Kluwer's
International Series

ADVANCING THE
STATE-OF-THE-ART

MULTIPLE CRITERIA OPTIMIZATION

State of the Art Annotated Bibliographic Surveys

Volume Contributors:

Masao Arakawa
Jean-Charles Billaut
Carlos A. Coello Coello
Matthias Ehrgott
Xavier Gandibleux
Alfred Göpfert
Dylan F. Jones
Hun Kuk
Kaisa Miettinen
Hirotaka Nakayama
Carlos E. Mariano Romero
Masatoshi Sakawa
Ralph E. Steuer
Vincent T'Kindt
Mehrdad Tamiz
Christiane Tammer
Tetsuzo Tanino
Ye Boon Yun

edited by
Matthias Ehrgott
Xavier Gandibleux

MULTIPLE CRITERIA OPTIMIZATION: STATE OF THE ART ANNOTATED BIBLIOGRAPHIC SURVEYS

INTERNATIONAL SERIES IN
OPERATIONS RESEARCH & MANAGEMENT SCIENCE
Frederick S. Hillier, Series Editor Stanford University

Ho, T.-H. & Tang, C. S. / *PRODUCT VARIETY MANAGEMENT*
El-Taha, M. & Stidham , S. / *SAMPLE-PATH ANALYSIS OF QUEUEING SYSTEMS*
Miettinen, K. M. / *NONLINEAR MULTIOBJECTIVE OPTIMIZATION*
Chao, H. & Huntington, H. G. / *DESIGNING COMPETITIVE ELECTRICITY MARKETS*
Weglarz, J. / *PROJECT SCHEDULING: Recent Models, Algorithms & Applications*
Sahin, I. & Polatoglu, H. / *QUALITY, WARRANTY AND PREVENTIVE MAINTENANCE*
Tavares, L. V. */ ADVANCED MODELS FOR PROJECT MANAGEMENT*
Tayur, S., Ganeshan, R. & Magazine, *M. / QUANTITATIVE MODELING FOR SUPPLY CHAIN MANAGEMENT*
Weyant, J./ *ENERGY AND ENVIRONMENTAL POLICY MODELING*
Shanthikumar, J.G. & Sumita, U./*APPLIED PROBABILITY AND STOCHASTIC PROCESSES*
Liu, B. & Esogbue, A.O. / *DECISION CRITERIA AND OPTIMAL INVENTORY PROCESSES*
Gal, T., Stewart, T.J., Hanne, T. */ MULTICRITERIA DECISION MAKING: Advances in MCDM Models, Algorithms, Theory, and Applications*
Fox, B. L./ *STRATEGIES FOR QUASI-MONTE CARLO*
Hall, R.W. / *HANDBOOK OF TRANSPORTATION SCIENCE*
Grassman, W.K./ *COMPUTATIONAL PROBABILITY*
Pomerol, J-C. & Barba-Romero, S. / *MULTICRITERION DECISION IN MANAGEMENT*
Axsäter, S. / *INVENTORY CONTROL*
Wolkowicz, H., Saigal, R., Vandenberghe, L./ *HANDBOOK OF SEMI-DEFINITE PROGRAMMING: Theory, Algorithms, and Applications*
Hobbs, B. F. & Meier, P. / *ENERGY DECISIONS AND THE ENVIRONMENT: A Guide to the Use of Multicriteria Methods*
Dar-El, E./ *HUMAN LEARNING: From Learning Curves to Learning Organizations*
Armstrong, J. S./ *PRINCIPLES OF FORECASTING: A Handbook for Researchers and Practitioners*
Balsamo, S., Personé, V., Onvural, R./ *ANALYSIS OF QUEUEING NETWORKS WITH BLOCKING*
Bouyssou, D. et al/ *EVALUATION AND DECISION MODELS: A Critical Perspective*
Hanne, T./ *INTELLIGENT STRATEGIES FOR META MULTIPLE CRITERIA DECISION MAKING*
Saaty, T. & Vargas, L./ *MODELS, METHODS, CONCEPTS & APPLICATIONS OF THE ANALYTIC HIERARCHY PROCESS*
Chatterjee, K. & Samuelson, W./ *GAME THEORY AND BUSINESS APPLICATIONS*
Hobbs, B. et al/ *THE NEXT GENERATION OF ELECTRIC POWER UNIT COMMITMENT MODELS*
Vanderbei, R.J./ *LINEAR PROGRAMMING: Foundations and Extensions, 2nd Ed.*
Kimms, A./ *MATHEMATICAL PROGRAMMING AND FINANCIAL OBJECTIVES FOR SCHEDULING PROJECTS*
Baptiste, P., Le Pape, C. & Nuijten, W./ *CONSTRAINT-BASED SCHEDULING*
Feinberg, E. & Shwartz, A./ *HANDBOOK OF MARKOV DECISION PROCESSES: Methods and Applications*
Ramík, J. & Vlach, M. / *GENERALIZED CONCAVITY IN FUZZY OPTIMIZATION AND DECISION ANALYSIS*
Song, J. & Yao, D. / *SUPPLY CHAIN STRUCTURES: Coordination, Information and Optimization*
Kozan, E. & Ohuchi, A./ *OPERATIONS RESEARCH/ MANAGEMENT SCIENCE AT WORK*
Bouyssou et al/ *AIDING DECISIONS WITH MULTIPLE CRITERIA : Essays in Honor of Bernard Roy*
Cox, Louis Anthony, Jr./ *RISK ANALYSIS: Foundations, Models and Methods*
Dror, M., L'Ecuyer, P. & Szidarovszky, F. / *MODELING UNCERTAINTY: An Examination of Stochastic Theory, Methods, and Applications*
Dokuchaev, N./ *DYNAMIC PORTFOLIO STRATEGIES: Quantitative Methods and Empirical Rules for Incomplete Information*
Sarker, R., Mohammadian, M. & Yao, X./ *EVOLUTIONARY OPTIMIZATION*
Demeulemeester, R. & Herroelen, W./ *PROJECT SCHEDULING: A Research Handbook*
Gazis, D.C. / *TRAFFIC THEORY*
Zhu/ *QUANTITATIVE MODELS FOR PERFORMANCE EVALUATION AND BENCHMARKING*

MULTIPLE CRITERIA OPTIMIZATION: STATE OF THE ART ANNOTATED BIBLIOGRAPHIC SURVEYS

Edited by
MATTHIAS EHRGOTT
University of Auckland

XAVIER GANDIBLEUX
Université de Valenciennes

KLUWER ACADEMIC PUBLISHERS
NEW YORK, BOSTON, DORDRECHT, LONDON, MOSCOW

eBook ISBN: 0-306-48107-3
Print ISBN: 1-4020-7128-0

Created in the United States of America

Visit Kluwer Online at: http://kluweronline.com
and Kluwer's eBookstore at: http://ebooks.kluweronline.com

Contents

This page intentionally left blank

List of Figures

List of Tables

Preface

The generalized area of multiple criteria decision making (MCDM) can be defined as the body of methods and procedures by which the concern for multiple conflicting criteria can be formally incorporated into the analytical process. MCDM consists mostly of two branches, multiple criteria optimization and multi-criteria decision analysis (MCDA). While MCDA is typically concerned with multiple criteria problems that have a small number of alternatives often in an environment of uncertainty (location of an airport, type of drug rehabilitation program), multiple criteria optimization is typically directed at problems formulated within a mathematical programming framework, but with a stack of objectives instead of just one (river basin management, engineering component design, product distribution). It is about the most modern treatment of multiple criteria optimization that this book is concerned.

I look at this book as a nicely organized and well-rounded presentation of what I view as "new wave" topics in multiple criteria optimization. Looking back to the origins of MCDM, most people agree that it was not until about the early 1970s that multiple criteria optimization congealed as a field. At this time, and for about the following fifteen years, the focus was on theories of multiple objective linear programming that subsume conventional (single criterion) linear programming, algorithms for characterizing the efficient set, theoretical vector-maximum developments, and interactive procedures. While much important work still needs to be done in these areas, since about the early 1990s a new wave of innovative ideas has begun to overlay the field. Included among these are fuzzy multiple objective programming, multiple criteria heuristics, evolutionary algorithms in multiple criteria optimization, multiple criteria applications in scheduling, and the integration of data envelopment analysis from economics into the MCDM picture. Moreover, there have been recent advancements that have broken through the difficulties that had been holding back areas such as in interactive nonlinear procedures and multiple criteria combinatorial optimization, thus now allowing new bursts in progress on these topics. Capturing these latest ideas and

advancements in the unique bibliographic/source-literature style of this book, the book should well serve researchers as a comprehensive reference volume and teachers with an ideal text for courses at the advanced undergraduate and graduate levels in which research is a focus.

Other aspects of multiple criteria optimization that are reflected across this book are that the contributions that have built and continue to sustain the field have come not only from a cross-section of disciplines (mathematics, engineering, computers, business, operational research, environmental studies), but also from nations all over the world. We see this from the affiliations and nationalities of the authors of the papers listed in the comprehensive bibliographies at the ends of the chapters as well as from the authors of the chapter contributions to this volume. In my mind, it is indeed the blend of multidisciplinary and multicultural ideas and perspectives that makes multiple criteria optimization in general, and this book in particular, so intriguing, fascinating and rewarding to study. I hope you feel the same and welcome to the new face of multiple criteria optimization as presented by this book.

Ralph E. Steuer
Athens, Georgia

Introduction

Matthias Ehrgott, Xavier Gandibleux

Human beings constantly make decisions. By adopting an optimizing behavior, their desire is to perform a given task in the best possible way with respect to some unique criterion to minimize costs or maximize benefits. In an economic environment this might be, e.g., cost of raw materials, return on investment, volume of production, delivery time, etc.

The optimum has remarkable properties. It is unique and proven. It asserts and imposes itself as the best solution without possibility for doubt. Consequently, one understands the reasons which motivate decision makers to be in possession of such an undeniable concept for making a decision.

Because human culture is often dominated by the exact sciences, decision makers have difficulties to renounce to the concept of optimum. B. Roy disagrees with this "paradigm of optimization", which incites to believe that an optimum must exist in all circumstances. Making a decision based solely on a single criterion appears insufficient as soon as the decision-making process deals with complex organizational environments: It is difficult if not impossible to summarize in a single objective the complexity of opinions, the motivations and the goals found in organizations. Thus we may assume that decisions, no matter if made by a group or an individual, most often involve several conflicting objectives.

It seems, therefore, that in many environments it is more realistic to endeavor achieving several objectives simultaneously. This observation implies that real world problems have to be solved optimally according to criteria which prohibit an "ideal" solution – optimal for each decision-maker under each of the criteria considered. Consequently, one must acknowledge the presence of several criteria which are at least partially

contradictory and often noncommensurable, leading to the development of multicriteria optimization.

Multiple Criteria Decision Making: A Young Discipline with Tradition

From its first roots, which were laid by Pareto at the end of the 19th century the discipline has prospered and grown, especially during the last three decades. Today, many decision support systems incorporate methods to deal with conflicting objectives. The foundation for such systems is a mathematical theory of optimization under multiple objectives.

The subject of multicriteria optimization is, generally spoken, the selection of good decisions from a set of alternatives with respect to multiple criteria or objective functions. Therefore it is not surprising that its origin lies in economic theory. The earliest precursors date back to the nineteenth century, when economic welfare and utility theory have been considered first by Walras and others. Edgeworth [10] introduced utility functions and indifference curves which have been used by Pareto [24] to define an economic equilibrium. Nowadays such a situation would be called a local Pareto optimum. Since these early years utility theory has been studied and developed as a branch of economics.

From a mathematical point of view, multicriteria or vector optimization is concerned with the determination of maximal (or minimal) elements of ordered sets. Therefore vector optimization may also be traced back to the work of Cantor [5] and Hausdorff [13].

However, the research in the area which is understood today as multicriteria optimization is much younger. It was necessary to await the second half of the twentieth century to witness the rise of multicriteria optimization. The term "efficient" appears in the work of Koopmans [19] for the first time. The definition of a vector maximum problem has been given by Kuhn and Tucker [21]. Mathematical investigation in this field has finally been established by the work on vector optimization problems in topological vector spaces by Hurwicz [14]. A decade after the definition of vector maximum problems algorithmic aspects have been considered for the first time by Charnes and Cooper [7].

At the end of the Sixties the foundations of goal programming have been laid, an area of research which today is sometimes considered as a separate field. We refer to [22] for a first monograph on the subject (see also [16]). The outranking notion and the discrete multicriteria decision-aid method, ELECTRE [27] appeared in 1968. Almost 20 years later the first monograph [28] was published (see also [29] for methods and applications).

The first interactive methods, the STEM method [4] and the Geoffrion-Dyer-Feinberg method [11], have been introduced in the 1970s. The absence of canonical orders in vector spaces led to the investigation of efficiency for orders defined by cones [42]. Concerning algorithmic aspects the development of methods for the solution of multiple criteria linear programs by Zeleny [43] and Isermann [17] has to be mentioned. Multi-Attribute Utility Theory – MAUT – was popularized in 1976 by Keeney and Raiffa [18].

In 1980, Saaty [30] published his book about the Analytic Hierarchy Process (AHP). At the end of the 70's and in the 80's monographs and textbooks summarizing the state of the art knowledge in the field appeared for the first time. We only mention the books [6, 9, 12, 15, 33, 36, 39, 44, 31, 34]. The Eighties were also distinguished by the use of the possibilities offered by micro-computers in the design and implementation of methods. Visualization and interactivity became subjects of study [2, 3, 20] (see also [25]). The first encounter between metaheuristics and multiobjective optimization is recorded for 1984, when Schaffer [32] presented the VEGA method, an extension of genetic algorithms for problems with multiple objective. Today, thousands of papers give evidence of the blossoming of the investigation of vector optimization in the last three decades. A survey of the activities in Multiple Criteria Decision Making (MCDM) [35] lists 1216 papers between 1987 and 1992. The same survey mentions a total of 208 books, 31 journal special issues and 143 conferences concerned with the subject.

Annotated Bibliographies in Multiple Criteria Optimization

Despite, or because of, this vast body of literature we have noticed the lack of a reliable guide to provide an access to this knowledge. Over the years, many literature surveys and bibliographies have been published. With the ever rapidly increasing rate of publications in the area and the development of subfields, these were mostly devoted to particular aspects of multicriteria optimization, e.g. Multiobjective Integer Programming [26, 45], Multiobjective Combinatorial Optimization [38], Vector Optimization [1, 23], Multiobjective Evolutionary Methods [8], Applications of MCDM [41], MCDM Software [40], Goal Programming [37].

Eventually we decided that it was a good time to provide a more comprehensive overview of the literature in multicriteria optimization that could serve as a state of the art survey and guide to the vast amount of publications. A collection of annotated bibliographies seemed to be

the ideal format. We contacted experts in various areas of multicriteria optimization and asked them to contribute to the present volume. The book on hand is the result of this work.

The chapters in this book roughly follow a thread from most general to more specific. Some of them are about particular types of problems (Theory of Vector Optimization, Nonlinear Multiobjective Programming, Fuzzy Multiobjective Programming, Multiobjective Combinatorial Optimization, Multicriteria Scheduling Problems) the others are focused on multiobjective methodologies (Goal Programming, Interactive Methods, Evolutionary Algorithms, Data Envelopment Analysis). All contributing authors invested great effort to produce comprehensive overviews and bibliographies and to have references that are as precise as possible. In general, highest importance was given to papers published in scientific journals and conference proceedings. Occasional references to technical reports were impossible to avoid, though.

The volume was eventually completed. We were surprised to find that the nine chapters list an amazing 2217 references on almost 500 pages. Now it is up to you, as the readers and users of this book, to judge if the project can be considered a success. We hope that you find this book (and the efforts of authors and editors) worthwhile.

References

[1] A. Achilles, K.-H. Elster, and R. Nehse. Bibliographie zur Vektoroptimierung (Theorie und Anwendungen) (in German). *Mathematische Operationsforschung und Statistik, Series Optimization,* 10:277–321, 1979.

[2] A. Angehrn and H. Luthi. Intelligent decision support systems: A visual interactive approach. *Interfaces,* 20(6):17–28, 1990.

[3] V. Belton and S. Vickers. Use of a simple multi-attribute value function incorporating visual interactive sensitivity analysis for multiple criteria decision making. In C. Bana e Costa, editor, *Readings in Multiple Criteria Decision Aid,* pages 319–334. Springer Verlag, Berlin, 1990.

[4] R. Benayoun, J. de Montgolfier, J. Tergny, and O.I. Larichev. Linear programming with multiple objective functions: Step method (STEM). *Mathematical Programming,* l(3):366–375, 1971.

[5] G. Cantor. Beiträge zur Begründung der transfiniten Mengenlehre (in German). *Mathematische Annalen,* 46:481–512, 1895.

[6] V. Chankong and Y.Y. Haimes. *Multiobjective Decision Making: Theory and Methodology.* Elsevier Science Publishing Co., New

York, NY, 1983.

[7] A. Charnes and W. Cooper. *Management Models and Industrial Applications of Linear Programming.* John Wiley & Sons, New York, NY, 1961.

[8] C.A. Coello, List of references on evolutionary multiobjective optimization. http://www.lania.mx/~ccoello/EMOO/, 2001.

[9] J. de Montgolfier and P. Bertier. *Approche multicritère des problèmes de décision (in French).* Collection Afcet. Editions Hommes et Techniques, Surenes, 1978.

[10] F.Y. Edgeworth. *Mathematical Psychics.* P. Keagan, London, 1881.

[11] A. Geoffrion, J. Dyer, and A. Feinberg. An interactive approach for multi-criterion optimization, with an application to the operation of an academic department. *Management Science,* 19(4):357–368, 1972.

[12] A. Göpfert and R. Nehse. *Vektoroptimierung (in German).* B.G. Teubner, Leipzig, 1990.

[13] F. Hausdorff. Untersuchungen über Ordnungstypen (in German). *Berichte über die Verhandlungen der Königlich Sächsischen Gesellschaft der Wissenschaften zu Leipzig,* 58:106–169, 1906.

[14] L. Hurwicz. Programming in linear spaces. In K.J. Arrow, L. Hurwicz, and H. Uzawa, editors, *Studies in Linear and Nonlinear Programming,* pages 38–102. Stanford University Press, Stanford, 1958.

[15] C.L. Hwang and A.S.M. Masud. *Multiple Objective Decision Making – Methods and Applications,* volume 164 of *Lecture Notes in Economics and Mathematical Systems.* Springer Verlag, Berlin, 1979.

[16] J.P. Ignizio. *Goal Programming and Extensions.* Heath (Lexington Books), Lexington, MA, 1976.

[17] H. Isermann. *Lineare Vektoroptimierung (in German).* PhD thesis, Universität Regensburg, 1974.

[18] R.L. Keeney and H. Raiffa. *Decisions with Multiple Objectives: Preferences and Value Trade-offs.* John Wiley & Sons, New York, NY, 1976.

[19] T.C. Koopmans. Analysis of production as an efficient combination of activities. In T.C. Koopmans, editor, *Activity Analysis of Production and Allocation,* number 13 in Cowles Commission Monographs, pages 33–97. John Wiley & Sons, New York, 1951.

[20] P. Korhonen and J. Laakso. A visual interactive method for solving the multiple criteria problem. *European Journal of Operational Research,* 24:277–287, 1986.

[21] H.W. Kuhn and A.W. Tucker. Nonlinear programming. In J. Neyman, editor, *Proceedings of the second Berkeley Symposium on on Mathematical Statistics and Probability,* pages 481–492. University of California Press, Berkeley, 1951.

[22] S.M. Lee. *Goal Programming for Decision Analysis.* Auerbach Publications, Philadelphia, PA, 1972.

[23] R. Nehse. Bibliographie zur Vektoroptimierung – Theorie und Anwendungen (in German). *Mathematische Operationsforschung und Statistik, Series Optimization,* 13:593–625, 1982.

[24] V. Pareto. *Manual d'économie politique (in French).* F. Rouge, Lausanne, 1896.

[25] J.C. Pomerol and S. Barba-Romero. *Choix multicritère dans l'entreprise; principe et pratique (in French).* Collection Informatique. Hermes, Paris, 1993.

[26] L.M. Rasmussen. Zero-one programming with multiple criteria. *European Journal of Operational Research,* 26:83–95, 1986.

[27] B. Roy. Classement et choix en présence de points de vue multiples, la méthode ELECTRE (in French). *R.A.I.R.O. Revue Française d'Informatique et de Recherche Opérationnelle,* 2(8):57–75, 1968.

[28] B. Roy. *Méthodologie multicritère d'aide à la décision (in French).* Collection Gestion. Economica, Paris, 1985.

[29] B. Roy and D. Bouyssou. *Aide multicritère à la décision: méthodes et cas (in French).* Collection Gestion. Economica, Paris, 1993.

[30] T.L. Saaty. *The Analytic Hierarchy Process.* McGraw-Hill, New York, NY, 1980.

[31] Y. Sawaragi, H. Nakayama, and T. Tanino. *Theory of Multiobjective Optimization.* Academic Press, Orlando, FL, 1985.

[32] J.D. Schaffer. *Multiple Objective Optimization with Vector Evaluated Genetic Algorithms.* PhD thesis, Vanderbilt University, Nashville, TN, 1984.

[33] A. Schärlig. *Décider sur plusieurs critères. Panorama de l'aide à la décision multicritére (in French),* volume 1 of *Collection Diriger l'Entreprise.* Presses Polytechniques Romandes, Lausanne, 1985

[34] R.E. Steuer. *Multiple Criteria Optimization: Theory, Computation and Application.* John Wiley & Sons, New York, NY, 1985.

[35] R.E. Steuer, L.R. Gardiner, and J. Gray. A bibliographic survey of the activities and international nature of multiple criteria decision making. *Journal of Multi-Criteria Decision Analysis,* 5:195–217, 1996.

[36] M. Tabucanon. *Multiple Criteria Decision Making in Industry,* volume 8 of *Studies in Production and Engineering Economics.* Elsevier Science, Amsterdam, 1988.

[37] M. Tamiz, D.F. Jones, and E. ElDarzi. A review of goal programming and its applications. *Annals of Operations Research,* 58:39–53, 1995.

[38] E.L. Ulungu and J. Teghem. Multi-objective combinatorial optimization problems: A survey. *Journal of Multi-Criteria Decision Analysis,* 3:83–104, 1994.

[39] P. Vincke. *L'aide multicritère à la décision (in French).* Collection Statistique et Mathématiques Appliquées. Editions de l'Université de Bruxelles, Bruxelles, 1989.

[40] H.R. Weistroffer and S.C. Narula. The state of multiple criteria decision support software. *Annals of Operations Research,* 72:299–313, 1997.

[41] D.J. White. A bibliography on the application of mathematical programming multiple-objective methods. *Journal of the Operational Research Society,* 41(8):669–691, 1990.

[42] P.L. Yu. Cone convexity, cone extreme points and nondominated solutions in decision problems with multiobjectives. *Journal of Optimization Theory and Applications,* 14:319–377, 1974.

[43] M. Zeleny. *Linear Multiobjective Programming,* volume 95 of *Lecture Notes in Economics and Mathematical Systems.* Springer-Verlag, Berlin, 1974.

[44] M. Zeleny. *Multiple Criteria Decision Making.* Mc Graw-Hill, 1982.

[45] S. Zionts. A survey of multiple criteria integer programming methods. *Annals of Discrete Mathematics,* 5:389–398, 1979.

This page intentionally left blank

Chapter 1

THEORY OF VECTOR OPTIMIZATION

Christiane Tammer, Alfred Göpfert
Department of Mathematics and Computer Science
Martin-Luther University Halle- Wittenberg
D-06099 Halle an der Saale
Germany
{ tammer.goepfert }@mathematik.uni-halle.de

Abstract We introduce several solution concepts for multicriteria optimization problems, give a characterization of approximately efficient elements and discuss a general scalarization procedure. Furthermore, we derive necessary and sufficient optimality conditions, a minimal point theorem, a vector-valued variational principle of Ekeland's type, Lagrangean multiplier rules and duality statements. An overview on vector variational inequalities and vector equilibria is given. Moreover, we discuss the results for special classes of vector optimization problems (vector-valued location and approximation problems, multicriteria fractional programming and optimal control problems).

1. Solution Concepts

Keywords: Vector optimization, Approximately efficient elements, Scalarization.

1.1. Minimality Notions in Partially Ordered Spaces

Many practically important problems can be described by a multicriteria optimization problem with more than one objective function. In order to introduce a solution concept for such problems we consider a nonempty subset M of a linear topological space Y and a reflexive, transitive and antisymmetric relation $\mathcal{R}$ on Y (and so also on M) which gives an order structure (a partial order) on Y (and M). If two elements m_1, m_2 of M are comparable with respect to $\mathcal{R}$ we write $(m_1, m_2) \in \mathcal{R}$ or $m_1 \leq m_2$.

With help of such an order relation one is able to define what a **minimal element** $m^0 \in M$ is:

$$\nexists\, m \in M \setminus \{m^0\} \text{ with } m \leq m^0.$$

Normally, $M \subset Y$ is a set which is given (in any way) by the decision maker. But how to choose really a preference relation $\mathcal{R}$ is in a wide sense an interesting question and touches also questions of sensitivity of the problem dealt with. A **vector minimization problem or vector optimization problem** (v.o.p.) is a problem to determine efficient elements of a certain set in the following sense:

Definition 1 (Efficiency) *Given $Y, M, \mathcal{R}$ as above, minimal elements of M with respect to $\mathcal{R}$ are called efficient and the set (possibly empty) of all efficient elements on M with respect to $\mathcal{R}$ is denoted by* $\operatorname{Min}(M, \mathcal{R})$.

It is often the case that a relation $\mathcal{R}_C$ on Y is generated by a proper convex pointed cone $C \subset Y$. Then $\mathcal{R}_C := \{(y_1, y_2) \mid y_2 - y_1 \in C\}$ is antisymmetric and we write $\operatorname{Eff}(M, C)$ *for* $\operatorname{Min}(M, \mathcal{R}_C)$.

For a v.o.p. with an objective function $f : U \longrightarrow Y$ and a feasible set U we write:

$$(P) \textit{ Determine the set } \operatorname{Eff}(f[U], C),$$

or sometimes simply $f(x) \longrightarrow v\text{-}\min, x \in U$, suppressing the order defining cone C.

Throughout this chapter Y is a linear topological space, Y^* is its topological dual, $C \subset Y$ is a convex cone, i.e., $C + C \subset C$ and $[0, \infty[\, C \subset C$, $C^* = \{y^* \in Y^* \mid y^*(y) \geq 0 \ \forall y \in C\}$ is the dual cone of C and $C^{*0} = \{y^* \in Y^* \mid y^*(y) > 0 \ \forall y \in C \setminus \{0\}\}$. Furthermore, we assume that C is pointed, i.e., $C \cap -C = \{0\}$. We denote for a subset B of a linear topological space Y the topological interior of the set B by *int B*, the topological closure of B by *cl B* and the topological boundary of B by *bd B*.

A functional $z\colon Y \longrightarrow R$ is called B-monotone if $y^1 \in y^2 + B$ implies $z(y^1) \geq z(y^2)$ for all $y^1, y^2 \in Y$. We say that a B-monotone functional z is even strictly B-monotone if, additionally, $y^1 \in y^2 + (B \setminus \{0\})$ implies $z(y^1) > z(y^2)$.

The importance of the mentioned general efficiency concept for decision making was pointed out by Yu [299] for convex cones in R^n (compare also [11, 195, 283, 284]). Yu's concept has been extended by various authors to more general spaces (see [136, 137, 138]), and by Gerstewitz and Iwanow [106] and by Weidner [279] for general sets D. Several efficiency concepts in stochastic multiple objective programming are given in [41]. For definitions on approximate efficiency see Definition 3.

Definition 2 (Weak, proper and strongly proper efficiency) *Let Y, M, C as above.*

(i) *Assume int C is nonempty and* $y^0 \in M$. *If* $y^0 \in \mathrm{Eff}(M, \{0\} \cup int\ C)$, *then* y^0 *is called* **weakly efficient** *and we write* $y^0 \in w\text{-}\mathrm{Eff}(M, C)$.

(ii) *Assume that there is a cone* $H \subset Y$, $H \neq Y$, $H \neq C$ *such that* $C \setminus \{0\} \subset int\ H$ *and* $y^0 \in M$. *If* $y^0 \in \mathrm{Eff}(M, H)$ *then* y^0 *is called* **properly efficient** *on M with respect to C and H and we write* $y^0 \in p\text{-}\mathrm{Eff}(M, C)$ *(having H in mind).*

(iii) *Assume that D is a proper convex cone in Y and* U_Y *a fundamental system of neighbourhoods of zero in Y such that the pair (C, D) has the property*

$$\forall W \in U_Y\ \exists V \in U_Y : (C \setminus W) + V \subset D, \tag{1.1}$$

then $y^0 \in M$ *is called* **strongly proper efficient** *on M with respect to C, if* $y^0 \in \mathrm{Eff}(M, D)$. *We write* $y^0 \in SP\,\mathrm{Eff}(M, C)$.

Dependant on H **and** D there are a lot of special proper-efficiency concepts in the literature, cf. [27, 103, 106, 133, 145, 150, 171, 235, 302].

Obviously

$$SP\,\mathrm{Eff}(M, C) \subset p\text{-}\mathrm{Eff}(M, C) \subset \mathrm{Eff}(M, C) \subset w\text{-}\mathrm{Eff}(M, C) \tag{1.2}$$

and this chain of inclusions, giving in some sense a set-valued estimation of the wanted set Eff(M, C), justifies Definition 2. Further results (generalizations of the theorem of Arrow, Barankin and Blackwell) are given by Jahn [150] and Ferro [97]. But much more can be proved, so $w\text{-}\mathrm{Eff}(M, C)$ often possesses properties like connectedness (cf. Luc [190]) or closedness (whereas the efficient set does not), and *p-Eff* (M, C) permits dependent on the choice of H an interpretation by **scalarization,** which can be used for real calculations of elements of Eff (M, C), compare Section 1.2. Generally, the last two inclusions in (1.2) are not equalities as can be seen by the following example.

