as the electron motion. One has then

$$\max \frac{E_1'}{E_1} = \frac{1 + (V_e/c)}{1 - (V_e/c)}.$$

If the electron is relativistic, i.e. if its Lorentz factor Γ_e with respect to $\mathscr{O}$ is large, the Taylor expansion of formula $V_e/c = \sqrt{1 - \Gamma_e^{-2}}$ yields $V_e/c \simeq 1 - 1/(2\Gamma_e^2)$, so that the above formula becomes

$$\boxed{\max \frac{E_1'}{E_1} \simeq 4\Gamma_e^2}_{\Gamma_e \gg 1}. \tag{9.87}$$

The inverse Compton scattering on relativistic electrons is thus a way for considerably increasing the energy of photons.

Remark 9.16. From the point of view of a comoving observer (cf. Fig. 9.11b), the Compton scattering and the inverse Compton scattering are exactly the same process.

The inverse Compton scattering plays an important role in astrophysics. It is notably involved in the ***cosmic microwave background (CMB)***, i.e. the famous thermal radiation at 3 K that fills the universe. The CMB has been emitted during the formation of the first atoms, $\sim 3 \times 10^5$ years after the Big Bang, when the universe became transparent to radiation; it appears today as an almost perfect black-body radiation at the temperature $T = 2.725$ K. However the CMB photons are submitted to inverse Compton scattering when moving through a galaxy cluster. Indeed galaxy clusters are filled with a very dilute but very hot hydrogen plasma. The electrons of this plasma are very energetic, and by the inverse Compton effect, they increase the energy of the CMB photons, which results in some distortion with respect to the black-body radiation. This is called the ***Sunyaev–Zel'dovich effect*** (Peter and Uzan 2009; Bernardeau 2007), from the names of the two Soviet astrophysicists who predicted it in 1969 (Zeldovich and Sunyaev 1969). The first observations of this effect have been performed in 1983, and the Planck satellite launched in 2009 is creating a catalogue of galaxy clusters through this effect (Aghanim et al. 2012).

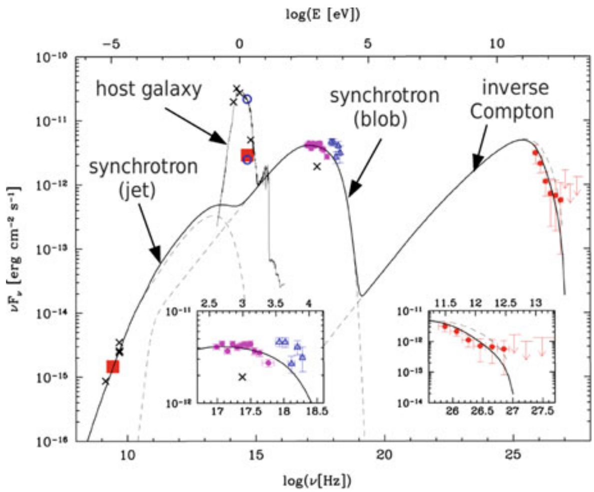

Fig. 9.12 Energy spectrum of the active galactic nucleus RGB J0152+017. The abscissa marks the frequency ν of the photons, and the ordinate the flux spectral density F_ν multiplied by ν ($F_\nu\, d\nu$ is the energy received on Earth by unit area and unit time in the frequency bandwidth $[\nu, \nu + d\nu]$) [Adapted from Aharonian et al. (2008)]

Another astrophysical context where inverse Compton scattering occurs is the generation of high-energy gamma photons, in the TeV regime (1 TeV $= 10^{12}$ eV), in ***active galactic nuclei***.[9] This is illustrated in Fig. 9.12, which shows the energy distribution of photons emitted by the blazar RGB J0152+017, ranging from radio waves (detected by the Nançay radio telescope, France) up to high-energy gamma rays (detected by the Cherenkov telescope HESS in Namibia). A ***blazar*** is an active galactic nuclei whose jet points towards the Earth. On Fig. 9.12, crosses correspond to observations (down arrows mark upper limits), and the continuous or dashed lines are the results of an emission computation performed in 2008 by researchers from the HESS collaboration (Aharonian et al. 2008), by assuming synchrotron emission[10] by the electrons in the jet (the whole jet in the radio domain and a more dense blob in the optical, UV and X-ray domains) and some inverse Compton emission. The latter results from the encounter of the relatively low-energy photons of the synchrotron radiation with the relativistic electrons of the jet, the very same that generate the synchrotron radiation.

[9]One says that a galaxy has an *active nucleus* when it harbours in its core a supermassive black hole in the vicinity of which a relativistic jet is emitted, as we shall discuss in more details in Sect. 21.7.1.

[10]Synchrotron radiation will be studied in Sect. 20.4.

9.4.6 Inelastic Collisions

Let us move to the study of inelastic collisions (cf. Sect. 9.4.1) of the type

$$\mathscr{P}_1 + \mathscr{P}_2 \longrightarrow \mathscr{P}'_1 + \cdots + \mathscr{P}'_N, \tag{9.88}$$

where $N \geq 1$ is the total number of produced particles. We aim at determining the minimal energy of the system $\{\mathscr{P}_1, \mathscr{P}_2\}$ with respect to a given observer $\mathscr{O}$ so that the reaction (9.88) is possible. Such an energy is called the ***threshold energy*** of the reaction with respect to $\mathscr{O}$. We shall take here a pure dynamical point of view, i.e. we shall determine whether (9.88) is possible under the sole condition of 4-momentum conservation, which from (9.39) and (9.40) is expressed as

$$\vec{\boldsymbol{p}}_1 + \vec{\boldsymbol{p}}_2 = \vec{\boldsymbol{p}}'_1 + \cdots + \vec{\boldsymbol{p}}'_N. \tag{9.89}$$

conservation of electric charge (Sect. 18.4) or conservation of baryon number (Sect. 21.3.2), for instance.

The system formed by $\{\mathscr{P}_1, \mathscr{P}_2\}$ before the collision and by $\{\mathscr{P}'_1, \ldots, \mathscr{P}'_N\}$ after it is supposed to be isolated. It is simpler to study the constraint (9.89) from the point of view of an observer $\mathscr{O}_*$ comoving with the system because it reduces to a single component: that of energy, since by definition of a comoving observer, the total linear momentum of the system with respect to $\mathscr{O}_*$ vanishes. The energy of the system measured by $\mathscr{O}_*$ is $E_* = mc^2 = c\sqrt{s}$, where m is the total mass of the system and s the Mandelstam variable defined by (9.52a). We have, by (9.48), $E_* = E'_{1*} + \cdots + E'_{N*}$, where E'_{a*} is the energy of the particle $\mathscr{P}'_a$ measured by $\mathscr{O}_*$. We have $E'_{a*} \geq m'_a c^2$, so that

$$E_* \geq \sum_{a=1}^{N} m'_a c^2. \tag{9.90}$$

The threshold energy with respect to $\mathscr{O}_*$ corresponds to the equality in this formula: all the produced particles are then at rest with respect to $\mathscr{O}_*$. They have thus the same motion with respect to observer $\mathscr{O}$. In other words, no energy is "spoiled" into proper motions with respect to the comoving observer. Since $m = E_*/c^2$, the mass of the system is then the smallest possible and we can rewrite (9.90) as

$$\boxed{m \geq m_{\text{thres}} := \sum_{a=1}^{N} m'_a}. \tag{9.91}$$

This is the necessary and sufficient condition for the reaction (9.88) to be dynamically possible.

Let us express m in terms of quantities relative to observer $\mathscr{O}$ before the collision. Since $m = c^{-1}\|\vec{\boldsymbol{p}}_1 + \vec{\boldsymbol{p}}_2\|_{\boldsymbol{g}}$ [Eq. (9.41)] and $\vec{\boldsymbol{p}}_a \cdot \vec{\boldsymbol{p}}_a = -m_a^2 c^2$ $(a = 1, 2)$, we get

$$m = \sqrt{m_1^2 + m_2^2 - 2c^{-2}\vec{\boldsymbol{p}}_1 \cdot \vec{\boldsymbol{p}}_2}.$$

Now $\vec{\boldsymbol{p}}_a = (E_a/c)\vec{\boldsymbol{u}}_0 + \vec{\boldsymbol{P}}_a$ $(a = 1, 2)$, E_a and $\vec{\boldsymbol{P}}_a$ Being, respectively, the energy and the linear-momentum vector of $\mathscr{P}_a$ with respect to $\mathscr{O}$, whose 4-velocity is $\vec{\boldsymbol{u}}_0$. We deduce that

$$m = \sqrt{m_1^2 + m_2^2 + \frac{2}{c^4}\left(E_1 E_2 - \vec{\boldsymbol{P}}_1 \cdot \vec{\boldsymbol{P}}_2\, c^2\right)}. \tag{9.92}$$

Remark 9.17. In the case where $\mathscr{P}_2$ is at rest with respect to $\mathscr{O}$, $E_2 = m_2 c^2$, $\vec{\boldsymbol{P}}_2 = 0$ and we recover (9.60), as it should.

Substituting (9.92) for m in (9.91), we obtain a criterion involving only quantities measured by observer $\mathscr{O}$: the reaction (9.88) is possible iff

$$\boxed{E_1 E_2 - \vec{\boldsymbol{P}}_1 \cdot \vec{\boldsymbol{P}}_2\, c^2 \geq \frac{c^4}{2}\left(m_{\text{thres}}^2 - m_1^2 - m_2^2\right)}. \tag{9.93}$$

For a fixed target, $E_2 = m_2 c^2$ and $\vec{\boldsymbol{P}}_2 = 0$, this criterion becomes

$$\boxed{E_1 \geq \frac{c^2}{2m_2}\left(m_{\text{thres}}^2 - m_1^2 - m_2^2\right).}_{\,\mathscr{P}_2\ \text{fixed}} \tag{9.94}$$

Example 9.5. The antiparticle of the electron, the ***positron*** e^+, whose existence has been predicted by P.A.M. Dirac (cf. p. 372) in 1928, has been discovered during the study of cosmic rays by C. D. Anderson (cf. p. 110) in 1932 (Anderson 1932, 1933) (cf. also Crozon (2005)). The positron is indeed created in the Earth atmosphere via ***pair production***:

$$\gamma + \mathrm{p} \longrightarrow \mathrm{p} + \mathrm{e}^- + \mathrm{e}^+, \tag{9.95}$$

where $\gamma = \mathscr{P}_1$ is a cosmic gamma photon and $\mathrm{p} = \mathscr{P}_2$ a proton in a hydrogen atom of an atmospheric water molecule, which we may assume to be at rest with respect to the terrestrial observer $\mathscr{O}$. We have then $m_1 = 0$ and $m_2 = m_\mathrm{p}$. Besides, from (9.91), $m_{\text{thres}} = m_\mathrm{p} + 2m_\mathrm{e}$, the positron having the same mass as the electron. The criterion (9.94) becomes then

$$E_1 \geq \frac{c^2}{2m_\mathrm{p}}\left[(m_\mathrm{p} + 2m_\mathrm{e})^2 - m_\mathrm{p}^2\right] = 2m_\mathrm{e}c^2\left(1 + \frac{m_\mathrm{e}}{m_\mathrm{p}}\right) \simeq 2m_\mathrm{e}c^2.$$

Thus, a photon whose energy is larger than $2m_e c^2 \simeq 1.02$ MeV (gamma range) can produce electron-positron pairs in the atmosphere.

Remark 9.18. The proton in the reaction (9.95) is mandatory to ensure the conservation of 4-momentum. Indeed the reaction $\gamma \to e^- + e^+$ is not possible, for $\vec{\boldsymbol{p}}_1$ is a null vector and $\vec{\boldsymbol{p}}'_1 + \vec{\boldsymbol{p}}'_2$ a timelike one (as the sum of future-directed timelike vectors, cf. Sect. 9.3.5).

Example 9.6. One calls ***photoproduction of pions*** the reaction

$$\gamma + \mathrm{p} \longrightarrow \pi^0 + \mathrm{p}, \tag{9.96}$$

where π^0 is the neutral ***pi meson***, also called the neutral ***pion*** (cf. p. 110). Its mass is $m_{\pi^0} = 134.9766\ \mathrm{MeV/c^2}$. If the proton is at rest with respect to $\mathscr{O}$, then, as in the preceding example, $m_1 = 0$ and $m_2 = m_\mathrm{p}$. On the other side, $m_\mathrm{thres} = m_\mathrm{p} + m_{\pi^0}$, so that the criterion (9.94) becomes

$$E_1 \geq m_{\pi^0} c^2 \left(1 + \frac{m_{\pi^0}}{2m_\mathrm{p}}\right) \simeq 144\ \mathrm{MeV}. \tag{9.97}$$

Very energetic gamma photons are then required for the photoproduction of pions on protons at rest.

If, in addition of having $\mathscr{P}_2$ at rest, one has $m_1 = m_2$ (case where $\mathscr{P}_2$ is a particle of the same family than $\mathscr{P}_1$ or is $\mathscr{P}_1$'s antiparticle), formula (9.94) becomes

$$\boxed{E_1 \geq m_1 c^2 \left[\frac{1}{2}\left(\frac{m_\mathrm{thres}}{m_1}\right)^2 - 1\right].}_{\mathscr{P}_2\ \text{fixed},\ m_1 = m_2} \tag{9.98}$$

Example 9.7. A standard reaction for producing antiprotons is

$$\mathrm{p} + \mathrm{p} \longrightarrow \mathrm{p} + \mathrm{p} + \mathrm{p} + \bar{\mathrm{p}}. \tag{9.99}$$

We have then $m_1 = m_2 = m_\mathrm{p}$ and $m_\mathrm{thres} = 4m_\mathrm{p}$ (the antiproton having the same mass as the proton), so that (9.98) yields $E_1 \geq 7m_\mathrm{p}c^2 \simeq 6.57$ GeV.

Remark 9.19. The reactions $\mathrm{p} + \mathrm{p} \longrightarrow \bar{\mathrm{p}}$ and $\mathrm{p} + \mathrm{p} \longrightarrow \mathrm{p} + \mathrm{p} + \bar{\mathrm{p}}$, a priori simpler than (9.99), are forbidden by the law of electric-charge conservation (Sect. 18.4), as well as by the law of baryon number conservation (cf. Sect. 21.3.2), the baryon number of the proton being $+1$ and that of the antiproton -1.

Historical note: *At the beginning of the 1950s, the Bevatron*[11] *has been constructed in Berkeley (California). It was a synchrotron (to be discussed in Sect. 17.5) capable of accelerating protons up to a kinetic energy of* 6.2 GeV, *leading to the total energy*

[11]The name *Bevatron* stems from the abbreviation *BeV* for *billion of electronvolts*; today BeV has been replaced by the international abbreviation GeV.

7.1 GeV, *in order to produce antiprotons via the reaction (9.99). This is actually the way by which the antiproton has been discovered in 1955 by the American physicist Owen Chamberlain (1920–2006) and the Italian one Emilio Segrè (1905–1989) (Chamberlain et al. 1955). They have been awarded the Nobel Prize in Physics in 1959 for this discovery.*

Example 9.8. Let us consider again the photoproduction of pions (9.96), but this time with a high-velocity proton interacting with a low-energy photon. Since $\mathscr{P}_2 =$ p is no longer at rest, we must use the general formula (9.93). We still have $m_1 = 0$ and $m_2 = m_\mathrm{p}$, but this time $\vec{\boldsymbol{P}}_2 \neq 0$. The most favourable case occurs when $\vec{\boldsymbol{P}}_1$ and $\vec{\boldsymbol{P}}_2$ are anti-aligned: we have then $-\vec{\boldsymbol{P}}_1 \cdot \vec{\boldsymbol{P}}_2 c^2 = P_1 P_2 c^2$ with $P_1 c = E_1$ (photon) and $P_2 c = \sqrt{E_2^2 - m_\mathrm{p}^2 c^4}$. We obtain thus

$$E_1 \left(E_2 + \sqrt{E_2^2 - m_\mathrm{p}^2 c^4} \right) \geq m_{\pi^0} m_\mathrm{p} c^4 \left(1 + \frac{m_{\pi^0}}{2 m_\mathrm{p}} \right).$$

Let us fix the energy E_1 of the photon and determine the proton energy E_2 required for the reaction to take place. Let us assume moreover that $E_1 \ll m_{\pi^0} c^2 \simeq$ 135 MeV. The above equation shows then that the proton must be ultra-relativistic: $E_2 \gg m_\mathrm{p} c^2$. Accordingly $\sqrt{E_2^2 - m_\mathrm{p}^2 c^4} \simeq E_2$ and the criterion becomes

$$E_2 \geq \frac{m_{\pi^0} m_\mathrm{p} c^4}{2 E_1} \left(1 + \frac{m_{\pi^0}}{2 m_\mathrm{p}} \right). \tag{9.100}$$

Let us apply it to the CMB photons (cosmic microwave background; cf. p. 305)—since the latter is a black-body radiation at $T = 2.725$ K, the emission is peaked around the wavelength given by Wien's displacement law: $\lambda = 1.1$ mm (microwave radiation). The corresponding photon energy is $E_1 = 1.2 \times 10^{-3}$ eV. This is an energy much smaller than that given by (9.97). The proton must therefore have a very high velocity for the reaction to take place. Plugging E_1 in (9.100), we obtain indeed

$$E_2 \geq 5.8 \times 10^{19} \text{ eV}.$$

This energy is enormous: several joules in a single proton! We have seen above that such particles are observed in cosmic rays (cf. Fig. 9.3). This shows that a proton of energy E_2 larger than $E_\mathrm{crit} \sim 6 \times 10^{19}$ eV may interact with CMB photons to produce pions, thereby losing some of its energy. Given the density of photons in the CMB, it follows that cosmic rays should not contain protons of energy larger than E_crit originating from sources more distant than ~ 100 Mpc. This is called the ***GZK cut-off***, from the names of the three physicists who predicted it in 1966 : the American Kenneth Greisen (1966) and the Russians Georgiy Zatsepin and Vadim Kuzmin (1966). In 2008, two experiments devoted to ultrahigh-energy cosmic rays have announced to have observed the GZK cut-off: HiRes (Abbasi et al. 2008) and the Pierre Auger Observatory (Abraham et al. 2008) (Fig. 9.4).

Energy in the "Centre-of-Mass Frame"

From the point of view of a comoving observer $\mathscr{O}_*$ (often referred to as the "centre-of-mass frame"; cf. footnote p. 297), the threshold energy is easier to compute, since $E_1 + E_2 = mc^2$. One has simply

$$E_1 + E_2 \geq m_{\text{thres}} c^2 \quad \text{(comoving observ.).} \tag{9.101}$$

We can even express E_2 in terms of E_1, m_1 and m_2, starting from Einstein relation $E_2 = \sqrt{P_2^2 c^2 + m_2^2 c^4}$ and using the fact that for the comoving observer $\vec{\boldsymbol{P}}_2 = -\vec{\boldsymbol{P}}_1$, so that $P_2^2 c^2 = P_1^2 c^2 = E_1^2 - m_1^2 c^4$ and $E_2 = \sqrt{E_1^2 + (m_2^2 - m_1^2)c^4}$. The criterion (9.101) thus becomes

$$\boxed{E_1 + \sqrt{E_1^2 + (m_2^2 - m_1^2)c^4} \geq m_{\text{thres}} c^2.}_{\text{comoving observ.}} \tag{9.102}$$

If $m_1 = m_2$, this formula reduces to

$$\boxed{E_1 \geq \frac{1}{2} m_{\text{thres}} c^2,}_{\text{comoving observ., } m_1 = m_2} \tag{9.103}$$

which is natural since in this case $E_2 = E_1$ [cf. (9.101)].

Comparing (9.94) and (9.102) or (9.98) and (9.103) if $m_1 = m_2$, we observe that for a collision with a fixed target, the required energy E_1 increases as the square of m_{thres}—the sum of the masses of the reaction products, whereas for a collision in the "centre-of-mass frame", E_1 increases linearly with m_{thres} (at least if $E_1 \gg c^2 \sqrt{|m_2^2 - m_1^2|}$, which is achieved if $m_1 = m_2$). This fact motivates the construction of ***colliders***, i.e. particle accelerators in which the "centre-of-mass frame" is the laboratory frame (in our language, this means that the laboratory observer is comoving with the particle system). To achieve this, a collider accelerates two beams of particles towards each other, with equal magnitude but opposite linear momenta.

Example 9.9. In the Large Hadron Collider (LHC) at CERN, two proton beams are accelerated towards each other at the energy $E_1 = E_2 = 7$ TeV (cf. Table 17.1). The energy released during the collision is then $m_{\text{thres}} c^2 = 14$ TeV. To get such an energy with a fixed target, according to formula (9.98) with $m_1 c^2 = m_{\text{p}} c^2 \simeq 9.4 \times 10^{-4}$ TeV, one should instead accelerate protons up to the energy $9.4 \times 10^{-4}[(14/(9.4 \times 10^{-4}))^2/2 - 1] \simeq 10^5$ TeV! This shows clearly the benefit of the collider concept.

Example 9.10. The energy of protons in cosmic rays can reach $E_1 = 10^{20}$ eV with respect to the Earth (cf. Example 9.2 p. 277). If such a proton hits some terrestrial proton, the energy in the "centre-of-mass frame" is given by (9.92) with $m_1 = m_2$,

$\vec{\boldsymbol{P}}_2 = 0$ and $E_2 = m_2 c^2$ (target proton at rest):

$$E_* = mc^2 = m_1 c^2 \sqrt{2\left(1 + \frac{E_1}{m_1 c^2}\right)}.$$

Setting $m_1 c^2 = m_\mathrm{p} c^2 = 0.938$ GeV and $E_1 = 10^{20}$ eV in this formula, we get $E_* = 4.3 \times 10^{14}$ eV $= 430$ TeV. This is 30 times the energy $E_* = 14$ TeV achieved at LHC and refutes the questions raised before the start of the LHC regarding the creation of particles potentially dangerous for the whole Earth (strangelets or micro black holes; cf. Ellis et al. (2008)). Given the flux of cosmic rays shown in Fig. 9.3, one may say that for 4.5 billion years, the Earth has been hit by $\sim 10^{22}$ cosmic rays of energy in the "centre-of-mass frame" larger than that delivered by the LHC (Ellis et al. 2008) … and the Earth still exists!

9.5 Four-Force

9.5.1 Definition

Let us consider a particle $\mathscr{P}$, of mass $m > 0$, worldline $\mathscr{L}$, 4-velocity $\vec{\boldsymbol{u}}$ and proper time τ. We have seen in Sect. 9.3.4 that if $\mathscr{P}$ is isolated, its 4-momentum $\boldsymbol{p} = \boldsymbol{p}(\tau)$ is a constant field of linear forms along $\mathscr{L}$. If, on the contrary, $\mathscr{P}$ is not isolated, one calls ***four-force***, or ***4-force*** for short, ***acting on*** $\mathscr{P}$ the derivative of $\boldsymbol{p}$ along $\mathscr{L}$:

$$\boxed{\boldsymbol{f} := \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}\tau}}. \tag{9.104}$$

The quantity $\mathrm{d}\boldsymbol{p}/\mathrm{d}\tau$ is the linear form metric dual of the derivative vector $\mathrm{d}\vec{\boldsymbol{p}}/\mathrm{d}\tau$, the latter having been defined in Sect. 2.7.2. It is clear that if $\mathscr{P}$ is isolated, $\boldsymbol{f} = 0$.

We read on (9.104) that the dimension of a 4-force is the same as that of an "ordinary" force, namely, mass $\times$ length / (time)^2. In the SI system, the unit of 4-force is thus the ***newton***: $1\ \mathrm{N} := 1\ \mathrm{kg\,m\,s^{-2}}$.

In view of the relation (9.3) between 4-momentum and 4-velocity, we have $\mathrm{d}\boldsymbol{p}/\mathrm{d}\tau = mc\, \mathrm{d}\underline{\boldsymbol{u}}/\mathrm{d}\tau + c(\mathrm{d}m/\mathrm{d}\tau)\,\underline{\boldsymbol{u}}$. Now $c^{-1}\mathrm{d}\vec{\boldsymbol{u}}/\mathrm{d}\tau$ is the 4-acceleration $\vec{\boldsymbol{a}}$ of particle $\mathscr{P}$ [cf. Eq. (2.16)]. Equation (9.104) can be then written as

$$\boxed{\boldsymbol{f} = mc^2\,\underline{\boldsymbol{a}} + c\frac{\mathrm{d}m}{\mathrm{d}\tau}\,\underline{\boldsymbol{u}}}. \tag{9.105}$$

Since the 4-acceleration is always orthogonal to the 4-velocity [Eq. (2.17)] and $\langle \underline{\boldsymbol{u}}, \vec{\boldsymbol{u}} \rangle = -1$, we deduce immediately from (9.105) that

$$\boxed{\langle \boldsymbol{f}, \vec{\boldsymbol{u}} \rangle = -c \frac{\mathrm{d}m}{\mathrm{d}\tau}}. \tag{9.106}$$

One calls ***pure 4-force***, or sometimes ***Minkowski force***, any 4-force that is orthogonal to the 4-velocity of the particle, in the sense of

$$\langle \boldsymbol{f}, \vec{\boldsymbol{u}} \rangle = 0. \tag{9.107}$$

Equation (9.106) then shows that a pure 4-force preserves the particle's mass.

Example 9.11. An important example of a pure 4-force is the *Lorentz 4-force* that is exerted on a charged particle in some electromagnetic field. As we shall discuss in Chap. 17, this 4-force is written $\boldsymbol{f} = q\,\boldsymbol{F}(., \vec{\boldsymbol{u}})$, where q is the particle's electric charge and $\boldsymbol{F}$ the antisymmetric bilinear form representing the electromagnetic field. Thanks to the antisymmetry of $\boldsymbol{F}$, we check that $\langle \boldsymbol{f}, \vec{\boldsymbol{u}} \rangle = q\,\boldsymbol{F}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{u}}) = 0$.

Remark 9.20. In the above exposition, Eqs. (9.104) and (9.105) do not, strictly speaking, constitute relativistic generalizations of Newton's second law of mechanics, for they are empty of any physical content: they provide only the *definition* of the 4-force $\boldsymbol{f}$. It is only when the form of $\boldsymbol{f}$ is specified that a physical content is really given to these relations. In other words, from our point of view, the physical postulate to add to the principles already stated is not $\boldsymbol{f} = \mathrm{d}\boldsymbol{p}/\mathrm{d}\tau$ but $\boldsymbol{f} = \cdots$ with the dots replaced by an expression arising from the interaction under study. For instance, we shall see in Chap. 17 that, in the case of the electromagnetic interaction, the postulate consists in stating that $\boldsymbol{f}$ is the Lorentz 4-force (cf. the above example).

9.5.2 Orthogonal Decomposition of the 4-Force

Let us consider an observer $\mathscr{O}$ of worldline $\mathscr{L}_0$, 4-velocity $\vec{\boldsymbol{u}}_0$ and proper time t. Equation (9.104) can be transformed into

$$\boldsymbol{f} = \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}\tau} = \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} = \Gamma \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t}, \tag{9.108}$$

where $\Gamma := \mathrm{d}t/\mathrm{d}\tau$ is the Lorentz factor of $\mathscr{P}$ relative to $\mathscr{O}$ [Eq. (4.1)]. Introducing the orthogonal decomposition (9.8) of $\mathscr{P}$'s 4-momentum with respect to $\mathscr{O}$, (9.108) becomes

$$\boldsymbol{f} = \Gamma \left(\frac{1}{c}\frac{\mathrm{d}E}{\mathrm{d}t}\,\underline{\boldsymbol{u}}_0 + E\,\underline{\boldsymbol{a}}_0 + \frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t} \right), \tag{9.109}$$

where E and $\boldsymbol{P}$ are, respectively, $\mathscr{P}$'s energy and $\mathscr{P}$'s linear momentum, both measured by $\mathscr{O}$, and $\underline{\boldsymbol{a}}_0$ is the linear form metric dual of the 4-acceleration $\vec{\boldsymbol{a}}_0$

of observer $\mathscr{O}$: $\vec{\boldsymbol{a}}_0 := c^{-1}\,\mathrm{d}\vec{\boldsymbol{u}}_0/\mathrm{d}t$. Expression (9.109) does not constitute an orthogonal decomposition of the 4-force with respect to $\mathscr{O}$, for a priori the vector $\mathrm{d}\vec{\boldsymbol{P}}/\mathrm{d}t$ is not orthogonal to $\vec{\boldsymbol{u}}_0$, despite $\vec{\boldsymbol{P}}$ is. But the Fermi–Walker derivative of $\vec{\boldsymbol{P}}$ along $\mathscr{L}_0$ is, on its side, orthogonal to $\vec{\boldsymbol{u}}_0$, according to (3.72). This derivative is related to $\mathrm{d}\vec{\boldsymbol{P}}/\mathrm{d}t$ by (3.69):

$$\frac{\mathrm{d}\vec{\boldsymbol{P}}}{\mathrm{d}t} = \boldsymbol{D}^{\mathrm{FW}}_{\boldsymbol{u}_0}\vec{\boldsymbol{P}} + c(\vec{\boldsymbol{a}}_0\cdot\vec{\boldsymbol{P}})\,\vec{\boldsymbol{u}}_0.$$

Defining $\boldsymbol{D}^{\mathrm{FW}}_{\boldsymbol{u}_0}\underline{\boldsymbol{P}}$ as the linear form metric dual to the vector $\boldsymbol{D}^{\mathrm{FW}}_{\boldsymbol{u}_0}\vec{\boldsymbol{P}}$, we can then rewrite (9.109) as

$$\boldsymbol{f} = \Gamma\left[\left(\frac{1}{c}\frac{\mathrm{d}E}{\mathrm{d}t} + c\langle\underline{\boldsymbol{P}},\vec{\boldsymbol{a}}_0\rangle\right)\underline{\boldsymbol{u}}_0 + \boldsymbol{D}^{\mathrm{FW}}_{\boldsymbol{u}_0}\underline{\boldsymbol{P}} + E\,\underline{\boldsymbol{a}}_0\right]. \tag{9.110}$$

This constitutes the orthogonal decomposition of $\boldsymbol{f}$ with respect to $\mathscr{O}$. Equation (9.110) is equivalent to the system

$$\boxed{\frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{c}{\Gamma}\langle\boldsymbol{f},\vec{\boldsymbol{u}}_0\rangle - c^2\langle\underline{\boldsymbol{P}},\vec{\boldsymbol{a}}_0\rangle} \tag{9.111a}$$

$$\boxed{\boldsymbol{D}^{\mathrm{FW}}_{\boldsymbol{u}_0}\underline{\boldsymbol{P}} = \frac{1}{\Gamma}\boldsymbol{f}\circ\perp_{\boldsymbol{u}_0} - E\,\underline{\boldsymbol{a}}_0}. \tag{9.111b}$$

The particle $\mathscr{P}$ is isolated iff the 4-force $\boldsymbol{f}$ acting upon it vanishes. From the above equations, we have then the equivalence

$$\mathscr{P}\ \text{isolated} \iff \begin{cases} \dfrac{\mathrm{d}E}{\mathrm{d}t} = -c^2\langle\underline{\boldsymbol{P}},\vec{\boldsymbol{a}}_0\rangle \\ \boldsymbol{D}^{\mathrm{FW}}_{\boldsymbol{u}_0}\underline{\boldsymbol{P}} = -E\underline{\boldsymbol{a}}_0. \end{cases} \tag{9.112}$$

If $\mathscr{O}$ is an inertial observer, $\vec{\boldsymbol{a}}_0 = 0$, $\boldsymbol{D}^{\mathrm{FW}}_{\boldsymbol{u}_0}\underline{\boldsymbol{P}} = \mathrm{d}\underline{\boldsymbol{P}}/\mathrm{d}t$ and we recover the result (9.49): for an isolated particle, $\mathrm{d}E/\mathrm{d}t = 0$ and $\mathrm{d}\underline{\boldsymbol{P}}/\mathrm{d}t = 0$.

9.5.3 Force Measured by an Observer

We have introduced in Sect. 9.2.3 the quantity of motion $\boldsymbol{q}$ of a particle with respect to an observer. It is then natural to define the ***force acting on*** $\mathscr{P}$ ***and measured by observer*** $\mathscr{O}$ as the derivative of the quantity of motion with respect to $\mathscr{O}$:

$$\boxed{\boldsymbol{F} := \boldsymbol{D}_{\mathscr{O}}\,\boldsymbol{q}}. \tag{9.113}$$

The derivative $\boldsymbol{D}_{\mathscr{O}}$ has been defined for a vector in Sect. 3.6.2. The extension to linear forms, as in $\boldsymbol{D}_{\mathscr{O}}\,\boldsymbol{q}$, is straightforward by metric duality (cf. Sect. 1.6): $\boldsymbol{D}_{\mathscr{O}}\,\boldsymbol{q}$ is the linear form whose metric dual is the vector $\boldsymbol{D}_{\mathscr{O}}\,\vec{\boldsymbol{q}}$. Note that if $\mathscr{O}$ is inertial, $\boldsymbol{D}_{\mathscr{O}}\,\boldsymbol{q} = \mathrm{d}\boldsymbol{q}/\mathrm{d}t$.

Remark 9.21. Equation (9.113), which we consider as the definition of the force, is identical to the fundamental law of Newtonian dynamics (Newton's second law) expressed in terms of the quantity of motion. But if one expresses it in terms of the particle's acceleration, then the relativistic and Newtonian versions diverge, as we shall see below.

Since $\vec{\boldsymbol{q}} \in E_{\boldsymbol{u}_0}$ and the derivative $\boldsymbol{D}_{\mathscr{O}}$ preserves the orthogonality with respect to $\vec{\boldsymbol{u}}_0$ [property (3.66)], we have $\vec{\boldsymbol{F}} \in E_{\boldsymbol{u}_0}$, i.e.

$$\langle \boldsymbol{F}, \vec{\boldsymbol{u}}_0 \rangle = 0. \tag{9.114}$$

Let us relate $\boldsymbol{D}_{\mathscr{O}}\,\vec{\boldsymbol{q}}$ to the Fermi–Walker derivative of $\boldsymbol{P}$ introduced above, $\boldsymbol{D}^{\mathrm{FW}}_{\boldsymbol{u}_0}\boldsymbol{P}$. Equation (9.15) leads to

$$\boldsymbol{D}^{\mathrm{FW}}_{\boldsymbol{u}_0}\vec{\boldsymbol{q}} = \boldsymbol{D}^{\mathrm{FW}}_{\boldsymbol{u}_0}\vec{\boldsymbol{P}} - \Gamma m\, \boldsymbol{D}^{\mathrm{FW}}_{\boldsymbol{u}_0}\left(\vec{\boldsymbol{\omega}} \times_{\boldsymbol{u}_0} \overrightarrow{OM}\right) - m\frac{\mathrm{d}\Gamma}{\mathrm{d}t}\left(\vec{\boldsymbol{\omega}} \times_{\boldsymbol{u}_0} \overrightarrow{OM}\right),$$

with $\boldsymbol{D}^{\mathrm{FW}}_{\boldsymbol{u}_0}\left(\vec{\boldsymbol{\omega}} \times_{\boldsymbol{u}_0} \overrightarrow{OM}\right)$ given by (4.54) and $\boldsymbol{D}^{\mathrm{FW}}_{\boldsymbol{u}_0}\vec{\boldsymbol{q}}$ related to $\boldsymbol{D}_{\mathscr{O}}\,\vec{\boldsymbol{q}}$ by (3.70). We thus get

$$\begin{aligned}\boldsymbol{D}_{\mathscr{O}}\,\vec{\boldsymbol{q}} = {} & \boldsymbol{D}^{\mathrm{FW}}_{\boldsymbol{u}_0}\vec{\boldsymbol{P}} - m\frac{\mathrm{d}\Gamma}{\mathrm{d}t}\left(\vec{\boldsymbol{\omega}} \times_{\boldsymbol{u}_0} \overrightarrow{OM}\right)\\ & -\Gamma m\left[2\vec{\boldsymbol{\omega}} \times_{\boldsymbol{u}_0} \vec{\boldsymbol{V}} + \vec{\boldsymbol{\omega}} \times_{\boldsymbol{u}_0}\left(\vec{\boldsymbol{\omega}} \times_{\boldsymbol{u}_0} \overrightarrow{OM}\right) + \frac{\mathrm{d}\vec{\boldsymbol{\omega}}}{\mathrm{d}t} \times_{\boldsymbol{u}_0} \overrightarrow{OM}\right].\end{aligned}$$

Let us replace $\boldsymbol{D}_{\mathscr{O}}\,\vec{\boldsymbol{q}}$ by $\vec{\boldsymbol{F}}$ and $\boldsymbol{D}^{\mathrm{FW}}_{\boldsymbol{u}_0}\vec{\boldsymbol{P}}$ by (9.111b), to get the value of the (total) force acting on $\mathscr{P}$ and measured by $\mathscr{O}$:

$$\boxed{\begin{aligned}\vec{\boldsymbol{F}} = {} & \vec{\boldsymbol{F}}_{\mathrm{ext}} - E\,\underline{\boldsymbol{a}}_0 - \Gamma m\left[\vec{\boldsymbol{\omega}} \times_{\boldsymbol{u}_0}\left(\vec{\boldsymbol{\omega}} \times_{\boldsymbol{u}_0} \overrightarrow{OM}\right) + \frac{\mathrm{d}\vec{\boldsymbol{\omega}}}{\mathrm{d}t} \times_{\boldsymbol{u}_0} \overrightarrow{OM}\right]\\ & -2\Gamma m\,\vec{\boldsymbol{\omega}} \times_{\boldsymbol{u}_0} \vec{\boldsymbol{V}} - m\frac{\mathrm{d}\Gamma}{\mathrm{d}t}\left(\vec{\boldsymbol{\omega}} \times_{\boldsymbol{u}_0} \overrightarrow{OM}\right),\end{aligned}} \tag{9.115}$$

where

$$\vec{\boldsymbol{F}}_{\mathrm{ext}} := \Gamma^{-1} \perp_{\boldsymbol{u}_0} \vec{\boldsymbol{f}} \tag{9.116}$$

is the contribution of the 4-force $\boldsymbol{f}$ acting on the particle ("external" force). The other terms are called ***inertial forces*** and account for the non-inertial character of observer $\mathscr{O}$. More specifically, the term $-\Gamma m\,\vec{\boldsymbol{\omega}} \times_{u_0} \left(\vec{\boldsymbol{\omega}} \times_{u_0} \overrightarrow{OM}\right)$ is called ***centrifugal force*** and the term $-2\Gamma m\,\vec{\boldsymbol{\omega}} \times_{u_0} \vec{\boldsymbol{V}}$ is called ***Coriolis force***. These terms are generalizations of the well-known inertial forces of Newtonian mechanics.

If $\mathscr{O}$ is an inertial observer, (9.115) reduces to

$$\vec{\boldsymbol{F}} = \vec{\boldsymbol{F}}_{\text{ext}} = \Gamma^{-1} \perp_{u_0} \vec{\boldsymbol{f}} \qquad (\mathscr{O}\text{ inertial}). \tag{9.117}$$

Remark 9.22. Each vector appearing in (9.115) is a vector of $\mathscr{O}$'s local rest space, E_{u_0}.

Taking into account (9.116), expression (9.111b) for the Fermi–Walker of the linear momentum simplifies to

$$\boxed{\boldsymbol{D}^{\text{FW}}_{u_0} \boldsymbol{P} = \boldsymbol{F}_{\text{ext}} - E\,\underline{\boldsymbol{a}}_0}. \tag{9.118}$$

The orthogonal decomposition of the 4-force (9.110) is then

$$\boxed{\boldsymbol{f} = \Gamma\left[\left(\frac{1}{c}\frac{\mathrm{d}E}{\mathrm{d}t} + c\langle \boldsymbol{P}, \vec{\boldsymbol{a}}_0\rangle\right)\underline{\boldsymbol{u}}_0 + \boldsymbol{F}_{\text{ext}}\right]}. \tag{9.119}$$

9.5.4 Relativistic Version of Newton's Second Law

From the relations $\vec{\boldsymbol{q}} = \Gamma m \vec{\boldsymbol{V}}$ [Eq. (9.14)] and $\vec{\boldsymbol{F}} = \boldsymbol{D}_{\mathscr{O}}\,\vec{\boldsymbol{q}}$ [Eq. (9.113)], we get

$$\vec{\boldsymbol{F}} = \Gamma m \boldsymbol{D}_{\mathscr{O}}\,\vec{\boldsymbol{V}} + \frac{\mathrm{d}}{\mathrm{d}t}(\Gamma m)\,\vec{\boldsymbol{V}}.$$

Now $\boldsymbol{D}_{\mathscr{O}}\,\vec{\boldsymbol{V}}$ is nothing but the acceleration $\vec{\boldsymbol{\gamma}}$ of particle $\mathscr{P}$ relative to observer $\mathscr{O}$ [cf. Eq. (4.47)]. Consequently,

$$\boxed{\Gamma m \vec{\boldsymbol{\gamma}} + \frac{\mathrm{d}}{\mathrm{d}t}(\Gamma m)\,\vec{\boldsymbol{V}} = \vec{\boldsymbol{F}}}, \tag{9.120}$$

where $\vec{\boldsymbol{F}}$ is given by (9.115) and $\mathrm{d}\Gamma/\mathrm{d}t$ is expressible in terms of $\vec{\boldsymbol{V}}$ and $\vec{\boldsymbol{\gamma}}$ via (4.61), by substituting $\vec{\boldsymbol{u}}_0$ for $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{a}}_0$ for $\vec{\boldsymbol{a}}$. We recognize in (9.120) the relativistic generalization of Newton's second law of motion expressed in terms of the acceleration.

If $\mathscr{O}$ is an inertial observer, then $\vec{\boldsymbol{\omega}} = 0$ and $\vec{\boldsymbol{a}}_0 = 0$, (4.61) simplifies to $\mathrm{d}\Gamma/\mathrm{d}t = \Gamma^3 \vec{\boldsymbol{\gamma}} \cdot \vec{\boldsymbol{V}}/c^2$ and (9.120) becomes

$$\boxed{\Gamma m \left[\vec{\boldsymbol{\gamma}} + \frac{\Gamma^2}{c^2} (\vec{\boldsymbol{\gamma}} \cdot \vec{\boldsymbol{V}})\, \vec{\boldsymbol{V}} \right] = \vec{\boldsymbol{F}} - \Gamma \frac{\mathrm{d}m}{\mathrm{d}t}\, \vec{\boldsymbol{V}}} \qquad (\mathscr{O} \text{ inertial}). \tag{9.121}$$

Let us recall that if the 4-force is a pure one, then $\mathrm{d}m/\mathrm{d}t = 0$, which simplifies the right-hand side of this equation. The Newtonian limit of (9.121) is obtained by setting $\Gamma \to 1$, $c \to +\infty$ and $\mathrm{d}m/\mathrm{d}t = 0$. It is of course Newton's second law:

$$m \vec{\boldsymbol{\gamma}} = \vec{\boldsymbol{F}} \qquad \text{(nonrelativistic)}. \tag{9.122}$$

Except for the replacement of m by Γm and the possibility of a time-depending mass, the major difference between (9.121) and (9.122) is that in the relativistic case, the acceleration is no longer collinear to the force exerted onto the particle.

9.5.5 *Evolution of Energy*

We are going to deduce from (9.106), namely, $\langle \boldsymbol{f}, \vec{\boldsymbol{u}} \rangle = -c\, \mathrm{d}m/\mathrm{d}\tau$, a relation between the variation of the particle's energy, $\mathrm{d}E/\mathrm{d}t$, and the "work" of the force acting upon it. Let us start by writing the orthogonal decomposition of $\vec{\boldsymbol{u}}$ with respect to $\vec{\boldsymbol{u}}_0$; it is given by (4.27) (adapting the notations: $\vec{\boldsymbol{u}}' \to \vec{\boldsymbol{u}}$, $\vec{\boldsymbol{u}} \to \vec{\boldsymbol{u}}_0$ and $\vec{\boldsymbol{a}} \to \vec{\boldsymbol{a}}_0$). Combining with (9.13), we get

$$\vec{\boldsymbol{u}} = \Gamma (1 + \vec{\boldsymbol{a}}_0 \cdot \overrightarrow{OM})\, \vec{\boldsymbol{u}}_0 + \frac{1}{mc} \vec{\boldsymbol{P}}.$$

Equation (9.106) is then equivalent to

$$\Gamma (1 + \vec{\boldsymbol{a}}_0 \cdot \overrightarrow{OM}) \langle \boldsymbol{f}, \vec{\boldsymbol{u}}_0 \rangle + \frac{1}{mc} \langle \boldsymbol{f}, \vec{\boldsymbol{P}} \rangle = -c \frac{\mathrm{d}m}{\mathrm{d}\tau}.$$

Since $\vec{\boldsymbol{P}} \in E_{\boldsymbol{u}_0}$, we can write $\langle \boldsymbol{f}, \vec{\boldsymbol{P}} \rangle = \langle \boldsymbol{f} \circ \perp_{\boldsymbol{u}_0}, \vec{\boldsymbol{P}} \rangle$, i.e. by (9.116), $\langle \boldsymbol{f}, \vec{\boldsymbol{P}} \rangle = \Gamma \langle \boldsymbol{F}_{\mathrm{ext}}, \vec{\boldsymbol{P}} \rangle$. Using (9.111a) to replace $\langle \boldsymbol{f}, \vec{\boldsymbol{u}}_0 \rangle$ and to let $\mathrm{d}E/\mathrm{d}t$ appear, as well as (9.13) to express $\vec{\boldsymbol{P}}$, we get then

$$\boxed{\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{1}{1 + \vec{\boldsymbol{a}}_0 \cdot \overrightarrow{OM}} \left(\langle \boldsymbol{F}_{\mathrm{ext}}, \vec{\boldsymbol{V}} + \vec{\boldsymbol{\omega}} \times_{\boldsymbol{u}_0} \overrightarrow{OM} \rangle + \frac{c^2}{\Gamma} \frac{\mathrm{d}m}{\mathrm{d}t} \right) - c^2 \langle \boldsymbol{P}, \vec{\boldsymbol{a}}_0 \rangle}. \tag{9.123}$$

In the case where $\mathscr{O}$ is an inertial observer, this formula simplifies to

$$\boxed{\frac{\mathrm{d}E}{\mathrm{d}t} = \langle \boldsymbol{F}, \vec{\boldsymbol{V}} \rangle + \frac{c^2}{\Gamma} \frac{\mathrm{d}m}{\mathrm{d}t}} \qquad (\mathscr{O} \text{ inertial}). \tag{9.124}$$

For a pure 4-force, $\mathrm{d}m/\mathrm{d}t = 0$, and thanks to (9.20), one can replace E by the kinetic energy E_{kin} in the left-hand side. If so, then (9.124) is identical to the equation of Newtonian mechanics relating the variation of a particle's kinetic energy to the power of the force exerted on that particle. The power is usually expressed by the scalar product $\vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{V}}$, which is nothing but $\langle \boldsymbol{F}, \vec{\boldsymbol{V}} \rangle$.

Remark 9.23. The *linear form* aspect of a force, as opposed to the *vector* one, clearly appears in (9.124): $\boldsymbol{F}$ is the linear form that, once applied to the velocity vector $\vec{\boldsymbol{V}}$, yields a number: the energy provided to the particle by unit time.

9.5.6 Expression of the 4-Force

Inserting (9.123) in the orthogonal decomposition (9.119) of the 4-force, we obtain an expression of $\boldsymbol{f}$ in terms of the non-inertial part $\boldsymbol{F}_{\mathrm{ext}}$ of the force acting on $\mathscr{P}$ and measured by observer $\mathscr{O}$:

$$\boxed{\boldsymbol{f} = \Gamma \left[\frac{\langle \boldsymbol{F}_{\mathrm{ext}}, \vec{\boldsymbol{V}} + \vec{\boldsymbol{\omega}} \times_{\boldsymbol{u}_0} \overrightarrow{OM} \rangle + c^2/\Gamma \, \mathrm{d}m/\mathrm{d}t}{c(1 + \vec{\boldsymbol{a}}_0 \cdot \overrightarrow{OM})} \underline{\boldsymbol{u}}_0 + \boldsymbol{F}_{\mathrm{ext}} \right]}. \tag{9.125}$$

If $\mathscr{O}$ is inertial, this formula simplifies notably:

$$\boxed{\boldsymbol{f} = \Gamma \left[\left(\frac{\langle \boldsymbol{F}, \vec{\boldsymbol{V}} \rangle}{c} + \frac{c}{\Gamma} \frac{\mathrm{d}m}{\mathrm{d}t} \right) \underline{\boldsymbol{u}}_0 + \boldsymbol{F} \right]} \qquad (\mathscr{O} \text{ inertial}). \tag{9.126}$$

Remark 9.24. In the case of a pure 4-force ($\mathrm{d}m/\mathrm{d}t = 0$), the above formula shows that the 4-force $\boldsymbol{f}$ depends entirely of the force $\boldsymbol{F}$, which has only 3 degrees of freedom (it is a vector of $\mathscr{O}$'s rest space). This reflects the constraint (9.107) : $\langle \boldsymbol{f}, \vec{\boldsymbol{u}} \rangle = 0$, which restricts $\boldsymbol{f}$'s degrees of freedom from 4 to 3.

Historical note: *The concept of 4-force appeared as early as 1905 in Henri Poincaré's "Palermo memoir" (Poincaré 1906) (cf. p. 26), under the form of the four components of (9.126) (with* $\mathrm{d}m/\mathrm{d}t = 0$*). Poincaré showed that these components transform as the components of a four-dimensional vector under Lorentz transformations. The definition of the 4-force in the form* $\boldsymbol{f} = mc^2 \boldsymbol{a}$ *[Eq. (9.105) with* $\mathrm{d}m/\mathrm{d}\tau = 0$*] is due to Hermann Minkowski (cf. p. 26) in his 1908 article (Minkowski 1908); it has been repeated in his famous text on spacetime (Minkowski 1909).*

Chapter 10
Angular Momentum

10.1 Introduction

The preceding chapter being devoted to the conservation of 4-momentum, we turn now to the second principle that rules relativistic dynamics: that of conservation of angular momentum. After having defined the angular momentum for a particle (Sect. 10.2) and for a system (Sect. 10.3), we shall state the principle of its conservation (Sect. 10.4). We will then investigate the concepts of centre of inertia and spin (Sect. 10.5) and consider the evolution of angular momentum under a four-torque (Sect. 10.6). Finally, we shall discuss the concept of particle with spin and the notion of free gyroscope (Sect. 10.7).

10.2 Angular Momentum of a Particle

10.2.1 Definition

Let us consider a particle $\mathscr{P}$, either massive or massless, of worldline $\mathscr{L}$ and 4-momentum $\boldsymbol{p}$. Given an event $C \in \mathscr{E}$, one calls ***angular momentum 2-form of $\mathscr{P}$ with respect to C***, or simply ***angular momentum of $\mathscr{P}$ with respect to C***, the field of bilinear forms defined along $\mathscr{L}$ by

$$\forall M \in \mathscr{L}, \quad \boxed{\boldsymbol{J}_C(M) := \underline{CM} \wedge \boldsymbol{p}(M)}, \tag{10.1}$$

where $\underline{CM}$ is the linear form associated with the vector $\overrightarrow{CM}$ by metric duality (cf. Sect. 1.6) and $\wedge$ stands for the ***exterior product*** operator, which transforms

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any couple of linear forms $(\boldsymbol{a}, \boldsymbol{b}) \in E^* \times E^*$ into an antisymmetric bilinear form according to[1]

$$\boxed{\boldsymbol{a} \wedge \boldsymbol{b} := \boldsymbol{a} \otimes \boldsymbol{b} - \boldsymbol{b} \otimes \boldsymbol{a}}. \tag{10.2}$$

In view of the definitions (10.2) and (3.38), the explicit action of the bilinear form $\boldsymbol{J}_C(M)$ onto pairs of vectors is

$$\forall(\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}) \in E^2, \quad \boldsymbol{J}_C(M)(\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}) = (\overrightarrow{CM} \cdot \vec{\boldsymbol{v}}) \langle \boldsymbol{p}(M), \vec{\boldsymbol{w}} \rangle - \langle \boldsymbol{p}(M), \vec{\boldsymbol{v}} \rangle (\overrightarrow{CM} \cdot \vec{\boldsymbol{w}}).$$

$\boldsymbol{J}_C(M)$ is clearly an *antisymmetric* bilinear form:

$$\forall(\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}) \in E^2, \quad \boldsymbol{J}_C(M)(\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}) = -\boldsymbol{J}_C(M)(\vec{\boldsymbol{w}}, \vec{\boldsymbol{v}}). \tag{10.3}$$

For this reason, $\boldsymbol{J}_C$ is named a ***2-form***. More generally, a ***p-form*** is a multilinear form of p arguments (p vectors in E) that is fully antisymmetric. We shall study p-forms in detail in Chap. 14 and shall generalize the exterior product to them.

Remark 10.1. The reader familiar with Newtonian mechanics could have been surprised by the definition of the angular momentum as a *bilinear form*, whereas in nonrelativistic mechanics, it is defined as a *vector*. The transition from 3 to 4 dimensions seems not sufficient to account for the "transformation" of a vector into a bilinear form. We shall see below that the angular momentum 2-form $\boldsymbol{J}_C$ actually combines the angular momentum vector $\vec{\boldsymbol{\sigma}}_C$ with the mass-energy dipole moment $\vec{\boldsymbol{D}}$. Let us stress that $\vec{\boldsymbol{\sigma}}_C$ and $\vec{\boldsymbol{D}}$ are quantities relative to an observer, while $\boldsymbol{J}_C$ is intrinsic to the particle $\mathscr{P}$ (and to the point C), as the 4-momentum $\boldsymbol{p}$.

Remark 10.2. By analogy with 4-momentum, it would have been preferable to name $\boldsymbol{J}_C$ *angular 4-momentum*, in order to leave the appellation *angular momentum* to the vector $\vec{\boldsymbol{\sigma}}_C$ relative to an observer. Nevertheless, we follow the standard usage that amounts to calling $\boldsymbol{J}_C$ *angular momentum*, or *angular momentum 2-form*.

10.2.2 Angular Momentum Vector Relative to an Observer

Let us consider an observer $\mathscr{O}$ of worldline $\mathscr{L}_0$ and 4-velocity $\vec{\boldsymbol{u}}_0$. Let $M(t) \in \mathscr{E}$ be the event intersection of particle $\mathscr{P}$'s worldline with $\mathscr{O}$'s local rest space at proper time t, $\mathscr{E}_{\boldsymbol{u}_0}(t)$ (cf. Fig. 10.1). Let $O(t) = \mathscr{L}_0 \cap \mathscr{E}_{\boldsymbol{u}_0}(t)$ be the position of $\mathscr{O}$ at the instant t and $C_*(t)$ the orthogonal projection of C onto $\mathscr{E}_{\boldsymbol{u}_0}(t)$. The following orthogonal decomposition then holds:

$$\overrightarrow{CM}(t) = h(t)\, \vec{\boldsymbol{u}}_0(t) + \vec{\boldsymbol{X}}(t) \tag{10.4}$$

[1]Let us recall that the tensor product $\otimes$ has been defined in Sect. 3.5.2.

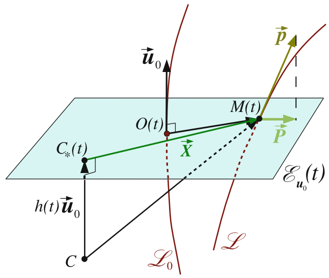

Fig. 10.1 Decomposition of the angular momentum of a particle (worldline $\mathscr{L}$) with respect to an observer (worldline $\mathscr{L}_0$)

with

$$h(t) := -\vec{u}_0(t) \cdot \overrightarrow{CM}(t) = -\vec{u}_0 \cdot \overrightarrow{CO}(t) \quad \text{and} \quad \vec{X}(t) := \overrightarrow{C_*(t)M(t)} \in E_{u_0}(t). \tag{10.5}$$

Note that if $\mathscr{O}$ is an inertial observer, then

$$h(t) = ct + h_0 \qquad (\mathscr{O} \text{ inertial}), \tag{10.6}$$

where h_0 is some constant.

The orthogonal decomposition of the second linear form involved in the angular momentum definition (10.1), namely, the 4-momentum $\boldsymbol{p}$, is provided by (9.8): $\boldsymbol{p} = (E/c)\,\underline{\boldsymbol{u}}_0 + \boldsymbol{P}$, E and $\boldsymbol{P}$ being, respectively, the energy and the linear momentum of $\mathscr{P}$ measured by $\mathscr{O}$. Introducing the decompositions (10.4) and (9.8) into (10.1), we get (we shall omit the mention of dependence in t or $M(t)$, except for $h(t)$)

$$\boldsymbol{J}_C = \underline{\boldsymbol{X}} \wedge \boldsymbol{P} + \left[\frac{E}{c}\underline{\boldsymbol{X}} - h(t)\boldsymbol{P}\right] \wedge \underline{\boldsymbol{u}}_0. \tag{10.7}$$

Moreover, as an antisymmetric bilinear form, $\boldsymbol{J}_C$ can be decomposed with respect to $\vec{u}_0$ according to the general rule established in Sect. 3.5.2 [Eq. (3.37)]. The linear form $\boldsymbol{q}$ involved in this decomposition is, from (3.40) and (10.7), $\boldsymbol{q} = \boldsymbol{J}_C(., \vec{u}_0) = -(E/c)\,\underline{\boldsymbol{X}} + h(t)\boldsymbol{P}$. It follows that there exists a unique vector $\vec{\sigma}_C \in E_{u_0}$ (the vector denoted by $\vec{b}$ in Sect. 3.5.2) such that

$$\boxed{\boldsymbol{J}_C = \boldsymbol{\epsilon}(\vec{u}_0, \vec{\sigma}_C, ., .) + \left[\frac{E}{c}\underline{\boldsymbol{X}} - h(t)\boldsymbol{P}\right] \wedge \underline{\boldsymbol{u}}_0}. \tag{10.8}$$

The vector $\vec{\sigma}_C$, which belongs to the hyperplane $E_{u_0}(t)$, is called ***angular momentum vector of $\mathscr{P}$ with respect to the point C and measured by observer $\mathscr{O}$ at the time t***.

Comparing expressions (10.7) and (10.8), we get

$$\boldsymbol{\epsilon}(\vec{u}_0, \vec{\sigma}_C, ., .) = \underline{\boldsymbol{X}} \wedge \boldsymbol{P}, \tag{10.9}$$

By definition, the three vectors $\vec{\boldsymbol{\sigma}}_C$, $\vec{\boldsymbol{P}}$ and $\vec{\boldsymbol{X}}$ belong to the hyperplane $E_{\boldsymbol{u}_0}$. They can therefore be expressed in terms of the spatial vectors of $\mathscr{O}$'s local frame $(\vec{\boldsymbol{e}}_\alpha)$: $\vec{\boldsymbol{\sigma}}_C = \sigma^i_C \vec{\boldsymbol{e}}_i$, $\vec{\boldsymbol{P}} = P^i \vec{\boldsymbol{e}}_i$ and $\vec{\boldsymbol{X}} = X^i \vec{\boldsymbol{e}}_i$. Let us apply (10.9), which is an equality between two bilinear forms, to a couple of vectors $(\vec{\boldsymbol{e}}_i, \vec{\boldsymbol{e}}_j)$ of $\mathscr{O}$'s local frame; we get successively

$$\begin{aligned}
\boldsymbol{\epsilon}(\vec{\boldsymbol{u}}_0, \sigma^k_C \vec{\boldsymbol{e}}_k, \vec{\boldsymbol{e}}_i, \vec{\boldsymbol{e}}_j) &= \langle \underline{\boldsymbol{X}}, \vec{\boldsymbol{e}}_i\rangle\langle \boldsymbol{P}, \vec{\boldsymbol{e}}_j\rangle - \langle \boldsymbol{P}, \vec{\boldsymbol{e}}_i\rangle\langle \underline{\boldsymbol{X}}, \vec{\boldsymbol{e}}_j\rangle \\
&= (\vec{\boldsymbol{X}} \cdot \vec{\boldsymbol{e}}_i)(\vec{\boldsymbol{P}} \cdot \vec{\boldsymbol{e}}_j) - (\vec{\boldsymbol{P}} \cdot \vec{\boldsymbol{e}}_i)(\vec{\boldsymbol{X}} \cdot \vec{\boldsymbol{e}}_j) \\
\sigma^k_C\, \boldsymbol{\epsilon}(\vec{\boldsymbol{u}}_0, \vec{\boldsymbol{e}}_k, \vec{\boldsymbol{e}}_i, \vec{\boldsymbol{e}}_j) &= X^i P^j - P^i X^j .
\end{aligned} \tag{10.10}$$

The second line is obtained by metric duality and the third one by the multilinearity of Levi–Civita tensor and the fact that $(\vec{\boldsymbol{e}}_i)$ is an orthonormal basis of $E_{\boldsymbol{u}_0}$. Now, since $(\vec{\boldsymbol{e}}_\alpha)$ is a direct orthonormal basis of E (with $\vec{\boldsymbol{e}}_0 = \vec{\boldsymbol{u}}_0$), the definition of the Levi–Civita tensor (cf. Sect. 1.5) yields $\boldsymbol{\epsilon}(\vec{\boldsymbol{u}}_0, \vec{\boldsymbol{e}}_k, \vec{\boldsymbol{e}}_i, \vec{\boldsymbol{e}}_j) = 1$ (resp. -1) if $(\vec{\boldsymbol{e}}_k, \vec{\boldsymbol{e}}_i, \vec{\boldsymbol{e}}_j)$ is an even (resp. odd) permutation of $(\vec{\boldsymbol{e}}_1, \vec{\boldsymbol{e}}_2, \vec{\boldsymbol{e}}_3)$, and $\boldsymbol{\epsilon}(\vec{\boldsymbol{u}}_0, \vec{\boldsymbol{e}}_k, \vec{\boldsymbol{e}}_i, \vec{\boldsymbol{e}}_j) = 0$ otherwise. We deduce then from (10.10) that the vector $\vec{\boldsymbol{\sigma}}_C$ is nothing but the cross product of the vectors $\vec{\boldsymbol{X}}$ and $\vec{\boldsymbol{P}}$ induced by $\boldsymbol{\epsilon}$ on $E_{\boldsymbol{u}_0}$, according to (3.46): $\vec{\boldsymbol{\sigma}}_C = \vec{\boldsymbol{X}} \times_{\boldsymbol{u}_0} \vec{\boldsymbol{P}}$. Since $\vec{\boldsymbol{u}}_0 \times_{\boldsymbol{u}_0} \vec{\boldsymbol{P}} = \vec{\boldsymbol{\epsilon}}(\vec{\boldsymbol{u}}_0, \vec{\boldsymbol{u}}_0, \vec{\boldsymbol{P}}, .) = 0$, $\vec{\boldsymbol{X}}$ can be replaced by $\overrightarrow{CM}$ in this formula. Hence,

$$\boxed{\vec{\boldsymbol{\sigma}}_C = \vec{\boldsymbol{X}} \times_{\boldsymbol{u}_0} \vec{\boldsymbol{P}} = \overrightarrow{CM} \times_{\boldsymbol{u}_0} \vec{\boldsymbol{P}}}. \tag{10.11}$$

Thus, we recover the classical expression of the angular momentum vector with respect to the point C and measured by a given observer.

10.2.3 Components of the Angular Momentum

The matrix $(J_{\alpha\beta})$ of the bilinear form $\boldsymbol{J}_C$ with respect to the orthonormal basis $(\vec{\boldsymbol{e}}_\alpha)$ ($\mathscr{O}$'s local frame) is defined by[2]

$$J_{\alpha\beta} := \boldsymbol{J}_C(\vec{\boldsymbol{e}}_\alpha, \vec{\boldsymbol{e}}_\beta). \tag{10.12}$$

We have then

$$\forall (\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}) \in E^2, \quad \boldsymbol{J}_C(\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}) = J_{\alpha\beta}\, v^\alpha w^\beta, \tag{10.13}$$

where the (v^α) (resp. (w^α)) are the components of the vector $\vec{\boldsymbol{v}}$ (resp. $\vec{\boldsymbol{w}}$) in the basis $(\vec{\boldsymbol{e}}_\alpha)$: $\vec{\boldsymbol{v}} = v^\alpha \vec{\boldsymbol{e}}_\alpha$ and $\vec{\boldsymbol{w}} = w^\alpha \vec{\boldsymbol{e}}_\alpha$. Given that $\vec{\boldsymbol{e}}_0 = \vec{\boldsymbol{u}}_0$, we deduce from (10.12) and (10.8) that

[2]Compare with (1.12).

$$J_{\alpha\beta} = \begin{pmatrix} 0 & \frac{E}{c}X^1 - h(t)P^1 & \frac{E}{c}X^2 - h(t)P^2 & \frac{E}{c}X^3 - h(t)P^3 \\ -\frac{E}{c}X^1 + h(t)P^1 & 0 & \sigma_C^3 & -\sigma_C^2 \\ -\frac{E}{c}X^2 + h(t)P^2 & -\sigma_C^3 & 0 & \sigma_C^1 \\ -\frac{E}{c}X^3 + h(t)P^3 & \sigma_C^2 & -\sigma_C^1 & 0 \end{pmatrix}. \tag{10.14}$$

with, according to (10.11),

$$\sigma_C^1 = X^2 P^3 - X^3 P^2, \quad \sigma_C^2 = X^3 P^1 - X^1 P^3, \quad \sigma_C^3 = X^1 P^2 - X^2 P^1. \tag{10.15}$$

10.3 Angular Momentum of a System

10.3.1 Definition

Given a system $\mathscr{S}$ of particles of worldlines $\mathscr{L}_a$ and 4-momenta $\boldsymbol{p}_a$, an oriented hypersurface $\Sigma \subset \mathscr{E}$ and a point $C \in \mathscr{E}$, we define the ***angular momentum of system*** $\mathscr{S}$ ***with respect to*** C ***on*** Σ in a manner analogous to what we did for the total 4-momentum in Chap. 9 [cf. Eq. (9.34)]:

$$\boldsymbol{J}_C|_\Sigma := \sum_{a=1}^{N} \sum_{M \in \mathscr{L}_a \cap \Sigma} \varepsilon\, \underline{CM} \wedge \boldsymbol{p}_a(M), \tag{10.16}$$

where we have expressed the angular momentum of each particle according to (10.1) and where, as in (9.34), $\varepsilon = +1$ (resp. $\varepsilon = -1$) if the 4-momentum vector $\vec{\boldsymbol{p}}_a(M)$ has the direction (resp. the opposite direction) set by the positive orientation of Σ. As a sum of antisymmetric bilinear forms, $\boldsymbol{J}_C|_\Sigma$ is itself an antisymmetric bilinear form, i.e. a 2-form.

If some observer $\mathscr{O}$ is prescribed, a natural choice for the hypersurface Σ is the local rest space of $\mathscr{O}$ at some instant t of his proper time: $\Sigma = \mathscr{E}_{\boldsymbol{u}_0}(t)$ ($\vec{\boldsymbol{u}}_0$ stands for $\mathscr{O}$'s 4-velocity). We shall then call $\boldsymbol{J}_C|_{\mathscr{E}_{\boldsymbol{u}_0}(t)}$ the ***total angular momentum of the system*** $\mathscr{S}$ ***with respect to*** C ***at the instant*** t ***of observer*** $\mathscr{O}$. In this case, one has always $\varepsilon = 1$. More, since $\mathscr{E}_{\boldsymbol{u}_0}(t)$ is a spacelike hypersurface, the intersection $\mathscr{L}_a \cap \mathscr{E}_{\boldsymbol{u}_0}(t)$ is necessarily limited to a single point (cf. Sect. 9.3.5). Equation (10.16) becomes then

$$\boxed{\boldsymbol{J}_C|_{\mathscr{E}_{\boldsymbol{u}_0}(t)} := \sum_{a=1}^{N} \underline{CM_a} \wedge \boldsymbol{p}_a(M_a)}, \tag{10.17}$$

where $M_a = M_a(t) := \mathscr{L}_a \cap \mathscr{E}_{\boldsymbol{u}_0}(t)$. We can also write

$$\boxed{\boldsymbol{J}_C|_{\mathscr{E}_{\boldsymbol{u}_0}(t)} = \sum_{a=1}^{N} \boldsymbol{J}_C^a(M_a(t))}, \tag{10.18}$$

where $\boldsymbol{J}_C^a$ is the angular momentum 2-form of particle a with respect to the point C.

Historical note: *As for the total 4-momentum (cf. Sect. 9.3.1), the definition (10.16) of the total angular momentum of a system on a hypersurface has been introduced in 1935 by John L. Synge (cf. p. 74) (1935).*

10.3.2 Change of Origin

If the point $C \in \mathscr{E}$ in (10.16) is changed to another point $C' \in \mathscr{E}$, we may write, thanks to Chasles' relation $\overrightarrow{C'M} = \overrightarrow{C'C} + \overrightarrow{CM}$,

$$\begin{aligned} \boldsymbol{J}_{C'}|_{\Sigma} &= \sum_{a=1}^{N} \sum_{M \in \mathscr{L}_a \cap \Sigma} \varepsilon\, (\underline{C'C} + \underline{CM}) \wedge \boldsymbol{p}_a(M) \\ &= \underline{C'C} \wedge \left(\sum_{a=1}^{N} \sum_{M \in \mathscr{L}_a \cap \Sigma} \varepsilon\, \boldsymbol{p}_a(M) \right) + \boldsymbol{J}_C|_{\Sigma} \,. \end{aligned}$$

In view of (9.34), the term inside the parentheses is the total 4-momentum of $\mathscr{S}$ on the hypersurface Σ. Hence, we get the change-of-origin formula:

$$\boxed{\boldsymbol{J}_{C'}|_{\Sigma} = \boldsymbol{J}_C|_{\Sigma} + \underline{C'C} \wedge \boldsymbol{p}|_{\Sigma}}. \tag{10.19}$$

10.3.3 Angular Momentum Vector and Mass-Energy Dipole Moment

Let us consider an observer $\mathscr{O}$ of worldline $\mathscr{L}_0$ and 4-velocity $\vec{\boldsymbol{u}}_0$. Let us denote by $O(t)$ the position of $\mathscr{O}$ on $\mathscr{L}_0$ at the proper time t. By combining (10.18) and (10.8), we can write the angular momentum of the system $\mathscr{S}$ with respect to C at the instant t of $\mathscr{O}$ as

$$\boldsymbol{J}_C|_{\mathscr{E}_{\boldsymbol{u}_0}(t)} = \sum_{a=1}^{N} \left\{ \boldsymbol{\epsilon}(\vec{\boldsymbol{u}}_0, \vec{\boldsymbol{\sigma}}_C^a, ., .) + \left[\frac{E_a}{c} \underline{\boldsymbol{X}}_a - h(t) \boldsymbol{P}_a \right] \wedge \underline{\boldsymbol{u}}_0 \right\}, \tag{10.20}$$

where $\vec{\sigma}^a_C$ is the angular momentum vector of particle a with respect to C and measured by $\mathscr{O}$; E_a and $\boldsymbol{P}_a$ are, respectively, the energy and linear momentum of the particle a, both measured by $\mathscr{O}$; $\vec{X}_a$ is the vector $\overrightarrow{C_*(t)M_a(t)}$, $C_*(t)$ being the orthogonal projection of C onto $\mathscr{E}_{u_0}(t)$; and $h(t) = -\vec{u}_0 \cdot \overrightarrow{CO}(t)$ [cf. Eq. (10.5)] is independent of the particle a. Thanks to the multilinearity of the Levi–Civita tensor $\boldsymbol{\epsilon}$, we can rewrite (10.20) as

$$\boldsymbol{J}_C|_{\mathscr{E}_{u_0}(t)} = \boldsymbol{\epsilon}(\vec{u}_0, \vec{\sigma}_C, ., .) + \left[\sum_{a=1}^{N} \frac{E_a}{c} \underline{\boldsymbol{X}}_a - h(t)\boldsymbol{P}\right] \wedge \underline{\boldsymbol{u}}_0, \tag{10.21}$$

where $\boldsymbol{P} = \sum_{a=1}^{N} \boldsymbol{P}_a$ is the linear momentum of the system $\mathscr{S}$ measured by $\mathscr{O}$ at the instant t [cf. Eq. (9.48)] and

$$\boxed{\vec{\sigma}_C := \sum_{a=1}^{N} \vec{\sigma}^a_C(M_a)} \tag{10.22}$$

is, by definition, the ***total angular momentum vector of the system $\mathscr{S}$ with respect to the point C and measured by $\mathscr{O}$ at the instant t***. Note that $\vec{\sigma}_C \in E_{u_0}(t)$.

Let us define the ***mass-energy dipole moment of the system $\mathscr{S}$ relative to observer $\mathscr{O}$ at the instant t*** as the following vector of the local rest space $E_{u_0}(t)$:

$$\boxed{\vec{\boldsymbol{D}} := \frac{1}{c^2} \sum_{a=1}^{N} E_a\, \overrightarrow{O(t)M_a(t)}}. \tag{10.23}$$

We have then

$$\begin{aligned}\sum_{a=1}^{N} \frac{E_a}{c} \vec{X}_a &= \sum_{a=1}^{N} \frac{E_a}{c} \overrightarrow{C_*(t)M_a(t)} = \sum_{a=1}^{N} \frac{E_a}{c} \left[\overrightarrow{C_*(t)O(t)} + \overrightarrow{O(t)M_a(t)}\right] \\ &= \frac{E}{c} \overrightarrow{C_*(t)O(t)} + c\vec{\boldsymbol{D}},\end{aligned}$$

where $E = \sum_{a=1}^{N} E_a$ is the total energy of the system measured by $\mathscr{O}$ [cf. Eq. (9.48)]. Equation (10.21) can then be written as

$$\boxed{\boldsymbol{J}_C|_{\mathscr{E}_{u_0}(t)} = \boldsymbol{\epsilon}(\vec{u}_0, \vec{\sigma}_C, ., .) + \left[c\underline{\boldsymbol{D}} + \frac{E}{c} \underline{C_* O} - h(t)\boldsymbol{P}\right] \wedge \underline{\boldsymbol{u}}_0}. \tag{10.24}$$

Remark 10.3. Since $\overrightarrow{CO} = \overrightarrow{CC_*} + \overrightarrow{C_*O} = h(t)\,\vec{u}_0 + \overrightarrow{C_*O}$, we have $\underline{CO} \wedge \underline{\boldsymbol{u}}_0 = \underline{C_*O} \wedge \underline{\boldsymbol{u}}_0$ so that we can replace C_* by C in the above expression.

Remark 10.4. The expression (10.24) is the orthogonal decomposition of the antisymmetric bilinear form $\boldsymbol{J}_C|_{\mathscr{E}_{u_0}(t)}$ with respect to $\vec{\boldsymbol{u}}_0$, as established in Sect. 3.5.2.

Remark 10.5. Whereas the 4-momentum combines energy and linear momentum in a same object (cf. Remark 9.5 p. 275), expression (10.24) shows that the angular momentum 2-form combines the angular momentum vector $\vec{\boldsymbol{\sigma}}_C$ with the mass-energy dipole moment $\vec{\boldsymbol{D}}$.

10.4 Conservation of Angular Momentum

10.4.1 Principle of Angular Momentum Conservation

In addition to that of 4-momentum conservation (Sect. 9.3.3), the second fundamental principle of the dynamics of relativistic particles is:

If a particle system is isolated, its angular momentum with respect to any point $C \in \mathscr{E}$ and on any closed hypersurface vanishes:

$$\boxed{\mathscr{S} \text{ isolated and } \Sigma \text{ closed} \implies \boldsymbol{J}_C|_\Sigma = 0}. \tag{10.25}$$

The same comments as those made in Sect. 9.3.3 regarding the principle conservation of 4-momentum could be repeated mutatis mutandis here.

As for the 4-momentum in Sect. 9.3.4, the principle (10.25) applied to a system reduced to a single particle implies that the angular momentum of an isolated particle with respect to a fixed point $C \in \mathscr{E}$ is a constant field of bilinear forms along the particle's worldline $\mathscr{L}$:

$$\forall M \in \mathscr{L}, \quad \boldsymbol{J}_C(M) = \text{const.} \tag{10.26}$$

Remark 10.6. For an isolated massive particle, the property (10.26) can also be obtained as a consequence of the law of inertia (9.37) and therefore of the principle of 4-momentum conservation. Indeed, using (10.1) and denoting by τ the proper time along $\mathscr{L}$, the derivative of $\boldsymbol{J}_C(M)$ along the worldline $\mathscr{L}$ is

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\boldsymbol{J}_C(M(\tau)) = \frac{\mathrm{d}\underline{CM}}{\mathrm{d}\tau} \otimes \boldsymbol{p} + \underline{CM} \otimes \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}\tau} - \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}\tau} \otimes \underline{CM} - \boldsymbol{p} \otimes \frac{\mathrm{d}\underline{CM}}{\mathrm{d}\tau}. \tag{10.27}$$

Now, from the definition of 4-velocity, $d\underline{CM}/\mathrm{d}\tau = c\,\underline{\boldsymbol{u}}$. Since $\underline{\boldsymbol{u}} = (mc)^{-1}\boldsymbol{p}$ [Eq. (9.3)], there comes $\mathrm{d}\underline{CM}/\mathrm{d}\tau \otimes \boldsymbol{p} - \boldsymbol{p} \otimes \mathrm{d}\underline{CM}/\mathrm{d}\tau = m^{-1}(\boldsymbol{p} \otimes \boldsymbol{p} - \boldsymbol{p} \otimes \boldsymbol{p}) = 0$, so that (10.27) reduces to

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\boldsymbol{J}_C(M(\tau)) = \underline{CM} \wedge \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}\tau}.$$

The law of inertia (9.37) implies $\mathrm{d}\boldsymbol{p}/\mathrm{d}\tau = 0$ and thus $\mathrm{d}\boldsymbol{J}_C(M(\tau))/\mathrm{d}\tau = 0$. □

Remark 10.7. In Newtonian mechanics, the conservation of the angular momentum of an isolated system is deduced from the action-reaction principle (Newton's third law) *in its strong version*, i.e. not only the force exerted by particle A onto particle B is the exact opposite of the force exerted by B onto A, but this force is in addition aligned with the vector connecting A and B (cf., e.g. Deruelle and Uzan (2006)). In the relativistic case, there is no principle of the type "action-reaction". The conservation of angular momentum of a system comprising at least two particles (cf. the above remark for a single particle) appears then as a first principle, at the same level as the conservation of 4-momentum.

Remark 10.8. As for the 4-momentum (cf. Sect. 9.3.3), the conservation of angular momentum appears, in a Lagrangian formulation, as a consequence of Noether theorem, and not a first principle. We shall see it explicitly in Chap. 11.

10.4.2 Angular Momentum of an Isolated System

As a consequence of the conservation law (10.25), the following property holds:

If the particle system $\mathscr{S}$ is isolated, its angular momentum $\boldsymbol{J}_C|_{\Sigma}$ does not depend on the choice of the hypersurface Σ, provided the latter is spacelike and intersects all the worldlines of $\mathscr{S}$'s particles. One can therefore define

$$\boxed{\boldsymbol{J}_C := \sum_{a=1}^{N} \underline{CM_a} \wedge \boldsymbol{p}_a(M_a)}, \qquad (10.28)$$

where M_a stands for the unique intersection of the worldline $\mathscr{L}_a$ with Σ. $\boldsymbol{J}_C$ is called the ***total angular momentum of the isolated system*** $\mathscr{S}$ ***with respect to the point*** C.

The proof is identical to that given in Sect. 9.3.5 for the 4-momentum.

In particular, the total angular momentum at the instant t of an observer [cf. Eq. (10.17)] is independent of t and of the observer:

$$\boldsymbol{J}_C|_{\mathscr{E}_{\boldsymbol{u}_0}(t)} = \boldsymbol{J}_C|_{\mathscr{E}_{\boldsymbol{u}'_0}(t')} = \boldsymbol{J}_C. \tag{10.29}$$

10.4.3 Conservation of the Angular Momentum Vector Relative to an Inertial Observer

Let us consider an isolated system $\mathscr{S}$. If $\mathscr{O}$ is an inertial observer, its 4-velocity $\vec{\boldsymbol{u}}_0$ is constant, as well as the hyperplane $E_{\boldsymbol{u}_0}$. Let then $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{w}}$ be two generic vectors in $E_{\boldsymbol{u}_0}$. We have, thanks to (10.24),

$$\boldsymbol{J}_C(\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}) = \boldsymbol{\epsilon}(\vec{\boldsymbol{u}}_0, \vec{\boldsymbol{\sigma}}_C, \vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}).$$

Since the system $\mathscr{S}$ is isolated, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\boldsymbol{J}_C(\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}})\right] = 0 = \boldsymbol{\epsilon}\left(\vec{\boldsymbol{u}}_0, \frac{\mathrm{d}}{\mathrm{d}t}\vec{\boldsymbol{\sigma}}_C, \vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}\right),$$

where we have used the multilinearity of the Levi–Civita tensor and the constant character of the vectors $\vec{\boldsymbol{u}}_0$, $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{w}}$. The above equality being valid for any pair of vectors $(\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}})$, we deduce that $\mathrm{d}\vec{\boldsymbol{\sigma}}_C/\mathrm{d}t$ is necessarily collinear to $\vec{\boldsymbol{u}}_0$. Now since $\vec{\boldsymbol{\sigma}}_C \in E_{\boldsymbol{u}_0}$ and $E_{\boldsymbol{u}_0}$ is independent of t, we have $\mathrm{d}\vec{\boldsymbol{\sigma}}_C/\mathrm{d}t \in E_{\boldsymbol{u}_0}$. The collinearity with $\vec{\boldsymbol{u}}_0$ implies then $\mathrm{d}\vec{\boldsymbol{\sigma}}_C/\mathrm{d}t = 0$:

> The angular momentum vector of an *isolated* system with respect to any point C and measured by any *inertial* observer is constant:
>
> $$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\vec{\boldsymbol{\sigma}}_C = 0}. \tag{10.30}$$

This conservation law is of course similar to the laws (9.49) obtained in Chap. 9 for the energy and the linear momentum of the system.

10.5 Centre of Inertia and Spin

10.5.1 Centroid of a System

Let us consider a particle system $\mathscr{S}$ and an observer $\mathscr{O}$ of worldline $\mathscr{L}_0$ and 4-velocity $\vec{\boldsymbol{u}}_0$. As in Sect. 10.3.3, let us denote by $O(t) \in \mathscr{L}_0$ the position of $\mathscr{O}$ on its worldline at the proper time t. One calls ***centroid of the system*** $\mathscr{S}$ ***relative to observer*** $\mathscr{O}$ the point $G_{\mathscr{O}}(t)$ of $\mathscr{O}$'s local rest space $\mathscr{E}_{\boldsymbol{u}_0}(t)$ such that

$$\boxed{\overrightarrow{O(t)G_{\mathscr{O}}(t)} := \frac{1}{E}\sum_{a=1}^{N} E_a\, \overrightarrow{O(t)M_a(t)} = \frac{c^2}{E}\vec{\boldsymbol{D}}}, \tag{10.31}$$

where (i) $M_a(t)$ is the position of particle a at time t with respect to $\mathscr{O}$, i.e. the intersection of the worldline $\mathscr{L}_a$ of particle a with the hyperplane $\mathscr{E}_{\boldsymbol{u}_0}(t)$; (ii) E_a and E are, respectively, the energy of particle a and the total energy of the system, both measured by $\mathscr{O}$; and (iii) $\vec{\boldsymbol{D}}$ is the mass-energy dipole moment of $\mathscr{S}$ relative to $\mathscr{O}$, as defined by (10.23). Since $E = \sum_{a=1}^{N} E_a$ [Eq. (9.48)], $G_{\mathscr{O}}(t)$ is the weighted mean of the positions of the particles with respect to observer $\mathscr{O}$, the weights being the energies of the particles relative to $\mathscr{O}$.

Remark 10.9. At the nonrelativistic limit, $E \to mc^2$, $E_a \to m_a c^2$, and we recover the definition of the centre of mass of the system. In the relativistic case, it must be noticed that the definition of the centroid depends upon the observer $\mathscr{O}$ and this is in two manners: via the local rest space $\mathscr{E}_{\boldsymbol{u}_0}(t)$, which defines the points $O(t)$ and $M_a(t)$, and via the energies E_a and E, which are both relative to $\mathscr{O}$.

Although the centroid depends upon the choice of the observer, it is easy to see that two inertial observers, $\mathscr{O}$ and $\mathscr{O}'$ say, sharing the same 4-velocity, $\vec{\boldsymbol{u}}_0$ say, will agree on the centroid. Indeed, the worldlines of the two observers are in this case parallel straight lines in $\mathscr{E}$, and their rest spaces coincide (cf. Fig. 8.3). Up to some choice of proper time origin, we can then write $t' = t$. Denoting by $G_{\mathscr{O}'}$ the centroid of system $\mathscr{S}$ with respect to $\mathscr{O}'$ at some instant t, we have

$$\begin{aligned}
\overrightarrow{O'G_{\mathscr{O}'}} &:= \frac{1}{E'}\sum_{a=1}^{N} E'_a\, \overrightarrow{O'M_a} = \frac{1}{E'}\sum_{a=1}^{N} E'_a \left(\overrightarrow{O'O} + \overrightarrow{OM_a}\right) \\
&= \underbrace{\frac{1}{E'}\left(\sum_{a=1}^{N} E'_a\right)}_{1} \overrightarrow{O'O} + \underbrace{\frac{1}{E}\sum_{a=1}^{N} E_a\, \overrightarrow{OM_a}}_{\overrightarrow{OG_{\mathscr{O}}}} \\
\overrightarrow{O'G_{\mathscr{O}'}} &= \overrightarrow{O'G_{\mathscr{O}}},
\end{aligned} \tag{10.32}$$

where we have used $E'_a = E_a$ et $E' = E$ (this follows from the fact that $\mathscr{O}$ and $\mathscr{O}'$ have the same 4-velocity $\vec{\boldsymbol{u}}_0$: $E'_a = -c\,\langle \boldsymbol{p}_a, \vec{\boldsymbol{u}}_0\rangle = E_a$ and $E' = -c\,\langle \boldsymbol{p}, \vec{\boldsymbol{u}}_0\rangle = E$). The result (10.32) allows us to conclude that $G_{\mathscr{O}'} = G_{\mathscr{O}}$. Hence, we have shown:

The centroid of a system is the same for all inertial observers having the same 4-velocity, i.e. for all observers who belong to the same rigid array of inertial observers, as defined in Sect. 8.2.4.

10.5.2 Centre of Inertia of an Isolated System

In all what follows, we suppose that (i) the system $\mathscr{S}$ is isolated and (ii) the observer $\mathscr{O}$ is inertial. From the relation (4.24) with $\vec{\boldsymbol{a}} = 0$ and $\vec{\boldsymbol{\omega}} = 0$ (since $\mathscr{O}$ is inertial), the derivative vector

$$\vec{\boldsymbol{V}}_{G_{\mathscr{O}}} := \frac{\mathrm{d}}{\mathrm{d}t}\overrightarrow{O(t)G_{\mathscr{O}}(t)} \tag{10.33}$$

is the velocity of the centroid $G_{\mathscr{O}}$ relative to observer $\mathscr{O}$. To evaluate it, one must thus derive (10.31) with respect to t. According to the result of Sect. 9.3.6 and to the assumptions (i) and (ii) here above, E is constant, so that we get

$$\vec{\boldsymbol{V}}_{G_{\mathscr{O}}} = \frac{c^2}{E}\frac{\mathrm{d}}{\mathrm{d}t}\vec{\boldsymbol{D}}. \tag{10.34}$$

To compute the time derivative of $\vec{\boldsymbol{D}}$, it suffices to recall that $\vec{\boldsymbol{D}}$ is involved in the decomposition (10.24) of the angular momentum and to invoke the conservation of the latter, since $\mathscr{S}$ is an isolated system. Given a point $C \in \mathscr{E}$, let us consider the angular momentum $\boldsymbol{J}_C$ of $\mathscr{S}$ with respect to C. Using $\mathscr{O}$'s 4-velocity as the first argument of bilinear form $\boldsymbol{J}_C$, one defines the linear form $\boldsymbol{J}_C(\vec{\boldsymbol{u}}_0, .)$. Its expression follows from (10.24), using the antisymmetry of $\boldsymbol{\epsilon}$ as well as the properties $\vec{\boldsymbol{D}} \cdot \vec{\boldsymbol{u}}_0 = 0$, $\overrightarrow{C_* O} \cdot \vec{\boldsymbol{u}}_0 = 0$, $\langle \boldsymbol{P}, \vec{\boldsymbol{u}}_0\rangle = 0$ and $\vec{\boldsymbol{u}}_0 \cdot \vec{\boldsymbol{u}}_0 = -1$:

$$\boldsymbol{J}_C(\vec{\boldsymbol{u}}_0, .) = c\underline{\boldsymbol{D}} + \frac{E}{c}\underline{C_* O} - h(t)\boldsymbol{P}. \tag{10.35}$$

Since $\boldsymbol{J}_C$ does not depend on t ($\mathscr{S}$ isolated) nor $\vec{\boldsymbol{u}}_0$ does ($\mathscr{O}$ inertial), the above relation yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{J}_C(\vec{\boldsymbol{u}}_0, .) = 0 = c\frac{\mathrm{d}}{\mathrm{d}t}\underline{\boldsymbol{D}} + \frac{E}{c}\frac{\mathrm{d}\underline{C_* O}}{\mathrm{d}t} - \frac{\mathrm{d}h}{\mathrm{d}t}\boldsymbol{P}. \tag{10.36}$$

Note that we have used the properties $\mathrm{d}E/\mathrm{d}t = 0$ and $\mathrm{d}\boldsymbol{P}/\mathrm{d}t = 0$ [Eq. (9.49)]. Now, since $\mathscr{O}$ is inertial [cf. in particular (10.6)],

$$\overrightarrow{C_*O} = \overrightarrow{C_*(t)O(t)} = \overrightarrow{C_*(t)C} + \overrightarrow{CO(t)} = -h(t)\,\vec{\boldsymbol{u}}_0 + \overrightarrow{CO(t)} = -(ct + h_0)\,\vec{\boldsymbol{u}}_0 + \overrightarrow{CO(t)},$$

which leads to

$$\frac{\mathrm{d}\overrightarrow{C_*O}}{\mathrm{d}t} = -c\,\vec{\boldsymbol{u}}_0 + \underbrace{\frac{\mathrm{d}\overrightarrow{CO(t)}}{\mathrm{d}t}}_{c\,\vec{\boldsymbol{u}}_0} = 0.$$

By metric duality and taking into account $\mathrm{d}\boldsymbol{g}/\mathrm{d}t = 0$, we deduce that the second term in the right-hand side of (10.36) vanishes. Since from (10.6), $\mathrm{d}h/\mathrm{d}t = c$, we conclude that

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\underline{\boldsymbol{D}} = \boldsymbol{P}}. \tag{10.37}$$

Plugging this result into (10.34), we obtain the velocity of the centroid of system $\mathscr{S}$ relative to $\mathscr{O}$:

$$\vec{\boldsymbol{V}}_{G_{\mathscr{O}}} = \frac{c^2}{E}\,\vec{\boldsymbol{P}}. \tag{10.38}$$

The vector $\vec{\boldsymbol{P}}$ being constant, this means that $G_{\mathscr{O}}$ has a constant-velocity motion relative to $\mathscr{O}$. But there is more: in Sect. 9.3.5, we have introduced the notion of 4-velocity $\vec{\boldsymbol{u}}$ for an isolated system whose total mass m is nonvanishing—case we shall consider henceforth. By combining (9.42), (9.43) and (9.45), we get $E = \Gamma mc^2$ and $\vec{\boldsymbol{P}} = mc \perp_{\boldsymbol{u}_0} \vec{\boldsymbol{u}}$, with $\Gamma := -\vec{\boldsymbol{u}}_0 \cdot \vec{\boldsymbol{u}}$, so that we can rewrite (10.38) as

$$\boxed{\vec{\boldsymbol{V}}_{G_{\mathscr{O}}} = \frac{c}{\Gamma} \perp_{\boldsymbol{u}_0} \vec{\boldsymbol{u}}}. \tag{10.39}$$

Comparing with (4.32), we observe that the velocity $\vec{\boldsymbol{V}}_{G_{\mathscr{O}}}$ is nothing but the velocity relative to $\mathscr{O}$ of a point particle that would have the constant vector $\vec{\boldsymbol{u}}$ as 4-velocity. We therefore conclude:

> The centroid with respect to an *inertial* observer of an *isolated* system $\mathscr{S}$ of nonvanishing total mass follows a timelike straight line in Minkowski spacetime $\mathscr{E}$. This line is parallel to the 4-velocity $\vec{\boldsymbol{u}}$ of the system. Since $\vec{\boldsymbol{u}}$ is independent of any observer, the centroids of $\mathscr{S}$ with respect to various inertial observers follow parallel lines in $\mathscr{E}$.

We have seen in Sect. 10.5.1 that the centroid of a system is the same for all inertial observers who share a given 4-velocity. If one chooses this common 4-velocity to be the 4-velocity of the system, $\vec{\boldsymbol{u}}$ (comoving observers; cf. Sect. 9.3.5), then the corresponding centroid is intrinsic to the system. We shall call it ***centre of inertia of the isolated system*** $\mathscr{S}$ and will denote it by G, without any index. The alternative names ***centre of mass*** and ***barycentre*** are also used. G follows a worldline $\mathscr{L}_G$ that is a straight line of $\mathscr{E}$ with tangent vector $\vec{\boldsymbol{u}}$. We shall call ***barycentric observer*** of the system $\mathscr{S}$ any inertial observer having $\mathscr{L}_G$ as a worldline. The frame $(\vec{\boldsymbol{e}}_\alpha)$ of such an observer is then called ***centre-of-inertia frame*** or sometimes ***centre-of-mass frame*** or ***centre-of-momentum frame***. Barycentric observers differ only by a fixed rotation of their spatial frame $(\vec{\boldsymbol{e}}_i)$. A barycentric observer is of course a particular case of an observer *comoving* with the system, in the sense defined in Sect. 9.3.5, i.e. an inertial observer of 4-velocity $\vec{\boldsymbol{u}}$.

Remark 10.10. The centre of inertia has been defined only for an isolated system. For a general system, we have only the notion of centroid, which depends upon the choice of the observer.

Remark 10.11. In the above demonstration, we have used the principle of angular momentum conservation (Sect. 10.4.1) to establish the constancy of the velocity of system $\mathscr{S}$'s centroid relative to the inertial observer $\mathscr{O}$. Actually, we have used only the constancy of the part $\boldsymbol{J}_C(\vec{\boldsymbol{u}}_0, .)$ of the angular momentum [cf. Eq. (10.36)]. We have seen in Sect. 10.4.3 that the constancy of the part fully orthogonal to $\vec{\boldsymbol{u}}_0$ [i.e. the part $\boldsymbol{\epsilon}(\vec{\boldsymbol{u}}_0, \vec{\boldsymbol{\sigma}}_C, ., .)$ in the decomposition (10.24)] leads to the conservation of the angular momentum vector relative to $\mathscr{O}$, $\vec{\boldsymbol{\sigma}}_C$ [Eq. (10.30)]. To summarize, the angular momentum 2-form of an isolated system, $\boldsymbol{J}_C$, has 6 independent components (its matrix $J_{\alpha\beta}$ with respect to $\mathscr{O}$ is a 4×4 antisymmetric matrix), and the law of angular momentum conservation yields the conservation of the angular momentum vector $\vec{\boldsymbol{\sigma}}_C$ (3 components) and the conservation of the velocity of the centroid of the system $\vec{\boldsymbol{V}}_{G_\mathscr{O}}$ (3 components), these two vectors being relative to the inertial observer $\mathscr{O}$.

Since $O(t) = G(t)$ for a barycentric observer, we deduce, respectively, from (10.31) and (10.33) that

$$\sum_{a=1}^{N} E_a \, \overrightarrow{G(t)M_a(t)} = 0 \qquad \text{and} \qquad \vec{\boldsymbol{V}}_G = 0, \tag{10.40}$$

where E_a is the energy of particle a measured by the barycentric observer. The first of these two relations implies that the mass-energy dipole moment of $\mathscr{S}$ with respect to any barycentric observer vanishes:

$$\vec{\boldsymbol{D}} = 0 \qquad (\mathscr{O} \text{ barycentric}). \tag{10.41}$$

Via (10.37), we deduce that the total linear momentum of $\mathscr{S}$ measured by any barycentric observer also vanishes:

$$\boldsymbol{P} = 0 \qquad (\mathscr{O} \text{ barycentric}). \tag{10.42}$$

The Einstein relation (9.47) shows then that the total energy measured by a barycentric observer reduces to the total mass of $\mathscr{S}$:

$$E = m\, c^2 \qquad (\mathscr{O} \text{ barycentric}). \tag{10.43}$$

The property (10.39), namely, that all the centroids of an isolated system have the 4-velocity $\vec{\boldsymbol{u}}$ of the system can be recast as follows: the centroid of an isolated system with respect to a given inertial observer is *fixed* with respect to a barycentric observer, in the sense defined in Sect. 3.4.3.

10.5.3 Spin of an Isolated System

Let us consider an isolated system $\mathscr{S}$. Its angular momentum with respect to a point $C \in \mathscr{E}$, $\boldsymbol{J}_C$, is then independent of any observer. Let us decompose it relatively to a barycentric observer, according to formula (10.24). In the present case, $\vec{\boldsymbol{u}}_0 = \vec{\boldsymbol{u}}$, $O(t) = G(t)$, $\underline{\boldsymbol{D}} = 0$ [Eq. (10.41)] and $\boldsymbol{P} = 0$ [Eq. (10.42)]. We get therefore

$$\boldsymbol{J}_C = \boldsymbol{\epsilon}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{\sigma}}_C, ., .) + c\, m\, \underline{CG} \wedge \underline{\boldsymbol{u}},$$

where we have used (10.43) and replaced C_* by C, following Remark 10.3 p. 325. Thanks to (9.42), we can replace $c\, m\, \underline{\boldsymbol{u}}$ by the total 4-momentum of the system, $\boldsymbol{p}$. We obtain then

$$\boldsymbol{J}_C = \boldsymbol{\epsilon}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{\sigma}}_C, ., .) + \underline{CG} \wedge \boldsymbol{p}. \tag{10.44}$$

If one considers the angular momentum of $\mathscr{S}$ with respect to a second point $C' \in \mathscr{E}$, it follows from (10.44) and the multilinearity of $\boldsymbol{\epsilon}$ that

$$\boldsymbol{J}_{C'} = \boldsymbol{J}_C + \boldsymbol{\epsilon}\left(\vec{\boldsymbol{u}},\ \vec{\boldsymbol{\sigma}}_{C'} - \vec{\boldsymbol{\sigma}}_C,\ .\ ,\ .\right) + \underline{C'C} \wedge \boldsymbol{p}.$$

By comparing with the general law for a change of origin, namely, formula (10.19), we obtain immediately the vanishing of the bilinear form $\boldsymbol{\epsilon}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{\sigma}}_{C'} - \vec{\boldsymbol{\sigma}}_C, ., .)$:

$$\boldsymbol{\epsilon}\left(\vec{\boldsymbol{u}},\ \vec{\boldsymbol{\sigma}}_{C'} - \vec{\boldsymbol{\sigma}}_C,\ .\ ,\ .\right) = 0.$$

This is possible only if the vector $\vec{\boldsymbol{\sigma}}_{C'} - \vec{\boldsymbol{\sigma}}_C$ is itself zero; hence,

$$\vec{\boldsymbol{\sigma}}_{C'} = \vec{\boldsymbol{\sigma}}_C. \tag{10.45}$$

We obtain thus a result well known in Newtonian mechanics:

> For an isolated system, the angular momentum vector relative to a barycentric observer does not depend on the point C with respect to which it is defined. We shall call $\vec{\boldsymbol{\sigma}}_C$ (independent of C) the ***spin vector of the system*** $\mathscr{S}$ and denote it by $\vec{\boldsymbol{\sigma}}$.

From (10.44), the angular momentum of $\mathscr{S}$ with respect to its centre of inertia G is

$$\boldsymbol{J}_G = \boldsymbol{S}, \tag{10.46}$$

where the 2-form $\boldsymbol{S}$ is defined by

$$\boxed{\boldsymbol{S} := \boldsymbol{\epsilon}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{\sigma}}, ., .)} \tag{10.47}$$

and is called the ***spin of the system*** $\mathscr{S}$.

10.5.4 König Theorem

In view of (10.47) and the independence of $\vec{\boldsymbol{\sigma}}_C$ from C, we may rewrite (10.44) as

$$\boxed{\boldsymbol{J}_C = \underbrace{\boldsymbol{S}}_{\text{spin}} + \underbrace{\underline{CG} \wedge \boldsymbol{p}}_{\substack{\text{orbital angular}\\ \text{momentum}}}}. \tag{10.48}$$

The term named ***orbital angular momentum*** encompasses all the dependence in C. Comparing with (10.1), we observe that it is identical to the angular momentum of a particle that would have the same worldline as the centre of inertia G and the same 4-momentum as the system. The decomposition (10.48) is the relativistic version of the famous ***König theorem***.

Since the total 4-momentum vector of the system $\mathscr{S}$, $\vec{\boldsymbol{p}}$, is collinear to the 4-velocity $\vec{\boldsymbol{u}}$ [by the very definition of the latter, cf. Eq. (9.42)], the alternate character of the Levi–Civita tensor leads to $\boldsymbol{\epsilon}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{\sigma}}, \vec{\boldsymbol{p}}, .) = 0$, i.e. by virtue of (10.47),

$$\boxed{\boldsymbol{S}(\vec{\boldsymbol{p}}, .) = 0}. \tag{10.49}$$

From (10.46), this relation is equivalent to

$$\boxed{\boldsymbol{J}_G(\vec{\boldsymbol{p}}, .) = 0}. \tag{10.50}$$

Remark 10.12. The centre of inertia G of an isolated system $\mathscr{S}$ has been defined as the centroid of the system with respect to the observers whose 4-velocity coincides with that of the system ($\vec{u}$). One can alternatively define the centre of inertia without invoking the notion of centroid by starting from (10.50): the worldline of the centre of inertia is formed by all the points $G \in \mathscr{E}$ such that the equality (10.50) is fulfilled. To show it, let us consider an inertial observer $\mathscr{O}$ comoving with $\mathscr{S}$, i.e. of 4-velocity $\vec{u}$. Let $O(t) \in \mathscr{E}$ be the position of that observer at the instant t of his proper time. From the change-of-origin formula (10.19), we have

$$\boldsymbol{J}_{O(t)} = \boldsymbol{J}_G + \underline{O(t)G} \wedge \boldsymbol{p},$$

so that property (10.50) is equivalent to

$$\boldsymbol{J}_{O(t)}(\vec{\boldsymbol{p}}, .) = [\overrightarrow{O(t)G} \cdot \vec{\boldsymbol{p}}]\, \boldsymbol{p} - \langle \boldsymbol{p}, \vec{\boldsymbol{p}} \rangle\, \underline{O(t)G}.$$

Writing $\vec{\boldsymbol{p}} = mc\, \vec{\boldsymbol{u}}$, we get

$$\boldsymbol{J}_{O(t)}(\vec{\boldsymbol{u}}, .) = mc \left[(\vec{\boldsymbol{u}} \cdot \overrightarrow{O(t)G})\, \underline{\boldsymbol{u}} + \underline{O(t)G} \right]. \tag{10.51}$$

Now from the very definition of $\mathscr{S}$'s angular momentum,

$$\boldsymbol{J}_{O(t)} = \sum_{a=1}^{N} \underline{O(t)M_a(t)} \wedge \boldsymbol{p}_a(t),$$

so that

$$\begin{aligned} \boldsymbol{J}_{O(t)}(\vec{\boldsymbol{u}}, .) &= \sum_{a=1}^{N} \underbrace{[\overrightarrow{O(t)M_a(t)} \cdot \vec{\boldsymbol{u}}]}_{0}\, \boldsymbol{p}_a(t) - \underbrace{\langle \boldsymbol{p}_a(t), \vec{\boldsymbol{u}} \rangle}_{-E_a/c}\, \underline{O(t)M_a(t)} \\ &= \frac{1}{c} \sum_{a=1}^{N} E_a\, \underline{O(t)M_a(t)}. \end{aligned}$$

Substituting in (10.51), we get, by metric duality

$$\frac{1}{mc^2} \sum_{a=1}^{N} E_a\, \overrightarrow{O(t)M_a(t)} = \perp_{\boldsymbol{u}} \overrightarrow{O(t)G}, \tag{10.52}$$

where the expression (3.11) of the orthogonal projector $\perp_{\boldsymbol{u}}$ onto $E_{\boldsymbol{u}}$ has been used. The total energy of the system measured by $\mathscr{O}$ being $E = mc^2$, we recognize in the left-hand side of the above equality the position vector of the system's centroid, $G_{\mathscr{O}}(t)$, in $\mathscr{E}_{\boldsymbol{u}}(t)$ [cf. Eq. (10.31)]. But since $\mathscr{O}$ is comoving, the centroid coincides with the centre of inertia of the system. Equation (10.52) shows that the point G is necessarily on the straight line going through $G_{\mathscr{O}}(t)$ and orthogonal to the hyperplane $\mathscr{E}_{\boldsymbol{u}}(t)$. This line being the worldline of the centre of inertia, this completes the demonstration that any point G satisfying (10.50) is the centre of inertia of the system at some instant.

10.5.5 Minimal Size of a System with Spin

Let $\mathscr{S}$ be an isolated system and $\mathscr{L}_G$ the worldline of its centre of inertia ($\mathscr{L}_G$ is necessarily a straight line of $\mathscr{E}$). Let $\vec{\boldsymbol{u}}$ be the 4-velocity of $\mathscr{S}$, as defined by (9.42), assuming that the total mass of $\mathscr{S}$ is nonvanishing. Let us denote by $\mathscr{O}_G$ a barycentric observer: his worldline is $\mathscr{L}_G$, and his 4-velocity is $\vec{\boldsymbol{u}}$. Let us consider as well an inertial observer $\mathscr{O}$ of 4-velocity $\vec{\boldsymbol{u}}_0$, proper time t, and position $O(t)$. Let $G(t)$ be the position with respect to $\mathscr{O}$ of the centre of inertia of $\mathscr{S}$ at the instant t, i.e. $G(t) := \mathscr{L}_G \cap \mathscr{E}_{\boldsymbol{u}_0}(t)$. Let $G_{\mathscr{O}}(t)$ be the centroid of the system $\mathscr{S}$ with respect to $\mathscr{O}$. Let us consider the linear form $\boldsymbol{S}(\vec{\boldsymbol{u}}_0, .)$, obtained by selecting $\mathscr{O}$'s 4-velocity as the first argument of the bilinear form $\boldsymbol{S}$, spin of the system $\mathscr{S}$. Given the expression (10.47) of $\boldsymbol{S}$ in terms of the spin vector $\vec{\boldsymbol{\sigma}}$ of $\mathscr{S}$, we have

$$\boldsymbol{S}(\vec{\boldsymbol{u}}_0, .) = \boldsymbol{\epsilon}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{\sigma}}, \vec{\boldsymbol{u}}_0, .). \tag{10.53}$$

Let $\vec{\boldsymbol{V}}_{\mathscr{O}}$ be the velocity of $\mathscr{O}$ relative to $\mathscr{O}_G$: $\vec{\boldsymbol{V}}_{\mathscr{O}}$ appears in the orthogonal decomposition of $\vec{\boldsymbol{u}}_0$ with respect to $\vec{\boldsymbol{u}}$ according to (4.31):

$$\vec{\boldsymbol{u}}_0 = \Gamma\left(\vec{\boldsymbol{u}} + \frac{1}{c}\vec{\boldsymbol{V}}_{\mathscr{O}}\right), \qquad \text{with} \quad \Gamma = \left(1 - \frac{1}{c^2}\vec{\boldsymbol{V}}_{\mathscr{O}} \cdot \vec{\boldsymbol{V}}_{\mathscr{O}}\right)^{-1/2}. \tag{10.54}$$

Plugging this expression into (10.53) and taking into account the alternate character of $\boldsymbol{\epsilon}$, we get

$$\boldsymbol{S}(\vec{\boldsymbol{u}}_0, .) = \frac{\Gamma}{c}\boldsymbol{\epsilon}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{\sigma}}, \vec{\boldsymbol{V}}_{\mathscr{O}}, .) = \frac{\Gamma}{c}\,\boldsymbol{g}(\vec{\boldsymbol{\sigma}} \times_u \vec{\boldsymbol{V}}_{\mathscr{O}}, .). \tag{10.55}$$

An alternative expression for $\boldsymbol{S}(\vec{\boldsymbol{u}}_0, .)$ can be derived from (10.46): for any value of t, $\boldsymbol{S} = \boldsymbol{J}_{G(t)}$. We deduce that $\boldsymbol{S}(\vec{\boldsymbol{u}}_0, .) = \boldsymbol{J}_{G(t)}(\vec{\boldsymbol{u}}_0, .)$. Let us express $\boldsymbol{J}_{G(t)}(\vec{\boldsymbol{u}}_0, .)$ via formula (10.35) with $C = G(t)$:

$$\boldsymbol{S}(\vec{\boldsymbol{u}}_0, .) = \boldsymbol{J}_{G(t)}(\vec{\boldsymbol{u}}_0, .) = c\underline{\boldsymbol{D}}(t) + \frac{E}{c}\,\underline{G(t)O(t)} - h(t)\boldsymbol{P}, \tag{10.56}$$

where $\underline{\boldsymbol{D}}(t)$, E and $\boldsymbol{P}$ are, respectively, the mass-energy dipole moment of $\mathscr{S}$, the total energy of $\mathscr{S}$ and the total linear momentum of $\mathscr{S}$, these three quantities being relative to $\mathscr{O}$. $h(t)$ is the component of the vector $\overrightarrow{G(t)O(t)}$ along $\vec{\boldsymbol{u}}_0$ [cf. Eq. (10.5)]: $h(t) := -\vec{\boldsymbol{u}}_0 \cdot \overrightarrow{G(t)O(t)}$. In the present case, $\overrightarrow{G(t)O(t)} \in E_{u_0}(t)$, so that $h(t) = 0$. Besides, $\vec{\boldsymbol{D}}$ is related to the centroid $G_{\mathscr{O}}(t)$ by (10.31): $\overrightarrow{O(t)G_{\mathscr{O}}(t)} = (c^2/E)\,\vec{\boldsymbol{D}}$. We have thus

$$\boldsymbol{S}(\vec{\boldsymbol{u}}_0, .) = \frac{E}{c}\, \underline{G(t)G_{\mathscr{O}}(t)}. \tag{10.57}$$

Equating (10.55) and (10.57), we get $\overrightarrow{G(t)G_{\mathscr{O}}(t)} = (\Gamma/E)\vec{\boldsymbol{\sigma}} \times_{\boldsymbol{u}} \vec{\boldsymbol{V}}_{\mathscr{O}}$. Now, given the relations $E = -c\,\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{u}}_0$ [Eq. (9.43)], $\vec{\boldsymbol{p}} = mc\,\vec{\boldsymbol{u}}$ [Eq. (9.42)] and $\Gamma = -\vec{\boldsymbol{u}}\cdot\vec{\boldsymbol{u}}_0$ [Eq. (10.54)], we have $E = \Gamma mc^2$, where m is the total mass of the system $\mathscr{S}$, as defined in Sect. 9.3.5. Hence, we can write

$$\boxed{\overrightarrow{G(t)G_{\mathscr{O}}(t)} = \frac{1}{mc^2}\, \vec{\boldsymbol{\sigma}} \times_{\boldsymbol{u}} \vec{\boldsymbol{V}}_{\mathscr{O}}}. \tag{10.58}$$

Remark 10.13. The vector $\overrightarrow{G(t)G_{\mathscr{O}}(t)}$ belongs to $\mathscr{O}$'s rest space, $E_{\boldsymbol{u}_0}$, and the vector $\vec{\boldsymbol{\sigma}} \times_{\boldsymbol{u}} \vec{\boldsymbol{V}}_{\mathscr{O}}$ to the rest space of the barycentric observer $\mathscr{O}_G$, $E_{\boldsymbol{u}}$. But since $\vec{\boldsymbol{\sigma}} \times_{\boldsymbol{u}} \vec{\boldsymbol{V}}_{\mathscr{O}}$ is orthogonal to $\vec{\boldsymbol{V}}_{\mathscr{O}}$ (by definition of $\times_{\boldsymbol{u}}$) and $\vec{\boldsymbol{V}}_{\mathscr{O}}$ is $\mathscr{O}$'s velocity relative to $\mathscr{O}_G$, we have actually $\vec{\boldsymbol{\sigma}} \times_{\boldsymbol{u}} \vec{\boldsymbol{V}}_{\mathscr{O}} \in E_{\boldsymbol{u}} \cap E_{\boldsymbol{u}_0}$, so that the equality (10.58) is possible.

A first interesting property can be drawn from (10.58):

> The centroids of an isolated system with respect to all possible observers coincide with the system's centre of inertia iff the spin of the system vanishes.

Proof. If $G_{\mathscr{O}} = G$ for any observer $\mathscr{O}$, then (10.58) implies $\vec{\boldsymbol{\sigma}} = 0$, which, via (10.47), leads to $\boldsymbol{S} = 0$. The converse is straightforward. □

A second consequence of (10.58) regards the size of the system. Denoting by θ the angle between vectors $\vec{\boldsymbol{\sigma}}$ and $\vec{\boldsymbol{V}}_{\mathscr{O}}$ in $E_{\boldsymbol{u}}$ (cf. Sect. 3.2.6), formula (10.58) leads to

$$\left\|\overrightarrow{G(t)G_{\mathscr{O}}(t)}\right\|_{\boldsymbol{g}} = \frac{1}{mc^2}\, \|\vec{\boldsymbol{\sigma}}\|_{\boldsymbol{g}} \left\|\vec{\boldsymbol{V}}_{\mathscr{O}}\right\|_{\boldsymbol{g}} |\sin\theta|.$$

Since the barycentric observer is inertial, one has always $\|\vec{\boldsymbol{V}}_{\mathscr{O}}\|_{\boldsymbol{g}} < c$, so that

$$\left\|\overrightarrow{G(t)G_{\mathscr{O}}(t)}\right\|_{\boldsymbol{g}} < R_0, \qquad \text{with} \quad R_0 := \frac{1}{mc}\, \|\vec{\boldsymbol{\sigma}}\|_{\boldsymbol{g}}\,. \tag{10.59}$$

Let us assume that $\vec{\boldsymbol{\sigma}} \neq 0$ and consider, in the rest space of the barycentric observer, the disk $\mathscr{D}$, centred on the centre of inertia, of radius R_0 and perpendicular to $\vec{\boldsymbol{\sigma}}$:

$$\mathscr{D}(t_G) := \left\{ A \in \mathscr{E}_{\boldsymbol{u}}(t_G), \quad \vec{\boldsymbol{\sigma}}\cdot\overrightarrow{G(t_G)A} = 0 \quad \text{and} \quad \left\|\overrightarrow{G(t_G)A}\right\|_{\boldsymbol{g}} < R_0 \right\},$$

where t_G stands for the proper time of the barycentric observer, with $G(t_G) = G(t)$. From (10.58), it is clear that any centroid is located in that disk. Conversely, if $A \in \mathscr{D}(t_G)$, there exists a unique vector $\vec{V}_\perp \in E_{\boldsymbol{u}} \cap \mathrm{Span}(\vec{\boldsymbol{\sigma}})^\perp$ such that

$$\overrightarrow{G(t_G)A} = \frac{1}{mc^2}\,\vec{\boldsymbol{\sigma}} \times_{\boldsymbol{u}} \vec{V}_\perp. \tag{10.60}$$

Moreover, since $A \in \mathscr{D}(t_G)$, we have $\|\vec{V}_\perp\|_{\boldsymbol{g}} < c$. Then for any $V_\parallel \in \mathbb{R}$ obeying $V_\parallel^2 < c^2 - \vec{V}_\perp \cdot \vec{V}_\perp$, the vector

$$\vec{V} = V_\parallel\,\vec{\boldsymbol{\sigma}} + \vec{V}_\perp \tag{10.61}$$

constitutes the velocity vector $\vec{V}_{\mathscr{O}}$ of an observer $\mathscr{O}$ relative to $\mathscr{O}_G$. The centroid of the system $\mathscr{S}$ for this observer satisfies then $\overrightarrow{G(t)G_{\mathscr{O}}(t)} = (mc^2)^{-1}\vec{\boldsymbol{\sigma}} \times_{\boldsymbol{u}} \vec{V}_{\mathscr{O}} = (mc^2)^{-1}\vec{\boldsymbol{\sigma}} \times_{\boldsymbol{u}} \vec{V}_\perp = \overrightarrow{G(t)A}$, hence $A = G_{\mathscr{O}}(t)$. In other words, any point of the disk $\mathscr{D}(t_G)$ corresponds to the centroid of the system for an infinite number of inertial observers (who differ only by the value of $V_\parallel$).

When t_G varies, $\mathscr{D}(t_G)$ spans a cylinder in Minkowski spacetime, centred on the line $\mathscr{L}_G$. We shall call this cylinder the ***tube of centroids*** of system $\mathscr{S}$.

Let us suppose that for the barycentric observer $\mathscr{O}_G$, all the particles forming the system $\mathscr{S}$ are, at each instant of the proper time t_G, contained within a ball $\mathscr{B}_R(t_G)$ of $\mathscr{E}_{\boldsymbol{u}}(t_G)$ centred on $G(t_G)$ and of rayon R, independent of t_G. When t_G varies, this ball spans a worldtube $\mathscr{T}$ of axis $\mathscr{L}_G$. This is a four-dimensional tube, contrary to the tube of centroids, which is three-dimensional. Let us show that the disk of centroids, $\mathscr{D}(t_G)$, is totally included in the ball $\mathscr{B}_R(t_G)$. Let $G_{\mathscr{O}}$ be a point of $\mathscr{D}(t_G)$ and $\mathscr{O}$ an inertial observer such that the centroid of $\mathscr{S}$ with respect to $\mathscr{O}$ is $G_{\mathscr{O}}$. At some instant t of $\mathscr{O}$'s proper time, let us consider the volume of $\mathscr{O}$'s rest space $\mathscr{E}_{\boldsymbol{u}_0}(t)$ occupied by the system: $\tilde{\mathscr{B}}(t) := \mathscr{T} \cap \mathscr{E}_{\boldsymbol{u}_0}(t)$. The positions $M_a(t)$ of the system particles at the instant t (i.e. the intersections of their worldlines with $\mathscr{E}_{\boldsymbol{u}_0}(t)$) are all contained in $\tilde{\mathscr{B}}(t)$. Now the centroid $G_{\mathscr{O}}(t)$ is, by definition, the barycentre of the positions $M_a(t)$ weighted by the energies E_a of particles with respect to $\mathscr{O}$ [cf. (10.31)]. Since $\mathscr{O}$ is inertial, these energies are all positive ($E_a = \Gamma_a m_a c^2$), so that the centroid is located inside the system, i.e. inside $\tilde{\mathscr{B}}(t)$. The worldline of the centroid being a straight having $\vec{\boldsymbol{u}}$ as a direction vector, as the axis of the tube $\mathscr{T}$, we obtain that at each instant of the proper time t_G, $G_{\mathscr{O}}(t_G)$ is within the ball $\mathscr{B}_R(t_G)$. We have thus shown that $\mathscr{D}(t_G) \subset \mathscr{B}_R(t_G)$. The radius of the disk $\mathscr{D}(t_G)$ being R_0, we conclude that the radius R of the ball $\mathscr{B}_R(t_G)$ that gives the size of the system for the barycentric observer must satisfy $R \geq R_0$, i.e. given (10.59),

$$\boxed{R \geq \frac{1}{mc}\,\|\vec{\boldsymbol{\sigma}}\|_{\boldsymbol{g}}}. \tag{10.62}$$

Hence, the norm of the spin vector provides a lower bound of the size of a particle system. In other words, a system with spin cannot have an arbitrary small size.

Remark 10.14. This results can be understood intuitively by observing that if one would like to maintain a finite angular momentum with respect to the centre of inertia (i.e. a finite spin) while reducing the system size, the particles have to "rotate" at larger and larger velocities. The speed of light being an upper bound on the particle velocities, we realize that the system size cannot be arbitrarily reduced.

Historical note: *The definition (10.31) of the centroid of a system, as well as that of the centre of inertia, has been introduced in 1929 by Adriaan D. Fokker*[3] *(1929a) (cf. also Pryce (1948)), who used the term* invariant centre of mass *for what we call* centre of inertia*. The alternative definition of the centre of inertia, based on the identity $\boldsymbol{J}_G(\vec{\boldsymbol{p}}, .) = 0$ [Eq. (10.50)], has been given in 1935 by John L. Synge (cf. p. 74) (1935). The result (10.62), regarding the minimal size of a system with spin, is due to Christian Møller*[4] *(1949a; 1949b).*

10.6 Angular Momentum Evolution

10.6.1 Four-Torque

As in Sect. 9.5, let us consider a particle $\mathscr{P}$ of mass $m > 0$, worldline $\mathscr{L}$, 4-velocity $\vec{\boldsymbol{u}}$ and proper time τ. We have seen in Sect. 10.4 that, if $\mathscr{P}$ is isolated, its angular momentum $\boldsymbol{J}_C$ with respect to any point $C \in \mathscr{E}$ is a constant field of bilinear forms along $\mathscr{L}$. If $\mathscr{P}$ is not isolated, we shall define the derivative of $\boldsymbol{J}_C$ along $\mathscr{L}$ as the ***four-torque***, or ***4-torque*** for short, ***with respect to the point C and acting on particle*** $\mathscr{P}$:

$$\boxed{\boldsymbol{N}_C := \frac{\mathrm{d}\boldsymbol{J}_C}{\mathrm{d}\tau}}. \tag{10.63}$$

As $\boldsymbol{J}_C$, $\boldsymbol{N}_C$ is a field of antisymmetric bilinear forms (2-forms) defined along $\mathscr{L}$.

Replacing $\boldsymbol{J}_C$ by its expression (10.1), we get

$$\boldsymbol{N}_C = \frac{\mathrm{d}\underline{CM}}{\mathrm{d}\tau} \wedge \boldsymbol{p} + \underline{CM} \wedge \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}\tau}.$$

[3]**Adriaan D. Fokker** (1887–1972): Dutch physicist and musician, cousin of the aircraft manufacturer Anthony Fokker. He is mostly known for the *Fokker–Planck equation*, which is involved in the study of Brownian motion. Fokker has also designed a new type of organ, known today as the *Fokker organ*.

[4]**Christian Møller** (1904–1980): Danish physicist, who contributed to relativity and particle physics; he authored in 1952 a textbook about special and general relativity (Møller 1952), which remained famous for many years.

Now, by definition of $\mathscr{P}$'s 4-velocity and of the 4-force $\boldsymbol{f}$ acting of $\mathscr{P}$ [Eq. (9.104)],

$$\frac{\mathrm{d}\underline{CM}}{\mathrm{d}\tau} = c\,\underline{\boldsymbol{u}} \qquad \text{and} \qquad \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}\tau} = \boldsymbol{f}\,.$$

On the other side, $\boldsymbol{p} = mc\,\underline{\boldsymbol{u}}$ [Eq. (9.3)], so that $\underline{\boldsymbol{u}} \wedge \boldsymbol{p} = 0$. Hence,

$$\boxed{\boldsymbol{N}_C = \underline{CM} \wedge \boldsymbol{f}}\,. \tag{10.64}$$

10.6.2 Evolution of the Angular Momentum Vector

Let us derive an evolution law for the angular momentum vector $\vec{\boldsymbol{\sigma}}_C$ of particle $\mathscr{P}$ with respect to a point C and measured by to an observer $\mathscr{O}$. We denote by t the proper time of $\mathscr{O}$ and by $\vec{\boldsymbol{u}}_0$ his 4-velocity. Let us recall that $\vec{\boldsymbol{\sigma}}_C = \vec{\boldsymbol{\sigma}}_C(t) \in E_{\boldsymbol{u}_0}(t)$, where $E_{\boldsymbol{u}_0}(t)$ is $\mathscr{O}$'s local rest space at the proper time t. We would like to evaluate $\mathrm{d}\vec{\boldsymbol{\sigma}}_C/\mathrm{d}t$. Two cases can be distinguished: (i) C is a fixed point in spacetime $\mathscr{E}$ and (ii) C evolves with t; we shall suppose then that $C(t)$ follows a timelike worldline in $\mathscr{E}$. In practice, the latter is the most interesting case; this is the one usually considered in Newtonian mechanics. One deals often with the particular case where $C(t)$ is fixed with respect to $\mathscr{O}$ (i.e. has constant coordinates in $\mathscr{O}$'s reference space; cf. Sect. 3.4.3), with, as a subcase, $C(t) = O(t)$, origin of $\mathscr{O}$'s local coordinates. We shall focus on the case (ii). The computation starting point is relation (10.11), which expresses $\vec{\boldsymbol{\sigma}}_C$ as the moment of the particle's linear momentum $\vec{\boldsymbol{P}}(t)$ measured by $\mathscr{O}$ with respect to the point C:

$$\vec{\boldsymbol{\sigma}}_C = \overrightarrow{CM} \times_{\boldsymbol{u}_0} \vec{\boldsymbol{P}} = \vec{\boldsymbol{\epsilon}}\left(\vec{\boldsymbol{u}}_0(t), \overrightarrow{C(t)M(t)}, \vec{\boldsymbol{P}}(t), .\right).$$

Using the multilinearity of Levi–Civita tensor, we obtain

$$\frac{\mathrm{d}\vec{\boldsymbol{\sigma}}_C}{\mathrm{d}t} = \vec{\boldsymbol{\epsilon}}\left(\frac{\mathrm{d}\vec{\boldsymbol{u}}_0}{\mathrm{d}t}, \overrightarrow{CM}, \vec{\boldsymbol{P}}, .\right) + \vec{\boldsymbol{\epsilon}}\left(\vec{\boldsymbol{u}}_0, \frac{\mathrm{d}\overrightarrow{CM}}{\mathrm{d}t}, \vec{\boldsymbol{P}}, .\right) + \vec{\boldsymbol{\epsilon}}\left(\vec{\boldsymbol{u}}_0, \overrightarrow{CM}, \frac{\mathrm{d}\vec{\boldsymbol{P}}}{\mathrm{d}t}, .\right). \tag{10.65}$$

Let us evaluate separately each of the three terms. For the first one, the definition of $\mathscr{O}$'s 4-acceleration $\vec{\boldsymbol{a}}_0$ enables us to write

$$\vec{\boldsymbol{\epsilon}}\left(\frac{\mathrm{d}\vec{\boldsymbol{u}}_0}{\mathrm{d}t}, \overrightarrow{CM}, \vec{\boldsymbol{P}}, .\right) = c\,\vec{\boldsymbol{\epsilon}}\left(\vec{\boldsymbol{a}}_0, \overrightarrow{CM}, \vec{\boldsymbol{P}}, .\right).$$

$\vec{\boldsymbol{a}}_0$, $\overrightarrow{CM}$ and $\vec{\boldsymbol{P}}$ being three vectors of $E_{\boldsymbol{u}_0}(t)$, the same computation as that yielding (4.52) in Chap. 4 leads to

$$\vec{\boldsymbol{\epsilon}}\left(\frac{\mathrm{d}\vec{\boldsymbol{u}}_0}{\mathrm{d}t}, \overrightarrow{CM}, \vec{\boldsymbol{P}}, .\right) = c\left[\vec{\boldsymbol{a}}_0 \cdot \left(\overrightarrow{CM} \times_{\boldsymbol{u}_0} \vec{\boldsymbol{P}}\right)\right] \vec{\boldsymbol{u}}_0 = c\,(\vec{\boldsymbol{a}}_0 \cdot \vec{\boldsymbol{\sigma}}_C)\,\vec{\boldsymbol{u}}_0. \tag{10.66}$$

In order to evaluate the second term in the right-hand side of (10.65), let us write

$$\frac{\mathrm{d}\overrightarrow{CM}}{\mathrm{d}t} = \frac{\mathrm{d}\overrightarrow{CO}}{\mathrm{d}t} + \frac{\mathrm{d}\overrightarrow{OM}}{\mathrm{d}t} = -\frac{\mathrm{d}\overrightarrow{OC}}{\mathrm{d}t} + \frac{\mathrm{d}\overrightarrow{OM}}{\mathrm{d}t}$$

and use relation (4.24) for the points $M(t)$ (particle $\mathscr{P}$) and $C(t)$ (point with respect to which the angular momentum is considered):

$$\frac{\mathrm{d}\overrightarrow{CM}}{\mathrm{d}t} = -\vec{\boldsymbol{V}}_C - \vec{\boldsymbol{\omega}} \times_{\boldsymbol{u}_0} \overrightarrow{OC} - c(\vec{\boldsymbol{a}}_0 \cdot \overrightarrow{OC})\,\vec{\boldsymbol{u}}_0 + \vec{\boldsymbol{V}} + \vec{\boldsymbol{\omega}} \times_{\boldsymbol{u}_0} \overrightarrow{OM} + c(\vec{\boldsymbol{a}}_0 \cdot \overrightarrow{OM})\,\vec{\boldsymbol{u}}_0,$$

where $\vec{\boldsymbol{V}}$ (resp. $\vec{\boldsymbol{V}}_C$) is the velocity of $\mathscr{P}$ (resp. of point $C(t)$) relative to observer $\mathscr{O}$ and $\vec{\boldsymbol{\omega}}$ is $\mathscr{O}$'s 4-rotation. We have thus

$$\vec{\boldsymbol{\epsilon}}\left(\vec{\boldsymbol{u}}_0, \frac{\mathrm{d}\overrightarrow{CM}}{\mathrm{d}t}, \vec{\boldsymbol{P}}, .\right) = \vec{\boldsymbol{\epsilon}}(\vec{\boldsymbol{u}}_0, \vec{\boldsymbol{V}} + \vec{\boldsymbol{\omega}} \times_{\boldsymbol{u}_0} \overrightarrow{OM}, \vec{\boldsymbol{P}}, .) - \vec{\boldsymbol{\epsilon}}(\vec{\boldsymbol{u}}_0, \vec{\boldsymbol{V}}_C + \vec{\boldsymbol{\omega}} \times_{\boldsymbol{u}_0} \overrightarrow{OC}, \vec{\boldsymbol{P}}, .).$$

Now, from (9.27), the vectors $\vec{\boldsymbol{V}} + \vec{\boldsymbol{\omega}} \times_{\boldsymbol{u}_0} \overrightarrow{OM}$ and $\vec{\boldsymbol{P}}$ are collinear. The first term of the right-hand side of the above equation thus vanishes (antisymmetry of $\boldsymbol{\epsilon}$), and there remains only the second one, which can be written as a cross product:

$$\vec{\boldsymbol{\epsilon}}\left(\vec{\boldsymbol{u}}_0, \frac{\mathrm{d}\overrightarrow{CM}}{\mathrm{d}t}, \vec{\boldsymbol{P}}, .\right) = \vec{\boldsymbol{P}} \times_{\boldsymbol{u}_0} \left(\vec{\boldsymbol{V}}_C + \vec{\boldsymbol{\omega}} \times_{\boldsymbol{u}_0} \overrightarrow{OC}\right). \tag{10.67}$$

Regarding the third term in the right-hand side of (10.65), let us write $\mathrm{d}\vec{\boldsymbol{P}}/\mathrm{d}t = \boldsymbol{D}^{\mathrm{FW}}_{\boldsymbol{u}_0}\vec{\boldsymbol{P}} + c(\vec{\boldsymbol{a}}_0 \cdot \vec{\boldsymbol{P}})\,\vec{\boldsymbol{u}}_0$ and use expression (9.118) of $\boldsymbol{D}^{\mathrm{FW}}_{\boldsymbol{u}_0}\vec{\boldsymbol{P}}$ to get

$$\begin{aligned}
\vec{\boldsymbol{\epsilon}}\left(\vec{\boldsymbol{u}}_0, \overrightarrow{CM}, \frac{\mathrm{d}\vec{\boldsymbol{P}}}{\mathrm{d}t}, .\right) &= \vec{\boldsymbol{\epsilon}}\left(\vec{\boldsymbol{u}}_0, \overrightarrow{CM}, \vec{\boldsymbol{F}}_{\mathrm{ext}} - E\,\vec{\boldsymbol{a}}_0 + c\langle \boldsymbol{P}, \vec{\boldsymbol{a}}_0\rangle\,\vec{\boldsymbol{u}}_0, .\right) \\
&= \vec{\boldsymbol{\epsilon}}\left(\vec{\boldsymbol{u}}_0, \overrightarrow{CM}, \vec{\boldsymbol{F}}_{\mathrm{ext}} - E\,\vec{\boldsymbol{a}}_0, .\right) \\
&= \overrightarrow{CM} \times_{\boldsymbol{u}_0} (\vec{\boldsymbol{F}}_{\mathrm{ext}} - E\,\vec{\boldsymbol{a}}_0).
\end{aligned} \tag{10.68}$$

By plugging (10.66), (10.67) and (10.68) into (10.65), we obtain the expression of the time derivative of the angular momentum vector:

$$\frac{\mathrm{d}\vec{\boldsymbol{\sigma}}_C}{\mathrm{d}t} = c\,(\vec{\boldsymbol{a}}_0 \cdot \vec{\boldsymbol{\sigma}}_C)\,\vec{\boldsymbol{u}}_0 + \vec{\boldsymbol{P}} \times_{\boldsymbol{u}_0} \left(\vec{\boldsymbol{V}}_C + \vec{\boldsymbol{\omega}} \times_{\boldsymbol{u}_0} \overrightarrow{OC}\right) + \overrightarrow{CM} \times_{\boldsymbol{u}_0} (\vec{\boldsymbol{F}}_{\mathrm{ext}} - E\,\vec{\boldsymbol{a}}_0).$$

We recognize in $\mathrm{d}\vec{\boldsymbol{\sigma}}_C/\mathrm{d}t - c\,(\vec{\boldsymbol{a}}_0 \cdot \vec{\boldsymbol{\sigma}}_C)\,\vec{\boldsymbol{u}}_0$ the Fermi–Walker derivative of $\vec{\boldsymbol{\sigma}}_C$ along the worldline of observer $\mathscr{O}$ [cf. Eq. (3.69) with $\vec{\boldsymbol{u}}_0 \cdot \vec{\boldsymbol{\sigma}}_C = 0$], so that the above expression can be recast as

$$\boxed{\boldsymbol{D}^{\mathrm{FW}}_{\boldsymbol{u}_0}\vec{\boldsymbol{\sigma}}_C = \overrightarrow{CM} \times_{\boldsymbol{u}_0} (\vec{\boldsymbol{F}}_{\mathrm{ext}} - E\,\vec{\boldsymbol{a}}_0) + \vec{\boldsymbol{P}} \times_{\boldsymbol{u}_0} \left(\vec{\boldsymbol{V}}_C + \vec{\boldsymbol{\omega}} \times_{\boldsymbol{u}_0} \overrightarrow{OC}\right)}. \tag{10.69}$$

The term $\overrightarrow{CM} \times_{\boldsymbol{u}_0} (\vec{\boldsymbol{F}}_{\mathrm{ext}} - E\,\vec{\boldsymbol{a}}_0)$ is called ***torque exerted on the particle $\mathscr{P}$ with respect to the point C and with respect to observer $\mathscr{O}$.***

Remark 10.15. The right-hand side of (10.69) is clearly a vector orthogonal to $\vec{\boldsymbol{u}}_0$; in the left-hand side, contrary to $\mathrm{d}/\mathrm{d}t$, the Fermi–Walker derivative ensures that $\vec{\boldsymbol{\sigma}}_C$ will remain orthogonal to $\vec{\boldsymbol{u}}_0$.

Remark 10.16. If $\mathscr{O}$ is an inertial observer [$\boldsymbol{D}^{\mathrm{FW}}_{\boldsymbol{u}_0} = \mathrm{d}/\mathrm{d}t$, $\vec{\boldsymbol{a}}_0 = 0$ $\vec{\boldsymbol{\omega}} = 0$ and $\vec{\boldsymbol{F}} = \vec{\boldsymbol{F}}_{\mathrm{ext}}$; cf. Eq. (9.117)] and, moreover, if C is fixed with respect to $\mathscr{O}$ ($\vec{\boldsymbol{V}}_C = 0$), formula (10.69) reduces to

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\vec{\boldsymbol{\sigma}}_C = \overrightarrow{CM} \times_{\boldsymbol{u}_0} \vec{\boldsymbol{F}}.}_{\ \mathscr{O}\ \text{inertial and } C \text{ fixed}} \tag{10.70}$$

In particular, in the absence of any force ($\vec{\boldsymbol{F}} = 0$), we recover the conservation law (10.30).

Remark 10.17. If we had considered the point C as fixed in spacetime [case (i) discussed at the beginning of this section], we would have obtained the formula

$$\boldsymbol{D}^{\mathrm{FW}}_{\boldsymbol{u}_0}\vec{\boldsymbol{\sigma}}_C = \overrightarrow{CM} \times_{\boldsymbol{u}_0} (\vec{\boldsymbol{F}}_{\mathrm{ext}} - E\,\vec{\boldsymbol{a}}_0) + c\,h(t)\vec{\boldsymbol{P}} \times_{\boldsymbol{u}_0} \vec{\boldsymbol{a}}_0, \tag{10.71}$$

instead of (10.69).

10.7 Particle with Spin

10.7.1 Definition

The concept of a particle with spin is fundamentally a quantum notion (Le Bellac 2006; Penrose 2007). We have defined the spin $\boldsymbol{S}$ of an isolated system as the angular momentum with respect to its centre of inertia [cf. Eq. (10.46)]. However, if the system is reduced to a single particle, the comparison of Eq. (10.1) with $M = G$ and Eq. (10.48) leads to $\boldsymbol{S} = 0$. Another argument against the concept of "classical" (i.e. non-quantum) spin relies on the existence of a minimal size for any system with nonvanishing spin, as we have seen in Sect. 10.5.5: one can therefore not reduce the size of the system to zero to take the "particle limit".

Nevertheless, one can extend the concept of particle beyond the simple particle model introduced in Chap. 9 to include the spin. We define formally a ***particle with spin*** as the following data:

1. A worldline $\mathscr{L} \subset \mathscr{E}$, either timelike or null
2. A field of linear forms $\boldsymbol{p}$ defined along $\mathscr{L}$ such that the vector $\vec{\boldsymbol{p}}(M)$ is tangent to $\mathscr{L}$ at any point $M \in \mathscr{L}$
3. A field of antisymmetric bilinear forms (2-forms) $\boldsymbol{S}$ defined along $\mathscr{L}$ such that

$$\boxed{\boldsymbol{S}(\vec{\boldsymbol{p}}, .) = 0} \tag{10.72}$$

The 2-form $\boldsymbol{S}$ is called ***spin of the particle***. Items 1 and 2 are those already considered in the definition of a simple particle in Sect. 9.2.1. The extension is thus constituted by item 3. The relation (10.72) is motivated by the result (10.49) obtained for a system of particles.

Remark 10.18. This model of particle with spin is that considered by John L. Synge (cf. p. 74) in 1956 (Synge 1956). There exist other models, where the 4-momentum vector is not supposed to be tangent to the worldline (Corben 1968).

In the case where $\mathscr{P}$ is a particle with spin of nonzero mass, it is natural to introduce its 4-velocity $\vec{\boldsymbol{u}}$ and to decompose the 2-form $\boldsymbol{S}$ with respect to $\vec{\boldsymbol{u}}$ following (3.37) :

$$\boldsymbol{S} = \boldsymbol{\epsilon}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{s}}, ., .) + \underline{\boldsymbol{u}} \wedge \boldsymbol{q}, \tag{10.73}$$

where $\vec{\boldsymbol{s}}$ is a vector orthogonal to $\vec{\boldsymbol{u}}$: $\vec{\boldsymbol{s}} \in E_{\boldsymbol{u}}$ and $\boldsymbol{q}$ is a linear form such that $\langle \boldsymbol{q}, \vec{\boldsymbol{u}} \rangle = 0$. In the present case, the constraint (10.72) implies $\boldsymbol{q} = 0$. Indeed, the vector $\vec{\boldsymbol{p}}$ is collinear to $\vec{\boldsymbol{u}}$ ($\vec{\boldsymbol{p}} = mc\, \vec{\boldsymbol{u}}$), so that (10.73) and the alternate character of the Levi–Civita tensor lead to

$$\boldsymbol{S}(\vec{\boldsymbol{p}}, .) = mc \underbrace{\boldsymbol{\epsilon}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{s}}, \vec{\boldsymbol{u}}, .)}_{0} + mc \underbrace{\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{u}}}_{-1}\, \boldsymbol{q} - mc \underbrace{\langle \boldsymbol{q}, \vec{\boldsymbol{u}} \rangle}_{0} \underline{\boldsymbol{u}} = -mc\, \boldsymbol{q}.$$

The condition (10.72) implies then immediately $\boldsymbol{q} = 0$. Consequently, the decomposition (10.73) reduces to

$$\boxed{\boldsymbol{S} = \boldsymbol{\epsilon}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{s}}, ., .)} \quad \text{with} \quad \boxed{\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{s}} = 0}. \tag{10.74}$$

The vector $\vec{\boldsymbol{s}} \in E_{\boldsymbol{u}}$, which is unique for a given 2-form $\boldsymbol{S}$, is called the ***spin vector*** of particle $\mathscr{P}$. The relation (10.74) shows that $\vec{\boldsymbol{s}}$ fully determines the spin 2-form $\boldsymbol{S}$. In other words, we could have replaced item 3 in the above definition of a particle with spin by "a vector field $\vec{\boldsymbol{s}}$ defined along $\mathscr{L}$ and orthogonal to $\vec{\boldsymbol{p}}$" (cf. Fig. 10.2).

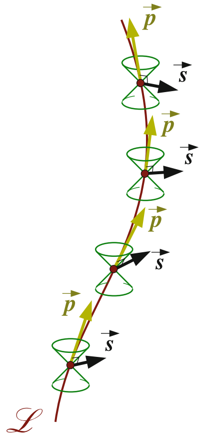

Fig. 10.2 Particle with spin: the spin vector $\vec{s}$ is orthogonal to the 4-momentum vector $\vec{p}$ at any point of the particle's worldline. In particular, $\vec{s}$ is a spacelike vector

The ***angular momentum of a particle with spin with respect to a point*** $C \in \mathscr{E}$ is defined by

$$\forall M \in \mathscr{L}, \quad \boxed{\boldsymbol{J}_C(M) := \boldsymbol{S}(M) + \underline{CM} \wedge \boldsymbol{p}(M)}. \tag{10.75}$$

This formula generalizes (10.1) and has exactly the same structure as the decomposition of the angular momentum of a system provided by König theorem [Eq. (10.48)].

Historical note: *The notion of a (classical) particle with spin has been introduced by Yacov I. Frenkel*[5] *in 1926 (Frenkel 1926). Frenkel considered the 2-form* $\boldsymbol{S}$*. On the other hand, the spin 4-vector* $\vec{s}$ *has been introduced by Igor I. Tamm*[6] *in 1929 (Tamm 1929). An important contribution has been provided by Myron Mathisson*[7] *who obtained in 1937 (Mathisson 1937) the equations of motion of a particle with spin from the multipole expansion of an extended body.*

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[5] **Yacov Ilich Frenkel** (1894–1952): Soviet physicist, who worked in solid-state physics (*Frenkel defects* in crystals), physics of liquid and semiconductors.

[6] **Igor Yevgenyevich Tamm** (1895–1971): Soviet physicist, 1958 Nobel Prize in Physics for the discovery and interpretation of the Cherenkov effect; coinventor of the tokamak for the controlled thermonuclear fusion.

[7] **Myron Mathisson** (1897–1940): Polish theoretical physicist, who made important contributions to the problem of motion in general relativity; he was corresponding (in French) with Albert Einstein. His career was brief, for he died from tuberculosis at 43 years old. He impressed so much the mathematician Jacques Hadamard that the latter published an article to his memory in 1942.

10.7.2 Spin Evolution

The derivative of the angular momentum $\boldsymbol{J}_C$ with respect to the proper time τ of particle $\mathscr{P}$ defines the 4-couple $\boldsymbol{N}_C$ acting on $\mathscr{P}$ [cf. Eq. (10.63)]. By deriving (10.75), we get thus

$$\frac{\mathrm{d}\boldsymbol{S}}{\mathrm{d}\tau} + \underline{CM} \wedge \boldsymbol{f} = \boldsymbol{N}_C, \tag{10.76}$$

where $\boldsymbol{f} = \mathrm{d}\boldsymbol{p}/\mathrm{d}\tau$ is the 4-force acting on the particle, and we have used $d\underline{CM}/\mathrm{d}\tau = c\,\underline{\boldsymbol{u}}$ and $\boldsymbol{p} = mc\,\underline{\boldsymbol{u}}$ to write $d\underline{CM}/\mathrm{d}\tau \wedge \boldsymbol{p} = 0$. In view of (10.76), it is natural to split $\boldsymbol{N}_C$ in two parts: $\boldsymbol{N}_C = \boldsymbol{N}_{\mathrm{spin}} + \boldsymbol{N}_C^{\mathrm{orb}}$, such that

$$\frac{\mathrm{d}\boldsymbol{S}}{\mathrm{d}\tau} = \boldsymbol{N}_{\mathrm{spin}} \qquad \text{and} \qquad \boldsymbol{N}_C^{\mathrm{orb}} = \underline{CM} \wedge \boldsymbol{f}. \tag{10.77}$$

We shall call $\boldsymbol{N}_{\mathrm{spin}}$ the ***four-torque on the spin*** and $\boldsymbol{N}_C^{\mathrm{orb}}$ the ***orbital four-torque***. Note that $\boldsymbol{N}_{\mathrm{spin}}$ is independent of the point C.

Let us derive from (10.77) an evolution law for the spin vector $\vec{\boldsymbol{s}}$. Using (10.74) and the multilinearity of the Levi–Civita tensor, we get

$$\frac{\mathrm{d}\boldsymbol{S}}{\mathrm{d}\tau} = \boldsymbol{\epsilon}\left(\frac{\mathrm{d}\vec{\boldsymbol{u}}}{\mathrm{d}\tau}, \vec{\boldsymbol{s}}, ., .\right) + \boldsymbol{\epsilon}\left(\vec{\boldsymbol{u}}, \frac{\mathrm{d}\vec{\boldsymbol{s}}}{\mathrm{d}\tau}, ., .\right) = c\,\boldsymbol{\epsilon}\,(\vec{\boldsymbol{a}}, \vec{\boldsymbol{s}}, ., .) + \boldsymbol{\epsilon}\left(\vec{\boldsymbol{u}}, \frac{\mathrm{d}\vec{\boldsymbol{s}}}{\mathrm{d}\tau}, ., .\right),$$

where $\vec{\boldsymbol{a}} = c^{-1}\mathrm{d}\vec{\boldsymbol{u}}/\mathrm{d}\tau$ is the particle's 4-acceleration. The first of equations (10.77) becomes thus

$$c\,\boldsymbol{\epsilon}\,(\vec{\boldsymbol{a}}, \vec{\boldsymbol{s}}, ., .) + \boldsymbol{\epsilon}\left(\vec{\boldsymbol{u}}, \frac{\mathrm{d}\vec{\boldsymbol{s}}}{\mathrm{d}\tau}, ., .\right) = \boldsymbol{N}_{\mathrm{spin}}. \tag{10.78}$$

Now, $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{s}}$ being two vectors de $E_{\boldsymbol{u}}$, the following property holds, equivalent to the identity (4.52) in Chap. 4:

$$\forall \vec{\boldsymbol{v}} \in E_{\boldsymbol{u}}, \quad \boldsymbol{\epsilon}(\vec{\boldsymbol{a}}, \vec{\boldsymbol{s}}, \vec{\boldsymbol{v}}, .) = [\vec{\boldsymbol{a}} \cdot (\vec{\boldsymbol{s}} \times_{\boldsymbol{u}} \vec{\boldsymbol{v}})]\,\underline{\boldsymbol{u}} = [\vec{\boldsymbol{v}} \cdot (\vec{\boldsymbol{a}} \times_{\boldsymbol{u}} \vec{\boldsymbol{s}})]\,\underline{\boldsymbol{u}} = \langle \underline{\vec{\boldsymbol{a}} \times_{\boldsymbol{u}} \vec{\boldsymbol{s}}}, \vec{\boldsymbol{v}} \rangle\,\underline{\boldsymbol{u}}.$$

By antisymmetry, we deduce that $\boldsymbol{\epsilon}(\vec{\boldsymbol{a}}, \vec{\boldsymbol{s}}, ., .) = (\underline{\vec{\boldsymbol{a}} \times_{\boldsymbol{u}} \vec{\boldsymbol{s}}}) \wedge \underline{\boldsymbol{u}}$, so that (10.78) becomes

$$\boldsymbol{\epsilon}\left(\vec{\boldsymbol{u}}, \frac{\mathrm{d}\vec{\boldsymbol{s}}}{\mathrm{d}\tau}, ., .\right) + c\,(\underline{\vec{\boldsymbol{a}} \times_{\boldsymbol{u}} \vec{\boldsymbol{s}}}) \wedge \underline{\boldsymbol{u}} = \boldsymbol{N}_{\mathrm{spin}}. \tag{10.79}$$

As for any 2-form, $\boldsymbol{N}_{\mathrm{spin}}$ can be decomposed with respect to $\vec{\boldsymbol{u}}$ via (3.37):

$$\boldsymbol{N}_{\mathrm{spin}} = \boldsymbol{\epsilon}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{C}}, ., .) + \underline{\boldsymbol{u}} \wedge \boldsymbol{B}, \tag{10.80}$$

where $\vec{C} \in E_u$ and $\boldsymbol{B}$ is a linear form obeying $\langle \boldsymbol{B}, \vec{u} \rangle = 0$. The vector $\vec{C}$ is called the ***torque on the spin***. By combining (10.79) and (10.80), we get

$$\boldsymbol{\epsilon}\left(\vec{u}, \frac{\mathrm{d}\vec{s}}{\mathrm{d}\tau}, ., .\right) = \boldsymbol{\epsilon}(\vec{u}, \vec{C}, ., .) \qquad \text{and} \qquad \boldsymbol{B} = -c\, \underline{\vec{a} \times_u \vec{s}}.$$

The first of these two equations implies

$$\frac{\mathrm{d}\vec{s}}{\mathrm{d}\tau} = \vec{C} + \lambda\, \vec{u}, \tag{10.81}$$

with λ a scalar field defined along $\mathscr{L}$. λ is determined by taking the derivative of the condition $\vec{u} \cdot \vec{s} = 0$ [Eq. (10.74)]:

$$0 = \frac{\mathrm{d}}{\mathrm{d}\tau}(\vec{u} \cdot \vec{s}) = \frac{\mathrm{d}\vec{u}}{\mathrm{d}\tau} \cdot \vec{s} + \vec{u} \cdot \frac{\mathrm{d}\vec{s}}{\mathrm{d}\tau} = c\, \vec{a} \cdot \vec{s} + \underbrace{\vec{u} \cdot \vec{C}}_{0} + \lambda\, \underbrace{\vec{u} \cdot \vec{u}}_{-1},$$

hence $\lambda = c\, \vec{a} \cdot \vec{s}$ and (10.81) becomes

$$\boxed{\frac{\mathrm{d}\vec{s}}{\mathrm{d}\tau} = \vec{C} + c\, (\vec{a} \cdot \vec{s})\, \vec{u}}. \tag{10.82}$$

By comparing with the definition (3.69) and taking into account $\vec{u} \cdot \vec{s} = 0$, we recognize in $\mathrm{d}\vec{s}/\mathrm{d}\tau - c\, (\vec{a} \cdot \vec{s})\, \vec{u}$ the Fermi–Walker derivative of $\vec{s}$ along $\mathscr{P}$'s worldline. We get thus a very simple formula:

$$\boxed{\boldsymbol{D}^{\mathrm{FW}}_{\boldsymbol{u}} \vec{s} = \vec{C}} \qquad \text{with} \quad \vec{u} \cdot \vec{C} = 0 \tag{10.83}$$

We shall now treat two particular cases of this evolution law.

10.7.3 Free Gyroscope

We shall say that the particle constitutes a ***free gyroscope*** if its spin is not subject to any torque: $\vec{C} = 0$. Equation (10.83) reduces then to

$$\boxed{\boldsymbol{D}^{\mathrm{FW}}_{\boldsymbol{u}} \vec{s} = 0}. \tag{10.84}$$

In other words, the spin vector $\vec{s}$ is Fermi–Walker transported along $\mathscr{P}$'s worldline $\mathscr{L}$ (cf. Sect. 3.6.3). A nice property is that the norm of the vector is constant along $\mathscr{L}$:

$$\boxed{\|\vec{s}\|_{\boldsymbol{g}} = \text{const}}. \tag{10.85}$$

Proof. $\|\vec{s}\|_{\boldsymbol{g}} := \sqrt{\vec{s} \cdot \vec{s}}$ and $\mathrm{d}(\vec{s} \cdot \vec{s})/\mathrm{d}\tau = 2\vec{s} \cdot \mathrm{d}\vec{s}/\mathrm{d}\tau = 2\vec{s} \cdot [\boldsymbol{D}^{\mathrm{FW}}_{\boldsymbol{u}} \vec{s} + c(\vec{a} \cdot \vec{s})\, \vec{u}] = 0 + 0 = 0$ since $\vec{s} \cdot \vec{u} = 0$. □

In a general manner, the motion of a vector $\vec{s}$ along a worldline such that the norm of $\vec{s}$ is preserved is called ***precession***. The spin of a free gyroscope is thus precessing.

10.7.4 BMT Equation

As a second example of the evolution law (10.83), let us consider a charged particle with spin $\mathscr{P}$ moving in some electromagnetic field. The torque on the spin is then

$$\vec{C} = \frac{gq}{2mc} \perp_u \vec{F}(., \vec{s}), \tag{10.86}$$

where F is the electromagnetic field 2-form to be introduced in Chap. 17, $\vec{F}(., \vec{s})$ is the vector metric dual of the linear form $E \to \mathbb{R}$, $\vec{v} \mapsto F(\vec{v}, \vec{s})$, $\perp_u$ is the orthogonal projector onto E_u, m is the mass of particle $\mathscr{P}$, q its electric charge and g some dimensionless constant called the ***Landé factor*** of the particle. The coefficient $gq/(2m)$ is called the ***gyromagnetic ratio*** of particle $\mathscr{P}$. For an electron,[8] $g = 2$. Inserting (10.86) into (10.83), we obtain the following evolution law for $\vec{s}$:

$$\boxed{D_u^{\mathrm{FW}} \vec{s} = \frac{gq}{2mc} \perp_u \vec{F}(., \vec{s})}. \tag{10.87}$$

It is interesting to express the Fermi–Walker derivative in terms of the derivative with respect to $\mathscr{P}$'s proper time τ, according to (3.69):

$$D_u^{\mathrm{FW}} \vec{s} = \frac{d\vec{s}}{d\tau} - c(\vec{a} \cdot \vec{s})\, \vec{u}, \tag{10.88}$$

and to express the 4-acceleration $\vec{a}$ in terms of the 4-force $\vec{f}$ acting on $\mathscr{P}$ via (9.105): $\underline{a} = (mc^2)^{-1} f$. The linear form f is the *Lorentz 4-force* already encountered in Example 9.11 p. 313 and that we shall discuss extensively in Chap. 17:

$$f = qF(., \vec{u}). \tag{10.89}$$

It satisfies $\langle f, \vec{u}\rangle = 0$, thanks to the antisymmetry of F, which guarantees the vanishing of the term $dm/d\tau$ in (9.105) [cf. (9.106)]. Hence, given that $\vec{a} \cdot \vec{s} = \langle \underline{a}, \vec{s}\rangle = m^{-1}c^{-2}\langle f, \vec{s}\rangle$, (10.88) becomes

$$D_u^{\mathrm{FW}} \vec{s} = \frac{d\vec{s}}{d\tau} - \frac{q}{mc} F(\vec{s}, \vec{u})\, \vec{u}.$$

[8] Actually, quantum electrodynamics corrections make g being not exactly equal to 2: for the electron $g - 2 \simeq 2.3 \times 10^{-3}$.

Copying this expression into (10.87) and expressing the orthogonal projector $\perp_{\boldsymbol{u}}$ via (3.12), we get

$$\boxed{\frac{\mathrm{d}\vec{\boldsymbol{s}}}{\mathrm{d}\tau} = \frac{q}{mc}\left[\frac{g}{2}\vec{\boldsymbol{F}}(.,\vec{\boldsymbol{s}}) + \left(\frac{g}{2}-1\right)\boldsymbol{F}(\vec{\boldsymbol{u}},\vec{\boldsymbol{s}})\,\vec{\boldsymbol{u}}\right]}. \tag{10.90}$$

This relation is called ***BMT equation***, in reference to a study by V. Bargmann, L. Michel and V.L. Telegdi published in 1959 (Bargmann et al. 1959). The BMT equation guarantees that the norm of the spin vector is preserved:

$$\boxed{\|\vec{\boldsymbol{s}}\|_{\boldsymbol{g}} = \mathrm{const}}. \tag{10.91}$$

Proof. We have

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(\vec{\boldsymbol{s}}\cdot\vec{\boldsymbol{s}}) = 2\vec{\boldsymbol{s}}\cdot\frac{\mathrm{d}\vec{\boldsymbol{s}}}{\mathrm{d}\tau} = \frac{2q}{mc}\Big[\frac{g}{2}\underbrace{\boldsymbol{F}(\vec{\boldsymbol{s}},\vec{\boldsymbol{s}})}_{0} + \left(\frac{g}{2}-1\right)\boldsymbol{F}(\vec{\boldsymbol{u}},\vec{\boldsymbol{s}})\,\underbrace{\vec{\boldsymbol{s}}\cdot\vec{\boldsymbol{u}}}_{0}\Big] = 0,$$

where the antisymmetry of $\boldsymbol{F}$ has been used. □

$\vec{\boldsymbol{s}}$ has therefore a precession motion, as the spin vector of a free gyroscope (Sect. 10.7.3).

For an electron, the BMT equation simplifies considerably since $g = 2$:

$$\boxed{\frac{\mathrm{d}\vec{\boldsymbol{s}}}{\mathrm{d}\tau} = \frac{q}{mc}\vec{\boldsymbol{F}}(.,\vec{\boldsymbol{s}})}_{g=2}. \tag{10.92}$$

Historical note: *The BMT equation has been actually derived for the first time by Llewellyn H. Thomas (cf. p. 215) in 1927 (Thomas 1927), explicitly for the case $g = 2$ [Eq. (10.92)] and under the form of an equation equivalent to (10.90) in the general case.*

Chapter 11
Principle of Least Action

11.1 Introduction

Most of the modern physical theories are based on a *principle of least action*, also called *variational principle*. Such an approach leads naturally to the determination of conserved quantities from the symmetries of the system under study. Moreover, it is usually a first step to the quantum version of a classical theory. Here we reformulate in this framework the dynamics of relativistic particles developed in the two preceding chapters.

11.2 Principle of Least Action for a Particle

11.2.1 Reminder of Nonrelativistic Lagrangian Mechanics

In pre-relativistic Lagrangian mechanics, also named *analytical mechanics*, a system with N degrees of freedom is entirely described by a scalar function having the dimension of an energy:

$$L = L(q_1, \ldots, q_N, \dot{q}_1, \ldots, \dot{q}_N, t), \tag{11.1}$$

where t stands for the Newtonian absolute time, $(q_a)_{1 \le a \le N}$ for the N ***generalized coordinates*** of the system and $(\dot{q}_a)_{1 \le a \le N}$ the N ***generalized velocities***, i.e. the time derivatives of the generalized coordinates: $\dot{q}_a = \mathrm{d}q_a/\mathrm{d}t$. The configuration of the system at some instant t is defined by the N functions $q_a(t)$, which span a part of $\mathbb{R}^N$ called the ***configuration space***. The function L, whose precise form defines the physical problem to be studied, is called the ***Lagrangian*** of the system. For a system made of M particles and for which the force exerted on each particle arises from a potential V, we have $N = 3M$, and a standard choice of Lagrangian is $L = T - V$, where T is the total kinetic energy of the system.

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The ***principle of least action*** , also called ***Hamilton principle*** (cf. p. 245), states that the evolution of the system between two fixed configurations $(q_a(t_1))$ and $(q_a(t_2))$ is such that the ***action***, defined by

$$S := \int_{t_1}^{t_2} L(q_1, \ldots, q_N, \dot{q}_1, \ldots, \dot{q}_N, t)\, \mathrm{d}t, \qquad (11.2)$$

takes the smallest value among all the trajectories connecting $(q_a(t_1))$ and $(q_a(t_2))$ in the configuration space. If the N generalized coordinates q_a are independent, the principle of least action leads to the ***Euler–Lagrange equations***:

$$\frac{\partial L}{\partial q_a} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_a}\right) = 0, \qquad a \in \{1, \ldots, N\}. \qquad (11.3)$$

Conversely, if the Euler–Lagrange equations are satisfied, then for fixed initial and final states, the action S is extremal on the path actually taken by the system in the configuration space.

For a detailed exposition of nonrelativistic Lagrangian mechanics and examples, the reader is referred to the textbooks by Basdevant (2007), Deruelle and Uzan (2006), Goldstein, Safko and Poole (2002), and Hakim (1995) (to mention only recent books).

11.2.2 Relativistic Generalization

The generalization of the principle of least action to a relativistic system faces a conceptual difficulty from the very beginning: there does not exist any absolute time t in relativity. This raises the issue of defining the generalized velocities by $\dot{q}_a := \mathrm{d}q_a/\mathrm{d}t$, as well as the action S by an integral over t, as in (11.2). For particles, one might think about using the proper time, but there is no uniqueness of the latter as soon as there is more than one particle. Even for a system reduced to a single particle, this choice is not directly applicable. Indeed, if one picks for generalized coordinates the coordinates (x^α) of the particle in some affine frame of $\mathscr{E}$, then the generalized velocities $\dot{x}^\alpha := \mathrm{d}x^\alpha/\mathrm{d}t$, with t the particle's proper time, are nothing but (up to a c factor) the components of the particle's 4-velocity $\vec{u}$ in the basis associated with the affine frame [cf. Eq. (2.12)]. They are thus submitted to the constraint $g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = -c^2$, arising from $\vec{u}\cdot\vec{u} = -1$. This constraint restricts the possible variations in the principle of least action. It is possible to take it into account by the Lagrange multipliers technique,[1] but this is not the most widespread method, and we shall not employ it here.

[1]Cf. p. 65 of Barut's book (Barut 1964) for some example.

11.2.3 Lagrangian and Action for a Particle

Let us consider a system reduced to a single particle $\mathscr{P}$. The solution to the problem mentioned above consists in replacing the Newtonian absolute time by some parameter λ that increases along $\mathscr{P}$'s worldline $\mathscr{L}$. This amounts to introducing a *parametrization* of $\mathscr{L}$, as defined in Sect. 2.2. Note that any parametrization is a priori valid and has not to coincide with the proper time. Given an affine coordinate system (x^α) of $\mathscr{E}$, $\mathscr{P}$'s worldline is determined by the equations

$$\mathscr{L}: \qquad x^\alpha = x^\alpha(\lambda), \quad \lambda \in \mathbb{R}, \qquad \alpha \in \{0, 1, 2, 3\}, \tag{11.4}$$

where the x^α's are four functions[2] (at least twice differentiable) $\mathbb{R} \to \mathbb{R}$. Let us then set $\dot{x}^\alpha := \mathrm{d}x^\alpha/\mathrm{d}\lambda$. The $(\dot{x}^\alpha)$ are nothing but the components in the considered affine frame of the vector tangent to $\mathscr{L}$ associated with the parameter λ [cf. Eq. (2.4)]:

$$\vec{\boldsymbol{v}} = \frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\, \vec{\boldsymbol{e}}_\alpha = \dot{x}^\alpha\, \vec{\boldsymbol{e}}_\alpha, \tag{11.5}$$

where $(\vec{\boldsymbol{e}}_\alpha)$ is the basis of E associated with the affine coordinates (x^α).

One calls ***Lagrangian*** of particle $\mathscr{P}$ any differentiable function $L : \mathbb{R}^8 \to \mathbb{R}$ such that between any two events A_1 and A_2 of $\mathscr{P}$'s worldline (of respective parameters λ_1 and λ_2), the integral

$$\boxed{S := \int_{\lambda_1}^{\lambda_2} L(x^\alpha(\lambda), \dot{x}^\alpha(\lambda))\, \mathrm{d}\lambda} \tag{11.6}$$

has the following properties:

(i) S has the dimension of an energy multiplied by a time.
(ii) S is independent of the parametrization λ.

The quantity S is called ***action*** of the particle between the events A_1 and A_2. The explicit form of L will define the physical settings (for instance, a free particle or a charged particle in an electromagnetic field).

The independence of the value of S with respect to the parametrization of $\mathscr{L}$ induces a constraint on the function L. Indeed, let us consider a second parametrization $\tilde{\lambda}$ of $\mathscr{L}$:

$$\mathscr{L}: \qquad x^\alpha = \tilde{x}^\alpha(\tilde{\lambda}), \quad \tilde{\lambda} \in \mathbb{R}, \qquad \alpha \in \{0, 1, 2, 3\}, \tag{11.7}$$

[2] The same symbol x^α is employed to denote the affine coordinates on $\mathscr{E}$ and the functions of λ defining $\mathscr{P}$'s worldline. This constitutes a slight abuse of notation, quite common in physics. Rigorously, one should write something like $x^\alpha = X^\alpha(\lambda)$ rather than (11.4).

where the $\tilde{x}^\alpha$'s are four functions $\mathbb{R} \to \mathbb{R}$ a priori different from the functions x^α's introduced in (11.4). From the definition of a parametrization (cf. Sect. 2.2), there exists an invertible map $f : \mathbb{R} \to \mathbb{R}$ such that $\lambda = f(\tilde{\lambda})$. By combining (11.4) and (11.7), we get

$$\tilde{x}^\alpha(\tilde{\lambda}) = x^\alpha(\lambda); \tag{11.8}$$

hence,

$$\dot{\tilde{x}}^\alpha := \frac{d\tilde{x}^\alpha}{d\tilde{\lambda}} = \frac{dx^\alpha}{d\lambda}\frac{d\lambda}{d\tilde{\lambda}} = \frac{d\lambda}{d\tilde{\lambda}}\,\dot{x}^\alpha, \tag{11.9}$$

with $d\lambda/d\tilde{\lambda} = f'(\tilde{\lambda})$. The invariance of the action (11.6) with respect to the parametrization is equivalent to

$$L(\tilde{x}^\alpha, \dot{\tilde{x}}^\alpha)\, d\tilde{\lambda} = L(x^\alpha, \dot{x}^\alpha)\, d\lambda.$$

By means of (11.8) and (11.9), we obtain

$$L\left(x^\alpha(\lambda), \frac{d\lambda}{d\tilde{\lambda}}\,\dot{x}^\alpha\right) = \frac{d\lambda}{d\tilde{\lambda}}\, L(x^\alpha, \dot{x}^\alpha).$$

Since this relation must be fulfilled for any value of $d\lambda/d\tilde{\lambda}$, we conclude that L is a positive homogeneous function of degree 1 with respect to each of its four last arguments:

$$\forall \mu > 0,\ \forall (x^\alpha, \dot{x}^\alpha) \in \mathbb{R}^8, \quad \boxed{L(x^\alpha, \mu\dot{x}^\alpha) = \mu L(x^\alpha, \dot{x}^\alpha)}. \tag{11.10}$$

The Euler theorem about homogeneous functions implies then that the Lagrangian L must obey

$$\boxed{\dot{x}^\alpha \frac{\partial L}{\partial \dot{x}^\alpha} = L}. \tag{11.11}$$

11.2.4 Principle of Least Action

The ***principle of least action*** states:

> If the Lagrangian L describes the dynamics of particle $\mathscr{P}$ correctly, the worldline followed by $\mathscr{P}$ between the events A_1 and A_2 is that for which the action S is minimal.

More precisely, let us consider a variation of $\mathscr{P}$'s worldline, keeping the events A_1 and A_2 fixed. This amounts to consider a worldline $\mathscr{L}'$ close to $\mathscr{L}$, whose equation within the affine coordinates (x^α) is

$$\mathscr{L}' : \qquad x^\alpha = x^\alpha(\lambda) + \delta x^\alpha(\lambda), \qquad \alpha \in \{0, 1, 2, 3\}, \tag{11.12}$$

with $\delta x^\alpha(\lambda)$ infinitely small and such that

$$\delta x^\alpha(\lambda_1) = 0 \qquad \text{and} \qquad \delta x^\alpha(\lambda_2) = 0, \tag{11.13}$$

so that A_1 and A_2 are held fixed. The corresponding variation of the action is

$$\delta S = \int_{\lambda_1}^{\lambda_2} \left[\frac{\partial L}{\partial x^\alpha} \delta x^\alpha + \frac{\partial L}{\partial \dot{x}^\alpha} \delta \dot{x}^\alpha \right] \mathrm{d}\lambda. \tag{11.14}$$

$\delta \dot{x}^\alpha$ being obtained by taking the derivative of (11.12) with respect to λ, we have $\delta \dot{x}^\alpha = \mathrm{d}(\delta x^\alpha)/\mathrm{d}\lambda$. The second term of (11.14) can be then integrated by parts, yielding

$$\delta S = \int_{\lambda_1}^{\lambda_2} \left[\frac{\partial L}{\partial x^\alpha} \delta x^\alpha - \frac{\mathrm{d}}{\mathrm{d}\lambda} \left(\frac{\partial L}{\partial \dot{x}^\alpha} \right) \delta x^\alpha \right] \mathrm{d}\lambda + \left[\frac{\partial L}{\partial \dot{x}^\alpha} \delta x^\alpha \right]_{\lambda_1}^{\lambda_2}.$$

Given (11.13), the last term in the above equation vanishes, and there remains

$$\delta S = \int_{\lambda_1}^{\lambda_2} \left[\frac{\partial L}{\partial x^\alpha} - \frac{\mathrm{d}}{\mathrm{d}\lambda} \left(\frac{\partial L}{\partial \dot{x}^\alpha} \right) \right] \delta x^\alpha \, \mathrm{d}\lambda. \tag{11.15}$$

The principle of least action stipulates that S reaches a minimum on the worldline actually followed by the particle, which implies

$$\boxed{\delta S = 0} \tag{11.16}$$

whatever the variation δx^α around $\mathscr{L}$. In view of (11.15), we conclude that

$$\boxed{\frac{\partial L}{\partial x^\alpha} - \frac{\mathrm{d}}{\mathrm{d}\lambda} \left(\frac{\partial L}{\partial \dot{x}^\alpha} \right) = 0}, \qquad \alpha \in \{0, 1, 2, 3\}. \tag{11.17}$$

In other words, the Lagrangian L must fulfil ***Euler–Lagrange equations*** identical to equations (11.3), except that the Newtonian time has been replaced by a generic parameter along the particle's worldline.

Remark 11.1. There are four Euler–Lagrange equations (11.17), whereas there are only three of them in Newtonian mechanics [Eqs. (11.3) with $a \in \{1, 2, 3\}$]. One should of course not conclude that relativity adds a degree of freedom to a system made of a single particle! The four equations (11.17) are indeed not independent, because of the identity (11.11), which must be fulfilled by the relativistic Lagrangian. To make this explicit, let us evaluate the following expression, *without assuming that (11.17) holds*:

$$\dot{x}^\alpha \left[\frac{\partial L}{\partial x^\alpha} - \frac{\mathrm{d}}{\mathrm{d}\lambda} \left(\frac{\partial L}{\partial \dot{x}^\alpha} \right) \right] = \dot{x}^\alpha \frac{\partial L}{\partial x^\alpha} - \frac{\mathrm{d}}{\mathrm{d}\lambda} \left(\dot{x}^\alpha \frac{\partial L}{\partial \dot{x}^\alpha} \right) + \frac{\mathrm{d}\dot{x}^\alpha}{\mathrm{d}\lambda} \frac{\partial L}{\partial \dot{x}^\alpha}$$
$$= \frac{\mathrm{d}L}{\mathrm{d}\lambda} - \frac{\mathrm{d}}{\mathrm{d}\lambda} \left(\dot{x}^\alpha \frac{\partial L}{\partial \dot{x}^\alpha} \right) = \frac{\mathrm{d}}{\mathrm{d}\lambda} \left(L - \dot{x}^\alpha \frac{\partial L}{\partial \dot{x}^\alpha} \right).$$

The identity (11.11) gives then

$$\dot{x}^\alpha \left[\frac{\partial L}{\partial x^\alpha} - \frac{\mathrm{d}}{\mathrm{d}\lambda} \left(\frac{\partial L}{\partial \dot{x}^\alpha} \right) \right] = 0. \tag{11.18}$$

This identity reduces the number of independent Euler–Lagrange equations (11.17) from four to three.

11.2.5 Action of a Free Particle

The principle of least action must of course be completed by some specific choice of the Lagrangian function. The simplest case is that of a free (i.e. isolated) massive particle $\mathscr{P}$. We already know that $\mathscr{P}$'s worldline is a straight line of $\mathscr{E}$ (Sect. 9.3.4). Moreover, we have noticed in Sect. 2.7.1 that the timelike straight lines are timelike geodesics of Minkowski spacetime: they achieve a maximum proper time between two given events. It is then natural to consider for the action of a free particle a quantity proportional to the proper time elapsed along the worldline. The proportionality constant α must be negative to turn the maximum of proper time into a minimum of the action. In addition, α must have the dimension of an energy in order for S to have the dimension of an action. There is then only one (simple) possible choice from the sole particle data: $\alpha = -mc^2$, where m is the mass of $\mathscr{P}$, assumed to be constant along $\mathscr{L}$ (this is indeed the case for a free particle; cf. Sect. 9.3.4). Consequently, the action of a free massive particle between two events A_1 and A_2 of its worldline is

$$\boxed{S = -mc^2 \int_{\tau_1}^{\tau_2} \mathrm{d}\tau = -mc^2(\tau_2 - \tau_1)}, \tag{11.19}$$

where τ_1 (resp. τ_2) is the proper time of $\mathscr{P}$ at A_1 (resp. A_2) and $\mathrm{d}\tau$ is the increment of proper time along $\mathscr{L}$. Let us express the latter in terms of the increment of the parameter λ according to (2.9):

$$\mathrm{d}\tau = \frac{1}{c}\sqrt{-\boldsymbol{g}(\vec{\boldsymbol{v}}, \vec{\boldsymbol{v}})}\, \mathrm{d}\lambda,$$

$\vec{\boldsymbol{v}}$ being the vector tangent to $\mathscr{L}$ associated with the parameter λ. Thanks to (11.5), this relation can be written as

$$\mathrm{d}\tau = \frac{1}{c}\sqrt{-g_{\alpha\beta}\, \dot{x}^\alpha\, \dot{x}^\beta}\, \mathrm{d}\lambda, \tag{11.20}$$

where the $g_{\alpha\beta}$'s are the components of the metric tensor with respect to the basis $(\vec{\boldsymbol{e}}_\alpha)$. Inserting (11.20) into the expression (11.19) of the action and comparing with the definition (11.6), we obtain the expression of the Lagrangian of a free particle:

$$\boxed{L(x^\alpha, \dot{x}^\alpha) = -mc\sqrt{-g_{\alpha\beta}\, \dot{x}^\alpha\, \dot{x}^\beta}}. \tag{11.21}$$

We check that this Lagrangian is a positive homogeneous function of degree 1 with respect to $\dot{x}^\alpha$, i.e. that it obeys property (11.10).

Let us also check that the Euler–Lagrange equations derived from (11.21) lead to worldlines that are straight lines of $\mathscr{E}$. We have

$$\frac{\partial L}{\partial \dot{x}^\alpha} = \frac{mc}{2\sqrt{-g_{\mu\nu}\, \dot{x}^\mu\, \dot{x}^\nu}}\, g_{\mu\nu}\Bigg(\underbrace{\frac{\partial \dot{x}^\mu}{\partial \dot{x}^\alpha}}_{\delta^\mu{}_\alpha}\, \dot{x}^\nu + \dot{x}^\mu\, \underbrace{\frac{\partial \dot{x}^\nu}{\partial \dot{x}^\alpha}}_{\delta^\nu{}_\alpha}\Bigg) = \frac{mc}{\sqrt{-g_{\mu\nu}\, \dot{x}^\mu\, \dot{x}^\nu}}\, g_{\alpha\beta}\, \dot{x}^\beta.$$

Now, from (11.5) and (2.13),

$$\frac{\dot{x}^\alpha}{\sqrt{-g_{\mu\nu}\, \dot{x}^\mu\, \dot{x}^\nu}} = u^\alpha, \tag{11.22}$$

where the u^α's are the components of the particle's 4-velocity $\vec{\boldsymbol{u}}$ in the basis $(\vec{\boldsymbol{e}}_\alpha)$. The expression of $\partial L/\partial \dot{x}^\alpha$ can thus be recast as

$$\boxed{\frac{\partial L}{\partial \dot{x}^\alpha} = mcu_\alpha}, \tag{11.23}$$

where $u_\alpha = g_{\alpha\beta}u^\beta = \vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{e}}_\alpha$ are the components within the basis dual to $(\vec{\boldsymbol{e}}_\alpha)$ of the linear form $\underline{\boldsymbol{u}}$ associated with $\vec{\boldsymbol{u}}$ by metric duality. In view of (11.23) and $\partial L/\partial x^\alpha = 0$, the Euler–Lagrange equation (11.17) reduces to

$$\frac{\mathrm{d}u_\alpha}{\mathrm{d}\lambda} = 0.$$

Multiplying by the matrix $(g^{\alpha\beta})$—the inverse of $(g_{\alpha\beta})$ (cf. §. 1.3.2)—we get $\mathrm{d}u^\beta/\mathrm{d}\lambda = 0$, which implies that $\vec{\boldsymbol{u}}$ is constant along $\mathscr{L}$. We conclude that the worldline $\mathscr{L}$ is a straight line of $\mathscr{E}$.

Example 11.1. Let us choose for (x^α) the coordinates with respect to some inertial observer $\mathscr{O}$: $(x^\alpha) = (ct, x, y, z)$, where t is $\mathscr{O}$'s proper time. The basis $(\vec{e}_\alpha)$ is then orthonormal, and the matrix $(g_{\alpha\beta})$ is the Minkowski matrix $(\eta_{\alpha\beta})$, as given by (1.17). Moreover, let us choose the proper time of observer $\mathscr{O}$ as parameter along the worldline $\mathscr{L}$: $\lambda = t$. Then

$$\dot{x}^0 = \frac{\mathrm{d}x^0}{\mathrm{d}\lambda} = \frac{\mathrm{d}(ct)}{\mathrm{d}t} = c \qquad \text{and} \qquad \dot{x}^i = \frac{\mathrm{d}x^i}{\mathrm{d}\lambda} = \frac{\mathrm{d}x^i}{\mathrm{d}t} = V^i, \tag{11.24}$$

where the V^i's are the components in the basis $(\vec{e}_i)$ of $\mathscr{P}$'s velocity relative to $\mathscr{O}$, $\vec{V}$. The Lagrangian (11.21) becomes thus

$$L = -mc\sqrt{-\eta_{\alpha\beta}\,\dot{x}^\alpha\,\dot{x}^\beta} = -mc\sqrt{c^2 - \delta_{ij}\,V^i\,V^j}, \tag{11.25}$$

i.e.

$$\boxed{L = -mc^2\sqrt{1 - \frac{\vec{V}\cdot\vec{V}}{c^2}}}. \tag{11.26}$$

Remark 11.2. The Lagrangian (11.26), which refers to an inertial observer, is that generally considered in introductory textbooks, as those of Landau and Lifshitz (1975), Feynman (2011), Basdevant (2007) or Pérez (2005).

Remark 11.3. If one chooses as a parameter, instead of the proper time t of an inertial observer, the proper time of the particle itself, the numerical value of the Lagrangian is constant, since, in this case, $\dot{x}^\alpha = \mathrm{d}x^\alpha/\mathrm{d}\tau = cu^\alpha$ and the relation

$$g_{\alpha\beta}u^\alpha u^\beta = -1 \tag{11.27}$$

reduces (11.21) to $L = -mc^2$. A constant Lagrangian leads obviously to Euler–Lagrange equations without any content of the type "$0 = 0$". We recover hence the problem underlined in Sect. 11.2.2. The parameter λ from which the Lagrangian (11.21) is formed must not be constrained. One may use the particle's proper time but without assuming that (11.27) with $u^\alpha = \dot{x}^\alpha/c$ is fulfilled a priori. It is only *a posteriori*, i.e. after the Euler–Lagrange equations have been written and solved, that (11.27) must be enforced.

Historical note: *It is Henri Poincaré (cf. p. 26) who, in 1905, wrote the Lagrangian of a free relativistic particle in the famous "Palermo memoir" (Poincaré 1906). He obtained a form equivalent to (11.26) by postulating the invariance of the action under the Lorentz group (cf. Bracco and Provost (2009) for a detailed discussion). In 1906, Max Planck (cf. p. 279) (1906) also obtained the Lagrangian (11.26) but with an additive constant—which prevented the action from being invariant under the Lorentz group. He suppressed the constant the year after (Planck 1907).*

11.2.6 Particle in a Vector Field

We shall say that a particle $\mathscr{P}$ is submitted to a ***vector field***, or equivalently to a ***vectorial interaction*** with an external field, if there exists a real constant q and a field of linear forms on Minkowski spacetime, $\boldsymbol{A} : \mathscr{E} \to E^*$, such that $\mathscr{P}$ obeys the principle of least action with the Lagrangian

$$\boxed{L(x^\alpha, \dot{x}^\alpha) = -mc\sqrt{-g_{\alpha\beta}\,\dot{x}^\alpha\,\dot{x}^\beta} + \frac{q}{c}\,A_\beta(x^\alpha)\,\dot{x}^\beta}, \tag{11.28}$$

where $A_\beta(x^\alpha) = A_\beta(x^0, x^1, x^2, x^3)$ stands for the β^{th} component of the linear form $\boldsymbol{A}(M)$ in the basis dual to $(\vec{e}_\alpha)$, $M \in \mathscr{E}$ being the point of affine coordinates (x^0, x^1, x^2, x^3): $A_\beta(x^\alpha) = \langle \boldsymbol{A}(M), \vec{e}_\beta \rangle$. The constant q is called ***charge of the particle*** in the vector field and $\boldsymbol{A}$ the ***potential 1-form*** of the field. An important example of a vector field is the electromagnetic field, which we shall study in Chap. 17. Thanks to (11.5), we can reexpress the Lagrangian (11.28) with vectorial notations:

$$L = -mc\sqrt{-\vec{v}\cdot\vec{v}} + \frac{q}{c}\,\langle \boldsymbol{A}, \vec{v} \rangle. \tag{11.29}$$

Note that the function (11.28) is eligible for a Lagrangian since it is a homogeneous function of degree 1 with respect to $\dot{x}^\alpha$, in agreement with (11.10).

Remark 11.4. The first term in (11.28) is nothing but the Lagrangian of a free particle, as given by (11.21). The second term is the simplest construction of a homogeneous function of degree 1 with respect to $\dot{x}^\beta$ from the components (A_β) of the linear form $\boldsymbol{A}$.

Deriving (11.28) and using the already computed expression for the derivative with respect to $\dot{x}^\alpha$ of the free-particle term in L [Eq. (11.23)], we get

$$\frac{\partial L}{\partial x^\alpha} = \frac{q}{c}\frac{\partial A_\beta}{\partial x^\alpha}\,\dot{x}^\beta \qquad \text{and} \qquad \frac{\partial L}{\partial \dot{x}^\alpha} = mc\,u_\alpha + \frac{q}{c}A_\alpha. \tag{11.30}$$

Plugging these two expressions into the Euler–Lagrange equations (11.17) and using $\mathrm{d}A_\alpha/\mathrm{d}\lambda = \mathrm{d}/\mathrm{d}\lambda[A_\alpha(x^\beta)] = (\partial A_\alpha/\partial x^\beta)\,\dot{x}^\beta$, we obtain

$$\frac{q}{c}\left(\frac{\partial A_\beta}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\beta}\right)\dot{x}^\beta - mc\frac{\mathrm{d}u_\alpha}{\mathrm{d}\lambda} = 0. \tag{11.31}$$

Let us express the derivative $\mathrm{d}u_\alpha/\mathrm{d}\lambda$ in terms of the derivative with respect to $\mathscr{P}$'s proper time, τ, via the relation (11.20) between $\mathrm{d}\tau$ and $\mathrm{d}\lambda$:

$$\frac{\mathrm{d}u_\alpha}{\mathrm{d}\lambda} = \frac{\mathrm{d}u_\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}\lambda} = \frac{1}{c}\sqrt{-g_{\mu\nu}\,\dot{x}^\mu\,\dot{x}^\nu}\,\frac{\mathrm{d}u_\alpha}{\mathrm{d}\tau} = \sqrt{-g_{\mu\nu}\,\dot{x}^\mu\,\dot{x}^\nu}\,a_\alpha, \tag{11.32}$$

where the a_α's are the components in the basis dual to $(\vec{e}_\alpha)$ of the linear form $\underline{a}$ associated with $\mathscr{P}$'s 4-acceleration by metric duality: $\underline{a} = c^{-1}\,\mathrm{d}\underline{u}/\mathrm{d}\tau$ [cf. Eq. (2.16)]. Equation (11.31) becomes then

$$\frac{q}{c}\left(\frac{\partial A_\beta}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\beta}\right)\dot{x}^\beta - mc\sqrt{-g_{\mu\nu}\,\dot{x}^\mu\,\dot{x}^\nu}\,a_\alpha = 0.$$

Dividing by $\sqrt{-g_{\mu\nu}\,\dot{x}^\mu\,\dot{x}^\nu}$ and using (11.22), this equation can be rewritten as

$$mc^2\,a_\alpha = q\,F_{\alpha\beta}u^\beta, \tag{11.33}$$

with

$$\boxed{F_{\alpha\beta} := \frac{\partial A_\beta}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\beta}}. \tag{11.34}$$

Since clearly $F_{\alpha\beta} = -F_{\beta\alpha}$, the quantities $(F_{\alpha\beta})$ constitute the matrix within the basis $(\vec{e}_\alpha)$ of an antisymmetric bilinear form $\boldsymbol{F}$. We can then write (11.33) as

$$\boxed{\boldsymbol{f} = q\,\boldsymbol{F}(., \vec{\boldsymbol{u}})}, \tag{11.35}$$

where we have let the 4-force $\boldsymbol{f} = mc^2\,\underline{\boldsymbol{a}}$ appear [Eq. (9.105) with $m = \text{const}$]. The principle of least action applied to the Lagrangian (11.28) rules thus the motion of a particle submitted to a 4-force of the type (11.35). This is a pure 4-force (cf. Sect. 9.5) for $\langle \boldsymbol{f}, \vec{\boldsymbol{u}}\rangle = q\,\boldsymbol{F}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{u}}) = 0$ by the antisymmetry of $\boldsymbol{F}$. If the field under consideration is an electromagnetic one, q can be interpreted as the electric charge, and we recover the Lorentz 4-force encountered in Example 9.11 p. 313 and in Sect. 10.7.4 [Eq. (10.89)]. We shall discuss it in more details in Chap. 17.

11.2.7 Other Examples of Lagrangians

Example 11.2. **Particle in a scalar field**. We shall say that a particle $\mathscr{P}$ is submitted to a ***scalar field*** if its Lagrangian takes the form

$$L(x^\alpha, \dot{x}^\alpha) = -\left[mc + \frac{q}{c}\Phi(x^\alpha)\right]\sqrt{-g_{\alpha\beta}\,\dot{x}^\alpha\,\dot{x}^\beta}, \tag{11.36}$$

where $\Phi : \mathscr{E} \to \mathbb{R}$ is a scalar field on Minkowski spacetime and q is a constant that represents the ***scalar charge*** of the particle: if $q = 0$, $\mathscr{P}$ is not sensitive to the scalar field. The dimensions of q and Φ must be such that the product $q\Phi$ has the dimension of an energy, so that the sum $mc + q\Phi/c$ appearing in (11.36) is well defined.

Remark 11.5. The Lagrangian (11.36) can be split in two parts: $L = L_{\rm free} + L_{\rm inter}$, with $L_{\rm free} = -mc\sqrt{-g_{\alpha\beta}\,\dot{x}^\alpha\,\dot{x}^\beta}$ [Lagrangian (11.21) for a free particle] and

$$L_{\rm inter} = -\frac{q}{c}\Phi(x^\alpha)\sqrt{-g_{\alpha\beta}\,\dot{x}^\alpha\,\dot{x}^\beta}. \tag{11.37}$$

$L_{\rm inter}$ describes the interaction with the scalar field. It is the simplest one yielding a homogeneous function of degree 1 with respect to $\dot{x}^\alpha$ from the scalar field Φ.

In the present case,

$$\frac{\partial L}{\partial x^\alpha} = -\frac{q}{c}\sqrt{-g_{\mu\nu}\,\dot{x}^\mu\,\dot{x}^\nu}\,\frac{\partial \Phi}{\partial x^\alpha} \qquad \text{and} \qquad \frac{\partial L}{\partial \dot{x}^\alpha} = \left(mc + \frac{q}{c}\Phi\right) u_\alpha,$$

where use has been made of the computation leading to (11.23) to get the second equation. Consequently, the Euler–Lagrange equations (11.17) yield

$$-\frac{q}{c}\sqrt{-g_{\mu\nu}\,\dot{x}^\mu\,\dot{x}^\nu}\,\frac{\partial \Phi}{\partial x^\alpha} - \frac{q}{c}\frac{\mathrm{d}\Phi}{\mathrm{d}\lambda}u_\alpha - \left(mc + \frac{q}{c}\Phi\right)\frac{\mathrm{d}u_\alpha}{\mathrm{d}\lambda} = 0.$$

Writing $\mathrm{d}\Phi/\mathrm{d}\lambda = \partial\Phi/\partial x^\beta\,\dot{x}^\beta$ and using (11.32) and (11.22), we get

$$\left(mc^2 + q\Phi\right)a_\alpha = -q\frac{\partial \Phi}{\partial x^\beta}\left(\delta^\beta_{\ \alpha} + u^\beta u_\alpha\right). \tag{11.38}$$

We recognize in $\delta^\beta_{\ \alpha} + u^\beta u_\alpha$ the orthogonal projector $\perp_{\boldsymbol{u}}$ onto the particle's local rest space. We conclude that the particle is subjected to the 4-force

$$\boldsymbol{f} = -q\boldsymbol{\nabla}\Phi \circ \perp_{\boldsymbol{u}} - q\Phi\,\underline{\boldsymbol{a}}, \tag{11.39}$$

where $\boldsymbol{\nabla}\Phi$ stands for the gradient[3] of the scalar field Φ, i.e. the linear form $E \to \mathbb{R}$, $\vec{\boldsymbol{v}} \mapsto v^\alpha\partial\Phi/\partial x^\alpha$.

Remark 11.6. The 4-force (11.39) is a pure one : $\langle \boldsymbol{f}, \vec{\boldsymbol{u}}\rangle = 0$ [Eq. (9.107)] for $\perp_{\boldsymbol{u}}\vec{\boldsymbol{u}} = 0$ and $\langle \underline{\boldsymbol{a}}, \vec{\boldsymbol{u}}\rangle = \vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{u}} = 0$. Besides, let us note that this 4-force involves the 4-acceleration.

Example 11.3. **Particle in a tensor field**. An example of interaction of a particle with a tensor field of rank 2 is provided by the Lagrangian

$$L(x^\alpha, \dot{x}^\alpha) = -mc\sqrt{-\left[g_{\alpha\beta} + \frac{q}{m}h_{\alpha\beta}(x^\mu)\right]\dot{x}^\alpha\,\dot{x}^\beta}, \tag{11.40}$$

[3] The gradient will be studied in detail in Chap. 15.

where the $h_{\alpha\beta}$'s are the components with respect to the coordinates (x^α) of a field $\boldsymbol{h}$ of symmetric bilinear forms on $\mathscr{E}$ and q is the charge of the particle with respect to the field (q has the dimension of a mass). An alternative Lagrangian is

$$L(x^\alpha, \dot{x}^\alpha) = -mc\sqrt{-g_{\alpha\beta}\,\dot{x}^\alpha\,\dot{x}^\beta} + \frac{1}{2}qc\,\frac{h_{\alpha\beta}(x^\mu)\dot{x}^\alpha\dot{x}^\beta}{\sqrt{-g_{\mu\nu}\dot{x}^\mu\,\dot{x}^\nu}}. \tag{11.41}$$

The above two Lagrangians are homogeneous functions of degree 1 in $\dot{x}^\alpha$, as they should; they appear in attempts to treat gravitation in Minkowski spacetime, which will be discussed in Chap. 22 (then $q = m$).

Remark 11.7. The Lagrangian (11.41) is a first-order series expansion of the Lagrangian (11.40) when $|h_{\alpha\beta}| \ll |g_{\alpha\beta}|$.

11.3 Noether Theorem

The Noether theorem is one of the pillars of theoretical physics. It relates quantities conserved during the motion to the symmetries of the Lagrangian. We shall establish it for a relativistic particle and apply it to the case of a free particle.

11.3.1 Noether Theorem for a Particle

Let us consider an infinitesimal change of the affine coordinates on $\mathscr{E}$:

$$x'^\alpha = x^\alpha + \varepsilon G^\alpha(x^\beta), \tag{11.42}$$

with ε infinitely small. The functions $G^\alpha(x^\beta)$ are called ***generators of the coordinate change***. The equation of the worldline of particle $\mathscr{P}$ in these new coordinates is

$$\mathscr{L}: \qquad x'^\alpha = x'^\alpha(\lambda) = x^\alpha(\lambda) + \varepsilon G^\alpha(\lambda), \quad \lambda \in \mathbb{R}, \qquad \alpha \in \{0, 1, 2, 3\}, \tag{11.43}$$

with the notation $G^\alpha(\lambda) := G^\alpha(x^\beta(\lambda))$.

Let us suppose that the Lagrangian is invariant under the coordinate change (11.42), i.e. that

$$\forall \lambda \in \mathbb{R}, \quad L(x'^\alpha(\lambda), \dot{x}'^\alpha(\lambda)) = L\left(x^\alpha(\lambda), \dot{x}^\alpha(\lambda)\right). \tag{11.44}$$

Taking into account (11.43), this hypothesis becomes

$$L(x^\alpha + \varepsilon G^\alpha, \dot{x}^\alpha + \varepsilon \dot{G}^\alpha) = L(x^\alpha, \dot{x}^\alpha).$$

Let us expand to first order in ε, subtract $L(x^\alpha, \dot{x}^\alpha)$ from both sides and divide by ε; we are left with

$$\frac{\partial L}{\partial x^\alpha} G^\alpha + \frac{\partial L}{\partial \dot{x}^\alpha} \frac{\mathrm{d}G^\alpha}{\mathrm{d}\lambda} = 0.$$

Using the Euler–Lagrange equations (11.17) to replace $\partial L / \partial x^\alpha$, we get

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \left(\frac{\partial L}{\partial \dot{x}^\alpha} G^\alpha \right) = 0.$$

Hence, the quantity $(\partial L / \partial \dot{x}^\alpha)\, G^\alpha$ is constant along the particle's worldline. This is ***Noether theorem*** for a relativistic particle. We shall state it as

$$\boxed{L(x^\alpha + \varepsilon G^\alpha, \dot{x}^\alpha + \varepsilon \dot{G}^\alpha) = L(x^\alpha, \dot{x}^\alpha) \quad \Longrightarrow \quad \frac{\partial L}{\partial \dot{x}^\alpha} G^\alpha = \mathrm{const}}\,. \tag{11.45}$$

Historical note: *The above result is actually a particular case of the theorem established in 1918 by Emmy Noether*[4] *(1918) (cf. Byers (1999)). She was considering very general variational principles, based on multidimensional integrals, while the action (11.6) considered here is unidimensional ($\lambda \in \mathbb{R}$). The actions involving multidimensional integrals (notably four-dimensional) are those used in field theory (we shall see an example in Sect. 18.7). The work of Emmy Noether was motivated by the variational formulation of general relativity developed in 1915 by David Hilbert.*[5] *Emmy Noether joined him in Göttingen the same year. Let us stress that the particular case (but fundamental for relativity!) where the coordinate changes are Poincaré transformations has been treated as soon as 1911 by Gustav Herglotz*[6] *(1911). Regarding Newtonian mechanics, the German mathematician*

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[4]**Emmy Noether** (1882–1935): German mathematician, known for her works in algebra and topology and for her famous theorem in mathematical physics. Banned from the University of Göttingen by the Nazis, she emigrated to the United States in 1934, where she died one year after.

[5]**David Hilbert** (1862–1943): one of the greatest mathematicians of all times; founder of the Göttingen school, which was in the beginning of the twentieth century the world centre of mathematics. Notably, he hired Hermann Minkowski and Emmy Noether.

[6]**Gustav Herglotz** (1881–1953): German mathematician and astronomer; student of Ludwig Boltzmann, he applied elaborated mathematics to solve astronomical and geophysical problems.

Carl Jacobi (1804–1851) had already shown, in a lecture given in 1842–1843 and published in 1866 (Jacobi 1866), that the conservations of linear momentum and angular momentum result from the invariance by spatial translation and by rotation, respectively.

11.3.2 Application to a Free Particle

In what follows, we suppose that the affine coordinates (x^α) are *inertial* coordinates. The basis $(\vec{e}_\alpha)$ is then orthonormal, and the matrix of the metric tensor is Minkowski matrix: $g_{\alpha\beta} = \eta_{\alpha\beta}$ [Eq. (1.17)]. The Lagrangian of a free particle, as given by (11.21), becomes then

$$L = -mc\sqrt{-\eta_{\alpha\beta}\,\dot{x}^\alpha\,\dot{x}^\beta}. \tag{11.46}$$

An infinitesimal change of inertial coordinates $(x^\alpha) \mapsto (x'^\alpha)$ is by definition an infinitesimal Poincaré transformation [cf. Eq. (8.12)]:

$$x'^\alpha = x^\alpha + \varepsilon G^\alpha(x^\beta) = \Lambda^\alpha{}_\beta x^\beta + c^\alpha, \tag{11.47}$$

where $\Lambda^\alpha{}_\beta$ is a (infinitesimal) Lorentz matrix and the c^α's are four (infinitesimal) constants. We deduce that $\dot{x}'^\alpha(\lambda) = \Lambda^\alpha{}_\beta \dot{x}^\beta(\lambda)$. Consequently, using the property (6.10) of Lorentz matrices,

$$\eta_{\alpha\beta}\,\dot{x}'^\alpha\,\dot{x}'^\beta = \eta_{\alpha\beta}\Lambda^\alpha{}_\mu\Lambda^\beta{}_\nu\,\dot{x}^\mu\,\dot{x}^\nu = \eta_{\mu\nu}\,\dot{x}^\mu\,\dot{x}^\nu. \tag{11.48}$$

This identity shows clearly that the Lagrangian (11.46) is invariant under any Poincaré transformation. The Noether theorem yields then the following conserved quantity:

$$\frac{\partial L}{\partial \dot{x}^\alpha}G^\alpha = mc\,u_\alpha G^\alpha = \text{const}, \tag{11.49}$$

the first equality resulting from expression (11.23) for $\partial L/\partial\dot{x}^\alpha$.

Since (11.47) is an infinitesimal Poincaré transformation, the generators $G = (G^\alpha)$ of the coordinate change are members of the Lie algebra of the Poincaré group studied in Sect. 8.3.4. This algebra being of dimension 10, the Noether theorem provides 10 independent conserved quantities. Let us exhibit each of them when G equals successively the 10 generators of the Poincaré group listed in Sect. 8.3.4:

- $G = P_{\alpha_0}$, $\alpha_0 \in \{0, 1, 2, 3\}$: generator of translations along the vector $\vec{e}_{\alpha_0}$ of the basis associated with the inertial coordinates (x^α). In this case, $\Lambda^\alpha{}_\beta = \delta^\alpha{}_\beta$

and $c^\alpha = \varepsilon \delta^\alpha_{\ \alpha_0}$, so that $G^\alpha(x^\beta) = \delta^\alpha_{\ \alpha_0}$. The conserved quantity (11.49) is then (writing α instead of α_0)

$$\boxed{mc\, u_\alpha = \text{const}}, \qquad \alpha \in \{0, 1, 2, 3\}. \tag{11.50}$$

- $G = K_i$, $i \in \{1, 2, 3\}$: generator of Poincaré transformations associated with Lorentz boosts of plane $\mathrm{Span}(\vec{e}_0, \vec{e}_i)$. In this case, $G^\alpha(x^\beta) = (K_i)^\alpha_{\ \beta} x^\beta$, where the matrix $(K_i)^\alpha_{\ \beta}$ is given by (7.16a)–(7.16b). The conserved quantity (11.49) is then $mc\, (K_i)^\alpha_{\ \beta} x^\beta u_\alpha$, i.e.

$$\boxed{mc(u_0 x^i + u_i x^0) = \text{const}}, \qquad i \in \{1, 2, 3\}. \tag{11.51}$$

- $G = J_i$, $i \in \{1, 2, 3\}$: generator of spatial rotations in the plane $\mathrm{Span}(\vec{e}_0, \vec{e}_i)^\perp$. In this case, $G^\alpha(x^\beta) = (J_i)^\alpha_{\ \beta} x^\beta$, where the matrix $(J_i)^\alpha_{\ \beta}$ is given by (7.16b)–(7.16c). The conserved quantity (11.49) is then $mc\, (J_i)^\alpha_{\ \beta} x^\beta u_\alpha$, i.e.

$$\boxed{mc(u_3 x^2 - u_2 x^3) = \text{const}} \qquad (i = 1) \tag{11.52a}$$

$$\boxed{mc(u_1 x^3 - u_3 x^1) = \text{const}} \qquad (i = 2) \tag{11.52b}$$

$$\boxed{mc(u_2 x^1 - u_1 x^2) = \text{const}} \qquad (i = 3). \tag{11.52c}$$

Let us interpret the 10 conserved quantities (11.50)–(11.52) in view of the results of Chaps. 9 and 10. We recognize in (11.50) the components of the 4-momentum $\boldsymbol{p}$ of particle $\mathscr{P}$ [cf. Eq. (9.3)], so that (11.50) expresses nothing but the conservation of the 4-momentum of an isolated particle. We recover thus the result (9.37). Note in passing that the components $p_\alpha = mc\, u_\alpha$ that are involved in (11.50) are related by (9.31) to the energy E and to the components P_i of $\mathscr{P}$'s linear momentum, both measured by the inertial observer $\mathscr{O}$ whose coordinates are (x^α) :

$$mc\, u_0 = -\frac{E}{c} \qquad \text{and} \qquad mc\, u_i = P_i. \tag{11.53}$$

We can rewrite the four conservation laws (11.50) as

$$E = \text{const} \qquad \text{and} \qquad P_i = \text{const}, \quad i \in \{1, 2, 3\}. \tag{11.54}$$

Let us now focus on the three conservation laws (11.51). Given (11.53) and the relation $x^0 = ct$ between the coordinate x^0 and the proper time t of the inertial observer, these laws are written as

$$-\frac{E}{c} x^i + P_i\, ct = \text{const}, \quad i \in \{1, 2, 3\}. \tag{11.55}$$

Comparing with (10.14), with $h(t) = ct$ (for $\mathscr{O}$ is inertial), we observe that the above equation is nothing but

$$J_{i0} = \text{const}, \quad i \in \{1, 2, 3\}, \tag{11.56}$$

$J_{\alpha\beta}$ standing for the components of $\mathscr{P}$'s angular momentum $\boldsymbol{J}_O$ with respect to the origin O of the coordinate system (x^α).

Remark 11.8. Since E is constant [Eq. (11.54)], one can divide (11.55) by $-E/c$ and use relation (9.28) between $\boldsymbol{P}$, E and $\mathscr{P}$'s velocity $\vec{\boldsymbol{V}} = V^i \vec{\boldsymbol{e}}_i$ with respect to $\mathscr{O}$ to obtain

$$x^i - V^i t = \text{const}, \tag{11.57}$$

which does correspond to the uniform rectilinear motion of an isolated particle.

Finally, the three conservation laws (11.52) can be rewritten, thanks to (11.53) as $x^2 P_3 - x^3 P_2 = \text{const}$, $x^3 P_1 - x^1 P_3 = \text{const}$ and $x^1 P_2 - x^2 P_1 = \text{const}$. Comparing with (10.14) and (10.15) and taking into account that $P^i = P_i$, we conclude that they are equivalent to

$$J_{ij} = \text{const}, \quad i, j \in \{1, 2, 3\}, \ j > i, \tag{11.58}$$

or to

$$\sigma^i_{\ O} = \text{const}, \quad i \in \{1, 2, 3\}, \tag{11.59}$$

the $\sigma^i_{\ O}$'s being the components of $\mathscr{P}$'s angular momentum vector $\vec{\boldsymbol{\sigma}}_O$ with respect to the point O and measured by observer $\mathscr{O}$. We thus recover the law (10.30) of conservation of the angular momentum vector for an isolated particle and an inertial observer.

11.3.2.1 Summary

The invariance of the Lagrangian of a free particle under the Poincaré group leads, via the Noether theorem, to the conservation of the 4-momentum linear form $\boldsymbol{p}$ (invariance under translations) and of the angular momentum 2-form $\boldsymbol{J}_O$ (invariance under the Lorentz boosts and spatial rotations).

Remark 11.9. For a system reduced to a single particle, we have seen in Chap. 10 that these two conservation laws are not independent, the conservation of $\boldsymbol{p}$ leading that of $\boldsymbol{J}_O$ (cf. Remark 10.6 p. 326).

11.4 Hamiltonian Formulation

The Hamiltonian formulation of any classical[7] theory is worth investigating, since it is the starting point for the canonical quantization of the theory. After a few reminders about the Hamiltonian formulation of nonrelativistic mechanics, we shall examine the case of a relativistic particle. Systems of particles will be discussed in Sect. 11.5.2.

11.4.1 Reminder of Nonrelativistic Hamiltonian Mechanics

Let us extend the reminder of nonrelativistic mechanics of Sect. 11.2.1 from the Lagrangian formulation to the Hamiltonian one. From the Lagrangian (11.1), one defines the N ***generalized momenta***, also called ***conjugate momenta***, by

$$p_a := \frac{\partial L}{\partial \dot{q}_a}, \qquad a \in \{1, \ldots, N\}. \tag{11.60}$$

Each p_a is a function of t. Denoting $\dot{p}_a := \mathrm{d}p_a/\mathrm{d}t$, the Euler–Lagrange equations (11.3) become

$$\dot{p}_a = \frac{\partial L}{\partial q_a}, \qquad a \in \{1, \ldots, N\}. \tag{11.61}$$

One then introduces the ***Hamiltonian*** of the system by

$$H := \sum_{a=1}^{N} p_a \dot{q}_a - L. \tag{11.62}$$

In the right-hand side, the variables $\dot{q}_a$'s are assumed to be functions of the q_a's and the p_a's, i.e. one assumes that (11.60) can be inverted to get $\dot{q}_a = \dot{q}_a(q_b, p_b)$. Equation (11.62) is named a ***Legendre transformation***, and one considers that H is a function of the generalized coordinates and generalized momenta (as well as of time if L depends explicitly on t):

$$H = H(q_1, \ldots, q_N, p_1, \ldots, p_N, t).$$

Taking the differential of H and taking into account (11.60) and (11.61) (i.e. the equations of motion resulting from the principle of least action), we obtain the ***canonical equations of Hamilton***:

[7] That is to say, *non-quantum*.

$$\dot{q}_a = \frac{\partial H}{\partial p_a}, \qquad \dot{p}_a = -\frac{\partial H}{\partial q_a}, \qquad a \in \{1, \ldots, N\} \tag{11.63}$$

as well as the relation $\partial H/\partial t = -\partial L/\partial t$. The $2N$ first-order equations (11.63) are equivalent to the N second-order Euler–Lagrange equations (11.3). Note that Eqs. (11.63) imply

$$\frac{\mathrm{d}H}{\mathrm{d}t} = -\frac{\partial L}{\partial t}. \tag{11.64}$$

Hence, if L is not an explicit function of time, H is a constant of motion.

The Lagrangian formalism is based on the variables $(q_a(t))$, which span an N-dimensional space *(the configuration space)*. On the other hand, the Hamiltonian formalism is based on the variables $(q_a(t), p_a(t))$, which span a $2N$-dimensional space (a part of $\mathbb{R}^{2N}$ or more generally a *manifold* (cf. Sect. 7.2.1) of dimension $2N$) called the ***phase space*** and denoted by P. For any couple (f, g) of functions $\mathsf{P} \to \mathbb{R}$, one defines the ***Poisson bracket*** of f and g by

$$\{f, g\} := \sum_{a=1}^{N} \left(\frac{\partial f}{\partial q_a}\frac{\partial g}{\partial p_a} - \frac{\partial f}{\partial p_a}\frac{\partial g}{\partial q_a} \right). \tag{11.65}$$

Hence, $\{f, g\}$ is a map $\mathsf{P} \to \mathbb{R}$, as are f and g. The Poisson bracket is clearly antisymmetric and bilinear. Moreover, it is easy to show that it satisfies the ***Jacobi identity***:

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0. \tag{11.66}$$

The Poisson bracket fulfils then the three axioms of the definition of a *Lie bracket* given in Sect. 7.3.2: it is a Lie bracket on the (infinite-dimensional) vector space formed by scalar functions on P. Consequently, the Poisson bracket provides this vector space with a Lie algebra structure.

From the definition (11.65), it is immediate to check that

$$\{q_a, q_b\} = 0, \qquad \{q_a, p_b\} = \delta_{ab}, \quad \text{and} \quad \{p_a, p_b\} = 0. \tag{11.67}$$

The canonical equations of Hamilton (11.63) become then

$$\dot{q}_a = \{q_a, H\} \qquad \text{and} \qquad \dot{p}_a = \{p_a, H\}. \tag{11.68}$$

Note that contrary to (11.63), this writing is symmetric in q_a and p_a. We deduce from this fact that for any function $f : \mathsf{P} \to \mathbb{R}$, $(q_a, p_a) \mapsto f(q_a, p_a)$, the following relation holds:

$$\frac{\mathrm{d}f}{\mathrm{d}t} := \frac{\mathrm{d}}{\mathrm{d}t} f(q_a(t), p_a(t)) = \{f, H\}. \tag{11.69}$$

One calls ***canonical transformation*** any coordinate change in phase space, $(q_a, p_a) \mapsto (q'_a, p'_a)$, that preserves the canonical equations of Hamilton, i.e. any coordinate change such that if $H(q_a, p_a)$ satisfies (11.63), there exists a function $H'(q'_a, p'_a)$ so that

$$\dot{q}'_a = \frac{\partial H'}{\partial p'_a}, \qquad \dot{p}'_a = -\frac{\partial H'}{\partial q'_a}, \qquad a \in \{1, \ldots, N\}. \tag{11.70}$$

It can be shown that any canonical transformation is entirely defined by a function $F : \mathsf{P} \to \mathbb{R}$, called ***generating function of the canonical transformation***, such that

$$H'(q'_a, p'_a) = H(q_a, p_a) + \sum_{a=1}^{N} p'_a \dot{q}'_a - \sum_{a=1}^{N} p_a \dot{q}_a + \frac{\mathrm{d}F}{\mathrm{d}t}. \tag{11.71}$$

By means of (11.62), this relation takes a simpler form in terms of Lagrangians:

$$L'(q'_a, \dot{q}'_a, t) = L(q_a, \dot{q}_a, t) - \frac{\mathrm{d}F}{\mathrm{d}t}. \tag{11.72}$$

For example, if $F = F_1(q_a, q'_a, t)$, then necessarily

$$p_a = \frac{\partial F_1}{\partial q_a}(q_a, q'_a, t) \qquad \text{and} \qquad p'_a = -\frac{\partial F_1}{\partial q'_a}(q_a, q'_a, t), \tag{11.73}$$

$$H'(q'_a, p'_a) = H(q_a, p_a) + \frac{\partial F_1}{\partial t}(q_a, q'_a, t). \tag{11.74}$$

If F_1 is given, the first of the relations (11.73) must be inverted to get $q'_a = q'_a(q_b, p_b)$. Inserting the obtained value in the second relation (11.73), one obtains $p'_a = p'_a(q_b, p_b)$, which shows that the choice of F_1 fully determines the canonical transformation $(q_a, p_a) \mapsto (q'_a, p'_a)$, hence the name *generating function*.

Another possible choice is $F = F_2(q_a, p'_a, t) - \sum_{a=1}^{N} q'_a p'_a$. We have then

$$p_a = \frac{\partial F_2}{\partial q_a}(q_a, p'_a, t) \qquad \text{and} \qquad q'_a = \frac{\partial F_2}{\partial p'_a}(q_a, p'_a, t), \tag{11.75}$$

$$H'(q'_a, p'_a) = H(q_a, p_a) + \frac{\partial F_2}{\partial t}(q_a, p'_a, t). \tag{11.76}$$

Remark 11.10. The Euler–Lagrange equations (11.3) are invariant by any coordinate change $(q_a) \mapsto (q'_a)$ in the configuration space, provided that one considers as new Lagrangian $L'(q'_a, \dot{q}'_a, t) := L(q_a, \dot{q}_a, t)$ with $q_a = q_a(q'_b)$ and $\dot{q}_a = \sum_{b=1}^{N}(\partial q_a/\partial q'_b)\dot{q}'_b$. The canonical transformations are more general than the transformations $(q_a) \mapsto (q'_a)$ since they "mix" the q's and the p's. The coordinate changes $(q_a) \mapsto (q'_a)$ are actually particular canonical transformations of the type

$$q'_a = q'_a(q_b) \quad \text{and} \quad p'_a = \sum_{b=1}^{N} p_b \frac{\partial q_b}{\partial q'_a}. \tag{11.77}$$

They are generated by the function $F_2(q_a, p'_a, t) = \sum_{a=1}^{N} q'_a(q_b) p'_a$. An example of a canonical transformation that is not a coordinate change in the configuration space is $q'_a = p_a,\ p'_a = -q_a$. This transformation is generated by the function $F_1 = \sum_{a=1}^{N} q_a q'_a$ and amounts to swapping the generalized coordinates and generalized momenta, these two quantities being on the same footing in the Hamiltonian formalism.

We have the fundamental theorem:

A change of variables in phase space, $(q_a, p_a) \mapsto (q'_a, p'_a)$, is a canonical transformation iff it *preserves the Poisson bracket*. This last property means that for any couple (f, g) of functions defined on the phase space,

$$\sum_{a=1}^{N} \left(\frac{\partial f}{\partial q_a}\frac{\partial g}{\partial p_a} - \frac{\partial f}{\partial p_a}\frac{\partial g}{\partial q_a} \right) = \sum_{a=1}^{N} \left(\frac{\partial f}{\partial q'_a}\frac{\partial g}{\partial p'_a} - \frac{\partial f}{\partial p'_a}\frac{\partial g}{\partial q'_a} \right).$$

It is easy to see that it is equivalent to demand that the variables (q'_a, p'_a) obey the same relations as (q_a, p_a), namely, (11.67):

$$\{q'_a, q'_b\} = 0, \qquad \{q'_a, p'_b\} = \delta_{ab}, \quad \text{and} \quad \{p'_a, p'_b\} = 0. \tag{11.78}$$

An infinitesimal canonical transformation is generated by a function F_2 of the type

$$F_2(q_a, p'_a, t) = \sum_{a=1}^{N} q_a p'_a + \varepsilon G(q_a, p'_a, t), \tag{11.79}$$

where $\varepsilon \in \mathbb{R}$ is some infinitesimal parameter and G is any function $\mathbb{R}^{2N+1} \to \mathbb{R}$. Indeed, from (11.75), this choice leads to

$$q'_a = q_a + \varepsilon \frac{\partial G}{\partial p'_a}(q_a, p'_a, t) \quad \text{and} \quad p'_a = p_a - \varepsilon \frac{\partial G}{\partial q_a}(q_a, p'_a, t), \tag{11.80}$$

which does correspond to an infinitesimal transformation $(q_a, p_a) \mapsto (q'_a, p'_a)$. The function G is called ***generator of the infinitesimal canonical transformation***. We can express (11.80) in terms of the Poisson bracket with G:

$$q'_a = q_a + \varepsilon\{q_a, G\} \quad \text{and} \quad p'_a = p_a + \varepsilon\{p_a, G\}. \tag{11.81}$$

The relation between the Hamiltonian in the new coordinates, $H'(q'_a, p'_a)$, and that in the old ones, $H(q_a, p_a)$, is given by (11.76). By means of (11.79) and (11.80), as well as the equations of motion (11.63), we obtain

$$H'(q'_a, p'_a) = H(q'_a, p'_a) + \varepsilon \frac{\mathrm{d}G}{\mathrm{d}t}. \tag{11.82}$$

We conclude that G is a constant of motion ($\mathrm{d}G/\mathrm{d}t = 0$) iff the Hamiltonian is invariant under the infinitesimal canonical transformation generated by G ($H'(q'_a, p'_a) = H(q'_a, p'_a)$). This result, which relates conserved quantities to the symmetries of the system, is a "Hamiltonian version" of the Noether theorem discussed in Sect. 11.3.

11.4.2 Generalized Four-Momentum of a Relativistic Particle

Let us consider the description of a relativistic particle by means of a Lagrangian, as performed in Sect. 11.2.3. One defines the ***generalized four-momentum***, or ***generalized 4-momentum*** for short, of particle $\mathscr{P}$ as the field of linear forms $\boldsymbol{p}$ along $\mathscr{L}$ whose components $p_\alpha = \langle \boldsymbol{p}, \vec{e}_\alpha \rangle$ in the basis dual[8] to $(\vec{e}_\alpha)$ are given at any point by

$$\boxed{p_\alpha := \frac{\partial L}{\partial \dot{x}^\alpha}}. \tag{11.83}$$

This formula is the exact analogue of (11.60). Let us show that the quantity $\boldsymbol{p}$ hence defined is independent of the parametrization of $\mathscr{L}$, i.e. depends only on the considered point of $\mathscr{L}$. This is not obvious a priori since, at any point $M \in \mathscr{L}$ of parameter λ, the definition (11.83) can be written explicitly as $p_\alpha = \partial L/\partial \dot{x}^\alpha \, (x^\mu(\lambda), \dot{x}^\mu(\lambda))$ and if the numerical value of $x^\mu(\lambda)$ at M is independent of λ (this is some coordinate of M in the considered affine frame), things are different

[8] Cf. Sect. 1.6.1.

for $\dot{x}^\mu(\lambda)$. The independence of p_α from λ results actually from the homogeneous character of the Lagrangian function with respect to the generalized velocities, i.e. from the independence of the action with respect to the worldline parametrization (cf. Sect. 11.2.3). Indeed, by deriving (11.10) with respect to $\dot{x}^\alpha$, one gets

$$\forall \mu > 0,\ \forall (x^\beta, \dot{x}^\beta) \in \mathbb{R}^8, \quad \frac{\partial L}{\partial \dot{x}^\alpha}(x^\beta, \mu \dot{x}^\beta) = \frac{\partial L}{\partial \dot{x}^\alpha}(x^\beta, \dot{x}^\beta). \tag{11.84}$$

In other words, $\partial L / \partial \dot{x}^\alpha$ is a homogeneous function of degree 0 with respect to the variables $(\dot{x}^\alpha)$. If one performs a change of parametrization, $\lambda \mapsto \tilde{\lambda}$, (11.8), (11.9) and (11.84) lead to

$$\frac{\partial L}{\partial \dot{x}^\alpha}\left(\tilde{x}^\beta(\tilde{\lambda}), \dot{\tilde{x}}^\beta(\tilde{\lambda})\right) = \frac{\partial L}{\partial \dot{x}^\alpha}\left(x^\beta(\lambda), \dot{x}^\beta(\lambda)\right),$$

which proves the independence of p_α with respect to λ. □

The generalized 4-momentum takes its value from the Noether theorem (11.45), since the latter can be recast as

$$\boxed{L(x^\alpha + \varepsilon G^\alpha, \dot{x}^\alpha + \varepsilon \dot{G}^\alpha) = L(x^\alpha, \dot{x}^\alpha) \quad \Longrightarrow \quad p_\alpha G^\alpha = \text{const}}. \tag{11.85}$$

In particular, if the Lagrangian is invariant under translations, $G^\beta(x^\mu) = \delta^\beta_\alpha$ and the generalized 4-momentum is constant along the worldline of the particle.

The generalized 4-momentum of a free particle is given by (11.23):

$$\boxed{\boldsymbol{p} = mc\,\underline{\boldsymbol{u}}} \qquad \text{(free particle)}. \tag{11.86}$$

We note that it coincides with the "ordinary" 4-momentum defined in Chap. 9 [cf. Eq. (9.3)]. In particular, it obeys (9.2):

$$\langle \boldsymbol{p}, \vec{\boldsymbol{p}} \rangle = g^{\alpha\beta} p_\alpha p_\beta = -m^2 c^2. \tag{11.87}$$

For a particle in a vector field, Eq. (11.30) gives immediately

$$\boxed{\boldsymbol{p} = mc\,\underline{\boldsymbol{u}} + \frac{q}{c}\boldsymbol{A}} \qquad \text{(particle in a vector field)}. \tag{11.88}$$

Thus, in this case, the generalized 4-momentum differs from the 4-momentum defined in Chap. 9 by a term proportional to the potential 1-form $\boldsymbol{A}$. The 4-velocity normalization yields

$$\left\langle \boldsymbol{p} - \frac{q}{c}\boldsymbol{A},\ \vec{\boldsymbol{p}} - \frac{q}{c}\vec{\boldsymbol{A}} \right\rangle = -m^2 c^2. \tag{11.89}$$

Remark 11.11. In Chap. 9, we have stressed that the 4-momentum is fundamentally a linear form and not a vector. The Noether theorem, under the form (11.85), is one of the main justifications for this feature: the G^α's are fundamentally the components of a vector, $\vec{G}$, that connects the point $\vec{x}$ to the point $\vec{x}\,' = \vec{x} + \epsilon\vec{G}$ [an active viewpoint is adopted here regarding the coordinate change (11.42)], and the conserved quantity provided by (11.85) is simply the linear form $\boldsymbol{p}$ applied to the vector $\vec{G}$: $p_\alpha G^\alpha = \langle \boldsymbol{p}, \vec{G}\rangle$. Another justification is the equality (11.88), where $\boldsymbol{A}$ is fundamentally a linear form: those from which the antisymmetric bilinear form "electromagnetic field" $\boldsymbol{F}$ is deduced via (11.34). The latter identity involves the *exterior derivative* operator, which is well defined for linear forms, but not for vectors, as we shall see in Chap. 15.

11.4.3 Hamiltonian of a Relativistic Particle

Once the generalized 4-momentum is introduced, it would seem natural to define the Hamiltonian of a relativistic particle by a formula analogous to (11.62): $H := p_\alpha \dot{x}^\alpha - L$. There are however two problems with this formula, both related to the homogeneity of L with respect to $\dot{x}^\alpha$. The first one is that substituting p_α by its definition (11.83) results in an identically vanishing Hamiltonian, by virtue of Euler theorem (11.11): $H = (\partial L/\partial \dot{x}^\alpha)\,\dot{x}^\alpha - L = 0$. Secondly, in the Hamiltonian formalism, the variables are (x^α, p_α), and the relation (11.83) must be invertible to express the $\dot{x}^\alpha$'s as functions of (x^α, p_α); this is necessary in order to replace them in L at the right-hand side of $H := p_\alpha \dot{x}^\alpha - L$ to obtain $H = H(x^\alpha, p_\alpha)$. Now, the Jacobian matrix $(\partial p_\alpha/\partial \dot{x}^\beta)$ deduced from (11.83) is

$$\frac{\partial p_\alpha}{\partial \dot{x}^\beta} = \frac{\partial^2 L}{\partial \dot{x}^\alpha \partial \dot{x}^\beta}.$$

But taking the derivative of (11.11) with respect to $\dot{x}^\beta$ leads to

$$\dot{x}^\alpha \frac{\partial^2 L}{\partial \dot{x}^\alpha \partial \dot{x}^\beta} = 0.$$

This shows that $(\dot{x}^\alpha)$ is a nonvanishing vector in the kernel of the Jacobian matrix $(\partial p_\alpha/\partial \dot{x}^\beta)$. This matrix is thus not invertible. By virtue of the local inversion theorem, we conclude that, at fixed (x^α), the map $(\dot{x}^\alpha) \mapsto (p_\alpha)$ is not invertible. This is not surprising: a priori, the $x^\alpha(\lambda)$'s are 4 independent functions, whereas the $p_\alpha(\lambda)$'s are linked by (11.87) (free particle), (11.89) (particle in a vector field) or a similar relation (other cases).

Remark 11.12. The fact that only 3 of the 4 components of the generalized 4-momentum are independent reflects the 3 degrees of freedom of a particle. The independence of the 4 functions $(x^\alpha(\lambda))$ does not yield 4 physical degrees of freedom, because of the freedom in choosing the parameter λ.

Relations of the type (11.87) or (11.89) are called ***primary constraints*** on the considered system; this means that they do not depend upon the equations of motion. The standard procedure, developed by Dirac[9] (1964), consists in choosing an Hamiltonian proportional to the constraint.[10] In the case of a particle in a vector field, one defines the ***Hamiltonian*** by

$$\boxed{H(x^\alpha, p_\alpha) = \frac{1}{2m} g^{\alpha\beta} \left[p_\alpha - \frac{q}{c} A_\alpha(x^\mu) \right] \left[p_\beta - \frac{q}{c} A_\beta(x^\mu) \right]}. \tag{11.90}$$

The Hamiltonian of a free particle is deduced from the above formula by setting $q = 0$. Note that H has the dimension of an energy. By virtue of the primary constraint (11.89), the value of H is constant: $H = -\frac{1}{2}mc^2$. Nevertheless, what matters for the canonical equations of motion is the functional dependency of the Hamiltonian with respect to (x^α, p_α) and not its numerical value.

The canonical equations of Hamilton are

$$\boxed{\dot{x}^\alpha = \frac{\partial H}{\partial p_\alpha}}, \qquad \boxed{\dot{p}_\alpha = -\frac{\partial H}{\partial x^\alpha}}, \qquad \alpha \in \{0, 1, 2, 3\}. \tag{11.91}$$

Let us check that they yield the usual equations of motion. We have, from (11.90),

$$\frac{\partial H}{\partial x^\alpha} = -\frac{q}{mc} g^{\mu\nu} \frac{\partial A_\mu}{\partial x^\alpha} \left(p_\nu - \frac{q}{c} A_\nu \right) \qquad \text{and} \qquad \frac{\partial H}{\partial p_\alpha} = \frac{1}{m} g^{\alpha\mu} \left(p_\mu - \frac{q}{c} A_\mu \right),$$

so that the canonical equations (11.91) become

$$\dot{x}^\alpha = \frac{1}{m} g^{\alpha\mu} \left(p_\mu - \frac{q}{c} A_\mu \right) \qquad \text{and} \qquad \dot{p}_\alpha = \frac{q}{mc} g^{\mu\nu} \frac{\partial A_\mu}{\partial x^\alpha} \left(p_\nu - \frac{q}{c} A_\nu \right).$$

Multiplying the first equation by the matrix $(g_{\alpha\beta})$, this system can be rewritten as

$$m\dot{x}_\alpha = p_\alpha - \frac{q}{c} A_\alpha \qquad \text{and} \qquad \dot{p}_\alpha = \frac{q}{c} \frac{\partial A_\beta}{\partial x^\alpha} \dot{x}^\beta,$$

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[9]**Paul A. M. Dirac** (1902–1984): British theoretical physicist, well known for his contributions to quantum mechanics and quantum electrodynamics; notably, he has proposed the equation ruling the dynamics of relativistic electrons in quantum mechanics and has predicted the existence of antimatter; he was awarded the 1933 Nobel Prize in Physics.

[10]Cf., e.g. Chap. 8 of the treatise by Sudarshan and Mukunda (1974).

with $\dot{x}_\alpha := g_{\alpha\beta}\dot{x}^\beta$. The derivative of the first equation with respect to λ leads to $\dot{p}_\alpha = m\ddot{x}_\alpha + \frac{q}{c}\partial A_\alpha/\partial x^\beta\, \dot{x}^\beta$, so that we can rewrite the second equation in terms of $\ddot{x}_\alpha$ and obtain the system

$$m\dot{x}_\alpha = p_\alpha - \frac{q}{c}A_\alpha \tag{11.92a}$$

$$m\ddot{x}_\alpha = \frac{q}{c}F_{\alpha\beta}\dot{x}^\beta, \tag{11.92b}$$

where $F_{\alpha\beta}$ is defined by (11.34). Equation (11.92b) greatly resembles the equation of motion (11.33) deduced from the Euler–Lagrange equations. To have a complete identity, $\ddot{x}_\alpha$ should be related to the 4-acceleration of the particle and $\dot{x}^\alpha$ to its 4-velocity. We are going to see that is necessarily the case. Indeed, multiplying (11.92b) by $\dot{x}^\alpha$ and summing on α, we get

$$m\ddot{x}_\alpha\dot{x}^\alpha = \frac{q}{c}F_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = 0$$

thanks to the antisymmetry of $F_{\alpha\beta}$. Using $\ddot{x}_\alpha = g_{\alpha\beta}\ddot{x}^\beta$, we obtain

$$g_{\alpha\beta}\dot{x}^\alpha\ddot{x}^\beta = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta\right) = 0,$$

where the first equality results from the symmetry of $g_{\alpha\beta}$. This means that $v^2 := -\vec{v}\cdot\vec{v} = -g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta$ is constant along the particle's worldline. The relation (11.20) between the parameter λ and the particle's proper time τ is thus $\mathrm{d}\lambda = (c/v)\mathrm{d}\tau$ with $(c/v) = \text{const}$. This shows that λ is essentially the proper time:

$$\lambda = \alpha\tau + \lambda_0, \qquad \alpha = \text{const} \quad \text{and} \quad \lambda_0 = \text{const}. \tag{11.93}$$

For simplicity, let us choose $\alpha = 1$ and $\lambda_0 = 0$, so that

$$\boxed{\lambda = \tau}. \tag{11.94}$$

We have then $\dot{x}^\alpha = \mathrm{d}x^\alpha/\mathrm{d}\tau = cu^\alpha$ and $\ddot{x}^\alpha = \mathrm{d}^2x^\alpha/\mathrm{d}\tau^2 = c^2a^\alpha$. Consequently, (11.92a) gives the relation (11.88) between the 4-velocity and the generalized 4-momentum and (11.92b) gives the equation of motion (11.33), obtained within the Lagrangian formalism. Let us conclude:

> The canonical equations (11.91) applied to the Hamiltonian (11.90) lead to (i) the identification of the parameter λ with the particle's proper time and (ii) to the motion under the action of the Lorentz 4-force (11.35).

Remark 11.13. If the Lagrangian had not been a homogeneous function of degree 1 of the $\dot{x}^\alpha$'s, we could have applied the standard procedure of building the Hamiltonian via the Legendre transformation: $H := p_\alpha \dot{x}^\alpha - L$. Now, as we have seen in Sect. 11.2.3, the homogeneity of the Lagrangian is the consequence of the independence of the action S from the parametrization of the worldline. An alternative is to consider an action that is not independent of the worldline parametrization, arguing that what matters after all is that the principle of least action leads to the correct equations of motion. This is notably the point of view adopted in the textbooks by Goldstein et al. (2002) and Gruber and Benoit (1998) (cf. also the discussion in Leubner (1986)). For instance, one may consider the following Lagrangian to describe a particle in a vector field:

$$L = \frac{1}{2} m\, g_{\alpha\beta}\, \dot{x}^\alpha \dot{x}^\beta + \frac{q}{c} A_\beta(x^\alpha)\, \dot{x}^\beta. \tag{11.95}$$

It differs from the Lagrangian (11.28) solely by the "free particle" part: $1/2\, m\, g_{\alpha\beta}\, \dot{x}^\alpha \dot{x}^\beta$ instead of $-mc\sqrt{-g_{\alpha\beta}\, \dot{x}^\alpha \dot{x}^\beta}$. This Lagrangian is not homogeneous in the $\dot{x}^\alpha$'s. The four Euler–Lagrange equations are then independent, contrary to the case where L is a homogeneous function of degree 1 (cf. Remark 11.1 p. 353). They lead to the four equations of motions (11.92b): $m\ddot{x}_\alpha = (q/c)\, F_{\alpha\beta}\dot{x}^\beta$. We have shown above that these equations imply that the parameter λ coincides with the proper time τ, up to some constant factors [Eq. (11.93)]. The three remaining degrees of freedom give the motion under the action of a Lorentz force. The generalized 4-momentum deduced from the Lagrangian (11.95) via the definition (11.83) is

$$p_\alpha = m\dot{x}_\alpha + \frac{q}{c} A_\alpha. \tag{11.96}$$

The Hamiltonian formed from L via the Legendre transformation $H := p_\alpha \dot{x}^\alpha - L$ is then

$$H = \frac{1}{2} m\, g_{\alpha\beta}\, \dot{x}^\alpha \dot{x}^\beta. \tag{11.97}$$

Taking into account (11.96), we observe that it coincides with the Hamiltonian (11.90).

11.5 Systems of Particles

Up to now, we have only considered systems reduced to a single particle, evolving in a given (scalar, vector or tensor) field. Here we briefly discuss the extension to systems of many particles.

11.5.1 Principle of Least Action

As we have underlined in Sect. 11.2.2, the principle of least action of the pre-relativistic analytical mechanics is not directly transposable to a system of relativistic particles, for there is no uniqueness of the time t. A principle of least action can nevertheless be formulated by considering as many time integration parameters as the number N of particles. More precisely, if the equation of the worldline $\mathscr{L}_a$ of particle no. a in an affine frame (x^α) of $\mathscr{E}$ is written as[11]

$$\mathscr{L}_a : \qquad x^\alpha = x_a^\alpha(\lambda_a), \qquad \lambda_a \in \mathbb{R}, \qquad \alpha \in \{0, 1, 2, 3\}, \tag{11.98}$$

a rather general form of the action is

$$\begin{aligned} S = & -\sum_{a=1}^{N} m_a c \int_{\lambda_a^-}^{\lambda_a^+} \sqrt{-g_{\alpha\beta}\dot{x}_a^\alpha \dot{x}_a^\beta}\, \mathrm{d}\lambda_a \\ & + \sum_{a=1}^{N} \sum_{b=a+1}^{N} q_a q_b \int_{\lambda_a^-}^{\lambda_a^+} \int_{\lambda_b^-}^{\lambda_b^+} K(x_a^\alpha, \dot{x}_a^\alpha, x_b^\alpha, \dot{x}_b^\alpha)\, \mathrm{d}\lambda_a\, \mathrm{d}\lambda_b, \end{aligned} \tag{11.99}$$

where $x_a^\alpha = x_a^\alpha(\lambda_a)$, $\dot{x}_a^\alpha = \dot{x}_a^\alpha(\lambda_a) := \mathrm{d}x_a^\alpha/\mathrm{d}\lambda_a$, m_a and q_a are constants and K is a function $\mathbb{R}^4 \to \mathbb{R}$. We recognize in the first term of S the sum of the individual actions of free particles [cf. Eq. (11.21)], m_a being interpreted as the "free mass" of particle a. The second term, involving the double integral, describes the interaction between the particles, q_a being the ***coupling constant***, or ***interaction charge***, of particle a: if $q_a = 0$, particle a does not interact with any other. The action (11.99) is called ***Tetrode–Fokker action***. The only dynamical variables that it contains are the particle positions $(x_a^\alpha(\lambda_a))$. This means that the interaction between the particles is described without appealing to the notion of field. One says that it is an ***action at a distance***.

We shall not derive here the equations of motion by applying the principle of least action to the Tetrode–Fokker action (11.99), referring the interested reader to the book by Barut (1964) (Chap. VI) or that by Sudarshan and Mukunda (1974) (Chap. 22). Let us stress that the obtained equations are not second-order differential equations but *integro-differential equations*. Finding a solution to these equations is much more complicated than solving differential equations. In particular, integro-differential equations cannot be formulated as a *Cauchy problem*, for which the uniqueness of the solution would have been guaranteed, given some initial data on positions and velocities.

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[11]Contrary to Sect. 11.2.3, we must now use distinct notations for the affine coordinates on $\mathscr{E}$, (x^α), and the functions defining the worldline, (x_a^α).

Two specific examples of Tetrode–Fokker action are interesting:

11.5.1.1 Scalar Interaction at a Distance

One describes a ***scalar interaction*** propagating between the particles at the speed of light when one chooses the following function K in (11.99):

$$K(x_a^\alpha, \dot{x}_a^\alpha, x_b^\alpha, \dot{x}_b^\alpha) := \frac{1}{4\pi}\sqrt{-g_{\alpha\beta}\dot{x}_a^\alpha \dot{x}_a^\beta}\sqrt{-g_{\alpha\beta}\dot{x}_b^\alpha \dot{x}_b^\beta}\;\delta\Big(g_{\alpha\beta}(x_a^\alpha - x_b^\alpha)(x_a^\beta - x_b^\beta)\Big), \tag{11.100}$$

where δ is the Dirac delta function. The qualifier *scalar* means that the dependency of K with respect to the generalized 4-velocities of the particles, $(\dot{x}_a^\alpha)$, takes place only via their norms. Incidentally, let us notice that the terms in $\dot{x}_a^\alpha$ in K are the simplest ones to guarantee the invariance of the action by a reparametrization of the worldlines, since $\sqrt{-g_{\alpha\beta}\dot{x}_a^\alpha \dot{x}_a^\beta}\,\mathrm{d}\lambda_a$ is independent of the parameter λ_a (this is the element of proper time along $\mathscr{L}_a$). The δ term in K ensures that the interaction propagates at the speed of light. Indeed, if an event $A(\lambda_a) \in \mathscr{L}_a$ is fixed, the δ term will keep only the events that obey $g_{\alpha\beta}(x_a^\alpha - x_b^\alpha)(x_a^\beta - x_b^\beta) = 0$ in the integral on λ_b, i.e. only the events belonging to the null cone of $A(\lambda_a)$. Note that the two sheets of the null cone, past and future, are concerned.

11.5.1.2 Wheeler–Feynman Electrodynamics

The case of a ***vectorial interaction*** is described by the function

$$K(x_a^\alpha, \dot{x}_a^\alpha, x_b^\alpha, \dot{x}_b^\alpha) := \frac{1}{4\pi} g_{\alpha\beta}\, \dot{x}_a^\alpha \dot{x}_b^\beta\;\delta\Big(g_{\alpha\beta}(x_a^\alpha - x_b^\alpha)(x_a^\beta - x_b^\beta)\Big). \tag{11.101}$$

With respect to the scalar interaction (11.100), we observe that, this time, K depends on the generalized 4-velocities via the scalar product $g_{\alpha\beta}\,\dot{x}_a^\alpha \dot{x}_b^\beta$ of the generalized 4-velocity of particle a with that of particle b. The δ term is, on its side, identical to that of (11.100); the interaction propagates thus at the speed of light.

The choice (11.101), inserted into the action (11.99), leads to a theory of electromagnetic interactions called ***Wheeler–Feynman electrodynamics***, the constants q_a being then the ***electric charges*** of the particles. *If one disregards the electromagnetic radiation reaction*, the Wheeler–Feynman electrodynamics is physically equivalent to Maxwell electromagnetism: it leads to particle motions that are identical to that resulting from the half-sum of retarded and advanced solutions of Maxwell equations (the so-called *Liénard–Wiechert potentials*, which we shall discuss in Chap. 18). It is necessary to take into account the advanced potentials to get the equivalence with Maxwell theory because of the δ term in (11.101), which is symmetric with respect to the future and past sheets of the light cone.

Let us stress that Wheeler–Feynman electrodynamics does not use the notion of field, contrary to Maxwell electromagnetism. It describes directly the interaction between the particles via the action given by (11.99) and (11.101). In particular, there is no electromagnetic radiation in this theory. However, one can take into account what is usually called the *reaction to electromagnetic radiation* and whose existence is shown by the experiment. To this aim, the electric charges of the radiation detectors are added to the system; they are called ***absorbers*** and are assumed to be distributed at the periphery of the system. The acceleration of the absorbers can then be interpreted as the effect of electromagnetic radiation, and the action at a distance of the absorbers onto the particles of the initial system can be interpreted as the radiation reaction.

Remark 11.14. In Wheeler–Feynman theory, there is no issue of self-interaction (action of a particle onto itself), along with the associated divergences, since b is always different from a in the double sum (11.99).

Historical note: *In 1903, i.e. two years before the advent of special relativity, Karl Schwarzschild*[12] *expressed the mutual action of two moving electrons as an interaction at a distance (Schwarzschild 1903a,b). He was however not aiming at founding all the electrodynamics on this principle. A general formulation of the interaction at a distance within the relativistic framework has been obtained by Hugo Tetrode*[13] *in 1922 (Tetrode 1922) and Adriaan D. Fokker (cf. p. 339) in 1929 (Fokker 1929b). They introduced the action (11.99) with the expression (11.101) for K. In 1949, John A. Wheeler (cf. p. 79) and Richard Feynman*[14] *extended the Tetrode–Fokker theory and showed notably that it does not violate causality, despite the presence of the future null cone induced by the δ term in (11.101) (Wheeler and Feynman 1949). The concept of absorber was introduced by Tetrode (1922) and developed by Wheeler and Feynman (1945). The solution to the problem of two electric charges in circular motion within Wheeler–Feynman electrodynamics was*

[12]**Karl Schwarzschild** (1873–1916): German astrophysicist, known for having found in 1915 the first (non-trivial) exact solution of the equations of general relativity—a solution which will be recognized later on as describing a static black hole. He died from some disease the year after, while serving as a soldier on the Russian front.

[13]**Hugo Tetrode** (1895–1931): Dutch physicist who authored works in quantum mechanics; he died from tuberculosis at 35.

[14]**Richard Feynman** (1918–1988): American theoretical physicist, student of John Wheeler (cf. p. 79); he invented the path integral in quantum mechanics, as well as the famous diagrams bearing his name; he was awarded the 1965 Nobel Prize in Physics for his fundamental contribution to quantum electrodynamics. He is also well known for his lectures on physics (Feynman et al. 2011).

obtained by Alfred Schild[15] *in 1963 (Schild 1963). In 1973, Pierre Ramond*[16] *gave the most general form that the function K can take assuming that the action must be invariant under a Poincaré transformation (Ramond 1973).*

11.5.2 Hamiltonian Formulation

On general grounds, a ***relativistic Hamiltonian theory*** consist in (i) a phase space P endowed with a Poisson bracket $\{,\}$ and a Hamiltonian (the set of all canonical transformations is then denoted by Canon(P)) and (ii) an ***action of the Poincaré group*** on P via canonical transformations, i.e. a mapping

$$\begin{array}{lcl} \mathrm{IO}(3,1) & \longrightarrow & \mathrm{Canon}(\mathsf{P}) \\ \Lambda & \longmapsto & f_\Lambda, \end{array}$$

such that the image of the identity is the identity and

$$\forall(\Lambda_1, \Lambda_2) \in \mathrm{IO}(3,1)^2, \quad f_{\Lambda_1} \circ f_{\Lambda_2} = f_{\Lambda_1 \circ \Lambda_2}. \tag{11.102}$$

Condition (ii) guarantees the invariance of physical laws, expressed via the canonical equations of Hamilton on P, under a change of inertial observer. Actually, it suffices to ensure that the *infinitesimal* Poincaré transformations lead to (necessarily infinitesimal) canonical transformations. The Poisson brackets of the generators of infinitesimal canonical transformations [cf. Eq. (11.79)–(11.80)] must then obey the same structure relations as the Lie brackets of the generators of Poincaré group, i.e. (8.38).

To treat a system of particles in this framework, it would seem natural to use as canonical coordinates on P the positions (x_a^α) of each particle in a system of inertial coordinates on $\mathscr{E}$ and the conjugate momenta (p_α^a). However, according to a theorem established by D.G. Currie, J.T. Jordan and E.C.G. Sudarshan (1963), the conditions

(i) Invariance of the Hamiltonian structure under the action of the Poincaré group (cf. the above definition).
(ii) Using the spacetime coordinates of the particles as canonical coordinates.

are not compatible, except if there is no interaction between the particles. This result has been called the ***no-interaction theorem***.

[15]**Alfred Schild** (1921–1977): American physicist, mainly known for his studies in general relativity; he also contributed to the development of the first atomic clocks.

[16]**Pierre Ramond**: American physicist of French origin, born in 1943; one of the founder of string theory.

A solution of this problem consists in abandoning condition (ii), i.e. to use canonical coordinates in the phase space that are not the particles' spacetime coordinates. This approach called ***a priori Hamiltonian formalism*** has been developed by P. Droz-Vincent (1970; 1975; 1977).

Historical note: *The formal bases of relativistic Hamiltonian theories, as sketched above, have been set by Paul A. M. Dirac (cf. p. 372) en 1949 (1949). A review of results obtained by the beginning of the 1980s is the collective book (Llosa 1982).*

Chapter 12
Accelerated Observers

12.1 Introduction

In this chapter and in the following one, we examine in detail non-inertial observers. There are essentially two ways for an observer to be non-inertial: having a nonvanishing 4-acceleration or a nonvanishing 4-rotation. In this chapter, we investigate the first case, Chap. 13 being devoted to the second one. Let us recall that the concepts of 4-acceleration and 4-rotation have been introduced in, respectively, Sect. 2.4.2 and Sect. 3.5.

12.2 Uniformly Accelerated Observer

12.2.1 Definition

The simplest configuration for an accelerated observer is that where:

1. His worldline lies in a plane Π of spacetime $\mathscr{E}$.
2. The metric norm of its 4-acceleration $\vec{\boldsymbol{a}}$ is constant along his worldline[1]:

$$\boxed{a := \|\vec{\boldsymbol{a}}\|_{\boldsymbol{g}} = \text{const}}. \tag{12.1}$$

3. His 4-rotation is identically zero:

$$\vec{\boldsymbol{\omega}} = 0. \tag{12.2}$$

[1]Let us recall that a has the dimension of the inverse of a length and that, $\vec{\boldsymbol{a}}$ being a spacelike vector, $\|\vec{\boldsymbol{a}}\|_{\boldsymbol{g}} = \sqrt{\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{a}}}$ [cf. Eq. (1.19)].

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Such an observer is said to be ***uniformly accelerated*** or ***in hyperbolic motion***. The name ***Rindler observer*** is also used. Note that Π is necessarily a timelike plane (cf. Sect. 6.4.2).

Remark 12.1. Naively, one might think defining a uniformly accelerated observer as an observer whose 4-acceleration *vector* is constant:

$$\vec{a} = \text{const}, \tag{12.3}$$

and not only its *norm* as in (12.1). Equation (2.16) defining the 4-acceleration, $\mathrm{d}\vec{u}/\mathrm{d}t = c\,\vec{a}$, is then straightforwardly integrated in $\vec{u}(t) = ct\,\vec{a} + \vec{u}_0$, where t is the observer's proper time and $\vec{u}_0$ a constant vector, which must be unit timelike, since $\vec{u}_0 = \vec{u}(0)$. The scalar square of the 4-velocity is then

$$\vec{u}(t)\cdot\vec{u}(t) = c^2t^2\,\vec{a}\cdot\vec{a} + 2ct\,\underbrace{\vec{u}_0\cdot\vec{a}}_{0} + \underbrace{\vec{u}_0\cdot\vec{u}_0}_{-1} = c^2t^2\,\vec{a}\cdot\vec{a} - 1,$$

where use has been made of the orthogonality of the 4-velocity and the 4-acceleration [Eq. (2.17)] to write $\vec{u}_0\cdot\vec{a} = \vec{u}(0)\cdot\vec{a}(0) = 0$. The above equation and the normalization $\vec{u}(t)\cdot\vec{u}(t) = -1$ imply that for $t \neq 0$, $\vec{a}\cdot\vec{a} = 0$. As $\vec{a}$ is either zero or a spacelike vector, this results necessarily in $\vec{a} = 0$. Hence, definition (12.3) leads to a trivial situation (inertial observer). The definition (12.1) is less restrictive and allows the 4-acceleration vector to vary, contrary to (12.3), in order to always stay orthogonal to the 4-velocity.

Example 12.1. A concrete example of uniformly accelerated motion is that of a charged particle in a uniform electric field. The norm of the 4-acceleration is then $a = |q|E/(mc^2)$, where q is the particle's electric charge, m its mass and E the (constant) norm of the electric field. This example will be treated in detail in Chap. 17.

The reader would have recognized the origin of the word *hyperbolic* in the above definition if she/he remembers that the Langevin's traveller introduced in Chap. 2 and in the examples of Chap. 4 has a worldline formed by arcs of hyperbola and that, on each of these arcs, the 4-acceleration has a constant norm [cf. Eq. (2.37)]. We are now going to show the converse, namely, that (12.1) and the hypothesis of worldline restricted to a plane lead to a hyperbolic worldline.

12.2.2 Worldline

Let $\mathscr{O}$ be a uniformly accelerated observer, of worldline $\mathscr{L}_0$, 4-velocity $\vec{u}$, 4-acceleration $\vec{a}$ and proper time t. The event of proper time t on $\mathscr{L}_0$ will be denoted by $O(t)$. By hypothesis, $\mathscr{L}_0$ is entirely contained in a plane $\Pi \subset \mathscr{E}$. Let us use the same symbol Π to denote the underlying vector plane: $\Pi \subset E$. The vector $\vec{u}$ being always tangent to $\mathscr{L}_0$, we have necessarily

$$\forall t \in \mathbb{R}, \quad \vec{u}(t) \in \Pi \quad \text{and} \quad \vec{a}(t) \in \Pi, \tag{12.4}$$

the second property resulting from $\vec{a} = c^{-1}\mathrm{d}\vec{u}/\mathrm{d}t$.

It is convenient to introduce an *inertial* observer $\mathscr{O}_*$, of frame $(\vec{e}^{\,*}_\alpha)$ and proper time t_*, such that

$$\Pi = \mathrm{Span}\,(\vec{e}^{\,*}_0, \vec{e}^{\,*}_1), \tag{12.5}$$

and such that at the instant $t_* = 0$, the worldlines of $\mathscr{O}$ and $\mathscr{O}_*$ are tangent at the point $O(0)$, origin of $\mathscr{O}$'s proper time t (cf. Fig. 12.1). The 4-velocities of $\mathscr{O}$ and $\mathscr{O}_*$ are then necessarily equal at $O(0)$:

$$\vec{u}(0) = \vec{e}^{\,*}_0. \tag{12.6}$$

Moreover, since $\vec{a}(0) \in \Pi$ is orthogonal to $\vec{u}(0)$ and $\vec{e}^{\,*}_1$ is a unit vector, we have

$$\vec{a}(0) = a\,\vec{e}^{\,*}_1. \tag{12.7}$$

A priori, we should write $\vec{a}(0) = \pm a\,\vec{e}^{\,*}_1$, but up to a change $\vec{e}^{\,*}_1 \mapsto -\vec{e}^{\,*}_1$, one can always select the + sign.

Since $(\vec{e}^{\,*}_0, \vec{e}^{\,*}_1)$ is a basis of Π [Eq. (12.5)] and $\vec{u}$ and $\vec{a}$ belong to that plane, we can write

$$\vec{u}(t) = u^0(t)\,\vec{e}^{\,*}_0 + u^1(t)\,\vec{e}^{\,*}_1 \quad \text{and} \quad \vec{a}(t) = a^0(t)\,\vec{e}^{\,*}_0 + a^1(t)\,\vec{e}^{\,*}_1. \tag{12.8}$$

Given the orthonormality of the basis $(\vec{e}^{\,*}_\alpha)$, the conditions $\vec{u}\cdot\vec{u} = -1$ and $\vec{a}\cdot\vec{a} = a^2$ are equivalent to

$$-[u^0(t)]^2 + [u^1(t)]^2 = -1 \qquad \text{and} \qquad -[a^0(t)]^2 + [a^1(t)]^2 = a^2, \tag{12.9}$$

hence,

$$a^1(t) = \pm\sqrt{a^2 + [a^0(t)]^2}.$$

In particular, $|a^1(t)| \geq a$. By continuity, we deduce that $a^1(t)$ cannot change its sign when t varies. Since $a^1(0) = a$ [Eq. (12.7)], we conclude that the + sign must be selected in the above expression. Besides, by definition of the 4-acceleration [cf. Eq. (2.16)],

$$a^0(t) = \frac{1}{c}\frac{\mathrm{d}u^0}{\mathrm{d}t} \qquad \text{and} \qquad a^1(t) = \frac{1}{c}\frac{\mathrm{d}u^1}{\mathrm{d}t}. \tag{12.10}$$

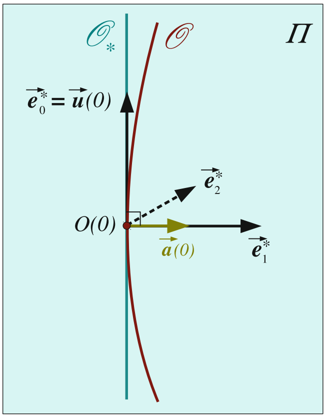


Fig. 12.1 Uniformly accelerated observer $\mathscr{O}$ and reference inertial observer $\mathscr{O}_*$. $\mathscr{O}$'s worldline is entirely contained in the plane Π

From (12.9), we get $u^0 = \sqrt{1 + (u^1)^2}$ (for $u^0 > 0$), so that

$$a^0(t) = \frac{1}{c} \frac{u^1}{\sqrt{1 + (u^1)^2}} \frac{\mathrm{d}u^1}{\mathrm{d}t}.$$

Inserting this relation, as well as expression (12.10) for $a^1(t)$, into (12.9), there comes

$$\frac{1}{\sqrt{1 + (u^1)^2}} \frac{\mathrm{d}u^1}{\mathrm{d}t} = ca,$$

where the property $\mathrm{d}u^1/\mathrm{d}t = ca^1 > 0$ has been taken into account. Given the initial condition $u^1(0) = 0$ [cf. Eq. (12.6)], the above equation is integrated into

$$u^1(t) = \sinh(act).$$

Since $u^0 = \sqrt{1 + (u^1)^2}$ and $1 + \sinh^2 x = \cosh^2 x$, we deduce that $u^0(t) = \cosh(act)$, hence the expression of $\mathscr{O}$'s 4-velocity:

$$\boxed{\vec{u}(t) = \cosh(act)\, \vec{e}^{\,*}_0 + \sinh(act)\, \vec{e}^{\,*}_1}. \tag{12.11}$$

The 4-acceleration is deduced immediately from (12.10):

$$\boxed{\vec{a}(t) = a \left[\sinh(act)\, \vec{e}^{\,*}_0 + \cosh(act)\, \vec{e}^{\,*}_1\right]}. \tag{12.12}$$

We check on these formulas that $\vec{u}(t) \cdot \vec{a}(t) = 0$, $\vec{u}(t) \cdot \vec{u}(t) = -1$ and $\vec{a}(t) \cdot \vec{a}(t) = a^2$.

Let us introduce the inertial coordinates $(x_*^\alpha) = (ct_*, x_*, y_*, z_*)$ associated with observer $\mathscr{O}_*$; let then

$$x_*^\alpha = X_*^\alpha(t) \tag{12.13}$$

be the equation of $\mathscr{O}$'s worldline ($\mathscr{L}_0$) in these coordinates. Note that the chosen parameter in the proper time t of $\mathscr{O}$ and that $X_*^2(t) = X_*^3(t) = 0$, since $\mathscr{O}$'s motion is confined to the plane Π. By definition of the 4-velocity, the components (u^α) of $\vec{u}$ within the basis $(\vec{e}^{\,*}_\alpha)$ are related to $X_*^\alpha(t)$ by $u^\alpha = c^{-1}\mathrm{d}X_*^\alpha/\mathrm{d}t$ [cf. (2.12)]. By virtue of (12.11), we have thus

$$\frac{\mathrm{d}X_*^0}{\mathrm{d}t} = c\cosh(act) \qquad \text{and} \qquad \frac{\mathrm{d}X_*^1}{\mathrm{d}t} = c\sinh(act).$$

Given the initial conditions $X_*^0(0) = 0$ and $X_*^1(0) = 0$ (point $O(0)$ on Fig. 12.1), these equations are easily integrated and lead to the following equation for the worldline $\mathscr{L}_0$:

$$\boxed{\begin{cases} ct_* = X_*^0(t) = a^{-1}\sinh(act) \\ x_* = X_*^1(t) = a^{-1}\left[\cosh(act) - 1\right] \\ y_* = X_*^2(t) = 0 \\ z_* = X_*^3(t) = 0, \end{cases}} \tag{12.14}$$

where the proper time t spans $\mathbb{R}$. Given the relation $\cosh^2 x - \sinh^2 x = 1$, we observe that on $\mathscr{L}_0$, the coordinates t_* and x_* are related by

$$\boxed{(ax_* + 1)^2 - (act_*)^2 = 1}. \tag{12.15}$$

We recognize the equation of an equilateral hyperbola in the plane (ct_*, x_*) of centre $(ct_* = 0, x_* = -a^{-1})$ and having for asymptotes the lines

$$\Delta_1 : \quad ct_* = x_* + a^{-1}, \quad y_* = 0, \quad z_* = 0 \tag{12.16a}$$

$$\Delta_2 : \quad ct_* = -x_* - a^{-1}, \quad y_* = 0, \quad z_* = 0. \tag{12.16b}$$

$\mathscr{L}_0$ is depicted in Fig. 12.2.

Remark 12.2. We recover through (12.11), (12.12) and (12.15) the formulas of Chap. 2: (2.32), (2.33) and (2.21) with $k = 0$, taking into account the changes of notation $\vec{u} \leftrightarrow \vec{u}'$, $\vec{a} \leftrightarrow \vec{a}'$, $a \leftrightarrow \alpha/(cT)$, $t \leftrightarrow t'$, $t_* \leftrightarrow t$ and $x_* \leftrightarrow x$.

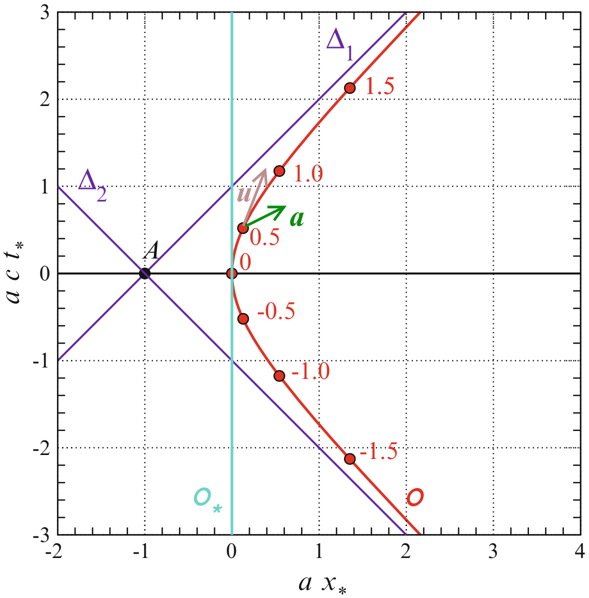


Fig. 12.2 Worldline $\mathscr{L}_0$ of the uniformly accelerated observer $\mathscr{O}$, drawn in the coordinates (ct_*, x_*) of the inertial observer $\mathscr{O}_*$: $\mathscr{L}_0$ is a branch of hyperbola with asymptotes Δ_1 and Δ_2. The numbers ranging from -1.5 to 1.5 along $\mathscr{L}_0$ mark $\mathscr{O}$'s proper time t, in units of $(ac)^{-1}$. The proper time of $\mathscr{O}_*$ is t_*

12.2.3 Change of the Reference Inertial Observer

The "top" $O(0)$ of the hyperbola depicted in Fig. 12.2 is not a peculiar point for the worldline $\mathscr{L}_0$. It appears as a top solely because the figure has been drawn within the coordinates (ct_*, x_*) linked to the inertial observer tangent to $\mathscr{L}_0$ at $O(0)$. If the figure is redrawn within the coordinates of an inertial observer tangent to $\mathscr{L}_0$ at another point, the latter appears as the top of the hyperbola, as illustrated in Fig. 12.3. The situation is perfectly similar to that of the hyperboloid $\mathscr{U}_O$ considered in Sect. 1.4.3: we had already noticed that the tops of $\mathscr{U}_O$ depicted in Figs. 1.6 and 1.7 are actually linked to the inertial coordinates used for the graphical representation and have no real physical meaning.

Let us show it by introducing a second inertial observer, $\mathscr{O}'_*$, tangent to $\mathscr{L}_0$ at $O(t')$ with $t' \neq 0$. By transposing (12.6) and (12.7) from $t = 0$ to $t = t'$, we observe that the frame $(\vec{e}^{\,*'}_\alpha)$ of $\mathscr{O}'_*$ is such that

$$\vec{e}^{\,*'}_0 = \vec{u}(t'), \quad \vec{e}^{\,*'}_1 = a^{-1}\,\vec{a}(t'), \quad \vec{e}^{\,*'}_2 = \vec{e}^{\,*}_2 \quad \text{and} \quad \vec{e}^{\,*'}_3 = \vec{e}^{\,*}_3,$$

where the last two conditions are nothing but a convenient choice of basis in the vector plane $\Pi^\perp$. Taking into account (12.11) and (12.12), there comes then

$$\begin{cases} \vec{e}^{\,*'}_0 = \cosh(act')\,\vec{e}^{\,*}_0 + \sinh(act')\,\vec{e}^{\,*}_1 \\ \vec{e}^{\,*'}_1 = \sinh(act')\,\vec{e}^{\,*}_0 + \cosh(act')\,\vec{e}^{\,*}_1, \end{cases}$$

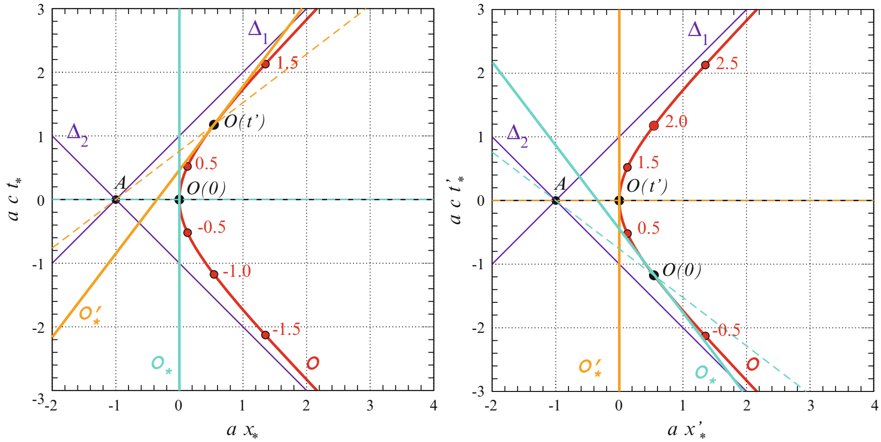


Fig. 12.3 Invariance of the hyperbola representing $\mathscr{O}$'s worldline $\mathscr{L}_0$ under a change of inertial observer: the left panel is based on the coordinates (ct_*, x_*) of the inertial observer $\mathscr{O}_*$, who is tangent to $\mathscr{O}$ at $O(0)$ (same as Fig. 12.2), while the right panel is based on the coordinates (ct'_*, x'_*) of the inertial observer $\mathscr{O}'_*$, who is tangent to $\mathscr{O}$ at $O(t')$ with $t' := (ac)^{-1}$. As in Fig. 12.2, the numbers along $\mathscr{L}_0$ mark the proper time t in units of $(ac)^{-1}$. The dashed lines show the rest spaces of the inertial observers $\mathscr{O}_*$ and $\mathscr{O}'_*$ at the considered instants. The change of coordinates $(ct_*, x_*) \mapsto (ct'_*, x'_*)$ is given by the Poincaré transformation (12.17)

as well as $\vec{e}^{\,*\prime}_2 = \vec{e}^{\,*}_2$ and $\vec{e}^{\,*\prime}_3 = \vec{e}^{\,*}_3$. We conclude that the frames of observers $\mathscr{O}_*$ and $\mathscr{O}'_*$ are related by a Lorentz boost of plane Π and rapidity $\psi = act'$ [cf. (6.43)]. By means of formulas (8.11) and (8.12) (with the inverse transformation), we deduce that the inertial coordinates $(ct'_*, x'_*, y'_*, z'_*)$ relative to $\mathscr{O}'_*$ are related to the inertial coordinates (ct_*, x_*, y_*, z_*) relative to $\mathscr{O}_*$ by the following Poincaré transformation:

$$\begin{cases} ct'_* = \cosh(act')\, ct_* - \sinh(act')\, x_* - a^{-1} \sinh(act') \\ x'_* = -\sinh(act')\, ct_* + \cosh(act')\, x_* + a^{-1} \cosh(act') - a^{-1} \end{cases}, \tag{12.17}$$

with $y'_* = y_*$ and $z'_* = z_*$. The change of coordinates is illustrated in Fig. 12.3 in the particular case $t' = (ac)^{-1}$. We note that in these new coordinates, $\mathscr{L}_0$ has exactly the same shape as in the old ones. Moreover, the coordinates of the point A where the two asymptotes intersect are invariant, as it can been seen by setting $(ct_*, x_*) = (0, -a^{-1})$ in (12.17):

$$(ct_*, x_*) = (0, -a^{-1}) \iff (ct'_*, x'_*) = (0, -a^{-1}).$$

The invariance of the hyperbola representing $\mathscr{O}$'s worldline under a change of tangent inertial observer reflects the fact that all events along $\mathscr{L}_0$ are equivalent. Indeed, from the point of view of observer $\mathscr{O}$, nothing happens as time passes, since

$\mathscr{O}$'s 4-rotation vanishes and the norm of his 4-acceleration stays constant. One says that $\mathscr{O}$ is a ***stationary observer***. Another example of stationary observer is of course an inertial observer.

12.2.4 Motion Perceived by the Inertial Observer

In the reference space of the inertial observer $\mathscr{O}_*$ (cf. Sect. 3.4.3), the accelerated observer $\mathscr{O}$ is moving along a straight line—the x_*-axis, with a time dependence deduced from (12.15):

$$x_*(t_*) = a^{-1}\left[\sqrt{1+(act_*)^2} - 1\right], \quad t_* \in \mathbb{R}. \tag{12.18}$$

$\mathscr{O}$ arrives from $x_* = +\infty$ ($t_* \to -\infty$), reaches $x_* = 0$ at $t_* = 0$ and moves back towards $x_* = +\infty$ as $t_* \to +\infty$ (cf. Fig. 12.2). The velocity of $\mathscr{O}$ relative to $\mathscr{O}_*$ is by definition [Eq. (4.19)]

$$\vec{\boldsymbol{V}} = \frac{\mathrm{d}x_*}{\mathrm{d}t_*}\,\vec{\boldsymbol{e}}^{\,*}_1 = V\,\vec{\boldsymbol{e}}^{\,*}_1, \qquad \text{with} \quad V := c\,\frac{act_*}{\sqrt{1+(act_*)^2}}. \tag{12.19}$$

For $|t_*| \ll (ac)^{-1}$, this expression reduces to $V \simeq \gamma_0 t_*$, with $\gamma_0 := ac^2$ [cf. (4.64)]. We recover the nonrelativistic value of the velocity as a function of time for a constant acceleration γ_0 and the initial condition $V(0) = 0$. On the other side, when $|t_*|$ is large, $\mathscr{O}$'s velocity has the following behaviour:

$$\lim_{t_*\to-\infty} V = -c \quad \text{and} \quad \lim_{t_*\to+\infty} V = c.$$

The velocity of $\mathscr{O}$ relative to the inertial observer $\mathscr{O}_*$ tends thus towards the velocity of light when t_* increases, which corresponds to the expected behaviour for a body undergoing a "constant acceleration" (cf. however Remark 12.3 below).

The acceleration of $\mathscr{O}$ relative to $\mathscr{O}_*$ is, by definition [Eq. (4.44)],

$$\vec{\boldsymbol{\gamma}} = \frac{\mathrm{d}^2x_*}{\mathrm{d}t_*^2}\,\vec{\boldsymbol{e}}^{\,*}_1 = \gamma\,\vec{\boldsymbol{e}}^{\,*}_1, \qquad \text{with} \quad \gamma := \frac{ac^2}{[1+(act_*)^2]^{3/2}}. \tag{12.20}$$

Thus,

$$\lim_{t_*\to-\infty} \gamma = 0, \qquad \gamma(t_*=0) = ac^2 \qquad \text{and} \qquad \lim_{t_*\to+\infty} \gamma = 0. \tag{12.21}$$

The velocity V and the acceleration γ, as given by (12.19) and (12.20), are plotted in Fig. 12.4.

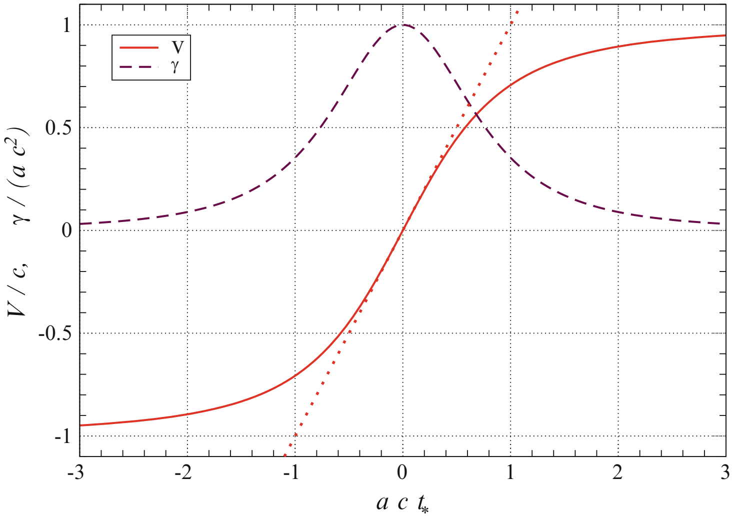


Fig. 12.4 Velocity V and acceleration γ of the uniformly accelerated observer $\mathscr{O}$, both relative to the inertial observer $\mathscr{O}_*$ tangent to $\mathscr{O}$ at $t_* = 0$, as functions of $\mathscr{O}_*$'s proper time t_*. The dotted line corresponds to the value of V at the nonrelativistic limit (γ is then constant and equal to ac^2)

Remark 12.3. The norm $\|\vec{\boldsymbol{\gamma}}\|_{\boldsymbol{g}} = \gamma$ of $\mathscr{O}$'s acceleration relative to the inertial observer $\mathscr{O}_*$ is not constant, whereas $\mathscr{O}$'s motion is qualified as *uniformly accelerated*. Actually, γ cannot be constant in order for V to be always lower than c. What remains constant is the norm a of $\mathscr{O}$'s 4-acceleration, which is a quantity independent of any observer, contrary to γ, which depends upon $\mathscr{O}_*$ (cf. Remark 4.7 p. 116).

12.2.5 Local Rest Spaces

Let M be a generic event in $\mathscr{E}$, of inertial coordinates (ct_*, x_*, y_*, z_*). M belongs to the local rest space of $\mathscr{O}$ at proper time t, $\mathscr{E}_{\boldsymbol{u}}(t)$, iff[2] $\vec{\boldsymbol{u}}(t) \cdot \overrightarrow{O(t)M} = 0$. Using (12.14) for the inertial coordinates of $O(t)$, we have

$$\overrightarrow{O(t)M} = \left[ct_* - a^{-1}\sinh(act)\right] \vec{\boldsymbol{e}}^{\,*}_0 + \left[x_* - a^{-1}\left(\cosh(act) - 1\right)\right] \vec{\boldsymbol{e}}^{\,*}_1 + y_*\, \vec{\boldsymbol{e}}^{\,*}_2 + z_*\, \vec{\boldsymbol{e}}^{\,*}_3.$$

The components of $\vec{\boldsymbol{u}}(t)$ in the basis $(\vec{\boldsymbol{e}}^{\,*}_\alpha)$ being given by (12.11), the condition $\vec{\boldsymbol{u}}(t) \cdot \overrightarrow{O(t)M} = 0$ becomes then

[2] Let us recall that $O(t)$ stands for the position of $\mathscr{O}$ at the proper time t.

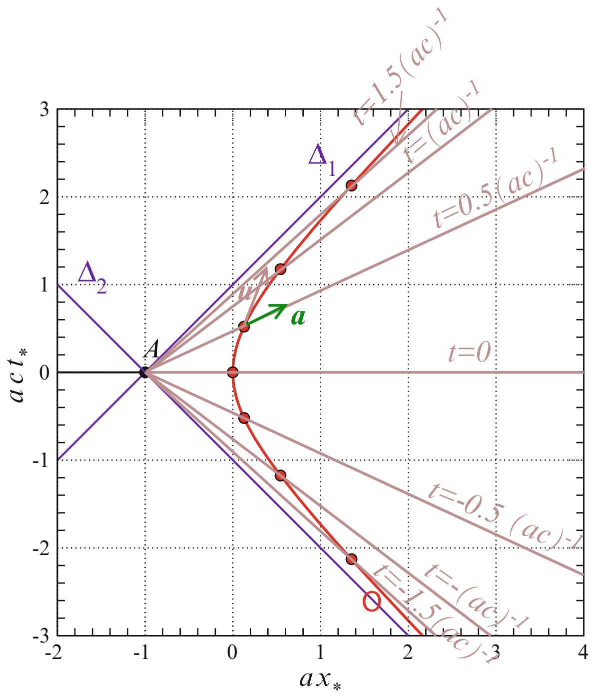


Fig. 12.5 Local rest spaces $\mathscr{E}_{\boldsymbol{u}}(t)$ of the uniformly accelerated observer $\mathscr{O}$. The plane of this figure is Π and each $\mathscr{E}_{\boldsymbol{u}}(t)$ appears as a straight line through A of slope $\tanh(act)$

$$-\cosh(act)\left[ct_* - a^{-1}\sinh(act)\right] + \sinh(act)\left[x_* - a^{-1}\left(\cosh(act) - 1\right)\right] = 0,$$

i.e. after simplification,

$$\boxed{ct_* = \tanh(act)(x_* + a^{-1})}. \tag{12.22}$$

This is the equation of the hyperplane $\mathscr{E}_{\boldsymbol{u}}(t)$ in the inertial coordinates (ct_*, x_*, y_*, z_*). $\mathscr{E}_{\boldsymbol{u}}(t)$ is depicted in Fig. 12.5 for various values of t. We observe that, when t varies, the hyperplanes $\mathscr{E}_{\boldsymbol{u}}(t)$ intersect at a common plane of $\mathscr{E}$, of equation $ct_* = 0$ and $x_* = -a^{-1}$. The trace of this plane on Fig. 12.5 (plane Π) is the point A, where the two asymptotes of hyperbola $\mathscr{L}_0$ intersect. That $\mathscr{O}$'s local rest spaces intersect should not be a surprise: this is a generic property of accelerated observers discussed in Sect. 3.7. In that section, we estimated the length scale a^{-1}: for an acceleration $\gamma = c^2 a = 10\ \mathrm{m\,s^{-2}}$, $a^{-1} \simeq 9 \times 10^{15}\ \mathrm{m} \simeq 1$ light-year.

12.2.6 Rindler Horizon

In the spacetime diagrams of Figs. 12.2, 12.3 and 12.5, which depict the plane Π with the inertial coordinates (ct_*, x_*), the worldlines of photons are straight lines inclined at $\pm 45°$. It is then clear that photons that are emitted in the domain located above the asymptote Δ_1 will never reach $\mathscr{L}_0$ (cf. Fig. 12.6). This domain is thus invisible for the observer $\mathscr{O}$.

More generally, i.e. away from the plane Π, let us determine the conditions under which an emitter $M \in \mathscr{E}$ can be perceived by $\mathscr{O}$ (cf. Fig. 12.7). Without any loss of generality, we may assume that $M \in \mathscr{E}_{\boldsymbol{u}}(0)$, since all the events along $\mathscr{L}_0$ are equivalent (cf. Sect. 12.2.3). This is the situation depicted in Fig. 12.7. Let $(ct_*^{\rm em}, x_*^{\rm em}, y_*^{\rm em}, z_*^{\rm em})$ be the inertial coordinates (relative to $\mathscr{O}_*$) of the emitter M. We have $t_*^{\rm em} = 0$, since $t = 0 \iff t_* = 0$ [cf. (12.14)]. A photon emitted from M reaches $\mathscr{O}$ iff there exists a null geodesic connecting M to a point $O(t) \in \mathscr{L}_0$ (dotted segment in Fig. 12.7). By definition, the inertial coordinates of $O(t)$ are the $X_*^\alpha(t)$'s given by (12.14), so that the vector $\overrightarrow{MO(t)}$ is

$$\overrightarrow{MO(t)} = a^{-1}\sinh(act)\,\vec{\boldsymbol{e}}^{\,*}_0 + [a^{-1}\cosh(act) - a^{-1} - x_*^{\rm em}]\,\vec{\boldsymbol{e}}^{\,*}_1 - y_*^{\rm em}\,\vec{\boldsymbol{e}}^{\,*}_2 - z_*^{\rm em}\,\vec{\boldsymbol{e}}^{\,*}_3.$$

Taking the scalar square and simplifying, we obtain that $\overrightarrow{MO(t)}$ is a null vector iff

$$2(ax_*^{\rm em} + 1)\cosh(act) = 1 + a^2\left[(x_*^{\rm em} + a^{-1})^2 + (y_*^{\rm em})^2 + (z_*^{\rm em})^2\right]. \qquad (12.23)$$

At fixed $(x_*^{\rm em}, y_*^{\rm em}, z_*^{\rm em})$, this is an equation for t. Given that $\cosh(act) \geq 1$, two cases are to be considered:

1. If $x_*^{\rm em} \leq -a^{-1}$, the term $(ax_*^{\rm em} + 1)$ is negative or zero and, since the right-hand side of (12.23) is greater than 1, there is no solution to the equation: a light ray emitted from M never reaches $\mathscr{O}$.
2. If $x_*^{\rm em} > -a^{-1}$, (12.23) can be recast as

$$\cosh(act) = 1 + a^2\frac{(x_*^{\rm em})^2 + (y_*^{\rm em})^2 + (z_*^{\rm em})^2}{2(ax_*^{\rm em} + 1)}.$$

   Since the right-hand side is clearly greater than or equal to 1, there exists a unique solution $t \geq 0$ to that equation: a light ray emitted from M reaches $\mathscr{O}$ at $\mathscr{O}$'s proper time t.

By reasoning on the intersection of the light cones with the hyperplane $t_* = \alpha \neq 0$, one finds that when $t_*^{\rm em} \neq 0$, the condition $x_*^{\rm em} > -a^{-1}$ is generalized to $x_*^{\rm em} - ct_*^{\rm em} > -a^{-1}$. Hence, the boundary between the domain of Minkowski spacetime that can send photons to $\mathscr{O}$ and the domain that cannot is the hyperplane $\mathscr{H}$ of equation

$$ct_* = x_* + a^{-1}. \qquad (12.24)$$

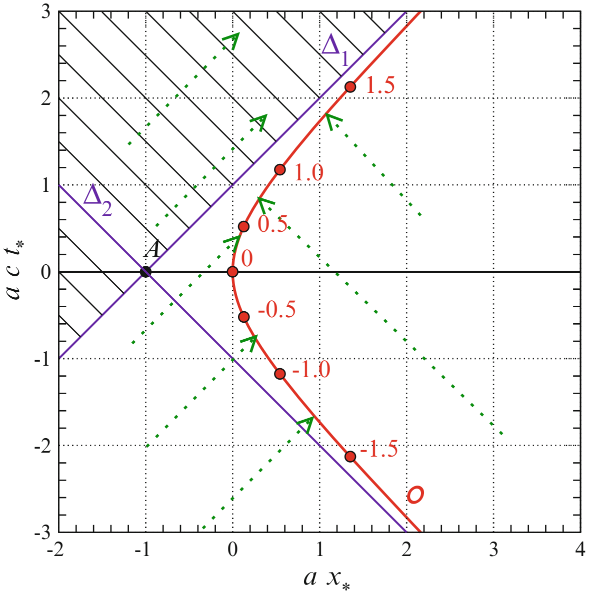


Fig. 12.6 Null geodesics (dotted lines) in the plane Π. Photons emitted in the hatched domain will never reach the accelerated observer $\mathscr{O}$

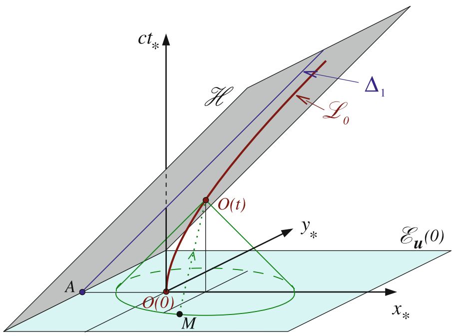


Fig. 12.7 Rindler horizon $\mathscr{H}$ of the uniformly accelerated observer (worldline $\mathscr{L}_0$, position $O(t)$ at proper time t)

$\mathscr{H}$ is called ***Rindler horizon of observer*** $\mathscr{O}$. It is depicted in Fig. 12.7. $\mathscr{H}$ is a ***null hyperplane*** in the sense that the metric induced by $\boldsymbol{g}$ onto $\mathscr{H}$ is degenerate (this is the three-dimensional analogue of the *null plane* defined in Sect. 6.4.5). Equivalently, any vector normal to $\mathscr{H}$ is also tangent to $\mathscr{H}$. It is thus necessarily null. In the present case, such a vector is collinear to $\vec{\boldsymbol{e}}^{\,*}_0 + \vec{\boldsymbol{e}}^{\,*}_1$.

Remark 12.4. The term *horizon* naming $\mathscr{H}$ has been introduced by analogy with the *event horizon* of a black hole (Rindler 1966). However, there is an important difference between the two concepts: the event horizon of a black hole is a structure intrinsic to spacetime, i.e. independent of any observer, while the Rindler horizon depends clearly upon the considered accelerated observer.

12.2.7 Local Frame of the Uniformly Accelerated Observer

We have not discussed yet the local frame $(\vec{e}_\alpha(t))$ of observer $\mathscr{O}$, except for demanding that its 4-rotation vanishes [condition (12.2)]. Let us choose this frame in the following manner: $\vec{e}_0(t) = \vec{u}(t)$ (by definition), $\vec{e}_1(t)$ is collinear to $\mathscr{O}$'s 4-acceleration:

$$\vec{e}_1(t) := \frac{1}{a}\,\vec{a}(t) \tag{12.25}$$

and $\vec{e}_2(t)$ and $\vec{e}_3(t)$ are constant vectors that coincide with the vectors $\vec{e}_2^{\,*}$ and $\vec{e}_3^{\,*}$ of the frame of the inertial observer $\mathscr{O}_*$:

$$\vec{e}_2(t) := \vec{e}_2^{\,*} \qquad \text{and} \qquad \vec{e}_3(t) := \vec{e}_3^{\,*}. \tag{12.26}$$

The tetrad $(\vec{e}_\alpha)$ defined above is an admissible local frame for $\mathscr{O}$, i.e. (i) it is an orthonormal basis of $(E, \boldsymbol{g})$ and (ii) its 4-rotation vanishes [condition (12.2)].

Proof. By construction, $\vec{e}_1$ is a spacelike unit vector; moreover, it is orthogonal to $\vec{u}$ (since $\vec{a}$ is). The plane Π of $\mathscr{O}$'s worldline being generated by $\vec{u}(t)$ and $\vec{a}(t)$, we have

$$\forall t \in \mathbb{R}, \quad \Pi = \mathrm{Span}(\vec{e}_0(t), \vec{e}_1(t)). \tag{12.27}$$

In view of (12.5), the vectors $\vec{e}_2^{\,*}$ and $\vec{e}_3^{\,*}$ form an orthonormal basis of the plane $\Pi^\perp$ (orthogonal complementary to Π). The same thing holds thus for $\vec{e}_2$ and $\vec{e}_3$:

$$\Pi^\perp = \mathrm{Span}(\vec{e}_2, \vec{e}_3). \tag{12.28}$$

Since $E = \Pi \overset{\perp}{\oplus} \Pi^\perp$, the properties (12.27) and (12.28) show that $(\vec{e}_\alpha(t))$ is an orthonormal basis of $(E, \boldsymbol{g})$ for any $t \in \mathbb{R}$. Moreover, this basis varies along worldline $\mathscr{L}_0$ according to

$$\frac{1}{c}\frac{\mathrm{d}\vec{e}_0}{\mathrm{d}t} = a\,\vec{e}_1, \qquad \frac{1}{c}\frac{\mathrm{d}\vec{e}_1}{\mathrm{d}t} = a\,\vec{e}_0 \quad \text{and} \quad \frac{1}{c}\frac{\mathrm{d}\vec{e}_2}{\mathrm{d}t} = \frac{1}{c}\frac{\mathrm{d}\vec{e}_3}{\mathrm{d}t} = 0. \tag{12.29}$$

The first equation is an immediate consequence of $\vec{e}_0 = \vec{u}$ and $a\,\vec{e}_1 = \vec{a}$, the second follows from (12.25): $\mathrm{d}\vec{e}_1/\mathrm{d}t = a^{-1}\mathrm{d}\vec{a}/\mathrm{d}t$, with $\mathrm{d}\vec{a}/\mathrm{d}t$ computed by taking the derivative of (12.12). Comparing (12.29) to the general law (3.52) for the evolution of a local frame, we conclude that $\vec{\boldsymbol{\omega}} = 0$ in the present case. □

Any other local frame compatible with the definition of a uniformly accelerated observer would be related to $(\vec{\boldsymbol{e}}_\alpha)$ by a *constant* rotation of the three spatial vectors $(\vec{\boldsymbol{e}}_i)$.

Remark 12.5. Since $\vec{\boldsymbol{\omega}} = 0$, one can say that the tetrad $(\vec{\boldsymbol{e}}_\alpha(t))$ is Fermi–Walker transported along $\mathscr{L}_0$ (cf. Sect. 3.6.3): $\boldsymbol{D}^{\mathrm{FW}}_{\boldsymbol{u}} \vec{\boldsymbol{e}}_\alpha = 0$.

The frames $(\vec{\boldsymbol{e}}_\alpha(t))$ and $(\vec{\boldsymbol{e}}^{\,*}_\alpha)$, associated, respectively, to observers $\mathscr{O}$ and $\mathscr{O}_*$, constitute two orthonormal bases of $(E, \boldsymbol{g})$. They are thus related by a Lorentz transformation depending upon t: $\vec{\boldsymbol{e}}_\alpha(t) = \boldsymbol{\Lambda}(t)(\vec{\boldsymbol{e}}^{\,*}_\alpha)$. The explicit form of $\boldsymbol{\Lambda}(t)$ follows from (12.11), (12.12), (12.25) and (12.26):

$$\boxed{\begin{cases} \vec{\boldsymbol{e}}_0(t) = \cosh(act)\,\vec{\boldsymbol{e}}^{\,*}_0 + \sinh(act)\,\vec{\boldsymbol{e}}^{\,*}_1 \\ \vec{\boldsymbol{e}}_1(t) = \sinh(act)\,\vec{\boldsymbol{e}}^{\,*}_0 + \cosh(act)\,\vec{\boldsymbol{e}}^{\,*}_1 \\ \vec{\boldsymbol{e}}_2(t) = \vec{\boldsymbol{e}}^{\,*}_2 \\ \vec{\boldsymbol{e}}_3(t) = \vec{\boldsymbol{e}}^{\,*}_3. \end{cases}} \tag{12.30}$$

Comparing with (6.43), we conclude that $\boldsymbol{\Lambda}(t)$ is a Lorentz boost, of rapidity $\psi = act$.

The coordinates $(x^0 = ct, x^1, x^2, x^3)$ associated with $\mathscr{O}$'s local frame $(\vec{\boldsymbol{e}}_\alpha(t))$ (cf. Sect. 3.4.2) are called ***Rindler coordinates***. Let us denote them by $(ct, x, y, z) := (x^0, x^1, x^2, x^3)$. By definition [cf. (3.25)], they are such that

$$\forall M(t, x, y, z) \in \mathscr{E}_{\boldsymbol{u}}(t), \quad \overrightarrow{O(t)M} = x\,\vec{\boldsymbol{e}}_1(t) + y\,\vec{\boldsymbol{e}}_2(t) + z\,\vec{\boldsymbol{e}}_3(t). \tag{12.31}$$

A some fixed instant t, the coordinates (x, y, z) coincide with the coordinates (x'_*, y'_*, z'_*) associated with an inertial observer $\mathscr{O}'_*$ of 4-velocity $\vec{\boldsymbol{u}}(t)$ and whose worldline is tangent to $\mathscr{L}_0$ at $O(t)$. Given the invariance of $\mathscr{L}_0$ with respect to t (cf. Sect. 12.2.3), we may apply the results of Sect. 12.2.6 (obtained for $t = 0$) and state that the domain $x = x'_* \leq -a^{-1}$ is invisible for observer $\mathscr{O}$, or equivalently, that only the domain

$$\boxed{x > -a^{-1}} \tag{12.32}$$

is perceivable by $\mathscr{O}$ via light signals. We shall thus limit the extension of Rindler coordinates to that part of the hyperplane $\mathscr{E}_{\boldsymbol{u}}(t)$. This is the part represented in Fig. 12.5 (the part "on the right of" A).

The inertial coordinates (ct_*, x_*, y_*, z_*) of M are defined by

$$\overrightarrow{O(0)M} = ct_*\,\vec{\boldsymbol{e}}^{\,*}_0 + x_*\,\vec{\boldsymbol{e}}^{\,*}_1 + y_*\,\vec{\boldsymbol{e}}^{\,*}_2 + z_*\,\vec{\boldsymbol{e}}^{\,*}_3. \tag{12.33}$$

Besides, the vector $\overrightarrow{O(0)M}$ can be written as

$$\begin{aligned}
\overrightarrow{O(0)M} &= \overrightarrow{O(0)O(t)} + \overrightarrow{O(t)M} \\
&= X_*^\alpha(t)\, \vec{e}^{\,*}_\alpha + x\, \vec{e}_1(t) + y\, \vec{e}_2(t) + z\, \vec{e}_3(t) \\
&= a^{-1}\sinh(act)\, \vec{e}^{\,*}_0 + a^{-1}[\cosh(act) - 1]\, \vec{e}^{\,*}_1 \\
&\quad + x[\sinh(act)\, \vec{e}^{\,*}_0 + \cosh(act)\, \vec{e}^{\,*}_1] + y\, \vec{e}^{\,*}_2 + z\, \vec{e}^{\,*}_3 \\
&= (x + a^{-1})\sinh(act)\, \vec{e}^{\,*}_0 + [(x + a^{-1})\cosh(act) - a^{-1}]\, \vec{e}^{\,*}_1 \\
&\quad + y\, \vec{e}^{\,*}_2 + z\, \vec{e}^{\,*}_3,
\end{aligned} \tag{12.34}$$

where the $X_*^\alpha(t)$'s are the functions introduced in (12.13) to define the equation of $\mathscr{O}$'s worldline, $O(t)$ being a generic point of the latter. To get the third line, use has been made of expression (12.14) for $X_*^\alpha(t)$ as well as (12.30). Comparing (12.33) with (12.34), we obtain the relation between the inertial coordinates (ct_*, x_*, y_*, z_*) and the Rindler coordinates (ct, x, y, z):

$$\boxed{\begin{cases} ct_* = (x + a^{-1})\sinh(act) \\ x_* = (x + a^{-1})\cosh(act) - a^{-1} \\ y_* = y \\ z_* = z. \end{cases}} \qquad \begin{array}{l} t \in \mathbb{R} \\ x > -a^{-1} \end{array} \tag{12.35}$$

The iso-coordinate lines $x = \text{const}$ are depicted in Fig. 12.8. In terms of the inertial coordinates (ct_*, x_*), they are branches of equilateral hyperbolas of centre A and asymptotes Δ_1 and Δ_2 (as $\mathscr{L}_0$). Indeed, by combining the first two equations of the system (12.35), we get

$$\left(\frac{ax_* + 1}{ax + 1}\right)^2 - \left(\frac{act_*}{ax + 1}\right)^2 = 1.$$

It is clear from Fig. 12.8 that the spacetime domain covered by Rindler coordinates is the domain located between the two hyperplanes of respective equations $ct_* = x_* + a^{-1}$ and $ct_* = -x_* - a^{-1}$, the trace of which in Fig. 12.8 being formed by the two straight lines Δ_1 and Δ_2.

Historical note: *Albert Einstein presented the first discussion of an accelerated observer in 1907 (Einstein 1907). After a correspondence with Max Planck (cf. p. 279) about his article, he was brought to make precise the definition of a uniformly accelerated observer (Einstein 1908). He pointed notably that the acceleration* $\vec{\gamma}$ *relative to an inertial observer depends on the latter (cf. Remark 4.7 p. 116), contrary to what happens in nonrelativistic mechanics, where* $\vec{\gamma}$ *is a Galilean invariant. Einstein defined then the constant acceleration* γ *as that measured by a tangent inertial observer. From (4.64),* $\mathscr{O}$*'s 4-acceleration is expressed in terms of*

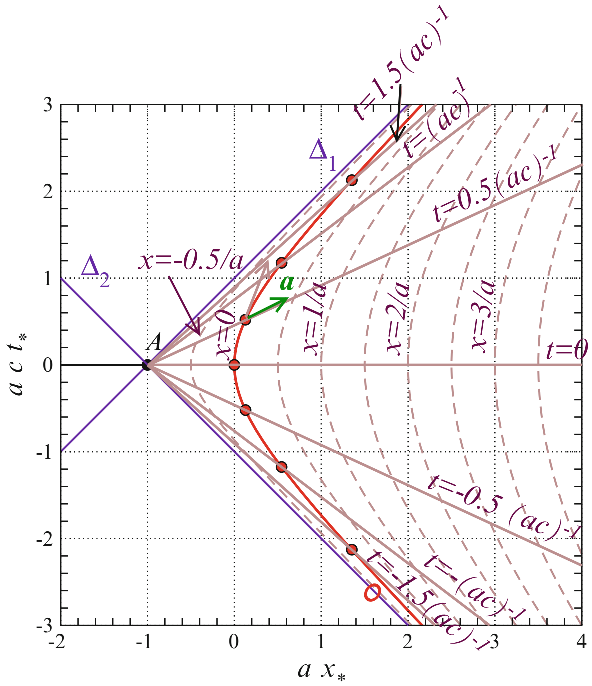


Fig. 12.8 Rindler coordinates (ct, x) in the plane Π containing the worldline of the uniformly accelerated observer $\mathscr{O}$: the curves $t = \text{const}$ are straight lines through A, whereas the curves $x = \text{const}$ (dashed lines) are hyperbola branches, the curve $x = 0$ coinciding with $\mathscr{O}$'s worldline and the "curve" $x = -a^{-1}$ with the point A

the acceleration $\vec{\boldsymbol{\gamma}}$ with respect to the tangent inertial observer as $\vec{\boldsymbol{a}} = c^{-2}\,\vec{\boldsymbol{\gamma}}$, and we notice that Einstein's definition $\gamma = \text{const}$ corresponds indeed to the definition $a = \text{const}$ given above for a uniformly accelerated observer. The spacetime point of view, with the introduction of the 4-acceleration, appeared only in 1909, with the famous article by Hermann Minkowski (cf. p. 26) (1909). Minkowski related notably the 4-acceleration to the "osculating" hyperbola of the worldline at the considered point, the inverse of the 4-acceleration norm being nothing but the distance between a point of the worldline and the centre of the hyperbola. Later in the same year Max Born (cf. p. 76) studied the uniformly accelerated motion and named it hyperbolic motion *(Born 1909). This work was followed the next year by a study of Arnold Sommerfeld (cf. p. 27) (1910b). Rindler coordinates bear their name from a detailed study of the uniformly accelerated observer performed by Wolfgang Rindler[3] in 1966 (Rindler 1966) (cf. also Sect. 37 of his book (Rindler 1969)). In the same article, he developed the analogy between the horizon $\mathscr{H}$, called since then after him, and the event horizon of a black hole. However, the Rindler coordinates were known well before, in particular, by Einstein, who used them in 1935 (Einstein and*

[3]**Wolfgang Rindler**: physicist born in 1924 in Austria, currently professor of physics at the University of Texas at Dallas and author of many textbooks about relativity, among them (Rindler 1969) and (Rindler 1991) (entirely devoted to special relativity).

Rosen 1935). Similarly, the existence of a horizon for any accelerated observer has been noticed quite soon, at least as early as 1938 by Edward Milne (cf. p. 6) and Gerald J. Whitrow[4] (1938).

12.3 Difference Between the Local Rest Space and the Simultaneity Hypersurface

We have seen in Sect. 3.2.3 that the set of all events simultaneous with respect to $\mathscr{O}$ to a given event $O(t) \in \mathscr{L}_0$ is a hypersurface of $\mathscr{E}$, denoted by $\Sigma_{\boldsymbol{u}}(t)$ and called the *simultaneity hypersurface* of $O(t)$ for $\mathscr{O}$. As an approximation to this hypersurface, we have introduced the *local rest space* $\mathscr{E}_{\boldsymbol{u}}(t)$, which is the hyperplane tangent to $\Sigma_{\boldsymbol{u}}(t)$ at $O(t)$ (cf. Fig. 3.3). We are going to investigate here the extent to which $\mathscr{E}_{\boldsymbol{u}}(t) = \Sigma_{\boldsymbol{u}}(t)$, and, when the two differ, we shall evaluate the distance from $\mathscr{L}_0$ for which the hyperplane $\mathscr{E}_{\boldsymbol{u}}(t)$ constitutes a good approximation of $\Sigma_{\boldsymbol{u}}(t)$. In what follows, we assume that the 4-acceleration of $\mathscr{O}$ does not vanish: $\vec{\boldsymbol{a}} \neq 0$. Otherwise, $\mathscr{L}_0$ is the worldline of an inertial observer, and we know that $\mathscr{E}_{\boldsymbol{u}}(t) = \Sigma_{\boldsymbol{u}}(t)$ in this case. To simplify the writing, we shall design $O(t)$ simply by O.

12.3.1 Case of a Generic Observer

Let us abandon for a while the uniformly accelerated observer, to deal with a generic observer, i.e. an observer $\mathscr{O}$ whose 4-acceleration norm is not necessarily constant and whose worldline is not necessarily confined to a plane of $\mathscr{E}$.

Let M be some point of $\Sigma_{\boldsymbol{u}}(t)$, i.e. an event simultaneous to O for $\mathscr{O}$ according to Einstein–Poincaré criterion (3.1). Let then $A_1 \in \mathscr{L}_0$ be the event of proper time t_1 corresponding to the emission of a photon towards M and $A_2 \in \mathscr{L}_0$ the event of proper time t_2 corresponding to the reception by $\mathscr{O}$ of the photon immediately reflected at M (cf. Fig. 12.9). From the simultaneity criterion (3.1), we can write $t_1 = t - T$ and $t_2 = t + T$, with $T \geq 0$. By construction, $\overrightarrow{A_1M}$ is a null vector; we have thus $\overrightarrow{A_1M} \cdot \overrightarrow{A_1M} = 0$. Writing $\overrightarrow{A_1M} = \overrightarrow{A_1O} + \overrightarrow{OM}$, we deduce

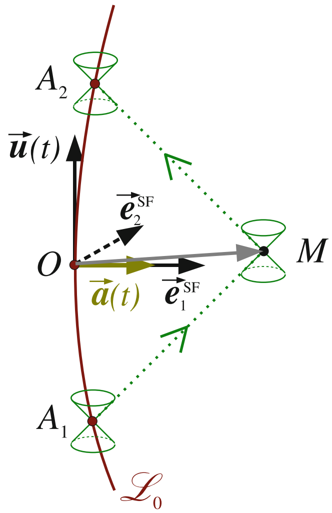
$$\overrightarrow{OA_1} \cdot \overrightarrow{OA_1} + 2\overrightarrow{A_1O} \cdot \overrightarrow{OM} + \overrightarrow{OM} \cdot \overrightarrow{OM} = 0.$$

Forming the same relation for A_2 and subtracting, we get

$$2\overrightarrow{A_1A_2} \cdot \overrightarrow{OM} = \overrightarrow{OA_2} \cdot \overrightarrow{OA_2} - \overrightarrow{OA_1} \cdot \overrightarrow{OA_1}. \tag{12.36}$$

[4] **Gerald J. Whitrow** (1912–2000): British cosmologist and historian of science.

Fig. 12.9 Event M simultaneous with the event O of proper time t on the worldline of observer $\mathscr{O}$



The vectors $\overrightarrow{OA_1}$ and $\overrightarrow{OA_2}$ can be expressed in terms of T, thanks to the Taylor series (2.60) obtained in Sect. 2.7.3. There comes

$$\overrightarrow{OA_1} = -\left(1 + \frac{a^2 s^2}{6}\right) s\, \vec{e}_0^{\,\mathrm{SF}} + \left(a - \frac{\dot{a}}{3} s\right) \frac{s^2}{2}\, \vec{e}_1^{\,\mathrm{SF}} - \frac{a T_1}{6} s^3\, \vec{e}_2^{\,\mathrm{SF}} + O((as)^4) \tag{12.37}$$

$$\overrightarrow{OA_2} = \left(1 + \frac{a^2 s^2}{6}\right) s\, \vec{e}_0^{\,\mathrm{SF}} + \left(a + \frac{\dot{a}}{3} s\right) \frac{s^2}{2}\, \vec{e}_1^{\,\mathrm{SF}} + \frac{a T_1}{6} s^3\, \vec{e}_2^{\,\mathrm{SF}} + O((as)^4), \tag{12.38}$$

where

- $s := cT$.
- $a := \|\vec{a}(t)\|_g = \sqrt{\vec{a}(t) \cdot \vec{a}(t)}$ is the curvature of $\mathscr{L}_0$ at O and $\dot{a} := c^{-1}\, \mathrm{d}a/\mathrm{d}t$.
- $as = acT$ is the dimensionless small quantity measuring the remoteness of M from $\mathscr{L}_0$.
- $(\vec{e}_\alpha^{\,\mathrm{SF}})$ is the Serret–Frenet tetrad of worldline $\mathscr{L}_0$ at O introduced in Sect. 2.7.3; in particular,

$$\vec{e}_0^{\,\mathrm{SF}} = \vec{u}(t) \qquad \text{and} \qquad \vec{e}_1^{\,\mathrm{SF}} = a^{-1}\, \vec{a}(t). \tag{12.39}$$

- T_1 is the first torsion of $\mathscr{L}_0$ at O.

Remark 12.6. At first order in as, (12.37) and (12.38) reduce to $\overrightarrow{OA_1} = -s\, \vec{u}(t)$ and $\overrightarrow{OA_2} = s\, \vec{u}(t)$. We recover thus expressions (3.3) of Sect. 3.2.3.

The vector $\overrightarrow{A_1A_2}$ that appears in (12.36) can be written $\overrightarrow{A_1A_2} = \overrightarrow{A_1O} + \overrightarrow{OA_2} = -\overrightarrow{OA_1} + \overrightarrow{OA_2}$, i.e. from (12.37) and (12.38),

$$\overrightarrow{A_1A_2} = 2\left(1 + \frac{a^2s^2}{6}\right) s\, \vec{e}_0^{\,\mathrm{SF}} + \frac{\dot{a}}{3} s^3\, \vec{e}_1^{\,\mathrm{SF}} + \frac{aT_1}{3} s^3\, \vec{e}_2^{\,\mathrm{SF}} + O((as)^4). \tag{12.40}$$

Besides, we infer from (12.37) to (12.38) and the orthonormality of the basis $(\vec{e}_\alpha^{\,\mathrm{SF}})$ that

$$\overrightarrow{OA_1} \cdot \overrightarrow{OA_1} = -s^2 \left[1 + \frac{(as)^2}{12}\right] + O((as)^5)$$
$$\overrightarrow{OA_2} \cdot \overrightarrow{OA_2} = -s^2 \left[1 + \frac{(as)^2}{12}\right] + O((as)^5).$$

We notice that

$$\overrightarrow{OA_1} \cdot \overrightarrow{OA_1} = \overrightarrow{OA_2} \cdot \overrightarrow{OA_2} + O((as)^5). \tag{12.41}$$

Substituting (12.40) and (12.41) in (12.36), we get

$$2\left(1 + \frac{a^2s^2}{6}\right) s\, \vec{e}_0^{\,\mathrm{SF}} \cdot \overrightarrow{OM} + \frac{\dot{a}}{3} s^3\, \vec{e}_1^{\,\mathrm{SF}} \cdot \overrightarrow{OM} + \frac{aT_1}{3} s^3\, \vec{e}_2^{\,\mathrm{SF}} \cdot \overrightarrow{OM} = O((as)^4).$$

Taking into account (12.39), we can rewrite this equation as

$$\boxed{\begin{gathered}\left[1+\frac{(as)^2}{6}\right] \vec{u}(t) \cdot \overrightarrow{OM} = -\frac{(as)^2}{6} \left[\frac{\dot{a}}{a^3} \vec{a}(t) \cdot \overrightarrow{OM} + \frac{T_1}{a} \vec{e}_2^{\,\mathrm{SF}} \cdot \overrightarrow{OM}\right] \\ +O((as)^3)\end{gathered}}\,. \tag{12.42}$$

If M was belonging to the local rest space $\mathscr{E}_{\boldsymbol{u}}(t)$, one would have $\vec{u}(t) \cdot \overrightarrow{OM} = 0$. The above relation thus shows that the discrepancy between the simultaneity hypersurface $\Sigma_{\boldsymbol{u}}(t)$ and the hyperplane $\mathscr{E}_{\boldsymbol{u}}(t)$ starts only at the second order in as. Moreover, if the variation of a and the first torsion are such that $|\dot{a}|/a^2 \ll as$ and $|T_1|/a \ll as$, then (12.42) reduces to

$$\boxed{\vec{u}(t) \cdot \overrightarrow{OM} = O((as)^3).}_{\ \max\left(\frac{|\dot{a}|}{a^2},\ \frac{|T_1|}{a}\right) \ll as} \tag{12.43}$$

This means that, at the second order in as, $\Sigma_{\boldsymbol{u}}(t)$ and $\mathscr{E}_{\boldsymbol{u}}(t)$ coincide.

12.3.2 Case of a Uniformly Accelerated Observer

If $\mathscr{O}$ is uniformly accelerated, then $\dot{a} = 0$ (a is constant) and $T_1 = 0$ (worldline confined in the plane Π), so that (12.43) holds: $\vec{u}(t) \cdot \overrightarrow{OM} = O((as)^3)$. But there is more: actually $\vec{u}(t) \cdot \overrightarrow{OM} = 0$ at any order in as. In other words,

For a uniformly accelerated observer, the simultaneity hypersurface $\Sigma_{\boldsymbol{u}}(t)$ coincides exactly with the local rest space $\mathscr{E}_{\boldsymbol{u}}(t)$:

$$\forall t \in \mathbb{R}, \qquad \boxed{\Sigma_{\boldsymbol{u}}(t) = \mathscr{E}_{\boldsymbol{u}}(t)}. \tag{12.44}$$

Proof. The result follows from simple symmetry considerations. Indeed let us choose $O = O(0)$. Within the inertial coordinates (ct_*, x_*, y_*, z_*), the hyperbola representing the worldline $\mathscr{L}_0$ is symmetric with respect to the hyperplane $\mathscr{E}_{\boldsymbol{u}}(0)$ (cf. Fig. 12.10). In the same coordinates, the null geodesics, which are used in the Einstein–Poincaré simultaneity criterion, are straight lines with a slope of $\pm 45°$. It is then clear that the points simultaneous to O are those of the hyperplane $\mathscr{E}_{\boldsymbol{u}}(0)$. We have thus $\Sigma_{\boldsymbol{u}}(0) = \mathscr{E}_{\boldsymbol{u}}(0)$. The point $O(0)$ being undistinguishable from the other points of $\mathscr{L}_0$ (cf. Sect. 12.2.3), except from the convention fixing the origin of $\mathscr{O}$'s proper time, we get the result. □

12.4 Physics in an Accelerated Frame

In all that follows, we consider for simplicity the uniformly accelerated observer $\mathscr{O}$ introduced in Sect. 12.2. All the investigated physical effects hold for a generic accelerated observer, possibly with more complicated explicit formulas.

12.4.1 Clock Synchronization

Let us consider a second observer, $\mathscr{O}'$, *fixed* with respect to $\mathscr{O}$, in the sense defined in Sect. 3.4.3—the coordinates (x, y, z) of $\mathscr{O}'$ relative to $\mathscr{O}$ (Rindler coordinates in the present case) are constant (cf. Fig. 12.11):

$$x = x_0 = \text{const}, \quad y = y_0 = \text{const} \quad \text{and} \quad z = z_0 = \text{const}. \tag{12.45}$$

Fig. 12.10 Simultaneity for the uniformly accelerated observer $\mathscr{O}$: the events simultaneous to O are the events of the hyperplane $\mathscr{E}_{\boldsymbol{u}}(0)$ through O (hyperplane $t_* = 0$ on the figure)

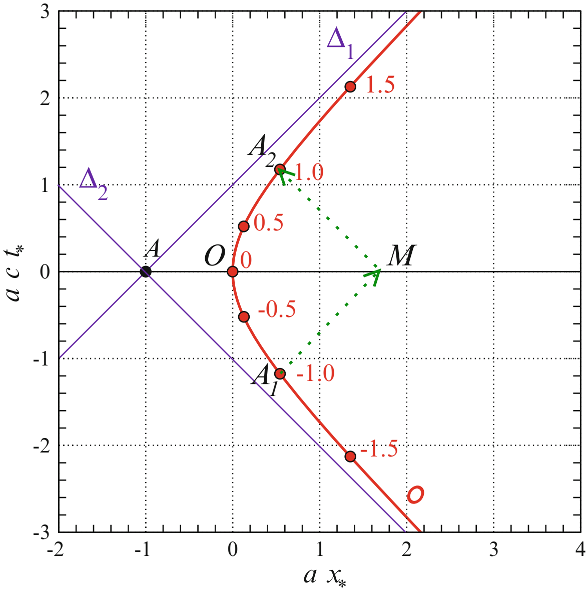
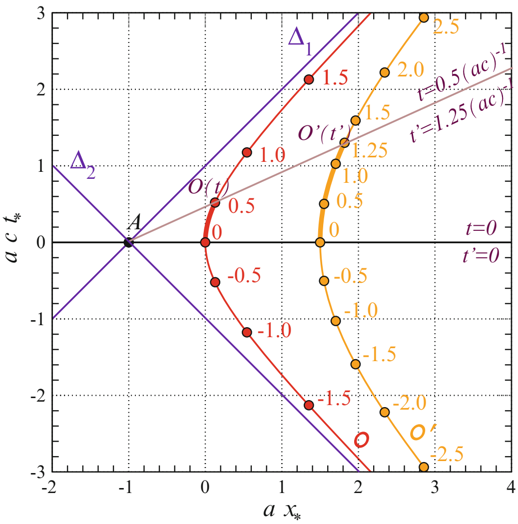


Fig. 12.11 Worldlines of the uniformly accelerated observer $\mathscr{O}$ and the observer $\mathscr{O}'$ who is fixed with respect to $\mathscr{O}$, at the coordinates (x_0, y_0, z_0). For the figure, $(x_0, y_0, z_0) = (1.5a^{-1}, 0, 0)$. The numbers along each worldline mark the proper time of each observer in units of $(ac)^{-1}$



One says also that $\mathscr{O}'$ is an observer ***comoving with*** $\mathscr{O}$. The worldline $\mathscr{L}'_0$ of $\mathscr{O}'$ is obtained from the system (12.35):

$$\begin{cases} ct_* = (x_0 + a^{-1}) \sinh(act) \\ x_* = (x_0 + a^{-1}) \cosh(act) - a^{-1} \\ y_* = y_0 \\ z_* = z_0 , \end{cases} \tag{12.46}$$

where $t \in \mathbb{R}$ appears as a parametrization of $\mathscr{L}'_0$. Equation (12.46) defines a branch of hyperbola in the timelike plane $(y_* = y_0,\ z_* = z_0)$ of $\mathscr{E}$. This implies that the observer $\mathscr{O}'$ is himself a uniformly accelerated observer.

The physical interpretation of the parameter t along $\mathscr{L}'_0$, as it appears in (12.46), is that the event of $\mathscr{L}'_0$ corresponding to a given value of t is simultaneous for $\mathscr{O}$ to the event of proper time t on $\mathscr{O}$'s worldline. A priori, t is different from the proper time of $\mathscr{O}'$, which we shall denote by t'. Let us relate these two proper times. We note $O'(t')$ the intersection of $\mathscr{L}'_0$ with the rest space[5] $\mathscr{E}_{\boldsymbol{u}}(t)$ of $\mathscr{O}$ at proper time t (cf. Fig. 12.11). The 4-velocity $\vec{\boldsymbol{u}}'(t')$ of $\mathscr{O}'$ at $O'(t')$ obeys (4.27). Since $\vec{\boldsymbol{\omega}} = 0$ [$\mathscr{O}$ is not rotating; cf. (12.2)] and $\vec{\boldsymbol{V}} = 0$ [$\mathscr{O}'$ is fixed with respect to $\mathscr{O}$], this formula reduces to

$$\vec{\boldsymbol{u}}'(t') = \Gamma \left[1 + \vec{\boldsymbol{a}}(t) \cdot \overrightarrow{O(t)O'(t')}\right] \vec{\boldsymbol{u}}(t),$$

where Γ is the Lorentz factor of $\mathscr{O}'$ with respect to $\mathscr{O}$. Now $\vec{\boldsymbol{u}}'(t')$ and $\vec{\boldsymbol{u}}(t)$ are two unit future-directed timelike vectors. If they are proportional, as above, the proportionality factor must be equal to one. We have thus necessarily

$$\vec{\boldsymbol{u}}'(t') = \vec{\boldsymbol{u}}(t) \tag{12.47}$$

and

$$\Gamma = \left[1 + \vec{\boldsymbol{a}}(t) \cdot \overrightarrow{O(t)O'(t')}\right]^{-1}. \tag{12.48}$$

Note that (12.48) can also be deduced directly from (4.30), by setting $\vec{\boldsymbol{\omega}} = 0$ and $\vec{\boldsymbol{V}} = 0$. The rest space of an observer being, by definition, the hyperplane orthogonal to his 4-velocity at the considered event, we deduce immediately from (12.47) that the rest spaces of observers $\mathscr{O}'$ and $\mathscr{O}$ coincide:

$$\boxed{\mathscr{E}_{\boldsymbol{u}'}(t') = \mathscr{E}_{\boldsymbol{u}}(t)}. \tag{12.49}$$

This common rest space is represented by the straight line through the points A, $O(t)$ and $O'(t')$ in Fig. 12.11.

Since $\vec{\boldsymbol{a}}(t) = a\, \vec{\boldsymbol{e}}_1(t)$ [Eq. (12.25)] and $\overrightarrow{O(t)O'(t')} = x_0\, \vec{\boldsymbol{e}}_1(t) + y_0\, \vec{\boldsymbol{e}}_2(t) + z_0\, \vec{\boldsymbol{e}}_3(t)$ [Eq. (12.31)], we have $\vec{\boldsymbol{a}}(t) \cdot \overrightarrow{O(t)O'(t')} = a x_0$, so that (12.48) reduces to

$$\boxed{\Gamma = (1 + a x_0)^{-1}}. \tag{12.50}$$

[5] In view of the result (12.44), we may omit the qualifier *local* in the denomination of $\mathscr{E}_{\boldsymbol{u}}(t)$.

By definition [Eq. (4.1)], $\Gamma = \mathrm{d}t/\mathrm{d}t'$, so that we obtain the relation between the proper times of $\mathscr{O}$ and $\mathscr{O}'$:

$$\boxed{\mathrm{d}t' = (1 + ax_0)\,\mathrm{d}t}. \tag{12.51}$$

Remark 12.7. It is instructive to get (12.51) without appealing to the results of Chap. 4, as done above, but by starting from the equation of the worldline of $\mathscr{O}'$ [Eq. (12.46)] and from the definition (2.6) of proper time. According to the latter, the increase in t' corresponding to a small displacement $\mathrm{d}\vec{x}$ along $\mathscr{L}'_0$ in the future direction is

$$c\,\mathrm{d}t' = \sqrt{-\boldsymbol{g}(\mathrm{d}\vec{x}, \mathrm{d}\vec{x})} = \sqrt{c^2\mathrm{d}t_*^2 - \mathrm{d}x_*^2 - \mathrm{d}y_*^2 - \mathrm{d}z_*^2}, \tag{12.52}$$

where we have expressed $\mathrm{d}\vec{x}$ in the frame of inertial observer $\mathscr{O}_*$: $\mathrm{d}\vec{x} = c\,\mathrm{d}t_*\,\vec{e}^{\,*}_0 + \mathrm{d}x_*\,\vec{e}^{\,*}_1 + \mathrm{d}y_*\,\vec{e}^{\,*}_2 + \mathrm{d}z_*\,\vec{e}^{\,*}_3$. Now, from (12.46), $\mathrm{d}\vec{x}$ is generated by a small increment $\mathrm{d}t$ of the parameter t according to

$$\begin{cases} c\,\mathrm{d}t_* = (ax_0 + 1)\cosh(act)\,c\,\mathrm{d}t \\ \mathrm{d}x_* = (ax_0 + 1)\sinh(act)\,c\,\mathrm{d}t \\ \mathrm{d}y_* = \mathrm{d}z_* = 0. \end{cases}$$

Substituting these values in (12.52), we get

$$c\,\mathrm{d}t' = c\,\mathrm{d}t|ax_0 + 1|\sqrt{\cosh^2(act) - \sinh^2(act)} = |ax_0 + 1|\,c\,\mathrm{d}t.$$

Given that $1 + ax_0 > 0$ [cf. Eq. (12.32)], we recover (12.51).

Since x_0 is constant along the worldline $\mathscr{L}'_0$, (12.51) is integrated in

$$\boxed{t' = (1 + ax_0)t}, \tag{12.53}$$

where the integration constant has been chosen in order to ensure $t' = 0$ when $t = 0$. This formula, which relates the proper time of $\mathscr{O}'$ to that of $\mathscr{O}$, reveals a major difference between an accelerated observer and an inertial one:

For an inertial observer, all the ideal clocks that are fixed with respect to him and synchronized with his own clock at $t = 0$ remain synchronized for any $t > 0$. On the contrary, for an accelerated observer, an ideal clock fixed with respect to him and located at $x_0 \neq 0$ is desynchronized as soon as $t > 0$: its proper time, t', does no longer coincide with the observer's proper time t, even for simultaneous events.

This can be clearly seen in Fig. 12.11: the events $O(t)$ and $O'(t')$ are simultaneous, from the point of view of $\mathscr{O}$ as well as that of $\mathscr{O}'$, but $\mathscr{O}$ attributes the date $t = 0.5(ac)^{-1}$ to them and $\mathscr{O}'$ the date $t' = 1.25(ac)^{-1}$, while the clocks of $\mathscr{O}$ and $\mathscr{O}'$ having been synchronized at $t = t' = 0$.

Historical note: *The relation (12.51) between the proper time of $\mathscr{O}$ and that of an observer fixed with respect to him has been obtained by Albert Einstein in 1907 (Einstein 1907).*

12.4.2 4-Acceleration of Comoving Observers

Substituting the value of t deduced from (12.53) in (12.46), we obtain the equation of the worldline of $\mathscr{O}'$ parametrized by his proper time t':

$$\begin{cases} ct_* = a'^{-1} \sinh(a'ct') \\ x'_* = a'^{-1} \left[\cosh(a'ct') - 1\right] \\ y'_* = 0 \\ z'_* = 0, \end{cases} \tag{12.54}$$

with

$$\boxed{a' := \frac{a}{1 + ax_0}} \tag{12.55}$$

and

$$x'_* := x_* - x_0, \quad y'_* := y_* - y_0, \quad z'_* := z_* - z_0. \tag{12.56}$$

The system (12.54) has exactly the same structure as (12.14). Since (ct_*, x'_*, y'_*, z'_*) are inertial coordinates, we conclude that $\mathscr{O}'$ is a uniformly accelerated observer, whose 4-acceleration norm is a'. We note that

$$x_0 \geq 0 \iff a' \leq a \qquad \text{and} \qquad \lim_{x_0 \to -a^{-1}} a' = +\infty. \tag{12.57}$$

The asymptotes to the hyperbola branch forming the worldline of $\mathscr{O}'$ are given by formulas similar to (12.16):

$$\Delta'_1 : \quad ct_* = x'_* + a'^{-1}, \quad y'_* = 0, \quad z'_* = 0$$
$$\Delta'_2 : \quad ct_* = -x'_* - a'^{-1}, \quad y'_* = 0, \quad z'_* = 0.$$

In terms of the coordinates (x_*, y_*, z_*), these equations become

$$\Delta_1' : \quad ct_* = x_* + a^{-1}, \quad y_* = y_0, \quad z_* = z_0$$
$$\Delta_2' : \quad ct_* = -x_* - a^{-1}, \quad y_*' = y_0, \quad z_*' = z_0.$$

Comparing with (12.16), we observe that if $y_0 = 0$ and $z_0 = 0$, then $\Delta_1' = \Delta_1$ and $\Delta_2' = \Delta_2$, which is not surprising since $\mathscr{L}_0$ and $\mathscr{L}_0'$ are then equilateral hyperbolas sharing the same centre: the point A, of coordinates

$$(ct_*, x_*) = (0, -a^{-1}) \iff (ct_*, x'_*) = (0, -a'^{-1}).$$

The Rindler horizon of the accelerated observer $\mathscr{O}'$ is given by (12.24), with the coordinates (x_*, y_*, z_*) replaced (x_*', y_*', z_*') and a replaced by a': $ct_* = x_*' + a'^{-1}$. Now, thanks to (12.55) and (12.56), $x_*' + a'^{-1} = x_* + a^{-1}$. We obtain then the same equation as (12.24). We thus conclude:

All observers fixed with respect to $\mathscr{O}$ have the same Rindler horizon $\mathscr{H}$.

12.4.3 Rigid Ruler in Accelerated Motion

The above discussion can be used to treat the problem of an accelerated rigid ruler in relativity. Let us recall that we have defined an *infinitesimal* rigid ruler in Sect. 3.3.2, from Born's rigidity criterion. In the present case (uniform acceleration), we can extend this criterion to rulers of finite extent. Let us consider the ruler whose extremities are the worldlines $\mathscr{L}_0$ and $\mathscr{L}_0'$ of the accelerated observers $\mathscr{O}$ and $\mathscr{O}'$ considered above. We assume that $\mathscr{L}_0$ and $\mathscr{L}_0'$ are coplanar, i.e. that $y_0 = z_0 = 0$. Such a ruler is drawn in Fig. 12.12. It is accelerated along its length (x-axis of observer $\mathscr{O}$). For observer $\mathscr{O}$ as well as for observer $\mathscr{O}'$, who shares the same rest space, the (metric) length of the ruler at a given instant of proper time t is $\ell_0 = \|\overrightarrow{O(t)O'(t')}\|_g$, where $t'$is related to t by (12.53). The segment $O(t)O'(t')$ is depicted at various instants t by the dashed elongated rectangles in Fig. 12.12. Now, from the definition of Rindler coordinates, $\overrightarrow{O(t)O'(t')} = x_0\, \vec{e}_1(t)$. We have thus

$$\boxed{\ell_0 = |x_0|}. \tag{12.58}$$

Since $x_0 = \text{const}$ (from the very definition of $\mathscr{O}'$), we conclude that the length ℓ_0 of the ruler measured by a comoving observer is constant: this is the reason why the ruler is qualified of ***rigid***. The quantity ℓ_0 is called the ***rest length*** of the ruler.

On the other side, for the inertial observer $\mathscr{O}_*$, the ruler at some instant of his proper time t_* is perceived as aligned with the x_*-axis (horizontal rectangles, drawn with solid lines, in Fig. 12.12). The ruler's length measured by $\mathscr{O}_*$ is thus

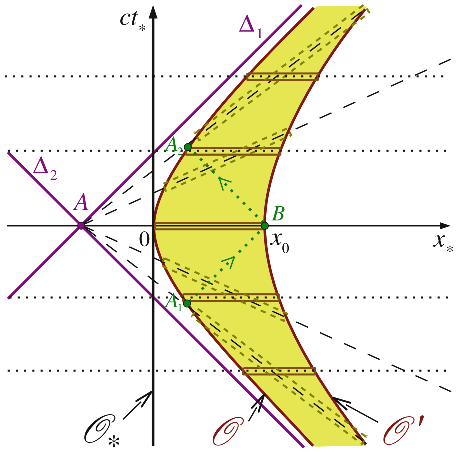


Fig. 12.12 Ruler in uniformly accelerated motion. The coloured zone is the spacetime domain covered by the ruler. The horizontal rectangles (solid lines) represent the ruler as perceived by the inertial observer $\mathscr{O}_*$ at various instants t_* of his proper time. The inclined rectangles (dashed lines) represent the rule as perceived by an observer at rest with respect to it, as, for instance, the observer $\mathscr{O}$ located at its left end or the observer $\mathscr{O}'$ located at its right end

$$\ell(t_*) = |x_*(O') - x_*(O)|,$$

where $x_*(O)$ and $x_*(O')$ are related to t_* by, respectively, (12.15) and (12.46):

$$x_*(O) = a^{-1}\sqrt{1+(act_*)^2} - a^{-1}, \quad x_*(O') = (x_0 + a^{-1})\sqrt{1+\left(\frac{act_*}{1+ax_0}\right)^2} - a^{-1}.$$

We conclude that

$$\boxed{\ell(t_*) = \ell_0 \frac{2+ax_0}{\sqrt{(1+ax_0)^2+(act_*)^2}+\sqrt{1+(act_*)^2}}}. \tag{12.59}$$

It is clear that $\ell(t_*)$ is not constant and that we have

$$\ell(0) = \ell_0, \qquad \lim_{t_*\to\pm\infty} \ell(t_*) = 0 \tag{12.60}$$

as well as

$$\boxed{\ell(t_*) \leq \ell_0}. \tag{12.61}$$

This last result reflects the FitzGerald–Lorentz contraction discussed in Sect. 5.2.2.

Remark 12.8. The motion of the accelerated ruler, as depicted in Fig. 12.12, illustrates clearly the "phenomenon" of length contraction: the ruler at different instants $t_* \geq 0$ appears as horizontal segments of smaller and smaller length. All these segments represent the ruler as perceived by the same inertial observer, $\mathscr{O}_*$, so the noticed graphical contraction can be interpreted unambiguously as a contraction of the metric length of the ruler. On the contrary, in Fig. 5.3, the two depicted segments correspond to the ruler as perceived by two distinct inertial observers. Given the Lorentzian nature of the metric $\boldsymbol{g}$, one cannot convert directly the Euclidean lengths of the segments drawn in Fig. 5.3 into metric lengths. Another difference with the situation contemplated in Chap. 5 is that, in the present case, the coefficient of length contraction is not a Lorentz factor, as in (5.17), for the different points of the ruler do not have the same velocity relative to $\mathscr{O}_*$ (the left end moves faster than the right end). There is therefore not a unique Lorentz factor of the ruler with respect to $\mathscr{O}_*$.

Remark 12.9. Although the ruler is qualified of rigid, it appears for $\mathscr{O}_*$ more and more compressed as t_* increases from $t_* = 0$. So the rigidity is effective only for a comoving observer.

Let us compute the round-trip time of a photon between the two ends of the ruler, as measured by observer $\mathscr{O}$, which is "fixed" at the left end. Since $\mathscr{O}$ is a stationary observer (cf. Sect. 12.2.3), we can, without any loss of generality, consider that the photon is emitted by $\mathscr{O}$ at some instant $t = -T$ with $T > 0$ (event A_1), reflected by $\mathscr{O}'$ (at the second end) at the instant $t = 0$ (event B) and received by $\mathscr{O}$ at the instant $t = T$ (event A_2). The three events A_1, A_2 and B are depicted in Fig. 12.12. Setting $t = -T$ in (12.14), we get the inertial coordinates of event A_1: $ct_*(A_1) = -a^{-1}\sinh(acT)$, $x_*(A_1) = a^{-1}\cosh(acT) - a^{-1}$. Those of B being $ct_*(B) = 0$ and $x_*(B) = x_0$, we obtain the components of vector $\overrightarrow{A_1B}$ in the orthonormal basis $(\vec{\boldsymbol{e}}^{\,*}_{\alpha})$ associated with the inertial observer $\mathscr{O}_*$:

$$\overrightarrow{A_1B} = a^{-1}\sinh(acT)\,\vec{\boldsymbol{e}}^{\,*}_0 + [x_0 + a^{-1} - a^{-1}\cosh(acT)]\,\vec{\boldsymbol{e}}^{\,*}_1. \tag{12.62}$$

Now, as being tangent to a photon worldline, $\overrightarrow{A_1B}$ is a null vector. Expressing $\overrightarrow{A_1B}\cdot\overrightarrow{A_1B} = 0$ with (12.62), we get

$$\cosh(acT) = \frac{1}{2}\left(1 + ax_0 + \frac{1}{1 + ax_0}\right).$$

Setting $1+ax_0 =: \mathrm{e}^{\xi}$, we recognize $\cosh\xi$ in the right-hand side, hence $acT = \pm\xi$, i.e.

$$\boxed{T = \pm(ac)^{-1}\ln(1 + ax_0)}, \tag{12.63}$$

where $\pm = \mathrm{sgn}(x_0)$ (sign of x_0); one has thus always $T \geq 0$. The photon round-trip time is $2T = t(A_2) - t(A_1)$. It depends only on the norm a of $\mathscr{O}$'s 4-acceleration and on the position x_0 (relatively to $\mathscr{O}$) of the ruler's second end. Since a and x_0 are constant along $\mathscr{L}_0$, we conclude that the time interval T is the same whatever the emission point A_1 on $\mathscr{O}$'s worldline: Born's rigidity criterion is thus fulfilled.

Formula (12.63) is easily inverted, yielding x_0 in terms of T. Since the ruler's rest length is $\ell_0 = |x_0|$ [Eq. (12.58)], we get

$$\boxed{\ell_0 = \pm a^{-1}\left(\mathrm{e}^{\pm acT} - 1\right)}, \tag{12.64}$$

still with $\pm = \mathrm{sgn}(x_0)$. A second-order Taylor expansion with respect to acT leads to

$$\ell_0 \simeq cT\left(1 \pm \frac{acT}{2}\right). \tag{12.65}$$

If $acT \ll 1$, the above formula results in $\ell_0 \simeq cT$, i.e. one recovers formula (3.21) of Chap. 3. The latter had been obtained by neglecting the curvature of the worldline of the ruler's end, which amounts exactly to $acT \ll 1$. Formula (12.65) shows that the light round-trip distance cT provides an underestimate of the rest length of the ruler when the observer who is sending and receiving the light signal is located at the left end ($x_0 > 0$) and an overestimate when he is located at the right end ($x_0 < 0$), the "left" and "right" being defined by the orientation of the 4-acceleration vector $\vec{\boldsymbol{a}}$: given that $\vec{\boldsymbol{a}}$ belongs to $\mathscr{O}$'s rest space and is parallel to the ruler, $\mathscr{O}$ is said to be *at left* of the ruler if $\vec{\boldsymbol{a}}$ is oriented towards the other end and to be *at right* in the opposite case.

Remark 12.10. We have seen in Sect. 12.3.2 that acceleration, provided that it stays uniform, does not change anything to the geometrical criterion for simultaneity introduced in Chap. 3, namely, the orthogonality to the worldline. On the other hand, formula (12.65) reveals that the procedure of chronometrical measure of distances introduced in Chap. 3 is no longer valid for length scales of the order of a^{-1} or larger: the chronometrical measure cT differs from the actual metric length ℓ_0 by a quantity proportional to acT.

12.4.4 Photon Trajectories

Let us investigate the null geodesics in the plane Π of the uniformly accelerated observer $\mathscr{O}$. They are the straight lines of Π whose equations within the inertial coordinates (ct_*, x_*) spanning Π are

$$ct_* = \pm(x_* - b), \qquad b \in \mathbb{R},$$

with the sign $+$ for photons moving rightward (increasing x_*) and $-$ for those moving leftward (decreasing x_*). The parameter b is the photon's abscissa x_* at

$t_* = 0$. Thanks to the transformation law (12.35) between the inertial coordinates (ct_*, x_*, y_*, z_*) and $\mathscr{O}$'s coordinates (ct, x, y, z), the above equation becomes

$$(x + a^{-1})\sinh(act) = \pm\left[(x + a^{-1})\cosh(act) - a^{-1} - b\right],$$

hence, via formulas $\cosh u = (e^u + e^{-u})/2$ and $\sinh u = (e^u - e^{-u})/2$,

$$\boxed{ct = \pm a^{-1}\ln\left(\frac{1+ax}{1+ab}\right)}. \tag{12.66}$$

By means of this equation, a few null geodesics are plotted in Fig. 12.13, which describes the plane Π in terms of the coordinates (ct, x). Although being straight lines of Minkowski spacetime, they appear curved when plotted in Rindler coordinates. In the vicinity of $\mathscr{O}$'s worldline ($x = 0$), we observe however that the null geodesics can be approximated by straight lines inclined at ±45°, as if they were drawn in inertial coordinates. Setting $b = 0$ and $|x| \ll a^{-1}$ in (12.66), there comes indeed

$$ct \simeq \pm x \quad (|x| \ll a^{-1}).$$

We notice also on Fig. 12.13 that no geodesic arises from the Rindler horizon $\mathscr{H}$ and that no geodesic reaches $\mathscr{H}$ within a finite time t. Let us also stress that Fig. 12.13 is invariant under a translation in t: this reflects the stationary character of $\mathscr{O}$ discussed in Sect. 12.2.3.

12.4.5 *Spectral Shift*

Let us consider the reception by $\mathscr{O}$ of a photon emitted by observer $\mathscr{O}'$, who is fixed with respect to $\mathscr{O}$ and located in the plane Π, at coordinates[6] $(x, y, z) = (x_{em}, 0, 0)$. Without any loss of generality, we may consider that the emission takes place at the instant $t = 0$ (cf. Fig. 12.14). The instant t_{rec} of the photon's reception by $\mathscr{O}$ is then deduced from (12.66), setting $x = 0$ ($\mathscr{O}$'s position) and $b = x_{em}$:

$$ct_{rec} = \pm a^{-1}\ln(1 + ax_{em}), \tag{12.67}$$

with the sign $-$ if $x_{em} \le 0$ (rightward photon's propagation) and $+$ otherwise.

Let E_{em} be the photon's energy measured by $\mathscr{O}'$ at the emission point. From (9.22), the photon's 4-momentum is

[6] In the previous sections, we have denoted by x_0 the x-coordinate of observer $\mathscr{O}'$; here we rather use x_{em}, which recalls that he is an emitter.

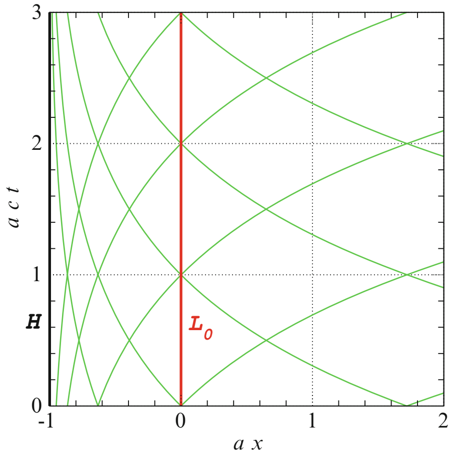


Fig. 12.13 Null geodesics in the plane Π in terms of the coordinates (ct, x) associated with the uniformly accelerated observer $\mathscr{O}$ (worldline $\mathscr{L}_0$). $\mathscr{H}$ (at $ax = -1$) is the Rindler horizon

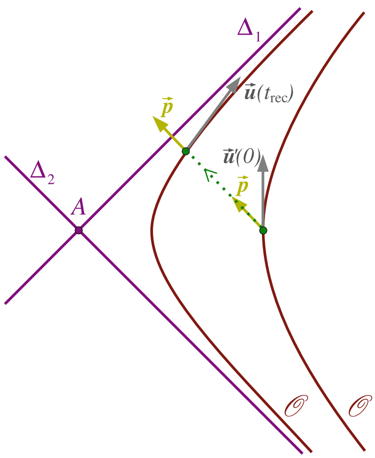


Fig. 12.14 Reception by $\mathscr{O}$ of a photon emitted by observer $\mathscr{O}'$, fixed with respect to $\mathscr{O}$. $\vec{p}$ is the photon's 4-momentum vector

$$\vec{p} = \frac{E_{\mathrm{em}}}{c}\left(\vec{u}'(0) + \vec{n}'\right),$$

where $\vec{u}'(0)$ is the 4-velocity of $\mathscr{O}'$ and $\vec{n}'$ the unit vector giving the photon's propagation direction with respect to $\mathscr{O}'$. From (12.47), $\vec{u}'(0) = \vec{u}(0) = \vec{e}^{\,*}_0$. We have then necessarily $\vec{n}' = \pm\vec{e}^{\,*}_1$ (since the photon is propagating in the plane Π), so that

$$\vec{p} = \frac{E_{\mathrm{em}}}{c}\left(\vec{e}^{\,*}_0 \pm \vec{e}^{\,*}_1\right), \tag{12.68}$$

with the $+$ sign if $x_{\mathrm{em}} \leq 0$ and the $-$ sign if $x_{\mathrm{em}} \geq 0$ (case depicted in Fig. 12.14).

The photon's energy measured by $\mathscr{O}$ at the reception point is given by (9.4): $E_{\mathrm{rec}} = -c\,\vec{p}\cdot\vec{u}(t_{\mathrm{rec}})$, where $\vec{p}$ is the same vector as in (12.68), by conservation of the photon's 4-momentum [Eq. (9.37)] and $\vec{\boldsymbol{u}}(t_{\rm rec})$ is $\mathscr{O}$'s 4-velocity at the proper time $t_{\rm rec}$. The latter is given by (12.11), with $t = t_{\rm rec}$. Thus,

$$\begin{aligned} E_{\rm rec} &= -E_{\rm em}\left(\vec{\boldsymbol{e}}^{\,*}_0 \pm \vec{\boldsymbol{e}}^{\,*}_1\right)\cdot\left[\cosh(act_{\rm rec})\,\vec{\boldsymbol{e}}^{\,*}_0 + \sinh(act_{\rm rec})\,\vec{\boldsymbol{e}}^{\,*}_1\right] \\ &= -E_{\rm em}\left[-\cosh(act_{\rm rec}) \pm \sinh(act_{\rm rec})\right] = E_{\rm em}e^{\pm act_{\rm rec}}, \end{aligned}$$

with $\pm = \mathrm{sgn}(x_{\rm em})$. Combining with (12.67), we obtain a formula that does no longer depend on the sign of $x_{\rm em}$:

$$E_{\rm rec} = E_{\rm em}(1 + ax_{\rm em}). \tag{12.69}$$

Moreover, this formula does not involve $t_{\rm rec}$, in agreement with the stationary character of $\mathscr{O}$. By means of the Planck–Einstein formula (9.24), the above relation can be expressed in terms of the photon's frequency:

$$\boxed{f_{\rm rec} = f_{\rm em}(1 + ax_{\rm em})}, \tag{12.70}$$

or, in terms of the radiation period $T = 1/f$:

$$\boxed{T_{\rm rec} = \frac{T_{\rm em}}{1 + ax_{\rm em}}}. \tag{12.71}$$

The same relation holds for the wavelengths, since $\lambda = cT$. The ***redshift factor*** is defined by $z := \lambda_{\rm rec}/\lambda_{\rm em} - 1$. In view of (12.71), we get

$$\boxed{z = \frac{1}{1 + ax_{\rm em}} - 1}. \tag{12.72}$$

Therefore, if the emitter is at the left of $\mathscr{O}$ ($x_{\rm em} \le 0$), $z \ge 0$: the spectral shift is towards the red, with $z \to +\infty$ when the emission point approaches the Rindler horizon ($x_{\rm em} \to -a^{-1}$). Conversely, for an emission at right of $\mathscr{O}$, $z \le 0$ and we have a blueshift. We shall see in Chap. 22 that the result (12.72) leads to a well-known effect in general relativity: the *gravitational redshift*. This follows from the *equivalence principle*, to be discussed in Sect. 22.3, according to which a gravitational field is locally equivalent to an accelerated frame.

Note that the measure of z allows observer $\mathscr{O}$ to determine $ax_{\rm em}$. If, in addition, $\mathscr{O}$ measures the round-trip time $2T$ of a photon between $x = 0$ and $x_{\rm em}$, formula (12.63) (with $x_0 = x_{\rm em}$), combined to (12.72), leads to an expression of a that involves only quantities measured by $\mathscr{O}$:

$$\boxed{a = \mp\frac{\ln(1+z)}{cT}}, \tag{12.73}$$

with $\mp = -\mathrm{sgn}(x_{\rm em})$, so that a is always positive. Equation (12.73) shows that the amplitude of the 4-acceleration is a measurable quantity, as announced in Sect. 3.5.3.

Remark 12.11. We shall see in Sect. 13.2.2 that the 4-rotation of an observer is also a measurable quantity. On the contrary, the 4-velocity is not measurable (incidentally its norm is always 1).

12.4.6 Motion of Free Particles

Let us consider a free particle $\mathscr{P}$. According to the law of inertia obtained in Sect. 9.3.4, its worldline $\mathscr{L}$ is a straight line of $\mathscr{E}$. Let us consider the case where $\mathscr{L}$ is parallel to $\vec{e}^{\,*}_0$, which means that $\mathscr{P}$ is at rest with respect to the inertial observer $\mathscr{O}_*$ and its 4-velocity is $\vec{e}^{\,*}_0$. Moreover, we assume that $\mathscr{L}$ lies in the plane Π (cf. Fig. 12.15). The equation of $\mathscr{L}$ in the inertial coordinates associated with $\mathscr{O}_*$ is then

$$x_* = b, \quad y_* = 0, \quad z_* = 0,$$

where the parameter b is chosen within $]-a^{-1}, +\infty[$.

Let $M(t)$ be the intersection of $\mathscr{L}$ with the local rest space $\mathscr{E}_{\boldsymbol{u}}(t)$ of the accelerated observer $\mathscr{O}$, whose position at the proper time t is denoted by $O(t)$. In view of equation (12.22) for $\mathscr{E}_{\boldsymbol{u}}(t)$, the inertial coordinates of $M(t)$ are $ct_* = \tanh(act)(b + a^{-1})$ and $x_* = b$. Those of $O(t)$ being given by (12.14), we deduce the expression of the vector $\overrightarrow{O(t)M(t)}$ in the basis $(\vec{e}^{\,*}_\alpha)$:

$$\begin{aligned}\overrightarrow{O(t)M(t)} &= \left[\tanh(act)(b + a^{-1}) - a^{-1}\sinh(act)\right]\vec{e}^{\,*}_0 \\ &\quad + \left[b - a^{-1}\cosh(act) + a^{-1}\right]\vec{e}^{\,*}_1 \\ &= \left[\frac{b + a^{-1}}{\cosh(act)} - a^{-1}\right]\left[\sinh(act)\,\vec{e}^{\,*}_0 + \cosh(act)\,\vec{e}^{\,*}_1\right].\end{aligned}$$

With the help of (12.30), we recognize in this expression the vector $\vec{e}_1(t)$ of $\mathscr{O}$'s local frame; we may thus write

$$\overrightarrow{O(t)M(t)} = x(t)\,\vec{e}_1(t), \tag{12.74}$$

with

$$\boxed{x(t) = \frac{b + a^{-1}}{\cosh(act)} - a^{-1}}. \tag{12.75}$$

By definition, $x(t)$ is the Rindler coordinate x of the point $M(t)$. We check that $x(0) = b$, which is consistent with $x = x_*$ at $t = 0$. The curve $x = x(t)$ corresponding to (12.75) ($\mathscr{P}$'s worldline) is depicted in the right panel of Fig. 12.15

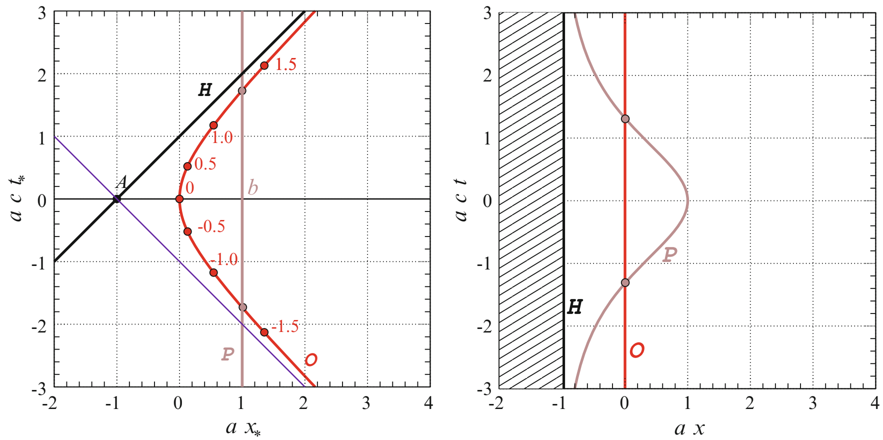


Fig. 12.15 Worldlines of the free particle $\mathscr{P}$ and of the uniformly accelerated observer $\mathscr{O}$, depicted in the inertial coordinates (ct_*, x_*) (left panel) and in the Rindler coordinates (ct, x) associated with $\mathscr{O}$ (right panel). $\mathscr{H}$ is $\mathscr{O}$'s Rindler horizon

in the case $b = a^{-1}$. The particle $\mathscr{P}$ encounters observer $\mathscr{O}$ iff $x(t_0) = 0$ for a certain value t_0 of t, i.e. iff

$$\cosh(act_0) = 1 + ab.$$

This equation admits two solutions:

$$t_0 = \pm(ac)^{-1}\text{arcosh}(1 + ab) \tag{12.76}$$

iff $b \geq 0$. In the case $b = a^{-1}$ depicted in Fig. 12.15, this corresponds to $t_0 \simeq \pm 1.317(ac)^{-1}$.

By definition, $\mathscr{P}$'s velocity relative to observer $\mathscr{O}$ is $\vec{V}(t) = (\mathrm{d}x/\mathrm{d}t)\,\vec{e}_1(t)$. We get

$$\vec{V}(t) = V(t)\,\vec{e}_1(t) \qquad \text{with} \qquad \boxed{V(t) = -c(1 + ab)\frac{\sinh(act)}{\cosh^2(act)}}. \tag{12.77}$$

The Lorentz factor of $\mathscr{P}$ relative to $\mathscr{O}$ is given by (4.9), with $\vec{u}\cdot\vec{u}' = \vec{u}(t)\cdot\vec{e}^{\,*}_0 = -\cosh(act)$ and $1 + \vec{a}\cdot\overrightarrow{OM} = 1 + ax(t)$. Hence,

$$\boxed{\Gamma = \frac{\cosh^2(act)}{1 + ab}}. \tag{12.78}$$

The 4-velocity of $\mathscr{P}$ is the constant vector $\vec{\boldsymbol{u}}_{\mathscr{P}} = \vec{\boldsymbol{e}}^{\,*}_0$. From (12.30), we can express it in $\mathscr{O}$'s local frame as $\vec{\boldsymbol{u}}_{\mathscr{P}} = \cosh(act)\,\vec{\boldsymbol{u}}(t) - \sinh(act)\,\vec{\boldsymbol{e}}_1(t)$. $\mathscr{P}$'s 4-momentum vector is $\vec{\boldsymbol{p}} = mc\,\vec{\boldsymbol{u}}_{\mathscr{P}}$ [Eq. (9.3)], m being $\mathscr{P}$'s mass. We have thus

$$\vec{\boldsymbol{p}} = mc\left[\cosh(act)\,\vec{\boldsymbol{u}}(t) - \sinh(act)\,\vec{\boldsymbol{e}}_1(t)\right]. \tag{12.79}$$

By comparing with the orthogonal decomposition (9.7), we read the energy of particle $\mathscr{P}$ measured by $\mathscr{O}$:

$$\boxed{E = mc^2\cosh(act)}, \tag{12.80}$$

as well as the linear momentum of $\mathscr{P}$ measured by $\mathscr{O}$:

$$\boxed{\vec{\boldsymbol{P}} = -mc\sinh(act)\,\vec{\boldsymbol{e}}_1(t)}. \tag{12.81}$$

By means of (12.75), we can express the energy in terms of the position of the particle:

$$\boxed{E = mc^2\frac{1+ab}{1+ax(t)}}. \tag{12.82}$$

Note that neither E nor $\vec{\boldsymbol{P}}$ are constant, while the particle is free. This reflects the fact that $\mathscr{O}$ is not an inertial observer.

Remark 12.12. By combining (12.77), (12.78) and (12.81), we check relation (9.13) (with $\vec{\boldsymbol{\omega}} = 0$): $\vec{\boldsymbol{P}} = \Gamma m\vec{\boldsymbol{V}}$.

At the limit of small accelerations, or of distances from $\mathscr{O}$ small in front of a^{-1}, i.e. $|act| \ll 1$, $|ax(t)| \ll 1$ and $|ab| \ll 1$, Taylor expansions of formulas (12.75), (12.77), (12.78), (12.80), (12.81) and (12.82) lead, respectively, to

$$x(t) \simeq b - \frac{\gamma}{2}t^2 \tag{12.83}$$

$$V \simeq -\gamma\,t \tag{12.84}$$

$$\Gamma \simeq 1 + \frac{1}{2}\frac{V^2}{c^2} - \frac{\gamma b}{c^2} \tag{12.85}$$

$$E \simeq mc^2 + \frac{1}{2}mV^2 \tag{12.86}$$

$$\vec{\boldsymbol{P}} = mV\,\vec{\boldsymbol{e}}_1(t) \tag{12.87}$$

$$E \simeq mc^2 - m\gamma[x(t) - b], \tag{12.88}$$

where we have let appear the norm $\gamma := c^2 a$ of $\mathscr{O}$'s acceleration relative to the tangent inertial observer $\mathscr{O}_*$ at $t = t_* = 0$ [cf. (12.20)]. We recognize in (12.83) and (12.84) the equations of motion of Newtonian mechanics describing the free fall of a particle in a uniform gravitational field of amplitude γ oriented towards the negative x's. Moreover, (12.88) shows that the quantity

$$E' := E + E_{\rm pot}, \qquad \text{with} \quad E_{\rm pot} := m\gamma\, x(t), \tag{12.89}$$

is a constant of motion (equal to $mc^2 + b$). The expression of $E_{\rm pot}$ is of course reminiscent of the potential energy in a uniform gravitational field. We shall come back to this fundamental point in Chap. 22, which is devoted to gravitation.

12.5 Thomas Precession

Thomas precession is a relativistic phenomenon that consists in the rotation of the frame of a *nonrotating* accelerated observer (i.e. an observer whose 4-rotation $\vec{\boldsymbol{\omega}}$ vanishes) when this frame is compared to that of a given inertial observer. This phenomenon has no equivalent in Newtonian mechanics. It also does not occur when, with respect to the inertial observer, the acceleration is collinear to the velocity, which is the case of the uniformly accelerated observer considered in Sect. 12.2 and 12.4.

12.5.1 Derivation

Thomas precession is actually a manifestation of *Thomas rotation* studied in Sect. 6.7.2, namely, of the fact that the product of two Lorentz boosts of different planes is not a boost but the product of a boost by a spatial rotation.