Example 1 *Assume* $Y = R^2$, $M = \{A_1 = (0,1),\ A_2 = (0,0),\ A_3 = (1,-1)\} \subset Y$, $C = R^2_+$, $H = \{(y_1, y_2) \in Y : y_2 > 0\} \cup \{(y_1, 0) \in Y : y_1 \geq 0\}$. *Then C and H are pointed cones and*

$$\begin{aligned} \mathrm{Eff}(M, C) &= \{A_2, A_3\}, \\ w\text{-}\mathrm{Eff}(M, C) &= M = \mathrm{Eff}(M, 0 \cup int\ C), \\ \mathrm{Eff}(M, H) &= \{A_3\} \in p\text{-}\mathrm{Eff}(M, C). \end{aligned}$$

A geometrical characterization of weakly efficient elements is given by Carrizosa and Plastria [50]. Existence results for efficient elements are shown by Borwein [31, 32], Cesari and Suryanarayana [52], Chew [66], Corley [70], Dedieu [77], Dolecki and Malivert [84], Gajek and Zagrodny [101, 102], Ha [127, 128], Hartley [130], Hazen and Morin [131], Henig [132], Isac [136, 137, 138], Jahn [147], Krasnosel'skij [164], Luc [189], Postolica [223, 224], Sonntag and Zalinescu [246], Sterna-Karwat [250, 251], Takahashi [252]. Furthermore, stability results for set-valued mappings and vector optimization problems are presented by [9, 13, 14, 15, 84, 95, 96, 135, 160, 166, 167, 191, 205, 206, 213, 214, 216, 217, 218, 238, 239, 243, 248, 249, 258, 268, 270, 303, 304].

1.2. Scalarization Methods and Separation Theorems

The following separation theorems given in [110] play an important role in the theory of multicriteria optimization since they permit (under conditions) scalarizing of vector optimization problems, compare [105, 106, 110, 153, 190, 211, 235, 280]. We give a first separation theorem in linear topological spaces without any convexity assumptions (see [110]).

Theorem 1 *Let us assume that:*

(i) Y *is a linear topological space;*

(ii) $C \subset Y$ *is a cone with* $int\ C \neq \emptyset$*;*

(iii) D *is a proper subset of* Y *with non-empty interior such that* $cl\ D + int\ C \subset int\ D$*;*

(iv) $A \subset Y$ *is a non-empty subset of* Y*.*

Then the following statements are true:

(a) $A \cap (-int\ D) = \emptyset$ *implies that there exists a continuous functional* $z : Y \longrightarrow R$*, which is strictly* (*int* C)*-monotone with the range* $(-\infty, +\infty)$ *and*

$$\begin{aligned} z(A) &\geq 0, \\ z(-cl\ D) &\leq 0, \\ z(-bd\ D) &= 0, \\ z(-int\ D) &< 0, \ and \\ z(int\ A) &> 0 \ if\ int\ A \neq \emptyset. \end{aligned}$$

(b) *If cl D is a convex set, then the functional z in (a) can be chosen such that it is also convex.*

(c) *One can construct the functional z in (a) such that z is B-monotone for each set $B \subset Y$ with bd $D+B \subset$ cl D and strictly B- monotone, if bd $D + (B \setminus \{0\}) \subset int\ D$.*

(d) *If bd D + bd $D \subset$ cl D, then z in* (a) *can be chosen such that it is subadditive on Y.*

In the proof of this separation theorem we use a functional $z : Y \longrightarrow R$ introduced by Gerstewitz [105] (compare also Gerth and Weidner [110]) which is defined for an arbitrary set $D \subset Y$ given by (iii) with $int\ D \neq \emptyset$, $cl\ D + int\ C \subset int\ D$ and for a fixed vector $k^0 \in int\ D$ in the following way:

$$z(y) := \inf\{t \in R \mid y \in tk^0 - cl\ D\}, \quad y \in Y. \tag{1.3}$$

This functional has the following nice properties (cf. [110]):

$$z(y) < r \iff y \in -int\ D + rk^0, \tag{1.4}$$
$$z(y) \leq r \iff y \in -cl\ D + rk^0, \tag{1.5}$$
$$z(y) = r \iff y \in -bd\ D + rk^0, \tag{1.6}$$
$$z(y) \geq r \iff y \notin -int\ D + rk^0, \tag{1.7}$$
$$z(y) > r \iff y \notin -cl\ D + rk^0. \tag{1.8}$$

From Theorem 1 we derive the following corollary.

Corollary 1 *Assume that:*

(i) *Y is a linear topological space;*

(ii) *$C \subset Y$ is a proper convex cone with non-empty interior;*

(iii) *$A \subset Y$ is a proper nonempty subset of Y.*

Then $A \cap -int\ C = \emptyset$ implies that there exists a continuous sublinear functional $z : Y \longrightarrow R$ which is strictly int C-monotone with the range

$(-\infty, +\infty)$ *and*

$$\begin{aligned} z(A) &\geq 0, \\ z(-cl\ C) &\leq 0, \\ z(-bd\ C) &= 0, \\ z(-int\ C) &< 0, \\ z(C) &\geq 0, \\ z(int\ C) &> 0, \ and \\ z(int\ A) &> 0 \ if\ int\ A \neq \emptyset. \end{aligned}$$

In Theorem 1 and Corollary 1 we deal with strictly *int C*- monotone functional. If $C \neq Y$ is a convex cone with non-empty interior, then the strict *int C*-monotonicity implies the *C*-monotonicity.

In Theorem 1 we have supposed that *C* is a cone with non-empty interior. This property is not guaranteed in many important cases; for example, even the usual ordering cone $L_2^+ := \{x \in L_2[0,1] : x(t) \geq 0$ almost everywhere $\in [0,1]\ \}$ in $L_2[0,1]$ has no interior point. Therefore, we formulate a separation theorem in which the assumptions on *D* do not depend on a cone *C* with non-empty interior. In this case we have to demand the directedness of *Y* (this is (iv) in the following theorem) with respect to the closure of D explicitly.

Theorem 2 (Second Separation Theorem) *Let us assume:*

(i) *Y is a linear topological space;*

(ii) *D is a proper convex and open subset of Y;*

(iii) *there exists an element* $k^0 \in Y$ *such that* $cl\ D + \alpha k^0 \subset cl\ D$ *for each* $\alpha \geq 0$*;*

(iv) $Y = \cup\{-cl\ D + \alpha k^0 \mid \alpha \in R\}$*;*

(v) $A \subset Y$ *is a non-empty subset of Y.*

Then the following statements are true:

(a) $A \cap (-D) = \emptyset$ *implies that there exists a continuous convex functional* $z : Y \longrightarrow R$ *with the range* $(-\infty, +\infty)$ *and*

$$\begin{aligned} z(A) &\geq 0, \\ z(-bd\ D) &= 0, \\ z(-D) &< 0, \ and \\ z(int\ A) &> 0 \ if\ int\ A \neq \emptyset. \end{aligned}$$

(b) *One can construct the functional z in (a) such that z is B-monotone for each set $B \subset Y$ with $bd\ D + B \subset cl\ D$.*

(c) *If $B \subset Y$ and $bd\ D + (B \setminus \{0\}) \subset D$ then one can construct the functional in (a) such that it is also strictly B-monotone.*

(d) *If $bd\ D + bd\ D \subset cl\ D$, then z in (a) can be chosen such that it is subadditive on Y.*

The separation theorems generalize assertions by Gerstewitz and Iwanow [105] for a not necessarily convex set A and a convex set D which were proved in another way by using stronger assumptions.

Example 2 *Choose $Y = R^2$, $k^0 = (1,1)^T$,*

$$\begin{aligned} D \;:=\; & \{(-y_1,-y_2)^T \mid y_1 + y_2 < 1\} \;\cup \\ & \{(-y_1,-y_2)^T \mid \exists \lambda \in R:\ ||(y_1,y_2)^T - \lambda k^0||_2 < 1\}. \end{aligned}$$

Y, k^0 and D meet all the assumptions of Theorem 2 except the convexity of D. The functional z defined by (1.3) has no finite value in $k^0 \in -D$.

Example 3 *Choose Y, k^0 as in the example above and*

$$D := \{(y_1,y_2)^T \mid \exists \lambda \in R:\ ||(y_1,y_2)^T - \lambda k^0||_2 < 1\}.$$

Y, k^0 and D meet all conditions in Theorem 2 except

$$Y = \cup\{-cl\ D + \alpha k^0 \mid \alpha \in R\}.$$

The functional z constructed in (1.3) takes no finite value in $k^0 \in D$; moreover, it attains nowhere finite values on D.

Scalarization of a given v.o.p. means converting that problem into an optimization problem (or a family of problems) with a real valued function to be minimized, cf. [110, 144, 149, 146, 190]. If solutions of the latter problems (often called scalarized problems) are also solutions of the given v.o.p., then the scalarization seems to be advantageous in order to solve the v.o.p. since methods of common "scalar" optimization (nonlinear programming) can be used.

Mostly advantageous scalarizing is done by using suitable monotone functionals, where the following propositions serve as theoretical background.

Proposition 1 *Let Y be a linear space partially ordered by a nontrivial pointed convex cone C, M a nonvoid set in Y and $s : M \longrightarrow R$ a functional with $dom\ s = M$. If for some $y^0 \in M$*

$$s(y) \geq s(y^0) \quad \forall y \in M \tag{1.9}$$

then $y^0 \in \mathrm{Eff}(M, C)$ *if one of the following conditions is fulfilled:*

(i) *s is monotone increasing on M and* y^0 *is uniquely determined,*

(ii) *s is strictly monotone increasing on M.*

Proof: We consider $y^0 \in M$ such that (1.9) is valid. If $y^0 \notin \mathrm{Eff}(M, C)$ then there exists $y^1 \in (y^0 - (C \setminus \{0\})) \cap M$, $y^1 \neq y^0$. So $-(y^1 - y^0) \in C \setminus \{0\}$, and if s is monotone increasing, then $s(y^0) \geq s(y^1)$, which together with (1.9) gives $s(y^1) = s(y^0)$, $y^1 \neq y^0$, in contradiction to (i). If s is strictly monotone we get $s(y^1) < s(y^0)$, a contradiction, too. □

So Proposition 1 seems especially to be advantageous if one has strictly monotone functional. In particular this is the case if Y is a Banach space and $C^{*0} \neq \emptyset$, obviously. Then efficiency can even be characterized by scalarization:

Proposition 2 *Let Y be a Banach space, partially ordered by a non-trivial convex pointed cone* $C \subset Y$ *with* $C^{*0} \neq \emptyset$ *and* $\emptyset \neq M \subset Y$. $y^0 \in \mathrm{Eff}(M, C)$ *iff there are* $c \in Y$ *and* $y^* \in C^{*0}$ *such that* y^0 *solves the scalarized optimization problem*

$$y^*(y) \longrightarrow min s.t. y \in M,\ y \leq_C c. \tag{1.10}$$

Proof: Let y^0 be a solution of (1.10), then $y^0 \in \mathrm{Eff}(M, C)$. Otherwise there exists $\bar{y} \in M$ such that $\bar{y} \in y^0 - (C \setminus \{0\})$. So $y^* \in C^{*0}$ gives $y^*(\bar{y}) < y^*(y^0)$ in contradiction to (1.10) since $\bar{y} \in y^0 - C \in c - C - C = c - C$,i.e. $\bar{y} \leq c$. If $y^0 \in \mathrm{Eff}(M, C)$ put $c := y^0$. Take $y^* \in C^{*0}$, then y^0 solves (1.10) with the chosen c. Indeed, if there were a feasible $\bar{y}$ such that $y^*(\bar{y}) < y^*(y^0)$ it would follow $\bar{y} \neq y^0$ since $y^* \in C^{*0}$. But also $\bar{y} \leq_C c = y^0$ and so $y^0 \notin \mathrm{Eff}(M, C)$, a contradiction. □

We assume for the rest of this section

(i) Y is a linear topological space;

(ii) C is a cone in Y with nonempty interior; $k^0 \in int\ C$;

(iii) D is a proper subset of Y with nonempty interior such that $cl\ D + int\ C \subset int\ D$.

The properties of the functional z defined by (1.3) can easily be used in order to characterize weak efficiency. Theorem 3 is an example of such a result. Furthermore, if we suppose additional assumptions for the set D then additional properties follow for z. Theorem 3 and 4 both collect such results.

Theorem 3 *Suppose additionally to the assumptions given above:*

(A_1) *Let $D \subset Y$ be a set with int $D \neq \emptyset$ and $0 \in D \backslash int\ D$ for which there exists a cone C with nonempty interior such that cl D + int $C \subset$ int D.*

Then $y^0 \in w\text{-}\mathrm{Eff}(M, D)$ if and only if $y^0 \in M$ and there exists a continuous functional $z : Y \longrightarrow R$ which is strictly int C-monotone (even strictly int D-monotone if D is a convex cone) with the range $(-\infty, +\infty)$ and with the properties

$$z(y^0) = 0,\ z(M) \geq 0,\ z(int\ M) > 0,\ z(y^0 - bd\ D) = 0 \qquad (1.11)$$

$$z(y^0 - D) \leq 0,\ z(y^0 - int\ D) < 0. \qquad (1.12)$$

If cl D is convex, z can be chosen convex.

(A_2) *If int D is convex, $0 \in cl\ (int\ D) \setminus int\ D$ and if there is $k^0 \in Y$ with*

$$cl\ (int\ D) + \alpha k^0 \subset cl\ (int\ D),\ for\ each\ \alpha \geq 0,$$

and

$$Y = \cup\{cl\ (int\ D) + \alpha k^0 \mid \alpha \in R\},$$

then $y^0 \in w\text{-}\mathrm{Eff}(M, D)$ if and only if $y^0 \in M$ and there exists a continuous convex functional $z : Y \longrightarrow R$ with the range $(-\infty, +\infty)$ such that (1.11) and (1.12) hold.

Theorem 4 *Let Y, M, D, y^0 be as in Theorem 3. If condition (A_1) (or (A_2)) holds, we have for z*

(a) *z can be chosen int D-monotone, if there holds bd D + int $D \subset$ cl D (or bd (int D) + int $D \subset$ cl (int D)).*

(b) *z can be chosen D-monotone, if there holds bd $D + D \subset$ cl D (or bd (int D) + $D \subset$ cl (int D)).*

(c) *z can be chosen strictly (int D)-monotone, if bd D + int $D \subset$ int D (or bd (int D) + int $D \subset$ (int D)).*

(d) *z can be chosen subadditive, if there holds bd D + bd $D \subset y^0$ + cl D (or bd (int D) + bd (int D) $\subset y^0$ + cl (int D)).*

In some sense the following Theorem 5 is converse to Theorem 4. Furthermore, the following theorem gives a characterization of properly efficient elements.

Theorem 5 *Assume that D is as in Theorem 3 and that z is a strictly D-monotone functional and let $y^0 \in M$ be a point in which z attains its minimum on M. Then, these results hold:*

(a) y^0 is *an efficient element of M with respect to D.*

(b) *If z is continuous, there exists an open set $H \supset (D \setminus \{0\})$, with $cl\ H + (D \setminus \{0\}) \subset H$ and $y^0 \in \mathrm{Eff}(M, H)$.*

(c) *If z is convex, there exists a convex set $H \supset (D \setminus \{0\})$, such that $y^0 \in \mathrm{Eff}(M, H)$.*

(d) *If z is continuous and convex, there exists an open convex set $H \supset (D \setminus \{0\})$ with $cl\ H + (D \setminus \{0\}) \subset H$, such that $y^0 \in \mathrm{Eff}(M, H)$.*

(e) *If $0 \in cl\ (D \setminus \{0\})$ holds, H in (b), (c), (d) can be chosen such that $0 \in cl\ H \setminus H$.*

(f) *If z is linear, there exists a set $H \supset (D \setminus \{0\})$ such that $H \cup \{0\}$ is a convex cone and $y^0 \in \mathrm{Eff}(M, H)$.*

(g) *If z is linear and continuous, H in (b) can be chosen such that $H \cup \{0\}$ is a convex cone.*

Example 4 *If $0 \notin cl\ (D \setminus \{0\})$, then the set H constructed in the proof of Theorem 5 can fail to fulfill $0 \in cl\ H$. For example, take*

$$Y = R,\ D = R_+ \setminus (0,1),\ M = [1,4],$$

and define

$$z(y) = \begin{cases} y + \frac{1}{2}: & y < \frac{1}{2}, \\ 1: & \frac{1}{2} \le y \le 1, \\ y: & y > 1. \end{cases}$$

z is continuous and strictly D-monotone and attains its minimum on F in $y^0 := 1$. Note that 0 is not a cluster point of the set

$$H = \{y \in Y \mid z(-y + y^0) < z(y^0)\} = (1/2, \infty).$$

Considering properly efficient points instead of weakly efficient ones one can prove connections to scalarizing functionals z similar as in the theorems above. As an example we give

Theorem 6 *Let be $y^0 \in \mathrm{Eff}(M, H)$ for an open convex set $H \supset (D \setminus \{0\})$ such that $0 \in cl\ H \setminus H$. Assume either condition*

(A_1) *There exists an element* $k^0 \in Y$ *with* $cl\ H + \alpha k^0 \subset cl\ H$ *for each* $\alpha \in R_+$ *and* $Y = \cup\{cl\ H + \alpha k^0 \mid \alpha \in R\}$.

or condition

(A_2) *Let* D *be a cone with nonempty interior and* $cl\ H + int\ D \subset H$.

hold. Then the following hold:

(a) *There exists a continuous functional* $z : Y \longrightarrow R$ *with the range* $(-\infty, +\infty)$ *such that (1.11) and (with H instead of D) (1.12) hold. Under* (A_1) z *can be chosen convex, under* (A_2) *strictly int D-monotone. In the last case convexity of* z *follows if cl H is convex.*

(b) *If* $bd\ H + D \subset cl\ H$, z *can be chosen such that it is D-monotone.*

(c) *If* $bd\ H + (D \setminus \{0\}) \subset H$, z *can be chosen such that it is strictly D-monotone.*

(e) *If* $bd\ H + bd\ H \subset y^0 + cl\ H$, z *can be chosen such that it is subadditive.*

Theorems 1 – 6 in full generality stem from a paper by Gerth and Weidner [110]. But of course special cases of those theorems had been proved earlier, e.g., scalarization results for weakly efficient points for the case $Y = R^n$ and $B = y^0 - R^n_+$, $k^0 \in int\ R^n_+$ with the functional

$$z(y) = \inf\left\{t \in R \mid y \in y^0 - R^n_+ + tk^0\right\},$$

had been obtained for $k^0 = (1, ..., 1)^T$ by Brosowski and Conci [39].

1.3. Approximate Minimality

In the last years several concepts for approximately efficient solutions of a vector optimization problem were published, compare [79, 80, 118, 119, 120, 134, 139, 159, 185, 201, 202, 208, 247, 253, 254, 258, 274]). The reason for introducing approximately efficient solutions is the fact that numerical algorithms usually generate only approximative solutions anyhow and moreover, the efficient point set may be empty, whereas approximately efficient points always exist under very weak assumptions.

To find an access to approximate efficiency we will use our concept given in [108] by means of nonempty subsets M and B of a linear topological space Y, a proper, convex pointed cone $C \subset Y$, $C \neq Y$, with $int\ C \neq \emptyset$, $cl\ B + int\ C \subset int\ B$, $k^0 \in int\ C$ and a real number $\epsilon \geq 0$. Relations to some other concepts follow afterwards.

Definition 3 (Approximate efficiency, ϵk^0-efficiency) *An element $y_\epsilon \in M$ is said to be ϵk^0-efficient on M with respect to B as above if*

$$M \cap (y_\epsilon - \epsilon k^0 - (B \setminus \{0\})) = \emptyset.$$

We denote the set of ϵk^0-efficient points of M with respect to B by $\mathrm{Eff}(M, B_{\epsilon k^0})$. For the case $\epsilon = 0$ and $B = C$ the set $\mathrm{Eff}(M, B_{\epsilon k^0})$ coincides with the usual set $\mathrm{Eff}(M, C)$ of efficient points of M with respect to C *as* given in Definition 1.

Obviously for $\epsilon k^0 + (B \setminus \{0\}) \subset B \setminus \{0\}$ it holds

$$\mathrm{Eff}(M, B) \subseteq \mathrm{Eff}(M, B_{\epsilon k^0})$$

for any $\epsilon \geq 0$ and $k^0 \in int\ C$.

Approximate efficiency can also be defined by scalarization:

Definition 4 *Let $z : Y \longrightarrow R$ be any C-monotone functional. An element $y_\epsilon \in M$ is said to be ϵ-efficient with respect to z if*

$$y \in y_\epsilon - C \Longrightarrow z(y_\epsilon) \leq z(y) + \epsilon$$

holds for each $y \in M$. We denote the set of such ϵ-efficient points of M with respect to z by $\epsilon - \mathrm{Eff}^z(M)$.

Simple examples show that generally $\epsilon - \mathrm{Eff}^z(M) \neq \mathrm{Eff}(M, B_{\epsilon k^0})$. So it seems to be very interesting to study relations between the Definitions 3 and 4 (cf. [258]). We use the functional $z : Y \longrightarrow R$ introduced in Section 1.2 by (1.3).

Theorem 7 *Let $B \subset Y$ with $int\ B \neq \emptyset$, $C \subset Y$ a cone with $int\ C \neq \emptyset$, $cl\ B + int\ C \subseteq int\ B$, $bd\ B + bd\ B \subset cl\ B$, $0 \in bd\ B$, $k^0 \in int\ C$ and $\epsilon > 0$. Then z given in (1.3) is a continuous, strictly int C-monotone. functional with the range $(-\infty, \infty)$ and for any $y_\epsilon \in \mathrm{Eff}(M, B_{\epsilon k^0})$ and $y \in M$ we have*

$$\hat{z}(y_\epsilon) \leq \hat{z}(y) + \epsilon$$

for $\hat{z}(y) := z(y - y_\epsilon)$, where $\hat{z}$ is again a continuous, strictly int C-monotone functional with the range $(-\infty, \infty)$.

Additionally, if $cl\ B + (C \setminus \{0\}) \subset int\ B$, then the functional z (and hence $\hat{z}$ also) is strictly C-monotone.

Theorem 8 *Let us consider a cone $C \subset Y$ with $int\ C \neq \emptyset$, a set $B \subset Y$ with $int\ B \neq \emptyset$, $cl\ B + int\ C \subseteq int\ B$, an element $k^0 \in int\ C$, $\epsilon \geq 0$*

and assume that z given in (1.3) is strictly C-monotone, subadditive and continuous on Y. If $y_\epsilon \in M$ fulfills the inequality

$$z(y_\epsilon) \leq z(y) + \epsilon \tag{1.13}$$

for each $y \in M$, then there is an open set $D \subset Y$ with $C \setminus \{0\} \subset D$, $0 \in cl\ D \setminus D$ and $cl\ D + (C \setminus \{0\}) \subset D$ such that

$$y_\epsilon \in \mathrm{Eff}(M, D_{\epsilon k^0}).$$

Corollary 2

(1) *Under the assumptions of Theorem 7 it holds:*

$$y_\epsilon \in \mathrm{Eff}(M, B_{\epsilon k^0}) \Longrightarrow y_\epsilon \in \epsilon - \mathrm{Eff}^{\hat{z}}(M)$$

with $\hat{z}(y) = z(y - y_\epsilon)$ and z from (1.3).

(2) *Under the assumptions of Theorem 8 it holds:*

$$y_\epsilon \in \epsilon - \mathrm{Eff}^{z}(M) \Longrightarrow y_\epsilon \in \mathrm{Eff}(M, D_{\epsilon k^0}),$$

with z given by (1.3).

1.4. Conclusions

In the last years several concepts of approximately efficient elements have been introduced, followed, recently, by characterizations of approximately efficient elements and necessary and sufficient conditions for such elements. In order to describe solution procedures for multicriteria optimization problems it is important to discuss useful scalarization procedures, especially monotonicity properties of scalarizing functionals (or utility functions). Of course, studies of dependence of solutions if data of the given problem vary, are always basic tasks.

2. Optimality Conditions

Keywords: Maximal point theorem, Variational principle, Lagrangean multipliers, Saddle point assertions.

2.1. Maximal Point Theorems and Variational Principles

Dealing with extremal problems one of the main objectives is to derive optimality conditions. Therefore very often variational methods are used

skillfully linked with the introduction of a perturbation of the problem. Important results of duality, optimality and saddle point theory have been obtained in this way. Since the seventies with Ekeland's variational principle and some equivalent results an essentially new approach was available for deriving optimality conditions.

Ekeland's variational principle [89] is a momentous assertion about the existence of an exact solution of a slightly perturbed optimization problem in a neighbourhood of an approximate solution of the original problem. Many authors have published extensions and applications of Ekeland's variational principle as well as equivalent statements (cf. Brezis and Browder [37], Brondsted [38], Rockafellar [229], Oettli [208], Penot [215], Danes [75], Borwein and Preiss [33], Isac [139], Rolewicz [230], Georgiev [104], De Figueiredo [78], Takahashi [252], Phelps [221], Attouch and Riahi [9], Gajek and Zagrodny [102], Göpfert, Tammer and Zalinescu [121, 122]).

In the background of the mentioned variational principle there exist conical support points of arbitrary closed sets in the considered spaces. Of course, conical support points are closely related with supporting of convex sets by halfspaces, being the base of convex duality as well as the background of a lot of optimality conditions in convex analysis.

The results of Bishop and Phelps [21] from 1962 concerning supporting points of (convex) sets can also be seen as a first contribution to the topic of Ekeland's variational principle, given in the form of a maximal point theorem.

Extremal problems do not only exist for goal functional having R^1 as image space but a more general partially ordered space, where chiefly the order is given by a cone in this space. Such problems are the so called multiobjective extremal problems. Also for such problems (and even more general ones, think of set valued or stochastic functions which may be involved in extremal problems) optimality conditions have been derived.

Loridan [185] in 1984 was the first who successfully made use of a special multiobjective variational principle, discovered by himself, to get optimality conditions. He [185] has presented a vector-valued variational principle for the finite-dimensional case using a scalarization and Ekeland's original result. Further, Chen and Huang [61, 62], Chen, Huang and Lee [63] Dentscheva and Helbig [79], Göpfert, Tammer, Zalinescu [121, 122], Huang [135], Isac [139], Nemeth [202], Khanh [159], Tammer [253] have derived vector-valued variational principles for an objective function which takes its values in general spaces.

We remember that a Banach space Y is **partially ordered** if in Y a reflexive, transitive, and antisymmetrical relation is given. This is

exactly realized if in Y a convex, pointed cone C with $0 \in C$ exists, and this means that exactly the elements of C **dominate** $0 \in Y$. Therefore, given a nonempty set A in Y, a point $z^0 \in A$ is called a **maximal point** of A if

$$\{z^0\} = A \cap (z^0 + C)$$

(where $z^0 + C$ means the set $\{z^0 + k : k \in C\}$), which means that z^0 is the only point being simultaneously in A as well as in $z^0 + C$.

Using the cone

$$\mathcal{K}_\epsilon^R := \{(x, r) \in X \times R : r + \sqrt{\epsilon} \mid\mid x \mid\mid \leq 0\}, \tag{1.14}$$

where $\epsilon > 0$ is a parameter and X is a Banach space, Phelps showed that Ekeland's variational principle [89] is a direct consequence of a maximal point theorem (Phelps [221]). This maximal point theorem simply says, that under the given boundedness and closedness conditions for A in the partial ordering (reflexive, transitive, antisymmetrical) defined by $\mathcal{K}_\epsilon^R$ (i.e. that exactly the elements of $\mathcal{K}_\epsilon^R$ dominate $0 \in X \times R$) any point (ξ, β) of A is dominated by at least one maximal point (x^o, r^o) of A. It is also possible to interpret Phelps in the form, that (x^o, r^o) solve the multicriteria optimization problem

$$(x, r) \longrightarrow \max \text{ s.t. } (x, r) \in A, \tag{1.15}$$

where $X \times R$ is partially ordered by $\mathcal{K}_\epsilon^R$. With other words, there is no element (x, r), $(x, r) \neq (x^o, r^o)$, which dominates (x^o, r^o) relative to the cone $\mathcal{K}_\epsilon^R$ as in (1.14).

To find an access to vector optimization instead of .R we consider a more general space *Y:* Let X and Y be Banach spaces, Y partially ordered by a given cone $C \subset Y$, where C is convex, pointed and closed with $int\ C \neq \emptyset$. Furthermore, choose $k^o \in int\ C$ such that $\mid\mid k^o \mid\mid = 1$. For any ϵ with $0 < \epsilon < 1$ we define

$$\mathcal{K}_\epsilon := \left\{(x, y) \in X \times Y : y + \sqrt{\epsilon} k^o \mid\mid x \mid\mid \in -C\right\}. \tag{1.16}$$

$\mathcal{K}_\epsilon$ is clearly a cone, it is the hypograph of the mapping $-\sqrt{\epsilon} k^o \mid\mid . \mid\mid$: $X \longrightarrow Y$, recalling

$$\text{hypo } \left(-\sqrt{\epsilon} k^o \mid\mid . \mid\mid\right) := \left\{(x, y) : y \in -\sqrt{\epsilon} k^o \mid\mid x \mid\mid -C\right\}.$$

$\mathcal{K}_\epsilon$ is pointed and closed, because C has those properties. The triangle inequality gives that $\mathcal{K}_\epsilon$ is convex. Finally, $(0, -k^o) \in \mathcal{K}_\epsilon$ and $(0, y) \in \mathcal{K}_\epsilon\ \forall y \in -C$.

Taking $Y = R$, $C = R_+$ and $k^o = 1$ (1.16) coincides with (1.14). Given $\bar{y} \in -C$, the second inclusion in (1.16) is fulfilled for a norm bounded

set in X. Otherwise there is a sequence $\{x_n\}$ with $\{|| x_n ||\} \longrightarrow \infty$, and so

$$\begin{aligned} - || x_n || k^o &\in C + \frac{\bar{y}}{\sqrt{\epsilon}} \quad \forall n, \\ -k^o &\in C + \frac{1}{|| x_n ||}\frac{\bar{y}}{\sqrt{\epsilon}} \quad \forall n \end{aligned}$$

but $-k^o \notin C$.