Let us consider an accelerated observer $\mathscr{O}$, of worldline $\mathscr{L}$, proper time t, 4-velocity $\vec{\boldsymbol{u}}(t)$, 4-acceleration $\vec{\boldsymbol{a}}(t) \neq 0$ and vanishing 4-rotation: $\vec{\boldsymbol{\omega}}(t) = 0$. Let us denote by $(\vec{\boldsymbol{e}}_\alpha(t))$ the local frame of $\mathscr{O}$ (cf. Fig. 12.16). In addition, let $\mathscr{O}_*$ be an inertial observer, of worldline $\mathscr{L}_*$, proper time t_* and frame $(\vec{\boldsymbol{e}}^{\,*}_\alpha)$. Denoting by Γ the Lorentz factor of $\mathscr{O}$ relative to $\mathscr{O}_*$, by $\vec{V}$ and $\vec{\boldsymbol{\gamma}}$ the velocity and acceleration of $\mathscr{O}$ relative to $\mathscr{O}_*$, the following relations hold:

$$\Gamma = \mathrm{d}t_*/\mathrm{d}t, \tag{12.90}$$

$$\Gamma = \left(1 - \frac{1}{c^2}\vec{V}\cdot\vec{V}\right)^{-1/2}, \tag{12.91}$$

$$\vec{\boldsymbol{u}} = \vec{\boldsymbol{e}}_0 = \Gamma\left(\vec{\boldsymbol{e}}^{\,*}_0 + \frac{1}{c}\vec{V}\right), \tag{12.92}$$

$$\vec{\boldsymbol{\gamma}} = \Gamma^{-2}\left[c^2\,\vec{\boldsymbol{a}} - (\vec{\boldsymbol{a}}\cdot\vec{V})\left(\vec{V} + c\,\vec{\boldsymbol{e}}^{\,*}_0\right)\right]. \tag{12.93}$$

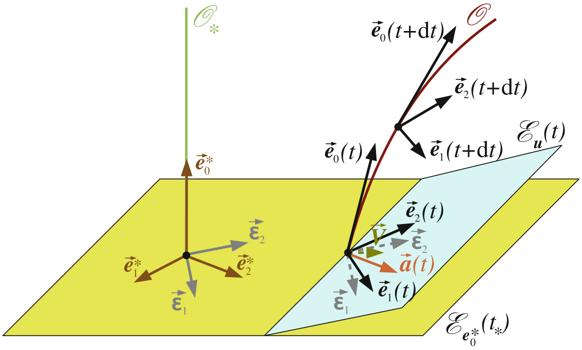


Fig. 12.16 Accelerated observer $\mathscr{O}$ [local frame $(\vec{e}_\alpha(t))$] and inertial observer $\mathscr{O}_*$ [frame $(\vec{e}^{\,*}_\alpha)$]. The change from the tetrad $(\vec{e}^{\,*}_0, \vec{\varepsilon}_1, \vec{\varepsilon}_2, \vec{\varepsilon}_3)$ to the tetrad $(\vec{e}_\alpha(t))$ is performed via a Lorentz boost, whose velocity parameter $\vec{V}$ is nothing but the velocity of $\mathscr{O}$ relative to $\mathscr{O}_*$

They are nothing but formulas (4.1), (4.33), (4.31) and (4.72) established in Chap. 4 and adapted to the present notations.

Let $\boldsymbol{S}$ be the unique Lorentz boost sending the 4-velocity of $\mathscr{O}_*$ to that of $\mathscr{O}$ at the instant t (cf. Sect. 6.6.1):

$$\vec{e}_0(t) = \boldsymbol{S}(\vec{e}^{\,*}_0). \tag{12.94}$$

Let us define

$$\vec{\varepsilon}_\alpha(t_*) := \boldsymbol{S}^{-1}(\vec{e}_\alpha(t)), \tag{12.95}$$

noticing that, by construction, $\vec{\varepsilon}_0(t_*) = \vec{e}^{\,*}_0 = \text{const}$. Since $\boldsymbol{S}$ is a Lorentz transformation, $(\vec{\varepsilon}_\alpha(t_*))$ is an orthonormal basis of $(E, \boldsymbol{g})$. It is such that the three vectors $(\vec{\varepsilon}_i(t_*)) = (\vec{\varepsilon}_1(t_*), \vec{\varepsilon}_2(t_*), \vec{\varepsilon}_3(t_*))$ form an orthonormal basis (depending upon time) of $\mathscr{O}_*$'s rest space, $E_{\boldsymbol{e}^*_0}$. The triad $(\vec{\varepsilon}_i(t_*))$ can be seen as "representing" the spatial frame $(\vec{e}_i(t))$ of $\mathscr{O}$ with respect to $\mathscr{O}_*$: this is the triad of $\mathscr{O}_*$'s rest space, $E_{\boldsymbol{e}^*_0}$, that is the "most parallel" to $(\vec{e}_i(t))$: the two triads coincide if $\vec{e}_0(t) = \vec{e}^{\,*}_0$ ($\mathscr{O}$ momentarily at rest with respect to $\mathscr{O}_* \iff \boldsymbol{S} = \mathrm{Id}$); if $\vec{e}_0(t) \neq \vec{e}^{\,*}_0$, the two triads cannot be parallel (i.e. coincide) for they belong to different vector hyperplanes ($E_{\boldsymbol{e}^*_0}$ and $E_{\boldsymbol{u}}(t)$; cf. Fig. 12.16). However, if one of the vectors $\vec{e}_i(t)$, for instance, $\vec{e}_1(t)$, is in the plane of the boost $\boldsymbol{S}$, then $\vec{\varepsilon}_2(t_*) = \vec{e}_2(t)$ and $\vec{\varepsilon}_3(t_*) = \vec{e}_3(t)$, so that the triads $(\vec{e}_i(t))$ and $(\vec{\varepsilon}_i(t_*))$ are quasiparallel (cf. Sect. 8.3.1).

Thomas precession concerns the evolution of the triad $(\vec{\varepsilon}_i(t_*))$ during the motion of $\mathscr{O}$, i.e. when t (and thus t_*) varies. The local frame of $\mathscr{O}$ evolves according to the law (3.52) with $\vec{\boldsymbol{\omega}} = 0$ (since $\mathscr{O}$ is nonrotating). We shall use this law under the form (7.27) obtained in Chap. 7:

$$\vec{e}_\alpha(t + \mathrm{d}t) = \boldsymbol{\Lambda}(\vec{e}_\alpha(t)), \tag{12.96}$$

$\boldsymbol{\Lambda}$ being the endomorphism of E defined by

$$\boldsymbol{\Lambda} := \mathrm{Id} + c\,\mathrm{d}t\, a^i \boldsymbol{K}_i, \tag{12.97}$$

where (i) the a^i's are the components of $\vec{\boldsymbol{a}}$ in the basis $(\vec{\boldsymbol{e}}_i)$ of $E_{\boldsymbol{u}}$: $\vec{\boldsymbol{a}} = a^i\, \vec{\boldsymbol{e}}_i$ (let us remind that $\vec{\boldsymbol{a}} \in E_{\boldsymbol{u}}$) and (ii) the three endomorphisms $\boldsymbol{K}_i$ are the generators of Lorentz boosts introduced in Sect. 7.3.3. It is clear that $\boldsymbol{\Lambda}$ is an infinitesimal Lorentz boost. At first order in $\mathrm{d}t$, its Lorentz factor is $\Gamma \simeq 1$ and its velocity is[7]

$$\vec{\boldsymbol{W}} = c^2 \mathrm{d}t\, \vec{\boldsymbol{a}}. \tag{12.98}$$

Let us look for the evolution of the triad $\vec{\boldsymbol{\varepsilon}}_i(t_*)$ starting from (12.95), which we transpose from (t_*, t) to $(t_* + \mathrm{d}t_*, t + \mathrm{d}t)$. Care must be taken that a priori $\boldsymbol{S}$ depends on t. Let us thus denote by $\boldsymbol{S}_{t+\mathrm{d}t}$ the value of $\boldsymbol{S}$ at the instant $t + \mathrm{d}t$. Equation (12.94) becomes then $\vec{\boldsymbol{e}}_0(t + \mathrm{d}t) = \boldsymbol{S}_{t+\mathrm{d}t}(\vec{\boldsymbol{e}}^{\,*}_0)$ and (12.95) leads to

$$\begin{aligned}
\vec{\boldsymbol{\varepsilon}}_\alpha(t_* + \mathrm{d}t_*) &= \boldsymbol{S}^{-1}_{t+\mathrm{d}t}(\vec{\boldsymbol{e}}_\alpha(t + \mathrm{d}t)) \\
&= \boldsymbol{S}^{-1}_{t+\mathrm{d}t} \circ \boldsymbol{\Lambda}(\vec{\boldsymbol{e}}_\alpha(t)) \\
&= \boldsymbol{S}^{-1}_{t+\mathrm{d}t} \circ \boldsymbol{\Lambda} \circ \boldsymbol{S}(\vec{\boldsymbol{\varepsilon}}_\alpha(t_*)),
\end{aligned} \tag{12.99}$$

where use has been made of (12.96) to get the second line and (12.95) for the third one. Since $\boldsymbol{\Lambda}$ and $\boldsymbol{S}$ are two Lorentz boosts whose planes intersect along $\vec{\boldsymbol{e}}_0(t) = \boldsymbol{S}(\vec{\boldsymbol{\varepsilon}}_0(t_*)) = \boldsymbol{S}(\vec{\boldsymbol{e}}^{\,*}_0)$, we know, from the study performed in Sect. 6.7.2, that the composite function $\boldsymbol{\Lambda} \circ \boldsymbol{S}$ is the product of a boost $\boldsymbol{S}'$ whose plane contains $\vec{\boldsymbol{e}}^{\,*}_0$ by a spatial rotation $\boldsymbol{R}$, of plane orthogonal to $\vec{\boldsymbol{e}}^{\,*}_0$—Thomas rotation [cf. Eq. (6.100)]:

$$\boldsymbol{\Lambda} \circ \boldsymbol{S} = \boldsymbol{S}' \circ \boldsymbol{R}. \tag{12.100}$$

Let us show that actually $\boldsymbol{S}' = \boldsymbol{S}_{t+\mathrm{d}t}$. Since $\boldsymbol{R}(\vec{\boldsymbol{e}}^{\,*}_0) = \vec{\boldsymbol{e}}^{\,*}_0$, the identity (12.100) leads to

$$\boldsymbol{S}'(\vec{\boldsymbol{e}}^{\,*}_0) = \boldsymbol{\Lambda} \circ \underbrace{\boldsymbol{S}(\vec{\boldsymbol{e}}^{\,*}_0)}_{\vec{\boldsymbol{e}}_0(t)} = \boldsymbol{\Lambda}(\vec{\boldsymbol{e}}_0(t)) = \vec{\boldsymbol{e}}_0(t + \mathrm{d}t) = \boldsymbol{S}_{t+\mathrm{d}t}(\vec{\boldsymbol{e}}^{\,*}_0),$$

where we have used successively (12.94), (12.96) and again (12.94) with $t \to t + \mathrm{d}t$ and $\boldsymbol{S} \to \boldsymbol{S}_{t+\mathrm{d}t}$. The above result shows that $\boldsymbol{S}'$ and $\boldsymbol{S}_{t+\mathrm{d}t}$ are two boosts whose planes contain $\vec{\boldsymbol{e}}^{\,*}_0$ and that give the same image of $\vec{\boldsymbol{e}}^{\,*}_0$. They necessarily coincide, so that (12.100) can be written $\boldsymbol{\Lambda} \circ \boldsymbol{S} = \boldsymbol{S}_{t+\mathrm{d}t} \circ \boldsymbol{R}$. By inserting this relation in (12.99), there comes $\vec{\boldsymbol{\varepsilon}}_\alpha(t_* + \mathrm{d}t_*) = \boldsymbol{R}(\vec{\boldsymbol{\varepsilon}}_\alpha(t_*))$; hence, in particular,

$$\boxed{\vec{\boldsymbol{\varepsilon}}_i(t_* + \mathrm{d}t_*) = \boldsymbol{R}(\vec{\boldsymbol{\varepsilon}}_i(t_*))}. \tag{12.101}$$

[7] To see it, it suffices to express $\boldsymbol{\Lambda}$ in terms of its rapidity $\delta\psi$ according to (7.22), to compare with (12.97) and to write the velocity of $\boldsymbol{\Lambda}$ as $W = c\tanh(\delta\psi) \simeq c\,\delta\psi$.

Thus, the evolution of the triad $(\vec{\boldsymbol{\varepsilon}}_i(t_*))$, which represents the triad $(\vec{\boldsymbol{e}}_i(t))$ of $\mathscr{O}$ with respect to $\mathscr{O}_*$, takes place via a rotation: the Thomas rotation $\boldsymbol{R}$ arising from the composition of $\boldsymbol{\Lambda}$ by $\boldsymbol{S}$.

Let us determine the parameters of $\boldsymbol{R}$ from the results of Sect. 6.7.2. The notations used here are related to those of Sect. 6.7.2 in Table 12.1. The plane of $\boldsymbol{R}$ is [cf. Eq. (6.107)]

$$\Pi_{\boldsymbol{R}} = \mathrm{Span}(\vec{\boldsymbol{n}}, \vec{\boldsymbol{m}}) \subset E_{\boldsymbol{e}_0^*}, \tag{12.102}$$

where

- $\vec{\boldsymbol{n}} \in E_{\boldsymbol{e}_0^*}$ is the unit vector in the direction of the velocity $\vec{\boldsymbol{V}}$:

$$\vec{\boldsymbol{V}} =: V\vec{\boldsymbol{n}}, \quad \text{with} \quad \vec{\boldsymbol{n}}\cdot\vec{\boldsymbol{n}} = 1. \tag{12.103}$$

- $\vec{\boldsymbol{m}} \in E_{\boldsymbol{e}_0^*}$ is the unit vector in the plane orthogonal to $\vec{\boldsymbol{V}}$ such that

$$\vec{\boldsymbol{a}} =: a\left[\cos\theta\, \boldsymbol{S}(\vec{\boldsymbol{n}}) + \sin\theta\, \vec{\boldsymbol{m}}\right], \quad \text{with} \quad \vec{\boldsymbol{m}}\cdot\vec{\boldsymbol{m}} = 1 \quad \text{and} \quad \vec{\boldsymbol{n}}\cdot\vec{\boldsymbol{m}} = 0. \tag{12.104}$$

$\vec{\boldsymbol{n}}$ and $\vec{\boldsymbol{m}}$ are, respectively, noted $\vec{\boldsymbol{e}}_1$ and $\vec{\boldsymbol{e}}_2$ in Sect. 6.7.2 (cf. Table 12.1). θ is the angle between the planes of the two Lorentz transformations $\boldsymbol{S}$ and $\boldsymbol{\Lambda}$, as defined in Sect. 6.7.2; this is also the angle between $\boldsymbol{S}(\vec{\boldsymbol{V}})$ and $\vec{\boldsymbol{W}}$ or between $\boldsymbol{S}(\vec{\boldsymbol{V}})$ and $\vec{\boldsymbol{a}}$, all these vectors belonging to $E_{\boldsymbol{u}}(t)$. Besides, $\vec{\boldsymbol{m}}$ being orthogonal to $\vec{\boldsymbol{V}}$, $\boldsymbol{S}(\vec{\boldsymbol{m}}) = \vec{\boldsymbol{m}}$. We deduce from (12.103) and (12.104) that $\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{V}} = \Gamma a V\cos\theta$. Substituting this value in (12.93) and using again (12.103) and (12.104), we get

$$\vec{\boldsymbol{\gamma}} = \frac{c^2 a}{\Gamma^2}\left(\frac{\cos\theta}{\Gamma}\,\vec{\boldsymbol{n}} + \sin\theta\,\vec{\boldsymbol{m}}\right). \tag{12.105}$$

The relative velocity and acceleration vectors $\vec{\boldsymbol{V}}$ and $\vec{\boldsymbol{\gamma}}$ both belong to the hyperplane $E_{\boldsymbol{e}_0^*}$. If they are not collinear, in other words if $\theta \neq 0$, (12.103) and (12.105) show that they form a basis of the plane $\mathrm{Span}(\vec{\boldsymbol{n}}, \vec{\boldsymbol{m}})$. We may thus rewrite (12.102) as

$$\boxed{\Pi_{\boldsymbol{R}} = \mathrm{Span}(\vec{\boldsymbol{V}}, \vec{\boldsymbol{\gamma}})}. \tag{12.106}$$

Having determined the plane of the Thomas rotation, let us turn to its angle φ_{T}. It is given by formula (6.111b) obtained in Chap. 6. In the present case, numerous simplifications occur because $\boldsymbol{\Lambda}_2$ is an infinitesimal transformation, of velocity $V_2 = W = ac^2\mathrm{d}t$ [cf. Eq. (12.98) and Table 12.1]. Remaining at first order in $\mathrm{d}t$, we can set $\Gamma_2 \simeq 1$, $\Gamma_1\Gamma_2/(1+\Gamma)\,V_1V_2 \simeq \Gamma_1/(1+\Gamma_1)\,V_1V_2$ and replace the term in parentheses in (6.111b) by 1. Since $\Gamma_1 = \Gamma$, $V_1 = V$ (cf. Table 12.1), formula (6.111b) reduces then to

$$\sin \varphi_{\mathrm{T}} = -\sin\theta \frac{\Gamma}{1+\Gamma} aV \,\mathrm{d}t.$$

The angle φ_{T} is clearly infinitesimal, of the same order as $\mathrm{d}t$. Let us denote it rather by $\mathrm{d}\varphi_{\mathrm{T}}$. We have of course $\sin \mathrm{d}\varphi_{\mathrm{T}} \simeq \mathrm{d}\varphi_{\mathrm{T}}$, so that the above formula becomes

$$\mathrm{d}\varphi_{\mathrm{T}} = -\frac{\Gamma}{1+\Gamma} aV \sin\theta \,\mathrm{d}t. \tag{12.107}$$

θ is the angle between $\boldsymbol{S}(\vec{V})$ and $\vec{\boldsymbol{a}}$ in the vector hyperplane $E_{\boldsymbol{u}}(t)$ associated with observer $\mathscr{O}$. Let us express $\mathrm{d}\varphi_{\mathrm{T}}$ rather in terms of the angle θ_* between $\vec{V}$ and $\vec{\boldsymbol{\gamma}}$ in the vector hyperplane $E_{\boldsymbol{e}_0^*}$ associated with the inertial observer $\mathscr{O}_*$. We deduce from (12.105) and (12.91) that the norm of $\vec{\boldsymbol{\gamma}}$ is

$$\gamma := \|\vec{\boldsymbol{\gamma}}\|_{\boldsymbol{g}} = \sqrt{\vec{\boldsymbol{\gamma}}\cdot\vec{\boldsymbol{\gamma}}} = \frac{c^2 a}{\Gamma^2}\sqrt{1-(V^2/c^2)\cos^2\theta}. \tag{12.108}$$

Relation (12.105) is then rewritten as

$$\vec{\boldsymbol{\gamma}} = \gamma\left(\cos\theta_*\,\vec{\boldsymbol{n}} + \sin\theta_*\,\vec{\boldsymbol{m}}\right), \tag{12.109}$$

with

$$\cos\theta_* = \frac{\cos\theta}{\Gamma\sqrt{1-(V^2/c^2)\cos^2\theta}} \quad \text{and} \quad \sin\theta_* = \frac{\sin\theta}{\sqrt{1-(V^2/c^2)\cos^2\theta}}. \tag{12.110}$$

We deduce from (12.105) and (12.109) that

$$a\sin\theta = \Gamma^2\frac{\gamma}{c^2}\sin\theta_*. \tag{12.111}$$

Substituting this value into (12.107) and substituting $\mathrm{d}t$ by $\Gamma^{-1}\mathrm{d}t_*$ [cf. Eq. (12.90)], there comes

$$\boxed{\frac{\mathrm{d}\varphi_{\mathrm{T}}}{\mathrm{d}t_*} = -\frac{\Gamma^2}{1+\Gamma}\frac{\gamma V}{c^2}\sin\theta_*}. \tag{12.112}$$

Table 12.1 Correspondence between the notations used in this chapter and those used in Sect. 6.7.2

Here	$\boldsymbol{S}$	$\boldsymbol{\Lambda}_0$	$\vec{\boldsymbol{e}}_0^* = \vec{\boldsymbol{\varepsilon}}_0$	$\vec{\boldsymbol{e}}_0(t)$	$\vec{\boldsymbol{e}}_0(t + \mathrm{d}t)$	$\vec{\boldsymbol{V}}$	$\boldsymbol{S}(\vec{\boldsymbol{V}})$	$\vec{\boldsymbol{W}}$
Sect. 6.7.2	$\boldsymbol{\Lambda}_1$	$\boldsymbol{\Lambda}_2$	$\vec{\boldsymbol{e}}_0$	$\vec{\boldsymbol{e}}'_0$	$\vec{\boldsymbol{e}}''_0$	$V_1\vec{\boldsymbol{e}}_1$	$\vec{\boldsymbol{V}}_1$	$\vec{\boldsymbol{V}}_2$
Here	$\vec{\boldsymbol{n}}$	$\vec{\boldsymbol{m}}$	$\boldsymbol{S}(\vec{\boldsymbol{n}})$	$a^{-1}\vec{\boldsymbol{a}}$	V	$a\,c^2\,\mathrm{d}t$	Γ	$1 + O(\mathrm{d}t^2)$
Sect. 6.7.2	$\vec{\boldsymbol{e}}_1$	$\vec{\boldsymbol{e}}_2$	$\vec{\boldsymbol{e}}'_1$	$\vec{\boldsymbol{\varepsilon}}_1$	V_1	V_2	Γ_1	Γ_2

We may conclude:

The triad $(\vec{\boldsymbol{\varepsilon}}_i(t_*))$, which for the inertial observer $\mathscr{O}_*$ "represents" the spatial triad $(\vec{\boldsymbol{e}}_i(t))$ of the accelerated observer $\mathscr{O}$ (in the sense of being quasiparallel to it), has a motion of rotation, of plane $\Pi_{\boldsymbol{R}} = \mathrm{Span}(\vec{\boldsymbol{V}}, \vec{\boldsymbol{\gamma}})$ [Eq. (12.106)] and angular velocity $\mathrm{d}\varphi_{\mathrm{T}}/\mathrm{d}t_*$ given by (12.112), where $\vec{\boldsymbol{V}}$ and $\vec{\boldsymbol{\gamma}}$ are $\mathscr{O}$'s velocity and acceleration relative to $\mathscr{O}_*$. This property is called ***Thomas precession***.

Remark 12.13. For a Newtonian physicist, the surprising aspect of Thomas precession arises from the fact that the accelerated observer $\mathscr{O}$ is nonrotating: his local frame does not rotate, since $\vec{\boldsymbol{\omega}} = 0$. In particular, $\mathscr{O}$ does not feel any Coriolis or centrifugal force, as they appear in (9.120). Nevertheless, the spatial triad formed by the vectors $\vec{\boldsymbol{\varepsilon}}_i$, which are quasiparallel to the vectors of $\mathscr{O}$'s spatial frame, undergoes some rotation with respect to the inertial observer $\mathscr{O}_*$. That this is a pure relativistic effect is clear on (12.112): $\mathrm{d}\varphi_{\mathrm{T}}/\mathrm{d}t_* \to 0$ if $\gamma V/c^2 \to 0$.

Remark 12.14. The rotation plane, $\Pi_{\boldsymbol{R}}$, varies with time, since it is determined by vectors $\vec{\boldsymbol{V}}$ and $\vec{\boldsymbol{\gamma}}$, which are a priori functions of t_*.

Remark 12.15. If the relative acceleration $\vec{\boldsymbol{\gamma}}$ is collinear to the velocity $\vec{\boldsymbol{V}}$, as it occurs, for instance, for the Langevin's traveller treated in Sect. 2.6 or for the uniformly accelerated observer considered in Sects. 12.2 and 12.4, then $\theta_* = 0$ and (12.112) leads to $\mathrm{d}\varphi_{\mathrm{T}}/\mathrm{d}t_* = 0$. There is thus no Thomas precession in this case.

Relation (12.101) leads to the following expression of the time derivative of the triad $(\vec{\boldsymbol{\varepsilon}}_i(t_*))$:

$$\boxed{\frac{\mathrm{d}\vec{\boldsymbol{\varepsilon}}_i}{\mathrm{d}t_*} = \vec{\boldsymbol{\omega}}_{\mathrm{T}} \times_{\boldsymbol{e}_0^*} \vec{\boldsymbol{\varepsilon}}_i}, \qquad (12.113)$$

where $\vec{\boldsymbol{\omega}}_{\mathrm{T}} \in E_{e_0^*}$ is the vector orthogonal to the rotation plane $\Pi_{\boldsymbol{R}}$ and whose norm is the absolute value of the angular velocity $\mathrm{d}\varphi_{\mathrm{T}}/\mathrm{d}t_*$ given by (12.112). Because of the $-$ sign in (12.112), $\vec{\boldsymbol{\omega}}_{\mathrm{T}}$ has an orientation opposite to that of the cross product $\vec{\boldsymbol{V}} \times_{\boldsymbol{e}_0^*} \vec{\boldsymbol{\gamma}}$. Since the norm of the latter is $\gamma V \sin\theta_*$, formula (12.112) allows one to write $\vec{\boldsymbol{\omega}}_{\mathrm{T}}$ in terms of the cross product of acceleration by velocity:

$$\boxed{\vec{\omega}_{\mathrm{T}} = \frac{\Gamma^2}{c^2(1+\Gamma)}\,\vec{\gamma} \times_{e_0^*} \vec{V}}. \tag{12.114}$$

For small velocities, we can set $\Gamma \simeq 1$ and obtain

$$\vec{\omega}_{\mathrm{T}} \simeq \frac{1}{2c^2}\,\vec{\gamma} \times_{e_0^*} \vec{V} \qquad (|V| \ll c). \tag{12.115}$$

Remark 12.16. From (12.91), it is easy to derive the identity $\Gamma^2/(1+\Gamma) = (\Gamma - 1)c^2/V^2$, so that some authors present the result (12.114) as

$$\vec{\omega}_{\mathrm{T}} = \frac{\Gamma - 1}{V^2}\,\vec{\gamma} \times_{e_0^*} \vec{V}. \tag{12.116}$$

12.5.2 Application to a Gyroscope

Let us consider a free gyroscope, as defined in Sect. 10.7.3, carried by the accelerated observer $\mathscr{O}$. The gyroscope's spin vector obeys the law (10.84): $\boldsymbol{D}^{\mathrm{FW}}_{\boldsymbol{u}}\vec{s} = 0$ ($\vec{s}$ is Fermi–Walker transported along $\mathscr{L}$). Since $\mathscr{O}$ is nonrotating, this implies that $\vec{s}$ is fixed with respect to $\mathscr{O}$ [cf. Eq. (3.71)], i.e. that the components (s^i) of $\vec{s}$ in $\mathscr{O}$'s local frame are constant:

$$\vec{s}(t) = s^i\,\vec{e}_i(t), \quad \text{with} \quad s^i = \text{const}. \tag{12.117}$$

Since $\vec{e}_i(t) = \boldsymbol{S}\,(\vec{\varepsilon}_i(t_*))$ [Eq. (12.95)], we deduce that

$$\boxed{\vec{s}(t) = \boldsymbol{S}(\vec{s}_*(t_*))}, \quad \text{with} \quad \vec{s}_*(t_*) := s^i\,\vec{\varepsilon}_i(t_*). \tag{12.118}$$

The vector $\vec{s}_*(t_*)$ hence defined belongs to the rest space of the inertial observer: $\vec{s}_*(t_*) \in E_{e_0^*}$. As the coefficients s^i are constant, we deduce immediately from (12.113) that $\vec{s}_*(t_*)$ obeys the following evolution law:

$$\boxed{\frac{\mathrm{d}\vec{s}_*}{\mathrm{d}t_*} = \vec{\omega}_{\mathrm{T}} \times_{e_0^*} \vec{s}_*}. \tag{12.119}$$

Hence, $\vec{s}_*(t_*)$ is submitted to Thomas precession, at the angular velocity $\vec{\omega}_{\mathrm{T}}$ given by (12.114).

Remark 12.17. The vector $\vec{s}_*$ belongs to the rest space of the inertial observer; it is therefore distinct from the spin vector $\vec{s}$, which belongs to the local rest space of the accelerated observer. However, the vectors $\vec{s}_*$ and $\vec{s}$ have the same components (s^i), one with respect to the basis $(\vec{\varepsilon}_i(t_*))$ and the other one with respect to $(\vec{e}_i(t))$, these two basis being related by the Lorentz boost $\boldsymbol{S}$. In particular, $\vec{s}_*$ and $\vec{s}$ have the same norm. Authors who use a three-dimensional point of view, such as Jackson (1998), implicitly identify $\vec{s}_*$ and $\vec{s}$. The vector $\vec{s}_*$, as defined by (12.118), is employed by Rowe (1984; 1996) (who calls it *"representative of the spin"*), by Jantzen, Carini and Bini (1992) (who call it *"boosted spin vector"*) and by Jonsson (2006; 2007) (who calls it *"stopped spin vector"*).

Remark 12.18. Some authors, as, for instance, Misner, Thorne and Wheeler (1973) (cf. p. 79), do not derive the equation of motion for $\vec{s}_*$, but those for the orthogonal projection of the spin onto the rest space of the inertial observer $\mathscr{O}_*$, namely, the vector $\perp_{e_0^*}\vec{s}$. The two vectors are related by

$$\perp_{e_0^*}\vec{s} = \vec{s}_* + \frac{\Gamma^2}{c^2(1+\Gamma)}(\vec{V}\cdot\vec{s}_*)\,\vec{V}. \tag{12.120}$$

The equation of motion for $\perp_{e_0^*}\vec{s}$ is more complicated than that for $\vec{s}_*$ and does not reduce to a mere (Thomas) precession (cf. Eq. (6.27) in Misner et al. (1973)).

12.5.3 Gyroscope in Circular Orbit

Let us examine the particular case of a free gyroscope in a circular uniform motion in the plane (x_*, y_*), where $(x_*^\alpha) = (ct_*, x_*, y_*, z_*)$ are the inertial coordinates associated with observer $\mathscr{O}_*$. Such a motion corresponds to Example 4.3 studied in Chap. 4. The worldline of observer $\mathscr{O}$ carrying the gyroscope (cf. Fig. 4.4) obeys (4.6), where t has to be replaced by t_* : $x_*(t_*) = R\cos\Omega t_*$ and $y_*(t_*) = R\sin\Omega t_*$, the constants R and Ω being positive and such that $R\Omega < c$. The gyroscope's velocity and acceleration relative to the inertial observer are given by (4.23) and (4.46):

$$\begin{aligned}\vec{V} &= R\Omega\left(-\sin\Omega t_*\ \vec{e}_1^{\,*} + \cos\Omega t_*\ \vec{e}_2^{\,*}\right)\\ \vec{\gamma} &= -R\Omega^2\left(\cos\Omega t_*\ \vec{e}_1^{\,*} + \sin\Omega t_*\ \vec{e}_2^{\,*}\right).\end{aligned}$$

Inserting these formulas into (12.116) and using $\vec{e}_1^{\,*} \times_{\boldsymbol{e}_0^*} \vec{e}_2^{\,*} = \vec{e}_3^{\,*}$ and $V^2 = R^2\Omega^2$, we get

$$\boxed{\vec{\omega}_{\mathrm{T}} = -(\Gamma-1)\Omega\,\vec{e}_3^{\,*}}. \tag{12.121}$$

Thus, the frequency of Thomas precession is nothing by the rotation frequency multiplied by $\Gamma - 1$. We recover that Thomas precession is a pure relativistic effect, since $\Gamma - 1 = 0$ at the nonrelativistic limit. The $-$ sign in (12.121) means that Thomas precession occurs in the sense opposite to the gyroscope's rotational motion (cf. Fig. 12.17).

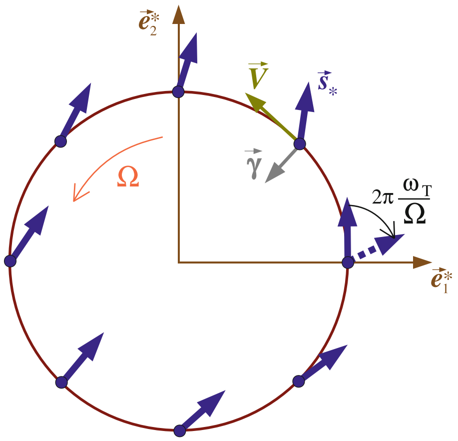


Fig. 12.17 Thomas precession of the vector $\vec{s}_*$ associated with the spin of a free gyroscope in uniform circular motion

The Lorentz factor that appears in (12.121) is given by (4.7). In the limit of a small rotation velocity, $R\Omega \ll c$, we can write $\Gamma - 1 \simeq 1/2\,(R\Omega/c)^2$, so that (12.121) becomes

$$\vec{\omega}_{\mathrm{T}} \simeq -\frac{1}{2}\left(\frac{R\Omega}{c}\right)^2 \Omega\, \vec{e}^{\,*}_3 \qquad (R\Omega \ll c). \tag{12.122}$$

12.5.4 Thomas Equation

Let us consider now the case where the spin is submitted to some torque $\vec{C}$. We have seen in Chap. 10 that the spin evolves then according to the law (10.82):

$$\frac{\mathrm{d}\vec{s}}{\mathrm{d}t} = \vec{C} + c\,(\vec{a}\cdot\vec{s})\,\vec{u}, \tag{12.123}$$

where $\vec{u}$, $\vec{a}$ and t are the 4-velocity, 4-acceleration and proper time of the accelerated observer that we consider from now on as a *particle with spin* (cf. Sect. 10.7). We are going to deduce from (12.123) an evolution law for the vector $\vec{s}_*$ defined by (12.118); we should recover (12.119) as the particular case $\vec{C} = 0$. Let us start by making explicit the relation between $\vec{s}$ and $\vec{s}_*$, starting from (12.118): $\vec{s}_* = \boldsymbol{S}^{-1}(\vec{s})$. Since $\boldsymbol{S}^{-1}$ is a Lorentz boost of velocity $-\vec{V}$ with respect to $\vec{e}^{\,*}_0$, it can be expressed according to formula (6.69) (making $\vec{u} \to \vec{e}^{\,*}_0$ and $\vec{V} \to -\vec{V}$):

$$\begin{aligned}\vec{s}_* = \boldsymbol{S}^{-1}(\vec{s}) = -\Gamma(\vec{e}^{\,*}_0\cdot\vec{s})\,\vec{e}^{\,*}_0 + \frac{\Gamma}{c}\left[-(\vec{V}\cdot\vec{s})\,\vec{e}^{\,*}_0 + (\vec{e}^{\,*}_0\cdot\vec{s})\,\vec{V}\right] \\ +\vec{s} + (\vec{e}^{\,*}_0\cdot\vec{s})\,\vec{e}^{\,*}_0 + \frac{\Gamma^2}{c^2(1+\Gamma)}(\vec{V}\cdot\vec{s})\,\vec{V}.\end{aligned} \tag{12.124}$$

Given the property $\vec{u} \cdot \vec{s} = 0$ [Eq. (10.74)] and the decomposition (12.92) of $\vec{u}$, we deduce immediately that

$$\vec{e}_0^{\,*} \cdot \vec{s} = -\frac{1}{c}\vec{V} \cdot \vec{s}. \tag{12.125}$$

Inserting this expression into (12.124), we get, after simplification,

$$\vec{s}_* = \vec{s} - \frac{1}{c}(\vec{V} \cdot \vec{s})\left[\vec{e}_0^{\,*} + \frac{\Gamma}{c(1+\Gamma)}\vec{V}\right]. \tag{12.126}$$

This implies

$$\begin{aligned}\frac{\mathrm{d}\vec{s}_*}{\mathrm{d}t_*} = \frac{\mathrm{d}\vec{s}}{\mathrm{d}t_*} - \frac{1}{c}\left(\frac{\mathrm{d}\vec{V}}{\mathrm{d}t_*} \cdot \vec{s} + \vec{V} \cdot \frac{\mathrm{d}\vec{s}}{\mathrm{d}t_*}\right)\left[\vec{e}_0^{\,*} + \frac{\Gamma}{c(1+\Gamma)}\vec{V}\right] \\ -\frac{1}{c^2}(\vec{V} \cdot \vec{s})\left[\frac{\mathrm{d}}{\mathrm{d}t_*}\left(\frac{\Gamma}{1+\Gamma}\right)\vec{V} + \frac{\Gamma}{1+\Gamma}\frac{\mathrm{d}\vec{V}}{\mathrm{d}t_*}\right].\end{aligned} \tag{12.127}$$

Now, from (12.90) and (12.123),

$$\frac{\mathrm{d}\vec{s}}{\mathrm{d}t_*} = \frac{1}{\Gamma}\frac{\mathrm{d}\vec{s}}{\mathrm{d}t} = \frac{1}{\Gamma}\left[\vec{C} + c\,(\vec{a} \cdot \vec{s})\;\vec{u}\right],$$

with

$$\vec{a} \cdot \vec{s} = \frac{\Gamma^2}{c^2}\Big[\vec{\gamma} \cdot \vec{s} + \frac{\Gamma^2}{c^2}(\vec{\gamma} \cdot \vec{V})(\underbrace{\vec{V} \cdot \vec{s} + c\,\vec{e}_0^{\,*} \cdot \vec{s}}_{0})\Big] = \frac{\Gamma^2}{c^2}\vec{\gamma} \cdot \vec{s}.$$

To get the last relation, we have used expression (4.63) of the 4-acceleration in terms of the acceleration $\vec{\gamma}$ relative to the inertial observer as well as property (12.125). Given (12.92), we have thus

$$\frac{\mathrm{d}\vec{s}}{\mathrm{d}t_*} = \frac{1}{\Gamma}\vec{C} + \frac{\Gamma^2}{c}(\vec{\gamma} \cdot \vec{s})\left(\vec{e}_0^{\,*} + \frac{1}{c}\vec{V}\right). \tag{12.128}$$

In particular,

$$\frac{\mathrm{d}\vec{V}}{\mathrm{d}t_*} \cdot \vec{s} + \vec{V} \cdot \frac{\mathrm{d}\vec{s}}{\mathrm{d}t_*} = \vec{\gamma} \cdot \vec{s} + \frac{1}{\Gamma}(\vec{V} \cdot \vec{C}) + \frac{\Gamma^2}{c^2}(\vec{\gamma} \cdot \vec{s})(\vec{V} \cdot \vec{V}) = \frac{1}{\Gamma}(\vec{V} \cdot \vec{C}) + \Gamma^2(\vec{\gamma} \cdot \vec{s}), \tag{12.129}$$

where we have used $\vec{V} \cdot \vec{e}_0^{\,*} = 0$, $\mathrm{d}\vec{V}/\mathrm{d}t_* = \vec{\gamma}$ (for $\mathscr{O}_*$ is inertial) and (12.91).

In addition, (12.91) leads to

$$\frac{\mathrm{d}}{\mathrm{d}t_*}\left(\frac{\Gamma}{1+\Gamma}\right) = \frac{1}{(1+\Gamma)^2}\,\frac{\mathrm{d}\Gamma}{\mathrm{d}t_*} = \frac{\Gamma^3}{c^2(1+\Gamma)^2}\,\vec{\boldsymbol{\gamma}}\cdot\vec{\boldsymbol{V}}. \tag{12.130}$$

Inserting (12.128), (12.129) and (12.130) in (12.127) and using $\mathrm{d}\vec{\boldsymbol{V}}/\mathrm{d}t_* = \vec{\boldsymbol{\gamma}}$, there comes, after simplification,

$$\begin{aligned}\frac{\mathrm{d}\vec{\boldsymbol{s}}_*}{\mathrm{d}t_*} = \frac{1}{\Gamma}\left[\vec{\boldsymbol{C}} - \frac{(\vec{\boldsymbol{V}}\cdot\vec{\boldsymbol{C}})}{c}\left(\vec{\boldsymbol{e}}^{\,*}_0 + \frac{\Gamma}{c(1+\Gamma)}\vec{\boldsymbol{V}}\right)\right] + \frac{\Gamma^2}{c^2(1+\Gamma)}\Bigg\{(\vec{\boldsymbol{\gamma}}\cdot\vec{\boldsymbol{s}})\vec{\boldsymbol{V}} \\ -(\vec{\boldsymbol{V}}\cdot\vec{\boldsymbol{s}})\left[\frac{\Gamma}{c^2(1+\Gamma)}(\vec{\boldsymbol{V}}\cdot\vec{\boldsymbol{\gamma}})\vec{\boldsymbol{V}} + \frac{1}{\Gamma}\vec{\boldsymbol{\gamma}}\right]\Bigg\}.\end{aligned} \tag{12.131}$$

The vector $\vec{\boldsymbol{C}}$ sharing with $\vec{\boldsymbol{s}}$ the property of being orthogonal to $\vec{\boldsymbol{u}}$, the expression of $\boldsymbol{S}^{-1}(\vec{\boldsymbol{C}})$ is the same as that of $\boldsymbol{S}^{-1}(\vec{\boldsymbol{s}})$, with $\vec{\boldsymbol{s}}$ replaced by $\vec{\boldsymbol{C}}$. Comparing with $\boldsymbol{S}^{-1}(\vec{\boldsymbol{s}}) = \vec{\boldsymbol{s}}_*$ as given by (12.126), we recognize then $\boldsymbol{S}^{-1}(\vec{\boldsymbol{C}})$ in the first term of (12.131):

$$\vec{\boldsymbol{C}} - \frac{(\vec{\boldsymbol{V}}\cdot\vec{\boldsymbol{C}})}{c}\left(\vec{\boldsymbol{e}}^{\,*}_0 + \frac{\Gamma}{c(1+\Gamma)}\vec{\boldsymbol{V}}\right) = \boldsymbol{S}^{-1}(\vec{\boldsymbol{C}}). \tag{12.132}$$

Besides, the scalar product of (12.126) by $\vec{\boldsymbol{V}}$ leads to

$$\vec{\boldsymbol{V}}\cdot\vec{\boldsymbol{s}}_* = \vec{\boldsymbol{V}}\cdot\vec{\boldsymbol{s}} - \frac{1}{c}(\vec{\boldsymbol{V}}\cdot\vec{\boldsymbol{s}})\frac{\Gamma}{c(1+\Gamma)}\vec{\boldsymbol{V}}\cdot\vec{\boldsymbol{V}} = \vec{\boldsymbol{V}}\cdot\vec{\boldsymbol{s}}\underbrace{\left(1 - \frac{\Gamma}{c^2(1+\Gamma)}\vec{\boldsymbol{V}}\cdot\vec{\boldsymbol{V}}\right)}_{1/\Gamma},$$

hence,

$$\vec{\boldsymbol{V}}\cdot\vec{\boldsymbol{s}} = \Gamma\,\vec{\boldsymbol{V}}\cdot\vec{\boldsymbol{s}}_*. \tag{12.133}$$

Similarly, (12.126) leads to

$$\vec{\boldsymbol{\gamma}}\cdot\vec{\boldsymbol{s}} = \vec{\boldsymbol{\gamma}}\cdot\vec{\boldsymbol{s}}_* + \frac{\Gamma^2}{c^2(1+\Gamma)}(\vec{\boldsymbol{\gamma}}\cdot\vec{\boldsymbol{V}})(\vec{\boldsymbol{V}}\cdot\vec{\boldsymbol{s}}_*). \tag{12.134}$$

Inserting (12.132), (12.133) and (12.134) in (12.131), we get

$$\frac{\mathrm{d}\vec{\boldsymbol{s}}_*}{\mathrm{d}t_*} = \frac{1}{\Gamma}\,\boldsymbol{S}^{-1}(\vec{\boldsymbol{C}}) + \frac{\Gamma^2}{c^2(1+\Gamma)}\left[(\vec{\boldsymbol{\gamma}}\cdot\vec{\boldsymbol{s}}_*)\,\vec{\boldsymbol{V}} - (\vec{\boldsymbol{V}}\cdot\vec{\boldsymbol{s}}_*)\,\vec{\boldsymbol{\gamma}}\right]. \tag{12.135}$$

One can write $(\vec{\gamma} \cdot \vec{s}_*)\, \vec{V} - (\vec{V} \cdot \vec{s}_*)\, \vec{\gamma}$ as a double cross product:

$$(\vec{\gamma} \cdot \vec{s}_*)\, \vec{V} - (\vec{V} \cdot \vec{s}_*)\, \vec{\gamma} = \vec{s}_* \times_{e_0^*} (\vec{V} \times_{e_0^*} \vec{\gamma}) = (\vec{\gamma} \times_{e_0^*} \vec{V}) \times_{e_0^*} \vec{s}_*.$$

We recognize then in the right-hand side of (12.135) the Thomas rotation vector $\vec{\omega}_{\mathrm{T}}$, as given by (12.114), so that we obtain finally ***Thomas equation***:

$$\boxed{\frac{\mathrm{d}\vec{s}_*}{\mathrm{d}t_*} = \frac{1}{\Gamma}\, \boldsymbol{S}^{-1}(\vec{C}) + \vec{\omega}_{\mathrm{T}} \times_{e_0^*} \vec{s}_*}. \tag{12.136}$$

If $\vec{C} = 0$ (free gyroscope), we recover the precession law (12.119).

Remark 12.19. Contrary to the computation of Sect. 12.5.1, which led to (12.119), we have not used in the above derivation the Thomas rotation obtained in Chap. 6. In other words, we have obtained Thomas equation by a direct computation, from the "Fermi–Walker" evolution law (12.123), without appealing explicitly to the product of two Lorentz boosts.

Historical note: *Equation (12.136) has been derived in 1926 at the lowest order in V/c by Llewellyn H. Thomas (cf. p. 215) (1926), in the case where $\vec{C}$ is the torque exerted on the spin of an electron moving in a uniform electric field. It should be noticed that Thomas did make the distinction between the vectors $\vec{s}_*$ and $\vec{s}$, without however formalizing it as in (12.118), when he wrote "the precession which an observer at rest with respect to the nucleus*[8] *would observe, and which should be summed to give the secular precession, is that precession which would turn the direction of the spin axis at time t in (2) into its direction at time $t + \mathrm{d}t$ in (3)* if both directions were regarded as direction in (1)[9]*" (Thomas 1926). In a more detailed article published in 1927 (Thomas 1927), Thomas obtained the exact form (12.136), still in the case where $\vec{C}$ is the torque exerted on the spin of an electron moving in a uniform electric field. The equation written by Thomas (Eq. (4.121) in Thomas (1927)) is actually (12.136) multiplied by Γ, i.e. the equation for $\mathrm{d}\vec{s}_*/\mathrm{d}t$ and not for $\mathrm{d}\vec{s}_*/\mathrm{d}t_*$.*

[8] The inertial observer $\mathscr{O}_*$ in our language.

[9] Underlined by us.

Chapter 13
Rotating Observers

13.1 Introduction

After the accelerated observers, let us now examine the rotating ones, i.e. the observers whose 4-rotation is nonzero. We start by the physical interpretation of the 4-rotation vector (Sect. 13.2) and the treatment of the rotating disk (Sect. 13.3). We shall then discuss the issue of synchronizing clocks in a rotating frame, a concrete application being the definition of a timescale at the surface of the Earth (International Atomic Time) (Sect. 13.4). Next, we shall discuss the famous Ehrenfest paradox regarding the rotating disk, not so much for its historical importance but rather because it provides a very instructive example (Sect. 13.5). Finally, we shall investigate the principal relativistic effect induced by rotation: the Sagnac effect, which is used today in high-precision gyrometers for air and space navigation (Sect. 13.6).

13.2 Rotation Velocity

We have defined the 4-rotation of an observer in Sect. 3.5 as the vector $\vec{\boldsymbol{\omega}}$ that is involved in the law (3.52) ruling the evolution of the observer's local frame $(\vec{\boldsymbol{e}}_\alpha)$. Here we show that this quantity is directly measurable, by comparison with a nonrotating observer. Let us start by discussing the latter.

13.2.1 Physical Realization of a Nonrotating Observer

Let us consider an observer $\mathscr{O}$, of worldline $\mathscr{L}_0$, proper time t, 4-velocity $\vec{\boldsymbol{u}}$ and 4-acceleration $\vec{\boldsymbol{a}}$. Let $(\vec{\boldsymbol{e}}_\alpha(t))$ be the local frame of $\mathscr{O}$; we have thus $\vec{\boldsymbol{e}}_0 = \vec{\boldsymbol{u}}$.

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By definition, the derivative of each of the vectors $\vec{e}_i$ ($i \in \{1, 2, 3\}$) with respect to observer $\mathscr{O}$ is zero [Eq. (3.64)]. Combining with (3.70), we can then write

$$\boldsymbol{D}^{\rm FW}_{\boldsymbol{u}} \vec{e}_i = \vec{\omega} \times_{\boldsymbol{u}} \vec{e}_i, \tag{13.1}$$

where $\boldsymbol{D}^{\rm FW}_{\boldsymbol{u}}$ stands for the Fermi–Walker derivative along $\mathscr{L}_0$ and $\vec{\omega}$ for $\mathscr{O}$'s 4-rotation. We have thus

$$\vec{\omega} = 0 \iff \forall i \in \{1, 2, 3\},\ \boldsymbol{D}^{\rm FW}_{\boldsymbol{u}} \vec{e}_i = 0. \tag{13.2}$$

Hence, an observer is nonrotating iff the spatial vectors of his local frame are Fermi–Walker transported along his worldline.

Now, we have seen in Sect. 10.7.3 that the spin vector of a free gyroscope is Fermi–Walker transported along a worldline [Eq. (10.84)]. This provides us with the following physical criterion:

To set up a nonrotating observer, one must equip it with three free gyroscopes in three orthogonal directions (the orthogonality being checked by the procedure described in Sect. 3.4.1) and orient each of the three basis vectors $(\vec{e}_i)$ in the direction of the spin vector of one of the gyroscopes (cf. Fig. 13.1).

13.2.2 Measurement of the Rotation Velocity

The preceding construction also leads to a way of measuring the 4-rotation $\vec{\omega}$ of a given observer. Indeed, any observer can, by means of gyroscopes, define a nonrotating spatial frame $(\vec{e}\,'_i)$, in addition to his own spatial frame $(\vec{e}_i)$. Formally, this amounts to considering two observers, $\mathscr{O}$ and $\mathscr{O}'$, whose local frames are, respectively, $(\vec{e}_\alpha) = (\vec{u}, \vec{e}_1, \vec{e}_2, \vec{e}_3)$ and $(\vec{e}\,'_\alpha) = (\vec{u}, \vec{e}\,'_1, \vec{e}\,'_2, \vec{e}\,'_3)$. Observers $\mathscr{O}$ and $\mathscr{O}'$ share the same worldline $\mathscr{L}_0$. Consequently, they have the same 4-velocity $(\vec{u})$ and the same 4-acceleration. They differ only by their 4-rotations: $\vec{\omega}$ for $\mathscr{O}$ and zero for $\mathscr{O}'$. The derivative of $\vec{e}_i$ with respect to observer $\mathscr{O}'$ is expressed according to (3.70):

$$\boldsymbol{D}_{\mathscr{O}'} \vec{e}_i = \boldsymbol{D}^{\rm FW}_{\boldsymbol{u}} \vec{e}_i - \underbrace{\vec{\omega}'}_{0} \times_{\boldsymbol{u}} \vec{e}_i = \boldsymbol{D}^{\rm FW}_{\boldsymbol{u}} \vec{e}_i.$$

Substituting (13.1) for the Fermi–Walker derivative of $\vec{e}_i$, we get

$$\boxed{\boldsymbol{D}_{\mathscr{O}'} \vec{e}_i = \vec{\omega} \times_{\boldsymbol{u}} \vec{e}_i}. \tag{13.3}$$

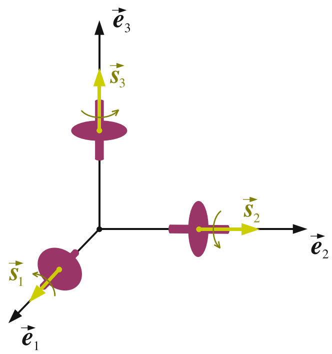

Fig. 13.1 Spatial frame $(\vec{e}_i)$ of a nonrotating observer, elaborated by aligning the vectors $\vec{e}_i$ along the spin vectors of three free gyroscopes

Hence,

> The 4-rotation vector of $\mathscr{O}$ appears as the rotation vector of his spatial frame with respect to the nonrotating observer $\mathscr{O}'$ who follows the same worldline as him.

Conversely, if $\mathscr{O}$ carries a free gyroscope along his worldline, the gyroscope's spin vector $\vec{s}$ obeys (3.70) with $\boldsymbol{D}^{\mathrm{FW}}_{\boldsymbol{u}}\vec{s} = 0$ [Eq. (10.84)]. We have thus

$$\boxed{\boldsymbol{D}_{\mathscr{O}}\vec{s} = -\vec{\omega} \times_u \vec{s}}. \tag{13.4}$$

The 4-rotation vector $\vec{\omega}$ is thus the opposite of the rotation velocity of a free gyroscope as measured by observer $\mathscr{O}$. However, a single free gyroscope is not sufficient to measure $\vec{\omega}$ because (13.4) does not constrain the part of $\vec{\omega}$ that is parallel to $\vec{s}$.

13.3 Rotating Disk

We examine here the simplest case of a rotating observer: that of a constant 4-rotation. This case is related to the problem of the rotating disk, which generated a lot of discussions during the development of relativity (see, e.g. Grøn (2004)).

13.3.1 Uniformly Rotating Observer

An observer $\mathscr{O}$ is said to be ***uniformly rotating*** iff:

1. Its 4-acceleration vanishes: $\vec{a} = 0$.
2. Its 4-rotation is constant: $\vec{\omega} = \text{const}$.

The first property implies that $\mathscr{O}$'s 4-velocity, $\vec{u}$, is constant and that $\mathscr{O}$'s worldline, $\mathscr{L}_0$, is a straight line of $\mathscr{E}$. In particular, the vector hyperplane E_{u} underlying $\mathscr{O}$'s local rest frame $\mathscr{E}_{u}(t)$ is independent of t, the proper time of $\mathscr{O}$.

$\mathscr{O}$'s local frame $(\vec{e}_\alpha(t))$ obeys the evolution law (3.52) without any Fermi–Walker part since $\vec{a} = 0$. For the index $\alpha = 0$, this equation reduces to $0 = 0$, because $\vec{e}_0 = \vec{u}$ is constant and $\vec{\omega} \times_{u} \vec{u} = 0$. For the other indices, it reduces to

$$\frac{\mathrm{d}\vec{e}_i}{\mathrm{d}t} = \vec{\omega} \times_{u} \vec{e}_i. \tag{13.5}$$

Let us introduce an inertial observer $\mathscr{O}_*$ whose 4-velocity and worldline coincide with those of $\mathscr{O}$ (cf. Fig. 13.2). Its proper time t_* is then the same as that of $\mathscr{O}$ as well as his rest spaces. $\mathscr{O}_*$ can always be chosen so that the last vector of his frame $(\vec{e}^{\,*}_\alpha)$ is collinear to the constant vector $\vec{\omega}$ and oriented in the same direction:

$$\vec{\omega} = \omega\, \vec{e}^{\,*}_3 \qquad \text{with} \quad \omega := \|\vec{\omega}\|_{g} \geq 0. \tag{13.6}$$

Let $(\vec{\varepsilon}_i(t))$ be the orthonormal basis of E_{u} defined by

$$\begin{aligned}
\vec{\varepsilon}_1(t) &= \cos\omega t\; \vec{e}^{\,*}_1 + \sin\omega t\; \vec{e}^{\,*}_2 \\
\vec{\varepsilon}_2(t) &= -\sin\omega t\; \vec{e}^{\,*}_1 + \cos\omega t\; \vec{e}^{\,*}_2 \\
\vec{\varepsilon}_3(t) &= \vec{e}^{\,*}_3.
\end{aligned}$$

It is easy to see that the basis $(\vec{\varepsilon}_i)$ obeys the relation

$$\frac{\mathrm{d}\vec{\varepsilon}_i}{\mathrm{d}t} = \vec{\omega} \times_{u} \vec{\varepsilon}_i. \tag{13.7}$$

Given that $(\vec{e}_i(t))$ and $(\vec{\varepsilon}_i(t))$ are two right-handed orthonormal bases of $(E_{u}, \boldsymbol{g})$, there exists necessarily a spatial rotation $\boldsymbol{R} = \boldsymbol{R}(t)$ such that

$$\vec{e}_i = \boldsymbol{R}(\vec{\varepsilon}_i) = R^j_{\ i}\, \vec{\varepsilon}_j.$$

We have then, by means of (13.7),

$$\frac{\mathrm{d}\vec{e}_i}{\mathrm{d}t} = \frac{\mathrm{d}R^j_{\ i}}{\mathrm{d}t}\, \vec{\varepsilon}_j + R^j_{\ i} \underbrace{\frac{\mathrm{d}\vec{\varepsilon}_j}{\mathrm{d}t}}_{\vec{\omega} \times_{u} \vec{\varepsilon}_j} = \frac{\mathrm{d}R^j_{\ i}}{\mathrm{d}t}\, \vec{\varepsilon}_j + \vec{\omega} \times_{u} \vec{e}_i,$$

which, via (13.5), implies $\mathrm{d}R^j_{\ i}/\mathrm{d}t = 0$. In other words, the bases $(\vec{e}_i(t))$ and $(\vec{\varepsilon}_i(t))$ are linked by a *constant* rotation. Without any loss of generality, we may then assume that $\vec{e}_i(t) = \vec{\varepsilon}_i(t)$, i.e. that the spatial frame of the uniformly rotating observer is given by

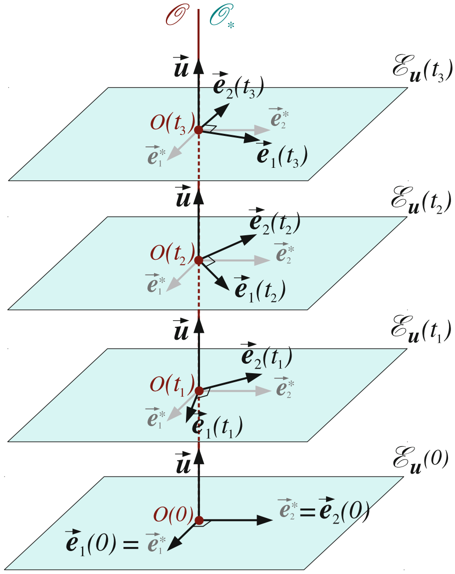

Fig. 13.2 Uniformly rotating observer $\mathscr{O}$ [frame $(\vec{e}_\alpha(t))$] and inertial observer $\mathscr{O}_*$ sharing the same worldline as $\mathscr{O}$ [frame $(\vec{e}^{\,*}_\alpha)$]. The dimension in the direction of the rotation vector $\vec{\omega}$ has been suppressed

$$\vec{e}_1(t) = \cos\omega t\, \vec{e}^{\,*}_1 + \sin\omega t\, \vec{e}^{\,*}_2 \tag{13.8a}$$

$$\vec{e}_2(t) = -\sin\omega t\, \vec{e}^{\,*}_1 + \cos\omega t\, \vec{e}^{\,*}_2 \tag{13.8b}$$

$$\vec{e}_3(t) = \vec{e}^{\,*}_3 = \omega^{-1}\,\vec{\omega}. \tag{13.8c}$$

This relation between the frames $(\vec{e}_\alpha(t))$ and $(\vec{e}^{\,*}_\alpha)$ is illustrated in Fig. 13.2. Let us denote by $(x^\alpha) = (ct, x, y, z)$ the coordinates of observer $\mathscr{O}$ and by $(x^\alpha_*) = (ct, x_*, y_*, z_*)$ those of observer $\mathscr{O}_*$. The point $O(t)$ being the position of $\mathscr{O}$ at the instant t on $\mathscr{L}_0$, any event M in $\mathscr{E}_{\boldsymbol{u}}(t)$ fulfils $\overrightarrow{O(t)M} = x^i\,\vec{e}_i(t) = x^i_*\,\vec{e}^{\,*}_i$. We deduce then from (13.8) the following relation between the two coordinate systems:

$$\boxed{\begin{cases} x = x_*\cos\omega t + y_*\sin\omega t \\ y = -x_*\sin\omega t + y_*\cos\omega t \\ z = z_*. \end{cases}} \iff \boxed{\begin{cases} x_* = x\cos\omega t - y\sin\omega t \\ y_* = x\sin\omega t + y\cos\omega t \\ z_* = z. \end{cases}} \tag{13.9}$$

13.3.2 Corotating Observers

Let us define a ***corotating observer*** with respect to the uniformly rotating observer $\mathscr{O}$ as an observer $\mathscr{O}'$ such that:

1. $\mathscr{O}'$ is *fixed* with respect to $\mathscr{O}$, in the sense defined in Sect. 3.4.3, i.e. its spatial coordinates (x, y, z) relative to $\mathscr{O}$ are constant.

2. Each vector $\vec{e}'_\alpha$ of the local frame of $\mathscr{O}'$ is *fixed* with respect to $\mathscr{O}$, still in the sense of Sect. 3.4.3: at any intersection point between the worldline of $\mathscr{O}'$ and the local rest space $\mathscr{E}_{\boldsymbol{u}}(t)$ of $\mathscr{O}$, $\vec{e}'_\alpha = e'^\beta{}_\alpha \, \vec{e}_\beta(t)$, where the $e'^\beta{}_\alpha$'s are independent of t.

Remark 13.1. Using the vocabulary of Sect. 12.4.1, we could have named $\mathscr{O}'$ a *comoving* observer (with respect to $\mathscr{O}$). However, the term *corotating* seems more appropriate here.

The first condition in the above definition regards the worldline of $\mathscr{O}'$. We shall only consider corotating observers having $z = 0$. It is natural to introduce cylindrical coordinates (r, φ) to write the coordinates (x, y, z) of $\mathscr{O}'$ with respect to $\mathscr{O}$ as

$$x = r\cos\varphi \qquad \text{and} \qquad y = r\sin\varphi. \tag{13.10}$$

As for x and y, the coordinates r and φ are constant along the worldline of $\mathscr{O}'$.

Remark 13.2. One shall not attribute any direct physical meaning to the coordinates (r, φ), but rather conceive (r, φ) as a mere label to identify each corotating observer. In Sect. 13.5, we shall discuss the physical measure of the disk radius by corotating observers and shall see that it is indeed equal to r. On the other side, we shall see that the element of disk circumference is *not* equal to $r\,\mathrm{d}\varphi$.

By combining (13.9) and (13.10) and using the trigonometric identities $\cos(\omega t + \varphi) = \cos\omega t\cos\varphi - \sin\omega t\sin\varphi$ and $\sin(\omega t + \varphi) = \cos\omega t\sin\varphi + \sin\omega t\cos\varphi$, we obtain the expression of the inertial coordinates of the corotating observer $\mathscr{O}'$:

$$\begin{cases} x_*(t) = r\cos(\omega t + \varphi) \\ y_*(t) = r\sin(\omega t + \varphi) \\ z_*(t) = 0. \end{cases} \tag{13.11}$$

r and φ being constant, we recognize, up to some azimuthal shift, Example 4.3 of Chap. 4 (uniform circular motion). More precisely, Chap. 4 dealt with the case $\varphi = 0$. The case $\varphi \neq 0$ is deduced from it by a mere rotation of constant angle φ in the plane (x_*, y_*). When drawn with respect to the inertial coordinates (x_*^α), the worldline $\mathscr{L}'$ of $\mathscr{O}'$ is then a helix (cf. Fig. 4.4). It is depicted in Fig. 13.3.

The velocity $\vec{V}$ of the corotating observer relative to the inertial observer $\mathscr{O}_*$ is obtained by taking the derivative of (13.11) with respect to[1] t [see also (4.23)]:

$$\vec{V} = r\omega\left[-\sin(\omega t + \varphi)\,\vec{e}^{\,*}_1 + \cos(\omega t + \varphi)\,\vec{e}^{\,*}_2\right].$$

Thanks to (13.8a)–(13.8b), this velocity can be reexpressed as

$$\boxed{\vec{V} = r\omega\,\vec{n}}, \qquad \text{with} \quad \vec{n} := -\sin\varphi\,\vec{e}_1 + \cos\varphi\,\vec{e}_2. \tag{13.12}$$

[1] Let us recall that t, the proper time of $\mathscr{O}$, is also the proper time of $\mathscr{O}_*$.

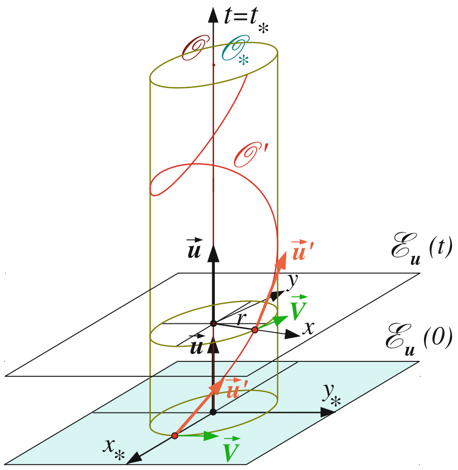

Fig. 13.3 Worldline of the corotating observer $\mathscr{O}'$

$\vec{n} = \vec{n}(t)$ is by construction a spacelike unit vector and the condition $\|V\|_g < c$ [Eq. (4.37)] implies an upper bound on the coordinate r of the corotating observer:

$$\boxed{r < \frac{c}{\omega}}. \tag{13.13}$$

Actually for $r > c/\omega$, we deduce from (13.11) that the worldline of $\mathscr{O}'$ would be spacelike, which is of course not admissible for an observer worldline. For $r = c/\omega$, (13.11) leads to a null curve, i.e. a curve with a null tangent vector at any point (cf. Remark 2.12 p. 39).

For a fixed $R \in]0, c/\omega[$, we shall call ***rotating disk of radius*** R the set of all corotating observers satisfying $r \in [0, R]$ and $z = 0$.

13.3.3 4-Acceleration and 4-Rotation of the Corotating Observer

Let us examine now the local frame $(\vec{e}\,'_\alpha)$ of $\mathscr{O}'$. We have of course $\vec{e}\,'_0 = \vec{u}'$ (4-velocity of $\mathscr{O}'$). Regarding the spatial vectors $(\vec{e}\,'_i)$, they must be fixed with respect to $\mathscr{O}$, according to the definition of a corotating observer. One can always introduce a constant spatial rotation to achieve the following configuration:

1. The vector $\vec{e}\,'_1$ lies in the plane Span$(\vec{u}, \vec{u}')$ (cf. Fig. 13.4).
2. The vector $\vec{e}\,'_3$ equals the vector $\vec{e}_3$ of $\mathscr{O}$'s local frame, which is itself equal to the vector $\vec{e}^{\,*}_3$ of the frame of the inertial observer $\mathscr{O}_*$:

$$\vec{e}\,'_3 = \vec{e}_3 = \vec{e}^{\,*}_3. \tag{13.14}$$

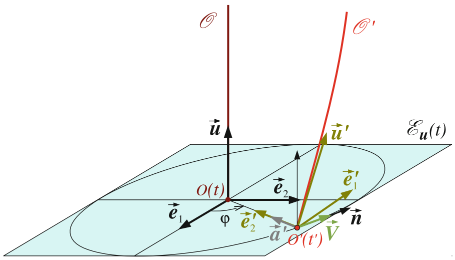

Fig. 13.4 Frame $(\vec{e}'_\alpha) = (\vec{u}', \vec{e}'_1, \vec{e}'_2, \vec{e}'_3)$ of the corotating observer $\mathscr{O}'$. The dimension along $\vec{e}'_3 = \vec{e}_3$ has been suppressed. $\vec{a}'$ is the 4-acceleration of $\mathscr{O}'$ and $\vec{V}$ his velocity relative to the inertial observer $\mathscr{O}_*$, whose worldline coincides with that of $\mathscr{O}$

The vector $\vec{e}'_2$ is then necessarily such that (cf. Fig. 13.4)

$$\overrightarrow{O'(t')O(t)} = r\,\vec{e}'_2, \tag{13.15}$$

where $O(t)$ is the position of $\mathscr{O}$ at the instant t of his proper time and $O'(t')$ is the position of $\mathscr{O}'$ at the instant t' of his proper time where he encounters $\mathscr{E}_{\boldsymbol{u}}(t)$. From the point of view of $\mathscr{O}$, $O'(t')$ is the event of worldline $\mathscr{L}'$ that is simultaneous to $O(t)$.

Let us denote by $\boldsymbol{\Lambda}$ the Lorentz boost that relates the 4-velocities of observers $\mathscr{O}$ (or $\mathscr{O}_*$) and $\mathscr{O}'$:

$$\vec{u}' = \boldsymbol{\Lambda}(\vec{u}) = \Gamma\left(\vec{u} + \frac{1}{c}\vec{V}\right) = \Gamma\left(\vec{u} + \frac{r\omega}{c}\,\vec{n}\right), \tag{13.16}$$

where use has been made of expression (13.12) for $\vec{V}$. In the above formula, Γ is the Lorentz factor of $\mathscr{O}'$ relative to $\mathscr{O}$ (or $\mathscr{O}_*$) [cf. Eq. (4.7)]:

$$\boxed{\Gamma = \left(1 - \frac{r^2\omega^2}{c^2}\right)^{-1/2}}. \tag{13.17}$$

Since $\vec{e}'_1 \in \mathrm{Span}(\vec{u}, \vec{u}')$, we have necessarily (cf. Fig. 13.4)

$$\vec{e}'_1 = \boldsymbol{\Lambda}(\vec{n}) = \Gamma\left(\frac{r\omega}{c}\,\vec{u} + \vec{n}\right) = \Gamma\left(\frac{r\omega}{c}\,\vec{u} - \sin\varphi\,\vec{e}_1 + \cos\varphi\,\vec{e}_2\right). \tag{13.18}$$

Besides, along with $\overrightarrow{O(t)O'(t')} = x\,\vec{e}_1 + y\,\vec{e}_2 = r\cos\varphi\,\vec{e}_1 + r\sin\varphi\,\vec{e}_2$, Eq. (13.15) leads to

$$\vec{e}'_2 = -\cos\varphi\,\vec{e}_1 - \sin\varphi\,\vec{e}_2. \tag{13.19}$$

Equations (13.16), (13.18), (13.19) and (13.14) fully specify the local frame of $\mathscr{O}'$ in terms of that of $\mathscr{O}$.

The 4-acceleration $\vec{a}'$ of the corotating observer $\mathscr{O}'$ is given by (4.67)–(4.46):

$$\boxed{\vec{a}' = \frac{\Gamma^2}{c^2}\, r\omega^2\, \vec{e}'_2}. \tag{13.20}$$

Let us determine now the 4-rotation of $\mathscr{O}'$. To this aim, we evaluate the variation of $\vec{e}'_1$ along $\mathscr{L}'$, via the relation $\mathrm{d}t = \Gamma\, \mathrm{d}t'$ between the proper times of $\mathscr{O}$ and $\mathscr{O}'$ [Eq. (4.1)]:

$$\frac{\mathrm{d}\vec{e}'_1}{\mathrm{d}t'} = \frac{\mathrm{d}\vec{e}'_1}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}t'} = \Gamma\frac{\mathrm{d}\vec{e}'_1}{\mathrm{d}t},$$

Thanks to (13.18) and given the constant character of Γ, r, ω, $\vec{u}$ and φ, there comes

$$\begin{aligned}\frac{\mathrm{d}\vec{e}'_1}{\mathrm{d}t'} &= \Gamma^2\left(-\sin\varphi\,\frac{\mathrm{d}\vec{e}_1}{\mathrm{d}t} + \cos\varphi\,\frac{\mathrm{d}\vec{e}_2}{\mathrm{d}t}\right)\\ &= \Gamma^2\left(-\sin\varphi\,\vec{\omega}\times_u\vec{e}_1 + \cos\varphi\,\vec{\omega}\times_u\vec{e}_2\right)\\ &= \Gamma^2\,\vec{\omega}\times_u\vec{n} = \Gamma^2\vec{\epsilon}(\vec{u},\vec{\omega},\vec{n},.),\end{aligned} \tag{13.21}$$

where we have used (13.5) and (13.12) to write, respectively, the second and third lines. Finally, the last equality, which involves the Levi–Civita tensor ϵ, results from the very definition of the cross product [Eq. (3.46)]. Now, from (13.18), $\vec{n} = \Gamma^{-1}\vec{e}'_1 - (r\omega/c)\vec{u}$. The alternate character of ϵ gives then $\epsilon(\vec{u},\vec{\omega},\vec{n},.) = \Gamma^{-1}\epsilon(\vec{u},\vec{\omega},\vec{e}'_1,.)$. Moreover,

$$\vec{u} = \Lambda^{-1}(\vec{u}') = \Gamma\left(\vec{u}' - \frac{r\omega}{c}\vec{e}'_1\right),$$

so that $\epsilon(\vec{u},\vec{\omega},\vec{e}'_1,.) = \Gamma\epsilon(\vec{u}',\vec{\omega},\vec{e}'_1,.)$. We conclude thus that $\epsilon(\vec{u},\vec{\omega},\vec{n},.) = \epsilon(\vec{u}',\vec{\omega},\vec{e}'_1,.)$. Inserting this result into (13.21), we get

$$\frac{\mathrm{d}\vec{e}'_1}{\mathrm{d}t'} = \Gamma^2\vec{\epsilon}(\vec{u}',\vec{\omega},\vec{e}'_1,.) = \Gamma^2\,\vec{\omega}\times_{u'}\vec{e}'_1. \tag{13.22}$$

Computing similarly $\mathrm{d}\vec{e}'_2/\mathrm{d}t'$ (cf. Remark 13.3 below) and comparing with the general law (3.52) ruling the evolution of a local frame vector (with $\vec{a}'\cdot\vec{e}'_1 = 0$ from (13.20) and $\vec{u}'\cdot\vec{e}'_1 = 0$), we read the value of the 4-rotation of observer $\mathscr{O}'$:

$$\boxed{\vec{\omega}' = \Gamma^2\vec{\omega}}. \tag{13.23}$$

Thus, while it is qualified as "corotating", the observer $\mathscr{O}'$ has not exactly the same 4-rotation as the central observer, except at the nonrelativistic limit, where the Lorentz factor Γ tends to 1. Note that, as for $\vec{\omega}$, the 4-rotation $\vec{\omega}'$ is constant along the worldline of $\mathscr{O}'$.

Remark 13.3. As an exercise, we may check, by taking the derivative of (13.19) with respect to t', that

$$\frac{\mathrm{d}\vec{e}'_2}{\mathrm{d}t'} = \frac{\Gamma^2}{c} r\omega^2 \,\vec{u}' + \vec{\omega}' \times_{u'} \vec{e}'_2,$$

in full agreement with the general law (3.52), taking into account that $\vec{a}' \cdot \vec{e}'_2 = \Gamma^2 r\omega^2/c^2$ [Eq. (13.20)].

13.3.4 Simultaneity for a Corotating Observer

The proper time t' of the corotating observer $\mathscr{O}'$ is of course different from that of $\mathscr{O}$ (denoted t) for $r \neq 0$. Both times are related by the Lorentz factor determined in Sect. 4.2.1: between the hyperplanes $\mathscr{E}_{\boldsymbol{u}}(t)$ and $\mathscr{E}_{\boldsymbol{u}}(t + \mathrm{d}t)$, the elapsed proper time along $\mathscr{L}'$ is $\mathrm{d}t' = \Gamma^{-1}\,\mathrm{d}t$, with $\Gamma = [1 - (r\omega/c)^2]^{-1/2}$ [Eq. (13.17)]. Since Γ is constant, we can write (choosing the same origin of the proper times for the two observers)

$$\boxed{t' = \Gamma^{-1}\, t = t\sqrt{1 - (r\omega/c)^2}}. \tag{13.24}$$

This relation is however valid only along the worldline of $\mathscr{O}'$. As soon as one moves away from it, t' is determined by the Einstein–Poincaré simultaneity criterion introduced in Sect. 3.2.2. Let us then determine the simultaneity hypersurfaces with respect to $\mathscr{O}'$, denoted $by\Sigma_{\boldsymbol{u}'}(t')$ according to the convention defined in Sect. 3.2.3.

Without any loss of generality, we can set $\varphi = 0$ and $t' = t = 0$. Let $M \in \mathscr{E}$ be an event simultaneous to $O'(0)$ from the point of view of $\mathscr{O}'$: this means that $\mathscr{O}'$ could emit an electromagnetic signal at the proper time $t'_1 = -T'$, with $T' > 0$, (event A_1) and that this signal has been reflected by M to come back to $\mathscr{O}'$ at the proper time $t'_2 = T'$ (event A_2) (cf. Fig. 13.5). According to (3.1), the date attributed to M by $\mathscr{O}'$ is $t' = (t'_1 + t'_2)/2 = 0$, so that M is simultaneous to $O'(0)$ from the point of view of $\mathscr{O}'$. In view of (13.24), the time coordinates of the events A_1 and A_2 with respect to $\mathscr{O}$ are, respectively, $t_1 = -T$ and $t_2 = T$, with $T := \Gamma T'$. The inertial coordinates of A_1 and A_2 are then given by (13.11) with $\varphi = 0$:

$$A_1 \begin{cases} t_* = -T = -\Gamma T' \\ x_* = r\cos(\omega T) \\ y_* = -r\sin(\omega T) \\ z_* = 0, \end{cases} \qquad A_2 \begin{cases} t_* = T = \Gamma T' \\ x_* = r\cos(\omega T) \\ y_* = r\sin(\omega T) \\ z_* = 0. \end{cases} \tag{13.25}$$

For a fixed T', let us consider the set $\Sigma_{T'}$ formed by the events M that are simultaneous to $O'(0)$ for $\mathscr{O}'$ and have the same half-time T' between the departure

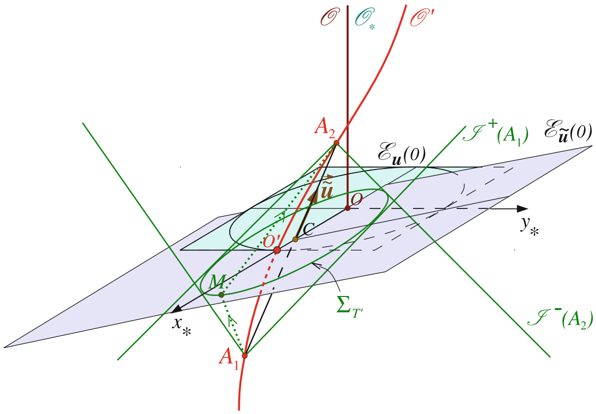

Fig. 13.5 Set $\Sigma_{T'}$ formed by the events M simultaneous to $O' = O'(0)$ for $\mathscr{O}'$ and of elapsed time $2T'$ between the departure A_1 ($t' = -T'$) and the return A_2 ($t' = +T'$) of the electromagnetic signal used to establish the simultaneity. $\Sigma_{T'}$ is the intersection of the future light cone of A_1, $\mathscr{I}^+(A_1)$, with the past light cone of A_2, $\mathscr{I}^-(A_2)$; it is a sphere in the hyperplane $\mathscr{E}_{\tilde{u}}(0)$ orthogonal to $\overrightarrow{A_1A_2}$

A_1 and the return A_2 of the photon. $\Sigma_{T'}$ is the intersection of the future light cone of A_1 with the past light cone of A_2 (cf. Fig. 13.5):

$$\Sigma_{T'} = \mathscr{I}^+(A_1) \cap \mathscr{I}^-(A_2). \tag{13.26}$$

To study $\Sigma_{T'}$, it is appropriate to introduce the inertial observer $\tilde{\mathscr{O}}$ whose worldline is the line A_1A_2. Let us first check that this is always possible, i.e. that the vector $\overrightarrow{A_1A_2}$ is always timelike. In view of (13.25), we have

$$\overrightarrow{A_1A_2} = 2cT\, \vec{e}^{\,*}_0 + 2r\sin(\omega T)\, \vec{e}^{\,*}_2,$$

so that

$$\overrightarrow{A_1A_2} \cdot \overrightarrow{A_1A_2} = -4c^2T^2 + 4r^2\sin^2(\omega T) = -4c^2T^2\left(1 - \frac{\tilde{V}^2}{c^2}\right), \tag{13.27}$$

with

$$\tilde{V} := \frac{r}{T}\sin(\omega T). \tag{13.28}$$

$\overrightarrow{A_1A_2}$ is timelike iff $|\tilde{V}|/c < 1$. Now, since $r < c/\omega$ [Eq. (13.13)],

$$\frac{|\tilde{V}|}{c} = \frac{r}{cT}|\sin(\omega T)| < \left|\frac{\sin(\omega T)}{\omega T}\right| < 1 \quad \text{if} \quad T \neq 0.$$

Hence, $\overrightarrow{A_1A_2}$ is indeed timelike, and we may introduce the 4-velocity

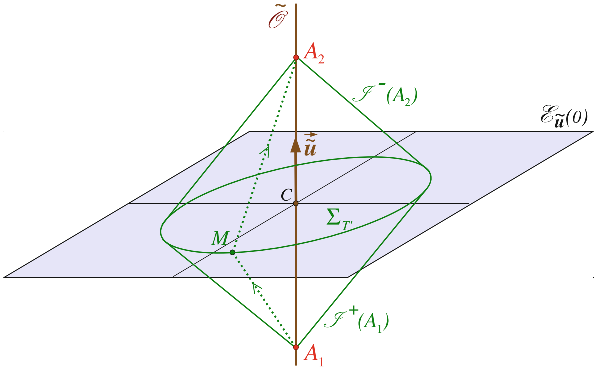

Fig. 13.6 Intersection $\Sigma_{T'}$ of the light cones $\mathscr{I}^+(A_1)$ and $\mathscr{I}^-(A_2)$, as in Fig. 13.5, but with a representation based on the coordinates of the inertial observer $\tilde{\mathscr{O}}$ whose worldline goes through the events A_1 and A_2

$$\vec{\tilde{u}} := \left\| \overrightarrow{A_1 A_2} \right\|_{g}^{-1} \overrightarrow{A_1 A_2} = \tilde{\Gamma} \left(\vec{e}^{\,*}_0 + \frac{\tilde{V}}{c} \vec{e}^{\,*}_2 \right), \quad \tilde{\Gamma} := \left(1 - \frac{\tilde{V}^2}{c^2} \right)^{-1/2}. \tag{13.29}$$

$\vec{\tilde{u}}$ is the 4-velocity of the inertial observer $\tilde{\mathscr{O}}$, whose worldline is the line $A_1 A_2$, and $\tilde{V} \vec{e}^{\,*}_2$ is nothing but $\tilde{\mathscr{O}}$'s velocity relative to the inertial observer $\mathscr{O}_*$.

Remark 13.4. We have

$$\lim_{T \to 0} \tilde{V} = \lim_{T \to 0} r\omega \frac{\sin(\omega T)}{\omega T} = r\omega.$$

Thus, if $T \to 0$, $\tilde{V}$ tends towards the velocity of the corotating observer $\mathscr{O}'$ relative to $\mathscr{O}_*$, as it should since $A_1 A_2$ tends then to the tangent to the worldline of $\mathscr{O}'$.

In a spacetime diagram based on $\tilde{\mathscr{O}}$, the light cones $\mathscr{I}^+(A_1)$ and $\mathscr{I}^-(A_2)$ are aligned (cf. Fig. 13.6), and it is clear that $\Sigma_{T'}$ is a sphere in the rest space of $\tilde{\mathscr{O}}$, $\mathscr{E}_{\tilde{u}}(\tilde{t}_0)$ (for a certain value $\tilde{t}_0$ of $\tilde{\mathscr{O}}$'s proper time $\tilde{t}$). By symmetry, $\mathscr{E}_{\tilde{u}}(\tilde{t}_0)$ contains both $O(0)$ and $O'(0)$. We can then choose the origin of $\tilde{t}$ to ensure $\tilde{t}_0 = 0$. The equation of the hyperplane $\mathscr{E}_{\tilde{u}}(0)$, in terms of the inertial coordinates (x^α_*), is given by the condition $\vec{\tilde{u}} \cdot \overrightarrow{O(0)M} = 0$. Using (13.29), we get $-\tilde{\Gamma} ct_* + \tilde{\Gamma}(\tilde{V}/c) y_* = 0$, i.e. $ct_* = (\tilde{V}/c) y_*$, or, given (13.28),

$$\mathscr{E}_{\tilde{u}}(0) \ : \qquad ct_* = \frac{r \sin(\omega T)}{cT} \, y_*. \tag{13.30}$$

The centre C of the sphere $\Sigma_{T'}$ is the middle of the segment $A_1 A_2$. Its coordinates are directly deduced from (13.25):

$$C \ : \quad (ct_* = 0, \ x_* = r \cos(\omega T), \ y_* = 0, \ z_* = 0). \tag{13.31}$$

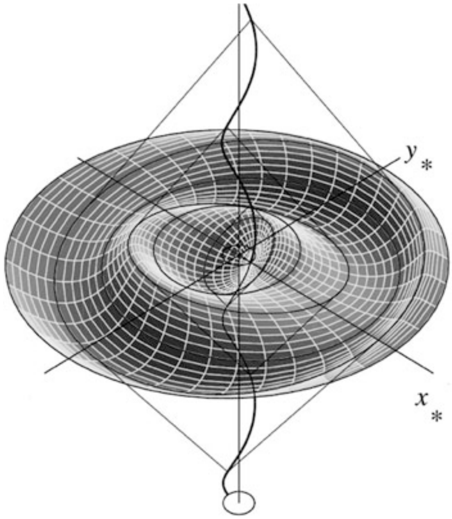

Fig. 13.7 Simultaneity hypersurface $\Sigma_{u'}(0)$ of the corotating observer $\mathscr{O}'$ at time $t' = 0$. Each ellipse represents the sphere $\Sigma_{T'}$ for a different time T' (figure adapted from Pauri and Vallisneri (2000))

The radius of $\Sigma_{T'}$ is equal to half the norm of $\overrightarrow{A_1 A_2}$ (cf. Fig. 13.6); thanks to (13.27),

$$R = \sqrt{c^2 T^2 - r^2 \sin^2(\omega T)}. \tag{13.32}$$

The simultaneity hypersurface of the event $O'(0)$ with respect to observer $\mathscr{O}'$ is obtained by varying T':

$$\Sigma_{u'}(0) = \bigcup_{T' \in \mathbb{R}^+} \Sigma_{T'}. \tag{13.33}$$

For each value of T', $\Sigma_{T'}$ belongs to a different hyperplane [cf. Eq. (13.30) with $T = \Gamma T'$]. Each hyperplane is aligned with $\mathscr{E}_{\boldsymbol{u}}(0)$ in the directions x_* and z_*; but in the direction y_*, it is inclined with respect to $\mathscr{E}_{\boldsymbol{u}}(0)$ with an oscillating slope (which tends to zero when $T' \to +\infty$), given by (13.30). Accordingly, if $\Sigma_{u'}(0)$ is drawn with respect to the inertial coordinates (x_*^α), it takes an "undulate" aspect (cf. Fig. 13.7). In particular, $\Sigma_{u'}(0)$ does not coincide with the simultaneity hypersurface $\mathscr{E}_{\boldsymbol{u}}(0)$ of observer $\mathscr{O}$.

13.4 Clock Desynchronization

13.4.1 Introduction

The relation (13.24) between the proper times t and t' shows that, for $r \neq 0$, it is not possible to synchronize a clock carried by a corotating observer with that of the central uniformly rotating observer. We recover the same situation as that encountered in Sect. 12.4.1 for accelerated observers: the fact that $\mathscr{O}'$ is fixed with respect to $\mathscr{O}$ does not suffice to guarantee the synchronization of clocks.

Remark 13.5. An important difference with the case of the uniformly accelerated motion treated in Chap. 12 is that, for the latter, $\mathscr{O}$ and $\mathscr{O}'$ share the same simultaneity hypersurfaces (which incidentally coincide with their local rest spaces) [cf. Eqs. (12.44) and (12.49)]. Here, we have just seen (Sect. 13.3.4) that the simultaneity hypersurfaces of $\mathscr{O}'$ (which are "undulate"; cf. Fig. 13.7) are different from those of $\mathscr{O}$ (which are hyperplanes).

Let us investigate the clock synchronization within a continuous 1-parameter family of corotating observers. The parameter λ will span [0, 1] and the observers in the family will be denoted by $\mathscr{O}'_{(\lambda)}$. For instance, if all observers have the same coordinate r, one can choose $\lambda = \varphi/(2\pi)$. In this case, all observers $\mathscr{O}'_{(\lambda)}$ have the same Lorentz factor Γ with respect to the central observer $\mathscr{O}$ [Γ is given by (13.17)]. We could then hope to synchronize their clocks. We are going to see that this is possible locally, but not globally.

13.4.2 Local Synchronization

Let us start by examining the local synchronization. To this aim, let us consider two infinitely close observers in the family: $\mathscr{O}'_{(\lambda)}$ and $\mathscr{O}'_{(\lambda+\mathrm{d}\lambda)}$. Let $A_{(\lambda)}$ be an event on the worldline of $\mathscr{O}'_{(\lambda)}$ and let us denote by $A_{(\lambda+\mathrm{d}\lambda)}$ the event on the worldline of $\mathscr{O}'_{(\lambda+\mathrm{d}\lambda)}$ that is simultaneous to $A_{(\lambda)}$ from the point of view of $\mathscr{O}'_{(\lambda)}$ (cf. Fig. 13.8). Let $x^i_{(\lambda)} = (x_{(\lambda)}, y_{(\lambda)}, z_{(\lambda)})$ be the (fixed) coordinates of $\mathscr{O}'_{(\lambda)}$ with respect to the central observer $\mathscr{O}$, and t be the date of event $A_{(\lambda)}$ with respect to $\mathscr{O}$ and $t + \mathrm{d}t$ that of $A_{(\lambda+\mathrm{d}\lambda)}$. We have then, from the very definition of the coordinates $x^i_{(\lambda)}$,

$$\overrightarrow{O(t)A_{(\lambda)}} = x^i_{(\lambda)}\, \vec{\boldsymbol{e}}_i(t) \quad \text{and} \quad \overrightarrow{O(t+\mathrm{d}t)A_{(\lambda+\mathrm{d}\lambda)}} = x^i_{(\lambda+\mathrm{d}\lambda)}\, \vec{\boldsymbol{e}}_i(t+\mathrm{d}t), \tag{13.34}$$

where $O(t)$ stands for the position of observer $\mathscr{O}$ at the instant t of his proper time. The condition of simultaneity of $A_{(\lambda+\mathrm{d}\lambda)}$ and $A_{(\lambda)}$ from the point of view of $\mathscr{O}'_{(\lambda)}$ is equivalent to the orthogonality of the vector $\overrightarrow{A_{(\lambda)}A_{(\lambda+\mathrm{d}\lambda)}}$ and the 4-velocity $\vec{\boldsymbol{u}}'_{(\lambda)}$ of $\mathscr{O}'_{(\lambda)}$ [cf. Eq. (3.8)]:

$$\vec{\boldsymbol{u}}'_{(\lambda)} \cdot \overrightarrow{A_{(\lambda)}A_{(\lambda+\mathrm{d}\lambda)}} = 0. \tag{13.35}$$

Now, from (13.34),

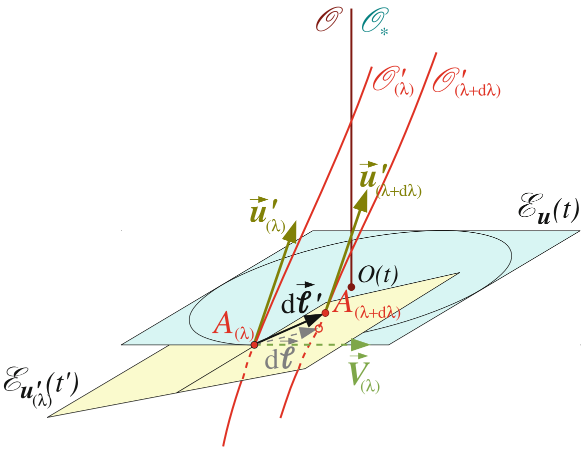

Fig. 13.8 Neighbouring observers $\mathscr{O}'_{(\lambda)}$ and $\mathscr{O}'_{(\lambda+\mathrm{d}\lambda)}$ in a family of corotating observers. The event $A_{(\lambda+\mathrm{d}\lambda)}$ of the worldline of $\mathscr{O}'_{(\lambda+\mathrm{d}\lambda)}$ is simultaneous to the event $A_{(\lambda)}$ from the point of view of observer $\mathscr{O}'_{(\lambda)}$

$$\begin{aligned}\overrightarrow{A_{(\lambda)}A_{(\lambda+\mathrm{d}\lambda)}} &= \overrightarrow{A_{(\lambda)}O(t)} + \overrightarrow{O(t)O(t+\mathrm{d}t)} + \overrightarrow{O(t+\mathrm{d}t)A_{(\lambda+\mathrm{d}\lambda)}} \\ &= -x^i_{(\lambda)}\,\vec{e}_i(t) + c\,\mathrm{d}t\,\vec{u} + x^i_{(\lambda+\mathrm{d}\lambda)}\,\vec{e}_i(t+\mathrm{d}t) \\ &= -x^i_{(\lambda)}\,\vec{e}_i(t) + c\,\mathrm{d}t\,\vec{u} + x^i_{(\lambda+\mathrm{d}\lambda)}\,[\vec{e}_i(t) + \mathrm{d}t\,\vec{\omega}\times_{\boldsymbol{u}}\vec{e}_i] \\ &= c\,\mathrm{d}t\,\vec{u} + \mathrm{d}x^i\,\vec{e}_i(t) + \mathrm{d}t\,x^i_{(\lambda)}\,\vec{\omega}\times_{\boldsymbol{u}}\vec{e}_i\,,\end{aligned} \tag{13.36}$$

where use has been made of (13.5) to write the third line and the notation $\mathrm{d}x^i := x^i_{(\lambda+\mathrm{d}\lambda)} - x^i_{(\lambda)}$ has been introduced for the difference of coordinates between $A_{(\lambda+\mathrm{d}\lambda)}$ and $A_{(\lambda)}$. Note that, in the second line, we have neglected a term in $\mathrm{d}t\,\mathrm{d}x^i$, as being of second order. Since $\vec{\omega} = \omega\,\vec{e}_3$ [Eq. (13.8c)], we have

$$x^i_{(\lambda)}\,\vec{\omega}\times_{\boldsymbol{u}}\vec{e}_i = \omega\left[x^1_{(\lambda)}\,\vec{e}_2 - x^2_{(\lambda)}\,\vec{e}_1\right] = \vec{V},$$

where $\vec{V}$ (we shall write $\vec{V}_{(\lambda)}$) is the velocity of $\mathscr{O}'_{(\lambda)}$ relative to the inertial observer $\mathscr{O}_*$, as given by (13.12). Let us introduce the separation vector between $\mathscr{O}'_{(\lambda)}$ and $\mathscr{O}'_{(\lambda+\mathrm{d}\lambda)}$ from the point of view of $\mathscr{O}$:

$$\boxed{\mathrm{d}\vec{\ell} := \mathrm{d}x^i\,\vec{e}_i(t) = \frac{\mathrm{d}x^i_{(\lambda)}}{\mathrm{d}\lambda}\,\vec{e}_i(t)\,\mathrm{d}\lambda}. \tag{13.37}$$

We can then write (13.36) as

$$\boxed{\mathrm{d}\vec{\ell}' := \overrightarrow{A_{(\lambda)}A_{(\lambda+\mathrm{d}\lambda)}} = c\,\mathrm{d}t\,\vec{u} + \mathrm{d}\vec{\ell} + \mathrm{d}t\,\vec{V}}. \tag{13.38}$$

The events $A_{(\lambda)}$ and $A_{(\lambda+\mathrm{d}\lambda)}$ being simultaneous for $\mathscr{O}'_{(\lambda)}$, we can say that $\mathrm{d}\vec{\ell}'$ is the ***proper separation vector*** between the two corotating observers. Besides, from (13.16), $\vec{u}'_{(\lambda)} = \Gamma(\vec{u} + c^{-1}\vec{V})$. The simultaneity condition (13.35) becomes then

$$\left(\vec{u} + \frac{1}{c}\vec{V}\right) \cdot \left(c\,\mathrm{d}t\,\vec{u} + \mathrm{d}\vec{\ell} + \mathrm{d}t\,\vec{V}\right) = 0.$$

Expanding and taking into account $\vec{u} \cdot \vec{u} = -1$, $\vec{u} \cdot \mathrm{d}\vec{\ell} = 0$, $\vec{u} \cdot \vec{V} = 0$ and $1 - \vec{V} \cdot \vec{V}/c^2 = \Gamma^{-2}$, we get

$$\boxed{\mathrm{d}t = \Gamma^2 \frac{\vec{V} \cdot \mathrm{d}\vec{\ell}}{c^2}}. \tag{13.39}$$

Hence, the event $A_{(\lambda+\mathrm{d}\lambda)}$ of date $t + \mathrm{d}t$ with respect to $\mathscr{O}$ is simultaneous to $A_{(\lambda)}$ from the point of view of $\mathscr{O}'_{(\lambda)}$ provided that $\mathrm{d}t$ obeys (13.39). A natural question is then: is $A_{(\lambda)}$ simultaneous to $A_{(\lambda+\mathrm{d}\lambda)}$ from the point of view of $\mathscr{O}'_{(\lambda+\mathrm{d}\lambda)}$? The answer is yes, because observers $\mathscr{O}'_{(\lambda)}$ and $\mathscr{O}'_{(\lambda+\mathrm{d}\lambda)}$ have close 4-velocities: they differ only by terms of the order $\mathrm{d}x^i$, so that the scalar product $\vec{u}'_{(\lambda+\mathrm{d}\lambda)} \cdot \overrightarrow{A_{(\lambda)}A_{(\lambda+\mathrm{d}\lambda)}}$ contains only second-order terms in $\mathrm{d}x^i$ if (13.35) is fulfilled. Hence, at first order in $\mathrm{d}x^i$,

$$\vec{u}'_{(\lambda+\mathrm{d}\lambda)} \cdot \overrightarrow{A_{(\lambda)}A_{(\lambda+\mathrm{d}\lambda)}} = 0.$$

This shows that $A_{(\lambda)}$ is simultaneous to $A_{(\lambda+\mathrm{d}\lambda)}$ from the point of view of $\mathscr{O}'_{(\lambda+\mathrm{d}\lambda)}$. The events $A_{(\lambda)}$ and $A_{(\lambda+\mathrm{d}\lambda)}$ are thus simultaneous, from the point of view of $\mathscr{O}'_{(\lambda)}$ as well as from that of $\mathscr{O}'_{(\lambda+\mathrm{d}\lambda)}$. Consequently, observers $\mathscr{O}'_{(\lambda)}$ and $\mathscr{O}'_{(\lambda+\mathrm{d}\lambda)}$ can synchronize their clocks and define their respective proper times $t'_{(\lambda)}$ and $t'_{(\lambda+\mathrm{d}\lambda)}$ such that $t'_{(\lambda)} = 0$ at $A_{(\lambda)}$ and $t'_{(\lambda+\mathrm{d}\lambda)} = 0$ at $A_{(\lambda+\mathrm{d}\lambda)}$.

13.4.3 Impossibility of a Global Synchronization

We can extend the above synchronization procedure from point to point and define a curve of simultaneity $t'_{(\lambda)} = 0$ for all the corotating observers of the family parametrized by λ. We shall denote by $\mathscr{S}$ this curve in the spacetime $\mathscr{E}$:

$$\mathscr{S} := \left\{A_{(\lambda)}, \quad \lambda \in [0, 1]\right\}. \tag{13.40}$$

The problem of *global* synchronization arises when we consider a *closed* family, i.e. a family of corotating observers such that the final observer is the same as the first one: $\mathscr{O}'_{(1)} = \mathscr{O}'_{(0)}$. We can represent such a family by a closed curve $\mathscr{C}$ in the reference space of observer $\mathscr{O}$, $R_{\mathscr{O}}$ (cf. Sect. 3.4.3): each point of $\mathscr{C}$ corresponds to an observer $\mathscr{O}_{(\lambda)}$ (cf. Fig. 13.9). In $R_{\mathscr{O}}$, the equation of $\mathscr{C}$ is the parametric equation

$$\mathscr{C}: \qquad x^i = x^i_{(\lambda)}, \quad \lambda \in [0, 1]. \tag{13.41}$$

The variation Δt of the coordinate t along the simultaneity curve $\mathscr{S}$ is evaluated by integrating (13.39) from $\lambda = 0$ to $\lambda = 1$:

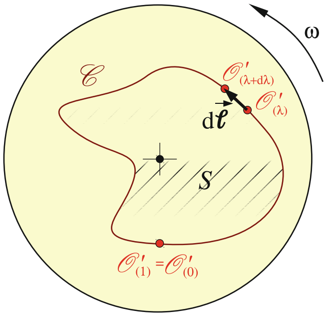

Fig. 13.9 Curve $\mathscr{C}$ representing all the corotating observers of a closed family in the reference space of the uniformly rotating observer $\mathscr{O}$

$$\Delta t = \frac{1}{c^2} \oint_{\mathscr{C}} \Gamma^2 \vec{V} \cdot d\vec{\ell}. \tag{13.42}$$

This represents the increase of coordinate t between the events $A_{(0)}$ and $A_{(1)}$ on the worldline of $\mathscr{O}'_{(0)}$. From (13.24), the corresponding change of $\mathscr{O}'_{(0)}$'s proper time is obtained by dividing by the Lorentz factor $\Gamma_{(0)}$ of $\mathscr{O}'_{(0)}$ with respect to $\mathscr{O}_*$:

$$\boxed{\Delta t'_{\text{desync}} = \frac{1}{c^2 \Gamma_{(0)}} \oint_{\mathscr{C}} \Gamma^2 \vec{V} \cdot d\vec{\ell}}, \tag{13.43}$$

where the index "desync" stands for *desynchronization*. The fact that $\Delta t'_{\text{desync}} \neq 0$ shows indeed that

It is not possible to synchronize clocks along a closed loop on the rotating disk. This can be achieved locally, as shown in Sect. 13.4.2, but not globally: the events $A_{(0)}$ and $A_{(1)}$, despite belonging to the simultaneity curve $\mathscr{S}$ of corotating observers, have dates with respect to $\mathscr{O}'_{(0)}$ which are separated by $\Delta t'_{\text{desync}}$ (cf. Fig. 13.10).

Remark 13.6. In formula (13.43), $\vec{V}$ and $d\vec{\ell}$ are two vectors of $\mathscr{O}$'s local rest space, $E_{\boldsymbol{u}}$ [cf. (13.12) and (13.37)]. We can thus identify them with two vectors of $\mathscr{O}$'s reference space, $R_{\mathscr{O}}$ [via the isomorphism (3.27)]. Moreover, $\mathscr{C}$ is a curve defined in $R_{\mathscr{O}}$, and, as seen easily from (13.37) and (13.41), $d\vec{\ell}$ is tangent to $\mathscr{C}$ (cf. Fig. 13.9). The integral in (13.43) is thus nothing but the circulation of the vector $\Gamma^2 \vec{V}$ along $\mathscr{C}$.

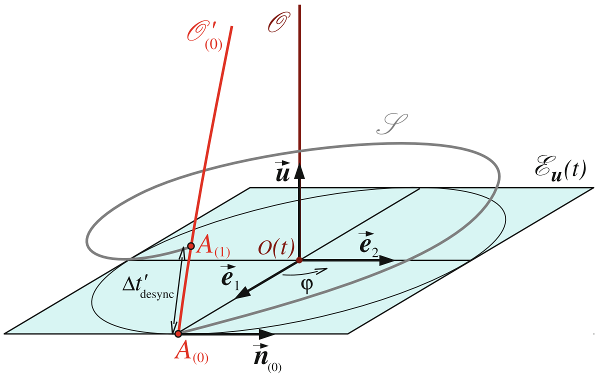

Fig. 13.10 Simultaneity curve $\mathscr{S}$, marking $t' = 0$ for the corotating observers having a fixed value of r

Remark 13.7. Since $\vec{V} = r\omega\,\vec{n}$ [Eq. (13.12)] and $\Gamma = (1 - r^2\omega^2/c^2)^{-1/2}$ [Eq. (13.17)], (13.43) can be written as

$$\Delta t'_{\rm desync} = \frac{\omega}{c^2}\sqrt{1 - r_{(0)}^2\omega^2/c^2}\oint_{\mathscr{C}} \frac{r}{1 - r^2\omega^2/c^2}\,\vec{n}\cdot \mathrm{d}\vec{\ell}, \tag{13.44}$$

where $r_{(0)}$ stands for the coordinate r of $\mathscr{O}_{(0)}$.

Two particular cases are interesting. Let us first consider a family of observers at *constant radius*, i.e. having the same coordinate r. We can then choose $\lambda = \pm\varphi/(2\pi)$, with $+$ for λ varying in the same sense as φ and $-$ otherwise. We have then $\Gamma = \text{const} = \Gamma_{(0)}$ and $\mathrm{d}\vec{\ell} = \pm r\mathrm{d}\varphi\,\vec{n}$, which yields $\vec{V}\cdot\mathrm{d}\vec{\ell} = \pm r^2\omega\mathrm{d}\varphi$. Since Γ, r and ω are constant, formula (13.43) is easily integrated in

$$\boxed{\Delta t'_{\rm desync} = \pm 2\pi\Gamma\frac{r^2\omega}{c^2}.}_{\;r=\rm const} \tag{13.45}$$

It is also easy to integrate (13.39) to get the simultaneity curve $\mathscr{S}$: in terms of the inertial coordinates (ct, x_*, y_*, z_*), the equation of $\mathscr{S}$ is found to be

$$\mathscr{S}: \begin{cases} ct = \pm\dfrac{r^2\omega}{c}\,\Gamma^2\varphi \\ x_* = r\cos\left(\Gamma^2\varphi\right) \\ y_* = r\sin\left(\Gamma^2\varphi\right). \end{cases} \tag{13.46}$$

We recognize the equation of a helix, parametrized by $\Gamma^2\varphi$. This helix is depicted in Fig. 13.10.

The second particular case is that of small velocities: $\|\vec{V}\|_g = r\omega \ll c$. We can then set $\Gamma_{(0)} \simeq 1$ and $\Gamma \simeq 1$ in (13.43), which reduces to

$$\Delta t'_{\text{desync}} \simeq \frac{1}{c^2} \oint_{\mathscr{C}} \vec{V} \cdot \mathrm{d}\vec{\ell} = \frac{1}{c^2} \int_S \operatorname{curl} \vec{V} \cdot \mathrm{d}\vec{\mathscr{A}}. \tag{13.47}$$

The second equality follows from Stokes theorem [cf. Eq. (16.55)]: S is the surface of $R_{\mathscr{O}}$ inside the contour $\mathscr{C}$ (cf. Fig. 13.9) and $\mathrm{d}\vec{\mathscr{A}}$ the element of area of S oriented according to the sense of variation of λ along $\mathscr{C}$. In the present case (rigid rotation), $\operatorname{curl} \vec{V} = 2\vec{\omega}$. Since $\vec{\omega}$ is constant, there comes

$$\boxed{\Delta t'_{\text{desync}} \simeq \frac{2}{c^2}\, \vec{\omega} \cdot \vec{\mathscr{A}}}_{\,r\omega \ll c}, \tag{13.48}$$

where $\vec{\mathscr{A}}$ is the area vector of the surface S, oriented according to the sense of variation of λ along $\mathscr{C}$. In other words, $\vec{\omega} \cdot \vec{\mathscr{A}} \geq 0$ if $\mathrm{d}\vec{\ell}$ is in the direction of rotation (as in Fig. 13.9) and $\vec{\omega} \cdot \vec{\mathscr{A}} \leq 0$ otherwise.

Remark 13.8. The result (13.48) does not constitute a Newtonian limit of the desynchronization time but rather an approximation at small velocities of a proper relativistic feature. Indeed, in Newtonian theory, we would have $\Delta t'_{\text{desync}} = 0$, since all observers, corotating or not, measure the same time, namely, the absolute Newtonian time. To strengthen this, we notice that if formula (13.48) would be a real Newtonian limit, it would not involve any c factor.

Historical note: *The expression (13.42) for Δt is present in the second volume of the famous treatise on theoretical physics by Lev D. Landau*[2] *and Evgeny M. Lifshitz*[3] *(Landau and Lifshitz 1975, Sect. 89), whose first edition dates from the 1950s (it was derived by a method different from that presented here). The approximate formula (13.48) is also found there. The synchronization helix $\mathscr{S}$, which allows to visualize $\Delta t'_{\text{desync}}$ in Minkowski spacetime, as in Fig. 13.10, has been introduced by Theodor Kaluza*[4] *in 1910 (Kaluza 1910); Kaluza certainly knew an expression equivalent to (13.42), but he did not write it down. The synchronization*

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[2]**Lev D. Landau** (1908–1968): Soviet theoretical physicist, 1962 Nobel Prize in Physics for the explanation of superfluidity; Landau contributed to many fields of physics, among which relativistic hydrodynamics. He wrote with Evgeny Lifshitz a course covering all theoretical physics of the twentieth century (Landau and Lifshitz 1975).

[3]**Evgeny M. Lifshitz** (1915–1985): Soviet theoretical physicist, former student of Landau, specialist of solid state physics and general relativity.

[4]**Theodor Kaluza** (1885–1954): German mathematician, mostly known for his work in theoretical physics, especially for the so-called Kaluza–Klein theory (1921)—an attempt to unify gravitation and electromagnetism (the only known interactions at that time) in a five-dimensional space. Polyglot, Kaluza was speaking not less than 17 languages!

helix has been discussed in detail by Vittorio Cantoni in 1968 (Cantoni 1968) (cf. also Anandan (1981), Rizzi and Tartaglia (1998) and Rizzi and Serafini (2004)).

13.4.4 Clock Transport on the Rotating Disk

The synchronization procedure described above is based on the orthogonality to the worldline [Eq. (13.35)], which is the geometrical translation of the Einstein–Poincaré simultaneity criterion discussed in Sect. 3.2.2. The physical realization of the latter involves measuring round-trip times of light signals. One may think about a second synchronization procedure: that of ***slow clock transport***. The various corotating observers are actually at rest with respect to each other and one can propagate time by carrying from observer to observer the same clock at very low velocity, to minimize the time dilation effect. For a rigid array of inertial observers (cf. Sect. 8.2.4), such a process is equivalent to the Einstein–Poincaré synchronization. What happens for an array of corotating observers?

To answer, let us consider an observer $\mathscr{O}''$ who travels by meeting successively all the members $\mathscr{O}'_{(\lambda)}$ of a closed family of corotating observers. The trajectory of $\mathscr{O}''$ in the reference space of $\mathscr{O}$ is thus the curve $\mathscr{C}$ defined above (cf. Fig. 13.9). Let $\vec{\boldsymbol{v}}$ be the velocity of $\mathscr{O}''$ relative to each observer $\mathscr{O}'_{(\lambda)}$ that he encounters. We set $v := \|\vec{\boldsymbol{v}}\|_{\boldsymbol{g}}$. Eventually, $\mathscr{O}''$ will be the observer carrying the synchronization clock, and v will be assumed to be small, to achieve the slow transport. But at present, we shall not make any hypothesis on the value of v. Even, we shall use $v \to c$ in Sect. 13.6.1. By definition, observer $\mathscr{O}''$ quits the corotating observer $\mathscr{O}'_{(0)}$ at event A, encounters successively each $\mathscr{O}'_{(\lambda)}$ at the event $M_{(\lambda)}$ say and is back to observer $\mathscr{O}'_{(0)}$ at event B (cf. Fig. 13.11). We have then $M_{(0)} = A$ and $M_{(1)} = B$.

When $\mathscr{O}''$ moves from $M_{(\lambda)}$ to $M_{(\lambda+\mathrm{d}\lambda)}$, the proper time measured by $\mathscr{O}'_{(\lambda)}$ is, by definition of the velocity $\vec{\boldsymbol{v}}$,

$$\mathrm{d}t' = \frac{\mathrm{d}\ell'}{v}, \tag{13.49}$$

where $\mathrm{d}\ell' := \|\mathrm{d}\vec{\boldsymbol{\ell}}'\|_{\boldsymbol{g}}$, $\mathrm{d}\vec{\boldsymbol{\ell}}'$ being the proper separation vector (13.38) between $\mathscr{O}'_{(\lambda)}$ and $\mathscr{O}'_{(\lambda+\mathrm{d}\lambda)}$ (cf. Fig. 13.12). Combining (13.38) and (13.39) yields a relation between $\mathrm{d}\vec{\boldsymbol{\ell}}'$ and $\mathrm{d}\vec{\boldsymbol{\ell}}$:

$$\mathrm{d}\vec{\boldsymbol{\ell}}' = \mathrm{d}\vec{\boldsymbol{\ell}} + \frac{\Gamma^2}{c}(\vec{\boldsymbol{V}} \cdot \mathrm{d}\vec{\boldsymbol{\ell}})\left(\vec{\boldsymbol{u}} + \frac{1}{c}\vec{\boldsymbol{V}}\right).$$

Taking the scalar square of this relation and using $\vec{\boldsymbol{u}} \cdot \mathrm{d}\vec{\boldsymbol{\ell}} = 0$, $\vec{\boldsymbol{u}} \cdot \mathrm{d}\vec{\boldsymbol{V}} = 0$ and $1 - V^2/c^2 = \Gamma^{-2}$, we get

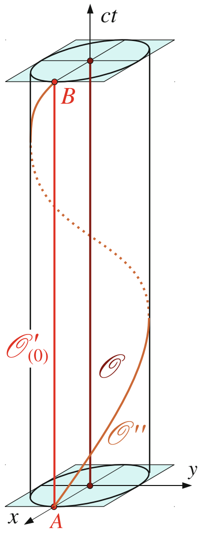


Fig. 13.11 Worldline of the synchronization clock (observer $\mathscr{O}''$) slowly transported among the corotating observers, in the case where they are located at the same coordinate r. Note that this spacetime diagram is drawn in the coordinates (ct, x, y, z) of the uniformly rotating observer $\mathscr{O}$, contrary to the diagrams of Figs. 13.3 and 13.10, which are based on the inertial coordinates (ct, x_*, y_*, z_*). Accordingly, the worldline of the corotating observer $\mathscr{O}'_{(0)}$ appears as a vertical straight line, whereas in Figs. 13.3 and 13.10, it was a helix

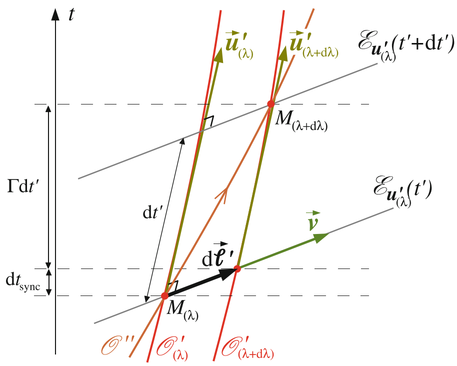


Fig. 13.12 Observer $\mathscr{O}''$ moving from a corotating observer $\mathscr{O}'_{(\lambda)}$ to the neighbouring observer $\mathscr{O}'_{(\lambda+\mathrm{d}\lambda)}$

$$\mathrm{d}\ell'^2 = \mathrm{d}\ell^2 + \frac{\Gamma^2}{c^2}(\vec{\boldsymbol{V}} \cdot \mathrm{d}\vec{\boldsymbol{\ell}})^2. \tag{13.50}$$

Let us denote by θ the angle between vectors $\mathrm{d}\vec{\boldsymbol{\ell}}$ and $\vec{\boldsymbol{V}} = r\omega\, \vec{\boldsymbol{n}}$ in $E_{\boldsymbol{u}}$: $\vec{\boldsymbol{V}} \cdot \mathrm{d}\vec{\boldsymbol{\ell}} = r\omega \mathrm{d}\ell \cos\theta$. Equation (13.50) can then be recast as

$$\mathrm{d}\ell' = \Gamma \sqrt{1 - \frac{r^2\omega^2}{c^2}\sin^2\theta}\,\mathrm{d}\ell \tag{13.51}$$

and (13.49) becomes

$$\mathrm{d}t' = \frac{\Gamma}{v}\sqrt{1 - \frac{r^2\omega^2}{c^2}\sin^2\theta}\,\mathrm{d}\ell. \tag{13.52}$$

The proper time $\mathrm{d}t''$ measured by $\mathscr{O}''$ between $M_{(\lambda)}$ and $M_{(\lambda+\mathrm{d}\lambda)}$ is deduced from $\mathrm{d}t'$ via the Lorentz factor corresponding to the velocity $\vec{\boldsymbol{v}}$:

$$\mathrm{d}t'' = \sqrt{1 - \frac{v^2}{c^2}}\,\mathrm{d}t' = \frac{\Gamma}{v}\sqrt{\left(1 - \frac{v^2}{c^2}\right)\left(1 - \frac{r^2\omega^2}{c^2}\sin^2\theta\right)}\,\mathrm{d}\ell. \tag{13.53}$$

If $v \to 0$, $\mathrm{d}t'' = \mathrm{d}t'$, which shows that the synchronization of the corotating observers' clocks by means of the clock transported by $\mathscr{O}''$ coincides with the Einstein–Poincaré synchronization if $\mathscr{O}''$ moves infinitely slowly.

With respect to the inertial observer $\mathscr{O}_*$, $\mathrm{d}t'$ corresponds to a time lapse $\Gamma\mathrm{d}t'$ [cf. (13.24)]. The total elapsed time measured by $\mathscr{O}_*$ between the events $M_{(\lambda)}$ and $M_{(\lambda+\mathrm{d}\lambda)}$ is then

$$\mathrm{d}t = \Gamma\mathrm{d}t' + \mathrm{d}t_{\mathrm{sync}}, \tag{13.54}$$

where $\mathrm{d}t_{\mathrm{sync}}$ is the increase (13.39) of the coordinate t between two events that are simultaneous from the point of view of corotating observers (cf. Fig. 13.12). Using (13.52) and (13.39), we have thus

$$\mathrm{d}t = \Gamma^2\left(\sqrt{1 - \frac{r^2\omega^2}{c^2}\sin^2\theta}\,\frac{\mathrm{d}\ell}{v} + \frac{\vec{\boldsymbol{V}}\cdot\mathrm{d}\vec{\boldsymbol{\ell}}}{c^2}\right). \tag{13.55}$$

For the three observers $\mathscr{O}_*$, $\mathscr{O}'_{(0)}$ and $\mathscr{O}''$, let us set to zero the date of the event A where $\mathscr{O}''$ leaves $\mathscr{O}'_{(0)}$:

$$t_A = t'_A = t''_A = 0. \tag{13.56}$$

The elapsed time for the journey of $\mathscr{O}''$ from A to B, as measured by $\mathscr{O}_*$, is obtained by integrating (13.55) between $\lambda = 0$ and $\lambda = 1$, i.e. by integrating over the contour $\mathscr{C}$:

$$t_B = \oint_{\mathscr{C}} \Gamma^2\left(\sqrt{1 - \frac{r^2\omega^2}{c^2}\sin^2\theta}\,\frac{\mathrm{d}\ell}{v} + \frac{\vec{\boldsymbol{V}}\cdot\mathrm{d}\vec{\boldsymbol{\ell}}}{c^2}\right). \tag{13.57}$$

By virtue of (13.24), the elapsed time for $\mathscr{O}'_{(0)}$ follows by dividing by the Lorentz factor $\Gamma_{(0)}$. Using (13.43), we get

$$t'_B = \frac{1}{\Gamma_{(0)}} \oint_{\mathscr{C}} \frac{\Gamma^2}{v} \sqrt{1 - \frac{r^2\omega^2}{c^2} \sin^2\theta}\, \mathrm{d}\ell + \Delta t'_{\mathrm{desync}}. \tag{13.58}$$

On the other side, the time measured by $\mathscr{O}''$ between A and B is obtained by integrating (13.53):

$$t''_B = \oint_{\mathscr{C}} \frac{\Gamma}{v} \sqrt{\left(1 - \frac{v^2}{c^2}\right)\left(1 - \frac{r^2\omega^2}{c^2} \sin^2\theta\right)}\, \mathrm{d}\ell. \tag{13.59}$$

The difference between the travel time measured by $\mathscr{O}'_{(0)}$ and that measured by $\mathscr{O}''$ is

$$\boxed{t'_B - t''_B = \oint_{\mathscr{C}} \frac{\Gamma}{v} \sqrt{1 - \frac{r^2\omega^2}{c^2} \sin^2\theta} \left(\frac{\Gamma}{\Gamma_{(0)}} - \sqrt{1 - \frac{v^2}{c^2}}\right) \mathrm{d}\ell + \Delta t'_{\mathrm{desync}}}. \tag{13.60}$$

In all what follows, we shall assume that the norm v of the velocity of $\mathscr{O}''$ relative to each corotating observer is constant.

It is worth expressing (13.60) in the particular case where the corotating observers are all located at the same r-coordinate, which implies that the trajectory of $\mathscr{O}''$ is a circle of radius r with respect to $\mathscr{O}$ (or $\mathscr{O}_*$). We have then $\theta = 0$, $\Gamma = \Gamma_{(0)} = \text{const}$ and $\Delta t'_{\mathrm{desync}}$ is given by (13.45). Since both Γ and v are constant, all the terms can be extracted from the integral, which reduces to $\oint \mathrm{d}\ell = 2\pi r$. Using the identity $1 - \sqrt{1 - x^2} = x^2/[1 + \sqrt{1 - x^2}]$, there comes then

$$\boxed{t'_B - t''_B = \Gamma \frac{2\pi r}{c^2} \frac{v}{1 + \sqrt{1 - v^2/c^2}} + \Delta t'_{\mathrm{desync}}.}_{\;r=\mathrm{const}} \tag{13.61}$$

If $\mathscr{O}''$ is moving slowly from a corotating observer to the other one, then $v \to 0$ and the above formula reduces to

$$\boxed{t'_B - t''_B = \Delta t'_{\mathrm{desync}},}_{\;r=\mathrm{const},\ v\to 0} \tag{13.62}$$

with $\Delta t'_{\mathrm{desync}}$ given by (13.45). Thus, when it is back to $\mathscr{O}_{(0)}$, the clock transported by $\mathscr{O}''$ does not show the same time as a clock that stayed with $\mathscr{O}_{(0)}$, the two clocks having been synchronized at the departure of $\mathscr{O}''$ [$t'_A = t''_A = 0$, Eq. (13.56)].

We conclude that the synchronization of corotating observers with a slowly transported clock faces the same problem as the synchronization by exchange of light signals: after a complete turn, the clocks are desynchronized by the quantity $\Delta t'_{\rm desync}$ given by (13.45).

Remark 13.9. The result (13.62) might appear surprising at first glance, since it has been obtained by taking the limit $v \to 0$, and observers $\mathscr{O}''$ and $\mathscr{O}'_{(0)}$ coincide if $v = 0$, so that one could have expected $t''_B = t'_B$ in this limit. Actually, when $v \to 0$, the travel time of $\mathscr{O}''$ between A and B becomes infinitely large, as seen on (13.59). We have thus $t''_B \to +\infty$ and $t'_B \to +\infty$, the difference between the two approaching the finite number $\Delta t'_{\rm desync}$. Moreover, let us stress that if v was strictly zero, the event B would be ill defined.

13.4.5 *Experimental Measures of the Desynchronization*

13.4.5.1 Hafele–Keating Experiment

Let us reconsider the Hafele–Keating experiment (1971) described in Sect. 2.6.6, namely, the comparison of an atomic clock after a flight around the Earth with an atomic clock stayed on the ground (Hafele 1972b; Hafele and Keating 1972a,b). In this experiment, the rotating system is of course the Earth. $\mathscr{O}'$ is then the ground observer and $\mathscr{O}''$ the observer travelling in the plane. To simplify the computation, we shall assume that a single plane is employed and that it follows an exactly circular trajectory, i.e. it performs the trip all around the Earth by staying at the same latitude and keeping a constant velocity with respect to the ground (the parameter v). In this case, the formula giving the difference in proper time between the ground clock and the onboard clock is (13.61):

$$t_{\rm ground} - t_{\rm plane} = \frac{\pi r v}{c^2} + \Delta t'_{\rm desync}, \tag{13.63}$$

where we have set[5] $t_{\rm ground} := t'_B$ and $t_{\rm plane} = t''_{\rm B}$ and have of course taken the limit of small velocities: $\Gamma = 1$ and $1 + \sqrt{1 - v^2/c^2} = 2$. $\Delta t'_{\rm desync}$ is given by (13.45) (still with $\Gamma = 1$):

$$\Delta t'_{\rm desync} = \pm 2\pi \frac{r^2 \omega}{c^2}, \tag{13.64}$$

[5] $t_{\rm ground}$ and $t_{\rm plane}$ were noted, respectively, T and T' in Sect. 2.6.6.

with the + sign for a journey in the same sense as Earth rotation, i.e. eastward, and the − sign otherwise. In formulas (13.63) and (13.64), $r = R\cos\lambda$, where $R = 6.37\times10^6$ m is the Earth radius and λ is the latitude of the plane trajectory: $\lambda = 30°$ (the altitude of the plane is neglected in front of R). In (13.63), v is the plane velocity relative to the ground. For the airline jets employed by Hafele and Keating $v = 830\,\mathrm{km\,h^{-1}} = 230\,\mathrm{m\,s^{-1}}$. Finally, ω is the angular velocity of the Earth with respect to an inertial frame: $\omega = \omega_\oplus = 2\pi/(23\,\mathrm{h}\;56\,\mathrm{min}) = 7.29\times10^{-5}\,\mathrm{rad\,s^{-1}}$. With the above numerical values, we get

$$\Delta t'_{\mathrm{desync}} = 155\,\mathrm{ns}\ (\mathrm{east}) \quad \text{and} \quad \Delta t'_{\mathrm{desync}} = -155\,\mathrm{ns}\ (\mathrm{west}), \tag{13.65}$$

$$t_{\mathrm{ground}} - t_{\mathrm{plane}} = 199\,\mathrm{ns}\ (\mathrm{east}) \quad \text{and} \quad t_{\mathrm{ground}} - t_{\mathrm{plane}} = -111\,\mathrm{ns}\ (\mathrm{west}). \tag{13.66}$$

Given the simplifications on the plane trajectories, we recover the values mentioned in Sect. 2.6.6[6]: $t_{\mathrm{ground}} - t_{\mathrm{plane}} = 184 \pm 18$ ns for the eastward journey and $t_{\mathrm{ground}} - t_{\mathrm{plane}} = -96 \pm 10$ ns for the westward one. The latter values, which take into account the actual trajectories of the planes (hence the error bars), are in very good agreement with (13.66).

As we have already discussed in Sect. 2.6.6, to get the time shift actually measured by Hafele and Keating, one must add to (13.63) a term arising from general relativity, reflecting the fact that the plane is higher is the Earth gravitational potential than the ground station: the *gravitational-redshift* term, which will be discussed in Chap. 22. The full formula is then

$$t_{\mathrm{ground}} - t_{\mathrm{plane}} = \frac{\pi r v}{c^2} + \Delta t'_{\mathrm{desync}} + \Delta t_{\mathrm{grav}} \tag{13.67}$$

with

$$\Delta t_{\mathrm{grav}} = -\frac{GM}{c^2 R}\frac{h}{R}\, t_{\mathrm{ground}}, \tag{13.68}$$

where $G = 6.67\times10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$ is the gravitation constant, $M = 6.0\times10^{24}$ kg is the Earth mass and h is the plane altitude. Taking $h = 9$ km and $t_{\mathrm{ground}} = 2\pi r/v$, the amplitude of the gravitational-redshift term is

$$\Delta t_{\mathrm{grav}} = -148\,\mathrm{ns}. \tag{13.69}$$

Finally,

$$t_{\mathrm{ground}} - t_{\mathrm{plane}} = 51\,\mathrm{ns}\ (\mathrm{east}) \quad \text{and} \quad t_{\mathrm{ground}} - t_{\mathrm{plane}} = -259\,\mathrm{ns}\ (\mathrm{west}), \tag{13.70}$$

to be compared with the experimental values:

[6] In Sect. 2.6.6, the considered quantity was $T' - T = t_{\mathrm{plane}} - t_{\mathrm{ground}}$, rather than $t_{\mathrm{ground}} - t_{\mathrm{plane}}$.

$$t_{\text{ground}} - t_{\text{plane}} = 59 \pm 10 \text{ ns (east)} \quad \text{and} \quad t_{\text{ground}} - t_{\text{plane}} = -273 \pm 7 \text{ ns (west)}. \tag{13.71}$$

Given the approximation on the trajectories, there is a good agreement between the predicted values and the measured ones. Since the three terms in (13.67) are of the same order of magnitude, we conclude that Hafele–Keating experiment constitutes an experimental confirmation of the synchronization defect $\Delta t'_{\text{desync}}$ caused by the rotation of a system of observers.

13.4.5.2 Synchronization of Atomic Clocks on the Earth

The modern timescale used on Earth is based on the ***International Atomic Time*** (TAI, from the French *temps atomique international*), which combines data from a few hundred atomic clocks around the world. Each atomic clock is at rest at the surface of the Earth; it therefore gives the proper time of an observer $\mathscr{O}'$ rotating at the angular velocity $\omega_\oplus = 7.29 \times 10^{-5}$ rad s^{-1} with respect to an inertial observer. When comparing the atomic clocks among themselves to define the TAI, one faces the synchronization problem of corotating observers. The solution amounts to correcting the output of each clock to bring it to the time t of the central inertial observer $\mathscr{O}_*$. In this context, t is called ***Geocentric Coordinate Time*** (TCG, from the French *temps-coordonnée géocentrique*) and $\mathscr{O}_*$ the ***Geocentric Celestial Reference System*** (GCRS). The advantage of t, with respect to the measured atomic time t', is that it allows one to bypass the synchronization problem on the rotating Earth. The fundamental equation to make the transfer from the atomic clock time t' to the coordinate time t is (13.54). The correction to apply[7] to the output t' of an atomic clock is thus to multiply by the Lorentz factor Γ and to add the integral of $\mathrm{d}t_{\text{sync}}$. This last correction has to be applied any time one wants to combine the outputs of clocks at different locations on the Earth surface. The necessity of this term has been shown experimentally in the synchronization of two atomic clocks (i) by the transport of a third atomic clock between the two clocks by Neil Ashby and David W. Allan in 1979 (Ashby and Allan 1979) and (ii) by means of the analysis of the electromagnetic signals from the satellites of the Global Positioning System (GPS) by David W. Allan, Marc A. Weiss and Neil Ashby in 1985 (Allan et al. 1985). For more details on this topic, the reader is invited to consult (Ashby 2003, 2004; Blanchet et al. 2001; Petit and Wolf 2005).

[7] Additionally, one must correct from the general relativistic redshift mentioned above, because $\mathscr{O}'$ is higher in the Earth gravitational potential than $\mathscr{O}_*$, who is located at the Earth centre.

13.5 Ehrenfest Paradox

13.5.1 Circumference of the Rotating Disk

We have defined in Sect. 13.3.2 the *rotating disk* as the set of all corotating observers $\mathscr{O}'$ whose r coordinate ranges from 0 (central observer $\mathscr{O}$) and some upper bound $R < c/\omega$, called the *disk radius*. The set of corotating observers of coordinate $r = R$ constitutes the ***disk circumference***. This is a closed family, of parameter $\lambda = \varphi/(2\pi)$. Relation (13.37) becomes then

$$\mathrm{d}\vec{\boldsymbol{\ell}} = R\mathrm{d}\varphi\, \vec{\boldsymbol{n}}. \tag{13.72}$$

The distance $\mathrm{d}\ell'$ between two neighbouring corotating observers on the disk circumference, $\mathscr{O}'_{(\lambda)}$ and $\mathscr{O}'_{(\lambda+\mathrm{d}\lambda)}$, as measured by $\mathscr{O}'_{(\lambda)}$ [i.e. the norm of the vector $\overrightarrow{A_{(\lambda)}A_{(\lambda+\mathrm{d}\lambda)}}$; cf. (13.38)] is given by (13.51). Now, in the present case, $\theta = 0$ ($\vec{\boldsymbol{V}}$ and $\mathrm{d}\vec{\boldsymbol{\ell}}$ are aligned) and, using (13.72), $\mathrm{d}\ell = R\,\mathrm{d}\varphi$. Thus, (13.51) reduces to

$$\mathrm{d}\ell' = \Gamma R\,\mathrm{d}\varphi. \tag{13.73}$$

From the point of view of corotating observers, the circumference of the rotating disk is then

$$L' = \int_{\varphi=0}^{\varphi=2\pi} \mathrm{d}\ell' = \int_0^{2\pi} \Gamma R\,\mathrm{d}\varphi.$$

Since $\Gamma = (1 - R^2\omega^2/c^2)^{-1/2}$ is independent of φ, there comes immediately

$$\boxed{L' = \Gamma\, 2\pi R}. \tag{13.74}$$

13.5.2 Disk Radius

To define the disk radius measured by the corotating observers, let us consider a family of observers sharing the same φ coordinate with respect to $\mathscr{O}$ (cf. Fig. 13.13). The parameter of this family is then $\lambda = r/R$. Equation (13.37) becomes

$$\mathrm{d}\vec{\boldsymbol{\ell}} = -\mathrm{d}r\, \vec{\boldsymbol{e}}'_2. \tag{13.75}$$

The distance $\mathrm{d}\ell'$ between $\mathscr{O}'_{(\lambda)}$ and $\mathscr{O}'_{(\lambda+\mathrm{d}\lambda)}$, as measured by $\mathscr{O}'_{(\lambda)}$, is still given by (13.51), with this time $\theta = \pi/2$ ($\vec{\boldsymbol{V}}$ and $\mathrm{d}\vec{\boldsymbol{\ell}}$ are orthogonal) and, from (13.75), $\mathrm{d}\ell = \mathrm{d}r$. Thus, $\mathrm{d}\ell' = \Gamma\sqrt{1 - r^2\omega^2/c^2}\,\mathrm{d}r = \mathrm{d}r$. From the point of view of corotating observers, the length of the radius of the rotating disk is thus

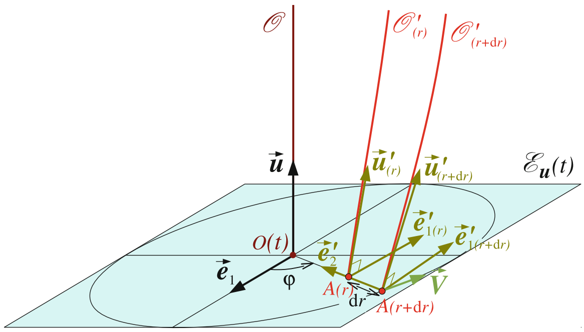


Fig. 13.13 Worldlines of the corotating observers $\mathscr{O}'_{(r)}$ and $\mathscr{O}'_{(r+\mathrm{d}r)}$ having the same φ coordinate and infinitely close r coordinates

$$R' = \int_{r=0}^{r=R} \mathrm{d}\ell' = \int_0^R \mathrm{d}r,$$

hence,

$$\boxed{R' = R}. \tag{13.76}$$

Comparing with (13.74), we conclude that the circumference L' and the radius R' of the disk, both measured by corotating observers, are linked by

$$\boxed{L' = \Gamma\, 2\pi R' = \frac{2\pi R'}{\sqrt{1-(R'\omega/c)^2}}}. \tag{13.77}$$

For $\omega \neq 0$, this relation differs from the standard formula $L' = 2\pi R'$. Consequently, the corotating observers "perceive" a *non-Euclidean* spatial geometry.

13.5.3 The "Paradox"

In 1909, Paul Ehrenfest[8] (1909) noticed that if one considers a disk[9] initially at rest with respect to an inertial observer $\mathscr{O}_*$, its circumference is related to its radius by

$$L_0 = 2\pi R_0. \tag{13.78}$$

[8]**Paul Ehrenfest** (1880–1933): Austrian physicist (naturalized Dutch in 1922), known for his work in quantum mechanics and famous for the clarity of his physics lecture at the University of Leiden. He got depressed and committed suicide, as his thesis advisor, Ludwig Boltzmann, did 27 years before.

[9]Actually, Ehrenfest considered a cylinder, rather than a disk; but this changes nothing to the present discussion, since the height of the cylinder plays no role.

If the disk is set to rotation, then the circumference and radius measured by the inertial observer still satisfy $L = 2\pi R$ with $R = R_0$. Because of the length contraction (cf. Sect. 5.2.2), the circumference L' measured by observers at rest on the disk must obey $L < L'$ for the velocity is tangent to the circumference. On the other hand, there is no length contraction in the radial direction, the latter being orthogonal to the disk velocity with respect to $\mathscr{O}_*$. The radius measured by the disk observers is thus $R' = R = R_0$. One infers from $L = 2\pi R$ and $L < L'$ that

$$L' > 2\pi R_0. \tag{13.79}$$

This relation is of course in agreement with the result (13.77), which stipulates that the factor between L' and $2\pi R_0 = 2\pi R'$ is $\Gamma > 1$. The paradox appears if one assumes that the disk remained rigid during all the stage of rotation increase, from zero to the final value of ω. Indeed, according to Born's rigidity criterion (cf. Sect. 3.3.2), the distance between any two elements of the disk must remain constant. This amounts to considering the worldlines of the adjacent observers $\mathscr{O}'_{(\lambda)}$ and $\mathscr{O}'_{(\lambda+\mathrm{d}\lambda)}$ on the disk circumference as the extremities of a ruler and to asking that this ruler is rigid in the sense defined in Sect. 3.3.2. Then $L' = L_0$ holds and Eqs. (13.78) and (13.79) are contradictory. This is the so-called ***Ehrenfest paradox***.

The solution of this "paradox" relies on the relaxation of the rigidity hypothesis, i.e. on the equality $L' = L_0$. Indeed, as detailed hereafter, it turns out that the disk cannot remain rigid during the set up of rotation.

13.5.4 Setting the Disk into Rotation

Let us consider a disk in the plane $z_* = 0$ of the inertial observer $\mathscr{O}_*$. The disk is assumed to be at rest with respect to $\mathscr{O}_*$ for $t < t_0$. At the instant $t = t_0$, each point of the disk is submitted to the same angular acceleration[10] $\mathrm{d}^2\varphi_*/\mathrm{d}t^2 \neq 0$. The value of $\mathrm{d}^2\varphi_*/\mathrm{d}t^2$ is kept constant until the desired angular velocity $\omega = \mathrm{d}\varphi_*/\mathrm{d}t$ is reached at $t = t_1$. The whole process is represented in the spacetime diagram of Fig. 13.14. A rigid ruler (in Born sense) carried by the corotating observer $\mathscr{O}'_A$ is depicted in this figure. At $t = t_0$, $\mathscr{O}'_A$ is at the end A_0 of the ruler, and the other end marks the position B_0 of a neighbouring corotating observer, $\mathscr{O}'_B$ say. At the end of the angular acceleration phase ($t = t_1$), $\mathscr{O}'_A$ is in A_1 and $\mathscr{O}'_B$ in B_1. In the local rest space of $\mathscr{O}'_A$, the second end of the ruler, B'_1 say, is located along the vector $\vec{e}\,'_1$. Since the ruler is supposed rigid, its proper length is always equal to $\ell_0 := \|\overrightarrow{A_0B_0}\|_g$. The point B'_1 is thus determined by

$$\overrightarrow{A_1B'_1} = \ell_0\,\vec{e}\,'_1. \tag{13.80}$$

[10] φ_* stands for the azimuthal coordinates related to the inertial coordinates (x_*, y_*) by $x_* = r\cos\varphi_*$, $y_* = r\sin\varphi_*$ and $r := \sqrt{x_*^2 + y_*^2}$.

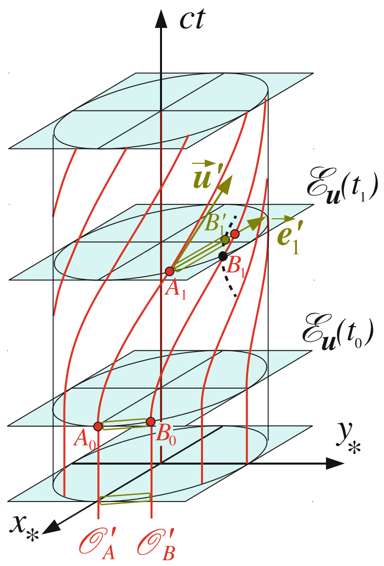


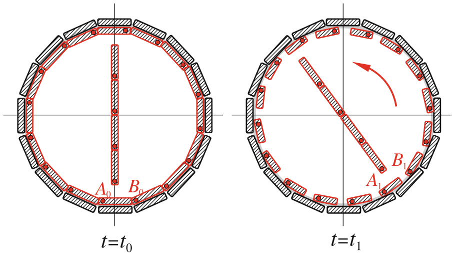
Fig. 13.14 Setting the disk into rotation: the disk is at rest (with respect to $\mathscr{O}_*$) for $t < t_0$, it accelerates uniformly for $t_0 \le t < t_1$ and has a uniformly circular motion at the velocity ω for $t \ge t_1$. Each *solid line* is the worldline of a corotating observer. A rigid ruler between two adjacent corotating observers is depicted (extremities A and B)

The angular acceleration having been the same (from the point of view of $\mathscr{O}_*$) for all the points of the disk, we have $\|\overrightarrow{A_1 B_1}\|_{\boldsymbol{g}} = \|\overrightarrow{A_0 B_0}\|_{\boldsymbol{g}} = \ell_0$. We deduce that the point B_1' is located on the hyperboloid $\mathscr{H}$ through B_1 that defines the extremities of the vectors arising from A_1 and having a norm equal to ℓ_0 (this hyperboloid is homothetic to the hyperboloid of one sheet $\mathscr{S}_{A_1}$ introduced in Sect. 1.4.3). The trace of $\mathscr{H}$ in the plane tangent to the spacetime cylinder formed by the rotating disk is the branch of hyperbola depicted by a dashed line in Fig. 13.14. We see clearly that B_1' is not located on the worldline of $\mathscr{O}_B'$. In other words, the ruler has lost the contact with $\mathscr{O}_B'$. Since the ruler is rigid (i.e. has a constant metric length), this implies that the distance between observers $\mathscr{O}_A'$ and $\mathscr{O}_B'$ has increased during the acceleration phase: the disk has not obeyed Born rigidity. We recover actually the result (13.73), which leads to the increase of the disk circumference, by the Lorentz factor Γ: $L' = \Gamma 2\pi R$.

Another illustration of the circumference increase measured by the corotating observers is given in Fig. 13.15. The latter represents the rotating disk as perceived by the inertial observer $\mathscr{O}_*$ at the instants t_0 and t_1. This is nothing but the drawing of the plane sections defined by $\mathscr{E}_{\boldsymbol{u}}(t_0)$ and $\mathscr{E}_{\boldsymbol{u}}(t_1)$ in Fig. 13.14. Three types of rigid rulers have been depicted:

- Rulers $\mathscr{R}_*$ located slightly outside the disk and fixed with respect to $\mathscr{O}_*$.
- Rulers $\mathscr{R}_1'$ located at the disk periphery and following the disk in its motion, in the following sense: (i) an extremity of the ruler is attached to the disk, i.e. follows the worldline of a corotating observer (this is the extremity marked by a dot in Fig. 13.15), and (ii) the ruler is always tangent to the periphery of the disk.

Fig. 13.15 Disk at rest with respect to the inertial observer $\mathscr{O}_*$ ($t = t_0$) and in uniform rotation ($t = t_1$). The dots mark the fixations of the rulers to the disk. The points A_0, B_0, A_1 and B_1 are the same as in Fig. 13.14



- Rulers $\mathscr{R}'_2$ located along a diameter of the disk and at rest with respect to it in the following sense: (i) an extremity of the ruler is attached to the disk (extremity marked by a dot in Fig. 13.15) and (ii) the ruler is always aligned along a diameter of the disk.

All these rulers are supposed to be identical, i.e. have the same length (in the frame where they are at rest). The rulers $\mathscr{R}_*$ are of course at the same position in the two panels of Fig. 13.15. At $t = t_0$, when the disk is still at rest, the rulers $\mathscr{R}'_1$ coincide (except for a slight radial shift) with the rulers $\mathscr{R}_*$. On the other hand, at $t = t_1$, when the disk is in stationary rotation, the rulers $\mathscr{R}'_1$ as perceived by $\mathscr{O}_*$ are affected by the FitzGerald–Lorentz contraction, since they are aligned in the direction of motion. Each ruler has then lost contact with its neighbours. More rulers would be thus required to close the circumference. This shows clearly that the circumference length for corotating observers, namely, L', is larger than the length L measured by $\mathscr{O}_*$. On the other side, the rulers $\mathscr{R}'_2$, which at each instant are perpendicular to the direction of motion, do not suffer the FitzGerald–Lorentz contraction and keep contact with each other. This means that the radius R' measured by corotating observers is unchanged and illustrates the non-Euclidean relation (13.77): $L' > 2\pi R'$.

If the rotating disk is a solid body, then the stretch (in the direction parallel to the circumference) caused by setting the disk into rotation induces some elastic constraints (tensions) in the material. These constraints could lead to the breakup of the disk if the rotation velocity was too high.

Historical note: *Since its formulation in 1909 (Ehrenfest 1909), the Ehrenfest paradox has generated an important literature, with many debates. The non-Euclidean character of the geometry perceived by the corotating observers has been established by Theodor Kaluza (cf. p. 445) in 1910 (Kaluza 1910), as well as by Albert Einstein in 1916 (Einstein 1916, 1919), and developed, among others, by Paul Langevin (cf. p. 40) in 1935 (Langevin 1935), Carlton W. Berenda*[11] *in 1942 (Berenda 1942) and Nathan Rosen*[12] *in 1947 (Rosen 1947). Let us point*

[11]**Carlton W. Berenda** (1911–1980): American physicist and philosopher of sciences.

[12]**Nathan Rosen** (1909–1995): American–Israeli physicist, assistant of Einstein at Princeton; he is the *"R"* of the famous *EPR paradox* in quantum mechanics; he is also known for the *Einstein–Rosen bridge* in general relativity.

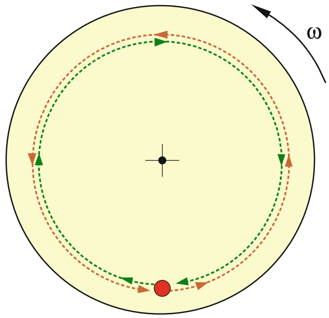


Fig. 13.16 Principle of a Sagnac experiment

out that the problem of the rotating disk seems to have played an important role in Albert Einstein's thoughts towards general relativity (Stachel 1980), notably regarding the difference between mathematical coordinates and measured lengths and times. This distinction was not done in the first articles about special relativity, which were restricted to inertial observers. But, as stressed above, for corotating observers, the coordinates (r, φ) *do not correspond directly to physical measures. For instance, relation (13.73) between the physical length* $\mathrm{d}\ell'$ *and* $r\,\mathrm{d}\varphi$ *is not simple, because of the* Γ *factor. The impossibility to set the disk into rotation while preserving Born rigidity has been underlined as early as 1909 by Gustav Herglotz (cf. p. 361) (1909). For more details about the history and the solution to Ehrenfest paradox, some useful references are Grøn (2004), Rizzi and Ruggiero (2002) and Walter (1996).*

13.6 Sagnac Effect

The most interesting aspect of rotating observers is certainly the Sagnac effect. It has been put forward in 1913 by Georges Sagnac[13] (1913a; 1913b) as a phase shift proportional to the angular velocity in an optical interferometer attached to a rotating table. The effect is however not limited to electromagnetic waves; actually it results from the difference between the instants of return of two signals emitted at some location of a rotating disk and making one turn of the disk in opposite directions (cf. Fig. 13.16). The nature of the signals is not important (electromagnetic waves, electrons, neutrons, atoms, etc.), provided that the norms of the propagation speed of the two signals are the same with respect to corotating observers.

[13]**Georges Sagnac** (1869–1928): French physicist, one of the pioneers of X-ray studies in France (he notably discovered X-ray fluorescence); he got interested in the optics of moving bodies, in the framework of the aether theory, of which he was a proponent. He was a friend of Paul Langevin (cf. p. 40), Émile Borel (cf. p. 215) and Pierre and Marie Curie.

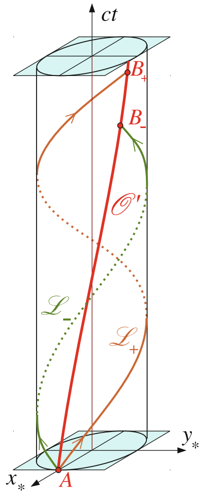


Fig. 13.17 Circular signals emitted in A by the corotating observer $\mathscr{O}'$, in the sense of rotation (worldline $\mathscr{L}_+$) and in the reverse one (worldline $\mathscr{L}_-$). The first signal reaches $\mathscr{O}'$ at B_+ and the second one at B_-

13.6.1 Sagnac Delay

Let us consider a corotating observer $\mathscr{O}'$ who emits at the same event A two "pulses" or "signals", $\mathscr{S}_+$ and $\mathscr{S}_-$. These two signals travel on the same path on the disk, but in opposite directions: in the same sense than the disk rotation for $\mathscr{S}_+$ (***prograde signal***) and in the opposite sense for $\mathscr{S}_-$ (***retrograde signal***). After one turn, the two signals encounter again observer $\mathscr{O}'$, at events B_+ and B_-, respectively. We shall model these signals as two particles of respective worldlines $\mathscr{L}_+$ and $\mathscr{L}_-$ (cf. Fig. 13.17). $\mathscr{L}_+$ and $\mathscr{L}_-$ can be timelike curves (signal carried by massive particles) or null ones (electromagnetic signal). In both cases, the rotating disk is equipped with "mirrors" to make the trajectory a closed loop.

Since they are coming back to their departure point, each of the signals obeys the equations established for observer $\mathscr{O}''$ in Sect. 13.4.4. The quantity v is then the norm of the signal velocity relative to corotating observers. Stating that the two signals travel the same path, but in opposite directions, amounts to considering that their trajectories in $\mathscr{O}$'s reference space is the same closed curve $\mathscr{C}$ (cf. Fig. 13.9). If the signals are lightlike, $\mathscr{O}'' = \mathscr{S}_+$ or $\mathscr{S}_-$ is no longer strictly speaking an observer, but the kinematical formulas of Sect. 13.4.4 still hold with $v = c$. We have then $t''_B = 0$.

Let us set the origin of the proper time of $\mathscr{O}'$ at event $A : t'_A = 0$. The duration t'_+ (resp. t'_-) measured by $\mathscr{O}'$ between the departure of the signal $\mathscr{S}_+$ (resp. $\mathscr{S}_-$) at A and its return in B_+ (resp. B_-) is given by (13.58):

$$t'_\pm = \frac{1}{\Gamma_{(0)}} \oint_{\mathscr{C}} \frac{\Gamma^2}{v_\pm} \sqrt{1 - \frac{r^2\omega^2}{c^2} \sin^2\theta}\; \mathrm{d}\ell + \Delta t'^{\pm}_{\text{desync}}, \tag{13.81}$$

where v_+ (resp. v_-) if the norm of the velocity of $\mathscr{S}_+$ (resp. $\mathscr{S}_-$) relative to each corotating observer $\mathscr{O}'_{(\lambda)}$ encountered by the signal and where $\Delta t'^{\pm}_{\text{desync}}$ obeys (13.43):

$$\Delta t'^{\pm}_{\text{desync}} = \frac{1}{c^2 \Gamma_{(0)}} \oint_{\mathscr{C}} \Gamma^2 \vec{\boldsymbol{V}} \cdot \mathrm{d}\vec{\boldsymbol{\ell}}^{\pm}, \tag{13.82}$$

$\mathrm{d}\vec{\boldsymbol{\ell}}^{+}$ (resp. $\mathrm{d}\vec{\boldsymbol{\ell}}^{-}$) being the element of the contour $\mathscr{C}$ oriented in the sense of propagation of $\mathscr{S}_+$ (resp. $\mathscr{S}_-$). At any point of $\mathscr{C}$, $\mathrm{d}\vec{\boldsymbol{\ell}}^{-} = -\mathrm{d}\vec{\boldsymbol{\ell}}^{+}$, so that

$$\Delta t'^{-}_{\text{desync}} = -\Delta t'^{+}_{\text{desync}}. \tag{13.83}$$

Let us assume now that, from the point of view of the corotating observer $\mathscr{O}'$, the velocities of the two signal are identical:

$$\boxed{v_+ = v_- =: v}. \tag{13.84}$$

In particular, if the signals are electromagnetic ones, $v = c$. Indeed, we have seen in Sect. 4.6.2 that the light velocity measured locally by an observer is always c, even if the observer is accelerated or rotating, which is the case for $\mathscr{O}'$. We deduce from (13.81), (13.83) and (13.84) that the difference between the arrival times of signals $\mathscr{S}_+$ and $\mathscr{S}_-$ is

$$\Delta t' := t'_+ - t'_- = 2\Delta t'^{+}_{\text{desync}}. \tag{13.85}$$

Making $\Delta t'^{+}_{\text{desync}}$ explicit via (13.82), we get

$$\boxed{\Delta t' = \frac{2}{c^2 \Gamma_{(0)}} \oint_{\mathscr{C}} \Gamma^2 \vec{\boldsymbol{V}} \cdot \mathrm{d}\vec{\boldsymbol{\ell}}}, \tag{13.86}$$

where we have noted $\mathrm{d}\vec{\boldsymbol{\ell}} = \mathrm{d}\vec{\boldsymbol{\ell}}^{+}$ (length element of the path $\mathscr{C}$ oriented in the sense of rotation, so that $\vec{\boldsymbol{V}} \cdot \mathrm{d}\vec{\boldsymbol{\ell}} \geq 0$) .

The fact that $\Delta t' \neq 0$, i.e. that the signal emitted in the sense of rotation does not come back to $\mathscr{O}'$ at the same instant than the signal emitted in the opposite sense, constitutes the ***Sagnac effect***. The quantity $\Delta t'$ is called ***Sagnac delay***. It is always positive since $\vec{\boldsymbol{V}} \cdot \mathrm{d}\vec{\boldsymbol{\ell}} = \vec{\boldsymbol{V}} \cdot \mathrm{d}\vec{\boldsymbol{\ell}}^{+} \geq 0$. Hence, the signal in the sense of rotation arrives always after that propagating in the reverse sense (cf. Fig. 13.17). Formula (13.85) shows that the Sagnac effect is related to the impossibility of a global synchronization of corotating observers discussed in Sect. 13.4.

Remark 13.10. At the nonrelativistic limit, $\|V\|_g/c = r\omega/c \to 0$ and (13.86) reduces to

$$\Delta t' = 0 \quad \text{(nonrelativistic).} \tag{13.87}$$

Thus, there is no Sagnac effect in Newtonian theory, which is expected since the signals have to travel the same distance to come back to their starting point and they have the same velocity $v_+ = v_-$ relative to the rotating disk.

Remark 13.11. The Sagnac delay $\Delta t'$ is independent of the velocity v of the signals relative to the corotating observer $\mathscr{O}'$. It will therefore be the same for photons and massive particles. In particular, the c factor that appears (squared) in (13.86) does not arise from the signal propagation speed but from the metric of Minkowski spacetime.

The two particular cases treated in Sect. 13.4.3 lead to simplified formulas for the Sagnac delay: for a circular signal trajectory, at fixed radius r, Eqs. (13.45) and (13.85) lead to

$$\boxed{\Delta t' = 4\pi\Gamma\frac{r^2\omega}{c^2}.}_{\,r=\text{const}}, \tag{13.88}$$

whereas for small rotation velocities, Eqs. (13.48) and (13.85) lead to

$$\boxed{\Delta t' \simeq \frac{4}{c^2}\,\vec{\boldsymbol{\omega}}\cdot\overrightarrow{\mathscr{A}}}_{\,r\omega\ll c}, \tag{13.89}$$

where $\overrightarrow{\mathscr{A}}$ is the area vector of the surface delimited by the trajectory $\mathscr{C}$ of the signals, oriented in the sense of rotation, i.e. such that $\vec{\boldsymbol{\omega}}\cdot\overrightarrow{\mathscr{A}} \geq 0$.

Remark 13.12. For circular signals, $\|\overrightarrow{\mathscr{A}}\|_g = \pi r^2$, so that at small rotation velocity ($\Gamma \to 1$), (13.88) reduces to (13.89), as it should.

13.6.2 Alternative Derivation

In the particular case of purely circular signals (propagation at constant r), it is instructive to recover the Sagnac delay (13.88) by a method that does not rely on the desynchronization time $\Delta t'_{\text{desync}}$ computed in Sect. 13.4. Indeed, formula (13.88) can be obtained directly from the equations of the signals' worldlines, which are simple in the present case, and the relativistic law of velocity composition.

The worldlines of the signals $\mathscr{S}_+$ and $\mathscr{S}_-$, once expressed in terms of the inertial coordinates (ct, x_*, y_*, z_*), are helices of equations

$$\mathscr{L}_+ : \begin{cases} x_* = r\cos\Omega_+ t \\ y_* = r\sin\Omega_+ t \end{cases} \quad \text{and} \quad \mathscr{L}_- : \begin{cases} x_* = r\cos\Omega_- t \\ y_* = r\sin\Omega_- t \end{cases}, \tag{13.90}$$

where r is the radial coordinate of $\mathscr{O}'$ and $\Omega_+ > 0$ and$\Omega_- < 0$ are two constants (cf. Fig. 13.17). The equation of $\mathscr{O}'$ in the same coordinates being given by (13.11) (we choose $\varphi = 0$), we deduce that the worldline $\mathscr{L}_+$ encounters again the worldline of $\mathscr{O}'$ at the event B_+ where $t = t_{B_+}$ is such that $\Omega_+ t_{B_+} = \omega t_{B_+} + 2\pi$, i.e.

$$t_{B_+} = \frac{2\pi}{\Omega_+ - \omega}. \tag{13.91}$$

Similarly, the coordinate $t = t_{B_-}$ of the event B_- where $\mathscr{L}_-$ encounters again the worldline of $\mathscr{O}'$ fulfils $\Omega_- t_{B_-} = \omega t_{B_-} - 2\pi$ (the $-$ sign taking count of the retrograde motion of $\mathscr{S}_-$); hence,

$$t_{B_-} = \frac{2\pi}{\omega - \Omega_-}. \tag{13.92}$$

For $\mathscr{O}'$, the proper time interval between the events B_- and B_+ is, from (13.24), $\Delta t' = \Gamma^{-1}(t_{B_+} - t_{B_-})$; hence,

$$\Delta t' = \frac{2\pi}{\Gamma}\left(\frac{1}{\Omega_+ - \omega} + \frac{1}{\Omega_- - \omega}\right). \tag{13.93}$$

With respect to the inertial observer $\mathscr{O}_*$, the velocities of $\mathscr{O}'$, of the prograde signal and the retrograde one, at the emission at A, are, respectively,

$$\vec{V} = r\omega\,\vec{n}, \quad \vec{V}_+ = r\Omega_+\,\vec{n} \quad \text{and} \quad \vec{V}_- = r\Omega_-\,\vec{n}. \tag{13.94}$$

Rather than $\vec{V}_+$ and $\vec{V}_-$, let us introduce the velocities of the two signals with respect to observer $\mathscr{O}'$, still at the instant of emission at A [14]:

$$\vec{v}_+ = v_+\,\vec{e}\,'_1 \quad \text{and} \quad \vec{v}_- = -v_-\,\vec{e}\,'_1. \tag{13.95}$$

Hence, $v_+ = \|\vec{v}_+\|_g$ and $v_- = \|\vec{v}_-\|_g$. The velocities $\vec{v}_+$ and $\vec{v}_-$ are related to $\vec{V}_+$ and $\vec{V}_-$ by the law of velocity composition specified to the case of collinear velocities [the involved velocities relative to $\mathscr{O}_*$ are all along $\vec{n}$; cf. (13.94)]. The formula to apply is then (5.45) with $V' = V_+ = r\Omega_+$, $V = v_+$ and $U' = V = r\omega$:

$$r\Omega_+ = \frac{v_+ + r\omega}{1 + r\omega v_+/c^2}. \tag{13.96}$$

[14] Let us recall that the unit vectors $\vec{n}$ and $\vec{e}\,'_1$ have been defined in Sect. 13.3.3.

Let us recall that this formula still holds if $v_+ = c$ (case of electromagnetic signals). It leads then to $r\Omega_+ = c$. Using the relation $1 - r^2\omega^2/c^2 = \Gamma^{-2}$, we deduce from (13.96) that

$$\Omega_+ - \omega = \Gamma^{-2}\left(1 + \frac{r\omega v_+}{c^2}\right)^{-1} \frac{v_+}{r}. \tag{13.97}$$

Similarly, we obtain, when replacing v_+ by $-v_-$,

$$\Omega_- - \omega = -\Gamma^{-2}\left(1 - \frac{r\omega v_-}{c^2}\right)^{-1} \frac{v_-}{r}. \tag{13.98}$$

Substituting (13.97) and (13.98) in (13.93), we get

$$\Delta t' = 2\pi\Gamma r\left(\frac{2r\omega}{c^2} + \frac{1}{v_+} - \frac{1}{v_-}\right). \tag{13.99}$$

By virtue of hypothesis (13.84), namely, the equality of the propagation velocities v_+ and v_- relative to corotating observers, it is clear that (13.99) yields the Sagnac delay (13.88).

13.6.3 Proper Travelling Time for Each Signal

The proper times T_+ and T_- elapsed for each signal between A and $B_\pm$ are given by formula (13.59), in which the terms in v can be extracted from the integral, since they are constant:

$$T_\pm = \frac{1}{v}\sqrt{1 - \frac{v^2}{c^2}} \oint_{\mathscr{C}} \Gamma \sqrt{1 - \frac{r^2\omega^2}{c^2}\sin^2\theta}\, \mathrm{d}\ell. \tag{13.100}$$

Given that the propagation velocity v has the same value for both signals, $v = v_+ = v_-$ [cf. Eq. (13.84)], and that the integral on $\mathscr{C}$ is independent of the sense of the course, we obtain

$$\boxed{T_+ = T_-}. \tag{13.101}$$

Thus, although from the point of view of the emitter and receiver of the signals ($\mathscr{O}'$), the arrival times t'_+ and t'_- of the two signals are different —separated by the Sagnac delay (13.86)—the proper travelling times are equal.

Remark 13.13. If $v \to c$ in (13.100), then $T_+ = T_- = 0$, as it should be for light signals.

Remark 13.14. In the case of circular signals, $\theta = 0$, $\Gamma = \text{const}$ and (13.100) can be recast as

$$T_+ = T_- = \frac{1}{\tilde{\Gamma}} \frac{L'}{v}, \tag{13.102}$$

where $L' = \Gamma\, 2\pi r$ is the length travelled by the two signals from the point of view of corotating observers [cf. (13.77) and (13.76)] and where $\tilde{\Gamma} := 1/\sqrt{1-(v/c)^2}$ is the Lorentz factor between $\mathscr{S}_+$ and $\mathscr{O}'$ (or equivalently between $\mathscr{S}_-$ and $\mathscr{O}'$). v being the propagation speed of the signals with respect to $\mathscr{O}'$, we observe that (13.102) is identical to the formula giving the proper travelling time of an inertial observer between events A and B, in terms of the time L'/v between A and B measured by a second inertial observer and the Lorentz factor $\tilde{\Gamma}$ between the two observers.

13.6.4 Optical Sagnac Interferometer

An ***optical Sagnac interferometer*** is a device that exhibits the Sagnac effect by means of light interferences. All the elements of the interferometer are attached to a rotating table (cf. Fig. 13.18). A source of monochromatic light sends a beam onto a semi-transparent mirror, which splits the beams in two parts: the first one propagates in the sense of rotation and the second one in the reserve sense. Thanks to various mirrors, the two beams make a complete turn before being recombined on the semi-transparent mirror. Provided that the light is sufficiently monochromatic, there appears interference fringes which are recorded on a detector. With respect to the situation where the interferometer does not rotate (with respect to the lab frame, which is assumed to be inertial), one observes a shift of the fringes that is proportional to the rotation velocity. This expresses the phase shift between the two signals resulting from the Sagnac effect.

To understand this, let us consider a component E of the electric field of the electromagnetic wave at the level of the semi-transparent mirror, the latter being identified with the corotating observer $\mathscr{O}'$ of Sect. 13.6.1. We assume that the wave in monochromatic, at the frequency f with respect to $\mathscr{O}'$:

$$E(t') = \sin(2\pi f t').$$

The "worldlines" of the nodes of the component E, i.e. the spacetime locations where E vanishes, are depicted in the spacetime diagram of Fig. 13.19. When they are back at the semi-transparent mirror, the signals are

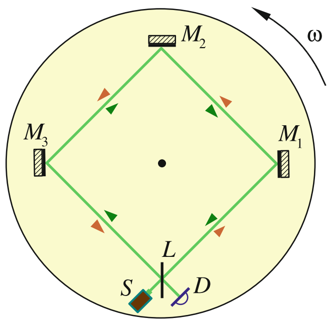


Fig. 13.18 Simplified scheme of a Sagnac interferometer: S is a monochromatic light source; L a semi-transparent mirror; M_1, M_2 and M_3 three mirrors; and D a detector monitoring the interference fringes. This scheme involves three mirrors, but their number can be arbitrary (at least larger or equal to two)

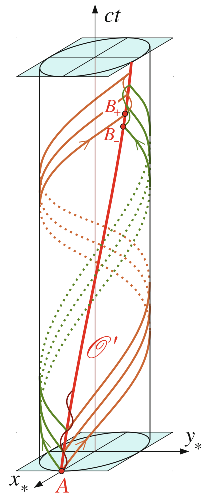


Fig. 13.19 Spacetime diagram of a Sagnac interferometer: the two signals, which have the same phase at the emission point A, present a phase shift when they are combined after one turn, because of the Sagnac delay

$$E_+(t') = \sin\left[2\pi f(t' - t'_+)\right] \tag{13.103a}$$

$$E_-(t') = \sin\left[2\pi f(t' - t'_-)\right], \tag{13.103b}$$

where t'_+ and t'_- are the dates attributed by $\mathscr{O}'$ to the events B_+ and B_-, which are the return of the node emitted at A for, respectively, the prograde signal and the retrograde one. The fact that the frequency f in (13.103) is the same than that at the emission is due to the stationary character of observer $\mathscr{O}'$. Indeed, as shown explicitly by (13.20) and (13.23), the norms of the 4-acceleration and the 4-rotation of $\mathscr{O}'$ are constant along his worldline, which implies that $\mathscr{O}'$ is a stationary observer (cf. Sect. 12.2.3). The worldlines of successive nodes in Fig. 13.19 can then be deduced one from each other by a constant time translation: the period of the nodes at the arrival is thus the same as at departure, hence the equality of the frequencies.

The phase shift between $E_+(t')$ and $E_-(t')$ read on (13.103) is $\Delta\phi = 2\pi f(t'_+ - t'_-) = 2\pi f \Delta t'$, where $\Delta t'$ is the Sagnac delay given by (13.86). We obtain thus

$$\boxed{\Delta\phi = \frac{4\pi f}{c^2 \Gamma_{(0)}} \oint_{\mathscr{C}} \Gamma^2 \vec{V} \cdot \mathrm{d}\vec{\ell}}. \tag{13.104}$$

In actual experiments, the rotation velocities are small and formula (13.89) can be used for $\Delta t'$; it involves the area $\vec{\mathscr{A}}$ delimited by the interferometer branches. There comes then

$$\boxed{\Delta\phi = \frac{8\pi f}{c^2} \vec{\omega} \cdot \vec{\mathscr{A}}.}_{r\omega \ll c} \tag{13.105}$$

In the experiment performed by Sagnac in 1913 (Sagnac 1913a,b), the light wavelength was $\lambda = c/f = 436$ nm, the rotation frequency was $\omega/(2\pi) = 2$ Hz and the area was $\mathscr{A} = 0.0866$ m^2. Formula (13.105) yields then $\Delta\phi \simeq 0.21$ rad. This value has been measured by Sagnac with a relative accuracy of 4 %.

It is worth noticing that it is not necessary to install the interferometer onto a table rotating with respect to the laboratory to exhibit the Sagnac effect: the quantity ω in (13.105) can very well be the angular velocity of the Earth with respect to an inertial frame: $\omega = \omega_\oplus \simeq 7.29 \times 10^{-5}$ rad s^{-1}. This velocity is of course much smaller than that of the Sagnac experiment, but this can be compensated by increasing the size of the interferometer, represented by the term $\vec{\mathscr{A}}$ in (13.105). This was done by Albert A. Michelson (cf. p. 125), Henry G. Gale[15] and Fred Pearson in 1925 (Michelson et al. 1925), who built in Illinois a rectangular interferometer of 613 m by 339 m, yielding $\mathscr{A} = 2.08 \times 10^5$ m^2. The Sagnac shift due to the Earth rotation in such an instrument is $\Delta\phi = 1.44$ rad ($\lambda = c/f = 570$ nm). The value measured by Michelson, Gale and Pearson agrees with it within 3%.

Remark 13.15. Formula (13.105) for the Sagnac shift is often presented with the radiation frequency f expressed in terms of the wavelength: $f = c/\lambda$; one obtains then

$$\Delta\phi = \frac{8\pi}{c\lambda} \vec{\omega} \cdot \vec{\mathscr{A}}. \tag{13.106}$$

This formula is obviously correct for an interferometer using optical beams propagating in vacuum, but it hides somewhat the relativistic origin of the Sagnac effect: the c factor in (13.106) does not stand for the propagation velocity of the waves in the interferometer. For an interferometer using other waves than light ones in vacuum, it would not be correct to replace (13.106) by $\Delta\phi = 8\pi\vec{\omega} \cdot \vec{\mathscr{A}}/(v\lambda)$, where v is the phase velocity of the waves. Setting $f = v/\lambda$ in (13.105), we see that the correct formula is actually

[15] **Henry G. Gale** (1874–1942): American astrophysicist, editor of the *Astrophysical Journal* from 1912 to 1940.

$$\Delta\phi = \frac{8\pi v}{c^2\lambda}\,\vec{\omega}\cdot\vec{\mathscr{A}}. \qquad (13.107)$$

Formula (13.105), which involves only the wave frequency, and not both their wavelength and phase velocity, is therefore more didactic.

Historical note: *The Sagnac effect for light waves has been predicted before the advent of relativity, on the basis of the aether theory. We have stressed in Sect. 13.6.1 that the Sagnac effect does not exist in Newtonian theory [Eq. (13.87)]. But this result relies on the hypothesis (13.84), namely, the equality of the propagation velocities of the two beams* relative to the rotating emitter. *Now in the theory of electromagnetism based on the aether, the velocities of the beams* relative to the aether*, and not to the emitter, must be identical, each of them being equal to c: using the notations of Sect. 13.6.2, we must have in this framework*

$$r\Omega_+ = -r\Omega_- = c \quad \textit{(aether).}$$

The Galilean law of composition of velocities yields then

$$v_+ = c - r\omega \quad \textit{and} \quad v_- = -c - r\omega \quad \textit{(aether),}$$

instead of (13.84). Substituting these relations into the nonrelativistic limit ($\Gamma \to 1$, $r\omega/c \to 0$) of the general formula (13.99), one obtains

$$\Delta t' = \frac{4\pi r^2\omega}{c^2} \quad \textit{(aether).} \qquad (13.108)$$

Thus, one ends with the same formula as the relativistic prediction (13.89) with $\mathscr{A} = \pi r^2$. The aether-based prediction (13.108) has been established by Oliver J. Lodge[16] in 1893 for an interferometer dragged along by the Earth rotation (Lodge 1893) and four years later for an interferometer on a rotating table (Lodge 1897) (cf. Anderson et al. (1994)). The prediction has also performed by Albert A. Michelson (cf. p. 125) in 1904 (Michelson 1904) and by Georges Sagnac (cf. p. 458) in 1911 (Sagnac 1911), still in the framework of the aether theory.

The Sagnac effect has been first observed in 1911 by a young German scientist, Franz Harress, while working for his thesis (Harress 1912). He was trying to measure the dispersion properties of various types of glass by means of a rotating annular interferometer. Actually, Harress could not explain the observed displacement of the fringes for he was explicitly supposing that the Sagnac effect was not existing (cf. Post (1967)). This is thus Georges Sagnac who, in 1913 (Sagnac 1913a,b), both measured the effect and interpreted it in view of formula (13.108).

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[16]**Oliver J. Lodge** (1851–1940): British physicist and writer, who performed important studies in electromagnetism, notably on wireless telegraphy; he also invented a type of spark ignition for internal combustion engines (the so-called *Lodge Igniter*).

The experiment has been described succinctly above. Note that Sagnac presented his result as a demonstration of the existence of aether! In 1914, Paul Harzer[17] *(1914) rediscussed the results from Harress experiment and showed that they stemmed from the Sagnac effect, with an accuracy even better than that obtained by Sagnac himself. Even if the optical Sagnac effect can be derived within the theory of aether [cf. Eq. (13.108)], it was of course logically incorrect to conclude, as Sagnac did, that the effect demonstrates the existence of aether. Before the first observations, Max Laue (cf. p. 146) had shown in 1911 (Laue 1911a) that relativity also predicts the Sagnac effect. The demonstration of the Sagnac effect in relativity has been reformulated by Laue himself in 1920 (von Laue 1920) (at first order in v/c only) and by Paul Langevin (cf. p. 40) in 1921 (Langevin 1921) and 1937 (Langevin 1937). For more details about the history of Sagnac effect, the reader is referred to Anderson et al. (1994), Malykin (2000) and Post (1967).*

13.6.5 Matter-Wave Sagnac Interferometer

We have already underlined that the Sagnac effect is independent of the nature of the signal and in particular of its velocity of propagation. Sagnac experiments can thus be realized by interferometers involving matter waves, instead of light waves. By ***matter waves***, it is meant de Broglie waves, i.e. waves arising from the quantum nature of massive particles, via the de Broglie wavelength. The frequency f that appears in (13.104) is then related to the energy of the particles with respect to observer $\mathscr{O}'$ by the Planck–Einstein formula $E = hf$, where h is Planck constant [Eq. (9.24)]. The energy being expressed in terms of the mass m of the particles and their Lorentz factor $\Gamma_{\rm p}$ relative to $\mathscr{O}'$ according to $E = \Gamma_{\rm p} mc^2$ [Eq. (9.16)], (13.104) yields

$$\boxed{\Delta\phi = 2\,\frac{\Gamma_{\rm p}}{\Gamma_{(0)}}\frac{m}{\hbar}\oint_{\mathscr{C}} \Gamma^2\,\vec{V}\cdot \mathrm{d}\vec{\ell}}\,, \tag{13.109}$$

where $\hbar := h/(2\pi)$ is the reduced Planck constant [cf. (9.25)]. In practice, the rotation velocities are small and (13.105) leads to

$$\boxed{\Delta\phi = 4\Gamma_{\rm p}\frac{m}{\hbar}\,\vec{\omega}\cdot\vec{\mathscr{A}}.}_{\, r\omega\ll c} \tag{13.110}$$

Of course, a rigorous derivation of (13.109) or (13.110) requires a quantum mechanical computation, based on the Schrödinger equation (nonrelativistic case)

[17] **Paul Harzer** (1857–1932): German astronomer working at Kiel Observatory.

or the Dirac equation (relativistic case). We refer the reader to Bordé (1991); Bordé et al. (2000).

Remark 13.16. In practice, the matter-wave interferometers are often of the Bordé-Ramsey type, with a geometry equivalent to a Mach–Zehnder optical interferometer (cf. Le Bellac (2006)): the two signals are recombined after a half turn, instead of a complete one. Consequently, the phase shift is the half of that given by (13.110): $\Delta\phi = 2\Gamma_{\mathrm{p}}(m/\hbar)\,\vec{\omega}\cdot\vec{\mathscr{A}}$.

Let us compare the Sagnac shifts between a matter-wave interferometer and an optical one, in laboratory conditions, i.e. for nonrelativistic rotation velocities. The ratio of formulas (13.110) and (13.105) yields, for the same rotation velocity ω and the same interferometer area $\mathscr{A}$,

$$\frac{\Delta\phi_{\mathrm{mat}}}{\Delta\phi_{\mathrm{opt}}} = \Gamma_{\mathrm{p}}\frac{mc^2}{hf} \sim 4\times 10^8, \tag{13.111}$$

where the numerical value has been evaluated for a proton ($mc^2 \sim 0.9\,\mathrm{GeV}$) with $\Gamma_{\mathrm{p}} \sim 1$ and visible radiation ($hf \sim 2\,\mathrm{eV}$). This result shows that to exhibit the Sagnac effect, it is a priori much more favourable to use a matter-wave interferometer than an optical one. However, it must be mentioned that it is easier to construct an optical interferometer of large size and the signal-to-noise ratio of an optical interferometer is usually much larger than that of a matter-wave interferometer.

The first measure of the Sagnac effect by a matter-wave interferometer was obtained in 1965 by James E. Zimmerman (1923–1999) and James E. Mercereau (1965) with superconducting electrons. Since then, numerous experiments using electrons, neutrons or atoms have been performed. The principal of them are listed in Table 13.1. Note that five of them have measured the Sagnac effect due to the Earth rotation. They can therefore be considered as matter-wave equivalents to the experiment of Michelson, Gale and Pearson mentioned in Sect. 13.6.4.

Historical note: *It is Fernand Prunier who underlined in 1935 (Prunier 1935) that the Sagnac effect could be obtained with material particles (he gave the example of electrons), rather than with light, and that in this case, this would be a confirmation of relativity, since the aether theory predicts no effect for particles other than photons.*

13.6.6 Application: Gyrometers

The Sagnac effect is a relativistic feature that has a great practical value since it allows one to devise high-precision ***gyrometers***, i.e. devices for measuring the angular velocity. With the accelerometer, the gyrometer is the key element of a ***inertial navigation system***, notably for aircraft navigation.

Table 13.1 Measure of the Sagnac effect with matter-wave interferometers. $\omega_\oplus$ stands for the angular velocity of the Earth with respect to an inertial frame: $\omega_\oplus = 7.29 \times 10^{-5}$ rad s^{-1}

Experiment	Type of matter	ω [rad s^{-1}]	Relative difference with the theoretical $\Delta\phi$
Zimmerman and Mercereau (1965)	Superconducting electrons	10	5 %
Hasselbach and Nicklaus (1993)	Free electrons	3	30 %
Atwood et al. (1984)	Neutrons	7×10^{-4}	0.4 %
Werner et al. (1979)	Neutrons	$\omega_\oplus$	0.4 %
Riehle et al. (1991)	Ca atoms	0.1	20 %
Lenef et al. (1997)	Na atoms	$\omega_\oplus$	1 %
Gustavson et al. (1997)	Cs atoms	$\omega_\oplus$	2 %
Gustavson et al. (2000)	Cs atoms	$\omega_\oplus$	1 %
Canuel et al. (2007; 2006)	Cold Cs atoms	$\omega_\oplus$	1 %

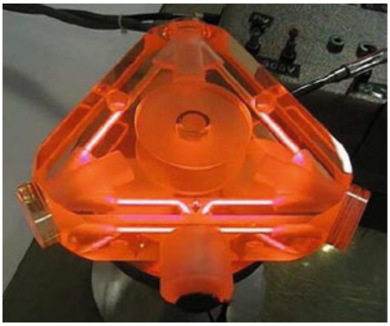

Fig. 13.20 Helium-neon gyrolaser elaborated by Thales Aerospace [Source: Sylvain Schwartz (2006) and Thales Aerospace]

Modern gyrometers are no longer based on mechanical gyroscopes (spinning tops) but on optical Sagnac interferometers. The latter ones have indeed the advantages of not involving moving mechanical pieces (hence have no friction) and of being light and compact. The matter-wave Sagnac interferometers are not in the industrial stage yet because those based on neutrons require a particle accelerator and those based on atoms require very low temperatures.

There exists two types of gyrometers based on the optical Sagnac effect:

- ***Optical-fibre gyrometer***: the effective area of the interferometer is substantially increased by propagating the light beam in an optical fibre that is wound many times (the total length of the fibre can reach a few kilometres!). The phase shift (13.105) is then multiplied by the number N of loops of the optical fibre, which makes the measure of ω much easier (cf. Arditty and Lefevre (1981) or Appendix D of Holleville (2001) for more details).

- ***Ring-laser gyrometer***, also called ***gyrolaser***: it is based on an annular laser cavity (cf. Fig. 13.20), involving usually a helium-neon laser. The Sagnac shift (13.105) induces a difference of the frequencies of the cavity proper modes, and it is this frequency difference that is measured (cf. Chow et al. (1985) and Stedman (1997) for more details).

Chapter 14
Tensors and Alternate Forms

14.1 Introduction

After Chaps. 1, 6 and 7, here is again a purely mathematical chapter. Up to now, the mathematical objects introduced on Minkowski spacetime are:

- *Vectors* (for example, the 4-velocity of a particle)
- *Linear forms* (for example, the 4-momentum of a particle)
- *Bilinear forms*, either symmetric (the metric tensor $\boldsymbol{g}$) or antisymmetric (the angular momentum of a particle)
- *Trilinear forms* (the mixed product $\boldsymbol{\varepsilon_u}$ acting on three vectors in the local rest space of an observer)
- A *four-linear form*: the Levi–Civita tensor $\boldsymbol{\varepsilon}$
- Linear maps $E \to E$ *(endomorphisms)* (for example, Lorentz transformations)

All these objects belong actually to the same family: that of *tensors*. We shall describe them in detail (Sect. 14.2), as well as operations on them (Sect. 14.3). Then we shall focus on a subfamily that is very important for physics: the subfamily of *fully antisymmetric forms*, also called *alternate forms* (Sect. 14.4); we shall present an operation specific to these forms: the *Hodge star* (Sect. 14.5), which will turn to be useful in the next chapters.

14.2 Tensors: Definition and Examples

14.2.1 Definition

Let us first recall that E^* stands for the set of all linear forms on the vector space E underlying Minkowski spacetime; E^* is the dual space of E (cf. Sect. 1.6); in particular, E^* is a four-dimensional vector space on $\mathbb{R}$ (as E).

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For $(k,\ell) \in \mathbb{N}^2$ and $(k,\ell) \neq (0,0)$, a ***tensor of type*** (k,ℓ) is a map

$$\boxed{\begin{array}{rcl} \boldsymbol{T}: \underbrace{E^* \times \cdots \times E^*}_{k \text{ times}} \times \underbrace{E \times \cdots \times E}_{\ell \text{ times}} & \longrightarrow & \mathbb{R} \\ (\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_k, \vec{\boldsymbol{v}}_1, \ldots, \vec{\boldsymbol{v}}_\ell) & \longmapsto & \boldsymbol{T}(\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_k, \vec{\boldsymbol{v}}_1, \ldots, \vec{\boldsymbol{v}}_\ell) \end{array}} \tag{14.1}$$

that is linear with respect to each of its arguments; for $\lambda \in \mathbb{R}$, $m \in \{0,\ldots,k\}$ and $n \in \{0,\ldots,\ell\}$, the following identities hold:

$$\begin{aligned} \boldsymbol{T}(\boldsymbol{\omega}_1, \ldots, \lambda\boldsymbol{\omega}_m + \boldsymbol{\omega}'_m, \ldots, \boldsymbol{\omega}_k, \vec{\boldsymbol{v}}_1, \ldots, \vec{\boldsymbol{v}}_\ell) &= \lambda \boldsymbol{T}(\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_m, \ldots, \boldsymbol{\omega}_k, \vec{\boldsymbol{v}}_1, \ldots, \vec{\boldsymbol{v}}_\ell) \\ &\quad + \boldsymbol{T}(\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}'_m, \ldots, \boldsymbol{\omega}_k, \vec{\boldsymbol{v}}_1, \ldots, \vec{\boldsymbol{v}}_\ell) \\ \boldsymbol{T}(\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_k, \vec{\boldsymbol{v}}_1, \ldots, \lambda\vec{\boldsymbol{v}}_n + \vec{\boldsymbol{v}}'_n, \ldots, \vec{\boldsymbol{v}}_\ell) &= \lambda \boldsymbol{T}(\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_k, \vec{\boldsymbol{v}}_1, \ldots, \vec{\boldsymbol{v}}_n, \ldots, \vec{\boldsymbol{v}}_\ell) \\ &\quad + \boldsymbol{T}(\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_k, \vec{\boldsymbol{v}}_1, \ldots, \vec{\boldsymbol{v}}'_n, \ldots, \vec{\boldsymbol{v}}_\ell). \end{aligned}$$

The integer $k+\ell$ is called ***valence***, or ***order***, or even ***rank***, of the tensor. One says also that $\boldsymbol{T}$ is a ***tensor*** k ***times contravariant and*** ℓ ***times covariant***.

We shall denote by $\mathscr{T}_{(k,\ell)}(E)$ the set of all tensors of type (k,ℓ) on E. This is a vector space over $\mathbb{R}$, for any linear combination of two tensors of type (k,ℓ) is a tensor of the same type. The dimension of $\mathscr{T}_{(k,\ell)}(E)$ is $4^{k+\ell}$.

By convention, we shall denote by $\mathscr{T}_{(0,0)}(E)$ the base field of the vector space E, namely, $\mathbb{R}$:

$$\mathscr{T}_{(0,0)}(E) = \mathbb{R}. \tag{14.2}$$

This convention allows one to extend the validity of certain formulas involving $\mathscr{T}_{(k,\ell)}(E)$ to the case $(k,\ell) = (0,0)$.

14.2.2 Tensors Already Met

According to the above definition, a linear form is a tensor of type $(0,1)$ and a bilinear form is a tensor of type $(0,2)$. In particular, the metric tensor $\boldsymbol{g}$ is a tensor of type $(0,2)$. On its side, the Levi–Civita tensor $\boldsymbol{\varepsilon}$ introduced in Sect. 1.5 is a tensor of type $(0,4)$.

Since E^* is a vector space, we may consider its dual space, E^{**}. The latter is canonically identified with E: indeed any vector of E can be considered as a linear form on E^*, according to

$$\begin{aligned} \vec{\boldsymbol{v}} : E^* &\longrightarrow \mathbb{R} \\ \boldsymbol{\omega} &\longmapsto \langle \boldsymbol{\omega}, \vec{\boldsymbol{v}} \rangle. \end{aligned} \tag{14.3}$$

Comparing (14.3) with (14.1), we may say that a vector is a tensor of type $(1, 0)$.

Besides, we can identify any endomorphism $\boldsymbol{L} : E \to E$ with a tensor of type $(1, 1)$, according to

$$\begin{aligned} \boldsymbol{L} : E^* \times E &\longrightarrow \mathbb{R} \\ (\boldsymbol{\omega}, \vec{\boldsymbol{v}}) &\longmapsto \langle \boldsymbol{\omega}, \boldsymbol{L}(\vec{\boldsymbol{v}}) \rangle. \end{aligned} \tag{14.4}$$

Since $\boldsymbol{L}$ is linear, the above map does define a tensor. The starting space being $E^* \times E$, it is a tensor of type $(1, 1)$. We shall use the same symbol (here $\boldsymbol{L}$) to denote the endomorphism or the tensor associated via (14.4).

In particular, Lorentz transformations (Chap. 6) or the members of the Lie algebra of the Lorentz group (Chap. 7) are tensors of type $(1, 1)$.

The above identifications allow one to write

$$\mathscr{T}_{(0,1)}(E) = E^*, \quad \mathscr{T}_{(1,0)}(E) = E \quad \text{and} \quad \mathscr{T}_{(1,1)}(E) = \mathscr{L}(E), \tag{14.5}$$

where $\mathscr{L}(E)$ stands for the space of endomorphisms on E (cf. Sect. 7.3.1).

14.3 Operations on Tensors

14.3.1 Tensor Product

Given a tensor $\boldsymbol{A}$ of type (k, ℓ) and a tensor $\boldsymbol{B}$ of type (m, n), one calls ***tensor product of A by B***, the tensor of type $(k + m, \ell + n)$ denoted by $\boldsymbol{A} \otimes \boldsymbol{B}$ and defined by

$$\begin{aligned} \underbrace{E^* \times \cdots \times E^*}_{k+m \text{ times}} \times \underbrace{E \times \cdots \times E}_{\ell+n \text{ times}} &\longrightarrow \mathbb{R} \\ (\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_{k+m}, \vec{\boldsymbol{v}}_1, \ldots, \vec{\boldsymbol{v}}_{\ell+n}) &\longmapsto \boldsymbol{A}(\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_k, \vec{\boldsymbol{v}}_1, \ldots, \vec{\boldsymbol{v}}_\ell) \\ &\qquad \times \boldsymbol{B}(\boldsymbol{\omega}_{k+1}, \ldots, \boldsymbol{\omega}_{k+m}, \vec{\boldsymbol{v}}_{\ell+1}, \ldots, \vec{\boldsymbol{v}}_{\ell+n}), \end{aligned} \tag{14.6}$$

where $\times$ stands for the multiplication within $\mathbb{R}$. This definition generalizes that introduced in Sect. 3.5.2 for the tensor product of two linear forms $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$: $\boldsymbol{\omega}_1 \otimes \boldsymbol{\omega}_2$ is a tensor of type $(0, 2)$ (hence a bilinear form) defined by

$$\forall(\vec{v}, \vec{w}) \in E^2, \quad \boldsymbol{\omega}_1 \otimes \boldsymbol{\omega}_2 (\vec{v}, \vec{w}) = \langle \boldsymbol{\omega}_1, \vec{v} \rangle \langle \boldsymbol{\omega}_2, \vec{w} \rangle. \tag{14.7}$$

It is clear that the tensor product is associative:

$$\boldsymbol{A} \otimes (\boldsymbol{B} \otimes \boldsymbol{C}) = (\boldsymbol{A} \otimes \boldsymbol{B}) \otimes \boldsymbol{C}. \tag{14.8}$$

14.3.2 Components in a Vector Basis

Let $(\vec{e}_\alpha)$ be a basis of E (not necessarily orthonormal). We have seen in Sect. 1.6.1 that there exists a unique basis of E^* canonically associated with $(\vec{e}_\alpha)$: the dual basis $(\boldsymbol{e}^\alpha)$, which satisfies

$$\langle \boldsymbol{e}^\alpha, \vec{e}_\beta \rangle = \delta^\alpha{}_\beta. \tag{14.9}$$

From definition (14.6), the tensor product $\vec{e}_{\alpha_1} \otimes \ldots \otimes \vec{e}_{\alpha_k} \otimes \boldsymbol{e}^{\beta_1} \otimes \ldots \otimes \boldsymbol{e}^{\beta_\ell}$ is then the tensor of type (k, ℓ) by which the image of $(\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_k, \vec{v}_1, \ldots, \vec{v}_\ell)$ is the real number:

$$\prod_{p=1}^{k} \langle \boldsymbol{\omega}_p, \vec{e}_{\alpha_p} \rangle \times \prod_{q=1}^{\ell} \langle \boldsymbol{e}^{\beta_q}, \vec{v}_q \rangle.$$

One can expand any tensor $\boldsymbol{T}$ of type (k, ℓ) according to

$$\boxed{\boldsymbol{T} = T^{\alpha_1 \ldots \alpha_k}{}_{\beta_1 \ldots \beta_\ell}\, \vec{e}_{\alpha_1} \otimes \cdots \otimes \vec{e}_{\alpha_k} \otimes \boldsymbol{e}^{\beta_1} \otimes \cdots \otimes \boldsymbol{e}^{\beta_\ell}}. \tag{14.10}$$

The $4^{k+\ell}$ real numbers $T^{\alpha_1 \ldots \alpha_k}{}_{\beta_1 \ldots \beta_\ell}$ are called the ***components of the tensor $\boldsymbol{T}$ in the basis*** $(\vec{e}_\alpha)$. An index of the type α_p is qualified as ***contravariant*** and an index of the type β_q is qualified as ***covariant***.

Example 14.1. For a vector $\vec{v} \in E$ [tensor of type $(1, 0)$], (14.10) gives the standard definition of the components in a basis:

$$\vec{v} = v^\alpha \, \vec{e}_\alpha. \tag{14.11}$$

Example 14.2. For a linear form $\boldsymbol{\omega} \in E^*$ [tensor of type $(0, 1)$], we also recover the definition (1.39) of the components:

$$\boldsymbol{\omega} = \omega_\alpha \, \boldsymbol{e}^\alpha. \tag{14.12}$$

By virtue of (14.9) and the multilinearity of $\boldsymbol{T}$, the components can be expressed in terms of the basis vectors $(\vec{e}_\alpha)$ and the dual-basis forms $(\boldsymbol{e}^\alpha)$ according to

$$\boxed{T^{\alpha_1\ldots\alpha_k}{}_{\beta_1\ldots\beta_\ell} = \boldsymbol{T}(\boldsymbol{e}^{\alpha_1},\ldots,\boldsymbol{e}^{\alpha_k},\vec{\boldsymbol{e}}_{\beta_1},\ldots,\vec{\boldsymbol{e}}_{\beta_\ell})}. \tag{14.13}$$

Example 14.3. For a linear form, (14.13) reduces to

$$\omega_\alpha = \langle \boldsymbol{\omega}, \vec{\boldsymbol{e}}_\alpha \rangle. \tag{14.14}$$

Remark 14.1. In view of (14.11) and (14.12), the index labelling the components of a vector is contravariant and that labelling the components of a linear form is covariant.

The combination of (14.11), (14.12) and (14.9) shows that the action of the linear form $\boldsymbol{\omega}$ on the vector $\vec{\boldsymbol{v}}$ takes a simple expression in terms of the components:

$$\boxed{\langle \boldsymbol{\omega}, \vec{\boldsymbol{v}} \rangle = \omega_\alpha v^\alpha}. \tag{14.15}$$

We thus recover Eq. (1.40) of Chap. 1. More generally, for a tensor of type (k,ℓ), we have

$$\boxed{\boldsymbol{T}(\boldsymbol{\omega}_1,\ldots,\boldsymbol{\omega}_k,\vec{\boldsymbol{v}}_1,\ldots,\vec{\boldsymbol{v}}_\ell) = T^{\alpha_1\ldots\alpha_k}{}_{\beta_1\ldots\beta_\ell}(\omega_1)_{\alpha_1}\ldots(\omega_k)_{\alpha_k} v_1^{\beta_1}\ldots v_\ell^{\beta_\ell}}, \tag{14.16}$$

where the $(\omega_p)_\alpha$'s are the components of the linear form $\boldsymbol{\omega}_p$ $(1 \leq p \leq k)$ and the v_q^β's the components of the vector $\vec{\boldsymbol{v}}_q$ $(1 \leq q \leq \ell)$.

For the metric tensor $\boldsymbol{g}$ [tensor of type $(0,2)$], the components $g_{\alpha\beta}$ are nothing but the elements of the matrix of $\boldsymbol{g}$ with respect to the basis $(\vec{\boldsymbol{e}}_\alpha)$ as defined in Sect. 1.3.2. Indeed, for $\boldsymbol{g}$, the expansion (14.10) is

$$\boxed{\boldsymbol{g} = g_{\alpha\beta}\, \boldsymbol{e}^\alpha \otimes \boldsymbol{e}^\beta}, \tag{14.17}$$

with, according to (14.13), $g_{\alpha\beta} = \boldsymbol{g}(\vec{\boldsymbol{e}}_\alpha, \vec{\boldsymbol{e}}_\beta)$. We recover thus relation (1.12) defining $g_{\alpha\beta}$. Moreover, formula (14.16) for $\boldsymbol{T} = \boldsymbol{g}$ coincides with (1.13): $\boldsymbol{g}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}) = g_{\alpha\beta}\, u^\alpha v^\beta$.

14.3.3 *Change of Basis*

Let us examine the transformation of the components of a tensor under a change of basis. Let $(\vec{\boldsymbol{e}}_\alpha)$ and $(\vec{\boldsymbol{e}}'_\alpha)$ be two bases of E and P be the ***change-of-basis matrix***, i.e. the matrix defined by

$$\boxed{\vec{\boldsymbol{e}}'_\alpha = P^\beta{}_\alpha\, \vec{\boldsymbol{e}}_\beta}. \tag{14.18}$$

Example 14.4. If $(\vec{e}_\alpha)$ and $(\vec{e}\,'_\alpha)$ are the frames of two inertial observers, P is the matrix of a restricted Lorentz transformation. Using the notations of Sect. 8.3.1, $P = \Lambda^{-1}$ [cf. Eq. (8.11)].

It is easy to see that the change-of-basis matrix between the dual bases $(\boldsymbol{e}^\alpha)$ and $(\boldsymbol{e}'^\alpha)$ is nothing but the transpose of the inverse of P:

$$\boxed{\boldsymbol{e}'^\alpha = (P^{-1})^\alpha{}_\beta\, \boldsymbol{e}^\beta}. \tag{14.19}$$

Proof. Let us consider (14.19) as the definition of $\boldsymbol{e}'^\alpha$ and compute the action of this linear form on the vector $\vec{e}\,'_\beta$, by means of (14.18):

$$\langle \boldsymbol{e}'^\alpha, \vec{e}\,'_\beta\rangle = \langle (P^{-1})^\alpha{}_\mu\, \boldsymbol{e}^\mu,\ P^\nu{}_\beta\, \vec{e}_\nu\rangle = (P^{-1})^\alpha{}_\mu P^\nu{}_\beta \underbrace{\langle \boldsymbol{e}^\mu, \vec{e}_\nu\rangle}_{\delta^\mu{}_\nu} = (P^{-1})^\alpha{}_\mu P^\mu{}_\beta = \delta^\alpha{}_\beta.$$

This shows that $(\boldsymbol{e}'^\alpha)$ is indeed the dual basis of $(\vec{e}\,'_\alpha)$. □

The components $T'^{\cdots}_{\ \cdots}$ of a tensor $\boldsymbol{T}$ of type (k, ℓ) in the basis $(\vec{e}\,'_\alpha)$ are given by (14.13):

$$T'^{\alpha_1\ldots\alpha_k}{}_{\beta_1\ldots\beta_\ell} = \boldsymbol{T}(\boldsymbol{e}'^{\alpha_1}, \ldots, \boldsymbol{e}'^{\alpha_k}, \vec{e}\,'_{\beta_1}, \ldots, \vec{e}\,'_{\beta_\ell}).$$

Replacing each $\boldsymbol{e}'^{\alpha_p}$ by (14.19) and each $\vec{e}\,'_{\beta_q}$ by (14.18), using the multilinearity of $\boldsymbol{T}$, and then (14.13), we find the relation between the two sets of components:

$$\boxed{T'^{\alpha_1\ldots\alpha_k}{}_{\beta_1\ldots\beta_\ell} = (P^{-1})^{\alpha_1}{}_{\mu_1} \cdots (P^{-1})^{\alpha_k}{}_{\mu_k} P^{\nu_1}{}_{\beta_1} \cdots P^{\nu_\ell}{}_{\beta_\ell}\, T^{\mu_1\ldots\mu_k}{}_{\nu_1\ldots\nu_\ell}}. \tag{14.20}$$

Remark 14.2. In rather old books, a *tensor* is not defined as a multilinear map of the kind (14.1), but as an "array of numbers" $T^{\alpha_1\ldots\alpha_k}{}_{\beta_1\ldots\beta_\ell}$ that transforms according to the law (14.20) under a change of basis.

Remark 14.3. The notation $T^{\alpha'_1\ldots\alpha'_k}{}_{\beta'_1\ldots\beta'_\ell}$ is often used instead of $T'^{\alpha_1\ldots\alpha_k}{}_{\beta_1\ldots\beta_\ell}$, i.e. the prime is put of the indices rather than on the letter denoting the tensor. We shall not use such a notation here.

Example 14.5. For a vector $\vec{v} = v^\alpha\, \vec{e}_\alpha = v'^\alpha\, \vec{e}\,'_\alpha$, (14.20) reduces to

$$v'^\alpha = (P^{-1})^\alpha{}_\beta v^\beta. \tag{14.21}$$

Example 14.6. For a linear form $\boldsymbol{\omega} = \omega_\alpha\, \boldsymbol{e}^\alpha = \omega'_\alpha\, \boldsymbol{e}'^\alpha$, (14.20) reduces to

$$\omega'_\alpha = P^\beta{}_\alpha \omega_\beta. \tag{14.22}$$

Example 14.7. For a bilinear form $\boldsymbol{T} = T_{\alpha\beta}\, \boldsymbol{e}^\alpha \otimes \boldsymbol{e}^\beta = T'_{\alpha\beta}\, \boldsymbol{e}'^\alpha \otimes \boldsymbol{e}'^\beta$, (14.20) gives

$$T'_{\alpha\beta} = P^{\mu}{}_{\alpha} P^{\nu}{}_{\beta} T_{\mu\nu} = P^{\mu}{}_{\alpha} T_{\mu\nu} P^{\nu}{}_{\beta}, \tag{14.23}$$

which can be written in matrix form as

$$T' = {}^{\mathrm{t}}P\, T\, P. \tag{14.24}$$

We recognize the standard change-of-basis law for the matrix of a bilinear form.

Example 14.8. For a tensor of type (1, 1), $\boldsymbol{T} = T^{\alpha}{}_{\beta}\, \vec{\boldsymbol{e}}_{\alpha} \otimes \boldsymbol{e}^{\beta} = T'^{\alpha}{}_{\beta}\, \vec{\boldsymbol{e}}'_{\alpha} \otimes \boldsymbol{e}'^{\beta}$, (14.20) gives

$$T'^{\alpha}{}_{\beta} = (P^{-1})^{\alpha}{}_{\mu} P^{\nu}{}_{\beta} T^{\mu}{}_{\nu} = (P^{-1})^{\alpha}{}_{\mu} T^{\mu}{}_{\nu} P^{\nu}{}_{\beta}, \tag{14.25}$$

which can be written in matrix form as

$$T' = P^{-1}\, T\, P. \tag{14.26}$$

We recognize the standard change-of-basis law for the matrix of an endomorphism.

14.3.4 Components and Metric Duality

As we have seen in Sect. 1.6.2, the metric duality consists in associating with any vector $\vec{\boldsymbol{v}} \in E$ a unique linear form $\underline{\boldsymbol{v}} \in E^*$ via $\forall \vec{\boldsymbol{w}} \in E$, $\langle \underline{\boldsymbol{v}}, \vec{\boldsymbol{w}} \rangle = \boldsymbol{g}(\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}})$. Let us denote by (v_{α}) the components of $\underline{\boldsymbol{v}}$ in a basis $(\vec{\boldsymbol{e}}_{\alpha})$ [cf. (14.12)] and (v^{α}) those of $\vec{\boldsymbol{v}}$:

$$\underline{\boldsymbol{v}} = v_{\alpha}\, \boldsymbol{e}^{\alpha} \qquad \text{and} \qquad \vec{\boldsymbol{v}} = v^{\alpha}\, \vec{\boldsymbol{e}}_{\alpha}. \tag{14.27}$$

We have then

$$\forall \vec{\boldsymbol{w}} \in E, \quad \langle \underline{\boldsymbol{v}}, \vec{\boldsymbol{w}} \rangle = \boldsymbol{g}(\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}) = g_{\alpha\beta}\, v^{\alpha} w^{\beta} = (g_{\alpha\beta}\, v^{\beta}) w^{\alpha}.$$

Comparing with (14.15), we get

$$\boxed{v_{\alpha} = g_{\alpha\beta}\, v^{\beta}}. \tag{14.28}$$

Conversely, let $\vec{\boldsymbol{\omega}} \in E$ be the vector associated with the linear form $\boldsymbol{\omega} \in E^*$ by metric duality. The components (ω^{α}) of $\vec{\boldsymbol{\omega}}$ in the basis $(\vec{\boldsymbol{e}}_{\alpha})$ are given by (1.43):

$$\boxed{\omega^{\alpha} = g^{\alpha\beta}\, \omega_{\beta}}, \tag{14.29}$$

where $(g^{\alpha\beta})$ is the inverse matrix of $(g_{\alpha\beta})$ (cf. Sect. 1.3.2).

Remark 14.4. The $g^{\alpha\beta}$'s can be considered as the components of a tensor of type $(2, 0)$:

$$\boldsymbol{g}^{-1} := g^{\alpha\beta}\, \vec{\boldsymbol{e}}_\alpha \otimes \vec{\boldsymbol{e}}_\beta. \tag{14.30}$$

$\boldsymbol{g}^{-1}$ is called the ***inverse metric***.

In view of (14.28) and (14.29), the metric duality is often expressed by saying that indices are ***lowered*** by means of $g_{\alpha\beta}$ and ***raised*** by means of $g^{\alpha\beta}$.

14.3.5 Contraction

The operation of contraction is a linear map $\mathscr{T}_{(k,\ell)}(E) \to \mathscr{T}_{(k-1,\ell-1)}(E)$ defined as follows. Given a tensor $\boldsymbol{T}$ of type (k, ℓ) with $k \geq 1$ and $\ell \geq 1$, and two integers $p \in \{1, \dots, k\}$ and $q \in \{1, \dots, \ell\}$, the ***contraction of $\boldsymbol{T}$ on the indices of rank p and q*** is the tensor $C^p_q \boldsymbol{T}$ of type $(k-1, \ell-1)$ defined by

$$\begin{aligned}
&\forall (\boldsymbol{\omega}_1, \dots, \boldsymbol{\omega}_{k-1}, \vec{\boldsymbol{v}}_1, \dots, \vec{\boldsymbol{v}}_{\ell-1}) \in (E^*)^{k-1} \times E^{\ell-1}, \\
&C^p_q \boldsymbol{T}(\boldsymbol{\omega}_1, \dots, \boldsymbol{\omega}_{k-1}, \vec{\boldsymbol{v}}_1, \dots, \vec{\boldsymbol{v}}_{\ell-1}) \\
&\qquad := \boldsymbol{T}(\boldsymbol{\omega}_1, \dots, \boldsymbol{e}^\alpha, \dots, \boldsymbol{\omega}_{k-1}, \vec{\boldsymbol{v}}_1, \dots, \vec{\boldsymbol{e}}_\alpha, \dots, \vec{\boldsymbol{v}}_{\ell-1}),
\end{aligned} \tag{14.31}$$

where $(\vec{\boldsymbol{e}}_\alpha)$ stands for a basis of E, $(\boldsymbol{e}^\alpha)$ for its dual basis, $\boldsymbol{e}^\alpha$ is located at the p^{th} position among the linear-form arguments of $\boldsymbol{T}$, $\vec{\boldsymbol{e}}_\alpha$ is located at the q^{th} position among the vector arguments and the summation is taking place on the index α. Because of (14.18)-(14.19) and the linearity of $\boldsymbol{T}$ with respect to each of its arguments, it is clear that the above definition does not depend upon the choice of the basis $(\vec{\boldsymbol{e}}_\alpha)$.

The components of $C^p_q \boldsymbol{T}$ are deduced from those of $\boldsymbol{T}$ via the formula

$$\boxed{(C^p_q T)^{\alpha_1 \dots \alpha_{k-1}}{}_{\beta_1 \dots \beta_{\ell-1}} = T^{\alpha_1 \dots \overset{\overset{p}{\downarrow}}{\mu} \dots \alpha_{k-1}}{}_{\beta_1 \dots \underset{\underset{q}{\uparrow}}{\mu} \dots \beta_{\ell-1}}}, \tag{14.32}$$

where the arrows indicate the position of the summation index μ.

Example 14.9. For a tensor $\boldsymbol{T}$ of type $(1, 1)$, there is only one possible contraction: $p = q = 1$; $C^1_1 \boldsymbol{T}$ is then a tensor of type $(0, 0)$, i.e. a real number, according to the convention (14.2):

$$C^1_1 \boldsymbol{T} = T^\mu{}_\mu. \tag{14.33}$$

If $\boldsymbol{T}$ is considered as an endomorphism of E [cf. (14.5)], we note that the contraction yields nothing but the ***trace of*** $\boldsymbol{T}$. A particular case of tensor of type $(1, 1)$ is the tensor product of a vector by a linear form: $\boldsymbol{T} = \vec{\boldsymbol{v}} \otimes \boldsymbol{\omega}$. We have then, according to (14.33) and (14.15),

$$C_1^1(\vec{\boldsymbol{v}} \otimes \boldsymbol{\omega}) = v^\mu \omega_\mu = \langle \boldsymbol{\omega}, \vec{\boldsymbol{v}} \rangle. \tag{14.34}$$

14.4 Alternate Forms

14.4.1 Definition and Examples

A particular class of tensors is constituted by the multilinear forms that are fully antisymmetric, i.e. by the tensors of type $(0, p)$ that change their sign whenever any two of their arguments are switched:

$$\begin{aligned}
p = 2: &\quad \boldsymbol{A}(\vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_2) = -\boldsymbol{A}(\vec{\boldsymbol{v}}_2, \vec{\boldsymbol{v}}_1) \\
p = 3: &\quad \boldsymbol{A}(\vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_2, \vec{\boldsymbol{v}}_3) = -\boldsymbol{A}(\vec{\boldsymbol{v}}_2, \vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_3) = -\boldsymbol{A}(\vec{\boldsymbol{v}}_3, \vec{\boldsymbol{v}}_2, \vec{\boldsymbol{v}}_1), \text{ etc.} \\
p = 4: &\quad \boldsymbol{A}(\vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_2, \vec{\boldsymbol{v}}_3, \vec{\boldsymbol{v}}_4) = -\boldsymbol{A}(\vec{\boldsymbol{v}}_2, \vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_3, \vec{\boldsymbol{v}}_4), \text{ etc.}
\end{aligned}$$

These multilinear forms are called ***alternate forms*** because they vanish if any two of their arguments are equal. For any integer $p \geq 2$, one calls ***p-form*** any alternate form of valence p. One denotes by $\mathscr{A}_p(E)$ the set of all p-forms. We have of course

$$\mathscr{A}_p(E) \subset \mathscr{T}_{(0,p)}(E). \tag{14.35}$$

Moreover, any linear combination of p-forms being a p-form, $\mathscr{A}_p(E)$ is clearly a vector subspace of $\mathscr{T}_{(0,p)}(E)$.

The concept of antisymmetry is a priori meaningless for a form of valence 1 (linear form), but we shall extend the definition of p-forms to the case $p = 1$ by setting

$$\mathscr{A}_1(E) := \mathscr{T}_{(0,1)}(E) = E^*. \tag{14.36}$$

Moreover, we shall define $\mathscr{A}_0(E)$ as the base field of the vector space E, namely, $\mathbb{R}$:

$$\mathscr{A}_0(E) := \mathbb{R}. \tag{14.37}$$

Remark 14.5. With the above convention, any linear form is a 1-form. On the other side, note that not all bilinear forms are 2-forms: they must be antisymmetric for this.

Example 14.10. Examples of alternate forms encountered up to now are:

- The Levi–Civita tensor (Chap. 1): $\boldsymbol{\varepsilon} \in \mathscr{A}_4(E)$.
- The mixed product in the local rest space of an observer (Chap. 3): $\boldsymbol{\varepsilon_u} \in \mathscr{A}_3(E)$.
- The bilinear form associated with the variation of the local frame of an observer (Chap. 3) : $\underline{\boldsymbol{\Omega}} \in \mathscr{A}_2(E)$.
- The angular momentum of a particle with respect to a point C (Chap. 10): $\boldsymbol{J}_C \in \mathscr{A}_2(E)$.
- The spin of an isolated particle system (Chap. 10): $\boldsymbol{S} \in \mathscr{A}_2(E)$.
- The 4-torque with respect to a point C and acting on a particle (Chap. 10): $\boldsymbol{N}_C \in \mathscr{A}_2(E)$.
- The bilinear form describing the electromagnetic field (Chaps. 10 and 11, as well as Chap. 17): $\boldsymbol{F} \in \mathscr{A}_2(E)$.

A p-form being a tensor of type $(0, p)$, its expansion with respect to a basis $(\vec{\boldsymbol{e}}_\alpha)$ of E takes the form

$$\boldsymbol{A} = A_{\alpha_1\ldots\alpha_p}\, \boldsymbol{e}^{\alpha_1} \otimes \cdots \otimes \boldsymbol{e}^{\alpha_p}. \tag{14.38}$$

It is immediate to see that, for $p \geq 2$, $\boldsymbol{A}$ is a p-form iff the components $A_{\alpha_1\ldots\alpha_p}$ are antisymmetric under any index permutation.

The dimension of E being 4, there cannot exist any alternate form but the zero one whenever $p > 4$:

$$\mathscr{A}_p(E) = \{0\} \quad \text{if} \quad p > 4. \tag{14.39}$$

Proof. The components of an alternate form $\boldsymbol{A}$ in a basis $(\vec{\boldsymbol{e}}_\alpha)$ are given by (14.13): $A_{\alpha_1\ldots\alpha_p} = \boldsymbol{A}(\vec{\boldsymbol{e}}_{\alpha_1}, \ldots, \vec{\boldsymbol{e}}_{\alpha_p})$. The right-hand side cannot involve more than 4 different vectors $\vec{\boldsymbol{e}}_{\alpha_i}$; it is thus necessarily zero if $p > 4$. □

For $p \leq 4$, the components $(A_{\alpha_1\ldots\alpha_p})$ vanish if two of the indices α_i are equal. The number of nonvanishing components is thus at most $4 \times 3 \times \ldots \times (5 - p)$. In addition, thanks to the antisymmetry of $\boldsymbol{A}$, the components that are related to a same p-tuple $(\alpha_1, \ldots, \alpha_p)$ by a permutation are equal, up to the sign. We conclude that the number of independent components is $4 \times 3 \times \ldots \times (5 - p)/p! = \binom{4}{p}$. The dimension of the vector space $\mathscr{A}_p(E)$ is equal to this number. Explicitly, taking into account (14.37),

$$\begin{aligned} &\dim \mathscr{A}_0(E) = 1, \quad \dim \mathscr{A}_1(E) = 4, \quad \dim \mathscr{A}_2(E) = 6, \\ &\dim \mathscr{A}_3(E) = 4, \quad \dim \mathscr{A}_4(E) = 1. \end{aligned} \tag{14.40}$$

Remark 14.6. The dimensions of $\mathscr{A}_0(E)$ and $\mathscr{A}_1(E)$ are immediate since $\mathscr{A}_0(E) = \mathbb{R}$ and $\mathscr{A}_1(E) = E^*$. Regarding the dimension de $\mathscr{A}_2(E)$, it is easily recovered by considerations on components: a 2-form is represented in a basis of E by a 4×4 antisymmetric matrix: $(\boldsymbol{A} = A_{\alpha\beta}\, \boldsymbol{e}^\alpha \otimes \boldsymbol{e}^\beta)$. Now such a matrix has only 6 independent components.

Remark 14.7. We have already underlined in Sect. 1.5 that $\dim \mathscr{A}_4(E) = 1$: all the 4-forms are proportional to the Levi–Civita tensor: $\forall \boldsymbol{A} \in \mathscr{A}_4(E),\ \exists\lambda \in \mathbb{R},\ \boldsymbol{A} = \lambda\boldsymbol{\varepsilon}$.

14.4.2 Exterior Product

The tensor product of two alternate forms does not yield an alternate form. But it can be antisymmetrized to achieve this. One defines thus the ***exterior product***, also called ***wedge product***, as the mapping

$$\begin{array}{rcl} \wedge : \mathscr{A}_p(E) \times \mathscr{A}_q(E) & \longrightarrow & \mathscr{A}_{p+q}(E) \\ (\boldsymbol{A}, \boldsymbol{B}) & \longmapsto & \boldsymbol{A} \wedge \boldsymbol{B} \end{array} \tag{14.41}$$

such that

$$\boxed{\begin{aligned} \boldsymbol{A} \wedge \boldsymbol{B}(\vec{\boldsymbol{v}}_1, \ldots, \vec{\boldsymbol{v}}_{p+q}) := \frac{1}{p!q!} \sum_{\sigma\in\mathfrak{S}_{p+q}} (-1)^{k(\sigma)}\ & \boldsymbol{A}(\vec{\boldsymbol{v}}_{\sigma(1)}, \ldots, \vec{\boldsymbol{v}}_{\sigma(p)}) \\ & \times \boldsymbol{B}(\vec{\boldsymbol{v}}_{\sigma(p+1)}, \ldots, \vec{\boldsymbol{v}}_{\sigma(p+q)}) \end{aligned}}, \tag{14.42}$$

where $(\vec{\boldsymbol{v}}_1, \ldots, \vec{\boldsymbol{v}}_{p+q}) \in E^{p+q}$, $\mathfrak{S}_{p+q}$ stands for the group of permutations of $p+q$ elements and $k(\sigma)$ is the number of transpositions (permutations of two elements) in product of which σ can be decomposed. In the above formula, it is obvious that $\boldsymbol{A} \wedge \boldsymbol{B}$ is an alternate form of valence $p + q$, so that the mapping (14.41) is well defined.

Let us specify formula (14.42) for some particular cases. First of all, if $p = 0$, then $\boldsymbol{A} = \lambda \in \mathbb{R}$ [cf. (14.37)] and (14.42) reduces to

$$\boldsymbol{A} \wedge \boldsymbol{B}(\vec{\boldsymbol{v}}_1, \ldots, \vec{\boldsymbol{v}}_q) = \frac{\lambda}{q!} \sum_{\sigma\in\mathfrak{S}_q} (-1)^{k(\sigma)}\, \underbrace{\boldsymbol{B}(\vec{\boldsymbol{v}}_{\sigma(1)}, \ldots, \vec{\boldsymbol{v}}_{\sigma(q)})}_{(-1)^{k(\sigma)}\boldsymbol{B}(\vec{\boldsymbol{v}}_1, \ldots, \vec{\boldsymbol{v}}_q)} = \lambda \boldsymbol{B}(\vec{\boldsymbol{v}}_1, \ldots, \vec{\boldsymbol{v}}_q),$$

since the cardinal of $\mathfrak{S}_q$ is $q!$. Hence, for $p = 0$, the exterior product reduces to the mere multiplication of an element of the vector space $\mathscr{A}_q(E)$ by a scalar.

In the case $p = 1$ and $q = 1$, $\boldsymbol{A}$ and $\boldsymbol{B}$ are two linear forms and according to (14.42), $\boldsymbol{A} \wedge \boldsymbol{B}$ is the antisymmetric bilinear form defined by

$$\forall(\vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_2) \in E^2, \quad \boldsymbol{A} \wedge \boldsymbol{B}(\vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_2) = \langle \boldsymbol{A}, \vec{\boldsymbol{v}}_1\rangle\langle \boldsymbol{B}, \vec{\boldsymbol{v}}_2\rangle - \langle \boldsymbol{A}, \vec{\boldsymbol{v}}_2\rangle\langle \boldsymbol{B}, \vec{\boldsymbol{v}}_1\rangle.$$

By definition of the tensor product, this formula can be rewritten as

$$\forall(\boldsymbol{A}, \boldsymbol{B}) \in \mathscr{A}_1(E)^2, \quad \boxed{\boldsymbol{A} \wedge \boldsymbol{B} = \boldsymbol{A} \otimes \boldsymbol{B} - \boldsymbol{B} \otimes \boldsymbol{A}}. \tag{14.43}$$

We recover the definition of the exterior product given in Chap. 10 for 1-forms [Eq. (10.2)].

In the case $p = 1$ and $q = 2$, (14.42) leads to

$$\boxed{\boldsymbol{A} \wedge \boldsymbol{B}(\vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_2, \vec{\boldsymbol{v}}_3) = \langle \boldsymbol{A}, \vec{\boldsymbol{v}}_1\rangle \boldsymbol{B}(\vec{\boldsymbol{v}}_2, \vec{\boldsymbol{v}}_3) + \langle \boldsymbol{A}, \vec{\boldsymbol{v}}_2\rangle \boldsymbol{B}(\vec{\boldsymbol{v}}_3, \vec{\boldsymbol{v}}_1) + \langle \boldsymbol{A}, \vec{\boldsymbol{v}}_3\rangle \boldsymbol{B}(\vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_2)} \tag{14.44}$$

for any 3-tuple $(\vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_2, \vec{\boldsymbol{v}}_3)$ of vectors of E.

Remark 14.8. The exterior product owes its name from the fact that if $\boldsymbol{A}$ and $\boldsymbol{B}$ are two elements of the vector space $\mathscr{A}_p(E)$, $\boldsymbol{A} \wedge \boldsymbol{B}$ is not an element of $\mathscr{A}_p(E)$ but of another vector space: $\mathscr{A}_{2p}(E)$.

Note that, from (14.43) and (14.44), $\boldsymbol{B} \wedge \boldsymbol{A} = -\boldsymbol{A} \wedge \boldsymbol{B}$ if $p = q = 1$ and $\boldsymbol{B} \wedge \boldsymbol{A} = \boldsymbol{A} \wedge \boldsymbol{B}$ if $p = 1$ and $q = 2$. More generally, we have

$$\forall (\boldsymbol{A}, \boldsymbol{B}) \in \mathscr{A}_p(E) \times \mathscr{A}_q(E), \quad \boxed{\boldsymbol{B} \wedge \boldsymbol{A} = (-1)^{pq} \boldsymbol{A} \wedge \boldsymbol{B}}. \tag{14.45}$$

Another property of the exterior product is to be associative: for any alternate forms $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$,

$$\boldsymbol{A} \wedge (\boldsymbol{B} \wedge \boldsymbol{C}) = (\boldsymbol{A} \wedge \boldsymbol{B}) \wedge \boldsymbol{C}. \tag{14.46}$$

14.4.3 Basis of the Space of *p*-Forms

The exterior product allows one to construct a basis of the vector space $\mathscr{A}_p(E)$ from the linear forms $(\boldsymbol{e}^\alpha)$ of the dual basis of some basis of E. Let us consider indeed a 2-form $\boldsymbol{A}$ and its components $(A_{\alpha\beta})$ in the basis $\boldsymbol{e}^\alpha \otimes \boldsymbol{e}^\beta$ of $\mathscr{T}_{(0,2)}(E)$ [cf. (14.10) with $k = 0$ and $\ell = 2$]: $\boldsymbol{A} = A_{\alpha\beta}\, \boldsymbol{e}^\alpha \otimes \boldsymbol{e}^\beta$. We have then, since $(A_{\alpha\beta})$ is antisymmetric,

$$\begin{aligned}
\boldsymbol{A} &= A_{\alpha\beta}\, \boldsymbol{e}^\alpha \otimes \boldsymbol{e}^\beta = \frac{1}{2}\left(A_{\alpha\beta}\, \boldsymbol{e}^\alpha \otimes \boldsymbol{e}^\beta + A_{\beta\alpha}\, \boldsymbol{e}^\beta \otimes \boldsymbol{e}^\alpha\right) \\
&= \frac{1}{2} A_{\alpha\beta} \left(\boldsymbol{e}^\alpha \otimes \boldsymbol{e}^\beta - \boldsymbol{e}^\beta \otimes \boldsymbol{e}^\alpha\right) = \frac{1}{2} A_{\alpha\beta}\, \boldsymbol{e}^\alpha \wedge \boldsymbol{e}^\beta.
\end{aligned}$$

Since $A_{\alpha\alpha} = 0$, the double sum on α and β can be split in two parts:

$$\boldsymbol{A} = \frac{1}{2} \sum_{\alpha<\beta} A_{\alpha\beta} \boldsymbol{e}^\alpha \wedge \boldsymbol{e}^\beta + \frac{1}{2} \sum_{\beta<\alpha} \underbrace{A_{\alpha\beta}}_{-A_{\beta\alpha}} \underbrace{\boldsymbol{e}^\alpha \wedge \boldsymbol{e}^\beta}_{-\boldsymbol{e}^\beta \wedge \boldsymbol{e}^\alpha} = \sum_{\alpha<\beta} A_{\alpha\beta} \boldsymbol{e}^\alpha \wedge \boldsymbol{e}^\beta.$$

We have thus shown that

$$\boldsymbol{A} = A_{\alpha\beta}\, \boldsymbol{e}^{\alpha} \otimes \boldsymbol{e}^{\beta} = \frac{1}{2} A_{\alpha\beta}\, \boldsymbol{e}^{\alpha} \wedge \boldsymbol{e}^{\beta} = \sum_{\alpha<\beta} A_{\alpha\beta} \boldsymbol{e}^{\alpha} \wedge \boldsymbol{e}^{\beta}. \tag{14.47}$$

More generally, for a p-form,

$$\boxed{\begin{aligned} \boldsymbol{A} &= A_{\alpha_1\ldots\alpha_p}\, \boldsymbol{e}^{\alpha_1} \otimes \ldots \otimes \boldsymbol{e}^{\alpha_p} = \frac{1}{p!} A_{\alpha_1\ldots\alpha_p}\, \boldsymbol{e}^{\alpha_1} \wedge \ldots \wedge \boldsymbol{e}^{\alpha_p} \\ &= \sum_{\alpha_1<\ldots<\alpha_p} A_{\alpha_1\ldots\alpha_p}\, \boldsymbol{e}^{\alpha_1} \wedge \ldots \wedge \boldsymbol{e}^{\alpha_p}. \end{aligned}} \tag{14.48}$$

We conclude that $(\boldsymbol{e}^{\alpha_1} \wedge \ldots \wedge \boldsymbol{e}^{\alpha_p})_{\alpha_1<\ldots<\alpha_p}$ is a basis of the vector space $\mathscr{A}_p(E)$. Moreover, the components of an element of $\mathscr{A}_p(E)$ in this basis are identical to its components as an element of $\mathscr{T}_{(0,p)}(E)$. Explicitly, the bases are the following ones:

- $\mathscr{A}_1(E) : (\boldsymbol{e}^0, \boldsymbol{e}^1, \boldsymbol{e}^2, \boldsymbol{e}^3)$
- $\mathscr{A}_2(E) : (\boldsymbol{e}^0 \wedge \boldsymbol{e}^1,\ \boldsymbol{e}^0 \wedge \boldsymbol{e}^2,\ \boldsymbol{e}^0 \wedge \boldsymbol{e}^3,\ \boldsymbol{e}^1 \wedge \boldsymbol{e}^2,\ \boldsymbol{e}^1 \wedge \boldsymbol{e}^3,\ \boldsymbol{e}^2 \wedge \boldsymbol{e}^3)$
- $\mathscr{A}_3(E) : (\boldsymbol{e}^0 \wedge \boldsymbol{e}^1 \wedge \boldsymbol{e}^2,\ \boldsymbol{e}^0 \wedge \boldsymbol{e}^1 \wedge \boldsymbol{e}^3,\ \boldsymbol{e}^0 \wedge \boldsymbol{e}^2 \wedge \boldsymbol{e}^3,\ \boldsymbol{e}^1 \wedge \boldsymbol{e}^2 \wedge \boldsymbol{e}^3)$
- $\mathscr{A}_4(E) : (\boldsymbol{e}^0 \wedge \boldsymbol{e}^1 \wedge \boldsymbol{e}^2 \wedge \boldsymbol{e}^3)$

The sizes of the bases are of course in agreement with the dimensions (14.40).

14.4.4 Components of the Levi–Civita Tensor

The Levi–Civita tensor $\boldsymbol{\varepsilon}$ introduced in Sect. 1.5 is a 4-form. Its components $(\varepsilon_{\alpha\beta\gamma\delta})$ in a basis $(\vec{\boldsymbol{e}}_\alpha)$ of E (not necessarily orthonormal) obey (14.48) with the sum on $\alpha < \beta < \gamma < \delta$ necessarily limited to a single term: $(\alpha, \beta, \gamma, \delta) = (0, 1, 2, 3)$. Hence,

$$\boldsymbol{\varepsilon} = \varepsilon_{\alpha\beta\gamma\delta}\, \boldsymbol{e}^{\alpha} \otimes \boldsymbol{e}^{\beta} \otimes \boldsymbol{e}^{\gamma} \otimes \boldsymbol{e}^{\delta} = \varepsilon_{0123}\, \boldsymbol{e}^0 \wedge \boldsymbol{e}^1 \wedge \boldsymbol{e}^2 \wedge \boldsymbol{e}^3. \tag{14.49}$$

Besides, for the Levi–Civita tensor, formula (14.16) becomes

$$\forall (\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}, \vec{\boldsymbol{z}}) \in E^4, \quad \boldsymbol{\varepsilon}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}, \vec{\boldsymbol{z}}) = \varepsilon_{\alpha\beta\gamma\delta}\, u^{\alpha}\, v^{\beta}\, w^{\gamma}\, z^{\delta}. \tag{14.50}$$

If $(\vec{\boldsymbol{e}}_\alpha)$ is a right-handed orthonormal basis, the definition (1.33) of $\boldsymbol{\varepsilon}$ results in the following components: $\varepsilon_{\alpha\beta\gamma\delta} = [\alpha, \beta, \gamma, \delta]$, where the symbol $[\alpha, \beta, \gamma, \delta]$ means

$$\begin{cases} 0 \text{ if any two of the indices } (\alpha, \beta, \gamma, \delta) \text{ are equal} \\ 1 \text{ if } (\alpha, \beta, \gamma, \delta) \text{ is deduced from } (0, 1, 2, 3) \text{ by an even permutation} \\ -1 \text{ if } (\alpha, \beta, \gamma, \delta) \text{ is deduced from } (0, 1, 2, 3) \text{ by an odd permutation.} \end{cases}$$

More generally, in any basis,

$$\boxed{\varepsilon_{\alpha\beta\gamma\delta} = \pm\sqrt{-\det g}\;[\alpha, \beta, \gamma, \delta]}\,. \tag{14.51}$$

where (i) $\det g$ is the determinant of the matrix $(g_{\alpha\beta})$ of $\boldsymbol{g}$'s components in the basis $(\vec{\boldsymbol{e}}_\alpha)$ and (ii) the $\pm$ sign must be $+$ (resp. $-$) for a right-handed basis (resp. left-handed).

Proof. Formula (14.51) is equivalent to

$$\boldsymbol{\varepsilon}(\vec{\boldsymbol{e}}_0, \vec{\boldsymbol{e}}_1, \vec{\boldsymbol{e}}_2, \vec{\boldsymbol{e}}_3) = \pm\sqrt{-\det g}. \tag{14.52}$$

Let us then introduce a right-handed orthonormal basis $(\vec{\boldsymbol{e}}^*_\alpha)$, i.e. an orthonormal basis such that

$$\boldsymbol{\varepsilon}(\vec{\boldsymbol{e}}^*_0, \vec{\boldsymbol{e}}^*_1, \vec{\boldsymbol{e}}^*_2, \vec{\boldsymbol{e}}^*_3) = 1. \tag{14.53}$$

Let us reconsider the definition of the Levi–Civita tensor: in Sect. 1.5, we have admitted that if a 4-form $\boldsymbol{\varepsilon}$ takes the value 1 on an orthonormal basis of $(E, \boldsymbol{g})$, then it takes necessarily the value ± 1 on any other orthonormal basis. We shall prove it here by taking (14.53) as a starting point, without assuming anything of the value of $\boldsymbol{\varepsilon}$ on any other orthonormal basis. Let $P^\mu_{\ \alpha}$ be the change-of-basis matrix from $(\vec{\boldsymbol{e}}^*_\alpha)$ to $(\vec{\boldsymbol{e}}_\alpha)$: $\vec{\boldsymbol{e}}_\alpha = P^\mu_{\ \alpha}\vec{\boldsymbol{e}}^*_\mu$. The 4-linearity of $\boldsymbol{\varepsilon}$ yields then

$$\boldsymbol{\varepsilon}(\vec{\boldsymbol{e}}_0, \vec{\boldsymbol{e}}_1, \vec{\boldsymbol{e}}_2, \vec{\boldsymbol{e}}_3) = P^\mu_{\ 0}\, P^\nu_{\ 1}\, P^\rho_{\ 2}\, P^\lambda_{\ 3}\, \boldsymbol{\varepsilon}(\vec{\boldsymbol{e}}^*_\mu, \vec{\boldsymbol{e}}^*_\nu, \vec{\boldsymbol{e}}^*_\rho, \vec{\boldsymbol{e}}^*_\lambda). \tag{14.54}$$

Now (14.53) implies $\boldsymbol{\varepsilon}(\vec{\boldsymbol{e}}^*_\mu, \vec{\boldsymbol{e}}^*_\nu, \vec{\boldsymbol{e}}^*_\rho, \vec{\boldsymbol{e}}^*_\lambda) = [\mu, \nu, \rho, \lambda]$, so that (14.54) becomes

$$\boldsymbol{\varepsilon}(\vec{\boldsymbol{e}}_0, \vec{\boldsymbol{e}}_1, \vec{\boldsymbol{e}}_2, \vec{\boldsymbol{e}}_3) = \sum_{\sigma\in\mathfrak{S}_4} (-1)^{k(\sigma)}\, P^{\sigma(0)}_{\ \ 0}\; P^{\sigma(1)}_{\ \ 1}\; P^{\sigma(2)}_{\ \ 2}\; P^{\sigma(3)}_{\ \ 3}. \tag{14.55}$$

We recognize in the right-hand side the determinant of the matrix P; hence,

$$\boldsymbol{\varepsilon}(\vec{\boldsymbol{e}}_0, \vec{\boldsymbol{e}}_1, \vec{\boldsymbol{e}}_2, \vec{\boldsymbol{e}}_3) = \det P. \tag{14.56}$$

If $(\vec{\boldsymbol{e}}_\alpha)$ is an orthonormal basis, then P is necessarily a Lorentz matrix, which implies $\det P = \pm 1$ [property (6.9)], and we get $\boldsymbol{\varepsilon}(\vec{\boldsymbol{e}}_0, \vec{\boldsymbol{e}}_1, \vec{\boldsymbol{e}}_2, \vec{\boldsymbol{e}}_3) = \pm 1$. We have thus demonstrated the property (1.33). Let us consider again a general (i.e. not necessarily orthonormal) basis $(\vec{\boldsymbol{e}}_\alpha)$. The components of the metric tensor with respect to $(\vec{\boldsymbol{e}}_\alpha)$ are

$$g_{\alpha\beta} = \vec{e}_\alpha \cdot \vec{e}_\beta = (P^\mu{}_\alpha \vec{e}^{\,*}_\mu) \cdot (P^\nu{}_\beta \vec{e}^{\,*}_\nu) = P^\mu{}_\alpha \, P^\nu{}_\beta \, \vec{e}^{\,*}_\mu \cdot \vec{e}^{\,*}_\nu = P^\mu{}_\alpha \, \eta_{\mu\nu} P^\nu{}_\beta, \tag{14.57}$$

where we have used $\vec{e}^{\,*}_\mu \cdot \vec{e}^{\,*}_\nu = \eta_{\mu\nu}$ since $(\vec{e}^{\,*}_\alpha)$ is an orthonormal basis. The above formula can be rewritten in terms of matrix products:

$$g = {}^t P \, \eta \, P. \tag{14.58}$$

This implies

$$\det g = \det {}^t P \; \det \eta \; \det P = -(\det P)^2, \tag{14.59}$$

since $\det {}^t P = \det P$ and $\det \eta = -1$ [cf. Eq. (1.17)]. Equations (14.56) and (14.59) establish (14.52) and thus (14.51). □

Remark 14.9. Since $(\det P)^2 > 0$, an immediate consequence of (14.59) is that the determinant of the components of the metric tensor with respect to any basis of E is always negative:

$$\det g < 0. \tag{14.60}$$

Accordingly, formula (14.51), which involves $\sqrt{-\det g}$, is well posed.

Remark 14.10. As a tensor of valence 4, $\boldsymbol{\varepsilon}$ has a priori $4^4 = 256$ components. The second equality in (14.49) shows that it has actually only one independent component: ε_{0123}. This reflects the fact that the vector space $\mathscr{A}_4(E)$ is one-dimensional. From (14.51), this unique component equals $\sqrt{-\det g}$ if the considered basis is right-handed and $-\sqrt{-\det g}$ otherwise.

14.5 Hodge Duality

For a fixed $p \in \{0, 1, 2, 3, 4\}$, Hodge duality performs a correspondence between the p-forms and the $(4-p)$-forms, by means of an isomorphism between the vector spaces $\mathscr{A}_p(E)$ and $\mathscr{A}_{4-p}(E)$, which, as already seen, have the same dimension [Eq. (14.40)]. Hodge duality is based on the Levi–Civita tensor and on tensors that can be associated with it by metric duality. We start therefore by introducing these tensors.

14.5.1 Tensors Associated with the Levi–Civita Tensor

From the Levi–Civita tensor $\boldsymbol{\varepsilon}$ and the metric tensor $\boldsymbol{g}$, one may define four tensors of valence 4 according to

$$\begin{array}{rl} {}^1\boldsymbol{\varepsilon}: E^* \times E \times E \times E & \longrightarrow \mathbb{R} \\ (\boldsymbol{\omega}, \vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_2, \vec{\boldsymbol{v}}_3) & \longmapsto \boldsymbol{\varepsilon}(\vec{\boldsymbol{\omega}}, \vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_2, \vec{\boldsymbol{v}}_3) \end{array} \tag{14.61a}$$

$$\begin{array}{rl} {}^2\boldsymbol{\varepsilon}: E^* \times E^* \times E \times E & \longrightarrow \mathbb{R} \\ (\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_2) & \longmapsto \boldsymbol{\varepsilon}(\vec{\boldsymbol{\omega}}_1, \vec{\boldsymbol{\omega}}_2, \vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_2) \end{array} \tag{14.61b}$$

$$\begin{array}{rl} {}^3\boldsymbol{\varepsilon}: E^* \times E^* \times E^* \times E & \longrightarrow \mathbb{R} \\ (\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\omega}_3, \vec{\boldsymbol{v}}) & \longmapsto \boldsymbol{\varepsilon}(\vec{\boldsymbol{\omega}}_1, \vec{\boldsymbol{\omega}}_2, \vec{\boldsymbol{\omega}}_3, \vec{\boldsymbol{v}}) \end{array} \tag{14.61c}$$

$$\begin{array}{rl} {}^4\boldsymbol{\varepsilon}: E^* \times E^* \times E^* \times E^* & \longrightarrow \mathbb{R} \\ (\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\omega}_3, \boldsymbol{\omega}_4) & \longmapsto \boldsymbol{\varepsilon}(\vec{\boldsymbol{\omega}}_1, \vec{\boldsymbol{\omega}}_2, \vec{\boldsymbol{\omega}}_3, \vec{\boldsymbol{\omega}}_4), \end{array} \tag{14.61d}$$

where, following the notation introduced in Sect. 1.6.2, $\vec{\boldsymbol{\omega}}$ stands for the vector associated with the linear form $\boldsymbol{\omega}$ by metric duality. ${}^1\boldsymbol{\varepsilon}$ is a tensor of type $(1,3)$, ${}^2\boldsymbol{\varepsilon}$ a tensor of type $(2,2)$, ${}^3\boldsymbol{\varepsilon}$ a tensor of type $(3,1)$ and ${}^4\boldsymbol{\varepsilon}$ a tensor of type $(4,0)$. Moreover each tensor ${}^p\boldsymbol{\varepsilon}$ ($p \in \{1,2,3,4\}$) is antisymmetric with respect to all its vector arguments, as well as with respect to all its linear-form arguments.

The components of the tensors ${}^p\boldsymbol{\varepsilon}$ in a basis $(\vec{\boldsymbol{e}}_\alpha)$ of E are easily deduced from those of $\boldsymbol{\varepsilon}$ via expression (14.29) for the metric duality:

$${}^1\boldsymbol{\varepsilon}: \varepsilon^{\alpha}{}_{\beta\gamma\delta} = g^{\alpha\mu}\, \varepsilon_{\mu\beta\gamma\delta} \tag{14.62a}$$

$${}^2\boldsymbol{\varepsilon}: \varepsilon^{\alpha\beta}{}_{\gamma\delta} = g^{\alpha\mu} g^{\beta\nu}\, \varepsilon_{\mu\nu\gamma\delta} \tag{14.62b}$$

$${}^3\boldsymbol{\varepsilon}: \varepsilon^{\alpha\beta\gamma}{}_{\delta} = g^{\alpha\mu} g^{\beta\nu} g^{\gamma\rho}\, \varepsilon_{\mu\nu\rho\delta} \tag{14.62c}$$

$${}^4\boldsymbol{\varepsilon}: \varepsilon^{\alpha\beta\gamma\delta} = g^{\alpha\mu} g^{\beta\nu} g^{\gamma\rho} g^{\delta\sigma}\, \varepsilon_{\mu\nu\rho\sigma}. \tag{14.62d}$$

Note that the prefix $p = 1,2,3,4$ has been suppressed in the writing of the components of ${}^p\boldsymbol{\varepsilon}$, since the position of the indices allows one to distinguish without any ambiguity the four tensors.

${}^4\boldsymbol{\varepsilon}$ is a tensor of type $(4,0)$ which is fully antisymmetric with respect to its 4 arguments. The set of all such tensors is a one-dimensional vector subspace of $\mathscr{T}_{(4,0)}(E)$, as $\mathscr{A}_4(E)$ is a one-dimensional vector subspace of $\mathscr{T}_{(0,4)}(E)$. Accordingly, all the components of ${}^4\boldsymbol{\varepsilon}$ can be deduced from a single one, which we choose to be ε^{0123}, via the formula $\varepsilon^{\alpha\beta\gamma\delta} = \varepsilon^{0123}[\alpha,\beta,\gamma,\delta]$. Combining (14.62d) and (14.51), there comes

$$\varepsilon^{0123} = \pm g^{0\mu} g^{1\nu} g^{2\rho} g^{3\sigma} \sqrt{-\det g}\, [\mu,\nu,\rho,\sigma],$$

which can be written as

$$\varepsilon^{0123} = \pm\sqrt{-\det g} \sum_{\sigma\in\mathfrak{S}_4} (-1)^{k(\sigma)} g^{0\sigma(0)} g^{1\sigma(1)} g^{2\sigma(2)} g^{3\sigma(3)}.$$

We recognize in the above sum the expression of the determinant of the matrix $g^{-1} = (g^{\alpha\beta})$. The latter being the inverse of $g = (g_{\alpha\beta})$, its determinant is $1/\det g$. We have thus, since $\det g < 0$, $\varepsilon^{0123} = \mp 1/\sqrt{-\det g}$. Consequently,

$$\boxed{\varepsilon^{\alpha\beta\gamma\delta} = \mp \frac{1}{\sqrt{-\det g}} [\alpha, \beta, \gamma, \delta]}, \tag{14.63}$$

with the $-$ sign if $(\vec{e}_\alpha)$ is a right-handed basis and the $+$ sign otherwise.

Let us consider the tensor product ${}^4\boldsymbol{\varepsilon} \otimes \boldsymbol{\varepsilon}$. It is a tensor of type $(4, 4)$, fully antisymmetric in its first four arguments, as well as in its last four arguments. The expression of ${}^4\boldsymbol{\varepsilon} \otimes \boldsymbol{\varepsilon}$ is relatively simple:

$$\begin{aligned}
&\forall (\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\omega}_3, \boldsymbol{\omega}_4, \vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4) \in (E^*)^4 \times E^4,\\
&{}^4\boldsymbol{\varepsilon} \otimes \boldsymbol{\varepsilon}\, (\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\omega}_3, \boldsymbol{\omega}_4, \vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4) =\\
&\quad - \sum_{\sigma \in \mathfrak{S}_4} (-1)^{k(\sigma)} \langle \boldsymbol{\omega}_{\sigma(1)}, \vec{v}_1 \rangle \langle \boldsymbol{\omega}_{\sigma(2)}, \vec{v}_2 \rangle \langle \boldsymbol{\omega}_{\sigma(3)}, \vec{v}_3 \rangle \langle \boldsymbol{\omega}_{\sigma(4)}, \vec{v}_4 \rangle.
\end{aligned} \tag{14.64}$$

Proof. The right-hand side of (14.64) defines clearly a tensor of type $(4, 4)$, fully antisymmetric in its first four arguments and in its last four ones, as ${}^4\boldsymbol{\varepsilon} \otimes \boldsymbol{\varepsilon}$. The set of all such tensors is a one-dimensional vector space because the subspaces of $\mathscr{T}_{(4,0)}(E)$ and $\mathscr{T}_{(0,4)}(E)$ formed by the fully antisymmetric tensors are each one-dimensional vector spaces. We deduce that ${}^4\boldsymbol{\varepsilon} \otimes \boldsymbol{\varepsilon}$ is necessarily proportional to the tensor appearing in the right-hand side of (14.64). To get the proportionality factor, it suffices to evaluate each tensor on the same 8-tuple, for instance, $(\boldsymbol{e}^\alpha, \vec{e}_\beta)$ where $(\vec{e}_\alpha)$ is a right-handed basis of $(E, \boldsymbol{g})$ and $(\boldsymbol{e}^\alpha)$ its dual basis. By definition of a tensor product, we have

$${}^4\boldsymbol{\varepsilon} \otimes \boldsymbol{\varepsilon}\, (\boldsymbol{e}^0, \boldsymbol{e}^1, \boldsymbol{e}^2, \boldsymbol{e}^3, \vec{e}_0, \vec{e}_1, \vec{e}_2, \vec{e}_3) = \underbrace{{}^4\boldsymbol{\varepsilon}(\boldsymbol{e}^0, \boldsymbol{e}^1, \boldsymbol{e}^2, \boldsymbol{e}^3)}_{-1}\ \underbrace{\boldsymbol{\varepsilon}(\vec{e}_0, \vec{e}_1, \vec{e}_2, \vec{e}_3)}_{1} = -1, \tag{14.65}$$

where ${}^4\boldsymbol{\varepsilon}(\boldsymbol{e}^0, \boldsymbol{e}^1, \boldsymbol{e}^2, \boldsymbol{e}^3) = -1$ follows from

$${}^4\boldsymbol{\varepsilon}(\boldsymbol{e}^0, \boldsymbol{e}^1, \boldsymbol{e}^2, \boldsymbol{e}^3) = \boldsymbol{\varepsilon}(\vec{e}^{\,0}, \vec{e}^{\,1}, \vec{e}^{\,2}, \vec{e}^{\,3}) = \boldsymbol{\varepsilon}(-\vec{e}_0, \vec{e}_1, \vec{e}_2, \vec{e}_3),$$

each $\vec{e}^{\,\alpha}$ being the vector associated with the linear form $\boldsymbol{e}^0$ by metric duality, so that $\vec{e}^{\,0} = -\vec{e}_0$ and $\vec{e}^{\,i} = \vec{e}_i$ $(1 \leq i \leq 3)$. On the other hand,

$$\sum_{\sigma \in \mathfrak{S}_4} (-1)^{k(\sigma)} \underbrace{\langle \boldsymbol{e}^{\sigma(0)}, \vec{e}_0 \rangle}_{\delta^{\sigma(0)}{}_0} \dots \underbrace{\langle \boldsymbol{e}^{\sigma(3)}, \vec{e}_3 \rangle}_{\delta^{\sigma(3)}{}_3} = \underbrace{\langle \boldsymbol{e}^0, \vec{e}_0 \rangle}_{1} \dots \underbrace{\langle \boldsymbol{e}^3, \vec{e}_3 \rangle}_{1} = 1. \tag{14.66}$$

In view of (14.65) and (14.66), we deduce that the proportionality factor in front the summation sign in (14.64) is -1. $\square$

In terms of components, (14.64) is expressed as

$$\boxed{\varepsilon^{\alpha_1\,\alpha_2\,\alpha_3\,\alpha_4}\ \varepsilon_{\beta_1\,\beta_2\,\beta_3\,\beta_4} = -\sum_{\sigma\in\mathfrak{S}_4}(-1)^{k(\sigma)}\ \delta^{\alpha_{\sigma(1)}}{}_{\beta_1}\ \delta^{\alpha_{\sigma(2)}}{}_{\beta_2}\ \delta^{\alpha_{\sigma(3)}}{}_{\beta_3}\ \delta^{\alpha_{\sigma(4)}}{}_{\beta_4}}. \tag{14.67}$$

Contracting successively on the indices α_1 and β_1, we obtain a series of useful formulas:

$$\varepsilon^{\mu\,\alpha_1\,\alpha_2\,\alpha_3}\ \varepsilon_{\mu\,\beta_1\,\beta_2\,\beta_3} = -\sum_{\sigma\in\mathfrak{S}_3}(-1)^{k(\sigma)}\ \delta^{\alpha_{\sigma(1)}}{}_{\beta_1}\ \delta^{\alpha_{\sigma(2)}}{}_{\beta_2}\ \delta^{\alpha_{\sigma(3)}}{}_{\beta_3}. \tag{14.68}$$

$$\varepsilon^{\mu\nu\,\alpha_1\,\alpha_2}\ \varepsilon_{\mu\nu\,\beta_1\,\beta_2} = -2\left(\delta^{\alpha_1}{}_{\beta_1}\ \delta^{\alpha_2}{}_{\beta_2} - \delta^{\alpha_2}{}_{\beta_1}\ \delta^{\alpha_1}{}_{\beta_2}\right). \tag{14.69}$$

$$\varepsilon^{\mu\nu\rho\alpha}\ \varepsilon_{\mu\nu\rho\beta} = -6\,\delta^{\alpha}{}_{\beta}. \tag{14.70}$$

$$\varepsilon^{\mu\nu\rho\sigma}\ \varepsilon_{\mu\nu\rho\sigma} = -24. \tag{14.71}$$

Identities (14.67)-(14.71) can be re-expressed in a single writing, valid for $p \in \{0, 1, 2, 3, 4\}$:

$$\boxed{\varepsilon^{\mu_1\ldots\mu_{4-p}\alpha_1\ldots\alpha_p}\ \varepsilon_{\mu_1\ldots\mu_{4-p}\beta_1\ldots\beta_p} = -(4-p)!\sum_{\sigma\in\mathfrak{S}_p}(-1)^{k(\sigma)}\ \delta^{\alpha_{\sigma(1)}}{}_{\beta_1}\ldots\delta^{\alpha_{\sigma(p)}}{}_{\beta_p}}. \tag{14.72}$$

14.5.2 Hodge Star

For $p \in \{0, 1, 2, 3, 4\}$, the ***Hodge star*** is the mapping

$$\begin{aligned} \star : \mathscr{A}_p(E) &\longrightarrow \mathscr{A}_{4-p}(E) \\ \boldsymbol{A} &\longmapsto \star\boldsymbol{A} \end{aligned} \tag{14.73}$$

defined by

$$\boxed{\star A_{\alpha_1\ldots\alpha_{4-p}} := \frac{1}{p!}\,\varepsilon_{\mu_1\ldots\mu_p\alpha_1\ldots\alpha_{4-p}}\ g^{\mu_1\nu_1}\ldots g^{\mu_p\nu_p}\ A_{\nu_1\ldots\nu_p}}. \tag{14.74}$$

Explicitly,

$$p = 0: \quad \star A_{\alpha\beta\gamma\delta} := A\,\varepsilon_{\alpha\beta\gamma\delta} \tag{14.75a}$$

$$p = 1: \quad \star A_{\alpha\beta\gamma} := \varepsilon_{\mu\alpha\beta\gamma}\ g^{\mu\rho}\ A_\rho = A_\mu\varepsilon^{\mu}{}_{\alpha\beta\gamma} \tag{14.75b}$$

$$p = 2: \quad \star A_{\alpha\beta} := \frac{1}{2}\,\varepsilon_{\mu\nu\alpha\beta}\ g^{\mu\rho}g^{\nu\sigma}\ A_{\rho\sigma} = \frac{1}{2}\,A_{\mu\nu}\varepsilon^{\mu\nu}{}_{\alpha\beta} \tag{14.75c}$$

$$p = 3: \quad \star A_\alpha := \frac{1}{6}\, \varepsilon_{\mu\nu\lambda\alpha}\, g^{\mu\rho} g^{\nu\sigma} g^{\lambda\tau}\, A_{\rho\sigma\tau} = \frac{1}{6}\, A_{\mu\nu\rho}\, \varepsilon^{\mu\nu\rho}{}_{\alpha} \tag{14.75d}$$

$$p = 4: \quad \star A := \frac{1}{24}\, \varepsilon_{\mu\nu\lambda\kappa}\, g^{\mu\rho} g^{\nu\sigma} g^{\lambda\tau} g^{\kappa\iota}\, A_{\rho\sigma\tau\iota} = \frac{1}{24} A_{\mu\nu\rho\sigma}\, \varepsilon^{\mu\nu\rho\sigma}. \tag{14.75e}$$

The second equalities on each line let appear the tensors ${}^{p}\boldsymbol{\varepsilon}$ according to (14.62). We observe that if $\boldsymbol{A}$ is a p-form, $\star\boldsymbol{A}$ is a $(4-p)$-form, so the mapping (14.73) is well defined. Moreover, $\star$ is clearly a linear mapping.

For any p-form $\boldsymbol{A}$, $\star\star\boldsymbol{A}$ is again a p-form. Let us evaluate it by applying (14.74) twice:

$$\begin{aligned}
\star\star A_{\alpha_1\ldots\alpha_p} &= \frac{1}{(4-p)!}\, \varepsilon_{\mu_1\ldots\mu_{4-p}\alpha_1\ldots\alpha_p}\, g^{\mu_1\nu_1}\ldots g^{\mu_{4-p}\nu_{4-p}}\, \star A_{\nu_1\ldots\nu_{4-p}} \\
&= \frac{1}{(4-p)!}\frac{1}{p!}\, \varepsilon_{\mu_1\ldots\mu_{4-p}\alpha_1\ldots\alpha_p}\, g^{\mu_1\nu_1}\ldots g^{\mu_{4-p}\nu_{4-p}} \times \\
&\qquad \times \varepsilon_{\rho_1\ldots\rho_p\nu_1\ldots\nu_{4-p}}\, g^{\rho_1\lambda_1}\ldots g^{\rho_p\lambda_p}\, A_{\lambda_1\ldots\lambda_p} \\
&= \frac{1}{p!(4-p)!}\, \varepsilon_{\mu_1\ldots\mu_{4-p}\alpha_1\ldots\alpha_p}\, \underbrace{\varepsilon^{\lambda_1\ldots\lambda_p\mu_1\ldots\mu_{4-p}}}_{(-1)^p\varepsilon^{\mu_1\ldots\mu_{4-p}\lambda_1\ldots\lambda_p}}\, A_{\lambda_1\ldots\lambda_p} \\
&= \frac{(-1)^p}{p!(4-p)!}\, \varepsilon^{\mu_1\ldots\mu_{4-p}\lambda_1\ldots\lambda_p}\, \varepsilon_{\mu_1\ldots\mu_{4-p}\alpha_1\ldots\alpha_p}\, A_{\lambda_1\ldots\lambda_p}.
\end{aligned}$$

Using (14.72), there comes

$$\star\star A_{\alpha_1\ldots\alpha_p} = \frac{(-1)^{p+1}}{p!} \sum_{\sigma\in\mathfrak{S}_p} (-1)^{k(\sigma)}\, \delta^{\lambda_{\sigma(1)}}{}_{\alpha_1}\ldots\delta^{\lambda_{\sigma(p)}}{}_{\alpha_p}\, A_{\lambda_1\ldots\lambda_p}.$$

Now, since $\boldsymbol{A}$ is fully antisymmetric, for any permutation $\sigma \in \mathfrak{S}_p$

$$(-1)^{k(\sigma)}\, \delta^{\lambda_{\sigma(1)}}{}_{\alpha_1}\ldots\delta^{\lambda_{\sigma(p)}}{}_{\alpha_p}\, A_{\lambda_1\ldots\lambda_p} = \delta^{\lambda_1}{}_{\alpha_1}\ldots\delta^{\lambda_p}{}_{\alpha_p}\, A_{\lambda_1\ldots\lambda_p} = A_{\alpha_1\ldots\alpha_p}.$$

The cardinal of $\mathfrak{S}_p$ being $p!$, we deduce that

$$\star\star A_{\alpha_1\ldots\alpha_p} = (-1)^{p+1} A_{\alpha_1\ldots\alpha_p}. \tag{14.76}$$

In other words,

$$\forall \boldsymbol{A} \in \mathscr{A}_p(E), \quad \boxed{\star\star\boldsymbol{A} = (-1)^{p+1}\boldsymbol{A}}. \tag{14.77}$$

This relation shows that the Hodge star is an invertible mapping, its inverse being itself, up to a factor $(-1)^{p+1}$. We conclude that, for a given $p \in \{0, 1, 2, 3, 4\}$, the Hodge star is an isomorphism between the vector spaces $\mathscr{A}_p(E)$ and $\mathscr{A}_{4-p}(E)$. This isomorphism implements the ***Hodge duality*** between the p-forms and the $(4-p)$-forms. One says that the $(4-p)$-form $\star\boldsymbol{A}$ is the ***Hodge dual*** of the p-form $\boldsymbol{A}$.

Remark 14.11. For $p = 2$, $4 - p = 2$, so that the Hodge star is an isomorphism of $\mathscr{A}_2(E)$ into itself, i.e. an automorphism.

Remark 14.12. We have encountered three types of duality that should not be confused:

- The *canonical duality* between the vector spaces E and E^{**}, according to which any vector in E can be considered as a linear form on E^* [cf. (14.3)]
- The *metric duality*, which, by means of the metric tensor, establishes a bijective correspondence between vectors in E and linear forms on E (cf. Sect. 1.6.2)
- The *Hodge duality*, which, by means of the Levi–Civita tensor and the metric tensor, establishes a bijective correspondence between p-forms and $(4-p)$-forms

Note that the first duality is independent of any structure on E (such as the metric tensor), hence the qualifier *canonical*.

14.5.3 Hodge Star and Exterior Product

Given two linear forms $\boldsymbol{a}$ and $\boldsymbol{b}$, the exterior product $\boldsymbol{a} \wedge \boldsymbol{b}$ is a 2-form. Its Hodge dual, $\star(\boldsymbol{a} \wedge \boldsymbol{b})$, is a 2-form as well. Let us express its components in a basis $(\vec{\boldsymbol{e}}_\alpha)$ of E, via the successive use of (14.75c), (14.43) and (14.29):

$$\begin{aligned} \star(a \wedge b)_{\alpha\beta} &= \frac{1}{2}\,\varepsilon_{\mu\nu\alpha\beta}\; g^{\mu\rho} g^{\nu\sigma}\; (a \wedge b)_{\rho\sigma} = \frac{1}{2}\,\varepsilon_{\mu\nu\alpha\beta}\; g^{\mu\rho} g^{\nu\sigma}\; (a_\rho b_\sigma - a_\sigma b_\rho) \\ &= \frac{1}{2}\,\varepsilon_{\mu\nu\alpha\beta}(a^\mu b^\nu - a^\nu b^\mu) \\ &= \varepsilon_{\mu\nu\alpha\beta} a^\mu b^\nu, \end{aligned} \tag{14.78}$$

where we have used the antisymmetry of $\boldsymbol{\varepsilon}$ to get the last line. We conclude thus that

$$\forall (\boldsymbol{a}, \boldsymbol{b}) \in \mathscr{A}_1(E)^2, \quad \boxed{\star(\boldsymbol{a} \wedge \boldsymbol{b}) = \boldsymbol{\varepsilon}(\vec{\boldsymbol{a}}, \vec{\boldsymbol{b}}, ., .)}. \tag{14.79}$$

14.5.4 Orthogonal Decomposition of 2-Forms

An interesting application of (14.79) regards the orthogonal decomposition of 2-forms established in Sect. 3.5.2: given a 2-form $\boldsymbol{A}$ and a unit timelike vector $\vec{\boldsymbol{u}}$, there exists a unique linear form $\boldsymbol{q} \in E^*$ and a unique vector $\vec{\boldsymbol{b}} \in E$ such that $\boldsymbol{A}$ can be written as (3.37). We recognize in the first term of the right-hand side of (3.37) the exterior product of the linear forms $\underline{\boldsymbol{u}}$ and $\boldsymbol{q}$. According to (14.79), the second term is nothing but the Hodge dual of the exterior product of the linear forms $\underline{\boldsymbol{u}}$ and $\underline{\boldsymbol{b}}$. We can thus rewrite the decomposition (3.37) in the compact form

$$\boxed{\boldsymbol{A} = \underline{\boldsymbol{u}} \wedge \boldsymbol{q} + \star(\underline{\boldsymbol{u}} \wedge \underline{\boldsymbol{b}})}, \quad \boxed{\langle \boldsymbol{q}, \vec{\boldsymbol{u}} \rangle = 0} \quad \text{and} \quad \boxed{\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{b}} = 0}. \tag{14.80}$$

The Hodge star will allow us to express the vector $\vec{\boldsymbol{b}}$ in terms of $\boldsymbol{A}$ and $\vec{\boldsymbol{u}}$, as we have already expressed $\boldsymbol{q}$ in terms of $\boldsymbol{A}$ and $\vec{\boldsymbol{u}}$ [Eq. (3.40)]. Indeed, let us take the Hodge star of (14.80); using (14.79) and (14.77) with $p = 2$, we get

$$\star\boldsymbol{A} = \star(\underline{\boldsymbol{u}} \wedge \boldsymbol{q}) + \star\star(\underline{\boldsymbol{u}} \wedge \underline{\boldsymbol{b}}) = \boldsymbol{\varepsilon}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{q}}, ., .) - \underline{\boldsymbol{u}} \wedge \underline{\boldsymbol{b}}.$$

Setting the first argument of this 2-form to $\vec{\boldsymbol{u}}$, we obtain a linear form:

$$\star\boldsymbol{A}(\vec{\boldsymbol{u}}, .) = \underbrace{\boldsymbol{\varepsilon}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{q}}, \vec{\boldsymbol{u}}, .)}_{0} - \underbrace{\langle \underline{\boldsymbol{u}}, \vec{\boldsymbol{u}} \rangle}_{-1} \underline{\boldsymbol{b}} + \underbrace{\langle \underline{\boldsymbol{b}}, \vec{\boldsymbol{u}} \rangle}_{\vec{\boldsymbol{b}} \cdot \vec{\boldsymbol{u}} = 0} \underline{\boldsymbol{u}} = \underline{\boldsymbol{b}}.$$

We have thus

$$\boxed{\underline{\boldsymbol{b}} = \star\boldsymbol{A}(\vec{\boldsymbol{u}}, .)}. \tag{14.81}$$

It is instructive to compare this relation with formula (3.40) expressing $\boldsymbol{q}$:

$$\boxed{\boldsymbol{q} := \boldsymbol{A}(., \vec{\boldsymbol{u}})}. \tag{14.82}$$

Hence, in the decomposition (14.80), the linear form $\boldsymbol{q}$ is obtained directly from $\boldsymbol{A}$, whereas the vector $\vec{\boldsymbol{b}}$ is obtained from the Hodge dual of $\boldsymbol{A}$.

In terms of components, (14.81) can be written, via (14.75c), as $b_\alpha = A_{\mu\nu}\, \varepsilon^{\mu\nu}{}_{\rho\alpha}\, u^\rho/2$. This can be rearranged as

$$\boxed{b^\alpha = -\frac{1}{2}\, \varepsilon^{\alpha\mu\nu}{}_{\rho}\, A_{\mu\nu}\, u^\rho}. \tag{14.83}$$

Example 14.11. The angular momentum vector $\vec{\boldsymbol{\sigma}}_C$ of a particle with respect to a point C and measured by an observer, as defined by (10.8), can be expressed in terms of the angular momentum 2-form $\boldsymbol{J}_C$ according to

$$\underline{\boldsymbol{\sigma}}_C = \star \boldsymbol{J}_C(\vec{\boldsymbol{u}}_0, .) \qquad \Longleftrightarrow \qquad \sigma^\alpha_C = -\frac{1}{2}\,\varepsilon^{\alpha\mu\nu}{}_\rho\,(J_C)_{\mu\nu}\,u^\rho_0, \tag{14.84}$$

where $\vec{\boldsymbol{u}}_0$ is the observer's 4-velocity.

Example 14.12. The spin vector $\vec{\boldsymbol{s}}$ of a particle, as defined by (10.74), can be expressed in terms of the spin 2-form $\boldsymbol{S}$ according to

$$\underline{\boldsymbol{s}} = \star \boldsymbol{S}(\vec{\boldsymbol{u}}, .) \qquad \Longleftrightarrow \qquad s^\alpha = -\frac{1}{2}\,\varepsilon^{\alpha\mu\nu}{}_\rho\,S_{\mu\nu}\,u^\rho, \tag{14.85}$$

where $\vec{\boldsymbol{u}}$ is the particle's 4-velocity.

Chapter 15
Fields on Spacetime

15.1 Introduction

The preceding chapter having introduced tensors on the vector space E underlying Minkowski spacetime $\mathscr{E}$, we move now to the notion of *tensor field*, i.e. to the prescription of a tensor at each point of the affine space $\mathscr{E}$. This chapter and the following one, dealing with the integration of tensor fields, are purely mathematical. They introduce the basic tools for the subsequent physical chapters devoted to electromagnetism, hydrodynamics and gravitation.

15.2 Arbitrary Coordinates on Spacetime

15.2.1 Coordinate System

Up to now, we have considered as coordinates on the whole spacetime $\mathscr{E}$ only *affine coordinates* (Sect. 1.2.3).[1] If the associated vector basis is orthonormal, affine coordinates are coordinates of some inertial observer and are then called *inertial coordinates* (Sect. 8.2.3). At the local level, in the vicinity of a worldline, we have defined the *coordinates with respect to an observer* (Sect. 3.4.2). From a pure mathematical point of view, one can however introduce on $\mathscr{E}$ any type of coordinates, i.e. general curvilinear coordinates, not necessarily related to some observer. More precisely, one calls ***coordinate system on*** $\mathscr{E}$ any mapping

$$\begin{aligned} \Phi : \mathscr{E} &\longrightarrow \mathbb{R}^4 \\ M &\longmapsto (x^0, x^1, x^2, x^3) \end{aligned} \tag{15.1}$$

[1] In Chap. 12, we have introduced, on a part of $\mathscr{E}$, Rindler coordinates, which differ from affine coordinates.

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that is injective (Φ is then bijective between $\mathscr{E}$ and $\Phi(\mathscr{E})$) and such that both Φ and Φ^{-1} are differentiable (one says that Φ is a ***diffeomorphism*** between $\mathscr{E}$ and $\Phi(\mathscr{E})$).

Example 15.1. Any affine coordinate system on $\mathscr{E}$, as defined in Sect. 1.2.3, is of course a coordinate system in the above sense.

Example 15.2. Given a system of inertial coordinates on $\mathscr{E}$, $(x_*^\alpha) = (ct, x, y, z)$, one defines the associated ***spherical coordinates*** $(x^\alpha) = (ct, r, \theta, \varphi)$ by $x^0 = x_*^0 = ct$ and

$$\begin{cases} x = r \sin\theta \cos\varphi \\ y = r \sin\theta \sin\varphi \\ z = r \cos\theta . \end{cases} \tag{15.2}$$

We have then $r \in [0, +\infty[$, $\theta \in [0, \pi]$ and $\varphi \in [0, 2\pi[$. Note that these coordinates are singular in the (timelike) plane of $\mathscr{E}$ defined by $x = y = 0$.

15.2.2 Coordinate Basis

Let (x^α) be a coordinate system on $\mathscr{E}$. At any point $M \in \mathscr{E}$ and for any $\alpha \in \{0, 1, 2, 3\}$, one defines a vector $\vec{\boldsymbol{e}}_\alpha(M) \in E$ that describes the increase of the coordinate x^α in the vicinity of M as follows: if (x^0, x^1, x^2, x^3) are the coordinates of M and if M_0 is the point of coordinates $(x^0 + \mathrm{d}x^0, x^1, x^2, x^3)$ with $\mathrm{d}x^0$ infinitely small, then

$$\overrightarrow{MM_0} = \mathrm{d}x^0\, \vec{\boldsymbol{e}}_0(M). \tag{15.3}$$

Similarly, if M_1, M_2 and M_3 are the points of respective coordinates $(x^0, x^1 + \mathrm{d}x^1, x^2, x^3)$, $(x^0, x^1, x^2 + \mathrm{d}x^2, x^3)$ and $(x^0, x^1, x^2, x^3 + \mathrm{d}x^3)$, then

$$\overrightarrow{MM_1} = \mathrm{d}x^1\, \vec{\boldsymbol{e}}_1(M), \qquad \overrightarrow{MM_2} = \mathrm{d}x^2\, \vec{\boldsymbol{e}}_2(M), \qquad \overrightarrow{MM_3} = \mathrm{d}x^3\, \vec{\boldsymbol{e}}_3(M). \tag{15.4}$$

More generally, if M' is any point close to M of coordinates $(x^\alpha + \mathrm{d}x^\alpha)$, we have

$$\boxed{\overrightarrow{MM'} = \mathrm{d}x^\alpha\, \vec{\boldsymbol{e}}_\alpha(M)}. \tag{15.5}$$

Example 15.3. If (x^α) is a system of affine coordinates of $\mathscr{E}$ of origin O, then $\overrightarrow{OM} = x^\alpha\, \vec{\boldsymbol{\varepsilon}}_\alpha$, where $(\vec{\boldsymbol{\varepsilon}}_\alpha)$ is the vector basis of E associated with the affine coordinates (x^α); it is clear that $\overrightarrow{MM'} = \mathrm{d}x^\alpha \vec{\boldsymbol{\varepsilon}}_\alpha$. We conclude that the vectors $\vec{\boldsymbol{e}}_\alpha(M)$ defined by (15.3)–(15.4) are constant and equal to the basis vectors of the affine frame: $\vec{\boldsymbol{e}}_\alpha(M) = \vec{\boldsymbol{\varepsilon}}_\alpha$.

For any coordinate system (x^α), the vectors $(\vec{e}_\alpha(M))$ defined by (15.3)–(15.4) constitute a basis of the vector space E at any point $M \in \mathscr{E}$. This basis is called the ***coordinate basis***, or ***natural basis***, associated with coordinates (x^α).

Moreover, if (x'^α) is another coordinate system on $\mathscr{E}$, its coordinate basis $(\vec{e}'_\alpha)$ is related to that of (x^α) by

$$\boxed{\vec{e}_\alpha(M) = \frac{\partial x'^\beta}{\partial x^\alpha}\, \vec{e}'_\beta(M)}\,. \tag{15.6}$$

Proof. For a fixed $\alpha \in \{0, 1, 2, 3\}$, let M_α be the point deduced from M by an infinitesimal change ε of the coordinate x^α. We have, from (15.3)–(15.4),

$$\overrightarrow{MM_\alpha} = \varepsilon\, \vec{e}_\alpha(M). \tag{15.7}$$

In the second system, if (x'^β) are the coordinates of M, those of M_α are $(x'^\beta + \mathrm{d}x'^\beta)$ with $\mathrm{d}x'^\beta = \partial x'^\beta/\partial x^\alpha\, \varepsilon$. Formula (15.5) yields then

$$\overrightarrow{MM_\alpha} = \mathrm{d}x'^\beta\, \vec{e}'_\beta(M) = \frac{\partial x'^\beta}{\partial x^\alpha}\, \varepsilon\, \vec{e}'_\beta(M).$$

Comparing with (15.7), we get (15.6). If one picks for (x'^α) a system of affine coordinates, then $(\vec{e}'_\beta(M))$ is a vector basis of E (cf. Example 15.3). Moreover, provided that the coordinate system (x^α) is regular around M, the Jacobian matrix $(\partial x'^\beta/\partial x^\alpha)$ is invertible. We deduce then from (15.6) that $(\vec{e}_\alpha(M))$ is a vector basis of E. □

Example 15.4. Let us consider again the spherical coordinated introduced in Example 15.2. The Jacobian matrix $(\partial x_*^\beta/\partial x^\alpha)$ is easily computed from (15.2), and (15.6) (with $x'^\beta = x_*^\beta$) leads to

$$\begin{cases} \vec{e}_0(M) = \vec{e}_{ct} \\ \vec{e}_r(M) = \sin\theta\cos\varphi\, \vec{e}_x + \sin\theta\sin\varphi\, \vec{e}_y + \cos\theta\, \vec{e}_z \\ \vec{e}_\theta(M) = r\cos\theta\cos\varphi\, \vec{e}_x + r\cos\theta\sin\varphi\, \vec{e}_y - r\sin\theta\, \vec{e}_z \\ \vec{e}_\varphi(M) = -r\sin\theta\sin\varphi\, \vec{e}_x + r\sin\theta\cos\varphi\, \vec{e}_y, \end{cases} \tag{15.8}$$

with the notations $\vec{e}_r := \vec{e}_1$, $\vec{e}_\theta := \vec{e}_2$, $\vec{e}_\varphi := \vec{e}_3$, $\vec{e}_{ct} = \vec{e}^{\,*}_0$, $\vec{e}_x := \vec{e}^{\,*}_1$, $\vec{e}_y := \vec{e}^{\,*}_2$ and $\vec{e}_z := \vec{e}^{\,*}_3$. The vectors $\vec{e}_r$ and $\vec{e}_\varphi$ are depicted in Fig. 15.1. We notice that the coordinate basis $(\vec{e}_\alpha)$ is not orthonormal (albeit orthogonal).

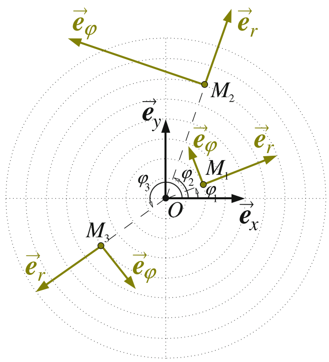

Fig. 15.1 Vectors $\vec{e}_r$ and $\vec{e}_\varphi$ of the coordinate basis associated with spherical coordinates at three points M_1, M_2 and M_3 of the plane $(t = 0, \theta = \pi/2)$

Remark 15.1. In differential geometry, the vectors of the coordinate basis associated with (x^α) are denoted by $\partial/\partial x^\alpha$. This notation stems from the definition of a vector on a manifold as a differential operator on scalar fields. We had not to use this definition in the present case because the concept of vector has been provided from the very beginning by the structure of affine space for $\mathscr{E}$. Note however that, by the chain rule, the operators $\partial/\partial x^\alpha$ and $\partial/\partial x'^\beta$ obey the same relation as that between the vectors $\vec{e}_\alpha$ and $\vec{e}\,'_\beta$ [Eq. (15.6)]:

$$\frac{\partial}{\partial x^\alpha} = \frac{\partial x'^\beta}{\partial x^\alpha}\frac{\partial}{\partial x'^\beta},$$

which is fully compatible with the identification $\vec{e}_\alpha = \partial/\partial x^\alpha$.

15.2.3 Components of the Metric Tensor

Let (x^α) and (x'^α) be two coordinate systems on $\mathscr{E}$ and $(\vec{e}_\alpha)$ and $(\vec{e}\,'_\alpha)$ the associated coordinate bases. At each point $M \in \mathscr{E}$, the components $g_{\alpha\beta}(M)$ of the metric tensor $\boldsymbol{g}$ in the basis $(\vec{e}_\alpha(M))$ are given by (1.12). Substituting $\vec{e}_\alpha(M)$ by (15.6) in this formula, we get

$$g_{\alpha\beta}(M) = \vec{e}_\alpha(M)\cdot\vec{e}_\beta(M) = \frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial x'^\nu}{\partial x^\beta}\underbrace{\vec{e}\,'_\mu(M)\cdot\vec{e}\,'_\nu(M)}_{g'_{\mu\nu}(M)}.$$

Hence, the relation between the components of $\boldsymbol{g}$ in the bases associated with the two coordinate systems:

$$\boxed{g_{\alpha\beta}(M) = \frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial x'^\nu}{\partial x^\beta}\,g'_{\mu\nu}(M)}. \tag{15.9}$$

Example 15.5 (Spherical coordinates). For the spherical coordinates $(x^\alpha) = (ct, r, \theta, \varphi)$ introduced in Example 15.2 p. 496, the direct computation of $g_{\alpha\beta} = \vec{e}_\alpha \cdot \vec{e}_\beta$ from (15.8) [using the orthonormality of the basis $(\vec{e}^{\,*}_\alpha)$] results in

$$g_{\alpha\beta}(M) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix}, \tag{15.10}$$

where r and θ are the coordinates x^1 and x^2 of M. That $g_{\alpha\beta}(M) \neq \eta_{\alpha\beta}$ indicates that the coordinate basis $(\vec{e}_0, \vec{e}_r, \vec{e}_\theta, \vec{e}_\varphi)$ associated with spherical coordinates is not orthonormal, which had already been noticed on Fig. 15.1.

Example 15.6 (Null coordinates). From the spherical coordinates (ct, r, θ, φ) of the previous example, denoted hereafter by (x'^α), one defines the ***null coordinates*** $(x^\alpha) = (u, v, \theta, \varphi)$ by

$$\begin{cases} u := ct - r \\ v := ct + r \end{cases} \iff \begin{cases} ct = (u+v)/2 \\ r \;\, = (v-u)/2 \end{cases} \tag{15.11}$$

These coordinates are depicted in Fig. 15.2. We observe that the hypersurfaces obtained by fixing u and letting (v, θ, φ) vary (resp. by fixing v and letting $(u, \theta, \varphi$ vary)) are the future (resp. past) light cones centred on $r = 0$, hence the name given to these coordinates. The Jacobian matrix of the coordinate change (15.11) is

$$\frac{\partial x'^\beta}{\partial x^\alpha} = \begin{array}{cc} & \beta \rightarrow \\ \begin{array}{c} \alpha \\ \downarrow \\ \\ \\ \end{array} & \begin{pmatrix} 1/2 & -1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

We deduce then from (15.6) the first two vectors $\vec{e}_u := \vec{e}_0$ and $\vec{e}_v := \vec{e}_1$ of the coordinate basis:

$$\vec{e}_u(M) = \frac{1}{2}\left[\vec{e}_{ct} - \vec{e}_r(M)\right] \quad \text{and} \quad \vec{e}_v(M) = \frac{1}{2}\left[\vec{e}_{ct} + \vec{e}_r(M)\right], \tag{15.12}$$

the last two vectors being nothing but $\vec{e}_\theta$ and $\vec{e}_\varphi$. Vectors $\vec{e}_u$ and $\vec{e}_v$ are depicted in Fig. 15.2. The components of the metric tensor in the null coordinates are easily obtained from the formula $g_{\alpha\beta} = \vec{e}_\alpha \cdot \vec{e}_\beta$:

$$g_{\alpha\beta}(M) = \begin{pmatrix} 0 & -1/2 & 0 & 0 \\ -1/2 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix}. \tag{15.13}$$

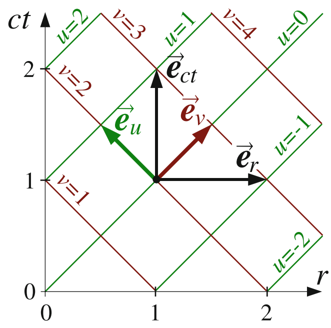

Fig. 15.2 Null coordinates (u, v) in a plane $(\theta = \mathrm{const}, \varphi = \mathrm{const})$

That the diagonal of the above matrix starts by two zeros means that $\vec{e}_u$ and $\vec{e}_v$ are null vectors. Null coordinates are much used in quantum chromodynamics (cf., e.g. Brodsky et al. (1998)).

Remark 15.2. The basis $(\vec{e}_u, \vec{e}_v, \vec{e}_\theta, \vec{e}_\varphi)$ is not made of one timelike vector and three spacelike vectors, as all the bases encountered up to now. In particular, it is not obvious when reading (15.13) that the signature of $\boldsymbol{g}$ is $(-, +, +, +)$.

Example 15.7 (Rindler coordinates). Rindler coordinates have been introduced in Sect. 12.2.7 as the coordinates with respect to a uniformly accelerated observer $\mathscr{O}$. They are depicted in Fig. 12.8. Here, we shall denote them by $(x^\alpha) = (c\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z})$ in order to keep the notation (ct, x, y, z) for the inertial coordinates (x'^α). The relation between the two coordinate systems is given by (12.35):

$$\begin{cases} ct = (\tilde{x} + a^{-1}) \sinh(ac\tilde{t}) & \\ x = (\tilde{x} + a^{-1}) \cosh(ac\tilde{t}) - a^{-1} & \qquad \tilde{t} \in \mathbb{R} \\ y = \tilde{y} & \qquad \tilde{x} > -a^{-1} \\ z = \tilde{z}, & \end{cases}$$

where the constant a is the metric norm of the 4-acceleration of observer $\mathscr{O}$. The Jacobian matrix describing the transition from Rindler coordinates (x^α) to inertial coordinates (x'^β) is then

$$\frac{\partial x'^\beta}{\partial x^\alpha} = \begin{array}{c} \\ \alpha \\ \downarrow \\ \\ \\ \end{array} \begin{array}{c} \beta \rightarrow \\ \begin{pmatrix} (1 + a\tilde{x}) \cosh(ac\tilde{t}) & (1 + a\tilde{x}) \sinh(ac\tilde{t}) & 0 & 0 \\ \sinh(ac\tilde{t}) & \cosh(ac\tilde{t}) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

Formula (15.6) provides the first two vectors $\vec{e}_{c\tilde{t}} := \vec{e}_0$ and $\vec{e}_{\tilde{x}} := \vec{e}_1$ of the coordinate basis associated with Rindler coordinates:

$$\vec{e}_{c\tilde{t}}(M) = (1 + a\tilde{x}) \left[\cosh(ac\tilde{t})\, \vec{e}_{ct} + \sinh(ac\tilde{t})\, \vec{e}_x\right] \tag{15.14}$$

$$\vec{e}_{\tilde{x}}(M) = \sinh(ac\tilde{t})\, \vec{e}_{ct} + \cosh(ac\tilde{t})\, \vec{e}_x, \tag{15.15}$$

the last two vectors being equal to $\vec{e}_y$ and $\vec{e}_z$. Comparing with (12.30), we can relate $\vec{e}_{c\tilde{t}}$ and $\vec{e}_{\tilde{x}}$ to the first two vectors of the local frame of observer $\mathscr{O}$, $\vec{e}_0^{\,obs}$ and $\vec{e}_1^{\,obs}$, taken at the same instant $\tilde{t}$ as M:

$$\vec{e}_{c\tilde{t}}(M) = (1 + a\tilde{x})\, \vec{e}_0^{\,obs}(\tilde{t}) \qquad \text{and} \qquad \vec{e}_{\tilde{x}}(M) = \vec{e}_1^{\,obs}(\tilde{t}). \tag{15.16}$$

If $\tilde{x} \neq 0$, i.e. if M is not located on $\mathscr{O}$'s worldline, we observe that $\vec{e}_{c\tilde{t}}(M) \neq \vec{e}_0^{\,obs}(\tilde{t})$. Since $\vec{e}_{\tilde{y}}(M) = \vec{e}_y = \vec{e}_2^{\,obs}(\tilde{t})$, $\vec{e}_{\tilde{z}}(M) = \vec{e}_z = \vec{e}_3^{\,obs}(\tilde{t})$ and $(\vec{e}_\alpha^{\,obs}(\tilde{t}))$ is an orthonormal basis, we deduce immediately from (15.16) the components of the metric tensor in the Rindler coordinate basis:

$$g_{\alpha\beta}(M) = \begin{pmatrix} -(1 + a\tilde{x})^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{15.17}$$

Note that this coordinate basis is not orthonormal as soon as the point M is not on the worldline of the accelerated observer ($\tilde{x} \neq 0$).

Example 15.8 (Rotating coordinates). From the coordinates (ct, x, y, z) with respect to a uniformly rotating observer introduced in Sect. 13.3.1 [cf. Eq. (13.9)], we can construct a spherical coordinate system $(x^\alpha) = (ct, r, \theta, \varphi)$ by the standard formulas (15.2). It is then easy to see that these coordinates are related to the spherical coordinates $(x'^\alpha) = (ct', r', \theta', \varphi')$ associated with an inertial observer (cf. Example 15.5) by $t' = t$, $r' = r$, $\theta' = \theta$ and

$$\varphi' = \varphi + \omega t,$$

where ω is the norm of the 4-rotation of the rotating observer. The Jacobian matrix linking the two spherical coordinate systems is then

$$\frac{\partial x'^\beta}{\partial x^\alpha} = \begin{array}{c} \\ \alpha \\ \downarrow \end{array} \overset{\beta \rightarrow}{\begin{pmatrix} 1 & 0 & 0 & \omega \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}.$$

The components $(g'_{\alpha\beta})$ of the metric tensor with respect to the inertial spherical coordinates being given by (15.10), the law (15.9) of change of coordinate basis leads to

$$g_{\alpha\beta}(M) = \begin{pmatrix} -1 + \omega^2 r^2 \sin^2\theta & 0 & 0 & \omega r^2 \sin^2\theta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ \omega r^2 \sin^2\theta & 0 & 0 & r^2 \sin^2\theta \end{pmatrix}. \tag{15.18}$$

These components are sometimes called ***Langevin metric***, referring to studies of Paul Langevin (cf. p. 40) (1921; 1935). This name is somewhat misleading, for (15.18) does not represent a new metric but the metric $\boldsymbol{g}$ of Minkowski spacetime expressed in rotating coordinates.

Historical note: *The description of Minkowski spacetime by means of arbitrary coordinates has been developed in 1955 by Vladimir A. Fock*[2] *in his famous treatise on relativity (Fock 1955). In this respect, it is worth mentioning that Fock, who was professor at Leningrad University, greatly contributed to the spreading of relativity in USSR.*

15.3 Tensor Fields

15.3.1 Definitions

Let us recall that for $(k,\ell) \in \mathbb{N}^2$, $\mathscr{T}_{(k,\ell)}(E)$ stands for the set of all tensors of type (k,ℓ) on the vector space E, with the convention $\mathscr{T}_{(0,0)}(E) = \mathbb{R}$ (cf. Sect. 14.2). One calls ***tensor field of type*** (k,ℓ) ***on*** $\mathscr{E}$ any mapping

$$\begin{aligned} \boldsymbol{T} : \mathscr{E} &\longrightarrow \mathscr{T}_{(k,\ell)}(E) \\ M &\longmapsto \boldsymbol{T}(M). \end{aligned} \tag{15.19}$$

Unless explicitly mentioned, we shall always assume that this mapping is smooth (infinitely differentiable). If $(k,\ell) = (0,0)$, the arrival set of (15.19) is $\mathbb{R}$ and $\boldsymbol{T}$ is called a ***scalar field***. If $(k,\ell) = (1,0)$, $\boldsymbol{T}$ is called a ***vector field***, since $\mathscr{T}_{(1,0)}(E) = E$.

Example 15.9. If (x^α) is a coordinate system on $\mathscr{E}$, the mapping that assigns to each point $M \in \mathscr{E}$ the vector $\vec{\boldsymbol{e}}_0(M)$ of the coordinate basis associated with (x^α) is a vector field on $\mathscr{E}$ (the same property holds for $\vec{\boldsymbol{e}}_1(M)$, $\vec{\boldsymbol{e}}_2(M)$ and $\vec{\boldsymbol{e}}_3(M)$).

Remark 15.3. Except for the coordinate bases introduced in Sect. 15.2.2, we have encountered up to now only tensor fields defined along the worldline $\mathscr{L}$ of some

[2]**Vladimir A. Fock** (1898–1974) : Soviet theoretical physicist, known for his work in quantum mechanics (*Fock space*, *Hartree–Fock approximation*); he contributed also to geophysics and general relativity.

particle, i.e. mappings $\mathscr{L} \longrightarrow \mathscr{T}_{(k,\ell)}$: vector fields (cf. Sect. 2.7.2) (4-velocity, 4-acceleration, spin vector), fields of linear forms (4-momentum, 4-force) and fields of 2-forms (angular momentum with respect to a point, spin).

One calls ***field of bases***, or ***moving frame***, any set of four vector fields on $\mathscr{E}$, $(\vec{e}_\alpha)$, such that at every point M, $(\vec{e}_\alpha(M))$ is a basis of E. To avoid any confusion, we shall sometimes use the expression ***fixed basis*** for a basis of E that is not considered as the value at some point of a moving frame. A field of *orthonormal* bases is called a ***tetrad***.

Example 15.10. Any coordinate basis constitutes a field of bases, but the converse is not true: there exists fields of bases that cannot be associated with a coordinate system on $\mathscr{E}$. For example, from the vectors $(\vec{e}_0, \vec{e}_r, \vec{e}_\theta, \vec{e}_\varphi)$ of the spherical coordinate basis [cf. Eq. (15.8)], one may construct a new basis by setting $\vec{e}\,'_0 := \vec{e}_0$, $\vec{e}\,'_1 := \vec{e}_r$, $\vec{e}\,'_2 := r^{-1}\vec{e}_\theta$ and $\vec{e}\,'_3 := (r\sin\theta)^{-1}\vec{e}_\varphi$. Contrary to $(\vec{e}_0, \vec{e}_r, \vec{e}_\theta, \vec{e}_\varphi)$, the basis $(\vec{e}\,'_\alpha)$ is orthonormal [this is immediate from (15.10)]. It is the orthonormal basis associated usually to spherical coordinates.[3] It can be shown that $(\vec{e}\,'_\alpha)$ is not a coordinate basis: there does not exist any coordinate system on $\mathscr{E}$ whose infinitesimal coordinate changes are described by the vectors $(\vec{e}\,'_\alpha)$.

15.3.2 Scalar Field and Gradient

The simplest example of a tensor field is of course a scalar field: $f : \mathscr{E} \longrightarrow \mathbb{R}$. If f is differentiable, one can associate with it a field of linear forms

$$\nabla f : \mathscr{E} \longrightarrow \mathscr{T}_{(0,1)}(E)$$

defined as follows. Between two infinitely close points of $\mathscr{E}$, M and M', the variation $\mathrm{d}f(M) := f(M') - f(M)$ of f is a linear function of the separation between M and M', which is represented by the vector $\overrightarrow{MM'}$. One defines thus $\nabla f(M)$ as the linear form that, applied to the vector $\overrightarrow{MM'}$, gives the variation of f:

$$\mathrm{d}f(M) = \langle \nabla f(M), \overrightarrow{MM'} \rangle.$$

As often in physics, we shall omit the argument M of the functions. Denoting by $\mathrm{d}\vec{x}$ the infinitesimal vector $\overrightarrow{MM'}$, we can then rewrite the above formula as

$$\boxed{\mathrm{d}f = \langle \nabla f, \mathrm{d}\vec{x} \rangle}. \qquad (15.20)$$

[3] In many textbooks, the vectors of this basis are denoted by $(\vec{e}_r, \vec{e}_\theta, \vec{e}_\varphi)$, while we reserve here this notation for the vectors of the coordinate basis.

The field of linear forms ∇f defined in this way is called the ***gradient of*** f. The symbol ∇ is named ***nabla***.

Remark 15.4. It is clear on the definition (15.20) that the gradient of a scalar field is fundamentally a linear form and not a vector. If, as often done in nonrelativistic physics, one defines the gradient as a vector by writing $\mathrm{d}f = \vec{\nabla} f \cdot \mathrm{d}\vec{x}$, one considers implicitly an additional structure on space (spacetime): the scalar product. On the contrary, the definition (15.20) is independent of any scalar product and thus of the metric tensor $\boldsymbol{g}$): it relies only on the primitive notion of vector. Of course, from the gradient linear form ∇f, one may always construct the vector $\vec{\nabla} f$ by metric duality, according to the procedure described in Sect. 1.6.2.

15.3.3 Gradients of Coordinates

Let (x^α) be a coordinate system on $\mathscr{E}$. For a fixed α, we may consider x^α as a scalar field on $\mathscr{E}$. The variation of x^α induced by a small displacement $\mathrm{d}\vec{x}$ around a point M is then given by (15.20):

$$\mathrm{d}x^\alpha = \langle \nabla x^\alpha, \mathrm{d}\vec{x} \rangle. \tag{15.21}$$

On the other side, the vector $\mathrm{d}\vec{x}$ can be expanded on the coordinate basis $(\vec{e}_\alpha)$ associated with (x^α) according to (15.5): $\mathrm{d}\vec{x} = \mathrm{d}x^\beta\, \vec{e}_\beta$. Substituting this expression into (15.21), we get $\mathrm{d}x^\alpha = \langle \nabla x^\alpha, \vec{e}_\beta \rangle\, \mathrm{d}x^\beta$. We deduce that

$$\boxed{\langle \nabla x^\alpha, \vec{e}_\beta \rangle = \delta^\alpha{}_\beta}. \tag{15.22}$$

In other words, at each point $M \in \mathscr{E}$, the 4-tuple of linear forms (∇x^α) constitutes the dual basis of the coordinate basis associated with the coordinates (x^α): according to the notation introduced in Sect. 1.6.1,

$$\boxed{\boldsymbol{e}^\alpha = \nabla x^\alpha}. \tag{15.23}$$

Let f be a scalar field on $\mathscr{E}$. Let us determine the components $\nabla_\alpha f := (\nabla f)_\alpha$ of the gradient of f in the coordinate basis $(\vec{e}_\alpha)$. By the definition of tensor components [Eq. (14.10) with $(k, \ell) = (0, 1)$],

$$\nabla f = \nabla_\alpha f\, \boldsymbol{e}^\alpha = \nabla_\alpha f\, \nabla x^\alpha. \tag{15.24}$$

For any infinitesimal displacement $\mathrm{d}\vec{x} = \mathrm{d}x^\alpha\, \vec{e}_\alpha$, the variation of f given by (15.20) is $\mathrm{d}f = \langle \nabla f, \mathrm{d}\vec{x} \rangle$. Now, with respect to the basis $(\vec{e}_\alpha)$, the components of ∇f are $\nabla_\alpha f$, and those of $\mathrm{d}\vec{x}$ are $\mathrm{d}x^\alpha$. Formula (14.15) leads then to

$$\mathrm{d}f = \langle \nabla f, \mathrm{d}\vec{x} \rangle = \nabla_\alpha f\, \mathrm{d}x^\alpha.$$

Besides, we have obviously $\mathrm{d}f = (\partial f/\partial x^\alpha)\,\mathrm{d}x^\alpha$. Identifying the two formulas, we get

$$\boxed{\nabla_\alpha f = \frac{\partial f}{\partial x^\alpha}}. \tag{15.25}$$

Hence, the components of the gradient in a coordinate basis are nothing by the partial derivatives with respect to the coordinates.

15.4 Covariant Derivative

15.4.1 Covariant Derivative of a Vector

Let $\vec{\boldsymbol{v}}$ be a vector field on $\mathscr{E}$. The variation of $\vec{\boldsymbol{v}}$ between two infinitely close points of $\mathscr{E}$, M and M' is

$$\mathrm{d}\vec{\boldsymbol{v}} := \vec{\boldsymbol{v}}(M') - \vec{\boldsymbol{v}}(M).$$

At first order, this variation must be linear in the separation vector $\mathrm{d}\vec{\boldsymbol{x}} := \overrightarrow{MM'}$. There exists thus an endomorphism of E that we shall denote by $\boldsymbol{\nabla}\vec{\boldsymbol{v}}$, such that $\mathrm{d}\vec{\boldsymbol{v}} = \boldsymbol{\nabla}\vec{\boldsymbol{v}}(\mathrm{d}\vec{\boldsymbol{x}})$. It is customary to write the argument of this endomorphism as an index: $\boldsymbol{\nabla}_{\mathrm{d}\vec{\boldsymbol{x}}}\,\vec{\boldsymbol{v}} := \boldsymbol{\nabla}\vec{\boldsymbol{v}}(\mathrm{d}\vec{\boldsymbol{x}})$. There comes then

$$\boxed{\mathrm{d}\vec{\boldsymbol{v}} = \boldsymbol{\nabla}_{\mathrm{d}\vec{\boldsymbol{x}}}\,\vec{\boldsymbol{v}}}. \tag{15.26}$$

As discussed in Sect. 14.2.2, the endomorphisms of E are identified to type $(1,1)$ tensors. One thus calls ***covariant derivative of*** $\vec{\boldsymbol{v}}$ the tensor field of type $(1,1)$ $\boldsymbol{\nabla}\vec{\boldsymbol{v}}$ that at any point $M \in \mathscr{E}$ gives the variation of $\vec{\boldsymbol{v}}$ resulting from an infinitesimal displacement $\mathrm{d}\vec{\boldsymbol{x}}$ according to formula (15.26).

Example 15.11. Let $(\vec{\boldsymbol{e}}_\alpha)$ be a fixed basis of E and (v^α) the components of the vector field $\vec{\boldsymbol{v}}$ in this basis:

$$\forall M \in \mathscr{E}, \quad \vec{\boldsymbol{v}}(M) = v^\alpha(M)\,\vec{\boldsymbol{e}}_\alpha.$$

This expression gives immediately $\mathrm{d}\vec{\boldsymbol{v}} = \mathrm{d}v^\alpha\,\vec{\boldsymbol{e}}_\alpha$. Since $\mathrm{d}v^\alpha = (\partial v^\alpha/\partial x^\beta)\,\mathrm{d}x^\beta$, we deduce the components, denoted by $\nabla_\beta v^\alpha$, of the covariant derivative $\boldsymbol{\nabla}\vec{\boldsymbol{v}}$ in the basis $(\vec{\boldsymbol{e}}_\alpha)$:

$$\nabla_\beta v^\alpha = \frac{\partial v^\alpha}{\partial x^\beta} \qquad \text{(fixed basis)}. \tag{15.27}$$

Let us stress that this formula is valid only because the basis $(\vec{e}_\alpha)$ is *fixed* (cf. Sect. 15.3.1). For a field of bases (moving frame), a corrective term must be added, as we shall see below.

A vector field $\vec{v}$ on $\mathscr{E}$ induces naturally a vector field along any worldline $\mathscr{L} \subset \mathscr{E}$, in the sense specified in Sect. 2.7.2. We may then consider the derivative $\mathrm{d}\vec{v}/\mathrm{d}t$ of $\vec{v}$ along $\mathscr{L}$, as defined by (2.53) ($\mathscr{L}$ is assumed to be timelike, and its proper time is denoted by t). $\mathrm{d}\vec{v}/\mathrm{d}t$ has been also called the *absolute derivative* of $\vec{v}$ along $\mathscr{L}$ in Sect. 3.6.1 to distinguish it from the derivative with respect to an observer. It is easy to relate $\mathrm{d}\vec{v}/\mathrm{d}t$ to the covariant derivative of the field $\vec{v}$. Indeed, let $A(t)$ and $A(t+\mathrm{d}t)$ be two infinitely close points of $\mathscr{L}$ and $\mathrm{d}\vec{x}$ the vector connecting these two points. From the definition (2.53), $\mathrm{d}\vec{v}/\mathrm{d}t$ is the limit taken for $\mathrm{d}t \to 0$ of the variation of $\vec{v}$ between $A(t)$ and $A(t+\mathrm{d}t)$ divided by $\mathrm{d}t$. Using (15.26) for $\mathrm{d}\vec{v}$, we get

$$\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \frac{\boldsymbol{\nabla}_{\mathrm{d}\vec{x}}\,\vec{v}}{\mathrm{d}t} = \boldsymbol{\nabla}_{\frac{\mathrm{d}\vec{x}}{\mathrm{d}t}}\,\vec{v},$$

where the second equality stems from the linearity of the endomorphism $\boldsymbol{\nabla}\vec{v}$. Now, by the very definition of a 4-velocity, $\mathrm{d}\vec{x}/\mathrm{d}t$ is c times the 4-velocity of the particle having $\mathscr{L}$ as worldline: $\mathrm{d}\vec{x}/\mathrm{d}t = c\,\vec{u}$ [Eq. (2.12)]. We can thus rewrite the above formula as

$$\boxed{\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = c\,\boldsymbol{\nabla}_{\vec{u}}\,\vec{v}}. \tag{15.28}$$

15.4.2 Generalization to All Tensors

The definition of the covariant derivative can be generalized to any kind of tensor field:

If $\boldsymbol{T}$ is a tensor field of type (k,ℓ), its ***covariant derivative*** $\boldsymbol{\nabla}\boldsymbol{T}$ is a tensor field of type $(k,\ell+1)$ such that the variation of $\boldsymbol{T}$ between two infinitely close points M and M' is given by

$$\boxed{\mathrm{d}\boldsymbol{T} := \boldsymbol{T}(M') - \boldsymbol{T}(M) = \boldsymbol{\nabla}_{\mathrm{d}\vec{x}}\,\boldsymbol{T}}, \quad \text{where} \quad \mathrm{d}\vec{x} := \overrightarrow{MM'}. \tag{15.29}$$

The notation $\nabla_{\mathrm{d}\vec{x}} \boldsymbol{T}$ means that the vector $\mathrm{d}\vec{x}$ is the *last* argument of the tensor $\boldsymbol{\nabla T}$:

$$\forall \vec{v} \in E, \quad \boxed{\nabla_{\vec{v}} \boldsymbol{T} := \boldsymbol{\nabla T}(.,\ldots,.,\vec{v})}. \tag{15.30}$$

One says that $\nabla_{\vec{v}} \boldsymbol{T}$ is the ***covariant derivative of*** $\boldsymbol{T}$ ***along the vector*** $\vec{v}$.

Remark 15.5. If $\vec{v}$ is a vector field on $\mathscr{E}$, $\nabla_{\vec{v}} \boldsymbol{T}$ is a tensor field of the same type (k, ℓ) as $\boldsymbol{T}$.

Example 15.12. For a scalar field, the comparison of (15.29) with (15.20) shows that the covariant derivative coincides with the gradient. This explains why the same symbol nabla has been used for the two operators.

The components of the covariant derivative in a basis $(\vec{e}_\alpha)$ are not denoted by $(\nabla T)^{\alpha_1\ldots\alpha_k}{}_{\beta_1\ldots\beta_{\ell+1}}$ but rather by $\nabla_{\beta_{\ell+1}} T^{\alpha_1\ldots\alpha_k}{}_{\beta_1\ldots\beta_\ell}$:

$$\boxed{\boldsymbol{\nabla T} = \nabla_{\beta_{\ell+1}} T^{\alpha_1\ldots\alpha_k}{}_{\beta_1\ldots\beta_\ell}\, \boldsymbol{e}_{\alpha_1} \otimes \ldots \otimes \boldsymbol{e}_{\alpha_k} \otimes \boldsymbol{e}^{\beta_1} \otimes \ldots \otimes \boldsymbol{e}^{\beta_\ell} \otimes \boldsymbol{e}^{\beta_{\ell+1}}}. \tag{15.31}$$

With this convention and (15.30), the components of the covariant derivative along a vector can be written as

$$(\nabla_{\vec{v}} \boldsymbol{T})^{\alpha_1\ldots\alpha_k}{}_{\beta_1\ldots\beta_\ell} = v^\mu \nabla_\mu T^{\alpha_1\ldots\alpha_k}{}_{\beta_1\ldots\beta_\ell}. \tag{15.32}$$

By the definitions (14.31) and (15.29), it is clear that the covariant derivative commutes with the contraction:

$$\boldsymbol{\nabla}(C_q^p \boldsymbol{T}) = C_q^p(\boldsymbol{\nabla T}). \tag{15.33}$$

The covariant derivative along a vector obeys the ***Leibniz rule*** with respect to the tensor product $\otimes$: for any vector $\vec{v}$ and any pair of tensor fields $(\boldsymbol{A}, \boldsymbol{B})$:

$$\boxed{\nabla_{\vec{v}}(\boldsymbol{A} \otimes \boldsymbol{B}) = \nabla_{\vec{v}} \boldsymbol{A} \otimes \boldsymbol{B} + \boldsymbol{A} \otimes \nabla_{\vec{v}} \boldsymbol{B}}. \tag{15.34}$$

This property follows easily from the definition (14.6) of the tensor product and the Leibniz rule regarding the multiplication in $\mathbb{R}$. In the particular case where $\boldsymbol{A}$ is a scalar field, $\boldsymbol{A} = f$, the above formula becomes

$$\nabla_{\vec{v}}(f\boldsymbol{B}) = (\nabla_{\vec{v}} f)\,\boldsymbol{B} + f\,\nabla_{\vec{v}} \boldsymbol{B}. \tag{15.35}$$

Remark 15.6. In differential geometry, where the affine space $\mathscr{E}$ appears as a special case of manifold (cf. Sect. 7.2.1), the covariant derivative is called a ***connection***. In the present case, the metric tensor $\boldsymbol{g}$ is a fixed tensor, of type $(0,2)$, on E. If considered as tensor field on $\mathscr{E}$, it is then a constant field, and we have trivially

$$\boldsymbol{\nabla} \boldsymbol{g} = 0. \tag{15.36}$$

On general grounds, any connection $\boldsymbol{\nabla}$ that satisfies (15.36), as well as the property $\nabla_\alpha \nabla_\beta f = \nabla_\beta \nabla_\alpha f$ for any scalar field f (the so-called *torsion-free* condition), is called a ***Levi–Civita connection*** or ***Riemannian connection*** on the manifold. If the metric $\boldsymbol{g}$ is nondegenerate, such a connection is unique.

15.4.3 Connection Coefficients

Given a field of bases $(\vec{\boldsymbol{e}}_\alpha)$ on $\mathscr{E}$, the covariant derivative of a basis vector along another such vector, $\boldsymbol{\nabla}_{\vec{\boldsymbol{e}}_\beta} \vec{\boldsymbol{e}}_\alpha$, is a vector field on $\mathscr{E}$. It can therefore be expanded at each point $M \in \mathscr{E}$ on the basis $(\vec{\boldsymbol{e}}_\alpha(M))$:

$$\boxed{\boldsymbol{\nabla}_{\vec{\boldsymbol{e}}_\beta} \vec{\boldsymbol{e}}_\alpha =: \Gamma^\mu_{\ \alpha\beta} \, \vec{\boldsymbol{e}}_\mu}. \tag{15.37}$$

The coefficients $\Gamma^\mu_{\ \alpha\beta}$ are called the ***connection coefficients relative to the moving frame*** $(\vec{\boldsymbol{e}}_\alpha)$. They are functions of the considered spacetime point. In other words, the $\Gamma^\mu_{\ \alpha\beta}$'s constitute $4^3 = 64$ scalar fields on $\mathscr{E}$.

Writing the action of the 1-form $\boldsymbol{e}^\alpha$ on the vector $\vec{\boldsymbol{e}}_\gamma$ as the contraction of the tensor product $\vec{\boldsymbol{e}}_\gamma \otimes \boldsymbol{e}^\alpha$ [Eq. (14.34)] and using the commutation property (15.33) as well as the Leibniz rule (15.34), we get

$$\boldsymbol{\nabla}_{\vec{\boldsymbol{e}}_\beta} \langle \boldsymbol{e}^\alpha, \vec{\boldsymbol{e}}_\gamma \rangle = \langle \boldsymbol{\nabla}_{\vec{\boldsymbol{e}}_\beta} \boldsymbol{e}^\alpha, \vec{\boldsymbol{e}}_\gamma \rangle + \langle \boldsymbol{e}^\alpha, \boldsymbol{\nabla}_{\vec{\boldsymbol{e}}_\beta} \vec{\boldsymbol{e}}_\gamma \rangle.$$

Now, by definition of a dual basis, $\langle \boldsymbol{e}^\alpha, \vec{\boldsymbol{e}}_\gamma \rangle$ is the constant field $\delta^\alpha_{\ \gamma}$, so that $\boldsymbol{\nabla}_{\vec{\boldsymbol{e}}_\beta} \langle \boldsymbol{e}^\alpha, \vec{\boldsymbol{e}}_\gamma \rangle = 0$. Moreover, from (15.37), $\boldsymbol{\nabla}_{\vec{\boldsymbol{e}}_\beta} \vec{\boldsymbol{e}}_\gamma = \Gamma^\mu_{\ \gamma\beta} \, \vec{\boldsymbol{e}}_\mu$. We obtain thus $\langle \boldsymbol{\nabla}_{\vec{\boldsymbol{e}}_\beta} \boldsymbol{e}^\alpha, \vec{\boldsymbol{e}}_\gamma \rangle = -\Gamma^\alpha_{\ \gamma\beta}$, from which

$$\boxed{\boldsymbol{\nabla}_{\vec{\boldsymbol{e}}_\beta} \boldsymbol{e}^\alpha = -\Gamma^\alpha_{\ \mu\beta} \, \boldsymbol{e}^\mu}. \tag{15.38}$$

From (15.37) and (15.38), we obtain[4] the following formula for the components of the covariant derivative of a tensor field of type (k, ℓ) (we use the notation $\rho = \beta_{\ell+1}$):

[4] It suffices to take the covariant derivative of the expansion (14.10) of a tensor in terms of its components, to apply the Leibniz rule, to use (15.37) and (15.38) and to compare the result with (15.31).

$$\boxed{\begin{aligned}\nabla_\rho T^{\alpha_1\ldots\alpha_k}{}_{\beta_1\ldots\beta_\ell} = \langle \boldsymbol{\nabla} T^{\alpha_1\ldots\alpha_k}{}_{\beta_1\ldots\beta_\ell},\ \vec{\boldsymbol{e}}_\rho\rangle &+ \sum_{p=1}^{k} \Gamma^{\alpha_p}{}_{\mu\rho} T^{\alpha_1\ldots \overset{p}{\underset{\mu}{\downarrow}} \ldots\alpha_{k-1}}{}_{\beta_1\ldots\beta_\ell} \\ &- \sum_{q=1}^{\ell} \Gamma^{\mu}{}_{\beta_q\rho} T^{\alpha_1\ldots\alpha_k}{}_{\beta_1\ldots \underset{q}{\overset{\mu}{\uparrow}} \ldots\beta_{\ell-1}}\end{aligned}}, \tag{15.39}$$

where $\boldsymbol{\nabla} T^{\alpha_1\ldots\alpha_k}{}_{\beta_1\ldots\beta_\ell}$ is the gradient of the component $T^{\alpha_1\ldots\alpha_k}{}_{\beta_1\ldots\beta_\ell}$, considered as a scalar field on $\mathscr{E}$. Note that if $(\vec{\boldsymbol{e}}_\alpha)$ is a coordinate basis, (15.25) allows one to write the first term of the right-hand side of (15.39) as

$$\langle \boldsymbol{\nabla} T^{\alpha_1\ldots\alpha_k}{}_{\beta_1\ldots\beta_\ell},\ \vec{\boldsymbol{e}}_\rho\rangle = \frac{\partial}{\partial x^\rho} T^{\alpha_1\ldots\alpha_k}{}_{\beta_1\ldots\beta_\ell} \qquad \text{(coordinate basis).} \tag{15.40}$$

Formula (15.39) shows that the covariant derivative of any tensor field can be computed from the connection coefficients $\Gamma^{\gamma}{}_{\alpha\beta}$.

Example 15.13. In the case of a vector field, (15.39) simplifies to

$$\nabla_\rho v^\alpha = \langle \boldsymbol{\nabla} v^\alpha,\ \vec{\boldsymbol{e}}_\rho\rangle + \Gamma^{\alpha}{}_{\mu\rho} v^\mu, \tag{15.41}$$

whereas for a field of linear forms, it reduces to

$$\nabla_\rho \omega_\alpha = \langle \boldsymbol{\nabla} \omega_\alpha,\ \vec{\boldsymbol{e}}_\rho\rangle - \Gamma^{\mu}{}_{\alpha\rho} \omega_\mu. \tag{15.42}$$

In view of (15.40), (15.41) and (15.42), we shall retain

$$\boxed{\nabla_\beta v^\alpha = \frac{\partial v^\alpha}{\partial x^\beta} + \Gamma^{\alpha}{}_{\mu\beta} v^\mu} \quad \text{and} \quad \boxed{\nabla_\beta \omega_\alpha = \frac{\partial \omega_\alpha}{\partial x^\beta} - \Gamma^{\mu}{}_{\alpha\beta} \omega_\mu} \quad \text{(coordinate basis).} \tag{15.43}$$

In the case where $(\vec{\boldsymbol{e}}_\alpha)$ is a basis of E associated with affine coordinates of $\mathscr{E}$ (for instance, inertial coordinates, cf. Sect. 8.2.3), then $(\vec{\boldsymbol{e}}_\alpha)$ is a fixed basis, and $\boldsymbol{\nabla}\vec{\boldsymbol{e}}_\alpha = 0$. The definition (15.37) of the connection coefficients leads to $\Gamma^{\mu}{}_{\alpha\beta} = 0$. Equations (15.39) and (15.40) then show that the components of the covariant derivative of a tensor are nothing but the partial derivatives of the components:

$$\nabla_\rho T^{\alpha_1\ldots\alpha_k}{}_{\beta_1\ldots\beta_\ell} = \frac{\partial}{\partial x^\rho} T^{\alpha_1\ldots\alpha_k}{}_{\beta_1\ldots\beta_\ell} \qquad \text{(affine coordinates).} \tag{15.44}$$

This relation generalizes that obtained above for a vector [Eq. (15.27)].

15.4.4 Christoffel Symbols

Let us compute the value of the connection coefficients in terms of the components $(g_{\alpha\beta})$ of the metric tensor in the case where $(\vec{e}_\alpha)$ is a basis associated with some coordinates (x^α).

Let us first show that, in this case, the coefficients $\Gamma^\gamma{}_{\alpha\beta}$ are symmetric in the indices α and β. From (15.37), it is equivalent to show that

$$\nabla_{\vec{e}_\beta} \vec{e}_\alpha = \nabla_{\vec{e}_\alpha} \vec{e}_\beta. \tag{15.45}$$

Let us show it for two specific values of α and β: $\alpha = 0$ and $\beta = 1$. Let $M \in \mathscr{E}$, $M_0 \in \mathscr{E}$ and $M_1 \in \mathscr{E}$ such that $\overrightarrow{MM_0} = \varepsilon\,\vec{e}_0$ and $\overrightarrow{MM_1} = \varepsilon\,\vec{e}_1$ with ε infinitely small. From the definition of a covariant derivative,

$$\nabla_{\varepsilon\vec{e}_1} \vec{e}_0 = \vec{e}_0(M_1) - \vec{e}_0(M) \quad \text{and} \quad \nabla_{\varepsilon\vec{e}_0} \vec{e}_1 = \vec{e}_1(M_0) - \vec{e}_1(M);$$

hence,

$$\varepsilon\,(\nabla_{\vec{e}_1} \vec{e}_0 - \nabla_{\vec{e}_0} \vec{e}_1) = \vec{e}_1(M) + \vec{e}_0(M_1) - \vec{e}_0(M) - \vec{e}_1(M_0). \tag{15.46}$$

Now, by definition of a natural basis, $\varepsilon[\vec{e}_1(M) + \vec{e}_0(M_1)]$ is the vector connecting M to the point of coordinates $(x^0 + \varepsilon, x^1 + \varepsilon, x^2, x^3)$, (x^0, x^1, x^2, x^3) being the coordinates of M. Similarly, $\varepsilon[\vec{e}_0(M) + \vec{e}_1(M_0)]$ is the vector connecting M to the point of coordinates $(x^0 + \varepsilon, x^1 + \varepsilon, x^2, x^3)$. This is actually the same point as above, so that the vectors $\varepsilon[\vec{e}_1(M) + \vec{e}_0(M_1)]$ and $\varepsilon[\vec{e}_0(M) + \vec{e}_1(M_0)]$ must coincide. We conclude that the right-hand side of (15.46) vanishes, which establishes (15.45) for $\alpha = 0$ and $\beta = 1$ and more generally for any pair (α, β) with $\alpha \neq \beta$. We have thus shown the symmetry of the connection coefficients relative to a coordinate basis:

$$\boxed{\Gamma^\gamma{}_{\alpha\beta} = \Gamma^\gamma{}_{\beta\alpha}} \qquad \text{(coordinate basis)}. \tag{15.47}$$

Let us now move to the explicit computation of $\Gamma^\gamma{}_{\alpha\beta}$ from the components $g_{\alpha\beta}$ of the metric tensor $\boldsymbol{g}$. If one considers the latter as a tensor field on $\mathscr{E}$, it is a constant one and we have $\nabla\boldsymbol{g} = 0$ [Eq. (15.36)]. On the other side, in general, the components $(g_{\alpha\beta})$ are not constant over $\mathscr{E}$, as shown by the various examples of Sect. 15.2.3: $\partial g_{\alpha\beta}/\partial x^\gamma \neq 0$. From (15.39) and the fact that $\boldsymbol{g}$ is a tensor of type $(0, 2)$, the property $\nabla\boldsymbol{g} = 0$ implies

$$\nabla_\gamma g_{\alpha\beta} = 0 = \frac{\partial g_{\alpha\beta}}{\partial x^\gamma} - \Gamma^\mu{}_{\alpha\gamma} g_{\mu\beta} - \Gamma^\mu{}_{\beta\gamma} g_{\alpha\mu}.$$

Multiplying by the matrix $(g^{\alpha\beta})$—the inverse of $(g_{\alpha\beta})$ —and changing the names of indices, we obtain

$$g^{\gamma\mu}\frac{\partial g_{\mu\beta}}{\partial x^\alpha} = \Gamma^\sigma{}_{\mu\alpha} g_{\sigma\beta} g^{\gamma\mu} + \Gamma^\sigma{}_{\beta\alpha} \underbrace{g_{\mu\sigma} g^{\gamma\mu}}_{\delta^\gamma{}_\sigma} = \Gamma^\sigma{}_{\mu\alpha} g_{\sigma\beta} g^{\gamma\mu} + \Gamma^\gamma{}_{\beta\alpha},$$

$$g^{\gamma\mu}\frac{\partial g_{\alpha\beta}}{\partial x^\mu} = \Gamma^\sigma{}_{\alpha\mu} g_{\sigma\beta} g^{\gamma\mu} + \Gamma^\sigma{}_{\beta\mu} g_{\alpha\sigma} g^{\gamma\mu},$$

hence

$$g^{\gamma\mu}\left(\frac{\partial g_{\mu\beta}}{\partial x^\alpha} + \frac{\partial g_{\mu\alpha}}{\partial x^\beta} - \frac{\partial g_{\alpha\beta}}{\partial x^\mu}\right) = \Gamma^\gamma{}_{\beta\alpha} + \Gamma^\gamma{}_{\alpha\beta}.$$

The symmetry property (15.47) allows us then to conclude

$$\boxed{\Gamma^\gamma{}_{\alpha\beta} = \frac{1}{2} g^{\gamma\mu}\left(\frac{\partial g_{\mu\beta}}{\partial x^\alpha} + \frac{\partial g_{\alpha\mu}}{\partial x^\beta} - \frac{\partial g_{\alpha\beta}}{\partial x^\mu}\right)} \qquad \text{(coordinate basis).} \tag{15.48}$$

The connection coefficients given by this formula are called ***Christoffel symbols***. Formula (15.48) enables one to compute the $\Gamma^\gamma{}_{\alpha\beta}$'s from the sole data of the components of the metric tensor. Let us stress that it is valid only for the components in a coordinate basis.

Example 15.14. For the spherical coordinates $(x^\alpha) = (ct, r, \theta, \varphi)$ considered in Example 15.5 p. 499, expression (15.10) for the components of $\boldsymbol{g}$ leads to

$$\Gamma^r{}_{\theta\theta} = -r \qquad \Gamma^r{}_{\varphi\varphi} = -r\sin^2\theta \tag{15.49a}$$

$$\Gamma^\theta{}_{r\theta} = \Gamma^\theta{}_{\theta r} = \frac{1}{r} \qquad \Gamma^\theta{}_{\varphi\varphi} = -\cos\theta\sin\theta \tag{15.49b}$$

$$\Gamma^\varphi{}_{r\varphi} = \Gamma^\varphi{}_{\varphi r} = \frac{1}{r} \qquad \Gamma^\varphi{}_{\theta\varphi} = \Gamma^\varphi{}_{\varphi\theta} = \frac{1}{\tan\theta}, \tag{15.49c}$$

all the other Christoffel symbols being zero. Let us consider the vector field $\vec{\boldsymbol{v}} := \vec{\boldsymbol{e}}_r$ (second vector of the coordinate basis). The components $v^\alpha = (0, 1, 0, 0)$ are constant, so that $\partial v^\alpha / \partial x^\beta = 0$. On the contrary, $\nabla_\beta v^\alpha \neq 0$. We check indeed with the help of (15.43) and the above Christoffel symbols that $\nabla_\theta v^\theta = 1/r$ and $\nabla_\varphi v^\varphi = 1/r$. We have thus $\boldsymbol{\nabla}\vec{\boldsymbol{v}} \neq 0$, in perfect agreement with the fact that the field $\vec{\boldsymbol{v}} = \vec{\boldsymbol{e}}_r$ is not constant on $\mathscr{E}$, as clearly seen on Fig. 15.1.

Example 15.15. Still within spherical coordinates, let us consider now the vector field $\vec{\boldsymbol{w}} := \vec{\boldsymbol{e}}_x$. From (15.8), we obtain the following components in the basis associated with spherical coordinates:

$$w^\alpha = \left(0,\ \sin\theta\cos\varphi,\ \frac{\cos\theta\cos\varphi}{r},\ -\frac{\sin\varphi}{r\sin\theta}\right). \tag{15.50}$$

We have thus clearly $\partial w^\alpha/\partial x^\beta \neq 0$. However, we check that (15.43) and (15.49) lead to $\nabla_\beta w^\alpha = 0$, as it should since $\vec{w} = \vec{e}_x$ is a constant vector field on $\mathscr{E}$.

15.4.5 Divergence of a Vector Field

Given a vector field $\vec{v}$ on $\mathscr{E}$, its covariant derivative $\nabla\vec{v}$ is a tensor field of type $(1,1)$. The contraction of the latter, as defined in Sect. 14.3.5, yields a scalar field [cf. Eq. (14.33)], called ***divergence of*** $\vec{v}$ and denoted by $\nabla\cdot\vec{v}$:

$$\boxed{\nabla\cdot\vec{v} := \nabla_\mu v^\mu}. \tag{15.51}$$

In terms of components with respect to a coordinate system (x^α), we get, from (15.43),

$$\nabla\cdot\vec{v} = \frac{\partial v^\mu}{\partial x^\mu} + \Gamma^\nu{}_{\mu\nu} v^\mu. \tag{15.52}$$

Now, if one performs the contraction of (15.48) on the indices γ and β, one gets

$$\begin{aligned}\Gamma^\nu{}_{\mu\nu} &= \frac{1}{2} g^{\rho\sigma}\frac{\partial g_{\rho\sigma}}{\partial x^\mu} = \frac{1}{2}\mathrm{tr}\left(g^{-1}\frac{\partial}{\partial x^\mu} g\right) = \frac{1}{2}\frac{\partial}{\partial x^\mu}\ln|\det g| \\ &= \frac{1}{\sqrt{-\det g}}\frac{\partial}{\partial x^\mu}\sqrt{-\det g},\end{aligned} \tag{15.53}$$

where (7.90) has been used to express the derivative of the determinant of the matrix g of the components of $\boldsymbol{g}$ with respect to the coordinates (x^α). We deduce then from (15.52) that the divergence of a vector field can be expressed solely in terms of partial derivatives and the determinant of the components of the metric tensor:

$$\boxed{\nabla\cdot\vec{v} = \frac{1}{\sqrt{-\det g}}\frac{\partial}{\partial x^\mu}\left(\sqrt{-\det g}\; v^\mu\right)}. \tag{15.54}$$

Example 15.16. For the spherical coordinates considered in Example 15.5 p. 499, the value (15.10) of the matrix g leads to $\det g = -r^4\sin^2\theta$, so that (15.54) becomes

$$\nabla\cdot\vec{v} = \frac{1}{c}\frac{\partial v^0}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v^r\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, v^\theta\right) + \frac{\partial v^\varphi}{\partial\varphi}. \tag{15.55}$$

Applying this formula to the vector $\vec{v} = \vec{e}_r$ considered in Example 15.14, there comes $\nabla\cdot\vec{v} = r^{-2}\partial/\partial r(r^2) = 2/r$. On the other side, for the vector $\vec{w} = \vec{e}_x$ of

Example 15.15, we deduce from components (15.50) that $\nabla\cdot \vec{w} = 0$, in agreement with the fact that $\vec{e}_x$ is a constant field on $\mathscr{E}$.

15.4.6 Divergence of a Tensor Field

The divergence operator can be generalized to any tensor field $\boldsymbol{T}$ of type (k, ℓ) with $k \geq 1$ by defining the ***divergence of*** $\boldsymbol{T}$ as the contraction on the last contravariant index and the derivation index (covariant index of rank $\ell + 1$) of the covariant derivative of $\boldsymbol{T}$:

$$\nabla\cdot \boldsymbol{T} := C^k_{\ell+1} \nabla \boldsymbol{T}. \tag{15.56}$$

In terms of components, we obtain the following formula, which generalizes (15.51):

$$\boxed{(\nabla\cdot \boldsymbol{T})^{\alpha_1\ldots\alpha_{k-1}}{}_{\beta_1\ldots\beta_\ell} = \nabla_\mu T^{\alpha_1\ldots\alpha_{k-1}\mu}{}_{\beta_1\ldots\beta_\ell}}. \tag{15.57}$$

In the case where $\boldsymbol{T}$ is an antisymmetric tensor of type $(2, 0)$, one can express the divergence by a formula similar to (15.54), which has been established for a vector. Indeed, in the present case, Eqs. (15.39)–(15.40) lead to

$$\nabla_\mu T^{\alpha\mu} = \frac{\partial T^{\alpha\mu}}{\partial x^\mu} + \Gamma^\alpha{}_{\nu\mu} T^{\nu\mu} + \Gamma^\mu{}_{\nu\mu} T^{\alpha\nu},$$

where (x^α) is a coordinate system on $\mathscr{E}$ and $\Gamma^\gamma{}_{\alpha\beta}$ the corresponding Christoffel symbols. Now, since $\boldsymbol{T}$ is antisymmetric and $\Gamma^\alpha{}_{\nu\mu}$ is symmetric with respect to the indices μ and ν [Eq. (15.47)], $\Gamma^\alpha{}_{\nu\mu} T^{\nu\mu} = 0$. Since, in addition, $\Gamma^\mu{}_{\nu\mu}$ is expressible in terms of $\det g$ via (15.53), we obtain

$$\boxed{\nabla_\mu T^{\alpha\mu} = \frac{1}{\sqrt{-\det g}} \frac{\partial}{\partial x^\mu} \left(\sqrt{-\det g}\; T^{\alpha\mu} \right)} \qquad (\boldsymbol{T} \text{ antisymmetric}). \tag{15.58}$$

15.5 Differential Forms

15.5.1 Definition

For $p \in \mathbb{N}$, one calls ***differential*** p***-form*** any smooth field of p-forms, i.e. any smooth field of alternate forms of valence p, as defined in Sect. 14.4. Hence, a differential 0-form is a scalar field [cf. (14.37)], and a differential 1-form is a field of linear forms [cf. (14.36)]. Differential forms play a fundamental role to define integrals on parts of $\mathscr{E}$, as we shall see in Chap. 16.

15.5.2 Exterior Derivative

A differential p-form $\boldsymbol{A}$ being a tensor field of type $(0, p)$, its covariant derivative $\boldsymbol{\nabla A}$ is a tensor field of type $(0, p+1)$, i.e. a field of multilinear forms of valence $p+1$. But, in general, $\boldsymbol{\nabla A}$ is not fully antisymmetric; it is therefore not a differential $(p+1)$-form. To get a differential $(p+1)$-form, it suffices to antisymmetrize it. Thus one defines,[5] for any $(p+1)$-tuple of vectors $(\vec{\boldsymbol{v}}_1, \ldots, \vec{\boldsymbol{v}}_{p+1})$,

$$\boxed{\mathbf{d}\boldsymbol{A}(\vec{\boldsymbol{v}}_1, \ldots, \vec{\boldsymbol{v}}_{p+1}) := \frac{1}{p!} \sum_{\sigma \in \mathfrak{S}_{p+1}} (-1)^{k(\sigma)}\, \boldsymbol{\nabla}_{\vec{\boldsymbol{v}}_{\sigma(1)}} \boldsymbol{A}(\vec{\boldsymbol{v}}_{\sigma(2)}, \ldots, \vec{\boldsymbol{v}}_{\sigma(p+1)})}\,. \tag{15.59}$$

By construction, $\mathbf{d}\boldsymbol{A}$ is a differential $(p+1)$-form; it is called the ***exterior derivative*** of the differential p-form $\boldsymbol{A}$. Explicitly:

- If $\boldsymbol{A} = f$ is a 0-form (scalar field), its exterior derivative is nothing but its gradient:

$$\mathbf{d}f = \boldsymbol{\nabla} f \qquad \text{(scalar field)}. \tag{15.60}$$

- If $\boldsymbol{A}$ is a 1-form, the definition (15.59) reduces to

$$\mathbf{d}\boldsymbol{A}(\vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_2) = \langle \boldsymbol{\nabla}_{\vec{\boldsymbol{v}}_1} \boldsymbol{A}, \vec{\boldsymbol{v}}_2 \rangle - \langle \boldsymbol{\nabla}_{\vec{\boldsymbol{v}}_2} \boldsymbol{A}, \vec{\boldsymbol{v}}_1 \rangle. \tag{15.61}$$

  The components of $\mathbf{d}\boldsymbol{A}$ in a basis $(\vec{\boldsymbol{e}}_\alpha)$ of E are then

$$(\mathrm{d}A)_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha. \tag{15.62}$$

- If $\boldsymbol{A}$ is a 2-form, the definition (15.59) yields

$$\mathbf{d}\boldsymbol{A}(\vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_2, \vec{\boldsymbol{v}}_3) = \boldsymbol{\nabla}_{\vec{\boldsymbol{v}}_1} \boldsymbol{A}(\vec{\boldsymbol{v}}_2, \vec{\boldsymbol{v}}_3) + \boldsymbol{\nabla}_{\vec{\boldsymbol{v}}_2} \boldsymbol{A}(\vec{\boldsymbol{v}}_3, \vec{\boldsymbol{v}}_1) + \boldsymbol{\nabla}_{\vec{\boldsymbol{v}}_3} \boldsymbol{A}(\vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_2), \tag{15.63}$$

  where we have used the fact that, for any vector $\vec{\boldsymbol{v}}$, $\boldsymbol{\nabla}_{\vec{\boldsymbol{v}}} \boldsymbol{A}$ is antisymmetric: $\boldsymbol{\nabla}_{\vec{\boldsymbol{v}}_1} \boldsymbol{A}(\vec{\boldsymbol{v}}_2, \vec{\boldsymbol{v}}_3) = -\boldsymbol{\nabla}_{\vec{\boldsymbol{v}}_1} \boldsymbol{A}(\vec{\boldsymbol{v}}_3, \vec{\boldsymbol{v}}_2)$, etc. This property follows from the very definition of a covariant derivative [Eq. (15.29)], since the difference between

[5] Cf. Sect. 14.4 for the notations.

two antisymmetric forms is antisymmetric. In terms of components, (15.63) can be written as

$$(\mathrm{d}A)_{\alpha\beta\gamma} = \nabla_\alpha A_{\beta\gamma} + \nabla_\beta A_{\gamma\alpha} + \nabla_\gamma A_{\alpha\beta}. \tag{15.64}$$

- If $\boldsymbol{A}$ is a 3-form, one shows similarly that

$$(\mathrm{d}A)_{\alpha\beta\gamma\delta} = \nabla_\alpha A_{\beta\gamma\delta} - \nabla_\beta A_{\gamma\delta\alpha} + \nabla_\gamma A_{\delta\alpha\beta} - \nabla_\delta A_{\alpha\beta\gamma}. \tag{15.65}$$

If the basis $(\vec{e}_\alpha)$ is a coordinate basis associated with some coordinate system (x^α), the components of the covariant derivatives can be expressed via (15.39) and (15.40). Given the symmetry of Christoffel symbols [property (15.47)], we observe that all the terms involving them vanish. For instance, (15.62) becomes [cf. (15.43)]

$$\begin{aligned}(\mathrm{d}A)_{\alpha\beta} &= \frac{\partial A_\beta}{\partial x^\alpha} - \Gamma^\mu_{\ \beta\alpha} A_\mu - \left(\frac{\partial A_\alpha}{\partial x^\beta} - \Gamma^\mu_{\ \alpha\beta} A_\mu\right)\\ &= \frac{\partial A_\beta}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\beta} + \underbrace{\left(\Gamma^\mu_{\ \alpha\beta} - \Gamma^\mu_{\ \beta\alpha}\right)}_{0} A_\mu.\end{aligned}$$

One can thus replace the nabla symbols in (15.62)–(15.65) by partial derivatives:

$$\boxed{(\mathrm{d}f)_\alpha = \frac{\partial f}{\partial x^\alpha}} \qquad \text{(scalar field)} \tag{15.66}$$

$$\boxed{(\mathrm{d}A)_{\alpha\beta} = \frac{\partial A_\beta}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\beta}} \qquad \text{(1-form)} \tag{15.67}$$

$$\boxed{(\mathrm{d}A)_{\alpha\beta\gamma} = \frac{\partial A_{\beta\gamma}}{\partial x^\alpha} + \frac{\partial A_{\gamma\alpha}}{\partial x^\beta} + \frac{\partial A_{\alpha\beta}}{\partial x^\gamma}} \qquad \text{(2-form)} \tag{15.68}$$

$$\boxed{(\mathrm{d}A)_{\alpha\beta\gamma\delta} = \frac{\partial A_{\beta\gamma\delta}}{\partial x^\alpha} - \frac{\partial A_{\gamma\delta\alpha}}{\partial x^\beta} + \frac{\partial A_{\delta\alpha\beta}}{\partial x^\gamma} - \frac{\partial A_{\alpha\beta\gamma}}{\partial x^\delta}} \qquad \text{(3-form).} \tag{15.69}$$

Remark 15.7. Formulas (15.66)–(15.69) show that the notion of exterior derivative is independent of that of covariant derivative. Indeed these formulas involve only the partial derivative with respect to the coordinates. In the more general framework of the theory of manifolds, a metric tensor must be provided to define the (Levi–Civita) covariant derivative $\boldsymbol{\nabla}$ (cf. Remark 15.6 p. 508), whereas the exterior derivative $\mathbf{d}$ does not depend on any structure but that of manifold. On the other side, $\mathbf{d}$ applies only to differential forms, while $\boldsymbol{\nabla}$ applies to all tensor fields.

Example 15.17. In Chap. 11, we have seen that the 4-force exerted on a particle in a vector field results from the action of a 2-form $\boldsymbol{F}$ on the particle's 4-velocity [cf. Eq. (11.35)]. The comparison of (11.34) and (15.67) shows that $\boldsymbol{F}$ is nothing but the exterior derivative of the potential 1-form $\boldsymbol{A}$ that is involved in the Lagrangian (11.28): $\boldsymbol{F} = \mathbf{d}\boldsymbol{A}$.

Example 15.18 (Link between the exterior derivative of a 1-form and the curl of a vector). Formula (15.67) reminds that expressing the curl of a vector in the three-dimensional Euclidean space. Let us make this relation concrete. Let $\vec{\boldsymbol{v}}$ be a vector field on a spacelike affine hyperplane $\Sigma \subset \mathscr{E}$. A differential 1-form $\underline{\boldsymbol{v}}$ is associated with $\vec{\boldsymbol{v}}$ by metric duality, and we can take its exterior derivative to get a 2-form, $\mathbf{d}\underline{\boldsymbol{v}}$. Let us consider the Hodge dual of $\mathbf{d}\underline{\boldsymbol{v}}$, $\star\mathbf{d}\underline{\boldsymbol{v}}$ (cf. Sect. 14.5). This is also a 2-form. Let us set as the first argument of this 2-form the future-directed unit normal Σ, $\vec{\boldsymbol{u}}$; we obtain then a 1-form: $\boldsymbol{w} := \star\mathbf{d}\underline{\boldsymbol{v}}(\vec{\boldsymbol{u}}, .)$. Its metric dual, $\vec{\boldsymbol{w}}$, is a vector tangent to Σ, since by construction, $\vec{\boldsymbol{w}} \cdot \vec{\boldsymbol{u}} = \langle \boldsymbol{w}, \vec{\boldsymbol{u}} \rangle = \star\mathbf{d}\underline{\boldsymbol{v}}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{u}}) = 0$. We shall define this vector as the ***curl*** of $\vec{\boldsymbol{v}}$ and will denote it by $\boldsymbol{\nabla}\mathbf{x}_{\boldsymbol{u}}\, \vec{\boldsymbol{v}}$:

$$\boldsymbol{\nabla}\mathbf{x}_{\boldsymbol{u}}\, \vec{\boldsymbol{v}} := \vec{\boldsymbol{w}}, \qquad \boldsymbol{w} := \star\mathbf{d}\underline{\boldsymbol{v}}(\vec{\boldsymbol{u}}, .). \tag{15.70}$$

To show that this definition gives indeed the usual curl, let us express the components of $\vec{\boldsymbol{w}}$ with respect to a coordinate system (x^α) one $\mathscr{E}$. From (14.75c) and (15.67), the components of $\star\mathbf{d}\underline{\boldsymbol{v}}$ are

$$(\star \mathrm{d}v)_{\alpha\beta} = \frac{1}{2}\, \epsilon^{\mu\nu}{}_{\alpha\beta} \left(\frac{\partial v_\nu}{\partial x^\mu} - \frac{\partial v_\mu}{\partial x^\nu} \right) = \epsilon^{\mu\nu}{}_{\alpha\beta}\, \frac{\partial v_\nu}{\partial x^\mu}.$$

The components of $\vec{\boldsymbol{w}}$ are then

$$w^\alpha = g^{\alpha\beta}\, \epsilon^{\mu\nu}{}_{\rho\beta}\, \frac{\partial v_\nu}{\partial x^\mu}\, u^\rho = \epsilon^{\mu\nu\rho\alpha}\, \frac{\partial v_\nu}{\partial x^\mu}\, u_\rho.$$

In other words,

$$w^\alpha = (\boldsymbol{\nabla}\mathbf{x}_{\boldsymbol{u}}\, \vec{\boldsymbol{v}})^\alpha = u_\rho\, \epsilon^{\rho\alpha\mu\nu}\, \frac{\partial v_\nu}{\partial x^\mu} = u_\rho\, \epsilon^{\rho\alpha\mu\nu}\, \nabla_\mu v_\nu. \tag{15.71}$$

The third equality, where the partial derivative has been replaced by a covariant derivative, results from the symmetry of the Christoffel symbols [Eq. (15.47)] and the antisymmetry of the Levi–Civita tensor.

Let us assume now that (x^α) are inertial coordinates such that the equation of Σ is $x^0 = 0$. The coordinate basis $(\vec{\boldsymbol{e}}_\alpha)$ associated with these coordinates is an orthonormal basis and satisfies $\vec{\boldsymbol{e}}_0 = \vec{\boldsymbol{u}}$. We will moreover assume that $(\vec{\boldsymbol{e}}_\alpha)$ is right-handed, which is always possible, thanks to some coordinate permutation. $(\vec{\boldsymbol{e}}_i)$ is then an orthonormal basis of the Euclidean vector space of vectors tangent to Σ. In particular, $\vec{\boldsymbol{v}} = v^i\, \vec{\boldsymbol{e}}_i$. Since $u_\rho = g_{\rho\sigma} u^\sigma = \eta_{\rho\sigma} u^\sigma = \eta_{\rho 0} = -\delta^0{}_\rho$, (15.71) leads to $w^0 = 0$ and

$$w^i = -\epsilon^{0ijk}\, \frac{\partial v_k}{\partial x^j}. \tag{15.72}$$

Now, since the basis $(\vec{e}_\alpha)$ is orthonormal, $v_k = v^k$, and from (14.63), $\epsilon^{0ijk} = -[i, j, k]$. We have thus

$$w^i = [i, j, k] \frac{\partial v^k}{\partial x^j}. \tag{15.73}$$

We recognize the standard expression of the components of a curl in Cartesian coordinates.

We shall retain from the above example that the curl of a vector field is one of the components of the Hodge dual of the exterior derivative of the 1-form associated with the vector field by metric duality. The exterior derivative can thus be perceived as some generalization of the concept of curl.

15.5.3 Properties of the Exterior Derivative

If f is a scalar field, we have, by combining (15.66) and (15.67),

$$(\mathrm{d}\,\mathrm{d}f)_{\alpha\beta} = \frac{\partial^2 f}{\partial x^\alpha \partial x^\beta} - \frac{\partial^2 f}{\partial x^\beta \partial x^\alpha} = 0,$$

for the partial derivatives commute. More generally, it is easy to see that for any differential p-form $\boldsymbol{A}$,

$$\boxed{\mathbf{d}\,\mathbf{d}\boldsymbol{A} = 0}. \tag{15.74}$$

In other words, the exterior derivative is nilpotent: $\mathbf{d}^2 = 0$.

A differential p-form $\boldsymbol{A}$ is said ***closed*** iff $\mathbf{d}\boldsymbol{A} = 0$. It is said ***exact*** iff there exists a differential $(p-1)$-form $\boldsymbol{B}$ such that $\boldsymbol{A} = \mathbf{d}\boldsymbol{B}$. The property (15.74) implies that any exact p-form is closed. The converse is true, provided that $\boldsymbol{A}$ is defined on the whole $\mathscr{E}$ or on a star-shaped subdomain of $\mathscr{E}$:

$$\boxed{\mathbf{d}\boldsymbol{A} = 0 \implies \exists\, \boldsymbol{B},\ \boldsymbol{A} = \mathbf{d}\boldsymbol{B}}. \tag{15.75}$$

This property is known as ***Poincaré lemma***.

The exterior derivative of an exterior product (cf. Sect. 14.4.2) obeys

$$\mathbf{d}(\boldsymbol{A} \wedge \boldsymbol{B}) = \mathbf{d}\boldsymbol{A} \wedge \boldsymbol{B} + (-1)^p \boldsymbol{A} \wedge \mathbf{d}\boldsymbol{B}, \tag{15.76}$$

where p is the valence of the differential form $\boldsymbol{A}$ [*Exercise:* show (15.76)]. We observe that the exterior derivative obeys the Leibniz rule with respect to the exterior product only if p is even.

15.5.4 Expansion with Respect to a Coordinate System

Let (x^α) be a coordinate system on $\mathscr{E}$ and $(\vec{e}_\alpha)$ the associated coordinate basis. We have seen in Sect. 15.3.3 that the dual basis, $(\boldsymbol{e}^\alpha)$, is made of the gradients of the coordinates [Eq. (15.23)]. Given (15.60), the dual basis can be written as

$$\boxed{\boldsymbol{e}^\alpha = \mathbf{d}x^\alpha}. \tag{15.77}$$

Remark 15.8. One should not confuse $\mathrm{d}x^\alpha$, the infinitesimal increase of the coordinate x^α, with $\mathbf{d}x^\alpha$, the exterior derivative of the coordinate x^α considered as a scalar field on $\mathscr{E}$. Thanks to the equality between the exterior derivative and the gradient for a scalar field [Eq. (15.60)], the link between the two quantities is $\mathrm{d}x^\alpha = \langle \mathbf{d}x^\alpha, \overrightarrow{MM'}\rangle$, where M' is the point that differs from M by the increase $\mathrm{d}x^\alpha$ of the coordinate x^α.

The expansion (14.48) of alternate forms implies the following expansion of differential forms:

For any differential p-form $\boldsymbol{A}$,

$$\boxed{\boldsymbol{A} = A_{\alpha_1\ldots\alpha_p}\, \mathbf{d}x^{\alpha_1} \otimes \ldots \otimes \mathbf{d}x^{\alpha_p} = \sum_{\alpha_1<\ldots<\alpha_p} A_{\alpha_1\ldots\alpha_p}\, \mathbf{d}x^{\alpha_1} \wedge \ldots \wedge \mathbf{d}x^{\alpha_p}}. \tag{15.78}$$

Explicitly:

$$\boldsymbol{A} = A_0\, \mathbf{d}x^0 + A_1\, \mathbf{d}x^1 + A_2\, \mathbf{d}x^2 + A_3\, \mathbf{d}x^3 \quad \text{(1-form)}, \tag{15.79}$$

$$\begin{aligned}\boldsymbol{A} = {} & A_{01}\, \mathbf{d}x^0 \wedge \mathbf{d}x^1 + A_{02}\, \mathbf{d}x^0 \wedge \mathbf{d}x^2 + A_{03}\, \mathbf{d}x^0 \wedge \mathbf{d}x^3 + A_{12}\, \mathbf{d}x^1 \wedge \mathbf{d}x^2 \\ & + A_{13}\, \mathbf{d}x^1 \wedge \mathbf{d}x^3 + A_{23}\, \mathbf{d}x^2 \wedge \mathbf{d}x^3 \quad \text{(2-form)},\end{aligned} \tag{15.80}$$

$$\begin{aligned}\boldsymbol{A} = {} & A_{012}\, \mathbf{d}x^0 \wedge \mathbf{d}x^1 \wedge \mathbf{d}x^2 + A_{013}\, \mathbf{d}x^0 \wedge \mathbf{d}x^1 \wedge \mathbf{d}x^3 + A_{023}\, \mathbf{d}x^0 \wedge \mathbf{d}x^2 \wedge \mathbf{d}x^3 \\ & + A_{123}\, \mathbf{d}x^1 \wedge \mathbf{d}x^2 \wedge \mathbf{d}x^3 \quad \text{(3-form)},\end{aligned} \tag{15.81}$$

$$\boldsymbol{A} = A_{0123}\, \mathbf{d}x^0 \wedge \mathbf{d}x^1 \wedge \mathbf{d}x^2 \wedge \mathbf{d}x^3 \quad \text{(4-form)}. \tag{15.82}$$

Applying (15.82) to the Levi–Civita tensor, we get, taking (14.51) into account,

$$\boxed{\boldsymbol{\epsilon} = \pm\sqrt{-\det g}\; \mathbf{d}x^0 \wedge \mathbf{d}x^1 \wedge \mathbf{d}x^2 \wedge \mathbf{d}x^3}, \tag{15.83}$$

where $\det g$ stands for the determinant of the matrix $g = (g_{\alpha\beta})$ of the metric tensor components in the coordinates (x^α) and where $\pm$ is $+$ if the coordinate basis is right-handed (in which case, the coordinates (x^α) are said to be ***right-handed***) and $-$ otherwise (in which case, the coordinates (x^α) are said to be ***left-handed***).

Remark 15.9. In the above formula, one considers $\boldsymbol{\epsilon}$ as a tensor field on $\mathscr{E}$. It is then a constant field, as $\boldsymbol{g}$. In particular, $\boldsymbol{\nabla\epsilon} = 0$. On the other side, the unique component $\pm\sqrt{-\det g}$ of $\boldsymbol{\epsilon}$ in the basis $\mathbf{d}x^0 \wedge \mathbf{d}x^1 \wedge \mathbf{d}x^2 \wedge \mathbf{d}x^3$ is not constant in general (except if (x^α) are inertial coordinates). The property $\boldsymbol{\nabla\epsilon} = 0$ can be recovered by computing the components $\nabla_\rho \epsilon_{\alpha\beta\gamma\delta}$ from (15.39) and the Christoffel symbols (15.48).

15.5.5 Exterior Derivative of a 3-Form and Divergence of a Vector Field

If $\boldsymbol{A}$ is a differential 3-form on $\mathscr{E}$, its exterior derivative is a differential 4-form whose components with respect to a field of bases are given by (15.65). Let us evaluate the Hodge dual of this 4-form by means of (14.75e). We obtain the scalar field

$$\begin{aligned}\star\mathbf{d}\boldsymbol{A} &= \frac{1}{24}\epsilon^{\alpha\beta\gamma\delta}\,(\mathrm{d}A)_{\alpha\beta\gamma\delta} \\ &= \frac{1}{24}\epsilon^{\alpha\beta\gamma\delta}\left[\nabla_\alpha A_{\beta\gamma\delta} - \nabla_\beta A_{\gamma\delta\alpha} + \nabla_\gamma A_{\delta\alpha\beta} - \nabla_\delta A_{\alpha\beta\gamma}\right]. \qquad (15.84)\end{aligned}$$

Let us consider the first term of this expression; since $\boldsymbol{\epsilon}$ is a constant field on $\mathscr{E}$ (cf. Remark 15.9), we have $\nabla_\alpha \epsilon^{\alpha\beta\gamma\delta} = 0$, hence

$$\epsilon^{\alpha\beta\gamma\delta}\nabla_\alpha A_{\beta\gamma\delta} = \nabla_\alpha\left(\epsilon^{\alpha\beta\gamma\delta}A_{\beta\gamma\delta}\right).$$

Writing $\epsilon^{\alpha\beta\gamma\delta}A_{\beta\gamma\delta} = -\epsilon^{\beta\gamma\delta\alpha}A_{\beta\gamma\delta} = -A_{\beta\gamma\delta}\epsilon^{\beta\gamma\delta}{}_{\mu}\,g^{\mu\alpha} = -6(\star A)_\mu\,g^{\mu\alpha}$, we let appear the Hodge dual of the 3-form $\boldsymbol{A}$ [cf. Eq. (14.75d)], so that the above equation becomes

$$\epsilon^{\alpha\beta\gamma\delta}\nabla_\alpha A_{\beta\gamma\delta} = -6\nabla_\alpha(\star A)^\alpha = -6\,\boldsymbol{\nabla}\cdot\overrightarrow{\star\boldsymbol{A}},$$

where $(\star A)^\alpha = (\star A)_\mu\,g^{\mu\alpha}$ stands for the components of the vector $\overrightarrow{\star\boldsymbol{A}}$ associated with the 1-form $\star\boldsymbol{A}$ by metric duality. The last equality, which involves the divergence of $\overrightarrow{\star\boldsymbol{A}}$, results from (15.54). Similarly, the three other terms in (15.84) are each equal to $-6\,\boldsymbol{\nabla}\cdot\overrightarrow{\star\boldsymbol{A}}$. We end thus with the simple formula:

$$\boxed{\star\mathbf{d}\boldsymbol{A} = -\boldsymbol{\nabla}\cdot\overrightarrow{\star\boldsymbol{A}}}. \qquad (15.85)$$

The Hodge dual of this relation [cf. Eq. (14.77) with $p = 4$] leads to $\mathbf{d}\boldsymbol{A} = \star\nabla\cdot\overrightarrow{\star\boldsymbol{A}}$. Using (14.75a), we obtain thus the following formula for the exterior derivative of any differential 3-form:

$$\boxed{\mathbf{d}\boldsymbol{A} = \left[\nabla\cdot(\overrightarrow{\star\boldsymbol{A}})\right]\boldsymbol{\epsilon}} \qquad \text{(3-form).} \tag{15.86}$$

Remark 15.10. Since the exterior derivative of a 3-form is a 4-form and the space of 4-forms is of dimension one [cf. (14.40)], it was expected that $\mathbf{d}\boldsymbol{A}$ is proportional to the Levi–Civita tensor $\boldsymbol{\epsilon}$. The non-trivial content of (15.86) is thus the proportionality factor being the divergence of the vector field $\overrightarrow{\star\boldsymbol{A}}$.

Let us consider now a vector field $\vec{\boldsymbol{v}}$ on $\mathscr{E}$. The Hodge dual of the 1-form $\underline{\boldsymbol{v}}$ associated with $\vec{\boldsymbol{v}}$ by metric duality is the 3-form defined by (14.75b): $\star v_{\alpha\beta\gamma} := \epsilon_{\mu\alpha\beta\gamma} v^{\mu}$. In other words,

$$\boxed{\star\underline{\boldsymbol{v}} := \boldsymbol{\epsilon}(\vec{\boldsymbol{v}}, ., ., .)}. \tag{15.87}$$

Let us then apply formula (15.85) to this 3-form: if $\boldsymbol{A} := \star\underline{\boldsymbol{v}}$, then via Eq. (14.77) with $p = 1$, $\star\boldsymbol{A} = \star\star\,\underline{\boldsymbol{v}} = \underline{\boldsymbol{v}}$ and $\overrightarrow{\star\boldsymbol{A}} = \vec{\boldsymbol{v}}$. Equation (15.85) leads then to the following expression of the divergence of $\vec{\boldsymbol{v}}$:

$$\boxed{\nabla\cdot\vec{\boldsymbol{v}} = -\star\mathbf{d}\star\underline{\boldsymbol{v}}}. \tag{15.88}$$

Remark 15.11. More generally, the operator $-\star\mathbf{d}\star$ acting on a p-form, for any value of p, is called ***codifferential***. Contrary to the exterior derivative, which maps a p-form to a $(p+1)$-form, the codifferential maps a p-form to a $(p-1)$-form [in the above case, a 1-form to a 0-form].

The identity (15.86) can also be reexpressed is terms of $\vec{\boldsymbol{v}}$:

$$\boxed{\mathbf{d}\star\underline{\boldsymbol{v}} = (\nabla\cdot\vec{\boldsymbol{v}})\,\boldsymbol{\epsilon}}. \tag{15.89}$$

This formula will turn to be very useful in the next chapters.

Historical note: *The general concepts of differential p-form and exterior derivative have been introduced in 1899 by Élie Cartan (cf. p. 6) (1899; 1945). They constitute one of the bases of what is known as* Cartan calculus.

Chapter 16
Integration in Spacetime

16.1 Introduction

This chapter is entirely devoted to the integration of tensor fields over parts of spacetime. The aim is to prepare, among others, the discussion of conservation laws in chapters about electromagnetism and hydrodynamics. This is the last purely mathematical chapter of the book.

16.2 Integration Over a Four-Dimensional Volume

16.2.1 Volume Element

In the three-dimensional Euclidean space, the volume of an elementary parallelepiped constructed upon three infinitesimal vectors $\mathrm{d}\vec{\ell}_1$, $\mathrm{d}\vec{\ell}_2$ and $\mathrm{d}\vec{\ell}_3$ (cf. Fig. 16.1) is given by the mixed product of these vectors:

$$\mathrm{d}V = \mathrm{d}\vec{\ell}_1 \cdot (\mathrm{d}\vec{\ell}_2 \wedge \mathrm{d}\vec{\ell}_3). \tag{16.1}$$

In particular, if $\mathrm{d}\vec{\ell}_1 = \mathrm{d}x^1\,\vec{e}_1$, $\mathrm{d}\vec{\ell}_2 = \mathrm{d}x^2\,\vec{e}_2$, $\mathrm{d}\vec{\ell}_3 = \mathrm{d}x^3\,\vec{e}_3$ and $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ is an orthonormal basis, then $\mathrm{d}V = \mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}x^3$.