Of course, conical support points are related to supporting points of convex sets. An overview can be found in Phelps [221]. Especially, if X is a Banach space, X^* its continuous dual, then taking γ with $0 < \gamma < 1$, $x^* \in X^*$ with $|| x^* ||= 1$, the cone

$$\mathcal{K}(x^*; \gamma) := \{x \in X : \gamma || x ||\leq x^*(x)\} \subset X \tag{1.17}$$

is closed, line free, convex and $int\, \mathcal{K}(x^*, \gamma) \neq \emptyset$. It is just the cone which was used by Bishop and Phelps to prove density theorems for supporting points and functionals for convex sets (although the corresponding maximal point lemma for the supporting points doesn't use convexity). The cone (1.17) is related to the cones in (1.14) and (1.16) in an interesting kind: For clarifying we consider the cone (γ, X as above in (1.17))

$$\mathcal{K}_\gamma^X = \{x \in X : x - \gamma k^o || x ||\in C\}, \tag{1.18}$$

where C is a convex, pointed cone in X, $0 \in C$, $C \neq \{0\}$, and $k^o \in C \setminus \{0\}$, $|| k^o ||= 1$. Then $\mathcal{K}_\gamma^X$ is clearly a cone, it is convex (because of the triangle inequality) and line free. Obviously $k^o \in \mathcal{K}_\gamma^X$ and if $k \in \mathcal{K}_\gamma^X$, it follows

$$k \in \gamma k^o || k || + C \subset C,$$

that means $\mathcal{K}_\gamma^X \subset C$, and if $\gamma \longrightarrow 0$, then $\mathcal{K}_\gamma^X$ approximates C, and if $\gamma \longrightarrow 1$ then $\mathcal{K}_\gamma^X$ approximates the jet, given by k^o. Now we obtain two results relative to $\mathcal{K}_\gamma^X$ in (1.18):

(i) If $\bar{x} \in \mathcal{K}_\gamma^X$, it follows $\bar{x} - \gamma k^o || \bar{x} ||\in C$. Suppose $\bar{x}^* \in C^* \setminus \{0\}$ and $\bar{x}^*(k^o) = 1$, then it holds

$$\bar{x}^*(\bar{x} - \gamma k^o || \bar{x} ||) \geq 0$$

and therefore

$$\bar{x}^*(\bar{x}) \geq \gamma || \bar{x} ||,$$

that means, comparing with (1.17)

$$\mathcal{K}_\gamma^X \subset \mathcal{K}(\bar{x}^*, \gamma). \tag{1.19}$$

(ii) Using $\mathcal{K}_\gamma^X$ instead of $\mathcal{K}_\epsilon$ in (1.16) leads to another kind of maximal point theorems which generalize support properties of sets (see [118]). In $\mathcal{K}_\gamma^X$ linear functional x^* don't appear, so contrary to results obtained with help of (1.17), corresponding conical supporting points (with help of (1.18)) do not only belong to the convex hull of A but even to A itself.

Phelps [221] has shown a maximal point theorem in a product space $X \times R$. The following maximal point theorem in a product space $X \times Y$, where X and Y are Banach spaces (presented by Göpfert and Tammer in [119] and under weaker assumptions by Göpfert, Tammer and Zalinescu in [121] and [122]) says, that under certain conditions for a closed set $A \subset X \times Y$ and for a cone $C \subset Y$, any point of A is dominated by a maximal point, where we use the partial ordering defined by the cone (1.16):

$$\mathcal{K}_\epsilon = \{(x, y) \in X \times Y : y + \sqrt{\epsilon} k^o \mid\mid x \mid\mid \in -C\}.$$

Theorem 9 *Assume that A is a closed subset of $X \times Y$, where X and Y are Banach spaces. Further, suppose that $C \subset Y$ is a pointed, closed, convex cone with nonempty interior and bounded base, and assume $k^o \in int\, C$, $\bar{y} \in Y$, $\{y \in Y,\ \exists x \in X$ with $(x, y) \in A\} \subset \bar{y} + C$, $\epsilon > 0$.*

Then for any point $(x, y) \in A$ there exists $(x_o, y_o) \in A$ such that

$$(x_o, y_o) \in A \cap (\mathcal{K}_\epsilon + (x, y))$$

and

$$\{(x_o, y_o)\} = A \cap (\mathcal{K}_\epsilon + (x_o, y_o)).$$

The essential idea in the proof of Theorem 9 is the following: Consider a sequence $\{A_n\}_{n \in N}$ of sets:

$$A_n := A \cap (\mathcal{K}_\epsilon + (x_n, y_n)). \tag{1.20}$$

Under the given assumptions the sets A_n are closed. Define the sequence $\{(x_n, y_n)\}$ inductively as follows:

$$(x_1, y_1) = (x, y).$$

When we have obtained (x_1, y_1), (x_2, y_2),...,(x_n, y_n), then we choose $(x_{n+1}, y_{n+1}) \in A_n$ such that using a cone $\tilde{C}$ with $(C \setminus \{0\}) \subset int\, \tilde{C}$

$$\nexists (x, y) \in A_n \text{ with } y \in y_{n+1} - \frac{1}{n+1} k^o - (\tilde{C} \setminus \{0\}). \tag{1.21}$$

It is possible to show that *diam* $A_n \longrightarrow 0$. Applying Cantor's Intersection Theorem the assertions in the theorem follow.

A variational principle for a vector optimization problem is a direct consequence of our Theorem 9. This following Theorem is an assertion about the existence of an efficient solution of a slightly perturbed vector optimization problem in a certain neighbourhood of an approximately efficient element of the original vector optimization problem.

Variational principles for vector optimization problems were presented by Loridan [185], Nemeth [202], Khanh [159], Tammer [253], Göpfert and Tammer [119], Göpfert, Tammer, Zalinescu [121, 122], Isac [139], Dentscheva and Helbig [79], Chen and Huang [61, 62, 63], Huang [135] and others.

In the following X and Y are considered to be real Banach spaces, U is a nonempty closed subset of X and $C \subset Y$ is a pointed, closed, convex cone with nonempty interior and bounded base, $k^0 \in int\, C$.

Now, we introduce a function $f : U \longrightarrow Y$ and assume that f is bounded from below. A function $f : U \longrightarrow Y$ is said to be bounded from below on U if there exists an element $z \in Y$ with $f[U] \subset z + C$.

Theorem 10 *Suppose that U is a closed subset of X. Assume that* $f : U \longrightarrow Y$ *is bounded from below and the epigraph of f is closed.*

Then for any $\epsilon > 0$ *and any* $f(x^o) \in \mathrm{Eff}(f[U], C_{\epsilon k^o})$ *there exists an element* $x_\epsilon \in U$ *with*

(1) $f(x_\epsilon) \in f(x^o) - \sqrt{\epsilon} \mid\mid x_\epsilon - x^o \mid\mid k^o - C$,

(2) $\mid\mid x_\epsilon - x^o \mid\mid \leq \sqrt{\epsilon}$,

(3) $f_{\epsilon k^o}(x_\epsilon) \in \mathrm{Eff}(f_{\epsilon k^o}[U], C)$, *where* $f_{\epsilon k^o}(x) := f(x) + \sqrt{\epsilon} \mid\mid x - x_\epsilon \mid\mid k^o$.

2.2. Lagrangean Multipliers and Saddle Point Assertions

Consider a convex vector minimization problem

$$(P) \quad \mathrm{Eff}(f[\mathcal{A}], C_Y),$$

where $f : M \longrightarrow Y$, $g : M \longrightarrow Z$, Y and Z are normed spaces, C_Z and C_Y are closed convex pointed cones in Z, Y, respectively, and

$$\mathcal{A} := \{x \in M \mid g(x) \in -C_Z\}.$$

As in ordinary scalar optimization Lagrange multipliers can be used for different purposes as for duality, sensitivity, or for numerical approaches (compare Amahroq and Taa [2], Clarke [68], El Abdouni and Thibault

[91], Li and Wang [181], Miettinen, [196], Minami [197], Tanaka [264, 265, 266, 267], Thibault [271], Wang [275]).

In the following we derive existence results for Lagrangean multipliers. These results extend well known theorems on Lagrange multipliers in nonlinear programming considerably.

Lemma 1 *Let X be a linear space, M a convex subset of X, Y and Z normed spaces, C_Z and C_Y are closed convex pointed cones in Z, Y, respectively, and int $C_Y \neq \emptyset$. Assume that the mappings $f : M \longrightarrow Y$, $g : M \longrightarrow Z$ are C_Y-convex, C_Z-convex, respectively, for which the following regularity assumptions are fulfilled:*

(A1) *int* $\{(y,z) \in Y \times Z \mid \exists x \in M : y \in f(x) + C_Y$ *and* $z \in g(x) + C_Z\} \neq \emptyset$,

(A2) $\exists y_0 \in cl\, f[\mathcal{A}]$ *and* $f[\mathcal{A}] \cap (y_0 - (C_Y \setminus \{0\})) = \emptyset$.

Then there exist $y_0^* \in C_{Y^*}$, $z_0^* \in C_{Z^*}$ *with* $(y_0^*, z_0^*) \neq (0,0)$ *and it holds*

$$(y_0^*, y_0) = \inf\{(y_0^*, f(x)) + (z_0^*, g(x)) \mid x \in M\}.$$

Proof: Consider the following sets:

$$A := \{(y,z) \in Y \times Z \mid \exists x \in M \ : \ y \in f(x) + C_Y,\ z \in g(x) + C_Z\}$$

and

$$B := \{(y,z) \in Y \times Z \mid y \in y_0 - C_Y,\ z \in -C_Z\}.$$

In order to apply a separation theorem for convex sets we show that the assumptions of the separation theorem are fulfilled.

A is convex since we get for $(y^1, z^1) \in A$, $(y^2, z^2) \in A$, $0 \leq \lambda \leq 1$ and corresponding elements $x^1, x^2 \in M$:

$$\begin{aligned} \lambda y^1 + (1-\lambda)y^2 &\in \lambda f(x^1) + C_Y + (1-\lambda) f(x^2) + C_Y \\ &\subset f(\lambda x^1 + (1-\lambda)x^2) + C_Y + C_Y \\ &\subset f(\lambda x^1 + (1-\lambda)x^2) + C_Y, \end{aligned}$$

since C_Y is a convex cone and f a C_Y-convex mapping. Together with

$$\begin{aligned} \lambda z^1 + (1-\lambda)z^2 &\in \lambda g(x^1) + (1-\lambda) g(x^2) + C_Z \\ &\subset g(\lambda x^1 + (1-\lambda)x^2) + C_Z, \end{aligned}$$

because C_Z is a convex cone and g a C_Z-convex mapping we can conclude

$$\left(\lambda y^1 + (1-\lambda)y^2, \lambda z^1 + (1-\lambda)z^2\right) \in A.$$

Moreover, B is convex regarding the convexity of C_Y and C_Z. Under the assumption *(A1)* it holds $int\ A \neq \emptyset$.

In order to show $int\ A \cap B = \emptyset$ we suppose: $\exists (y, z) \in int\ A \cap B$. This implies there exists $x \in M$ with

$$g(x) \in z - C_Z \subset -C_Z \text{ and } f(x) \in y - C_Y \subset y_0 - C_Y$$

such that we get $y_0 = y = f(x)$ because of the definition of y_0 in (A2) and since C_Y is a pointed convex cone.

Regarding $(y, z) \in int\ A$ it follows that there are an $\epsilon > 0$ and $U_\epsilon(y) \subset Y$, $V_\epsilon(z) \subset Z$ with $U_\epsilon(y) \times V_\epsilon(z) \subset A$. Especially for $k^0 \in C_Y \setminus \{0\}$, $||k^0|| = 1$ we consider $(y - \frac{\epsilon}{2}k^0, z) \in A$, i.e., for some $x' \in M$ it holds

$$y_0 - \frac{\epsilon}{2}k^0 = y - \frac{\epsilon}{2}k^0 \in f(x') + C_Y$$

and

$$g(x') \in z - C_Z \subset -C_Z.$$

This means $x' \in \mathcal{A}$ and $f(x') \in y_0 - (C_Y \setminus \{0\})$ in contradiction to the definition of y_0 in (A2).

Now, it is possible to apply a separation theorem for convex sets. This separation theorem implies the existence of $(y_0^*, z_0^*) \in (Y^* \times Z^*) \setminus \{0\}$ such that

$$\begin{gathered}(z_0^*, z^1) + (y_0^*, y^1) \geq (z_0^*, z^2) + (y_0^*, y^2) \\ \forall (y^1, z^1) \in A \text{ and } \forall (y^2, z^2) \in B.\end{gathered} \tag{1.22}$$

In the following we show that $y_0^* \in C_{Y^*}$ and $z_0^* \in C_{Z^*}$.

If we suppose $y_0^* \notin C_{Y^*}$, i.e., $(y_0^*, \bar{y}) < 0$ for an element $\bar{y} \in C_Y$, we get for $y := -\bar{y} \in -C_Y$ regarding that C_Y is a cone

$$\sup\{(y_0^*, ny) \mid n \in N\} = \sup\{n(y_0^*, y) \mid n \in N\} = \infty$$

in contradiction to the separation property (1.22). Analogously we can show $z_0^* \in C_{Z^*}$.

For all $x \in M$ it holds $(f(x), g(x)) \in A$ and with $(y_0, 0) \in B$ we get

$$\inf\{(z_0^*, g(x)) + (y_0^*, f(x)) \mid x \in M\} \geq (y_0^*, y_0).$$

Now, consider a sequence $\{x_n\}_{n \in N}$ in $\mathcal{A} = \{x \in M \mid g(x) \in -C_Z\}$ with

$$\lim_{n \to \infty} f(x_n) = y_0.$$

Then we get

$$\begin{aligned}
&\inf\{(z_0^*, g(x)) + (y_0^*, f(x)) \mid x \in M\} \\
&\quad \leq \inf\{(z_0^*, g(x)) + (y_0^*, f(x)) \mid x \in \mathcal{A}\} \\
&\quad \leq \inf\{(y_0^*, f(x)) \mid x \in \mathcal{A}\} \\
&\quad \leq \lim_{n\to\infty} (y_0^*, f(x_n)) = (y_0^*, y_0)
\end{aligned}$$

such that the equation holds. □

Lemma 2 *Additionally to the assumptions of Lemma 1 we suppose*

(A3) (Generalized Slater condition) *There exists an element* $x^1 \in M$ *such that for all* $z^* \in C_{Z^*} \setminus \{0\}$ *it holds:* $(z^*, g(x^1)) < 0$.

(i) *Then there exist elements* $y_0^* \in C_{Y^*} \setminus \{0\}$ *and* $z_0^* \in C_{Z^*}$ *with*

$$(y_0^*, y_0) = \inf\{(y_0^*, f(x)) + (z_0^*, g(x)) \mid x \in M\}.$$

(ii) *If* $x_0 \in \mathcal{A}$ *and* $f(x_0) = y_0$, *i.e.,* $f(x_0) \in \mathrm{Eff}(f[\mathcal{A}], C_Y)$, *then* x_0 *is also a minimal solution of* $(y_0^*, f(.)) + (z_0^*, g(.))$ *on* M *and it holds*

$$(z_0^*, g(x_0)) = 0.$$

Proof:

(i) From Lemma 1 we can conclude that there exist $y_0^* \in C_{Y^*}$, $z_0^* \in C_{Z^*}$ with $(y_0^*, z_0^*) \neq 0$ and

$$(y_0^*, y_0) = \inf\{(y_0^*, f(x)) + (z_0^*, g(x)) \mid x \in M\}. \tag{1.23}$$

Under the assumption (A3) we suppose $y_0^* = 0$. Then we get in (1.22) with $z^1 = g(x^1)$, $z^2 = 0$

$$(z_0^*, g(x^1)) \geq (z_0^*, 0) = 0. \tag{1.24}$$

Regarding $(y_0^*, z_0^*) \neq 0$ it holds $z_0^* \neq 0$ and now together with the assumption (A3) a contradiction:

$$0 > (z_0^*, g(x^1)) \geq 0,$$

because of (1.24).

(ii) If $x_0 \in \mathcal{A}$ and $f(x_0) = y_0 \in \mathrm{Eff}(f[\mathcal{A}], C_Y)$ then (1.23) implies

$$(y_0^*, y_0) \leq (y_0^*, f(x_0)) + (z_0^*, g(x_0)) \leq (y_0^*, f(x_0)) = (y_0^*, y_0)$$

such that $(y_0^*, f(x_0)) + (z_0^*, g(x_0)) = \inf\{(y_0^*, f(x)) + (z_0^*, g(x)) \mid x \in M\}$ and $(z_0^*, g(x_0)) = 0$. □

Remark 1 *Conversely, if $x_0 \in M$ is a minimal solution of the Lagrangean $(y_0^*, f(.)) + (z_0^*, g(.))$ with $g(x_0) \in -C_Z$ and $(z_0^*, g(x_0)) = 0$, then*

$$f(x_0) \in w\text{-}\mathrm{Eff}(f[\mathcal{A}], C_Y)$$

follows without regularity assumption:

$$\begin{aligned}(y_0^*, f(x_0)) &= (y_0^*, f(x_0)) + (z_0^*, g(x_0)) \\ &\leq (y_0^*, f(x)) + (z_0^*, g(x)) \leq (y_0^*, f(x))\end{aligned}$$

for all $x \in M$ with $g(x) \in -C_Z$ and $f(x_0) \in w\text{-}\mathrm{Eff}(f[\mathcal{A}], C_Y)$ regarding $y_0^* \in C_{Y^*} \setminus \{0\}$.

Theorem 11 *Suppose that (A1), (A2), and (A3) are fulfilled. Assume $x_0 \in M$. Then it holds:*

(i) *If $f(x_0) \in \mathrm{Eff}(f[\mathcal{A}], C_Y)$ then there exist $y_0^* \in C_{Y^*} \setminus \{0\}$ and $z_0^* \in C_{Z^*}$ such that the following saddle point assertion is fulfilled:*

$$\begin{aligned}(y_0^*, f(x_0)) + (z^*, g(x_0)) &\leq (y_0^*, f(x_0)) + (z_0^*, g(x_0)) \quad (1.25) \\ &\leq (y_0^*, f(x)) + (z_0^*, g(x)) \; \forall x \in M, \forall z^* \in C_{Z^*}.\end{aligned}$$

(ii) *Conversely, if there are $y_0^* \in C_{Y^*} \setminus \{0\}$ and $(x_0, z_0^*) \in M \times C_{Z^*}$ such that the saddle point assertion (1.25) is fulfilled for all $x \in M$ and $z^* \in C_{Z^*}$, then $f(x_0) \in w\text{-}Eff(f[\mathcal{A}], C_Y)$.*

Proof:

(i) Assume $f(x_0) \in \mathrm{Eff}(f[\mathcal{A}], C_Y)$. Using Lemma 2(ii), we get that there exist $y_0^* \in C_{Y^*} \setminus \{0\}$ and $z_0^* \in C_{Z^*}$ with

$$(y_0^*, f(x_0)) + (z_0^*, g(x_0)) \leq (y_0^*, f(x)) + (z_0^*, g(x)) \; \forall \, x \in M.$$

Furthermore, regarding $-g(x_0) \in C_Z$, it follows again with Lemma 2(ii),

$$(z^*, g(x_0)) \leq 0 = (z_0^*, g(x_0)) \quad \forall z^* \in C_{Z^*}.$$

This yields

$$(y_0^*, f(x_0)) + (z^*, g(x_0)) \leq (y_0^*, f(x_0)) + (z_0^*, g(x_0)) \; \forall \, z^* \in C_{Z^*}.$$

Then both inequalities are fulfilled.

(ii) Suppose $y_0^* \in C_{Y^*} \setminus \{0\}$ and assume that the saddle point assertion is fulfilled for $(x_0, z_0^*) \in M \times C_{Z^*}$. Then the first inequality implies

$$(z^*, g(x_0)) \leq (z_0^*, g(x_0)) \quad \forall z^* \in C_{Z^*}$$

such that we get regarding that C_{Z^*} is a convex cone

$$\begin{aligned}(z^* + z_0^*, g(x_0)) \leq (z_0^*, g(x_0)) \quad \forall z^* \in C_{Z^*},\\ (z^*, g(x_0)) \leq 0 \quad \forall z^* \in C_{Z^*}\end{aligned}$$

and $g(x_0) \in -C_Z$. This implies

$$0 \geq (z_0^*, g(x_0)) \geq (0, g(x_0)) = 0$$

since $0 \in C_{Z^*}$ and so $(z_0^*, g(x_0)) = 0$. Consider now $x \in M$ with $g(x) \leq 0$, then we conclude from the second inequality in the saddle point assertion

$$\begin{aligned}(y_0^*, f(x_0)) &= (y_0^*, f(x_0)) + (z_0^*, g(x_0))\\ &\leq (y_0^*, f(x)) + (z_0^*, g(x)) \leq (y_0^*, f(x))\end{aligned}$$

and

$$(y_0^*, f(x_0)) \leq (y_0^*, f(x)) \qquad \forall x \in \mathcal{A}.$$

This means $f(x_0) \in w\text{-}\operatorname{Eff}(f[\mathcal{A}], C_Y)$. □

Remark 2 *A point $(x_0, z_0^*) \in M \times C_{Z^*}$ satisfying the property (1.25) for an element $y_0^* \in C_{Y^*} \setminus \{0\}$ is called a* **y_0^*-saddle point** *of the Lagrangean*

$$\Phi(x, z^*) := (y_0^*, f(x)) + (z^*, g(x)), \quad x \in M,\ z^* \in C_{Z^*}.$$

The relation (1.25) can be described by

$$\begin{aligned}\Phi(x_0, z_0^*) &\in \operatorname{Min}(\{\Phi(x, z_0^*) : x \in M\}, y_0^*),\\ \Phi(x_0, z_0^*) &\in \operatorname{Max}(\{\Phi(x_0, z^*) : z^* \in C_{Z^*}\}, y_0^*).\end{aligned}$$

Remark 3 *Taking $M = R_+^2 \subset Y = R^2$, $C_Y = C_Z = R_+^2$, $Y = Z = R^2$, $f = I$ (identity), $g(x) = -x\ \forall x \in M$, we have $\mathcal{A} = \{x \in R_+^2\}$ and all assumptions of Theorem 11 are satisfied. Then $x_0 = (0,1)^T$, $y_0^* = (1,0)^T$, $z_0^* = (0,0)^T$ is a y_0^*-saddle point of the Lagrangean Φ : $\Phi(x, z^*) = (y_0^*, f(x)) + (z^*, g(x))$, $x \in M$, $z^* \in C_{Z^*}$ since $0 + (z^*, -x_0) \leq 0 \leq (x)_1$. x_0 is only weakly efficient as proved in the theorem. So we cannot expect a symmetrical assertion of the kind "saddle-point iff efficiency".*

2.3. Conclusions

The application of vector-valued variational principles of Ekeland's type for the characterization of approximately efficient elements for special classes of multicriteria optimization problems seems to be a successful and prospective research direction. Furthermore, the study of set-valued variants of Ekeland's variational principle including applications is a young and growing field.

3. Duality

Keywords: Conjugation, Lagrangean, Axiomatic duality, Dual problem.

It is an old idea to try to complement a given optimization problem ($f(x) \to \min$ with minimal value I) by a dual problem ($g(y) \to \max$ with supremal value S, $S \leq I$), remember the dual variational principles of Dirichlet and Thompson (cf. Zeidler [305]) or e.g. the paper of K.O. Friedrichs [98] or simply the pair of dual programs in linear optimization. The reasons for the introduction of a useful dual problem are the following:

- The dual problem has (under additional conditions) the same optimal value as the given "primal" optimization problem, but solving the dual problem could be done with other methods of analysis or numerical mathematics.

- An approximate solution of the given minimization problem gives an estimation of the minimal value I from above, whereas an approximate solution of the dual problem is an estimation of I from below, so that one gets intervals which contain I.

- Recalling Lagrange method, saddle points, equilibrium points of two person games, shadow prices in economics, perturbation methods or dual variational principles, it becomes clear that optimal dual variables often have a special meaning for the given problem.

Of course, the just listed advantages require a skillfully chosen dual program. Nevertheless, the mentioned points are motivation enough to look for dual problems in multicriteria optimization, too. There are a lot of papers, which are dedicated to that aim, also a lot of survey papers (see Jahn [146], Luc [190]). There are different approaches to duality:

Conjugation: Schönfeld [237], Breckner [34, 35], Zowe [306, 307], Nehse [200], Rosinger [231], Tanino and Sawaragi [269], Brumelle [40],

Kawasaki [156, 157], Gerstewitz and Göpfert [107], Sawaragi, Nakayama and Tanino [233], Luc [190], Zalinescu [300].

Lagrangean: Corley [71, 72], Bitran [22], Gerstewitz and Iwanow [106], Göpfert and Gerth [116], Nehse [200], Jahn [143, 146, 148], Iwanow and Nehse [141], Nakayama [198, 199], Sawaragi, Nakayama, and Tanino [233], Luc [190].

Axiomatic Duality: Luc [190], Luc and Jahn [189].

We explain some ideas and give examples.

3.1. Duality Without Scalarization

Let Y be a linear topological space partially ordered by a convex pointed cone C, $C^* := \{y^* \in Y^* \mid y^*(c) \geq 0 \ \forall\, c \in C\}$ the dual cone to C, $\mathcal{P}, \mathcal{D}$ nonempty subsets of Y, and let us consider the multicriteria problems

$$(P) \qquad \mathrm{Eff}_{min}(\mathcal{P}, C),$$
$$(D) \qquad \mathrm{Eff}_{max}(\mathcal{D}, C) := \mathrm{Eff}_{min}(\mathcal{D}, -C).$$

We speak of a pair of **weakly dual problems**, if

$$\mathcal{P} \cap (\mathcal{D} - (C \setminus \{0\})) = \emptyset. \tag{1.26}$$

Since C is pointed, this is equivalent to $(\mathcal{P}+(C\setminus\{0\}))\cap(\mathcal{D}-(C\setminus\{0\})) = \emptyset$. (P) and (D) are called strongly dual, if (1.26) holds together with $0 \in cl(\mathcal{P} - \mathcal{D})$ or equivalently $(\mathcal{P} + O) \cap (\mathcal{D} + O) \neq \emptyset$ for all open neighbourhoods O of zero in Y. So strong duality means that $\mathcal{P}$ and $\mathcal{D}$ touch each other or with other words, $\mathcal{P}$ and $\mathcal{D}$ don't overlap. Otherwise we speak of a pair of dual problems with a duality gap (in the scalar case at the beginning of this chapter that would mean $I > S$). Having a pair of strongly dual multicriteria programs their feasible elements respectively give estimations of the set of efficient elements of (P) "from above and below" (with respect to C).

Lemma 3 *Assume (P), (D) to be weakly dual. If there are $z^0 \in \mathcal{P}, \zeta^0 \in \mathcal{D}$ such that $z^0 = \zeta^0$, then z^0 is minimal for (P), ζ^0 maximal for (D) and (P), (D) are strongly dual.*

Proof: z^0 not minimal means that there is $z^1 \in \mathcal{P}$ such that $z^1 \in z^0 - (C \setminus \{0\}) = \zeta^0 - (C \setminus \{0\})$, which contradicts (1.26). ζ^0 to be maximal follows similarly. From $z^0 = \zeta^0$ it follows that (P) and (D) are strongly dual. □

To construct dual programs one uses – similar to ordinary "scalar" programming – Lagrange technique. We apply such methods, for other

approaches compare Jahn [146] and Luc [190]. The question is whether to apply scalarization methods or not. We present at first duality theorems without taking into account scalarization of the objective function.

Instead of (P) we consider the following multicriteria problem with side constraints

$$(P_1) \quad \mathrm{Eff}_{min}(f(\mathcal{A}), C), \tag{1.27}$$

with

$$\mathcal{A} := \{x \in M, g(x) \in C_V\},$$

where X is a linear space, M a non-empty set in X, (V, C_V) a partially ordered linear topological space, (Y, C) as above. We add to Y (V) an element ∞ not belonging to the space Y (V), obtaining thus the space $Y^\bullet$: $Y^\bullet = Y \cup \{\infty\}$ $(V^\bullet$: $V^\bullet = V \cup \{\infty\})$. We consider $f : X \to Y^\bullet, dom f = M, g : X \to V^\bullet, dom\ g = M$. So a vector optimization problem is given as usual. To deal with it, we suppose, that for C *in* Y we choose an open and convex set $B \subset Y, B \neq Y$ such that

$$0 \in bdB,\ cl\ B + (C \setminus \{0\}) \subset B. \tag{1.28}$$

In particular, $cl\ B$ could be a convex cone, such that B contains $C \setminus \{0\}$, because the second inclusion in (1.28) is equivalent to $C \setminus \{0\} \subset B$. So if $dim\ Y < \infty$ such a cone B exists provided C is closed (and pointed).

We introduce the following generalized Lagrangean, having another set M^{0*},

$$L : M \times M^{0*} \longrightarrow Y \cup \{+\infty_Y\}$$

and assume

$$L(x, y) \in f(x) - cl\ B$$

for all $x \in \mathcal{A}$ and for all $y \in M^{0*}$.

Now we are able to write down a problem (D_1) which can be considered as dual problem to (P_1) :

$$(D_1) \quad \mathrm{Eff}_{max}(f^*(M^*), C), \tag{1.29}$$

with

$$M^* := \left\{y \in M^{0*} \ :\ Eff_{min}(L(M, y), B) \neq \emptyset\right\},$$

where B is as explained above, and $f^*(y) \in \text{Eff}_{min}(\{L(x,y) : x \in M\}, B) \neq \emptyset \subset Y$.

(D_1) is a so-called Lagrange dual problem to (P_1) as is easy to see, if we reduce to the finite dimensional scalar case $(Y, C) = (R^1, R^1_+), x \in R^n, (V, C_V) = (R^m, R^m_+), B = int\ R^1_+$ and if we take y as linear mapping λ. Then instead of (P_1) we get

$$f(x) \to \min \text{ s.t. } x \in \mathcal{A} = \{x \in M, g(x) \in R^m_+\}.$$

So, with $M^* = \{\lambda \in -R^m_+ : \exists x_0 \in M : (f+\lambda^T g)(x) \geq (f+\lambda^T g)(x_0)\ \forall x \in M$ and $f^*(\lambda) = \min\limits_{x \in M} (f+\lambda^T g)(x) > -\infty\}$ (D_1) has the well-known max-min form:

$$\max_{\lambda \in M^*} f^*(\lambda) = \max_{\lambda \in M^*} \min_{x \in M} (f + \lambda^T g)(x).$$

Lemma 4 (Weak duality) *(P_1) and (D_1) are weakly dual, i.e.,*

$$f(\mathcal{A}) \cap (f^*(M^*) - (C \setminus \{0\})) = \emptyset. \tag{1.30}$$

A strong duality theorem holds, too. Therefore we assume a condition (V1):

$$(f + y \circ g)(M) \subset \text{Eff}_{min}((f + y \circ g)(M), B) + (B \cup \{0\})$$

for all $y \in M^*$. Since one can choose the wrapping set B (or cone B) very close to $C/\{0\}$, (V1) means, that $(f+y \circ g)\ (M)$ cannot have improperly efficient elements with respect to C. This is reflected by the formulation of the next theorem which takes into account only efficient points of (P_1) with respect to a set B which fulfills (1.28) referring to C.

Theorem 12 (Strong duality theorem) *For (P_1) and (D_1) as given above, let (V1) be fulfilled and assume $f(\overline{x}) \in \text{Eff}_{min}(f(\mathcal{A}), B)$ to be in $\text{Eff}_{min}(f(\mathcal{A}), C)$. Then $f(\overline{x}) \in \text{Eff}_{max}(f^*(M^*), C)$.*

Sometimes results like Theorem 12 are called a strong **direct** duality theorem, because a primal optimal solution is shown to be dually optimal. The converse direction is also interesting. This leads to converse duality theorems. To get such a theorem for our pair of optimization problems, we state a condition (V2):

(V2) Every solution of (D_1) is to be dominated by a properly efficient solution of (P_1), that means with a set B as in (1.28), for all $\tilde{d} \in \text{Eff}_{max}(f^*(M^*), C)$ there is an $f(\tilde{x}) \in \text{Eff}_{max}(\mathcal{A}), B)$ such that $f(\tilde{x}) \in \tilde{d} + C$.

Theorem 13 (Converse duality theorem) *Assume that both (V1) and (V2) hold and* $\tilde{d} \in \mathrm{Eff}_{max}(f^*(M^*), C)$. *Then there are* $\overline{x} \in \mathcal{A}, B' \subset Y$ *with* $clB + B' \subset B$ *such that* $\tilde{d} = f(\overline{x}) \in \mathrm{Eff}_{min}(f(\mathcal{A}), B')$.

The duality in the last section is an abstract and nonscalarized Lagrangean formalism for very general optimization problems. In Section 3.3 we will present pairs of dual problems for special classes of vector optimization problems.

3.2. Duality by Scalarization

Consider (P) in the following form

$$(P_2) \quad \mathrm{Eff}(f_{min}[\mathcal{A}], C),$$

with

$$\mathcal{A} = \{x \in M, g(x) \in C_V\} \neq \emptyset,$$

where $f : M \to Y, \emptyset \neq M \subset X, g : M \to V, C_V$ is a nonempty set in V and X, Y, V are topological vector spaces, Y Hausdorff, Y partially ordered by C closed and with nonempty interior. Additionally, Y is to be directed with respect to $k^0 \in C \setminus \{0\}$, i.e., $Y = \cup\{C + \lambda k_0 : \lambda \in R\}$. Let S be the set of sublinear surjective continuous strictly monotone (with respect to C) decreasing functional $s : Y \to R$. Assume S is nonempty. Again we introduce a generalized Lagrangean $L : M \times M^* \longrightarrow Y \cup \{+\infty_Y\}$ and assume for $s \in S$

$$\sup_{y \in M^*} s(L(x, y)) = s(f(x)) \text{ if } g(x) \in C_V.$$

Using the functionals $s \in S$ we define a dual problem to (P_2) :

$$(D_2) \quad \mathrm{Eff}_{max}(\mathcal{A}_D, C) \tag{1.31}$$

where

$$\begin{aligned} \mathcal{A}_D \quad = \quad & \{h \in Y : \exists s \in S, \exists y \in M^* \text{ with} \\ & s(h) = \inf\{s(L(x, y)) : x \in M\} > -\infty\}. \end{aligned}$$

It is possible to prove for the pair (P_2), (D_2) weak, strong direct and strong converse duality:

Lemma 5 (Weak duality) *The pair* $(P_2), (D_2)$ *fulfills*

$$f(\mathcal{A}) \cap (\mathcal{A}_D - (C \setminus \{0\})) = \emptyset.$$

For **properly efficient elements** $z_0 = f(x_0)$, i.e., there is an $s_0 \in S$ such that $s_0(f(x_0)) = \inf\{s_0(f(x)), x \in \mathcal{A}\} > -\infty$, a **strong duality theorem** can be proved.

Theorem 14 (Strong (direct)duality)) *If $z_0 = f(x_0)$ with $x_0 \in \mathcal{A}$ is properly efficient for (P_2), then it is efficient for (D_2), i.e., $z_0 \in \mathrm{Eff}_{max}(\mathcal{A}_D, C)$.*

Under some more conditions also a converse duality theorem holds, that is, having z_0 dually efficient then there is $x_0 \in \mathcal{A}$ such that $f(x_0)$ is efficient for (P_2). In order to prove such a converse duality theorem we use a characterization of efficient elements $h \in \mathcal{A}_D$ by scalarization.

Lemma 6 *$h^0 \in \mathrm{Eff}_{max}(\mathcal{A}_D, C)$ iff $s_h(h) \leq s_h(h^0)\ \forall\ h \in \mathcal{A}_D$ where s_h corresponds to h according to the definition of $\mathcal{A}_D$ in (D_2).*

Additionally a strong converse duality statement holds.

Theorem 15 (Strong converse duality) *Under the conditions given for the pair $(P_2), (D_2)$ and if, additionally, $P := f(\mathcal{A}) + C$ is closed and if for all $s \in S$ such that $\inf\{s(f(x)) : x \in \mathcal{A}\} > -\infty$ there is an $x_0 \in \mathcal{A}$ with $\inf\{s(f(x)) : x \in \mathcal{A}\} = s(f(x_0))$ then a dually efficient element is properly primal efficient.*

3.3. Duality Assertions for Vector-valued Approximation Problems

For special classes of vector optimization problems it is possible to derive a useful dual problem explicitly. Consider a vector-valued approximation problems of the following form:

$$(P_3) \quad \mathrm{Eff}(f_{min}[\mathcal{A}], C),$$

with

$$\mathcal{A} := \{x \in C_X : Bx - b \in C_V\},$$

where now X is a linear normed space, partially ordered by a cone C_X, V is a reflexive Banach space, C_V a cone in V, $B \in L(X, V), Y = R^p$, partially ordered by a cone C such that $C + R^p_+ \subset C$, all cones closed, convex, pointed, and $int\, C^* \neq \emptyset$, and f may have the form

$$f(x) = f_1(x) + \begin{pmatrix} \alpha_1 \|A^1 x - a^1\| \\ \dots \\ \alpha_p \|A^p x - a^p\| \end{pmatrix} \tag{1.32}$$

with $f_1 \in L(X, R^p), \alpha_i \geq 0$ real $(i = 1, \ldots, p)$, and $a^i \in U$, a given real normed space, $A^i \in L(X, U), i = 1, ..., p$.

(P_3), although a special case of (P_1) or (P_2) contains itself important special cases:

(i) (P_3) is a semi-infinite linear problem, if $\alpha_i = 0 \; \forall \; i = 1, ..., p$.

(ii) f can be interpreted as Lipschitz perturbed linear problem,

(iii) $f_1 = 0$ gives a multicriteria location problem.

(iv) Consider $f_1(x) \to \min, x \in \mathcal{A}_0 = \{x \in \mathcal{A} \neq \emptyset, A^i x = a^i, i = 1, ..., p\} = \emptyset$. Then (P_3) is a parameterized surrogate problem.

(v) The general multicriteria approximation problem or location problem (cf. Gerth and Pöhler [109], Jahn [146], Tammer, K. and Tammer, C. [261] and Wanka [276, 277, 278]).

In order to use the above ideas one introduces a suitable Lagrangean L_{λ^0} to (P_3) for a given $\lambda^0 \in int\, C^*$:

$$L_{\lambda^0}(x, Y, u^*) = \lambda^0 \left(\left(\begin{array}{c} \cdots \\ \alpha_i Y^i (a^i - A^i x) \\ \cdots \end{array} \right) + \left(\begin{array}{c} \cdots \\ f_{1i}(x) \\ \cdots \end{array} \right) \right) + u^*(b - Bx),$$

where

$$x \in M = C_X$$

and

$$\begin{aligned} (Y, u^*) \in M^{0*} \quad := \quad & \{(Y, u^*) : \; Y = (Y^1, ..., Y^p), \; Y^i \in L(U_i, R), \\ & u^* \in L(U_i, R), u^* \in C_V^*, \\ & \alpha_i ||Y^i||_* \leq \alpha_i, \; i = 1, ..., p\}. \end{aligned}$$

From this setting and using Jahn's descalarization result (cf. Jahn [146]) one gets a dual program (D_3) to (P_3) with (strong duality if $b \neq 0$):

$$(D_3) \quad \text{Eff}_{max}(\mathcal{A}_D, C),$$

where

$$\mathcal{A}_D := \left\{ d = \left(\begin{array}{c} \cdots \\ \alpha_i Y^i (a^i) \\ \cdots \end{array} \right) + Zb \text{ for } (Y, Z) \in M^* \right\}$$

and

$$M^* = \Big\{ (Y,Z) : Y = (Y^1,...,Y^p), Y^i \in L(U_i,R),\ i = 1,...,p,$$
$$Z \in L(V,R^p),\ \exists \lambda^* \in int\, C^* \text{ such that}$$
$$\sum_{i=1}^{p} \lambda_i^*(-\alpha_i A^{i*} Y^i + f_{1i}) - (ZB)^* \lambda^* \in C_X^*,$$
$$Z^*\lambda^* \in C_V^*, \alpha_i \|Y^i\|_* \leq \alpha_i,\ i = 1,...,p \Big\}.$$

We add some simple scalar examples.

Example 5 *Consider instead of* (P_3) *the following scalar optimization problem*

$$(P_4) \quad f_1(x) + \sum_{i=1}^{p} \alpha_i \|A^i x - a^i\| \to \min$$
$$s.t.\ x \in C_X, Bx - b \in C_V,$$

where $X, V, C_X, C_V, A^i, a^i (i = 1,...,p)$ *are as above,* $f_1 \in L(X, R^1)$ *and* $\alpha_i > 0$ *for all* $i = 1, \ldots, p$. *Since the sum in* (P_4) *is really a norm,* (P_4) *is an example of* (P_3). *Then we get a dual problem* (D_4) *as a special case of* (D_3) :

$$(D_4) \quad \sum_{i=1}^{p} \alpha_i y^i(a^i) + z(b) \to \max$$
$$s.t.\ y = (y^1,...,y^p) \in L(U,R)^p$$
$$\|y^i\|_* \leq 1\ \forall\, i, z \in L(V,R), z \in C_V^*,$$
$$\sum_{i=1}^{p} (-\alpha_i A^{i*} y^i + f_1) - B^* z \in C_X^*.$$

(D_4) *follows immediately from* (D_3) *since* $\|y\|_* \leq 1$ *in* (D_3) *means* $\max_{i=1,...,p} \|y^i\|_* \leq 1$ *and so* $\|y^i\|_* \leq 1$ *for all* i, *because the maximum norm is suitable as dual norm to a sum of norms.*

Example 6 *The ordinary scalar classic location problem is contained in* $(P_4) : X, \alpha_1, ..., \alpha_p, a_1, ..., a_p$ *as above,* $a_1, ..., a_p$ *fixed.*

$$(P_5) \quad \sum_{i=1}^{p} \alpha_i \|x - a^i\| \to \min_{x \in X} \quad \textit{(Fermat's problem)},$$

$$(D_5) \quad \sum_{i=1}^{p} \alpha_i y^i(a^i) \to \max$$

$$s.t.\ y^i \in X^*, \|y^i\|_* \leq 1\ \forall i,\ \sum_{1}^{p} \alpha_i y^i = 0.$$

Also the following linear scalar approximation problem (P_6) *together with* (D_6) *is a special case of the pair* $(P_4), (D_4)$ *: Let* A *be a convex closed subset of* X *and* $a \in X$ *fixed. Consider the problem* $\|x - a\| \to \min$ *s.t.* $x \in \mathcal{A}$. *Then* (D_4) *reduces to*

$$y(a) \to \max y \in X^*, \|y\|_* \leq 1.$$

Taking as $\mathcal{A}$ *a closed linear subspace* $M \subset X$, *we get*

$$(P_6) \quad \|x - a\| \to \min \ \ s.t. x \in M$$
$$(D_6) \quad y(a) \to \max \ \ s.t.\ y \in M^{\perp}, \|y\|_* \leq 1,$$

where $M^{\perp}$ *is the* **annihilator** *to* $M : M^{\perp} = \{y \in X^* : y(x) = 0\ \forall\ x \in M\}$.

Example 7 *We consider the following vector-valued location problem*

$$(P_7) \quad \mathrm{Eff}_{min}(f[R^2], C),$$

with

$$f(x) := \begin{pmatrix} || x - a^1 ||_{max} \\ || x - a^2 ||_{max} \\ \cdots \\ || x - a^p ||_{max} \end{pmatrix},$$

where $x, a^i \in R^2$ $(i = 1, ..., p)$ *and*

$$|| x ||_{max} = max\{| x_1 |, | x_2 |\}.$$

For applications in town planning it is important that we can choose different norms in the formulation of (P_7). *The decision which of the norms will be used depends on the course of the roads in the city or in the district.*

In the following we study the problem (P_7) *with* $X = R^2$, $C = R^p_+$, *where* R^p_+ *denotes the usual ordering cone in the p-dimensional Euclidean space.*

Using duality assertions we will present an algorithm for solving (P_7) *(compare Chalmet, Francis and Kolen [54], Gerth (Tammer) and Pöhler [109]). In [109] we have derived the following dual problem for* (P_7)*:*

$$(D_7): \quad \mathrm{Eff}_{max}(f^*[\mathcal{B}], C),$$

with

$$f^*(y) = \begin{pmatrix} Y^1 a^1 \\ \cdots \\ Y^n a^n \end{pmatrix},$$

where

$$\mathcal{B} = \left\{ Y = (Y^1, ..., Y^p), Y^i \in \mathcal{L}(R^2, R) : \exists \lambda \in int\ R^p_+ \text{ with } \sum_{i=1}^{n} \lambda_i^* Y^i = 0, \text{ and } \| Y^i \|_* \leq 1 \quad (i = 1, ..., p) \right\}.$$

Here $\| \cdot \|_*$ *denotes the Lebesgue-norm. We can use the conditions* $\sum_{i=1}^p \lambda_i^* Y^i = 0$, *and* $\| Y^i \|_* \leq 1 \quad (i = 1, ..., p)$ *in order to derive an algorithm (cf. [109]). Consider the following sets with respect to the given facilities* $a^i \in R^2$ $(i = 1, ..., p)$ *which are related to the structure of the subdifferential of the maximum norm:*

$$\begin{aligned}
s_1(a^i) &= \{x \in R^2 \mid a_1^i - x_1 = a_2^i - x_2 \geq 0\}, \\
s_2(a^i) &= \{x \in R^2 \mid a_1^i - x_1 = a_2^i - x_2 \leq 0\}, \\
s_3(a^i) &= \{x \in R^2 \mid a_1^i - x_1 = x_2 - a_2^i \geq 0\}, \\
s_4(a^i) &= \{x \in R^2 \mid a_1^i - x_1 = x_2 - a_2^i \leq 0\}, \\
s_5(a^i) &= \{x \in R^2 \mid a_2^i - x_2 > \mid a_1^i - x_1 \mid\}, \\
s_6(a^i) &= \{x \in R^2 \mid x_2 - a_2^i > \mid a_1^i - x_1 \mid\}, \\
s_7(a^i) &= \{x \in R^2 \mid a_1^i - x_1 > \mid a_2^i - x_2 \mid\}, \\
s_8(a^i) &= \{x \in R^2 \mid x_1 - a_1^i > \mid a_2^i - x_2 \mid\}.
\end{aligned}$$

Moreover, we introduce the sets

$$\mathcal{S}_r := \{x \in \mathcal{N} \mid \exists i \in \{1, ..., p\} \text{ and } x \in s_r(a^i)\}$$

$(r = 5,6,7,8)$, *where* $\mathcal{N}$ *denotes the smallest level set of the dual norm to the maximum-norm (Lebesgue-norm) containing the points* a^i $(i = 1, ..., p)$.

Now, we are able to describe the following algorithm for solving the vector-valued location problem (see Gerth and Pöhler [109]):

$$\begin{aligned}\mathrm{Eff}(f(X), C) \;=\; & \{(cl\ \mathcal{S}_5 \cap cl\ \mathcal{S}_6) \cup [(\mathcal{N} \setminus \mathcal{S}_5) \cap (\mathcal{N} \setminus \mathcal{S}_6)]\} \;\cap \\ & \{(cl\ \mathcal{S}_7 \cap cl\ \mathcal{S}_8) \cup [(\mathcal{N} \setminus \mathcal{S}_7) \cap (\mathcal{N} \setminus \mathcal{S}_8)]\}.\end{aligned}$$

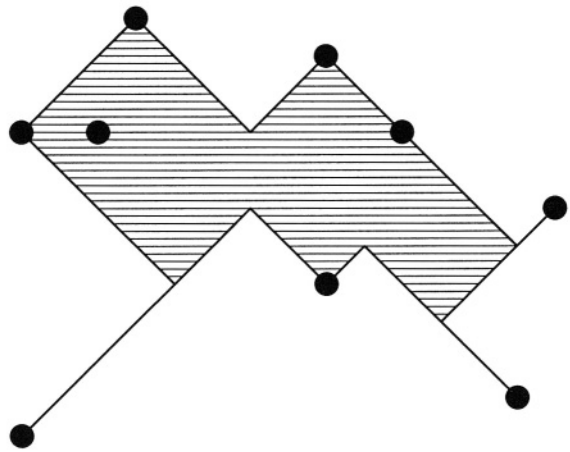


Figure 1.1. The set of efficient elements of a vector-valued location problem (P_7) with the maximum norm.

3.4. Conclusions

There are many papers dealing with duality concepts (conjugation, Lagrangean and axiomatic duality) on the base of scalarization as well as without scalarization. In order to derive algorithms or inclusions for solutions (bounds) it is important to have duality assertions for special classes of vector optimization problems. Especially, it is essential to find useful dual problems, corresponding duality assertions and their application for the construction of primal-dual algorithms in the case of not necessarily convex multicriteria optimization problems. Very often a skillful interpretation of found dual solutions or Lagrange variables can be useful.

4. Vector Variational Inequalities and Vector Equilibria

Keywords: Existence results, Mathematical Economy, Game theory.

4.1. Vector Variational Inequalities and Vector Equilibrium Problems

It is well known that optimization and nonlinear analysis are two branches of modern mathematics much developed lately. The results obtained in these areas use certain kinds of differentiability (directional derivative, subgradient, generalized subgradients, etc.), certain generalizations of convexity, geometrical methods (cone of feasible directions, normal and tangent cones, etc.), game theory, fixed point theory, topological degree, etc.

In recent years equilibrium problems have come to play a central role in the development and unification of diverse areas of mathematics, economics and physical sciences. Thus various problems of practical interest in optimization, variational inequalities, complementarity, economics, Nash equilibria and engineering involve equilibrium in their description.

The vector variational inequalities have been widely developed in recent years, and various solutions have been characterized and computed. These were first introduced by Giannessi [111] and further developed by many authors in different areas.

Recent topics attracting considerable attention are equilibrium problems for vector-valued mappings. Inspired by the scalar case, such problems have received different developments depending on the kind of order space where these have been considered.

There are different approaches to establish the existence of solutions of equilibrium problems in the vector case. The first one directly used a generalization of the well known lemma of Knaster, Kuratowski, and Mazurkiewicz. The second one, as proposed by Oettli, leads to deduce, in a straightforward way, existence results for vector equilibrium problems from the results about scalar case. A key tool for the study of such problems is an appropriate gauge function.

The published papers could be grouped in the following way:

- Theory of vector optimization, vector equilibrium problems and vector variational inequalities (Ansari [3, 4, 5, 6]; Ansari and Siddiqi [7]; Ansari, Siddiqi and Yao [8]; Bianchi, Hadjisavvas and Schaible [19]; Blum and Oettli [25, 26]; Chen and Craven [57, 58]; Conway [69]; Fan [94]; Fu [99]; Konnov and Yao [163]; Lee and Kum [176]; Lee, G.M., Kim, and Lee, B.S. [172, 173, 179, 180]; Lee, G.M., Kim, Lee, B.S., and Cho [174]; Lee and Kum [178]; Li, Yang and Chen [182]; Lin [182]; Lin, Yang, and Yao [184]; Noor [207]; Oettli [209]; Siddiqi, Ansari, and Ahmad [240]; Rapcsak [228]; Yang [285, 286, 287, 288, 289]; Yang and Chen [290]; Yang and Goh [291], Yao [293, 294]; Yu and Yao [298]).

- Existence of solutions for generalized vector variational inequalities and complementarity problems (Chang, Thompson, and Yuan [54]; Chen [55]; Chen and Hou [60]; Chen and Yang [64, 65]; Danilidis and Hadjisavvas [76]; Isac and Yuan [140]; Kazmi [158]; Lai and Yao [170]; Yin and Xu [297]; Qun [226]).

- Vector variational inequalities and vector equilibrium problems with set-valued mappings (Ding and Tarafdar [83]; Fu [100]; Parida and Sen [210]; Siddiqi, Ansari, and Khan [241]; Song [244])

- Vector Variational Inequalities, Vector Optimization and Scalarization (Chen, G.-Y. and Chen, G.M. [56]; Chen and Craven [58]; Giannessi, Mastroeni and Pellegrini [112]; Goh and Yang [114]; Lee, G.M., Kim, Lee, B.S., and Yen [175]).

- Stability of the solution sets (Yen and Phuong [296] and Yen [295]).

- Monotone vector variational inequalities (Chowdhury and Tan [67]; Hadjisavvas and Schaible [129], Ding and Tarafdar [83]; Yen and Lee [295]).

4.2. Conclusions

Vector variational inequalities, vector equilibrium problems and vector complementarity problems contain vector optimization or Lagrangean problems or game theoretical problems as special cases, so they become more and more important for the modelling of more complicated practical problems in Economy, Engineering and Biosciences. Many authors have proved existence theorems for solutions of such problems. In recent papers generalizations of the vector variational inequalities are considered. Further research on these topics will concern:

- Refinements of the existence conditions for solutions of vector variational inequalities, vector equilibrium problems and vector complementarity problems;

- Solution methods and properties of solution sets such as sensitivity, stability or a-priori-estimations;

- Practical applications, e.g. in Mathematical Economy.

5. Multicriteria Fractional Programming

Keywords: Multicriteria fractional programming, Dinkelbach-transformed problem, Dialogue procedure.

5.1. Approximate Solutions in Multicriteria Fractional Programming

Many aims in real decision problems can be expressed by a fractional objective function (cf. [234]) so that the field of fractional optimization is very up to date.

Example 8 *The problem in economics to minimize a cost functional* $f_1(x) = a^T x$ *and to maximize profitability* $f_2(x) = b^T x / c^T x$, *where* $x, a, b, c \in R^n$, *can be formulated as the problem to determine the set of efficient elements of a multicriteria fractional programming problem*

$$\mathrm{Eff}(f[X], R^2_+),$$

with $f(x) = (f_1(x), f_2(x))$ *and* $X \subseteq R^n$.

For the case of optimization problems with only one fractional objective function Dinkelbach [83] has proposed a parametric solution approach. This approach is based on the relation to a special parametric problem, which is described without the original ratios. However, it requires, additionally, the generation of that unknown parameter value, for which equivalence to our original problem holds. Many other authors have already published results to generalize Dinkelbach's idea also for efficient and properly efficient solutions of vector optimization problems with *m* fractional objective functions (cf. Bector and Chandra [12], Kaul and Lyall [155], Tammer, K., Tammer, C., and Ohlendorf [262] and Weir [282]). But some of those results are not entirely correct.

The aim of this section is to extend the results of Dinkelbach and other authors to different sets of approximately efficient and properly efficient solutions of multicriteria fractional optimization problems, which were introduced by Tammer [253] as well as Dentcheva and Helbig [80]. As a by-product we obtain the corrected formulations of corresponding results for the exact solutions.

The main part of this section is devoted to the mentioned relations between the (approximate) solutions of the original multicriteria fractional problem and the transformed one. Moreover, we discuss possibilities to solve the transformed problem by a three-level dialogue approach following ideas of the book [126].

Consider for $C = R^m_+$ a vectorial fractional optimization problem Eff ($f(X)$, C), i.e., the problem

$$(\mathcal{P}_f) \quad f(x) = \frac{g(x)}{h(x)} \rightarrow v\text{-}\min \text{ subject to } x \in X \subseteq R^n,$$

where

$$\frac{g(x)}{h(x)} := \left(\frac{g_1(x)}{h_1(x)}, ..., \frac{g_m(x)}{h_m(x)}\right)^T$$

and $h_i(x) > 0 \; \forall \, x \in X, \; i = 1, ..., m.$

We show, that $(\mathcal{P}_f)$ is closely related to a multiparametric vector optimization problem $\mathcal{P}(\lambda)$, which we call the corresponding Dinkelbach-transformed problem, namely

$$\mathcal{P}(\lambda) \quad H(x, \lambda) \; \rightarrow v\text{-min subject to } x \in X,$$

where $H_i(x, \lambda) = g_i(x) - \lambda_i h_i(x), \; i = 1, ..., m$ and $\lambda \in R^m$ is a parameter which must be chosen in a suitable way.

The original result of Dinkelbach [83] from 1967 (and also the foregoing result of Jagannathan [142] from 1966 for linear fractional problems) concerns the case $m = 1$ with only one objective function and says that a given point $\bar{x}$ is optimal for $(\mathcal{P}_f)$ iff it is optimal for $\mathcal{P}(\bar{\lambda})$ with $\bar{\lambda} = g(\bar{x})/h(\bar{x})$.

Corresponding results for the sets of efficient and properly efficient solutions, respectively, of both problems in the case $m \geq 1$ were given by Bector and Chandra [12], Kaul and Lyall [155], Weir [282] and others. Note that the formulation as well as the proof of the corresponding Lemma 1 in [155] and Theorem 4 in [282] are not entirely correct in the given form. Above all, the authors disregarded the fact that in the case of proper efficiency it is essential to assume, additionally, that all ratios h_i/h_j are bounded below by positive bounds (and not only by zero).

In the following two theorems we formulate the relations between the sets of approximate solutions of $(\mathcal{P}_f)$ and $\mathcal{P}(\lambda)$.

Theorem 16 *Let $k^0 \in int \, R^m_+, \epsilon \geq 0$ and $\bar{x} \in X$. Then we have $f(\bar{x}) \in \mathrm{Eff}(f(X), C_{\epsilon k^0}) \iff H(\bar{x}, \bar{\lambda}) \in \mathrm{Eff}(H(X, \bar{\lambda}), C_{\epsilon \bar{k}})$ for*

$$\bar{\lambda}_i = \frac{g_i(\bar{x})}{h_i(\bar{x})} - \epsilon k_i^0 \quad \text{and} \quad \bar{k}_i = k_i^0 h_i(\bar{x}), \quad \forall i = 1, ..., m. \tag{1.33}$$

For the special case $\epsilon = 0$ we get the already mentioned result of [12] and [155] in the corrected form (namely, including the essential condition (1.33) for $\epsilon = 0$, which actually was used there in the proofs but had been forgotten in the formulation of the statement).

Corollary 3 $f(\bar{x}) \in \mathrm{Eff}(f(X), C) \iff H(\bar{x}, \bar{\lambda}) \in \mathrm{Eff}(H(X, \bar{\lambda}), C)$ *for $\bar{\lambda}$ according to (1.33) with $\epsilon = 0$.*

We denote the set of approximately properly efficient elements regarding Definitions 2(ii) and 3 (compare [262]) by $p\text{-}\mathrm{Eff}(f(X), C_{\epsilon k^0})$.

Theorem 17 *Let $k^0 \in int\, R_+^m$, $\epsilon \geq 0$ and $\bar{x} \in X$ and assume that there is a positive number γ such that for all i,j= 1,..., m and all $x \in X$ it holds $h_i(x)/h_j(x) \geq \gamma$. Then we have*

$$f(\bar{x}) \in p\text{-Eff}(f(X), C_{\epsilon k^0}) \Leftrightarrow H(\bar{x}, \bar{\lambda}) \in p\text{-Eff}(H(X, \bar{\lambda}), C_{\epsilon \bar{k}})$$

for $\bar{\lambda}$ and $\bar{k}$ according to (1.33).

Note that the assertion of Theorem 17 does not remain true if we only assume (as it was done in [155] and [282]) $h_i(x) > 0$ on X for $i = 1, \ldots, m$, since then $\inf\{h_i(x)/h_j(x) : x \in X\} = 0$ is not excluded.

This can be seen by the following small example. Let $n = 1$, $m = 2$, $X = \{x \in R | x \geq 0\}$, $g_1(x) = e^{-x}$, $h_1 = 1$, $g_2(x) = x^2 + x + 1$, $h_2(x) = x^2 + 1$. Then, for instance, $\bar{x} = 0$ yields a properly efficient element (in the sense of Geoffrion) $H(\bar{x}, \bar{\lambda}) = (0, 0)^T$ for $\mathcal{P}(\bar{\lambda})$ with $\bar{\lambda} = (1, 1)^T$ but $(g_1(\bar{x})/h_1(\bar{x}), g_2(\bar{x})/h_2(\bar{x}))^T = (1, 1)^T$ is not properly efficient in the sense of Geoffrion for $(\mathcal{P}_f)$. Similar examples can also be constructed for the other direction of Theorem 17.

Of course, if all functions h_i are equal or if there are positive lower and upper bounds for all functions h_i on X, the required boundedness of all ratios h_i/h_j by positive bounds is satisfied.

5.2. Possibilities for a Solution Approach

The reason for using models of vector optimization for solving concrete decision problems is the fact that very often it is impossible to formulate the interests of the decision maker a priori by only one objective function. As a natural consequence of such an incomplete knowledge about the underlying decision problem we can observe the phenomenon that in vector optimization we get a great number of "solutions", enjoying a priori the same rights. Of course, in practical decision problems the final aim must be to find such a feasible decision which corresponds to the decision maker's interests in a certain "optimal" way.

As already described in [126] this can often be realized by organizing a learning process in form of a dialogue procedure in which one can compute and compare as many solutions as necessary to help the decision maker to express his individual interests more precisely. Such a dialogue procedure is usually a certain kind of a two-level algorithm and needs essentially a suitable parametric surrogate optimization problem related to the underlying vector optimization problem.

Theoretically, all these ideas can also be applied directly to the fractional vector optimization problem $(\mathcal{P}_f)$ studied in the foregoing section. However, there may be computational difficulties to handle problems with complicated fractional objective functions. Moreover, there are

also theoretical difficulties to ensure convexity properties of the surrogate problem which are to be solved in such a dialogue procedure. Note that even linear fractionals are not convex but only pseudoconvex and that for sums of fractionals even generalized convexity properties do not hold anymore.

For this reason we want to discuss here possibilities to apply a dialogue procedure not directly to the original fractional problem $(\mathcal{P}_f)$ but to the corresponding Dinkelbach-transformed problem $\mathcal{P}(\lambda)$. However, in such an approach we have to overcome another difficulty, namely the generation of the essential parameter value $\bar{\lambda}$ satisfying (1.33). Hence, different to dialogue procedures in the usual case, we propose a three-level dialogue procedure for our considered case of fractional vector optimization problems.