In spacetime $\mathscr{E}$, we may consider a four-dimensional elementary parallelepiped, which we shall call an ***hyperparallelepiped***, constructed upon four infinitesimal vectors $(\mathrm{d}\vec{\boldsymbol{\ell}}_\alpha)_{0\le\alpha\le3}$. We have already noticed in Sect. 1.5 that within $\mathscr{E}$ the role of the mixed product is played by the Levi–Civita tensor $\boldsymbol{\epsilon}$. We shall then define the ***four-volume***, or ***4-volume*** for short, of the elementary hyperparallelepiped by

$$\boxed{\mathrm{d}U := \boldsymbol{\epsilon}(\mathrm{d}\vec{\boldsymbol{\ell}}_0, \mathrm{d}\vec{\boldsymbol{\ell}}_1, \mathrm{d}\vec{\boldsymbol{\ell}}_2, \mathrm{d}\vec{\boldsymbol{\ell}}_3)}. \tag{16.2}$$

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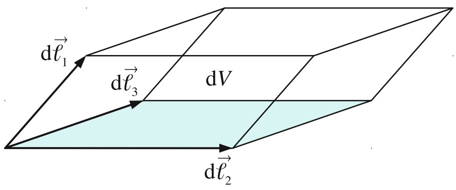

Fig. 16.1 Elementary parallelepiped constructed upon three infinitesimal vectors $d\vec{\ell}_1$, $d\vec{\ell}_2$ and $d\vec{\ell}_3$

If the vector $d\vec{\boldsymbol{\ell}}_\alpha$ corresponds to the infinitesimal increase dx^α of the coordinate x^α of some coordinate system on $\mathscr{E}$, i.e. if

$$d\vec{\boldsymbol{\ell}}_0 = dx^0\,\vec{\boldsymbol{e}}_0, \quad d\vec{\boldsymbol{\ell}}_1 = dx^1\,\vec{\boldsymbol{e}}_1, \quad d\vec{\boldsymbol{\ell}}_2 = dx^2\,\vec{\boldsymbol{e}}_2, \quad d\vec{\boldsymbol{\ell}}_3 = dx^3\,\vec{\boldsymbol{e}}_3, \tag{16.3}$$

where $(\vec{\boldsymbol{e}}_\alpha)$ is the coordinate basis associated with (x^α), then (16.2) gives, thanks to the multilinearity of $\boldsymbol{\epsilon}$, $dU = dx^0dx^1dx^2dx^3\boldsymbol{\epsilon}(\vec{\boldsymbol{e}}_0, \vec{\boldsymbol{e}}_1, \vec{\boldsymbol{e}}_2, \vec{\boldsymbol{e}}_3)$. Substituting (14.52) and assuming that (x^α) are right-handed coordinates (so that the basis $(\vec{\boldsymbol{e}}_\alpha)$ is right-handed; cf. Sect. 15.5.4), there comes

$$\boxed{dU = \sqrt{-\det g}\;dx^0dx^1dx^2dx^3}. \tag{16.4}$$

In particular, if $(x^\alpha) = (ct, x, y, z)$ are inertial coordinates, $dU = c\,dt\,dx\,dy\,dz$—a formula that generalizes the expression of the three-dimensional volume element: $dV = dx\,dy\,dz$.

16.2.2 Four-Volume of a Part of Spacetime

Let $\mathscr{V}$ be a compact four-dimensional domain of spacetime $\mathscr{E}$. In view of (16.4), one defines naturally the ***four-volume of*** $\mathscr{V}$, or ***4-volume*** for short, by

$$\boxed{\operatorname{vol}\mathscr{V} := \int_{\mathscr{V}} \sqrt{-\det g}\;dx^0dx^1dx^2dx^3}, \tag{16.5}$$

where the coordinates (x^α) are assumed to be right-handed and the integral is limited to the coordinate ranges that cover $\mathscr{V}$ and is a Lebesgue integral[1] over $\mathbb{R}^4$, $\sqrt{-\det g}$ being considered as a function of (x^0, x^1, x^2, x^3).

The key point is that the definition of $\operatorname{vol}\mathscr{V}$ is independent of the choice of the coordinates (x^α) or, equivalently, is independent of the system of elementary hyperparallelepipeds $(d\vec{\boldsymbol{\ell}}_\alpha)$ used to decompose $\mathscr{V}$.

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[1] Or a Riemann–Darboux integral: since we shall consider only piecewise continuous functions, we shall not make any distinction.

Proof. Let us perform a change of coordinates $(x^\alpha) \mapsto (x'^\alpha)$. The well-known formula ruling the change of variable in a Lebesgue integral allows then to rewrite (16.5) as

$$\mathrm{vol}\,\mathscr{V} = \int_{\mathscr{V}} \sqrt{-\det g}\;|J|\,\mathrm{d}x'^0\mathrm{d}x'^1\mathrm{d}x'^2\mathrm{d}x'^3, \tag{16.6}$$

where J is the Jacobian of the coordinate change: $J = \det\left(\partial x^\beta/\partial x'^\alpha\right)$. In addition, according to the law (15.9) for the change $g \mapsto g'$ of the matrix of the metric tensor components, we have $g' = {}^t P\, g\, P$ with $P^\beta_{\ \alpha} := \partial x^\beta/\partial x'^\alpha$. We note that $J = \det P$, so that $\det g' = (\det P)^2 \det g = J^2 \det g$. Hence,

$$\sqrt{-\det g'} = |J|\sqrt{-\det g}. \tag{16.7}$$

In view of (16.7), we conclude that (16.6) has the same form as (16.5), with each $\mathrm{d}x^\alpha$ replaced by $\mathrm{d}x'^\alpha$ and g replaced by g', which shows the independence of (16.5) with respect to the coordinate system. □

16.2.3 Integral of a Differential 4-Form

From (16.2), we can write

$$\mathrm{vol}\,\mathscr{V} = \int_{\mathscr{V}} \boldsymbol{\epsilon}(\mathrm{d}\vec{\boldsymbol{\ell}}_0, \mathrm{d}\vec{\boldsymbol{\ell}}_1, \mathrm{d}\vec{\boldsymbol{\ell}}_2, \mathrm{d}\vec{\boldsymbol{\ell}}_3), \tag{16.8}$$

with the infinitesimal vectors $\mathrm{d}\vec{\boldsymbol{\ell}}_\alpha$ given by (16.3). This formula can be seen as *defining* the integral of the 4-form $\boldsymbol{\epsilon}$ over $\mathscr{V}$, the result being independent of the choice of the $\mathrm{d}\vec{\boldsymbol{\ell}}_\alpha$'s.

More generally, for any differential 4-form $\boldsymbol{A}$ on $\mathscr{E}$, we define the ***integral of A over*** $\mathscr{V}$ by

$$\boxed{\int_{\mathscr{V}} \boldsymbol{A} := \int_{\mathscr{V}} \boldsymbol{A}(\mathrm{d}\vec{\boldsymbol{\ell}}_0, \mathrm{d}\vec{\boldsymbol{\ell}}_1, \mathrm{d}\vec{\boldsymbol{\ell}}_2, \mathrm{d}\vec{\boldsymbol{\ell}}_3) = \int_{\mathscr{V}} A_{0123}\,\mathrm{d}x^0\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3}, \tag{16.9}$$

where (x^α) is a right-handed coordinate system on the part of $\mathscr{E}$ containing $\mathscr{V}$ and $(\mathrm{d}\vec{\boldsymbol{\ell}}_\alpha)$ are the associated "elementary hyperparallelepiped" vectors defined by (16.3). In the second equality, $A_{0123} = \boldsymbol{A}(\vec{\boldsymbol{e}}_0, \vec{\boldsymbol{e}}_1, \vec{\boldsymbol{e}}_2, \vec{\boldsymbol{e}}_3)$ is the component of $\boldsymbol{A}$ with respect to the coordinates (x^α) [cf. (15.82)] and the integral of the right-hand side is a Lebesgue integral over $\mathbb{R}^4$.

As for (16.8), the definition (16.9) does not depend upon the choice of the right-handed coordinate system (x^α).

Proof. The space $\mathscr{A}_4(E)$ of valence-4 alternate forms being of dimension one [Eq. (14.40)], there exists necessarily a scalar field $\alpha : \mathscr{E} \to \mathbb{R}$ such that $\boldsymbol{A} = \alpha\,\boldsymbol{\epsilon}$. Combining (15.82) and (15.83) (with the $+$ sign since the coordinates (x^α) are right-handed), there comes

$$A_{0123}\,\mathrm{d}x^0\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3 = \alpha\sqrt{-\det g}\,\mathrm{d}x^0\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3.$$

The argument is then identical to that of Sect. 16.2.2: since α is invariant under a coordinate change, the identity (16.7) leads to the independence of the definition (16.9) with respect to the coordinates (x^α). □

Remark 16.1. The notion of integral of a differential 4-form over a part of $\mathscr{E}$ does not depend upon the metric tensor $\boldsymbol{g}$, since the right-hand side of (16.9) does not involve $\boldsymbol{g}$. In particular, there is no need to rely on the Levi–Civita tensor $\boldsymbol{\epsilon}$ (which does depend upon $\boldsymbol{g}$) in the demonstration of the invariance of the integral by a change of coordinates. We did it here in order not to repeat the demonstration of Sect. 16.2.2.

As shown by (16.8), the definition (16.9) applied to the 4-form $\boldsymbol{\epsilon}$ yields

$$\mathrm{vol}\,\mathscr{V} = \int_{\mathscr{V}} \boldsymbol{\epsilon}. \tag{16.10}$$

Remark 16.2. Because of this identity, the Levi–Civita tensor is often called the ***volume element*** of Minkowski spacetime.

16.3 Submanifolds of $\mathscr{E}$

The integration can also be defined on parts of $\mathscr{E}$ of dimension lower than 4: curves, surfaces and hypersurfaces. Technically, these parts are called *submanifolds*[2] of $\mathscr{E}$. Let us start by defining them in full generality.

16.3.1 Definition of a Submanifold

A part $\mathscr{V}$ of $\mathscr{E}$ is called a ***submanifold of $\mathscr{E}$ of dimension*** $p \in \{1, 2, 3\}$ iff in the vicinity of any point of $\mathscr{V}$, there exists a coordinate system of $\mathscr{E}$, (x^α), such that $\mathscr{V}$ is defined by the $4 - p$ equations

$$\mathscr{V}: \quad x^A = \text{const}, \quad A \in \{0, \ldots, 3 - p\}. \tag{16.11}$$

[2] Let us recall that the concept of *manifold* has been defined in Sect. 7.2.1.

The coordinate system (x^α) is then said to be ***adapted to*** $\mathscr{V}$. The three possible cases are

- $p = 1$: $\mathscr{V}$ is a ***curve*** of $\mathscr{E}$ (e.g. a worldline); it obeys $x^0 = \text{const}$, $x^1 = \text{const}$ and $x^2 = \text{const}$, and the coordinate x^3 can be chosen as a parameter along $\mathscr{V}$.
- $p = 2$: $\mathscr{V}$ is a ***surface*** of $\mathscr{E}$; it obeys $x^0 = \text{const}$ and $x^1 = \text{const}$, and the coordinates (x^2, x^3) are those labelling the points of $\mathscr{V}$.
- $p = 3$: $\mathscr{V}$ is a ***hypersurface*** of $\mathscr{E}$; it is defined by $x^0 = \text{const}$, and (x^1, x^2, x^3) can be chosen as coordinates internal to $\mathscr{V}$.

Remark 16.3. More than one adapted coordinate system can be required, covering different parts, to define a submanifold of $\mathscr{E}$.

For the remaining of this chapter, we adopt the following convention to name coordinates adapted to $\mathscr{V}$: an upper-case Latin letter from the beginning of alphabet, A, B, etc. is used for the indices of the coordinates that are constant on $\mathscr{V}$, as in (16.11), and a lower-case Latin letter, still from the beginning of alphabet, a, b, etc., for the indices of the remaining coordinates (coordinates "internal" to $\mathscr{V}$). Hence,

- If $p = 1$, $A \in \{0, 1, 2\}$ and $a = 3$.
- If $p = 2$, $A \in \{0, 1\}$ and $a \in \{2, 3\}$.
- If $p = 3$, $A = 0$ and $a \in \{1, 2, 3\}$.

Example 16.1. Let us consider the spherical coordinates $(x^\alpha) = (ct, r, \theta, \varphi)$ introduced in Example 15.2 p. 496.Then the conditions $ct = 0$ and $r = R > 0$ determine a sphere $\mathscr{S}$ of radius R. It is a submanifold of $\mathscr{E}$ of dimension $p = 2$. In this case $(x^A) = (ct, r)$ and $(x^a) = (\theta, \varphi)$.

Example 16.2. Still with the same spherical coordinates, the condition $ct = 0$ defines an hyperplane of $\mathscr{E}$: it is the rest space of the inertial observer from which the spherical coordinates are defined. In this case, $(x^A) = (ct)$ and $(x^a) = (r, \theta, \varphi)$.

Example 16.3. Let us consider now the null coordinates introduced in Example 15.6 p. 499: $(x^\alpha) = (u, v, \theta, \varphi)$. The condition $u = 0$ defines a hypersurface of $\mathscr{E}$, which is nothing but the future light cone of the event defined by $r = 0$ and $t = 0$. In this case, $(x^A) = (u)$ and $(x^a) = (v, \theta, \varphi)$.

Let us determine the condition that change of coordinates on $\mathscr{E}$, $(x^\alpha) \mapsto (x'^\alpha)$, must fulfil in order for the new coordinates to be adapted to $\mathscr{V}$ if the old coordinates are. Let us start from the general formula $\mathrm{d}x'^\alpha = (\partial x'^\alpha / \partial x^\beta)\mathrm{d}x^\beta$. On $\mathscr{V}$, by definition, $\mathrm{d}x^\beta = 0$ for $\beta \in \{0, \ldots, 3 - p\}$. The summation over β is thus restricted to coordinates that vary on $\mathscr{V}$:

$$\mathrm{d}x'^\alpha\big|_{\mathscr{V}} = \left.\frac{\partial x'^\alpha}{\partial x^a}\right|_{\mathscr{V}} \mathrm{d}x^a.$$

The new coordinates are adapted to $\mathscr{V}$ iff $x'^A|_{\mathscr{V}} = \text{const}$, i.e. $\mathrm{d}x'^A|_{\mathscr{V}} = 0$. From the above expression, this condition is equivalent to

$$\left.\frac{\partial x'^A}{\partial x^a}\right|_{\mathscr{V}} \mathrm{d}x^a = 0.$$

This relation must be satisfied whatever the infinitesimal variations $\mathrm{d}x^a$ of the coordinates (x^a) on $\mathscr{V}$. We deduce the necessary and sufficient condition for the new coordinates to be adapted to $\mathscr{V}$:

$$\left.\frac{\partial x'^A}{\partial x^a}\right|_{\mathscr{V}} = 0, \qquad 0 \leq A \leq 3-p, \quad 4-p \leq a \leq 3. \tag{16.12}$$

16.3.2 Submanifold with Boundary

As submanifolds have been defined, via (16.11), they cannot admit a boundary. For example, a disk does not obey this definition, contrary to a sphere. One defines then a ***submanifold with boundary*** $\mathscr{V}$ by adding to (16.11) the condition that the first internal coordinate, i.e. x^{4-p}, can take only values lower than a given constant $K \in \mathbb{R}$:

$$\mathscr{V}: \quad x^A = \text{const}, \quad A \in \{0, \ldots, 3-p\} \quad \text{and} \quad x^{4-p} \leq K. \tag{16.13}$$

Moreover, $p = 4$ is authorized in this definition; it reduces then to $x^0 \leq K$ and allows one to encompass the four-dimensional volumes considered in Sects. 16.2.2 and 16.2.3.

The ***boundary of*** $\mathscr{V}$ is the part of $\mathscr{V}$ for which the coordinate x^{4-p} reaches the value K; it is denoted by $\partial\mathscr{V}$. Since K is a constant, we observe that the equation of $\partial\mathscr{V}$ is

$$\partial\mathscr{V}: \quad x^A = \text{const}, \quad A \in \{0, \ldots, 4-p\}. \tag{16.14}$$

In view of (16.11), this means that $\partial\mathscr{V}$ is a submanifold (without boundary) of $\mathscr{E}$ of dimension $p-1$. Note that on $\partial\mathscr{V}$, the vector $\vec{\boldsymbol{e}}_{4-p}$ of the coordinate basis associated with (x^α) is directed towards the exterior of $\mathscr{V}$.

Example 16.4. Let us consider the spherical coordinates $(x^\alpha) = (ct, r, \theta, \varphi)$ introduced in Example 15.2 p. 496. For any constant $R > 0$, the conditions

$$t = 0 \quad \text{and} \quad r \leq R$$

define a ball in the hyperplane $t = 0$. By the definition (16.13), this a manifold with boundary of dimension $p = 3$. The boundary of this manifold obeys the equations $t = 0$ and $r = R$; it is a sphere of radius R.

16.3.3 Orientation of a Submanifold

To define properly the integration on a submanifold $\mathscr{V}$, an *orientation* of $\mathscr{V}$ must be first prescribed. As we did for the whole spacetime by the choice of the 4-form $\boldsymbol{\epsilon}$, defining an orientation on $\mathscr{V}$ amounts to selecting a differential p-form $\boldsymbol{\rho}$ (p being $\mathscr{V}$'s dimension) that never vanishes for any p-tuple of linearly independent vectors tangent to $\mathscr{V}$. If it is possible to choose $\boldsymbol{\rho}$ in a continuous manner over $\mathscr{V}$, one says that $\mathscr{V}$ is ***orientable*** and that the p-form $\boldsymbol{\rho}$ constitutes an ***orientation*** of $\mathscr{V}$. The pair $(\mathscr{V}, \boldsymbol{\rho})$ is then called an ***oriented submanifold***.[3] One says that a p-tuple $(\vec{\boldsymbol{v}}_1, \ldots, \vec{\boldsymbol{v}}_p)$ of vectors tangents to $\mathscr{V}$ is ***right-handed*** (resp. ***left-handed***) iff $\boldsymbol{\rho}(\vec{\boldsymbol{v}}_1, \ldots, \vec{\boldsymbol{v}}_p) > 0$ (resp. $\boldsymbol{\rho}(\vec{\boldsymbol{v}}_1, \ldots, \vec{\boldsymbol{v}}_p) < 0$).

Example 16.5. Every curve ($p = 1$) is orientable. The plane or the sphere is orientable surfaces. More generally, any simply connected submanifold is orientable. A standard example of non-orientable surface is Moebius strip.

Given a coordinate system (x^α) adapted to $\mathscr{V}$, the last p vectors $(\vec{\boldsymbol{e}}_a)_{4-p\le a\le 3}$ of the associated coordinate basis are tangent to $\mathscr{V}$. This is clear on the very definition of the coordinate basis $(\vec{\boldsymbol{e}}_\alpha)$ (cf. Sect. 15.2.2). If $\mathscr{V}$ is oriented, we shall say that the coordinates (x^α) are ***right-handed with respect to*** $\mathscr{V}$ iff the vectors $(\vec{\boldsymbol{e}}_a)$ are right-handed in the oriented manifold $\mathscr{V}$.

If a manifold with boundary $\mathscr{V}$ is endowed with an orientation $\boldsymbol{\rho}$, the latter induces an orientation of the boundary $\partial\mathscr{V}$, as follows. Given a coordinate system (x^α) adapted to $\mathscr{V}$ and to its boundary, i.e. a coordinate system obeying (16.13), the vector $\vec{\boldsymbol{e}}_{4-p}$ of the associated coordinate basis is tangent to $\mathscr{V}$, but not to $\partial\mathscr{V}$ and is directed towards the exterior of $\mathscr{V}$. The $(p-1)$-form defined at any point of $\partial\mathscr{V}$ by

$$\boldsymbol{\rho}(\vec{\boldsymbol{e}}_{4-p}, ., \ldots, .) : \quad \begin{array}{ccc} E^{p-1} & \longrightarrow & \mathbb{R} \\ (\vec{\boldsymbol{v}}_1, \ldots, \vec{\boldsymbol{v}}_{p-1}) & \longmapsto & \boldsymbol{\rho}(\vec{\boldsymbol{e}}_{4-p}, \vec{\boldsymbol{v}}_1, \ldots, \vec{\boldsymbol{v}}_{p-1}) \end{array} \tag{16.15}$$

is then an orientation of $\partial\mathscr{V}$, called ***induced orientation***.

16.4 Integration on a Submanifold of $\mathscr{E}$

16.4.1 Integral of Any Differential Form

Let $\mathscr{V}$ be a submanifold of $\mathscr{E}$ (with or without boundary), of dimension $p \in \{1, 2, 3, 4\}$ and oriented. Given a coordinate system (x^α) adapted to $\mathscr{V}$ and right-handed with respect to $\mathscr{V}$, the p infinitesimal vectors $(\mathrm{d}\vec{\boldsymbol{\ell}}_{4-p}, \ldots, \mathrm{d}\vec{\boldsymbol{\ell}}_3)$ associated

[3]In practice, the explicit mention of $\boldsymbol{\rho}$ is often skipped, and one speaks about the "oriented submanifold $\mathscr{V}$".

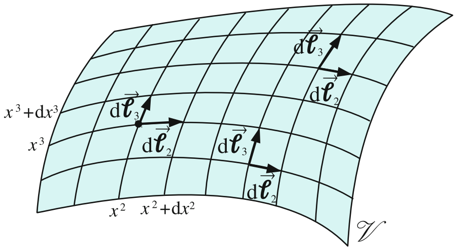

Fig. 16.2 Mesh of a submanifold $\mathscr{V}$ of $\mathscr{E}$ induced by the infinitesimal vectors $(\mathrm{d}\vec{\ell}_{4-p}, \ldots, \mathrm{d}\vec{\ell}_3)$ associated with coordinates (x^α) adapted to $\mathscr{V}$ (in this figure, $p = 2$)

with the coordinates by (16.3) provide a ***mesh*** of $\mathscr{V}$. In particular, at each point, these vectors are tangent to $\mathscr{V}$ and are linearly independent (cf. Fig. 16.2). We may then generalize the definition (16.9), which holds only for the case $p = 4$, by defining the ***integral of a differential*** p***-form*** $\boldsymbol{A}$ ***over*** $\mathscr{V}$ as the integral of the value of $\boldsymbol{A}$ taken on the p vectors of the mesh[4]

$$\boxed{\int_{\mathscr{V}} \boldsymbol{A} := \int_{\mathscr{V}} \boldsymbol{A}(\mathrm{d}\vec{\ell}_{4-p}, \ldots, \mathrm{d}\vec{\ell}_3) = \int_{\mathscr{V}} A_{4-p\cdots 3}\, \mathrm{d}x^{4-p} \ldots \mathrm{d}x^3}, \qquad (16.16)$$

where $A_{4-p\cdots 3} = \boldsymbol{A}(\vec{e}_{4-p}, \ldots, \vec{e}_3)$ is the component of indices $(4-p, \ldots, 3)$ of $\boldsymbol{A}$ with respect to the coordinates (x^α) and the integral of $A_{4-p\cdots 3}$ is the Lebesgue integral in $\mathbb{R}^p$. Indeed, the coordinates $(x^0, \ldots, x^{3-p})$ being constant on $\mathscr{V}$, we may consider that the component $A_{4-p\cdots 3}$ is a function of only the p coordinates $(x^{4-p}, \ldots, x^3)$ and take its Lebesgue integral over the domain of $\mathbb{R}^p$ covering $\mathscr{V}$. Explicitly, the definition (16.16) is

$$\boxed{\int_{\mathscr{V}} \boldsymbol{A} := \int_{\mathscr{V}} \langle \boldsymbol{A}, \mathrm{d}\vec{\ell}_3 \rangle = \int_{\mathscr{V}} A_3\, \mathrm{d}x^3} \qquad (p = 1), \qquad (16.17a)$$

$$\boxed{\int_{\mathscr{V}} \boldsymbol{A} := \int_{\mathscr{V}} \boldsymbol{A}(\mathrm{d}\vec{\ell}_2, \mathrm{d}\vec{\ell}_3) = \int_{\mathscr{V}} A_{23}\, \mathrm{d}x^2\, \mathrm{d}x^3} \qquad (p = 2), \qquad (16.17b)$$

$$\boxed{\int_{\mathscr{V}} \boldsymbol{A} := \int_{\mathscr{V}} \boldsymbol{A}(\mathrm{d}\vec{\ell}_1, \mathrm{d}\vec{\ell}_2, \mathrm{d}\vec{\ell}_3) = \int_{\mathscr{V}} A_{123}\, \mathrm{d}x^1\, \mathrm{d}x^2\, \mathrm{d}x^3} \ (p = 3). \quad (16.17c)$$

For $p = 4$, (16.16) coincides with the definition (16.9).

In order for the definition (16.16) to be well posed, it must not depend upon the choice of the adapted coordinates (x^α) or, equivalently, it must not depend upon the mesh $(\mathrm{d}\vec{\ell}_{4-p}, \ldots, \mathrm{d}\vec{\ell}_3)$ of $\mathscr{V}$. Let us show it explicitly in the case $p = 2$,

[4] This definition assumes that $\mathscr{V}$ is entirely covered by a single adapted coordinate system. Now, for certain submanifolds, various coordinate systems can be required. The integral must then be decomposed in a sum via a process called *partition of unity*. We shall not enter in these technical considerations here; cf. e.g. Berger and Gostiaux (1988).

the cases $p = 1$ and $p = 3$ being similar and the case $p = 4$ having been treated in Sect. 16.2.3. If (x'^α) is a second coordinate system adapted to $\mathscr{V}$ and right-handed with respect to $\mathscr{V}$, the standard formula of change of variable in the Lebesgue integral leads to

$$\int_{\mathscr{V}} A_{23}\, \mathrm{d}x^2\, \mathrm{d}x^3 = \int_{\mathscr{V}} A_{23}\, J\, \mathrm{d}x'^2\, \mathrm{d}x'^3, \tag{16.18}$$

where $J = \det(\partial x^a / \partial x'^b)$ is the Jacobian of the change of coordinates on $\mathscr{V}$:

$$J = \frac{\partial x^2}{\partial x'^2} \frac{\partial x^3}{\partial x'^3} - \frac{\partial x^3}{\partial x'^2} \frac{\partial x^2}{\partial x'^3}. \tag{16.19}$$

Note that a priori the absolute value of J should appear in (16.18), but since both coordinate systems (x^a) and (x'^a) are right-handed with respect to $\mathscr{V}$, one has $J > 0$. Besides, the component A'_{23} of $\boldsymbol{A}$ with respect to the new coordinates is deduced from the components $A_{\alpha\beta}$ with respect to the old ones via formula (14.23), in which the change-of-basis matrix is given by (15.6):

$$A'_{23} = A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x'^2} \frac{\partial x^\beta}{\partial x'^3} = A_{ab} \frac{\partial x^a}{\partial x'^2} \frac{\partial x^b}{\partial x'^3} = A_{23} \frac{\partial x^2}{\partial x'^2} \frac{\partial x^3}{\partial x'^3} + A_{32} \frac{\partial x^3}{\partial x'^2} \frac{\partial x^2}{\partial x'^3},$$

where, to get the second equality, we have used property (16.12) [in which we permute the roles of (x^α) and (x'^α)], which allows us to limit the summations on α and β to the values 2 and 3 of the indices. Since $\boldsymbol{A}$ is antisymmetric, $A_{32} = -A_{23}$ and we recognize in the above expression the Jacobian J given by (16.19). There comes thus $A'_{23} = A_{23}\, J$. Equation (16.18) becomes then

$$\int_{\mathscr{V}} A_{23}\, \mathrm{d}x^2\, \mathrm{d}x^3 = \int_{\mathscr{V}} A'_{23}\, \mathrm{d}x'^2\, \mathrm{d}x'^3. \tag{16.20}$$

We conclude that the definition (16.17b) is independent of the choice of the coordinates adapted to $\mathscr{V}$. □

Remark 16.4. The differential p-forms are really the mathematical objects for which one can define the integral over a p-dimensional submanifold in an intrinsic manner, i.e. independently of any coordinate system or any other structure on $\mathscr{E}$ (as, for instance, the metric tensor). In particular, if $\boldsymbol{A}$ was a generic (i.e. not necessarily antisymmetric) tensor field of type $(0, p)$, the argument that led to $A'_{23} = A_{23}\, J$ in the above demonstration would not hold, so that the integral defined by (16.17b) would depend upon the choice of the coordinates (x^α).

Remark 16.5. The definitions (16.17) involve only one component of $\boldsymbol{A}$: A_3, A_{23} or A_{123} according to the value of p. This is actually the only component that plays a role when $\boldsymbol{A}$ is applied to vectors tangent to $\mathscr{V}$. Indeed, any vector $\vec{\boldsymbol{v}}$ tangent to $\mathscr{V}$ can be written as $\vec{\boldsymbol{v}} = v^a\, \vec{\boldsymbol{e}}_a$ where $(\vec{\boldsymbol{e}}_\alpha)$ is the basis associated with the coordinates adapted to $\mathscr{V}$ and $a \in \{4-p, \ldots, 3\}$. We deduce that the first $4-p$ linear forms of the dual basis $(\boldsymbol{e}^\alpha)$ vanish on $\vec{\boldsymbol{v}}$:

$$\langle \boldsymbol{e}^A, \vec{\boldsymbol{v}} \rangle = v^a \langle \boldsymbol{e}^A, \vec{\boldsymbol{e}}_a \rangle = v^a \delta^A_{\ a} = 0.$$

Consequently, (14.48) leads to the following expression for the action of $\boldsymbol{A}$ on vectors $\vec{\boldsymbol{v}}$, $\vec{\boldsymbol{w}}$ and $\vec{\boldsymbol{z}}$ tangent to $\mathscr{V}$:

$$\begin{aligned}
\langle \boldsymbol{A}, \vec{\boldsymbol{v}} \rangle &= A_3 v^3 \qquad (p = 1) \\
\boldsymbol{A}(\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}) &= A_{23}(v^2 w^3 - v^3 w^2) \qquad (p = 2) \\
\boldsymbol{A}(\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}, \vec{\boldsymbol{z}}) &= A_{123}\big[v^1(w^2 z^3 - w^3 z^2) + w^1(z^2 v^3 - z^3 v^2) \\
&\qquad + z^1(v^2 w^3 - v^3 w^2)\big] \qquad (p = 3),
\end{aligned}$$

which demonstrates the above statement.

16.4.2 Volume Element of a Hypersurface

Let us consider a hypersurface $\mathscr{V}$ of $\mathscr{E}$ (submanifold of dimension $p = 3$). In an adapted coordinate system (x^α), $\mathscr{V}$ obeys[5] $x^0 = \text{const}$ [Eq. (16.11) with $p = 3$]. This implies that the gradient of the coordinate x^0 (considered as a scalar field on $\mathscr{E}$) vanishes for any vector $\vec{\boldsymbol{v}}$ tangent to $\mathscr{V}$: $\langle \boldsymbol{\nabla} x^0, \vec{\boldsymbol{v}} \rangle = 0$. Introducing the vector $\boldsymbol{m}$ associated with the linear form $\boldsymbol{\nabla} x^0$ by metric duality, this property becomes

$$\forall \vec{\boldsymbol{v}} \in E, \quad \vec{\boldsymbol{v}} \text{ tangent to } \mathscr{V} \iff \vec{\boldsymbol{m}} \cdot \vec{\boldsymbol{v}} = 0. \tag{16.21}$$

One says that the vector $\vec{\boldsymbol{m}}$ is ***normal*** to the hypersurface $\mathscr{V}$. Three cases can occur:

- If $\vec{\boldsymbol{m}}$ is timelike, all vectors tangent to $\mathscr{V}$ are necessarily spacelike; one says that $\mathscr{V}$ is ***spacelike hypersurface***.[6]
- If $\vec{\boldsymbol{m}}$ is spacelike, vectors tangent to $\mathscr{V}$ can be timelike, spacelike or null; one says that $\mathscr{V}$ is a ***timelike hypersurface***.
- If $\vec{\boldsymbol{m}}$ is null, vectors tangents to $\mathscr{V}$ are either spacelike or null; one says that $\mathscr{V}$ is a ***null hypersurface***. The vector $\vec{\boldsymbol{m}}$ is then both normal and tangent to $\mathscr{V}$, since it fulfils criterion (16.21) : $\vec{\boldsymbol{m}} \cdot \vec{\boldsymbol{m}} = 0$.

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[5] Let us recall that, in the present context, x^0 stands for the first coordinate of a system adapted to $\mathscr{V}$; it is thus not necessarily a time coordinate.

[6] We have already encountered this type of hypersurface in Sect. 9.3.5.

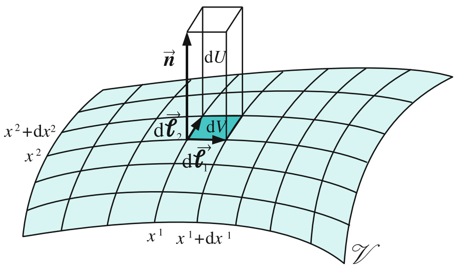

Fig. 16.3 Hyperparallelepiped constructed from the elementary parallelepiped $(\mathrm{d}\vec{\boldsymbol{\ell}}_1, \mathrm{d}\vec{\boldsymbol{\ell}}_2, \mathrm{d}\vec{\boldsymbol{\ell}}_3)$ of an hypersurface $\mathscr{V}$ and the unit normal $\vec{\boldsymbol{n}}$ to $\mathscr{V}$

Let us assume that $\mathscr{V}$ is either spacelike or timelike. Then $\vec{\boldsymbol{m}} \cdot \vec{\boldsymbol{m}} \neq 0$, and we can introduce the vector

$$\vec{\boldsymbol{n}} := \frac{\vec{\boldsymbol{m}}}{\|\vec{\boldsymbol{m}}\|_g}. \tag{16.22}$$

$\vec{\boldsymbol{n}}$ is a unit vector normal to $\mathscr{V}$. It is unique up to a sign. If $\mathscr{V}$ is spacelike, we shall assume that $\vec{\boldsymbol{n}}$ is future-directed, unless otherwise stated. Let us consider an elementary parallelepiped in $\mathscr{V}$ defined by three infinitesimal vectors $(\mathrm{d}\vec{\boldsymbol{\ell}}_1, \mathrm{d}\vec{\boldsymbol{\ell}}_2, \mathrm{d}\vec{\boldsymbol{\ell}}_3)$ according to (16.3), and let us erect a hyperparallelepiped by adding the vector $\vec{\boldsymbol{n}}$ to $(\mathrm{d}\vec{\boldsymbol{\ell}}_1, \mathrm{d}\vec{\boldsymbol{\ell}}_2, \mathrm{d}\vec{\boldsymbol{\ell}}_3)$ (cf. Fig. 16.3). The 4-volume of this hyperparallelepiped is $\mathrm{d}U = \boldsymbol{\epsilon}(\vec{\boldsymbol{n}}, \mathrm{d}\vec{\boldsymbol{\ell}}_1, \mathrm{d}\vec{\boldsymbol{\ell}}_2, \mathrm{d}\vec{\boldsymbol{\ell}}_3)$. Since $\vec{\boldsymbol{n}}$ is a unit vector, we may consider that $\mathrm{d}U$ is the numerical value of the volume of the base parallelepiped $(\mathrm{d}\vec{\boldsymbol{\ell}}_1, \mathrm{d}\vec{\boldsymbol{\ell}}_2, \mathrm{d}\vec{\boldsymbol{\ell}}_3)$. We shall then define the ***volume element 3-form*** of the hypersurface $\mathscr{V}$ as

$$\boxed{\boldsymbol{\epsilon}_{\mathscr{V}} := \boldsymbol{\epsilon}(\vec{\boldsymbol{n}}, ., ., .)} \qquad (p = 3). \tag{16.23}$$

$\boldsymbol{\epsilon}_{\mathscr{V}}$ is a field of 3-forms on $\mathscr{V}$. If $\mathscr{V}$ is the rest space of an observer ($\vec{\boldsymbol{n}}$ = observer's 4-velocity $\vec{\boldsymbol{u}}$), $\boldsymbol{\epsilon}_{\mathscr{V}}$ is nothing but the 3-form $\boldsymbol{\epsilon}_{\boldsymbol{u}}$ introduced in Chap. 3 [cf. Eq. (3.45)] and used to define the cross product $\times_{\boldsymbol{u}}$. The ***volume*** of the elementary parallelepiped $(\mathrm{d}\vec{\boldsymbol{\ell}}_1, \mathrm{d}\vec{\boldsymbol{\ell}}_2, \mathrm{d}\vec{\boldsymbol{\ell}}_3)$ in $\mathscr{V}$ is then

$$\mathrm{d}V = \boldsymbol{\epsilon}_{\mathscr{V}}(\mathrm{d}\vec{\boldsymbol{\ell}}_1, \mathrm{d}\vec{\boldsymbol{\ell}}_2, \mathrm{d}\vec{\boldsymbol{\ell}}_3). \tag{16.24}$$

If (x^α) is a coordinate system adapted to $\mathscr{V}$ and the $\mathrm{d}\vec{\boldsymbol{\ell}}_i$'s are vectors related to the infinitesimal coordinate increases $\mathrm{d}x^i$ by (16.3), then

$$\mathrm{d}V = \boldsymbol{\epsilon}(\vec{\boldsymbol{n}}, \mathrm{d}x^1\, \vec{\boldsymbol{e}}_1, \mathrm{d}x^2\, \vec{\boldsymbol{e}}_2, \mathrm{d}x^3\, \vec{\boldsymbol{e}}_3) = \epsilon_{\mu 123}\, n^\mu\, \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3 = \epsilon_{0123}\, n^0\, \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3.$$

Using the components (14.51) of the Levi–Civita tensor, we get

$$\boxed{\mathrm{d}V = n^0 \sqrt{-\det g}\, \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3}. \tag{16.25}$$

Example 16.6. Let us take for $\mathscr{V}$ the ball considered in Example 16.4 p. 526. Within the spherical coordinates $(x^\alpha) = (ct, r, \theta, \varphi)$ adapted to $\mathscr{V}$, we have $\sqrt{-\det g} =$

$r^2 \sin\theta$ [cf. (15.10)] and $\vec{n} = \vec{e}_0$; hence, $n^0 = 1$, so that (16.25) yields the standard volume element

$$\mathrm{d}V = r^2 \sin\theta \, dr \, \mathrm{d}\theta \, \mathrm{d}\varphi.$$

By the definition (16.23), the components of the volume element 3-form in a generic basis (not necessarily adapted to $\mathscr{V}$) are

$$(\epsilon_{\mathscr{V}})_{\alpha\beta\gamma} = n^\mu \epsilon_{\mu\alpha\beta\gamma}. \tag{16.26}$$

Comparing with (14.75b), we observe that $\boldsymbol{\epsilon}_{\mathscr{V}}$ is the Hodge dual of the 1-form $\underline{\boldsymbol{n}}$:

$$\boxed{\boldsymbol{\epsilon}_{\mathscr{V}} = \star\underline{\boldsymbol{n}}}. \tag{16.27}$$

16.4.3 Area Element of a Surface

Let us consider now the case where $\mathscr{V}$ is a two-dimensional surface, which we shall assume to be ***spacelike***, in the sense that all vectors tangent to $\mathscr{V}$ are spacelike. At each point $M \in \mathscr{V}$, the vector space E is decomposed in the direct sum $E = \Pi \oplus \Pi^\perp$ where Π is the vector plane generated by vectors tangent to $\mathscr{V}$ and $\Pi^\perp$ the vector plane generated by vectors orthogonal to $\mathscr{V}$ at M. One can always find an orthonormal basis of $(E, \boldsymbol{g})$ with two vectors within Π. The two other vectors are then necessarily within $\Pi^\perp$. Since $\mathscr{V}$ is a spacelike surface, the two vectors in Π are spacelike. By virtue of the signature of $\boldsymbol{g}$, one of the two remaining vectors, which we shall call $\vec{n}$, must be timelike and the other vector, which we shall call $\vec{s}$, must be spacelike. The pair $(\vec{n}, \vec{s})$ is then an orthonormal basis of $(\Pi^\perp, \boldsymbol{g})$: $\vec{n} \cdot \vec{n} = -1$, $\vec{s} \cdot \vec{s} = 1$, $\vec{n} \cdot \vec{s} = 0$. Let us consider an elementary parallelogram of $\mathscr{V}$ constructed upon two vectors $(\mathrm{d}\vec{\boldsymbol{\ell}}_2, \mathrm{d}\vec{\boldsymbol{\ell}}_3)$, and let us erect a hyperparallelepiped by adding the vectors $\vec{n}$ and $\vec{s}$. The 4-volume of this hyperparallelepiped is

$$\mathrm{d}U = \boldsymbol{\epsilon}(\vec{n}, \vec{s}, \mathrm{d}\vec{\boldsymbol{\ell}}_2, \mathrm{d}\vec{\boldsymbol{\ell}}_3). \tag{16.28}$$

The vectors $\vec{n}$ and $\vec{s}$ being unitary, we shall define the area $\mathrm{d}S$ of the elementary parallelogram $(\mathrm{d}\vec{\boldsymbol{\ell}}_2, \mathrm{d}\vec{\boldsymbol{\ell}}_3)$ as $\mathrm{d}S = \mathrm{d}U$. This justifies the introduction of the ***area-element 2-form*** of the surface $\mathscr{V}$ by

$$\boxed{\boldsymbol{\epsilon}_{\mathscr{V}} := \boldsymbol{\epsilon}(\vec{n}, \vec{s}, ., .)} \qquad (p = 2). \tag{16.29}$$

$\boldsymbol{\epsilon}_{\mathscr{V}}$ if a field of 2-forms on $\mathscr{V}$. The ***area*** of the elementary parallelogram of $\mathscr{V}$ constructed upon the vectors $\mathrm{d}\vec{\boldsymbol{\ell}}_2$ and $\mathrm{d}\vec{\boldsymbol{\ell}}_3$ is then

$$\boxed{\mathrm{d}S = \boldsymbol{\epsilon}_{\mathscr{V}}(\mathrm{d}\vec{\boldsymbol{\ell}}_2, \mathrm{d}\vec{\boldsymbol{\ell}}_3)}. \tag{16.30}$$

By construction, we have $\mathrm{d}S = \mathrm{d}U$, $\mathrm{d}U$ being given by (16.28). If (x^α) is a coordinate system adapted to $\mathscr{V}$ and the vectors $\mathrm{d}\vec{\boldsymbol{\ell}}_a$ are related to the infinitesimal coordinate increases $\mathrm{d}x^a$ by (16.3), then

$$\mathrm{d}U = \boldsymbol{\epsilon}(\vec{\boldsymbol{n}}, \vec{\boldsymbol{s}}, \mathrm{d}x^2\, \vec{\boldsymbol{e}}_2, \mathrm{d}x^3\, \vec{\boldsymbol{e}}_3) = \epsilon_{\mu\nu 23}\, n^\mu s^\nu\, \mathrm{d}x^2 \mathrm{d}x^3 = \epsilon_{0123}\,(n^0 s^1 - n^1 s^0)\, \mathrm{d}x^2 \mathrm{d}x^3.$$

Using the components (14.51) of the Levi–Civita tensor, we get

$$\boxed{\mathrm{d}S = (n^0 s^1 - n^1 s^0)\sqrt{-\det g}\, \mathrm{d}x^2 \mathrm{d}x^3}. \tag{16.31}$$

Example 16.7. Let us choose for $\mathscr{V}$ the sphere considered in Example 16.1 p. 525. Within the spherical coordinates $(x^\alpha) = (ct, r, \theta, \varphi)$ adapted to $\mathscr{V}$, we have the following components: $n^\alpha = (1, 0, 0, 0)$ and $s^\alpha = (0, 1, 0, 0)$; hence, $n^0 s^1 - n^1 s^0 = 1$. Since, on the other side, $\sqrt{-\det g} = r^2 \sin\theta = R^2 \sin\theta$ [cf. (15.10)], we recover from (16.31) the standard area element for a sphere of radius R:

$$\mathrm{d}S = R^2 \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\varphi.$$

Contrary to the case of a hypersurface, for which the unit future-directed normal vector $\vec{\boldsymbol{n}}$ is unique, the pair $(\vec{\boldsymbol{n}}, \vec{\boldsymbol{s}})$ of unit vectors normal to a surface $\mathscr{V}$ is not unique in the four-dimensional space $\mathscr{E}$. Indeed, at any point $M \in \mathscr{V}$, the only characteristic feature of $(\vec{\boldsymbol{n}}, \vec{\boldsymbol{s}})$ is to be an orthonormal basis of the plane $\Pi^\perp$ orthogonal to $\mathscr{V}$ at M, and there exists an infinite number of such bases. The key point is that the definition of $\boldsymbol{\epsilon}_{\mathscr{V}}$ does not depend upon the choice of the orthonormal basis of $\Pi^\perp$, provided that it has the same orientation as $(\vec{\boldsymbol{n}}, \vec{\boldsymbol{s}})$.

Proof. Let us consider a second orthonormal basis of $\Pi^\perp$, $(\vec{\boldsymbol{n}}', \vec{\boldsymbol{s}}')$ say. It can be deduced from $(\vec{\boldsymbol{n}}, \vec{\boldsymbol{s}})$ by a Lorentz transformation acting in the plane $\Pi^\perp$. If the orientation of the bases is preserved, this must be a Lorentz boost, since $\Pi^\perp$ is a timelike plane (cf. Sect. 6.4.4). Denoting by ψ the rapidity of this boost, we have then, by (6.43),

$$\begin{cases} \vec{\boldsymbol{n}}' = \cosh\psi\, \vec{\boldsymbol{n}} + \sinh\psi\, \vec{\boldsymbol{s}} \\ \vec{\boldsymbol{s}}' = \sinh\psi\, \vec{\boldsymbol{n}} + \cosh\psi\, \vec{\boldsymbol{s}}. \end{cases}$$

Thanks to the antisymmetry of the Levi–Civita tensor, we get

$$\begin{aligned} \boldsymbol{\epsilon}(\vec{\boldsymbol{n}}', \vec{\boldsymbol{s}}', .,.) &= \cosh^2\psi\, \boldsymbol{\epsilon}(\vec{\boldsymbol{n}}, \vec{\boldsymbol{s}}, .,.) + \sinh^2\psi\, \boldsymbol{\epsilon}(\vec{\boldsymbol{s}}, \vec{\boldsymbol{n}}, .,.) \\ &= \underbrace{(\cosh^2\psi - \sinh^2\psi)}_{1}\, \boldsymbol{\epsilon}(\vec{\boldsymbol{n}}, \vec{\boldsymbol{s}}, .,.) = \boldsymbol{\epsilon}(\vec{\boldsymbol{n}}, \vec{\boldsymbol{s}}, .,.). \end{aligned}$$ □

By the definition (16.29), the components of the area element 2-form in a general basis are

$$(\epsilon_{\mathscr{V}})_{\alpha\beta} = n^\mu s^\nu \epsilon_{\mu\nu\alpha\beta}. \tag{16.32}$$

Besides, let us apply the Hodge star to the 2-form resulting from the exterior product of the 1-forms $\underline{\boldsymbol{n}}$ and $\underline{\boldsymbol{s}}$, metric duals of the vectors $\vec{\boldsymbol{n}}$ and $\vec{\boldsymbol{s}}$. From (14.75c) and (14.43), we get

$$\begin{aligned}\star\,(\underline{\boldsymbol{n}}\wedge\underline{\boldsymbol{s}})_{\alpha\beta} &= \frac{1}{2}\,\epsilon_{\mu\nu\alpha\beta}\; g^{\mu\rho}g^{\nu\sigma}\;(\underline{\boldsymbol{n}}\wedge\underline{\boldsymbol{s}})_{\rho\sigma} = \frac{1}{2}\,\epsilon_{\mu\nu\alpha\beta}\; g^{\mu\rho}g^{\nu\sigma}(n_\rho s_\sigma - s_\rho n_\sigma)\\ &= \frac{1}{2}\,\epsilon_{\mu\nu\alpha\beta}\,(n^\mu s^\nu - s^\mu n^\nu) = \epsilon_{\mu\nu\alpha\beta}\,n^\mu s^\nu .\end{aligned}$$

Comparing with (16.32), we conclude that the area-element 2-form is nothing but the Hodge dual of the exterior product of $\underline{\boldsymbol{n}}$ by $\underline{\boldsymbol{s}}$:

$$\boxed{\boldsymbol{\epsilon}_{\mathscr{V}} = \star(\underline{\boldsymbol{n}}\wedge\underline{\boldsymbol{s}})}. \tag{16.33}$$

16.4.4 Length-Element of a Curve

Let $\mathscr{V}$ be an oriented submanifold of $\mathscr{E}$ of dimension $p = 1$, i.e. a curve in $\mathscr{E}$. At any point $M \in \mathscr{V}$, we shall assume that $\mathscr{V}$ is either timelike or spacelike, but neither null. We can then find an orthonormal basis of E containing a vector, $\vec{\boldsymbol{u}}$ say, that is tangent to $\mathscr{V}$ and shares the orientation of $\mathscr{V}$. Let $\vec{\boldsymbol{n}}_1$, $\vec{\boldsymbol{n}}_2$ and $\vec{\boldsymbol{n}}_3$ be the remaining three vectors of the orthonormal basis. If $\mathscr{V}$ is timelike at M, $\vec{\boldsymbol{u}}$ is a timelike unit vector: $\vec{\boldsymbol{u}}\cdot\vec{\boldsymbol{u}} = -1$. It can then be interpreted as the 4-velocity of the particle that would have $\mathscr{V}$ for worldline (hence the notation $\vec{\boldsymbol{u}}$). The other vectors can be ordered so that the orthonormal basis $(\vec{\boldsymbol{u}}, \vec{\boldsymbol{n}}_1, \vec{\boldsymbol{n}}_3, \vec{\boldsymbol{n}}_2)$ is right-handed (take care about the order of $\vec{\boldsymbol{n}}_3$ and $\vec{\boldsymbol{n}}_2$, which is chosen to ensure that $(\vec{\boldsymbol{n}}_1, \vec{\boldsymbol{n}}_2, \vec{\boldsymbol{n}}_3, \vec{\boldsymbol{u}})$ is right-handed). If, on the other side, $\mathscr{V}$ is spacelike at M, then $\vec{\boldsymbol{u}}$ is a spacelike unit vector: $\vec{\boldsymbol{u}}\cdot\vec{\boldsymbol{u}} = 1$. There exists then among the $\vec{\boldsymbol{n}}_i$'s a timelike vector, which we will choose to be $\vec{\boldsymbol{n}}_1$. The other vectors can be ordered so that the orthonormal basis $(\vec{\boldsymbol{n}}_1, \vec{\boldsymbol{n}}_2, \vec{\boldsymbol{n}}_3, \vec{\boldsymbol{u}})$ is right-handed. By the same reasoning on the erected hyperparallelepiped as that leading to (16.23) and (16.29), we are led to define the ***length element 1-form*** of the curve $\mathscr{V}$ by

$$\boxed{\boldsymbol{\epsilon}_{\mathscr{V}} := \boldsymbol{\epsilon}(\vec{\boldsymbol{n}}_1, \vec{\boldsymbol{n}}_2, \vec{\boldsymbol{n}}_3, .)} \qquad (p = 1). \tag{16.34}$$

$\boldsymbol{\epsilon}_{\mathscr{V}}$ is a field of 1-forms along $\mathscr{V}$. Since either $(\vec{\boldsymbol{u}}, \vec{\boldsymbol{n}}_1, \vec{\boldsymbol{n}}_3, \vec{\boldsymbol{n}}_2)$ or $(\vec{\boldsymbol{n}}_1, \vec{\boldsymbol{n}}_2, \vec{\boldsymbol{n}}_3, \vec{\boldsymbol{u}})$ is a right-handed orthonormal basis, we have $\boldsymbol{\epsilon}(\vec{\boldsymbol{n}}_1, \vec{\boldsymbol{n}}_2, \vec{\boldsymbol{n}}_3, \vec{\boldsymbol{u}}) = 1$; hence,

$$\langle \boldsymbol{\epsilon}_{\mathscr{V}}, \vec{\boldsymbol{u}}\rangle = 1. \tag{16.35}$$

Besides, by definition of $\vec{\boldsymbol{u}}$:

$$\langle \underline{\boldsymbol{u}}, \vec{\boldsymbol{u}}\rangle = \pm 1, \tag{16.36}$$

with the $+$ (resp. $-$) sign if $\vec{\boldsymbol{u}}$ is spacelike (resp. timelike). $\boldsymbol{\epsilon}_{\mathscr{V}}$ and $\underline{\boldsymbol{u}}$ being two linear forms that vanish on all the vectors orthogonal to $\vec{\boldsymbol{u}}$, we deduce from (16.35) and (16.36) that

$$\boxed{\boldsymbol{\epsilon}_{\mathscr{V}} = \pm\underline{\boldsymbol{u}}}, \tag{16.37}$$

with the same sign convention as above.

Let $\mathrm{d}\vec{\boldsymbol{\ell}}$ be an infinitesimal displacement vector along the curve $\mathscr{V}$, oriented in the sense of $\vec{\boldsymbol{u}}$; the length corresponding to $\mathrm{d}\vec{\boldsymbol{\ell}}$ is

$$\boxed{\mathrm{d}\ell = \langle \boldsymbol{\epsilon}_{\mathscr{V}}, \mathrm{d}\vec{\boldsymbol{\ell}}\rangle = \left\|\mathrm{d}\vec{\boldsymbol{\ell}}\right\|_{\boldsymbol{g}}}, \tag{16.38}$$

where $\|\mathrm{d}\vec{\boldsymbol{\ell}}\|_{\boldsymbol{g}}$ is the norm of the vector $\mathrm{d}\vec{\boldsymbol{\ell}}$ as defined by (1.19). The second equality follows from $\mathrm{d}\vec{\boldsymbol{\ell}} = \|\mathrm{d}\vec{\boldsymbol{\ell}}\|_{\boldsymbol{g}}\,\vec{\boldsymbol{u}}$ and (16.35). It shows that the notion of length induced by the Levi–Civita tensor on a curve $\mathscr{V}$ coincides with that defined by the metric tensor.

16.4.5 Integral of a Scalar Field on a Submanifold

Having defined for each submanifold $\mathscr{V}$ of dimension $p \in \{1, 2, 3, 4\}$ the volume element p-form $\boldsymbol{\epsilon}_{\mathscr{V}}$ (for $p = 4$, we set $\boldsymbol{\epsilon}_{\mathscr{V}} := \boldsymbol{\epsilon}$, cf. Sect. 16.2.2), we are in position to define the ***integral of a scalar field*** f ***over*** $\mathscr{V}$ by

$$\boxed{\mathrm{int}_{\mathscr{V}}(f) := \int_{\mathscr{V}} f\,\boldsymbol{\epsilon}_{\mathscr{V}}}. \tag{16.39}$$

This formula is meaningful: $\boldsymbol{\epsilon}_{\mathscr{V}}$ being a p-form, so is the product $f\boldsymbol{\epsilon}_{\mathscr{V}}$, and (16.39) is the integral of a p-form over a submanifold of dimension p, as defined in Sect. 16.4.1. Explicitly, in terms of coordinates adapted to $\mathscr{V}$ and Lebesgue integrals over $\mathbb{R}^p$, formulas (16.4), (16.25), (16.31) and (16.38) lead to

$$p = 4: \quad \mathrm{int}_{\mathscr{V}}(f) = \int_{\mathscr{V}} f\,\mathrm{d}U = \int_{\mathscr{V}} f\sqrt{-\det g}\,\mathrm{d}x^0\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3 \tag{16.40a}$$

$$p = 3: \quad \mathrm{int}_{\mathscr{V}}(f) = \int_{\mathscr{V}} f\,\mathrm{d}V = \int_{\mathscr{V}} f n^0\sqrt{-\det g}\,\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3 \tag{16.40b}$$

$$p = 2: \quad \mathrm{int}_{\mathscr{V}}(f) = \int_{\mathscr{V}} f\,\mathrm{d}S = \int_{\mathscr{V}} f(n^0s^1 - n^1s^0)\sqrt{-\det g}\,\mathrm{d}x^2\mathrm{d}x^3 \tag{16.40c}$$

$$p = 1: \quad \mathrm{int}_{\mathscr{V}}(f) = \int_{\mathscr{V}} f\,\mathrm{d}\ell. \tag{16.40d}$$

If $\mathscr{V}$ is compact, the choice $f = 1$ gives, respectively, the 4-volume, the volume, the area and the length of $\mathscr{V}$.

16.4.6 Integral of a Tensor Field

To define the integral of a tensor field $\boldsymbol{T}$ of type (k, ℓ) over a submanifold $\mathscr{V} \subset \mathscr{E}$, we introduce a *fixed* basis of E (in the sense specified in Sect. 15.3.1), $(\vec{\boldsymbol{e}}_\alpha)$. The components $T^{\alpha_1\ldots\alpha_k}{}_{\beta_1\ldots\beta_\ell}$ of $\boldsymbol{T}$ within this basis are given by the expansion (14.10). We define then the ***integral of $\boldsymbol{T}$ over $\mathscr{V}$*** as the tensor of type (k, ℓ) whose components within the basis $(\vec{\boldsymbol{e}}_\alpha)$ are the integrals of the components of $\boldsymbol{T}$:

$$\int_{\mathscr{V}} \boldsymbol{T}\, \mathrm{d}V := \left(\int_{\mathscr{V}} T^{\alpha_1\ldots\alpha_k}{}_{\beta_1\ldots\beta_\ell}\, \mathrm{d}V\right) \vec{\boldsymbol{e}}_{\alpha_1} \otimes \ldots \otimes \vec{\boldsymbol{e}}_{\alpha_k} \otimes \boldsymbol{e}^{\beta_1} \otimes \ldots \otimes \boldsymbol{e}^{\beta_\ell}, \quad (16.41)$$

where $\mathrm{d}V$ is the element of 4-volume/volume/area or length of $\mathscr{V}$ and the integral of the right-hand side is that of $T^{\alpha_1\ldots\alpha_k}{}_{\beta_1\ldots\beta_\ell}$ considered as a scalar field on $\mathscr{E}$, according to the definition (16.39). It is obvious that this definition does not depend upon the choice of the basis $(\vec{\boldsymbol{e}}_\alpha)$ of E.

Remark 16.6. If $\boldsymbol{T}$ is a differential p-form and $\mathscr{V}$ a submanifold of dimension p, we have at disposal two definitions of the integral of $\boldsymbol{T}$ over $\mathscr{V}$: that provided by (16.16) and the above one. One shall take care to distinguish these two definitions: the first one is independent of the metric tensor and yields a real number, whereas the second one depends upon the metric tensor (via the volume element $\mathrm{d}V$) and yields a p-form on E.

16.4.7 Flux Integrals

Let $\mathscr{V} \subset \mathscr{E}$ be an oriented three-dimensional submanifold (hypersurface), either spacelike or timelike. Let $\vec{\boldsymbol{n}}$ be the field of unit vectors normal to $\mathscr{V}$, such that $\boldsymbol{\epsilon}_{\mathscr{V}} := \boldsymbol{\epsilon}(\vec{\boldsymbol{n}}, ., ., .)$ is compatible with the orientation of $\mathscr{V}$. For any vector field $\vec{\boldsymbol{v}}$ on $\mathscr{E}$, one defines the ***flux of $\vec{\boldsymbol{v}}$ through $\mathscr{V}$*** by

$$\boxed{\Phi_{\mathscr{V}}(\vec{\boldsymbol{v}}) := \pm \int_{\mathscr{V}} \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{n}}\, \mathrm{d}V}, \quad (16.42)$$

where the volume element $\mathrm{d}V$ is given by (16.25) and the sign $\pm$ must be $+$ (resp. $-$) if $\vec{\boldsymbol{n}}$ is spacelike (resp. timelike). According to (16.40b), $\Phi_{\mathscr{V}}(\vec{\boldsymbol{v}})$ is nothing but the integral of the scalar field $\pm\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{n}}$ over $\mathscr{V}$.

Remark 16.7. If $\vec{\boldsymbol{v}}$ is tangent to $\mathscr{V}$ at all points, $\Phi_{\mathscr{V}}(\vec{\boldsymbol{v}}) = 0$.

Let (x^α) be a coordinate system adapted to $\mathscr{V}$ and $(\vec{e}_\alpha)$ the associated coordinate basis. At every point of $\mathscr{V}$, $(\vec{n}, \vec{e}_1, \vec{e}_2, \vec{e}_3)$ is then a basis of E, with $\vec{n}$ normal to $\mathscr{V}$ and the $\vec{e}_i$'s tangent to $\mathscr{V}$. Let us expand $\vec{v}$ onto this basis:

$$\vec{v} = v^0\, \vec{n} + v^i\, \vec{e}_i \qquad \text{with } v^0 = \pm \vec{n} \cdot \vec{v}, \tag{16.43}$$

the meaning of $\pm$ being the same as in (16.42). Introducing $\mathrm{d}\vec{\ell}_i = \mathrm{d}x^i\, \vec{e}_i$ [Eq. (16.3)], the integrand of (16.42) can be recast as

$$\vec{v} \cdot \vec{n}\, \mathrm{d}V = \underbrace{\vec{v} \cdot \vec{n}}_{\pm v^0}\, \boldsymbol{\epsilon}_{\mathscr{V}}(\mathrm{d}\vec{\ell}_1, \mathrm{d}\vec{\ell}_2, \mathrm{d}\vec{\ell}_3) = \pm \mathrm{d}x^1\, \mathrm{d}x^2\, \mathrm{d}x^3\, \boldsymbol{\epsilon}(v^0 \vec{n}, \vec{e}_1, \vec{e}_2, \vec{e}_3).$$

Substituting $\vec{v} - v^i\, \vec{e}_i$ for $v^0\, \vec{n}$ in this formula [cf. (16.43)] and using the alternate character of the Levi–Civita tensor, we get

$$\begin{aligned} \pm\, \vec{v} \cdot \vec{n}\, \mathrm{d}V &= \mathrm{d}x^1\, \mathrm{d}x^2\, \mathrm{d}x^3\, \boldsymbol{\epsilon}(\vec{v}, \vec{e}_1, \vec{e}_2, \vec{e}_3) = \boldsymbol{\epsilon}(\vec{v}, \mathrm{d}\vec{\ell}_1, \mathrm{d}\vec{\ell}_2, \mathrm{d}\vec{\ell}_3) \\ &= \star\underline{\boldsymbol{v}}(\mathrm{d}\vec{\ell}_1, \mathrm{d}\vec{\ell}_2, \mathrm{d}\vec{\ell}_3), \end{aligned}$$

where we have let appear the Hodge dual of the 1-form $\underline{\boldsymbol{v}}$, according to (15.87). Substituting this identity in (16.42), we conclude that the flux of the vector $\vec{v}$ through the hypersurface $\mathscr{V}$ is the integral of the 3-form $\star\underline{\boldsymbol{v}}$ over $\mathscr{V}$:

$$\boxed{\Phi_{\mathscr{V}}(\vec{v}) = \int_{\mathscr{V}} \star\underline{\boldsymbol{v}}}. \tag{16.44}$$

Remark 16.8. From the very definition of the flux [Eq. (16.42)], the volume of the hypersurface $\mathscr{V}$ can be seen as the flux of the unit normal vector $\vec{n}$. As a check, we recover this result from (16.44) combined with the identity (16.27).

If, instead of being three-dimensional, $\mathscr{V}$ is a spacelike two-dimensional surface, the flux of a vector can no longer be defined by an equation of the type (16.42) for the normal vector is not unique: in Minkowski spacetime, the set of all vectors normal at a point to a spacelike surface is a two-dimensional vector space (cf. Sect. 16.4.3). On the other side, one can start from expression (16.44) and define the ***flux of a 2-form A through*** $\mathscr{V}$ by an analogous formula:

$$\boxed{\Phi_{\mathscr{V}}(\boldsymbol{A}) := \int_{\mathscr{V}} \star\boldsymbol{A}}. \tag{16.45}$$

Indeed, the Hodge dual of $\boldsymbol{A}$ is itself a 2-form, and its integral over the surface $\mathscr{V}$ is well defined.

16.5 Stokes' Theorem

The fundamental theorem of the theory of integration is *Stokes' theorem*. As we shall see in the next chapters, it forms the roots of the local expression of many conservation laws in physics.

16.5.1 Statement and Examples

Let $\mathscr{V}$ be a submanifold with boundary of $\mathscr{E}$, of dimension p, oriented and compact. Its boundary $\partial\mathscr{V}$ is then a submanifold of dimension $p-1$. If $\boldsymbol{A}$ is a differential $(p-1)$-form, it may be integrated over $\partial\mathscr{V}$. Its exterior derivative $\mathbf{d}\boldsymbol{A}$ is, on its side, a differential p-form and may be integrated over $\mathscr{V}$. ***Stokes' theorem*** states that the two integrals are equal:

$$\boxed{\int_{\mathscr{V}} \mathbf{d}\boldsymbol{A} = \int_{\partial\mathscr{V}} \boldsymbol{A}}, \tag{16.46}$$

where $\partial\mathscr{V}$ is endowed with the orientation induced by that of $\mathscr{V}$ (cf. Sect. 16.3.3).

Proof. See, e.g. Berger and Gostiaux's book (Berger and Gostiaux 1988).

Example 16.8. Let us set $p = 1$ and choose for $\mathscr{V}$ the segment of a straight line; there exists an inertial coordinate system of $\mathscr{E}$, $(x^\alpha) = (ct, x, y, z)$, so that $\mathscr{V}$ is defined by $t = 0$, $x = 0$, $y = 0$ and $a \leq z \leq b$. The boundary of $\mathscr{V}$ is then constituted by the two points $A(0,0,0,a)$ and $B(0,0,0,b)$. This constitutes a submanifold of dimension 0. Let then $\boldsymbol{A} = f$ be a scalar field on $\mathscr{E}$. Let us assume that the orientation of $\mathscr{V}$ is given by the increase of z. Given the definition (16.17a) of the integral over a one-dimensional submanifold and expression (15.66) of the components of the gradient of f, we have

$$\int_{\mathscr{V}} \mathbf{d}f = \int_a^b (\mathrm{d}f)_z \, \mathrm{d}z = \int_a^b \frac{\partial f}{\partial z} \, \mathrm{d}z. \tag{16.47}$$

Besides, the integral of f over $\partial\mathscr{V} = \{A, B\}$ endowed with the induced orientation is reduced to

$$\int_{\partial\mathscr{V}} f = f(B) - f(A). \tag{16.48}$$

Stokes' theorem yields thus

$$\int_a^b \frac{\partial f}{\partial z}\,\mathrm{d}z = f(B) - f(A), \tag{16.49}$$

which is nothing but the fundamental theorem of calculus.

Example 16.9. Let us set $p = 2$ and choose for $\mathscr{V}$ a compact submanifold with boundary of the plane defined by $(t = 0, z = 0)$ within the inertial coordinates $(x^\alpha) = (ct, z, x, y)$, taking care about the coordinate order: $x^1 =: z$, $x^2 =: x$ and $x^3 =: y$. (ct, z, x, y) constitutes a coordinate system adapted to $\mathscr{V}$; we may choose the orientation of $\mathscr{V}$ such that the coordinates (x, y) are right-handed. Let $\boldsymbol{A}$ be the 1-form defined by

$$\boldsymbol{A} = P(x, y)\,\mathbf{d}x + Q(x, y)\,\mathbf{d}y, \tag{16.50}$$

where $P(x, y)$ and $Q(x, y)$ are arbitrary functions. By (16.17b) and (15.67), the integral of the 2-form $\mathbf{d}\boldsymbol{A}$ over $\mathscr{V}$ can be expressed as

$$\int_{\mathscr{V}} \mathbf{d}\boldsymbol{A} = \int_{\mathscr{V}} \left(\frac{\partial A_3}{\partial x^2} - \frac{\partial A_2}{\partial x^3}\right) \mathrm{d}x^2\,\mathrm{d}x^3 = \int_{\mathscr{V}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y.$$

Stokes' theorem (16.46) leads then to

$$\int_{\mathscr{V}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y = \int_{\partial\mathscr{V}} P(x, y)\,\mathbf{d}x + Q(x, y)\,\mathbf{d}y. \tag{16.51}$$

We recognize ***Green–Riemann formula***.

Example 16.10. Still in the case $p = 2$, let us choose for $\mathscr{V}$ an arbitrary surface with boundary in a spacelike hyperplane $\Sigma \subset \mathscr{E}$. $\boldsymbol{A}$ is still a 1-form. By (16.17b) and (15.67), the integral of the 2-form $\mathbf{d}\boldsymbol{A}$ over $\mathscr{V}$ can be expressed in a coordinate system adapted to $\mathscr{V}$ as

$$\int_{\mathscr{V}} \mathbf{d}\boldsymbol{A} = \int_{\mathscr{V}} \left(\frac{\partial A_3}{\partial x^2} - \frac{\partial A_2}{\partial x^3}\right) \mathrm{d}x^2\,\mathrm{d}x^3. \tag{16.52}$$

In the neighbourhood of any point $M \in \mathscr{V}$, one can always choose adapted coordinates of Cartesian type: $(x^\alpha) = (ct, x, y, z)$. $(x^2, x^3) = (y, z)$ are then Cartesian coordinates in the plane tangent to $\mathscr{V}$ at M. We recognize in $\partial A_3/\partial x^2 - \partial A_2/\partial x^3$ the x-component of $\mathrm{curl}\,\vec{\boldsymbol{A}}$, the curl of the metric dual of $\boldsymbol{A}$ in the Euclidean space $(\Sigma, \boldsymbol{g})$ (cf. Example 15.18 p. 516). Besides $\mathrm{d}x^2\,\mathrm{d}x^3 = \mathrm{d}y\,\mathrm{d}z$ is the area element of $\mathscr{V}$ around M. Introducing in $(\Sigma, \boldsymbol{g})$ the area-element vector normal to $\mathscr{V}$ by $\mathrm{d}\vec{\boldsymbol{S}} := \mathrm{d}y\,\mathrm{d}z\,\vec{\boldsymbol{e}}_x$, we may write

$$\left(\frac{\partial A_3}{\partial x^2} - \frac{\partial A_2}{\partial x^3}\right) \mathrm{d}x^2\,\mathrm{d}x^3 = \mathrm{curl}\,\vec{\boldsymbol{A}} \cdot \mathrm{d}\vec{\boldsymbol{S}}. \tag{16.53}$$

On another side, by (16.17a),

$$\int_{\partial\mathscr{V}} \boldsymbol{A} = \int_{\partial\mathscr{V}} A_3 \,\mathrm{d}x^3 = \int_{\partial\mathscr{V}} \vec{\boldsymbol{A}} \cdot \mathrm{d}\vec{\boldsymbol{\ell}}, \tag{16.54}$$

where $\mathrm{d}\vec{\boldsymbol{\ell}} = \mathrm{d}z\, \vec{\boldsymbol{e}}_z$ (locally) is the length-element vector along the boundary of $\mathscr{V}$. In view of (16.52), (16.53) and (16.54), Stokes's theorem (16.46) gives

$$\int_{\mathscr{V}} \mathrm{curl}\, \vec{\boldsymbol{A}} \cdot \mathrm{d}\vec{\boldsymbol{S}} = \int_{\partial\mathscr{V}} \vec{\boldsymbol{A}} \cdot \mathrm{d}\vec{\boldsymbol{\ell}}. \tag{16.55}$$

We recognize the "elementary" two-dimensional form of Stokes' theorem, known as ***Kelvin–Stokes theorem***.

16.5.2 Applications

In addition to the above examples, let us investigate two applications of Stokes' theorem that are particularly important.

16.5.2.1 Three-Dimensional Gauss–Ostrogradsky Theorem

Let $\mathscr{V}$ be a three-dimensional compact submanifold with boundary entirely contained in a spacelike hyperplane Σ. Let us choose inertial coordinates $(x^\alpha) = (ct, x, y, z)$ so that Σ is defined by $t = 0$. Let $\vec{\boldsymbol{v}}$ be a vector field tangent to Σ. Let us consider the differential 2-form

$$\boldsymbol{A} := \boldsymbol{\epsilon}(\vec{\boldsymbol{e}}_0, \vec{\boldsymbol{v}}, ., .), \tag{16.56}$$

where $\vec{\boldsymbol{e}}_0$ is the first vector of the coordinate basis $(\vec{\boldsymbol{e}}_\alpha)$ associated with (x^α). Since (x^α) are inertial coordinates, $(\vec{\boldsymbol{e}}_\alpha)$ is an orthonormal basis of $(E, \boldsymbol{g})$. The components of $\boldsymbol{A}$ in this basis are $A_{\alpha\beta} = \epsilon_{\mu\nu\alpha\beta}\delta^\mu{}_0 v^\nu = \epsilon_{0\nu\alpha\beta} v^\nu$. Given the value (14.51) of the components of $\boldsymbol{\epsilon}$, with $\det g = -1$ (orthonormal basis), we deduce that

$$A_{0\alpha} = 0 \qquad \text{and} \qquad A_{ij} = [0, k, i, j]\, v^k = [i, j, k]\, v^k, \tag{16.57}$$

where $[i, j, k] = 1$ (resp. -1) if (i, j, k) is an even (resp. odd) permutation of $(1, 2, 3)$ and $[i, j, k] = 0$ otherwise. (x^α) being a coordinate system adapted to $\mathscr{V}$, formulas (16.17c) and (15.68) yield

$$\int_{\mathscr{V}} \mathbf{d}\boldsymbol{A} = \int_{\mathscr{V}} (\mathrm{d}A)_{123}\, \mathrm{d}x^1\, \mathrm{d}x^2\, \mathrm{d}x^3 = \int_{\mathscr{V}} \left(\frac{\partial A_{23}}{\partial x^1} + \frac{\partial A_{31}}{\partial x^2} + \frac{\partial A_{12}}{\partial x^3} \right) \mathrm{d}x^1\, \mathrm{d}x^2\, \mathrm{d}x^3.$$

Now, by (16.57), $A_{23} = v^1$, $A_{31} = v^2$ and $A_{12} = v^3$, so that we can write

$$\int_{\mathscr{V}} \mathbf{d}\boldsymbol{A} = \int_{\mathscr{V}} \boldsymbol{\nabla}\cdot \vec{\boldsymbol{v}}\, \mathrm{d}V, \tag{16.58}$$

where $\boldsymbol{\nabla}\cdot \vec{\boldsymbol{v}} = \partial v^\mu/\partial x^\mu = \partial v^i/\partial x^i = \partial v^x/\partial x + \partial v^y/\partial y + \partial v^z/\partial z$ is the divergence of the vector field $\vec{\boldsymbol{v}}$ (cf. Sect. 15.4.5) and $\mathrm{d}V = \mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}x^3 = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ is the volume element within the Euclidean hyperplane $(\Sigma, \boldsymbol{g})$. Let us introduce now a coordinate system $(x'^\alpha) = (ct, w, u, v)$ adapted to the boundary of $\mathscr{V}$, i.e. such that $\partial\mathscr{V}$ is the surface defined by $t = 0$ and $w = 0$, $w < 0$ corresponding to the interior of $\mathscr{V}$. We can use the definition (16.17b) of the integral of a 2-form over a surface:

$$\int_{\partial\mathscr{V}} \boldsymbol{A} = \int_{\partial\mathscr{V}} A'_{uv}\, \mathrm{d}u\, \mathrm{d}v. \tag{16.59}$$

The component A'_{uv} of $\boldsymbol{A}$ within coordinates (x'^α) is related to the components $A_{\alpha\beta}$ within coordinates (x^α) via (14.23), with the change-of-basis matrix $P^\alpha{}_\beta = \partial x^\alpha/\partial x'^\beta$:

$$A'_{uv} = A_{\alpha\beta} \frac{\partial x^\alpha}{\partial u} \frac{\partial x^\beta}{\partial v}.$$

Given the value (16.57) of $A_{\alpha\beta}$, we get

$$A'_{uv} = v^x \left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial y}{\partial v} \right) + v^y \left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial x}{\partial u}\frac{\partial z}{\partial v} \right) + v^z \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} \right). \tag{16.60}$$

Besides, the area-element vector normal to $\partial\mathscr{V}$ within the Euclidean space $(\Sigma, \boldsymbol{g})$ is

$$\mathrm{d}\vec{\boldsymbol{S}} = \mathrm{d}\vec{\boldsymbol{\ell}}_u \times \mathrm{d}\vec{\boldsymbol{\ell}}_v = (\mathrm{d}u\, \vec{\boldsymbol{e}}'_u) \times (\mathrm{d}v\, \vec{\boldsymbol{e}}'_v),$$

with by (15.6),

$$\vec{\boldsymbol{e}}'_u = \frac{\partial x}{\partial u}\vec{\boldsymbol{e}}_x + \frac{\partial y}{\partial u}\vec{\boldsymbol{e}}_y + \frac{\partial z}{\partial u}\vec{\boldsymbol{e}}_z \quad \text{and} \quad \vec{\boldsymbol{e}}'_v = \frac{\partial x}{\partial v}\vec{\boldsymbol{e}}_x + \frac{\partial y}{\partial v}\vec{\boldsymbol{e}}_y + \frac{\partial z}{\partial v}\vec{\boldsymbol{e}}_z.$$

Forming the cross product $\vec{\boldsymbol{e}}'_u \times \vec{\boldsymbol{e}}'_v$, we let appear the vector whose components are between the parentheses in (16.60), so that we can write

$$A'_{uv}\, \mathrm{d}u\, \mathrm{d}v = v^x \mathrm{d}S^x + v^y \mathrm{d}S^y + v^z \mathrm{d}S^z = \vec{\boldsymbol{v}} \cdot \mathrm{d}\vec{\boldsymbol{S}}. \tag{16.61}$$

Taking into account (16.58), (16.59) and (16.61), Stokes' theorem (16.46) leads to

$$\boxed{\int_{\mathscr{V}} \boldsymbol{\nabla}\cdot \vec{\boldsymbol{v}}\, \mathrm{d}V = \int_{\partial\mathscr{V}} \vec{\boldsymbol{v}} \cdot \mathrm{d}\vec{\boldsymbol{S}}}. \tag{16.62}$$

We recognize ***Gauss–Ostrogradsky theorem***, relating the flux of a vector through a closed surface to the integral of the divergence of the vector over the volume bounded by the surface.

16.5.2.2 Four-Dimensional Gauss–Ostrogradsky Theorem

Let us consider now the case where $\mathscr{V}$ is a four-dimensional compact submanifold of $\mathscr{E}$, bounded by $\partial\mathscr{V}$. The latter is a three-dimensional submanifold of $\mathscr{E}$, and we can consider the flux of a vector field $\vec{\boldsymbol{v}}$ through $\partial\mathscr{V}$ (cf. Sect. 16.4.7). Expression (16.44) for the flux, combined with Stokes's theorem (16.46), leads to

$$\Phi_{\partial\mathscr{V}}(\vec{\boldsymbol{v}}) = \int_{\partial\mathscr{V}} \star\underline{\boldsymbol{v}} = \int_{\mathscr{V}} \mathbf{d}\star\underline{\boldsymbol{v}}. \tag{16.63}$$

Now, $\mathbf{d}\star\underline{\boldsymbol{v}}$ is related to the divergence of $\vec{\boldsymbol{v}}$ by (15.89): $\mathbf{d}\star\underline{\boldsymbol{v}} = (\boldsymbol{\nabla}\cdot\vec{\boldsymbol{v}})\,\boldsymbol{\epsilon}$, so that the above integral is nothing but the integral of the scalar field $\boldsymbol{\nabla}\cdot\vec{\boldsymbol{v}}$ over $\mathscr{V}$. We obtain thus the ***four-dimensional Gauss–Ostrogradsky theorem***:

$$\boxed{\Phi_{\partial\mathscr{V}}(\vec{\boldsymbol{v}}) = \int_{\mathscr{V}} \boldsymbol{\nabla}\cdot\vec{\boldsymbol{v}}\,\mathrm{d}U}. \tag{16.64}$$

Remark 16.9. Stokes' theorem, in its general form (16.46), is independent of the metric tensor $\boldsymbol{g}$, since it relates the integral of a differential p-form ($\mathbf{d}\boldsymbol{A}$) over a submanifold of dimension p ($\mathscr{V}$) to the integral of a differential $(p-1)$-form ($\boldsymbol{A}$) over a submanifold of dimension $p-1$ ($\partial\mathscr{V}$), these two integrals being defined independently of $\boldsymbol{g}$. On the other side, the Gauss–Ostrogradsky theorem depends on the metric $\boldsymbol{g}$ in various aspects: (i) the definition of the divergence, (ii) the integration of a scalar field over $\mathscr{V}$ and (iii) the flux integrals.

Historical note: *In a famous encyclopedia article published in 1921 (Pauli 1921), 2-forms were called* surface tensors *by Wolfgang Pauli,*[7] *thereby underlying their role in the theory of integration over surfaces. Stokes' theorem owes its name to George G. Stokes,*[8] *who was asking the demonstration of the two-dimensional version (16.55) as an exam for getting some prizes at the University of Cambridge.*

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[7]**Wolfgang Pauli** (1900–1958): Austrian theoretical physicist, who authored fundamental works in quantum mechanics and received the Nobel Prize in Physics in 1945 for the discovery of the exclusion principle. His contribution to relativity is mostly the large encyclopedia article (Pauli 1921), which he wrote at the age of 21, at the request of his thesis advisor Arnold Sommerfeld (cf. p. 27). He also got interested in the relativistic treatment of gravitation (cf. Sect. 22.2.4).

[8]**George G. Stokes** (1819–1903): British physicist and mathematician, of Irish origin, known for his works in fluid dynamics (*Navier–Stokes equation*) and optics.

It seems that the first demonstration of the theorem is due to William Thomson (the future Lord Kelvin) (1824–1907), as indicated by a letter that he wrote to Stokes in 1850. For this reason, Roger Penrose (cf. p. 162) even suggested to suppress any reference to Stokes in the naming of the theorem and to call it the fundamental theorem of exterior calculus *(Penrose 2007).*

Chapter 17
Electromagnetic Field

17.1 Introduction

Without any doubt, special relativity originates from investigations about the electromagnetic field. Conversely, Minkowski spacetime is the ideal framework to express the classical (Maxwell) theory of electromagnetism but also quantum electrodynamics. Notably, as we shall see in the next chapter, once expressed in terms of tensor fields on Minkowski spacetime, Maxwell equations take a much simpler form than the group of four equations on $\vec{E}$ and $\vec{B}$ presented in elementary courses of electromagnetism. Before this, this chapter is devoted to the definition of the electromagnetic field (Sect. 17.2), to the transformation laws of its components under a change of observer (Sect. 17.3) and to the motion of a charged particle in a given field (Sect. 17.4), with some applications to particle accelerators (Sect. 17.5).

17.2 Electromagnetic Field Tensor

17.2.1 Electromagnetic Field and Lorentz 4-Force

Historically, the concept of *electromagnetic field* has appeared progressively, after many experiments and theoretical constructions,[1] which have lead to the relatively elaborated notions of *electric field vector* $\vec{E}$ and *magnetic field vector* $\vec{B}$. In the present case, namely, that of Minkowski spacetime $(\mathscr{E}, \boldsymbol{g})$, the definition of the electromagnetic field is simpler. It is provided by the four-dimensional description of the action of the electromagnetic field onto a charged particle, $\mathscr{P}$ say. The basic hypothesis is that the electromagnetic interaction is a *vector interaction*. This means

[1] For details about the history of electromagnetism, the reader is referred to the books by O. Darrigol (2000; 2005).

É. Gourgoulhon, *Special Relativity in General Frames*, Graduate Texts in Physics,
DOI 10.1007/978-3-642-37276-6_17, © Springer-Verlag Berlin Heidelberg 2013

that the force exerted onto $\mathscr{P}$ must depend on a direction associated with $\mathscr{P}$ and not only on a number (like its mass) characterizing it, as it would be the case of a *scalar interaction*. In Minkowski spacetime, the force is described independently of any observer by the 4-force linear form $\boldsymbol{f}$, introduced in Sect. 9.5, and the only "direction" vector that is intrinsic to the particle $\mathscr{P}$ is its 4-velocity $\vec{\boldsymbol{u}}$, which is tangent to its worldline. The simplest assumption that we may make is that the 4-force exerted onto $\mathscr{P}$ is linear in $\vec{\boldsymbol{u}}$. There is then only one possible relation to describe the electromagnetic interaction: there must exist a field of bilinear forms $\boldsymbol{F}$, which we shall call the ***electromagnetic field tensor***, or simply ***electromagnetic field***, such that at any point of $\mathscr{P}$'s worldline,[2]

$$\boxed{\boldsymbol{f} = q\,\boldsymbol{F}(.,\vec{\boldsymbol{u}})}, \tag{17.1}$$

where the coefficient q is a constant that characterizes the particle and is called ***electric charge***. If $q = 0$, the particle $\mathscr{P}$ does not feel the electromagnetic field; one says that it is ***electrically neutral***. If $q \neq 0$, $\mathscr{P}$ is said to be ***electrically charged***.[3]

The 4-force (17.1) is called ***Lorentz 4-force***. In terms of components in some basis of E, (17.1) takes the form

$$f_\alpha = q\,F_{\alpha\beta}u^\beta. \tag{17.2}$$

Remark 17.1. The electromagnetic field tensor $\boldsymbol{F}$ is sometimes named ***Faraday tensor*** (Misner et al. 1973), ***field-strength tensor*** (Jackson 1998) or ***Maxwell field tensor*** (Penrose 2007).

Remark 17.2. We have already encountered the concept of vector interaction in the framework of the Lagrangian formalism, in Sect. 11.2.6. Since the Lagrangian L of a particle is a scalar (and not a linear form as the 4-force), the vector interaction involves a linear form $\boldsymbol{A}$, rather than a bilinear form, that acts on $\vec{\boldsymbol{u}}$, or more precisely on a vector collinear to $\vec{\boldsymbol{u}}$: the tangent vector $\vec{\boldsymbol{v}}$ associated with some parametrization of the worldline. This leads to the term $q/c\ \langle \boldsymbol{A}, \vec{\boldsymbol{v}}\rangle$ in the Lagrangian [cf. Eq. (11.28)]. This concept of vector interaction is fully compatible with that considered above: Eq. (11.34) makes the link between $\boldsymbol{A}$ and $\boldsymbol{F}$ explicit: $\boldsymbol{F} = \mathbf{d}\boldsymbol{A}$ (cf. Example 15.17 p. 515). We shall see in Chap. 18 that the converse is true for Maxwell electromagnetism: the electromagnetic field tensor can always be expressed (at least locally) as the exterior derivative of a differential 1-form.

In the International System of Units (SI), the electric charge has a dimension and its unit is the ***coulomb*** (symbol: C). It equals one ampere times one second, the ***ampere*** (symbol: A) being a base unit of the SI system: $1\ \mathrm{C} = 1\ \mathrm{A\,s}$. The coulomb is actually a macroscopic unit. At the level of particles, the elementary charge is that

[2] Let us recall that the notation $\boldsymbol{F}(.,\vec{\boldsymbol{u}})$ means that $\forall \vec{\boldsymbol{v}} \in E$, $\langle \boldsymbol{f}, \vec{\boldsymbol{v}}\rangle = q\,\boldsymbol{F}(\vec{\boldsymbol{v}}, \vec{\boldsymbol{u}})$.

[3] The qualifier *electrically* is often omitted, and one speaks simply about a *neutral particle* and a *charged particle*.

of the electron, denoted by $-e$, with (Yao et al. 2006)

$$e = 1.602\,176\,487(40) \times 10^{-19}\ \mathrm{C}. \tag{17.3}$$

By virtue of (17.1) and the dimensionless character of the 4-velocity $\vec{\boldsymbol{u}}$, the dimension of the electromagnetic field tensor $\boldsymbol{F}$ is that of a force per unit charge. Its SI unit is thus the newton per coulomb: $1\ \mathrm{N\,C^{-1}} = 1\ \mathrm{kg\,m\,s^{-3}\,A^{-1}}$. Defining the unit ***volt*** (symbol: V) as a watt per ampere: $1\ \mathrm{V} = 1\ \mathrm{W\,A^{-1}} = 1\ \mathrm{kg\,m^2\,s^{-3}\,A^{-1}}$, we note that the SI unit of $\boldsymbol{F}$ is the volt per metre ($\mathrm{V\,m^{-1}}$).

17.2.2 The Electromagnetic Field as a 2-Form

In addition to being of the form (17.1), the Lorentz 4-force is postulated to be a *pure* 4-force, in the sense defined in Sect. 9.5.1: $\langle \boldsymbol{f}, \vec{\boldsymbol{u}} \rangle = 0$ [Eq. (9.107)]. Given (17.1), this implies $q\,\boldsymbol{F}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{u}}) = 0$, i.e. if $q \neq 0$, $\boldsymbol{F}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{u}}) = 0$. This relation having to be fulfilled for any 4-velocity $\vec{\boldsymbol{u}}$, we conclude that the bilinear form $\boldsymbol{F}$ is alternate, $\forall \vec{\boldsymbol{v}} \in E,\ \boldsymbol{F}(\vec{\boldsymbol{v}}, \vec{\boldsymbol{v}}) = 0$ or, equivalently, that it is antisymmetric,

$$\forall (\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}) \in E^2, \quad \boxed{\boldsymbol{F}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}) = -\boldsymbol{F}(\vec{\boldsymbol{v}}, \vec{\boldsymbol{u}})}. \tag{17.4}$$

The electromagnetic field tensor is thus a *differential 2-form* on $\mathscr{E}$, as studied in Sect. 15.5.

17.2.3 Electric and Magnetic Fields

Let $\mathscr{O}$ be an observer of worldline $\mathscr{L}_0$, 4-velocity $\vec{\boldsymbol{u}}_0$ and proper time t. Let M be some event in the local rest space $\mathscr{E}_{\boldsymbol{u}_0}(t)$ of $\mathscr{O}$ (t is then the date set by $\mathscr{O}$ to M) (cf. Fig. 17.1). Let us consider the value of the field $\boldsymbol{F}$ at M. It is a 2-form, which can be orthogonally decomposed with respect to $\vec{\boldsymbol{u}}_0(t)$ via formula (3.37): there exists a unique linear form $\boldsymbol{E} \in E^*$ and a unique vector $\vec{\boldsymbol{B}} \in E$ such that

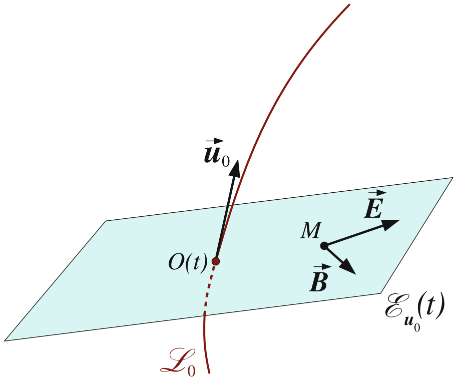
$$\boxed{\boldsymbol{F} = \underline{\boldsymbol{u}}_0 \otimes \boldsymbol{E} - \boldsymbol{E} \otimes \underline{\boldsymbol{u}}_0 + \boldsymbol{\epsilon}(\vec{\boldsymbol{u}}_0, c\vec{\boldsymbol{B}}, ., .)}, \tag{17.5a}$$

$$\boxed{\langle \boldsymbol{E}, \vec{\boldsymbol{u}}_0 \rangle = 0}, \qquad \boxed{\vec{\boldsymbol{u}}_0 \cdot \vec{\boldsymbol{B}} = 0}, \tag{17.5b}$$

where $\boldsymbol{F} = \boldsymbol{F}(M)$ and $\vec{\boldsymbol{u}}_0 = \vec{\boldsymbol{u}}_0(t)$. As we have noticed in Sect. 14.5.4, this decomposition can be expressed in terms of exterior products and the Hodge star [cf. (14.80)]:

$$\boxed{\boldsymbol{F} = \underline{\boldsymbol{u}}_0 \wedge \boldsymbol{E} + \star(\underline{\boldsymbol{u}}_0 \wedge c\underline{\boldsymbol{B}})}. \tag{17.6}$$

Fig. 17.1 Electric field vector $\vec{\boldsymbol{E}}$ and magnetic field vector $\vec{\boldsymbol{B}}$ at some point M of the local rest space of observer $\mathscr{O}$ at proper time t. $\vec{\boldsymbol{E}}$ is the metric dual to the linear form $\boldsymbol{E}$



The linear form $\boldsymbol{E}$ is called the ***electric field relative to observer*** $\mathscr{O}$ and the vector $\vec{\boldsymbol{B}}$ the ***magnetic field relative to observer*** $\mathscr{O}$. The name *field* reminds one that $\boldsymbol{E}$ and $\vec{\boldsymbol{B}}$ are functions of t and of the point M in $\mathscr{E}_{\boldsymbol{u}_0}(t)$. These fields are thus defined in all the spacetime domain where $\mathscr{O}$'s local rest spaces constitute a regular slicing of $\mathscr{E}$. If $\mathscr{O}$ is inertial, this is the whole $\mathscr{E}$; otherwise, the size of this domain is limited by the inverse of the norm of $\mathscr{O}$'s 4-acceleration (cf. Sect. 3.7).

$\boldsymbol{E}$ and $\vec{\boldsymbol{B}}$ are expressed in terms of the electromagnetic field tensor and $\mathscr{O}$'s 4-velocity via (14.81)–(14.82):

$$\boxed{\boldsymbol{E} = \boldsymbol{F}(., \vec{\boldsymbol{u}}_0)} \quad (17.7)$$

$$\boxed{c\underline{\boldsymbol{B}} = \star\boldsymbol{F}(\vec{\boldsymbol{u}}_0, .)}. \quad (17.8)$$

This last equation enables one to link the components of the vector $\vec{\boldsymbol{B}}$ to those of the 2-form $\boldsymbol{F}$ via the tensor ${}^3\boldsymbol{\epsilon}$ introduced in Sect. 14.5.1 [cf. Eq. (14.83)]:

$$B^\alpha = -\frac{1}{2c}\,\epsilon^{\alpha\mu\nu}{}_\rho\, F_{\mu\nu}\, u_0^\rho. \quad (17.9)$$

In view of (17.7)–(17.8) and of the dimensionless character of the 4-velocity $\vec{\boldsymbol{u}}_0$ and the Levi–Civita tensor, the electric field $\boldsymbol{E}$ has the same dimension as $\boldsymbol{F}$ and the magnetic field $\vec{\boldsymbol{B}}$ has the dimension of an electric field divided by a velocity. In the SI system, the unit for $\boldsymbol{E}$ is thus the volt per metre ($\mathrm{V\,m^{-1}}$) and that for $\vec{\boldsymbol{B}}$ the $\mathrm{V\,m^{-2}\,s}$, which is called ***tesla*** (symbol: T): $1\ \mathrm{T} = 1\ \mathrm{V\,m^{-2}\,s} = 1\ \mathrm{kg\,s^{-2}\,A^{-1}}$.

Remark 17.3. The electric field $\boldsymbol{E}$ and the magnetic field $\vec{\boldsymbol{B}}$ are *relative* quantities, since they depend on the observer $\mathscr{O}$ [Eqs. (17.7)–(17.8)]. In particular, they are orthogonal to $\mathscr{O}$'s 4-velocity. The only *absolute* quantity (i.e. independent of any observer) that characterizes the electromagnetic field is the tensor $\boldsymbol{F}$.

Let us express the components of $\boldsymbol{F}$ within $\mathscr{O}$'s local frame $(\vec{\boldsymbol{e}}_\alpha(t))$ in terms of the components of $\boldsymbol{E}$ and $\vec{\boldsymbol{B}}$. Equation (17.5a) leads to

$$F_{\alpha\beta} = (u_0)_\alpha E_\beta - E_\alpha (u_0)_\beta + c\,\epsilon_{\mu\nu\alpha\beta} u_0^\mu B^\nu. \tag{17.10}$$

Now, by definition of the local frame, $\vec{\boldsymbol{u}}_0 = \vec{\boldsymbol{e}}_0$, so that $u_0^\alpha = (1, 0, 0, 0)$. The basis $(\vec{\boldsymbol{e}}_\alpha)$ being orthonormal, we deduce that $(u_0)_\alpha = (-1, 0, 0, 0)$. The orthogonality conditions (17.5b) are then expressed via $E_0 = 0$ and $B^0 = 0$; hence,

$$E_\alpha = (0, E_1, E_2, E_3) \qquad \text{and} \qquad B^\alpha = (0, B^1, B^3, B^3). \tag{17.11}$$

Besides, since $(\vec{\boldsymbol{e}}_\alpha)$ is a right-handed orthonormal basis, $\epsilon_{\mu\nu\alpha\beta} = [\mu, \nu, \alpha, \beta]$ [cf. (14.51) with $\det g = -1$]. Equation (17.10) becomes then $F_{\alpha\beta} = -\delta^0{}_\alpha E_\beta + E_\alpha \delta^0{}_\beta + c[0, k, \alpha, \beta] B^k$ or, in matrix form:

$$\boxed{F_{\alpha\beta} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & cB^3 & -cB^2 \\ E_2 & -cB^3 & 0 & cB^1 \\ E_3 & cB^2 & -cB^1 & 0 \end{pmatrix}}. \tag{17.12}$$

Historical note: *The tensor $\boldsymbol{F}$ has been introduced explicitly in 1907 by Hermann Minkowski (cf. p. 26) (Minkowski 1907, 1908). It seems, however, that the tensorial aspect of $(\boldsymbol{E}, \vec{\boldsymbol{B}})$ gathered as in (17.12) was known to Henri Poincaré (cf. p. 26) in 1905 (Poincaré 1906) (cf. the discussion in Damour (2008)).*

17.2.4 Lorentz Force Relative to an Observer

Substituting the decomposition (17.5) into (17.1), we can express the 4-force exerted onto a particle $\mathscr{P}$ of charge q and 4-velocity $\vec{\boldsymbol{u}}$ as

$$\boldsymbol{f} = q\left[\langle \boldsymbol{E}, \vec{\boldsymbol{u}}\rangle\, \underline{\boldsymbol{u}}_0 - \langle \underline{\boldsymbol{u}}_0, \vec{\boldsymbol{u}}\rangle\, \boldsymbol{E} + \boldsymbol{\epsilon}(\vec{\boldsymbol{u}}_0, \vec{\boldsymbol{u}}, c\vec{\boldsymbol{B}}, .)\right], \tag{17.13}$$

where use has been made of the identity $\boldsymbol{\epsilon}(\vec{\boldsymbol{u}}_0, c\vec{\boldsymbol{B}}, ., \vec{\boldsymbol{u}}) = \boldsymbol{\epsilon}(\vec{\boldsymbol{u}}_0, \vec{\boldsymbol{u}}, c\vec{\boldsymbol{B}}, .)$ (even permutation of the arguments of $\boldsymbol{\epsilon}$). Let us express $\mathscr{P}$'s 4-velocity in terms of $\mathscr{P}$'s Lorentz factor Γ with respect to $\mathscr{O}$ and $\mathscr{P}$'s velocity $\vec{\boldsymbol{V}}$ relative to $\mathscr{O}$, according to (4.27):

$$\vec{\boldsymbol{u}} = \Gamma\left[(1 + \vec{\boldsymbol{a}}_0 \cdot \overrightarrow{OM})\, \vec{\boldsymbol{u}}_0 + \frac{1}{c}\left(\vec{\boldsymbol{V}} + \vec{\boldsymbol{\omega}} \times_{\boldsymbol{u}_0} \overrightarrow{OM}\right)\right],$$

where $\vec{\boldsymbol{a}}_0$ and $\vec{\boldsymbol{\omega}}$ are, respectively, the 4-acceleration and 4-rotation of observer $\mathscr{O}$, M the position of $\mathscr{P}$ on its worldline and O the event of $\mathscr{O}$'s worldline simultaneous to M from $\mathscr{O}$'s point of view. Inserting this relation into (17.13), we get, thanks to (17.5b) and the alternate character of $\boldsymbol{\epsilon}$:

$$\boldsymbol{f} = \Gamma q\Big[\frac{1}{c}\langle \boldsymbol{E}, \vec{V} + \vec{\omega} \times_{\boldsymbol{u}_0} \overrightarrow{OM}\rangle \underline{\boldsymbol{u}}_0 \\ +(1 + \vec{\boldsymbol{a}}_0 \cdot \overrightarrow{OM})\boldsymbol{E} + \boldsymbol{\epsilon}(\vec{\boldsymbol{u}}_0, \vec{V} + \vec{\omega} \times_{\boldsymbol{u}_0} \overrightarrow{OM}, \vec{\boldsymbol{B}}, .)\Big].$$

Let us then introduce the mixed product 3-form within $E_{\boldsymbol{u}_0}$, $\boldsymbol{\epsilon}_{\boldsymbol{u}_0} := \boldsymbol{\epsilon}(\vec{\boldsymbol{u}}_0, ., ., .)$ [cf. Eq. (3.45)] and compare with the orthogonal decomposition (9.119) of the 4-force. We obtain

$$\frac{\mathrm{d}\mathfrak{E}}{\mathrm{d}t} + c^2\langle \boldsymbol{P}, \vec{\boldsymbol{a}}_0\rangle = q\langle \boldsymbol{E}, \vec{V} + \vec{\omega} \times_{\boldsymbol{u}_0} \overrightarrow{OM}\rangle \tag{17.14a}$$

$$\mathfrak{F} = q\left[(1 + \vec{\boldsymbol{a}}_0 \cdot \overrightarrow{OM})\boldsymbol{E} + \boldsymbol{\epsilon}_{\boldsymbol{u}_0}(\vec{V} + \vec{\omega} \times_{\boldsymbol{u}_0} \overrightarrow{OM}, \vec{\boldsymbol{B}}, .)\right], \tag{17.14b}$$

where we have set $\mathfrak{E} := E$ and $\mathfrak{F} := \boldsymbol{F}_{\text{ext}}$ to keep the symbols E and $\boldsymbol{F}$ for, respectively, the electric field and the electromagnetic field tensor. In these relations, $\mathfrak{E}$ and $\boldsymbol{P}$ are, respectively, the energy and linear momentum of particle $\mathscr{P}$, both measured by observer $\mathscr{O}$, and $\mathfrak{F}$ is the force of non-inertial origin (in the present case electromagnetic origin) acting on $\mathscr{P}$ and measured by $\mathscr{O}$. If $\mathscr{O}$ is inertial ($\vec{\boldsymbol{a}}_0 = 0$ and $\vec{\omega} = 0$), the above formulas simplify to

$$\boxed{\frac{\mathrm{d}\mathfrak{E}}{\mathrm{d}t} = q\langle \boldsymbol{E}, \vec{V}\rangle} \quad (\mathscr{O} \text{ inertial}) \tag{17.15a}$$

$$\boxed{\mathfrak{F} = q\left[\boldsymbol{E} + \boldsymbol{\epsilon}_{\boldsymbol{u}_0}(\vec{V}, \vec{\boldsymbol{B}}, .)\right]} \quad (\mathscr{O} \text{ inertial}). \tag{17.15b}$$

The force (17.15b) is called ***Lorentz force relative to observer*** $\mathscr{O}$. We recover the classical expression of this force. In particular, the vector version of (17.15b) obtained by metric duality is

$$\boxed{\vec{\mathfrak{F}} = q\left(\vec{\boldsymbol{E}} + \vec{V} \times_{\boldsymbol{u}_0} \vec{\boldsymbol{B}}\right)}. \tag{17.16}$$

17.2.5 Metric Dual and Hodge Dual

By, respectively, metric duality and Hodge duality, one associates with the electromagnetic field tensor $\boldsymbol{F}$ two tensors of the same valence: $\boldsymbol{F}^{\sharp}$ and $\star\boldsymbol{F}$.

Whereas $\boldsymbol{F}$ is a tensor of type $(0, 2)$, $\boldsymbol{F}^{\sharp}$ is the tensor of type $(2, 0)$ defined by

$$\begin{aligned} \boldsymbol{F}^{\sharp} : E^* \times E^* &\longrightarrow \mathbb{R} \\ (\boldsymbol{\omega}_1, \boldsymbol{\omega}_2) &\longmapsto \boldsymbol{F}(\vec{\omega}_1, \vec{\omega}_2), \end{aligned} \tag{17.17}$$

where $\vec{\boldsymbol{\omega}}_{1,2}$ stands for the dual metric vector of the linear form $\boldsymbol{\omega}_{1,2}$. We may thus consider $\boldsymbol{F}^{\sharp}$ as the "double metric dual" of $\boldsymbol{F}$. In a given basis of E, the components of $\vec{\boldsymbol{\omega}}$ are related to those of $\boldsymbol{\omega}$ via (1.43): $\omega^{\alpha} = g^{\alpha\mu}\omega_{\mu}$. We deduce that the components of $\boldsymbol{F}^{\sharp}$ are

$$\boxed{\boldsymbol{F}^{\sharp}: \quad F^{\alpha\beta} = g^{\alpha\mu} g^{\beta\nu} F_{\mu\nu}}. \tag{17.18}$$

Remark 17.4. The symbol $\sharp$ is omitted on the components of $\boldsymbol{F}^{\sharp}$ since the position of the indices (both contravariant) is sufficient to distinguish from the components of $\boldsymbol{F}$ (covariant indices).

If the basis of E is chosen to be the local frame of some inertial observer $\mathscr{O}$, then $g^{\alpha\beta} = \eta^{\alpha\beta}$, and one deduces from (17.18) and (17.12) the following components of $\boldsymbol{F}^{\sharp}$:

$$F^{\alpha\beta} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & cB^3 & -cB^2 \\ -E_2 & -cB^3 & 0 & cB^1 \\ -E_3 & cB^2 & -cB^1 & 0 \end{pmatrix}. \tag{17.19}$$

Since $\boldsymbol{F}$ is a 2-form, another valence-2 tensor can be associated with it, beside $\boldsymbol{F}^{\sharp}$: its *Hodge dual* $\star\boldsymbol{F}$ (cf. Sect. 14.5). It is defined by (14.75c), which can be recast in terms of the components of $\boldsymbol{F}^{\sharp}$ via (17.18) and $\epsilon_{\alpha\beta\mu\nu} = \epsilon_{\mu\nu\alpha\beta}$:

$$\boxed{\star F_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta\mu\nu} F^{\mu\nu}}. \tag{17.20}$$

Given some observer $\mathscr{O}$ and the resulting split of $\boldsymbol{F}$ into an electric field $\boldsymbol{E}$ and a magnetic field $\vec{\boldsymbol{B}}$, let us express $\star\boldsymbol{F}$ by taking the Hodge star of (17.6). Since $\underline{\boldsymbol{u}}_0 \wedge c\underline{\boldsymbol{B}}$ is a 2-form, the property (14.77) yields $\star\star(\underline{\boldsymbol{u}}_0 \wedge c\underline{\boldsymbol{B}}) = -\underline{\boldsymbol{u}}_0 \wedge c\underline{\boldsymbol{B}}$, so that we get

$$\boxed{\star\boldsymbol{F} = -\underline{\boldsymbol{u}}_0 \wedge c\underline{\boldsymbol{B}} + \star(\underline{\boldsymbol{u}}_0 \wedge \boldsymbol{E})}. \tag{17.21}$$

Comparing with (17.6), we observe that $\star\boldsymbol{F}$ is deduced from $\boldsymbol{F}$ by substituting $-c\underline{\boldsymbol{B}}$ for $\boldsymbol{E}$ and $\boldsymbol{E}$ for $c\underline{\boldsymbol{B}}$. Consequently, the components of $\star\boldsymbol{F}$ in $\mathscr{O}$'s local frame are easily deduced from (17.12):

$$\star F_{\alpha\beta} = \begin{pmatrix} 0 & cB^1 & cB^2 & cB^3 \\ -cB^1 & 0 & E_3 & -E_2 \\ -cB^2 & -E_3 & 0 & E_1 \\ -cB^3 & E_2 & -E_1 & 0 \end{pmatrix}. \tag{17.22}$$

17.3 Change of Observer

As we have already stressed, for a given electromagnetic field $\boldsymbol{F}$, the electric field $\boldsymbol{E}$ and the magnetic field $\vec{\boldsymbol{B}}$ depend on the considered observer. Let us then examine the way $\boldsymbol{E}$ and $\vec{\boldsymbol{B}}$ transform when the observer is changed.

17.3.1 Transformation Law of the Electric and Magnetic Fields

Let us consider two observers, $\mathscr{O}$ and $\mathscr{O}'$, of respective 4-velocities $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{u}}'$. We shall restrict ourselves to the case where the worldlines of $\mathscr{O}$ and $\mathscr{O}'$ intersect at some event O. We shall then use the same notations as in Sect. 5.2: $\vec{\boldsymbol{U}}$ is the velocity of $\mathscr{O}'$ relative to $\mathscr{O}$ and $\vec{\boldsymbol{U}}'$ that of $\mathscr{O}$ relative to $\mathscr{O}'$. We shall however denote Γ, and not Γ_0, the Lorentz factor between $\mathscr{O}$ and $\mathscr{O}'$: $\Gamma = -\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{u}}'$. Let us also introduce the unit vectors $\vec{\boldsymbol{e}} \in E_{\boldsymbol{u}}$ and $\vec{\boldsymbol{e}}\,' \in E_{\boldsymbol{u}'}$ in the direction of $\vec{\boldsymbol{U}}$ and $\vec{\boldsymbol{U}}'$, respectively, (cf. Fig. 5.2). We may then decompose the electric and magnetic fields relative to $\mathscr{O}$ into a part parallel to $\vec{\boldsymbol{U}}$ and a part orthogonal to $\vec{\boldsymbol{U}}$, according to (cf. Fig. 17.2)

$$\boldsymbol{E} = E_{\parallel}\,\underline{\boldsymbol{e}} + \boldsymbol{E}_{\perp} \qquad \text{and} \qquad \vec{\boldsymbol{B}} = B_{\parallel}\,\vec{\boldsymbol{e}} + \vec{\boldsymbol{B}}_{\perp}, \tag{17.23}$$

with $\langle \boldsymbol{E}_{\perp}, \vec{\boldsymbol{e}}\rangle = 0$ and $\vec{\boldsymbol{B}}_{\perp} \cdot \vec{\boldsymbol{e}} = 0$. Expressing the unit vector $\vec{\boldsymbol{e}}$ in terms of $\vec{\boldsymbol{e}}\,'$ and $\vec{\boldsymbol{u}}'$ according to (5.13), we may write

$$\boldsymbol{E} = \Gamma E_{\parallel}\left(\underline{\boldsymbol{e}}' - \frac{U}{c}\underline{\boldsymbol{u}}'\right) + \boldsymbol{E}_{\perp} \quad \text{and} \quad \vec{\boldsymbol{B}} = \Gamma B_{\parallel}\left(\vec{\boldsymbol{e}}\,' - \frac{U}{c}\vec{\boldsymbol{u}}'\right) + \vec{\boldsymbol{B}}_{\perp}. \tag{17.24}$$

Note that the vectors $\vec{\boldsymbol{E}}_{\perp}$ and $\vec{\boldsymbol{B}}_{\perp}$ are orthogonal to $\vec{\boldsymbol{u}}$, $\vec{\boldsymbol{e}}$, $\vec{\boldsymbol{u}}'$ and $\vec{\boldsymbol{e}}\,'$ (cf. Fig. 17.2). We can also express the 4-velocity $\vec{\boldsymbol{u}}$ in terms of $\vec{\boldsymbol{u}}'$ and $\vec{\boldsymbol{e}}\,'$, via (5.2) and (5.10):

$$\vec{\boldsymbol{u}} = \Gamma\left(\vec{\boldsymbol{u}}' - \frac{U}{c}\vec{\boldsymbol{e}}\,'\right). \tag{17.25}$$

Let us then consider the decomposition (17.5) of the electromagnetic field tensor with respect to $\mathscr{O}$: $\boldsymbol{F} = \underline{\boldsymbol{u}} \wedge \boldsymbol{E} + \boldsymbol{\epsilon}(\vec{\boldsymbol{u}}, c\vec{\boldsymbol{B}}, .,.)$. Substituting expressions (17.24) and (17.25) for $\boldsymbol{E}$, $\vec{\boldsymbol{B}}$ and $\vec{\boldsymbol{u}}$, and expanding, taking into account the identities $\underline{\boldsymbol{u}}' \wedge \underline{\boldsymbol{u}}' = 0$, $\underline{\boldsymbol{e}}' \wedge \underline{\boldsymbol{e}}' = 0$ and $\Gamma^2(1 - U^2/c^2) = 1$, we get

$$\begin{aligned}\boldsymbol{F} = {} & E_{\parallel}\,\underline{\boldsymbol{u}}' \wedge \underline{\boldsymbol{e}}' + \Gamma\,\underline{\boldsymbol{u}}' \wedge \boldsymbol{E}_{\perp} - \frac{\Gamma U}{c}\underline{\boldsymbol{e}}' \wedge \boldsymbol{E}_{\perp} + cB_{\parallel}\,\boldsymbol{\epsilon}(\vec{\boldsymbol{u}}', \vec{\boldsymbol{e}}', .,.) \\ & + c\Gamma\,\boldsymbol{\epsilon}(\vec{\boldsymbol{u}}', \vec{\boldsymbol{B}}_{\perp}, .,.) - \Gamma U\,\boldsymbol{\epsilon}(\vec{\boldsymbol{e}}\,', \vec{\boldsymbol{B}}_{\perp}, .,.).\end{aligned} \tag{17.26}$$

Let us decompose the 2-form $\underline{\boldsymbol{e}}' \wedge \boldsymbol{E}_{\perp}$ with respect to the unit vector $\vec{\boldsymbol{u}}'$ according to (14.80). Since both $\vec{\boldsymbol{e}}\,'$ and $\boldsymbol{E}_{\perp}$ are orthogonal to $\vec{\boldsymbol{u}}'$, the "electric" part of this

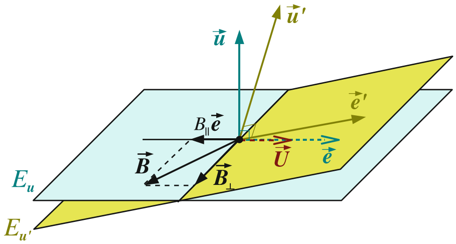


Fig. 17.2 Decomposition of the magnetic field $\vec{\boldsymbol{B}}$ relative to observer $\mathscr{O}$ (4-velocity $\vec{\boldsymbol{u}}$) into a part $B_\parallel \vec{\boldsymbol{e}}$ collinear to the velocity $\vec{U}$ of $\mathscr{O}'$ relative to $\mathscr{O}$ and a part $\vec{\boldsymbol{B}}_\perp$ orthogonal to $\vec{U}$

decomposition vanishes: $\boldsymbol{q} = 0$. There remains thus only the "magnetic" part $\vec{\boldsymbol{b}}$, which is evaluated via (14.81): $\underline{\boldsymbol{b}} = \star(\underline{\boldsymbol{e}}' \wedge \boldsymbol{E}_\perp)(\vec{\boldsymbol{u}}', .)$. Using the property (14.79), we get $\underline{\boldsymbol{b}} = \boldsymbol{\epsilon}(\vec{\boldsymbol{e}}\,', \vec{\boldsymbol{E}}_\perp, \vec{\boldsymbol{u}}', .) = \boldsymbol{\epsilon}(\vec{\boldsymbol{u}}', \vec{\boldsymbol{e}}\,', \vec{\boldsymbol{E}}_\perp, .) = \boldsymbol{\epsilon}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{e}}, \vec{\boldsymbol{E}}_\perp, .)$, the last equality resulting from expressions (5.1) and (5.12) for $\vec{\boldsymbol{u}}'$ and $\vec{\boldsymbol{e}}\,'$ and the identity $\Gamma^2(1 - U^2/c^2) = 1$. We have thus $\vec{\boldsymbol{b}} = \vec{\boldsymbol{e}} \times_{\boldsymbol{u}} \vec{\boldsymbol{E}}_\perp$; hence,

$$\underline{\boldsymbol{e}}' \wedge \boldsymbol{E}_\perp = \boldsymbol{\epsilon}(\vec{\boldsymbol{u}}', \vec{\boldsymbol{e}} \times_{\boldsymbol{u}} \vec{\boldsymbol{E}}_\perp, ., .). \tag{17.27}$$

Similarly, let us decompose with respect to $\vec{\boldsymbol{u}}'$ the 2-form $\boldsymbol{\epsilon}(\vec{\boldsymbol{e}}\,', \vec{\boldsymbol{B}}_\perp, ., .)$ that appears in (17.26). This time, the "magnetic" part vanishes, since for any pair $(\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}})$ of vectors orthogonal to $\vec{\boldsymbol{u}}'$, $\boldsymbol{\epsilon}(\vec{\boldsymbol{e}}\,', \vec{\boldsymbol{B}}_\perp, \vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}) = 0$. Indeed the four-vectors $\vec{\boldsymbol{e}}\,'$, $\vec{\boldsymbol{B}}_\perp$, $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{w}}$ belong to the three-dimensional vector space $E_{\boldsymbol{u}'}$ and therefore cannot be linearly independent. The "electric" part is obtained by means of (14.82): $\boldsymbol{q} = \boldsymbol{\epsilon}(\vec{\boldsymbol{e}}\,', \vec{\boldsymbol{B}}_\perp, ., \vec{\boldsymbol{u}}') = \boldsymbol{\epsilon}(\vec{\boldsymbol{u}}', \vec{\boldsymbol{B}}_\perp, \vec{\boldsymbol{e}}\,', .) = \boldsymbol{\epsilon}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{B}}_\perp, \vec{\boldsymbol{e}}, .)$ (cf. the above computation of $\underline{\boldsymbol{b}}$). Hence,

$$\boldsymbol{\epsilon}(\vec{\boldsymbol{e}}\,', \vec{\boldsymbol{B}}_\perp, ., .) = \underline{\boldsymbol{u}}' \wedge \boldsymbol{\epsilon}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{B}}_\perp, \vec{\boldsymbol{e}}, .). \tag{17.28}$$

Substituting (17.27) and (17.28) in (17.26), we get

$$\begin{aligned} \boldsymbol{F} = & \, \underline{\boldsymbol{u}}' \wedge \left[E_\parallel \, \underline{\boldsymbol{e}}' + \Gamma \left(\boldsymbol{E}_\perp - U \boldsymbol{\epsilon}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{B}}_\perp, \vec{\boldsymbol{e}}, .) \right) \right] \\ & + \boldsymbol{\epsilon} \left(\vec{\boldsymbol{u}}', c B_\parallel \, \vec{\boldsymbol{e}}\,' + c\Gamma \left(\vec{\boldsymbol{B}}_\perp - \frac{U}{c^2} \vec{\boldsymbol{e}} \times_{\boldsymbol{u}} \vec{\boldsymbol{E}}_\perp \right), ., . \right). \end{aligned} \tag{17.29}$$

Now, by definition of the electric field $\boldsymbol{E}'$ and the magnetic field $\vec{\boldsymbol{B}}'$ relative to $\mathscr{O}'$,

$$\boldsymbol{F} = \underline{\boldsymbol{u}}' \wedge \boldsymbol{E}' + \boldsymbol{\epsilon} \left(\vec{\boldsymbol{u}}', c\, \vec{\boldsymbol{B}}', ., . \right). \tag{17.30}$$

Since the 1-form that appears in the exterior product with $\underline{\boldsymbol{u}}'$ in (17.29) vanishes on $\vec{\boldsymbol{u}}'$ and the vector second argument of $\boldsymbol{\epsilon}$ in (17.29) is orthogonal to $\vec{\boldsymbol{u}}'$, the comparison of (17.29) with (17.30) shows that these 1-form and vector are, respectively, equal to $\boldsymbol{E}'$ and $\vec{\boldsymbol{B}}'$. Hence, the transformation law of the fields under a change of observer:

$$\boxed{\boldsymbol{E}' = E_{\parallel}\, \underline{e}' + \Gamma\left(\boldsymbol{E}_{\perp} + \boldsymbol{\epsilon}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{U}}, \vec{\boldsymbol{B}}_{\perp}, .)\right)} \tag{17.31a}$$

$$\boxed{\vec{\boldsymbol{B}}' = B_{\parallel}\, \vec{\boldsymbol{e}}\,' + \Gamma\left(\vec{\boldsymbol{B}}_{\perp} - \frac{1}{c^2}\, \vec{\boldsymbol{U}} \times_{\boldsymbol{u}} \vec{\boldsymbol{E}}_{\perp}\right)} \tag{17.31b}$$

(let us recall that $\vec{\boldsymbol{U}} = U\vec{\boldsymbol{e}}$). We observe that the obtained equations provide the decomposition of $\boldsymbol{E}'$ and $\vec{\boldsymbol{B}}'$ into parts parallel and orthogonal to $\vec{\boldsymbol{e}}'$, so that the transformation law of the fields can be recast as

$$\boxed{E'_{\parallel} = E_{\parallel}} \qquad \boxed{\boldsymbol{E}'_{\perp} = \Gamma\left(\boldsymbol{E}_{\perp} + \boldsymbol{\epsilon}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{U}}, \vec{\boldsymbol{B}}_{\perp}, .)\right)} \tag{17.32a}$$

$$\boxed{B'_{\parallel} = B_{\parallel}} \qquad \boxed{\vec{\boldsymbol{B}}'_{\perp} = \Gamma\left(\vec{\boldsymbol{B}}_{\perp} - \frac{1}{c^2}\, \vec{\boldsymbol{U}} \times_{\boldsymbol{u}} \vec{\boldsymbol{E}}_{\perp}\right)}. \tag{17.32b}$$

Remark 17.5. The transformation law (17.32) can be obtained by an alternative method, based on the transformation law of the components $F_{\alpha\beta}$ of the tensor $\boldsymbol{F}$ from the local frame of $\mathscr{O}$, $(\vec{\boldsymbol{e}}_{\alpha})$, to the local frame of $\mathscr{O}'$, $(\vec{\boldsymbol{e}}\,'_{\alpha})$. More precisely, let us choose the local frames so that the velocities $\vec{\boldsymbol{U}}$ and $\vec{\boldsymbol{U}}'$ are collinear to $\vec{\boldsymbol{e}}_1$ and $\vec{\boldsymbol{e}}\,'_1$, respectively. We have then $\vec{\boldsymbol{e}}_1 = \vec{\boldsymbol{e}}$, $\vec{\boldsymbol{e}}\,'_1 = \vec{\boldsymbol{e}}\,'$, and the transition from $(\vec{\boldsymbol{e}}_{\alpha})$ to $(\vec{\boldsymbol{e}}\,'_{\alpha})$ is performed by a boost of Lorentz factor Γ: $\vec{\boldsymbol{e}}\,'_{\alpha} = \boldsymbol{\Lambda}(\vec{\boldsymbol{e}}_{\alpha})$, with the matrix of $\boldsymbol{\Lambda}$ given by (6.48) (with V replaced by U). The components of $\boldsymbol{F}$ in the basis $(\vec{\boldsymbol{e}}\,'_{\alpha})$ are then obtained via the law (14.23):

$$F'_{\alpha\beta} = \Lambda^{\mu}{}_{\alpha} F_{\mu\nu} \Lambda^{\nu}{}_{\beta}. \tag{17.33}$$

Using the form (17.12) of $F_{\mu\nu}$ and the form (6.48) of $\Lambda^{\mu}{}_{\alpha}$ and writing the components of $F'_{\alpha\beta}$ as (17.12) with E_i and B^i substituted by E'_i and B'^i, we obtain

$$\begin{aligned} &E'_1 = E_1, \quad E'_2 = \Gamma(E_2 - UB^3), \qquad E'_3 = \Gamma(E_3 + UB^2)\\ &B'^1 = B^1, \quad B'^2 = \Gamma(B^2 + UE_3/c^2), \quad B'^3 = \Gamma(B^3 - UE_2/c^2). \end{aligned} \tag{17.34}$$

Since with our choice of local frame, $E_{\parallel} = E_1$, $\boldsymbol{E}_{\perp} = E_2\boldsymbol{e}^2 + E_3\boldsymbol{e}^3$, $B_{\parallel} = B^1$, $\vec{\boldsymbol{B}}_{\perp} = B^2\vec{\boldsymbol{e}}_2 + B^3\vec{\boldsymbol{e}}_3$, $E'_{\parallel} = E'_1$, $\boldsymbol{E}'_{\perp} = E'_2\boldsymbol{e}^2 + E'_3\boldsymbol{e}^3$, $B'_{\parallel} = B'^1$ and $\vec{\boldsymbol{B}}'_{\perp} = B'^2\vec{\boldsymbol{e}}_2 + B'^3\vec{\boldsymbol{e}}_3$, and since $\epsilon'_{\alpha\beta\gamma\delta} = [\alpha, \beta, \gamma, \delta]$ (for $(\vec{\boldsymbol{e}}\,'_{\alpha})$ is a right-handed orthonormal basis), we do recover (17.32).

Remark 17.6. As already noticed in Sect. 5.3 when discussing the law of velocity composition, many authors consider that spacelike vectors such as $\vec{\boldsymbol{e}}$, $\vec{\boldsymbol{e}}\,'$, $\vec{\boldsymbol{U}}$, $\vec{\boldsymbol{U}}'$, $\vec{\boldsymbol{E}}$, $\vec{\boldsymbol{E}}'$, $\vec{\boldsymbol{B}}$ and $\vec{\boldsymbol{B}}'$ all belong to the same "abstract" three-dimensional vector space. This is clearly not our point of view here, since (i) $\vec{\boldsymbol{e}}$, $\vec{\boldsymbol{U}}$, $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$ belong to the local rest space of $\mathscr{O}$, $E_{\boldsymbol{u}}$; (ii) $\vec{\boldsymbol{e}}\,'$, $\vec{\boldsymbol{U}}'$, $\vec{\boldsymbol{E}}'$ and $\vec{\boldsymbol{B}}'$ belong to that of $\mathscr{O}'$, $E_{\boldsymbol{u}'}$; and (iii) $E_{\boldsymbol{u}}$ and $E_{\boldsymbol{u}'}$ are two distincts hyperplanes as soon as $\mathscr{O}'$ moves with respect to $\mathscr{O}$ (cf. Fig. 17.2). As underlined in Remark 5.4 p. 141, the "unique vector space" point of view amounts to identifying the vectors $\vec{\boldsymbol{e}}$ and $\vec{\boldsymbol{e}}\,'$ (cf. Fig. 17.2). Equation (17.31) leads then to Eq. (7.106) of Boratav and Kerner (1991), Eq. (6.5) of Rougé (2004), Eq. (10–26) of Simon (2004) or the equation on p. 223 of Pérez (2005), to mention only a few recent books.

The transformation law (17.32) shows that if an electromagnetic field is "purely electric" for observer $\mathscr{O}$, i.e. if $\vec{\boldsymbol{B}} = 0$, then it is not necessarily so for an observer $\mathscr{O}'$ moving with a velocity $\vec{\boldsymbol{U}}$ relative to $\mathscr{O}$ not collinear to $\vec{\boldsymbol{E}}$. Similarly the concept of "purely magnetic" electromagnetic field ($\boldsymbol{E} = 0$) depends upon the considered observer.

At the nonrelativistic limit, $|U| \ll c$, $\Gamma \simeq 1$, $\vec{\boldsymbol{e}}\,' \simeq \vec{\boldsymbol{e}}$ and (17.31) reduces to[4]

$$\vec{\boldsymbol{E}}' = \vec{\boldsymbol{E}} + \vec{\boldsymbol{U}} \times \vec{\boldsymbol{B}} \quad \text{and} \quad \vec{\boldsymbol{B}}' = \vec{\boldsymbol{B}} \qquad \text{(nonrelativistic).} \tag{17.35}$$

This is the so-called classical law of transformation of the electric and magnetic fields.

17.3.2 Electromagnetic Field Invariants

From $\boldsymbol{F}$ and the associated tensors $\boldsymbol{F}^{\sharp}$ and $\star\boldsymbol{F}$ introduced in Sect. 17.2.5, one can define the following scalar fields on $\mathscr{E}$:

$$\boxed{I_1 := \frac{1}{2} F_{\mu\nu} F^{\mu\nu}} \quad \text{and} \quad \boxed{I_2 := \frac{1}{4} \star F_{\mu\nu} F^{\mu\nu}}. \tag{17.36}$$

Given the antisymmetry of $F_{\mu\nu}$, $F^{\mu\nu}$ and $\star F_{\mu\nu}$, the double summations that appear in the above definitions actually contain only 6 terms:

$$I_1 = F_{01}F^{01} + F_{02}F^{02} + F_{03}F^{03} + F_{12}F^{12} + F_{13}F^{13} + F_{23}F^{23},$$
$$I_2 = \frac{1}{2}\left(\star F_{01}F^{01} + \star F_{02}F^{02} + \star F_{03}F^{03} + \star F_{12}F^{12} + \star F_{13}F^{13} + \star F_{23}F^{23}\right).$$

Using the components of $\boldsymbol{F}$, $\boldsymbol{F}^{\sharp}$ and $\star\boldsymbol{F}$ in the local frame of some observer $\mathscr{O}$, as given, respectively, by (17.12), (17.19) and (17.22), we relate I_1 and I_2 to the electric and magnetic fields relative to $\mathscr{O}$:

$$\boxed{I_1 = c^2 \vec{\boldsymbol{B}} \cdot \vec{\boldsymbol{B}} - \vec{\boldsymbol{E}} \cdot \vec{\boldsymbol{E}}} \quad \text{and} \quad \boxed{I_2 = c\,\langle \boldsymbol{E}, \vec{\boldsymbol{B}} \rangle}. \tag{17.37}$$

[4] For the electric field, the metric dual version is presented.

The scalar fields I_1 and I_2 are called ***electromagnetic field invariants***. This name is somewhat historical, reflecting the fact that I_1 and I_2 are combinations of $\boldsymbol{E}$ and $\vec{\boldsymbol{B}}$ that are observer-independent, while taken individually, $\boldsymbol{E}$ and $\vec{\boldsymbol{B}}$ are not. However it is obvious from the definition (17.36) that I_1 and I_2 are independent of any observer.

Remark 17.7. As an exercise, the reader may check, by means of the transformation law (17.32), that expressions (17.37) are invariant under a change of observer.

By means of the formulas on the exterior product and Hodge star given in Sects. 14.4 and 14.5, one can derive the following expressions of the electromagnetic field invariants:

$$I_1 = \star(\boldsymbol{F} \wedge \star\boldsymbol{F}) \qquad \text{and} \qquad I_2 = \frac{1}{2} \star (\boldsymbol{F} \wedge \boldsymbol{F}). \tag{17.38}$$

We shall not give the details here but simply remark that $\boldsymbol{F}$ and $\star\boldsymbol{F}$ being 2-forms, the exterior products $\boldsymbol{F} \wedge \star\boldsymbol{F}$ and $\boldsymbol{F} \wedge \boldsymbol{F}$ are 4-forms. Their Hodge duals $\star(\boldsymbol{F} \wedge \star\boldsymbol{F})$ and $\star(\boldsymbol{F} \wedge \boldsymbol{F})$ are then 0-forms, i.e. scalar fields, so that the writing (17.38) is admissible.

We have underlined above that the notion of "purely electric" or "purely magnetic" field is observer-dependent. On the other side, thanks to the invariant I_1, one may define various kinds of electromagnetic fields, independently of any observer:

- If $I_1 > 0$, the electromagnetic field is said to be ***mostly magnetic***, since for any observer $c\|\vec{\boldsymbol{B}}\|_{\boldsymbol{g}} > \|\vec{\boldsymbol{E}}\|_{\boldsymbol{g}}$.
- If $I_1 < 0$, the electromagnetic field is said to be ***mostly electric***, since for any observer $\|\vec{\boldsymbol{E}}\|_{\boldsymbol{g}} > c\|\vec{\boldsymbol{B}}\|_{\boldsymbol{g}}$.
- If $I_1 = 0$, the amplitude of $\vec{\boldsymbol{E}}$ is equal to that of $c\,\vec{\boldsymbol{B}}$ for all observers.

If $I_2 = 0$, the vectors $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$ are orthogonal for all observers. In particular, if one of the fields $\vec{\boldsymbol{E}}$ or $\vec{\boldsymbol{B}}$ vanishes for some observer, then $I_2 = 0$ and $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$ will be orthogonal for all observers.

If both $I_1 = 0$ and $I_2 = 0$, the electromagnetic field is called ***null***. The vectors $\vec{\boldsymbol{E}}$ and $c\,\vec{\boldsymbol{B}}$ have then the same amplitude and are orthogonal, whatever the observer. We shall see in Chap. 18 that the radiative part of the electromagnetic field generated by an accelerated charge is of this kind.

Remark 17.8. Some authors, like A. Lichnerowicz (1955), rather qualify an electromagnetic field with $I_1 = I_2 = 0$ as *singular*. The term *null* is preferred here for the field $\boldsymbol{F}$ does not exhibit any physical singularity (it does not diverge at any point).

Historical note: *The transformation law of the fields $\boldsymbol{E}$ and $\vec{\boldsymbol{B}}$, written as (17.34), has been obtained by Joseph Larmor (cf. p. 191) in 1900 (Larmor 1900) and Hendrik A. Lorentz (cf. p. 108) in 1904 (Lorentz 1904). In 1905, Henri Poincaré (cf. p. 26) noticed that the combinations $c^2\vec{\boldsymbol{B}}\cdot\vec{\boldsymbol{B}} - \vec{\boldsymbol{E}}\cdot\vec{\boldsymbol{E}}$ and $c\,\langle \boldsymbol{E}, \vec{\boldsymbol{B}}\rangle$ are invariant under a Lorentz transformation (Poincaré 1906).*

17.3.3 Reduction to Parallel Electric and Magnetic Fields

A property that simplifies the study of the electromagnetic field is

> If the electromagnetic field $\boldsymbol{F}$ is not null at some point $O \in \mathscr{E}$, it is always possible to find an observer through O for which the electric and the magnetic fields at O are parallel.

Proof. Let us suppose that for some observer $\mathscr{O}$ through O, $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$ are not parallel. We can then construct the unit vector normal to the plane generated by $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$:

$$\vec{\boldsymbol{e}} := (EB\sin\theta)^{-1}\,\vec{\boldsymbol{E}} \times_{\boldsymbol{u}} \vec{\boldsymbol{B}}, \tag{17.39}$$

where $\vec{\boldsymbol{u}}$ is $\mathscr{O}$'s 4-velocity, $E := \|\vec{\boldsymbol{E}}\|_{\boldsymbol{g}}$, $B := \|\vec{\boldsymbol{B}}\|_{\boldsymbol{g}}$ and $\theta \in]0,\pi[$ is the angle between $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$ in $E_{\boldsymbol{u}}$. Let us then consider a second observer, $\mathscr{O}'$, whose worldline also contains the event O and whose velocity $\vec{\boldsymbol{U}}$ relative to $\mathscr{O}$ is along $\vec{\boldsymbol{e}}$: $\vec{\boldsymbol{U}} = U\vec{\boldsymbol{e}}$. By definition, $\vec{\boldsymbol{e}}\cdot\vec{\boldsymbol{E}} = 0$ and $\vec{\boldsymbol{e}}\cdot\vec{\boldsymbol{B}} = 0$, so that in the orthogonal decomposition (17.23), $E_{\parallel} = 0$, $B_{\parallel} = 0$, $\vec{\boldsymbol{E}}_{\perp} = \vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}_{\perp} = \vec{\boldsymbol{B}}$. The transformation law (17.31) reduces then to $\vec{\boldsymbol{E}}' = \Gamma(\vec{\boldsymbol{E}} + U\vec{\boldsymbol{e}} \times_{\boldsymbol{u}} \vec{\boldsymbol{B}})$ and $\vec{\boldsymbol{B}}' = \Gamma(\vec{\boldsymbol{B}} - c^{-2}U\vec{\boldsymbol{e}} \times_{\boldsymbol{u}} \vec{\boldsymbol{E}})$. Accordingly[5]

$$\vec{\boldsymbol{E}}' \times_{\boldsymbol{u}} \vec{\boldsymbol{B}}' = \Gamma^2\left\{\vec{\boldsymbol{E}} \times_{\boldsymbol{u}} \vec{\boldsymbol{B}} + \frac{U}{c^2}\left[U\,\vec{\boldsymbol{e}}\cdot(\vec{\boldsymbol{E}} \times_{\boldsymbol{u}} \vec{\boldsymbol{B}}) - E^2 - c^2B^2\right]\vec{\boldsymbol{e}}\right\},$$

where the double cross products have been expanded and the orthogonality of $\vec{\boldsymbol{E}}$ (resp. $\vec{\boldsymbol{B}}$) and $\vec{\boldsymbol{e}}$ has been used. Substituting $EB\sin\theta\,\vec{\boldsymbol{e}}$ for $\vec{\boldsymbol{E}} \times_{\boldsymbol{u}} \vec{\boldsymbol{B}}$ [Eq. (17.39)], we get

$$\vec{\boldsymbol{E}}' \times_{\boldsymbol{u}} \vec{\boldsymbol{B}}' = \Gamma^2\left[EB\sin\theta\left(1 + \frac{U^2}{c^2}\right) - \frac{U}{c^2}(E^2 + c^2B^2)\right]\vec{\boldsymbol{e}}.$$

The fields $\vec{\boldsymbol{E}}'$ and $\vec{\boldsymbol{B}}'$ relative to $\mathscr{O}'$ are parallel iff $\vec{\boldsymbol{E}}' \times_{\boldsymbol{u}} \vec{\boldsymbol{B}}' = 0$. From the above equation, this is equivalent to the condition

$$x^2 - \frac{E^2 + c^2B^2}{cEB\sin\theta}\,x + 1 = 0, \tag{17.40}$$

[5] In the present case ($E_{\parallel} = 0$ and $B_{\parallel} = 0$), the vectors $\vec{\boldsymbol{E}}'$ and $\vec{\boldsymbol{B}}'$ belong to $E_{\boldsymbol{u}}$ (actually to the intersection of $E_{\boldsymbol{u}}$ and $E_{\boldsymbol{u}'}$—cf. Fig. 17.2), so that it is legitimate to form the cross product $\vec{\boldsymbol{E}}' \times_{\boldsymbol{u}} \vec{\boldsymbol{B}}'$.

where $x := U/c$. This second-order equation in x has for discriminant

$$\Delta = \left(\frac{E^2 + c^2 B^2}{cEB\sin\theta}\right)^2 - 4 = \frac{I_1^2 + 4I_2^2}{(cEB\sin\theta)^2},$$

where we have let appear the invariants $I_1 = c^2B^2 - E^2$ and $I_2 = cEB\cos\theta$ [cf. (17.37)]. It is clear that $\Delta \geq 0$, so that (17.40) admits real roots. Given the coefficients of (17.40), the sum of these roots is positive and their product is equal to 1. The two roots are thus positive and the inverse of each other. Only the root $x < 1$ is physically acceptable (for $U < c$). It always exist, except if (17.40) admits a double root, which is then $x = 1$. This occurs for $\Delta = 0$, i.e. $I_1 = I_2 = 0$, or in other words, if $\boldsymbol{F}$ is a null electromagnetic field. □

Remark 17.9. Observer $\mathscr{O}'$ is by no means unique, since from the transformation law (17.32), any observer $\mathscr{O}''$ whose velocity relative to $\mathscr{O}'$ is along the common direction of $\vec{\boldsymbol{E}}'$ and $\vec{\boldsymbol{B}}'$ measures fields $\vec{\boldsymbol{E}}''$ and $\vec{\boldsymbol{B}}''$ that are parallel (and of same norms as $\vec{\boldsymbol{E}}'$ and $\vec{\boldsymbol{B}}'$).

Remark 17.10. If the local frame $(\vec{\boldsymbol{e}}'_\alpha)$ of observer $\mathscr{O}'$ is such that the common direction of $\vec{\boldsymbol{E}}'$ and $\vec{\boldsymbol{B}}'$ is $\vec{\boldsymbol{e}}'_3$: $\vec{\boldsymbol{E}}' = E'\,\vec{\boldsymbol{e}}'_3$ and $\vec{\boldsymbol{B}}' = B'\,\vec{\boldsymbol{e}}'_3$, then the matrix of $\boldsymbol{F}$ in this frame takes the following antidiagonal form, obtained by setting $E'_1 = E'_2 = 0$ and $B'^1 = B'^2 = 0$ in (17.12):

$$F'_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & -E' \\ 0 & 0 & cB' & 0 \\ 0 & -cB' & 0 & 0 \\ E' & 0 & 0 & 0 \end{pmatrix}. \tag{17.41}$$

Particular Case $I_2 = 0$

If $I_2 = 0$ ($\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$ are orthogonal), then $\vec{\boldsymbol{E}}'$ and $\vec{\boldsymbol{B}}'$ must be orthogonal, in addition of being parallel. Since $(E_{\boldsymbol{u}'}, \boldsymbol{g})$ is a Euclidean space, we deduce that one of these two vector is necessarily zero. In other words, if $I_1 \neq 0$, the condition $I_2 = 0$ is necessary and sufficient for the existence of an observer with respect to which the electromagnetic field is purely magnetic (case $I_1 > 0$) or purely electric (case $I_1 < 0$).

The condition $I_2 = 0$ implies $\sin\theta = 1$. The root of (17.40) that is lower than 1 is then $x = (E^2 + c^2B^2 - |I_1|)/(2cEB)$. Since $I_1 = c^2B^2 - E^2$, we obtain the explicit value of the amplitude of the velocity of observer $\mathscr{O}'$:

$$U = \frac{E}{B} \quad \text{if} \quad I_1 > 0 \qquad \text{and} \qquad U = c^2\frac{B}{E} \quad \text{if} \quad I_1 < 0. \tag{17.42}$$

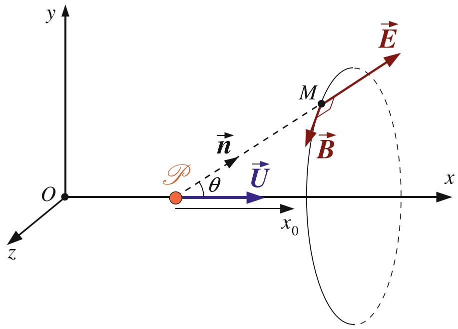


Fig. 17.3 Electric charge in uniform translation at the velocity $\vec{U}$ along the x-axis of some inertial observer

Since $\vec{U} = U\vec{e} = (U/EB)\,\vec{E} \times_u \vec{B}$ [Eq. (17.39) with $\theta = \pi/2$], the velocity vector is

$$\vec{U} = B^{-2}\,\vec{E} \times_u \vec{B} \quad \text{if} \quad I_1 > 0 \quad \text{and} \quad \vec{U} = (c^2/E^2)\,\vec{E} \times_u \vec{B} \quad \text{if} \quad I_1 < 0. \tag{17.43}$$

Substituting these values in the transformation law (17.32) leads to

$$\vec{E}' = 0 \quad \text{and} \quad \vec{B}' = \Gamma^{-1}\vec{B} \qquad (I_1 > 0) \tag{17.44}$$

$$\vec{E}' = \Gamma^{-1}\vec{E} \quad \text{and} \quad \vec{B}' = 0 \qquad (I_1 < 0). \tag{17.45}$$

We therefore get the explicit vanishing of the electric or magnetic field (according to the sign of I_1).

17.3.4 Field Created by a Charge in Translation

An interesting application of the transformation law of the electric and magnetic fields [Eq. (17.32)] is the determination of the electromagnetic field created by a charged particle in uniform rectilinear motion with respect to some inertial observer $\mathscr{O}$.

Let $(x^\alpha) = (ct, x, y, z)$ be the inertial coordinates associated with $\mathscr{O}$. The frame of $\mathscr{O}$ is accordingly denoted by $(\vec{u}, \vec{e}_x, \vec{e}_y, \vec{e}_z)$. Let us consider a particle $\mathscr{P}$ of electric charge q moving along the x-axis with the constant velocity $\vec{U} = U\vec{e}_x$ relative to $\mathscr{O}$ (cf. Fig. 17.3). $\vec{U}$ being constant, we may associate with $\mathscr{P}$ an inertial observer $\mathscr{O}'$, whose frame $(\vec{u}', \vec{e}\,'_x, \vec{e}\,'_y, \vec{e}\,'_z)$ is quasiparallel to that of $\mathscr{O}$: $\vec{e}\,'_y = \vec{e}_y$ and $\vec{e}\,'_z = \vec{e}_z$. Let $(x'^\alpha) = (ct', x', y', z')$ be the inertial coordinates associated with $\mathscr{O}'$. In the rest space of $\mathscr{O}'$, the particle $\mathscr{P}$ is at rest at the coordinate origin $(x', y', z') = (0, 0, 0)$. It generates then a vanishing magnetic field $\vec{B}'$ and an electric field $\vec{E}'$ obeying ***Coulomb's law***:

$$\vec{E}' = \frac{q}{4\pi\varepsilon_0\, r'^3}\,(x'\vec{e}\,'_x + y'\vec{e}\,'_y + z'\vec{e}\,'_z) \qquad \text{and} \qquad \vec{B}' = 0, \tag{17.46}$$

where ε_0 is some constant (*vacuum permittivity*), to be discussed in more details in Chap. 18, and $r' := \sqrt{x'^2 + y'^2 + z'^2}$. We shall provisionally admit Coulomb's law (17.46); it will be established as a consequence of Maxwell equations in Chap. 18 [cf. Eq. (18.113)].

The electric field $\vec{\boldsymbol{E}}$ and magnetic field $\vec{\boldsymbol{B}}$ measured by $\mathscr{O}$ are related to $\vec{\boldsymbol{E}}'$ and $\vec{\boldsymbol{B}}'$ by the law (17.32). Since $\vec{\boldsymbol{B}}' = 0$, we get immediately from (17.32b) that $B_\parallel = 0$ and $\vec{\boldsymbol{B}} = \vec{\boldsymbol{B}}_\perp = c^{-2}\vec{\boldsymbol{U}} \times_u \vec{\boldsymbol{E}}_\perp$. Thanks to the cross product with $\vec{\boldsymbol{U}}$, $\vec{\boldsymbol{E}}_\perp$ can be replaced by $\vec{\boldsymbol{E}}$ in the latter expression, yielding

$$\boxed{\vec{\boldsymbol{B}} = \frac{1}{c^2}\vec{\boldsymbol{U}} \times_u \vec{\boldsymbol{E}}}. \tag{17.47}$$

Remark 17.11. That $\vec{\boldsymbol{B}}$ is orthogonal to $\vec{\boldsymbol{E}}$ was expected since the invariant $I_2 = c\vec{\boldsymbol{E}} \cdot \vec{\boldsymbol{B}}$ is identically zero, because of $\vec{\boldsymbol{B}}' = 0$.

We deduce from the transformation law (17.32a) that

$$E_\parallel = E'_\parallel = \frac{q}{4\pi\varepsilon_0}\frac{x'}{r'^3} \tag{17.48}$$

and $\vec{\boldsymbol{E}}'_\perp = \Gamma(\vec{\boldsymbol{E}}_\perp + \vec{\boldsymbol{U}} \times_u \vec{\boldsymbol{B}})$. Substituting (17.47) for $\vec{\boldsymbol{B}}$, we get

$$\vec{\boldsymbol{E}}'_\perp = \Gamma\left[\vec{\boldsymbol{E}}_\perp + c^{-2}\vec{\boldsymbol{U}} \times_u (\vec{\boldsymbol{U}} \times_u \vec{\boldsymbol{E}}_\perp)\right] = \Gamma\left[\vec{\boldsymbol{E}}_\perp + c^{-2}\big(\underbrace{(\vec{\boldsymbol{U}} \cdot \vec{\boldsymbol{E}}_\perp)}_{0}\vec{\boldsymbol{U}} - U^2\vec{\boldsymbol{E}}_\perp\big)\right],$$

i.e. $\vec{\boldsymbol{E}}'_\perp = \Gamma(1 - U^2/c^2)\vec{\boldsymbol{E}}_\perp = \Gamma^{-1}\vec{\boldsymbol{E}}_\perp$, hence

$$\vec{\boldsymbol{E}}_\perp = \Gamma\vec{\boldsymbol{E}}'_\perp = \frac{\Gamma q}{4\pi\varepsilon_0\, r'^3}\,(y'\vec{\boldsymbol{e}}_y + z'\vec{\boldsymbol{e}}_z). \tag{17.49}$$

In the above expression, we have used the fact that $\vec{\boldsymbol{E}}'_\perp$ is given by the part along $\vec{\boldsymbol{e}}'_y$ and $\vec{\boldsymbol{e}}'_z$ of (17.46), with furthermore $\vec{\boldsymbol{e}}'_y = \vec{\boldsymbol{e}}_y$ and $\vec{\boldsymbol{e}}'_z = \vec{\boldsymbol{e}}_z$. Combining (17.48) and (17.49), we get

$$\vec{\boldsymbol{E}} = \frac{q}{4\pi\varepsilon_0\, r'^3}\left[x'\vec{\boldsymbol{e}}_x + \Gamma(y'\vec{\boldsymbol{e}}_y + z'\vec{\boldsymbol{e}}_z)\right]. \tag{17.50}$$

The relation between the coordinate systems (ct, x, y, z) and (ct', x', y', z') being given by the Poincaré transformation (8.14) (with $V = U$), we obtain the expression of the electric field measured by $\mathscr{O}$ in terms of the inertial coordinates associated with $\mathscr{O}$:

$$\boxed{\vec{\boldsymbol{E}} = \frac{\Gamma q}{4\pi\varepsilon_0\,[\Gamma^2(x - Ut)^2 + y^2 + z^2]^{3/2}}\left[(x - Ut)\vec{\boldsymbol{e}}_x + y\vec{\boldsymbol{e}}_y + z\vec{\boldsymbol{e}}_z\right]}. \tag{17.51}$$

Let us reexpress this result in terms of the spatial coordinates centred on the charged particle $\mathscr{P}$: $x_0 := x - Ut$, $R := \sqrt{x_0^2 + y^2 + z^2}$. The quantity involved in the denominator of (17.51) can be recast as

$$\Gamma^2 x_0^2 + y^2 + z^2 = \Gamma^2 R^2 \left(1 - \frac{U^2}{c^2} \sin^2\theta\right),$$

where θ is the angle between the velocity $\vec{U}$ and the radius vector centred on $\mathscr{P}$ (cf. Fig. 17.3): $y^2 + z^2 = R^2 \sin^2\theta$. Using the unit vector connecting the charge $\mathscr{P}$ to the generic point M (cf. Fig. 17.3),

$$\vec{n} := \frac{x_0}{R}\,\vec{e}_x + \frac{y}{R}\,\vec{e}_y + \frac{z}{R}\,\vec{e}_z, \tag{17.52}$$

the result (17.51) becomes

$$\boxed{\vec{E} = \frac{q}{4\pi\varepsilon_0\,\Gamma^2 R^2 \left[1 - (U/c)^2 \sin^2\theta\right]^{3/2}}\,\vec{n}}. \tag{17.53}$$

The magnetic field is then obtained via (17.47):

$$\boxed{\vec{B} = \frac{\mu_0}{4\pi}\,\frac{qU}{\Gamma^2 R^2 \left[1 - (U/c)^2 \sin^2\theta\right]^{3/2}}\,\vec{e}_x \times_u \vec{n}}, \tag{17.54}$$

where we have introduced the *vacuum permeability*, $\mu_0 = 1/(\varepsilon_0 c^2)$, which will be discussed in more details in Chap. 18. In the above formulas, R, θ and $\vec{n}$ are functions of the coordinates (ct, x, y, z) of the point M where $\vec{E}$ and $\vec{B}$ are evaluated: $R = \sqrt{(x - Ut)^2 + y^2 + z^2}$, $\sin^2\theta = (y^2 + z^2)/R^2$ and $\vec{n}$ is given by (17.52).

It is clear on (17.53) that the electric field at a point M stands in the radial direction with respect to the position of the charge at the considered instant t (vector $\vec{n}$), as for the Coulombian field (17.46). The difference with the latter lies in the amplitude of $\vec{E}$, which depends upon the direction θ with respect to the charge's velocity. This is illustrated in Fig. 17.4, where the electric field seems to be subject to the "FitzGerald–Lorentz contraction" in the direction of the charge's motion. Actually, with respect to a Coulombian field, $\vec{E}$ is smaller by a factor Γ^2 in the direction of motion ($\sin\theta = 0$) and larger by a factor Γ in the perpendicular direction ($\sin\theta = 1$).

Remark 17.12. It was not obvious from the transformation law (17.32a) of the electric field that the latter stays in the radial direction centred onto the electric charge. This results actually from the fact that the transverse and parallel parts of $\vec{E}$ let appear the same factor Γ: the transverse part via the transformation law of the electric field and the parallel part via the Lorentz transformation from x' to x, which allowed us to factorize by Γ to move from (17.50) to (17.51).

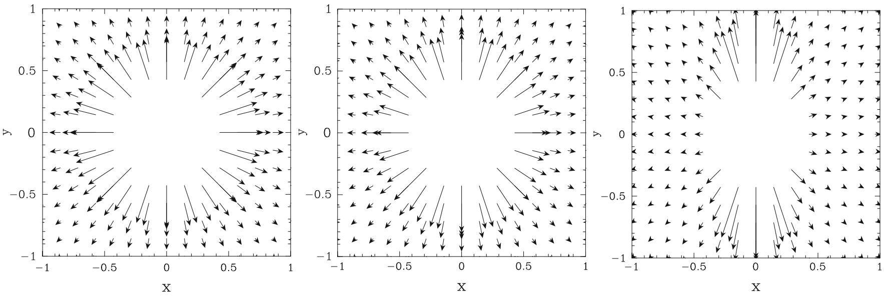


Fig. 17.4 Electric field $\vec{E}$ created by a charged particle in uniform translation with respect to some inertial observer, at the velocity $\vec{U} = U\vec{e}_x$. The three panels correspond to different values of U; from the left to the right: $U = 0$ (Coulombian field), $U = 0.5\,c$ and $U = 0.9\,c$. On each panel, the charged particle is located at $(x, y) = (0, 0)$, and the field $\vec{E}$ is not represented in the central area

Because of the cross product in (17.54), the magnetic field $\vec{B}$ is tangent to the circle through M and whose axis is the x-axis (cf. Fig. 17.3).

Remark 17.13. In the nonrelativistic limit ($\Gamma \simeq 1$, $U/c \simeq 0$), Eq. (17.54) reduces to

$$\vec{B} \simeq \frac{\mu_0}{4\pi}\frac{qU}{R^2}\,\vec{e}_x \times_u \vec{n} \qquad \text{(nonrelativistic).} \tag{17.55}$$

We recognize the ***Biot–Savart law***.

17.4 Particle in an Electromagnetic Field

Let us investigate now the motion of a particle $\mathscr{P}$ of mass $m > 0$ and electric charge q in a given electromagnetic field $\boldsymbol{F}$. The equation of motion is obtained by specifying the 4-force in (9.104) as the Lorentz 4-force (17.1). There comes then $\mathrm{d}\boldsymbol{p}/\mathrm{d}\tau = q\,\boldsymbol{F}(., \vec{u})$, where $\boldsymbol{p}$ is the particle's 4-momentum, τ its proper time and $\vec{u}$ its 4-velocity. The 4-momentum being related to the 4-velocity and the mass by $\boldsymbol{p} = mc\underline{\boldsymbol{u}}$ [Eq. (9.3)], this relation can be rewritten as

$$\frac{\mathrm{d}\underline{\boldsymbol{u}}}{\mathrm{d}\tau} = \frac{q}{mc}\,\boldsymbol{F}(., \vec{u}). \tag{17.56}$$

The simplest case, and of a great practical value, is that of a ***uniform field***, i.e. $\boldsymbol{F}$ constant on the whole spacetime $\mathscr{E}$.[6] We shall limit ourselves to this case in what follows.

[6]The qualifier *uniform* is thus employed with a spacetime perspective; from a nonrelativistic (i.e. three-dimensional) point of view, this field would be qualified as *uniform and static*.

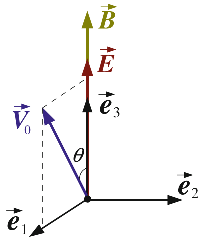


Fig. 17.5 Vectors $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ of observer $\mathscr{O}$'s frame adapted to the fields $\vec{E}$ and $\vec{B}$, as well as to the initial particle's velocity $\vec{V}_0$

17.4.1 Uniform Electromagnetic Field: Non-Null Case

Let us consider a uniform electromagnetic field $\boldsymbol{F}$. Let $\mathscr{O}$ be some inertial observer. Its 4-velocity $\vec{u}_0$ is constant over $\mathscr{E}$, as for $\boldsymbol{F}$; accordingly, the electric and magnetic fields measured by $\mathscr{O}$ are also constant over $\mathscr{E}$ [Eqs. (17.7)–(17.8)]. Let us assume that $\boldsymbol{F}$ is not null ($I_1 \neq 0$ or $I_2 \neq 0$); the null case will be treated in Sect. 17.4.2.2. As we have seen in Sect. 17.3.3, it is always possible to find an observer for which $\vec{E}$ and $\vec{B}$ parallel. We shall consider the case where $\mathscr{O}$ is such an observer, the general case being deduced from that one by a Poincaré transformation.[7] Note also that the cases $\vec{E} = 0$ and $\vec{B} = 0$ are particular cases of $\vec{E}$ and $\vec{B}$ parallel.

Let us choose $\mathscr{O}$'s frame $(\vec{e}_\alpha)$ so that the common direction of $\vec{E}$ and $\vec{B}$ is along $\vec{e}_3$:

$$\vec{E} = E\,\vec{e}_3 \quad \text{and} \quad \vec{B} = B\,\vec{e}_3. \tag{17.57}$$

In addition, let us choose $\vec{e}_2$ and $\vec{e}_3$ so that particle $\mathscr{P}$'s velocity relative to $\mathscr{O}$ at the instant $\tau = 0$ is in the plane $\mathrm{Span}(\vec{e}_1, \vec{e}_3)$ (cf. Fig. 17.5):

$$\vec{V}_0 = V_0 \sin\theta\,\vec{e}_1 + V_0 \cos\theta\,\vec{e}_3. \tag{17.58}$$

The matrix of $\boldsymbol{F}$ in this frame takes the antidiagonal form (17.41). The components of Eq. (17.56) are then

$$\begin{cases} \dfrac{\mathrm{d}u^0}{\mathrm{d}\tau} = \dfrac{qE}{mc} u^3 \\[2ex] \dfrac{\mathrm{d}u^3}{\mathrm{d}\tau} = \dfrac{qE}{mc} u^0 \end{cases} \tag{17.59a}$$

$$\begin{cases} \dfrac{\mathrm{d}u^1}{\mathrm{d}\tau} = \dfrac{qB}{m} u^2 \\[2ex] \dfrac{\mathrm{d}u^2}{\mathrm{d}\tau} = -\dfrac{qB}{m} u^1 \end{cases}, \tag{17.59b}$$

[7] We shall see it on a concrete example in Sect. 17.4.2.1.

where we have used the relations $u_0 = -u^0$, $u_1 = u^1$, $u_2 = u^2$ and $u_3 = u^3$ between the components u_α of $\underline{\boldsymbol{u}}$ and the components u^α of $\vec{\boldsymbol{u}}$ in the orthonormal basis $(\vec{\boldsymbol{e}}_\alpha)$. The two subsystems (17.59a) and (17.59b) are decoupled. The general solution of each of them is

$$\begin{cases} u^0(\tau) = k_1 \mathrm{e}^{qE\tau/mc} + k_2 \mathrm{e}^{-qE\tau/mc} \\ u^3(\tau) = k_1 \mathrm{e}^{qE\tau/mc} - k_2 \mathrm{e}^{-qE\tau/mc} \end{cases} \tag{17.60a}$$

$$\begin{cases} u^1(\tau) = k_3 \mathrm{e}^{\mathrm{i}qB\tau/m} + k_4 \mathrm{e}^{-\mathrm{i}qB\tau/m} \\ u^2(\tau) = i k_3 \mathrm{e}^{\mathrm{i}qB\tau/m} - i k_4 \mathrm{e}^{-\mathrm{i}qB\tau/m}, \end{cases} \tag{17.60b}$$

where k_1, k_2, k_3 and k_4 are four constant determined by the initial conditions. By virtue of (17.58), these conditions are

$$u^\alpha(0) = \left(\Gamma_0,\ \Gamma_0 \frac{V_0}{c} \sin\theta,\ 0,\ \Gamma_0 \frac{V_0}{c} \cos\theta \right), \tag{17.61}$$

with $\Gamma_0 := (1 - V_0^2/c^2)^{-1/2}$. We obtain $k_1 = \Gamma_0(1 + V_0/c\ \cos\theta)/2$, $k_2 = \Gamma_0(1 - V_0/c\ \cos\theta)/2$ and $k_3 = k_4 = \Gamma_0 V_0 \sin\theta/(2c)$. Then, we can rearrange the exponentials to let appear cosines and sines [hyperbolic ones for (17.60a)], yielding

$$u^0(\tau) = \Gamma_0 \left[\cosh\left(\frac{qE}{mc}\tau \right) + \frac{V_0}{c} \cos\theta \sinh\left(\frac{qE}{mc}\tau \right) \right] \tag{17.62a}$$

$$u^3(\tau) = \Gamma_0 \left[\sinh\left(\frac{qE}{mc}\tau \right) + \frac{V_0}{c} \cos\theta \cosh\left(\frac{qE}{mc}\tau \right) \right] \tag{17.62b}$$

$$u^1(\tau) = \Gamma_0 \frac{V_0}{c} \sin\theta \cos\left(\frac{qB}{m}\tau \right) \tag{17.62c}$$

$$u^2(\tau) = -\Gamma_0 \frac{V_0}{c} \sin\theta \sin\left(\frac{qB}{m}\tau \right). \tag{17.62d}$$

Two cases have to be distinguished.

17.4.1.1 Purely Magnetic Case ($\boldsymbol{E = 0}$)

If $E = 0$, (17.62a) and (17.62b) reduce to

$$u^0(\tau) = \Gamma_0 \quad \text{and} \quad u^3(\tau) = \Gamma_0 \frac{V_0}{c} \cos\theta, \tag{17.63}$$

i.e. u^0 and u^3 keep their initial values. Denoting by (ct, x, y, z) the inertial coordinates associated with observer $\mathscr{O}$, we have $u^0 = \mathrm{d}t/\mathrm{d}\tau$, $u^1 = c^{-1}\mathrm{d}x/\mathrm{d}\tau$,

$u^2 = c^{-1}\mathrm{d}y/\mathrm{d}\tau$ and $u^3 = c^{-1}\mathrm{d}z/\mathrm{d}\tau$. Starting from the initial conditions $(ct, x, y, z) = (0, 0, 0, 0)$ at $\tau = 0$, the integration of Eqs. (17.63), (17.62) and (17.62d) leads then to $t = \Gamma_0 \tau$ and

$$\boxed{\begin{cases} x = R \sin\left(\Gamma_0^{-1}\omega_B t\right) \\ y = R\left[\cos\left(\Gamma_0^{-1}\omega_B t\right) - 1\right], \\ z = V_0\, t\, \cos\theta \end{cases}} \qquad (17.64)$$

where we have introduced

$$\boxed{\omega_B := \frac{qB}{m}} \quad \text{and} \quad \boxed{R := \frac{\Gamma_0 V_0}{\omega_B} \sin\theta}. \qquad (17.65)$$

ω_B is called[8] ***cyclotron frequency***. It is a quantity that depends on the intensity B of the magnetic field and on the ratio of the electric charge by the particle's mass. Its values for an electron ($q = -e$ [Eq. (17.3)] and $m = 9.10938 \times 10^{-31}$ kg) and a proton ($q = e$ and $m = 1.67262 \times 10^{-27}$ kg) are

$$\omega_B^{\text{electron}} = -1.75882 \times 10^{11} \left(\frac{B}{1\,\mathrm{T}}\right) \text{ rad s}^{-1}, \qquad (17.66)$$

$$\omega_B^{\text{proton}} = 9.57883 \times 10^{7} \left(\frac{B}{1\,\mathrm{T}}\right) \text{ rad s}^{-1}. \qquad (17.67)$$

Remark 17.14. By convention, ω_B and R are algebraic quantities: $\omega_B < 0$ and $R < 0$ if the particle's charge is negative.

The frequency involved in (17.64) is actually

$$\boxed{\omega := \omega_B / \Gamma_0}. \qquad (17.68)$$

It is called ***synchrotron frequency*** or ***gyration frequency***. Contrary to ω_B, it depends on the velocity of the particle relative to $\mathscr{O}$, except at the nonrelativistic limit, since then $\Gamma_0 \simeq 1$.

R is called ***Larmor radius***, or ***gyration radius***. Its numerical value is

$$|R^{\text{electron}}| = 1.7045 \times 10^{-3} \ \sin\theta\ \Gamma_0 \left(\frac{V_0}{c}\right)\left(\frac{1\,\mathrm{T}}{B}\right) \text{ m}, \qquad (17.69)$$

$$R^{\text{proton}} = 3.12974\ \sin\theta\ \Gamma_0 \left(\frac{V_0}{c}\right)\left(\frac{1\,\mathrm{T}}{B}\right) \text{ m}. \qquad (17.70)$$

[8] Strictly speaking, one should say *cyclotron pulsation*, instead of *cyclotron frequency*.

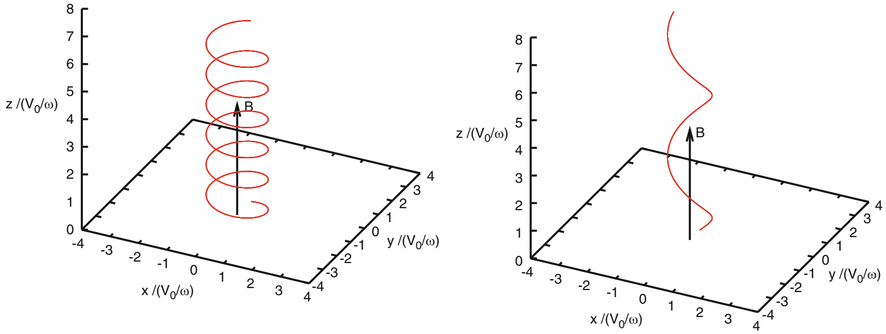


Fig. 17.6 Trajectory of a charged particle in a uniform magnetic field with respect to some inertial observer $\mathscr{O}$ ($\vec{\boldsymbol{B}} \neq 0$, $\vec{\boldsymbol{E}} = 0$). $\vec{\boldsymbol{V}}_0$ is the particle's velocity relative to $\mathscr{O}$ at $(x, y, z) = (0, 0, 0)$, and $\omega := \omega_B/\Gamma_0$ is the gyration frequency. The left panel corresponds to the angle $\theta = 80°$ between $\vec{\boldsymbol{V}}_0$ and $\vec{\boldsymbol{B}}$ and the right panel to $\theta = 45°$

The worldline obeying (17.64) is a *helix* of axis z (aligned with $\vec{\boldsymbol{B}}$) and of radius $|R|$ (cf. Fig. 17.6). The sign of q, which determines that of R, gives the handedness of the helix. The angle formed by the helix with the xy-plane is constant and equal to $\pi/2 - \theta$. In the particular case $\theta = \pi/2$, i.e. $\vec{\boldsymbol{V}}_0$ orthogonal to $\vec{\boldsymbol{B}}$, the trajectory is reduced to a *circle* in the plane $z = 0$. The sign of q gives the orientation of the motion on the circle.

Since $\mathscr{O}$ is inertial, $\mathscr{P}$'s velocity relative to $\mathscr{O}$ is $\vec{\boldsymbol{V}} = (\mathrm{d}x^i/\mathrm{d}t)\,\vec{\boldsymbol{e}}_i$, i.e.

$$\vec{\boldsymbol{V}} = V_0 \sin\theta \cos(\Gamma_0^{-1}\omega_B t)\,\vec{\boldsymbol{e}}_1 - V_0 \sin\theta \sin(\Gamma_0^{-1}\omega_B t)\,\vec{\boldsymbol{e}}_2 + V_0 \cos\theta\,\vec{\boldsymbol{e}}_3. \tag{17.71}$$

Note that the norm of $\vec{\boldsymbol{V}}$ is constant: $\|\vec{\boldsymbol{V}}\|_{\boldsymbol{g}} = V_0$. The linear momentum of $\mathscr{P}$ relative to $\mathscr{O}$ being $\vec{\boldsymbol{P}} = \Gamma m \vec{\boldsymbol{V}}$, we deduce that $P := \|\vec{\boldsymbol{P}}\|_{\boldsymbol{g}}$ is constant and takes the value $P = \Gamma_0 m V_0$. Relation (17.65) shows then that the radius R is expressible in terms of the component of $\vec{\boldsymbol{P}}$ that is transverse to $\vec{\boldsymbol{B}}$, according to

$$\boxed{R = \frac{P \sin\theta}{qB}}. \tag{17.72}$$

This formula shows that, knowing the charge q and the amplitude of the magnetic field B, the measure of R and θ provides the norm P of the particle's linear momentum.

17.4.1.2 Case $E \neq 0$

If $E \neq 0$, we can set (since $|V_0/c\ \cos\theta| < 1$)

$$\tau_0 := -\frac{mc}{qE}\,\mathrm{artanh}\left(\frac{V_0}{c}\cos\theta\right) = -\frac{mc}{2qE}\,\ln\left(\frac{1+(V_0/c)\cos\theta}{1-(V_0/c)\cos\theta}\right), \tag{17.73}$$

which allows us to rewrite (17.62a)–(17.62b) as

$$u^0(\tau) = \Gamma_0\sqrt{1-\frac{V_0^2}{c^2}\cos^2\theta}\ \cosh\left[\frac{qE}{mc}(\tau-\tau_0)\right] \tag{17.74a}$$

$$u^3(\tau) = \Gamma_0\sqrt{1-\frac{V_0^2}{c^2}\cos^2\theta}\ \sinh\left[\frac{qE}{mc}(\tau-\tau_0)\right]. \tag{17.74b}$$

The integration of these equations, along with that of (17.62)–(17.62d), leads to

$$\boxed{\begin{cases} t = (\tilde{a}c)^{-1}\left\{\sinh\left[ac(\tau-\tau_0)\right] + \sinh(ac\tau_0)\right\} \\ x = R\sin(\omega_B\tau) \\ y = R[\cos(\omega_B\tau)-1] \\ z = \tilde{a}^{-1}\left\{\cosh\left[ac(\tau-\tau_0)\right] - \cosh(ac\tau_0)\right\}, \end{cases}} \tag{17.75}$$

where R and ω_B are defined by (17.65),

$$\boxed{a := \frac{qE}{mc^2}} \qquad \text{and} \qquad \tilde{a} := a\sqrt{\frac{1-V_0^2/c^2}{1-V_0^2\cos^2\theta/c^2}}. \tag{17.76}$$

As for the case $E = 0$, the integration constants have been chosen to ensure $(ct, x, y, z) = (0, 0, 0, 0)$ at $\tau = 0$. The trajectory corresponding to (17.75) is depicted in Fig. 17.7. It is a helix whose pitch is increasing with time, due to the acceleration induced by the electric field. The latter appears clearly at the limit $B = 0$ (purely electric case).

17.4.1.3 Purely Electric Case ($B = 0$)

If $B \to 0$, $\omega_B \to 0$ and $R\sin(\omega_B\tau) \simeq \Gamma_0 V_0 \sin\theta/\omega_B \times (\omega_B\tau) = \Gamma_0 V_0 \sin\theta\ \tau$. Similarly $R[\cos(\omega_B\tau)-1] \simeq \Gamma_0 V_0\sin\theta/\omega_B \times (-\omega_B^2\tau^2/2) \to 0$. The system (17.75) reduces then to

$$\boxed{\begin{cases} t = (\tilde{a}c)^{-1}\left\{\sinh\left[ac(\tau-\tau_0)\right] + \sinh(ac\tau_0)\right\} \\ x = \Gamma_0 V_0\sin\theta\ \tau \\ y = 0 \\ z = \tilde{a}^{-1}\left\{\cosh\left[ac(\tau-\tau_0)\right] - \cosh(ac\tau_0)\right\}, \end{cases}} \tag{17.77}$$

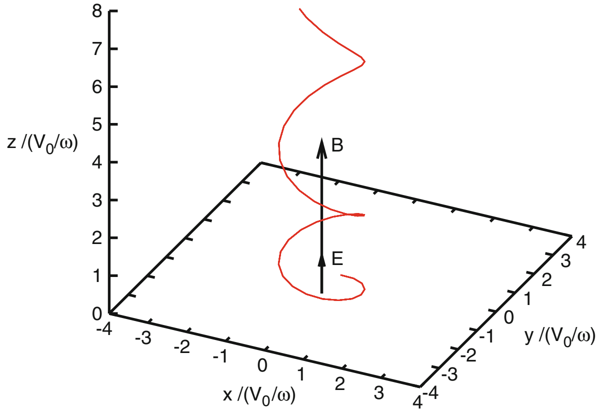


Fig. 17.7 Same as Fig. 17.6 but with a nonvanishing electric field parallel to the magnetic one and with the following parameters: $V_0 = c/2$, $\theta = 85°$ et $E = 0.026\,cB$

In the case where the initial velocity vanishes ($V_0 = 0$) or is aligned with $\vec{E}$ ($\theta = 0$), one has $\tilde{a} = a$ and the above system becomes

$$\begin{cases} t = (ac)^{-1}\{\sinh[ac(\tau - \tau_0)] + \sinh(ac\tau_0)\} \\ z = a^{-1}\{\cosh[ac(\tau - \tau_0)] - \cosh(ac\tau_0)\} \\ x = y = 0, \end{cases} \tag{17.78}$$

with $\tau_0 = 0$ if $V_0 = 0$ [cf. Eq. (17.73)]. We recognize a uniformly accelerated motion along the z-direction, with a 4-acceleration of norm a: up to some choice of origin of proper time τ and to the permutation $x \leftrightarrow z$, these equations are identical to those obtained in Chap. 12 while studying uniformly accelerated motions [cf. Eqs. (12.14)].

17.4.2 Orthogonal Electric and Magnetic Fields

Let us examine the case of a uniform electromagnetic field whose invariant I_2 vanishes. The fields $\vec{E}$ and $\vec{B}$ measured by any inertial observer $\mathscr{O}$ are then orthogonal. If $I_1 \neq 0$, we have seen in Sect. 17.3.3 that by a proper change of observer, we arrive at the cases $\vec{E} = 0$ ($I_1 > 0$) or $\vec{B} = 0$ ($I_1 < 0$) treated above. However, we shall suppose here a generic inertial observer $\mathscr{O}$, which will allow to encompass the case $I_1 = 0$. Let us determine the motion of a charged particle $\mathscr{P}$ whose initial velocity relative to $\mathscr{O}$, $\vec{V}_0$, is normal to the plane generated by $\vec{E}$ and $\vec{B}$. Without any loss of generality, we may assume that the frame $(\vec{e}_\alpha) = (\vec{e}_0, \vec{e}_x, \vec{e}_y, \vec{e}_z)$ of $\mathscr{O}$ is such that (cf. Fig. 17.8)

$$\vec{E} = E\,\vec{e}_y, \quad \vec{B} = B\,\vec{e}_z, \quad \vec{V}_0 = V_0\,\vec{e}_x, \tag{17.79}$$

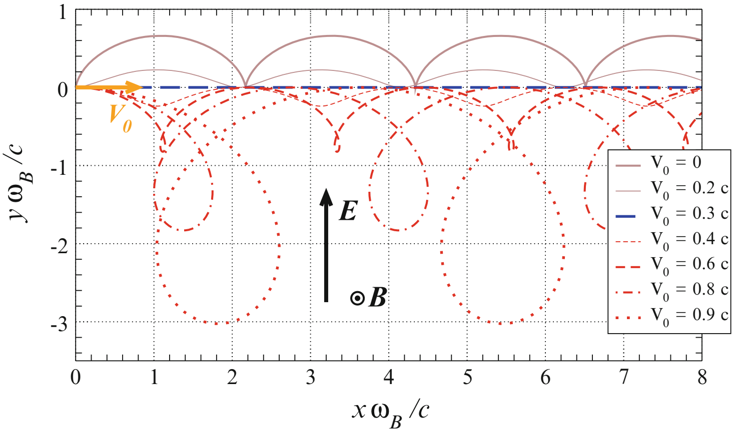


Fig. 17.8 Trajectory of a positively charged particle in an electromagnetic field with a vanishing invariant I_2 (orthogonal $\vec{E}$ and $\vec{B}$), in the case where the particle's initial velocity $\vec{V}_0$ is orthogonal to the plane defined by $\vec{E}$ and $\vec{B}$. For this figure, $E = 0.3\,cB$, which corresponds to $U = E/B = 0.3\,c$. Each axis is labelled in units of c/ω_B, where ω_B is the cyclotron frequency corresponding to the magnetic field B and to the particle's ratio q/m

with $E \geq 0$ and $B \geq 0$. Let us set

$$\beta := \frac{E}{cB}. \tag{17.80}$$

The matrix of the field $\boldsymbol{F}$ in the frame $(\vec{e}_\alpha)$ is given by (17.12) with $(E_1, E_2, E_3) = (0, \beta c B, 0)$ and $(B^1, B^2, B^3) = (0, 0, B)$, so that the components of the equation of motion (17.56) are

$$\mathrm{d}u^0/\mathrm{d}\tau = \beta\omega_B\, u^2 \tag{17.81a}$$

$$\mathrm{d}u^1/\mathrm{d}\tau = \omega_B\, u^2 \tag{17.81b}$$

$$\mathrm{d}u^2/\mathrm{d}\tau = \omega_B\, (\beta u^0 - u^1) \tag{17.81c}$$

$$\mathrm{d}u^3/\mathrm{d}\tau = 0, \tag{17.81d}$$

where $\omega_B := qB/m$ [definition (17.65)]. Since $\vec{V}_0 = V_0\,\vec{e}_x$, the initial conditions for the integration of the differential system (17.81) are

$$u^0(0) = \Gamma_0, \quad u^1(0) = \Gamma_0 V_0/c, \quad u^2(0) = 0, \quad u^3(0) = 0, \tag{17.82}$$

with $\Gamma_0 := (1 - V_0^2/c^2)^{-1/2}$. In addition, as before, we shall suppose that at $\tau = 0$, the particle is located at $(ct, x, y, z) = 0$.