Let us explain our ideas for the mostly used set Eff $(f(X), C)$ of efficient solutions $(\epsilon = 0)$ of $(\mathcal{P}_f)$ and the mostly used surrogate problem in which the artificial objective function is the weighted sum of the original objective functions. Applied to $\mathcal{P}(\lambda)$ our parametric surrogate problem has the form

$$\mathcal{P}(\lambda, \mu) \quad F(x, \lambda, \mu) \rightarrow \min \text{ subject to } x \in X,$$

where $\mu > 0$ and

$$F(x, \lambda, \mu) = \sum_{i=1}^{m} \mu_i (g_i(x) - \lambda_i h_i(x)).$$

The already mentioned three levels of a dialogue procedure for $(\mathcal{P}_f)$ may be characterized in the following way.

Level 1: Compare all stored results and decide to stop the procedure or not. If not, choose a new parameter vector μ.

Level 2: Find for the value of μ given from Level 1 a vector λ such that there exists a solution x of $\mathcal{P}(\lambda, \mu)$ satisfying

$$H(x, \lambda) = 0. \tag{1.34}$$

Level 3: Find for the values μ and λ given in the Levels 1 and 2 a solution x of $\mathcal{P}(\lambda, \mu)$ satisfying $H(x, \lambda) = 0$ and store x together with additional information on x (especially the vector $g(x) / h(x)$). Go to Level 1.

Level 1 is the pure dialogue part in which we have to generate a new parameter value μ as long as we are not satisfied with the generated results. Level 3 can often be realized successfully by pathfollowing

methods of parametric optimization. Because of the fact that possibilities to realize the Levels 1 and 3 are already described extensively in several papers (cf. [126]) we concentrate our considerations upon the second level. The typical difficulty in this level is the fact that the essential equation (1.34) is only given implicitly since the solution x of the third level is unknown at the time in which we have to solve the second level.

Let us study Level 2 under the following additional assumption (A). Here we take the symbol $\mathbf{PC}^r$, $(r \geq 1)$ to denote the class of those $\mathbf{C}^{r-1}$-functions for which the derivations of order r – 1 are piecewise $\mathbf{C}^1$, cf. [235, 263].

(A) $X = \{x \in R^n : q_l(x) \leq 0,\ l = 1,...,p\}$, $g_i, h_i, q_l \in \mathbf{PC}^2$ and there exists a strongly stable stationary point (x^*, u^*) of $\mathcal{P}(\lambda^*, \mu^*)$ satisfying condition (1.34).

Proposition 3 *Let us assume* **(A)**. *Then we have:*

(1) *There are neighbourhoods U of λ^*, V of μ^* and W of (x^*,u^*) such that for each $(\lambda, \mu) \in U \times V$ problem $\mathcal{P}(\lambda, \mu)$ has a unique stationary point $(\bar{x}(\lambda, \mu), \bar{u}(\lambda, \mu))$ in W.*

(2) *The vector function $(\bar{x}, \bar{u})$ belongs to the class $\mathbf{PC}^1$ on U.*

(3) *The vector function G, defined on U by*

$$G_i(\lambda) = g_i(\bar{x}(\lambda, \mu^*)) - \lambda_i h_i(\bar{x}(\lambda, \mu^*)),\ \ i = 1,...,m$$

belongs to the class $\mathbf{PC}^1$.

Obviously, under (A) condition (1.34) can be reformulated in the form

$$G(\lambda) = 0,\ \lambda \in U, \tag{1.35}$$

To solve (1.34) we can apply suitable generalizations of the Newton-method for nonsmooth equations using generalized derivatives. In the papers [235] and [263] one can find possibilities to generate the generalized Jacobian of the vector functions $\bar{x}$ and G. To ensure convergence to the (of course unknown) point λ^* from assumption **(A)** usually one needs a suitable initial point λ^0 in a sufficiently small neighbourhood of λ^*.

Among the great number of contributions concerning generalizations of the Newton-method to nonsmooth equations we refer here to the rather general results given in [168] and [227]. Useful ideas to guarantee convergence in the second level even in the case that only approximate

solutions of the third level may be generated (what may often be the case) can be found in [263].

Applying Lemma 2.1 in [152] the function $\bar{\varphi}(\lambda, \mu) = F(\bar{x}(\lambda, \mu), \lambda, \mu)$ belongs to the class $\mathbf{PC}^2$ and for each $(\lambda, \mu) \in U \times V$ it holds $\frac{\partial}{\partial \lambda_i} \bar{\varphi}(\lambda, \mu) = -\mu_i h_i(\bar{x}(\lambda, \mu))$ and, hence,

$$\sum_{j=1}^{m} \mu_j^* \frac{\partial}{\partial \lambda_i} G_j(\lambda) = -\mu_i^* h_i(\bar{x}(\lambda)) \tag{1.36}$$

with $\bar{x}(\lambda) = \bar{x}(\lambda, \mu^*)$. Hence, for the special case $m = 1$ (in which we can put without loss of generality $\mu = 1$) the function G belongs even to the class $\mathbf{PC}^2$ with $\nabla G(\lambda) = -h(\bar{x}(\lambda))$. In this way the iteration rule of the Newton-method has the very simple form

$$\lambda^{s+1} = \lambda^s - \frac{g(\bar{x}(\lambda^s)) - \lambda^s h(\bar{x}(\lambda^s))}{-h(\bar{x}(\lambda^s))} = f(\bar{x}(\lambda^s)), \tag{1.37}$$

which is nothing else than the iteration rule of Dinkelbach [83], who used this rule also under other assumptions as given in **(A)**, since convergence results are very much easier to obtain in the one-dimensional case.

Unfortunately, for $m > 1$ a formula of the type $\nabla G(\lambda) = -h(\bar{x}(\lambda))$ does not follow from (1.36).

5.3. Conclusions

Further research on multicriteria fractional programming includes:

- Duality assertions;
- Dependence of solutions or approximate solutions on variation of data of the problem;
- Solution procedures, convergence properties.

6. Multicriteria Control Problems

Keywords: ϵ-Pontrjagin's minimum principle, Structure of solutions.

6.1. Formulation of Multicriteria Control Problems

In control theory often one has the problem to minimize more than one objective function, for instance a cost functional as well as the distance between the final state and a given point.

To realize this task usually one takes as objective function a weighted sum of the different objectives. However, the more natural way would be to study the set of efficient points of a vector optimization problem with the given objective functions. It is well known that the weighted sum is only a special surrogate problem to find efficient points, which has the disadvantage that in the nonconvex case one cannot find all efficient elements in this way.

Necessary conditions for solutions of multiobjective dynamic programming or control problems were derived by several authors, see Klötzler [160], Benker and Kossert [17], Breckner [36], Gorochowik and Kirillowa [124], Gorochowik [123, 125] and Salukvadze [232]. It is difficult to show the existence of an optimal control (see Klötzler [162]), whereas suboptimal controls exist under very weak assumptions. So it is important to derive some assertions for suboptimal controls.

An ϵ-minimum principle in the sense of Pontrjagin for suboptimal controls of multicriteria control problems is derived by Tammer [260] applying a vector-valued variational principle.

Consider the system of differential equations

$$\begin{aligned} \frac{dx}{dt}(t) &= \varphi(t, x(t), u(t)), \\ x(0) &= x_0 \in R^n \end{aligned} \tag{1.38}$$

and the control restriction

$$u(t) \in U,$$

which must hold almost everywhere on [0, T] with $T > 0$.

We assume that

(C1) $\varphi : [0,T] \times R^n \times U \longrightarrow R^n$ is continuous and $U \subset R^s$ is a compact set.

The vector $x(t)$ describes the state of the system, $u(t)$ is the control at time t and belongs to the set U. Furthermore, suppose that

(C2) $\frac{\partial \varphi}{\partial x_i}, i = 1, ..., n$ are continuous on $[0,T] \times R^n \times U$,

(C3) $(x, \varphi(t, x, u)) \leq c(1 + ||x||^2)$ for some $c > 0$.

Remark 4 *Let $u : [0,T] \longrightarrow U$ be a measurable control. Condition (C2) and the continuity of φ ensure that there exists a unique solution x of the differential equation (1.38) on $[0,\tau]$ for a sufficiently small $\tau > 0$. By using Gronwall's inequality condition (C3) implies*

$$||x(t)||^2 \leq (||x_0||^2 + 2cT)e^{2cT},$$

and, hence, ensures the existence of the solution on the whole time interval [0,*T*].

Moreover, the last inequality yields

$$\left|\left|\frac{dx(t)}{dt}\right|\right| \leq \max\{\varphi(t,x,u) \mid (t,x,u) \in [0,T] \times B^0 \times U\},$$

where B^0 *denotes the ball of radius* $(||x_0||^2 + 2cT)^{\frac{1}{2}} e^{cT}$. *Applying Ascoli's theorem, we see that the family of all trajectories x of the control system (1.38) is equicontinuous and bounded and hence relatively compact in the uniform topology (compare Ekeland [89]).*

In order to formulate the multicriteria control problem we introduce the objective function $f : R^n \longrightarrow R^m$ and suppose that

(C4) f is a differentiable vector-valued function,

(C5) $C \subset R^m$ is a pointed, closed convex cone with bounded base, $k^0 \in int\ C$.

Now we formulate the **multiobjective optimal control problem** under the assumptions (C1) – (C5)

(P) Find some measurable control $\bar{u}$ such that the corresponding trajectory $\bar{x}$ satisfies

$$f(x(T)) \notin f(\bar{x}(T)) - (C \setminus \{0\})$$

for all solutions x of (1.38) (the pair $(\bar{x}, \bar{u})$ is denoted as process corresponding to (1.38)).

6.2. An ϵ-Minimum Principle for Multiobjective Optimal Control Problems

It is well known that it is difficult to show the existence of optimal (or efficient) controls of (P), whereas suboptimal controls exist under very weak conditions. So it is important to derive some assertions for suboptimal controls.

An application of a variational principle for vector optimization problems (see Section 2, Theorem 10) yields an ϵ-minimum principle for (P), which is closely related to Pontrjagin's minimum principle (for $\epsilon = 0$).

We introduce the space V of controls, defined as the set of all measurable functions $u : [0,T] \longrightarrow U$ with the metric

$$d(u_1, u_2) = measure\{t \in [0,T] : u_1(t) \neq u_2(t)\}.$$

Then it holds

Lemma 7 (Ekeland [89]) *(V, d) is a complete metric space.*

Lemma 8 (Ekeland [89]) *The function $F : u \longrightarrow f(x(T))$ is continuous on V, where $x(.)$ is the solution of (1.38) depending on $u \in V$.*

Theorem 18 *Consider the multicriteria control problem under the assumptions (C1) – (C5). For every $\epsilon > 0$, there exists a measurable control u_ϵ with the corresponding admissible trajectory x_ϵ such that*

(i) *$f(x(T)) \notin f(x_\epsilon(T)) - \epsilon k^0 - (C \setminus \{0\})$ for all solutions x of (1.38),*

(ii)

$$\begin{pmatrix} (\varphi(t, x_\epsilon(t), u(t)) \ , \ p_\epsilon^1(t)) \\ \cdots \\ (\varphi(t, x_\epsilon(t), u(t)) \ , \ p_\epsilon^m(t)) \end{pmatrix} \notin \begin{pmatrix} (\varphi(t, x_\epsilon(t), u_\epsilon(t)) \ , \ p_\epsilon^1(t)) \\ \cdots \\ (\varphi(t, x_\epsilon(t), u_\epsilon(t)) \ , \ p_\epsilon^m(t)) \end{pmatrix} - \epsilon k^0 - int\ C$$

for any $u \in U$ and almost all $t \in [0, T]$, where $p_\epsilon(.) = (p_\epsilon^1(.), ..., p_\epsilon^m(.))$ is the solution of the linear differential system:

$$\begin{array}{rcl} \frac{dp_{\epsilon i}^s}{dt}(t) & = & -\sum_{j=1}^n \frac{\partial \varphi_j}{\partial x_i}(t, x_\epsilon(t), u_\epsilon(t)) p_{\epsilon j}^s, \ i = 1, ..., n; \\ p_\epsilon^s(T) & = & f_s'(x_\epsilon(T)), \ s = 1, ..., m. \end{array} \tag{1.39}$$

Remark 5 *If we put $\epsilon = 0$ then Theorem 18 coincides with the following assertion: Whenever there is a measurable control u_ϵ and the corresponding admissible trajectory x_ϵ with*

(i) *$f(x(T)) \notin f(x_\epsilon(T)) - (C \setminus \{0\})$ for all solutions x of (1.38), then*

(ii)

$$\begin{pmatrix} (\varphi(t, x_\epsilon(t), u(t)) \ , \ p_\epsilon^1(t)) \\ \cdots \\ (\varphi(t, x_\epsilon(t), u(t)) \ , \ p_\epsilon^m(t)) \end{pmatrix} \notin \begin{pmatrix} (\varphi(t, x_\epsilon(t), u_\epsilon(t)) \ , \ p_\epsilon^1(t)) \\ \cdots \\ (\varphi(t, x_\epsilon(t), u_\epsilon(t)) \ , \ p_\epsilon^m(t)) \end{pmatrix} - int\ C$$

for any $u \in U$ and almost all $t \in [0, T]$, where $p_\epsilon(.)$ is the solution of linear differential system (1.39) *(the adjoint system).*

This means that u_ϵ fulfills a minimum principle for multicriteria control problems in the sense of Pontrjagin (compare Gorochowik and Kirillowa [124] and Gorochowik [123, 125]).

Remark 6 *Now let us study the special case* $Y = R^1$ *and put* $\epsilon = 0$. *Then Theorem 18 coincides with the following assertion: Whenever*

(i) $f(x_\epsilon(T)) \leq \inf\ f(x(T))$ *holds,*

then

(ii) $(\varphi(t, x_\epsilon(t), u_\epsilon(t)), p_\epsilon(t)) \leq \min_{u \in U} (\varphi(t, x_\epsilon(t), u(t)), p_\epsilon(t))$

almost everywhere on [0,T], where $p_\epsilon(.)$ *is the solution of (1.39).*

This is the statement of Pontrjagin's minimum principle. However, Theorem 18 holds even if optimal solutions do not exist.

6.3. Conclusions

There are only some papers dealing with vector-valued control problems. It is essential to find results, as for instance a minimum principle for approximate solutions, in order to have possibilities to analyse the structure of solutions or approximate solutions. Furthermore, it is important to consider additionally non-deterministic aspects, that means multicriteria stochastic optimal control problems, too.

References

[1] Aly, A. and Rahali, B. (1990). Analysis of a bicriteria location model. *Naval Research Logistics,* 37:937-944.

[2] Amahroq, T. and Taa, A. (1997). On Lagrange-Kuhn-Tucker multipliers for multiobjective optimization problems. *Optimization,* 41(2):159-172.

[3] Ansari, Q.H. (1995). On generalized vector variational-like inequalities. *Annales des Sciences Mathématiques du Québec,* 19:131-137.

[4] Ansari, Q.H. (1997). Extended generalized vector variational-like inequalities for nonmonotone multivalued maps. *Annales des Sciences Mathématiques du Québec,* 21:1-11.

[5] Ansari, Q.H. (1997). A note on generalized vector variational-like inequalities. *Optimization,* 41:197-205.

[6] Ansari, Q.H. (2000). Vector equilibrium problems and vector variational inequalities. In: Giannessi, F. (ed.) *Vector Variational Inequalities and Vector Equilibria,* 1-15. Nonconvex Optimization and its Applications 38. Kluwer Academic Publishers, Dordrecht.

[7] Ansari, Q.H. and Siddiqi, A.H. (1998). A generalized vector variational-like inequality and optimization over an efficient set. In: Brokate, M. and Siddiqi, A.H. (eds.) *Functional Analysis with Current Applications in Science, Technology and Industry,* 177-191. Pitman Research Notes in Mathematics Series 377. Longman, Harlow .

[8] Ansari, Q.H., Siddiqi A.H., and Yao, J.C. (2000). Generalized Vector Variational Like Inequalities and their Scalarization. In: Giannessi, F. (ed.) *Vector Variational Inequalities and Vector Equilibria,* 18-37. Nonconvex Optimization and its Applications 38. Kluwer Academic Publishers, Dordrecht.

[9] Attouch, H. and Riahi, H. (1993). Stability results for Ekeland ϵ - variational principle and cone extremal solutions. *Mathematics of Operations Research,* 18:173-201.

[10] Aubin, J.P. and Ekeland, I. (1984). *Applied Nonlinear Analysis.* John Wiley, New York.

[11] Ballestero, E. and Romero, C. (1998). *Multiple Criteria Decision Making and its Applications to Economic Problems.* Kluwer Academic Publishers, Boston.

[12] Bector, C.R. and Chandra, S. (1987). Multiobjective fractional programming duality: A parametric approach. Research Report, Faculty of Management, University of Manitoba.

[13] Bednarczuk, E. (1992), Some stability results for vector optimization problems in partially topological vector spaces. In: Lakshmikantham, V. (ed.) *Proceedings of the First World Congress of Nonlinear Analysts (Tampa, FL, 1992),* 2371-2382. De Gruyter, Berlin, 1996.

[14] Bednarczuk, E. (1994). An approach to well-posedness in vector optimization: consequences to stability *Control and Cybernetics,* 23:107-121.

[15] Bednarczuk, E. (1995). Berge-type theorems for vector optimization problems. *Optimization,* 32:373-384.

[16] Benker, H., Hamel, A., and Tammer, C. (1997). An algorithm for vectorial control approximation problems. In: Fandel, G.; Gal, T. (eds.) *Multiple Criteria Decision Making,* 3-12. Lecture Notes in Economics and Mathematical Systems 477. Springer-Verlag, Heidelberg.

[17] Benker, H. and Kossert, S. (1982). Remarks on quadratic optimal control problems in Hilbert spaces. *Zeitschrift für Analysis und ihre Anwendungen,* 1(3): 13-21.

[18] C. Berge (1963). *Topological Spaces.* Mcmillan Co., New York.

[19] Bianchi, M., Hadjisavvas, N., and Schaible, S. (1997). Vector equilibrium problems with generalized monotone bifunctions. *Journal of Optimization Theory and Applications,* 92:527-542.

[20] Bigi, G. and Pappalardo, M. (1999). Regularity conditions in vector optimization. *Journal of Optimization Theory and Applications,* 102:83-96.

[21] Bishop, E. and Phelps, R.R. (1962). The support functionals of a convex set. *Proceedings of Symposia in Pure Mathematics,* 7:27-35. American Mathematical Society.

[22] Bitran, G.R. (1981). Duality for nonlinear multiple-criteria optimization problems. *Journal of Optimization Theory and Applications,* 35:367-401.

[23] Bitran, G.R. and Magnanti, T.L. (1979). The structure of admissible points with respect to cone dominance. *Journal of Optimization Theory and Applications,* 29:573-614.

[24] Blum, E. and Oettli, W. (1975). *Mathematische Optimierung.* Springer-Verlag, Berlin.

[25] Blum, E. and Oettli, W. (1994). From optimization and variational inequalities to equilibrium problems. *The Mathematics Student,* 63:123-145.

[26] Blum, E. and Oettli, W. (1993). Variational principles for equilibrium problems. In: Guddat, J., Jongen, H.T., Kummer. B., and Nožička, F. (eds.) *Parametric Optimization and Related Topics III,* 79-88. Peter Lang, Frankfurt am Main.

[27] Borwein, J. (1977). Proper efficient points for maximizations with respect to cones. *SIAM Journal on Control and Optimization,* 15:57-63.

[28] Borwein, J.M. (1981). A Lagrange multiplier theorem and a sandwich theorem for convex relations. *Mathematica Scandinavica,* 48:198-204.

[29] Borwein, J.M. (1981) Convex relations in analysis and optimization. In: Schaible, S. and Ziemba, W. (eds.) *Generalized Concavity in Optimization and Economics,* 335-377. Academic Press, New York.

[30] Borwein, J.M. (1982). Continuity and differentiability properties of convex operators. *Proceedings of the London Mathematical Society,* 44:420 -444.

[31] Borwein, J.M. (1983). On the Existence of Pareto Efficient Points. *Mathematics of Operations Research,* 9:64-73.

[32] Borwein, J.M. (1987). Convex Cones, Minimality Notions and Consequences. In: Jahn, J. and Krabs, W. (eds.) *Recent Advances and Hystorical Development of Vector Optimization,* 64-73. Lecture Notes in Economics and Mathematical Systems 294. Springer-Verlag, Heidelberg.

[33] Borwein, J. M. and Preiss, D. (1987). A smooth variational principle with applications to subdifferentiability and to differentiability of convex functions. *Transactions of the American Mathematical Society,* 303:517-527.

[34] Breckner, W.W. (1972). Dualität bei Optimierungsaufgaben in halbgeordneten topologischen Vektorräumen (I). *Revue d'Analyse Numérique et de la Théorie de l'Approximation,* 1:5-35.

[35] Breckner, W.W. (1973). Dualität bei Optimierungsaufgaben in halbgeordneten topologischen Vektorräumen (II). *Revue d'Analyse Numérique et de la Théorie de l' Approximation,* 2:27-35.

[36] Breckner, W. W. (1997). Derived sets for weak multiobjective optimization problems with state and control variables. *Journal of Optimization Theory and Applications,* 93:73-102.

[37] Brezis, H. and Browder, F.E. (1976). A general principle on ordered sets in nonlinear functional analysis. *Advances in Mathematics,* 21:355-364.

[38] Brondsted, A. (1974). On a lemma of Bishop and Phelps. *Pacific Journal of Mathematics,* 55:335-341.

[39] Brosowski, B. and Conci, A. (1983). On Vector Optimization and Parametric Programming. *Segundas Jornados Latino Americans de Matematica Aplicada,* 2:483-495.

[40] Brumelle, S. (1981). Duality for Multiple Objective Convex Programs. *Mathematics of Operations Research,* 6(2):159-172.

[41] Caballero, R., Cerda Tena, E., Munoz, M., and Rey, L. (2000). Relations among Several Efficiency Concepts in Stochastic Multiple Objective Programming. In: Haimes, Y.Y. and Steuer, R.E. (eds.) *Research and Practice in Multiple Criteria Decision Making,* 57-69. Lecture Notes in Economics and Mathematical Systems 487. Springer-Verlag, Berlin, Heidelberg.

[42] Castellani, M., Mastroeni, G., and Pappalardo, M. (1996). On Regularity for Generalized Systems and Applications. In: Di Pillo, G. and Giannessi, F. (eds.) *Nonlinear Optimization and Applications,* 13-26. Plenum Press, New York.

[43] Carrizosa, E. (1992). *Problems de localization multiobjetivo (Multiobjective location problems* (in Spanish). Ph.D. Thesis, Faculdad de Matematicas, Universidad de Sevilla.

[44] Carrizosa, E., Conde, E., Fernandez, R., and Puerto, J. (1993). Efficiency in Euclidean constrained location problems *Operations Research Letters*, 14:291-295.

[45] Carrizosa, E. and Fernandez, R. (1990). El conjunto eficiente en problemas de localization con normas mixtas(lp) (The efficient set in location problems with mixed norms) (in Spanish). *Trabajos de Investigacion Operativa*, 6:61-69.

[46] Carrizosa, E. and Fernandez, R. (1993). A polygonal upper bound for the efficient set for single-location problems with mixed norms. *Top (Spanish Journal of Operations Research and Statistics)*, 1:107-116.

[47] Carrizosa, E., Fernandez, R., and Puerto, J. (1990). Determination of a pseudoefficient set for single-location problems with mixed polyhedral norms. In: Orban-Ferange, F. and Rasson, J.P. (eds.) *Proceedings of Meeting V of the EURO Working Group on Locational Analysis*, 27-39. Facultés Universitaires Notre-Dame de la Paix, Namur.

[48] Carrizosa, E., Fernandez, R., and Puerto, J. (1990). An axiomatic approach to location criteria. In: Orban-Ferange, F. and Rasson, J.P. (eds.) *Proceedings of Meeting V of the EURO Working Group on Locational Analysis*, 40-53. Facultés Universitaires Notre-Dame de la Paix, Namur.

[49] Carrizosa, E. and Plastria, F. (1993). A characterization of efficient points in constrained location problems with regional demand. *Operations Research Letters*, 19(3):129-131.

[50] Carrizosa, E. and Plastria, F. (1996). Geometrical characterization of weakly efficient points. *Journal of Optimization Theory and Applications*, 90(1):217-223.

[51] O. Cârjă (1998). *Elemente de Analiză Funcţională Neliniară.* Editura Universităţii "Al.I.Cuza" Iaşi, Iaşi, Romania.

[52] Cesari, L. and Suryanarayana, M.B. (1978). Existence theorems for Pareto optimization; multivalued and Banach space valued functionals. *Transactions of the American Mathematical Society,* 244:37-65.

[53] Chalmet, L.G., Francis, R.L., and Kolen, A. (1981). Finding efficient solutions for rectilinear distance location problem efficiently. *European Journal of Operational Research*, 6:117-124.

[54] Chang, S.S., Thompson H.B., and Yuan, G.X.Z. (2000). Existence of solutions for generalized vector variational-like inequalities. In: Giannessi, F. (ed.) *Vector Variational Inequalities and Vector Equilibria,* 39-53. Nonconvex Optimization and its Applications 38. Kluwer Academic Publishers, Dordrecht.

[55] Chen, G.Y. (1992). Existence of solutions for a vector variational inequality: An extension of Hartmann-Stampacchia theorem. *Journal of Optimization Theory and Applications,* 74:445-456.

[56] Chen, G.Y. and Chen, G.M. (1987). Vector variational inequality and vector optimization. In: Sawaragi, Y., Inoue, K. and Nakayama, H. (eds.) *Toward Interactive and Intelligent Decision Support Systems,* vol. 1, 408-416. Lecture Notes in Economics and Mathematical Systems 285. Springer-Verlag, Berlin.

[57] Chen, G.Y. and Craven, B.D. (1989). Approximate dual and approximate vector variational inequality for multiobjective optimization. *Journal of the Australian Mathematical Society, Series A,* 47:418-423.

[58] Chen, G.Y. and Craven, B. (1990). A vector variational inequality and optimization over an efficient set. *Zeitschrift für Operations Research,* 3:1-12.

[59] Chen, G.Y., Goh, C.J., and Yang, X.Q. (2000) On gap functions for vector variational inequalities. In: Giannessi, F. (ed.) *Vector Variational Inequalities and Vector Equilibria,* 55-72. Nonconvex Optimization and its Applications 38. Kluwer Academic Publishers, Dordrecht.

[60] Chen, G.Y. and Hou, S.H. (2000). Existence of solutions for vector variational inequalities. In: Giannessi, F. (ed.) *Vector Variational Inequalities and Vector Equilibria,* 73-86. Nonconvex Optimization and its Applications 38. Kluwer Academic Publishers, Dordrecht.

[61] Chen, G.Y. and Huang, X.X. (1998). A unified approach to the existing three types of variational principles for vector valued functions. *Mathematical Methods of Operations Research,* 48:349-357.

[62] Chen, G.Y. and Huang, X.X. (1998). Ekeland's ϵ-variational principle for set-valued mappings. *Mathematical Methods of Operations Research,* 48:181-186.

[63] Chen, G.Y., Huang, X.X., and Lee, G.M. (1999). Equivalents of an approximate variational principle for vector-valued functions and applications. *Mathematical Methods of Operations Research,* 49:125-136.

[64] Chen, G.Y. and Yang, X.Q. (1990). The vector complementarity problem and its equivalences with the weak minimal element in ordered spaces. *Journal of Mathematical Analysis and Applications,* 153:136-158

[65] Chen, G.Y. and Yang, X.Q. (2000). On the existence of solutions to vector complementarity problems. In: Giannessi, F. (ed.) *Vector Variational Inequalities and Vector Equilibria.,* 87-95 Nonconvex Optimization and its Applications 38. Kluwer Academic Publishers, Dordrecht.

[66] Chew, K.L. (1984) Maximal points with respect to cone dominance in Banach spaces and their existence. *Journal of Optimization Theory and Applications,* 44:1-53.

[67] Chowdhury, M.S.R. and Tan, K.K. (1997). Generalized variational inequalities for quasi-monotone operators and applications. *Bulletin of the Polish Academy of Sciences. Mathematics,* 45:25-54.

[68] Clarke, F.H. (1983). *Optimization and Nonsmooth Analysis.* John Wiley and Sons, New York.

[69] Conway, J.B. (1990). *A Course in Functional Analysis.* Springer-Verlag, New York.

[70] Corley, H.W. (1980). An existence result for maximizations with respect to cones. *Journal of Optimization Theory and Applications,* 31:277-281.

[71] Corley, H.W. (1981). Duality for maximization with respect to cones. *Journal of Mathematical Analysis and Applications,* 84:560-568.

[72] Corley, H.W. (1987). Existence and Lagrangian duality for maximization of set-valued maps. *Journal of Optimization Theory and Applications,* 54:489-501.

[73] Craven, B.D. (1989). Nonsmooth multiobjective programming. *Numerical Functional Analysis and Optimization,* 10:49-64.

[74] Daneš, J. (1972). A geometric theorem useful in nonlinear functional analysis. *Bolletino Unione Matematica Italiana,* 6:369-375.

[75] Daniele, P. and Maugeri, A. (2000). Vector variational inequalities and modelling of a continuum traffic equilibrium problem. In: Giannessi, F. (ed.) *Vector Variational Inequalities and Vector Equilibria,* 97-111. Nonconvex Optimization and its Applications 38. Kluwer Academic Publishers, Dordrecht.

[76] Danilidis, A. and Hadjisavvas, H. (1996). Existence theorems for vector variational inequalities. *Bulletin of the Australian Mathematical Society,* 54:473-481.

[77] Dedieu, J.P. (1978) Critères de fermeture pour l'image d'un fermé non convexe par une multiapplication. *Comptes Rendus des Séances de l'Académie des Sciences, Serie I*, 287:941-943.

[78] De Figueiredo, D.G. (1989). *Lectures on the Ekeland's variational principle with applications and detours*. Springer-Verlag, Berlin.

[79] Dentscheva, D. and Helbig, S. (1994). On several concepts for ϵ-efficiency. *Operations Research Spektrum*, 16:179-186.

[80] Dentcheva, D. and Helbig, S. (1996). On variational principles, level sets, well-posedness, and ε-solutions in vector optimization. *Journal of Optimization Theory and Applications*, 89:325-349.

[81] Ding, X.P. and Tarafdar, E. (2000). Gereralized vector variational-like inequalities. In: Giannessi, F. (ed.) *Vector Variational Inequalities and Vector Equilibria*, 113-124. Nonconvex Optimization and its Applications 38. Kluwer Academic Publishers, Dordrecht.