Given the initial condition $u^3(0) = 0$, Eq. (17.81d) is immediately integrated into $u^3 = 0$. Since $u^3 = c^{-1}\mathrm{d}z/\mathrm{d}\tau$ and $z = 0$ at $\tau = 0$, we deduce that $\mathscr{P}$'s motion is confined to the plane $z = 0$.

Deriving Eq. (17.81c) with respect to τ and substituting Eqs. (17.81a) and (17.81b) in the right-hand side, we get a differential equation that involves only $u^2(\tau)$:

$$\frac{\mathrm{d}^2u^2}{\mathrm{d}\tau^2} + (1-\beta^2)\omega_B^2\, u^2 = 0. \tag{17.83}$$

Three cases must then be distinguished: $1-\beta^2 > 0$, $1-\beta^2 = 0$ and $1-\beta^2 < 0$. Since $I_1 = c^2B^2 - E^2 = c^2B^2(1-\beta^2)$, these three cases correspond, respectively, to $I_1 > 0$, $I_1 = 0$ and $I_1 < 0$. Let us examine them successively.

17.4.2.1 Case $I_2 = 0$ and $I_1 > 0$ (Wien Filter)

If $I_1 > 0$, then $\beta < 1$ and we may set

$$U := c\beta = \frac{E}{B} \quad \text{and} \quad \Gamma := (1-\beta^2)^{-1/2} = \frac{cB}{\sqrt{I_1}}. \tag{17.84}$$

U is the velocity relative to $\mathscr{O}$ of an observer $\mathscr{O}'$ for whom the electric field identically vanishes [cf. Eq. (17.42)]. The factor $(1-\beta^2)\omega_B^2 = \omega_B^2/\Gamma^2$ in (17.83) is strictly positive: we obtain the equation of a harmonic oscillator. The solution that fulfils the initial condition $u^2(0) = 0$ [Eq. (17.82)] is

$$u^2(\tau) = A\sin\left(\Gamma^{-1}\omega_B\tau\right), \tag{17.85}$$

where A is some amplitude to be determined. Substituting the above value for $u^2(\tau)$ in (17.81a) and (17.81b) and integrating with the initial conditions (17.82), we obtain, respectively,

$$u^0(\tau) = \Gamma_0 + A\beta\Gamma\left[1-\cos\left(\frac{\omega_B}{\Gamma}\tau\right)\right] \quad \text{and}$$

$$u^1(\tau) = \frac{\Gamma_0 V_0}{c} + A\Gamma\left[1-\cos\left(\frac{\omega_B}{\Gamma}\tau\right)\right].$$

Substituting these values, as well as (17.85) for $u^2(\tau)$, in (17.81c), we get the constant A: $A = \Gamma_0\Gamma(\beta - V_0/c)$. Finally, we have

$$u^0(\tau) = \Gamma_0\Gamma^2\left[1-\beta\frac{V_0}{c} - \beta\left(\beta-\frac{V_0}{c}\right)\cos\left(\frac{\omega_B}{\Gamma}\tau\right)\right] \tag{17.86a}$$

$$u^1(\tau) = \Gamma_0\Gamma^2\left[\beta\left(1-\beta\frac{V_0}{c}\right) - \left(\beta-\frac{V_0}{c}\right)\cos\left(\frac{\omega_B}{\Gamma}\tau\right)\right] \tag{17.86b}$$

$$u^2(\tau) = \Gamma_0\Gamma\left(\beta-\frac{V_0}{c}\right)\sin\left(\frac{\omega_B}{\Gamma}\tau\right). \tag{17.86c}$$

The equation of the trajectory, under the form $(t(\tau), x(\tau), y(\tau))$, is obtained by integrating the relations $\mathrm{d}t/\mathrm{d}\tau = u^0(\tau)$, $\mathrm{d}x/\mathrm{d}\tau = cu^1(\tau)$ and $\mathrm{d}y/\mathrm{d}\tau = cu^2(\tau)$. Given $\beta = U/c$ and the initial conditions $(ct, x, y, z) = (0, 0, 0, 0)$, there comes

$$\boxed{\begin{cases} t = \Gamma_0 \Gamma^2 \left[\left(1 - \dfrac{UV_0}{c^2} \right) \tau - \dfrac{\Gamma U(U - V_0)}{\omega_B c^2} \sin\left(\dfrac{\omega_B}{\Gamma} \tau \right) \right] \\ x = \Gamma_0 \Gamma^2 \left[U \left(1 - \dfrac{UV_0}{c^2} \right) \tau - \dfrac{\Gamma(U - V_0)}{\omega_B} \sin\left(\dfrac{\omega_B}{\Gamma} \tau \right) \right] \\ y = \dfrac{\Gamma_0 \Gamma^2}{\omega_B} (U - V_0) \left[1 - \cos\left(\dfrac{\omega_B}{\Gamma} \tau \right) \right]. \end{cases}} \tag{17.87}$$

Along with the relation $z = 0$ obtained previously, these relations provide the equation of $\mathscr{P}$'s trajectory parametrized by its proper time τ. The trajectory is depicted in Fig. 17.8 for $U = E/B = 0.3\,c$ and different values of V_0. Various special cases are interesting:

- Case $U = 0$, i.e. $E = 0$: then $\Gamma = 1$ and the system (17.87) reduces to

$$\begin{cases} t = \Gamma_0 \tau \\ x = (\Gamma_0 V_0/\omega_B) \sin(\omega_B \tau) \\ y = (\Gamma_0 V_0/\omega_B) \left[\cos(\omega_B \tau) - 1\right]. \end{cases} \tag{17.88}$$

  We recover the circular motion in a constant magnetic field obtained in Sect. 17.4.1: the above equations are identical to (17.64) with $\theta = \pi/2$ ($\vec{V}_0$ orthogonal to $\vec{B}$).
- Case $V_0 = 0$: then $\Gamma_0 = 1$ and the system (17.87) reduces to

$$\begin{cases} t = \Gamma^3/\omega_B \,\left(\chi - U^2/c^2 \sin\chi\right) \\ x = (\Gamma^3 U/\omega_B)\,(\chi - \sin\chi) \\ y = (\Gamma^2 U/\omega_B)\,(1 - \cos\chi), \end{cases} \tag{17.89}$$

  with $\chi := \Gamma^{-1}\omega_B \tau$. We recognize the equation of a *cycloid*, stretched by a factor Γ in the x-direction. This trajectory is depicted in Fig. 17.8 (thick solid line in the domain $y \geq 0$).
- Case $V_0 = U$: then $\Gamma_0 = \Gamma$ and the system (17.87) reduces to

$$\begin{cases} t = \Gamma \tau \\ x = \Gamma U \tau \\ y = 0. \end{cases} \tag{17.90}$$

  The corresponding trajectory is very simple: it is the straight line $(y, z) = (0, 0)$. It is depicted as a dashed line in Fig. 17.8. Moreover the motion is uniform, at velocity V_0.

In the general case, $\mathscr{P}$'s trajectory is a periodic curve confined to the plane $z = 0$ of the *trochoid* kind. Various examples are shown in Fig. 17.8.

Remark 17.15. The shape of the trajectories depicted in Fig. 17.8 is easily understood from the Lorentz force relative to observer $\mathscr{O}$: $\vec{\mathfrak{F}} = q\left(\vec{E} + \vec{V} \times_{u_0} \vec{B}\right)$ [Eq. (17.16)]. If $\vec{V} = \vec{U} = B^{-2}\, \vec{E} \times_u \vec{B}$ [Eq. (17.43)], we get, thanks to $\vec{E} \cdot \vec{B} = 0$, $\vec{\mathfrak{F}} = 0$. The particle does not feel any force and its trajectory is a straight line (case $V_0 = 0.3\,c$ in Fig. 17.8). If $\|\vec{V}\|_g < U$, the electric term $q\vec{E}$ dominates over the magnetic term $q\vec{V} \times_{u_0} \vec{B}$ in the Lorentz force and the particle is deflected in the direction of $\vec{E}$ (if $q > 0$), hence towards the top of Fig. 17.8 (cases $V_0 = 0$ and $V_0 = 0.2\,c$). As the particle's velocity increases, due to the acceleration by the electric field, the magnetic term increases and finally curves the trajectory towards the bottom. Conversely, if initially $\|\vec{V}\|_g > U$, the magnetic term dominates over the electric one and the particle starts to move downwards (case $V_0 \geq 0.4\,c$ in Fig. 17.8). It suffers then some deceleration from the electric field, until the vertical motion is reversed, giving rise to a loop.

Remark 17.16. If q is changed into $-q$ in the trajectory equation (17.87), the sign of ω_B is changed [cf. Eq. (17.65)] and we notice that x is unchanged while y is turned to $-y$. The trajectory is thus obtained by a symmetry with respect to the line $y = 0$.

Remark 17.17. The result (17.87) can be recovered from that obtained in Sect. 17.4.1 for a pure magnetic field. Indeed, since $I_1 \neq 0$, there exists an inertial observer $\mathscr{O}'$ for which $\vec{E}' = 0$. His velocity $\vec{U}$ relative to $\mathscr{O}$ is given by (17.43); we have thus $\vec{U} = U\,\vec{e}_x$ with $U = E/B$. For $\mathscr{O}'$, the electric field vanishes and the magnetic field is $\vec{B}' = \Gamma^{-1}\vec{B}$ with $\Gamma := (1 - U^2/c^2)^{-1/2}$. The particle's initial velocity relative to $\mathscr{O}'$ is deduced from the law of velocity composition (5.47), $\vec{V}_0$ and $\vec{U}$ being collinear: $\vec{V}'_0 = V'_0\vec{e}_{x'}$ with

$$V'_0 = \frac{V_0 - U}{1 - U V_0/c^2}. \tag{17.91}$$

The Lorentz factor of $\mathscr{P}$ relative to $\mathscr{O}'$, Γ'_0, is expressed according to (5.25) (with the appropriate changes of notation): $\Gamma'_0 = \Gamma_0\Gamma(1 - U V_0/c^2)$; hence,

$$\Gamma'_0 V'_0 = \Gamma_0\Gamma(V_0 - U). \tag{17.92}$$

For $\mathscr{O}'$, $\mathscr{P}$'s trajectory is a circle since $\vec{V}'_0$ is orthogonal to $\vec{B}'$. Its equation is given by (17.64) with $\theta = \pi/2$:

$$\begin{cases} t' = \Gamma'_0\tau \\ x' = (\Gamma'_0 V'_0)/(\omega_{B'})\ \sin(\omega_{B'}\tau) \\ y' = (\Gamma'_0 V'_0)/(\omega_{B'})\ [\cos(\omega_{B'}\tau) - 1] \\ z' = 0, \end{cases} \tag{17.93}$$

with $\omega_{B'} = qB'/m = \Gamma^{-1}\omega_B$, since $B' = \Gamma^{-1}B$ [Eq. (17.44)]. The motion of $\mathscr{P}$ with respect to $\mathscr{O}$ is obtained by applying the transformation from the coordinates of $\mathscr{O}'$ to those of $\mathscr{O}$: this is a Poincaré transformation corresponding to a boost of plane Span$(\vec{e}_x, \vec{e}_{x'})$ and velocity parameter $-U$: it is given by (8.15) with $V = U$. Taking into account (17.92), we obtain exactly (17.87).

The fact that the particle is not deflected when its initial velocity is equal to $U = E/B$ (curve $V_0 = 0.3\,c$ in Fig. 17.8) can be used to set up a velocity selector by adjusting the ratio E/B to the desired velocity. This is the principle of the ***Wien filter***: a uniform electromagnetic field is created in a cavity $0 \le x \le L$ with $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$ orthogonal and, thanks to a small aperture at $(x, y, z) = (L, 0, 0)$, only particles with the desired velocity come out from the cavity.

17.4.2.2 Case $I_2 = 0$ and $I_1 = 0$ (Null Electromagnetic Field)

If $I_1 = 0$, as we already assumed $I_2 = 0$, the electromagnetic field is null. Note that $I_1 = 0$ is equivalent to $\beta = 1$ and $E = cB$. In this case, the differential equation (17.83) reduces to $\mathrm{d}^2u^2/\mathrm{d}\tau^2 = 0$. Given the initial condition $u^2(0) = 0$, it is readily integrated in $u^2(\tau) = \alpha\tau$, where α is a constant to be determined. Equations (17.81a) and (17.81b), along with the initial conditions (17.82), yield then

$$u^0(\tau) = \Gamma_0 + \alpha\frac{\omega_B}{2}\tau^2 \quad \text{and} \quad u^1(\tau) = \frac{\Gamma_0 V_0}{c} + \alpha\frac{\omega_B}{2}\tau^2.$$

α is determined by inserting these relations into (17.81c). There comes $\alpha = \Gamma_0(1 - V_0/c)\omega_B$, hence

$$\begin{cases} u^0(\tau) = \Gamma_0\left[1 + (1 - V_0/c)(\omega_B\tau)^2/2\right] \\ u^1(\tau) = \Gamma_0\left[V_0/c + (1 - V_0/c)(\omega_B\tau)^2/2\right] \\ u^2(\tau) = \Gamma_0(1 - V_0/c)\omega_B\tau. \end{cases} \tag{17.94}$$

Integrating with respect to τ, we get

$$\begin{cases} t = \Gamma_0\left[\tau + (1 - V_0/c)(\omega_B^2/6)\,\tau^3\right] \\ x = \Gamma_0\left[V_0\tau + (c - V_0)(\omega_B^2/6)\,\tau^3\right] \\ y = \Gamma_0(c - V_0)(\omega_B/2)\,\tau^2. \end{cases} \tag{17.95}$$

Expressing τ from the equation for y and replacing it in the equation for x, we obtain the equation of $\mathscr{P}$'s trajectory in the plane $z = 0$:

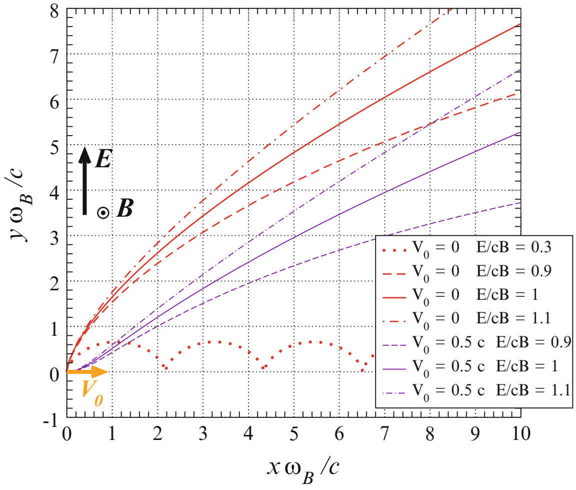


Fig. 17.9 Same as Fig. 17.8, but for values of E/cB in the vicinity of 1, encompassing the case of a null electromagnetic field ($E/cB = 1$). For comparison, the dotted line corresponds to $E = 0.3\,cB$ and is the cycloid labelled $V_0 = 0$ in Fig. 17.8

$$\boxed{x = \left(V_0 + \frac{\omega_B y}{3\Gamma_0}\right)\sqrt{\frac{2\Gamma_0 y}{(c - V_0)\omega_B}}}. \tag{17.96}$$

This is the equation of a *cubic elliptic curve*. It is depicted in Fig. 17.9 for $V_0 = 0$ and $V_0 = 0.5c$ (solid lines).

Remark 17.18. Let us recall that $\omega_B < 0$ for $q < 0$. Formula (17.96) implies then $y \leq 0$ (for the square root to be well defined). On the other side, one has always $x \geq 0$, whatever the sign of $\mathscr{P}$'s charge.

Remark 17.19. The system (17.95) can be obtained as some limit of the system (17.87) derived for $I_1 > 0$. It suffices to make $U \to c$ and $\Gamma \to +\infty$, since the case $I_1 = 0$ corresponds to $U = E/B = c$. Indeed the following Taylor expansions hold for $\Gamma \to +\infty$:

$$\sin\left(\frac{\omega_B}{\Gamma}\tau\right) \simeq \frac{\omega_B}{\Gamma}\tau - \frac{1}{6}\left(\frac{\omega_B}{\Gamma}\tau\right)^3, \qquad \cos\left(\frac{\omega_B}{\Gamma}\tau\right) \simeq 1 - \frac{1}{2}\left(\frac{\omega_B}{\Gamma}\tau\right)^2.$$

Using these expressions in (17.87), we obtain (17.95). That the case $I_1 = 0$ can be derived as the limit $I_1 \to 0$ of the case $I_1 > 0$ can also be seen in Fig. 17.9, by comparing the curves $E/cB = 0.9$ and $E/cB = 1$.

17.4.2.3 Case $I_2 = 0$ and $I_1 < 0$ (Mostly Electric Field)

If $I_1 < 0$, then $\beta > 1$ and we may set

$$U := \frac{c}{\beta} = c^2 \frac{B}{E} \quad \text{and} \quad \Gamma := (1 - U^2/c^2)^{-1/2} = \frac{E}{\sqrt{|I_1|}}. \tag{17.97}$$

U is the velocity relative to $\mathscr{O}$ of an observer $\mathscr{O}'$ for whom the magnetic field vanishes identically [cf. Eq. (17.42)]. The factor $(1 - \beta^2)\omega_B^2$ in the differential equation (17.83) is now strictly negative. Let us rewrite it in terms of the 4-acceleration a defined by (17.76):

$$(1 - \beta^2)\omega_B^2 = -\left(\frac{ac}{\Gamma}\right)^2. \tag{17.98}$$

The solution of (17.83) that satisfies the initial condition $u^2(0) = 0$ is then $u^2(\tau) = A \sinh(ac\tau/\Gamma)$. We can then repeat the reasoning of the case $I_1 > 0$ by replacing the sines/cosines by hyperbolic sines/cosines. Taking care about signs in the derivatives, we obtain the equation of $\mathscr{P}$'s trajectory parametrized by $\mathscr{P}$'s proper time τ:

$$\boxed{\begin{cases} t = \Gamma_0 \Gamma^2 \left[\dfrac{U}{c^2}(V_0 - U)\tau + \dfrac{\Gamma}{ac}\left(1 - \dfrac{UV_0}{c^2}\right) \sinh\left(\dfrac{ac}{\Gamma}\tau\right) \right] \\ x = \Gamma_0 \Gamma^2 \left[(V_0 - U)\tau + \dfrac{\Gamma U}{ac}\left(1 - \dfrac{UV_0}{c^2}\right) \sinh\left(\dfrac{ac}{\Gamma}\tau\right) \right] \\ y = \dfrac{\Gamma_0 \Gamma^2}{a}\left(1 - \dfrac{UV_0}{c^2}\right)\left[\cosh\left(\dfrac{ac}{\Gamma}\tau\right) - 1\right]. \end{cases}} \tag{17.99}$$

This trajectory is depicted in Fig. 17.9 for $E/cB = 1.1 \iff U = 0.909\,c$ and two values of V_0 (0 and $c/2$). As for $I_1 > 0$, let us focus on three special cases:

- Case $U = 0$, i.e. $B = 0$: then $\Gamma = 1$ and the system (17.99) reduces to

$$\begin{cases} t = \Gamma_0/(ac)\, \sinh(ac\tau) \\ x = \Gamma_0 V_0\, \tau \\ y = \Gamma_0/a\, [\cosh(ac\tau) - 1]. \end{cases} \tag{17.100}$$

  As a check, we recover the equation of motion in a pure electric field obtained above, namely, Eq. (17.77). If one sets $\theta = \pi/2$ in the latter, then $\tilde{a} = a/\Gamma_0$ and $\tau_0 = 0$, which, up to the permutation $z \leftrightarrow y$, yields the above equation. If $V_0 = 0$, the trajectory is the straight line $x = 0$: this is a uniformly accelerated motion in the y-direction (direction of $\vec{E}$). If $V_0 \neq 0$, we may get rid of τ and obtain the explicit equation of $\mathscr{P}$'s trajectory in the plane $z = 0$:

$$y = \frac{\Gamma_0}{a}\left[\cosh\left(\frac{ac}{\Gamma_0 V_0}\,x\right) - 1\right]. \tag{17.101}$$

We recognize the equation of a *catenary* having for axis the line $y = 0$.

- Case $V_0 = 0$: then $\Gamma_0 = 1$ and the system (17.99) reduces to

$$\begin{cases} t = \Gamma^3/(ac)\ (\sinh\chi - U^2/c^2\,\chi) \\ x = \Gamma^3 U/(ac)\ (\sinh\chi - \chi) \\ y = \Gamma^3/a\ (\cosh\chi - 1), \end{cases} \tag{17.102}$$

where $\chi := ac\tau/\Gamma$. By analogy with (17.89), such a curve could be called a "hyperbolic cycloid".

- Case $V_0 = U$: then $\Gamma_0 = \Gamma$ and the system (17.99) reduces to

$$\begin{cases} t = \Gamma^2/(ac)\ \sinh(ac\tau/\Gamma) \\ x = Ut \\ y = \Gamma/a\ [\cosh(ac\tau/\Gamma) - 1]\,. \end{cases} \tag{17.103}$$

If $U = 0$, the trajectory is the straight line $x = 0$ (case already considered above). If $U \neq 0$, we may get rid of τ via the identity $\cosh u = \sqrt{1 + \sinh^2 u}$ and obtain the explicit equation of the trajectory:

$$y = \frac{\Gamma}{a}\left[\sqrt{1 + \left(\frac{ac}{\Gamma^2 U}x\right)^2} - 1\right]. \tag{17.104}$$

We recognize a branch of *hyperbola* having for axis the line $y = 0$.

Remark 17.20. Contrary to the case $I_1 > 0$, $\mathscr{P}$'s trajectory is never the straight line $y = 0$, even if $V_0 = U$. This means that the Wien filter can work only with a mostly magnetic field.

Remark 17.21. As in the case $I_1 < 0$, we can recover the motion in a null electromagnetic field ($I_1 = 0$) by taking the limit $U \to c$ in (17.99). Since $\Gamma \to +\infty$, the expansion of the hyperbolic sines and cosines in the neighbourhood of 0 leads to (17.95).

17.5 Application: Particle Accelerators

17.5.1 Acceleration by an Electric Field

Let us consider a cavity at rest with respect to some inertial observer $\mathscr{O}$ and that harbours some uniform electromagnetic field. Let $\mathscr{P}$ be a charged particle that enters into the cavity with a vanishing velocity relative to $\mathscr{O}$: $V_0 = 0$. At each instant, the kinetic energy of $\mathscr{P}$ with respect to $\mathscr{O}$ is given by (9.19):

$$\mathfrak{E}_{\mathrm{kin}} = (\Gamma_{\mathscr{P}} - 1)mc^2. \tag{17.105}$$

We are using the notation $\mathfrak{E}_{\mathrm{kin}}$ instead of E_{kin} (Chap. 9) to avoid any confusion with the amplitude of the electric field. We have also denoted by $\Gamma_{\mathscr{P}}$ the Lorentz factor of $\mathscr{P}$ relative to $\mathscr{O}$, to distinguish it from the Lorentz factor Γ introduced in Sect. 17.4. By definition, $\Gamma_{\mathscr{P}} = \mathrm{d}t/\mathrm{d}\tau = u^0$, where t is $\mathscr{O}$'s proper time, τ that of $\mathscr{P}$ and u^0 is the first component of $\mathscr{P}$'s 4-velocity in $\mathscr{O}$'s frame.

If the electromagnetic field in the cavity is purely magnetic from the point of view of $\mathscr{O}$, $u^0 = \Gamma_{\mathscr{P}}$ is given by (17.63): $u^0(\tau) = \Gamma_0 = 1$ (since $V_0 = 0$). We have thus in this case $\mathfrak{E}_{\mathrm{kin}} = 0$. In other words, a magnetic field does not provide some kinetic energy to a charged particle. To do so, a nonvanishing electric field $\boldsymbol{E}$ is required. If $\vec{\boldsymbol{B}} = 0$ or $\vec{\boldsymbol{B}}$ is parallel to $\vec{\boldsymbol{E}}$, the Lorentz factor $\Gamma_{\mathscr{P}} = u^0$ is given by (17.62a), which, for $V_0 = 0$, reduces to $u^0 = \cosh[qE\tau/(mc)]$, so that

$$\mathfrak{E}_{\mathrm{kin}} = \left[\cosh\left(\frac{qE}{mc}\tau\right) - 1\right]mc^2. \tag{17.106}$$

The travelled distance z along the electric field is on its side given by (17.75) with $\tilde{a} = a$ and $\tau_0 = 0$ since $V_0 = 0$. We observe that $az = \cosh[qE\tau/(mc)] - 1$, so that $\mathfrak{E}_{\mathrm{kin}}$ has a simple expression in terms of z: $\mathfrak{E}_{\mathrm{kin}} = azmc^2$. Substituting $qE/(mc^2)$ for a [Eq. (17.76)], we get then

$$\boxed{\mathfrak{E}_{\mathrm{kin}} = qEz}. \tag{17.107}$$

Remark 17.22. According to (17.75), z has the same sign as qE, so that $\mathfrak{E}_{\mathrm{kin}}$ is always positive.

Remark 17.23. Expression (17.107) is identical to that obtained in nonrelativistic mechanics. On the other side, the expression of z in terms of t differs.

17.5.2 Linear Accelerators

The simplest way to accelerate charged particles by means of an electric field consists in creating some electrical potential difference[9] (electric tension) between two plates: this is the principle of the so-called ***electrostatic accelerator***. Formula (17.107) shows that the gain in kinetic energy is simply the product of the

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[9]The notion of electrical potential will be introduced formally in Chap. 18; here it suffices to know that for a uniform electric field, $Ez = V(0) - V(z)$.

particle's electric charge by the electrical potential difference ΔV:

$$\mathfrak{E}_{\text{kin}} = q\Delta V. \tag{17.108}$$

In practice, one cannot have $|\Delta V|$ much more larger than 10^7 V because of electric discharges, either by electrical breakdown (sparks) or resulting from some imperfect electric insulation.

Formula (17.108) with $q = -e$, combined with (17.3), leads to the value (9.11) for the electronvolt as a unit of energy, given in Sect. 9.2.2. For $\Delta V = 10^6$ V, the kinetic energy acquired by an electron or a proton is $\mathfrak{E}_{\text{kin}} = 10$ MeV. Via (17.105), this corresponds to the Lorentz factor $\Gamma_{\mathscr{P}} = 20.5$ ($\iff V = 0.998\,c$) for an electron and $\Gamma_{\mathscr{P}} = 1.01$ ($\iff V = 0.14\,c$) for a proton. Hence, for this kind of accelerator, electrons easily reach relativistic velocities, but not protons. Some examples of electrostatic accelerators are (i) the *cathode ray tubes (CRT)* ($\mathfrak{E}_{\text{kin}} \sim 10$ keV), which were used for TV and computer screens before the advent of LCD and plasma screens, and (ii) the *electron guns* in electron microscopes ($\mathfrak{E}_{\text{kin}} \sim 100$ keV).

To go beyond a few tens of MeV, a number of cavities must be added. The technique is then to use a variable electric field at high frequency (radio frequency). By synchronizing the frequency of $\boldsymbol{E}$ on the passages of the particles between the different cavities, important accelerations can be achieved. This type of device is called ***linac*** for *linear particle accelerator*. The most powerful linac to date is located at the Stanford Linear Accelerator Center (SLAC) at Stanford University in California. With a length of 3.2 km, it is capable of accelerating electrons and positrons up to $\mathfrak{E}_{\text{kin}} \sim 50$ GeV, which corresponds to Lorentz factors $\Gamma_{\mathscr{P}} \sim 10^5$.

17.5.3 Cyclotrons

To overcome the limitations of electrostatic accelerators, an alternative to the linac consists in making the particles cross many times the cavity hosting the accelerating electric field. To this purpose, the particle trajectories must be curved outside the cavity, thanks to some magnetic field, to make them return to the cavity: this is the principle of the so-called ***cyclotron***. A cyclotron is composed of two "D"-shaped cavities in which a uniform vertical magnetic field is created (cf. Fig. 17.10). An oscillating electric field is set between the cavities, at a well-chosen frequency, as we are going to see.

Let us indeed consider a particle $\mathscr{P}$ with positive charge at the point A of Fig. 17.10 with an initial velocity $\vec{V}$ directed from A to B. Let us assume that the electric field has the same orientation. The particle will then be accelerated between A and B. Between B and C, it is submitted to the action of the vertical magnetic field. As we have seen in Sect. 17.4.1, since $\vec{V}$ is orthogonal to $\vec{B}$, the trajectory of $\mathscr{P}$ is an arc of circle whose radius is proportional to the norm of $\mathscr{P}$'s velocity, which stays constant between B and C (there is no electric field in the cavity). If

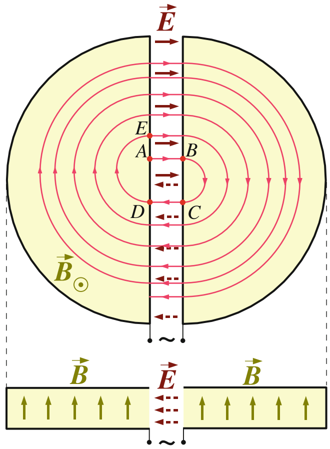

Fig. 17.10 Sketch of a cyclotron. *Top panel,* view from above; *bottom panel,* side view. The electric field is drawn in *solid lines* at the instant where the particles are emerging from the left cavity (points A, E, etc.) and in *dashed lines* half a period later

$\mathscr{P}$'s velocity is small in front of c, the angular velocity between B and C is the cyclotron pulsation $\omega_B = qB/m$ [Eq. (17.65)]. The basic principle of the cyclotron is then to adjust the frequency of the electric field $\vec{E}$ to the cyclotron frequency ω_B, the latter being constant for a fixed magnetic field and a given type of particle. Accordingly, half a period being elapsed between B and C, the particle is submitted along the path CD to an electric field of opposite sense than that it had between A and B. It has then the same sense as $\mathscr{P}$'s velocity at C and $\mathscr{P}$ is accelerated again. When it enters into the magnetic cavity at D, its velocity is larger than at B, so that the radius of its half-circle trajectory will be larger. On the other side, the angular velocity remains the same (cyclotron pulsation). Therefore, when $\mathscr{P}$ arrives at E, it feels again an accelerating electric field and a new iteration starts. The radius of the trajectory increasing at each half turn, the particle eventually reaches the apparatus' periphery, from which it is extracted.

From its very principle, the cyclotron is limited to the acceleration of particles up to velocities small in front of c. Indeed, as we have seen in Sect. 17.4.1, the angular velocity on the half-circle path in the magnetic cavities is not exactly the cyclotron pulsation ω_B, but rather the gyration pulsation $\omega = \Gamma^{-1}\omega_B$ [cf. Eq. (17.68)]. It depends therefore upon the particle's velocity at the cavity entry, via the Lorentz factor Γ. If one creates an electric field of constant frequency, one can adjust the latter to the gyration frequency only for $\Gamma \simeq 1$. This explains why in practice, cyclotrons are used to accelerate protons or ions (cf. Fig. 17.11) and not electrons. For the latter, electrostatic accelerators are sufficient to reach relativistic velocities, as we have noticed in Sect. 17.5.2. For a proton, if one set the nonrelativistic limit to $\Gamma_{\mathscr{P}} \leq 1.02$, then via (17.105), we note that the kinetic energy provided by a cyclotron cannot exceed ~ 20 MeV.

An important field of application of cyclotrons is medicine, either via *proton therapy* (the patient is exposed to the energetic protons produced by the cyclotron) or *nuclear medicine*, i.e. the making of radioactive compounds ingested by the patient and that will localize to the tumour to be treated (cf. Fig. 17.12).

Fig. 17.11 The two cyclotrons CSS1 and CSS2 of GANIL (Grand Accélérateur National d'Ions Lourds, Caen, France). Of a radius of 3 m each, they are connected in series. The intensity of the magnetic field is 0.4 to 0.95 T and the electric-field frequency is 7–14 MHz. The maximum kinetic energy at the output of CSS2 is 95 MeV par nucleon [Source: CNRS/IN2P3]

Fig. 17.12 Cyclotron Arronax installed at the Regional Center for Research in Cancerology in Nantes (France). This cyclotron accelerates protons, deuterons (^{2}H nuclei) and alpha particles (He nuclei) up to an energy of 70 MeV in order to produce radioisotopes for nuclear medicine [Source: Ion Beam Application]

17.5.4 Synchrotrons

To accelerate particles up to relativistic velocities, the first solution consists in modifying the cyclotron to lower the frequency of the electric field as the particles increase their velocity, according to the law (17.68): $\omega = \omega_B/\Gamma$. This is the principle of the ***synchrocyclotron***. Protons can thus be accelerated up to a few hundred MeV. Above that, the radius increasing as ΓV [cf. Eq. (17.65)], the size of the magnets becomes prohibitive.

The concept of ***synchrotron*** allows to get rid of this limit: the idea is to apply some magnetic field not on all the volume of "D"-shaped cavities, but only around a domain of fixed radius R (cf. Fig. 17.13). To maintain the radius constant, the value of B must be increased as the momentum P of the particles increases, according to formula (17.72) (with $\theta = \pi/2$):

$$B = \frac{P}{qR}. \tag{17.109}$$

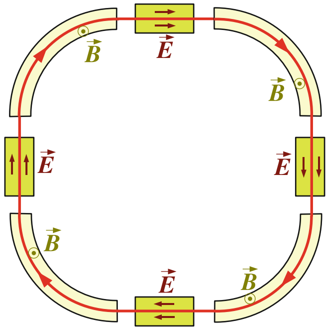

Fig. 17.13 Sketch of a synchrotron

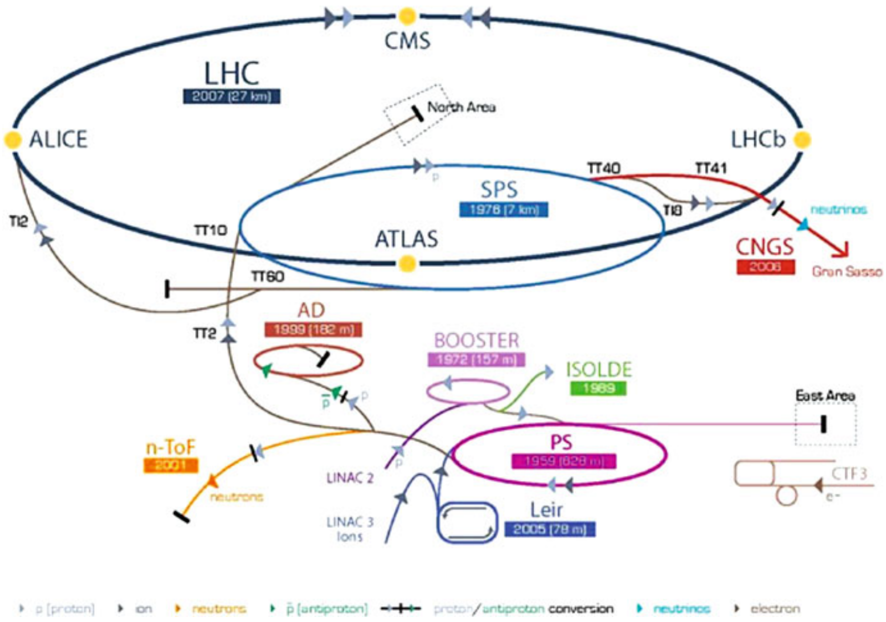

Fig. 17.14 CERN accelerator complex, at the French–Swiss border, near Geneva. It comprises several synchrotrons: LHC (in a tunnel having a circumference of 27 km, previously used for the LEP), SPS, PS, Booster, as well as various linacs [Source: CERN]

The acceleration is performed in cavities equipped with a radio frequency electric field. A synchrotron is thus essentially a succession of accelerating cavities (field $\vec{E}$) and deflecting cavities (field $\vec{B}$). Four cavities of each type are depicted in Fig. 17.13, but there may be much more.

The largest synchrotron operating in the world is the *LHC (Large Hadron Collider)* at CERN, across the French–Swiss border, near Geneva (cf. Fig. 17.14 and Table 17.1). It has a circumference of 27 km and is aimed to accelerate

Table 17.1 A sample of particle accelerators. LHC = Large Hadron Collider, CERN = Conseil Européen pour la Recherche Nucléaire, LEP = Large Electron–Positron Collider, SLC = Stanford Linear Collider, SLAC = Stanford Linear Accelerator Center, ILC = International Linear Collider, GANIL = Grand Accélérateur National d'Ions Lourds, RHIC = Relativistic Heavy Ion Collider, ESRF = European Synchrotron Radiation Facility

Name	Date	Type	Particles	Energy	Lorentz factor
LHC CERN, Geneva	2010–	Synchrotron $R = 4.3$ km	p Pb	7 TeV 2.8 TeV/nucl.	7.5×10^3 2.9×10^3
Tevatron Fermilab, Chicago	2001–	Synchrotron $R = 1.0$ km	p, $\bar{\mathrm{p}}$	0.98 TeV	1.0×10^3
LEP CERN, Geneva	1989–2000	Synchrotron $R = 4.3$ km	e^-, e^+	104 GeV	2×10^5
SLC SLAC, Stanford	1989–1998	Linac $L = 3.2$ km	e^-, e^+	50 GeV	9.8×10^4
ILC	Proposal	Linac $L = 31$ km	e^-, e^+	250 GeV	5×10^5
GANIL Caen	1983–	Cyclotron $R = 3$ m	C,..., U	95 MeV/nucl.	1.1
RHIC Brookhaven	2000–	Storage ring $R = 0.6$ km	p Au, Cu	250 GeV 100 GeV/nucl.	2.7×10^2 10^2
ESRF Grenoble	1994–	Storage ring $R = 134$ m	e^-	6 GeV	1.2×10^4
SOLEIL Saclay	2006–	Storage ring $R = 57$ m	e^-	2.75 GeV	5.4×10^3

protons up to the energy[10] $\mathfrak{E}_{\text{kin}} \simeq 7\,\text{TeV}$. The magnetic field required to maintain protons on the 27 km circumference is given by formula (17.109) with $R = 27/(2\pi) = 4.3\,\text{km}$, $q = e$ and $P \simeq \mathfrak{E}_{\text{kin}}/c$, since the protons are ultra-relativistic ($\mathfrak{E}_{\text{kin}} \gg m_{\text{p}}c^2 = 938\,\text{MeV}$). One obtains $B = 5.4\,\text{T}$. The particle trajectory in the LHC being not exactly a circle (it comprises some segments of straight lines; cf. Fig. 17.13), the required magnetic field is actually slightly larger: $B = 8.3\,\text{T}$. This remarkably large magnetic field is produced by means of superconducting magnets, cooled down to 1.9 K.

[10] At the time of writing of this book (2012), 4 TeV have been achieved; the full 7 TeV are expected for 2015.

17.5.5 Storage Rings

A ***storage ring*** is an annular tube within a magnetic field that maintains the particles on a circular trajectory, as in a synchrotron. The difference with the latter is that the particles are not accelerated in the ring itself, but before entering in it, generally by a linac in series with a synchrotron. Storage rings are used either to store particles before making them collide or to exploit the electromagnetic radiation emitted by particles in circular motion (*synchrotron radiation*, to be studied in Sect. 20.4). In Table 17.1, RHIC is a ring of the first kind, while SOLEIL and ESRF are rings of the second kind.

Historical note: *The first accelerator used in particle physics has been constructed in 1932 by the English physicist John D. Cockcroft (1897–1967) and the Irish one Ernest Walton (1903–1995). It was an electrostatic accelerator capable of accelerating protons up to* $\mathfrak{E}_{\text{kin}} = 0.7\,\text{MeV}$*. These protons have been used to break apart lithium nuclei, thereby demonstrating the nuclear fission by particle bombardment. For this achievement, Cockcroft and Walton have been awarded the 1951 Nobel Prize in Physics. Electrostatic accelerators have been subsequently constructed upon the model conceived in 1931 by the American physicist Robert Van de Graaff (1901–1967). This allowed to reach* $\mathfrak{E}_{\text{kin}} \sim 10\,\text{MeV}$*. The cyclotron has been invented in 1929 by the American physicist Ernest Lawrence (1901–1958), for which he received the Nobel Prize in Physics in 1939. On the first synchrotrons was the Bevatron, built in 1954 at the Lawrence Berkeley National Laboratory in California. It permitted to accelerate protons beyond the GeV, leading to the discovery of the antiproton, as we have seen in Sect. 9.4.6.*

Chapter 18
Maxwell Equations

18.1 Introduction

Having introduced the electromagnetic field $\boldsymbol{F}$ in the preceding chapter, we move now to the equations ruling it, namely, the famous *Maxwell equations*. They specify how $\boldsymbol{F}$ is generated by all the electric charges moving in Minkowski spacetime. We shall treat only the fundamental (microscopic) Maxwell equations and not Maxwell equations *in matter*. The latter are deduced from the former via microscopic models and averaging processes; this is out of the scope of the present text and rather pertains to a proper electromagnetism course.

This chapter starts by introducing the electric four-current vector, which describes globally the moving electric charges (Sect. 18.2). One is then in position to state Maxwell equations (Sect. 18.3), the four-current acting as a source. Sticking to our four-dimensional point of view, we first state Maxwell equations in terms of the tensor $\boldsymbol{F}$. The equations for the fields $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$, presented in elementary expositions of electromagnetism, are deduced in a second stage, after an arbitrary inertial observer is introduced. In Sect. 18.4, we shall see that Maxwell equations imply the conservation of electric charge. We shall deal with the solving of Maxwell equations in Sect. 18.5 via the introduction of the four-potential 1-form and the associated concept of gauge choice. Sect. 18.6 is devoted to the specific case where the source is a single charged particle in arbitrary motion (Liénard–Wiechert solution). At the end of this chapter, it is shown that Maxwell equations can be derived from a principle of least action (Sect. 18.7).

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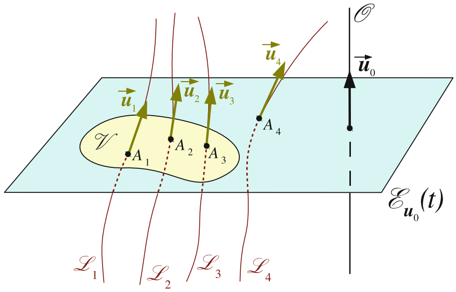

Fig. 18.1 Total electric charge of a three-dimensional domain $\mathscr{V}$ in the local rest space $\mathscr{E}_{\boldsymbol{u}_0}(t)$ of some observer $\mathscr{O}$, defined as a flux through $\mathscr{V}$

18.2 Electric Four-Current

18.2.1 Electric Four-Current Vector

Let us consider a finite set of charged particles, $(\mathscr{P}_a)_{1\le a\le N}$. Each $\mathscr{P}_a$ follows a worldline $\mathscr{L}_a$ of proper time τ_a and 4-velocity $\vec{\boldsymbol{u}}_a(\tau_a)$. In Chap. 17, we have defined the electric charge q_a of particle $\mathscr{P}_a$ as the coefficient involved in the expression of the Lorentz 4-force acting on $\mathscr{P}_a$. Let us now define the ***total electric charge*** Q of a three-dimensional compact domain (hypersurface) $\mathscr{V}$ in the rest space of some inertial observer $\mathscr{O}$. It is natural to introduce Q as the algebraic sum of the charges q_a of particles "contained" in $\mathscr{V}$, noticing that a particle $\mathscr{P}_a$ contributes to Q only if its worldline crosses $\mathscr{V}$ (cf. Fig. 18.1). This suggests to define Q as a *flux through* $\mathscr{V}$. More precisely, since $\mathscr{V}$ is a three-dimensional submanifold of $\mathscr{E}$, we may use the notion of *flux of a vector field through* $\mathscr{V}$ introduced in Sect. 16.4.7 and write, following (16.42),

$$\boxed{Q = \Phi_{\mathscr{V}}(\vec{\boldsymbol{j}}) := -\int_{\mathscr{V}} \vec{\boldsymbol{j}}\cdot\vec{\boldsymbol{u}}_0\,\mathrm{d}V}, \tag{18.1}$$

where $\vec{\boldsymbol{j}}$ is a vector field to be determined; $\vec{\boldsymbol{u}}_0$ is the unit normal to $\mathscr{V}$, which is nothing but the 4-velocity of observer $\mathscr{O}$; and $\mathrm{d}V$ is the volume element on $\mathscr{V}$: $\mathrm{d}V = \boldsymbol{\epsilon}(\vec{\boldsymbol{u}}_0, \mathrm{d}\vec{\boldsymbol{\ell}}_1, \mathrm{d}\vec{\boldsymbol{\ell}}_2, \mathrm{d}\vec{\boldsymbol{\ell}}_3)$ [cf. Eqs. (16.23) and (16.24)]. Since $\vec{\boldsymbol{u}}_0$ is timelike, we have selected the sign $-$ in the definition (16.42). Note that, according to (16.44), the charge Q can be written as the integral over the 3-volume $\mathscr{V}$ of the 3-form $\star\underline{\boldsymbol{j}}$, Hodge dual of the 1-form $\underline{\boldsymbol{j}}$, itself metric dual of $\vec{\boldsymbol{j}}$:

$$\boxed{Q = \int_{\mathscr{V}} \star\underline{\boldsymbol{j}}}. \tag{18.2}$$

It is clear that $\vec{j}$ cannot be a continuous field over $\mathscr{E}$: if the boundary of $\mathscr{V}$ varies so that one of the worldlines $\mathscr{L}_a$ enters $\mathscr{V}$ or leave it, then Q suddenly changes by the amount $\pm q_a$. Actually, to define $\vec{j}$, one must introduce a function "peaked" on the particle worldlines. This is achieved by means of the ***Dirac measure on*** $(\mathscr{E}, \boldsymbol{g})$ ***centred at a point*** $A \in \mathscr{E}$, δ_A. The latter satisfies

$$\forall M \in \mathscr{E}\backslash\{A\}, \quad \delta_A(M) = 0 \qquad \text{and} \qquad \int_{\mathscr{E}} \delta_A f \, \boldsymbol{\epsilon} = f(A) \tag{18.3}$$

for any scalar field $f : \mathscr{E} \to \mathbb{R}$. The above integral is the integral of the product $\delta_A f$ considered as a scalar field on $\mathscr{E}$—integral defined by (16.39) [cf. also (16.40a)]. δ_A can also be introduced as the limit of a sequence of continuous functions more and more "peaked" on A, but in all rigour, δ_A is a *distribution*[1] and not a scalar field on $\mathscr{E}$. For this reason, it is also called ***Dirac distribution on*** $(\mathscr{E}, \boldsymbol{g})$ ***centred at*** $A \in \mathscr{E}$. It is easy to see that, given a system of inertial coordinates (x^α) on $\mathscr{E}$, δ_A is expressible as

$$\delta_A(x^0, x^1, x^2, x^3) = \delta(x^0 - x^0_A)\, \delta(x^1 - x^1_A)\, \delta(x^2 - x^2_A)\, \delta(x^3 - x^3_A), \tag{18.4}$$

where δ is the "ordinary" Dirac distribution on $\mathbb{R}$.

Equipped with the tool δ_A, let us define the vector field $\vec{j}$ as

$$\forall M \in \mathscr{E}, \quad \boxed{\vec{j}(M) := \sum_{a=1}^{N} q_a \int_{-\infty}^{+\infty} \delta_{A_a(\tau)}(M)\, \vec{\boldsymbol{u}}_a(\tau)\, c\, \mathrm{d}\tau}, \tag{18.5}$$

where, in each integral, τ stands for the proper time of particle $\mathscr{P}_a$, $\vec{\boldsymbol{u}}_a(\tau)$ for $\mathscr{P}_a$'s 4-velocity and $A_a(\tau)$ for $\mathscr{P}_a$'s position in $\mathscr{E}$ at the instant τ. The vector field $\vec{j}$ is called ***electric four-current***, or ***electric 4-current*** for short. From (18.5) and the dimension $(\text{length})^{-4}$ of Dirac measure on $(\mathscr{E}, \boldsymbol{g})$, as well as the dimensionless character of a 4-velocity, the dimension of $\vec{j}$ is that of a volume density of electric charge. In the SI system, its unit is thus $\mathrm{C\,m^{-3}} = \mathrm{A\,s\,m^{-3}}$.

[1]From this point of view, denoting the action of δ_A on f by an integral as in (18.3) is an abuse of notation often used in physics.

The vector field defined by (18.5) has the wished property. Indeed, the flux of $\vec{j}$ through $\mathscr{V}$ is

$$\begin{aligned}\Phi_{\mathscr{V}}(\vec{j}) &= -\int_{\mathscr{V}} \vec{j}\cdot\vec{u}_0\,\mathrm{d}V = -\sum_{a=1}^{N} q_a \int_{\mathscr{V}}\int_{-\infty}^{+\infty} \delta_{A_a(\tau)}(M)\,\vec{u}_a(\tau)\cdot\vec{u}_0\,c\,\mathrm{d}\tau\,\mathrm{d}V \\ &= -\sum_{a=1}^{N} q_a \int_{\mathscr{V}}\int_{-\infty}^{+\infty} \delta(ct_0 - x_a^0(\tau))\,\delta(x^1 - x_a^1(\tau))\,\delta(x^2 - x_a^2(\tau))\,\times \\ &\qquad \times\delta(x^3 - x_a^3(\tau))\,\vec{u}_a(\tau)\cdot\vec{u}_0\,c\,\mathrm{d}\tau\,\mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}x^3,\end{aligned}$$

where t_0 is $\mathscr{O}$'s proper time corresponding to the rest space $\mathscr{E}_{\boldsymbol{u}_0}(t_0)$ in which $\mathscr{V}$ lies; (x^α) is the coordinate system associated with $\mathscr{O}$, with $x^0 = ct$; and $x_a^\alpha(\tau)$ are the coordinates in this system of the point $A_a(\tau)$ on the worldline $\mathscr{L}_a$. Now, from the relation (4.10), $-\vec{u}_a(\tau)\cdot\vec{u}_0 = \Gamma_a$—the Lorentz factor of $\mathscr{P}_a$ with respect to $\mathscr{O}$. By definition [Eq. (4.1)], $\Gamma_a = \mathrm{d}t/\mathrm{d}\tau$, so that we can write $-\vec{u}_a(\tau)\cdot\vec{u}_0\,c\,\mathrm{d}\tau = c\,\mathrm{d}t$. Using t rather than τ as a parameter along $\mathscr{L}_a$, there comes then

$$\begin{aligned}\Phi_{\mathscr{V}}(\vec{j}) &= \sum_{a=1}^{N} q_a \int_{\mathscr{V}}\int_{-\infty}^{+\infty} \delta(ct_0 - ct)\,\delta(x^1 - x_a^1(t))\,\delta(x^2 - x_a^2(t))\,\times \\ &\qquad \times\delta(x^3 - x_a^3(t))\,c\,\mathrm{d}t\,\mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}x^3. \\ &= \sum_{a=1}^{N} q_a \int_{\mathscr{V}} \delta(x^1 - x_a^1(t_0))\,\delta(x^2 - x_a^2(t_0))\delta(x^3 - x_a^3(t_0))\,\mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}x^3 \\ &= \sum_{a\,/\,A_a(t_0)\in\mathscr{V}} q_a. \end{aligned} \tag{18.6}$$

Hence the flux $\Phi_{\mathscr{V}}(\vec{j})$ is the sum of the electric charges carried by the particles whose worldlines intersect $\mathscr{V}$, which justifies the writing (18.1).

Historical note: *The expression (18.5) for the electric 4-current generated by a discrete distribution of charges is due to Paul A.M. Dirac (cf. p. 372), who used it (for a single particle) in a work published in 1938 (Dirac 1938, 1939).*

18.2.2 Electric Intensity

Let us consider an oriented (two-dimensional) surface $\mathscr{S}$ linked to the inertial observer $\mathscr{O}$: at each instant t, $\mathscr{S}(t)$ is a surface in the rest space $\mathscr{E}_{\boldsymbol{u}_0}(t)$ at a fixed

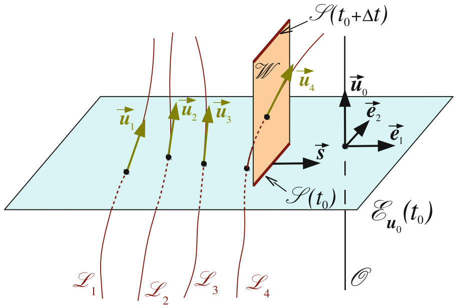

Fig. 18.2 Definition of the electric intensity from the spacetime 3-volume $\mathscr{W}$ swept by a surface $\mathscr{S}(t)$ in the rest space $\mathscr{E}_{u_0}(t)$ of observer $\mathscr{O}$. For the drawing, one dimension has been suppressed, so that $\mathscr{S}(t)$ appears as a segment and $\mathscr{W}$ as a surface

position in the spatial coordinates of $\mathscr{O}$, (x^i). One calls ***electric intensity through*** $\mathscr{S}$ and denotes by $I(\mathscr{S})$, the total electric charge that crosses $\mathscr{S}$ (in the sense of its orientation) per unit time. In the SI system, the unit of electric intensity is the ampere (symbol: A), with $1\ \mathrm{A} = 1\ \mathrm{C\,s^{-1}}$.

Let us show that between two instants $t = t_0$ and $t = t_0 + \Delta t$, the electric charge crossing $\mathscr{S}$ is equal to the flux of the 4-current $\vec{j}$ through the spacetime 3-volume $\mathscr{W}$ swept by $\mathscr{S}$ between t_0 and $t_0 + \Delta t$: $\mathscr{W} = \cup_{t=t_0}^{t_0+\Delta t} \mathscr{S}(t)$ (cf. Fig. 18.2). Denoting by $\vec{s} \in E_{u_0}$ the unit (spacelike) vector normal to $\mathscr{S}$ in $(\mathscr{E}_{u_0}(t), \boldsymbol{g})$, compatible with $\mathscr{S}(t)$'s orientation, we get

$$
\begin{aligned}
\Phi_{\mathscr{W}}(\vec{j}) &= \int_{\mathscr{W}} \vec{j} \cdot \vec{s}\, \mathrm{d}V = \sum_{a=1}^{N} q_a \int_{\mathscr{W}} \int_{\tau=-\infty}^{\tau=+\infty} \delta_{A_a(\tau)}(M)\, \vec{u}_a(\tau) \cdot \vec{s}\, c\, \mathrm{d}\tau\, \mathrm{d}V \\
&= \sum_{a=1}^{N} q_a \int_{\mathscr{W}} \int_{\tau=-\infty}^{\tau=+\infty} \delta_{A_a(\tau)}(M)\, \vec{s} \cdot \vec{V}_a\, \Gamma_a\, \mathrm{d}\tau\, \mathrm{d}V, \\
&= \sum_{a=1}^{N} q_a \int_{\mathscr{W}} \int_{t'=-\infty}^{t'=+\infty} \delta_{A_a(t')}(M)\, \vec{s} \cdot \vec{V}_a\, \mathrm{d}t'\, \mathrm{d}V, \qquad (18.7)
\end{aligned}
$$

where the second line results from the decomposition (4.31) of $\mathscr{P}_a$'s 4-velocity: $\vec{u}_a = \Gamma_a(\vec{u}_0 + c^{-1}\vec{V}_a)$, $\vec{V}_a$ being $\mathscr{P}_a$'s velocity relative to $\mathscr{O}$. Since $\vec{u}_0 \cdot \vec{s} = 0$, we have indeed $\vec{u}_a \cdot \vec{s} = \Gamma_a \vec{s} \cdot \vec{V}_a / c$. The last line has been obtained by means of the change of variable $\tau \mapsto t'$, where t' is $\mathscr{O}$'s proper time,[2] with $\Gamma_a\, \mathrm{d}\tau = \mathrm{d}t'$.

For simplicity, let us assume that $\mathscr{S}(t)$ is a planar surface contained in the plane $x^1 = x^1_{\mathscr{S}} =$ const and oriented so that $\vec{s} = \vec{e}_1$ ($(\vec{e}_\alpha)$ standing for $\mathscr{O}$'s frame).

[2] It is denoted by t' to keep t for the coordinate orthogonal to $\mathscr{S}$ in $\mathscr{W}$.

The coordinates internal to $\mathscr{S}(t)$ are then (x^2, x^3), and the volume element of $\mathscr{W}$ is $\mathrm{d}V = c\,\mathrm{d}t\,\mathrm{d}x^2\,\mathrm{d}x^3$. Moreover $\vec{s}\cdot\vec{V}_a = V_a^1$, so that (18.7) becomes

$$\begin{aligned}
\Phi_{\mathscr{W}}(\vec{j}) &= \sum_{a=1}^{N} q_a \int_{\mathscr{S}} \int_{t=t_0}^{t=t_0+\Delta t} \int_{t'=-\infty}^{t'=+\infty} \delta(ct - ct')\,\delta(x^1_{\mathscr{S}} - x^1_a(t'))\,\delta(x^2 - x^2_a(t')) \times \\
&\qquad \times \delta(x^3 - x^3_a(t'))\,V_a^1\,\mathrm{d}t'\,c\,\mathrm{d}t\,\mathrm{d}x^2\,\mathrm{d}x^3 \\
&= \sum_{a=1}^{N} q_a \int_{\mathscr{S}} \int_{t=t_0}^{t=t_0+\Delta t} \delta(x^1_{\mathscr{S}} - x^1_a(t))\,\delta(x^2 - x^2_a(t))\,\delta(x^3 - x^3_a(t)) \times \\
&\qquad \times V_a^1\,\mathrm{d}t\,\mathrm{d}x^2\,\mathrm{d}x^3.
\end{aligned}$$

Now $V_a^1 = \mathrm{d}x_a^1/\mathrm{d}t$. If $V_a^1 = 0$, the integral corresponding to $\mathscr{P}_a$ vanishes and therefore does not contribute to $\Phi_{\mathscr{W}}(\vec{j})$. If $V_a^1 \neq 0$, we can perform the change of variable $t \mapsto x_a^1$, with $V_a^1\,\mathrm{d}t = \mathrm{d}x_a^1$, and write

$$\Phi_{\mathscr{W}}(\vec{j}) = \sum_{a\,/\,V_a^1\neq 0} q_a \int_{\mathscr{S}} \int_{x_a^1(t_0)}^{x_a^1(t_0+\Delta t)} \delta(x^1_{\mathscr{S}} - x^1_a)\,\delta(x^2 - x^2_a(t))\,\delta(x^3 - x^3_a(t))\,\mathrm{d}x_a^1\,\mathrm{d}x^2\,\mathrm{d}x^3,$$

where $t = t(x_a^1)$. It is clear that the triple integral in the right-hand side takes the value 1 if there exists $x_a^1 \in [x_a^1(t_0),\ x_a^1(t_0 + \Delta t)]$ such that

$$(x_a^1,\ x_a^2(t),\ x_a^3(t)) = (x^1_{\mathscr{S}}, x^2, x^3) \quad \text{with } (x^2, x^3) \text{ coord. of a point of } \mathscr{S}$$

and the value 0 otherwise. The above condition being nothing but the crossing of $\mathscr{W}$ by the worldline $\mathscr{L}_a$, we conclude that

$$\Phi_{\mathscr{W}}(\vec{j}) = \sum_{a\,/\,\mathscr{L}_a\cap\mathscr{W}\neq\varnothing} q_a. \tag{18.8}$$

In other words, $\Phi_{\mathscr{W}}(\vec{j})$ is the sum of the electric charges that cross the surface $\mathscr{S}$ between t_0 and $t_0 + \Delta t$. In view of the definition of the intensity $I(\mathscr{S})$, we may thus write

$$I(\mathscr{S})\Delta t = \Phi_{\mathscr{W}}(\vec{j}) = \int_{\mathscr{W}} \vec{j}\cdot\vec{s}\,\mathrm{d}V = \int_{\mathscr{S}} \int_{t=t_0}^{t=t_0+\Delta t} \vec{j}\cdot\vec{s}\,c\,\mathrm{d}t\ \mathrm{d}S, \tag{18.9}$$

where $\mathrm{d}S$ is the area element of $\mathscr{S}$ [cf. (16.31)]. Setting $\Delta t \to 0$, we deduce the following formula for the electric intensity:

$$\boxed{I(\mathscr{S}) = c \int_{\mathscr{S}} \vec{j} \cdot \vec{s}\, \mathrm{d}S}. \tag{18.10}$$

18.2.3 Charge Density and Current Density

Given an observer $\mathscr{O}$ of 4-velocity $\vec{u}_0$, let us perform the orthogonal decomposition of the 4-current vector $\vec{j}$ with respect to $\vec{u}_0$:

$$\boxed{\vec{j} =: \rho\, \vec{u}_0 + \frac{1}{c}\vec{J}} \qquad \text{with} \quad \vec{u}_0 \cdot \vec{J} = 0. \tag{18.11}$$

We have then

$$\rho = -\vec{u}_0 \cdot \vec{j}. \tag{18.12}$$

In view of (18.1), we note that ρ is the quantity involved in the expression of the electric charge Q of a three-dimensional domain in the rest space of $\mathscr{O}$:

$$\boxed{Q = \int_{\mathscr{V}} \rho\, \mathrm{d}V}. \tag{18.13}$$

For this reason, we shall call ρ the ***electric charge density measured by*** $\mathscr{O}$. Its unit in the SI system is $\mathrm{C\,m^{-3}} = \mathrm{A\,s\,m^{-3}}$ (the same as for $\vec{j}$).

From (18.11), the vector $\vec{J}$ is, up to some c factor, nothing but the orthogonal projection of the electric 4-current in $\mathscr{O}$'s rest space:

$$\vec{J} = c \perp_{u_0} \vec{j} = c\left[\vec{j} + (\vec{u}_0 \cdot \vec{j})\, \vec{u}_0\right]. \tag{18.14}$$

In view of (18.10) and $\vec{s} \in E_{u_0}$, we observe that $\vec{J}$ is the part of $\vec{j}$ providing the intensity:

$$\boxed{I(\mathscr{S}) = \int_{\mathscr{S}} \vec{J} \cdot \vec{s}\, \mathrm{d}S}. \tag{18.15}$$

Hence the electric intensity through a surface is the flux of $\vec{J}$ through this surface. $\vec{J}$ is called the ***electric current density measured by*** $\mathscr{O}$. From (18.11), the dimension of $\vec{J}$ differs from that of $\vec{j}$ by a velocity. Its SI unit is thus $\mathrm{C\,m^{-2}s^{-1}} = \mathrm{A\,m^{-2}}$.

Remark 18.1. In many textbooks, one introduces first the quantities ρ and $\vec{J}$ relative to a given observer before combining them as in (18.11) to form the 4-vector $\vec{j}$.

Faithful to our four-dimensional point of view, we have first constructed $\vec{j}$, which is independent of any observer, and have deduced the observer-dependent quantities ρ and $\vec{J}$ in a second stage.

18.2.4 Four-Current of a Continuous Media

We have defined the electric 4-current $\vec{j}$ by adopting a microscopic point of view, i.e. by summing over all charged particles [Eq. (18.5)]. Now, if one considers a macroscopic region, the number N of particles is huge, and it is natural to take the continuous limit. $\vec{j}$ appears then as a continuous vector field on $\mathscr{E}$. In practice, it is even differentiable.

18.3 Maxwell Equations

18.3.1 Statement

Having introduced the electric 4-current $\vec{j}$, we are in position to state the famous Maxwell equations. They are expressed in terms of the exterior derivatives $\mathbf{d}\boldsymbol{F}$ and $\mathbf{d}{\star}\boldsymbol{F}$ of the electromagnetic field 2-form $\boldsymbol{F}$ and its Hodge dual[3] $\star\boldsymbol{F}$:

The electromagnetic field is governed by ***Maxwell equations***:

$$\boxed{\mathbf{d}\boldsymbol{F} = 0} \tag{18.16a}$$

$$\boxed{\mathbf{d}{\star}\boldsymbol{F} = \varepsilon_0^{-1} \star \underline{\boldsymbol{j}}}, \tag{18.16b}$$

where ε_0 is a universal constant, called ***vacuum permittivity***, and $\star\underline{\boldsymbol{j}}$ is the 3-form associated with the electric 4-current $\vec{j}$ by Hodge duality. More precisely, $\star\underline{\boldsymbol{j}}$ is the Hodge dual of the 1-form $\underline{\boldsymbol{j}}$ associated with the

[3] Let us recall that the exterior derivative has been introduced in Sect. 15.5, the Hodge dual in Sect. 14.5 and specifically for $\boldsymbol{F}$ in Sect. 17.2.5.

vector $\vec{\boldsymbol{j}}$ by metric duality. From (14.75b) [cf. also (15.87)],

$$\star\underline{\boldsymbol{j}} := \boldsymbol{\epsilon}(\vec{\boldsymbol{j}}, ., ., .). \tag{18.17}$$

As we have seen in Sect. 18.2.1, $\star\underline{\boldsymbol{j}}$ is the 3-form whose integration over a 3-volume yields the total electric charge contained in this volume [Eq. (18.2)].

The numerical value of the constant ε_0 is

$$\varepsilon_0 = \frac{1}{\mu_0 c^2} \simeq 8.854\,187\,817 \times 10^{-12}\ \mathrm{F\,m^{-1}}, \tag{18.18}$$

where the constant μ_0 is the ***vacuum permeability*** and has a well-defined value in SI units:

$$\mu_0 = 4\pi\ 10^{-7}\ \mathrm{N\,A^{-2}}. \tag{18.19}$$

Remark 18.2. The Maxwell equations (18.16) are independent of any observer, since $\boldsymbol{F}$ and $\vec{\boldsymbol{j}}$ are fields on $\mathscr{E}$ that do not make reference to any observer.

At this stage, we consider Maxwell equations (18.16) as a postulate at the foundation of the classical theory of electrodynamics. We shall see in Sect. 18.7 that they can actually *be derived* from a principle of least action.

18.3.2 Alternative Forms

The Maxwell equation (18.16b) can be recast in a form involving the divergence of the tensor $\boldsymbol{F}$ rather than the exterior derivative of $\star\boldsymbol{F}$. To this aim, one shall apply the Hodge star to (18.16b). Since from (14.77), $\star\star\underline{\boldsymbol{j}} = \underline{\boldsymbol{j}}$, there comes

$$\star\mathbf{d}\star\boldsymbol{F} = \varepsilon_0^{-1}\,\underline{\boldsymbol{j}}. \tag{18.20}$$

$\mathbf{d}\star\boldsymbol{F}$ being a 3-form, its Hodge dual, $\star\mathbf{d}\star\boldsymbol{F}$, is a 1-form. Let us consider the metric dual vector, $\overrightarrow{\star\mathbf{d}\star\boldsymbol{F}}$, and evaluate its components by combining (14.75d) and (15.64):

$$\begin{aligned}(\star\mathrm{d}\star F)^{\alpha} &= \frac{1}{6}\,\epsilon^{\mu\nu\rho\alpha}(\mathrm{d}\star F)_{\mu\nu\rho} \\ &= \frac{1}{6}\,\epsilon^{\mu\nu\rho\alpha}\left(\nabla_{\mu}\star F_{\nu\rho} + \nabla_{\nu}\star F_{\rho\mu} + \nabla_{\rho}\star F_{\mu\nu}\right)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{6}\left[\nabla_\mu\left(\epsilon^{\mu\nu\rho\alpha}\star F_{\nu\rho}\right) + \nabla_\nu\left(\epsilon^{\mu\nu\rho\alpha}\star F_{\rho\mu}\right) + \nabla_\rho\left(\epsilon^{\mu\nu\rho\alpha}\star F_{\mu\nu}\right)\right] \\
&= \frac{1}{6}\Big[\nabla_\mu\underbrace{(\epsilon^{\nu\rho\mu\alpha}\star F_{\nu\rho})}_{2\,(\star\star F)^{\mu\alpha}} + \nabla_\nu\underbrace{(\epsilon^{\rho\mu\nu\alpha}\star F_{\rho\mu})}_{2\,(\star\star F)^{\nu\alpha}} + \nabla_\rho\underbrace{(\epsilon^{\mu\nu\rho\alpha}\star F_{\mu\nu})}_{2\,(\star\star F)^{\rho\alpha}}\Big] \\
&= \nabla_\mu(\star\star F)^{\mu\alpha} = -\nabla_\mu F^{\mu\alpha} = \nabla_\mu F^{\alpha\mu},
\end{aligned}
$$

where to get the third line, we have used $\nabla_\mu\epsilon^{\mu\nu\rho\alpha} = 0$ since $\boldsymbol{\epsilon}$ is a constant field on $\mathscr{E}$ (cf. Remark 15.9 p. 519) and for the fourth and fifth lines, we have used the definition (14.75c) of the Hodge dual of $\star\boldsymbol{F}$, i.e. $\star\star\boldsymbol{F}$, as well as property (14.77): $\star\star\boldsymbol{F} = -\boldsymbol{F}$ and the antisymmetry of $\boldsymbol{F}$. We recognize in $\nabla_\mu F^{\alpha\mu}$ the components of the divergence $\boldsymbol{\nabla}\cdot\boldsymbol{F}^\sharp$ of the tensor $\boldsymbol{F}^\sharp$ introduced in Sect. 17.2.5 (cf. Sect. 15.4.6); hence,

$$
\overrightarrow{\star\mathbf{d}\star\boldsymbol{F}} = \boldsymbol{\nabla}\cdot\boldsymbol{F}^\sharp. \tag{18.21}
$$

Using this relation in (18.20), there comes $\boldsymbol{\nabla}\cdot\boldsymbol{F}^\sharp = \varepsilon_0^{-1}\,\vec{\boldsymbol{j}}$, so that the Maxwell equations (18.16) can be expressed in terms of $\boldsymbol{F}$ and $\boldsymbol{F}^\sharp$ as

$$
\boxed{\mathbf{d}\boldsymbol{F} = 0} \tag{18.22a}
$$

$$
\boxed{\boldsymbol{\nabla}\cdot\boldsymbol{F}^\sharp = \varepsilon_0^{-1}\,\vec{\boldsymbol{j}}}\,. \tag{18.22b}
$$

The components of these equations with respect to some coordinate system (x^α) on $\mathscr{E}$ are deduced from formulas (15.68) and (15.58) (the latter being applicable for $\boldsymbol{F}^\sharp$ is antisymmetric):

$$
\boxed{\frac{\partial F_{\beta\gamma}}{\partial x^\alpha} + \frac{\partial F_{\gamma\alpha}}{\partial x^\beta} + \frac{\partial F_{\alpha\beta}}{\partial x^\gamma} = 0} \tag{18.23a}
$$

$$
\boxed{\frac{1}{\sqrt{-\det g}}\,\frac{\partial}{\partial x^\mu}\left(\sqrt{-\det g}\;F^{\alpha\mu}\right) = \varepsilon_0^{-1}\,j^\alpha}\,. \tag{18.23b}
$$

Let us take now the Hodge star of the Maxwell equation (18.16a). Since $\star\star\boldsymbol{F} = -\boldsymbol{F}$, we have $\star\mathbf{d}\boldsymbol{F} = -\star\mathbf{d}\star(\star\boldsymbol{F})$. Now, by applying formula (18.21), which is valid for any 2-form, to $\star\boldsymbol{F}$ rather than $\boldsymbol{F}$, there comes $\star\mathbf{d}\star(\star\boldsymbol{F}) = \boldsymbol{\nabla}\cdot\star\boldsymbol{F}^\sharp$. We conclude that the Maxwell equations (18.16) can be recast solely in terms of the Hodge dual of $\boldsymbol{F}$ as

$$
\boxed{\boldsymbol{\nabla}\cdot\star\boldsymbol{F}^\sharp = 0} \tag{18.24a}
$$

$$
\boxed{\mathbf{d}\star\boldsymbol{F} = \varepsilon_0^{-1}\,\star\underline{\boldsymbol{j}}}\,. \tag{18.24b}
$$

Finally, gathering (18.24a) and (18.22b), we write Maxwell equations as

$$\boxed{\nabla \cdot \star \boldsymbol{F}^{\sharp} = 0} \tag{18.25a}$$

$$\boxed{\nabla \cdot \boldsymbol{F}^{\sharp} = \varepsilon_0^{-1}\, \vec{\boldsymbol{j}}}\,. \tag{18.25b}$$

Remark 18.3. These last two equations, which are equalities between vectors, are the Hodge duals of the original Maxwell equations (18.16), which were equalities between 3-forms.

Remark 18.4. In the form (18.16), as well as (18.25), we note some asymmetry between the two Maxwell equations: one has a source ($\star\underline{\boldsymbol{j}}$ or $\vec{\boldsymbol{j}}$) and the other has not (vanishing right-hand side). This asymmetry reflects the nonexistence of ***magnetic charge***, also called ***magnetic monopole***. Would magnetic monopoles exist, they would be described by a "magnetic 4-current" vector, $\vec{\boldsymbol{h}}$ say, analogous to the electric 4-current $\vec{\boldsymbol{j}}$ describing electric charges, and Maxwell equations would take the symmetric form

$$\begin{cases} \mathbf{d}\boldsymbol{F} = \varepsilon_0^{-1}\, \star\underline{\boldsymbol{h}} \\ \mathbf{d}\star\boldsymbol{F} = \varepsilon_0^{-1}\, \star\underline{\boldsymbol{j}} \end{cases} \qquad \Longleftrightarrow \qquad \begin{cases} \nabla \cdot \star\boldsymbol{F}^{\sharp} = \varepsilon_0^{-1}\, \vec{\boldsymbol{h}} \\ \nabla \cdot \boldsymbol{F}^{\sharp} = \varepsilon_0^{-1}\, \vec{\boldsymbol{j}}\,. \end{cases} \tag{18.26}$$

Some theories, like Grand Unified Theories or string theory, predict the existence of magnetic monopoles (see, e.g. Penrose (2007)) but none has been detected to date.

18.3.3 Expression in Terms of Electric and Magnetic Fields

We have noticed above that Maxwell equations are observer-independent. If however some observer is given, they can be written in terms of the couple $(\boldsymbol{E}, \vec{\boldsymbol{B}})$ resulting from the decomposition of $\boldsymbol{F}$ with respect to the observer. More precisely, let us consider a rigid array of inertial observers as defined in Sect. 8.2.4. Each observer of the array has the same constant 4-velocity $\vec{\boldsymbol{u}}$ and measures at each point of his worldline an electric field $\boldsymbol{E}$ and a magnetic field $\vec{\boldsymbol{B}}$, the total electromagnetic field $\boldsymbol{F}$ being reconstructible via (17.6). Let us start from the Maxwell equations under the form (18.25). The components of the tensor $\boldsymbol{F}^{\sharp}$ are deduced from (17.10) and (17.18):

$$F^{\alpha\beta} = u^{\alpha} E^{\beta} - E^{\alpha} u^{\beta} + c\, \epsilon^{\mu\nu\alpha\beta} u_{\mu} B_{\nu}. \tag{18.27}$$

Given that $\vec{\boldsymbol{u}}$ and ${}^4\boldsymbol{\epsilon}$ are constant tensor fields on $\mathscr{E}$, the divergence of $\boldsymbol{F}^{\sharp}$ is

$$\nabla_{\mu} F^{\alpha\mu} = u^{\alpha} \nabla_{\mu} E^{\mu} - u^{\mu} \nabla_{\mu} E^{\alpha} + c\, \epsilon^{\rho\nu\alpha\mu} u_{\rho} \nabla_{\mu} B_{\nu}. \tag{18.28}$$

As we have noticed in Sect. 17.2.5, $\star\boldsymbol{F}$ is deduced from $\boldsymbol{F}$ by substituting $-c\underline{\boldsymbol{B}}$ for $\boldsymbol{E}$ and $\boldsymbol{E}$ for $c\underline{\boldsymbol{B}}$. We obtain thus easily from (18.28) the components of $\nabla\cdot\star\boldsymbol{F}^{\sharp}$:

$$\nabla_\mu(\star F)^{\alpha\mu} = -cu^\alpha\nabla_\mu B^\mu + cu^\mu\nabla_\mu B^\alpha + \epsilon^{\rho\nu\alpha\mu}u_\rho\nabla_\mu E_\nu. \tag{18.29}$$

Let us split the Maxwell equation (18.25a) in two parts: one collinear to the 4-velocity $\vec{u}$ and the other one orthogonal to $\vec{u}$. The first part is obtained by the scalar product of (18.25a) with $\vec{u}$. From the expression (18.29), we get

$$-c\underbrace{u_\alpha u^\alpha}_{-1}\nabla_\mu B^\mu + cu^\mu\nabla_\mu[\underbrace{u_\alpha B^\alpha}_{0}] + \underbrace{\epsilon^{\rho\nu\alpha\mu}u_\rho u_\alpha}_{0}\nabla_\mu E_\nu = 0,$$

i.e.

$$\boxed{\nabla\cdot\vec{B} = 0}. \tag{18.30}$$

The part of (18.25a) orthogonal to $\vec{u}$ is obtained by means of the orthogonal projector $\perp_{\boldsymbol{u}}$:

$$(\delta^\alpha{}_\beta + u^\alpha u_\beta)\left[-cu^\beta\nabla_\mu B^\mu + cu^\mu\nabla_\mu B^\beta + \epsilon^{\rho\nu\beta\mu}u_\rho\nabla_\mu E_\nu\right] = 0$$
$$\Longrightarrow cu^\mu\nabla_\mu B^\alpha + \epsilon^{\rho\nu\alpha\mu}u_\rho\nabla_\mu E_\nu = 0.$$

Now $\epsilon^{\rho\nu\alpha\mu}u_\rho\nabla_\mu E_\nu = \epsilon^{\rho\alpha\mu\nu}u_\rho\nabla_\mu E_\nu$, and by comparing with (15.71), we recognize the curl of the field $\vec{E}$. We have thus

$$c\nabla_{\vec{u}}\vec{B} + \nabla\!\times_{\boldsymbol{u}}\vec{E} = 0.$$

But from (15.28), at any fixed point $M \in \mathscr{E}$, $c\nabla_{\vec{u}}\vec{B}$ is nothing but the derivative of the field $\vec{B}$ along the worldline of the inertial observer of the considered array going through this point. We shall denote it by a partial derivative with respect to the proper time t rather than by a full derivative as in (15.28), where there was only one worldline. We obtain thus

$$\boxed{\nabla\!\times_{\boldsymbol{u}}\vec{E} = -\frac{\partial\vec{B}}{\partial t}}. \tag{18.31}$$

This is the so-called ***Maxwell–Faraday equation***.

Let us now consider the orthogonal decomposition of the Maxwell equation (18.25b) with respect to $\vec{u}$. Taking into account (18.28) and (18.12), the scalar product of (18.25b) with $\vec{u}$ leads to

$$\underbrace{u_\alpha u^\alpha}_{-1}\nabla_\mu E^\mu - u^\mu\nabla_\mu[\underbrace{u_\alpha E^\alpha}_{0}] + c\underbrace{\epsilon^{\rho\nu\alpha\mu}u_\rho u_\alpha}_{0}\nabla_\mu B_\nu = \varepsilon_0^{-1}\underbrace{u_\alpha j^\alpha}_{-\rho},$$

i.e.

$$\boxed{\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}}. \tag{18.32}$$

This is the so-called ***Maxwell–Gauss equation***. The part of (18.25b) that is orthogonal to $\vec{u}$ is [cf. (18.14)]

$$-u^\mu \nabla_\mu E^\alpha + c\, \epsilon^{\rho\nu\alpha\mu} u_\rho \nabla_\mu B_\nu = \varepsilon_0^{-1} \underbrace{(\delta^\alpha{}_\beta + u^\alpha u_\beta) j^\beta}_{c^{-1} J^\alpha}.$$

Now, similarly to what we get for $\vec{B}$, $c\nabla_{\vec{u}} \vec{E} = \partial \vec{E}/\partial t$. Using $1/(\varepsilon_0 c^2) = \mu_0$ [Eq. (18.18)], we obtain then

$$\boxed{\nabla \times_{\boldsymbol{u}} \vec{B} = \mu_0 \vec{J} + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}}. \tag{18.33}$$

This is the so-called ***Maxwell–Ampère equation***.

We conclude that the two Maxwell equations expressed in terms of $\boldsymbol{F}$ and $\vec{j}$ are equivalent to the system of four equations (18.30), (18.31), (18.32) and (18.33) involving the fields $\boldsymbol{E}$, $\vec{B}$, ρ and $\vec{J}$ relative to a rigid array of inertial observers.

Remark 18.5. The number of components of the two systems of equations are of course the same: the original Maxwell equations, either in the form of identities between 3-forms [version (18.16)] or in the form of identities between vectors [version (18.25)], have $4 + 4 = 8$ components.[4] The Maxwell equations in terms of $(\boldsymbol{E}, \vec{B})$ are identities between scalars [Eqs. (18.30) and (18.32)] or vectors in the three-dimensional space $E_{\boldsymbol{u}}$ [Eqs. (18.31) and (18.33)]. They comprise thus $1 + 1 + 3 + 3 = 8$ components.

Remark 18.6. In connection with Remark 18.4 p. 595, if there existed magnetic monopoles, represented by a 4-current $\vec{h}$, the right-hand side of (18.30) would contain $\rho_{\rm m} := -\vec{u}_0 \cdot \vec{h}$ and that of (18.31) $\vec{J}_{\rm m} := c \perp_{u_0} \vec{h}$, making the Maxwell equations symmetric in $\vec{E}$ and $\vec{B}$.

Historical note: *The equations ruling the dynamics of the electromagnetic field have been published between 1861 and 1865 by James Clerk Maxwell*[5] *(1861; 1865). They were formulated in a preferred frame, that of aether, comprised*

---

[4] Let us recall that the space of 3-forms is four-dimensional; cf. (14.40).

[5] **James Clerk Maxwell** (1831–1879): Scottish theoretical physicist, famous for having unified electricity, magnetism and optics.

20 components and gave a privileged role to the magnetic potential[6] $\vec{\mathscr{A}}$. *The modern three-dimensional form of Maxwell equations, namely, Eqs. (18.30)–(18.33), which do not involve any potential, but only the vectors* $\vec{\boldsymbol{E}}$ *and* $\vec{\boldsymbol{B}}$, *is due to Oliver Heaviside*[7] *in 1885 (Heaviside 1885) and to Heinrich Hertz*[8] *in 1890 (Hertz 1890) (cf. (Darrigol 2005)). The four-dimensional formulation of Maxwell equations, in the form of the two equations (18.25), which involve the divergence of the tensor* $\boldsymbol{F}^\sharp$ *and its Hodge dual, dates from 1908: it is the work of Hermann Minkowski (cf. p. 26) (1908) in the framework of special relativity and no longer of the theory of aether.*

Also, let us stress that we have obtained the four equations (18.30)–(18.33) ruling $\vec{\boldsymbol{E}}$ *and* $\vec{\boldsymbol{B}}$ *for an* arbitrary *inertial observer. The fact that their shape is independent of the choice of this observer is a manifestation of the* relativity principle *discussed in the historical note p. 288. This confirms that the latter is not a first principle in the present exposition of special relativity but is deduced within the framework set at the beginning (Minkowski spacetime) and the four-dimensional Maxwell equations, based on the electromagnetic field tensor* $\boldsymbol{F}$.

18.4 Electric Charge Conservation

18.4.1 Derivation from Maxwell equations

Maxwell equation (18.16b) is an identity between two 3-forms. Applying the exterior derivative operator to each side and using $\mathbf{d}\mathbf{d}\star\boldsymbol{F} = 0$ [nilpotent character of the exterior derivative; cf. (15.74)], there comes immediately

$$\mathbf{d}\star\underline{\boldsymbol{j}} = 0. \tag{18.34}$$

In other words, the 3-form $\star\underline{\boldsymbol{j}}$ is closed. An important consequence is[9]

For any closed hypersurface $\Sigma \subset \mathscr{E}$, the flux of the electric 4-current through Σ vanishes:

[6]The magnetic potential will be introduced in Sect. 18.5.2.

[7]**Oliver Heaviside** (1850–1925): English physicist and mathematician; self-taught, he contributed to many domains of electromagnetism and mathematics (vector analysis, differential equations).

[8]**Heinrich Hertz** (1857–1894): German physicist, author of many works in electromagnetism and famous for having experimentally shown the existence of electromagnetic waves.

[9]Let us recall that for a hypersurface, *closed* means compact and without boundary.

$$\boxed{\Sigma \text{ closed} \Longrightarrow \Phi_\Sigma(\vec{j}) = \int_\Sigma \star\underline{j} = 0}, \qquad (18.35)$$

where (16.44) has been used to identify the flux of $\vec{j}$ with the integral of the 3-form $\star\underline{j}$.

Proof. If Σ is closed, it can be considered as the boundary of a four-dimensional domain $\mathscr{U}$, and Stokes theorem (16.46) gives

$$\int_\Sigma \star\underline{j} = \int_{\mathscr{U}} \mathbf{d}\star\underline{j} = 0,$$

the last equality resulting from (18.34). □

The property (18.35) is similar to that stated in Sect. 9.3.3 for the conservation of the 4-momentum of an isolated system [Eq. (9.36)] and expresses the ***electric charge conservation***. Indeed, let us consider two three-dimensional domains $\mathscr{V}$ and $\mathscr{V}'$ as the extremities of a four-dimensional domain $\mathscr{U}$ (cf. Fig. 18.3), such that $\mathscr{V}$ lies in the rest space $\mathscr{E}_{\boldsymbol{u}_0}(t)$ of some inertial observer $\mathscr{O}$ and $\mathscr{V}'$ in the rest space $\mathscr{E}_{\boldsymbol{u}'_0}(t')$ of some inertial observer $\mathscr{O}'$. $\mathscr{O}$ and $\mathscr{O}'$ can be the same observer, but then the instants t and t' must be different. The boundary $\Sigma := \partial\mathscr{U}$ of $\mathscr{U}$ is the union of $\mathscr{V}$, $\mathscr{V}'$ and a third hypersurface $\mathscr{W}$ (the "vertical wall" of the "tube", cf. Fig. 18.3). Σ is a closed hypersurface, so that the result (18.35) implies

$$\Phi_\Sigma(\vec{j}) = \Phi_{\mathscr{V}}(\vec{j}) + \Phi_{\mathscr{V}'}(\vec{j}) + \Phi_{\mathscr{W}}(\vec{j}) = 0. \qquad (18.36)$$

Now $Q' = \Phi_{\mathscr{V}'}(\vec{j})$ is nothing but the total electric charge of the domain $\mathscr{V}'$ [cf. Eq. (18.1)]. For $\mathscr{V}$, it is the same thing up to a sign, for the orientation of $\mathscr{V}$ as a boundary of $\mathscr{U}$ provides it with a past-directed normal (cf. Fig. 18.3), and the electric charge is obtained from the future-directed normal [4-velocity $\vec{u}_0$ in (18.1)]. We have thus $Q = -\Phi_{\mathscr{V}}(\vec{j})$. Let us assume that the tube $\mathscr{U}$ is ***electrically isolated***, in the sense that no charged particle crosses the boundary $\mathscr{W}$, then $\Phi_{\mathscr{W}}(\vec{j}) = 0$ and (18.36) reduces to $-Q + Q' = 0$, i.e.

$$\boxed{Q' = Q}. \qquad (18.37)$$

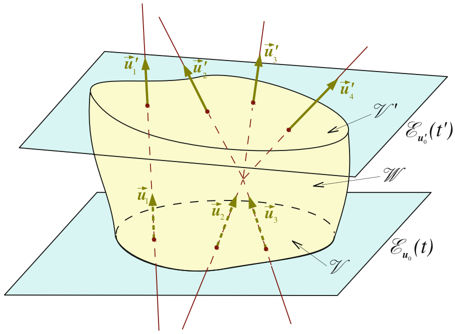

Fig. 18.3 Electric charge conservation between two three-dimensional domains $\mathscr{V}$ and $\mathscr{V}'$

This result can be interpreted in two ways:

- **Conservation law for a given observer:** if $\mathscr{O}'$ coincides with observer $\mathscr{O}$, the domain $\mathscr{V}'$ can be seen as resulting from the evolution of domain $\mathscr{V}$ from t to t', and (18.37) means that the electric charge stays constant. The insulation condition $\Phi_{\mathscr{W}}(\vec{j}) = 0$ can be then interpreted as the absence of any charged particle crossing the boundary of the volume $\mathscr{V}$ between t and t'.
- **Invariance under a change of observer:** if $\mathscr{O}$ and $\mathscr{O}'$ are two different observers, the result (18.37) expresses the invariance of the electric charge when moving from one observer to the other.

Remark 18.7. We have just shown that the law of conservation (or invariance) of the electric charge is a consequence of Maxwell equations; there is therefore no need to postulate it separately.

Remark 18.8. The above demonstration does not require the conservation of the number of particles between $\mathscr{V}$ and $\mathscr{V}'$: some reactions between particles can occur, as illustrated in Fig. 18.3. The number of charged particles may then vary, but the total charge stays constant.

In view of (15.88), the divergence of the vector field $\vec{j}$ is the opposite of the Hodge dual of the 4-form $\mathbf{d}\star\underline{j}$: $\nabla\cdot\vec{j} = -\star\mathbf{d}\star\underline{j}$. The property (18.34) from which the conservation of the electric charge has been deduced is thus equivalent to

$$\boxed{\nabla\cdot\vec{j} = 0}. \tag{18.38}$$

The electric 4-current is thus a divergence-free vector field.

Remark 18.9. An alternative way to obtain this result is to start from the expression of $\nabla\cdot\vec{j}$ in terms of the components j^α of $\vec{j}$ in a coordinate system (x^α), as given by (15.54): $\sqrt{-\det g}\,\nabla\cdot\vec{j} = \partial/\partial x^\mu(\sqrt{-\det g}\, j^\mu)$. Replacing j^μ via the Maxwell equation (18.23b), we get

$$\sqrt{-\det g}\,\nabla\cdot\vec{j} = \varepsilon_0 \frac{\partial^2}{\partial x^\mu \partial x^\nu}\left(\sqrt{-\det g}\, F^{\mu\nu}\right).$$

$\boldsymbol{F}$ being antisymmetric, we have $F^{\mu\nu} = -F^{\nu\mu}$. Since, on the contrary, $\partial^2/\partial x^\mu \partial x^\nu$ is symmetric, we deduce that the above expression vanishes, recovering (18.38).

Remark 18.10. The electric charge conservation in the form (18.35), namely, $\Phi_\Sigma(\vec{j}) = 0$ for any closed hypersurface Σ, can be deduced from $\nabla\cdot \vec{j} = 0$, thanks to the four-dimensional Gauss–Ostrogradsky theorem established in Sect. 16.5.2. Indeed, considering that Σ is the boundary of a four-dimensional domain $\mathscr{U}$, the Gauss–Ostrogradsky theorem (16.64) yields

$$\Phi_\Sigma(\vec{j}) = \int_{\mathscr{U}} \nabla\cdot \vec{j}\, \mathrm{d}U,$$

so that (18.38) does imply $\Phi_\Sigma(\vec{j}) = 0$.

18.4.2 Expression in Terms of Charge and Current Densities

Let us decompose the 4-current $\vec{j}$ into the charge density ρ and the current density $\vec{J}$ both relative to some inertial observer $\mathscr{O}$, according to (18.11). The components of $\vec{j}$ in $\mathscr{O}$'s frame are then $j^\alpha = (\rho, J^i/c)$. Accordingly (18.38) becomes (using (15.54) for expressing the divergence in terms of the coordinates (x^α) associated with $\mathscr{O}$):

$$\frac{\partial \rho}{\partial x^0} + \frac{\partial}{\partial x^i}\left(\frac{J^i}{c}\right) = 0,$$

where we have set $\det g = -1$ since the coordinates (x^α) are inertial. Writing $x^0 = ct$ and noticing that $\partial J^i/\partial x^i = \nabla\cdot \vec{J}$ for $J^0 = -\vec{u}_0 \cdot \vec{J} = 0$, there comes

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla\cdot \vec{J} = 0}. \tag{18.39}$$

We recover here a familiar equation in physics: that expressing the conservation of a quantity (electric charge, mass, baryon number, etc.) in terms of the volume density of that quantity (ρ) and the corresponding current density ($\vec{J}$).

18.4.3 Gauss Theorem

Let $\mathscr{S}$ be a closed[10] 2-surface delimiting a three-dimensional domain $\mathscr{V} \subset \mathscr{E}$: $\mathscr{S} = \partial\mathscr{V}$. $\mathscr{S}$ and $\mathscr{V}$ can lie in the rest space of some observer, but this is not necessary. The electric charge Q contained in $\mathscr{V}$ is expressed via (18.2) as the integral over

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[10] Compact and without boundary, as, for instance, a sphere.

$\mathscr{V}$ of the 3-form $\star\underline{\boldsymbol{j}}$. Since by the Maxwell equation (18.16b), $\star\underline{\boldsymbol{j}} = \varepsilon_0 \mathbf{d} \star \boldsymbol{F}$, we can write

$$Q = \varepsilon_0 \int_{\mathscr{V}} \mathbf{d}\star\boldsymbol{F} = \varepsilon_0 \int_{\mathscr{S}} \star\boldsymbol{F},$$

where the second equality results from Stokes theorem (16.46). Hence the integral of the 2-form $\star\boldsymbol{F}$ over the 2-surface $\mathscr{S}$ is (up to some ε_0 factor) the electric charge contained inside $\mathscr{S}$:

$$\boxed{\int_{\mathscr{S}} \star\boldsymbol{F} = \frac{Q}{\varepsilon_0}}. \tag{18.40}$$

This result is known as ***Gauss theorem***.

If we consider an inertial observer $\mathscr{O}$ and $\mathscr{S}$ in the rest space of $\mathscr{O}$, we can express $\star\boldsymbol{F}$ in terms of the electric and magnetic fields $(\boldsymbol{E}, \vec{\boldsymbol{B}})$ measured by $\mathscr{O}$ via (17.21). There comes then

$$\int_{\mathscr{S}} \star\boldsymbol{F} = \underbrace{-\int_{\mathscr{S}} \underline{\boldsymbol{u}}_0 \wedge c\underline{\boldsymbol{B}}}_{0} + \int_{\mathscr{S}} \star(\underline{\boldsymbol{u}}_0 \wedge \boldsymbol{E}) = \int_{\mathscr{S}} \boldsymbol{\epsilon}(\vec{\boldsymbol{u}}_0, \vec{\boldsymbol{E}}, ., .), \tag{18.41}$$

where the vanishing of the first integral results from the fact that $\vec{\boldsymbol{u}}_0$ is orthogonal to $\mathscr{S}$, so that for any elementary vector $\mathrm{d}\vec{\boldsymbol{\ell}}$ tangent to $\mathscr{S}$, $\langle \underline{\boldsymbol{u}}_0, \mathrm{d}\vec{\boldsymbol{\ell}} \rangle = 0$ [cf. the definition (16.17b) of an integral over a 2-surface]. The second equality in (18.41) arises from the expression (14.79) of the Hodge star applied to some exterior product. For any couple $(\mathrm{d}\vec{\boldsymbol{\ell}}_2, \mathrm{d}\vec{\boldsymbol{\ell}}_3)$ of elementary vectors tangent to $\mathscr{S}$, $\boldsymbol{\epsilon}(\vec{\boldsymbol{u}}_0, \vec{\boldsymbol{E}}, \mathrm{d}\vec{\boldsymbol{\ell}}_2, \mathrm{d}\vec{\boldsymbol{\ell}}_3)$ involves only the part of $\vec{\boldsymbol{E}}$ that is orthogonal to $\mathscr{S}$, i.e. the vector $(\vec{\boldsymbol{s}} \cdot \vec{\boldsymbol{E}})\,\vec{\boldsymbol{s}}$, where $\vec{\boldsymbol{s}}$ is the unit normal to $\mathscr{S}$ within $(\mathscr{E}_{\boldsymbol{u}_0}, \boldsymbol{g})$, oriented towards the exterior of $\mathscr{S}$. We have thus $\boldsymbol{\epsilon}(\vec{\boldsymbol{u}}_0, \vec{\boldsymbol{E}}, \mathrm{d}\vec{\boldsymbol{\ell}}_2, \mathrm{d}\vec{\boldsymbol{\ell}}_3) = (\vec{\boldsymbol{s}} \cdot \vec{\boldsymbol{E}})\,\boldsymbol{\epsilon}(\vec{\boldsymbol{u}}_0, \vec{\boldsymbol{s}}, \mathrm{d}\vec{\boldsymbol{\ell}}_2, \mathrm{d}\vec{\boldsymbol{\ell}}_3)$. Now, on $\mathscr{S}$, $\boldsymbol{\epsilon}(\vec{\boldsymbol{u}}_0, \vec{\boldsymbol{s}}, ., .)$ is nothing but the area element 2-form [cf. Eq. (16.29)]. We conclude that

$$\int_{\mathscr{S}} \star\boldsymbol{F} = \int_{\mathscr{S}} \vec{\boldsymbol{E}} \cdot \vec{\boldsymbol{s}}\, \mathrm{d}S, \tag{18.42}$$

so that the Gauss theorem (18.40) can be expressed in terms of the flux of the vector $\vec{\boldsymbol{E}}$ through the surface $\mathscr{S}$ [within the three-dimensional space $(\mathscr{E}_{\boldsymbol{u}_0}, \boldsymbol{g})$]:

$$\boxed{\int_{\mathscr{S}} \vec{\boldsymbol{E}} \cdot \vec{\boldsymbol{s}}\, \mathrm{d}S = \frac{Q}{\varepsilon_0}}. \tag{18.43}$$

Remark 18.11. This result can also be obtained from the Maxwell–Gauss equation (18.32), using the three-dimensional Gauss–Ostrogradsky theorem (16.62) and expression (18.13) of Q:

$$\int_{\mathscr{S}} \vec{\boldsymbol{E}} \cdot \vec{\boldsymbol{s}}\, \mathrm{d}S = \int_{\mathscr{V}} \nabla\cdot \vec{\boldsymbol{E}}\, \mathrm{d}V = \frac{1}{\epsilon_0} \int_{\mathscr{V}} \rho\, \mathrm{d}V = \frac{Q}{\epsilon_0}.$$

Note also that the same Gauss–Ostrogradsky theorem applied to the three-dimensional Maxwell equation (18.30), $\nabla \cdot \vec{\boldsymbol{B}} = 0$, leads to the vanishing of the flux of $\vec{\boldsymbol{B}}$ through any closed surface $\mathscr{S}$.

18.5 Solving Maxwell Equations

18.5.1 Four-Potential

The Maxwell equation $\mathbf{d}\boldsymbol{F} = 0$ [Eq. (18.16a)] means that the 2-form $\boldsymbol{F}$ is closed. From Poincaré lemma (cf. Sect. 15.5.3), there exists (at least locally) a 1-form $\boldsymbol{A}$ such that $\boldsymbol{F}$ is the exterior derivative of $\boldsymbol{A}$:

$$\boxed{\boldsymbol{F} = \mathbf{d}\boldsymbol{A}}. \tag{18.44}$$

$\boldsymbol{A}$ is called ***electromagnetic four-potential***, or ***electromagnetic 4-potential*** for short. Given a coordinate system (x^α) on $\mathscr{E}$, the components of $\boldsymbol{A}$ are related to that of $\boldsymbol{F}$ via (15.62) and (15.67):

$$\boxed{F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha = \frac{\partial A_\beta}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\beta}}. \tag{18.45}$$

The advantage to work with $\boldsymbol{A}$, rather than with $\boldsymbol{F}$, is that the first of the two Maxwell equations (18.22) is automatically satisfied since $\mathbf{dd}\boldsymbol{A} = 0$ (nilpotent character of the exterior derivative; cf. Sect. 15.5.3). The second Maxwell equation (18.22b) is expressed in terms of $\boldsymbol{F}^\sharp$, whose components are related to that of $\boldsymbol{A}$ via (17.18) and (18.45):

$$F^{\alpha\beta} = g^{\alpha\mu} g^{\beta\nu} \left(\nabla_\mu A_\nu - \nabla_\nu A_\mu\right) = \nabla^\alpha A^\beta - \nabla^\beta A^\alpha, \tag{18.46}$$

where $A^\alpha = g^{\alpha\mu} A_\mu$ are the components of the vector $\vec{\boldsymbol{A}}$, metric dual of the 1-form $\boldsymbol{A}$, and

$$\boxed{\nabla^\alpha := g^{\alpha\mu} \nabla_\mu}. \tag{18.47}$$

Extending the arrow notation introduced in Sect. 1.6, we shall write $\vec{\boldsymbol{\nabla}}$ for the operator whose components are ∇^α. While the covariant derivative operator $\boldsymbol{\nabla}$ maps a tensor field of type (k, ℓ) to a tensor field of type $(k, \ell + 1)$, the operator $\vec{\boldsymbol{\nabla}}$ maps it to a tensor field of type $(k + 1, \ell)$. In particular, for any scalar field f, $\vec{\boldsymbol{\nabla}} f$ is the vector field that is metric dual to the gradient of f [cf. Eq. (15.60)]:

$$\vec{\boldsymbol{\nabla}} f = \overrightarrow{\mathbf{d}f} \quad \Longleftrightarrow \quad \nabla^\alpha f = g^{\alpha\mu} \frac{\partial f}{\partial x^\mu}. \tag{18.48}$$

Inserting (18.46) into the Maxwell equation (18.22b), we get

$$\nabla_\mu \nabla^\alpha A^\mu - \nabla_\mu \nabla^\mu A^\alpha = \varepsilon_0^{-1} j^\alpha . \tag{18.49}$$

In this equation, there appears the ***d'Alembertian operator***

$$\boxed{\square := \nabla_\mu \nabla^\mu}. \tag{18.50}$$

If (x^α) are inertial coordinates, $\nabla_\mu = \partial/\partial x^\mu$ and $\nabla^\mu = \eta^{\mu\nu}\partial/\partial x^\nu$, so that $\nabla^\mu = (-\partial/\partial x^0, \partial/\partial x^1, \partial/\partial x^2, \partial/\partial x^3)$. Hence, setting $(x^\alpha) = (ct, x, y, z)$,

$$\boxed{\square = \eta^{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu} = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}} \quad \text{(inertial coord.)}. \tag{18.51}$$

Remark 18.12. Contrary to the operators covariant derivative or divergence, the d'Alembertian maps a tensor field of type (k, ℓ) to a tensor field of the same type.

Taking into account $\nabla_\mu \nabla^\alpha A^\mu = \nabla^\alpha \nabla_\mu A^\mu$, we can write the Maxwell equation (18.49) as

$$\boxed{\square \vec{A} - \vec{\nabla}(\nabla \cdot \vec{A}) = -\varepsilon_0^{-1} \vec{j}}. \tag{18.52}$$

18.5.2 Electric and Magnetic Potentials

Given an inertial observer $\mathscr{O}$, we may decompose the 1-form $\boldsymbol{A}$ orthogonally with respect to $\mathscr{O}$'s 4-velocity $\vec{u}_0$, according to

$$\boxed{\boldsymbol{A} =: V \underline{\boldsymbol{u}}_0 + c\, \mathscr{A}} \qquad \text{with} \quad \langle \mathscr{A}, \vec{u}_0 \rangle = 0. \tag{18.53}$$

The scalar field V thus introduced is called ***electric potential relative to*** $\mathscr{O}$, while the vector field $\vec{\mathscr{A}}$ metric dual to the 1-form $\mathscr{A}$ is called ***magnetic potential relative to*** $\mathscr{O}$. The components of $\boldsymbol{A}$ and $\vec{A}$ in $\mathscr{O}$'s frame $(\vec{e}_\alpha)$ are

$$A_\alpha = (-V, c\mathscr{A}_1, c\mathscr{A}_2, c\mathscr{A}_3) \qquad \text{and} \qquad A^\alpha = (V, c\mathscr{A}^1, c\mathscr{A}^2, c\mathscr{A}^3), \tag{18.54}$$

with $\mathscr{A}^i = \mathscr{A}_i$ since $(\vec{e}_\alpha)$ is an orthonormal basis.

Inserting (18.53) into (18.44) leads to the expression of the electromagnetic field in terms of the potentials V and $\mathscr{A}$:

$$\boldsymbol{F} = \mathbf{d}V \wedge \underline{\boldsymbol{u}}_0 + c\, \mathbf{d}\mathscr{A}. \tag{18.55}$$

Note that use has been made of (15.76) in the form $\mathbf{d}(V\underline{\boldsymbol{u}}_0) = \mathbf{d}(V \wedge \underline{\boldsymbol{u}}_0) = \mathbf{d}V \wedge \underline{\boldsymbol{u}}_0 + V \wedge \mathbf{d}\underline{\boldsymbol{u}}_0 = \mathbf{d}V \wedge \underline{\boldsymbol{u}}_0$, since $\mathbf{d}\underline{\boldsymbol{u}}_0 = 0$, $\underline{\boldsymbol{u}}_0$ being a constant field on $\mathscr{E}$ (for $\mathscr{O}$ is inertial).

The electric field measured by $\mathscr{O}$ is $\boldsymbol{E} = \boldsymbol{F}(., \vec{\boldsymbol{u}}_0)$ [Eq. (17.7)], so that (18.55) yields

$$\boldsymbol{E} = -\mathbf{d}V - \langle \mathbf{d}V, \vec{\boldsymbol{u}}_0 \rangle \, \underline{\boldsymbol{u}}_0 + c\, \mathbf{d}\mathscr{A}(., \vec{\boldsymbol{u}}_0). \tag{18.56}$$

The first two terms let appear the orthogonal projection of $\mathbf{d}V$ onto $E_{\boldsymbol{u}}$ [cf. Eq. (3.12)]: $\mathbf{d}V + \langle \mathbf{d}V, \vec{\boldsymbol{u}}_0 \rangle \, \underline{\boldsymbol{u}}_0 = \mathbf{d}V \circ \perp_{\boldsymbol{u}}$. Regarding the last term, its components in the inertial coordinates (x^α) associated with $\mathscr{O}$ are $(\mathbf{d}\mathscr{A}(., \vec{\boldsymbol{u}}_0))_\alpha = (\nabla_\alpha \mathscr{A}_\beta - \nabla_\beta \mathscr{A}_\alpha) u_0^\beta = -u_0^\beta \nabla_\beta \mathscr{A}_\alpha$, taking into account $\mathscr{A}_\beta u_0^\beta = 0$ and $\nabla_\alpha u_0^\beta = 0$. Thus we obtain

$$\boldsymbol{E} = -\mathbf{d}V \circ \perp_{\boldsymbol{u}} - c\, \boldsymbol{\nabla}_{\vec{\boldsymbol{u}}_0} \mathscr{A}. \tag{18.57}$$

Let us introduce the notation

$$\boldsymbol{\nabla}_{\perp_{\boldsymbol{u}}} V := \mathbf{d}V \circ \perp_{\boldsymbol{u}}. \tag{18.58}$$

$\boldsymbol{\nabla}_{\perp_{\boldsymbol{u}}}$ is the "purely spatial" gradient operator (with respect to $\mathscr{O}$): it contains only the components of the gradient of V in directions tangent to $\mathscr{O}$'s rest space. While the components of $\mathbf{d}V$ are $(\mathbf{d}V)_\alpha = \partial V / \partial x^\alpha$, those of $\boldsymbol{\nabla}_{\perp_{\boldsymbol{u}}} V$ are

$$(\nabla_{\perp_{\boldsymbol{u}}} V)_\alpha = \left(0, \frac{\partial V}{\partial x^1}, \frac{\partial V}{\partial x^2}, \frac{\partial V}{\partial x^3}\right). \tag{18.59}$$

Denoting the operator $c\, \boldsymbol{\nabla}_{\vec{\boldsymbol{u}}_0}$ by a partial derivative with respect to the coordinate $t := x^0/c$, as in Sect. 18.3.3, Eq. (18.57) becomes

$$\boxed{\boldsymbol{E} = -\boldsymbol{\nabla}_{\perp_{\boldsymbol{u}}} V - \frac{\partial \mathscr{A}}{\partial t}}, \tag{18.60}$$

whose components are

$$E_0 = 0 \quad \text{and} \quad E_i = -\frac{\partial V}{\partial x^i} - \frac{\partial \mathscr{A}_i}{\partial t}. \tag{18.61}$$

In a stationary regime, $\partial/\partial t = 0$ and $\boldsymbol{E} = -\boldsymbol{\nabla}_{\perp_{\boldsymbol{u}}} V$, which justifies the qualifier *electric* given to the potential V.

On its side, the magnetic field $\vec{\boldsymbol{B}}$ measured by $\mathscr{O}$ is obtained by substituting (18.55) for $\boldsymbol{F}$ in (17.8), with, according to (14.79),

$$\star(\mathbf{d}V \wedge \underline{\boldsymbol{u}}_0 + c\, \mathbf{d}\mathscr{A}) = \boldsymbol{\epsilon}(\vec{\boldsymbol{\nabla}} V, \vec{\boldsymbol{u}}_0, ., .) + c\, \star \mathbf{d}\mathscr{A}.$$

Since $\boldsymbol{\epsilon}(\vec{\boldsymbol{\nabla}} V, \vec{\boldsymbol{u}}_0, \vec{\boldsymbol{u}}_0, .) = 0$, we get

$$\underline{\boldsymbol{B}} = \star \mathbf{d}\mathscr{A}(\vec{\boldsymbol{u}}_0, .). \tag{18.62}$$

Now, from (15.70), the right-hand side is nothing but the 1-form metric dual of the curl of $\vec{\mathscr{A}}$. We may thus write

$$\boxed{\vec{\boldsymbol{B}} = \nabla \times_{u_0} \vec{\mathscr{A}}}. \tag{18.63}$$

This formula justifies the qualifier *magnetic* given to the potential $\vec{\mathscr{A}}$.

18.5.3 Gauge Choice

From its very definition, the 1-form $\boldsymbol{A}$ is not unique: for a given electromagnetic field $\boldsymbol{F}$, $\boldsymbol{A}$ is determined up to the gradient of a scalar field: if $\boldsymbol{A}$ fulfils $\mathbf{d}\boldsymbol{A} = \boldsymbol{F}$, then for any scalar field Ψ, we have also $\mathbf{d}\boldsymbol{A}' = \boldsymbol{F}$ with

$$\boxed{\boldsymbol{A}' := \boldsymbol{A} + \mathbf{d}\Psi}. \tag{18.64}$$

This results from the nilpotence of the exterior derivative: $\mathbf{dd}\Psi = 0$. The possibility to choose freely Ψ in (18.64) is named ***gauge freedom***, a specific choice of $\boldsymbol{A}$ being named a ***gauge choice***. Let us stress that different gauge choices lead to the same physical solution, since the latter is entirely described by the field $\boldsymbol{F}$. In particular, $\boldsymbol{A}$ is not a measurable quantity.[11] We may take advantage of the gauge freedom to simplify the only non-trivial Maxwell equation in terms of $\boldsymbol{A}$, namely, Eq. (18.52). Indeed, we can set to zero the divergence of $\vec{\boldsymbol{A}}$ which appears in the second term of this equation: the change of gauge relation (18.64) leads to[12]

$$\nabla \cdot \vec{\boldsymbol{A}}' = \nabla \cdot \vec{\boldsymbol{A}} + \nabla \cdot \vec{\nabla}\Psi = \nabla \cdot \vec{\boldsymbol{A}} + \Box\Psi. \tag{18.65}$$

If $\nabla \cdot \vec{\boldsymbol{A}} \neq 0$, it suffices to solve the scalar d'Alembert equation $\Box\Psi = -\nabla \cdot \vec{\boldsymbol{A}}$ to warrant $\nabla \cdot \vec{\boldsymbol{A}}' = 0$. The gauge choice

$$\boxed{\nabla \cdot \vec{\boldsymbol{A}} = 0} \tag{18.66}$$

is called ***Lorenz gauge***.

Remark 18.13. The gauge name is *Lorenz* (and not *Lorentz*), from the Danish physicist Ludvig Valentin Lorenz (1829–1891). The name *Lorentz*, omnipresent up to here, stands for the Dutch physicist Hendrik A. Lorentz (cf. p. 108), who gave

[11]This is true at least in classical electrodynamics; things are different in the quantum regime, where $\boldsymbol{A}$, or more precisely its integral, is directly involved in the measure of a phenomenon called the *Aharonov–Bohm effect* (cf., e.g. Sect. 12.3.3 of Le Bellac (2006)).

[12]Cf. (18.48) and Remark 15.4 p. 504.

his name to the *Lorentz transformation*, the *Lorentz group*, the *Lorentz factor* and the *Lorentz force*, but not to the gauge. The latter has been introduced in 1867 by L.V. Lorenz (see, e.g. Jackson and Okun (2001)). Many textbooks are incorrect on this point, among which the famous books by Landau and Lifshitz (1975), Feynman (2011) and the first two editions of Jackson's treatise but not the third one (Jackson 1998).

One must stress that the choice of Lorenz gauge does not fully determine the 4-potential $\boldsymbol{A}$. Indeed, Eq. (18.65) shows that if $\boldsymbol{A}$ fulfils Lorenz gauge, any other 4-potential $\boldsymbol{A}'$ related to $\boldsymbol{A}$ by (18.64) with Ψ such that $\Box\Psi = 0$ obeys Lorenz gauge too.

Within Lorenz gauge, the Maxwell equation (18.52) reduces to a ***d'Alembert equation*** for the vector $\vec{A}$:

$$\boxed{\Box\vec{A} = -\varepsilon_0^{-1}\,\vec{j}} \qquad \text{(Lorenz gauge)}. \tag{18.67}$$

18.5.4 Electromagnetic Waves

Expressing the exterior derivative in the relation $\boldsymbol{F} = \mathbf{d}\boldsymbol{A}$ in terms of the covariant derivative [cf. (15.62)] and using the fact that $\Box$ and $\boldsymbol{\nabla}$ commute (this is readily seen in inertial coordinates), we can write $\Box F_{\alpha\beta} = \Box(\nabla_\alpha A_\beta - \nabla_\beta A_\beta) = \nabla_\alpha(\Box A_\beta) - \nabla_\beta(\Box A_\alpha)$, i.e., invoking (18.67),

$$\Box\boldsymbol{F} = -\varepsilon_0^{-1}\,\mathbf{d}\underline{\boldsymbol{j}}. \tag{18.68}$$

In other words, the electromagnetic field tensor obeys a d'Alembert equation with the exterior derivative of $\underline{\boldsymbol{j}}$ as a source.

Remark 18.14. We have used the Lorenz gauge, in the form (18.67), to get (18.68), but the result is independent of any gauge choice, since it regards only the physical fields $\boldsymbol{F}$ and $\vec{j}$. It is by the way easy to derive (18.68) directly from the Maxwell equations $\mathbf{d}\boldsymbol{F} = 0$ and $\boldsymbol{\nabla}\cdot\boldsymbol{F}^\sharp = \varepsilon_0^{-1}\,\vec{j}$ [Eq. (18.22)].

In vacuum, $\vec{j} = 0$, and (18.68) reduces to

$$\boxed{\Box\boldsymbol{F} = 0} \qquad \text{(vacuum)}. \tag{18.69}$$

This is a ***wave equation*** for $\boldsymbol{F}$. For this reason, electromagnetic fields in regions free of electric charges are called ***electromagnetic waves***. The velocity of propagation of these waves with respect to an inertial observer is the velocity that appears in expression (18.51) of the d'Alembertian operator: it is the constant c, velocity of light.

Example 18.1. In the case of a field $\boldsymbol{F}$ that is constant in 2-planes $\{t = \text{const}, x = \text{const}\}$ of a given inertial observer $\mathscr{O}$, the general solution of (18.69) is

$$\boldsymbol{F}(ct, x, y, z) = \boldsymbol{F}_1(x - ct) + \boldsymbol{F}_2(x + ct), \tag{18.70}$$

where $\boldsymbol{F}_1(x-ct)$ stands for a field of 2-forms on $\mathscr{E}$ whose components $(F_1)_{\alpha\beta}$ with respect to $\mathscr{O}$ depend only on the variable $x-ct$ (idem for $\boldsymbol{F}_2(x+ct)$). This solution is called ***plane wave***. If $\boldsymbol{F}_2 = 0$, the wave propagates at the velocity $\mathrm{d}x/\mathrm{d}t = c$ in the direction of increasing x, whereas if $\boldsymbol{F}_1 = 0$, it propagates at the velocity c in the direction of decreasing x.

18.5.5 Solution for the 4-Potential in Lorenz Gauge

Within Lorenz gauge, the problem of solving Maxwell equations is reduced to finding a solution to the d'Alembert equation (18.67) for the 4-potential $\boldsymbol{A}$. The standard technique consists in introducing some ***Green function*** of the d'Alembertian operator, i.e. some function $G : \mathscr{E} \times \mathscr{E} \to \mathbb{R}$ such that for any point $N \in \mathscr{E}$, the scalar field $G(., N) : \mathscr{E} \to \mathbb{R}$, $M \mapsto G(M, N)$ is a solution of d'Alembert equation having as source the Dirac distribution on $(\mathscr{E}, \boldsymbol{g})$ centred on N (cf. Sect. 18.2.1):

$$\Box G(., N) = \delta_N. \tag{18.71}$$

The advantage of Green functions is that the general solution of the scalar d'Alembert equation with a given source S,

$$\Box \Phi = S, \tag{18.72}$$

is expressed as

$$\Phi(M) = \Phi_0(M) + \int_{\mathscr{E}} S(N)\, G(M, N)\, \mathrm{d}U, \tag{18.73}$$

where Φ_0 is a solution of the wave equation: $\Box\Phi_0 = 0$. In (18.73), N stands for the generic point in the integration over $\mathscr{E}$, and $\mathrm{d}U$ is the 4-volume element around N [cf. (16.2) and (16.4)]. The general solution (18.73) stems readily from the linearity of the operator $\Box$ and from the property (18.3) of Dirac distribution.

For a given operator, the Green function is not unique. The difference between two Green functions is a solution of the homogeneous equation (i.e. the equation with $S = 0$). In the case of the d'Alembertian operator, two standard Green functions are

$$\boxed{G_{\mathrm{ret}}(M, N) = -\frac{1}{2\pi}\, \delta\left(\overrightarrow{NM} \cdot \overrightarrow{NM}\right)\, \Upsilon(-\vec{\boldsymbol{u}}_0 \cdot \overrightarrow{NM})} \tag{18.74a}$$

$$\boxed{G_{\mathrm{adv}}(M, N) = -\frac{1}{2\pi}\, \delta\left(\overrightarrow{NM} \cdot \overrightarrow{NM}\right)\, \Upsilon(\vec{\boldsymbol{u}}_0 \cdot \overrightarrow{NM})}, \tag{18.74b}$$

where δ is the Dirac distribution over $\mathbb{R}$, Υ the ***Heaviside step function***: $\Upsilon : \mathbb{R} \to \mathbb{R}$, $x \mapsto 0$ if $x < 0$ and 1 if $x \geq 0$, and $\vec{u}_0$ is the 4-velocity of some inertial observer $\mathscr{O}$. The Green functions (18.74) are independent of the choice of that observer, the Heaviside step function involving only the sign of the scalar product $\vec{u}_0 \cdot \overrightarrow{NM}$: since the function δ is nonzero only for $\overrightarrow{NM} \cdot \overrightarrow{NM} = 0$, i.e. for $\overrightarrow{NM}$ null, it is easy to see that for any other observer of 4-velocity $\vec{u}'_0$, $\vec{u}'_0 \cdot \overrightarrow{NM}$ has the same sign as $\vec{u}_0 \cdot \overrightarrow{NM}$. For a fixed N, the Green function $G_{\mathrm{ret}}(., N)$ is zero everywhere except on the future light cone of N, where it shows a singularity of the Dirac distribution kind; G_{ret} is called the ***retarded Green function***. Conversely, the Green function $G_{\mathrm{adv}}(., N)$ vanishes everywhere except on the past light cone of N; it is called the ***advanced Green function***. The retarded Green function is *causal*: the source at N, $S(N)$, will contribute to $\Phi(M)$ via the integral (18.73) only if $-\vec{u}_0 \cdot \overrightarrow{NM} \geq 0$, i.e. if M is located on the *future* light cone of N.

Given a system of inertial coordinates $x^\alpha = (ct, x^1, x^2, x^3)$ on $\mathscr{E}$, expressions (18.74) can be recast as

$$G_{\mathrm{ret}}(M, N) = -\frac{1}{4\pi r_{NM}}\, \delta(ct_M - ct_N - r_{NM}) \tag{18.75a}$$

$$G_{\mathrm{adv}}(M, N) = -\frac{1}{4\pi r_{NM}}\, \delta(ct_M - ct_N + r_{NM}), \tag{18.75b}$$

where $r_{NM}^2 := \sum_{i=1}^3 (x_M^i - x_N^i)^2$. We shall not demonstrate here expression (18.74) or (18.75) for the Green functions of the d'Alembertian (cf., e.g. Jackson (1998)).

Let us go back to the problem of solving d'Alembert equation (18.67) for the 4-potential $\vec{A}$. If a system of inertial coordinates (x^α) is given on $\mathscr{E}$, this equation is reduced to four *scalar* d'Alembert equations [i.e. of the type (18.72)]: one for each component A^α of $\vec{A}$: $\Box A^\alpha = -\varepsilon_0^{-1} j^\alpha$, $\alpha \in \{0, 1, 2, 3\}$.

Remark 18.15. This would not be true if the coordinates (x^α) were not inertial. The components of the operator $\Box = \nabla_\mu \nabla^\mu$ would then let appear the Christoffel symbols, which would make the four equations a coupled system (cf. the examples of Sect. 15.4.4 based on spherical coordinates).

We may then write the solution in the form (18.73) with $\Phi = A^\alpha$ and $S = -\varepsilon_0^{-1} j^\alpha$. Using the retarded Green function (18.74a), there comes

$$A^\alpha(M) = A_0^\alpha(M) - \frac{1}{\varepsilon_0} \int_{\mathscr{E}} j^\alpha(N)\, G_{\mathrm{ret}}(M, N)\, \mathrm{d}U, \tag{18.76}$$

where A_0^α stands for the components of a general solution of the homogeneous d'Alembert equation (wave equation): $\Box \vec{A}_0 = 0$. The retarded Green function has been chosen to get a causal solution, but mathematically speaking, the solution with the advanced Green function would have been as much valid. The part $\vec{A}_0$ of the solution allows one to impose physical properties to the problem under study (initial or boundary conditions). In general, the system is assumed to be isolated, with no incoming wave: $\vec{A}_0 = 0$. We shall limit ourselves to this case. Making $G_{\rm ret}$ explicit via (18.74a), we obtain then

$$\boxed{A^\alpha(M) = \frac{1}{2\pi\varepsilon_0}\int_{\mathscr{E}} j^\alpha(N)\,\delta\left(\overrightarrow{NM}\cdot\overrightarrow{NM}\right)\,\Upsilon(-\vec{u}_0\cdot\overrightarrow{NM})\,\mathrm{d}U}. \tag{18.77}$$

Denoting by (x^α) the coordinates of M in the employed coordinate system, by (x'^α) those of N, and using for $\vec{u}_0$ the first vector of the corresponding coordinate basis, we get

$$A^\alpha(x^0,x^1,x^2,x^3) = \frac{1}{2\pi\varepsilon_0}\int_{\mathscr{E}} j^\alpha(x'^0,x'^1,x'^2,x'^3)\,\delta\left(\eta_{\mu\nu}(x^\mu - x'^\mu)(x^\nu - x'^\nu)\right)$$
$$\times\Upsilon(x^0 - x'^0)\,\mathrm{d}x'^0\mathrm{d}x'^1\mathrm{d}x'^2\mathrm{d}x'^3. \tag{18.78}$$

In order for (18.77) to be a solution of Maxwell equations, there remains to check that it obeys Lorenz gauge (18.66); if not, the d'Alembert equation (18.67) that it fulfils would not be equivalent to the Maxwell equation (18.52). We shall not check it here, except for the Liénard–Wiechert solution (cf. Remark 18.18 below), but this is actually the case (see, e.g. Barut (1964), p. 162).

By virtue of the relations $A^\alpha = (V, c\mathscr{A}^1, c\mathscr{A}^2, c\mathscr{A}^3)$ [Eq. (18.54)] and $j^\alpha = (\rho, J^1/c, J^2/c, J^3/c)$ [Eq. (18.11)], the solution (18.76) (with $A_0^\alpha = 0$) leads to the following expression of the electric and magnetic potentials:

$$V = \frac{1}{4\pi\varepsilon_0}\int_{\mathscr{E}}\frac{\rho(x'^0,x'^1,x'^2,x'^3)}{r(x^i,x'^i)}\,\delta\left(x^0 - x'^0 - r(x^i,x'^i)\right)\,\mathrm{d}x'^0\mathrm{d}x'^1\mathrm{d}x'^2\mathrm{d}x'^3$$

$$\mathscr{A}^i = \frac{\mu_0}{4\pi}\int_{\mathscr{E}}\frac{J^i(x'^0,x'^1,x'^2,x'^3)}{r(x^i,x'^i)}\,\delta\left(x^0 - x'^0 - r(x^i,x'^i)\right)\,\mathrm{d}x'^0\mathrm{d}x'^1\mathrm{d}x'^2\mathrm{d}x'^3$$

where V and $\mathscr{A}^i$ are evaluated at the point of coordinates (x^0,x^1,x^2,x^3), $r(x^i,x'^i) := \sqrt{\sum_{i=1}^3 (x^i - x'^i)^2}$ and use has been made of the form (18.75a) of $G_{\rm ret}$ rather than (18.74a) as in (18.78). Integrating on x'^0, there comes, given the definition of Dirac distribution,

$$\boxed{V(ct,x^1,x^2,x^3) = \frac{1}{4\pi\varepsilon_0}\int_{\mathbb{R}^3}\frac{\rho\left(ct - r(x^i,x'^i), x'^1, x'^2, x'^3\right)}{r(x^i,x'^i)}\,\mathrm{d}x'^1\mathrm{d}x'^2\mathrm{d}x'^3} \tag{18.79}$$

$$\boxed{\mathscr{A}^i(ct,x^1,x^2,x^3) = \frac{\mu_0}{4\pi}\int_{\mathbb{R}^3}\frac{J^i\left(ct - r(x^i,x'^i), x'^1, x'^2, x'^3\right)}{r(x^i,x'^i)}\,\mathrm{d}x'^1\mathrm{d}x'^2\mathrm{d}x'^3}. \tag{18.80}$$

The fields V and $\vec{\mathscr{A}}$ given by the above formulas are called ***retarded potentials***.

Remark 18.16. The integrals (18.79)–(18.80) are integrals on the past light cone of the point M where the potential is evaluated, each point on the cone being specified by the coordinates (x'^1, x'^2, x'^3) of its orthogonal projection onto the hyperplane $t = 0$.

18.6 Field Created by a Moving Charge

Let us apply the results of the above section to the computation of the electromagnetic field created by a particle $\mathscr{P}$ of charge q and arbitrary worldline $\mathscr{L}$ (in particular, $\mathscr{P}$ can be accelerated).

18.6.1 Liénard–Wiechert 4-Potential

The electric 4-current corresponding to particle $\mathscr{P}$ is given by formula (18.5) with $N = 1$:

$$\forall M \in \mathscr{E}, \quad \vec{j}(M) = q \int_{-\infty}^{+\infty} \delta_{X(\tau)}(M)\, \vec{u}(\tau)\, c\, \mathrm{d}\tau, \tag{18.81}$$

where τ is $\mathscr{P}$'s proper time, $\vec{u}(\tau)$ its 4-velocity and $X(\tau) \in \mathscr{L}$ its position on the worldline at the instant τ. Let us substitute this value for $\vec{j}$ in the expression (18.77) of the 4-potential in Lorenz gauge:

$$A^\alpha(M) = \frac{q}{2\pi\varepsilon_0} \int_{\mathscr{E}} \int_{-\infty}^{+\infty} \delta_{X(\tau)}(N)\, u^\alpha(\tau)\, \delta\left(\overrightarrow{NM} \cdot \overrightarrow{NM}\right) \Upsilon(-\vec{u}_0 \cdot \overrightarrow{NM})\, c\, \mathrm{d}\tau\, \mathrm{d}U.$$

Performing the integration over $\mathscr{E}$ (generic point N and volume element $\mathrm{d}U$) yields, taking into account the term $\delta_{X(\tau)}(N)$,

$$A^\alpha(M) = \frac{q}{2\pi\varepsilon_0} \int_{-\infty}^{+\infty} u^\alpha(\tau)\, \delta\left(\overrightarrow{X(\tau)M} \cdot \overrightarrow{X(\tau)M}\right) \Upsilon(-\vec{u}_0 \cdot \overrightarrow{X(\tau)M})\, c\, \mathrm{d}\tau.$$

To evaluate this integral, let us appeal to a well-known property of Dirac distribution on $\mathbb{R}$: for any couple of functions (f, g), such that g has m zeros, $(\tau_a)_{1\le a\le m}$, and $g'(\tau_a) \neq 0$ for all of them,

$$\int_{-\infty}^{\infty} f(\tau)\, \delta(g(\tau))\, \mathrm{d}\tau = \sum_{a=1}^{m} \frac{f(\tau_a)}{|g'(\tau_a)|}. \tag{18.82}$$

In the present case, $f(\tau) := c\, u^\alpha(\tau)\, \Upsilon(-\vec{u}_0 \cdot \overrightarrow{X(\tau)M})$ and $g(\tau) := \overrightarrow{X(\tau)M} \cdot \overrightarrow{X(\tau)M}$. M being fixed, the function $g(\tau)$ has only two zeros: the proper times τ_P and τ_Q of the intersections P and Q of the light cone of M, $\mathscr{I}(M)$, with the

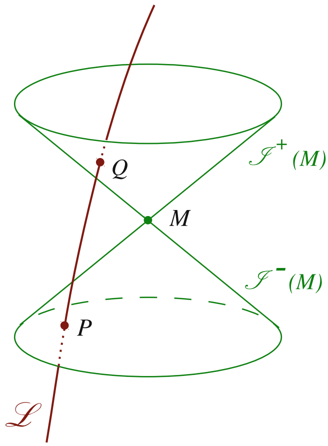


Fig. 18.4 Intersections P and Q of the light cone of M with the worldline $\mathscr{L}$ of the charged particle

worldline $\mathscr{L}$. $\mathscr{L}$ being a timelike curve, it is easy to see that the intersection of $\mathscr{I}(M)$ with $\mathscr{L}$ is formed by exactly two points (except if $M \in \mathscr{L}$): $P \in \mathscr{I}^-(M)$ (past light cone) and $Q \in \mathscr{I}^+(M)$ (future light cone) (cf. Fig. 18.4). The derivative of g is

$$g'(\tau) = 2\overrightarrow{X(\tau)M} \cdot \frac{\mathrm{d}}{\mathrm{d}\tau}\overrightarrow{X(\tau)M} = -2c\,\overrightarrow{X(\tau)M} \cdot \vec{\boldsymbol{u}}(\tau),$$

where we have used the fact that M is fixed to let appear $\mathscr{P}$'s 4-velocity according to (2.12). Formula (18.82) leads then to

$$A^\alpha(M) = \frac{q}{4\pi\varepsilon_0}\left[\frac{u^\alpha(\tau_P)\Upsilon(-\vec{\boldsymbol{u}}_0 \cdot \overrightarrow{PM})}{|\vec{\boldsymbol{u}}(\tau_P) \cdot \overrightarrow{PM}|} + \frac{u^\alpha(\tau_Q)\Upsilon(-\vec{\boldsymbol{u}}_0 \cdot \overrightarrow{QM})}{|\vec{\boldsymbol{u}}(\tau_Q) \cdot \overrightarrow{QM}|}\right]. \tag{18.83}$$

Now, Q being in the future of M, $\Upsilon(-\vec{\boldsymbol{u}}_0 \cdot \overrightarrow{QM}) = 0$, whereas for P, $\Upsilon(-\vec{\boldsymbol{u}}_0 \cdot \overrightarrow{PM}) = 1$. Besides, $|\vec{\boldsymbol{u}}(\tau_P) \cdot \overrightarrow{PM}| = -\vec{\boldsymbol{u}}(\tau_P) \cdot \overrightarrow{PM}$. We obtain thus a very simple formula for the 4-potential created at M by the charge $\mathscr{P}$[13]:

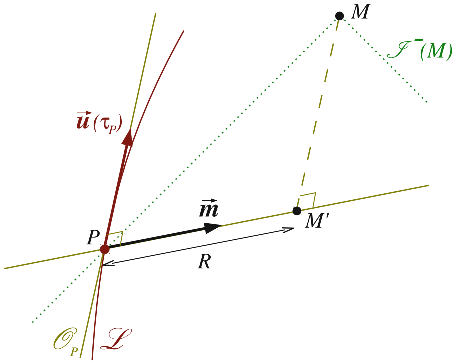
$$\boxed{\boldsymbol{A}(M) = -\frac{q}{4\pi\varepsilon_0}\frac{\underline{\boldsymbol{u}}(\tau_P)}{\vec{\boldsymbol{u}}(\tau_P) \cdot \overrightarrow{PM}}} \quad \text{with} \quad \{P\} = \mathscr{L} \cap \mathscr{I}^-(M). \tag{18.84}$$

The electromagnetic 4-potential given by (18.84) is called ***Liénard–Wiechert 4-potential***. The denominator in (18.84),

$$R := -\vec{\boldsymbol{u}}(\tau_P) \cdot \overrightarrow{PM}, \tag{18.85}$$

[13]Note that we have taken the metric dual to get the 1-form $\boldsymbol{A}$ instead of the vector $\vec{\boldsymbol{A}}$.

Fig. 18.5 Interpretation of $R = -\vec{u}(\tau_P)\cdot\overrightarrow{PM}$ as the distance between P and the point M' that, for observer $\mathscr{O}_P$, is (i) simultaneous to P and (ii) at the same spatial position as M



can be interpreted as follows (cf. Fig. 18.5). Since M lies in the future of P, we have $R \geq 0$. Let us then consider the inertial observer $\mathscr{O}_P$ whose worldline is tangent to that of $\mathscr{P}$ at P. His 4-velocity is then $\vec{u}(\tau_P)$, and R appears as the spatial distance between P and the point M' that is simultaneous to P for $\mathscr{O}_P$ and has the same spatial coordinates as M with respect to $\mathscr{O}_P$. In particular, we can write

$$\overrightarrow{PM} = R\,[\vec{u}(\tau_P) + \vec{m}]\,, \quad \text{with} \quad \vec{u}(\tau_P)\cdot\vec{m} = 0 \quad \text{and} \quad \vec{m}\cdot\vec{m} = 1, \qquad (18.86)$$

the last condition ensuring $\overrightarrow{PM}\cdot\overrightarrow{PM} = 0$, since $\overrightarrow{PM}\cdot\overrightarrow{PM} = R(-1+\vec{m}\cdot\vec{m})$.

Let us now introduce an arbitrary inertial observer $\mathscr{O}$ (proper time t, 4-velocity $\vec{u}_0$, coordinates $(x^\alpha) = (ct, x^i)$) and express from (18.84) the electric and magnetic potentials, V and $\vec{\mathscr{A}}$, relative to $\mathscr{O}$. To this aim, we perform the orthogonal decomposition of $\overrightarrow{PM}$ with respect to $\mathscr{O}$ and not with respect to $\mathscr{O}_P$ as in (18.86):

$$\overrightarrow{PM} =: r\,(\vec{u}_0 + \vec{n})\,, \quad \text{with} \quad \vec{u}\cdot\vec{n} = 0 \quad \text{and} \quad \vec{n}\cdot\vec{n} = 1. \qquad (18.87)$$

As for $\vec{m}$ in (18.86), the unit character of $\vec{n}$ results from the fact that $\overrightarrow{PM}$ is null. Similarly, r is positive and can be interpreted as the distance between the events P' and M in $\mathscr{O}$'s rest space at time t, $\mathscr{E}_{u_0}(t)$, P' being the position that particle $\mathscr{P}$ would have at time t if it was at rest with respect to $\mathscr{O}$ (cf. Fig. 18.6). In other words, the coordinates (x^i) of P' are identical to the coordinates (x^i) of P. We have $\overrightarrow{PP'} = r\,\vec{u}_0$, hence

$$r = c(t - t_P) \iff t_P = t - \frac{r}{c}. \qquad (18.88)$$

Let us express the 4-velocity of $\mathscr{P}$ in terms of its velocity $\vec{V}$ relative to $\mathscr{O}$ via (4.31): $\vec{u} = \Gamma(\vec{u}_0 + c^{-1}\vec{V})$, where $\Gamma = (1-\vec{V}\cdot\vec{V}/c^2)^{-1/2}$. Equation (18.85) becomes then

$$R = r\,\Gamma(\tau_P)\left[1 - \frac{\vec{n}\cdot\vec{V}(\tau_P)}{c}\right], \qquad (18.89)$$

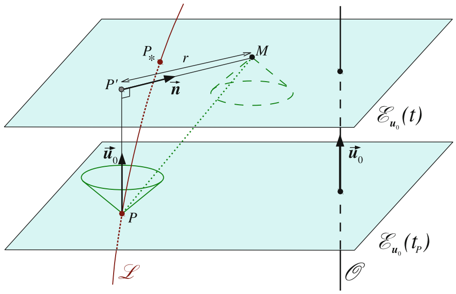


Fig. 18.6 Orthogonal decomposition of the null vector $\overrightarrow{PM}$ with respect to the inertial observer $\mathscr{O}$: $\overrightarrow{PM} = r\,(\vec{u}_0 + \vec{n})$ with $\vec{n}$ being a unit vector and $r = c(t - t_P)$, t and t_P being the time coordinates of, respectively, M and P relative to $\mathscr{O}$

so that (18.84) can be written as

$$\boldsymbol{A}(M) = \frac{1}{4\pi\varepsilon_0}\,\frac{q}{r\left[1 - \frac{\vec{n}\cdot\vec{V}(\tau_P)}{c}\right]}\left[\underline{\boldsymbol{u}}_0 + \frac{1}{c}\underline{\boldsymbol{V}}(\tau_P)\right]. \tag{18.90}$$

Comparing with (18.53), we obtain immediately

$$\boxed{V(M) = \frac{1}{4\pi\varepsilon_0}\,\frac{q}{r\left[1 - \frac{\vec{n}\cdot\vec{V}(\tau_P)}{c}\right]}}, \tag{18.91}$$

$$\boxed{\vec{\mathscr{A}}(M) = \frac{\mu_0}{4\pi}\,\frac{q}{r\left[1 - \frac{\vec{n}\cdot\vec{V}(\tau_P)}{c}\right]}\,\vec{V}(\tau_P)}. \tag{18.92}$$

The potentials V and $\vec{\mathscr{A}}$ given by the above formulas are called ***Liénard–Wiechert potentials***.

Remark 18.17. We have written $\vec{V}$ as a function of τ_P, but it can as well be considered as a function of the time coordinate t of the event P with respect to $\mathscr{O}$, i.e. t_P. The latter is related to the coordinate t of M by (18.88) *(retarded time).*

Historical note: *Formulas (18.91) and (18.92) have been derived in 1898 by Alfred-Marie Liénard*[14] *(1898). They have been reobtained independently by Emil Wiechert*[15] *in 1900 (Wiechert 1900). The four-dimensional version, i.e. expression (18.84) for the 4-potential* $\boldsymbol{A}$*, has been given by Arnold Sommerfeld (p. 27) in 1910 (Sommerfeld 1910b).*

[14]**Alfred-Marie Liénard** (1869–1958): French physicist and engineer; director of École des Mines in Paris from 1929 to 1936.

[15]**Emil Wiechert** (1861–1928): German geophysicist; he knew quite well Hilbert (p. 361), Minkowski (p. 26) and Sommerfeld (p. 27), because the four of them made their studies at the University of Königsberg and later on were professors at the University of Göttingen.

18.6.2 Electromagnetic Field

The electromagnetic field $\boldsymbol{F}$ is obtained as the exterior derivative of $\boldsymbol{A}$. To evaluate it, let us write from (18.84) and (18.85) the components of $\boldsymbol{A}$ in a system of inertial coordinates (x^α):

$$A_\alpha = \frac{q}{4\pi\varepsilon_0}\frac{u_\alpha(\tau_P)}{R}. \tag{18.93}$$

We have then

$$\frac{\partial A_\alpha}{\partial x^\beta} = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{R}\frac{\mathrm{d}u_\alpha}{\mathrm{d}\tau}\frac{\partial\tau_P}{\partial x^\beta} - \frac{u_\alpha}{R^2}\frac{\partial R}{\partial x^\beta}\right). \tag{18.94}$$

Now, up to a c factor, $\mathrm{d}u_\alpha/\mathrm{d}\tau$ is nothing but the component a_α of the 4-acceleration $\vec{\boldsymbol{a}}$ of particle $\mathscr{P}$. Regarding $\partial\tau_P/\partial x^\beta$, it is evaluated from $\overrightarrow{PM}\cdot\overrightarrow{PM} = 0$. Indeed, this relation can be written as

$$\eta_{\mu\nu}(x^\mu - X^\mu(\tau_P))(x^\nu - X^\nu(\tau_P)) = 0, \tag{18.95}$$

where the $X^\mu(\tau)$ are the coordinates of $\mathscr{P}$ along its worldline. We have $\partial X^\mu/\partial x^\beta(\tau_P) = \mathrm{d}X^\mu/\mathrm{d}\tau \times \partial\tau_P/\partial x^\beta = cu^\mu\,\partial\tau_P/\partial x^\beta$, so that by deriving (18.95) with respect to x^β, we get

$$\eta_{\mu\nu}(x^\mu - X^\mu(\tau_P))\left(\delta^\nu{}_\beta - cu^\nu\frac{\partial\tau_P}{\partial x^\beta}\right) = 0, \quad \text{hence} \quad \frac{\partial\tau_P}{\partial x^\beta} = -\frac{1}{cR}(PM)_\beta. \tag{18.96}$$

Finally, we compute from (18.85),

$$\begin{aligned}
\frac{\partial R}{\partial x^\beta} &= -\frac{\partial}{\partial x^\beta}\left\{u_\mu(\tau_P)[x^\mu - X^\mu(\tau_P)]\right\} \\
&= -\frac{\mathrm{d}u_\mu}{\mathrm{d}\tau}\frac{\partial\tau_P}{\partial x^\beta}[x^\mu - X^\mu(\tau_P)] - u_\mu\left[\delta^\mu{}_\beta - cu^\mu(\tau_P)\frac{\partial\tau_P}{\partial x^\beta}\right] \\
&= -u_\beta(\tau_P) - c\left[1 + a_\mu(\tau_P)(PM)^\mu\right]\frac{\partial\tau_P}{\partial x^\beta}.
\end{aligned} \tag{18.97}$$

Inserting (18.96) and (18.97) in (18.94), we get

$$\frac{\partial A_\alpha}{\partial x^\beta} = \frac{q}{4\pi\varepsilon_0 R^2}\left\{u_\alpha u_\beta - \left[a_\alpha + \frac{1 + a_\mu(PM)^\mu}{R}u_\alpha\right](PM)_\beta\right\}. \tag{18.98}$$

Note that the explicit dependence of u_α and a_α in τ_P has been omitted.

Remark 18.18. We may use this expression of $\partial A_\alpha/\partial x^\beta$ to check that the Liénard–Wiechert 4-potential (18.84) does obey Lorenz gauge. Since the coordinates (x^α) are inertial, we have indeed

$$\nabla\cdot\vec{\boldsymbol{A}} = \eta^{\alpha\beta}\frac{\partial A_\alpha}{\partial x^\beta} \propto \underbrace{u_\alpha u^\alpha}_{-1} - a_\alpha(PM)^\alpha - \frac{1 + a_\mu(PM)^\mu}{R}\underbrace{u_\alpha(PM)^\alpha}_{-R} = 0.$$

We are now in position to compute the electromagnetic field via (18.45); the term $u_\alpha u_\beta$ disappears in the antisymmetrization, leaving only

$$F_{\alpha\beta} = \frac{q}{4\pi\varepsilon_0 R^2}\left\{\left[a_\alpha + \frac{1 + a_\mu (PM)^\mu}{R}\, u_\alpha\right](PM)_\beta \right.$$
$$\left. - \left[a_\beta + \frac{1 + a_\mu (PM)^\mu}{R}\, u_\beta\right](PM)_\alpha\right\}, \qquad (18.99)$$

i.e., given the definition (14.43) of the exterior product of two 1-forms:

$$\boxed{\boldsymbol{F}(M) = \frac{q}{4\pi\varepsilon_0 R^2}\left[\underline{\boldsymbol{a}}(\tau_P) + \frac{1 + \vec{\boldsymbol{a}}(\tau_P)\cdot\overrightarrow{PM}}{R}\,\underline{\boldsymbol{u}}(\tau_P)\right] \wedge \underline{\boldsymbol{PM}}}. \qquad (18.100)$$

This formula gives the electromagnetic field created at a point $M \in \mathscr{E}$ by a particle of charge q following an arbitrary worldline. We note that $\boldsymbol{F}(M)$ depends only from the characteristics of the particle (4-velocity $\vec{\boldsymbol{u}}$ and 4-acceleration $\vec{\boldsymbol{a}}$) at the event P, intersection of the past light cone of M with the particle's worldline. P is thus a function of M, as well as the quantities τ_P and $R = -\vec{\boldsymbol{u}}(\tau_P)\cdot\overrightarrow{PM}$.

Given the expression (14.79) of the Hodge dual of an exterior product, we deduce readily from (18.100) the value of $\star\boldsymbol{F}$:

$$\star\boldsymbol{F}(M) = \frac{q}{4\pi\varepsilon_0 R^2}\,\boldsymbol{\epsilon}\left(\vec{\boldsymbol{a}}(\tau_P) + \frac{1 + \vec{\boldsymbol{a}}(\tau_P)\cdot\overrightarrow{PM}}{R}\,\vec{\boldsymbol{u}}(\tau_P),\ \overrightarrow{PM}, \ldots\right). \qquad (18.101)$$

The structure of the electromagnetic field created by a moving charge is remarkable: the 2-form $\boldsymbol{F}$ is the exterior product of two 1-forms: (18.100) shows that $\boldsymbol{F} = \boldsymbol{p}\wedge\boldsymbol{q}$ with $\boldsymbol{p} := q/(4\pi\varepsilon_0 R^2)[\underline{\boldsymbol{a}} + (1 + \vec{\boldsymbol{a}}\cdot\overrightarrow{PM})/R\ \underline{\boldsymbol{u}}]$ and $\boldsymbol{q} := \underline{\boldsymbol{PM}}$. An immediate consequence is the vanishing of the invariant I_2 (cf. Sect. 17.3.2). We have indeed $F^{\mu\nu} = p^\mu q^\nu - p^\nu q^\mu$ and, by (14.79), $\star F_{\mu\nu} = \epsilon_{\rho\sigma\mu\nu}\, p^\rho q^\sigma$, so that the definition (17.36) of I_2 leads to

$$I_2 = \frac{1}{4}\star F_{\mu\nu}F^{\mu\nu} = \frac{1}{4}\epsilon_{\rho\sigma\mu\nu}\, p^\rho q^\sigma (p^\mu q^\nu - p^\nu q^\mu) = \frac{1}{2}\epsilon_{\rho\sigma\mu\nu}\, p^\rho q^\sigma p^\mu q^\nu = 0.$$

We have thus

$$\boxed{I_2 = 0}. \qquad (18.102)$$

On its side, the invariant I_1 is

$$I_1 = \frac{1}{2}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}(p_\mu q_\nu - p_\nu q_\mu)(p^\mu q^\nu - p^\nu q^\mu) = p_\mu p^\mu \underbrace{q_\nu q^\nu}_{0} - (p_\mu q^\mu)^2 = -(p_\mu q^\mu)^2.$$

Now, given the values of $\boldsymbol{p}$ and $\boldsymbol{q}$ and the definition (18.85) of R, $p_\mu q^\mu = -q/(4\pi\varepsilon_0 R^2)$. We have thus

$$\boxed{I_1 = -\frac{q^2}{(4\pi\varepsilon_0)^2 R^4}}. \tag{18.103}$$

In addition, a remarkable property of the dual field, which we read directly on (18.101), is to be *transverse*, in the sense that

$$\star\boldsymbol{F}(\overrightarrow{PM}, .) = 0. \tag{18.104}$$

18.6.3 Electric and Magnetic Fields

The electric field $\boldsymbol{E}$ measured by an inertial observer $\mathscr{O}$, of 4-velocity $\vec{\boldsymbol{u}}_0$, is deduced from (18.100) via $\boldsymbol{E} = \boldsymbol{F}(., \vec{\boldsymbol{u}}_0)$ [Eq. (17.7)]; hence

$$\begin{aligned}\boldsymbol{E} = \frac{q}{4\pi\varepsilon_0 R^2}\Bigg[&(\overrightarrow{PM}\cdot\vec{\boldsymbol{u}}_0)\left(\underline{\boldsymbol{a}} + \frac{1+\vec{\boldsymbol{a}}\cdot\overrightarrow{PM}}{R}\,\underline{\boldsymbol{u}}\right)\\ &-\left(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{u}}_0 + \frac{1+\vec{\boldsymbol{a}}\cdot\overrightarrow{PM}}{R}\vec{\boldsymbol{u}}\cdot\vec{\boldsymbol{u}}_0\right)\underline{\boldsymbol{PM}}\Bigg].\end{aligned} \tag{18.105}$$

Now, using the same notations as in the end of Sect. 18.6.1, $\overrightarrow{PM}\cdot\vec{\boldsymbol{u}}_0 = -r$ [Eq. (18.87)], $\vec{\boldsymbol{u}}\cdot\vec{\boldsymbol{u}}_0 = -\Gamma$ and R is related to r by (18.89). Besides, the 4-acceleration $\vec{\boldsymbol{a}}$ of particle $\mathscr{P}$ is expressed in terms of its acceleration $\vec{\boldsymbol{\gamma}}$ and its velocity $\vec{\boldsymbol{V}}$, both relative to $\mathscr{O}$, according to (4.63):

$$\vec{\boldsymbol{a}} = \frac{\Gamma^2}{c^2}\left[\vec{\boldsymbol{\gamma}} + \frac{\Gamma^2}{c^2}(\vec{\boldsymbol{\gamma}}\cdot\vec{\boldsymbol{V}})\left(\vec{\boldsymbol{V}} + c\,\vec{\boldsymbol{u}}_0\right)\right].$$

Inserting all these relations in the above expression of $\boldsymbol{E}$, there comes, after simplification,[16]

$$\boxed{\vec{\boldsymbol{E}} = \frac{q}{4\pi\varepsilon_0\left(1-\frac{\vec{\boldsymbol{n}}\cdot\vec{\boldsymbol{V}}}{c}\right)^3 r}\left\{\frac{1}{\Gamma^2 r}\left(\vec{\boldsymbol{n}} - \frac{\vec{\boldsymbol{V}}}{c}\right) + \frac{1}{c^2}\,\vec{\boldsymbol{n}}\times_{u_0}\left[\left(\vec{\boldsymbol{n}} - \frac{\vec{\boldsymbol{V}}}{c}\right)\times_{u_0}\vec{\boldsymbol{\gamma}}\right]\right\}}. \tag{18.106}$$

[16]Notably via the identity $\vec{\boldsymbol{n}}\times_{u_0}[(\vec{\boldsymbol{n}} - \vec{\boldsymbol{V}}/c)\times_{u_0}\vec{\boldsymbol{\gamma}}] = (\vec{\boldsymbol{n}}\cdot\vec{\boldsymbol{\gamma}})(\vec{\boldsymbol{n}} - \vec{\boldsymbol{V}}/c) - (1 - \vec{\boldsymbol{n}}\cdot\vec{\boldsymbol{V}}/c)\vec{\boldsymbol{\gamma}}$.

In this formula, all the quantities relative to particle $\mathscr{P}$, namely, the velocity $\vec{\boldsymbol{V}}$, the acceleration $\vec{\boldsymbol{\gamma}}$ and the Lorentz factor Γ, are to be taken at the proper time τ_P, or equivalently, at the retarded time t_P given by (18.88). More precisely, in terms of the coordinates associated with $\mathscr{O}$, if (ct, x^1, x^2, x^3) are the coordinates of the point M where $\vec{\boldsymbol{E}}$ is evaluated and $(ct_P, x^1_P, x^2_P, x^3_P)$ are those of P, then

$$r = \sqrt{\sum_{i=1}^{3}(x^i - x^i_P)^2}, \quad n^i = \frac{x^i - x^i_P}{r} \quad \text{and} \quad t_P = t - \frac{r}{c}. \tag{18.107}$$

The magnetic field measured by observer $\mathscr{O}$ is computed according to (17.8): $\underline{\boldsymbol{B}} = c^{-1} \star \boldsymbol{F}(\vec{\boldsymbol{u}}_0, .)$, with $\star\boldsymbol{F}$ given by (18.101):

$$\underline{\boldsymbol{B}} = \frac{q}{4\pi\varepsilon_0 c\, R^2}\, \boldsymbol{\epsilon}\left(\vec{\boldsymbol{u}}_0,\ \vec{\boldsymbol{a}} + \frac{1 + \vec{\boldsymbol{a}} \cdot \overrightarrow{PM}}{R}\, \vec{\boldsymbol{u}},\ \overrightarrow{PM},\ .\right). \tag{18.108}$$

Now, from (18.105) and $\overrightarrow{PM} = r(\vec{\boldsymbol{u}}_0 + \vec{\boldsymbol{n}})$,

$$\begin{aligned}\boldsymbol{\epsilon}(\vec{\boldsymbol{u}}_0, \vec{\boldsymbol{n}}, \vec{\boldsymbol{E}}, .) &= \frac{1}{r}\, \boldsymbol{\epsilon}(\vec{\boldsymbol{u}}_0, \overrightarrow{PM}, \vec{\boldsymbol{E}}, .) \\ &= \frac{q(\overrightarrow{PM} \cdot \vec{\boldsymbol{u}}_0)}{4\pi\varepsilon_0\, R^2 r}\, \boldsymbol{\epsilon}\left(\vec{\boldsymbol{u}}_0,\ \overrightarrow{PM},\ \vec{\boldsymbol{a}} + \frac{1 + \vec{\boldsymbol{a}} \cdot \overrightarrow{PM}}{R}\, \vec{\boldsymbol{u}},\ .\right).\end{aligned}$$

Since $\overrightarrow{PM} \cdot \vec{\boldsymbol{u}}_0 = -r$, we note, by comparing with (18.108), that $\boldsymbol{\epsilon}(\vec{\boldsymbol{u}}_0, \vec{\boldsymbol{n}}, \vec{\boldsymbol{E}}, .) = c\, \underline{\boldsymbol{B}}$, i.e.

$$\boxed{\vec{\boldsymbol{B}} = \frac{1}{c}\, \vec{\boldsymbol{n}} \times_{u_0} \vec{\boldsymbol{E}}}. \tag{18.109}$$

In particular, $\vec{\boldsymbol{B}}$ is orthogonal to $\vec{\boldsymbol{E}}$. This result is not surprising since we have already seen that the electromagnetic field invariant I_2 is vanishing [Eq. (18.102)] [remember that $I_2 = c\vec{\boldsymbol{E}} \cdot \vec{\boldsymbol{B}}$, Eq. (17.37)].

Remark 18.19. Note also that $\vec{\boldsymbol{B}}$ is transverse, i.e. is orthogonal to $\vec{\boldsymbol{n}}$. This appears as an immediate consequence of the transverse character of $\star\boldsymbol{F}$ [Eq. (18.104)], for we deduce from (17.8) that $\vec{\boldsymbol{B}} \cdot \vec{\boldsymbol{n}} = c^{-1} \star\boldsymbol{F}(\vec{\boldsymbol{u}}_0, \vec{\boldsymbol{n}}) = (cr)^{-1} \star\boldsymbol{F}(\vec{\boldsymbol{u}}_0, \overrightarrow{PM})$.

18.6.4 Charge in Inertial Motion

If the motion of the charged particle $\mathscr{P}$ is inertial, its worldline is a straight line of $\mathscr{E}$, its 4-velocity $\vec{\boldsymbol{u}}$ is constant and its 4-acceleration $\vec{\boldsymbol{a}}$ vanishes. The expression (18.100) of $\boldsymbol{F}$ simplifies then drastically:

$$\boldsymbol{F}(M) = \frac{q}{4\pi\varepsilon_0\, R^3}\, \underline{\boldsymbol{u}} \wedge \underline{\boldsymbol{PM}} \qquad (\vec{\boldsymbol{a}} = 0). \tag{18.110}$$

Remark 18.20. $\underline{\boldsymbol{u}}$ being constant, it is no longer necessary to specify $\underline{\boldsymbol{u}} = \underline{\boldsymbol{u}}(\tau_P)$ as in the general expression (18.100).

Since $\vec{\boldsymbol{a}} = 0$, the acceleration $\vec{\boldsymbol{\gamma}}$ relative to the inertial observer $\mathscr{O}$ vanishes [cf. Eq. (4.72)]. Expression (18.106) for the electric field reduces then to

$$\vec{\boldsymbol{E}} = \frac{q}{4\pi\varepsilon_0 \Gamma^2 \left(1 - \frac{\vec{\boldsymbol{n}}\cdot\vec{\boldsymbol{V}}}{c}\right)^3 r^2} \left(\vec{\boldsymbol{n}} - \frac{\vec{\boldsymbol{V}}}{c}\right) \qquad (\vec{\boldsymbol{\gamma}} = 0). \tag{18.111}$$

The magnetic field is deduced via (18.109):

$$\vec{\boldsymbol{B}} = \frac{\mu_0}{4\pi} \frac{q}{\Gamma^2 \left(1 - \frac{\vec{\boldsymbol{n}}\cdot\vec{\boldsymbol{V}}}{c}\right)^3 r^2} \vec{\boldsymbol{V}} \times_{\boldsymbol{u}_0} \vec{\boldsymbol{n}} \qquad (\vec{\boldsymbol{\gamma}} = 0). \tag{18.112}$$

In the particular case where $\mathscr{P}$ is fixed with respect to $\mathscr{O}$, $\vec{\boldsymbol{V}} = 0$ and $\Gamma = 1$, so that the above formula reduces to

$$\boxed{\vec{\boldsymbol{E}} = \frac{q}{4\pi\varepsilon_0\, r^2}\, \vec{\boldsymbol{n}}} \quad \text{and} \quad \boxed{\vec{\boldsymbol{B}} = 0} \qquad (\vec{\boldsymbol{V}} = 0,\ \vec{\boldsymbol{\gamma}} = 0). \tag{18.113}$$

This is the famous ***Coulomb's law***, which gives the electric field created by a charge at rest with respect to an inertial observer.

When $\vec{\boldsymbol{V}} \neq 0$, formulas (18.111)–(18.112) must yield the results that we have already obtained in Sect. 17.3.4 by admitting Coulomb's law and using the transformation laws of the electric and magnetic fields. This seems however not obvious when comparing directly (17.53)–(17.54) with (18.111)–(18.112). But it should be noticed that in Sect. 17.3.4, $\vec{\boldsymbol{n}}$ does not stand for the same vector as here. In $\mathscr{O}$'s reference space (cf. Sect. 3.4.3), the vector $\vec{\boldsymbol{n}}$ hereabove is the unit vector from the position P' of charge $\mathscr{P}$ at the retarded time $t_P = t - r/c$ to the point M where $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$ are evaluated, whereas in Sect. 17.3.4, $\vec{\boldsymbol{n}}$ stands for the unit vector from the position of $\mathscr{P}$ *at the instant* t, i.e. P_* (cf. Fig. 18.6), to the point M. To avoid the confusion, let us denote by $\vec{\boldsymbol{n}}_*$ this last vector (cf. Fig. 18.7). By definition of $\mathscr{P}$'s velocity relative to $\mathscr{O}$, we have $\overrightarrow{P'P_*} = (t - t_P)\vec{\boldsymbol{V}} = (r/c)\vec{\boldsymbol{V}}$, the last equality resulting from (18.88). Chasles' relation $\overrightarrow{P'M} = \overrightarrow{P'P_*} + \overrightarrow{P_*M}$ leads then to $r\,\vec{\boldsymbol{n}} = (r/c)\vec{\boldsymbol{V}} + R_*\,\vec{\boldsymbol{n}}_*$, hence

$$r\left(\vec{\boldsymbol{n}} - \frac{\vec{\boldsymbol{V}}}{c}\right) = R_*\,\vec{\boldsymbol{n}}_*. \tag{18.114}$$

Besides, Pythagoras' theorem applied to the right triangle P_*AM (cf. Fig. 18.7) yields $R_*^2 = (V/c\, r\sin\theta)^2 + r^2(1 - \vec{\boldsymbol{n}}\cdot\vec{\boldsymbol{V}}/c)^2$. Taking into account $r\sin\theta = R_*\sin\theta_*$, this relation can be rewritten as

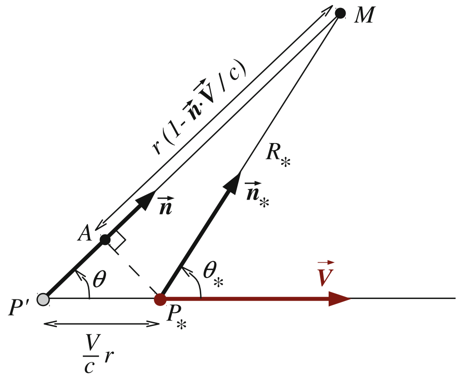


Fig. 18.7 Positions of the charged particle $\mathscr{P}$ in the reference space of the inertial observer $\mathscr{O}$ at instants t (point P_*) and $t_P = t - r/c$ (point P'), as well as the unit vectors $\vec{n}$ and $\vec{n}_*$

$$r\left(1 - \frac{\vec{n}\cdot\vec{V}}{c}\right) = R_*\sqrt{1 - \frac{V^2}{c^2}\sin^2\theta_*}. \tag{18.115}$$

Inserting (18.114) and (18.115) in (18.111), we get

$$\vec{E} = \frac{q}{4\pi\varepsilon_0\Gamma^2 R_*^2\left[1-(V/c)^2\sin^2\theta_*\right]^{3/2}}\,\vec{n}_*, \tag{18.116}$$

which, given the changes of notation $\vec{n}_* \to \vec{n}$, $R_* \to R$, $\theta_* \to \theta$ and $V \to U$, is identical to Eq. (17.53) obtained in Sect. 17.3.4. Similarly, expression (18.112) for the magnetic field is equivalent to the result (17.53) obtained in Sect. 17.3.4.

18.6.5 Radiative Part

In view of formulas (18.100) and (18.110), it is natural to split the electromagnetic field in two parts:

$$\boxed{\boldsymbol{F} = \boldsymbol{F}_{\text{Coul}} + \boldsymbol{F}_{\text{rad}}}, \tag{18.117}$$

with

$$\boxed{\boldsymbol{F}_{\text{Coul}}(M) := \frac{q}{4\pi\varepsilon_0\,R^3}\,\underline{\boldsymbol{u}}(\tau_P)\wedge\underline{\boldsymbol{PM}}}, \tag{18.118}$$

$$\boxed{\boldsymbol{F}_{\text{rad}}(M) := \frac{q}{4\pi\varepsilon_0\,R^2}\left[\underline{\boldsymbol{a}}(\tau_P) + \frac{\vec{a}(\tau_P)\cdot\overrightarrow{PM}}{R}\,\underline{\boldsymbol{u}}(\tau_P)\right]\wedge\underline{\boldsymbol{PM}}}. \tag{18.119}$$

$\boldsymbol{F}_{\text{Coul}}$ and $\boldsymbol{F}_{\text{rad}}$ are, respectively, called ***Coulombian part*** and ***radiative part*** of the electromagnetic field. Provided that $q \neq 0$, the Coulombian part never vanishes. On the other side, the radiative part is nonzero only if the particle is accelerated

($\vec{\boldsymbol{a}} \neq 0$). In this case, if M where is far from the charged particle $\mathscr{P}$, in the sense $|\vec{\boldsymbol{a}}(\tau_P) \cdot \overrightarrow{PM}| \gg 1$, $\boldsymbol{F}_{\text{Coul}}$ is negligible in front of $\boldsymbol{F}_{\text{rad}}$:

$$\boldsymbol{F} \simeq \boldsymbol{F}_{\text{rad}} \qquad \text{if} \quad |\vec{\boldsymbol{a}}(\tau_P) \cdot \overrightarrow{PM}| \gg 1. \tag{18.120}$$

The form (18.119) of the 2-form $\boldsymbol{F}_{\text{rad}}$ is remarkable: not only it is the exterior product of two 1-forms, as $\boldsymbol{F}$ itself (cf. Sect. 18.6.2), but moreover these two 1-forms are orthogonal to each other. Indeed, we can write $\boldsymbol{F}_{\text{rad}} = \boldsymbol{p} \wedge \boldsymbol{q}$ with $\boldsymbol{p} := q/(4\pi\varepsilon_0 R^2)\,[\underline{\boldsymbol{a}} + \vec{\boldsymbol{a}} \cdot \overrightarrow{PM}/R\,\underline{\boldsymbol{u}}]$, $\boldsymbol{q} := \underline{\boldsymbol{PM}}$ and $\langle \boldsymbol{p}, \vec{\boldsymbol{q}} \rangle = q/(4\pi\varepsilon_0 R^2)[\vec{\boldsymbol{a}} \cdot \overrightarrow{PM} + (\vec{\boldsymbol{a}} \cdot \overrightarrow{PM}/R)\, \vec{\boldsymbol{u}} \cdot \overrightarrow{PM}] = 0$, given that $R = -\vec{\boldsymbol{u}} \cdot \overrightarrow{PM}$ [Eq. (18.85)]. This property yields the cancellation of the invariant I_1 associated with $\boldsymbol{F}_{\text{rad}}$, since by the definition (17.36),

$$I_1 = \frac{1}{2}(F_{\text{rad}})_{\mu\nu} F^{\mu\nu}_{\text{rad}} = \frac{1}{2}(p_\mu q_\nu - p_\nu q_\mu)(p^\mu q^\nu - p^\nu q^\mu) = p_\mu p^\mu \underbrace{q_\nu q^\nu}_{0} - (\underbrace{p_\mu q^\mu}_{0})^2 = 0,$$

the property $q_\nu q^\nu = 0$ being nothing but the expression of the null character of vector $\overrightarrow{PM}$. As we had already $I_2 = 0$, thanks to the fact that $\boldsymbol{F}_{\text{rad}}$ is an exterior product (cf. Sect. 18.6.2), we conclude that

$$\boxed{\boldsymbol{F}_{\text{rad}}: \quad I_1 = 0 \quad \text{and} \quad I_2 = 0}. \tag{18.121}$$

In other words, the radiative part of the electromagnetic field is null, in the sense defined in Sect. 17.3.2. Moreover, $\boldsymbol{F}_{\text{rad}}$ is transverse, as $\star\boldsymbol{F}$ [cf. (18.104)]:

$$\boldsymbol{F}_{\text{rad}}(\overrightarrow{PM}, .) = 0. \tag{18.122}$$

This is an immediate consequence of the property $\langle \boldsymbol{p}, \vec{\boldsymbol{q}} \rangle = 0$ mentioned above and of $\overrightarrow{PM} \cdot \overrightarrow{PM} = 0$.

In terms of the electric and magnetic fields measured by an observer $\mathscr{O}$, the Coulombian part is given by (18.111)–(18.112), while the radiative part is the part involving $\vec{\boldsymbol{\gamma}}$ in (18.106) and (18.109):

$$\boxed{\vec{\boldsymbol{E}}_{\text{rad}} = \frac{q}{4\pi\varepsilon_0 c^2 \left(1 - \frac{\vec{\boldsymbol{n}}\cdot\vec{\boldsymbol{V}}}{c}\right)^3 r}\, \vec{\boldsymbol{n}} \times_{\boldsymbol{u}_0} \left[\left(\vec{\boldsymbol{n}} - \frac{\vec{\boldsymbol{V}}}{c}\right) \times_{\boldsymbol{u}_0} \vec{\boldsymbol{\gamma}}\right]} \tag{18.123}$$

$$\boxed{\vec{\boldsymbol{B}}_{\text{rad}} = \frac{1}{c}\, \vec{\boldsymbol{n}} \times_{\boldsymbol{u}_0} \vec{\boldsymbol{E}}_{\text{rad}}}, \tag{18.124}$$

where the quantities $\vec{V}$ and $\vec{\gamma}$ are to be taken at the retarded time $t_P = t - r/c$ and the unit vector $\vec{n}$ gives the direction from the particle's position P' at the retarded time to the point M where the field is evaluated. It is clear on (18.111)–(18.112) and (18.123)–(18.124) that the Coulombian part of $(\vec{E}, \vec{B})$ decays as $1/r^2$ far from the charge, whereas the radiative part decays only as $1/r$. It is thus the latter that dominates at large distance, as mentioned above.

Remark 18.21. We observe on (18.123) that $\vec{E}_{\mathrm{rad}}$ is orthogonal to $\vec{n}$. Equation (18.124) implies then that $c\vec{B}_{\mathrm{rad}}$ has the same amplitude than $\vec{E}_{\mathrm{rad}}$ (in addition to be orthogonal to it). Since $I_1 = c^2\|\vec{B}_{\mathrm{rad}}\|^2_{\boldsymbol{g}} - \|\vec{E}_{\mathrm{rad}}\|^2_{\boldsymbol{g}}$ [Eq. (17.37)], we recover $I_1 = 0$ and hence the null character of $\boldsymbol{F}_{\mathrm{rad}}$ [Eq. (18.121)].

In the nonrelativistic limit, $\|V\|_{\boldsymbol{g}} \ll c$, formulas (18.123)–(18.124) reduce to

$$\vec{E}_{\mathrm{rad}} \simeq \frac{q}{4\pi\varepsilon_0 c^2 r}\,\vec{n} \times_{\boldsymbol{u}_0} (\vec{n} \times_{\boldsymbol{u}_0} \vec{\gamma}) \qquad \text{(nonrelativistic)} \tag{18.125}$$

$$\vec{B}_{\mathrm{rad}} \simeq \frac{q}{4\pi\varepsilon_0 c^3 r}\,\vec{\gamma} \times_{\boldsymbol{u}_0} \vec{n} \qquad \text{(nonrelativistic).} \tag{18.126}$$

18.7 Maxwell Equations from a Principle of Least Action

In Sect. 18.3.1, we have presented Maxwell equations as a *postulate* on which the theory of classical electrodynamics is constructed. Another point of view consists in *deriving* them from another postulate, namely, a *principle of least action* (also called *variational principle*). Before describing this approach, let us first introduce the principle of least action on general grounds, i.e. for any classical field theory.

18.7.1 Principle of Least Action in a Classical Field Theory

We have already encountered the principle of least action in Chap. 11 for the dynamics of a relativistic particle in a given field. Here we aim to formulate a principle of least action for the dynamics of the field itself. This implies the transition from a finite number of degrees of freedom (the particle's coordinates and generalized velocities) to an infinite number (the field values at each spacetime point). For simplicity, let us assume that the theory under study involves only one tensor field $\boldsymbol{\varphi}$ of valence n: $\boldsymbol{\varphi}$ can be a scalar field ($n = 0$), a vector field or a differential 1-form ($n = 1$) or a tensor field of higher valence.

Let us set an inertial coordinate system (x^α) on spacetime $\mathscr{E}$ and denote by φ_A the components of $\boldsymbol{\varphi}$ in it; A is then some "multi-index": if $n = 0$, $A = \varnothing$, and if $\boldsymbol{\varphi}$ is a tensor of type (k, ℓ) ($k + \ell = n \geq 1$), $\varphi_A = \varphi^{\alpha_1\ldots\alpha_k}{}_{\beta_1\ldots\beta_\ell}$. Given a function of class C^1

$$L:\ \mathbb{R}^{5\times 4^n} \longrightarrow \mathbb{R},$$

one calls ***Lagrangian density of the field $\boldsymbol{\varphi}$***[17] the scalar field

$$\begin{aligned} \mathscr{L} : \mathscr{E} &\longrightarrow \mathbb{R} \\ M &\longmapsto \mathscr{L}(M) := L(\varphi_A(M), \nabla\varphi_B(M)), \end{aligned} \tag{18.127}$$

where the abridged notation $L(\varphi_A(M), \nabla\varphi_B(M))$ means that the 4^n first arguments of L are filled by the components of $\boldsymbol{\varphi}$ at M and the remaining 4^{n+1} arguments by the components of the covariant derivative of $\boldsymbol{\varphi}$ at M. Note that the latter are equal to the partial derivatives $\partial\varphi_A/\partial x^\alpha$, for the coordinates (x^α) are inertial. Two important particular cases are:

- $\varphi =$ scalar field ($n = 0$):

$$\mathscr{L}(M) = L\left(\varphi(M),\ \frac{\partial\varphi}{\partial x^0}(M),\ \frac{\partial\varphi}{\partial x^1}(M),\ \frac{\partial\varphi}{\partial x^2}(M),\ \frac{\partial\varphi}{\partial x^3}(M)\right);$$

- $\varphi =$ 1-form ($n = 1$):

$$\mathscr{L}(M) = L\left(\varphi_0(M), \dots, \varphi_3(M),\ \frac{\partial\varphi_0}{\partial x^0}(M), \frac{\partial\varphi_0}{\partial x^1}(M), \dots, \frac{\partial\varphi_3}{\partial x^3}(M)\right).$$

All functions $L : \mathbb{R}^{5\times 4^n} \longrightarrow \mathbb{R}$ are not acceptable, for one does not want the field theory for $\boldsymbol{\varphi}$ to depend on the choice of the inertial coordinates (x^α). It is then demanded that the value of $\mathscr{L}(M)$ at each point $M \in \mathscr{E}$ is independent of the coordinates (x^α). In other words, if (x'^α) is a second system of inertial coordinates on $\mathscr{E}$,

$$L(\varphi'_A, \nabla\varphi'_B) = L(\varphi_A, \nabla\varphi_B), \tag{18.128}$$

φ'_A and $\nabla\varphi'_B$ being the components of $\boldsymbol{\varphi}$ and $\boldsymbol{\nabla\varphi}$ in the coordinates (x'^α). Since (x'^α) and (x^α) are related by a Poincaré transformation, $x^\alpha = \Lambda^\alpha{}_\beta x'^\beta + x_0^\alpha$ [Eq. (8.12)], the components are related by some matrix products with Λ or Λ^{-1}, and the condition (18.128) is explicitly written as follows:

- For a scalar field,

$$L\left(\varphi,\ \Lambda^\mu{}_0\frac{\partial\varphi}{\partial x^\mu}, \dots\right) = L\left(\varphi, \frac{\partial\varphi}{\partial x^0}, \dots\right); \tag{18.129}$$

- For a 1-form,

$$L\left(\Lambda^\mu{}_0\,\varphi_\mu, \dots, \Lambda^\mu{}_0\Lambda^\nu{}_0\frac{\partial\varphi_\mu}{\partial x_\nu}, \dots\right) = L\left(\varphi_0, \dots, \frac{\partial\varphi_0}{\partial x^0}, \dots\right). \tag{18.130}$$

[17] One often says simply *Lagrangian*, instead of *Lagrangian density*.

Because of the properties (18.129) or (18.130), one says that the considered field theory is ***invariant under the action of Poincaré group***, or ***Poincaré invariant*** for short. To fulfil this invariance, it suffices that L is a scalar function obtained by purely tensorial operations on $\boldsymbol{\varphi}$ and $\boldsymbol{\nabla}\boldsymbol{\varphi}$ (contractions or scalar products via the metric tensor $\boldsymbol{g}$).

Example 18.2. A theory of the ***Klein–Gordon***[18] type is based on a scalar field φ and the following Lagrangian density

$$\mathscr{L} = -\frac{1}{2}\left(\langle \boldsymbol{\nabla}\varphi, \vec{\boldsymbol{\nabla}}\varphi\rangle + \ell^{-2}\varphi^2\right) = -\frac{1}{2}\left(\eta^{\mu\nu}\frac{\partial\varphi}{\partial x^\mu}\frac{\partial\varphi}{\partial x^\nu} + \ell^{-2}\varphi^2\right), \qquad (18.131)$$

where ℓ is some constant having the dimension of a length; in quantum field theory, ℓ is related to the ***mass*** m ***of the scalar field*** by $\ell = \hbar/(mc)$. The function $L : \mathbb{R}^5 \to \mathbb{R}$ corresponding to (18.131) is $L(y_1, y_2, y_3, y_4, y_5) = -1/2\ (-y_2^2 + y_3^2 + y_4^2 + y_5^2 + y_1^2/\ell^2)$.