[82] Ding, X.P. and Tarafdar, E. (2000). Generalized vector variational-like inequalities with $C_x - \eta$-pseudomonotone set-valued mappings. In: Giannessi, F. (ed.) *Vector Variational Inequalities and Vector Equilibria*, 125-140. Nonconvex Optimization and its Applications 38. Kluwer Academic Publishers, Dordrecht.

[83] Dinkelbach, W. (1967). On nonlinear fractional programming. *Management Science*, 13(7):492-498.

[84] Dolecki, S. and Malivert, C. (1987). Polarities and stability in vector optimization. In: Jahn, J. and Krabs, W. (eds.) *Recent Advances and Historical Development in Vector Optimization*, 96-113 Lecture Notes in Economics and Mathematical Systems 294. Springer-Verlag, Berlin.

[85] Durier, R. (1984). On efficient points and Fermat-Weber problem. Working Paper, Université de Bourgogne, Dijon.

[86] Durier, R. (1990). On Pareto optima, the Fermat-Weber problem and polyhedral gauges. *Mathematical Programming*, 47:65-79.

[87] Durier, R. and Michelot, C. (1986). Sets of efficient points in a normed space. *Jounal of Mathematical Analysis and Applications*, 117:506-528.

[88] Durier, R. and Michelot, C. (1985). Geometrical properties of the Fermat-Weber problem. *European Journal of Operational Research*, 20:332-343.

[89] Ekeland, I. (1974). On the variational principle. *Journal of Mathematical Analysis and Applications* 47:324-353.

[90] Ekeland, I. (1979). Nonconvex minimization problems. *Bulletin of the American Mathematical Society*, 1(3):443-474

[91] El Abdouni, B. and Thibault, L. (1992). Lagrange multipliers for Pareto nonsmooth programming problems in Banach spaces. *Optimization,* 26(3-4):277-285.

[92] Elster, K.H. and Elster, R. (1993). Vector approximation problems in geometric vector optimization. *Optimization,* 27:321-342.

[93] Ester, J. (1987). *Systemanalyse und mehrkriterielle Entscheidung.* Verlag Technik, Berlin.

[94] Fan, K. (1961). A generalization of Tychonoff's fixed point theorem. *Mathematische Annalen,* 142:305-310.

[95] Ferro, F. (1996) An optimization result for set-valued mappings and a stability property in vector problems with constraints. *Journal of Optimization Theory and Applications,* 90:63-77.

[96] Ferro, F. (1997) Optimization and stability results through cone lower semicontinuity. *Set-Valued Analysis,* 5:365-375.

[97] Ferro, F. (1999) A new ABB theorem in Banach spaces. *Optimization,* 46(4):353-362.

[98] Friedrichs, K.O. (1929). Ein Verfahren der Variationsrechnung, das Minimum eines Integrals als das Maximum eines anderen Ausdruckes darzustellen. *Nachrichten der Gesellschaft der Wissenschaften zu Göttingen,* 13-20.

[99] Fu, J. (1997). Simultaneous vector variational inequalities and vector implicit complementarity problem. *Journal of Optimization Theory and Applications,* 93:141-151.

[100] Fu, J. (2000). A vector variational-like inequalitiy for compact acyclic multifunctions and its applications. In: Giannessi, F. (ed.) *Vector Variational Inequalities and Vector Equilibria,* 141-151. Nonconvex Optimization and its Applications 38. Kluwer Academic Publishers, Dordrecht.

[101] Gajek, L. and Zagrodny, D. (1992). Countably orderable sets and their applications in optimization. *Optimization,* 26:287-301.

[102] Gajek, L. and Zagrodny, D, (1995). Geometric variational principle. *Dissertationes Mathematicae,* 340:55-71.

[103] Geoffrion, A.M. (1968). Proper efficiency and the theory of vector maximization. *Journal of Mathematical Analysis and Applications,* 22:618-630.

[104] Georgiev, P.G. (1988). The strong Ekeland variational principle, the strong drop theorem and applications. *Journal of Mathematical Analysis and Applications,* 131:1-21.

[105] Gerstewitz (Tammer), C. (1983). Nichtkonvexe Dualität in der Vektoroptimierung. *Wissenschaftliche Zeitschrift der TH Leuna-Merseburg,* 25:357-364.

[106] Gerstewitz, C. and Iwanow, E. (1985). Dualität für nichtkonvexe Vektoroptimierungsprobleme. *Wissenschaftliche Zeitschrift der TH Ilmenau,* 2:61-81.

[107] Gerstewitz (Tammer), C. and Göpfert, A. (1981). Zur Dualität in der Vektoroptimierung. *Seminarberichte der Sektion Mathematik der Humboldt-Universität zu Berlin,* 39:67-84.

[108] Gerth (Tammer), C. (1987). Näherungslösungen in der Vektoroptimierung. *Seminarberichte der Sektion Mathematik der Humboldt-Universität zu Berlin* 90:67-76.

[109] Gerth (Tammer), C. and Pöhler, K. (1988). Dualität und algorithmische Anwendung beim vektoriellen Standortproblem. *Optimization,* 19:491-512.

[110] Gerth (Tammer), C. and Weidner, P. (1990). Nonconvex separation theorems and some applications in vector optimization. *Journal of Optimization Theory and Applications,* 67(2):297-320.

[111] Giannessi, F. (1980). Theorems of alternative, quadratic programs and complementarity problems. In: Cottle, R.W., Giannessi, F., and Lions, J.-L. (eds.) *Variational Inequalities and Complementarity Problems,* 151-186. J. Wiley and Sons, Chichester.

[112] Giannessi, F., Mastroeni, G., and Pelligrini, L. (2000). On the theory of vector optimization and variational inequalities. Image space analysis and separation. In: Giannessi, F. (ed.) *Vector Variational Inequalities and Vector Equilibria.,* 153-215. Nonconvex Optimization and its Applications 38. Kluwer Academic Publishers, Dordrecht.

[113] Glover, B.M., Jeyakumar, V., and Rubinov, A.M. (1999). Dual conditions characterizing optimality for convex multi-objective programs. *Mathematical Programming,* 84:201-217.

[114] Goh, C.J. and Yang, X.Q. (2000). Scalarization methods for vector variational inequality. In: Giannessi, F. (ed.) *Vector Variational Inequalities and Vector Equilibria,* 217-232. Nonconvex Optimization and its Applications 38. Kluwer Academic Publishers, Dordrecht.

[115] Gong, X.H., Fu, W.T., and Liu, W. (2000). Super efficiency for a vector equilibrium in locally convex topological vector spaces. In: Giannessi, F. (ed.) *Vector Variational Inequalities and Vector Equilibria,* 233-252. Nonconvex Optimization and its Applications 38. Kluwer Academic Publishers, Dordrecht.

[116] Göpfert, A. and Gerth (Tammer), C. (1986). Über die Skalarisierung und Dualisierung von Vektoroptimierungsproblemen. *Zeitschrift für Analysis und ihre Anwendungen,* 5(4):377-384.

[117] Göpfert, A. and Nehse, R. (1990). *Vektoroptimierung. Theorie, Verfahren und Anwendungen.* BSB B.G. Teubner Verlagsgesellschaft, Leipzig.

[118] Göpfert, A. and Tammer, C. (1995). ϵ-approximate solutions and conical support points. A new maximal point theorem. *Zeitschrift für angewandte Mathematik und Mechanik,* 75:595-596.

[119] Göpfert, A. and Tammer, C. (1995). A new maximal point theorem. *Zeitschrift für Analysis und ihre Anwendungen,* 14(2):379-390.

[120] Göpfert, A. and Tammer. C. (2000). Maximal point theorems in product spaces and applications for multicriteria approximation problems. In: Haimes, Y.Y. and Steuer, R.E. (eds.) *Research and Practice in Multiple Criteria Decision Making,* 93-104. Lecture Notes in Economics and Mathematical Systems 487. Springer-Verlag, Berlin, Heidelberg.

[121] Göpfert, A., Tammer, C., and Zalinescu, C. (2000). On the vectorial Ekeland's variational principle and minimal points in product spaces. *Nonlinear Analysis,* 39:909-922.

[122] Göpfert, A., Tammer, C., and Zalinescu, C. (1999). A new minimal point theorem in product spaces. *Zeitschrift für Analysis und ihre Anwendungen,* 18(3):767-770.

[123] Gorochowik, B. B. (1972). About multiobjective optimization problems (in Russian). *Doklady Akademii Nauk SSSR,* 6.

[124] Gorochowik, B.B. and Kirillowa, F. M. (1975). About scalarization of vector optimization problems (in Russian). *Doklady Akademii Nauk SSSR,* 19(7):588-591.

[125] Gorochowik, B. B. (1976). Conditions for weakly efficient elements in multiobjective optimization (in Russian). Minsk, IN-T Matem. AN Soviet Union, Preprint.

[126] Guddat, J., Guerra Vasquez, F., Tammer, K., and Wendler, K. (1985). *Multiobjective and Stochastic Optimization Based on Parametric Optimization.* Akademie-Verlag, Berlin.

[127] Ha, T.X.D. (1994). A note on a class of cones ensuring the existence of efficient points in bounded complete sets. *Optimization,* 31:141-152.

[128] Ha, T.X.D. (1994). On the existence of efficient points in locally convex spaces. *Journal of Global Optimization,* 4:265-278.

[129] Hadjisavvas, N. and Schaible, S. (1998). From scalar to vector equilibrium problems in the quasimonotone case. *Journal of Optimization Theory and Applications,* 96(2):297-309.

[130] Hartley, R. (1978) On cone-efficiency, cone-convexity and cone-compactness. *SIAM Journal on Applied Mathematics,* 34:211-222.

[131] Hazen, G.B. and Morin, T.L. (1983) Optimality conditions in non-conical multiple-objective programming. *Journal of Optimization Theory and Applications,* 40:25-60.

[132] Henig, M.I. (1982) Existence and characterization of efficient decisions with respect to cones. *Mathematical Programming,* 23:111-116.

[133] Henig, M.I. (1982) Proper efficiency with respect to cones. *Journal of Optimization Theory and Applications,* 36:387-407.

[134] Henkel, E.C. and Tammer, C. (1996). ϵ-variational inequalities in partially ordered spaces. *Optimization,* 36:105-118.

[135] Huang, X.X. (2000). Equivalents of a general approximate variational principle for set-valued maps and application to efficiency. *Mathematical Methods of Operations Research* 51:433-442.

[136] Isac, G. (1983). Sur l'existence de l'optimum de Pareto. *Rivista di Matematica della Universita di Parma,* 9:303-325.

[137] Isac, G. (1987). Supernormal cones and fixed point theory. *The Rocky Mountain Journal of Mathematics,* 17:219-226.

[138] Isac, G. (1994). Pareto optimization in infinite dimensional spaces: The importance of nuclear cones. *Journal of Mathematical Analysis and Applications,* 182:393-404.

[139] Isac, G. (1996). The Ekeland's principle and the Pareto ϵ-efficiency. In: Tamiz, M. (ed.) *Multi-Objective Programming and Goal Programming: Theories and Applications,* 148-163. Lecture Notes in Economics and Mathematical Systems 432. Springer-Verlag, Berlin.

[140] Isac, G. and Yuan, G.X.Z. (2000). The existence of essentially connected components of solutions for variational inequalities. In: Giannessi, F. (ed.) *Vector Variational Inequalities and Vector Equilibria,* 253-265. Nonconvex Optimization and its Applications 38. Kluwer Academic Publishers, Dordrecht.

[141] Iwanow, E. and Nehse, R. (1985). Some results on dual vector optimization problems. *Mathematische Operationsforschung und Statistik. Series Optimization,* 16(4):505-517.

[142] Jagannathan, R. (1966). On some properties of programming problems in parametric form pertaining to fractional programming. *Management Sciences,* 12:609-615.

[143] Jahn, J. (1983). Duality in vector optimization. *Mathematical Programming,* 25:343-353.

[144] Jahn, J. (1984). Scalarization in vector optimization. *Mathematical Programming,* 29:203-218.

[145] Jahn, J. (1985). A Characterization of properly minimal elements of a set. *SIAM Journal on Control and Optimization,* 23(5):649-656.

[146] Jahn, J. (1986). *Mathematical Vector Optimization in Partially Ordered Spaces.* Peter Lang Verlag, Frankfurt am Main, Bern, New York.

[147] Jahn, J. (1986). Existence theorems in vector optimization. *Journal of Optimization Theory and Applications,* 50:397-406.

[148] Jahn, J. (1987). Duality in partially ordered sets. In: Jahn,J. and Krabs.W. (eds.) *Recent advances and Historical Development of Vector Optimization,* 160-172. Lecture Notes in Economics and Mathematical Systems 294. Springer-Verlag, Berlin.

[149] Jahn, J. (1987). Parametric approximation problems arising in vector optimization. *Journal of Optimization Theory and Applications,* 54:503-516.

[150] Jahn, J. (1988). A generalization of a theorem of Arrow, Barankin and Blackwell. *SIAM Journal on Control and Optimization,* 26(5):999-1005.

[151] Jahn, J. and Krabs, W. (1988). Applications of multicriteria optimization in approximation theory. In: Stadler, W. (ed.) *Multicriteria Optimization in Engineering and the Sciences,* 49-75. Plenum Press, New York.

[152] Jongen, H.T., Möbert, T. and Tammer, K. (1986). On iterated minimization in nonconvex optimization. *Mathematics of Operations Research* 11:679-691.

[153] Kaliszewski, I. (1987). A modified weighted Tchebycheff metric for multiple objective programming. *Computers and Operations Research,* 14:315-323.

[154] Kantorowitsch, L.W. and Akilow, G.P. (1978). *Funktionalanalysis in normierten Räumen.* Akademie-Verlag, Berlin.

[155] Kaul, R.N. and Lyall, V. (1989). A note on nonlinear fractional vector maximization. *Operations Research,* 26(2):108-121.

[156] Kawasaki, H. (1981). Conjugate relations and weak subdifferentials. *Mathematics of Operations Research* 6:593-607.

[157] Kawasaki, H. (1982). A duality theorem in multiobjective nonlinear programming. *Mathematics of Operations Research,* 7:95-110.

[158] Kazmi, K.R. (2000). Existence of solutions for vector saddle-point problems. In: Giannessi, F. (ed.) *Vector Variational Inequalities and Vector Equilibria,* 267-275. Nonconvex Optimization and its Applications 38. Kluwer Academic Publishers, Dordrecht.

[159] Khanh, P.Q. (1989). On Caristi-Kirk's theorem and Ekeland's variational principle for Pareto extrema. *Bulletin of the Polish Academy of Sciences, Mathematics,* 37:33-39.

[160] Klose, J. (1992). Sensitivity analysis using the tangent derivative. *Numerical Functional Analysis and Optimization,* 13:143-153.

[161] Klötzler, R. (1978). Multiobjective dynamic programming. *Mathematische Operationsforschung und Statistik. Series Optimization,* 9(3):423-426.

[162] Klötzler, R. (1979). On a general conception of duality in optimal control. In: F'abera, J. (ed.) *Equadiff IV,* 183-196. Lecture Notes in Mathematics 703. Springer-Verlag, Berlin.

[163] Konnov, I.V. and Yao, J.C. (1997). On the generalized vector variational inequality problem. *Journal of Mathematical Analysis and Applications,* 206:42-58.

[164] Krasnosel'skij, M.A. (1962). *Positive Solutions of Operator Equations* (in Russian). Fizmatgiz, Moskow.

[165] Kuhpfahl, I., Patz, R., and Tammer, C. (1996). Location problems in town planning. In: D. Schweigert (ed.) *Methods of Multicriteria Decision Theory,* 101-112. University of Kaiserslautern, Kaiserslautern.

[166] Kuk, H., Tanino, T., and Tanaka, M. (1996). Sensitivity analysis in vector optimization. *Journal of Optimization Theory and Applications,* 89:713-730.

[167] Kuk, H., Tanino, T., and Tanaka, M. (1996). Sensitivity analysis in parametrized convex vector optimization. *Journal of Mathematical Analysis and Applications,* 202:511-522.

[168] Kummer, B. (1992). Newton's method based on generalized derivatives for nonsmooth functions: Convergence analysis. In: Oettli, W. and Pallaschke, D. (eds.) *Advances in Optimization, Proceedings Lambrecht 1991,* 171-194. Springer-Verlag, Berlin.

[169] Kuratowski, K. (1966). *Topology.* Academic Press, New York and Polish Scientific Publishers, Warsaw.

[170] Lai, T.C. and Yao, J.C. (1996). Existence results for VVIP. *Applied Mathematical Letters,* 9:17-19.

[171] Lampe, U. (1981). Dualität und eigentliche Effizienz in der Vektoroptimierung. *Seminarberichte der Sektion Mathematik der Humboldt- Universität zu Berlin,* 37:45-54.

[172] Lee, G.M., Kim, D.S., and Lee, B.S. (1996). On noncooperative vector equilibrium. *Indian Journal of Pure and Applied Mathematics,* 27(8):735-739.

[173] Lee, G.M., Kim, D.S., and Lee, B.S. (1996). Generalized vector variational inequality. *Applied Mathematical Letters,* 9:39-42.

[174] Lee, G.M., Kim, D.S., Lee, B.S., and Cho, S.J. (1993). Generalized vector variational inequality and fuzzy extension. *Applied Mathematical Letters,* 6:47-51.

[175] Lee, G.M., Kim, D.S., Lee, B.S., and Yen, N.D. (1998). Vector variational inequality as a tool for studying vector optimization problems. *Nonlinear Analysis,* 34:745-765.

[176] Lee, G.M., Kim, D.S., Lee, B.S., and Yen, N.D. (2000). Vector variational inequality as a tool for studying vector optimization problems. In: Giannessi, F. (ed.) *Vector Variational Inequalities and Vector Equilibria,* 277-305. Nonconvex Optimization and its Applications 38. Kluwer Academic Publishers, Dordrecht.

[177] Lee, G.M. and Kum, S. (2000). Vector variational inequalities in a Hausdorff topological vector space. In: Giannessi, F. (ed.) *Vector Variational Inequalities and Vector Equilibria,* 307-320 Nonconvex Optimization and its Applications 38. Kluwer Academic Publishers, Dordrecht.

[178] Lee, G.M. and Kum, S. (2000). On implicit vector variational inequalities. *Journal of Optimization Theory and Applications,* 104(2), 409-425.

[179] Lee, B.S., Lee, G.M., and Kim, D.S. (1996). Generalized vector-valued variational inequalities and fuzzy extension. *Journal of the Korean Mathematical Society,* 33:609-624.

[180] Lee, B.S., Lee, G.M., and Kim, D.S. (1997). Generalized vector variational-like inequalities in locally convex Hausdorff topological vector spaces. *Indian Journal of Pure and Applied Mathematics,* 28:33-41.

[181] Li, Z.F. and Wang, S.Y. (1994). Lagrange multipliers and saddle points in multiobjective programming. *Journal of Optimization Theory and Applications,* 83(1):63-81.

[182] Li, S.J., Yang, S.Q., and Chen, G.Y. (2000). Vector Ekeland variational principle. In: Giannessi, F. (ed.) *Vector Variational Inequalities and Vector Equilibria,* 321-333. Nonconvex Optimization and its Applications 38. Kluwer Academic Publishers, Dordrecht.

[183] Lin, L.J. (1996). Pre-vector variation inequalities. *Bulletin of the Australian Mathematical Society,* 53:63-70.

[184] Lin, K.L., Yang, D.P., and Yao, J.C. (1997). Generalized vector variational inequalities. *Journal of Optimization Theory and Applications,* 92:117-125.

[185] Loridan, P. (1984). ϵ-solutions in vector minimization problems. *Journal of Optimization Theory and Applications,* 43(2):265-276.

[186] Loridan, P. (1984). A dual approach to the generalized Weber problem under locational uncertainty. *Cahiers du Centre dÉtudes de Recherche Opérationnelle,* 20(3-4):241-253.

[187] Loridan, P. and Morgan, J. (2000). Convergence of approximate solutions and values in parametric vector optimization. In: Giannessi, F. (ed.) *Vector Variational Inequalities and Vector Equilibria,* 335-350. Nonconvex Optimization and its Applications 38. Kluwer Academic Publishers, Dordrecht.

[188] Loridan, P., Morgan, J., and Raucci, R. (1999). Convergence of minimal and approximate minimal elements of sets in partially ordered vector spaces. *Journal of Mathematical Analysis and Applications,* 239(2):427-439.

[189] Luc, D.T. (1989). An existence theorem in vector optimization. *Mathematics of Operations Research,* 14:693-699.

[190] Luc, D.T. (1989). *Theory of Vector Optimization.* Lecture Notes in Economics and Mathematical Systems 319. Springer Verlag Berlin Heidelberg.

[191] Luc, D.T., Lucchetti, R., and Malivert, C. (1994). Convergence of efficient sets. *Set-Valued Analysis,* 2:207-218.

[192] Malivert, C. (1992). Fenchel duality in vector optimization. In: Oettli, W. and Pallaschke, D. (eds.) *Advances in Optimization,* 420-438. Lecture Notes in Economics and Mathematical Systems 382. Springer-Verlag, Berlin.

[193] Mastroeni, G. (2000). On Minty vector variational inequality. In: Giannessi, F. (ed.) *Vector Variational Inequalities and Vector Equilibria,* 351-361. Nonconvex Optimization and its Applications. Kluwer Academic Publishers, Dordrecht.

[194] Michelot, C. and Lefebvre, O. (1987). A primal-dual algorithm for the Fermat-Weber problem involving mixed gauges. *Mathematical Programming,* 39:319-335.

[195] Miettinen, K. (1999). *Nonlinear Multiobjective Optimization.* Kluwer Academic Publishers, Boston.

[196] Miettinen, K. and Mäkelä, M.M. (2000). Tangent and normal cones in nonconvex multiobjective optimization. In: Haimes, Y.Y. and Steuer, R. E. (eds.) *Research and Practice in Multiple Criteria Decision Making,* 114-124. Lecture Notes in Economics and Mathematical Systems 487. Springer-Verlag, Berlin, Heidelberg.

[197] Minami, M. (1983). Weak Pareto-optimal necessary conditions in a nondifferentiable multiobjective program on a Banach space. *Journal of Optimization Theory and Applications* 41(3):451-461.

[198] Nakayama, H. (1984). Geometric consideration of duality in vector optimization. *Journal of Optimization Theory and Applications,* 44:625-655.

[199] Nakayama, H. (1985). Duality theory in vector optimization: An overview. In: Haimes, Y.Y. and Chankong, V. (eds.) *Decision making with multiple objectives,* 109-125. Lecture Notes in Economics and Mathematical Systems 242. Springer-Verlag, Berlin.

[200] Nehse, R. (1981). Duale Vektoroptimierungsprobleme vom Wolfe-Typ. *Seminarberichte der Sektion Mathematik der Humboldt-Universität zu Berlin,* 37:55-60.

[201] Nemeth, A.B. (1981). The nonconvex minimization principle in ordered regular Banach spaces. *Revue d'Analyse Numérique et de la Théorie de l'Approximation,* 23(46):43-48.

[202] Nemeth, A.B. (1986). A nonconvex vector minimization problem. *Nonlinear Analysis,* 10(7):669-678.

[203] Nemeth, A.B. (1989). Between Pareto efficiency and Pareto ϵ-efficiency. *Optimization,* 20(5):615-637.

[204] Nieuwenhuis, J.W. (1980). Supremal points and generalized duality. *Mathematische Operationsforschung und Statistik, Series Optimization,* 11:41-59.

[205] Nikodem, K. (1986). Continuity of K-convex set-valued functions. *Bulletin of the Polish Academy of Sciences. Mathematics,* 34:393-400.

[206] Nikodem, K. (1987). On midpoint convex set-valued functions. *Aequationes Mathematicae,* 33:46-56.

[207] Noor, M.A. (1982). Strongly nonlinear variational inequalities. *Comptes Rendus Mathématiques. Reports of the Academy of Sciences of Canada,* 4:213-218.

[208] Oettli, W. (1977). Approximate solutions of variational inequalities. In: Albach, H., Helmstädter, E., and Henn, R. (eds.) *Quantitative Wirtschaftsforschung,* 535-538. Verlag J.C.B. Mohr, Tübingen.

[209] Oettli, W. (1997). A remark on vector-valued equilibria and generalized monotonicity. *Acta Mathematica Vietnamica,* 22:213-221.

[210] Parida, J. and Sen, A. (1987). A variational-like inequality for multifunctions with applications. *Journal of Mathematical Analysis and Applications,* 124:73-81.

[211] Pascoletti, A. and Serafini, P. (1984). Scalarizing vector optimization problems. *Journal of Optimization Theory and Applications,* 42:499-524.

[212] Pelegrin, B. and Fernández, F. R. (1988). Determination of efficient points in multiple objective location problems. *Naval Research Logistics,* 35:697-705.

[213] Penot, J.P. (1984). Differentiability of relations and differential stability of perturbed optimization problems. *SIAM Journal on Control and Optimization,* 22:528–551.

[214] Penot, J.P. (1978). L'optimisation à la Pareto: Deux ou trois choses que je sais d'Elle. In: *Proceedings of "Structures Economiques et Econométric",* 4.14-4.33. Lyon.

[215] Penot, J.P. (1986). The drop theorem, the petal theorem and Ekeland's variational principle. *Nonlinear Analysis,* 10(9):813-822.

[216] Penot, J.P. and Sterna-Karwat, A. (1986). Parametrized multicriteria optimization: Continuity and closedness of optimal multifunctions. *Journal of Mathematical Analysis and Applications,* 120:150-168.

[217] Penot, J.P. and Sterna-Karwat, A. (1989). Parametrized multicriteria optimization: Order continuity of the marginal multifunctions. *Journal of Mathematical Analysis and Applications,* 144:1-15.

[218] Penot, J.P. and Théra, M. (1982). Semi-continuous mappings in general topology. *Archiv der Mathematik,* 32:158-166.

[219] Peressini, A.L. (1967). *Ordered Topological Vector Spaces.* Harper and Row Publishers, New York.

[220] Phelps, R.R. (1989). *Convex Functions, Monotone Operators and Differentiability.* Lecture Notes in Mathematics 1364. Springer-Verlag, Berlin.

[221] Phelps, R.R. (1993). *Convex Functions, Monotone Operators and Differentiability* (2nd edition). Lecture Notes in Mathematics 1364. Springer-Verlag, Berlin.

[222] Podhinovski, V.V. and Nogin, V.D. (1982). *Pareto Optimal Solutions of Multicriteria Problems* (in Russian). Nauka, Moscow.

[223] Postolică, V. (1989). Existence conditions of efficient points for multifunctions with values in locally convex spaces ordered by supernormal cones. *Studii şi Cercetări Matematice,* 41:325–339.

[224] Postolică, V. (1993). New existence results for efficient points in locally convex spaces. *Journal of Global Optimization,* 3:233-242.

[225] Puerto, F.P. and Fernandez, F.R. (1994). Multicriteria decisions in location. *Studies in Locational Analysis,* 7:185-199.

[226] Qun, L. (2000). Generalized vector variational-like inequalities. In: Giannessi, F. (ed.) *Vector Variational Inequalities and Vector Equilibria,* 363-369. Nonconvex Optimization and its Applications 38. Kluwer Academic Publishers, Dordrecht.

[227] Ralph, D. and Scholtes, S. (1997), Sensitivity analysis and Newton's method for composite piecewise smooth equations. *Mathematical Programming,* 76:593-612.

[228] Rapcsák, T. (2000). On vector complementarity systems and vector variational inequalities. In: Giannessi, F. (ed.) *Vector Variational Inequalities and Vector Equilibria,* 371-380. Nonconvex Optimization and its Applications 38. Kluwer Academic Publishers, Dordrecht.

[229] Rockafellar, R.T. (1979). Directionally Lipschitzian functions and subdifferential calculus. *Proceedings of the London Mathematical Society,* 39:331-355.

[230] Rolewicz, S. (1987). On drop property. *Studia Mathematica,* 85:27-35.

[231] Rosinger, E.E. (1978). Multiobjective duality without convexity. *Journal of Mathematical Analysis and Applications,* 66:442-450.

[232] Salukvadze, M.E. (1979). *Vector-valued Optimization Problems in Control Theory.* Academic Press, New York.

[233] Sawaragi, Y., Nakayama, H., and Tanino, T. (1985). *Theory of Multiobjective Optimization.* Academic Press, Orlando FL.

[234] Schaible, S. and Ibaraki, T. (1983). Fractional programming. *European Journal of Operations Research,* 12:325-338.

[235] Schandl, B., Klamroth, K., and Wiecek, M.M. (2000). Using block norms in bicriteria optimization. In: Haimes, Y.Y. and Steuer,

R.E. (eds.) *Research and Practice in Multiple Criteria Decision Making,* 149-160. Lecture Notes in Economics and Mathematical Systems 487. Springer-Verlag,Berlin, Heidelberg.

[236] Scholtes, S. (1994). Introduction to piecewise differentiable equations. Preprint No. 53/1994, Universität Karlsruhe, Institut für Statistik und Mathematische Wirtschaftstheorie.

[237] Schönfeld, P. (1970). Some duality theorems for the nonlinear vector maximum problem. *Unternehmensforschung* 14:51-63.

[238] Shi, D.S. (1991). Contingent derivatives of the perturbation in multiobjective optimization. *Journal of Optimization Theory and Applications,* 70:351-362.

[239] Shi, D.S. (1993). Sensitivity analysis in convex vector optimization. *Journal of Optimization Theory and Applications,* 77:145-159.

[240] Siddiqi, A.H., Ansari, Q.H., and Ahmad, R. (1997). On vector variational-like inequalities. *Indian Journal of Pure and Applied Mathematics,* 28:1009-1016.

[241] Siddiqi, A.H., Ansari, Q.H., and Khan, M.F. (1999). Variational-like inequalities for multivalued maps. *Indian Journal of Pure and Applied Mathematics,* 3(2):161-166.

[242] Sinh, C. (1987). Optimality conditions in multiobjective differentiable programming. *Journal of Optimization Theory and Applications,* 53:115-123.

[243] Smithson, R.E. (1975). Subcontinuity for multifunctions. *Pacific Journal of Mathematics,* 61:283-288.

[244] Song, W. (2000). Generalized vector variational inequalities. In: Giannessi, F. (ed.) *Vector Variational Inequalities and Vector Equilibria,* 381-401. Nonconvex Optimization and its Applications 38. Kluwer Academic Publishers, Dordrecht.