Given a compact four-dimensional domain with boundary $\mathscr{U} \subset \mathscr{E}$, one calls ***action*** of the field $\boldsymbol{\varphi}$ on $\mathscr{U}$ the real number

$$\boxed{S := \int_{\mathscr{U}} \mathscr{L}\,\boldsymbol{\epsilon} = \int_{\mathscr{U}} \mathscr{L}\ \mathrm{d}x^0\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3}. \qquad (18.132)$$

The above integral is that of the scalar field $\mathscr{L}$ (Lagrangian density) over $\mathscr{U}$, as defined in Sect. 16.4.5. Note that in the present case, $\sqrt{-\det g} = 1$, since the coordinates (x^α) are inertial. Moreover, as $\mathscr{L}$ and the volume element of $\mathscr{U}$ do not depend on the choice of the inertial coordinates (x^α), the same property holds for S.

The ***principle of least action*** amounts to postulating that, among all the configurations of the field $\boldsymbol{\varphi}$, the physical one is the one that minimizes the action S in any variation of $\boldsymbol{\varphi}$ that does not change the field values on the boundary of $\mathscr{U}$. Let us show that this principle leads to a system of partial differential equations for the field $\boldsymbol{\varphi}$. An infinitesimal field variation $\delta\boldsymbol{\varphi}$ implies the following variation of the action

$$\delta S = \int_{\mathscr{U}} \delta\mathscr{L}\,\boldsymbol{\epsilon}. \qquad (18.133)$$

Now, from (18.127) and the fact that the components of $\boldsymbol{\nabla}\boldsymbol{\varphi}$ are nothing but the partial derivatives of the components of $\boldsymbol{\varphi}$,

$$\delta\mathscr{L} = \frac{\partial L}{\partial\varphi_A}\delta\varphi_A + \frac{\partial L}{\partial(\partial_\alpha\varphi_A)}\delta\frac{\partial\varphi_A}{\partial x^\alpha},$$

[18] *Klein* stands for the Swedish physicist Oskar Klein (1894–1977) and not for the German mathematician Felix Klein mentioned in Chap. 7 (cf. p. 255).

where the partial derivatives of the function L have been denoted by $\partial L/\partial\varphi_A$ for the 4^n first arguments and by $\partial L/\partial(\partial_\alpha\varphi_A)$ for the remaining 4^{n+1} arguments. Moreover, in the above writing, the Einstein summation convention has been extended to the multi-index A, in addition to the index $\alpha \in \{0,1,2,3\}$. It is obvious that $\delta\partial\varphi_A/\partial x^\alpha = \partial\delta\varphi_A/\partial x^\alpha$, so that the second term can be integrated by parts, yielding

$$\delta\mathscr{L} = \frac{\partial L}{\partial\varphi_A}\delta\varphi_A - \frac{\partial}{\partial x^\alpha}\left(\frac{\partial L}{\partial(\partial_\alpha\varphi_A)}\right)\delta\varphi_A + \frac{\partial}{\partial x^\alpha}\left(\frac{\partial L}{\partial(\partial_\alpha\varphi_A)}\delta\varphi_A\right).$$

Inserting this relation in (18.133), we get

$$\delta S = \int_{\mathscr{U}}\left[\frac{\partial L}{\partial\varphi_A} - \frac{\partial}{\partial x^\alpha}\left(\frac{\partial L}{\partial(\partial_\alpha\varphi_A)}\right)\right]\delta\varphi_A\,\boldsymbol{\epsilon} + \int_{\mathscr{U}}\boldsymbol{\nabla}\cdot\vec{\boldsymbol{V}}\,\boldsymbol{\epsilon},$$

where $\vec{\boldsymbol{V}}$ stands for the vector field whose components are $V^\alpha = \partial L/\partial(\partial_\alpha\varphi_A)\,\delta\varphi_A$ and we have used the identity $\partial V^\alpha/\partial x^\alpha = \boldsymbol{\nabla}\cdot\vec{\boldsymbol{V}}$ [Eq. (15.54) with $\det g = -1$ since the coordinates (x^α) are inertial]. Thanks to the four-dimensional Gauss–Ostrogradsky theorem (16.64), we observe that the last term in the above equation is equal to the flux of $\vec{\boldsymbol{V}}$ through the boundary of $\mathscr{U}$. Since the value of $\boldsymbol{\varphi}$ is kept fixed on $\partial\mathscr{U}$, we have $\delta\boldsymbol{\varphi} = 0$ and hence $\vec{\boldsymbol{V}} = 0$ on $\partial\mathscr{U}$, so that the flux of $\vec{\boldsymbol{V}}$ vanishes. There remains thus

$$\delta S = \int_{\mathscr{U}}\left[\frac{\partial L}{\partial\varphi_A} - \frac{\partial}{\partial x^\alpha}\left(\frac{\partial L}{\partial(\partial_\alpha\varphi_A)}\right)\right]\delta\varphi_A\,\boldsymbol{\epsilon}. \tag{18.134}$$

The principle of least action implies that $\delta S = 0$ for any variation $\delta\boldsymbol{\varphi}$ around the physical solution, whatever the domain $\mathscr{U}$. Consequently, (18.134) leads to the following 4^n equations:

$$\boxed{\frac{\partial L}{\partial\varphi_A} - \frac{\partial}{\partial x^\alpha}\left(\frac{\partial L}{\partial(\partial_\alpha\varphi_A)}\right) = 0}. \tag{18.135}$$

These equations are called ***field equations***. They are the analogue of the Euler–Lagrange equations (11.17) obtained in Chap. 11 by applying the principle of least action to a particle, the particle's degrees of freedom (x^α) being replaced by the components (φ_A) of the field and the evolution parameter λ by the four spacetime coordinates (x^α).

Example 18.3. Let us consider the Klein–Gordon scalar field introduced in Example 18.2 p. 624. The Lagrangian density being (18.131), we have $\partial L/\partial\varphi = -\ell^{-2}\varphi$, $\partial L/\partial(\partial_0\varphi) = \partial\varphi/\partial x^0$ and $\partial L/\partial(\partial_i\varphi) = -\partial\varphi/\partial x^i$. The field equation (18.135), which has only one component in this case, becomes

$$\Box\varphi - \ell^{-2}\varphi = 0, \tag{18.136}$$

where we have let appear the d'Alembertian operator according to (18.51). Equation (18.136) is called ***Klein–Gordon equation***. Note that it is a linear equation in φ.

18.7.2 Case of the Electromagnetic Field

For the variational formulation of electromagnetism, the field $\boldsymbol{\varphi}$ is the 4-potential 1-form $\boldsymbol{A}$ introduced in Sect. 18.5.1 and related to the electromagnetic field tensor $\boldsymbol{F}$ by $\boldsymbol{F} = \mathbf{d}\boldsymbol{A}$ [Eq. (18.44)]. One postulates then that the Lagrangian density of the electromagnetic field generated by some electric 4-current $\vec{\boldsymbol{j}}$ is

$$\boxed{\mathscr{L} = -\frac{\varepsilon_0}{4} F_{\mu\nu} F^{\mu\nu} + A_\mu j^\mu}, \tag{18.137}$$

where $F_{\mu\nu}$ and $F^{\mu\nu}$ are considered as the functions of $\boldsymbol{A}$ given by (18.45) and (18.46). The explicit form of the Lagrangian density is thus

$$\mathscr{L} = L\left(A_\alpha, \frac{\partial A_\beta}{\partial x^\alpha}\right) = -\frac{\varepsilon_0}{4} \eta^{\mu\rho} \eta^{\nu\sigma} \left(\frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}\right) \left(\frac{\partial A_\sigma}{\partial x^\rho} - \frac{\partial A_\rho}{\partial x^\sigma}\right) + A_\mu j^\mu. \tag{18.138}$$

Remark 18.22. The quantity involved in the Lagrangian density (18.137) is nothing but the invariant $I_1 = F_{\mu\nu} F^{\mu\nu}/2$, introduced in Sect. 17.3.2. In particular, this fully satisfies the demand of invariance under the action of the Poincaré group discussed above.

The field equations (18.135) become

$$\frac{\partial L}{\partial A_\beta} - \frac{\partial}{\partial x^\alpha} \left(\frac{\partial L}{\partial(\partial_\alpha A_\beta)}\right) = 0. \tag{18.139}$$

The partial derivative of the Lagrangian density with respect to A_β is simple:

$$\frac{\partial L}{\partial A_\beta} = j^\beta.$$

The partial derivative with respect to $\partial_\alpha A_\beta = \partial A_\beta / \partial x^\alpha$ is more complicated but does represent any major difficulty. Expanding (18.138) and deriving, there comes

$$\begin{aligned}
-\frac{4}{\varepsilon_0} \frac{\partial L}{\partial(\partial_\alpha A_\beta)} &= \eta^{\alpha\rho} \eta^{\beta\sigma} \frac{\partial A_\sigma}{\partial x^\rho} + \eta^{\mu\alpha} \eta^{\nu\beta} \frac{\partial A_\nu}{\partial x^\mu} - \eta^{\alpha\rho} \eta^{\beta\sigma} \frac{\partial A_\rho}{\partial x^\sigma} - \eta^{\mu\beta} \eta^{\nu\alpha} \frac{\partial A_\nu}{\partial x^\mu} \\
&\quad -\eta^{\beta\rho} \eta^{\alpha\sigma} \frac{\partial A_\sigma}{\partial x^\rho} - \eta^{\mu\alpha} \eta^{\nu\beta} \frac{\partial A_\mu}{\partial x^\nu} + \eta^{\beta\rho} \eta^{\alpha\sigma} \frac{\partial A_\rho}{\partial x^\sigma} + \eta^{\mu\beta} \eta^{\nu\alpha} \frac{\partial A_\mu}{\partial x^\nu} \\
&= 4\eta^{\alpha\mu} \eta^{\beta\nu} \left(\frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}\right) = 4F^{\alpha\beta} = -4F^{\beta\alpha}.
\end{aligned}$$

Inserting this result, as well as the expression of $\partial L/\partial A_\beta$, in (18.139), we get

$$\frac{\partial F^{\beta\alpha}}{\partial x^\alpha} = \varepsilon_0^{-1}\, j^\beta. \tag{18.140}$$

We obtain thus the Maxwell equation (18.23b).[19] Hence

> The Maxwell equation "with source" $\nabla \cdot \boldsymbol{F}^\sharp = \varepsilon_0^{-1}\vec{\boldsymbol{j}}$ [Eq. (18.22b)] can be derived from a principle of least action, based on the Lagrangian density (18.137). The remaining Maxwell equation, $\mathbf{d}\boldsymbol{F} = 0$ [Eq. (18.22a)], is automatically satisfied in this approach since it is assumed from the beginning that $\boldsymbol{F}$ is the exterior derivative of the 1-form $\boldsymbol{A}$.

Remark 18.23. The term $A_\mu j^\mu$ in the Lagrangian density (18.137) expresses the interaction between the electromagnetic field and the system of charged particles. In the case of a system reduced to a single particle, if we replace $\vec{\boldsymbol{j}}$ by (18.5) (with $N = 1$) and we integrate over the hypersurface $x^0 = \text{const}$, we recover exactly the term $(q/c)A_\mu \dot{x}^\mu$ that appears in the expression (11.28) of the Lagrangian of a particle in a vector field.

[19] Let us recall that in the present case, $\det g = -1$ since we are using inertial coordinates.

Chapter 19
Energy–Momentum Tensor

19.1 Introduction

While Chaps. 9, 10 and 11 were devoted to the dynamics of a particle system (*microscopic* point of view), we tackle here the case where the number of particles is so large that it is natural to treat matter as a *continuous medium* (*macroscopic* point of view). The basic tool to describe the relativistic dynamics of such a medium is the *energy–momentum tensor*, which we introduce in Sect. 19.2. We shall then state the principle of conservation of energy–momentum for continuous media (Sect. 19.3). Finally, we shall discuss the concept of angular momentum of a continuous medium and its conservation (Sect. 19.4).

This chapter is a preparatory one for those devoted to the energy–momentum of the electromagnetic field (Chap. 20), relativistic hydrodynamics (Chap. 21) and relativistic gravitation (Chap. 22).

19.2 Energy–Momentum Tensor

19.2.1 Definition

Let us consider a system of N massive particles $(\mathscr{P}_a)_{1\le a\le N}$, having in mind N very large, of the order of Avogadro's number ($\sim 6\times 10^{23}$). Let $\mathscr{L}_a$ be the worldline of particle $\mathscr{P}_a$, τ_a its proper time, $\vec{\boldsymbol{u}}_a = \vec{\boldsymbol{u}}_a(\tau_a)$ its 4-velocity and $\boldsymbol{p}_a = \boldsymbol{p}_a(\tau_a)$ its 4-momentum. Given an oriented three-dimensional domain (hypersurface) $\mathscr{V}\subset\mathscr{E}$, we have defined in Chap. 9 the total 4-momentum of the system on $\mathscr{V}$ by formula (9.34) (cf. Fig. 9.6):

$$\boldsymbol{p}|_{\mathscr{V}} := \sum_{a=1}^{N} \sum_{A\in\mathscr{L}_a\cap\mathscr{V}} \varepsilon\ \boldsymbol{p}_a(A), \tag{19.1}$$

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where $\varepsilon = +1$ (resp. $\varepsilon = -1$) if the 4-momentum vector $\vec{\boldsymbol{p}}_a(A)$ associated with $\boldsymbol{p}_a(A)$ has the direction (resp. the opposite direction) corresponding to the positive orientation of $\mathscr{V}$.

As for the electric charge in Sect. 18.2.1, the transition to the continuous limit is obtained by interpreting $\boldsymbol{p}|_{\mathscr{V}}$ as a *flux* through $\mathscr{V}$. To define a flux, we shall assume in all what follows that $\mathscr{V}$ is a timelike or spacelike hypersurface (cf. Sect. 16.4.7). $\mathscr{V}$ can comprise spacelike parts and timelike ones, but no null parts. We may then introduce the unit normal[1] $\vec{\boldsymbol{n}}$ that is compatible with $\mathscr{V}$'s orientation, in the sense that $\boldsymbol{\epsilon}_{\mathscr{V}} := \boldsymbol{\epsilon}(\vec{\boldsymbol{n}}, ., ., .)$.

A difference with the case of the electric charge is that the flux introduced in Sect. 18.2.1 is that of a vector field (the electric 4-current $\vec{\boldsymbol{j}}$) and the result is a scalar—the electric charge. In the present case, the result must be a linear form (the 4-momentum $\boldsymbol{p}|_{\mathscr{V}}$); accordingly, the flux cannot be that of a vector field. It is actually the flux of a field of bilinear forms, $\boldsymbol{T}$, which is called the ***energy–momentum tensor*** of the considered system. More precisely, we shall write

$$\boxed{\boldsymbol{p}|_{\mathscr{V}} = \pm\frac{1}{c}\int_{\mathscr{V}} \boldsymbol{T}(., \vec{\boldsymbol{n}})\,\mathrm{d}V}, \tag{19.2}$$

where the sign $\pm$ is $+$ if $\vec{\boldsymbol{n}}$ is spacelike and $-$ if it is timelike [cf. (16.42)]. If $\mathscr{V}$ comprises regions of different kinds, formula (19.2) must be replaced by a sum of integrals with the proper sign for each region. The integral in (19.2) is that of the tensor field of type (0, 1) (differential 1-form) $\boldsymbol{T}(., \vec{\boldsymbol{n}})$, the integral of a tensor field having been defined in Sect. 16.4.6.

For the considered particle system, the energy–momentum tensor field is given by

$$\forall M \in \mathscr{E}, \quad \boxed{\boldsymbol{T}(M) := \sum_{a=1}^{N} \int_{-\infty}^{+\infty} \delta_{A_a(\tau)}(M)\ \boldsymbol{p}_a(\tau) \otimes \underline{\boldsymbol{u}}_a(\tau)\ c^2\,\mathrm{d}\tau}, \tag{19.3}$$

where τ stands for $\mathscr{P}_a$'s proper time and $\delta_{A_a(\tau)}$ the Dirac distribution on $(\mathscr{E}, \boldsymbol{g})$ centred on the point $A_a(\tau)$ of the worldline $\mathscr{L}_a$ (cf. Sect. 18.2.1).

Proof. We have to show that substituting (19.3) for $\boldsymbol{T}$ in (19.2), we recover (19.1). By definition of the tensor product $\otimes$, the 1-form $\boldsymbol{\omega} := \boldsymbol{T}(., \vec{\boldsymbol{n}})$ to be integrated over $\mathscr{V}$ is expressible as

[1] If $\mathscr{V}$ was null, the unit normal could not be defined.

$$\forall M \in \mathscr{E}, \quad \boldsymbol{\omega}(M) = \sum_{a=1}^{N} \int_{-\infty}^{+\infty} \delta_{A_a(\tau)}(M)\, (\vec{\boldsymbol{u}}_a(\tau) \cdot \vec{\boldsymbol{n}})\, \boldsymbol{p}_a(\tau)\, c^2\, \mathrm{d}\tau,$$

so that there comes

$$\int_{\mathscr{V}} \boldsymbol{T}(., \vec{\boldsymbol{n}})\, \mathrm{d}V = \sum_{a=1}^{N} \int_{\mathscr{V}} \int_{-\infty}^{+\infty} \delta_{A_a(\tau)}(M)\, (\vec{\boldsymbol{u}}_a(\tau) \cdot \vec{\boldsymbol{n}})\, \boldsymbol{p}_a(\tau)\, c^2\, \mathrm{d}\tau\, \mathrm{d}V.$$

Let us introduce on $\mathscr{E}$ a coordinate system (x^α) adapted to $\mathscr{V}$ (cf. Sect. 16.3): $x^0 = \text{const}$ on $\mathscr{V}$. In terms of the components in these coordinates, $\vec{\boldsymbol{u}}_a(\tau) \cdot \vec{\boldsymbol{n}} = u_a^\alpha(\tau) n_\alpha = u_a^0(\tau) n_0$ for $n_\alpha = (n_0, 0, 0, 0)$, since the normal $\vec{\boldsymbol{n}}$ to $\mathscr{V}$ is collinear to $\vec{\nabla} x^0$ (cf. Sect. 16.4.2). Moreover, from (16.25), $\mathrm{d}V = n^0 \sqrt{-g}\, \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3$. We have thus in the above integral,

$$(\vec{\boldsymbol{u}}_a(\tau) \cdot \vec{\boldsymbol{n}}) \mathrm{d}V = u_a^0(\tau) n_0 n^0 \sqrt{-g}\, \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3 = \pm u_a^0(\tau) \sqrt{-g}\, \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3$$

for $n_0 n^0 = n_\alpha n^\alpha = \pm 1$, $\vec{\boldsymbol{n}}$ being a unit vector. Hence

$$\int_{\mathscr{V}} \boldsymbol{T}(., \vec{\boldsymbol{n}})\, \mathrm{d}V = \pm \sum_{a=1}^{N} \int_{\mathscr{V}} \int_{-\infty}^{+\infty} \delta_{A_a(\tau)}(M)\, \boldsymbol{p}_a(\tau)\, \sqrt{-g}\, u_a^0(\tau)\, c^2\, \mathrm{d}\tau\, \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3.$$

We assume that the worldline $\mathscr{L}_a$ is never tangent to $\mathscr{V}$. Then, in the neighbourhood of an intersection A of $\mathscr{L}_a$ with $\mathscr{V}$, $\mathscr{L}_a$ can be parametrized by x^0, the first coordinate of the system adapted to $\mathscr{V}$. x^0 is constant on $\mathscr{V}$ and we may choose its value to be 0. Let us then perform the change of variable $\tau \mapsto x^0$ in the above integral. One should notice that, thanks to the term $\delta_{A_a(\tau)}(M)$ with $M \in \mathscr{V}$, the integral in x^0 can be limited to a finite interval $[-\alpha, \alpha]$ around $x^0 = 0$ (the value on $\mathscr{V}$). Moreover, by definition of $\mathscr{P}_a$'s 4-velocity, $u_a^0(\tau)\, c\, \mathrm{d}\tau = \mathrm{d}x^0$. If x^0 is a decaying function of τ, i.e. if $\vec{\boldsymbol{p}}_a$ does not cross $\mathscr{V}$ in the sense given by its orientation, one must change the sign in the integral, represented by the parameter $\varepsilon = -1$. In view of these considerations, there comes

$$\int_{\mathscr{V}} \boldsymbol{T}(., \vec{\boldsymbol{n}})\, \mathrm{d}V = \pm c \sum_{a=1}^{N} \sum_{A \in \mathscr{L}_a \cap \mathscr{V}} \varepsilon \int_{\mathscr{V}} \int_{-\alpha}^{+\alpha} \delta_{A_a(x^0)}(M)\, \boldsymbol{p}_a(x^0)\, \sqrt{-g}\, \mathrm{d}x^0 \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3.$$

Since $\sqrt{-g}\, \mathrm{d}x^0 \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3$ is exactly the element of 4-volume on $(\mathscr{E}, \boldsymbol{g})$, we deduce from the definition of δ_A that

$$\int_{\mathscr{V}} \boldsymbol{T}(., \vec{\boldsymbol{n}})\, \mathrm{d}V = \pm c \sum_{a=1}^{N} \sum_{A \in \mathscr{L}_a \cap \mathscr{V}} \varepsilon\, \boldsymbol{p}_a(A).$$

The comparison with (19.1) completes the proof. □

Formula (19.2) shows that the dimension of the energy–momentum tensor is that of a momentum density multiplied by a velocity; it has thus the dimension of an *energy density* (SI unit: $\mathrm{J\,m^{-3}}$).

19.2.2 Interpretation

Let us consider some observer $\mathscr{O}$ of 4-velocity $\vec{u}_0$ and let us choose for $\mathscr{V}$ some elementary volume of $\mathscr{O}$'s local rest space at some fixed instant t of $\mathscr{O}$'s proper time. Let $\mathrm{d}V$ be the volume of $\mathscr{V}$; the 4-momentum $\mathrm{d}\boldsymbol{p}$ in $\mathscr{V}$ is given by the infinitesimal version of (19.2):

$$\mathrm{d}\boldsymbol{p} = -\frac{1}{c}\boldsymbol{T}(.,\vec{u}_0)\,\mathrm{d}V. \tag{19.4}$$

The $-$ sign has been selected for the unit normal to $\mathscr{V}$ is $\vec{u}_0$, which is timelike. By virtue of (9.43), the energy $\mathrm{d}E$ of the matter in $\mathscr{V}$ measured by $\mathscr{O}$ is $\mathrm{d}E = -c\,\langle \mathrm{d}\boldsymbol{p}, \vec{u}_0\rangle$, i.e.

$$\mathrm{d}E = \boldsymbol{T}(\vec{u}_0, \vec{u}_0)\,\mathrm{d}V.$$

We deduce immediately that the ***energy density*** $\varepsilon := \mathrm{d}E/\mathrm{d}V$ ***measured by*** $\mathscr{O}$ in $\mathscr{V}$ is

$$\boxed{\varepsilon = \boldsymbol{T}(\vec{u}_0, \vec{u}_0)}. \tag{19.5}$$

Hence the energy density measured by an observer is obtained by setting the two arguments of the bilinear form $\boldsymbol{T}$ to the 4-velocity of that observer.

On the other side, according to (9.44), the linear-momentum $\mathrm{d}\boldsymbol{P}$ measured by $\mathscr{O}$ in $\mathscr{V}$ is $\mathrm{d}\boldsymbol{P} = \mathrm{d}\boldsymbol{p} \circ \perp_{u_0}$, i.e., given (19.4),

$$\mathrm{d}\boldsymbol{P} = -\frac{1}{c}\boldsymbol{T}(\perp_{u_0}, \vec{u}_0)\,\mathrm{d}V.$$

The ***linear-momentum density*** $\boldsymbol{\varpi} := \mathrm{d}\boldsymbol{P}/\mathrm{d}V$ ***measured by*** $\mathscr{O}$ is thus the 1-form

$$\boxed{\boldsymbol{\varpi} = -\frac{1}{c}\boldsymbol{T}(\perp_{u_0}, \vec{u}_0)}. \tag{19.6}$$

If $(\vec{e}_\alpha)$ is $\mathscr{O}$'s local frame (in particular, $\vec{e}_0 = \vec{u}_0$) and $(\boldsymbol{e}^\alpha)$ the associated dual basis, we can write

$$\boxed{\boldsymbol{\varpi} = -\frac{\boldsymbol{T}(\vec{e}_i, \vec{u}_0)}{c}\,\boldsymbol{e}^i}. \tag{19.7}$$

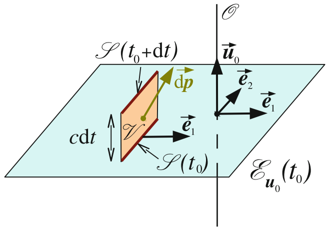

Fig. 19.1 3-volume $\mathscr{V}$ swept by an elementary 2-surface $\mathscr{S}$ during the infinitesimal increase $\mathrm{d}t$ of observer $\mathscr{O}$'s proper time. The surface $\mathscr{S}$ is fixed with respect to $\mathscr{O}$. The 4-momentum $\mathrm{d}\boldsymbol{p}$ in $\mathscr{V}$ is orthogonally split as $\mathrm{d}\boldsymbol{p} = (\mathrm{d}E/c)\,\underline{\boldsymbol{u}}_0 + \mathrm{d}\boldsymbol{P}$; $\mathrm{d}E/\mathrm{d}t$ is then the energy crossing $\mathscr{S}$ per unit time and $\mathrm{d}\boldsymbol{P}/\mathrm{d}t$ is the force exerted on $\mathscr{S}$

Since $\langle \boldsymbol{e}^i, \vec{\boldsymbol{u}}_0\rangle = 0$, it is clear that $\langle \boldsymbol{\varpi}, \vec{\boldsymbol{u}}_0\rangle = 0$ (this can also be seen directly on (19.6), because $\perp_{\boldsymbol{u}_0} \vec{\boldsymbol{u}}_0 = 0$).

Let us consider now some elementary 2-surface $\mathscr{S}$ fixed in the reference space of observer $\mathscr{O}$. To be specific, we choose $\mathscr{S}$ to be normal to one of the three spacelike vectors of $\mathscr{O}$'s local frame, $\vec{\boldsymbol{e}}_j$ say ($j = 1$ in Fig. 19.1). During some small increase $\mathrm{d}t$ of $\mathscr{O}$'s proper time, $\mathscr{S}$ sweeps an elementary 3-volume $\mathscr{V}$ in spacetime (cf. Fig. 19.1). $\mathscr{V}$ is a timelike hypersurface and its unit normal is $\vec{\boldsymbol{e}}_j$, which also sets $\mathscr{V}$'s orientation. The 4-momentum in $\mathscr{V}$ is given by the infinitesimal version of (19.2) with the spacelike normal $\vec{\boldsymbol{n}} = \vec{\boldsymbol{e}}_j$:

$$\mathrm{d}\boldsymbol{p} = \frac{1}{c}\boldsymbol{T}(., \vec{\boldsymbol{e}}_j)\,\mathrm{d}V = \boldsymbol{T}(., \vec{\boldsymbol{e}}_j)\,\mathrm{d}S\,\mathrm{d}t, \tag{19.8}$$

where the second equality results from the decomposition $\mathrm{d}V = \mathrm{d}S \times c\,\mathrm{d}t$ of $\mathscr{V}$'s volume, $\mathrm{d}S$ being the area of $\mathscr{S}$. For observer $\mathscr{O}$, the energy of the matter "contained" in $\mathscr{V}$—i.e. that crosses the surface $\mathscr{S}$ during the time lapse $\mathrm{d}t$—is $\mathrm{d}E = -c\,\langle \mathrm{d}\boldsymbol{p}, \vec{\boldsymbol{u}}_0\rangle$ [Eq. (9.43)], i.e.

$$\mathrm{d}E = -c\,\boldsymbol{T}(\vec{\boldsymbol{u}}_0, \vec{\boldsymbol{e}}_j)\,\mathrm{d}S\,\mathrm{d}t.$$

The ***energy flux***, i.e. the energy per unit time and per unit area that crosses $\mathscr{S}$, is thus $\varphi_j := \mathrm{d}E/(\mathrm{d}t\,\mathrm{d}S) = -c\,\boldsymbol{T}(\vec{\boldsymbol{u}}_0, \vec{\boldsymbol{e}}_j)$. The linear form $\boldsymbol{\varphi} := \varphi_j\,\boldsymbol{e}^j$, i.e.

$$\boxed{\boldsymbol{\varphi} = -c\,\boldsymbol{T}(\vec{\boldsymbol{u}}_0, \vec{\boldsymbol{e}}_j)\,\boldsymbol{e}^j}, \tag{19.9}$$

is then the ***energy-flux 1-form***: it gives the energy per unit time (power) that crosses a surface of normal $\vec{\boldsymbol{n}}$ according to

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \langle \boldsymbol{\varphi}, \vec{\boldsymbol{n}}\rangle\,\mathrm{d}S. \tag{19.10}$$

Finally, the linear momentum "contained" in $\mathscr{V}$ and measured by $\mathscr{O}$ —i.e. that crosses the surface $\mathscr{S}$ during the time lapse dt—is, by (9.44), d$\boldsymbol{P} = \mathrm{d}\boldsymbol{p} \circ \perp_{\boldsymbol{u}_0}$, i.e. from (19.8),

$$\mathrm{d}\boldsymbol{P} = \boldsymbol{T}(\perp_{\boldsymbol{u}_0}, \vec{\boldsymbol{e}}_j)\,\mathrm{d}S\,\mathrm{d}t = \boldsymbol{T}(\vec{\boldsymbol{e}}_i, \vec{\boldsymbol{e}}_j)\,\boldsymbol{e}^i\,\mathrm{d}S\,\mathrm{d}t.$$

The linear momentum per unit time, $\mathrm{d}\boldsymbol{P}/\mathrm{d}t$, that crosses $\mathscr{S}$ is the *force* $\mathrm{d}\boldsymbol{F}_j$ exerted on the surface $\mathscr{S}$ (relatively to $\mathscr{O}$). We have thus

$$\mathrm{d}\boldsymbol{F}_j = S_{ij}\,\boldsymbol{e}^i\,\mathrm{d}S, \tag{19.11}$$

with

$$\boxed{S_{ij} := \boldsymbol{T}(\vec{\boldsymbol{e}}_i, \vec{\boldsymbol{e}}_j)}. \tag{19.12}$$

The S_{ij}'s are actually the components of the tensor $\boldsymbol{S} = S_{ij}\,\boldsymbol{e}^i \otimes \boldsymbol{e}^j$, which is the orthogonal projection of the energy–momentum tensor onto $\mathscr{O}$'s local rest space:

$$\boldsymbol{S} := \boldsymbol{T}(\perp_{\boldsymbol{u}_0}, \perp_{\boldsymbol{u}_0}). \tag{19.13}$$

$\boldsymbol{S}$ is called the ***stress tensor*** relative to observer $\mathscr{O}$. It gives the force exerted on an elementary surface normal to $\vec{\boldsymbol{e}}_j$ via formula (19.11). More generally, on a surface of normal $\vec{\boldsymbol{n}} = n^j\,\vec{\boldsymbol{e}}_j$, the force is $\mathrm{d}\boldsymbol{F} = S_{ij}n^j\,\mathrm{d}S\,\boldsymbol{e}^i$, i.e.

$$\boxed{\mathrm{d}\boldsymbol{F} = \boldsymbol{S}(., \vec{\boldsymbol{n}})\,\mathrm{d}S}. \tag{19.14}$$

To summarize, the components $T_{\alpha\beta} = \boldsymbol{T}(\vec{\boldsymbol{e}}_\alpha, \vec{\boldsymbol{e}}_\beta)$ of the energy–momentum tensor in the local frame of observer $\mathscr{O}$ are

$$T_{\alpha\beta} = \begin{pmatrix} \varepsilon & -\varphi_1/c & -\varphi_2/c & -\varphi_3/c \\ -c\,\varpi_1 & S_{11} & S_{12} & S_{13} \\ -c\,\varpi_2 & S_{21} & S_{22} & S_{23} \\ -c\,\varpi_3 & S_{31} & S_{32} & S_{33} \end{pmatrix}, \tag{19.15}$$

where ε is the energy density, the ϖ_i's are the components of the linear-momentum density, the φ_i's those of the energy-flux 1-form and the S_{ij}'s those of the stress tensor. All these quantities are relative to observer $\mathscr{O}$.

19.2.3 Symmetry of the Energy–Momentum Tensor

For a system made of *simple* particles, in the sense defined in Sect. 9.2.1, the individual 4-momenta $\boldsymbol{p}_a$ are collinear to the 4-velocities: $\boldsymbol{p}_a = m_a c\, \underline{\boldsymbol{u}}_a$, m_a being the mass of particle $\mathscr{P}_a$. The energy–momentum tensor (19.3) takes then the following shape:

$$\forall M \in \mathscr{E}, \quad \boldsymbol{T}(M) = \sum_{a=1}^{N} m_a c^2 \int_{-\infty}^{+\infty} \delta_{A_a(\tau)}(M)\, \underline{\boldsymbol{u}}_a(\tau) \otimes \underline{\boldsymbol{u}}_a(\tau)\, c\, \mathrm{d}\tau. \tag{19.16}$$

The bilinear form $\underline{\boldsymbol{u}}_a(\tau) \otimes \underline{\boldsymbol{u}}_a(\tau)$ being obviously symmetric, we conclude that the same property holds for $\boldsymbol{T}$:

$$\forall (\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}) \in E^2, \quad \boxed{\boldsymbol{T}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}) = \boldsymbol{T}(\vec{\boldsymbol{v}}, \vec{\boldsymbol{u}})}. \tag{19.17}$$

We shall see in Sect. 19.4.2 that, as a consequence of the principle of angular momentum conservation, this property is fully general:

The energy–momentum tensor $\boldsymbol{T}$ of any physical system is a field of symmetric bilinear forms.

Remark 19.1. The energy–momentum tensor is thus sharing with the metric tensor $\boldsymbol{g}$ the property of being a symmetric bilinear form. However, the similitude does not go further: $\boldsymbol{T}$ can be a degenerate bilinear form (in particular, in vacuum, $\boldsymbol{T} = 0$), whereas $\boldsymbol{g}$ is never degenerate. Accordingly, $\boldsymbol{T}$ cannot be used to define a scalar product on E.

A consequence of the symmetry of $\boldsymbol{T}$ is that, with respect to an observer $\mathscr{O}$, $\boldsymbol{T}(\vec{\boldsymbol{u}}_0, \vec{\boldsymbol{e}}_i) = \boldsymbol{T}(\vec{\boldsymbol{e}}_i, \vec{\boldsymbol{u}}_0)$, which, from (19.7) and (19.9), implies the equality, up to a factor c^2, of the energy-flux 1-form and the linear-momentum density:

$$\boxed{\boldsymbol{\varphi} = c^2 \boldsymbol{\varpi}}. \tag{19.18}$$

Remark 19.2. This result can be seen as a consequence of the equivalence between mass and energy in relativity. Let us indeed express this equivalence as the relation (9.28) between the linear momentum $\boldsymbol{P}$ and the energy E of a particle: $E\, \vec{\boldsymbol{V}} = c^2 \vec{\boldsymbol{P}}$, where $\vec{\boldsymbol{V}}$ is the velocity of the particle relative to the considered observer (note that relation holds also for massless particles). If we consider a set of particles having the same velocity $\vec{\boldsymbol{V}}$ and divide by the volume of a matter element to let appear the energy density ε and the linear-momentum density $\vec{\boldsymbol{\varpi}}$, there comes

$$\varepsilon \vec{V} = c^2 \vec{\boldsymbol{\varpi}}.$$

$\varepsilon \vec{V}$ being nothing but the energy-flux vector, we recover (19.18).

Thanks to the symmetry of $\boldsymbol{T}$, the components (19.15) can be rewritten as

$$\boxed{T_{\alpha\beta} = \begin{pmatrix} \varepsilon & -c\varpi_1 & -c\varpi_2 & -c\varpi_3 \\ -c\varpi_1 & S_{11} & S_{12} & S_{13} \\ -c\varpi_2 & S_{12} & S_{22} & S_{23} \\ -c\varpi_3 & S_{13} & S_{23} & S_{33} \end{pmatrix}}. \tag{19.19}$$

From the very definition of the components of a bilinear form, $\boldsymbol{T} = T_{\alpha\beta}\boldsymbol{e}^{\alpha} \otimes \boldsymbol{e}^{\beta}$ [Eq. (14.10)], and given the relations $\boldsymbol{e}^0 = -\underline{\boldsymbol{u}}_0$, $\boldsymbol{\varpi} = \varpi_i\, \boldsymbol{e}^i$, $\boldsymbol{S} = S_{ij}\, \boldsymbol{e}^i \otimes \boldsymbol{e}^j$, we can write

$$\boxed{\boldsymbol{T} = \varepsilon\, \underline{\boldsymbol{u}}_0 \otimes \underline{\boldsymbol{u}}_0 + c\, \boldsymbol{\varpi} \otimes \underline{\boldsymbol{u}}_0 + c\, \underline{\boldsymbol{u}}_0 \otimes \boldsymbol{\varpi} + \boldsymbol{S}}. \tag{19.20}$$

Remark 19.3. This decomposition can be qualified as *orthogonal* since $\langle \boldsymbol{\varpi}, \vec{\boldsymbol{u}}_0 \rangle = 0$ and $\boldsymbol{S}(\vec{\boldsymbol{u}}_0, .) = \boldsymbol{S}(., \vec{\boldsymbol{u}}_0) = 0$. It is very general and applies to any symmetric bilinear form, provided that ε is defined as $\boldsymbol{T}(\vec{\boldsymbol{u}}_0, \vec{\boldsymbol{u}}_0)$, $\boldsymbol{\varpi}$ as $-c^{-1}\boldsymbol{T}(\perp_{\boldsymbol{u}_0}, \vec{\boldsymbol{u}}_0)$ and $\boldsymbol{S}$ as $\boldsymbol{T}(\perp_{\boldsymbol{u}_0}, \perp_{\boldsymbol{u}_0})$. This is the symmetric analogue of the orthogonal decomposition (3.37) of antisymmetric bilinear forms.

Historical note: *The concept of energy–momentum tensor has been introduced in 1908 by Hermann Minkowski (cf. p. 26) (1908). He however applied it only to the electromagnetic field.*[2] *It seems that the general use of the energy–momentum tensor to describe the dynamics of any type of matter or field is due to Max Laue (cf. p. 146). He gave the general decomposition (19.15) in 1911 (Laue 1911b). The property (19.18) expressing the equality (up to a factor c^2) of the energy flux and the linear-momentum density has been stated by Max Planck (cf. p. 279) in 1908 (Planck 1908). In the particular case of the electromagnetic field, it had been established in 1900 by Henri Poincaré (1900) (cf. the historical note p. 649).*

19.3 Energy–Momentum Conservation

In Chap. 9, we have stated the principle of 4-momentum conservation for any discrete system of particles. We shall now extend this principle to any continuous system described by an energy–momentum tensor.

[2] We shall discuss specifically the energy–momentum tensor of the electromagnetic field in Chap. 20.

19.3.1 Statement

As in Sect. 9.3.3, the principle of energy–momentum conservation is expressed by means of closed hypersurfaces:

> If a physical system $\mathscr{S}$ described by an energy–momentum tensor $\boldsymbol{T}$ is isolated, its 4-momentum $\boldsymbol{p}|_{\mathscr{V}}$ on any closed hypersurface $\mathscr{V} \subset \mathscr{E}$ vanishes:
>
> $$\boxed{\mathscr{S} \text{ isolated and } \mathscr{V} \text{ closed} \implies \boldsymbol{p}|_{\mathscr{V}} = 0}. \qquad (19.21)$$

Let us recall that $\boldsymbol{p}|_{\mathscr{V}}$ is related to $\boldsymbol{T}$ by (19.2). The same comments as in Sect. 9.3.3 are relevant. In particular, one may change the point of view and consider the above statement not as a principle but as the *definition* of a an isolated system.

19.3.2 Local Version

Let us apply the linear form $\boldsymbol{p}|_{\mathscr{V}}$ to a generic vector $\vec{\boldsymbol{v}} \in E$, using (19.2) :

$$\langle \boldsymbol{p}|_{\mathscr{V}}, \vec{\boldsymbol{v}}\rangle = \pm\frac{1}{c}\int_{\mathscr{V}} \boldsymbol{T}(\vec{\boldsymbol{v}}, \vec{\boldsymbol{n}})\,\mathrm{d}V = \pm\frac{1}{c}\int_{\mathscr{V}} \vec{\boldsymbol{w}}\cdot\vec{\boldsymbol{n}}\,\mathrm{d}V = \frac{1}{c}\Phi_{\mathscr{V}}(\vec{\boldsymbol{w}}), \qquad (19.22)$$

where $\vec{\boldsymbol{w}}$ is the vector field metric dual of the 1-form $\boldsymbol{T}(\vec{\boldsymbol{v}}, .)$; its components are $w^\alpha = g^{\alpha\nu}T_{\mu\nu}v^\mu$. The last equality in (19.22) involves the flux of $\vec{\boldsymbol{w}}$ through $\mathscr{V}$ (cf. Sect. 16.4.7). Let us choose for the hypersurface $\mathscr{V}$ the boundary of a compact four-dimensional domain $\mathscr{U} \subset \mathscr{E}$: $\mathscr{V} = \partial\mathscr{U}$. We may then apply Gauss–Ostrogradsky theorem (16.64) and write

$$\langle \boldsymbol{p}|_{\mathscr{V}}, \vec{\boldsymbol{v}}\rangle = \frac{1}{c}\int_{\mathscr{U}} \nabla\cdot\vec{\boldsymbol{w}}\,\mathrm{d}U. \qquad (19.23)$$

Now

$$\nabla\cdot\vec{\boldsymbol{w}} = \nabla_\nu w^\nu = \nabla^\nu w_\nu = \nabla^\nu(T_{\mu\nu}v^\mu) = \nabla^\nu T_{\mu\nu}\,v^\mu + T_{\mu\nu}\underbrace{\nabla^\nu v^\mu}_{0}, \qquad (19.24)$$

the vanishing of the last term resulting from the constant character of $\vec{\boldsymbol{v}}$. In the last but one term, there appears the divergence of the energy–momentum tensor. In Sect. 15.4.6, we have defined the divergence of a tensor field that is at list one time contravariant. Here, the divergence of $\boldsymbol{T}$, which is zero time contravariant and twice covariant, is defined as the 1-form metric dual of the divergence of $\boldsymbol{T}^\sharp$.

The latter is the metric dual of $\boldsymbol{T}$, namely, the analogue for $\boldsymbol{T}$ of the tensor $\boldsymbol{F}^\sharp$ introduced in Sect. 17.2.5 for $\boldsymbol{F}$. The divergence $\nabla\cdot\boldsymbol{T}^\sharp$ as defined in Sect. 15.4.6 has for components $\nabla_\mu T^{\alpha\mu}$. Its metric dual, that we shall define as the ***divergence of T*** and denote by $\vec{\nabla}\cdot\boldsymbol{T}$, has the following components:

$$\boxed{(\vec{\nabla}\cdot\boldsymbol{T})_\alpha = \nabla^\mu T_{\alpha\mu} = g^{\mu\nu}\nabla_\nu T_{\alpha\mu}}. \tag{19.25}$$

In view of (19.23) and (19.24), we can thus write

$$\langle \boldsymbol{p}|_{\mathscr{V}}, \vec{\boldsymbol{v}}\rangle = \frac{1}{c}\int_{\mathscr{U}} \langle \vec{\nabla}\cdot\boldsymbol{T}, \vec{\boldsymbol{v}}\rangle \, \mathrm{d}U.$$

This identity being valid for any vector $\vec{\boldsymbol{v}} \in E$, we deduce that

$$\boxed{\boldsymbol{p}|_{\mathscr{V}} = \frac{1}{c}\int_{\mathscr{U}} \vec{\nabla}\cdot\boldsymbol{T} \, \mathrm{d}U}. \tag{19.26}$$

As the boundary of $\mathscr{U}$, $\mathscr{V}$ is necessarily a closed hypersurface. If the system $\mathscr{S}$ is isolated, the principle of energy–momentum conservation leads to the vanishing of the above integral. This result being valid whatever $\mathscr{U}$, we may conclude:

For an isolated system,

$$\boxed{\vec{\nabla}\cdot\boldsymbol{T} = 0}_{\ \mathscr{S}\ \text{isolated}}. \tag{19.27}$$

This is the local expression of the principle of energy–momentum conservation.

19.3.3 Four-Force Density

If the system $\mathscr{S}$ is not isolated, we set

$$\boxed{\vec{\nabla}\cdot\boldsymbol{T} =: \mathscr{F}}. \tag{19.28}$$

The 1-form $\mathscr{F}$ is called ***four-force density***, or ***4-force density*** for short.

In order to interpret $\mathscr{F}$, let us consider some inertial observer $\mathscr{O}$ of proper time t and 4-velocity $\vec{\boldsymbol{u}}_0$. Let $\mathscr{S} = \mathscr{S}(t)$ be a closed surface in $\mathscr{O}$'s rest space $\mathscr{E}_{\boldsymbol{u}_0}(t)$

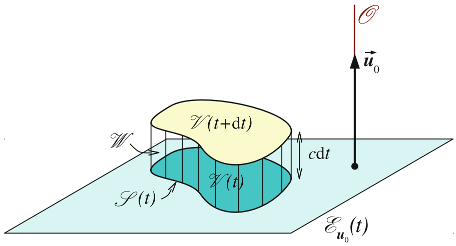

Fig. 19.2 Worldtube swept by the volume $\mathscr{V}(t)$, delimited by the surface $\mathscr{S}(t)$ in the local rest space of observer $\mathscr{O}$, between t and $t + \mathrm{d}t$

and let $\mathscr{V} = \mathscr{V}(t)$ be the volume delimited by $\mathscr{S}$ (cf. Fig. 19.2). $\mathscr{S}$ and $\mathscr{V}$ are assumed to be at rest with respect to $\mathscr{O}$. Let us define the system's 4-momentum in $\mathscr{V}$ at the instant t as $\boldsymbol{p}_{\mathscr{V}}(t) := \boldsymbol{p}|_{\mathscr{V}(t)}$, the orientation of $\mathscr{V}(t)$ being given by the normal $\vec{\boldsymbol{u}}_0$. For an infinitesimal increment $\mathrm{d}t$ of t, let us consider the worldtube $\mathscr{U}$ swept by $\mathscr{V}(t)$ during $\mathrm{d}t$. Its boundary is the union of $\mathscr{V}(t + \mathrm{d}t)$, $\mathscr{V}(t)$ and $\mathscr{W}$, the hypercylinder swept by $\mathscr{S}$ between t and $t + \mathrm{d}t$ (cf. Fig. 19.2). Using (19.26) with $\vec{\nabla} \cdot \boldsymbol{T}$ replaced by $\mathscr{F}$, there comes

$$\boldsymbol{p}_{\mathscr{V}}(t + \mathrm{d}t) - \boldsymbol{p}_{\mathscr{V}}(t) + \boldsymbol{p}|_{\mathscr{W}} = \frac{1}{c} \int_{\mathscr{U}} \mathscr{F} \,\mathrm{d}U = \frac{1}{c} \times c \,\mathrm{d}t \int_{\mathscr{V}} \mathscr{F} \,\mathrm{d}V, \qquad (19.29)$$

the $-$ sign in front of $\boldsymbol{p}_{\mathscr{V}}(t)$ resulting from the change of orientation of $\mathscr{V}(t)$ when considered as a part of $\mathscr{U}$'s boundary. Let us evaluate $\boldsymbol{p}|_{\mathscr{W}}$: the normal $\vec{\boldsymbol{n}}$ to $\mathscr{W}$ [and to $\mathscr{S}$ within $(\mathscr{E}_{\boldsymbol{u}_0}, \boldsymbol{g})$] being spacelike, formula (19.2) leads to

$$\boldsymbol{p}|_{\mathscr{W}} = \frac{1}{c} \int_{\mathscr{W}} \boldsymbol{T}(., \vec{\boldsymbol{n}}) \,\mathrm{d}V = \mathrm{d}t \int_{\mathscr{S}} \boldsymbol{T}(., \vec{\boldsymbol{n}}) \,\mathrm{d}S = \mathrm{d}t \int_{\mathscr{S}} \left[c \langle \boldsymbol{\varpi}, \vec{\boldsymbol{n}} \rangle \underline{\boldsymbol{u}}_0 + \boldsymbol{S}(., \vec{\boldsymbol{n}}) \right] \mathrm{d}S,$$

where we have used the decomposition (19.20) of $\boldsymbol{T}$ with respect to observer $\mathscr{O}$. Inserting this result in (19.29) and dividing by $\mathrm{d}t$, we get

$$\boxed{\frac{\mathrm{d}\boldsymbol{p}_{\mathscr{V}}}{\mathrm{d}t} = \int_{\mathscr{V}} \mathscr{F} \,\mathrm{d}V - c \int_{\mathscr{S}} \langle \boldsymbol{\varpi}, \vec{\boldsymbol{n}} \rangle \,\mathrm{d}S \,\underline{\boldsymbol{u}}_0 - \int_{\mathscr{S}} \boldsymbol{S}(., \vec{\boldsymbol{n}}) \,\mathrm{d}S}. \qquad (19.30)$$

The two integrals over $\mathscr{S}$ are, respectively, the energy flux through $\mathscr{S}$ and the surface force exerted on $\mathscr{S}$ [cf. Eq. (19.11)]. Identity (19.30) justifies the name *four-force density* given to $\mathscr{F}$: at a sufficiently small scale, so that the volume $\mathscr{V}$ can be considered as a particle, the worldline of this "particle" is parallel to that of $\mathscr{O}$; therefore, its proper time is t. According to the definition (9.104), $\mathrm{d}\boldsymbol{p}_{\mathscr{V}}/\mathrm{d}t$ appears then as the 4-force exerted on the particle. If no energy is entering into the volume $\mathscr{V}$ and no force is exerted at its surface, $\mathscr{F}$ is actually the 4-force volume density.

For a discrete system made of N particles, $(\mathscr{P}_a)_{1 \le a \le N}$, the 4-force density is of the type

$$\forall M \in \mathscr{E}, \quad \boxed{\mathscr{F}(M) = \sum_{a=1}^{N} \int_{-\infty}^{+\infty} \delta_{A_a(\tau)}(M)\ \boldsymbol{f}_a(\tau)\ c\,\mathrm{d}\tau}, \tag{19.31}$$

where $\boldsymbol{f}_a(\tau)$ is the 4-force exerted on particle $\mathscr{P}_a$ at the instant τ of its proper time.

Proof. Let us integrate (19.31) over a volume $\mathscr{V}$ associated with some inertial observer $\mathscr{O}$ and sufficiently large to encompass all the particles of the system:

$$\begin{aligned}\int_{\mathscr{V}} \mathscr{F}\,\mathrm{d}V &= \sum_{a=1}^{N} \int_{\mathscr{V}} \int_{-\infty}^{+\infty} \delta_{A_a(\tau)}(M)\ \boldsymbol{f}_a(\tau)\ c\,\mathrm{d}\tau\,\mathrm{d}V \\ &= \sum_{a=1}^{N} \int_{\mathscr{V}} \int_{-\infty}^{+\infty} \delta_{A_a(t)}(M)\ \boldsymbol{f}_a(t)\ \frac{c\,\mathrm{d}t}{\Gamma_a}\,\mathrm{d}V = \sum_{a=1}^{N} \frac{1}{\Gamma_a}\,\boldsymbol{f}_a.\end{aligned} \tag{19.32}$$

The computation leading to the last equality is very similar to that leading to (18.6), and we shall not repeat it here. Besides, we have, from (19.1),

$$\boldsymbol{p}_{\mathscr{V}}(t) = \sum_{a=1}^{N} \boldsymbol{p}_a(\tau_a(t)),$$

$\tau_a(t)$ being the proper time of particle $\mathscr{P}_a$ when its worldline encounters $\mathscr{O}$'s rest space at the instant t, $\mathscr{E}_{\boldsymbol{u}_0}(t)$. Deriving with respect to t, we get

$$\frac{\mathrm{d}\boldsymbol{p}_{\mathscr{V}}}{\mathrm{d}t} = \sum_{a=1}^{N} \underbrace{\frac{\mathrm{d}\boldsymbol{p}_a}{\mathrm{d}\tau_a}}_{\boldsymbol{f}_a} \underbrace{\frac{\mathrm{d}\tau_a}{\mathrm{d}t}}_{\Gamma_a^{-1}} = \sum_{a=1}^{N} \frac{1}{\Gamma_a}\,\boldsymbol{f}_a,$$

where $\Gamma_a = \Gamma_a(\tau_a)$ is the Lorentz factor of $\mathscr{P}_a$ with respect to $\mathscr{O}$. Comparing with (19.32), we recover (19.30), with the surface terms set to zero. This establishes that the 4-force density of a particle system is given by (19.31). □

19.3.4 Conservation of Energy and Momentum with Respect to an Observer

Let us consider an inertial observer $\mathscr{O}$ and a physical system $\mathscr{S}$ described by some energy–momentum tensor $\boldsymbol{T}$. Let us substitute the orthogonal decomposition (19.20) for $\boldsymbol{T}$ in the equation $\vec{\nabla} \cdot \boldsymbol{T} = \mathscr{F}$ [Eq. (19.28)]. Since $\vec{\boldsymbol{u}}_0$ is constant ($\mathscr{O}$ inertial), we obtain

$$(\nabla_{\vec{u}_0}\,\varepsilon)\underline{u}_0 + c\nabla_{\vec{u}_0}\,\boldsymbol{\varpi} + c(\vec{\nabla}\cdot\boldsymbol{\varpi})\,\underline{u}_0 + \vec{\nabla}\cdot\boldsymbol{S} = \mathscr{F}. \tag{19.33}$$

Projecting this relation on $\vec{u}_0$ (i.e. applying the linear form appearing at each side to the vector $\vec{u}_0$), there comes

$$-\nabla_{\vec{u}_0}\,\varepsilon + c\nabla_{\vec{u}_0}\,\underbrace{\langle\boldsymbol{\varpi},\vec{u}_0\rangle}_{0} - c\vec{\nabla}\cdot\boldsymbol{\varpi} + \vec{\nabla}\cdot\underbrace{\boldsymbol{S}(\vec{u}_0,.)}_{0} = \langle\mathscr{F},\vec{u}_0\rangle.$$

Since $\nabla_{\vec{u}_0}\,\varepsilon = c^{-1}\partial\varepsilon/\partial t$ [cf. (15.28)], we get

$$\boxed{\frac{\partial\varepsilon}{\partial t} + c^2\vec{\nabla}\cdot\boldsymbol{\varpi} = -c\langle\mathscr{F},\vec{u}_0\rangle}. \tag{19.34}$$

This is the equation expressing the ***energy conservation*** with respect to observer $\mathscr{O}$: it relates the time derivative of the energy density ε to the divergence of the energy flux $c^2\boldsymbol{\varpi}$ [cf. (19.18)] and to the power per unit volume provided to the system, $-c\langle\mathscr{F},\vec{u}_0\rangle$ [cf. (9.111a) with $\vec{a}_0 = 0$].

On the other side, if we project (19.33) on $\mathscr{O}$'s rest space E_{u_0} (i.e. if we compose the linear form on each side of (19.33) with the orthogonal projector $\perp_{u_0}$), we get, taking into account that $\underline{u}_0\circ\perp_{u_0} = 0$, $\boldsymbol{\varpi}\circ\perp_{u_0} = \boldsymbol{\varpi}$, $\boldsymbol{S}\circ\perp_{u_0} = \boldsymbol{S}$ and $\nabla_{\vec{u}_0}\,\boldsymbol{\varpi} = c^{-1}\partial\boldsymbol{\varpi}/\partial t$,

$$\boxed{\frac{\partial\boldsymbol{\varpi}}{\partial t} + \vec{\nabla}\cdot\boldsymbol{S} = \mathscr{F}\circ\perp_{u_0}}. \tag{19.35}$$

This is the equation expressing the ***linear-momentum conservation*** with respect to observer $\mathscr{O}$: it relates the time derivative of the linear-momentum density $\boldsymbol{\varpi}$ to the divergence of the stress tensor $\boldsymbol{S}$ and to the volume density of external forces exerted on the system, $\mathscr{F}\circ\perp_{u_0}$.

19.4 Angular Momentum

19.4.1 Definition

Given a physical system $\mathscr{S}$ of energy–momentum tensor $\boldsymbol{T}$, a point $C\in\mathscr{E}$ and an oriented hypersurface $\mathscr{V}\subset\mathscr{E}$ that is not null, one calls ***angular momentum of $\mathscr{S}$ with respect to C on $\mathscr{V}$*** the following antisymmetric bilinear form (2-form):

$$\boxed{\boldsymbol{J}_C|_{\mathscr{V}} := \pm\frac{1}{c}\int_{\mathscr{V}}\underline{CM}\wedge\boldsymbol{T}(.,\vec{n})\,\mathrm{d}V}, \tag{19.36}$$

where M is the generic point of $\mathscr{V}$, over which the integration is performed; $\vec{n} = \vec{n}(M)$ is the unit normal to $\mathscr{V}$ compatible with $\mathscr{V}$'s orientation; and the sign $\pm$ is a shorthand writing to indicate that the integral must be split into various parts depending on the kind of $\mathscr{V}$, with a $+$ sign where $\mathscr{V}$ is timelike ($\vec{n}$ is then spacelike) and a $-$ sign where $\mathscr{V}$ is spacelike ($\vec{n}$ is then timelike). Let us recall that $\underline{CM} \wedge \boldsymbol{T}(., \vec{n})$ stands for the exterior product of the 1-form $\underline{CM}$ by the 1-form $\boldsymbol{T}(., \vec{n})$, according to the definition (14.43): $\underline{CM} \wedge \boldsymbol{T}(., \vec{n}) := \underline{CM} \otimes \boldsymbol{T}(., \vec{n}) - \boldsymbol{T}(., \vec{n}) \otimes \underline{CM}$. As for (19.2), the above integral is the integral of a tensor field on $\mathscr{E}$ (here a tensor field of type $(0, 2)$), as defined in Sect. 16.4.6.

For a discrete system made of N particles, $(\mathscr{P}_a)_{1 \le a \le N}$, the energy–momentum tensor takes the form (19.3). Inserting it in (19.36), we obtain

$$\boldsymbol{J}_C|_{\mathscr{V}} = \pm \sum_{a=1}^{N} \int_{\mathscr{V}} \int_{-\infty}^{+\infty} [\underline{CM} \wedge \boldsymbol{p}_a(\tau)] \, \delta_{A_a(\tau)}(M) \, (\vec{\boldsymbol{u}}_a(\tau) \cdot \vec{\boldsymbol{n}}) \, c \, \mathrm{d}\tau \, \mathrm{d}V.$$

We can then introduce a coordinate system (x^α) adapted to $\mathscr{V}$, and by a computation similar to that performed in Sect. 19.2.1, we obtain

$$\boldsymbol{J}_C|_{\mathscr{V}} = \sum_{a=1}^{N} \sum_{A \in \mathscr{L}_a \cap \mathscr{V}} \varepsilon \, \underline{CA} \wedge \boldsymbol{p}_a(A), \tag{19.37}$$

where $\varepsilon = +1$ (resp. $\varepsilon = -1$) if the 4-momentum vector $\vec{\boldsymbol{p}}_a(A)$ associated with $\boldsymbol{p}_a(A)$ has the sense (resp. the opposite sense) given by the orientation of $\mathscr{V}$. Comparing with (10.16), we recover the definition of the angular momentum of a particle system given in Chap. 10. This fully justifies the definition (19.36).

19.4.2 Angular Momentum Conservation

In a manner similar to the principle of energy–momentum conservation (Sect. 19.3.1), we shall state the principle of angular momentum conservation as follows:

If a physical system $\mathscr{S}$ is isolated, its angular momentum $\boldsymbol{J}_C|_{\mathscr{V}}$ with respect to any point $C \in \mathscr{E}$ and on any closed hypersurface $\mathscr{V} \subset \mathscr{E}$ vanishes:

$$\boxed{\mathscr{S} \text{ isolated and } \mathscr{V} \text{ closed} \implies \boldsymbol{J}_C|_{\mathscr{V}} = 0}. \tag{19.38}$$

This principle is of course a generalization of that seen in Sect. 10.4.

Let us show that, in conjunction with the principle of energy–momentum conservation, the principle of angular momentum conservation leads to the symmetry of the energy–momentum tensor stated in Sect. 19.2.3. We start by applying the definition (19.36) to a couple $(\vec{v}, \vec{w})$ of (constant) vectors in E:

$$\begin{aligned} c\, \boldsymbol{J}_C|_{\mathscr{V}}(\vec{v}, \vec{w}) &= \pm \int_{\mathscr{V}} \left[(\overrightarrow{CM} \cdot \vec{v}) \boldsymbol{T}(\vec{w}, \vec{n}) - (\overrightarrow{CM} \cdot \vec{w}) \boldsymbol{T}(\vec{v}, \vec{n}) \right] \mathrm{d}V \\ &= \pm \int_{\mathscr{V}} \vec{a} \cdot \vec{n}\, \mathrm{d}V \mp \int_{\mathscr{V}} \vec{b} \cdot \vec{n}\, \mathrm{d}V = \Phi_{\mathscr{V}}(\vec{a}) - \Phi_{\mathscr{V}}(\vec{b}), \end{aligned}$$

where the vectors $\vec{a}$ and $\vec{b}$ are the metric duals of the following 1-forms:

$$\underline{a} := (\overrightarrow{CM} \cdot \vec{v}) \boldsymbol{T}(\vec{w}, .) \qquad \text{and} \qquad \underline{b} := (\overrightarrow{CM} \cdot \vec{w}) \boldsymbol{T}(\vec{v}, .).$$

If $\mathscr{V}$ is the boundary of a compact four-dimensional domain of $\mathscr{E}$, $\mathscr{V} = \partial \mathscr{U}$, Gauss–Ostrogradsky theorem (16.64) yields

$$c\, \boldsymbol{J}_C|_{\mathscr{V}}(\vec{v}, \vec{w}) = \int_{\mathscr{U}} \left(\nabla \cdot \vec{a} - \nabla \cdot \vec{b} \right) \mathrm{d}U.$$

Now, $\vec{v}$ and $\vec{w}$ being constant vectors, we have

$$\begin{aligned} \nabla \cdot \vec{a} &= \nabla_\rho a^\rho = \nabla_\rho \left[(CM)^\sigma v_\sigma T_{\mu\nu} w^\mu g^{\nu\rho} \right] \\ &= \underbrace{\nabla_\rho (CM)^\sigma}_{\delta^\sigma{}_\rho} v_\sigma T_{\mu\nu} w^\mu g^{\nu\rho} + (CM)^\sigma v_\sigma \nabla_\rho T_{\mu\nu} w^\mu g^{\nu\rho} \\ &= g^{\nu\sigma} v_\sigma T_{\mu\nu} w^\mu + (CM)^\sigma v_\sigma \nabla^\nu T_{\mu\nu} w^\mu = T_{\mu\nu} w^\mu v^\nu + (CM)^\sigma v_\sigma \nabla^\nu T_{\mu\nu} w^\mu. \end{aligned}$$

If the system is isolated, the energy–momentum conservation gives $\nabla^\nu T_{\mu\nu} = 0$, so that only the first term remains: $\nabla \cdot \vec{a} = \boldsymbol{T}(\vec{w}, \vec{v})$. Similarly, $\nabla \cdot \vec{b} = \boldsymbol{T}(\vec{v}, \vec{w})$. Thus,

$$c\, \boldsymbol{J}_C|_{\mathscr{V}}(\vec{v}, \vec{w}) = \int_{\mathscr{U}} \left[\boldsymbol{T}(\vec{w}, \vec{v}) - \boldsymbol{T}(\vec{v}, \vec{w}) \right] \mathrm{d}U.$$

Since $\mathscr{V}$ is a closed hypersurface, the principle of angular momentum conservation (19.38) leads then to the vanishing of the integral in the right-hand side. The domain $\mathscr{U}$ being arbitrary, we conclude that $\boldsymbol{T}(\vec{w}, \vec{v}) - \boldsymbol{T}(\vec{v}, \vec{w}) = 0$. This establishes the symmetry of the energy–momentum tensor stated in Sect. 19.2.3.

Remark 19.4. To establish the symmetry of $\boldsymbol{T}$ from the principle of angular momentum conservation, we have used only the property $\vec{\nabla} \cdot \boldsymbol{T} = 0$. The latter has been established in Sect. 19.3.2 without appealing to the symmetry of $\boldsymbol{T}$; there is thus no loophole in the above demonstration.

Chapter 20
Energy–Momentum of the Electromagnetic Field

20.1 Introduction

The study of the electromagnetic field, started in Chaps. 17 and 18, is completed in this chapter, which focuses on the energetic aspects. We shall see in Sect. 20.2 that some energy–momentum can be associated with the electromagnetic field and that it can be described by an energy–momentum tensor. We shall treat in detail the case of the field created by an accelerated electric charge (Sect. 20.3), computing the total radiated power as well as the radiation pattern. A particular case of an accelerated charge is that of a particle moving on a helical trajectory in a magnetic field, as studied in Chap. 17. It gives birth to *synchrotron radiation*, which we shall discuss in Sect. 20.4. This type of radiation plays an important role in astrophysics and has many applications on Earth.

20.2 Energy–Momentum Tensor of the Electromagnetic Field

20.2.1 Introduction

Let us consider a set of N charged particles $(\mathscr{P}_a, q_a)_{1\leq a\leq N}$ in some electromagnetic field $\boldsymbol{F}$. Each particle is submitted to the Lorentz 4-force $\boldsymbol{f}_a = q_a \boldsymbol{F}(., \vec{\boldsymbol{u}}_a)$ [Eq. (17.1)] ($\vec{\boldsymbol{u}}_a$ being $\mathscr{P}_a$'s 4-velocity). From (19.31), the 4-force density exerted by the electromagnetic field on the particle system is then

$$\begin{aligned}\mathscr{F}(M) &= \sum_{a=1}^{N} q_a \int_{-\infty}^{+\infty} \delta_{A_a(\tau)}(M)\ \boldsymbol{F}(., \vec{\boldsymbol{u}}_a(\tau))\ c\,\mathrm{d}\tau \\ &= \boldsymbol{F}\left(\ .\ , \sum_{a=1}^{N} q_a \int_{-\infty}^{+\infty} \delta_{A_a(\tau)}(M)\ \vec{\boldsymbol{u}}_a(\tau)\ c\,\mathrm{d}\tau\right),\end{aligned}$$

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the second equality resulting from the bilinearity of $\boldsymbol{F}$. We recognize in the second argument of $\boldsymbol{F}$ the electric 4-current $\vec{\boldsymbol{j}}$ of the particle system, as given by (18.5). The electromagnetic 4-force density takes therefore a very simple form:

$$\boxed{\mathscr{F} = \boldsymbol{F}(.,\vec{\boldsymbol{j}})}. \tag{20.1}$$

If the electromagnetic field is entirely created by the considered charge distribution, then $\boldsymbol{F}$ and $\vec{\boldsymbol{j}}$ are related by the Maxwell equation (18.22b): $\vec{\boldsymbol{j}} = \varepsilon_0 \boldsymbol{\nabla} \cdot \boldsymbol{F}^\sharp$, and (20.1) becomes, once written in terms of components in some basis $(\vec{\boldsymbol{e}}_\alpha)$,

$$\begin{aligned} \mathscr{F}_\alpha &= F_{\alpha\mu} j^\mu = \varepsilon_0 F_{\alpha\mu} \nabla_\beta F^{\mu\beta} = -\varepsilon_0 F_{\mu\alpha} \nabla_\beta F^{\mu\beta} \\ &= -\varepsilon_0 \left[\nabla_\beta (F_{\mu\alpha} F^{\mu\beta}) - F^{\mu\beta} \nabla_\beta F_{\mu\alpha} \right]. \end{aligned} \tag{20.2}$$

Let us rewrite the last term via the Maxwell equation $\mathbf{d}\boldsymbol{F} = 0$ [Eq. (18.16a)]. Expressing the exterior derivative by means of the covariant derivative according to (15.64), Eq. (18.16a) becomes $\nabla_\beta F_{\mu\alpha} + \nabla_\mu F_{\alpha\beta} + \nabla_\alpha F_{\beta\mu} = 0$; hence,

$$\begin{aligned} & F^{\mu\beta} \nabla_\beta F_{\mu\alpha} + F^{\mu\beta} \nabla_\mu F_{\alpha\beta} + F^{\mu\beta} \nabla_\alpha F_{\beta\mu} = 0, \\ & \underbrace{F^{\mu\beta} \nabla_\beta F_{\mu\alpha} + F^{\beta\mu} \nabla_\mu F_{\beta\alpha}}_{2F^{\mu\beta}\nabla_\beta F_{\mu\alpha}} - \underbrace{F^{\mu\beta} \nabla_\alpha F_{\mu\beta}}_{1/2\,\nabla_\alpha (F^{\mu\beta} F_{\mu\beta})} = 0, \\ \Longrightarrow\ & F^{\mu\beta} \nabla_\beta F_{\mu\alpha} = \frac{1}{4} \nabla_\alpha (F_{\mu\nu} F^{\mu\nu}). \end{aligned}$$

Thanks to this formula, (20.2) can be recast as

$$\begin{aligned} \mathscr{F}_\alpha &= -\varepsilon_0 \left[\nabla_\beta (F_{\mu\alpha} F^{\mu\beta}) - \frac{1}{4} \nabla_\alpha (F_{\mu\nu} F^{\mu\nu}) \right] \\ &= -\varepsilon_0 \left[\nabla^\beta (F_{\mu\alpha} F^\mu{}_\beta) - \frac{1}{4} g_{\alpha\beta} \nabla^\beta (F_{\mu\nu} F^{\mu\nu}) \right]. \end{aligned}$$

Consequently, by defining

$$\boxed{T^{\rm em}_{\alpha\beta} := \varepsilon_0 \left(F_{\mu\alpha} F^\mu{}_\beta - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}\, g_{\alpha\beta} \right)}, \tag{20.3}$$

we can write

$$\mathscr{F}_\alpha = -\nabla^\beta T^{\rm em}_{\alpha\beta}. \tag{20.4}$$

Hence the 4-force density exerted on the particle system by the electromagnetic field appears as minus the divergence of the tensor $\boldsymbol{T}^{\rm em}$, whose components are given by (20.3). Let us denote by $\boldsymbol{T}^{\text{mat}}$ the energy–momentum tensor of the particle system, i.e. the tensor given by (19.3), 'mat' standing for 'matter'. The energy–momentum conservation equation (19.28) is then $\vec{\nabla} \cdot \boldsymbol{T}^{\text{mat}} = \mathscr{F}$, i.e., in view of (20.4),

$$\boxed{\vec{\nabla} \cdot \left(\boldsymbol{T}^{\text{mat}} + \boldsymbol{T}^{\text{em}}\right) = 0}. \tag{20.5}$$

Since the system of charged particles is not isolated, but submitted to the electromagnetic field, it is not surprising that its energy–momentum is not conserved: $\vec{\nabla} \cdot \boldsymbol{T}^{\text{mat}} = \mathscr{F} \neq 0$. However, the result (20.5) shows that the sum $\boldsymbol{T}^{\text{mat}} + \boldsymbol{T}^{\text{em}}$ is a tensor with vanishing divergence; it leads then, by integration over a hypersurface, to a conserved quantity, as discussed in Sect. 19.3. This fully justifies to attribute some energy–momentum to the electromagnetic field and to consider $\boldsymbol{T}^{\text{em}}$ as the ***energy–momentum tensor of the electromagnetic field***. Equation (20.5) expresses then the conservation of the *total* energy–momentum of the system formed by the charged particles and the electromagnetic field. Moreover, it is clear on (20.3) that $\boldsymbol{T}^{\text{em}}$ is a symmetric bilinear form, as it should be for a valid energy–momentum tensor (cf. Sect. 19.2.3).

Remark 20.1. The quantity $F_{\mu\nu} F^{\mu\nu}/4$, which appears in the expression (20.3) of $\boldsymbol{T}^{\text{em}}$, is nothing but half the electromagnetic field invariant I_1 [cf. Eq. (17.36)].

Example 20.1. Let us consider the electromagnetic field created by a charged particle $\mathscr{P}$ in inertial motion (Coulombian field), as studied in Sect. 18.6.4. We are using the same notations as in that section: $\vec{\boldsymbol{u}}$ for the constant 4-velocity of $\mathscr{P}$ and P for the intersection of $\mathscr{P}$'s worldline with the past light cone of the point M where the field is evaluated. Given expression (18.110) for $\boldsymbol{F}$, we have

$$\begin{aligned} F_{\mu\alpha} F^{\mu}_{\ \ \beta} &= \left(\frac{q}{4\pi\varepsilon_0 R^3}\right)^2 [u_\mu (PM)_\alpha - u_\alpha (PM)_\mu][u^\mu (PM)_\beta - u_\beta (PM)^\mu] \\ &= \left(\frac{q}{4\pi\varepsilon_0 R^3}\right)^2 \left\{R[u_\alpha (PM)_\beta + u_\beta (PM)_\alpha] - (PM)_\alpha (PM)_\beta\right\}, \end{aligned}$$

where we have used $u_\mu (PM)^\mu = -R$ [Eq. (18.85)] and $(PM)_\mu (PM)^\mu = 0$ (P is located on the light cone of M). Substituting this value in (20.3) and using expression (18.103) for $F_{\mu\nu} F^{\mu\nu}/2$, we get

$$T^{\text{em}}_{\alpha\beta}(M) = \frac{q^2}{16\pi^2 \varepsilon_0 R^4} \left[\frac{u_\alpha (PM)_\beta + u_\beta (PM)_\alpha}{R} - \frac{(PM)_\alpha (PM)_\beta}{R^2} + \frac{1}{2} g_{\alpha\beta}\right],$$

i.e.

$$\boldsymbol{T}^{\text{em}}(M) = \frac{q^2}{16\pi^2 \varepsilon_0 R^4} \left[\frac{1}{R}(\underline{\boldsymbol{u}} \otimes \underline{PM} + \underline{PM} \otimes \underline{\boldsymbol{u}}) - \frac{1}{R^2} \underline{PM} \otimes \underline{PM} + \frac{1}{2} \boldsymbol{g}\right]. \tag{20.6}$$

20.2.2 Quantities Relative to an Observer

Let $\mathscr{O}$ be an observer of 4-velocity $\vec{\boldsymbol{u}}_0$ and local frame $(\vec{\boldsymbol{e}}_\alpha) = (\vec{\boldsymbol{u}}_0, \vec{\boldsymbol{e}}_i)$. The electromagnetic energy density measured by $\mathscr{O}$ is, according to (19.5), $\rho_{\rm em} = \boldsymbol{T}^{\rm em}(\vec{\boldsymbol{u}}_0, \vec{\boldsymbol{u}}_0)$. Substituting (20.3) for $\boldsymbol{T}^{\rm em}$, we get

$$\rho_{\rm em} = T^{\rm em}_{\alpha\beta} u_0^\alpha u_0^\beta = \varepsilon_0 \Bigg(\underbrace{F_{\mu\alpha} u_0^\alpha}_{E_\mu} \underbrace{F^\mu{}_\beta u_0^\beta}_{E^\mu} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \underbrace{g_{\alpha\beta} u_0^\alpha u_0^\beta}_{-1} \Bigg),$$

where we have let appear the electric field $\boldsymbol{E} = \boldsymbol{F}(., \vec{\boldsymbol{u}}_0)$ measured by $\mathscr{O}$ [cf. (17.7)]. Moreover, $F_{\mu\nu} F^{\mu\nu}/2$ is the invariant I_1 [cf. (17.36)], which can be expressed in terms of $\boldsymbol{E}$ and of the magnetic field $\vec{\boldsymbol{B}}$ measured by $\mathscr{O}$ according to (17.37): $I_1 = c^2 \vec{\boldsymbol{B}} \cdot \vec{\boldsymbol{B}} - \vec{\boldsymbol{E}} \cdot \vec{\boldsymbol{E}}$. Hence

$$\boxed{\rho_{\rm em} = \frac{\varepsilon_0}{2} \left(\vec{\boldsymbol{E}} \cdot \vec{\boldsymbol{E}} + c^2 \vec{\boldsymbol{B}} \cdot \vec{\boldsymbol{B}} \right)}. \tag{20.7}$$

The electromagnetic linear-momentum density measured by $\mathscr{O}$ is given by (19.7) : $\boldsymbol{\varpi}^{\rm em} = \varpi_i^{\rm em} \boldsymbol{e}^i$, with

$$\varpi_i^{\rm em} = -\frac{1}{c} \boldsymbol{T}^{\rm em}(\vec{\boldsymbol{e}}_i, \vec{\boldsymbol{u}}_0) = -\frac{\varepsilon_0}{c} \Bigg(F_{\mu\alpha} e_i^\alpha \underbrace{F^\mu{}_\beta u_0^\beta}_{E^\mu} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \underbrace{g_{\alpha\beta} e_i^\alpha u_0^\beta}_{0} \Bigg).$$

Now, from the decomposition (17.5a) of $\boldsymbol{F}$, $F_{\mu\alpha} e_i^\alpha E^\mu = \boldsymbol{F}(\vec{\boldsymbol{E}}, \vec{\boldsymbol{e}}_i) = \boldsymbol{\epsilon}(\vec{\boldsymbol{u}}_0, c\vec{\boldsymbol{B}}, \vec{\boldsymbol{E}}, \vec{\boldsymbol{e}}_i)$. We have thus $\varpi_i^{\rm em} = \varepsilon_0 \boldsymbol{\epsilon}(\vec{\boldsymbol{u}}_0, \vec{\boldsymbol{E}}, \vec{\boldsymbol{B}}, \vec{\boldsymbol{e}}_i)$; hence

$$\boxed{\boldsymbol{\varpi}^{\rm em} = \varepsilon_0 \, \boldsymbol{\epsilon}(\vec{\boldsymbol{u}}_0, \vec{\boldsymbol{E}}, \vec{\boldsymbol{B}}, .)}. \tag{20.8}$$

The electromagnetic energy-flux 1-form is related to the linear-momentum density by (19.18): $\boldsymbol{\varphi}^{\rm em} = c^2 \boldsymbol{\varpi}^{\rm em}$. Given that $\varepsilon_0 c^2 = \mu_0^{-1}$, we get $\boldsymbol{\varphi}^{\rm em} = \mu_0^{-1} \boldsymbol{\epsilon}(\vec{\boldsymbol{u}}_0, \vec{\boldsymbol{E}}, \vec{\boldsymbol{B}}, .)$. The vector associated with $\boldsymbol{\varphi}^{\rm em}$ by metric duality is then, from the definition (3.46) of the cross product,

$$\boxed{\vec{\boldsymbol{\varphi}}^{\rm em} = \frac{1}{\mu_0} \vec{\boldsymbol{E}} \times_{\boldsymbol{u}_0} \vec{\boldsymbol{B}}}. \tag{20.9}$$

This energy-flux vector is called the ***Poynting vector***.

Finally, the electromagnetic stress tensor is given by (19.12):

$$S_{ij}^{\rm em} = \boldsymbol{T}^{\rm em}(\vec{\boldsymbol{e}}_i, \vec{\boldsymbol{e}}_j) = \varepsilon_0 \Bigg(g^{\mu\nu} F_{\mu\alpha} e_i^\alpha F_{\nu\beta} e_j^\beta - \frac{1}{4} \underbrace{F_{\mu\nu} F^{\mu\nu}}_{2(c^2 \vec{\boldsymbol{B}} \cdot \vec{\boldsymbol{B}} - \vec{\boldsymbol{E}} \cdot \vec{\boldsymbol{E}})} \underbrace{g_{\alpha\beta} e_i^\alpha e_j^\beta}_{\delta_{ij}} \Bigg).$$

To compute the first term, let us consider the components in $\mathscr{O}$'s local frame $(\vec{e}_\alpha)$. We have then $g^{\mu\nu} = \eta^{\mu\nu}$ and $F_{\mu\alpha}$ given by (17.12), so that $F_{\mu\alpha} e_i^\alpha = (-E_i, W_1, W_2, W_3)$ with $\vec{W} := c\, \vec{e}_i \times_{u_0} \vec{B}$. We deduce that $g^{\mu\nu} F_{\mu\alpha} e_i^\alpha F_{\nu\beta} e_j^\beta = -E_i E_j + c^2 (\vec{e}_i \times_{u_0} \vec{B}) \cdot (\vec{e}_j \times_{u_0} \vec{B}) = -E_i E_j - c^2 B_i B_j + c^2 B_k B^k \delta_{ij}$; hence

$$\boxed{S^{\text{em}}_{ij} = \varepsilon_0 \left[\frac{1}{2} \left(\vec{E} \cdot \vec{E} + c^2 \vec{B} \cdot \vec{B} \right) \delta_{ij} - E_i E_j - c^2 B_i B_j \right]}, \qquad (20.10)$$

which can be written in tensor form as

$$\boxed{\boldsymbol{S}^{\text{em}} = \rho_{\text{em}} (\boldsymbol{g} + \underline{\boldsymbol{u}}_0 \otimes \underline{\boldsymbol{u}}_0) - \varepsilon_0 \left(\boldsymbol{E} \otimes \boldsymbol{E} + c^2 \underline{\boldsymbol{B}} \otimes \underline{\boldsymbol{B}} \right)}, \qquad (20.11)$$

with ρ_{em} given by (20.7). $\boldsymbol{S}^{\text{em}}$ is called the ***Maxwell stress tensor***.

Historical note: *The energy–momentum tensor of the electromagnetic field has been introduced in 1908 by Hermann Minkowski (cf. p. 26) (1908). Before that, expression (20.8) of the linear-momentum density of the electromagnetic field had been established by Henri Poincaré (cf. p. 26) in 1900 (Poincaré 1900), from considerations about momentum conservation. Poincaré stressed the identity of the linear-momentum density vector and the Poynting vector (up to some factor c^2), thereby establishing property (19.18) in the particular case of the electromagnetic field.*

20.3 Radiation by an Accelerated Charge

20.3.1 Electromagnetic Energy–Momentum Tensor

In Sect. 18.6, we have seen that the electromagnetic field created by a particle $\mathscr{P}$ of electric charge q can be split into a Coulombian part and a radiative part [cf. Eq. (18.117)], the former being negligible in front of the latter at large distance whenever the 4-acceleration $\vec{a}$ of $\mathscr{P}$ is not vanishing. We shall focus on this case and thus will only consider the energy–momentum tensor associated with the radiative field $\boldsymbol{F}_{\text{rad}}$, which is given by (18.119):

$$\boldsymbol{F}_{\text{rad}} = \frac{q}{4\pi\varepsilon_0 R^2}\, \boldsymbol{Q} \wedge \underline{PM} \quad \text{with} \quad \boldsymbol{Q} := \underline{\boldsymbol{a}} + \frac{\vec{a} \cdot \overrightarrow{PM}}{R}\, \underline{\boldsymbol{u}}, \qquad (20.12)$$

where M is the point where the field is evaluated, P the intersection of $\mathscr{P}$'s worldline with the past light cone of M, $\vec{u}$ and $\vec{a}$ are, respectively, the 4-velocity and the 4-acceleration of $\mathscr{P}$ at the point P and $R := -\vec{u} \cdot \overrightarrow{PM}$ is the distance to the particle defined by the orthogonal decomposition of $\overrightarrow{PM}$ with respect to $\vec{u}$

[cf. (18.85)-(18.86) and Fig. 18.5]. For the radiative field, $I_1 = 0$ [Eq. (18.121)], so that the electromagnetic energy–momentum tensor formed via (20.3) does not contain any term proportional to $\boldsymbol{g}$:

$$
\begin{aligned}
T^{\mathrm{em}}_{\alpha\beta} &= \varepsilon_0 (F_{\mathrm{rad}})_{\mu\alpha} (F_{\mathrm{rad}})^{\mu}{}_{\beta} \\
&= \frac{q^2}{16\pi^2\varepsilon_0\, R^4} \left[Q_\mu (PM)_\alpha - Q_\alpha (PM)_\mu \right] \left[Q^\mu (PM)_\beta - Q_\beta (PM)^\mu \right].
\end{aligned}
$$

Taking into account $Q_\mu (PM)^\mu = 0$ (cf. the above definitions of Q and R), $(PM)_\mu (PM)^\mu = 0$ (null character of $\overrightarrow{PM}$) and $Q_\mu \cdot Q^\mu = \vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{a}} - (\vec{\boldsymbol{a}} \cdot \overrightarrow{PM}/R)^2$, there comes

$$
\boxed{\boldsymbol{T}^{\mathrm{em}} = \frac{q^2}{16\pi^2\varepsilon_0\, R^4} \left[\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{a}} - (\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{m}})^2 \right] \underline{PM} \otimes \underline{PM}}\,, \tag{20.13}
$$

where we have introduced the unit spacelike vector $\vec{\boldsymbol{m}}$ according to the orthogonal decomposition (18.86) of $\overrightarrow{PM}$: $\overrightarrow{PM} = R[\vec{\boldsymbol{u}}(\tau_P) + \vec{\boldsymbol{m}}]$. Let us stress that in this equation, $\vec{\boldsymbol{a}} = \vec{\boldsymbol{a}}(\tau_P)$.

20.3.2 Radiated Energy

Given $\boldsymbol{T}^{\mathrm{em}}$, we may estimate the total energy–momentum radiated by particle $\mathscr{P}$ as follows. Let P be a point of $\mathscr{P}$'s worldline $\mathscr{L}$ and $\mathrm{d}\tau$ some infinitesimal increment of $\mathscr{P}$'s proper time, from $P = A(\tau)$ to $Q = A(\tau + \mathrm{d}\tau)$, A denoting the generic point of $\mathscr{L}$. Given the structure of the electromagnetic field generated by $\mathscr{P}$ (retarded potentials), one may consider that all the energy–momentum radiated between P and Q is localized between $\mathscr{I}^+(P)$ and $\mathscr{I}^+(Q)$, the future light cones of, respectively, P and Q (cf. Fig. 20.1). Let us then introduce the inertial observer $\mathscr{O}_P$ whose wordline is the tangent to $\mathscr{L}$ at P (cf. Figs. 18.5 and 20.1). Let $\mathscr{S}$ be the sphere formed by the intersection of the light cone $\mathscr{I}^+(P)$ with $\mathscr{O}_P$'s rest space at some instant sufficiently posterior to P so that the electromagnetic field on $\mathscr{S}$ can be assumed to be entirely radiative. We consider next the hypercylinder $\mathscr{V}$ of basis $\mathscr{S}$, having $\mathscr{O}_P$'s worldline for axis and connecting $\mathscr{I}^+(P)$ to $\mathscr{I}^+(Q)$ (cf. Fig. 20.1). The height of this hypercylinder is $\mathrm{d}\tau$. The electromagnetic 4-momentum crossing $\mathscr{S}$ during the time $\mathrm{d}\tau$ is, from (19.2),

$$
\mathrm{d}\boldsymbol{p}^{\mathrm{rad}} = \frac{1}{c} \int_{\mathscr{V}} \boldsymbol{T}^{\mathrm{em}}(., \vec{\boldsymbol{m}})\, \mathrm{d}V,
$$

where we have used the fact that the unit normal to $\mathscr{V}$ is $\vec{\boldsymbol{m}}$ and is spacelike. Substituting (20.13) for $\boldsymbol{T}^{\mathrm{em}}$ and using $\mathrm{d}V = c\, \mathrm{d}\tau\, \mathrm{d}S$ ($\mathrm{d}S = \mathscr{S}$'s area element), we get

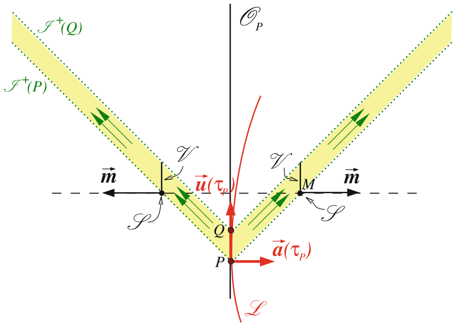

Fig. 20.1 Electromagnetic 4-momentum emitted by an accelerated particle between two events P and Q of its worldline $\mathscr{L}$ and radiated through the sphere $\mathscr{S}$. The latter is fixed with respect to the inertial observer $\mathscr{O}_P$ whose worldline is tangent to $\mathscr{L}$ at P

$$\mathrm{d}\boldsymbol{p}^{\rm rad} = \frac{q^2\,\mathrm{d}\tau}{16\pi^2\varepsilon_0\,R^4}\int_{\mathscr{S}}\left[\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{a}} - (\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{m}})^2\right]\langle\underline{PM},\vec{\boldsymbol{m}}\rangle\,\underline{PM}\,\mathrm{d}S.$$

Now, $\underline{PM} = R[\underline{\boldsymbol{u}}(\tau_P)+\underline{\boldsymbol{m}}]$ [Eq. (18.86)] and $\langle\underline{PM},\vec{\boldsymbol{m}}\rangle = R$. Introducing spherical coordinates (θ,φ) on $\mathscr{S}$ such that the polar axis is along $\vec{\boldsymbol{a}} = \vec{\boldsymbol{a}}(\tau_P)$, we have $\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{m}} = a\cos\theta$ with $a := \sqrt{\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{a}}}$, so that $\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{a}} - (\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{m}})^2 = a^2(1-\cos^2\theta) = a^2\sin^2\theta$. Since $\mathrm{d}S = R^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi$, we get

$$\mathrm{d}\boldsymbol{p}^{\rm rad} = \frac{q^2\,a^2\,\mathrm{d}\tau}{16\pi^2\varepsilon_0}\int_{\mathscr{S}}[\underline{\boldsymbol{u}}(\tau_P)+\underline{\boldsymbol{m}}]\,\sin^3\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi.$$

Now, using the Cartesian components of $\underline{\boldsymbol{m}}$ ($\underline{\boldsymbol{m}} = \sin\theta\cos\varphi\,\boldsymbol{e}^x + \sin\theta\sin\varphi\,\boldsymbol{e}^y + \cos\theta\,\boldsymbol{e}^z$), it is easy to see that the integral of $\underline{\boldsymbol{m}}\,\sin^3\theta$ vanishes. There remains thus only the integral of $\underline{\boldsymbol{u}}(\tau_P)\,\sin^3\theta$, which is actually the integral of $\sin^3\theta$ on the sphere, since $\underline{\boldsymbol{u}}(\tau_P)$ is constant. Given that $\int_0^\pi\int_0^{2\pi}\sin^3\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi = 8\pi/3$, we finally obtain

$$\boxed{\mathrm{d}\boldsymbol{p}^{\rm rad} = \frac{q^2\,\vec{\boldsymbol{a}}(\tau_P)\cdot\vec{\boldsymbol{a}}(\tau_P)\,\mathrm{d}\tau}{6\pi\varepsilon_0}\,\underline{\boldsymbol{u}}(\tau_P)}. \tag{20.14}$$

It is remarkable that this 4-momentum is independent from the radius R of $\mathscr{S}$, i.e. actually from the proper time instant τ of observer $\mathscr{O}_P$ at which it is evaluated (provided that τ is sufficiently in the future of τ_P for the approximation of purely radiative field to be valid).

For the inertial observer $\mathscr{O}_P$, the energy crossing the sphere $\mathscr{S}$ is given by (9.4): $\mathrm{d}E = -c\langle\mathrm{d}\boldsymbol{p}^{\rm rad},\vec{\boldsymbol{u}}(\tau_P)\rangle$. Using (20.14) and $\langle\underline{\boldsymbol{u}}(\tau_P),\vec{\boldsymbol{u}}(\tau_P)\rangle = -1$, we find the energy radiated per unit time (power) $\mathscr{P} = \mathrm{d}E/\mathrm{d}\tau$:

$$\mathscr{P} = \frac{c\,q^2\,\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{a}}}{6\pi\varepsilon_0}. \tag{20.15}$$

We may express the 4-acceleration $\vec{\boldsymbol{a}} = \vec{\boldsymbol{a}}(\tau_P)$ in terms of the acceleration $\vec{\boldsymbol{\gamma}}$ relative to the inertial observer $\mathscr{O}_P$. Since $\mathscr{P}$'s velocity with respect to $\mathscr{O}_P$ is, by definition of the latter, zero at the instant τ_P, the formula to apply is (4.64): $\vec{\boldsymbol{a}} = c^{-2}\,\vec{\boldsymbol{\gamma}}$; hence,

$$\boxed{\mathscr{P} = \frac{q^2\gamma^2}{6\pi\varepsilon_0 c^3}}, \tag{20.16}$$

with $\gamma^2 := \vec{\boldsymbol{\gamma}}\cdot\vec{\boldsymbol{\gamma}}$, the acceleration $\vec{\boldsymbol{\gamma}}$ being taken at the instant τ_P. Equation (20.16), which gives the power radiated by an accelerated particle through a sphere surrounding it, is named ***Larmor formula***.

20.3.3 Radiated 4-Momentum

We have stressed that expression (20.14) for the 4-momentum $\mathrm{d}\boldsymbol{p}^{\mathrm{rad}}$ radiated by a charged particle $\mathscr{P}$ between τ and $\tau + \mathrm{d}\tau$ does not depend upon the radius of the sphere $\mathscr{S}$ through which the inertial observer $\mathscr{O}_P$ measures it. But there is much more: this 4-momentum does not depend upon observer $\mathscr{O}_P$! We are indeed going to see that if one considers a sphere $\mathscr{S}'$ in the rest space of an arbitrary inertial observer $\mathscr{O}'$, then the radiated 4-momentum through $\mathscr{S}'$ when $\mathscr{P}$ moves from P to Q is exactly given by (20.14), even if at P, $\mathscr{P}$ has a nonvanishing velocity with respect to $\mathscr{O}'$.

Given $\mathscr{O}'$, the sphere $\mathscr{S}'$ is defined by the intersection of the light cone $\mathscr{I}^+(P)$ with the rest space of $\mathscr{O}'$ at a certain instant t' of his proper time. We shall assume that t' is sufficiently large so that $\mathscr{S}'$ is located outside $\mathscr{S}$ on $\mathscr{I}^+(P)$ (cf. Fig. 20.2). Let $\mathscr{V}'$ be the hypercylinder of basis $\mathscr{S}'$, having its axis along the 4-velocity of $\mathscr{O}'$ and connecting $\mathscr{I}^+(P)$ to $\mathscr{I}^+(Q)$, $\mathscr{V}$ still standing for the hypercylinder of basis $\mathscr{S}$ defined in Sect. 20.3.1. Finally, let $\mathscr{C}_P$ be the part of the light cone $\mathscr{I}^+(P)$ located between $\mathscr{S}$ and $\mathscr{S}'$ and $\mathscr{C}_Q$ that of the light cone $\mathscr{I}^+(Q)$ located between the upper extremities of $\mathscr{V}$ and $\mathscr{V}'$ (cf. Fig. 20.2). Let us then consider the four-dimensional volume $\mathscr{U}$ delimited by $\mathscr{V}$, $\mathscr{V}'$, $\mathscr{C}_P$ and $\mathscr{C}_Q$: $\partial\mathscr{U} = \mathscr{V}\cup\mathscr{V}'\cup\mathscr{C}_P\cup\mathscr{C}_Q$. For any vector $\vec{\boldsymbol{v}} \in E$, we define

$$I := \int_{\mathscr{U}} \langle \vec{\boldsymbol{\nabla}}\cdot\boldsymbol{T}^{\mathrm{em}},\ \vec{\boldsymbol{v}}\rangle\,\boldsymbol{\epsilon} = \int_{\mathscr{U}} (\boldsymbol{\nabla}\cdot\vec{\boldsymbol{w}})\,\boldsymbol{\epsilon} = \int_{\mathscr{U}} \mathbf{d}\star\underline{\boldsymbol{w}}, \tag{20.17}$$

where $\vec{\boldsymbol{w}}$ is the metric dual of the 1-form $\underline{\boldsymbol{w}} := \boldsymbol{T}^{\mathrm{em}}(\vec{\boldsymbol{v}},.)$; the second equality stems from (19.24) and the third one from the identity (15.89). Let us then apply Stokes theorem (16.46):

$$\begin{aligned} I &= \int_{\partial\mathscr{U}} \star\underline{\boldsymbol{w}} = \int_{\mathscr{V}} \star\underline{\boldsymbol{w}} - \int_{\mathscr{V}'} \star\underline{\boldsymbol{w}} - \int_{\mathscr{C}_P} \star\underline{\boldsymbol{w}} + \int_{\mathscr{C}_Q} \star\underline{\boldsymbol{w}} \\ &= \Phi_{\mathscr{V}}(\vec{\boldsymbol{w}}) - \Phi_{\mathscr{V}'}(\vec{\boldsymbol{w}}) - \int_{\mathscr{C}_P} \star\underline{\boldsymbol{w}} + \int_{\mathscr{C}_Q} \star\underline{\boldsymbol{w}}, \end{aligned}$$

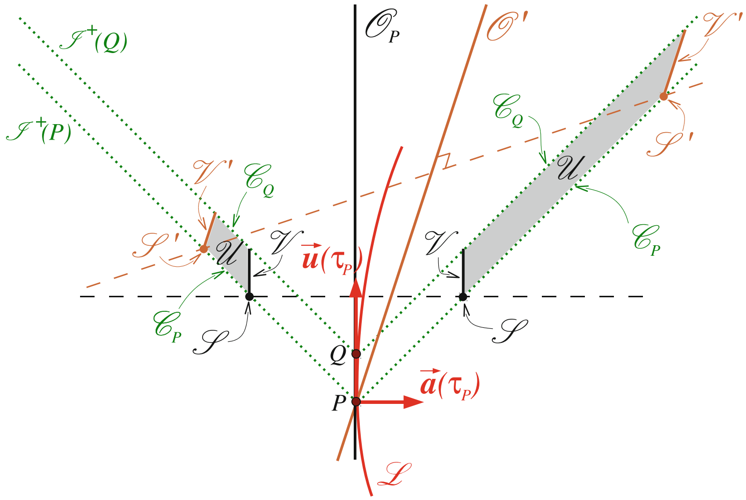

Fig. 20.2 Four-momentum radiated by an accelerated particle (worldline $\mathscr{L}$) through the sphere $\mathscr{S}$ for observer $\mathscr{O}_P$ and through the sphere $\mathscr{S}'$ for observer $\mathscr{O}'$. The dashed lines mark the rest spaces of each observer

where the $-$ signs result from the changes of orientation when $\mathscr{V}$ and $\mathscr{C}_P$ are considered as parts of $\mathscr{U}$'s boundary. Moreover, use has been made of identity (16.44) to let appear the fluxes through $\mathscr{V}$ and $\mathscr{V}'$. Using formulas (19.22) and (15.87), there comes

$$I = c\,\langle \mathrm{d}\boldsymbol{p}^{\mathrm{rad}}\big|_{\mathscr{V}}, \vec{v}\rangle - c\langle \mathrm{d}\boldsymbol{p}^{\mathrm{rad}}\big|_{\mathscr{V}'}, \vec{v}\rangle - \int_{\mathscr{C}_P} \boldsymbol{\epsilon}(\vec{w}, ., ., .) + \int_{\mathscr{C}_Q} \boldsymbol{\epsilon}(\vec{w}, ., ., .),$$

where $\mathrm{d}\boldsymbol{p}^{\mathrm{rad}}\big|_{\mathscr{V}}$ is the 4-momentum (20.14) radiated through $\mathscr{S}$ during $\mathrm{d}\tau$ and $\mathrm{d}\boldsymbol{p}^{\mathrm{rad}}\big|_{\mathscr{V}'}$ is the 4-momentum radiated through $\mathscr{S}'$, during the same proper time lapse $\mathrm{d}\tau$ of the charge at P. Now, according to the form (20.13) of $\boldsymbol{T}^{\mathrm{em}}$, one has, on $\mathscr{C}_P$,

$$\vec{w} = \overrightarrow{\boldsymbol{T}^{\mathrm{em}}(\vec{v}, .)} = \frac{q^2}{16\pi^2\varepsilon_0 R^4}\left[\vec{a}\cdot\vec{a} - (\vec{a}\cdot\vec{m})^2\right] (\overrightarrow{PM}\cdot\vec{v})\ \overrightarrow{PM}.$$

$\vec{w}$ is thus collinear to $\overrightarrow{PM}$, which implies that $\vec{w}$ is tangent to $\mathscr{I}^+(P)$ and therefore to $\mathscr{C}_P$. It follows that the 3-form $\boldsymbol{\epsilon}(\vec{w}, ., ., .)$ is identically zero on $\mathscr{C}_P$: it is not possible to find three vectors $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ tangent to $\mathscr{C}_P$ such that $(\vec{w}, \vec{e}_1, \vec{e}_2, \vec{e}_3)$ is a system of linearly independent vectors. Consequently,

$$I = c\,\langle \mathrm{d}\boldsymbol{p}^{\mathrm{rad}}\big|_{\mathscr{V}}, \vec{v}\rangle - c\langle \mathrm{d}\boldsymbol{p}^{\mathrm{rad}}\big|_{\mathscr{V}'}, \vec{v}\rangle. \tag{20.18}$$

Let us now come back to the definition (20.17) of I: since the domain $\mathscr{U}$ is free from matter (it does not contain $\mathscr{P}$'s worldline), the energy–momentum conservation (20.5) with $\boldsymbol{T}^{\mathrm{mat}} = 0$ leads to $\vec{\nabla} \cdot \boldsymbol{T}^{\mathrm{em}} = 0$ on $\mathscr{U}$. Consequently $I = 0$. The identity (20.18), which must be fulfilled for any vector $\vec{\boldsymbol{v}} \in E$, yields then

$$\boxed{\mathrm{d}\boldsymbol{p}^{\mathrm{rad}}\big|_{\mathscr{V}'} = \mathrm{d}\boldsymbol{p}^{\mathrm{rad}}\big|_{\mathscr{V}}}. \tag{20.19}$$

Hence the radiated 4-momentum is always given by formula (20.14).

The radiated energy measured by an arbitrary inertial observer $\mathscr{O}$, of 4-velocity $\vec{\boldsymbol{u}}_0$, is, according to (9.4), $\mathrm{d}E = -c\langle \mathrm{d}\boldsymbol{p}^{\mathrm{rad}}, \vec{\boldsymbol{u}}_0\rangle$, with $\mathrm{d}\boldsymbol{p}^{\mathrm{rad}}$ given by (20.14) from the above result. Hence

$$\mathrm{d}E = \Gamma \frac{c\, q^2\, \vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{a}}\, \mathrm{d}\tau}{6\pi\varepsilon_0}$$

where $\vec{\boldsymbol{a}} = \vec{\boldsymbol{a}}(\tau_P)$ and $\Gamma = \Gamma(\tau_P) = -\vec{\boldsymbol{u}}_0 \cdot \vec{\boldsymbol{u}}(\tau_P)$ is the Lorentz factor of $\mathscr{P}$ with respect to $\mathscr{O}$, at the instant τ_P. Relatively to $\mathscr{O}$, the radiated power is $\mathscr{P} = \mathrm{d}E/\mathrm{d}t$, where $\mathrm{d}t$ is the increment of $\mathscr{O}$'s proper time. Since $\mathrm{d}t = \Gamma\,\mathrm{d}\tau$, the factor Γ disappears and we obtain the same expression as (20.15):

$$\boxed{\mathscr{P} = \frac{c\, q^2\, \vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{a}}}{6\pi\varepsilon_0}}. \tag{20.20}$$

We may express the scalar square of the 4-acceleration in terms of the acceleration $\vec{\boldsymbol{\gamma}}$ and the velocity $\vec{\boldsymbol{V}}$ of $\mathscr{P}$ relative to $\mathscr{O}$, via formula (4.69). We obtain in this way

$$\boxed{\mathscr{P} = \frac{q^2}{6\pi\varepsilon_0\, c^3}\Gamma^4 \left[\vec{\boldsymbol{\gamma}} \cdot \vec{\boldsymbol{\gamma}} + \frac{\Gamma^2}{c^2}\left(\vec{\boldsymbol{\gamma}} \cdot \vec{\boldsymbol{V}}\right)^2\right]}. \tag{20.21}$$

The relation is called ***Liénard formula***. It generalizes Larmor formula (20.16) to the case $\vec{\boldsymbol{V}} \neq 0$. Using the alternative expressions (4.70) and (4.71) of $\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{a}}$, Liénard formula can be recast as

$$\boxed{\mathscr{P} = \frac{q^2}{6\pi\varepsilon_0\, c^3}\Gamma^6 \left[\vec{\boldsymbol{\gamma}} \cdot \vec{\boldsymbol{\gamma}} - \frac{1}{c^2}(\vec{\boldsymbol{\gamma}} \times_{\boldsymbol{u}} \vec{\boldsymbol{V}})^2\right] = \frac{q^2}{6\pi\varepsilon_0\, c^3}\Gamma^4 \left(\Gamma^2 \gamma_\parallel^2 + \gamma_\perp^2\right)}. \tag{20.22}$$

Remark 20.2. At the nonrelativistic limit, Liénard formula reduces obviously to Larmor formula (20.16). Furthermore, if $\vec{\boldsymbol{V}}$ is orthogonal to $\vec{\boldsymbol{\gamma}}$, both formulas differ only by a factor Γ^4, which becomes very large for relativistic particles.

Historical note: *Formula (20.16), which gives the electromagnetic power radiated by an accelerated charge in terms of the acceleration relative to an observer with respect to which the charge is momentarily at rest, has been obtained in 1897 by*

Joseph Larmor (cf. p. 191) (Larmor 1897). Its generalization to the case $\vec{V} \neq 0$ [Eq. (20.21)] has been given the year after by Alfred-Marie Liénard (cf. p. 614) (Liénard 1898).

20.3.4 Angular Distribution of Radiation

Liénard formula (20.21) provides the total power radiated by an accelerated charge, i.e. the power integrated over a sphere surrounding the particle. To get the power radiated in a given direction, one must evaluate the Poynting vector $\vec{\boldsymbol{\varphi}}^{\text{em}}$. We consider an arbitrary inertial observer $\mathscr{O}$, with respect to whom the charge $\mathscr{P}$ moves with the velocity $\vec{V} = \vec{V}(t)$ and the acceleration $\vec{\boldsymbol{\gamma}} = \vec{\boldsymbol{\gamma}}(t)$, t standing for $\mathscr{O}$'s proper time. The Poynting vector is expressed in terms of the electric and magnetic fields relative to $\mathscr{O}$, $\vec{E}$ and $\vec{B}$, according to (20.9). Since $\vec{B}$ is related to $\vec{E}$ by (18.109): $\vec{B} = c^{-1}\,\vec{n} \times_{u_0} \vec{E}$, we get

$$\vec{\boldsymbol{\varphi}}^{\text{em}} = \frac{1}{\mu_0}\vec{E} \times_{u_0} \vec{B} = \frac{1}{\mu_0 c}\vec{E} \times_{u_0} (\vec{n} \times_{u_0} \vec{E}) = \frac{1}{\mu_0 c}\left[(\vec{E}\cdot\vec{E})\,\vec{n} - (\vec{n}\cdot\vec{E})\,\vec{E}\right],$$

where $\vec{u}_0$ is the 4-velocity of $\mathscr{O}$ and $\vec{n}$ is the unit vector along the direction connecting the position P' of $\mathscr{P}$ at the retarded time $t_P = t - r/c$ to the point M where the field is evaluated (cf. Figs. 18.6 and 18.7). Now for the radiative field emitted by a particle, $\vec{n}\cdot\vec{E} = 0$, as it follows from (18.123). We have then

$$\vec{\boldsymbol{\varphi}}^{\text{em}} = \frac{\vec{E}\cdot\vec{E}}{\mu_0 c}\,\vec{n}. \tag{20.23}$$

The vector $\vec{E}$ being given by (18.123), we have to evaluate the scalar square of the vector $\vec{n} \times_{u_0} [(\vec{n} - \vec{V}/c) \times_{u_0} \vec{\boldsymbol{\gamma}}]$. Starting from the identity

$$\vec{n} \times_{u_0} \left[\left(\vec{n} - \frac{\vec{V}}{c}\right) \times_{u_0} \vec{\boldsymbol{\gamma}}\right] = (\vec{n}\cdot\vec{\boldsymbol{\gamma}})\left(\vec{n} - \frac{\vec{V}}{c}\right) - \left(1 - \frac{\vec{n}\cdot\vec{V}}{c}\right)\vec{\boldsymbol{\gamma}},$$

we get, after simplification,

$$\boxed{\vec{\boldsymbol{\varphi}}^{\text{em}} = \frac{q^2}{16\pi^2\varepsilon_0 c^3}\,\frac{\gamma^2 + \dfrac{\vec{n}\cdot\vec{\boldsymbol{\gamma}}}{1 - \frac{\vec{n}\cdot\vec{V}}{c}}\left[\dfrac{2}{c}\vec{V}\cdot\vec{\boldsymbol{\gamma}} - \dfrac{\vec{n}\cdot\vec{\boldsymbol{\gamma}}}{1 - \frac{\vec{n}\cdot\vec{V}}{c}}\left(1 - \dfrac{V^2}{c^2}\right)\right]}{r^2\left(1 - \dfrac{\vec{n}\cdot\vec{V}}{c}\right)^4}\,\vec{n}}. \tag{20.24}$$

We have set $\gamma := \|\vec{\boldsymbol{\gamma}}\|_g$ and $V := \|\vec{V}\|_g$. In this formula, the velocity $\vec{V}$ and the acceleration $\vec{\boldsymbol{\gamma}}$ of $\mathscr{P}$ are to be taken at the retarded time $t_P = t - r/c$.

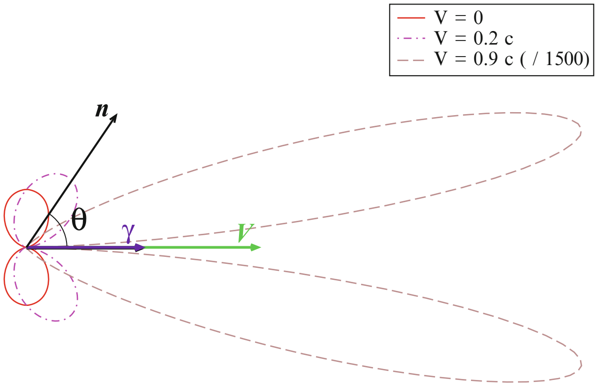

Fig. 20.3 Radiation pattern of an accelerated charge when the velocity $\vec{V}$ is collinear to the acceleration $\vec{\gamma}$. The diagram for $V = 0.9c$ has been reduced by a factor 1500 with respect to those for $V = 0$ and $V = 0.2c$

Remark 20.3. If $\vec{\gamma} = 0$, then $\vec{\varphi}^{\,\mathrm{em}} = 0$, as it should, since $\boldsymbol{F}_{\mathrm{rad}} = 0$ for a non-accelerated particle.

Various subcases of formula (20.24) are worth discussing:

20.3.4.1 Case $V(t_P) = 0$

If, at the retarded time t_P, the particle has a vanishing velocity with respect to $\mathscr{O}$ (but a nonzero acceleration), formula (20.24) simplifies considerably the numerator of the second fraction reducing to $\gamma^2 - (\vec{n}\cdot\vec{\gamma})^2 = \gamma^2(1-\cos^2\theta) = \gamma^2\sin^2\theta$, where θ is the angle between $\vec{\gamma}$ and the unit vector $\vec{n}$. We obtain thus

$$\boxed{\vec{\varphi}^{\,\mathrm{em}} = \frac{q^2\gamma^2\sin^2\theta}{16\pi^2\varepsilon_0 c^3 r^2}\,\vec{n}}_{\vec{V}(t_P)=0} \qquad \vec{n}\cdot\vec{\gamma} =: \gamma\cos\theta. \tag{20.25}$$

The corresponding radiation pattern, i.e. $\|\vec{\varphi}^{\,\mathrm{em}}\|_g$ plotted as a function of θ, is depicted in Fig. 20.3 (solid line). We recognize a characteristic dipole shape.

Computing the flux of $\vec{\varphi}^{\,\mathrm{em}}$ through a sphere of radius r in $\mathscr{O}$'s rest space yields Larmor formula (20.16), which is not surprising since the observer $\mathscr{O}_P$ considered in Sect. 20.3.1 is by definition an observer for which $\vec{V}(t_P) = 0$.

20.3.4.2 Case $V(t_P)$ Collinear to $\gamma(t_P)$

If $\vec{V}$ is collinear to $\vec{\gamma}$ at $t = t_P$, let us denote by θ the angle between $\vec{V}$ and $\vec{n}$. This is also the angle between $\vec{\gamma}$ and $\vec{n}$, up to a factor π if $\vec{\gamma}\cdot\vec{V} < 0$. We have then

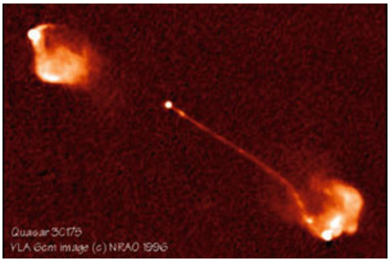

Fig. 20.4 Radio wave image of the quasar 3C 175 at the redshift $z = 0.768$ (resulting from the expansion of the universe), obtained by the American radio telescope VLA at the wavelength $\lambda = 6$ cm. 3C 175 is an active galactic nucleus that emits a relativistic jet. The jet is emitted in two opposite directions, but due to Doppler boosting, only the part of the jet coming towards us appears on the image. On the other side, the lobes at each extremity of the jet have a nonrelativistic velocity, so that they both appear with the same intensity [Source: Alan Bridle (NRAO)]

$\vec{\boldsymbol{n}} \cdot \vec{\boldsymbol{V}} = V\cos\theta$, $\vec{\boldsymbol{n}} \cdot \vec{\boldsymbol{\gamma}} = \pm\gamma\cos\theta$ and $\vec{\boldsymbol{V}} \cdot \vec{\boldsymbol{\gamma}} = \pm\gamma V$, with, in these last two relations, a $+$ sign if $\vec{\boldsymbol{\gamma}}$ has the same sense as $\vec{\boldsymbol{V}}$ and a $-$ sign otherwise. Expression (20.24) for the Poynting vector reduces then to

$$\left.\boxed{\vec{\boldsymbol{\varphi}}^{\,\mathrm{em}} = \frac{q^2\,\gamma^2\,\sin^2\theta}{16\pi^2\varepsilon_0 c^3\,r^2\left(1-\frac{V}{c}\cos\theta\right)^6}\,\vec{\boldsymbol{n}}}\right|_{\vec{\boldsymbol{V}}(t_P)\parallel\vec{\boldsymbol{\gamma}}(t_P)} \qquad \vec{\boldsymbol{n}}\cdot\vec{\boldsymbol{V}} =: V\cos\theta. \tag{20.26}$$

Because of the factor $(1 - V/c\ \cos\theta)^6$, the difference with the case $\vec{\boldsymbol{V}} = 0$ lies in some "focusing" of the radiation around the directions

$$\theta_\pm = \pm\arccos\left(\frac{6V/c}{1+\sqrt{1+24V^2/c^2}}\right), \tag{20.27}$$

which correspond to the maxima of the function $\theta \mapsto \sin^2\theta/[1-(V/c)\cos\theta)]^6$. The corresponding radiation pattern is depicted in Fig. 20.3 for $V = 0.2\,c$ and $V = 0.9\,c$. At the ultra-relativistic limit, $V/c \to 1$ and $\theta_\pm \to 0$. More precisely, the Taylor expansion of (20.27) leads to

$$\theta_\pm \simeq \pm\frac{1}{\sqrt{5}\Gamma} \qquad \text{(ultra-relativistic)}, \tag{20.28}$$

where $\Gamma := (1 - V^2/c^2)^{-1/2}$ is the Lorentz factor of $\mathscr{P}$ with respect to $\mathscr{O}$.

At the ultra-relativistic limit ($\Gamma \to +\infty$), we observe that most of the radiation is contained in a narrow cone around the direction defined by the particle's velocity. Moreover, the radiation amplitude becomes very large in this direction (in Fig. 20.3, the diagram for $V = 0.9c$ had to be reduced by a factor 1500 to fit on the plot!). This phenomenon is named ***Doppler boosting***. It is observed frequently in relativistic jets emitted by active galactic nuclei, as illustrated in Fig. 20.4.

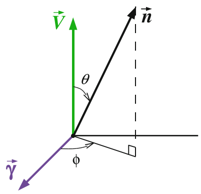

Fig. 20.5 Definition of the angles (θ, ϕ) setting the direction of observation $\vec{n}$ when $\vec{V}$ and $\vec{\gamma}$ are orthogonal

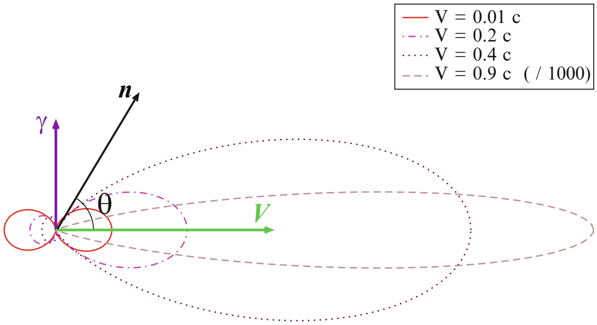

Fig. 20.6 Radiation pattern of an accelerated charge when (i) the velocity $\vec{V}$ is orthogonal to the acceleration $\vec{\gamma}$ and (ii) the azimuthal angle of the observing direction $\vec{n}$ is $\phi = 0$. The diagram for $V = 0.9\,c$ has been rescaled by a factor $1/1000$ to fit in the figure

20.3.4.3 Case $V(t_P)$ Orthogonal to $\gamma(t_P)$

If $\vec{V}$ is orthogonal to $\vec{\gamma}$, we still denote by θ the angle between $\vec{V}$ and $\vec{n}$ and introduce the azimuthal angle ϕ between the vector $\vec{n}$ and the plane $\mathrm{Span}(\vec{V}, \vec{\gamma})$ (cf. Fig. 20.5). We have then $\vec{n} \cdot \vec{V} = V \cos\theta$, $\vec{n} \cdot \vec{\gamma} = \gamma \sin\theta \cos\phi$ and $\vec{V} \cdot \vec{\gamma} = 0$, so that (20.24) leads to

$$\boxed{\vec{\varphi}^{\,\mathrm{em}} = \frac{q^2\,\gamma^2}{16\pi^2\varepsilon_0 c^3\, r^2 \left(1 - \frac{V}{c}\cos\theta\right)^4} \left[1 - \frac{\left(1 - \frac{V^2}{c^2}\right)\sin^2\theta\cos^2\phi}{\left(1 - \frac{V}{c}\cos\theta\right)^2}\right] \vec{n}.}_{\vec{V}(t_P)\perp\vec{\gamma}(t_P)} \tag{20.29}$$

At the ultra-relativistic limit, because of the term $(1 - V/c\ \cos\theta)^{-4}$, the emission takes place mostly for θ close to 0, i.e. in a narrow cone around the velocity direction (cf. Fig. 20.6), as in the case $\vec{V}$ collinear to $\vec{\gamma}$. More precisely, for V large and θ small, $V/c \simeq 1 - 1/(2\Gamma^2)$ and $\cos\theta \simeq 1 - \theta^2/2$, so that $1 - V/c\,\cos\theta \simeq (1 + \Gamma^2\theta^2)/(2\Gamma^2)$ and the Poynting vector (20.29) becomes

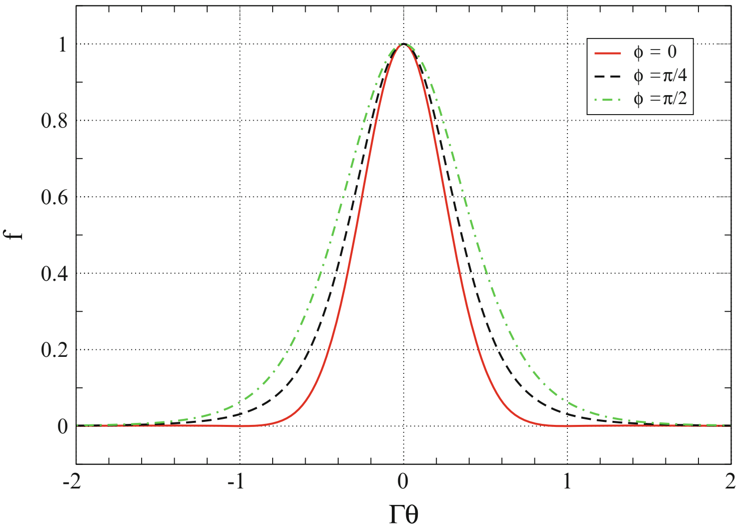

Fig. 20.7 Graph of the function $f(\Gamma\theta) = (1 - 2\Gamma^2\theta^2 \cos 2\phi + \Gamma^4\theta^4)/(1 + \Gamma^2\theta^2)^6$ ruling the angular dependence of the Poynting vector for an ultra-relativistic particle with an acceleration orthogonal to its velocity [Eq. (20.30)]

$$\vec{\varphi}^{\,\mathrm{em}} = \frac{q^2\gamma^2}{\pi^2\varepsilon_0 c^3 r^2}\,\frac{\Gamma^8}{(1+\Gamma^2\theta^2)^6}\left(1 - 2\Gamma^2\theta^2\cos 2\phi + \Gamma^4\theta^4\right)\vec{n}$$

$$(\Gamma \to +\infty \text{ and } |\theta| \ll 1). \qquad (20.30)$$

We note that the θ-dependency of $\vec{\varphi}^{\,\mathrm{em}}$ occurs via the product $\Gamma\theta$. The plot of the corresponding function is shown in Fig. 20.7. This function differs significantly from zero only for $|\Gamma\theta| \leq 1$. We conclude that the opening angle of the radiation cone is $\theta \sim \Gamma^{-1}$. We note also the phenomenon of Doppler boosting (cf. Fig. 20.6), as in the case $\vec{V}$ collinear to $\vec{\gamma}$.

20.4 Synchrotron Radiation

20.4.1 Introduction

An important example of electromagnetic radiation by an accelerated charge is that of a particle moving in a magnetic field (or more precisely in a mostly magnetic electromagnetic field; cf. Sect. 17.3.2). Assuming that the magnetic field is uniform for some inertial observer $\mathscr{O}$, we have shown in Sect. 17.4.1 that the particle's trajectory is a helix whose axis is parallel to the direction of the magnetic field (cf. Fig. 17.6). If $\mathscr{O}$ is an inertial observer with respect to which the electromagnetic field is reduced to a magnetic field $\vec{B}$ parallel to the vector $\vec{e}_3$ of his frame $(\vec{e}_\alpha)$, the particle's velocity with respect to $\mathscr{O}$ and at the retarded time $t_P = t - r/c$ is given by (17.71):

$$\vec{V} = V\left\{\sin\alpha\cos\left[\frac{\omega_B}{\Gamma}\left(t - \frac{r}{c}\right)\right]\vec{e}_1 - \sin\alpha\sin\left[\frac{\omega_B}{\Gamma}\left(t - \frac{r}{c}\right)\right]\vec{e}_2 + \cos\alpha\,\vec{e}_3\right\}, \tag{20.31}$$

where $V := \|\vec{V}\|_g$ is constant, as well as $\Gamma = (1 - V^2/c^2)^{-1/2}$, α is the angle between $\vec{V}$ and $\vec{B}$ (denoted by θ in Sect. 17.4.1) and ω_B is the cyclotron frequency: $\omega_B := qB/m$ [Eq. (17.65)]. The particle's acceleration with respect to $\mathscr{O}$ is found by deriving (17.71) with respect to t; its value at the retarded time is

$$\vec{\gamma} = -\frac{\omega_B}{\Gamma}V\sin\alpha\left\{\sin\left[\frac{\omega_B}{\Gamma}\left(t - \frac{r}{c}\right)\right]\vec{e}_1 + \cos\left[\frac{\omega_B}{\Gamma}\left(t - \frac{r}{c}\right)\right]\vec{e}_2\right\}. \tag{20.32}$$

We note that $\vec{\gamma}\cdot\vec{V} = 0$, i.e. we are in the configuration $\vec{V}$ orthogonal to $\vec{\gamma}$ considered in Sect. 20.3.4.3. The electromagnetic field radiated by $\mathscr{P}$ is given by formulas (18.123)–(18.124) in which $\vec{V}$ and $\vec{\gamma}$ have to be replaced by the above expressions. This electromagnetic field is called ***synchrotron radiation***, for it appeared experimentally in synchrotrons (cf. Sect. 17.5.4). At the nonrelativistic limit (in terms of the particle's velocity), it is named ***cyclotron radiation***. Formula (18.126) leads then to the following explicit form of the radiated magnetic field:

$$\vec{B}_{\rm rad} = \frac{q\omega_B V\sin\alpha}{4\pi\varepsilon_0 c^3 r}\left\{\sin\left[\omega_B\left(t - \frac{r}{c}\right)\right]\vec{n}\times_{u_0}\vec{e}_1 + \cos\left[\omega_B\left(t - \frac{r}{c}\right)\right]\vec{n}\times_{u_0}\vec{e}_2\right\}. \tag{20.33}$$

We note that this is a monochromatic radiation at the cyclotron frequency ω_B.

The total radiated power of synchrotron radiation is given by Liénard formula (20.21), with $\vec{\gamma}\cdot\vec{\gamma} = \omega_B^2 V^2\sin^2\alpha/\Gamma^2$ and $\vec{\gamma}\cdot\vec{V} = 0$. Accordingly

$$\boxed{\mathscr{P} = \frac{q^4 B^2\Gamma^2 V^2\sin^2\alpha}{6\pi\varepsilon_0 c^3 m^2}}. \tag{20.34}$$

This radiated power results in a loss of energy in particle accelerators of the cyclotron or synchrotron kind (cf. Sect. 17.5). For these accelerators, $\alpha = \pi/2$ ($\vec{V}$ is orthogonal to $\vec{B}$). The duration of one revolution in the accelerator being $T = 2\pi/(|\omega_B|/\Gamma)$, the energy lost by the particle via synchrotron radiation during one revolution is $\Delta\mathfrak{E} = \mathscr{P}\times 2\pi\Gamma/|\omega_B|$, i.e.

$$\Delta\mathfrak{E} = \frac{|q|^3 B\Gamma^3 V^2}{3\varepsilon_0 c^3 m} = \frac{q^2}{3\varepsilon_0 R}\left(\frac{V}{c}\right)^3\left(\frac{\mathfrak{E}}{mc^2}\right)^4. \tag{20.35}$$

In the second equality, we have let appear the trajectory radius via (17.72): $R = \Gamma mV/(|q|B)$, as well as the particle's energy $\mathfrak{E}$ via $\Gamma = \mathfrak{E}/(mc^2)$. For ultra-relativistic particles ($V/c \simeq 1$), of charge $q = \pm e$, the numerical values are

$$\Delta\mathfrak{E}_{\text{electron}} = 8.85 \times 10^{-8} \left(\frac{1\ \text{km}}{R}\right)\left(\frac{\mathfrak{E}}{1\ \text{GeV}}\right)^4 \text{GeV} \tag{20.36}$$

$$\Delta\mathfrak{E}_{\text{proton}} = 7.78 \times 10^{-21} \left(\frac{1\ \text{km}}{R}\right)\left(\frac{\mathfrak{E}}{1\ \text{GeV}}\right)^4 \text{GeV}. \tag{20.37}$$

Therefore, for the electrons of the former LEP synchrotron at CERN ($R = 4.3$ km and $\mathfrak{E} = 104$ GeV; cf. Table 17.1), $\Delta\mathfrak{E} = 2.4$ GeV $= 0.023\,\mathfrak{E}$, while for protons in the LHC ($R = 4.3$ km and $\mathfrak{E} = 7$ TeV; cf. Table 17.1), $\Delta\mathfrak{E} = 4.3$ keV $= 6 \times 10^{-10}\,\mathfrak{E}$. We conclude that the energy loss by synchrotron radiation is negligible for current proton synchrotrons, such as the LHC, but constitutes a limiting factor for electron synchrotrons. Future projects of electron or positron accelerators are therefore based on linear accelerators (linacs, cf. Sect. 17.5.2). This is notably the case of the ILC (International Linear Collider, cf. Table 17.1), which should accelerate electrons up to 250 GeV, to make them collide with positrons of the same energy (Barish et al. 2008).

20.4.2 Spectrum of Synchrotron Radiation

We have seen above that at the nonrelativistic limit, synchrotron radiation, then called *cyclotron radiation*, is emitted at a single frequency: the cyclotron frequency ω_B [cf. Eq. (20.33)]. How does this result change when the emitting particle moves with a relativistic velocity? In view of (20.31)–(20.32), a naive answer would be that the signal frequency is simply decreased by a factor Γ to become the synchrotron frequency ω_B/Γ [cf. (17.68)]. But considering the Doppler factor $(1 - \vec{\boldsymbol{n}} \cdot \vec{V})^{-3}$, as well as the cross product $\vec{V} \times_{u_0} \vec{\boldsymbol{\gamma}}$, in the expression (18.123) of the radiated electric field, we realize that the signal can no longer be monochromatic, even if $\vec{V}$ and $\vec{\boldsymbol{\gamma}}$ are periodic functions of t. Without going into the details of the computation (see, e.g. Rybicki and Lightman (1985)), let us show that the spectrum is extended, much beyond the frequency ω_B, as a consequence of the focusing effect described in Sect. 20.3.4.

Since most of the emission takes place in the direction of motion of the particle within a cone of opening angle $2\theta \simeq 2/\Gamma$ (cf. Sect. 20.3.4.3), the distant observer perceives only the radiation emitted on a fraction A_1A_2 of the trajectory, of length $d \simeq \tilde{R} \times 2\theta \simeq 2\tilde{R}/\Gamma$, where $\tilde{R}$ is the trajectory's curvature radius (cf. Fig. 20.8). The trajectory being a helix of radius $R = \Gamma V \sin\alpha/\omega_B$ [Eq. (17.65)] and angle α with respect to its axis, it is easy to see that the curvature radius is $\tilde{R} = R/\sin^2\alpha$. We have thus

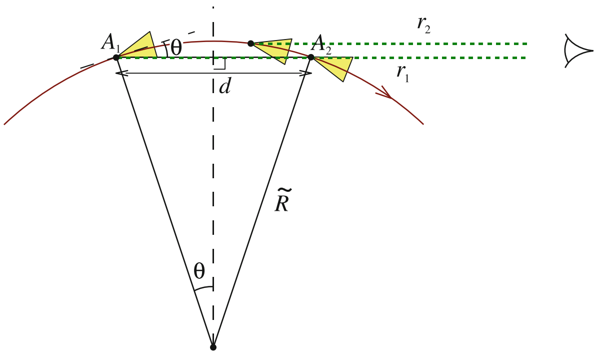

Fig. 20.8 Visibility window of synchrotron radiation by a distant observer. The opening half-angle of the emission cone is $\theta \simeq 1/\Gamma$, where Γ is the ultra-relativistic particle's Lorentz factor with respect to the observer

$$d \simeq \frac{2R}{\Gamma \sin^2 \alpha} = \frac{2V}{\omega_B \sin \alpha}.$$

For the distant observer, the duration of the signal emitted between the points A_1 and A_2, is

$$T = t_{A_2} + \frac{r_2}{c} - \left(t_{A_1} + \frac{r_1}{c}\right) = \underbrace{t_{A_2} - t_{A_1}}_{d/V} + \underbrace{\frac{r_1 - r_2}{c}}_{-d/c} = \frac{d}{V}\left(1 - \frac{V}{c}\right) = \frac{2(1 - V/c)}{\omega_B \sin \alpha}.$$

At the ultra-relativistic limit under consideration, $1 - V/c = (1 - V^2/c^2)/(1 + V/c) \simeq (1 - V^2/c^2)/2 \simeq 1/(2\Gamma^2)$, so that

$$T \simeq \frac{1}{\Gamma^2 \omega_B \sin \alpha}. \tag{20.38}$$

This duration is much shorter, by a factor of order Γ^2, than the cyclotron period $2\pi/\omega_B$. The corresponding frequency is $f_c = 1/T$, i.e.

$$\boxed{f_c = \Gamma^2 \omega_B \sin \alpha = \Gamma^2 \frac{qB}{m} \sin \alpha = \Gamma^3 \frac{c}{R} \sin^2 \alpha}, \tag{20.39}$$

where we have used the relation (17.65) between ω_B and (R, V) with $V \simeq c$. f_c is the characteristic frequency of the upper bound of the synchrotron spectrum, the lower bound being the synchrotron frequency $f_s = \omega_B/(2\pi\Gamma)$. Note that $f_c/f_s = 2\pi\Gamma^3 \sin \alpha \gg 1$. The spectrum of synchrotron radiation emitted by relativistic particles is therefore very wide.

Fig. 20.9 Synchrotron facility SOLEIL, in Saclay, near Paris. The circular building houses the electron acceleration chain (a linac + a synchrotron), the storage ring of radius 57 m and the various beamlines for the use of synchrotron radiation [Source: SOLEIL]

20.4.3 Applications

20.4.3.1 Use in Research and Industry

We have stressed above that synchrotron radiation is a drawback regarding circular accelerators for high-energy electrons. On the other side, synchrotron radiation presents unique features, leading to numerous applications: this is an intense source of photons with a broad energy spectrum, from microwaves up to X-rays, thanks to the factor Γ^3 in (20.39). Some facilities have thus been constructed, comprising an accelerator (usually a linac in series with a synchrotron) and a storage ring (cf. Sect. 17.5.5), with the only aim to exploit synchrotron radiation. The accelerated particles are electrons, and it is relatively easy to impart large Lorentz factor to them, above one thousand (cf. Table 17.1). For a storage ring, $\alpha = \pi/2$ and (20.39) yield the following characteristic energy:

$$\varepsilon_c = hf_c = 12.4 \left(\frac{\Gamma}{1000}\right)^3 \left(\frac{100\ \text{m}}{R}\right)\ \text{eV}. \tag{20.40}$$

One of the most efficient synchrotron facilities is SOLEIL, installed in Paris area and operating since 2006 (cf. Fig. 20.9). Substituting SOLEIL's parameters (R = 57 m and $\Gamma = 5.4 \times 10^3$; cf. Table 17.1) in (20.40), we get $\varepsilon_c = 3.4$ keV, i.e. X-ray photons. Actually, the energies reached in SOLEIL are ~ 20 keV. Formula (20.40) corresponds indeed to the emission by electrons submitted to the centripetal acceleration of a purely circular trajectory. However some magnetic devices, named *undulators*, have been inserted all along the ring, to make the electrons oscillate and thereby provide them with a supplementary acceleration, which allows one to extend the energy range to 20 keV. The low-energy part of SOLEIL synchrotron radiation is also exploited, with a beamline entirely devoted to the infrared radiation.

Synchrotron facilities, like SOLEIL or the European Synchrotron Radiation Facility (ESRF) in Grenoble, France (cf. Table 17.1), cover a wide field of applications: physics, chemistry, biology, materials science, geophysics, astrophysics and archaeology. Scientists from various horizons, as well as engineers specialized in the conception of materials, electronic compounds or pharmaceutical drugs, are using synchrotron radiation.

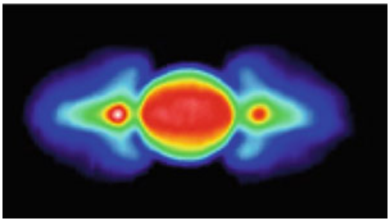

Fig. 20.10 Radio image of Jupiter, at the wavelength $\lambda = 13$ cm. The planet's disk appears via its thermal emission, while the belt surrounding it results from the synchrotron emission from relativistic electrons in the planet's magnetosphere. [Source: Y. Leblanc et al. (1997)]

One of the first applications is crystallography, using essentially the diffraction by a rather hard (10–20 keV) monochromatic radiation, to probe the structure of matter, at scales ranging from the angstrom to the micrometre. The structure of proteins is thus resolved, which is a major goal for biology and medicine. Other specific experiments take advantage of the unique features of synchrotron radiation. The extended spectrum allows one to measure the variations of the X-ray absorption by matter as a function of the wavelength near the absorption thresholds. The high brightness of synchrotron radiation allows for time-resolved kinematic studies (investigation of reaction mechanisms) or for studying materials under extreme conditions (pressure of a few tens of gigapascals, typical from the Earth's mantle). Finally, new X-ray imaging techniques are under development, using a synchrotron-radiation beam.

20.4.3.2 Synchrotron Radiation in Astrophysics

Synchrotron radiation plays an important role in astrophysics, due to the omnipresence of the magnetic field and the numerous situations where electrons are accelerated up to relativistic velocities. For instance, electrons trapped in Jupiter magnetosphere are producing an intense synchrotron emission in radio waves (cf. Fig. 20.10).

Another example is provided by the Crab Nebula, which is the remnant of the supernova observed in the year 1054. It harbours a neutron star, formed by the gravitational collapse of the core of the progenitor massive star during the supernova phenomenon. Being highly magnetized ($B \sim 10^8$ T) and rapidly rotating (30 rotations per second—thereby making the neutron star a *pulsar*), this star is a source of ultra-relativistic electrons. In the mean magnetic field of the nebula ($B \sim 10^{-8}$ T), the electrons generate an intense synchrotron radiation (cf. Fig. 20.11). The spectrum is very wide, from radio frequencies to X-rays. The maximum is around 10^{16} Hz, which is in the ultraviolet range and gives the bluish colour in Fig. 20.11. Setting $f_c = 10^{16}$ Hz in (20.39), as well as $B = 10^{-8}$ T, we find an estimate of the Lorentz factor of the electrons: $\Gamma \sim 10^6$. They are thus highly relativistic.

Fig. 20.11 Crab Nebula, remnant of the supernova that appeared in 1054. This picture has been taken by the Very Large Telescope (VLT) of the European Southern Observatory, in northern Chile. The intense diffuse emission near the centre (bluish on the colour image) is the synchrotron radiation by ultra-relativistic electrons moving in the nebula's magnetic field [Source: European Southern Observatory]

Finally, let us close this brief survey by mentioning the synchrotron emission of the jets emanating from active galactic nuclei, a nice example of which has been encountered in Sect. 9.4.5 (cf. Fig. 9.12). Another example is provided by Fig. 21.4 in the next chapter.

Chapter 21
Relativistic Hydrodynamics

21.1 Introduction

Within the framework of Minkowski spacetime, relativistic hydrodynamics can be defined as the study of fluids whose velocity relative to a reference observer is a non-negligible fraction of c or whose internal energy density and pressure are non-negligible with respect to the mass-energy density, the latter point meaning that the particles constituting the fluid have relativistic velocities. There are currently two growing application fields: (i) astrophysics, with the relativistic jets emitted by micro-quasars, active galactic nuclei or gamma-ray-burst sources, and (ii) heavy ion collisions in accelerators, which seem to generate a quark-gluon plasma for which a hydrodynamic description is appropriate. Of course, relativistic fluids are also present in neutron stars, in accretion disks around black holes, as well as in cosmology. Even if the complete study of these fluids requires general relativity (cf. Sect. 22.4), many results exposed in the present chapter are applicable to them.

We shall limit ourselves to perfect fluids, which are defined from their energy–momentum tensor in Sect. 21.2. We present then the equations of conservation of baryon number (Sect. 21.3) and of energy–momentum (Sect. 21.4), the latter leading notably to the relativistic generalization of the Euler equation. In Sect. 21.5, we provide an alternative formulation of relativistic fluid dynamics, which is based on exterior calculus and which gives an important role to a certain differential form: the vorticity 2-form. This approach has the advantage to lead simply to classical conservation laws (Bernoulli's theorem and Kelvin's circulation theorem), as we shall see in Sect. 21.6. Irrotational flows are discussed in the same section. Finally, in Sect. 21.7, we shall describe the applications mentioned above: astrophysical jets and quark-gluon plasma.

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21.2 The Perfect Fluid Model

21.2.1 Energy–Momentum Tensor

A ***perfect fluid*** can be defined formally as a medium whose energy–momentum tensor (cf. Chap. 19) takes the form

$$\boxed{\boldsymbol{T} = (\varepsilon + p)\, \underline{\boldsymbol{u}} \otimes \underline{\boldsymbol{u}} + p\, \boldsymbol{g}}, \qquad (21.1)$$

where

- $\underline{\boldsymbol{u}}$ a field of linear forms on $\mathscr{E}$, metric dual of a vector field $\vec{\boldsymbol{u}}$ that is timelike, future-directed and unit ($\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{u}} = -1$).
- ε and p are two scalar fields on $\mathscr{E}$.

At each point of $\mathscr{E}$, the field $\vec{\boldsymbol{u}}$ has all the properties of a 4-velocity. We may consider it as the 4-velocity of a so-called ***fluid particle***. The field lines of $\vec{\boldsymbol{u}}$ are then the worldlines of the fluid particles (cf. Fig. 21.1). We shall call them ***fluid lines***. At the microscopic level, each fluid particle comprises a large number of elementary particles, and $\vec{\boldsymbol{u}}$ is the mean 4-velocity of these particles. The set of all fluid lines forms a so-called ***congruence***. This means that through each point $M \in \mathscr{E}$, there is one, and only one, line: that whose tangent vector is $\vec{\boldsymbol{u}}(M)$.

Remark 21.1. The *fluid lines* defined above as curves in spacetime (worldlines) should not be confused with the *streamlines*, which are the field lines of the fluid velocity vector $\vec{V}$ in the rest space of a given observer at some fixed instant.

In order to interpret the scalar fields ε and p, let us consider an observer $\mathscr{O}$ linked to the fluid, i.e. an observer whose worldline is one of the fluid lines. We shall call such an observer a ***fluid-comoving observer***, or ***comoving observer*** for short. His 4-velocity is then $\vec{\boldsymbol{u}}$. The fluid energy density measured by $\mathscr{O}$ is given by (19.5):

$$\varepsilon_{\mathscr{O}} = \boldsymbol{T}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{u}}) = (\varepsilon + p)\, \underbrace{\langle \underline{\boldsymbol{u}}, \vec{\boldsymbol{u}} \rangle}_{-1}\, \underbrace{\langle \underline{\boldsymbol{u}}, \vec{\boldsymbol{u}} \rangle}_{-1} + p\, \underbrace{\boldsymbol{g}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{u}})}_{-1} = \varepsilon.$$

Hence ε is nothing but the energy density measured by a comoving observer. It is therefore called ***proper energy density*** of the fluid.

Furthermore, the density of linear momentum measured by $\mathscr{O}$ is (19.6):

$$\boldsymbol{\varpi} = -\frac{1}{c}\boldsymbol{T}(\perp_u, \vec{\boldsymbol{u}}) = -\frac{1}{c}\left[(\varepsilon + p)\underbrace{(\vec{\boldsymbol{u}} \cdot \perp_u)}_{0}(\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{u}}) + p\, \underbrace{\boldsymbol{g}(\perp_u, \vec{\boldsymbol{u}})}_{0}\right] = 0.$$

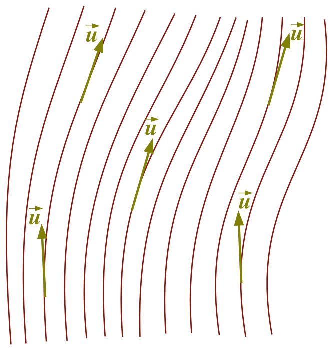

Fig. 21.1 Congruence formed by the worldlines of the fluid particles

It is not surprising to find zero for the fluid is at rest with respect to $\mathscr{O}$. Finally, the stress tensor relative to $\mathscr{O}$ is found through (19.13): $\boldsymbol{S} = \boldsymbol{T}(\perp_u, \perp_u)$. Substituting (21.1) for $\boldsymbol{T}$ and applying the outcome to a couple $(\vec{v}, \vec{w})$ of arbitrary vectors in E, we get

$$\begin{aligned}\boldsymbol{S}(\vec{v}, \vec{w}) &= (\varepsilon + p)\underbrace{\vec{u} \cdot \perp_u \vec{v}}_{0}\ \underbrace{\vec{u} \cdot \perp_u \vec{w}}_{0} + p\, \boldsymbol{g}(\perp_u \vec{v}, \perp_u \vec{w}) \\ &= p\,[\vec{v} + (\vec{u} \cdot \vec{v})\vec{u}] \cdot [\vec{w} + (\vec{u} \cdot \vec{w})\vec{u}] = p\,[\vec{v} \cdot \vec{w} + (\vec{u} \cdot \vec{v})(\vec{u} \cdot \vec{w})].\end{aligned}$$

We conclude that

$$\boxed{\boldsymbol{S} = p\,(\boldsymbol{g} + \underline{\boldsymbol{u}} \otimes \underline{\boldsymbol{u}})}. \tag{21.2}$$

After plugging this expression into relation (19.14), which gives the force $\mathrm{d}\boldsymbol{F}$ exerted onto a surface element of area $\mathrm{d}S$ and normal $\vec{n} \in E_u$ in $\mathscr{O}$'s local rest space, we obtain

$$\mathrm{d}\boldsymbol{F} = \boldsymbol{S}(., \vec{n})\, \mathrm{d}S = p[\underbrace{\boldsymbol{g}(., \vec{n})}_{\underline{\boldsymbol{n}}} + (\underbrace{\vec{u} \cdot \vec{n}}_{0})\underline{\boldsymbol{u}}]\, \mathrm{d}S,$$

i.e.

$$\boxed{\mathrm{d}\boldsymbol{F} = p\, \underline{\boldsymbol{n}}\, \mathrm{d}S}. \tag{21.3}$$

Therefore, the force exerted by the fluid on some surface element is directed along the normal to the surface, and p appears as the amplitude of the force per unit area: since $\vec{n}$ is a unit vector, one has indeed $p = \|\mathrm{d}\vec{F}\|_g / \mathrm{d}S$. The scalar field p is called ***pressure*** of the fluid. p being a function solely of the point $M \in \mathscr{E}$ where the elementary surface is considered (p is a scalar field!), and not of the direction normal to the surface, relation (21.3) is often expressed by stating that the stress tensor is ***isotropic***. This is the major characteristic of a perfect fluid. The components of $\boldsymbol{S}$ in $\mathscr{O}$'s local frame $(\vec{e}_\alpha)$ are $S_{ij} = \boldsymbol{S}(\vec{e}_i, \vec{e}_j)$ (the components $S_{0\alpha}$ are zero by definition of $\boldsymbol{S}$). Thanks to formula (21.2), there comes $S_{ij} = p[\vec{\boldsymbol{e}}_i \cdot \vec{\boldsymbol{e}}_j + (\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{e}}_i)(\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{e}}_j)]$. Now $\vec{\boldsymbol{e}}_i \cdot \vec{\boldsymbol{e}}_j = \delta_{ij}$ and $\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{e}}_i = 0$. We have thus

$$\boxed{S_{ij} = p\,\delta_{ij}}. \tag{21.4}$$

This expression clearly shows the isotropy of the stress tensor.

Remark 21.2. In view of (21.2), the energy–momentum tensor (21.1) can be rewritten as

$$\boldsymbol{T} = \varepsilon\,\underline{\boldsymbol{u}} \otimes \underline{\boldsymbol{u}} + \boldsymbol{S}. \tag{21.5}$$

Comparing with the general orthogonal decomposition (19.20) of an energy–momentum tensor with respect to an observer, we recover that $\boldsymbol{\varpi} = 0$.

21.2.2 Quantities Relative to an Arbitrary Observer

Let us now consider an arbitrary (i.e. not necessarily comoving) observer $\mathscr{O}$, of 4-velocity $\vec{\boldsymbol{u}}_0$. We may express the fluid characteristics as perceived by that observer via formulas of Sect. 19.2.2. First of all, the fluid energy density relative to $\mathscr{O}$ is given by (19.5): $E = \boldsymbol{T}(\vec{\boldsymbol{u}}_0, \vec{\boldsymbol{u}}_0)$. Substituting (21.1) for $\boldsymbol{T}$ yields $E = (\varepsilon + p)(\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{u}}_0)^2 + p(-1)$. Now $-\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{u}}_0 = \Gamma$—the fluid Lorentz factor with respect to $\mathscr{O}$ [cf. (4.10)]. We have thus

$$\boxed{E = \Gamma^2(\varepsilon + p) - p}. \tag{21.6}$$

If $\mathscr{O}$ is a comoving observer, $\Gamma = 1$ and we recover $E = \varepsilon$.

Remark 21.3. In view of Einstein formula $E = \Gamma m c^2$ [Eq. (9.16)], one could be surprised by the factor Γ^2, instead of Γ, in (21.6). But it should be reminded that in the present section, E is an energy *per unit volume*, so that an extra Γ factor arises from the length contraction in the direction of the fluid motion with respect to observer $\mathscr{O}$ (cf. Sect. 5.2.2).

The linear-momentum density of the fluid measured by $\mathscr{O}$ is found via (19.6):

$$\boldsymbol{\varpi} = -\frac{1}{c}\boldsymbol{T}(\perp_{u_0}, \vec{\boldsymbol{u}}_0) = -\frac{1}{c}\Big[(\varepsilon + p)(\vec{\boldsymbol{u}} \cdot \perp_{u_0})\underbrace{(\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{u}}_0)}_{-\Gamma} + p\,\underbrace{\boldsymbol{g}(\perp_{u_0}, \vec{\boldsymbol{u}}_0)}_{0}\Big].$$

It is appropriate to let appear the fluid velocity $\vec{\boldsymbol{V}}$ with respect to $\mathscr{O}$, via the orthogonal decomposition (4.31) of $\vec{\boldsymbol{u}}$:

$$\vec{u} = \Gamma \left(\vec{u}_0 + \frac{1}{c} \vec{V} \right). \tag{21.7}$$

We have then $\vec{u} \cdot \perp_{u_0} = (\Gamma/c)\vec{V} \cdot \perp_{u_0} = (\Gamma/c)\vec{V} \cdot \mathrm{Id} = (\Gamma/c)\underline{V}$, since $\vec{V} \in E_{u_0}$. Accordingly,

$$\boxed{\boldsymbol{\varpi} = \Gamma^2 \frac{\varepsilon + p}{c^2} \underline{V} = \frac{E + p}{c^2} \underline{V}}. \tag{21.8}$$

If $\mathscr{O}$ is a comoving observer, $\vec{V} = 0$ and we recover $\boldsymbol{\varpi} = 0$.

Finally, the components in $\mathscr{O}$'s local frame $(\vec{e}_\alpha)$ of the stress tensor relative to $\mathscr{O}$ are deduced from (19.12):

$$S_{ij} = \boldsymbol{T}(\vec{e}_i, \vec{e}_j) = (\varepsilon + p)(\vec{u} \cdot \vec{e}_i)(\vec{u} \cdot \vec{e}_j) + p \underbrace{\vec{e}_i \cdot \vec{e}_j}_{\delta_{ij}}.$$

Now, from (21.7), $\vec{u} \cdot \vec{e}_i = (\Gamma/c)\vec{V} \cdot \vec{e}_i = (\Gamma/c)V_i$. Hence

$$S_{ij} = p\,\delta_{ij} + \Gamma^2 \frac{\varepsilon + p}{c^2} V_i V_j = p\,\delta_{ij} + \frac{E + p}{c^2} V_i V_j. \tag{21.9}$$

Noticing that $\delta_{ij}\, \boldsymbol{e}^i \otimes \boldsymbol{e}^j = \boldsymbol{g} + \underline{u}_0 \otimes \underline{u}_0$ and $V_i\, \boldsymbol{e}^i = \underline{V}$, we can write

$$\boxed{\boldsymbol{S} = p\left(\boldsymbol{g} + \underline{u}_0 \otimes \underline{u}_0\right) + \frac{E + p}{c^2} \underline{V} \otimes \underline{V}}. \tag{21.10}$$

If $\mathscr{O}$ is a comoving observer, $\underline{u}_0 = \underline{u}$, $\underline{V} = 0$ and we recover (21.2).

21.2.3 Pressureless Fluid (Dust)

If the pressure field p is identically zero, the perfect-fluid energy–momentum tensor (21.1) reduces to

$$\boldsymbol{T} = \varepsilon\, \underline{u} \otimes \underline{u}. \tag{21.11}$$

This form can be recovered from the expression of the energy–momentum tensor derived in Chap. 19 for a system of particles [Eq. (19.3)]; by assuming that at a given event $M \in \mathscr{E}$, all particles have the same 4-velocity— that given by the vector field $\vec{u}$:

$$\vec{u}_a(\tau) = \vec{u}(A_a(\tau)), \tag{21.12}$$

$\vec{u}_a(\tau)$ being the 4-velocity of particle no. a, τ its proper time and $A_a(\tau)$ the event of its worldline at the instant τ. Assuming in addition that the particles are simple (cf. Sect. 9.2.1), so that $\boldsymbol{p}_a = mc\, \underline{u}_a$, expression (19.3) for the energy–momentum tensor becomes

$$\forall M \in \mathscr{E}, \quad \boldsymbol{T}(M) = \sum_{a=1}^{N} m_a c^2 \int_{-\infty}^{+\infty} \delta_{A_a(\tau)}(M)\, \underline{\boldsymbol{u}}(A_a(\tau)) \otimes \underline{\boldsymbol{u}}(A_a(\tau))\; c\, \mathrm{d}\tau.$$

Thanks to the Dirac distribution $\delta_{A_a(\tau)}(M)$, each $\underline{\boldsymbol{u}}(A_a(\tau))$ can be replaced by $\underline{\boldsymbol{u}}(M)$. The term $\underline{\boldsymbol{u}}(M) \otimes \underline{\boldsymbol{u}}(M)$ can be then extracted from the integral and even from the sum over a, since it is independent of a, leading to

$$\forall M \in \mathscr{E}, \quad \boldsymbol{T}(M) = \rho c^2\, \underline{\boldsymbol{u}}(M) \otimes \underline{\boldsymbol{u}}(M), \tag{21.13}$$

with

$$\rho := \sum_{a=1}^{N} m_a \int_{-\infty}^{+\infty} \delta_{A_a(\tau)}(M)\; c\, \mathrm{d}\tau. \tag{21.14}$$

The tensor (21.13) being of the form (21.11) (with $\varepsilon = \rho c^2$), we have thus justified the expression (21.1) for the energy–momentum tensor of a perfect fluid in the particular case of a system of noninteracting simple particles whose 4-velocities are identical at each spacetime event. In this case, one has necessarily $p = 0$. This model of pressureless fluid is called ***dust***.

Remark 21.4. It is natural to obtain $p = 0$ for a model where all the particles have the same 4-velocity, for within the framework of kinetic theory, the pressure corresponds to momentum transfer between two adjacent fluid particles by exchange of particles. If all the particles have the same 4-velocity, their worldlines are parallel to those of the fluid particles and there is no particle exchange.

21.2.4 Equation of State and Thermodynamic Relations

At the microscopic level, the fluid is made of different species of particles (for instance, electrons, protons, and nuclei). Let N be the total number of species[1] and n_a, $1 \leq a \leq N$, be the number density (i.e. number of particles per unit volume) of particles from species no. a evaluated by a comoving observer. We shall say that n_a is the ***proper particle density*** of species a. A fluid particle contains a large number of particles from each species, and we shall assume that the ***local thermodynamic equilibrium*** is achieved. This means that the mean free path of the particles is small compared to the size of the fluid particles. One may then use a thermodynamic description of the system and define the entropy and the temperature of each fluid particle. In particular, we shall use the ***proper entropy density*** s, i.e. the entropy per unit volume with respect to an comoving observer. Under the assumption of local thermodynamic equilibrium, the proper energy density is a function of the proper entropy density and the proper particle densities:

[1] Note that in Sect. 21.2.3, N stood rather for the total number of *particles*.

$$\boxed{\varepsilon = \varepsilon(s, n_1, \ldots, n_N)}. \tag{21.15}$$

This relation is called ***equation of state*** of the fluid. Its precise form depends on the model of matter. One defines the fluid's ***internal energy density*** ε_{int} as the quantity to add to the sum of all the mass-energy densities to get the total energy density ε:

$$\varepsilon = \sum_{a=1}^{N} n_a m_a c^2 + \varepsilon_{\text{int}}, \tag{21.16}$$

m_a being the mass of a particle of species a. To consider nonrelativistic limits, it is convenient to introduce the mass density $\rho := \sum_{a=1}^{N} n_a m_a$ and to write (21.16) as

$$\boxed{\varepsilon = \rho c^2 + \varepsilon_{\text{int}}}. \tag{21.17}$$

The ***temperature*** T of the fluid and the ***chemical potential*** μ_a of particles from species a are defined as the partial derivatives of the equation of state (21.15):

$$\boxed{T := \left(\frac{\partial \varepsilon}{\partial s}\right)_{n_a}} \quad \text{and} \quad \boxed{\mu_a := \left(\frac{\partial \varepsilon}{\partial n_a}\right)_{s, n_{b \neq a}}}. \tag{21.18}$$

We have then

$$\boxed{\mathrm{d}\varepsilon = T\,\mathrm{d}s + \sum_{a=1}^{N} \mu_a\,\mathrm{d}n_a}. \tag{21.19}$$

Remark 21.5. In view of (21.16), we rewrite the second equation in (21.18) as

$$\mu_a = m_a c^2 + \left(\frac{\partial \varepsilon_{\text{int}}}{\partial n_a}\right)_{s, n_{b \neq a}}. \tag{21.20}$$

It is then clear that the chemical potential μ_a comprises the single-particle mass energy $m_a c^2$ in addition to the term $\mu_a^{\text{int}} := (\partial \varepsilon_{\text{int}} / \partial n_a)_{s, n_{b \neq a}}$. At the nonrelativistic limit, it is μ_a^{int}, and not μ_a, that coincides with the "classical" chemical potential.

Let us consider a three-dimensional domain $\mathscr{V}$ that is (i) comoving with the fluid and (ii) sufficiently small so that the densities s and n_a can be considered as uniform over $\mathscr{V}$. Denoting by V the volume of $\mathscr{V}$ (as measured by a comoving observer), $\mathscr{V}$ contains then the entropy $S = sV$ and $N_a = n_a V$ particles from species a. The total fluid energy contained in $\mathscr{V}$ is $U = \varepsilon V$. Taking the differential of this relation and using (21.19) yields

$$
\begin{aligned}
\mathrm{d}U &= \mathrm{d}(\varepsilon V) = V\,\mathrm{d}\varepsilon + \varepsilon\,\mathrm{d}V = V\left[T\,\mathrm{d}s + \sum_{a=1}^{N}\mu_a\,\mathrm{d}n_a\right] + \varepsilon\,\mathrm{d}V \\
&= V\left[T\,\mathrm{d}\left(\frac{S}{V}\right) + \sum_{a=1}^{N}\mu_a\,\mathrm{d}\left(\frac{N_a}{V}\right)\right] + \varepsilon\,\mathrm{d}V \\
\mathrm{d}U &= T\,\mathrm{d}S + \left[\varepsilon - Ts - \sum_{a=1}^{N}\mu_a n_a\right]\mathrm{d}V + \sum_{a=1}^{N}\mu_a\,\mathrm{d}N_a.
\end{aligned}
\tag{21.21}
$$

Now the ***first law of thermodynamics*** states that

$$
\mathrm{d}U = T\,\mathrm{d}S - p\,\mathrm{d}V + \sum_{a=1}^{N}\mu_a\,\mathrm{d}N_a. \tag{21.22}
$$

Comparing with (21.21), we obtain the following expression of the fluid pressure:

$$
\boxed{p = -\varepsilon + Ts + \sum_{a=1}^{N}\mu_a n_a}. \tag{21.23}
$$

We recover here a familiar thermodynamic identity: that identifying the ***free enthalpy*** (or ***Gibbs free energy***) $G := U + PV - TS$ to the sum $\sum_{a=1}^{N}\mu_a N_a$. Indeed, (21.23) is nothing but the relation $G = \sum_{a=1}^{N}\mu_a N_a$ divided by V. Equation (21.23) shows that the pressure p involved in the energy–momentum tensor (21.1) of a perfect fluid is a function of $(s, n_1, \ldots, n_N)$ fully determined by the equation of state $\varepsilon(s, n_1, \ldots, n_N)$. Let us recall that T and μ_a are nothing but the partial derivatives of $\varepsilon(s, n_1, \ldots, n_N)$ [Eq. (21.18)]. Note also that the quantity $\varepsilon + p$ involved in the energy–momentum tensor (21.1) as well as in the expressions derived in Sect. 21.2.2 is the ***proper enthalpy density*** and that (21.23) provides the following expression for it:

$$
\varepsilon + p = Ts + \sum_{a=1}^{N}\mu_a n_a. \tag{21.24}
$$

21.2.5 Simple Fluids

In "ordinary" matter, most of the mass is contained in protons and neutrons, which belong to the family of ***baryons***, i.e. of subatomic particles made of three quarks. To each proton or neutron is assigned a ***baryon number*** of $+1$. We call ***proper baryon density*** and denote by n (without any index) the baryon number per unit volume measured by a comoving observer. We shall define a ***simple fluid*** as a perfect fluid whose equation of state depends only on the proper entropy density and on the proper baryon density:

$$\varepsilon = \varepsilon(s, n). \tag{21.25}$$

This model is valid in two opposite cases:

- The (chemical or nuclear) reactions between the various components are so fast with respect to the timescale of the studied problem that one may assume a complete chemical or nuclear equilibrium between the various species. Then the densities of each species are fully specified by n and s only: $n_a = Y_a^{\text{eq}}(s,n)\, n$, with some well-defined functions Y_a^{eq}. Accordingly, (21.15) takes the form (21.25).
- The characteristic timescales of the (chemical or nuclear) reactions between the various components are very large in comparison with the timescale of the studied problem, so that one may consider that the fluid composition is "frozen"; the densities of each species are then deduced from the baryon density according to $n_a = Y_a n$, with fixed particle to baryon ratios Y_a. The general equation of state (21.15) is then reduced to (21.25).

Remark 21.6. Actually, the simple fluid model holds even if n is not the density of baryons but of other (conserved) particles, the essential feature being that the equation of state (21.25) remains a function of two variables.

A particular case of simple fluid is that of a ***barotropic fluid***—the equation of state is then a function of the proper baryon density only:

$$\varepsilon = \varepsilon(n). \tag{21.26}$$

This is notably the case of cold dense matter (i.e. having a temperature below the Fermi temperature), which constitutes the interior of white dwarfs and neutron stars. For a barotropic fluid, $T = 0$ [given the definition (21.18)] and (21.24) takes a very simple form:

$$\varepsilon + p = \mu n \qquad \text{(barotropic)}. \tag{21.27}$$

In this case, the baryon chemical potential μ is thus equal to the enthalpy per baryon $(\varepsilon + p)/n$.

Example 21.1. An example of a barotropic fluid is a ***polytrope***. The corresponding equation of state is defined from three constants: κ, γ (the so-called ***adiabatic index***) and the mean baryon mass $m_{\text{b}} \simeq 1.66 \times 10^{-27}$ kg, according to

$$\varepsilon(n) = m_{\text{b}} c^2 n + \frac{\kappa}{\gamma - 1} n^{\gamma}. \tag{21.28}$$

Equation (21.27) along with $\mu = \mathrm{d}\varepsilon/\mathrm{d}n$ shows then that the pressure is

$$p(n) = \kappa\, n^{\gamma}. \tag{21.29}$$

A concrete example of polytrope is provided by matter composed of atomic nuclei and electrons, the latter being degenerate, forming an ideal Fermi gas and providing most of the pressure. This is notably the matter constituting stars of the white dwarf type. When the electrons are not relativistic (low density), the equation of state is polytropic with $\kappa = (3\pi^2)^{2/3}\hbar^2/(5m_e)Y_e^{5/3}$ and $\gamma = 5/3$, where m_e is the electron mass and $Y_e := n_e/n$ the number of electrons per baryon (in a white dwarf, $Y_e \simeq 0.5$). At the opposite, when the electrons are ultra-relativistic (high density), the equation of state is still polytropic, but with $\kappa = (3\pi^2)^{1/3}\hbar c Y_e^{4/3}/4$ and $\gamma = 4/3$. Between these two extreme regimes, the equation of state is not polytropic and exhibits a more complicated dependency in n (Diu et al. 1989).

21.3 Baryon Number Conservation

21.3.1 Baryon Four-Current

Given the proper baryon density n and the fluid 4-velocity $\vec{u}$, one defines the ***baryon four-current***, or ***baryon 4-current*** for short, as the vector field

$$\boxed{\vec{j}_b := n\vec{u}}. \tag{21.30}$$

The flux of $\vec{j}_b$ through any three-dimensional domain $\mathscr{V}$ in the rest space of an observer $\mathscr{O}$ gives the total number $\mathscr{N}$ of baryons contained in this domain (baryon number of $\mathscr{V}$):

$$\mathscr{N} = \Phi_{\mathscr{V}}(\vec{j}_b) = -\int_{\mathscr{V}} \vec{j}_b \cdot \vec{u}_0 \, dV = \int_{\mathscr{V}} \star \underline{j}_b, \tag{21.31}$$

where (i) $\vec{u}_0$ is the 4-velocity of $\mathscr{O}$, which is also the unit normal to the domain $\mathscr{V}$, (ii) the $-$ sign results from the timelike character of $\vec{u}_0$ and (iii) the last equality follows from formula (16.44), which allows one to write the flux of $\vec{j}_b$ as the integral over the 3-volume $\mathscr{V}$ of the 3-form $\star\underline{j}_b$, the Hodge dual of the 1-form $\underline{j}_b$ associated with $\vec{j}_b$ by metric duality.

Remark 21.7. The baryon 4-current $\vec{j}_b$ is the analogue regarding baryon number of the electric 4-current $\vec{j}$ regarding electric charge. In particular, (21.31) has exactly the same form as (18.1) and (18.2), which provide the total electric charge of the domain $\mathscr{V}$.

To be convinced that the quantity $\mathscr{N}$ defined by (21.31) does correspond to the number of baryons contained in $\mathscr{V}$, it suffices to replace $\vec{j}_b$ by $n\vec{u}$ and to notice that $-\vec{u} \cdot \vec{u}_0 = \Gamma$, the fluid Lorentz factor with respect to $\mathscr{O}$. Formula (21.31) becomes then

$$\mathcal{N} = \int_{\mathcal{V}} \Gamma n \, \mathrm{d}V.$$

Now, it is easy to see that

$$N := \Gamma n = -\vec{u}_0 \cdot \vec{j}_{\mathrm{b}} \tag{21.32}$$

is nothing but the baryon density measured by observer $\mathscr{O}$, the factor Γ taking into account the FitzGerald–Lorentz contraction [Eq. (5.17)] in the direction of the fluid motion with respect to $\mathscr{O}$ [compare (21.32) with the expression (18.12) of the electric-charge density]. As the integral of N over $\mathscr{V}$, we conclude that $\mathcal{N}$ is the total number of baryons in $\mathscr{V}$.

21.3.2 Principle of Baryon Number Conservation

In the standard model of particle physics, the baryon number is conserved[2] by the electromagnetic and strong interactions as well as by all the processes involving the weak interaction except for non-perturbative processes connected to the so-called *Adler–Bell–Jackiw anomaly*. Except in the primordial universe, the conditions of this anomaly are never fulfilled. We shall then postulate the conservation of the baryon number and state it in the same form as that of electric charge in Chap. 18 or that of energy–momentum in Chap. 19, namely:

If a fluid is isolated, the flux of the baryon 4-current through any closed hypersurface Σ vanishes:

$$\boxed{\text{isolated fluid and } \Sigma \text{ closed} \Longrightarrow \Phi_{\Sigma}(\vec{j}_{\mathrm{b}}) = \int_{\Sigma} \star \underline{j}_{\mathrm{b}} = 0}. \tag{21.33}$$

We can have exactly the same reasoning as in Sect. 18.4.1 and conclude that the principle (21.33) leads to a conservation law between two instants for a given observer as well as to the invariance of the baryon number under a change of observer.

---

[2]Some theories beyond the standard model, such as Grand Unified Theories, induce a violation of the baryon number conservation, leading to the proton decay; such a decay has however not been observed to date, the experimental lower bound on the proton life time being 10^{33} years.

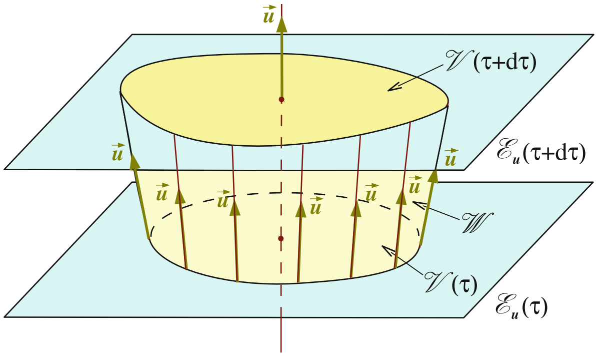

Fig. 21.2 Transport of a volume element $\mathscr{V}(\tau)$ by the fluid

By virtue of Stokes' theorem (16.46), (21.33) implies that the 4-form $\mathbf{d} \star \underline{j}_{\mathrm{b}}$ is identically zero. By Hodge duality, it follows that the 0-form (scalar field) $\nabla \cdot \vec{j}_{\mathrm{b}}$ vanishes as well [cf. Eq. (15.88)]. In other words, the local version of the principle of baryon number conservation is

$$\boxed{\nabla \cdot (n\vec{u}) = 0}. \tag{21.34}$$

Remark 21.8. Again, this equation is fully similar to (18.38), the local expression of electric-charge conservation.

It is instructive to establish (21.34) by another mean, based on the evolution of a fluid element. Let us consider a 3-volume $\mathscr{V}(\tau)$ transported by the fluid (τ being the fluid proper time) which is sufficiently small so that the densities n_a are constant over it. By *volume transported by the fluid*, it is understood that $\mathscr{V}(\tau) \subset \mathscr{E}_{\boldsymbol{u}}(\tau)$ and that the worldlines of the boundary of $\mathscr{V}(\tau)$ are tangent to $\vec{u}$ (i.e. they are fluid lines) (cf. Fig. 21.2). Let $\mathrm{d}\tau$ be some infinitesimal increment of the fluid proper time and $\mathscr{U}$ the four-dimensional domain bounded by $\mathscr{V}(\tau)$, $\mathscr{V}(\tau+\mathrm{d}\tau)$ and the set $\mathscr{W}$ of the fluid lines connecting $\mathscr{V}(\tau)$ to $\mathscr{V}(\tau+\mathrm{d}\tau)$ (cf. Fig. 21.2). The four-dimensional Gauss–Ostrogradsky theorem (16.64) applied to the vector field $\vec{u}$ in $\mathscr{U}$ leads to

$$\int_{\mathscr{U}} \nabla \cdot \vec{u} \, \mathrm{d}U = \Phi_{\mathscr{V}(\tau)}(\vec{u}) + \Phi_{\mathscr{W}}(\vec{u}) + \Phi_{\mathscr{V}(\tau+\mathrm{d}\tau)}(\vec{u}),$$

where the hypersurfaces $\mathscr{V}(\tau)$, $\mathscr{W}$ and $\mathscr{V}(\tau+\mathrm{d}\tau)$ are oriented as parts of $\mathscr{U}$'s boundary. Taking into account that $\mathscr{V}(\tau)$ and $\mathscr{V}(\tau+\mathrm{d}\tau)$ are spacelike and $\mathscr{W}$ is timelike, the above fluxes are

$$\Phi_{\mathscr{V}(\tau)}(\vec{\boldsymbol{u}}) = -\int_{\mathscr{V}(\tau)} \vec{\boldsymbol{u}} \cdot (-\vec{\boldsymbol{u}})\, \mathrm{d}V = -\int_{\mathscr{V}(\tau)} \mathrm{d}V = -V(\tau)$$

$$\Phi_{\mathscr{W}}(\vec{\boldsymbol{u}}) = \int_{\mathscr{W}} \underbrace{\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{n}}}_{0}\, \mathrm{d}V = 0$$

$$\Phi_{\mathscr{V}(\tau+\mathrm{d}\tau)}(\vec{\boldsymbol{u}}) = -\int_{\mathscr{V}(\tau+\mathrm{d}\tau)} \vec{\boldsymbol{u}} \cdot (+\vec{\boldsymbol{u}})\, \mathrm{d}V = \int_{\mathscr{V}(\tau+\mathrm{d}\tau)} \mathrm{d}V = V(\tau + \mathrm{d}\tau),$$

where $V(\tau)$ stands for the volume of $\mathscr{V}(\tau)$. We have thus

$$\int_{\mathscr{U}} \boldsymbol{\nabla}\cdot \vec{\boldsymbol{u}}\, \mathrm{d}U = V(\tau + \mathrm{d}\tau) - V(\tau).$$

Now, $\mathrm{d}\tau$ being an infinitesimal quantity and $\mathscr{V}(\tau)$ a volume sufficiently small so that $\boldsymbol{\nabla}\cdot \vec{\boldsymbol{u}}$ is uniform in it, we may write

$$\int_{\mathscr{U}} \boldsymbol{\nabla}\cdot \vec{\boldsymbol{u}}\, \mathrm{d}U = \int_{\mathscr{U}} \boldsymbol{\nabla}\cdot \vec{\boldsymbol{u}}\; c\mathrm{d}\tau \mathrm{d}V = c\mathrm{d}\tau \int_{\mathscr{V}(\tau)} \boldsymbol{\nabla}\cdot \vec{\boldsymbol{u}}\, \mathrm{d}V = c\mathrm{d}\tau\; (\boldsymbol{\nabla}\cdot \vec{\boldsymbol{u}}) V(\tau).$$

Finally, we get $c\mathrm{d}\tau\; (\boldsymbol{\nabla}\cdot \vec{\boldsymbol{u}}) V(\tau) = V(\tau + \mathrm{d}\tau) - V(\tau)$, i.e.

$$\boxed{c\, \boldsymbol{\nabla}\cdot \vec{\boldsymbol{u}} = \frac{1}{V} \frac{\mathrm{d}V}{\mathrm{d}\tau}}. \tag{21.35}$$

The divergence of the fluid 4-velocity corresponds thus to the relative time variation of a volume element dragged by the fluid. We can then easily evaluate the time variation of the number $\mathscr{N} = nV$ of baryons contained in $\mathscr{V}(\tau)$:

$$\frac{\mathrm{d}\mathscr{N}}{\mathrm{d}\tau} = \frac{\mathrm{d}}{\mathrm{d}\tau}(nV) = \underbrace{\frac{\mathrm{d}n}{\mathrm{d}\tau}}_{c\, \vec{\boldsymbol{u}}\cdot\vec{\boldsymbol{\nabla}} n} V + n \underbrace{\frac{\mathrm{d}V}{\mathrm{d}\tau}}_{Vc\boldsymbol{\nabla}\cdot\vec{\boldsymbol{u}}} = cV\boldsymbol{\nabla}\cdot(n\vec{\boldsymbol{u}}),$$

hence,

$$\frac{1}{V}\frac{\mathrm{d}\mathscr{N}}{\mathrm{d}\tau} = c\, \boldsymbol{\nabla}\cdot(n\vec{\boldsymbol{u}}). \tag{21.36}$$

The baryon number conservation expressed as $\mathrm{d}\mathscr{N}/\mathrm{d}\tau = 0$ is therefore equivalent to $\boldsymbol{\nabla}\cdot(n\vec{\boldsymbol{u}}) = 0$ [Eq. (21.34)].

21.3.3 Expression with Respect to an Inertial Observer

Given an inertial observer $\mathscr{O}$, of 4-velocity $\vec{\boldsymbol{u}}_0$, let us substitute the decomposition (21.7) for the fluid 4-velocity in the expression (21.34) of the baryon number conservation:

$$\nabla \cdot \left[\underbrace{n\Gamma}_{N} \left(\vec{\boldsymbol{u}}_0 + \frac{1}{c} \vec{\boldsymbol{V}} \right) \right] = \vec{\boldsymbol{u}}_0 \cdot \vec{\nabla} N + \frac{1}{c} \nabla \cdot (N \vec{\boldsymbol{V}}) = 0,$$

where we have let appear the baryon density with respect to $\mathscr{O}$, N, according to (21.32). Since $c\,\vec{\boldsymbol{u}}_0 \cdot \vec{\nabla} N = \partial N / \partial t$, t being $\mathscr{O}$'s proper time, there comes

$$\boxed{\frac{\partial N}{\partial t} + \nabla \cdot (N \vec{\boldsymbol{V}}) = 0}. \tag{21.37}$$

This relation can be expressed in terms of the components with respect to the inertial coordinates $(x^\alpha) = (ct, x^i)$ associated with $\mathscr{O}$ as (note that $V^0 = 0$)

$$\frac{\partial N}{\partial t} + \frac{\partial}{\partial x^i} (N V^i) = 0. \tag{21.38}$$

Remark 21.9. Equation (21.37), which links the time variation of the baryon density N to the divergence of the baryon flux, $N\vec{\boldsymbol{V}}$—these two quantities being defined relatively to observer $\mathscr{O}$—is identical to an equation of conservation in Newtonian physics: it does not involve any purely relativistic term, such as a Lorentz factor or a factor c^{-1}.