[245] Song, W. (2000). Vector equilibrium problems with set-valued mappings. In: Giannessi, F. (ed.): *Vector Variational Inequalities and Vector Equilibria,* 402-422. Nonconvex Optimization and its Applications 38. Kluwer Academic Publishers, Dordrecht.

[246] Sonntag, Y. and Zălinescu, C. (2000). A comparison of existence results for efficient points. *Journal of Optimization Theory and Applications,* 105:161-188.

[247] Staib, T. (1988). On generalizations of Pareto minimality. *Journal of Optimization Theory and Applications,* 59:289-306.

[248] Sterna-Karwat, A. (1989). Convexity of the optimal multifunctions and its consequences to vector optimization. *Optimization,* 20:799-808.

[249] Sterna-Karwat, A. (1987). Continuous dependence of solutions on a parameter in a scalarization method. *Journal of Optimization Theory and Applications,* 55:417-434.

[250] Sterna-Karwat, A. (1986). On existence of cone maximal points in real topological linear spaces. *Israel Journal of Mathematics,* 54:33-41.

[251] Sterna-Karwat, A. (1988). Remarks on convex cones. *Journal of Optimization Theory and Applications,* 59:335-340.

[252] Takahashi, W. (1991). Existence theorems generalizing fixed point theorems for multivalued mappings. In: Baillon, J.B. and Thera, M. (eds.) *Fixed point theory and applications,* 397-406. Pitman Research Notes in Mathematics 252. Longman, Harlow.

[253] Tammer, C. (1992). A generalization of Ekeland's variational principle. *Optimization,* 25:129-141.

[254] Tammer, C. (1993). Existence results and necessary conditions for ϵ-efficient elements. In: Brosowski, B., Ester, J., Helbig, S. and Nehse, R. (eds.) *Multicriteria Decision – Proceedings of the 14th Meeting of the German Working Group "Mehrkriterielle Entscheidung",* 97-110. Peter Lang Verlag, Frankfurt am Main, Bern.

[255] Tammer, C. (1993). Erweiterungen und Anwendungen des Variationsprinzips von Ekeland. *Zeitschrift für Angewandte Mathematik und Mechanik,* 73:832-826.

[256] Tammer, C. (1994). A variational principle and a fixed point theorem. In: Henry, J. and Yvon, J.P. (eds.) *System Modelling and Optimization,* 248-257. Lecture Notes in Control and Information Sciences 197. Springer, London.

[257] Tammer, C. (1994). A variational principle and applications for vectorial control approximation problems. *Mathematical Journal of the University of Bacau (Romania),* 4:85-98.

[258] Tammer, C. (1994). Stability results for approximately efficient solutions. *Operations Research Spektrum,* 16:47-52.

[259] Tammer, C. (1996). Approximate solutions of vector-valued control-approximation problems. *Studies in Locational Analysis,* 10:151-162.

[260] Tammer, C. (1998). Multiobjective optimal control problems. In: Schmidt, W.H. (ed.) *Variational Calculus, Optimal Control and Applications,* 97-106. International Series of Numerical Mathematics 124. Birkhäuser Verlag, Basel.

[261] Tammer, C. and Tammer, K. (1991). Generalization and sharpening of some duality relations for a class of vector optimization problems. *Zeitschrift für Operations Research,* 35:249-265.

[262] Tammer, K., Tammer, C., and Ohlendorf, E. (1996). Multicriterial fractional optimization. In: Guddat, J., Jongen, H.T., Nozicka, F., Still, G. and Twilt, F. (eds.) *Parametric Optimization and Related Topics IV,* 359-370. Verlag Peter Lang, Frankfurt am Main.

[263] Tammer, K. (1995). Two-level optimization with approximate solutions in the lower level. *Zeitschrift für Operations Research,* 41:231-249.

[264] Tanaka, T. (1988). Some minimax problems of vector valued functions. *Journal of Optimization Theory and Applications,* 59:505-524.

[265] Tanaka, T. (1989). Existence Theorems for Cone Saddle Points of Vector-Valued Functions in Infinite-Dimensional Spaces. *Journal of Optimization Theory and Applications,* 62:127-138.

[266] Tanaka, T. (1991). Two Types of Minimax Theorems for Vector-Valued Functions. *Journal of Optimization Theory and Applications,* 68:321–334.

[267] Tanaka, T. (1994). Generalized Quasiconvexities, Cone Saddle Points, and Minimax Theorem for Vector-Valued Functions. *Journal of Optimization Theory and Applications,* 81:355–377.

[268] Tanino, T. (1988). Sensitivity analysis in multiobjective optimization. *Journal of Optimization Theory and Applications,* 56:479-499.

[269] Tanino, T. and Sawaragi, Y. (1979). Duality theory in multiobjective programming. *Journal of Optimization Theory and Applications,* 30:229-253.

[270] Tanino, T. and Sawaragi, Y. (980). Stability of nondominated solutions in multicriteria decision-making. *Journal of Optimization Theory and Applications,* 30:229-253.

[271] Thibault, L. (1997). Sequential convex subdifferential calculus and sequential Lagrange multipliers. *SIAM Journal on Control and Optimization,* 35(4): 1434-1444.

[272] Turinici, M. (1991). Cone maximal points in topological linear spaces. *Analele Stiinţificeale Universităţii Al.I.Cuza din Iaşi, (Serie Nouă.) Secţiunea Ia matematică.*, 37:371-390.

[273] Valyi, I. (1985). On duality theory related to approximate solutions of vector-valued optimization problems. In: Demyanov, V.F.

and Pallaschke, D. (eds.) *Nondifferentiable Optimization: Motivations and Applications*, 150-162. Lecture Notes in Economics and Mathematical Systems 255. Springer-Verlag, Berlin.

[274] Valyi, I. (1987). Approximate saddle-point theorems in vector optimization. *Journal of Optimization Theory and Applications*, 55(3):435-448.

[275] Wang, S. (1984). Lagrange conditions in nonsmooth and multiobjective mathematical programming. *Mathematics in Economics*, 1:183-193.

[276] Wanka, G. (1991). Duality in vectorial control approximation problems with inequality restrictions. *Optimization*, 22:755-764.

[277] Wanka, G. (1991). On duality in the vectorial control-approximation problem. *Zeitschrift für Operations Research*, 35:309-320.

[278] Wanka, G. (1996). Characterization of approximately efficient solutions to multiobjective location problems using Ekeland's variational principle. *Studies in Locational Analysis*, 10:163-176.

[279] Weidner, P. (1985). Dominanzmengen und Optimalitätsbegriffe in der Vektoroptimierung. *Wissenschaftliche Zeitschrift der TH Ilmenau,* 31:133-146.

[280] Weidner, P. (1988). On the characterization of efficient points by means of monotone functionals. *Optimization*, 19:53-69.

[281] Weidner, P. (1991). *Ein Trennungskonzept und seine Anwendungen auf Vektoroptimierungsverfahren.* Dissertation B, Martin-Luther-Universität Halle-Wittenberg.

[282] Weir, T. (1986). A duality theorem for a multiple objective fractional optimization problem. *Bulletin of the Australian Mathematical Society,* 34:415-425.

[283] Wierzbicki, A.P. (1977). Basis properties of scalarizing functionals for multiobjective optimization. *Mathematische Operationsforschung und Statistik. Series Optimization*, 8:55-60.

[284] Yamada, S., Tanino, T., and Inuiguchi, M. (2000). An outer approximation method for optimization over the efficient set. In: Haimes, Y.Y. and Steuer, R.E. (eds.) *Research and Practice in Multiple Criteria Decision Making*, 196-207. Lecture Notes in Economics and Mathematical Systems 487. Springer-Verlag, Berlin, Heidelberg.

[285] Yang, X.Q. (1993). Generalized convex functions and vector variational inequalities. *Journal of Optimization Theory and Applications*, 79:563-580.

[286] Yang, X.Q. (1993). Vector complementarity and minimal element problems. *Journal of Optimization Theory and Applications,* 77:483-495.

[287] Yang, X.Q. (1993). Vector variational inequalitiy and its duality. *Nonlinear Analysis,* 21:869-877.

[288] Yang, X.Q. (1997). Vector variational inequalitiy and vector pseudolinear optimization. *Journal of Optimization Theory and Applications,* 95:729-734.

[289] Yang, X.Q. (2000). On some equilivalent conditions of vector variational inequalities. In: Giannessi, F. (ed.) *Vector Variational Inequalities and Vector Equilibria,* 423-432. Nonconvex Optimization and its Applications 38. Kluwer Academic Publishers, Dordrecht.

[290] Yang, X.Q. and Chen, G.Y. (2000). On inverse vector variational inequalities. In: Giannessi, F. (ed.) *Vector Variational Inequalities and Vector Equilibria,* 433-446. Nonconvex Optimization and its Applications 38. Kluwer Academic Publishers, Dordrecht.

[291] Yang, X.Q. and Goh, C.J. (1997). On vector variational inequalities: Application to vector equilibria. *Journal of Optimization Theory and Applications,* 95:431-443.

[292] Yang, X.Q. and Goh, C.J. (2000). Vector variational inequalities, vector equilibrium flow and vector optimization. In: Giannessi, F. (ed.) *Vector Variational Inequalities and Vector Equilibria,* 447-465. Nonconvex Optimization and its Applications 38. Kluwer Academic Publishers, Dordrecht.

[293] Yao, J.C. (1991). A basic theorem of complementarity for the generalized variational-like inequality problem. *Journal of Mathematical Analysis and Applications,* 158:124-138.

[294] Yao, J.C. (1993). Abstract variational inequality problem and a basic theorem of complementarity. *Computers and Mathematics with Applications,* 25:73-79.

[295] Yen, N.D. and Lee, G.M. (2000). On monotone and strongly monotone vector variational inequalities. In: Giannessi, F. (ed.) *Vector Variational Inequalities and Vector Equilibria,* 467-478. Nonconvex Optimization and its Applications 38. Kluwer Academic Publishers, Dordrecht.

[296] Yen, N.D. and Phuong, T.D. (2000). Connectedness and stability of the solution sets in linear fractional vector optimization problems. In: Giannessi, F. (ed.) *Vector Variational Inequalities and Vector Equilibria,* 479-489. Nonconvex Optimization and its Applications 38. Kluwer Academic Publishers, Dordrecht.

[297] Yin, H. and Xu, C. (2000). Vector variational inequality and implicit vector complementarity problems. In: Giannessi, F. (ed.) *Vector Variational Inequalities and Vector Equilibria*, 491-505. Nonconvex Optimization and its Applications 38. Kluwer Academic Publishers, Dordrecht.

[298] Yu, S.Y. and Yao, J.C. (1996). On vector variational inequalities. *Journal of Optimization Theory and Applications*, 89:749-769.

[299] Yu, P.L. (1974). Cone convexity, cone extreme points and nondominated solutions in decision problems with multiobjectives. *Journal of Optimization Theory and Applications*, 14:319-377.

[300] Zălinescu, C. (1983). Duality for vectorial nonconvex optimization by convexification and applications. *Analele Stiinţificeale Universităţii Al.I.Cuza din Iaşi, (Serie Nouă.) Secţiunea Ia matematică.*, 29(3):15-34.

[301] Zălinescu, C. (1987). Solvability results for sublinear functions and operators. *Zeitschrift für Operations Research,* 31:79-101.

[302] Zălinescu, C. (1987). On two notions of proper efficiency. In: Brosowski, B. and Martensen, E. (eds.) *Optimization in Mathematical Physics. Methoden und Verfahren der mathematischen Physik,* 34:77-86.

[303] Zălinescu, C. (1989). Stability for a class of nonlinear optimization problems and applications. In: Clarke, F.H., Demyanov, V.F., and Giannessi, F. (eds.) *Nonsmooth Optimization and Related Topics,* 437-458. Plenum Press, New York.

[304] Zălinescu, C. (1992). On a new stability condition in mathematical programming. In: Giannessi, F. (ed.) *Nonsmooth Optimization. Methods and Applications,* 429-438. Gordon and Breach Science Publishers, Philadelphia.

[305] Zeidler, E. (1986). *Nonlinear Functional Analysis and its Applications. Part III: Variational Methods and Optimization.* Springer-Verlag, New York.

[306] Zowe, J. (1974). Subdifferentiability of convex functions with values in a ordered vector space. *Mathematica Scandinavica,* 34:69-83.

[307] Zowe, J. (1975). A duality theorem for a convex programming problem in ordered complete vector lattices. *Journal of Mathematical Analysis and Applications,* 50:273-287.

Chapter 2

NONLINEAR MULTIOBJECTIVE PROGRAMMING

Tetsuzo Tanino
Department of Electronics and Informations Systems
Graduate School of Engineering, Osaka University
Suita, Osaka 565-0871
Japan
tanino@eie.eng.osaka-u.ac.jp

Hun Kuk
Department of Mathematics
School of Electronics and Information, Kyung Hee University
Yongin-si, Kyungki-do 449-701
Korea
kuk@khu.ac.kr

Abstract This chapter provides an annotated bibliography of nonlinear multiobjective programming. The list of references comprises more than 500 entries. First we explain some solution concepts which are fundamental and important in multiobjective optimization. Some basic properties of the solution sets are also discussed. The next section is devoted to scalarization techniques and optimality conditions for nonlinear multiobjective programming problems. The third topic is stability and sensitivity analysis, which discusses the behavior of the set of efficient points according to the change of parameter values in a nonlinear multiobjective programming problem qualitatively and quantitatively. The following section is devoted to several aspects of duality theory, i.e., Lagrange duality, conjugate duality and Wolfe type and Mond-Weir type duality with generalized convexity. Finally vector variational inequalities are also dealt with.

Keywords: Efficiency, Scalarization, Stability and sensitivity, Duality, Vector variational inequality.

1. Introduction

In this chapter we review several theoretical results concerning multiobjective optimization (vector optimization) or multiobjective programming. Since this volume contains another chapter dealing with vector optimization in abstract spaces (Tammer and Göpfert, Chapter), we mainly concentrate on nonlinear multiobjective programming problems which are defined on the ordinary Euclidean spaces and have a finite number of objective functions. In many cases, however, the results were obtained in more general forms or can be extended to more general cases. As for the results in more general spaces, the readers may refer to Chapter in this volume and [116]. Utility theory and several solution methods such as interactive methods are not included either. Miettinen provides a review of the latter topic in Chapter 9 of this volume.

Thus we consider the following nonlinear multiobjective optimization problem in this chapter:

$$\begin{array}{ll} \text{minimize} & f(x) = (f_1(x), f_2(x), \ldots, f_p(x)) \\ \text{subject to} & x \in X \subset \mathbb{R}^n \end{array}$$

Here x is an n-dimensional decision variable, $f_1(x), f_2(x), \ldots, f_p(x)$ are objective functions and X is a constraint set in the n-dimensional Euclidean space. Particularly in nonlinear multiobjective programming, the constraint set X is usually specified by a finite number of inequality and/or equality constraints as in the following:

$$X = \{x \in \mathbb{R}^n \mid g_i(x) \leq 0,\ i = 1, \ldots, m;\ h_j(x) = 0,\ j = 1, \ldots, l\}$$

This type of problem has been intensively studied in the last few decades and several books have been published (for example, White [483], Chankong and Haimes [80], Sawaragi, Nakayama and Tanino [397], Yu [508], Steuer [433], Guddat et al. [162], Jahn [202], Luc [313], Miettinen [332], Ehrgott [134] and so on). Some books deal with multiobjective programming or vector optimization partially (e.g. Pallaschke and Rolewicz [363] Chapter 10 and Shimizu et al. [404] Chapter 13).

In this chapter we briefly provide an annotated bibliography of nonlinear multiobjective programming. The contents of this chapter are as in the following:

2 Solution concepts

3 Scalarization and optimality conditions

4 Stability and sensitivity analysis

5 Duality

6 Vector variational inequality

2. Solution Concepts

In this section we will discuss some solution concepts in multiobjective optimization problems and consider properties of solution sets. An axiomatic approach to solution concepts can be seen in [482]. See also the monographs introduced in the first section and [206].

2.1. Efficiency and Weak Efficiency

An ordinary mathematical programming (or optimization) problem includes only one objective function, and our aim is to find an element which minimizes this function. In other words, the objective space is of one dimension and therefore the ordering in the objective space is trivial in ordinary mathematical programming. On the contrary, in a multiobjective problem, an element which minimizes an objective function does not generally minimize another objective function. Of course, if there exists a feasible solution $x^* \in X$ which minimizes all the objective functions simultaneously, it provides a solution to the problem. It is called the completely optimal solution and is defined formally by the following relation:

$$f_i(x^*) \leq f_i(x), \quad \text{for all } x \in X.$$

However, the completely optimal solution seldom exists and we introduce another solution concept. A feasible solution $x^* \in X$ is said to be a noninferior (a Pareto optimal, or an efficient) solution if there exists no $x \in X$ such that

$$\begin{aligned} &f_i(x) \leq f_i(x^*), \quad \text{for all } i = 1, \ldots, p; \\ &f_i(x) < f_i(x^*), \quad \text{for some } i \in \{1, \ldots, p\} \end{aligned}$$

[381, 512]. Moreover, $x^* \in X$ is said to be a weakly noninferior solution if there exists no $x \in X$ such that

$$f_i(x) < f_i(x^*), \quad \text{for all } i = 1, \ldots, p.$$

It is noted that $x^* \in X$ is noninferior if and only if $f(x^*)$ is a minimal element of the feasible set $f(X)$ in the objective space $\mathbb{R}^p$ with respect to the ordering cone $\mathbb{R}^p_+$. Unfortunately this ordering is not a linear (total) order, but a partial order and the minimal element is not unique. Of course, more general domination structures can be considered in the objective space [42, 53, 59, 507, 508, 509, 510]. Typically, let K be a pointed convex cone in $\mathbb{R}^p$. Then $x^* \in X$ is said to be K-efficient or K-minimal (resp. weakly K-efficient or weakly K-minimal) if there exists no $x \in X$ such that

$$\begin{aligned} &f(x^*) \in f(x) + K \setminus \{0\}, \\ &(\text{resp. } f(x^*) \in f(x) + \text{int } K). \end{aligned}$$

For a set $Y \in \mathbb{R}^p$, we often use the following notations:

$$\begin{aligned} \mathrm{Min}_K Y &= \{y \in \mathbb{R}^p \mid (y - K) \cap Y = \{y\}\}, \\ \mathrm{WMin}_K Y &= \{y \in \mathbb{R}^p \mid (y - \mathrm{int}\ K) \cap Y = \emptyset\}. \end{aligned}$$

Then $x^* \in X$ is K-efficient (respectively weakly K-efficient) if and only if $f(x^*) \in \mathrm{Min}_K f(X)$ (resp. $f(x^*) \in \mathrm{WMin}_K f(X)$). The cone K is often omitted in the case $K = \mathbb{R}^p_+$ in those notations.

The above definition provides globally efficient solutions. We may, of course, define locally efficient solutions as in the case of ordinary optimization. Any globally efficient solution is locally efficient, but the converse is true only under appropriate convexity assumptions [73, 318].

A characterization of efficiency and weak efficiency using level sets is given in [135], along with a new solution concept of strict Pareto optimality. In a convex multiobjective programming problem, the set of weakly efficient solutions is the union of the sets of efficient solutions of problems with parts of the original objective functions [134, 324].

We should also note that the bounds on the efficient set in the objective space are given by the ideal point and the Nadir point [134, 332].

Moreover, approximate solutions to the multiobjective optimization problem are considered. For example, an ε-efficient solution is defined in [303] as a feasible solution x^* for which there exists no $x \in X$ such that

$$\begin{aligned} &f_i(x) \leq f_i(x^*) - \varepsilon_i, \quad \text{for all } i = 1, \ldots, p; \\ &f_i(x) < f_i(x^*) - \varepsilon_i, \quad \text{for some } i \in \{1, \ldots, p\}, \end{aligned}$$

where ε is a vector in $\mathbb{R}^p_+$. Another type of ε-efficiency is given in [486]. Studies on approximate solutions can be found in [117, 284, 295, 296, 300, 301, 303, 304, 367, 458, 459, 505].

Recently, a problem of optimizing another objective function over the (weakly) efficient set of a multiobjective programming problem has been discussed by several authors [40, 56, 85, 112, 457, 493, 494]. This problem requires a combination of global optimization and multiobjective optimization.

2.2. Trade-off Rates and Proper Efficiency

At an efficient point $f(x^*)$ we may consider the trade-off rate between two objectives f_i and f_j, which is the change of the value of f_i concerning the unit change of the value of f_j on the noninferior surface (the set of noninferior points in the objective space). If this value is infinite we can improve f_i as much as we like with a small sacrifice of f_j, and therefore such a point might not be appropriate for a solution of the problem. The concept of proper efficiency is based on this consideration.

A vector $x^* \in X$ is said to be a properly efficient solution (in the sense of Geoffrion [152]) if it is efficient and there is some real number $M > 0$ such that for each f_i and each $x \in X$ satisfying $f_i(x) < f_i(x^*)$, there exists at least one f_j such that $f_j(x^*) < f_j(x)$ and

$$\frac{f_i(x^*) - f_i(x)}{f_j(x) - f_j(x^*)} \leq M.$$

The concept of proper efficiency was first introduced by Kuhn and Tucker [250]. It is extended to the case of more general K-efficient solutions and/or analyzed by several authors [36, 37, 39, 58, 90, 113, 163, 164, 179, 182, 241, 358, 431, 439, 517]. For example, Benson [36] defined x^* to be properly efficient if

$$\text{cl cone}(Y + K - f(x^*)) \cap (-K) = \{0\},$$

where cl denotes the closure, cone denotes the conical hull and $Y = f(X)$. Relationships among several definitions of proper efficiency can be found in [134, 397].

Proper efficiency is, of course, a stronger concept than ordinary efficiency, but the set of efficient points is included in the closure of the set of properly efficient points in the objective space under some assumptions [173]. [93] proved that every efficient solution is properly efficient under some mild conditions in pseudolinear multiobjective programming.

Another solution concept, super efficiency, was proposed in [62]. In finite dimensional spaces, it coincides with Borwein's proper efficiency [58]. Super efficiency is also considered in [55] along with the concept of super infima.

In ordinary scalar optimization, the infimum instead of the minimum is often considered, because the latter does not always exist. Several authors discussed an extension of the efficiency to the infimum (or supremum) in vector optimization [125, 225, 357, 447, 450]. Some kinds of closure operation on a set in $\mathbb{R}^n$ are introduced and/or the space itself is extended.

2.3. Existence, Domination Property and Connectedness

In any mathematical problem, existence of a solution is the first question which should be answered. The existence of efficient solutions has been discussed in [60, 86, 97, 173, 180, 203, 314, 320]. The simplest result is a direct extension of the fundamental theorem (Weierstrass' Theorem) in ordinary scalar optimization: If every f_i is lower semicontinuous and X is a compact set, then there exists an efficient solution.

These conditions guarantee that the feasible set in the objective space is cone compact. Weaker concepts such as cone semicompactness have been proposed. Comparison of existence results for efficient points is given in [426]. Existence of efficient or weakly efficient solutions are also discussed in connection with scalar optimization in [86, 118, 119]. Existence of efficient solutions with respect to general binary relations are stated in [145, 175, 426].

Another question is the following: For each feasible solution $x \in X$ which is not efficient, is there an efficient solution $x^* \in X$ which dominates x (i.e., which satisfies $f(x) \in f(x^*) + K$, K: domination cone)? The answer to this question depends on the problem. If it is affirmative, i.e., for the set of feasible values $Y = f(X)$ and the set of efficient values $W = \mathrm{Min}_K\, Y$ in the objective space, if the relation

$$Y \subset W + K$$

holds, then Y is said to have a domination property (or to be K-minimally complete or externally stable). The above sufficient conditions for the existence of efficient solutions (lower semicontinuity of f_i and compactness of X) also guarantee the domination property of Y. Further researches concerning the domination property are made in [38, 181, 306, 315]. The concept of dominators is introduced in [72].

The last question considered in this section is the connectedness of the set of efficient solutions. [348] provided sufficient conditions for the connectedness of the set of efficient values in the objective space. Moreover, [468] provided sufficient conditions for the connectedness of the set of efficient solutions in the decision space. Convexity plays an essential role in their results. [35, 188, 312] refine the results by weakening convexity to quasiconvexity. [35] also provides a review of connectedness results for efficiency, see also [134, Section 3.4]. The case of two objective functions is discussed in [456]. Moreover, conditions for the closedness of the efficient set are discussed in [41, 94].

3. Scalarization and Optimality Conditions

In this section we discuss scalarization of multiobjective optimization problems and necessary and/or sufficient conditions for efficiency, weak efficiency and proper efficiency.

3.1. Scalarization

In order to obtain efficient solutions in a multiobjective optimization problem, we generally transform it into a family of scalar optimization problems usually with a parameter vector ([148] proposes an approach

in which the objective functions are not scalarized). Scalarization has been studied by a number of authors. Some discuss general approaches to scalarization [150, 151, 154, 199, 309, 311, 364, 467, 471, 481, 488, 489, 504]. As is pointed out in [489], scalarization methods should have the following two properties:

1 An optimal solution of each scalarized problem is efficient (properly efficient or weakly efficient).

2 Every efficient solution can be obtained as an optimal solution of an appropriate scalarized problem by adjusting the parameter value.

Typical methods of scalarization are the following:

1 **Weighted sum minimization**
The weighted sum of all objective functions is taken as a new aggregated objective function.

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{i=1}^{p} w_i f_i(x) \\ \text{subject to} & x \in X \end{array}$$

where $w \in \mathbb{R}^p_+$, $w \neq 0$, is a weighting vector. Each optimal solution of the above problem is a weakly (resp. properly) efficient solution of the original multiobjective problem (resp. if $w > 0$). However, without convexity assumption, we cannot always obtain all efficient solutions by changing the value of w. A further research on this problem can be seen in [34]. [278, 487] deal with convexification of a noninferior frontier by applying the p-power to the objective functions.

2 **ε-constraint method** [78]
Converting all but one of the multiple objectives to inequality constraints leads to the following single-objective programming problem.

$$\begin{array}{ll} \text{minimize} & f_p(x) \\ \text{subject to} & x \in X \\ & f_i(x) \leq \varepsilon_i, \quad i = 1, \ldots, p-1 \end{array}$$

Every optimal solution of the above problem is weakly efficient in the original multiobjective problem. Moreover, we can compute the trade-off rates through the Lagrange multipliers obtained by

solving the above problem [169]. [127] discusses relationships between the set of efficient solutions and the set of properly efficient solutions through the weighted-sum scalarization.

Lin [288, 289] proposed a method of proper equality constraints by converting objective functions not to inequality constraints but to equality constraints.

3 **Tchebyshev scalarization**
A point called reference point or ideal point is chosen in the objective space. Then we try to find the nearest point to this reference point in the feasible objective set. The obtained point is often called the compromise solution [149]. In this case a norm is used to determine the distance in the objective space [231, 395, 509]. The so-called l_q norm is fundamental and l_∞ (maximum) norm is often used because it does not require convexity assumption to obtain all the efficient points [159, 217, 218, 495]. The Tchebyshev scalarizing function is a variant of this norm. The weighted Tchebyshev scalarization problem is formally described as

$$\begin{array}{ll} \text{minimize} & \max_{i=1,\dots,p} w_i(f_i(x) - y_i^*) \\ \text{subject to} & x \in X. \end{array}$$

Since an optimal solution to the above problem provides a weakly efficient solution, augmented Tchebyshev norm [123], augmented Tchebyshev scalarizing function or similar functions are usually used to focus on properly efficient solutions [94, 216, 490]. Discussions on the trade-off rates based on this type of scalarization methods are in [219, 220, 279, 355, 503]. This type of scalarization is fundamental in the reference point method for multiple criteria decision making [461, 490].

Actually, as can be seen in the above three approaches, proper efficiency and weak efficiency are more closely related to scalarization than ordinary efficiency [200, 201]. Other methods and studies on trade-off rates can be found in [183, 255, 393, 394].

3.2. Optimality Conditions

The most fundamental theorem in ordinary nonlinear programming is the Karush-Kuhn-Tucker theorem [250]. It is extended to the case of differentiable nonlinear multiobjective programming

$$\begin{array}{ll} \text{minimize} & f(x) = (f_1(x), f_2(x), \dots, f_p(x)) \\ \text{subject to} & g_j(x) \leq 0,\ j = 1, \dots, m \end{array}$$

where equality constraints are omitted for simplicity of description [252]. If $x^* \in X$ is an efficient (or a weakly efficient) solution and an appropriate constraint qualification is satisfied at x^*, then there exist nonzero vectors $\mu^* \in \mathbb{R}^p_+$, $\mu^* \neq 0$, and $\lambda^* \in \mathbb{R}^m_+$ such that

$$\sum_{i=1}^{p} \mu_i^* \nabla f_i(x^*) + \sum_{j=1}^{m} \lambda_j^* \nabla g_j(x^*) = 0,$$
$$\lambda_j^* g_j(x^*) = 0, \ j = 1, \ldots, m.$$

Moreover, if x^* is a properly efficient solution, the above μ^* can be taken a positive vector. These conditions are also sufficient for efficiency if the problem is convex, i.e., every function f_i, g_j is convex. This type of optimality conditions (usually under some appropriate constraint qualifications) have been investigated by a number of researchers [79, 81, 106, 170, 288, 321, 322, 410, 440, 480].

Second-order optimality conditions are also studied in [1, 57, 65, 71, 227, 465]. They require second-order approximation sets to the feasible region and new constraint qualifications. Optimality conditions for nondifferentiable problems are also discussed by several authors [51, 126, 193, 194, 195, 221, 371, 378, 387, 388, 428]. [193] provides optimality conditions in terms of directional derivatives. If all the functions are locally Lipschitz, we obtain the optimality conditions in which the gradients are replaced with the generalized gradients in the above KKT-type conditions [95, Theorem 6-1-3], [126, 193, 378]. See also [332, Chapter 3] and [404, Chapter 13]. In [51] upper Dini derivatives are used to derive optimality conditions. [230] deals with a partially differentiable and partially convex case.

For a function $f : \mathbb{R}^n \to \mathbb{R}$, the directional derivative of f at x in the direction d is defined by

$$f'(x; d) = \lim_{t \searrow 0} \frac{f(x + td) - f(x)}{t}$$

and the upper Dini derivative of f at x in the direction d is defined by

$$f^{\uparrow}(x; d) = \limsup_{t \searrow 0} \frac{f(x + td) - f(x)}{t}.$$

The generalized gradient of a Lipschitz function f at x is given by

$$\partial f(x) = \text{co} \{ \lim_{i \to \infty} \nabla f(x^k) \mid x^k \to x \}.$$

Optimality conditions for the problems with more general domination structure (nonconical dominance) can be found in [174, 175, 176].