21.4 Energy–Momentum Conservation

21.4.1 Introduction

The general form of the equation expressing the energy–momentum conservation has been given in Sect. 19.3.3: it relates the divergence of the energy–momentum tensor to the external 4-force density $\mathscr{F}_{\text{ext}}$ exerted onto the medium [Eq. (19.28)]:

$$\vec{\nabla} \cdot \boldsymbol{T} = \mathscr{F}_{\text{ext}}. \tag{21.39}$$

In what follows, we shall decompose this equation in various ways.

First of all, substituting the perfect fluid expression (21.1) for $\boldsymbol{T}$ in (21.39) and expanding, we obtain

$$[\nabla_{\vec{u}} (\varepsilon + p) + (\varepsilon + p) \nabla \cdot \vec{\boldsymbol{u}}]\, \underline{\boldsymbol{u}} + (\varepsilon + p)\, \underline{\boldsymbol{a}} + \nabla p = \mathscr{F}_{\text{ext}}, \tag{21.40}$$

where $\underline{\boldsymbol{a}}$ is the 1-form metric dual of the vector $\vec{\boldsymbol{a}} := \nabla_{\vec{u}} \vec{\boldsymbol{u}}$. The latter is nothing but the 4-acceleration of a fluid-comoving observer, as it appears when comparing the identities (2.16) and (15.28).

21.4.2 Projection onto the Fluid 4-Velocity

Equation (21.40) is an equality between two 1-forms. Applying it to the vector $\vec{u}$ yields, given that $\langle \underline{u}, \vec{u}\rangle = -1$, $\langle \underline{a}, \vec{u}\rangle = \vec{a}\cdot\vec{u} = 0$ and $\langle \nabla p, \vec{u}\rangle = \nabla_{\vec{u}}\, p$,

$$\nabla_{\vec{u}}\, \varepsilon + (\varepsilon + p)\nabla\cdot\vec{u} = -\langle \mathscr{F}_{\rm ext}, \vec{u}\rangle. \tag{21.41}$$

Now from (21.19),

$$\begin{aligned}\nabla_{\vec{u}}\, \varepsilon &= T\nabla_{\vec{u}}\, s + \sum_{a=1}^{N} \mu_a \nabla_{\vec{u}}\, n_a \\ &= T\left[\nabla\cdot(s\vec{u}) - s\nabla\cdot\vec{u}\right] + \sum_{a=1}^{N} \mu_a \left[\nabla\cdot(n_a\vec{u}) - n_a\nabla\cdot\vec{u}\right].\end{aligned}$$

Inserting this expression into (21.41) and using the identity (21.23) to set to zero the term in factor of $\nabla\cdot\vec{u}$, we get

$$\boxed{T\nabla\cdot(s\vec{u}) = -\langle \mathscr{F}_{\rm ext}, \vec{u}\rangle - \frac{1}{c}\sum_{a=1}^{N}\mu_a \mathscr{R}_a}, \tag{21.42}$$

where

$$\mathscr{R}_a := c\nabla\cdot(n_a\vec{u}) \tag{21.43}$$

is the ***volume creation rate of particles from species*** a, i.e. the number of particles from species a created per unit time and per unit volume relatively to the comoving observer. Indeed

$$\mathscr{R}_a = c\nabla\cdot(n_a\vec{u}) = c\nabla_{\vec{u}}\, n_a + n_a c\nabla\cdot\vec{u} = \frac{\mathrm{d}n_a}{\mathrm{d}\tau} + \frac{n_a}{V}\frac{\mathrm{d}V}{\mathrm{d}\tau},$$

where the last equality stems from (21.35). We have thus

$$\mathscr{R}_a = \frac{1}{V}\frac{\mathrm{d}(n_a V)}{\mathrm{d}\tau}, \tag{21.44}$$

which proves the above assertion. Similarly, the term $\nabla\cdot(s\vec{u})$, which appears in the left-hand side of (21.42), is c^{-1} times the volume creation rate of entropy.

Case of a Simple Fluid

For a simple fluid (cf. Sect. 21.2.5), the summation over a in (21.42) is limited to a single item: the baryons. Moreover, the volume creation rate of baryons is zero by virtue of the principle of baryon number conservation (21.34). Equation (21.42) reduces thus to

$$\boxed{\nabla\cdot(s\vec{\boldsymbol{u}}) = -\frac{1}{T}\langle\mathscr{F}_{\text{ext}}, \vec{\boldsymbol{u}}\rangle.}_{\text{simple fluid}} \tag{21.45}$$

In particular, if the simple fluid is isolated ($\mathscr{F}_{\text{ext}} = 0$), its entropy is conserved:

$$\boxed{\nabla\cdot(s\vec{\boldsymbol{u}}) = 0.}_{\text{isolated simple fluid}} \tag{21.46}$$

This equation is fully similar to that expressing the conservation of baryon number [Eq. (21.34)]. An immediate consequence of (21.46) and (21.34) is that the ***entropy per baryon***,

$$\boxed{S := \frac{s}{n}}, \tag{21.47}$$

stays constant along the fluid lines:

$$\boxed{\nabla_{\vec{u}}\, S = 0.}_{\text{isolated simple fluid}} \tag{21.48}$$

Indeed,

$$\begin{aligned}\nabla_{\vec{u}}\, S &= \nabla_{\vec{u}}\left(\frac{s}{n}\right) = \frac{1}{n}\left[\nabla_{\vec{u}}\, s - \frac{s}{n}\nabla_{\vec{u}}\, n\right]\\ &= \frac{1}{n}\left[\underbrace{\nabla\cdot(s\vec{\boldsymbol{u}})}_{0} - s\nabla\cdot\vec{\boldsymbol{u}} - \frac{s}{n}\underbrace{\nabla\cdot(n\vec{\boldsymbol{u}})}_{0} + s\nabla\cdot\vec{\boldsymbol{u}}\right] = 0.\end{aligned}$$

One translates the law of entropy conservation (21.46) or (21.48), by saying that the flow of an isolated simple fluid is ***adiabatic***: there is no diffusion of heat between the various fluid elements.

21.4.3 Part Orthogonal to the Fluid 4-Velocity

Equation (21.42) can be seen as one of the four components of (21.40). The three remaining components are found by combining each side of (21.40) with the orthogonal projector $\perp_u$. Since $\underline{\boldsymbol{u}} \circ \perp_u = \vec{\boldsymbol{u}} \cdot \perp_u = 0$, and $\underline{\boldsymbol{a}} \circ \perp_u = \underline{\boldsymbol{a}}$ (for $\vec{\boldsymbol{a}}$ is orthogonal to $\vec{\boldsymbol{u}}$), there comes

$$(\varepsilon + p)\underline{\boldsymbol{a}} = \mathscr{F}_{\mathrm{ext}} \circ \perp_{\boldsymbol{u}} - \nabla p \circ \perp_{\boldsymbol{u}}.$$

Using $\perp_{\boldsymbol{u}} = \mathrm{Id} + \langle \underline{\boldsymbol{u}}, .\rangle\, \vec{\boldsymbol{u}}$ [Eq. (3.12)], we can write

$$\boxed{(\varepsilon + p)\underline{\boldsymbol{a}} = -\nabla p - (\nabla_{\vec{\boldsymbol{u}}}\, p)\underline{\boldsymbol{u}} + \mathscr{F}_{\mathrm{ext}} + \langle \mathscr{F}_{\mathrm{ext}}, \vec{\boldsymbol{u}}\rangle\, \underline{\boldsymbol{u}}}. \tag{21.49}$$

This equation is clearly of the type "$m\underline{\boldsymbol{a}} = \boldsymbol{F}$". It is therefore sometimes called the *relativistic Euler equation* (cf., for instance, Misner et al. (1973) or Choquet-Bruhat (2009)), but we reserve this name to the equation involving the fluid velocity relative to some inertial observer [Eq. (21.55) below] and not involving the 4-velocity as in (21.49). Accordingly, we shall call (21.49) the ***four-dimensional Euler equation***.

Historical note: *Equations (21.41) and (21.49), which rule the dynamics of a relativistic perfect fluid, seem to have been first written in 1924 by Luther P. Eisenhart*[3] *(1924) (in the case* $\mathscr{F}_{\mathrm{ext}} = 0$*). They are found in the first synthetical article devoted to relativistic hydrodynamics written in 1937 by John L. Synge (cf. p. 74) (1937). The law of entropy conservation for an isolated simple fluid [Eq. (21.48)] has been obtained in 1954 by Abraham H. Taub*[4] *(1954).*

21.4.4 Evolution of the Fluid Energy Relative to Some Observer

Let us now perform the orthogonal decomposition of the energy–momentum conservation equation (21.39) with respect to an arbitrary inertial observer $\mathscr{O}$, of 4-velocity $\vec{\boldsymbol{u}}_0$. The right-hand side of (21.39), the external 4-force density, is orthogonally decomposed into the external power density P_{ext} and the external force density $\boldsymbol{F}_{\mathrm{ext}}$ according to

$$\mathscr{F}_{\mathrm{ext}} = \frac{P_{\mathrm{ext}}}{c}\underline{\boldsymbol{u}}_0 + \boldsymbol{F}_{\mathrm{ext}} \qquad \text{with} \quad \langle \boldsymbol{F}_{\mathrm{ext}}, \vec{\boldsymbol{u}}_0\rangle = 0. \tag{21.50}$$

We have seen in Sect. 19.3.4 that the orthogonal projection onto $\vec{\boldsymbol{u}}_0$ of the equation $\vec{\nabla}\cdot \boldsymbol{T} = \mathscr{F}_{\mathrm{ext}}$ leads to the following evolution equation for the energy density E measured by $\mathscr{O}$ [Eq. (19.34)]:

$$\frac{\partial E}{\partial t} + c^2 \vec{\nabla}\cdot \boldsymbol{\varpi} = P_{\mathrm{ext}}. \tag{21.51}$$

[3]**Luther P. Eisenhart** (1876–1965): American mathematician, known for his work in differential geometry.

[4]**Abraham H. Taub** (1911–1999): American mathematician and physicist, author of many works in relativity (more particularly in relativistic hydrodynamics) and in differential geometry.

For a perfect fluid, the linear-momentum density $\boldsymbol{\varpi}$ is expressed according to (21.8): $\boldsymbol{\varpi} = c^{-2}(E + p)\underline{\boldsymbol{V}}$, where $\vec{V}$ stands for the fluid velocity relative to $\mathscr{O}$. We obtain thus

$$\boxed{\frac{\partial E}{\partial t} + \nabla \cdot [(E + p)\vec{V}] = P_{\text{ext}}}. \tag{21.52}$$

Once rewritten in terms of the components with respect to the inertial coordinates $(x^\alpha) = (ct, x^i)$ associated with $\mathscr{O}$, this relation becomes

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x^i}[(E + p)V^i] = P_{\text{ext}}. \tag{21.53}$$

Remark 21.10. Like the equation of baryon number conservation [Eq. (21.37)], (21.52) does not involve any explicit relativistic term (Lorentz factor, etc.) and is identical to the corresponding equation of Newtonian hydrodynamics (see, e.g. Guyon et al. (2001); Rieutord (1997)).

21.4.5 Relativistic Euler Equation

The orthogonal projection of the equation $\vec{\nabla} \cdot \boldsymbol{T} = \mathscr{F}_{\text{ext}}$ onto $\mathscr{O}$'s rest space leads to (19.35), which expresses the conservation of linear momentum:

$$\frac{\partial \boldsymbol{\varpi}}{\partial t} + \vec{\nabla} \cdot \boldsymbol{S} = \boldsymbol{F}_{\text{ext}}. \tag{21.54}$$

Substituting the perfect fluid values (21.8) and (21.10) for, respectively, $\boldsymbol{\varpi}$ and $\boldsymbol{S}$, we get

$$\frac{1}{c^2}\frac{\partial}{\partial t}\,[(E + p)\underline{\boldsymbol{V}}] + \vec{\nabla} \cdot \left[p\left(\boldsymbol{g} + \underline{\boldsymbol{u}}_0 \otimes \underline{\boldsymbol{u}}_0\right) + c^{-2}(E + p)\,\underline{\boldsymbol{V}} \otimes \underline{\boldsymbol{V}}\right] = \boldsymbol{F}_{\text{ext}}.$$

Let us expand this expression, taking into account that $\nabla \boldsymbol{g} = 0$ and $\nabla \underline{\boldsymbol{u}}_0 = 0$ ($\mathscr{O}$ is inertial) and using (21.52) to replace $\partial E/\partial t + \vec{\nabla} \cdot [(E + p)\underline{\boldsymbol{V}}]$. We obtain then (considering the metric dual)

$$\boxed{\frac{\partial \vec{V}}{\partial t} + \nabla_{\vec{V}}\vec{V} = -\frac{c^2}{(E + p)}\left[\vec{\nabla}_{\perp_{u_0}} p + \frac{1}{c^2}\left(\frac{\partial p}{\partial t} + P_{\text{ext}}\right)\vec{V}\right] + \frac{c^2}{E + p}\vec{F}_{\text{ext}}}, \tag{21.55}$$

where $\vec{\nabla}_{\perp_{u_0}}$ stands for the purely spatial gradient operator with respect to $\mathscr{O}$ introduced in Sect. 18.5.2 [Eq. (18.58)]. The component $\alpha = 0$ of (21.55) in $\mathscr{O}$'s frame is identically zero (thanks to $V^0 = 0$, $F^0_{\text{ext}} = 0$ and expression (18.59) for the components of $\vec{\nabla}_{\perp_{u_0}}$). On the other hand, the three space components are

$$\frac{\partial V^i}{\partial t}+V^j\frac{\partial V^i}{\partial x^j}=-\frac{c^2}{(E+p)}\left[\frac{\partial p}{\partial x^i}+\frac{1}{c^2}\left(\frac{\partial p}{\partial t}+P_{\rm ext}\right)V^i\right]+\frac{c^2F^i_{\rm ext}}{E+p}. \qquad (21.56)$$

Equation (21.55) is the relativistic version of the ***Euler equation***, which rules the dynamics of perfect fluids. Let us indeed consider the nonrelativistic limit of (21.55). To this aim, in addition to the usual $\Gamma \to 1$ and $\vec{V}/c \to 0$, the following hypothesis have to be made:

$$\boxed{\frac{|\varepsilon_{\rm int}|}{c^2} \ll \rho}_{\rm nonrelativistic} \quad \text{and} \quad \boxed{\frac{p}{c^2} \ll \rho}_{\rm nonrelativistic}, \qquad (21.57)$$

where ρ is the mass density and $\varepsilon_{\rm int}$ the internal energy density, both introduced in Sect. 21.2.4. These properties express among others that, at the microscopic level, particles do not have relativistic velocities.

Example 21.2. For water at the atmospheric pressure, $p = 1$ atm $\simeq 10^5$ Pa, so that $p/c^2 \simeq 10^{-12}$ kg m^{-3}, which is fully negligible in front of the mass density $\rho = 10^3$ kg m^{-3}.

The nonrelativistic limit (21.57), along with (21.6) and (21.17), leads to

$$\frac{c^2}{E+p}=\frac{c^2}{\Gamma^2(\varepsilon+p)}=\frac{1}{\Gamma^2\left(\rho+\frac{\varepsilon_{\rm int}+p}{c^2}\right)}\simeq\frac{1}{\rho}, \qquad (21.58)$$

so that the nonrelativistic limit of (21.56) is

$$\frac{\partial V^i}{\partial t}+V^j\frac{\partial V^i}{\partial x^j}=-\frac{1}{\rho}\frac{\partial p}{\partial x^i}+\frac{F^i_{\rm ext}}{\rho}. \qquad (21.59)$$

We recognize the classical Euler equation (cf., e.g. Guyon et al. (2001) or Rieutord (1997)).

21.4.6 Speed of Sound

The equations derived above allow one to easily find the speed of sound in a relativistic fluid. Let us consider indeed an isolated homogeneous fluid, i.e. a fluid such that ε, p and $\vec{u}$ are constant fields on $\mathscr{E}$ (or at least on the part of $\mathscr{E}$ occupied by the fluid). Since $\vec{u}$ is constant, we may choose the inertial observer $\mathscr{O}$ to be a comoving observer. The fluid is then at rest with respect to $\mathscr{O}$, and we have $E = \varepsilon$, $\Gamma = 1$ and $\vec{V} = 0$. Let us suppose that at the instant $t = 0$ of $\mathscr{O}$'s proper time, the fluid is submitted to a small perturbation $n \to n + \delta n$, $\vec{V} = 0 \to \delta\vec{V}$, etc. We shall assume that the perturbation is adiabatic, so that $\delta S = 0$. Expanding (21.52) and (21.55) to the first order in the perturbations leads to the system

$$\frac{\partial \delta\varepsilon}{\partial t} + (\varepsilon + p)\frac{\partial \delta V^i}{\partial x^i} = 0 \tag{21.60a}$$

$$(\varepsilon + p)\frac{\partial \delta V^i}{\partial t} = -c^2 \frac{\partial \delta p}{\partial x^i}. \tag{21.60b}$$

Assuming that the equation of state is given in the form $p = p(\varepsilon, S)$, we may write

$$\delta p = \left(\frac{\partial p}{\partial \varepsilon}\right)_S \delta\varepsilon + \left(\frac{\partial p}{\partial S}\right)_\varepsilon \underbrace{\delta S}_{0} = \left(\frac{\partial p}{\partial \varepsilon}\right)_S \delta\varepsilon. \tag{21.61}$$

Then (21.60b) becomes

$$(\varepsilon + p)\frac{\partial \delta V^i}{\partial t} = -c_\mathrm{s}^2 \frac{\partial \delta\varepsilon}{\partial x^i}, \tag{21.62}$$

with

$$\boxed{c_\mathrm{s} := c\sqrt{\left(\frac{\partial p}{\partial \varepsilon}\right)_S}}. \tag{21.63}$$

Taking the time derivative of (21.60a) and the divergence of (21.62), and invoking the constant character of ε and p, we forge an equation that contains only $\delta\varepsilon$:

$$-\frac{1}{c_\mathrm{s}^2}\frac{\partial^2 \delta\varepsilon}{\partial t^2} + \sum_{i=1}^{3}\frac{\partial^2 \delta\varepsilon}{\partial {x^i}^2} = 0. \tag{21.64}$$

We recognize a wave equation, with c_s as the propagation velocity [compare with (18.69) and (18.51)]. From (21.61), the overpressure δp is propagating at the same velocity as $\delta\varepsilon$. For this reason, c_s is called ***speed of sound***. At the nonrelativistic limit, $\varepsilon \simeq \rho c^2$ [cf. (21.17) and (21.57)], and (21.63) results in the classical expression (cf., e.g. Sect. 5.2 of Rieutord (1997))

$$c_\mathrm{s} = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_S} \qquad \text{(nonrelativistic).} \tag{21.65}$$

21.4.7 Relativistic Hydrodynamics as a System of Conservation Laws

In this section, we consider an isolated fluid: $\mathscr{F}_\mathrm{ext} = 0$. Gathering the equations of conservation of baryon number, energy and linear momentum written with respect to some inertial observer $\mathscr{O}$, i.e. equations (21.37), (21.52) and (21.54) (with (21.10) substituted for $\boldsymbol{S}$), we obtain the system

$$\begin{cases} \dfrac{\partial N}{\partial t} + \dfrac{\partial}{\partial x^j}(NV^j) = 0 \\ \dfrac{\partial E}{\partial t} + \dfrac{\partial}{\partial x^j}[(E+p)V^j] = 0 \\ \dfrac{\partial \varpi_i}{\partial t} + \dfrac{\partial}{\partial x^j}\left(p\delta^j{}_i + \varpi_i V^j\right) = 0. \end{cases} \tag{21.66}$$

This is a system of conservation laws, which can be recast in the following condensed form:

$$\boxed{\frac{\partial U_A}{\partial t} + \frac{\partial}{\partial x^j} F^j_A = 0}, \tag{21.67}$$

where $A \in \{1,2,3,4,5\}$. The U_A are the components of a *state vector* in $\mathbb{R}^5$: $U_A = (N, E, \varpi_1, \varpi_2, \varpi_3)$ and the ***flux functions*** F^j_A are $F^j_A = (NV^j,\ (E+p)V^j,\ p\delta^j{}_1 + \varpi_1 V^j,\ p\delta^j{}_2 + \varpi_2 V^j,\ p\delta^j{}_3 + \varpi_3 V^j)$. The system is closed thanks to the relations

$$V^i = \frac{c^2 \varpi_i}{E+p},\ \Gamma = (1 - \delta_{ij} V^i V^j)^{-1/2},\ \varepsilon = \Gamma^{-2}(E+p) - p,\ n = \Gamma^{-1} N, \tag{21.68a}$$

$$p = p(n, \varepsilon). \tag{21.68b}$$

Equations (21.68a) are deduced from (21.8), (21.6) and (21.32). Regarding (21.68b), we assume that the fluid is simple and that the equation of state (21.25) can be inverted in order to express s as a function of n and ε, so that p can be written as a function of (n, ε).

The system (21.67) has the same structure as that of nonrelativistic hydrodynamics [the difference actually lies in (21.68a)]. It can be shown that if the equation of state is causal, i.e. if the speed of sound is smaller than c, the system (21.67) is ***hyperbolic*** (Anile 1989; Martí and Müller 2003), that is to say each of the three Jacobian matrices $J^j_{AB} := \partial F^j_A / \partial U_B$ $(1 \le j \le 3)$ has only real eigenvalues and is diagonalizable. Standard methods can then be applied to compute a solution to the system, the most efficient ones being the *high-resolution shock-capturing* methods (Martí et al. 1991; Font et al. 1994).

21.5 Formulation Based on Exterior Calculus

We are going to expose a formulation of relativistic hydrodynamics alternative to that presented above. It is based on the differential forms introduced in Sect. 15.5, for which we have already stressed the great usefulness in integral calculus (Chap. 16) and electromagnetism (Chap. 18). This approach has the advantage to lead in a simple way to the relativistic generalization of the standard conservation laws of the type of Bernoulli's theorem or Kelvin's circulation theorem.

In all this section, we shall assume that the fluid is isolated: $\mathscr{F}_{\text{ext}} = 0$.

21.5.1 Equation of Motion

The starting point is the four-dimensional Euler equation (21.49), in which we set $\mathscr{F}_{\text{ext}} = 0$:

$$(\varepsilon + p)\underline{\boldsymbol{a}} = -\boldsymbol{\nabla} p - (\boldsymbol{\nabla}_{\vec{u}}\, p)\underline{\boldsymbol{u}}. \tag{21.69}$$

We need the ***Gibbs–Duhem relation***, which is easily found by taking the differential of the thermodynamic identity (21.23) and substituting (21.19) for $\mathrm{d}\varepsilon$:

$$\boxed{\mathrm{d}p = s\,\mathrm{d}T + \sum_{a=1}^{N} n_a \mathrm{d}\mu_a}. \tag{21.70}$$

We may thus express $\boldsymbol{\nabla} p$ in terms of $\boldsymbol{\nabla} T$ and $\boldsymbol{\nabla}\mu_a$. Using (21.23) to replace $\varepsilon + p$, we can rewrite (21.69) as

$$\left(Ts + \sum_{a=1}^{N}\mu_a n_a\right)\underline{\boldsymbol{a}} = -s\boldsymbol{\nabla} T - \sum_{a=1}^{N} n_a \boldsymbol{\nabla}\mu_a - \left(s\boldsymbol{\nabla}_{\vec{u}} T + \sum_{a=1}^{N} n_a \boldsymbol{\nabla}_{\vec{u}}\,\mu_a\right)\underline{\boldsymbol{u}}.$$

Let us write $\underline{\boldsymbol{a}} = \boldsymbol{\nabla}_{\vec{u}}\,\underline{\boldsymbol{u}}$ and gather the various terms to obtain

$$s\,[\boldsymbol{\nabla}_{\vec{u}}(T\underline{\boldsymbol{u}}) + \boldsymbol{\nabla} T] + \sum_{a=1}^{N} n_a\,[\boldsymbol{\nabla}_{\vec{u}}(\mu_a\underline{\boldsymbol{u}}) + \boldsymbol{\nabla}\mu_a] = 0. \tag{21.71}$$

Now the 1-form $\boldsymbol{\nabla}_{\vec{u}}(T\underline{\boldsymbol{u}}) + \boldsymbol{\nabla} T$ can be written as $\mathbf{d}(T\underline{\boldsymbol{u}})(\vec{\boldsymbol{u}}, .)$, namely, the 2-form $\mathbf{d}(T\underline{\boldsymbol{u}})$, exterior derivative of $T\underline{\boldsymbol{u}}$, having the vector $\vec{\boldsymbol{u}}$ as first argument. Indeed, the component of the latter are, from (15.62),

$$\begin{aligned}
[\mathbf{d}(T\underline{\boldsymbol{u}})(\vec{\boldsymbol{u}}, .)]_\alpha &= [\nabla_\mu(Tu_\alpha) - \nabla_\alpha(Tu_\mu)]u^\mu \\
&= u^\mu\nabla_\mu(Tu_\alpha) - \nabla_\alpha(T\underbrace{u_\mu u^\mu}_{-1}) + T\underbrace{u_\mu\nabla_\alpha u^\mu}_{0} \\
&= u^\mu\nabla_\mu(Tu_\alpha) + \nabla_\alpha T,
\end{aligned}$$

where $u_\mu\nabla_\alpha u^\mu = 0$ results from the gradient of $u_\mu u^\mu = -1$. We have thus

$$\boldsymbol{\nabla}_{\vec{u}}(T\underline{\boldsymbol{u}}) + \boldsymbol{\nabla} T = \mathbf{d}(T\underline{\boldsymbol{u}})(\vec{\boldsymbol{u}}, .),$$

as well as a similar relation for the term with μ_a:

$$\nabla_{\vec{u}}(\mu_a \underline{u}) + \nabla \mu_a = \mathbf{d}(\mu_a \underline{u})(\vec{u}, .),$$

In view of these two relations, (21.71) becomes

$$\boxed{\left[\sum_{a=1}^{N} n_a \, \mathbf{d}(\mu_a \underline{u}) + s \, \mathbf{d}(T \underline{u})\right] (\vec{u}, .) = 0}. \tag{21.72}$$

This equation is equivalent to the four-dimensional Euler equation (21.69).

21.5.2 Vorticity of a Simple Fluid

In all what follows, we shall limit ourselves to a simple fluid (cf. Sect. 21.2.5). The summation over a in (21.72) reduces then to a single term (the baryons), so that (21.72) becomes

$$[\mathbf{d}(\mu \underline{u}) + S \, \mathbf{d}(T \underline{u})](\vec{u}, .) = 0, \tag{21.73}$$

where n is the proper baryon density, μ the baryon chemical potential and $S = s/n$ the entropy per baryon [cf. Eq. (21.47)]. In view of (21.73), let us introduce the 1-form $\boldsymbol{\pi}$ and the 2-form $\boldsymbol{\Omega}$ defined by

$$\boxed{\boldsymbol{\pi} := (\mu + TS)\underline{u}} \qquad \text{and} \qquad \boxed{\boldsymbol{\Omega} := \mathbf{d}\boldsymbol{\pi}}. \tag{21.74}$$

$\boldsymbol{\pi}$ is called the ***fluid momentum 1-form*** and $\boldsymbol{\Omega}$ the ***fluid vorticity 2-form***. Thanks to the thermodynamic relation (21.24), we have

$$\boxed{\mu + TS = \frac{\varepsilon + p}{n} =: h}, \tag{21.75}$$

where h is the ***enthalpy per baryon***. We may thus write

$$\boxed{\boldsymbol{\pi} = h \, \underline{u}} \qquad \text{and} \qquad \boxed{\boldsymbol{\Omega} = \mathbf{d}(h \, \underline{u})}. \tag{21.76}$$

The components of $\boldsymbol{\pi}$ and $\boldsymbol{\Omega}$ are, respectively, $\pi_\alpha = h u_\alpha$ and [cf. Eqs. (15.62) and (15.67)]

$$\Omega_{\alpha\beta} = \nabla_\alpha (h u_\beta) - \nabla_\beta (h u_\alpha) = \frac{\partial}{\partial x^\alpha}(h u_\beta) - \frac{\partial}{\partial x^\beta}(h u_\alpha). \tag{21.77}$$

The name *vorticity* arises from the link between $\boldsymbol{\Omega}$ and the ***kinematic vorticity vector*** $\vec{\boldsymbol{\omega}}$ defined as the curl of $\vec{u}$ in the fluid's local rest space [cf. (15.70)] (see e.g. Ehlers 1961):

$$\boxed{\vec{\boldsymbol{\omega}} := \nabla\!\times_u \vec{\boldsymbol{u}}}, \qquad \underline{\boldsymbol{\omega}} := \star\mathbf{d}\underline{\boldsymbol{u}}(\vec{\boldsymbol{u}}, .). \tag{21.78}$$

The link between the vorticity 2-form $\boldsymbol{\Omega}$ and the vector $\vec{\boldsymbol{\omega}}$ relies on Hodge duality. Indeed, expanding (21.76) yields $\boldsymbol{\Omega} = h\mathbf{d}\underline{\boldsymbol{u}} + \mathbf{d}h \wedge \underline{\boldsymbol{u}}$, whose Hodge dual is [use is made of the identity (14.79)]:

$$\star\boldsymbol{\Omega} = h \star \mathbf{d}\underline{\boldsymbol{u}} + \star(\mathbf{d}h \wedge \underline{\boldsymbol{u}}) = h \star \mathbf{d}\underline{\boldsymbol{u}} + \boldsymbol{\epsilon}(\vec{\nabla}h, \vec{\boldsymbol{u}}, ., .).$$

Since $\star\mathbf{d}\underline{\boldsymbol{u}}(\vec{\boldsymbol{u}}, .) = \underline{\boldsymbol{\omega}}$ and $\boldsymbol{\epsilon}(\vec{\nabla}h, \vec{\boldsymbol{u}}, \vec{\boldsymbol{u}}, .) = 0$, we find

$$\boxed{\underline{\boldsymbol{\omega}} = \frac{1}{h} \star \boldsymbol{\Omega}(\vec{\boldsymbol{u}}, .)}. \tag{21.79}$$

Comparing this expression with (14.81), we conclude that the orthogonal decomposition of the vorticity 2-form with respect to $\vec{\boldsymbol{u}}$, as given by (14.80), is

$$\boldsymbol{\Omega} = \underline{\boldsymbol{u}} \wedge \boldsymbol{q} + h\,\boldsymbol{\epsilon}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{\omega}}, ., .), \tag{21.80}$$

where $\boldsymbol{q}$ is a 1-form obeying $\langle \boldsymbol{q}, \vec{\boldsymbol{u}} \rangle = 0$ and that will be determined below from the equation of motion.

21.5.3 Canonical Form of the Equation of Motion

Let us apply the derivation rule for an exterior product (15.76) to the product of the 0-form S by the 1-form $T\underline{\boldsymbol{u}}$; we obtain

$$\mathbf{d}(ST\underline{\boldsymbol{u}}) = \mathbf{d}(S \wedge T\underline{\boldsymbol{u}}) = \mathbf{d}S \wedge (T\underline{\boldsymbol{u}}) + S\,\mathbf{d}(T\underline{\boldsymbol{u}}),$$

hence the expression of the 2-form involved in (21.73) [cf. (21.74)]:

$$\begin{aligned} \mathbf{d}(\mu\underline{\boldsymbol{u}}) + S\,\mathbf{d}(T\underline{\boldsymbol{u}}) &= \mathbf{d}(\mu\underline{\boldsymbol{u}}) + \mathbf{d}(ST\underline{\boldsymbol{u}}) - \mathbf{d}S \wedge (T\underline{\boldsymbol{u}}) \\ &= \mathbf{d}\left[(\mu + ST)\underline{\boldsymbol{u}}\right] - T\mathbf{d}S \wedge \underline{\boldsymbol{u}} = \boldsymbol{\Omega} - T\mathbf{d}S \wedge \underline{\boldsymbol{u}}. \end{aligned}$$

The equation of motion (21.73) can thus be recast as

$$\boldsymbol{\Omega}(\vec{\boldsymbol{u}}, .) - T\mathbf{d}S \wedge \underline{\boldsymbol{u}}(\vec{\boldsymbol{u}}, .) = 0. \tag{21.81}$$

By definition of an exterior product, the term in factor of T is

$$\mathbf{d}S \wedge \underline{\boldsymbol{u}}(\vec{\boldsymbol{u}}, .) = \langle \mathbf{d}S, \vec{\boldsymbol{u}} \rangle\, \underline{\boldsymbol{u}} - \langle \underline{\boldsymbol{u}}, \vec{\boldsymbol{u}} \rangle\, \mathbf{d}S = (\boldsymbol{\nabla}_{\vec{\boldsymbol{u}}}\, S)\underline{\boldsymbol{u}} + \mathbf{d}S.$$

But we have seen that for an isolated simple fluid, $\boldsymbol{\nabla}_{\vec{\boldsymbol{u}}}\, S = 0$ [Eq. (21.48)]. We have thus $\mathbf{d}S \wedge \underline{\boldsymbol{u}}(\vec{\boldsymbol{u}}, .) = \mathbf{d}S$, so that (21.81) reduces to

$$\boxed{\boldsymbol{\Omega}(\vec{\boldsymbol{u}}, .) = T\,\mathbf{d}S}. \tag{21.82}$$

Following B. Carter (1979; 1989) (cf. the historical note below), we call this relation the ***canonical equation of relativistic fluid dynamics***. It can be expressed in a single sentence:

> The 1-form obtained by setting the first argument of the vorticity 2-form to the fluid 4-velocity is equal to the temperature times the gradient of the entropy per baryon.

In terms of the components with respect to an arbitrary coordinate system (x^α) on $\mathscr{E}$, the canonical equation is expressed as [cf. (21.77)]

$$u^\mu \left[\frac{\partial}{\partial x^\mu}(hu_\alpha) - \frac{\partial}{\partial x^\alpha}(hu_\mu) \right] = T\frac{\partial S}{\partial x^\alpha}. \tag{21.83}$$

Remark 21.11. We have derived the canonical equation (21.82) from the equation of motion (21.69), which is the part orthogonal to $\vec{\boldsymbol{u}}$ of the energy–momentum conservation equation $\boldsymbol{\nabla}\cdot \boldsymbol{T} = 0$, using the principle of baryon number conservation and the equation $\nabla_{\vec{u}} S = 0$ [Eq. (21.48)]. The latter results from the part collinear to $\vec{\boldsymbol{u}}$ of the equation $\boldsymbol{\nabla}\cdot \boldsymbol{T} = 0$ (cf. Sect. 21.4.2). Now we recover $\nabla_{\vec{u}} S = 0$ by applying (21.82) to the vector $\vec{\boldsymbol{u}}$, since $\boldsymbol{\Omega}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{u}}) = 0$ by definition of a 2-form and $T\langle \mathbf{d}S, \vec{\boldsymbol{u}}\rangle = T\nabla_{\vec{u}} S$. To summarize, we have the equivalence

$$\begin{cases} \boldsymbol{\nabla}\cdot (n\vec{\boldsymbol{u}}) = 0 \\ \boldsymbol{\nabla}\cdot \boldsymbol{T} = 0 \end{cases} \iff \begin{cases} \boldsymbol{\nabla}\cdot (n\vec{\boldsymbol{u}}) = 0 \\ \boldsymbol{\Omega}(\vec{\boldsymbol{u}}, .) = T\,\mathbf{d}S \end{cases} \tag{21.84}$$

Remark 21.12. The canonical equation (21.82) does not involve the covariant derivative $\boldsymbol{\nabla}$, but only the exterior derivative $\mathbf{d}$, via the gradient of S and the definition of the vorticity 2-form: $\boldsymbol{\Omega} = \mathbf{d}(h\underline{\boldsymbol{u}})$. On the opposite, the form (21.69) of the equation of motion relies on $\boldsymbol{\nabla}$ (via $\underline{\boldsymbol{a}} = \nabla_{\vec{u}}\, \underline{\boldsymbol{u}}$). We shall see in Sect. 21.6 that the canonical equation has the advantage over (21.69) to lead easily to standard conservation laws.

From (14.82), the 1-form $\boldsymbol{q}$ that appears in the orthogonal decomposition (21.80) is $\boldsymbol{q} = \boldsymbol{\Omega}(., \vec{\boldsymbol{u}}) = -\boldsymbol{\Omega}(\vec{\boldsymbol{u}}, .)$. The canonical equation of motion (21.82) shows then that $\boldsymbol{q} = -T\,\mathbf{d}S$, so that we may rewrite (21.80) in the final form

$$\boxed{\boldsymbol{\Omega} = T\,\mathbf{d}S \wedge \underline{\boldsymbol{u}} + h\,\boldsymbol{\epsilon}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{\omega}}, ., .)}. \tag{21.85}$$

For a barotropic fluid, $T = 0$ (cf. Sect. 21.2.5) and the canonical equation (21.82) takes a very simple form:

$$\boxed{\boldsymbol{\Omega}(\vec{\boldsymbol{u}},.) = 0.}_{\text{barotropic}} \tag{21.86}$$

We obtain also the same form for an ***isentropic*** flow, i.e. a flow for which the entropy per baryon is uniform over all the fluid, in addition to stay constant along each fluid line, since then $\mathbf{d}S = 0$.

Historical note: *The canonical equation (21.82) has been first written in 1937 by John L. Synge (cf. p. 74) (1937) in the barotropic case, i.e. under the form (21.86). It has been reexpressed by André Lichnerowicz*[5] *in the language of differential form (Cartan's exterior calculus) in 1941 (Lichnerowicz 1941), stressing the independence with respect to the covariant derivative. The vorticity 2-form* $\boldsymbol{\Omega}$ *has been put forward in Lichnerowicz's relativity treatise published in 1955 (Lichnerowicz 1955). The general form (21.82) of the canonical equation has been obtained in 1959 by Abraham H. Taub (cf. p. 683) (1959) and is much employed in the relativistic hydrodynamics book by André Lichnerowicz (1967). In 1979, Brandon Carter*[6] *has shown that (21.82) can be considered as a canonical equation of Hamilton, of the type "*$\dot{\boldsymbol{\pi}} = -\boldsymbol{\nabla} H$*" [cf. Eq. (11.91)], by taking* $H(x^\alpha, \pi_\alpha) = 1/(2T)\,(g^{\alpha\beta}\pi_\alpha\pi_\beta/h + h) - S$ *as a Hamiltonian (Carter 1979),* $\boldsymbol{\pi}$ *being the fluid momentum 1-form defined by (21.74). The numerical value of the Hamiltonian is* $H = -S$ *since* $\vec{\boldsymbol{\pi}} \cdot \vec{\boldsymbol{\pi}} = -h^2$*. In the case of multi-component fluid, the equation of motion (21.72) has been derived by Carter in 1989 (Carter 1989) and called by him* standard formulation *of the dynamics of relativistic perfect fluids.*

21.5.4 Nonrelativistic Limit: Crocco Equation

In order to take the nonrelativistic limit of the canonical equation (21.82), let us introduce the ***specific internal enthalpy***

$$H := \frac{\varepsilon_{\text{int}} + p}{m_{\text{b}} n}, \tag{21.87}$$

where m_{b} is the baryon mass: $m_{\text{b}} = 1.66\times10^{-27}$ kg. H differs from the enthalpy per baryon h for (i) it does not take into account the mass energy and (ii) this is quantity per unit mass (hence, the qualifier *specific*) and not per baryon. The relation between H and h is found by combining (21.75) with (21.16):

[5] **André Lichnerowicz** (1915–1998): French mathematician, author of important works in general relativity and in relativistic hydrodynamics and magnetohydrodynamics.

[6] **Brandon Carter**: British theoretical physicist born in 1942 and working at Paris Observatory; author of major contributions in black hole theory and cosmology, he is also known for having formulated the anthropic principle.

$$h = m_{\mathrm{b}} c^2 \left(1 + \frac{H}{c^2}\right). \tag{21.88}$$

From (21.57), the nonrelativistic limit corresponds to $H/c^2 \ll 1$. Let us consider an inertial observer $\mathscr{O}$ and denote by $(x^0) = (ct, x^1, x^2, x^3)$ the associated coordinates. The components of the fluid 4-velocity are $u^\alpha = (\Gamma, \Gamma V^i/c)$, where Γ and the V^i's are, respectively, the Lorentz factor and the components of the fluid velocity relative to $\mathscr{O}$. At the limit of low velocities,

$$u^\alpha \simeq \left(1 + \frac{V^2}{2c^2}, \frac{V^i}{c}\right) \qquad \text{and} \qquad u_\alpha \simeq \left(-1 - \frac{V^2}{2c^2}, \frac{V^i}{c}\right). \tag{21.89}$$

Let us substitute (21.88) for h and (21.89) for u^α and u_α in the canonical equation (21.83). The component $\alpha = i$ becomes, after division by m_{b},

$$\begin{aligned} & u^0 \left\{ \frac{1}{c}\frac{\partial}{\partial t}\left[(c^2 + H)\frac{V^i}{c}\right] - \frac{\partial}{\partial x^i}\left[-(c^2 + H)\left(1 + \frac{V^2}{2c^2}\right)\right]\right\} \\ & \quad + \frac{V^j}{c}\left\{\frac{\partial}{\partial x^j}\left[(c^2 + H)\frac{V^i}{c}\right] - \frac{\partial}{\partial x^i}\left[(c^2 + H)\frac{V^j}{c}\right]\right\} = \frac{T}{m_{\mathrm{b}}}\frac{\partial S}{\partial x^i}. \end{aligned}$$

At the nonrelativistic limit, we may set $u^0 \simeq 1$ and neglect the terms in H/c^2 in this equation. There comes then

$$\frac{\partial V^i}{\partial t} + \frac{\partial}{\partial x^i}\left(H + \frac{V^2}{2}\right) + V^j \left(\frac{\partial V^i}{\partial x^j} - \frac{\partial V^j}{\partial x^i}\right) = T\frac{\partial \bar{S}}{\partial x^i}, \tag{21.90}$$

where we have introduced the specific entropy $\bar{S} := S/m_{\mathrm{b}}$. We recognize in the last term of the left-hand side the components of the cross product of the curl of $\vec{V}$ by $\vec{V}$ [cf. Eq. (15.70)]

$$V^j \left(\frac{\partial V^i}{\partial x^j} - \frac{\partial V^j}{\partial x^i}\right) = \left[(\boldsymbol{\nabla}\boldsymbol{\times}_{u_0} \vec{V}) \boldsymbol{\times}_{u_0} \vec{V}\right]_i.$$

Equation (21.90) can be thus written

$$\frac{\partial \vec{V}}{\partial t} + \vec{\nabla}\left(H + \frac{V^2}{2}\right) + (\boldsymbol{\nabla}\boldsymbol{\times}_{u_0} \vec{V}) \boldsymbol{\times}_{u_0} \vec{V} = T\vec{\nabla}\bar{S}. \tag{21.91}$$

This relation, which constitutes the nonrelativistic limit of the canonical equation of relativistic fluid dynamics, is known in Newtonian hydrodynamics as the ***Crocco equation***, or ***Crocco theorem*** (Rieutord 1997), from the Italian engineer Luigi Crocco (1909–1986). The Crocco equation is of course nothing but a rewriting of the classical Euler equation [Eq. (21.59) with $F^i_{\mathrm{ext}} = 0$].

21.6 Conservation Laws

We exploit here the power of the formulation presented in Sect. 21.5 to derive conservation laws that generalize to relativity some well-known laws of classical hydrodynamics. In all this part, we shall consider only *simple* fluids, in the sense defined in Sect. 21.2.5, namely, fluids whose equation of state depends only on the proper entropy and baryon densities: $\varepsilon = \varepsilon(s, n)$. The canonical equation (21.82) then holds and, along with the conservation laws (21.34) and (21.46) for baryon number and entropy, it fully rules the fluid motion.

21.6.1 Bernoulli's Theorem

A flow is said to be ***stationary*** with respect to some inertial observer $\mathscr{O}$ iff all the fluid quantities measured by $\mathscr{O}$ are independent of $\mathscr{O}$'s proper time t.

If the second argument of the vorticity 2-form $\boldsymbol{\Omega}$ is set to $\mathscr{O}$'s 4-velocity $\vec{\boldsymbol{u}}_0$, we get a 1-form: $\boldsymbol{\Omega}(., \vec{\boldsymbol{u}}_0)$. Let us evaluate its components with respect to the inertial coordinates $(x^\alpha) = (ct, x^1, x^2, x^3)$ associated with $\mathscr{O}$. Since $\boldsymbol{\Omega}$ is the exterior derivative of the fluid momentum 1-form $\boldsymbol{\pi}$, there comes

$$[\boldsymbol{\Omega}(., \vec{\boldsymbol{u}}_0)]_\alpha = \left(\frac{\partial \pi_\mu}{\partial x^\alpha} - \frac{\partial \pi_\alpha}{\partial x^\mu} \right) u_0^\mu = \frac{\partial}{\partial x^\alpha} (\pi_\mu u_0^\mu) - \pi_\mu \frac{\partial u_0^\mu}{\partial x^\alpha} - u_0^\mu \frac{\partial \pi_\alpha}{\partial x^\mu}.$$

Now by definition of the coordinates (x^α), $u_0^\mu = (1, 0, 0, 0)$. Hence,

$$[\boldsymbol{\Omega}(., \vec{\boldsymbol{u}}_0)]_\alpha = \frac{\partial}{\partial x^\alpha} (\pi_\mu u_0^\mu) - \frac{\partial \pi_\alpha}{\partial x^0}.$$

If the flow is assumed to be stationary, $\partial \pi_\alpha / \partial x^0 = c^{-1} \partial \pi_\alpha / \partial t = 0$ and the above relation results in

$$\boldsymbol{\Omega}(., \vec{\boldsymbol{u}}_0) = \boldsymbol{\nabla} \langle \boldsymbol{\pi}, \vec{\boldsymbol{u}}_0 \rangle. \tag{21.92}$$

In particular, $\boldsymbol{\Omega}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{u}}_0) = \boldsymbol{\nabla}_{\vec{\boldsymbol{u}}} \langle \boldsymbol{\pi}, \vec{\boldsymbol{u}}_0 \rangle$. The canonical equation (21.82) gives then

$$T \langle \mathbf{d}S, \vec{\boldsymbol{u}}_0 \rangle = \boldsymbol{\nabla}_{\vec{\boldsymbol{u}}} \langle \boldsymbol{\pi}, \vec{\boldsymbol{u}}_0 \rangle.$$

But for a stationary flow,

$$\langle \mathbf{d}S, \vec{\boldsymbol{u}}_0 \rangle = \boldsymbol{\nabla}_{\vec{\boldsymbol{u}}_0} S = u_0^\mu \frac{\partial S}{\partial x^\mu} = \frac{1}{c} \frac{\partial S}{\partial t} = 0.$$

Finally we obtain

$$\boxed{\boldsymbol{\nabla}_{\vec{\boldsymbol{u}}} \langle \boldsymbol{\pi}, \vec{\boldsymbol{u}}_0 \rangle = 0}. \tag{21.93}$$

This equation constitutes the

> **Relativistic Bernoulli's theorem:** Within a flow that is stationary with respect to an inertial observer $\mathscr{O}$ of 4-velocity $\vec{\boldsymbol{u}}_0$, the scalar $\langle \boldsymbol{\pi}, \vec{\boldsymbol{u}}_0 \rangle$ is constant along each fluid line. It is expressible as
>
> $$\langle \boldsymbol{\pi}, \vec{\boldsymbol{u}}_0 \rangle = h \vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{u}}_0 = -\Gamma h, \tag{21.94}$$
>
> where h is the enthalpy per baryon defined by (21.75) and $\Gamma = -\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{u}}_0$ is the fluid Lorentz factor with respect to $\mathscr{O}$.

Let us show that this is the relativistic generalization of the classical Bernoulli's theorem. Given (21.94) and the decomposition (21.7) of $\vec{\boldsymbol{u}}$ in terms of Γ and of the fluid velocity $\vec{\boldsymbol{V}}$ relative to $\mathscr{O}$, (21.93) is recast as

$$\underbrace{\nabla_{\vec{\boldsymbol{u}}_0}(\Gamma h)}_{0} + \frac{1}{c} \nabla_{\vec{\boldsymbol{V}}}(\Gamma h) = 0,$$

the vanishing of the first term resulting from the stationarity hypothesis. Introducing the specific internal enthalpy H via (21.88) and writing $\Gamma \simeq 1 + V^2/2$, we find the nonrelativistic limit of (21.93):

$$\vec{\boldsymbol{V}} \cdot \vec{\nabla} \left(H + \frac{V^2}{2} \right) = 0. \tag{21.95}$$

We recognize the classical Bernoulli's theorem: the sum of the enthalpy and the kinetic energy, both per unit mass, is constant along the streamlines (cf., e.g. Rieutord (1997)).

Remark 21.13. The relativistic Bernoulli's theorem is actually a particular case of the following general result—any symmetry of the flow gives birth to a conserved quantity along each fluid line, namely, the scalar $\langle \boldsymbol{\pi}, \vec{\boldsymbol{G}} \rangle$:

$$\nabla_{\vec{\boldsymbol{u}}} \langle \boldsymbol{\pi}, \vec{\boldsymbol{G}} \rangle = 0, \tag{21.96}$$

the vector field $\vec{\boldsymbol{G}}$ being the symmetry generator (cf. Sect. 11.3.1). In the case of Bernoulli's theorem, $\vec{\boldsymbol{G}} = \vec{\boldsymbol{u}}_0$, the symmetry being a time translation with respect to the inertial observer $\mathscr{O}$. Another example would be $\vec{\boldsymbol{G}}$ being the generator of spatial rotations about some axis (axisymmetric flow). The property (21.96) is similar to the Noether theorem discussed in Sect. 11.3 for a relativistic particle. In particular, it is worth noticing the analogy between $\langle \boldsymbol{\pi}, \vec{\boldsymbol{G}} \rangle$ and the conserved quantity (11.49):

$\langle mc\,\underline{\boldsymbol{u}}, \vec{\boldsymbol{G}}\rangle$, given that $mc\,\underline{\boldsymbol{u}}$ is the particle's 4-momentum, whose role is played here by the fluid momentum 1-form $\boldsymbol{\pi}$. To establish (21.96), the simplest calculation involves a kind of derivative along the field lines of $\vec{\boldsymbol{G}}$ called the *Lie derivative*. We shall not do it here, referring the reader to, e.g. Gourgoulhon (2006).

Historical note: *The relativistic Bernoulli's theorem (21.93) has been derived in 1937 by John L. Synge (cf. p. 74) (1937), from the "Eulerian" equation of motion (21.69), and by André Lichnerowicz (cf. p. 692) in 1940, from the canonical equation (21.82) (Lichnerowicz 1940). These two studies were limited to the barotropic case; the extension to an arbitrary simple fluid has been performed by Abraham H. Taub (cf. p. 683) in 1959 (Taub 1959).*

21.6.2 Irrotational Flow

A perfect fluid is said to be in ***irrotational flow*** iff the vorticity 2-form is identically zero:

$$\boxed{\boldsymbol{\Omega} = 0}. \tag{21.97}$$

This implies the vanishing of the kinematic vorticity vector $\vec{\boldsymbol{\omega}}$ defined by (21.78). More precisely, from the decomposition (21.85) of $\boldsymbol{\Omega}$, we have the equivalence

$$\text{(irrotational flow)} \iff (\,\vec{\boldsymbol{\omega}} = 0 \quad \text{and} \quad T\,\mathbf{d}S = 0\,). \tag{21.98}$$

The relation $T\,\mathbf{d}S = 0$ means that $T = 0$ (barotropic fluid) or that the entropy per baryon S is constant in all the fluid (isentropic fluid).

The property $\boldsymbol{\Omega} = 0$ is equivalent to $\mathbf{d}\boldsymbol{\pi} = 0$, i.e. the 1-form $\boldsymbol{\pi}$ is closed. From Poincaré lemma (cf. Sect. 15.5.3), we deduce that there exists (at least locally) some scalar field Ψ such that $\boldsymbol{\pi} = \mathbf{d}\Psi$, i.e. such that

$$\boxed{h\,\underline{\boldsymbol{u}} = \mathbf{d}\Psi}. \tag{21.99}$$

The field Ψ is called ***potential of the flow***.

Remark 21.14. The relativistic generalization of an irrotational flow is thus not $\underline{\boldsymbol{u}} = \mathbf{d}\Psi$, as one may naively infer from the relation $\underline{V} = \mathbf{d}\Psi$, which characterizes a Newtonian irrotational flow. It is indeed $\boldsymbol{\pi} = h\,\underline{\boldsymbol{u}}$ that is a gradient and not $\underline{\boldsymbol{u}}$. Of course, at the nonrelativistic limit, $h \to m_b c^2 = \text{const.}$, and we recover the Newtonian definition of irrotationality.

With $T = 0$ or S being constant, the canonical equation of relativistic fluid dynamics (21.82) is automatically satisfied by an irrotational flow. The only non-trivial equation is then the equation of baryon number conservation (21.34): $\nabla \cdot (n\vec{\boldsymbol{u}}) = 0$. Substituting $h^{-1}\vec{\nabla}\Psi$ for $\vec{\boldsymbol{u}}$ [cf. (21.99)], we turn it into an equation for the potential Ψ:

$$\Box\Psi + \vec{\nabla}\ln\left(\frac{n}{h}\right)\cdot\vec{\nabla}\Psi = 0, \tag{21.100}$$

where $\Box$ is the d'Alembertian operator defined by (18.51). In the general case, this equation is not linear in Ψ, for h is linked to Ψ by the equation

$$h = \left(-\vec{\nabla}\Psi\cdot\vec{\nabla}\Psi\right)^{1/2}, \tag{21.101}$$

which is deduced from (21.99) and the 4-velocity normalization ($\vec{\boldsymbol{u}}\cdot\vec{\boldsymbol{u}} = -1$). Moreover, n is related to Ψ via h and the equation of state. There exists however a case in which (21.100) is a linear equation, as shown by the following example.

Example 21.3. In the particular case $h = \alpha n$ with α constant, (21.100) reduces to a wave equation for Ψ:

$$\Box\Psi = 0. \tag{21.102}$$

If the equation of state is barotropic, $\varepsilon = \varepsilon(n)$ [Eq. (21.26)], then the enthalpy per baryon is equal to the chemical potential: $h = \mu = \mathrm{d}\varepsilon/\mathrm{d}n$ [Eq. (21.75) with $T = \partial\varepsilon/\partial s = 0$] and the condition $h = \alpha n$ implies

$$\varepsilon = \frac{\alpha}{2}n^2,$$

where the integration constant has been set to 0 to ensure $\varepsilon(0) = 0$. The pressure is then deduced from (21.75): $p = nh - \varepsilon$, which results in $p = (\alpha/2)\,n^2$. We observe that

$$p = \varepsilon.$$

From (21.63), this implies $c_{\mathrm{s}} = c$, i.e. the speed of sound equals the speed of light. The equation of state is thus the "hardest" one compatible with the causality constraint.

If, in addition to be irrotational, the flow is stationary with respect to an inertial observer $\mathscr{O}$ of 4-velocity $\vec{\boldsymbol{u}}_0$, Eq. (21.92) holds. Since $\boldsymbol{\Omega} = 0$, it reduces to $\nabla\langle\boldsymbol{\pi}, \vec{\boldsymbol{u}}_0\rangle = 0$, which implies that the scalar field $\langle\boldsymbol{\pi}, \vec{\boldsymbol{u}}_0\rangle$ is constant:

$$\boxed{\langle\boldsymbol{\pi}, \vec{\boldsymbol{u}}_0\rangle = \text{const}}. \tag{21.103}$$

Hence, for an irrotational and stationary flow, the quantity $\langle\boldsymbol{\pi}, \vec{\boldsymbol{u}}_0\rangle = -\Gamma h$ [cf. Eq. (21.94)], which is constant along each fluid line by virtue of Bernoulli's theorem (21.93), is actually uniform over the fluid lines.

Historical note: *The concept of irrotational relativistic fluid has been introduced in 1937 by John L. Synge (cf. p. 74) (Synge 1937) under the kinematical form* $\vec{\boldsymbol{\omega}} = 0$. *The dynamical definition* $\boldsymbol{\Omega} = 0$ *[Eq. (21.97)] has been given by André Lichnerowicz (cf. p. 692) in 1941 (Lichnerowicz 1941). For the barotropic case considered by Synge, the kinematical definition coincides with that of Lichnerowicz, given the equivalence (21.98) with* $T = 0$. *The mathematical properties of the equa-*

tion (21.100) for the flow potential have been studied by Yvonne Choquet-Bruhat[7] *in 1958 (Fourès-Bruhat 1958) (cf. also Chap. 9 of her textbook Choquet-Bruhat (2009)). The reduction of (21.100) to a wave equation for the equation of state* $p = \varepsilon$ *(Example 21.3) has been put forward by Vincent Moncrief in 1980 (Moncrief 1980).*

21.6.3 Kelvin's Circulation Theorem

Given a closed oriented curve $\mathscr{C} \subset \mathscr{E}$, one defines the ***fluid circulation along*** $\mathscr{C}$ as the integral

$$\boxed{C(\mathscr{C}) := \oint_{\mathscr{C}} \boldsymbol{\pi}}. \tag{21.104}$$

Since $\mathscr{C}$ is one-dimensional submanifold of $\mathscr{E}$ and $\boldsymbol{\pi}$ a 1-form, the above integral is well defined (cf. Sect. 16.4.1). In particular, it can be reexpressed via (16.17a) as

$$C(\mathscr{C}) = \oint_{\mathscr{C}} h\, \vec{\boldsymbol{u}} \cdot \mathrm{d}\vec{\boldsymbol{\ell}}, \tag{21.105}$$

where $\mathrm{d}\vec{\boldsymbol{\ell}}$ is an infinitesimal vector tangent to $\mathscr{C}$ and use has been made of the identity $\langle \boldsymbol{\pi}, \mathrm{d}\vec{\boldsymbol{\ell}} \rangle = \langle h\, \underline{\boldsymbol{u}}, \mathrm{d}\vec{\boldsymbol{\ell}} \rangle = h\, \vec{\boldsymbol{u}} \cdot \mathrm{d}\vec{\boldsymbol{\ell}}$. If the curve $\mathscr{C}$ is chosen to lie in the rest space of some inertial $\mathscr{O}$, we may also express the circulation in terms of the fluid velocity $\vec{\boldsymbol{V}}$ relative to $\mathscr{O}$ and the specific internal enthalpy H by substituting (21.7) and (21.88) for, respectively, $\vec{\boldsymbol{u}}$ and h in (21.105):

$$C(\mathscr{C}) = m_{\mathrm{b}} c \oint_{\mathscr{C}} \Gamma \left(1 + \frac{H}{c^2}\right) \vec{\boldsymbol{V}} \cdot \mathrm{d}\vec{\boldsymbol{\ell}}. \tag{21.106}$$

At the nonrelativistic limit, $\Gamma \to 1$, $|H|/c^2 \ll 1$ and we recover (up to some factor $m_{\mathrm{b}} c$) the classical expression for the fluid circulation.

Thanks to Stokes' theorem (16.46) and to the definition $\boldsymbol{\Omega} = \mathbf{d}\boldsymbol{\pi}$ of the vorticity 2-form, the circulation can be rewritten as the integral of $\boldsymbol{\Omega}$ over any surface $\mathscr{S}$ that admits $\mathscr{C}$ as boundary:

$$\boxed{C(\mathscr{C}) = \int_{\mathscr{S}} \boldsymbol{\Omega}}. \tag{21.107}$$

---

[7] **Yvonne Choquet-Bruhat**: French mathematician born in 1923; many of her works are applied to relativity; she has notably proved a famous existence and uniqueness theorem for the solution of the Einstein equation, which rules relativistic gravitation [Eq. (22.32) in the next chapter].

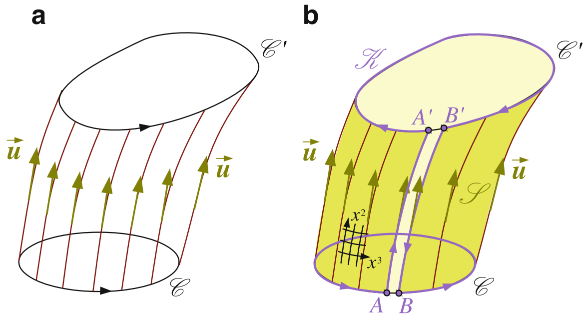

Fig. 21.3 (**a**) Transport of a curve $\mathscr{C}$ by the fluid. (**b**) Closed contour $\mathscr{K}$ defining the surface $\mathscr{S}$ onto the fluid tube

We deduced immediately that, in an irrotational flow, the fluid circulation is always zero [cf. (21.97)].

Let us assume that the curve $\mathscr{C}$ is not tangent at any point to the fluid 4-velocity $\vec{u}$. For instance, it suffices that $\mathscr{C}$ lies in the rest space of some observer, which is a spacelike domain, while $\vec{u}$ is always timelike. $\mathscr{C}$ can then be transported along the fluid lines, by an arbitrary distance along each line (but varying smoothly from one line to the next one), yielding to a new closed curve $\mathscr{C}'$ (cf. Fig. 21.3a). Such an operation is named ***transport of the curve*** $\mathscr{C}$ ***along the fluid lines***. Let us investigate the behaviour of the fluid circulation along $\mathscr{C}$ during this transport. The set of the fluid lines segments between $\mathscr{C}$ and $\mathscr{C}'$ forms a 2-surface of $\mathscr{E}$, which we shall call a ***fluid tube*** and denote by $\mathscr{T}$. Let us consider two points A and B infinitely close on $\mathscr{C}$, with the vector $\overrightarrow{AB}$ in the direction set by $\mathscr{C}$'s orientation. Let A' and B' denote their respective images on $\mathscr{C}'$ by the transport along the fluid lines (cf. Fig. 21.3b). Let us introduce the closed oriented curve

$$\mathscr{K} := \mathscr{L}_{A\to A'} \cup \mathscr{C}'_{A'\to B'} \cup \mathscr{L}_{B'\to B} \cup \mathscr{C}_{B\to A},$$

where $\mathscr{L}_{A\to A'}$ is the portion of the fluid line connecting A to A', $\mathscr{C}'_{A'\to B'}$ the portion of $\mathscr{C}'$ connecting A' to B' by covering the major part of $\mathscr{C}'$, $\mathscr{L}_{B'\to B}$ the portion of the fluid line connecting B' to B and $\mathscr{C}_{B\to A}$ the portion of $\mathscr{C}$ connecting B to A by covering the major part of $\mathscr{C}$ (cf. Fig. 21.3b). Let $\mathscr{S}$ be the part of the fluid tube $\mathscr{T}$ delimited by $\mathscr{K}$; $\mathscr{S}$ is a 2-surface, which, when A tends towards B, tends to cover the totality of $\mathscr{T}$. Since $\mathscr{K}$ is the boundary of $\mathscr{S}$, (21.107) leads to

$$\oint_{\mathscr{K}} \boldsymbol{\pi} = \int_{\mathscr{S}} \boldsymbol{\Omega} = \int_{\mathscr{S}} \boldsymbol{\Omega}(\mathrm{d}\vec{\boldsymbol{\ell}}_2, \mathrm{d}\vec{\boldsymbol{\ell}}_3),$$

the second equality being nothing but the definition (16.17b) of the integral of a 2-form over a 2-surface. Now since $\mathscr{S}$ is a part of the fluid tube and the latter is tangent at every point to the 4-velocity $\vec{u}$, the infinitesimal vector $\mathrm{d}\vec{\boldsymbol{\ell}}_2$ can always be chosen to be collinear to $\vec{u}$. We can even find a coordinate system (x^α) adapted to $\mathscr{S}$, in the sense that (x^2, x^3) span $\mathscr{S}$ (cf. Sect. 16.3.1), so that the vector $\vec{e}_2$ of the coordinate basis $(\vec{e}_\alpha)$ is exactly $\vec{u}$: the coordinate x^2 is then the proper time along the fluid lines and we have $\mathrm{d}\vec{\boldsymbol{\ell}}_2 = \mathrm{d}x^2\vec{\boldsymbol{u}}$ in addition to $\mathrm{d}\vec{\boldsymbol{\ell}}_3 = \mathrm{d}x^3\vec{\boldsymbol{e}}_3$ (cf. Fig. 21.3b). Consequently, we may invoke the canonical equation of relativistic fluid dynamics [Eq. (21.82)] to write the above integral as

$$\oint_{\mathscr{K}} \boldsymbol{\pi} = \int_{\mathscr{S}} \boldsymbol{\Omega}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{e}}_3)\,\mathrm{d}x^2\,\mathrm{d}x^3 = \int_{\mathscr{S}} T\langle \mathbf{d}S, \vec{\boldsymbol{e}}_3\rangle\,\mathrm{d}x^2\,\mathrm{d}x^3 = \int_{\mathscr{S}} T\boldsymbol{\nabla}_{\vec{\boldsymbol{e}}_3}S\,\mathrm{d}x^2\,\mathrm{d}x^3. \tag{21.108}$$

Now, by definition of $\mathscr{K}$,

$$\oint_{\mathscr{K}} \boldsymbol{\pi} = \int_{\mathscr{L}_{A\to A'}} \boldsymbol{\pi} + \int_{\mathscr{C}'_{A'\to B'}} \boldsymbol{\pi} + \int_{\mathscr{L}_{B'\to B}} \boldsymbol{\pi} + \int_{\mathscr{C}_{B\to A}} \boldsymbol{\pi}.$$

When A tends towards B, the sum of the integrals over $\mathscr{L}_{A\to A'}$ and $\mathscr{L}_{B'\to B}$ vanishes (the two integrals have the same amplitude but opposite signs), while the integral over $\mathscr{C}_{B\to A}$ tends towards the circulation $C(\mathscr{C})$ and that over $\mathscr{C}'_{A'\to B'}$ towards $-C(\mathscr{C}')$. In addition, in (21.108), the integration surface $\mathscr{S}$ tends towards the whole fluid tube $\mathscr{T}$. Therefore, the limit $A \to B$ yields

$$\boxed{C(\mathscr{C}') = C(\mathscr{C}) - \int_{\mathscr{T}} T\boldsymbol{\nabla}_{\vec{\boldsymbol{e}}_3}S\,\mathrm{d}x^2\,\mathrm{d}x^3}. \tag{21.109}$$

In view of this result, we may state the

Relativistic Kelvin's circulation theorem: The fluid circulation defined by (21.104) is conserved by transport along the fluid lines, that is to say $C(\mathscr{C}') = C(\mathscr{C})$, if $T = 0$ (barotropic fluid) or if the entropy per baryon S is constant on the initial contour $\mathscr{C}$.

The last assertion results from the conservation law (21.48) of the entropy per baryon: if $S = S_0 =$ const on $\mathscr{C}$, then $S = S_0$ on all the tube $\mathscr{T}$, since the latter is generated from $\mathscr{C}$ via the fluid lines. Then $\boldsymbol{\nabla}_{\vec{\boldsymbol{e}}_3}S = 0$ holds on $\mathscr{T}$, which yields the vanishing of the integral in the right-hand side of (21.109).

At the nonrelativistic limit, the classical Kelvin's circulation theorem (cf., e.g. Guyon et al. (2001), Rieutord (1997)) is easily recovered, since we have already noticed that (21.106) reduces to the nonrelativistic circulation in that limit. It suffices then to choose $\mathscr{C}$ in the rest space of some inertial observer at some instant t and $\mathscr{C}'$ in the rest space of the same observer at a subsequent instant t'.

Remark 21.15. The relativistic Kelvin's circulation theorem can be related to a local conservation law, that of the ***potential vorticity***, which is the scalar field defined by

$$e := \frac{h}{n}\boldsymbol{\nabla}_{\vec{\boldsymbol{\omega}}}\,S, \tag{21.110}$$

where $\vec{\omega}$ is the kinematic vorticity vector introduced in Sect. 21.5.2. From the Hodge dual of the canonical equation (21.82), it can be shown that $\nabla_{\vec{u}}\, e = 0$, i.e. that the potential vorticity is constant along the fluid lines. This result, obtained by Joseph Katz (1984), implies Kelvin's theorem, as it can be shown by taking the derivative of the circulation along the fluid line. We shall not go into details here and refer the reader to the article (Katz 1984).

Historical note: *For a barotropic fluid, Kelvin's circulation theorem has been proved in 1937 by John L. Synge (cf. p. 74) (1937). The case of an arbitrary simple fluid has been treated by Abraham H. Taub (cf. p. 683) in 1959 (Taub 1959).*

21.7 Applications

21.7.1 Astrophysics: Jets and Gamma-Ray Bursts

Astrophysics is obviously an important field of application for relativistic hydrodynamics. Numerous situations (cosmology, neutron stars, black hole environments) require hydrodynamics within *general* relativity, due to the amplitude of the gravitational field (cf. Sect. 22.4). There exist however some phenomena where the hydrodynamics within *special* relativity, as treated here, is fully sufficient. This regards the jets emitted by active galactic nuclei and micro-quasars as well as the "fireball" of gamma-ray bursts.

Relativistic astrophysical jets are made of matter ejected from the close surroundings of a black hole, which can be either a massive black hole at the centre of a galaxy, a so-called *active galactic nucleus*, or a stellar black hole, a so-called *micro-quasar* (Mirabel and Rodríguez 1999). The mechanism of formation and acceleration of the jet in the vicinity of the black hole is not completely elucidated, but almost certainly relies on the electromagnetic field and the rotation of the black hole (cf. Sauty et al. (2002) for a review). On the other hand, sufficiently far from the black hole, the jet, which is very weakly self-gravitating, is very well described by special relativistic hydrodynamics. The jet is composed of protons, electrons and positrons. The Lorentz factor Γ of the flow ranges from 3 to 10. An example of observation of such a jet is given in Fig. 21.4 (cf. also Fig. 20.4 in the preceding chapter as well as the spectrum of a jet displayed in Fig. 9.12).

Numerous numerical simulations of astrophysical jets have been performed by integrating the system of conservation laws (21.67) by means of high-resolution shock-capturing schemes (Martí and Müller 2003). An example is shown in Fig. 21.5. The result of the computed radio wave emission is to be compared with the observed data, as the image depicted in Fig. 20.4. It can also be compared with the image in Fig. 21.4, but it must be kept in mind that the latter has been obtained in the optical domain and not in the radio one.

Fig. 21.4 Relativistic jet emanating from the nucleus of the galaxy M87, with the Lorentz factor $\Gamma \sim 6$. The size of the jet visible on this picture taken by the Hubble Space Telescope is 5000 light-years. The optical emission of the jet is due to synchrotron radiation by electrons (cf. Sect. 20.4). The zone delimited by the small rectangle corresponds to Fig. 5.16. The jet emitted in the opposite direction is not seen because of the Doppler boosting effect discussed in Sect. 20.3.4 (cf. Fig. 20.4) [Source: NASA/HST]

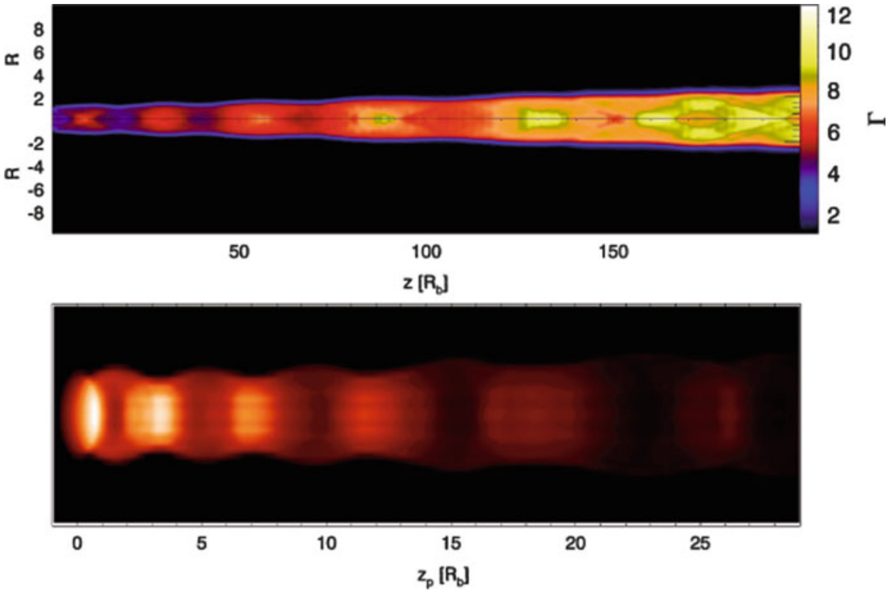

Fig. 21.5 Numerical simulation of the propagation of a relativistic jet in some external medium (atmosphere), according to the model "OP-L-AM" of Petar Mimica et al. (2009). The atmosphere (in black on the figures) is much more dense than the jet, the baryon density ratio being $n_{\rm jet}/n_{\rm atm} = 10^{-3}$ at the jet's basis. On the other hand, the pressure is larger in the jet: $p_{\rm jet}/p_{\rm atm} = 1.5$. *Upper figure:* fluid Lorentz factor Γ in the jet (it varies between 2 and 12; cf. the right scale). *Lower figure:* radio wave emission computed from a model of synchrotron emission (cf. Sect. 20.4). The horizontal axis corresponds to the distance z to the jet's basis in units of the jet's radius $R_{\rm b}$ at $z = 0$. *NB:* the scale is not the same in the two figures [Source: M.A. Aloy, J.M. Ibáñez and P. Mimica (2009)]

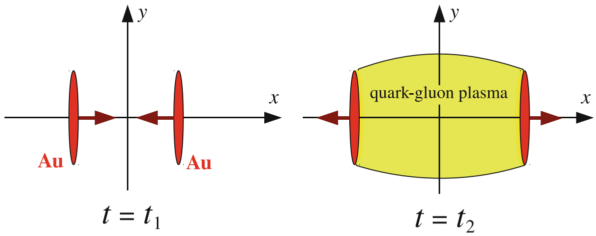

Fig. 21.6 Collision of two gold nuclei (symbol Au), in the rest space of the centre-of-mass observer $\mathscr{O}_0$, at the instant $t_1 \sim -3 \times 10^{-24}$ s (before the collision) at the instant $t_2 \sim +10^{-23}$ s (after the collision). The nuclei appear very flattened due to the length contraction in the direction of motion. Given the amplitude of their Lorentz factor ($\Gamma \sim 100$), they should even be drawn as thin vertical line segments in a more realistic picture

21.7.2 Quark-Gluon Plasma at RHIC and at LHC

In 2005, the four research teams of the Relativistic Heavy Ion Collider (RHIC) at the Brookhaven National Laboratory (cf. Table 17.1) have announced to have created a nearly "perfect liquid", i.e. a liquid of very low viscosity (Adams et al. 2005; Adcox et al. 2005; Arsene et al. 2005; Back et al. 2005). This liquid is made of deconfined[8] quarks and gluons—a so-called ***quark-gluon plasma***—and is formed during the collision of gold atomic nuclei accelerated to Lorentz factors of the order 100, which corresponds to an energy per nucleon of about 100 GeV. The collision can be described as an "interpenetration" of the nuclei since they do not lose their individuality (cf. Fig. 21.6) and generates thousands of particles. The particles interact among each other mostly via the strong interaction, and it is believed that these interactions are sufficient to reach a local thermodynamic equilibrium (cf. Sect. 21.2.4) roughly $1\ \mathrm{fm}/c = 3 \times 10^{-24}$ s after the collision. From this moment, the evolution of the system can be described by relativistic hydrodynamics as treated in this chapter. The fluid undergoes an ultrafast expansion, and after $\sim 3 \times 10^{-23}$ s, the system becomes too diluted for the hydrodynamic approximation to hold. One has then a system of free particles evolving without almost no interaction. It is these particles that are detected in the various RHIC detectors.

In the collisions performed at RHIC, the temperature at the beginning of the hydrodynamic phase is $T \sim 4 \times 10^{12}$ K ($\iff kT \sim 340$ MeV) and the energy density is $\varepsilon \sim 30\ \mathrm{GeV\,fm^{-3}}$. In these conditions, a quark-gluon plasma must be created, the deconfinement temperature predicted by quantum chromodynamics being $T \sim 2.0 - 2.2 \times 10^{12}$ K ($\iff kT \sim 170 - 190$ MeV) (Braun-Munzinger and Stachel 2007). The quark-gluon plasma obtained at RHIC is pretty well described by a perfect fluid of ultra-relativistic particles, whose equation of state is (Ollitrault 2008)

[8]Let us recall that in ordinary matter, quarks are confined three by three in protons and neutrons and that a free quark has never been observed.

$$n = 0 \tag{21.111}$$

$$\varepsilon = \varepsilon(s) = \frac{3\pi^{2/3}}{4(4g)^{1/3}} \hbar c \left(\frac{s}{k}\right)^{4/3}, \tag{21.112}$$

where $k := 1.3806505 \times 10^{-23}\,\mathrm{J\,K^{-1}}$ is the Boltzmann constant and g is the number of degrees of freedom of each particle (number of states for a fixed energy, resulting from different spins, colours or flavours): $g \simeq 40$. The property (21.111), namely, the vanishing of the baryon density, means that the thousands of particles created during the collision are equally spread between positive and negative baryon number.[9] The temperature is deduced from (21.112) via the definition (21.18): $T = \partial\varepsilon/\partial s$; there comes

$$T = \hbar c \left(\frac{\pi^2 s}{4gk^4}\right)^{1/3}. \tag{21.113}$$

The pressure is computed from (21.23), which in the present case ($n = 0$) is reduced to $p = -\varepsilon + Ts$. Thus,

$$p = \frac{\varepsilon}{3}. \tag{21.114}$$

This relation is standard for a gas of noninteracting ultra-relativistic particles.

Remark 21.16. The fact that at very high energy density, quarks have no interaction is called ***asymptotic freedom***. At lower temperature or higher energy density, it is necessary to take into account the strong interaction between the quarks. At first approximation, this can be done via a simple phenomenological approach called the ***bag model***. The equation of state is then

$$p = \frac{1}{3}(\varepsilon - 4B), \tag{21.115}$$

where B is a constant, of the order of 60 MeV fm^{-3}, named the ***bag constant***. When $\varepsilon \gg B$, we do recover (21.114).

For exactly head-on collisions, the worldlines of the centres of mass of the two nuclei are contained in a same plane $\Pi \subset \mathscr{E}$, which we shall name the ***collision plane***, and intersect each other at an event O (the “collision”; cf. Fig. 21.7). Let us call $\mathscr{O}_0$ the inertial observer linked to the centre of mass of the system (barycentric observer). If the two nuclei beams are symmetric, we may consider that $\mathscr{O}_0$ is the “lab observer”. His worldline is contained in Π and goes through O. Let $(\vec{e}_0, \vec{e}_x, \vec{e}_y, \vec{e}_z)$ denote $\mathscr{O}_0$'s frame and (ct, x, y, z) the associated coordinates, assuming that the collision takes place along the x-axis (cf. Fig. 21.6).

[9] Let us recall that the baryon number can be non-integer: it is $1/3$ for a quark and $-1/3$ for an antiquark.

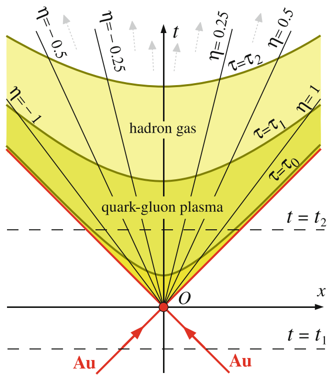

Fig. 21.7 Spacetime diagram (in the collision plane Π) of the head-on collision of two gold nuclei. The two nuclei having ultra-relativistic velocities ($\Gamma \sim 100$), their worldlines are very close to the light cone of O in this diagram. The two instants t_1 and t_2 are those considered in Fig. 21.6

As perceived by $\mathscr{O}_0$, the collision is symmetric. A simplifying hypothesis has been introduced by James D. Bjorken[10] in 1983 (Bjorken J.D. 1983). It consists in stating that the collision also appears symmetric and leads to the same relative evolution to all inertial observers that are "close" to $\mathscr{O}_0$, in the sense that:

(i) Their worldlines go through O and are contained in the collision plane Π.
(ii) For these observers, the two emerging nuclei recede from the collision point at velocities close to c.

Condition (ii) is obviously not fulfilled by an observer comoving with one of the nuclei, but it is for observers whose velocity $|U|$ relative to $\mathscr{O}_0$ is not too large. Bjorken's hypothesis is supported by experimental data. The worldlines of observers satisfying (i) are depicted in Fig. 21.7. They are labelled by their rapidity η with respect to $\mathscr{O}_0$ rather than by U. Let us recall that η is related to U by formula (6.74):

$$\eta = \operatorname{artanh} \frac{U}{c} = \frac{1}{2} \ln \left(\frac{1 + U/c}{1 - U/c} \right). \tag{21.116}$$

The domain of Π covered by the above observers whose rapidity is not too large, let us say $-2 \leq \eta \leq 2$, is called ***central rapidity region***. Accordingly the observers satisfying hypothesis (i) and (ii) are called ***central rapidity observers***. Let $\mathscr{O}$ be such an observer; his worldline goes through O, which implies that at each event on the worldline,

$$\frac{u^x_{\mathscr{O}}}{u^0_{\mathscr{O}}} = \frac{x}{ct},$$

where $u^0_{\mathscr{O}}$ and $u^x_{\mathscr{O}}$ are the components of $\mathscr{O}$'s 4-velocity in $\mathscr{O}_0$'s frame, $(\vec{e}_\alpha)$. Since $u^0_{\mathscr{O}} = \Gamma$ and $u^x_{\mathscr{O}} = \Gamma U/c$, we deduce

[10]**James D. Bjorken**: American theoretical physicist born in 1934; specialist of particle physics at Stanford University.

$$U = \frac{x}{t}. \tag{21.117}$$

The proper time spent since O for observer $\mathscr{O}$ is related to the coordinates (ct, x) by $\tau = t/\Gamma = t\sqrt{1 - U^2/c^2}$. Given the above expression for U, we get, for $t \geq 0$,

$$\tau = \sqrt{t^2 - \frac{x^2}{c^2}}. \tag{21.118}$$

Remark 21.17. The pair (τ, η) forms a coordinate system of the central rapidity region, related to the inertial coordinates (t, x) by (21.116)–(21.118). The coordinate η is named *spacetime rapidity* by particle physicists. The "grid" defined by the system (τ, η) is depicted in Fig. 21.7: the lines $\tau =$ const are hyperbola branches of vertical axis and the lines $\eta =$ const are straight lines through O. These coordinates are analogous to the Rindler coordinates introduced in Sect. 12.2.7; each coordinate system can be deduced from the other by symmetry with respect to the first diagonal $(x = ct)$, as it can be seen by comparing Figs. 21.7 and 12.8.

According to Bjorken's hypothesis, particles produced by the collision reach the local thermodynamic equilibrium after the same elapsed time τ_0 for all the central rapidity observers. The start of the hydrodynamical phase is therefore marked by the hypersurface parallel to $\vec{e}_y$ and $\vec{e}_z$ and whose intersection with the plane Π is the curve of all events that are distant from O by the same proper time τ_0 along the worldlines of the central rapidity observers. From (21.118), this is the hyperbola of equation $c^2t^2 - x^2 = c^2\tau_0^2$ (cf. Fig. 21.7).

As a first approximation, we may consider that the fluid motion takes place only in the collision plane Π and that it depends weakly on the transverse coordinates (y, z) (within the transverse extension of the nuclei of course). Such a motion is called ***longitudinal***. The fluid 4-velocity $\vec{u}$ is then parallel to the plane Π. At each event on $\mathscr{O}_0$'s worldline, $\vec{u}$ is equal to $\mathscr{O}_0$'s 4-velocity $\vec{e}_0$. In other words, the fluid velocity relative to $\mathscr{O}_0$, $V(t, x)$, vanishes for $x = 0$. This was expected from the symmetry of the problem (cf. Fig. 21.6). According to Bjorken's hypothesis, the same property must hold for any central rapidity observer $\mathscr{O}$—at each point of $\mathscr{O}$'s worldline $\mathscr{L}_{\mathscr{O}}$, the fluid 4-velocity must be equal to that of $\mathscr{O}$:

$$\forall M \in \mathscr{L}_{\mathscr{O}}, \quad \vec{u}(M) = \vec{u}_{\mathscr{O}}. \tag{21.119}$$

We deduce immediately that the fluid lines are the central rapidity observers' wordlines, namely, segments of straight lines through O and contained in Π (the lines labelled by η in Fig. 21.7). It follows also that the coordinate τ coincides with the fluid proper time and that the fluid velocity V relative to $\mathscr{O}_0$ is equal to $\mathscr{O}$'s velocity U relative to $\mathscr{O}_0$. We have thus, from (21.117),

$$V(t, x) = \frac{x}{t}. \tag{21.120}$$

The components of the fluid velocity field relative to $\mathscr{O}_0$ are then

$$u^\alpha = \left(\Gamma, \Gamma \frac{x}{ct}, 0, 0\right) = \frac{1}{\tau}\left(t, \frac{x}{c}, 0, 0\right), \tag{21.121}$$

where the second equality results from (21.118). Still according to Bjorken's hypothesis, at each point, the proper energy density ε, the proper entropy density s, the pressure p and the temperature T depend only on τ, for a dependency in η would imply a dependency with respect to the central rapidity observer. We have then

$$\varepsilon = \varepsilon(\tau), \quad s = s(\tau), \quad p = p(\tau), \quad T = T(\tau). \tag{21.122}$$

Let us now combine Bjorken's hypothesis with the equations of relativistic hydrodynamics. In the present case, the most convenient form is the entropy conservation law (21.46) and the four-dimensional Euler equation (21.49) [with $\mathscr{F}_{\rm ext} = 0$, i.e. actually Eq. (21.69)]. Let us rewrite (21.46) as

$$\boldsymbol{\nabla}_{\vec{u}}\, s + s\, \boldsymbol{\nabla}\cdot \vec{u} = 0. \tag{21.123}$$

Now, since τ coincide with the fluid proper time and that $s = s(\tau)$, we have

$$\boldsymbol{\nabla}_{\vec{u}}\, s = \frac{1}{c}\frac{\mathrm{d}s}{\mathrm{d}\tau}. \tag{21.124}$$

Besides, from (21.121) and the inertial character of coordinates $(x^\alpha) = (ct, x, y, z)$,

$$\boldsymbol{\nabla}\cdot \vec{u} = \frac{\partial u^\alpha}{\partial x^\alpha} = \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{t}{\tau}\right) + \frac{\partial}{\partial x}\left(\frac{x}{c\tau}\right).$$

Now, given expression (21.118) for τ as a function of (t, x),

$$\frac{\partial \tau}{\partial x^\alpha} = \left(\frac{1}{c}\frac{\partial \tau}{\partial t}, \frac{\partial \tau}{\partial x}, 0, 0\right) = \left(\frac{t}{c\tau}, -\frac{x}{c^2\tau}, 0, 0\right), \tag{21.125}$$

so that there comes

$$\boldsymbol{\nabla}\cdot \vec{u} = \frac{1}{c\tau}. \tag{21.126}$$

Thanks to (21.124) and (21.126), (21.123) becomes

$$\frac{\mathrm{d}s}{\mathrm{d}\tau} + \frac{s}{\tau} = 0. \tag{21.127}$$

This differential equation is easily integrated into

$$\boxed{s = s_0 \frac{\tau_0}{\tau}}, \tag{21.128}$$

where s_0 is the proper entropy density at the beginning of the hydrodynamical phase ($\tau = \tau_0$). The proper energy density ε is deduced from the equation of state (21.112):

$$\boxed{\varepsilon = \varepsilon_0 \left(\frac{\tau_0}{\tau}\right)^{4/3}}, \tag{21.129}$$

where $\varepsilon \sim 30\ \mathrm{GeV\,fm}^{-3}$ is the proper energy density at $\tau = \tau_0$. Similarly, the time evolution of pressure and temperature are found from (21.114) and (21.113):

$$\boxed{p = \frac{\varepsilon_0}{3}\left(\frac{\tau_0}{\tau}\right)^{4/3}} \quad \text{and} \quad \boxed{T = T_0 \left(\frac{\tau_0}{\tau}\right)^{1/3}}. \tag{21.130}$$

Let us now consider the four-dimensional Euler equation (21.49). In the present case, $\underline{\boldsymbol{a}} = 0$ for the fluid lines are straight line segments. The equation reduces then to

$$\boldsymbol{\nabla} p + (\nabla_{\vec{u}}\, p)\underline{\boldsymbol{u}} = 0.$$

Since p depends solely of τ, we have $\boldsymbol{\nabla} p = (\mathrm{d}p/\mathrm{d}\tau)\boldsymbol{\nabla}\tau$ and $\nabla_{\vec{u}}\, p = c^{-1}\,\mathrm{d}p/\mathrm{d}\tau$, so that the above equation becomes

$$\boldsymbol{\nabla}\tau + \frac{1}{c}\underline{\boldsymbol{u}} = 0. \tag{21.131}$$

The components $\nabla_\alpha \tau$ of $\boldsymbol{\nabla}\tau$ are given by (21.125) and those of $\underline{\boldsymbol{u}}$ are deduced from (21.121) via Minkowski matrix: $u_\alpha = \eta_{\alpha\mu}\, u^\mu = (-t/\tau, x/(c\tau), 0, 0)$. We note then that (21.131) is fulfilled. The four-dimensional Euler equation (21.49) is thus satisfied.

Finally, formulas (21.121), (21.128), (21.129) and (21.130) define a solution of the equation $\vec{\nabla}\cdot \boldsymbol{T} = 0$ ruling relativistic hydrodynamics. This solution corresponds to a quark-gluon plasma formed during the collision of two heavy ions, provided that the component of the motion transverse to the collision axis is neglected and that the invariance with respect to the central rapidity observers is satisfied (Bjorken's hypothesis). More complex solutions, relaxing the above hypotheses (notably describing non-head-on collisions), are presented in the review articles (Gelis 2008; Hirano et al. 2010; Huovinen and Ruuskanen 2006) and (Ollitrault 2008).

The study of the quark-gluon plasma is just beginning. The 2005 RHIC results have been confirmed in 2010, and in 2012 CERN scientists from the ALICE experiment have announced the creation of a quark-gluon plasma by collisions of lead nuclei in the LHC (CERN 2012) (cf. Table 17.1 and Fig. 17.14). Let us stress that the understanding of the quark-gluon plasma is of great importance for cosmology: this must have been the state of matter in the first microseconds after the Big Bang. It is only when the universe temperature decreased below $kT \sim 170$ MeV that the nucleons (protons and neutrons) could form.

Historical note: *It is Lev D. Landau (cf. p. 445) who, in 1953, had the idea to apply relativistic hydrodynamics to the study of particle collisions (Landau 1953). In Landau's model, the incident nuclei are stopped at the collision point with respect to the centre-of-mass observer $\mathscr{O}_0$. The initial conditions at $\tau = \tau_0$ (start of the hydrodynamical phase) are then not invariant with respect to the central rapidity observers, contrary to those of Bjorken's model described above: the initial velocity vanishes only for observer $\mathscr{O}_0$ (cf. Bialas et al. (2007) for a comparison of Landau and Bjorken models). After having been forgotten for some time, Landau's model was renewed in the beginning of the seventies to explain the proton-proton collisions observed at CERN, notably under the impulse of the American theoretical physicist Peter A. Carruthers (1935–1997). On the experimental side, clues regarding the creation of a quark-gluon plasma have been announced by CERN in 2000 (Heinz and Jacob 2000; Braun-Munzinger and Stachel 2007), from the analysis of lead nuclei collisions in the SPS synchrotron (cf. Fig. 17.14), at energies 10 times lower than at RHIC. The same year, RHIC started to operate, leading to the results described above.*

21.8 To Go Further…

In this chapter, we have not treated shock waves in relativistic fluids. Such a topic is discussed in Chap. 15 of Landau and Lifshitz textbook (Landau and Lifshitz 1987), and the exact solution to the relativistic Riemann shock tube problem is presented in the review article by J. M. Martí and E. Müller (2003). Besides, we have limited ourselves to perfect fluids. The relativistic treatment of dissipative (i.e. viscous or thermally conducting) fluids is delicate. The "naive" generalization of the Navier–Stokes equations, as presented, for instance, by Landau and Lifshitz (1987), leads to elliptic or parabolic equations, i.e. to an infinite velocity of information, and violates thereby relativistic causality. A discussion of causal theories for dissipative fluid is given in the review article by N. Andersson and G. L. Comer (2007). More generally, all the above topics are covered in the recent treatise about relativistic hydrodynamics written by L. Rezzolla and O. Zanotti (2013).

Chapter 22
What About Relativistic Gravitation?

22.1 Introduction

Among the four known fundamental interactions (electromagnetism, weak interaction, strong interaction and gravitation), only electromagnetism and gravitation are long-range ones and eligible for a non-quantum description. Electromagnetism fitting nicely into special relativity (Chaps. 17–20), it is natural to wonder about gravitation. In this chapter, we present some attempts to include it in Minkowski spacetime (Sect. 22.2), discussing successively theories where gravitation is described by a scalar, a vector or a valence-2 tensor field. We shall see in Sect. 22.3 that such an incorporation of gravitation in Minkowski spacetime is hardly bearable, because of the fundamental property that singularizes gravitation among all interactions, namely, the equality between inertial mass and gravitational mass. The (very!) good relativistic theory of gravitation turns out to be general relativity, which lies outside of the scope of this book, but about which we shall say a few words in Sect. 22.4.

22.2 Gravitation in Minkowski Spacetime

The Newtonian theory of gravitation is based on Poisson equation:

$$\Delta\Phi = 4\pi G\,\rho, \tag{22.1}$$

where Φ is the gravitational potential, ρ the mass density, $G = 6.6743 \times 10^{-11}$ m^3 kg^{-1} s^{-2} the gravitational constant and Δ the Laplace operator: $\Delta := \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ in Cartesian coordinates (x, y, z). The motion $\vec{r} = \vec{r}(t)$ of a massive particle in the gravitational field is ruled by Newton's second law:

$$\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} = -\vec{\nabla}\Phi. \tag{22.2}$$

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Any relativistic theory of gravitation must provide some generalization of (22.1)–(22.2). In what follows, we discuss various (unfruitful!) attempts to construct such generalizations in the framework of Minkowski spacetime $(\mathscr{E}, \boldsymbol{g})$, investigating successively scalar, vector and valence-2 tensor theories.

22.2.1 Nordström's Scalar Theory

It is clear that in Newtonian theory, gravitation is fully described by a scalar field: the potential Φ. As a first attempt, it is then natural to look for a scalar field Φ on Minkowski spacetime as a relativistic generalization. One has to find an equation for Φ, the nonrelativistic limit of which must be (22.1). The Laplace operator in the left-hand side of (22.1) is not an intrinsic differential operator on Minkowski spacetime $(\mathscr{E}, \boldsymbol{g})$: it depends on the inertial observer whose spatial coordinates $(x^i) = (x, y, z)$ are used to define the second-order partial derivatives. On the other hand, an operator intrinsic to $(\mathscr{E}, \boldsymbol{g})$ is the d'Alembertian defined by (18.51):

$$\square = \nabla_\mu \nabla^\mu = \eta^{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu} = -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}, \tag{22.3}$$

where the expression in terms of the inertial coordinates (ct, x, y, z) is independent of the choice of these coordinates. Moreover, $\square\Phi$ reduces to $\Delta\Phi$ when the speed of variation of Φ is small, in the sense that

$$\left| \frac{\partial^2 \Phi}{\partial t^2} \right| \ll c^2 \left| \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} \right|. \tag{22.4}$$

It is thus natural to propose as a relativistic extension of Poisson equation (22.1) the d'Alembert equation

$$\square\Phi = 4\pi G\, \mathscr{S}, \tag{22.5}$$

where $\mathscr{S}$ is a source that "generalizes" the mass density ρ and that remains to determine. Invoking the mass-energy equivalence, a first idea would be $\mathscr{S} = \varepsilon/c^2$ where ε is the total (i.e. including the mass) energy density of matter. However ε is not an invariant quantity: it depends on the observer measuring it, in two ways: as an energy and as a quantity per unit volume.

To determine a satisfactory $\mathscr{S}$, let us appeal to a principle of least action.[1] In general, using such a principle is often the way to get a well-posed physical

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[1] In this way, we do not follow Nordström's original approach (cf. historical note p. 718) but rather Einstein's reformulation of it (Einstein 1913b); see also D. Giulini (2008) or N. Deruelle (2011) for modern versions.

theory, though it does not guarantee it. In particular, we have seen in Sect. 18.7 that the equations for the electromagnetic field, namely, Maxwell equations, are deductible from a principle of least action. In the present case, the advantage of the principle of least action is not only to determine $\mathscr{S}$ but also to lead to the relativistic generalization of (22.2) for the motion of a particle in the gravitational field. To be specific, let us consider a system of N simple[2] particles, $(\mathscr{P}_a)_{1\le a\le N}$. Let m_a denote the mass of particle $\mathscr{P}_a$ and $\vec{\boldsymbol{u}}_a$ its 4-velocity. Let us assume that the gravitational field is entirely generated by these N particles and that the motion of each particle is solely ruled by the gravitational field. The considered problem is then the relativistic generalization of the *N-body problem* of Newtonian gravitation. Under these conditions, the total action of the system gravitational field + particles can be written

$$\boxed{S = S_{\text{field}} + S_{\text{inter}} + S_{\text{free part.}}}, \tag{22.6}$$

where S_{field} is the action of the free gravitational field, i.e. the action that would rule the dynamics of the gravitational field if there was no particle, and $S_{\text{free part.}}$ is the action for the free particles, i.e. the action governing the particles in the absence of gravitational field. S_{inter} is then the part of the action that describes the interaction between the particles and the gravitational field. Let us discuss these three terms separately:

- For S_{field}, we shall chose the simplest action for a scalar field, namely, the Klein–Gordon action introduced in Sect. 18.7.1, with a zero mass. Indeed, if we set to zero the parameter ℓ^{-2} in the Klein–Gordon equation (18.136), the latter reduces to $\Box\Phi = 0$, which is exactly the equation that we are looking for in the absence of particle. We shall thus postulate

$$S_{\text{field}} = -\frac{1}{8\pi Gc}\int_{\mathscr{U}} \langle \nabla\Phi, \vec{\nabla}\Phi\rangle\, \boldsymbol{\epsilon} = -\frac{1}{8\pi Gc}\int_{\mathscr{U}} g^{\mu\nu}\frac{\partial\Phi}{\partial x^{\mu}}\frac{\partial\Phi}{\partial x^{\nu}}\,\mathrm{d}U, \tag{22.7}$$

  where $\mathscr{U}$ is a four-dimensional domain of $\mathscr{E}$ and (x^{α}) a coordinate system on $\mathscr{U}$. The difference with the action based on the Klein–Gordon Lagrangian (18.131) is, besides $\ell^{-2} = 0$, the constant factor $4\pi Gc$. The latter plays the role of a ***coupling constant*** between the gravitational field and the particles.
- Regarding $S_{\text{free part.}}$, it is naturally the sum of the actions of free particles, as given by (11.6) and (11.21) [cf. also (11.99) with $K = 0$]:

$$S_{\text{free part.}} = -\sum_{a=1}^{N} m_a c\int_{\lambda_1}^{\lambda_2}\sqrt{-g_{\alpha\beta}\,\dot{x}_a^{\alpha}\,\dot{x}_a^{\beta}}\,\mathrm{d}\lambda, \tag{22.8}$$

[2] The concept of *simple* particle has been defined in Sect. 9.2.1.

where λ is some parameter along the worldline $\mathscr{L}_a$ of particle $\mathscr{P}_a$ and the functions $x_a^\alpha(\lambda)$ define the parametric equation of $\mathscr{L}_a$ in the coordinate system (x^α).

- Finally, for S_{inter}, we shall choose the interaction with a scalar field already encountered in Sect. 11.2.7, namely, that given by the Lagrangian (11.37). For each particle, we set the scalar charge to be equal to the mass: $q_a = m_a$. This implements the equality between the *gravitational mass* (q_a) and the *inertial mass* (m_a). We shall discuss this point with more details in Sect. 22.3. We have thus

$$S_{\text{inter}} = -\sum_{a=1}^{N} \frac{m_a}{c} \int_{\lambda_1}^{\lambda_2} \Phi(x_a^\alpha(\lambda)) \sqrt{-g_{\alpha\beta}\, \dot{x}_a^\alpha\, \dot{x}_a^\beta}\, \mathrm{d}\lambda . \tag{22.9}$$

In view of (22.6)–(22.9), the total action S is a functional of Φ and of the $x_a^\alpha(\lambda)$'s. Let us apply the principle of least action to the variations with respect to (x_a^α) for a given value of a. This implies only the part $S_{\text{free part.}} + S_{\text{inter}}$ of S and more particularly the term no. a in this sum. This is actually the action corresponding to the Lagrangian (11.36) presented in Example 11.2 p. 358. The equations of motion are thus given by (11.38) with $q = m$:

$$\boxed{\left(c^2 + \Phi\right) \underset{\rightarrow}{\boldsymbol{a}}_a = -\boldsymbol{\nabla}\Phi \circ \perp_{\boldsymbol{u}_a}}, \tag{22.10}$$

where $\vec{\boldsymbol{a}}_a$ stands for the 4-acceleration of particle $\mathscr{P}_a$ and $\perp_{\boldsymbol{u}_a}$ for the orthogonal projector onto its local rest space. In terms of components with respect to the inertial coordinates (x^α), this equation becomes

$$\boxed{\left(1 + \frac{\Phi}{c^2}\right) \frac{\mathrm{d}^2 x_a^\alpha}{\mathrm{d}\tau_a^2} = -\left(g^{\alpha\beta} + \frac{1}{c^2} \frac{\mathrm{d}x_a^\alpha}{\mathrm{d}\tau_a} \frac{\mathrm{d}x_a^\beta}{\mathrm{d}\tau_a}\right) \frac{\partial \Phi}{\partial x^\beta}}, \tag{22.11}$$

where τ_a is the proper time of particle $\mathscr{P}_a$. For the field Φ, the nonrelativistic limit is defined by

$$\boxed{\frac{|\Phi|}{c^2} \ll 1}_{\text{nonrelativistic}} . \tag{22.12}$$

Moreover, at the nonrelativistic limit, $\tau_a \to t$ and $|\mathrm{d}x_a^i/\mathrm{d}\tau_a| \ll c$. The nonrelativistic limit for the components $\alpha = i \in \{1, 2, 3\}$ of (22.11) is thus the Newtonian equation of motion (22.2), as desired.

Remark 22.1. For the gravitational field at the Earth surface, $\Phi \simeq -GM_\oplus/R_\oplus$ and $|\Phi|/c^2 \sim 10^{-10}$, so that the condition (22.12) is well satisfied. The same property holds for the gravitational field in the Solar System, where $|\Phi|/c^2$ reaches its maximum, of the order of 10^{-6}, in the vicinity of the Sun. On the other hand, for a neutron star, $|\Phi|/c^2 \sim 0.2$, which shows that such a star is a relativistic object.

Let us move now to the minimization of the action S with respect to variations of the field Φ. This regards only the term $S_{\text{field}} + S_{\text{inter}}$ for $S_{\text{free part.}}$ does not involve Φ. To apply the field equations obtained in Sect. 18.7.1, the action S_{inter} has first to be expressed as the integral of a Lagrangian density $\mathscr{L}_{\text{inter}}$ over a four-dimensional domain of $\mathscr{E}$, whereas (22.9) provides it as a sum of one-dimensional integrals. Let us start by rewriting (22.9) by selecting the worldline parameter to be the proper time τ of the considered particle: $\lambda = c\tau$; we get

$$S_{\text{inter}} = -\sum_{a=1}^{N} m_a \int_{\tau_1}^{\tau_2} \Phi(A_a(\tau))\, \mathrm{d}\tau, \tag{22.13}$$

where $A_a(\tau)$ is the generic point of worldline $\mathscr{L}_a$, i.e. the point of coordinates $(x_a^\alpha(\tau))$. It is then easy to let appear a four-dimensional integral, thanks to the Dirac measure introduced in Sect. 18.2.1 [Eq. (18.3)]:

$$S_{\text{inter}} = \int_{\mathscr{U}} \mathscr{L}_{\text{inter}}\, \mathrm{d}U, \tag{22.14}$$

with

$$\mathscr{L}_{\text{inter}}(M) = \Phi(M) \left[-\sum_{a=1}^{N} m_a \int_{\tau_1}^{\tau_2} \delta_{A_a(\tau)}(M)\, \mathrm{d}\tau \right]. \tag{22.15}$$

We recognize in the term in square brackets the trace of the energy–momentum tensor $\boldsymbol{T}$ of the particle system, up to some factor c^3. Indeed, $\boldsymbol{T}$ is given by (19.3). Using the simple particle relation $\boldsymbol{p}_a = m_a c\, \underline{\boldsymbol{u}}_a$ [Eq. (9.3)] and taking the trace with respect to the metric tensor $\boldsymbol{g}$ [i.e. performing the C_1^1 contraction of the tensor $\vec{\boldsymbol{T}}$, cf. (14.33)], there comes, since $u_a^\mu (u_a)_\mu = -1$,

$$T := T^\mu_{\ \mu} = g^{\mu\nu} T_{\mu\nu} = -\sum_{a=1}^{N} m_a c^3 \int_{-\infty}^{+\infty} \delta_{A_a(\tau)}(M)\, \mathrm{d}\tau. \tag{22.16}$$

The difference with (22.15) regards the boundaries on the integral over τ, but this does not matter for what follows, because the limits $\tau_1 \to -\infty$ and $\tau_2 \to +\infty$ can be taken in (22.15) without changing the content of the principle of least action. We have thus

$$\boxed{\mathscr{L}_{\text{inter}} = \frac{1}{c^3} \Phi T}. \tag{22.17}$$

The total Lagrangian density for the gravitational field is thus, in view of (22.7) and (22.17),

$$\mathscr{L}_{\rm field} + \mathscr{L}_{\rm inter} = -\frac{1}{8\pi Gc} g^{\mu\nu} \frac{\partial\Phi}{\partial x^\mu}\frac{\partial\Phi}{\partial x^\nu} + \frac{1}{c^3}\Phi T. \tag{22.18}$$

It is then easy to write the field equation (18.135) expressing the minimization of S with respect to the variations of Φ. We have seen in Sect. 18.7.1 that the term $\mathscr{L}_{\rm field}$ yields the contribution $(4\pi Gc)^{-1}\Box\Phi$ [cf. Eq. (18.136) with $\ell^{-2} = 0$]. As for the term $\mathscr{L}_{\rm inter}$, it yields the contribution $\partial/\partial\Phi(\Phi T/c^3) = T/c^3$. The field equation (18.135) is thus

$$\boxed{\Box\Phi = -\frac{4\pi G}{c^2} T}. \tag{22.19}$$

We do obtain an equation of the type (22.5) with $\mathscr{S} := -T/c^2$. Contrary to the energy density, the quantity T is independent of any observer. Moreover, at the nonrelativistic limit, (22.19) gives the Poisson equation (22.1). Indeed, at this limit, $\tau \to t$ and (22.16) yields [cf. (18.4)]

$$\begin{aligned} T(t,x,y,z) &= -\sum_{a=1}^{N} m_a c^2 \int_{-\infty}^{+\infty} \delta(ct-ct')\,\delta(x-x_a(t'))\,\delta(y-y_a(t')) \\ &\qquad\qquad \times \delta(z-z_a(t'))\; c\,\mathrm{d}t' \\ &= -c^2 \sum_{a=1}^{N} m_a \delta(x-x_a(t))\,\delta(y-y_a(t))\,\delta(z-z_a(t)). \end{aligned}$$

We recognize in the last term the Newtonian expression of the mass density ρ of the particle system, so that $T/c^2 = -\rho$. The nonrelativistic limit implies also a slowly varying field [Eq. (22.4)], so that $\Box\Phi \to \Delta\Phi$. It is then clear that (22.19) reduces to (22.1).

We have derived Eq. (22.19) for the scalar gravitational field in the specific case of a system of N simple particles, by making simple choices for the various parts of the action [Eqs. (22.7)-(22.9)]. Noticing that the final equation does not depend explicitly on this matter model, we may extend the theory to any type of matter by postulating the

Principle of universal coupling to gravitation: any kind of matter (for instance, a fluid) or any kind of non-gravitational field (for instance, the electromagnetic field) generates a gravitational field Φ according to Equation (22.19), where T is the trace of the energy–momentum tensor of the considered matter or field.

This principle is intimately related to the mass-energy equivalence and to the equality between gravitational and inertial masses. Thanks to it, we obtain a full scalar theory of gravitation in Minkowski spacetime.

Example 22.1. If the source of the gravitational field is a perfect fluid, the form (21.1) of the energy–momentum tensor leads to

$$T = T^{\mu}_{\ \mu} = (\varepsilon + p) \underbrace{u^{\mu} u_{\mu}}_{-1} + p \underbrace{g^{\mu\nu} g_{\mu\nu}}_{4} = 3p - \varepsilon,$$

so that the decomposition (21.17) of ε into a mass-energy density ρc^2 and an internal energy density $\varepsilon_{\rm int}$ leads to

$$-\frac{T}{c^2} = \rho + \frac{\varepsilon_{\rm int} - 3p}{c^2}.$$

The nonrelativistic limit (21.57) gives then $-T/c^2 \simeq \rho$, and the field equation (22.19) reduces to Poisson equation (22.1), as it should.

Remark 22.2. It can be shown (cf., e.g. Bergmann (1956); Giulini (2008)) that in the theory presented above, the conserved energy–momentum tensor, i.e. the tensor obeying $\vec{\nabla} \cdot \boldsymbol{T}_{\rm tot} = 0$ [Eq. (19.27)], is

$$\boldsymbol{T}_{\rm tot} = \boldsymbol{T}_{\rm mat} + \boldsymbol{T}_{\rm grav}, \tag{22.20}$$

with

$$\boldsymbol{T}_{\rm mat} := \left(1 + \frac{\Phi}{c^2}\right) \boldsymbol{T}, \tag{22.21}$$

$\boldsymbol{T}$ being the "ordinary" energy–momentum tensor of matter, i.e. the tensor given by (19.3) for a particle system or by (21.1) for a perfect fluid, and

$$\boldsymbol{T}_{\rm grav} := \frac{1}{4\pi G} \left[\nabla\Phi \otimes \nabla\Phi - \frac{1}{2} \langle \nabla\Phi, \vec{\nabla}\Phi \rangle \, \boldsymbol{g} \right]. \tag{22.22}$$

This last tensor is interpreted as the energy–momentum tensor of the gravitational field. We deduce from (22.21) the following relation between the traces of the tensors $\boldsymbol{T}$ and $\boldsymbol{T}_{\rm mat}$: $T = (1 + \Phi/c^2)^{-1} T_{\rm mat}$, so that we can rewrite the field equation (22.19) as

$$\left(1 + \frac{\Phi}{c^2}\right) \Box\Phi = -\frac{4\pi G}{c^2} T_{\rm mat}. \tag{22.23}$$

Written is this way, the gravitational field equation appears non-linear. This is actually the equation obtained in 1913 by G. Nordström (1913) (cf. the historical note below).

Remark 22.3. The gravitation theory described above is not the only possible scalar theory. For instance, in Exercise 7.1 of their textbook (Misner et al. 1973) (cf. Shapiro and Teukolsky (1993) for the solution), C.W. Misner, K.S. Thorne and J.A. Wheeler (cf. p. 79) propose a theory derived from a principle of least action as well but using

$$S_{\text{free part.}} + S_{\text{inter}} = -\sum_{a=1}^{N} m_a c \int_{\lambda_1}^{\lambda_2} \mathrm{e}^{\Phi(x_a^\alpha(\lambda))/c^2} \sqrt{-g_{\alpha\beta}\, \dot{x}_a^\alpha\, \dot{x}_a^\beta}\, \mathrm{d}\lambda, \tag{22.24}$$

instead of the sum of (22.8) and (22.9), which amounts to replace $(1 + \Phi/c^2)$ by e^{Φ/c^2} in the total action.

Historical note: *In the "Palermo memoir", published in 1906 (Poincaré 1906) and mentioned at many occasions in the previous chapters, Henri Poincaré (cf. p. 26) presented some attempts of relativistic extensions of Newton's $1/r^2$ gravitational force, focusing on the invariance with respect to the Lorentz group (cf. Walter (2007)). The treatment of gravitation as a scalar field on Minkowski spacetime has been developed between 1911 and 1913 by Max Abraham,[3] Gustav Mie,[4] Albert Einstein (cf. p. 26) and Gunnar Nordström[5] (cf. Norton (1992) and Pais (1982) for a detailed historical account). It is Max Laue (cf. p. 146) who suggested to Einstein that the source term in a scalar equation for the gravitational field must be the trace of the energy–momentum tensor. Accordingly, Einstein was calling T "Laue's scalar". The most achieved theory was that of Nordström, notably in the version published in 1913 (Nordström 1913). It is equivalent to the theory exposed above, except that Nordström did not derive it from a variational principle: he postulated the field equation to be of the form $\square\Phi = -4\pi G g(\Phi) T_{\text{mat}}/c^2$, and, via an argument based on the equality between inertial mass and gravitational mass, he determined the function $g(\Phi)$ as $g(\Phi) = (1 + \Phi/c^2)^{-1}$, thereby getting (22.23). In 1914, Albert Einstein and Adriaan Fokker (cf. p. 339) showed that Nordström's theory could be recast in a purely metric form, i.e. in a form such that the scalar field Φ does no longer appear but only the metric $\tilde{\boldsymbol{g}} := (1 + \Phi/c^2)^2\, \boldsymbol{g}$ (Einstein and Fokker 1914). One says that $\tilde{\boldsymbol{g}}$ is* ***conformal*** *to the metric $\boldsymbol{g}$. The physical metric on $\mathscr{E}$, in the sense of the metric providing the proper time along worldlines, is then $\tilde{\boldsymbol{g}}$ and no longer the Minkowski metric $\boldsymbol{g}$. In this approach, the concept of gravitational 4-force disappears and the particle trajectories in the gravitational*

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[3]**Max Abraham** (1875–1922): German physicist, student of Max Planck (cf. p. 279), supporter of the aether theory; he developed a model of the electron by considering it as a uniformly charged rigid sphere whose mass is entirely due to the electromagnetic energy.

[4]**Gustav Mie** (1869–1957): German physicist, known for his studies of the scattering of electromagnetic waves by spherical particles *(Mie scattering)*.

[5]**Gunnar Nordström** (1881–1923): Finish physicist, known for his scalar theory of gravitation and for a solution of Einstein equation [Eq. (22.32) below] corresponding to an electrically charged spherical source.

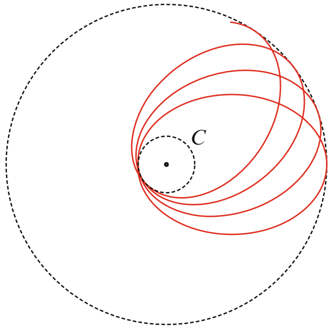

Fig. 22.1 Mercury perihelion advance. On this figure, both the eccentricity and the perihelion advance have been exaggerated

field are simply the geodesics of $\tilde{\boldsymbol{g}}$. *Note that the metrics* $\tilde{\boldsymbol{g}}$ *and* $\boldsymbol{g}$ *have the same light cones:* $\tilde{\boldsymbol{g}}(\vec{\boldsymbol{v}}, \vec{\boldsymbol{v}}) = 0 \iff \boldsymbol{g}(\vec{\boldsymbol{v}}, \vec{\boldsymbol{v}}) = 0$. *Within Einstein–Fokker approach, Nordström's theory became thus the first purely metric theory of gravitation, just before the advent of general relativity.*

22.2.2 Incompatibility with Observations

The scalar theory exposed above is well posed and generalizes Newtonian gravitation. But it suffers from a severe defect: it is not compatible with observations! Indeed, once applied to the motion of the planets around the Sun, it predicts a perihelion *retardation*, while the observations, notably of Mercury, show a perihelion *advance*, moreover with an amplitude larger by a factor 6.

The perihelion advance of Mercury is indeed a crucial test for any relativistic theory of gravitation. In Newtonian gravity, if the Sun is assumed to be perfectly spherical and the actions of the other planets are neglected, Mercury's orbit must be a perfect ellipse, as a solution to the Kepler problem. Any perturbation of the $1/r^2$ gravitational field (Sun flattening, gravitational field of the other planets, inclusion of relativistic effects) makes the orbit deviate from an ellipse. In particular, the orbit is no longer a closed curve and the ***perihelion*** (point of minimal distance to the Sun) is slightly displaced from one orbit to the other (cf. Fig. 22.1). This point moves on a circle (denoted by C in Fig. 22.1) with an angular velocity called ***perihelion advance***. The measured value is $574''$ per century. In 1859 the French astronomer Urbain Le Verrier (1811–1877) discovered that the Newtonian theory, including the perturbations from the other planets (the Sun flattening turning out to be negligible), does not account for the totality of this advance, leaving $43''$ per century unexplained.[6]

[6] This is the modern value, the value found by Le Verrier in 1859 being $38''$ per century.

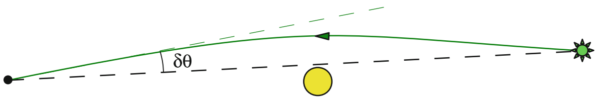

Fig. 22.2 Light deflection: discrepancy $\delta\theta$ between the apparent position of a star and the position that it would have if the light ray would not have travelled in the gravitational field of a massive body (the Sun)

By predicting a perihelion advance of $-7''$ per century (cf., e.g. Giulini (2008) for the detailed computation), Nordström's scalar theory clearly disagrees with the observations, by both the sign and the amplitude of the effect.

Remark 22.4. The perihelion advance of Mercury is small ($43''$ per century!), but there exist astrophysical systems with a much greater advance, thanks to a more intense gravitational field: the neutron-star binaries (Will 2006b). For instance, for the system called *double pulsar PSR J0737-3039*, the periastron advance reaches $17°$ per year !

Another source of disagreement with observations is the phenomenon of ***light deflection*** (Hakim 1999), i.e. the change of apparent direction of stars in the sky when the Sun moves in front of them (cf. Fig. 22.2). At optical wavelengths, this effect is observable only during a solar eclipse, but in radio waves it can be observed at any time. The deflection is all the more large as the light ray comes close to the Sun. For a ray grazing the Sun, the deflection reaches $\delta\theta = 1.75''$. This effect has been measured for the first time during the 1919 solar eclipse. Since then, it has been confirmed to a great accuracy on compact radio sources like quasars (Shapiro et al. 2004).

However, Nordström's scalar theory does not predict any light deflection because the electromagnetic field does not interact with the gravitational field in this theory. This can be seen on the interaction Lagrangian (22.17), according to which the coupling with the gravitational field Φ occurs only via the trace T of the energy–momentum tensor. But a remarkable feature of the electromagnetic field is to possess a traceless energy–momentum tensor, as it is clear from expression (20.3) for $\boldsymbol{T}^{\text{em}}$:

$$T^{\text{em}} = g^{\alpha\beta}\, T^{\text{em}}_{\alpha\beta} = \varepsilon_0 \Big(F_{\mu\alpha} \underbrace{g^{\alpha\beta} F^{\mu}{}_{\beta}}_{F^{\mu\alpha}} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \underbrace{g^{\alpha\beta} g_{\alpha\beta}}_{4} \Big) = 0.$$

Hence, for the electromagnetic field, the interaction Lagrangian (22.17) identically vanishes.

22.2.3 *Vector Theory*

The scalar theory of gravitation was the simplest relativistic extension of Newton's theory. As it must be rejected for disagreeing with experiment, it is natural to wonder about a vector theory, all the more that we know a very successful theory of this type in Minkowski spacetime: electromagnetism! It actually constitutes the prototype of vector theories (cf. Sects. 11.2.6 and 18.7.2). But there is a fundamental difference between electromagnetism and gravitation: in the former, two identical charges repel each other, while in the latter, two identical masses attract each other. This shows up via the change of sign in the constants: to obtain a formulation of gravitation analogous to electromagnetism, one must perform the substitution

$$\varepsilon_0 \longleftrightarrow -\frac{1}{4\pi G}. \tag{22.25}$$

The first component of Maxwell equation (18.67) yields then Poisson equation (22.1) for a slowly varying field ($\Box \to \Delta$).

The change of sign in (22.25), which might seem minor, has actually disastrous consequences for the theory. Let us consider, for instance, an accelerated particle $\mathscr{P}$ of mass $m > 0$. It emits some gravitational radiation, as an accelerated charged particle emits electromagnetic radiation (cf. Sect. 20.3). Let us assume that $\mathscr{P}$ is oscillating along some axis with respect to an inertial observer $\mathscr{O}$, so that its acceleration $\vec{\boldsymbol{\gamma}}$ is always collinear to its velocity $\vec{\boldsymbol{V}}$ (both being relative to $\mathscr{O}$). The radiation energy-flux vector ("gravitational Poynting vector") is then given by formula (20.26), with the substitution (22.25), along with $q \to m$:

$$\vec{\boldsymbol{\varphi}} = -\frac{G m^2\, \gamma^2\, \sin^2\theta}{4\pi c^3\, r^2 \left(1 - \frac{V}{c}\cos\theta\right)^6}\, \vec{\boldsymbol{n}}, \tag{22.26}$$

$\vec{\boldsymbol{n}}$ being the unit vector connecting $\mathscr{P}$'s position with respect to $\mathscr{O}$ at the retarded instant $t - r/c$ to the observation point where $\vec{\boldsymbol{\varphi}}$ is evaluated (cf. Fig. 18.6). We note that $\vec{\boldsymbol{\varphi}}$ is collinear to $\vec{\boldsymbol{n}}$ but oriented in the opposite direction. This means that the energy is radiated towards the particle and not away from it! In other words, the system gravitational field + particle gains some energy during the particle's oscillations. This feature leads to an instability and is not physically acceptable. Therefore, gravitation cannot be described by a vector theory, contrary to Maxwell electromagnetism.

Remark 22.5. The scalar theory of gravitation had to be rejected because it was not agreeing with observations. Here, it is worse: vector gravitation is even not viable on theoretical grounds.

Historical note: *As soon as 1865, James Clerk Maxwell (cf. p. 597) noticed that a theory of gravitation built on the model of electromagnetism would lead to a negative energy density of the gravitational field (Sect. 82 of Maxwell (1865)). Indeed, performing the substitution (22.25) in formula (20.7), which gives the field energy density with respect to an observer, yields*

$$\rho_{\rm grav} = -\frac{1}{8\pi G}\left(\vec{\boldsymbol{E}}\cdot\vec{\boldsymbol{E}} + c^2\vec{\boldsymbol{B}}\cdot\vec{\boldsymbol{B}}\right), \tag{22.27}$$

where $\vec{E}$ and $\vec{B}$ are the gravitational "electric" and "magnetic" field vectors. The negative energy density looked dubious to Maxwell, who concluded that it was not worth exploring further such a theory of gravitation.

22.2.4 Tensor Theory

Let us continue our search for a theory of gravitation in Minkowski spacetime by considering a tensor field of valence 2, after the unsuccessful attempts with a scalar field (valence 0) and a vector field (valence 1). Without going into details (cf. Sect. 8.10 of Anderson (1967), Box. 7.1 of Misner et al. (1973) and Sect. 1.2 of Straumann (2013)), let us simply mention that the main idea is to introduce on $\mathscr{E}$ a tensor field $\boldsymbol{h}$ of type $(0, 2)$ that is symmetric. The coupling to matter and to the non-gravitational fields is then naturally performed by contracting $\boldsymbol{h}$ with the total energy–momentum tensor $\boldsymbol{T}$ of the matter and the non-gravitational fields, to form the interaction Lagrangian

$$\mathscr{L}_{\text{inter}} = \frac{1}{2c} h_{\mu\nu} T^{\mu\nu}. \tag{22.28}$$

This formula is to be compared with that of the scalar case [Eq. (22.17)] and implements the principle of universal coupling to gravitation enounced in Sect. 22.2.1. In the case of a single particle, $\boldsymbol{T}$ takes the form (19.3) with $N = 1$ and (22.28) leads to the Lagrangian (11.41) with $q = m$.

Regarding the Lagrangian of the free field, $\mathscr{L}_{\text{field}}$, there is a natural way to write it, presented in 1939 by the Swiss physicist Markus Fierz (1912–2006) and Wolfgang Pauli (cf. p. 542) (1939). In the vocabulary of field theory, this is the Lagrangian of a so-called *massless spin-2 field*. However, the theory obtained in this way is such that matter does not feel gravity! More precisely, in this theory, the matter energy–momentum tensor $\boldsymbol{T}$ obeys by itself the conservation equation $\vec{\nabla} \cdot \boldsymbol{T} = 0$ (cf. Box. 7.1 of Misner et al. (1973) and Sect. 1.2 of Straumann (2013)). This implies that the density of gravitational 4-force is zero [cf. Eq. (19.28)]. To construct a more satisfactory theory, where matter is sensitive to gravitation, terms of order higher than that of (22.28) must be added to $\mathscr{L}_{\text{field}}$. This leads to a complicated theory, which is actually equivalent to general relativity (Deser 1970): the initial Minkowski metric $\boldsymbol{g}$ loses any physical signification, for the benefit of the metric

$$\boldsymbol{g}^* = \boldsymbol{g} + \boldsymbol{h}. \tag{22.29}$$

It is then simpler to use the framework of general relativity (Sect. 22.4) and not that of a field theory in Minkowski spacetime.

Historical note: *The approach, equivalent to general relativity, that treats gravitation as a valence-2 tensor field on a "background" Minkowski spacetime has* *been developed notably by the Greek physicist Achilles Papapetrou (1907–1997) (Papapetrou 1948), the Indian-American physicist Suraj N. Gupta (1954), Richard Feynman (cf. p. 377) (1995), Stanley Deser*[7] *(1970), Steven Weinberg*[8] *(1972) and the Russian physicist Leonid Grishchuk (1984; 1999).*

22.3 Equivalence Principle

22.3.1 The Principle

The fact that, in an acceptable relativistic theory of gravitation, Minkowski metric is stepping aside in favour of a more "physical" metric [Eq. (22.29)] is the geometrical counterpart of the so-called *equivalence principle*. The latter stems from the feature that singularizes out gravitation among all fundamental interactions: the equality between inertial and gravitational mass. It follows that the motion of a particle in a given gravitational field is independent of the nature of that particle. Things are different for the electromagnetic field: the motion depends upon the ratio of the particle's electric charge to its inertial mass. From an experimental point of view, the equality between inertial mass and gravitational mass has been checked to a very high accuracy: first at the level of 10^{-8} (in relative value) by the Hungarian physicist Loránd Eötvös (1848–1919) in 1909 (cf., e.g. Chap. 7 of Hakim (1999)) to 3×10^{-13} today (Will 2006b). The Microscope satellite, to be launched in 2014 by the French Space Agency (CNES), should improve the accuracy to 10^{-15} (cf. Fig. 22.3).

Because of the equality between the inertial and gravitational mass, a uniform gravitational force in a Galilean frame behaves exactly as the inertial force in a uniformly accelerated frame. This property allows one to generalize the principle of equality inertial mass-gravitational mass to relativity. One postulates the

Equivalence principle: as far as physical measurements are concerned, an inertial observer in a uniform gravitational field is equivalent to a uniformly accelerated observer in the absence of any gravitation field.

[7]**Stanley Deser** : American theoretical physicist born in 1931, known for his works in general relativity and quantum gravity. He notably developed, with Richard Arnowitt and Charles W. Misner (cf. p. 79), a Hamiltonian formulation of general relativity, famous under the name *ADM*.

[8]**Steven Weinberg** : American theoretical physicist born in 1933 and who received the 1979 Nobel Prize in Physics for the unification of electromagnetism and weak interaction; he authored a famous textbook about general relativity (Weinberg 1972).

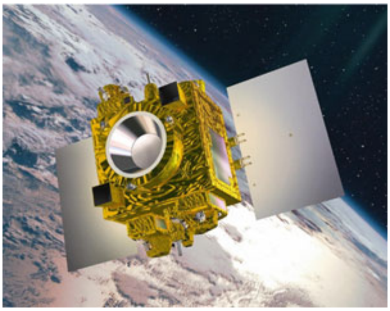

Fig. 22.3 The Microscope satellite will be placed on a polar orbit at the altitude of 700 km. Equipped with two differential accelerometers, it will test the equivalence principle by comparing the motion of two different test bodies, one made of titanium and the other of platinum, in the Earth gravitational field [W14] (Source: CNES, drawing by D. Ducros)

A word of caution is necessary: by *inertial* observer, it is meant an observer whose worldline is a straight line of Minkowski spacetime and whose 4-rotation is zero, according to the definition of Chap. 8. This is therefore not an inertial observer in the standard meaning of general relativity (i.e. an observer in free fall).

The equivalence principle is often illustrated by the following image: an observer located in a cabin without any window cannot determine, by any local experiment, whether the cabin is at rest at the surface of the Earth or travelling far from any gravitational influence, carried by a rocket imparting an acceleration equal to the Earth's gravity ($\gamma = 9.8\ \mathrm{m\,s^{-2}}$).

22.3.2 *Gravitational Redshift and Incompatibility with the Minkowski Metric*

Thanks to the results on accelerated observers obtained in Chap. 12, it is easy to see that the equivalence principle leads to abandon the Minkowski metric in any relativistic theory of gravitation, without having to go into the details of the theory. We shall indeed show that the equivalence principle implies a physical effect, the gravitational redshift, which cannot be accounted for by Minkowski metric.

Let us consider two inertial observers at rest with respect to each other in some uniform gravitational field, so that at the Newtonian limit, the gravitation field vector would be along the vector $\vec{e}_x$ of $\mathscr{O}$'s frame and in the opposite direction: $\vec{g} = -\gamma\,\vec{e}_x$, with $\gamma > 0$ (cf. Fig. 22.4). The x-coordinate of $\mathscr{O}$'s coordinate system measures then the "altitude" in the gravitational field. Let us assume that $\mathscr{O}'$ emits periodic light signals, of period $\Delta t'$ (in terms of this proper time). These signals are received by $\mathscr{O}$ with a period Δt (in terms of this proper time). According to the equivalence principle, things must be the same as if $\mathscr{O}$ and $\mathscr{O}'$ were uniformly accelerated observers in the absence of gravitational field, the 4-acceleration of $\mathscr{O}$

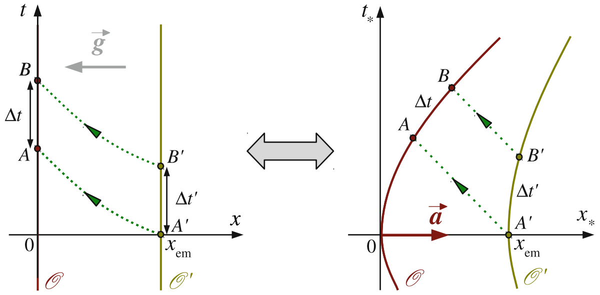

Fig. 22.4 *Left panel:* Inertial observers $\mathscr{O}$ and $\mathscr{O}'$ mutually at rest in a uniform gravitational field, $\mathscr{O}'$ emitting a periodic signal towards $\mathscr{O}$. *Right panel:* Corresponding situation according to the equivalence principle: $\mathscr{O}$ and $\mathscr{O}'$ are uniformly accelerated observers similar to those of Fig. 12.14

being $\vec{a} = (\gamma/c^2)\vec{e}_x$ [cf. (4.64)]. The situation is analogous to that depicted in Fig. 12.14, which is reproduced in the right panel of Fig. 22.4. The relation between Δt and $\Delta t'$ is then given by (12.71) with $a = \gamma/c^2$:

$$\boxed{\Delta t = \frac{\Delta t'}{1 + \gamma x_{\rm em}/c^2}}, \tag{22.30}$$

where $x_{\rm em}$ is the abscissa of $\mathscr{O}'$ in the frame of $\mathscr{O}$. It is clear on (22.30) that a nonvanishing gravitational field ($\gamma \neq 0$) induces $\Delta t \neq \Delta t'$. This phenomenon is called ***gravitational redshift***. If $x_{\rm em} > 0$ (the situation considered in Fig. 22.4), $\Delta t < \Delta t'$ and the corresponding frequency shift is actually a *blueshift*.

The gravitational redshift, which has been verified by many experiments as detailed below, leads to abandon the Minkowski metric when dealing with a gravitational field. Indeed, if the gravitational field is assumed to be stationary, the worldline of a given light signal (emitted at A' and received at A) differs from that of the next signal (emitted at B' and received at B) by a mere translation in the direction $\vec{e}_t$ (cf. Fig. 22.4). Consequently, the Minkowski-metric distance between events A and B must be the same as that between events A' and B'. But if the proper time is given by Minkowski metric, as assumed up to now (cf. Chap 2), the first distance is $c\Delta t$ and the second one $c\Delta t'$. We have thus $\Delta t = \Delta t'$, in contradiction with the relation (22.30) resulting from the equivalence principle. We conclude that

> If gravitation obeys the equivalence principle, then the proper time can no longer be given by Minkowski metric.

Note that this conclusion does not depend upon the detail of the photons' worldlines between $\mathscr{O}'$ and $\mathscr{O}$. For example, in Fig. 22.4, these worldlines are not assumed to be straight lines of $\mathscr{E}$, as they should be in the absence of gravitational field. Actually their shape has been copied from Fig. 12.13, which represents the null geodesics in terms of the coordinates (ct, x) associated with the accelerated observer $\mathscr{O}$ (Rindler coordinates).

Historical note: *The equivalence principle has been enounced by Albert Einstein in 1907 (Einstein 1907). It was the lead that conducted him to formulate general relativity eight years later. Einstein called it the "happiest thought" of his life (cf. Chap. 9 of Pais (1982)). The argument presented above regarding the necessity to abandon Minkowski metric is due to Alfred Schild (cf. p. 378), who wrote it in 1960 (Schild 1960).*

22.3.3 Experimental Verifications of the Gravitational Redshift

The gravitational redshift is the major prediction of any relativistic theory of gravitation based on the equivalence principle. It has been confirmed by various experiments, which are briefly described here.

22.3.3.1 Pound–Rebka Experiment (1960)

In the gravitational field of the Earth, the gravitational redshift is very small: setting $\gamma = g = 9.8 \text{ m s}^{-2}$ in (22.30) yields

$$\frac{\gamma x_{\text{em}}}{c^2} = 1.1 \times 10^{-16} \left(\frac{x_{\text{em}}}{1 \text{ m}}\right). \tag{22.31}$$

The corresponding tiny time shift can be observed by measuring the frequencies of nuclear emission lines. Hence the first experimental test of the gravitational redshift, due to the American physicists Robert V. Pound and Glen A. Rebka in 1960 (Pound and Rebka 1960), consisted in comparing the frequencies of the gamma line ($E = 14$ keV, $\lambda = 0.09$ nm) resulting from the decay of ^{57}Fe nuclei (an unstable isotope of iron, of life time 10^{-7} s) between the bottom and the top of a tower at Harvard University. The spectral shift is of the order of 10^{-15} [$x_{\text{em}} = 22$ m in (22.31)] and could be measured, thanks to the Mössbauer effect, which reduces considerably the Doppler broadening of the line. The found value turned out to agree with the general relativity prediction with an accuracy of around 10%. The experiment was redone in 1965 by R.V. Pound and J.L. Snider (1965), leading to a agreement at the 1% level with general relativity.

22.3.3.2 Atomic Clocks Aboard Aircrafts

The Hafele–Keating experiment (1971) and the Alley experiment (1975) described in Sect. 2.6.6 have verified the gravitational redshift with an accuracy of the order 10% for the former and 1% for the latter.

22.3.3.3 Vessot–Levine Experiment (1976)

To reach a better precision, it was necessary to increase the magnitude of the effect and thus x_{em}, i.e. the difference of altitude between the emitter and the receiver. The idea was then to launch a rocket carrying a clock and to compare its frequency with that of an identical clock on the ground. The experiment, called ***Gravity Probe A***, has been performed in 1976 under the supervision of Robert Vessot and Martin Levine, with a hydrogen-maser atomic clock ($\lambda = 21$ cm) loaded on a Scout D rocket (Vessot et al. 1980). In this case, the maximum altitude is 10^4 km and the gravitational redshift reaches 4×10^{-10}. The formula is not (22.30), for the gravitational field is not uniform at the scale of the rocket trajectory. Although it is considerably larger than that of Pound–Rebka experiment, this spectral shift is 10^5 times smaller than the Doppler shift due to the rocket motion (cf. Sect. 5.5)! The measure has been possible, thanks to the use of a transponder on the rocket that, receiving a signal from the ground, sends it back at exactly the same frequency. The frequency of the returned signal measured on the ground is twice shifted by the first-order Doppler effect: on the outward trip to the rocket and on the return trip. In both cases, the emitter and receiver are receding from each other, resulting in a redshift. On the contrary, the signal that comes back to the ground station is affected neither by the transverse Doppler effect nor by the gravitational redshift. Indeed the return trip cancels both shifts of this kind. Having measured the first-order Doppler effect by this way, one finds the rocket velocity. The transverse Doppler effect can then be computed. Both Doppler effects being known, they can be subtracted to the one-way signal, leaving only the gravitational redshift. The results of Vessot–Levine experiment have been sufficiently accurate to conclude that the relative discrepancy with the prediction resulting from the equivalence principle is at most of 7×10^{-5}.

22.3.3.4 Atomic Clock Ensemble in Space

The ESA experiment ACES (Atomic Clock Ensemble in Space) [W13] consists in a set of two atomic clocks to be installed onboard the International Space Station in 2013. It comprises the cold-caesium-atom clock PHARAO, developed at LNE-SYRTE (Observatoire de Paris) and at the Laboratoire Kastler Brossel (École Normale Supérieure, Paris), under the auspice of CNES (Reynaud et al. 2009) and the hydrogen-maser clock SHM developed by Swiss laboratories. The comparison by radio links with the best atomic clocks on the ground will allow to reach the level of 2×10^{-6} in the test of gravitational redshift, which represents an improvement by a factor 35 with respect to Vessot–Levine experiment.

22.3.3.5 Global Navigation Satellite Systems

The gravitational redshift plays a crucial role in the global navigation satellite systems (GNSS), among which the current American Global Positioning System

(GPS) and Russian GLONASS, as well as the future European Galileo. If the gravitational redshift was not taken into account, these navigation systems would be totally inoperative! (Ashby 2003). The principle is indeed the following one: an observer who is receiving the signal from at least four satellites of a GNSS constellation can compute its position $(t, \vec{r})$ in an inertial frame centred on the Earth[9] by solving the system of four equations :

$$\|\vec{r} - \vec{r}_i\| - c(t - t_i) = 0, \qquad i \in \{1, 2, 3, 4\},$$

where $(t_i, \vec{r}_i)$ is the emission date and position encoded in the signal from the satellite no. i, the date t_i being provided by the onboard atomic clock. The four unknowns are the three components of the position vector $\vec{r}$ and their date t of the simultaneous reception of the four signals. To achieve a precision of the order of a metre on $\vec{r}$, the precision on the dates t_i must be

$$\delta t \sim \frac{1\ \mathrm{m}}{c} \sim 3\ \mathrm{ns}.$$

Now, two relativistic effects lead to a variation δt much larger than the above value:

- The *time dilation* (cf. Sect. 4.2.3): the satellites are moving with respect to the Earth-centred inertial frame, on circular orbits of radius $r_{\mathrm{sat}} = 2.65 \times 10^4$ km. Their orbital velocity is $v = \sqrt{GM_\oplus / r_{\mathrm{sat}}} \simeq 3.87\ \mathrm{km\ s^{-1}}$ ($M_\oplus = 5.97 \times 10^{24}$ kg being the Earth mass), which yields $v/c \simeq 1.3 \times 10^{-5}$ and the Lorentz factor $\Gamma = 1 + 8.3 \times 10^{-11}$. Thanks to (4.1), there comes $\delta t / t = \Gamma - 1 \simeq 8.3 \times 10^{-11}$. If no correction was applied, $\delta t = 3$ ns would be reached within $t \sim$ half a minute!
- The *gravitational redshift*: the satellites are roughly four times higher in the Earth gravitational field than ground observers. The proper times arising from their clocks, once transmitted to the ground, are thus shifted with respect to ground clocks, by a value estimated[10] by setting $x_{\mathrm{em}} = r_{\mathrm{sat}} - R_\oplus \simeq 2 \times 10^4$ km in (22.31): $\delta t / t \sim 10^{-9}$. The exact value is $\delta t / t \simeq 5.3 \times 10^{-10}$. This effect is larger than the time dilation one. If no correction was applied, $\delta t = 3$ ns would be reached within 6 seconds! After one day, the time shift would reach $\delta t = 46\ \mu$s, which would result in a positioning error of 14 km!

The GNSS's constitute thus an application useful in everyday life (the only one to date!) for which the relativistic character of the gravitational field must be taken into account.

[9] In practice, one would like to know the position in an *Earth-fixed* frame, i.e. a frame rotating with respect to inertial frame. A rotation has then to be applied to get the final output (Ashby 2004).

[10] This is not the exact value, because the gravity acceleration γ is not constant between the observer and the satellite.

22.3.4 Light Deflection

Besides the gravitational redshift, another consequence of the equivalence principle is the ***light deflection*** or ***gravitational lensing***, i.e. the fact that photons do not propagate along straight lines in the presence of a gravitational field (cf. Sect. 22.2.2). We have seen indeed in Chap. 12 that, from the point of view of an accelerated observer, photons follow curved lines (cf. Fig. 12.13). The value of the curvature is given by (12.66). This formula is valid only at small scales, within a region where the gravitational field can be considered as homogeneous. It is therefore not sufficient to interpret the observations of light deflection in the Sun's vicinity mentioned in Sect. 22.2.2.

22.4 General Relativity

We have concluded in Sect. 22.3.2 that Minkowski metric cannot account for the gravitational redshift. We have also stressed in Sect. 22.2.4 that any consistent tensor theory of relativistic gravitation leads necessarily to the introduction of a "physical" metric, putting Minkowski metric in the background. All this means that the mathematical structure introduced in Chap. 1, namely, the affine space $\mathscr{E}$ and the metric tensor $\boldsymbol{g}$ on the underlying vector space E, is not adapted to gravitation. Special relativity must then be abandoned in favour of *general relativity*. This theory, elaborated by Albert Einstein in 1915 (Einstein 1915, 1916), is the simplest theory of relativistic gravitation that has passed all the observational and experimental tests to date (Will 2006b, 2010).

The mathematical framework of general relativity differs from that of special relativity by the following features:

- The base space $\mathscr{E}$ is not necessarily an affine space but a more general structure: a *differentiable manifold*, as defined in Sect. 7.2.1.
- The manifold structure implies that there is no longer a unique four-dimensional vector space E underlying $\mathscr{E}$ but an infinity of such spaces: one at each point $A \in \mathscr{E}$, with $E_A \neq E_B$ if $A \neq B$; E_A is called the ***tangent space to*** $\mathscr{E}$ ***at*** A (cf. Fig. 22.5).
- The metric tensor $\boldsymbol{g}$ is a field on $\mathscr{E}$: at each point $A \in \mathscr{E}$, $\boldsymbol{g}(A)$ is a bilinear form on E_A that is symmetric, nondegenerate and of signature $(-, +, +, +)$.
- In general, there does not exist any inertial coordinate system (x^α) over $\mathscr{E}$, i.e. a coordinate system where, at each point of $\mathscr{E}$, the components of $\boldsymbol{g}$ are given by the Minkowski matrix: $g_{\alpha\beta} = \eta_{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1)$.

General relativity implements the equality between inertial mass and gravitational mass—and thus the equivalence principle—by stipulating that the worldlines of particles submitted to a gravitational field are independent of the nature of the particles: they are well-defined lines of spacetime, namely, the

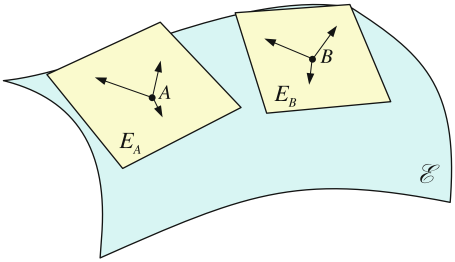

Fig. 22.5 Spacetime manifold $\mathscr{E}$ of general relativity, with the tangent spaces E_A and E_B at two points A and B (for graphical needs, the dimension of $\mathscr{E}$ has been reduced from 4 to 2). This figure is to be compared with Fig. 1.1, which describes the affine space of special relativity

geodesics of the metric $\boldsymbol{g}$. Massive particles follow timelike geodesics, while massless ones (photons) follow null geodesics.

Locally, gravitation cannot be distinguished from an acceleration (equivalence principle), but on a larger scale, the gravitational field is characterized by the variation of the (metric) distance between two nearby geodesics that were initially parallel. This property is called ***curvature*** in differential geometry, which explains why one often says that gravitation is described by the curvature of spacetime. In the case of Minkowski spacetime, two initially parallel geodesics always stay at the same distance since geodesics are straight lines of the affine space $\mathscr{E}$ (cf. Sect. 2.7.1). This means that the curvature vanishes (one says that Minkowski spacetime is ***flat***) and thus that there is no gravitational field in Minkowski spacetime.

The fundamental equation of general relativity is the ***Einstein equation***:

$$\boxed{\boldsymbol{R} - \frac{1}{2} R\, \boldsymbol{g} = \frac{8\pi G}{c^4} \boldsymbol{T}}. \tag{22.32}$$

This is an equality between two fields of symmetric bilinear forms on $\mathscr{E}$: $\boldsymbol{T}$ is the energy–momentum tensor of matter and all non-gravitational fields, $\boldsymbol{R}$ is a symmetric type-$(0,2)$ tensor that describes a part of the spacetime curvature and R is the trace of $\boldsymbol{R}$ with respect to $\boldsymbol{g}$: $R := g^{\mu\nu} R_{\mu\nu}$. $\boldsymbol{R}$ is called the ***Ricci tensor*** and R the ***scalar curvature***. At the Newtonian limit, one of the ten components of the Einstein equation reduces to the Poisson equation (22.1) and the nine others to the trivial "$0 = 0$". For a weakly relativistic gravitational field, i.e. for $\boldsymbol{g} = \bar{\boldsymbol{g}} + \boldsymbol{h}$ with $\bar{\boldsymbol{g}}$ being Minkowski metric and $\boldsymbol{h}$ some "small perturbation", the Einstein equation linearized at first order in $\boldsymbol{h}$ yields the dynamical equation of the tensor theory considered in Sect. 22.2.4 [Fierz–Pauli Lagrangian + interaction Lagrangian (22.28)].

Remark 22.6. Within Nordström's scalar theory studied in Sect. 22.2.1, particles follow geodesics of the metric $\boldsymbol{g} = (1 + \Phi/c^2)^2 \bar{\boldsymbol{g}}$, as shown by Einstein and Fokker (cf. historical note p. 718). The field equation (22.19) can then be recast as

$$R = \frac{24\pi G}{c^4} T, \tag{22.33}$$

where T is the trace of the energy–momentum tensor $\boldsymbol{T}$ with respect to $\boldsymbol{g}$: $T = g^{\mu\nu} T_{\mu\nu} = (1 + \Phi/c^2)^{-2} \bar{T}$, $\bar{T}$ being the trace of $\boldsymbol{T}$ with respect to Minkowski metric ($\bar{T}$ is denoted by T in Sect. 22.2.1). Equation (22.33), which involves the scalar curvature R in its left-hand side, is to be compared with the Einstein equation (22.32).

We stop at this point this small overview of general relativity, referring the reader to various books devoted to it: (Boratav and Kerner 1991; Hakim 1999; Heyvaerts 2006; Ludvigsen 1999; Tourrenc 1997; Choquet-Bruhat 2009; Carroll 2004; Hartle 2003) and (Straumann 2013).

Appendix A
Basic Algebra

A.1 Basic Structures

A.1.1 Group

A ***group*** is a set G endowed with a binary operation $*$, i.e. a map $G \times G \to G$, $(g_1, g_2) \mapsto g_1 * g_2$, such that

- $*$ is associative: $\forall (g_1, g_2, g_3) \in G^3,\ g_1 * (g_2 * g_3) = (g_1 * g_2) * g_3$.
- $*$ has an identity element $e \in G:\quad \forall g \in G,\ e * g = g * e = g$.
- Each element has an inverse: $\forall g \in G,\ \exists g^{-1} \in G,\ g^{-1} * g = g * g^{-1} = e$.

If the operation $*$ is commutative, i.e. if $\forall (g_1, g_2) \in G^2,\ g_1 * g_2 = g_2 * g_1$, the group $(G, *)$ is called ***abelian***.

Example A.1. In this book, we have considered the Lorentz group $\mathrm{O}(3,1)$ (Sect. 6.2.2), the restricted Lorentz group $\mathrm{SO_o}(3,1)$ (Sect. 6.3.3), the Poincaré group $\mathrm{IO}(3,1)$ (Sect. 8.3.3), the general linear group of E, GL(E) (Sect. 6.2.2), the group of rotations in the three-dimensional Euclidean space SO(3) (Sect. 7.5.2), the special linear group $\mathrm{SL}(2,\mathbb{C})$ (Sect. 7.5), the special unitary group SU(2) (Sect. 7.5.2), the Klein group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (Sect. 6.3.4) and the symmetric group $\mathfrak{S}_n$ (Sects. 1.5 and 14.4.2).

Given two groups $(G, *)$ and $(F, \star)$, a function $f: G \to F$ is a ***group homomorphism*** iff it preserves the group structure, i.e. iff

$$\forall (g_1, g_2) \in G^2, \quad f(g_1 * g_2) = f(g_1) \star f(g_2). \tag{A.1}$$

If moreover f is bijective, one says that f is a ***group isomorphism*** and that the groups G and F are ***isomorphic***, which is denoted by the symbol $\simeq$:

$$G \simeq F. \tag{A.2}$$

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An isomorphism $G \to G$ is called a ***group automorphism***.

If $(G, *)$ is a group, one calls ***subgroup*** of G any subset $H \subset G$ such that $(H, *)$ is a group. Moreover, H is said to be a ***normal subgroup*** iff

$$\forall (g, h) \in G \times H, \quad g * h * g^{-1} \in H. \tag{A.3}$$

If G is abelian, all subgroups are obviously normal. The importance of normal subgroups is to allow one to "divide" a given group to get a simpler group. Indeed, if H is a normal subgroup of G, an equivalence relation $\sim$ can be defined on G by

$$\forall (g_1, g_2) \in G^2, \quad g_1 \sim g_2 \iff \exists h \in H,\ g_2 = g_1 * h. \tag{A.4}$$

The equivalence class of an element $g \in G$ for this relation is the set of elements of G that differ from g only by the product with an element of H; it is denoted by gH:

$$gH := \{g * h,\ h \in H\}. \tag{A.5}$$

The ***quotient group*** G/H is the set of all equivalence classes endowed with the binary operation $\bar{*}$ defined by

$$(g_1 H)\, \bar{*}\, (g_2 H) := (g_1 * g_2) H. \tag{A.6}$$

If it is easy to see that this operation is well defined because H is a normal subgroup: the class $(g_1 * g_2)H$ is then independent of the choice of the elements g_1 and g_2 in the classes $g_1 H$ and $g_2 H$.

A group G is called ***simple*** iff it has no normal subgroups except itself and $\{e\}$. There is then no non-trivial quotient of G. On the contrary, if G is not a simple group and H is a normal subgroup of G, one can form the quotient group G/H and reduce the study of G to that of the "smaller" groups G/H and H.

Example A.2. The cyclic groups $\mathbb{Z}/p\mathbb{Z}$ with p prime are simple groups. On the other hand, the Klein group $\{\mathrm{Id}, \boldsymbol{I}, \boldsymbol{T}, \boldsymbol{P}\} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ considered in Sect. 6.3.4 is not a simple group: it admits $\{\mathrm{Id}, \boldsymbol{I}\}$, $\{\mathrm{Id}, \boldsymbol{T}\}$ and $\{\mathrm{Id}, \boldsymbol{P}\}$ as normal subgroups.

A.1.2 Fields

A ***field*** is a set K endowed with two binary operations, $+$ and $.$, say, such that

- $(K, +)$ is an abelian group.
- $(K \backslash \{0\}, .)$, where 0 stands for the identity element of $(K, +)$, is a group.
- The operation $.$ is distributive over $+$:

$$\forall (a, b, c) \in K^3, \quad a.(b + c) = a.b + a.c \quad \text{and} \quad (b + c).a = b.a + c.a.$$

If $(K\backslash\{0\}, .)$ is an abelian group, one says that K is a ***commutative field***.

Example A.3. Standard examples are the field of real numbers $\mathbb{R}$, the field of complex numbers $\mathbb{C}$ and the field of quaternions $\mathbb{H}$ (cf. Remark 7.17 p. 245), the latter being noncommutative, contrary to $\mathbb{R}$ and $\mathbb{C}$.

A.2 Linear Algebra

A.2.1 Vector Space

A ***vector space*** over a commutative field K is a set F endowed with a binary operation $+$ and an external operation of K over F, i.e. a function $K \times F \to F$, $(\lambda, x) \mapsto \lambda x$, such that

- $(F, +)$ is an abelian group.
- The external operation satisfies

$$\forall(\lambda,\mu)\in K^2,\ \forall(x,y)\in F^2,\quad (\lambda\mu)x = \lambda(\mu x) \tag{A.7a}$$

$$(\lambda+\mu)x = \lambda x + \mu x \tag{A.7b}$$

$$\lambda(x+y) = \lambda x + \lambda y \tag{A.7c}$$

$$1x = x, \tag{A.7d}$$

  where 1 stands for the identity element regarding multiplication in the field K.

A ***basis*** of F is a set $(e_i)_{i\in I}$ of elements F, indexed by some set I, such that any element $x \in F$ is expressed in a unique way as a finite linear combination of some of the e_i's: $x = \sum_{i\in I} \lambda_i e_i$, where the summation involves only a finite number of nonzero terms. It can be shown that if $(e_i)_{i\in I}$ and $(e'_j)_{j\in I'}$ are two bases, the sets I and I' are in bijection. One calls then the ***dimension*** of the vector space F and denotes by $\dim F$ the common cardinality of I and I'. F is thus of finite dimension iff I is a finite set.

Example A.4. A few examples of vector spaces considered in this book are the space E underlying Minkowski spacetime $\mathscr{E}$ (E is a four-dimensional vector space over $\mathbb{R}$, Sect. 1.2.1), the local rest space of an observer $E_{\boldsymbol{u}}$ (three-dimensional vector space over $\mathbb{R}$, Sect. 3.2.3), the set $\mathrm{Herm}(2,\mathbb{C})$ of Hermitian 2×2 matrices (four-dimensional vector space over $\mathbb{R}$, Sect. 7.5.1), the set $\mathscr{T}_{(k,\ell)}(E)$ of tensors of type (k,ℓ) on E (vector space of dimension $4^{k+\ell}$ over $\mathbb{R}$, Sect. 14.2.1), the set $\mathscr{A}_p(E)$ of linear p-forms on E (vector space of dimension $\binom{4}{p}$ over $\mathbb{R}$, Sect. 14.4.1) and the set of all tensor fields of type (k,ℓ) on $\mathscr{E}$ (vector space of infinite dimension over $\mathbb{R}$, Sect. 15.3.1).

A part of F that is itself a vector space over K for the same operations as for F is naturally called a ***vector subspace*** of F. If F has a finite dimension, a ***hyperplane*** of F is a vector subspace H satisfying $\dim H = \dim F - 1$. One calls ***vector subspace generated by a family*** $(x_1, \ldots, x_n)$ of n elements of F the smallest subspace of F that contains $x_1, \ldots, x_n$; it is denoted by $\mathrm{Span}(x_1, \ldots, x_n)$.

Given two vector spaces F and G over the same field K, one calls ***linear map*** from F to G any function $f : F \to G$ such that

$$\forall \lambda \in K,\ \forall (x, y) \in F^2, \quad f(\lambda x + y) = \lambda f(x) + f(y). \tag{A.8}$$

The ***kernel*** of f is the inverse image of 0 by f: $\mathrm{Ker}\, f := \{x \in F,\ f(x) = 0\}$ and the ***image*** of f is the set of all elements of G that are the output of some element of F by f: $\mathrm{Im}\, f := \{f(x),\ x \in F\}$. $\mathrm{Ker} f$ is a vector subspace of F and $\mathrm{Im}\, f$ is a vector subspace of G. The linear map f is injective iff $\mathrm{Ker}\, f = \{0\}$.

One calls ***isomorphism*** any linear map $f : F \to G$ that is bijective. If F and G have a finite dimension, then necessarily $\dim F = \dim G$.

A linear map of F to itself is called an ***endomorphism*** of F. If it is moreover bijective (i.e. if it is an isomorphism of F to itself), it is called an ***automorphism***. The set of all automorphisms of a given vector space F, endowed with the composition law $\circ$, is a group, called the ***general linear group of*** F and denoted by $\mathrm{GL}(F)$.

A.2.2 Algebra

An ***algebra*** over a commutative field K is a vector space F over K endowed with a binary operation $*$ satisfying

$$\forall \lambda \in K,\ \forall (x, y, z) \in F^3, \quad (\lambda x) * y = x * (\lambda y) = \lambda (x * y) \tag{A.9a}$$

$$x * (y + z) = x * y + x * z \tag{A.9b}$$

$$(y + z) * x = y * x + z * x. \tag{A.9c}$$

One says that F is an ***associative algebra*** iff the operation $*$ is associative: $x * (y * z) = (x * y) * z$.

Example A.5. Examples of algebras considered in this book are the set $\mathscr{L}(E)$ of all endomorphisms of E (associative algebra over $\mathbb{R}$ for $* = \circ$, Sect. 7.3.1), the Lie algebra of the Lorentz group, $\mathrm{so}(3, 1)$, and that of the Poincaré group, $\mathrm{iso}(3, 1)$ (non-associative algebras over $\mathbb{R}$ for $* = [,]$, Sects. 7.3.2 and 8.3.4), the set $\mathrm{SL}(2, \mathbb{C})$ of 2×2 complex matrices (associative algebra over $\mathbb{C}$ for $*$ = matrix multiplication, Sect. 7.5), the field of quaternions $\mathbb{H}$ (associative algebra over $\mathbb{R}$, Sect. 7.5.2), the Lie algebra $\mathrm{sl}(2, \mathbb{C})$ (non-associative algebra of dimension 3 over $\mathbb{C}$ and dimension 6 over $\mathbb{R}$ for $* = [,]$, Sect. 7.5.6) and the set of all tensors on E (associative algebra of infinite dimension over $\mathbb{R}$ for $* = \otimes$ (tensor product), Sect. 14.3.1).

Appendix B
Web Pages

There are many web pages devoted to special relativity. A small selection is presented here, as well as on the book web page: http://relativite.obspm.fr/sperel

General

[**W1**] Wikipedia: http://en.wikipedia.org/wiki/Special_relativity
[**W2**] Usenet Physics FAQ (D. Kock & J. Baez): http://math.ucr.edu/home/baez/physics/
[**W3**] Einstein Online (Albert Einstein Institute): http://www.einstein-online.info/

Visualization

[**W4**] Space Time Travel (U. Kraus & C. Zahn, Universität Hildesheim): http://www.spacetimetravel.org/
[**W5**] Seeing Relativity (A. Searle, Australian National University): http://www.anu.edu.au/Physics/Searle/
[**W6**] Guide to Special Relativistic Flight Simulators (A.J.S. Hamilton, University of Colorado): http://casa.colorado.edu/%7Eajsh/sr/srfs.html
[**W7**] Visualizing Relativity (D. Weiskopf, Universität Stuttgart): http://www.vis.uni-stuttgart.de/relativity/

É. Gourgoulhon, *Special Relativity in General Frames*, Graduate Texts in Physics,
DOI 10.1007/978-3-642-37276-6, © Springer-Verlag Berlin Heidelberg 2013

Experiments

[**W8**] What is the experimental basis of Special Relativity? (T. Roberts & S. Schleif, Fermilab): http://math.ucr.edu/home/baez/physics/Relativity/SR/experiments.html

[**W9**] Modern searches for Lorentz violation: http://en.wikipedia.org/wiki/Modern_searches_for_Lorentz_violation

[**W10**] Data Tables for Lorentz and CPT Violation: http://arxiv.org/abs/0801.0287

[**W11**] SYRTE, Observatoire de Paris (atomic clocks, time metrology): http://syrte.obspm.fr

[**W12**] Ring Laser Group (Univ. Canterbury) (Sagnac effect): http://www.ringlaser.org.nz/

[**W13**] ACES: Atomic Clock Ensemble in Space: http://www.esa.int/SPECIALS/HSF_Research/SEMJSK0YDUF_0.html

[**W14**] Microscope: exploring the limits of the equivalence principle: http://www.cnes.fr/web/CNES-en/2847-microscope.php

[**W15**] Synchrotron light source facilities: http://www.lightsources.org/

Historical Sources

[**W16**] The Albert Einstein Archives: http://www.albert-einstein.org/

[**W17**] Einstein's articles in *Annalen der Physik*: http://www.physik.uni-augsburg.de/annalen/history/Einstein-in-AdP.htm

[**W18**] Archives Henri Poincaré: http://poincare.univ-nancy2.fr/

[**W19**] Minkowski's papers on relativity: http://www.minkowskiinstitute.org/mip/books/minkowski.html

Miscellaneous

[**W20**] Null rotations (Greg Egan): http://www.gregegan.net/SCIENCE/GR2plus1/NullRotations.html

Appendix C
Special Relativity Books

There are numerous textbooks about special relativity. A small selection is presented here, indicating for each book the adopted signature of the metric tensor (cf. Remark 1.7 on p. 8). Let us stress that many general relativity textbooks have some chapters entirely devoted to special relativity: for instance, Hakim (1999), Hartle (2003), Misner et al. (1973) and Tourrenc (1997).

Geometrical Approach

- O. Costa de Beauregard (1949): *La théorie de la relativité restreinte*; (+,+,+,−)
- J.L. Synge (1956, 1965): *Relativity: the Special Theory*; (+,+,+,−)
- W.G. Dixon (1978): *Special relativity. The foundation of macroscopic physics*; (+,+,+,−)
- G.L. Naber (1992, 2012): *The Geometry of Minkowski Spacetime*; (+,+,+,−)
- A. Das (1993): *The Special Theory of Relativity—A Mathematical Exposition*; (+,+,+,−)
- E.G.P. Rowe (2001): *Geometrical Physics in Minkowski Spacetime*; (−,+,+,+)
- J. Parizet (2008): *La géométrie de la relativité restreinte*; (+,−,−,−)
- N. Dragon (2012): *The Geometry of Special Relativity – a Concise Course*; (+,−,−,−)

"Classical" Approach

Only books published after 1990 are listed:

- M. Boratav and R. Kerner (1991): *Relativité*; (+,+,+,−)
- W. Rindler (1991): *Introduction to Special Relativity*; (+,−,−,−)
- E.F. Taylor and J.A. Wheeler (1992): *Spacetime physics—introduction to special relativity*; (+,−,−,−)
- M. Hulin, N. Hulin and L. Mousselin (1998): *Relativité restreinte*; (+,+,+,−)

É. Gourgoulhon, *Special Relativity in General Frames*, Graduate Texts in Physics, DOI 10.1007/978-3-642-37276-6, © Springer-Verlag Berlin Heidelberg 2013

- M. Fayngold (2002): *Special Relativity and Motions Faster than Light*; (+,–,–,–)
- Y. Simon (2004): *Relativité restreinte*; (+,+,+,–)
- A. Rougé (2004): *Introduction à la relativité*; (+,–,–,–)
- J.-P. Pérez (2005): *Relativité et invariance*; (+,–,–,–)
- C. Semay and B. Silvestre-Brac B. (2005, 2010): *Relativité restreinte: bases et applications*; (+,–,–,–)
- D. Giulini (2005): *Special Relativity, a first encounter*; (+,–,–,–)
- R. Ferraro (2007): *Einstein's Spacetime*; (+,–,–,–)
- M. Fayngold (2008): *Special Relativity and How it Works*; (+,–,–,–)
- M. Tsamparlis (2010): *Special Relativity*; (–,+,+,+)

Advanced Textbooks

- A.O. Barut (1964): *Electrodynamics and classical theory of fields and particles*; (+,–,–,–)
- J.L. Anderson (1967): *Principles of Relativistic Physics*; (+,–,–,–)
- H. Bacry (1967): *Leçons sur la théorie des groupes et les symétries des particules élémentaires*; (+,–,–,–)
- A. Lichnerowicz (1967): *Relativistic hydrodynamics and magnetohydrodynamics*; (+,–,–,–)
- H.C. Corben (1968): *Classical and Quantum Theories of Spinning Particles*; (+,+,+,–)
- A.M. Anile (1989): *Relativistic Fluids and Magnetofluids*; (–,+,+,+)
- Y.Z. Zhang (1997): *Special Relativity and Its Experimental Foundation*; (+,–,–,–)
- R.U. Sexl and H.K. Urbantke (2001): *Relativity, Groups, Particles: Special Relativity and Relativistic Symmetry in Field and Particle Physics*; (+,–,–,–)

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When a reference includes some Internet address, this is the URL at which a copy of the article or the book can be freely downloaded. To avoid copying the address by hand, this URL is clickable in the list of references provided at http://relativite.obspm.fr/sperel/biblio.html

Most of the historical articles are nowadays accessible in this manner, from the articles laying the foundations of relativity at the beginning of the twentieth century to those of the "prehistory" of relativity back in the seventeenth century, such as Picard's article (Picard 1680) (first observation of stellar aberration) or Rømer's one (Rømer 1676) (first measure of the speed of light). When a bibliographic entry does not contain any Internet address, the article is generally downloadable from the journal web page, but this may require some subscription. Note however that most of recent articles are freely accessible as preprints at http://arxiv.org.

The abbreviation *Eng. tr.* stands for *English translation*.

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List of Symbols

$\vec{u} \cdot \vec{v}$:	Scalar product $\boldsymbol{g}(\vec{u}, \vec{v})$ of vectors $\vec{u}$ and $\vec{v}$ with respect to the metric tensor, p. 8
$\lVert\vec{v}\rVert_{g}$:	Norm of vector $\vec{v}$ with respect to the metric tensor, p. 11
$\langle\boldsymbol{\omega}, \vec{v}\rangle$:	Action of the linear form $\boldsymbol{\omega}$ on vector $\vec{v}$, p. 22
$\underline{\boldsymbol{v}}$:	Linear form associated with vector $\vec{v}$ by metric duality, p. 24
$\vec{\boldsymbol{\omega}}$:	Vector associated with the linear form $\boldsymbol{\omega}$ by metric duality, p. 24
$\perp_{\boldsymbol{u}}$:	Orthogonal projector onto the hyperplane normal to vector $\vec{u}$, p. 71
$\otimes$:	Tensor product, p. 83, 475
$\times_{\boldsymbol{u}}$:	Cross product in the vector subspace $E_{\boldsymbol{u}}$, p. 84
$[\,,\,]$:	Commutator or Lie bracket, p. 223
$\rtimes$:	Semidirect product, p. 265
$\{\,,\,\}$:	Poisson bracket, p. 366
$\wedge$:	Exterior product, p. 319, 483
$\star$:	Hodge star, p. 490
$\boldsymbol{\nabla}$:	Covariant derivative, p. 504, 506
$\vec{\boldsymbol{\nabla}}$:	Metric dual of the covariant derivative, p. 603
$\boldsymbol{\nabla}_{\vec{v}}$:	Covariant derivative along the vector field $\vec{v}$, p. 505, 507
$\boldsymbol{\nabla} \cdot \vec{v}$:	Divergence of the vector field $\vec{v}$, p. 512
$\boldsymbol{\nabla} \cdot \boldsymbol{T}$:	Divergence of the tensor field $\boldsymbol{T}$, p. 513
$\boldsymbol{\nabla}\times_{\boldsymbol{u}}$:	Curl in the hyperplane normal to $\vec{u}$, p. 516
$\boldsymbol{\nabla}_{\perp_{\boldsymbol{u}}}$:	Gradient projected onto the hyperplane normal to $\vec{u}$, p. 605
$\square$:	D'Alembertian operator, p. 604
$\simeq$:	Isomorphism, p. 733
$\mathscr{A}_p(E)$:	Space of p-forms on E, p. 20, 481
c:	Constant for time $\rightarrow$ length conversion $\equiv$ velocity of light, p. 4
$\Gamma^{\mu}{}_{\alpha\beta}$:	Connection coefficients, p. 508
$\mathbf{d}$:	Exterior derivative, p. 514
$\boldsymbol{D}_{\mathscr{O}}$:	Vector derivative with respect to observer $\mathscr{O}$, p. 90

É. Gourgoulhon, *Special Relativity in General Frames*, Graduate Texts in Physics, DOI 10.1007/978-3-642-37276-6, © Springer-Verlag Berlin Heidelberg 2013

$\boldsymbol{D}^{\mathrm{FW}}_{\boldsymbol{u}}$:	Fermi–Walker derivative along the worldline of 4-velocity $\vec{\boldsymbol{u}}$, p. 91
$\partial\mathscr{V}$:	Boundary of submanifold $\mathscr{V}$, p. 526
δ^{α}_{β}:	Kronecker symbol, p. 10
$\delta(x)$:	Dirac delta function on $\mathbb{R}$, p. 376
δ_A:	Dirac delta function on $(\mathscr{E}, \boldsymbol{g})$ centred at point A, p. 587
$\dim F$:	Dimension of the vector space F, p. 735
$\mathscr{E}$:	Affine space representing Minkowski spacetime, p. 3
$\mathscr{E}_{\boldsymbol{u}}(A)$:	Local rest space of the observer of 4-velocity $\vec{\boldsymbol{u}}$ at A, p. 68
$\mathscr{E}_{\boldsymbol{u}}(t)$:	Local rest space of the observer of 4-velocity $\vec{\boldsymbol{u}}$ at the date t of his proper time, p. 68
E:	Vector space underlying Minkowski spacetime $\mathscr{E}$, p. 3
E^*:	Dual of E, p. 22
$E_{\boldsymbol{u}}(A)$:	Vector space associated with the local rest space of the observer of 4-velocity $\vec{\boldsymbol{u}}$ at A, p. 68
$E_{\boldsymbol{u}}(t)$:	Vector space associated with the local rest space of the observer of 4-velocity $\vec{\boldsymbol{u}}$ at the date t of his proper time, p. 68
$\boldsymbol{\epsilon}$:	Levi–Civita tensor, p. 21
$\boldsymbol{\epsilon}_{\boldsymbol{u}}$:	Mixed product in the vector subspace $E_{\boldsymbol{u}}$, p. 84
$\boldsymbol{\epsilon}_{\mathscr{V}}$:	Volume element of the hypersurface $\mathscr{V}$, p. 531
$\boldsymbol{\epsilon}_{\mathscr{S}}$:	Area element of the surface $\mathscr{S}$, p. 532
${}^1\boldsymbol{\epsilon}$:	Tensor of type $(1, 3)$ associated with $\boldsymbol{\epsilon}$ by metric duality, p. 487
${}^2\boldsymbol{\epsilon}$:	Tensor of type $(2, 2)$ associated with $\boldsymbol{\epsilon}$ by metric duality, p. 487
${}^3\boldsymbol{\epsilon}$:	Tensor of type $(3, 1)$ associated with $\boldsymbol{\epsilon}$ by metric duality, p. 487
${}^4\boldsymbol{\epsilon}$:	Tensor of type $(4, 0)$ associated with $\boldsymbol{\epsilon}$ by metric duality, p. 487
$\eta_{\alpha\beta}$:	Minkowski matrix, p. 11
$\boldsymbol{F}$:	Electromagnetic field tensor, p. 546
$\boldsymbol{F}^{\sharp}$:	Metric dual of the electromagnetic field tensor, p. 550
$\Phi_{\mathscr{V}}(\vec{\boldsymbol{v}})$:	Flux of vector $\vec{\boldsymbol{v}}$ through the hypersurface $\mathscr{V}$, p. 536
$\boldsymbol{g}$:	Metric tensor, p. 7
$g_{\alpha\beta}$:	Matrix of the metric tensor in some basis, p. 9
$g^{\alpha\beta}$:	Inverse matrix of the metric tensor in some basis, p. 10
$\mathrm{GL}(E)$:	Group of automorphisms of the vector space E, p. 170
$\mathrm{Herm}(2, \mathbb{C})$:	Space of Hermitian matrices of size 2, p. 237
$\mathscr{I}$:	Null cone of the metric tensor, p. 15
$\mathscr{I}^+$, $\mathscr{I}^-$:	Future, past null cone of the metric tensor, p. 16
$\mathscr{I}(A)$:	Light cone of event A, p. 40
$\mathscr{I}^+(A)$, $\mathscr{I}^-(A)$:	Future, past light cone of event A, p. 40
$\mathbb{I}_n$:	Identity matrix of size n, p. 10
Id:	Identity operator, p. 71, 169

$\mathrm{IO}(3,1)$:	Poincaré group, p. 265
$\mathrm{ISO}_\mathrm{o}(3,1)$:	Restricted Poincaré group, p. 265
$\mathrm{iso}(3,1)$:	Lie algebra of Poincaré group, p. 266
$\mathscr{L}(E)$:	Space of linear maps $E \to E$, p. 221
$\mathrm{O}(3,1)$:	Lorentz group, p. 169
$\mathrm{O}_\mathrm{o}(3,1)$:	Orthochronous Lorentz group, p. 173
$\mathscr{S}$:	Set of spacelike unit vectors, p. 17
$\mathscr{S}_O$:	Hyperboloid of one sheet formed by all the points connected to point O by a spacelike unit vector, p. 17
$\mathrm{SL}(2,\mathbb{C})$:	Complex special linear group of index 2, p. 237
$\mathrm{sl}(2,\mathbb{C})$:	Lie algebra of $\mathrm{SL}(2,\mathbb{C})$, p. 251
$\mathrm{SO}(3,1)$:	Proper Lorentz group, p. 172
$\mathrm{SO}_\mathrm{o}(3,1)$:	Restricted Lorentz group, p. 174
$\mathrm{so}(3,1)$:	Lie algebra of the Lorentz group, p. 222
$\mathrm{Span}(\vec{\boldsymbol{v}}_1,\ldots,\vec{\boldsymbol{v}}_n)$:	Vector subspace of E generated by a family of n vectors $(1 \le n \le 4)$, p. 18, 736
$\mathrm{SU}(2)$:	Special unitary group of index 2, p. 243
$\mathfrak{S}_n$:	Group of permutations of n elements, p. 21, 483
$\Sigma_{\boldsymbol{u}}(A)$:	Simultaneity hypersurface of event A for the observer of 4-velocity $\vec{\boldsymbol{u}}$, p. 66
$\Sigma_{\boldsymbol{u}}(t)$:	Simultaneity hypersurface of date t for the observer of 4-velocity $\vec{\boldsymbol{u}}$, p. 66
σ_i:	Pauli matrix, p. 238
$\mathscr{T}_{(k,\ell)}(E)$:	Space of tensors of type (k,ℓ) on E p. 474
$\mathscr{U}$:	Set of timelike unit vectors, p. 17
$\mathscr{U}^+$:	Set of timelike future-directed unit vectors, p. 17
$\mathscr{U}^+_O$:	Future sheet of the hyperboloid formed by all the points connected to point O by a timelike unit vector, p. 17
$\mathscr{U}^-$:	Set of timelike past-directed unit vectors, p. 17
$\mathscr{U}^-_O$:	Past sheet of the hyperboloid formed by all the points connected to point O by a timelike unit vector, p. 17
Υ:	Heaviside step function, p. 609
vol:	4-volume of a domain of $\mathscr{E}$, p. 522

Index

É. Gourgoulhon, *Special Relativity in General Frames*, Graduate Texts in Physics,
DOI 10.1007/978-3-642-37276-6, © Springer-Verlag Berlin Heidelberg 2013