Optimality conditions for dynamic multiobjective optimization problems are studied in [63, 233, 234].

4. Stability and Sensitivity Analysis

Stability and sensitivity analysis aims to analyze qualitative and quantitative behavior of the efficient solutions and/or the efficient values according to changes of parameter values included in the original optimization problem. A rather broad review of stability and sensitivity analysis is made in [451]. See also [147] about stability and [114, 448] for a survey.

In this section we consider a family of parametrized multiobjective optimization problems

$$\begin{array}{ll} \text{minimize} & f(x,u) = (f_1(x,u), f_2(x,u), \ldots, f_p(x,u)) \\ \text{subject to} & x \in X(u) \subset I\!R^n \end{array}$$

where u is a parameter vector in $I\!R^m$ and X can be regarded as a set-valued mapping from $I\!R^m$ to $I\!R^n$. For each $u \in I\!R^m$, let

$$Y(u) = \{y \in I\!R^p | y = f(x,u),\ x \in X(u)\}$$

be the feasible set of the above problem. Then Y is a set-valued mapping from $I\!R^m$ to $I\!R^p$. Let the efficient value set of the parametrized problem be

$$W(u) = \mathrm{Min}_K\ Y(u) = \{y^* \in Y(u) |\ \nexists y \in Y(u) : y^* \in y + K \setminus \{0\}\}.$$

Then W is another set-valued mapping from $I\!R^m$ to $I\!R^p$, which is called the perturbation mapping. It is an extension of the perturbation function (optimal value function or marginal function) in the ordinary scalar optimization to the case of multiobjective optimization.

Stability, i.e., continuity of this set-valued mapping (or essentially the same one) is investigated in [143, 349, 368, 369, 453]. Given a set-valued mapping $F : I\!R^m \rightrightarrows I\!R^p$, it is said to be upper semicontinuous at $u \in I\!R^m$ if $u^k \to u$, $y^k \to y$ and $y^k \in F(u^k)$ imply $y \in F(u)$. On the other hand, F is said to be lower semicontinuous at u if $y \in F(u)$ and $u^k \to u$ imply the existence of a number K and a sequence y^k such that $y^k \in F(u^k)$ for all $k \geq K$. F is said to be continuous at u if it is both upper and lower semicontinuous at u Sufficient conditions for the semicontinuity of the perturbation mapping are considered.

[453] also studies the stability with respect to the change of domination structure of the decision maker. [215] studies stability of the compromise solution, not the whole efficient set. [430] investigates continuous dependence of solutions on a parameter in a scalarization method. [8] deals with stability of not only Pareto optimal solutions but also ε-approximate solutions. Well-posedness in vector optimization is discussed in [31, 120, 189, 305].

Sensitivity in multiobjective optimization is analyzed by considering derivatives of the perturbation mapping in [445]. Given a set-valued mapping $F : \mathbb{R}^m \rightrightarrows \mathbb{R}^p$ and a point $(\bar{u}, \bar{y}) \in \text{gph } F$, the graphical derivative (or contingent derivative) of F at $\bar{u}$ for $\bar{y}$ is another set-valued mapping $DF(\bar{u}|\bar{y}) : \mathbb{R}^m \rightrightarrows \mathbb{R}^p$ defined by

$$y \in DF(\bar{u}|\bar{y})(u) \iff (u, y) \in T_{\text{gph } F}(\bar{u}, \bar{y}).$$

Here

$$\text{gph } F = \{(u, y) \in \mathbb{R}^m \times \mathbb{R}^p \mid y \in F(u)\}$$

is the graph of F and $T_S(\bar{x})$ denotes the tangent cone of a set S at $\bar{x}$, i.e.,

$$T_S(\bar{x}) = \{x \mid x = \lim_{k \to \infty} x^k,\ x^k \in S\}.$$

Roughly speaking, when $(\bar{u}, \bar{y}) \in \text{gph } W$,

$$\text{Min}_K\ DY(\bar{u}|\bar{y})(u) \subset DW(\bar{u}|\bar{y})(u),\ \ \forall u$$

generally and the convexity assumptions guarantee that

$$\text{Min}_K\ DY(\bar{u}|\bar{y})(u) = DW(\bar{u}|\bar{y})(u)\ \ \forall u.$$

Moreover, $DY(\bar{u}|\bar{y})(u)$ can be obtained through the relationship

$$DY(\bar{u}|\bar{y})(u) = \nabla_x f(\bar{x}, \bar{u}) DX(\bar{u}|\bar{x})(u) + \nabla_u f(\bar{x}, \bar{u}) u$$

under some assumptions with $\bar{y} = f(\bar{x}, \bar{u})$.

A relationship between the derivative and the Lagrange multiplier vector is also established in [445]. Tanino [446] refines the results in convex multiobjective optimization. Shi [402, 403] further investigated the above results by introducing another set-valued derivative. If we consider properly efficient solutions or weakly efficient solutions instead of efficient solutions, we can consider other types of perturbation mappings. Kuk et al. [253, 254] provide sensitivity analysis concerning those perturbation mappings. [12, 13] provides sensitivity analysis in connection with duality theory in convex multiobjective programming with right-hand side perturbation. [101] also deals with sensitivity analysis in multicriteria optimization. [57] discusses differential sensitivity analysis along with second-order efficiency conditions.

5. Duality

Duality is not only theoretically very important but also practically very useful in nonlinear programming. Therefore it has been extended to the

multiobjective case by several authors in the last several decades (see, e.g. [356]). We will concentrate on duality in nonlinear multiobjective optimization (as to the linear case refer to, e.g., [192, 246]) and briefly review the Lagrange duality, the conjugate duality, the Wolfe [491] type, and the Mond-Weir [340] type duality with generalized convexity. Except those results, [121] developed nonconvex duality using a characterization of Pareto optima by means of generalized Tchebyshev norms. [331] discussed duality and reciprocity in multiobjective programming. [329] and [330] deal with other types of duality.

Results on saddle points of a vector-valued function are closely related to the duality results [444]. Several excellent results have been also obtained concerning saddle points and minimax theory [141, 142, 319, 359, 390, 441, 442, 443, 518]. See also Section 5.3 below.

5.1. Lagrange Duality

Corresponding to a nonlinear multiobjective programming problem,

$$\text{(P)} \qquad \begin{array}{ll} \text{minimize} & f(x) = (f_1(x), f_2(x), \ldots, f_p(x)) \\ \text{subject to} & g(x) = (g_1(x), \ldots, g_m(x))^T \leq 0, \\ & x \in X \subset \mathbb{R}^n \end{array}$$

we define the vector-valued Lagrangian function L by

$$L(x, \Lambda) = f(x) + \Lambda g(x)$$

where Λ is a $p \times m$ matrix. (In [452], the dual variable is an m-dimensional vector λ as in the ordinary duality. But the second term of the vector-valued Lagrangian is $(\lambda^T g(x), \ldots, \lambda^T g(x))^T$). The dual set-valued map Φ is defined by

$$\Phi(\Lambda) = \text{Min}\{L(x, \Lambda) | x \in X\}, \quad \Lambda \in \mathbb{R}^{p \times m}, \ \Lambda \mathbb{R}^m_+ \subset \mathbb{R}^p_+$$

where Min S enotes the set of efficient points in S with respect to the ordering cone $\mathbb{R}^p_+$. The dual problem is formally written by

$$\text{(D)} \qquad \begin{array}{ll} \text{maximize} & \Phi(\Lambda) \\ \text{subject to} & \Lambda \in \mathcal{L} = \{\Lambda \in \mathbb{R}^{p \times m} \mid \Lambda \mathbb{R}^m_+ \subset \mathbb{R}^p_+\} \end{array}$$

though Φ is not a function, but a set-valued map.

Unfortunately, Min (P) $\neq$ Max (D) generally even under the convexity assumptions, but several interesting relationships are obtained [452]. [10, 52, 98, 198, 307, 309, 466] also discuss Lagrange duality. [61, 198, 352, 353, 354] provides geometric considerations of duality in nonlinear multiobjective programming. [317] dealt with an axiomatic approach to duality.

Duality can be understood from the viewpoint of saddle-point theory as in [283, 386, 444]. In the above setting, a point $(x^*, \Lambda^*) \in X \times \mathcal{L}$ is a saddle point of L if

$$L(x^*, \Lambda^*) \in \mathrm{Min}\{L(x, \Lambda^*) | x \in X\} \cap \mathrm{Max}\{L(x^*, \Lambda) | \Lambda \in \mathcal{L}\}.$$

5.2. Conjugate Duality

[454] defined the conjugate map of a set-valued map, which is also a set-valued map, taking efficient values instead of the minimum value. Through conjugate maps, they define the conjugate dual problem and obtain the duality result. The concepts of subgradients and subdifferentials are also introduced. Introduction of the concept of infimum in multi-dimensional space led to advanced results in conjugate duality [226, 449]. Let Inf Y (resp. Sup Y) denote the infimum (resp. supremum) of a set $Y \subset \overline{\mathbb{R}}^p$, where $\overline{\mathbb{R}}^p$ is the extended p-dimensional Euclidean space. For a set-valued map $F : \mathbb{R}^n \rightrightarrows \overline{\mathbb{R}}^p$, its conjugate map $F^* : \mathbb{R}^{p \times n} \rightrightarrows \overline{\mathbb{R}}^p$ is defined by

$$F^*(\Lambda) = \mathrm{Sup} \bigcup_{x \in \mathbb{R}^n} [\Lambda x - F(x)].$$

The biconjugate map $F^{**} : \mathbb{R}^n \rightrightarrows \overline{\mathbb{R}}^p$ of F is defined by

$$F^{**}(x) = \mathrm{Sup} \bigcup_{\Lambda \in \mathbb{R}^{p \times n}} [\Lambda x - F^*(\Lambda)].$$

The primal multiobjective optimization problem

$$\text{(P)} \qquad \text{minimize } f(x)$$

with $f : \mathbb{R}^n \to \overline{\mathbb{R}}^p$ is embedded into a family of perturbed problems

$$\text{minimize } \phi(x, u)$$

with $\phi : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}^p$ satisfying $\phi(x, 0) = f(x)$ for all x. The dual problem is formally defined as

$$\text{(D)} \qquad \text{maximize } -\phi^*(0, U),$$

where $U \in \mathbb{R}^{p \times m}$, though ϕ is not a function but a set-valued map. Moreover, the perturbation map $W : \mathbb{R}^m \rightrightarrows \overline{\mathbb{R}}^p$ is defined by

$$W(u) = \mathrm{Inf}\{\phi(x, u) \mid x \in \mathbb{R}^n\}.$$

Clearly, Inf (P) = $W(0)$. It is proved that Sup (D) = $W^{**}(0)$. Moreover, if W is subdifferentiable at 0, then Inf (P) = Sup (D). Convexity assumptions essentially guarantee the subdifferentiability of W.

The results in [64] are based on another definition of infimum. Since [356] provides a concise introduction to conjugate duality, the reader may refer to it. [161] provides a generalization of Fenchel's duality theorem for convex vector optimization. Fenchel duality in vector optimization is also discussed in [323].

5.3. Generalized Convexity and Duality

Convexity plays a very important role in optimization theory. Various generalizations of convexity have been made in the literature. In this subsection we deal with applications of generalized convexity to multiobjective programming. For basic definitions, characterizations and properties of convexity and some of its early generalizations, the reader may consult Mangasarian [325], Roberts and Varberg [385]. Strict convexity, strong convexity, strict quasiconvexity, strict pseudoconvexity etc. of functions, their properties, characterizations, applications to economics and optimization theory and elaborate lists of references are presented in books by Schaible and Ziemba [398] and Avriel, Diewert, Schaible and Zang [9]. See also [105, 242, 370] for a survey of recent advances in generalized convexity.

Under pseudo/quasiconvexity, duality theory for multiobjective programming problems has been studied in [16, 17, 43, 129, 167, 471, 474, 477]. Under pseudolinearity of the components of the functions involved, [20] establishes duality theory, whereas assuming pseudolinearity of certain linear forms of the functions involved, [223] gives a fractional programming dual of the type of Weir [472].

[462, 463] define the concept of ρ-convexity, which is satisfied, if for arbitrary points x and y and any value $\lambda \in [0,1]$ the classical inequality of convex functions holds up to a term $\rho(1-\lambda)\lambda\|x-y\|^2$. It has been generalized in [210, 211] and applied to multiobjective programming problems in [131, 341].

[170] introduced into optimization theory a broad generalization of convexity for differentiable functions on $\mathbb{R}^n$, that for some vector function $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, the real-valued function f satisfies, for each $x, u \in \mathbb{R}^n$, $f(x) - f(u) \geq \nabla f(u)^T \eta(x,u)$, and showed that both weak duality and KKT sufficiency results, in constrained optimization, hold with the generalized convexity conditions, called invex by [100]. During the last twenty years, numerous articles have appeared in the literature reflecting further generalizations and applications in this category.

Duality theory in multiobjective programming problems has been discussed in [132, 236, 258, 415, 460] under invexity and pseudo-/ quasiinvexity. *F*-convexity has been defined by [171] replacing $\nabla f(u)^T \eta(x,u)$

by $F(x, u; \nabla f(u))$ in the above definition of invexity, where F is a sublinear function. Under F-convexity assumptions, applications to generalized F-convexity to multiobjective programming problems have been investigated in [46, 165, 235]. Other studies on duality in multiobjective programming can be seen in [418, 476] under preinvexity assumptions defined by [476] and in [15, 26] under b-invexity assumptions defined by [30].

[172] defines two new classes of functions called type I and type II functions. Let f be a real-valued differentiable function and let g be an m-dimensional vector-valued differentiable function defined on $\mathbb{R}^n$. The functions f and g are called type I with respect to $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ if for each $x, u \in \mathbb{R}^n$,

$$f(x) - f(u) \geq \nabla f(x)^T \eta(x, u),$$
$$-g(u) \geq \nabla g(u)^T \eta(x, u).$$

[224] considers Wolfe type and Mond-Weir type duality for multiobjective programming problem involving type I condition. [2] introduced classes of generalized type I vector-valued functions and derive Mond-Weir type duality results under generalized type I assumptions. Combining the concepts of type I and univex functions, [391] gives optimality conditions and duality in various settings (real valued, fractional, multiobjective). [435] defines generalized d-type I functions for a multiobjective nondifferentiable programming problem and derives Wolfe type and Mond-Weir type duality results.

KKT-type optimality conditions and duality were obtained by [382] under generalized cone invexity in a subdifferential setting. [383] extended the invexity to nonsmooth functions by the generalized directional derivative of Clarke [95] for Lipschitz functions. [261] extended the results of [383] to multiobjective programming problem involving nonsmooth Lipschitz invex functions.

[373] introduced generalized (F, ρ)-convexity, which is an extension of F-convexity defined by [171] and generalized ρ-convexity defined by [462, 463]. Let $\rho \in \mathbb{R}$ and $d(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that $d(x, x) = 0$ for any x. A real-valued differentiable function f defined on $\mathbb{R}^n$ is said to be (F, ρ)-convex if for each $x, u \in \mathbb{R}^n$

$$f(x) - f(u) \geq F(x, u; \nabla f(u)) + \rho d^2(x, u).$$

[373] used the generalized (F, ρ)-convexity to obtain Wolfe type and Mond-Weir type duality results for multiobjective programming problems. [47] defined (F, ρ)-convexity for nonsmooth functions, an extension of generalized (F, ρ)-convexity defined by [373], and [47] derived

some duality theorems for nonsmooth multiobjective programming problems. [492] introduced a mixed type dual for multiobjective programming problems and presented duality results under generalized (F, ρ)-convexity assumptions.

[212] defined generalized V-invexity of differentiable multiobjective programming problems which preserve the sufficient optimality conditions and duality results as in the scalar case, and avoid the major difficulty of verifying that the inequality holds for the same function $\eta(\cdot, \cdot)$ for the objective functions and the constraints. This relaxation allows us to treat nonlinear fractional programming problems also. [339, 296] further extended the results of [212] to nonsmooth multiobjective programming problems. Applications of generalized V-invexity to composite multiobjective nonsmooth programming are investigated in [334, 338]. See [22, 23] for minimax programming problems and [335, 344] for multiobjective variational problems involving V-invex functions. [251] defined V-ρ-invexity using Clarke's derivatives for locally Lipschitz functions and established sufficient optimality conditions and duality for nonsmooth multiobjective programming. See [380] for composite nonsmooth multiobjective programming problems involving V-ρ-invex functions.

Other studies on duality in multiobjective programming with generalized convex functions can be seen in [110, 158, 166, 342, 345, 374, 377, 379].

Applications of generalized convex functions to multiobjective fractional programming are discussed in [19, 21, 24, 45, 46, 75, 76, 96, 130, 260, 297, 298, 301, 341, 347, 365, 414, 415, 418, 434, 471, 472, 475]. Duality results for multiobjective variational problems with generalized convexity are established in [27, 49, 91, 92, 237, 238, 239, 263, 337, 345, 350, 351].

6. Vector Variational Inequalities

In this section we concentrate on the relations between solutions of vector variational inequalities and solutions of vector optimization problems (or multiobjective programming problems).

The concept of vector variational inequality in a finite dimensional Euclidean space was first introduced by Giannessi [155] in 1980.

Let X be a nonempty subset of $\mathbb{R}^n$ and let $F_i : X \to \mathbb{R}^n$ $(i = 1, \ldots, p)$ be vector-valued functions. Let

$$F := (F_1, \ldots, F_p) \quad \text{and} \quad F(x)(v) := (\langle F_1(x), v\rangle, \ldots, \langle F_p(x), v\rangle)$$

for every $x \in X$ and $v \in \mathbb{R}^n$. The scalar product in a Euclidean space is denoted by $\langle \cdot, \cdot \rangle$. The vector variational inequality, defined by the

function F and the set X, is the following problem:

$$\text{Find } \bar{x} \in X \text{ such that } F(\bar{x})(x - \bar{x}) \notin -\mathbb{R}^p_+ \setminus \{0\} \text{ for every } x \in X.$$

The existence theorems for solutions of vector variational inequalities or generalized vector variational inequalities are studied in [77, 82, 83, 84, 85, 89, 107, 122, 138, 144, 244, 265, 267, 268, 269, 270, 271, 273, 274, 275, 291, 294, 405, 406, 407, 408, 409, 497, 498, 499, 501, 511]. See also Giannessi [157] and the references cited therein about vector variational inequalities and their generalizations.

Vector variational inequalities and their generalizations have been used as a tool to solve vector optimization problems. Several authors have discussed relations between vector variational inequalities and vector optimization problems under some convexity or generalized convexity assumptions.

[272] showed that a necessary condition for a point to be a weakly efficient solution of a vector optimization problem for differentiable functions is that the point be a solution of a vector variational inequality.

Let $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, p$ be differentiable functions. We formulate the following Stampacchia type vector variational inequality for gradients:

(SVVI) Find $\bar{x} \in X$ such that for any $x \in X$,

$$(\nabla f_1(\bar{x})^T(x - \bar{x}), \ldots, \nabla f_p(\bar{x})^T(x - \bar{x})) \notin -\mathbb{R}^p_+ \setminus \{0\}.$$

[156] considered another type vector variational inequality, which is called the Minty type vector variational inequality for gradients:

(MVVI) Find $\bar{x} \in X$ such that for any $x \in X$,

$$(\nabla f_1(x)^T(x - \bar{x}), \ldots, \nabla f_p(x)^T(x - \bar{x})) \notin -\mathbb{R}^p_+ \setminus \{0\}.$$

[156] provided the equivalence between efficient solutions of a differentiable convex vector optimization problem and solutions of a Minty type vector variational inequality for gradients which is a vector version of the classical Minty variational inequality for gradients. Moreover, [156] proved the equivalence between solutions of weak Minty and Stampacchia type vector variational inequalities for gradients and weakly efficient solutions of a differentiable convex vector optimization problem. Following the approaches of [156], [262] studied the equivalence between nondifferentiable convex vector optimization problems and Minty type vector variational inequality and Stampacchia type vector variational inequality, both for subdifferentials as follows:

Find $\bar{x} \in X$ such that for any $x \in X$ and any $\xi_i \in \partial f_i(x)$, $i = 1, \ldots, p$,

$$(\xi_1^T \eta(x - \bar{x}), \ldots, \xi_p^T \eta(x - \bar{x})) \notin -I\!R_+^p \setminus \{0\},$$

where $\partial f_i(x)$ is the subdifferential of f_i at x;

Find $\bar{x} \in X$ such that for any $x \in X$, there exists $\xi_i \in \partial f_i(\bar{x})$, $i = 1, \ldots, p$,

$$(\xi_1^T \eta(x - \bar{x}), \ldots, \xi_p^T \eta(x - \bar{x})) \notin -I\!R_+^p \setminus \{0\}.$$

Let $f_i : I\!R^n \to I\!R$, $i = 1, \ldots, p$ be differentiable functions and let $\eta : X \times X \to R^p$ be a vector-valued function. Then we consider the following Minty type vector variational-like inequality and Stampacchia type vector variational-like inequality for gradients:

(MVVLI) Find $\bar{x} \in X$ such that for any $x \in X$,

$$(\nabla f_1(x)^T \eta(x, \bar{x}), \ldots, \nabla f_p(x)^T \eta(x, \bar{x})) \notin -\text{int} I\!R_+^p;$$

(SVVLI) Find $\bar{x} \in X$ such that for any $x \in X$,

$$(\nabla f_1(\bar{x})^T \eta(x, \bar{x}), \ldots, \nabla f_p(\bar{x})^T \eta(x, \bar{x})) \notin -\text{int} I\!R_+^p.$$

[229, 497] established the equivalence between a vector variational-like inequality with a multiobjective programming problem for generalized invex functions. The vector variational-like inequality approach was used in [264, 274] to prove some existence theorems for generalized efficient solutions of nondifferentiable invex vector optimization problems. The results in [264, 274] are generalizations of existence results established in [85, 86] for differentiable and convex vector optimization problems and in [228] for differentiable preinvex vector optimization problems. [6] proved the equivalence among the Minty vector variational-like inequality, Stampacchia vector variational-like inequality, and a nondifferentiable and nonconvex vector optimization problem. [6] also established an existence theorem for generalized weakly efficient solutions of nondifferentiable nonconvex vector optimization problems by using a fixed point theorem.

[499] gave the equivalence between solutions of a Stampacchia vector variational inequality for gradients and efficient solutions of a linear fractional vector optimization problem of which the numerators of the objective functions are linear and the denominators of the objective functions are the same linear functions.

Several existence results of solutions for vector equilibrium problems can be found in [5, 50, 74, 157, 168, 245] and the references cited therein.

7. Concluding Remarks

In this chapter we have reviewed a great number of papers concerning mathematical theory of nonlinear multiobjective programming. As we mentioned first, we have focused on the nonlinear case. Moreover, many articles mainly connected with theoretical results in abstract spaces are either omitted or only listed without citation if they are extended from those obtained in the Euclidean spaces. We have dealt with neither practical methods such as interactive methods for solving multiobjective problems from the viewpoint of decision making, nor multiattribute utility theory. The topics dealt with are solution concepts, scalarization, optimality conditions, stability, sensitivity analysis, duality, and vector variational inequalities. Since some topics have not been explained so far, we would like to mention a few of them briefly in this section.

[191, 248] demonstrated an equivalence between a linear complementarity problem with general data and finding a certain subset of the efficient points of a multiobjective programming problem. A new multiobjective programming based approach to solving linear complementarity problems is presented.

Recently set-valued optimization is an interesting topic, which is an extension of multiobjective optimization in a sense. Several results can be seen in [11, 33, 88, 99, 143, 205, 208, 281, 282, 285, 287, 302, 316, 421, 422, 423, 424, 425, 496] and the references cited therein. In set-valued optimization, mappings are set-valued. On the other hand some authors investigated multiobjective problems with set functions where decision variables are sets [18, 48, 184, 185, 186, 292, 293, 299, 376].

Though we tried to list as many papers as possible, some important ones might be missing because of the limit of our ability. The authors would highly appreciate it if the readers would be tolerant of this matter.

References

[1] B. Aghezzaf and M. Hachimi. Second-order optimality conditions in multiobjective optimization problems, *Journal of Optimization Theory and Applications,* 102(1):37-50, 1999.

[2] B. Aghezzaf and M. Hachimi. Generalized invexity and duality in multiobjective programming problems, *Journal of Global Optimization,* 18:91-101, 2000.

[3] T. Amahroq and A. Taa. Sufficient conditions of optimality for multiobjective optimization problems with γ-paraconvex data, *Studia, Mathematica,* 124(3):239-247, 1997.

[4] T. Amahroq and A. Taa. On Lagrange-Kuhn-Tucker multipliers for multiobjective optimization problems, *Optimization*, 41 (2): 159-172, 1997.

[5] Q.H. Ansari and J.C. Yao. An existence result for the generalized vector equilibrium problem, *Applied Mathematics Letters*, 12(8):53-56, 1999.

[6] Q.H. Ansari and J.C. Yao. On nondifferentiable and nonconvex vector optimization problems, *Journal of Optimization Theory and Applications*, 106(3):475-488, 2000.

[7] M. Athans and H.P. Greenberg. Necessary and sufficient conditions for differentiable nonscalar-valued functions to attain extrema, *IEEE Transactions on Automatic Control*, AC-18(2):132-138, 1973.

[8] H. Attouch and H. Riahi. Stability results for Ekeland epsilon-variational principle and cone extremal solutions, *Mathematics of Operations Research*, 18(1):173-201, 1993.

[9] M. Avriel, W.E. Diewert, S. Schaible and I. Zang. *Generalized Concavity*, Plenum Press, New York, 1987.

[10] A.Y. Azimov. Duality in nonconvex problems of vector optimization, *Problems of Control and Information Theory*, 12(3):209-218, 1983.

[11] J. Baier and J. Jahn. On subdifferentials of set-valued maps, *Journal of Optimization Theory and Applications*, 100 (1):233-240, 1999.

[12] A. Balbas, M. Ballve and P. Jimenez Guerra. Sensitivity in multiobjective programming under homogeneity assumptions, *Journal of Multi-Criteria Decision Analysis*, 8(3):133-138, 1999.

[13] A. Balbas and P. Jimenez Guerra. Sensitivity analysis for convex multiobjective programming in abstract spaces, *Journal of Mathematical Analysis and Applications*, 202:645-658, 1996.

[14] C.R. Bector. Wolfe-type duality involving (B,η)-invex functions for a minmax programming problem, *Journal of Mathematical Analysis and Applications*, 201:114-127, 1996.

[15] C.R. Bector, M.K. Bector, A. Gill and C. Singh. Duality for vector valued *b*-invex programming, in: S. Komlosi, T. Rapcsak and S. Schaible (eds.), *Generalized Convexity,* 358-373, Springer-Verlag, Berlin, 1994.

[16] C.R. Bector and B.L. Bhatia. Duality for multiple objective nonlinear nonconvex program, *Utilitas Mathematica*, 28:175-192, 1985.

[17] C.R. Bector, D. Bhatia and P. Jain. Generalized convexity and duality in multiobjective nonsmooth programming, *Utilitas Mathematica,* 43:71-78, 1993.

[18] C.R. Bector, D. Bhatia and S. Pandey. Duality for multiobjective fractional programming involving *n*–set functions, *Journal of Mathematical Analysis and Applications,* 186(3):747-768, 1994.

[19] C.R. Bector, S. Chandra and M.K. Bector. Generalized fractional programming duality: A parametric approach, *Journal of Optimization Theory and Applications,* 60:243-260, 1989.

[20] C.R. Bector, S. Chandra and M.V. Durga Prasad. Duality in pseudolinear multiobjective programming, *Asia-Pacific Journal of Operational Research,* 5(2):150-159, 1988.

[21] C.R. Bector, S. Chandra and I. Husain. Optimality conditions and duality in subdifferentiable multiobjective fractional programming, *Journal of Optimization Theory and Applications,* 79(1):105-125, 1993.

[22] C.R. Bector, S. Chandra and V. Kumar. Duality for minimax programming involving *V*-invex functions, *Optimization,* 30:93-103, 1994.

[23] C.R. Bector, S. Chandra and V. Kumar. Duality for minimax programming involving nonsmooth *V*-invex functions, *Optimization,* 38(3):209-221, 1996.

[24] C.R. Bector, S. Chandra and C. Singh. Duality in multiobjective fractional programming, in: A. Cambini, E. Castagnoli, L. Martein and S. Schaible (eds.), *Generalized Convexity and Fractional Programming with Economic Applications,* 232-241, Springer-Verlag, Berlin, 1990.

[25] C.R. Bector, S. Chandra and C. Singh. A linearization approach to multiobjective programming duality, *Journal of Mathematical Analysis and Applications,* 175:268-279, 1993.

[26] C.R. Bector, A. Gill and C. Singh. Duality for vector valued *b*-vex programming, *Asia-Pacific Journal of Operational Research,* 10:15-30, 1993.

[27] C.R. Bector and I. Husain. Duality for multiobjective variational problems, *Journal of Mathematical Analysis and Applications,* 166:214-229, 1992.

[28] M.K. Bector, I. Husain, S. Chandra and C.R. Bector. A duality model for a generalized minimax program, *Naval Research Logistics,* 35:493-501, 1988.

[29] C.R. Bector and S. Suneja. Duality in nondifferentiable generalized fractional programming, *Asia-Pacific Journal of Operational Research*, 5:134-139, 1988.

[30] C.R. Bector, S.K. Suneja and C.S. Lalitha. Generalized *B*-vex functions and generalized *B*-vex programming, *Journal of Optimization Theory and Applications*, 76:561-576, 1993.

[31] E.M. Bednarczuk. An approach to well-posedness in vector optimization: consequences to stability. Parametric optimization, *Control and Cybernetics*, 23(1-2):107-122, 1994.

[32] E.M. Bednarczuk. Berge-type theorems for vector optimization problems, *Optimization*, 32(4):373-384, 1995.

[33] E.M. Bednarczuk and W. Song. Contingent epiderivative and its applications to set-valued optimization, *Control and Cybernetics*, 27(3):375-386, 1998.

[34] K. Belkeziz and M. Pirlot. Proper efficiency in nonconvex vector-maximization problems, *European Journal of Operational Research*, 54(1):74-80, 1991.

[35] J. Benoist. Connectedness of the efficient set for strictly quasiconcave sets, *Journal of Optimization Theory and Applications*, 96(3):627-654, 1998.

[36] H.P. Benson. An improved definition of proper efficiency for vector maximization with respect to cones, *Journal of Mathematical Analysis and Applications*, 71:232-241, 1979.

[37] H.P. Benson. Efficiency and proper efficiency in vector maximization with respect to cones, *Journal of Mathematical Analysis and Applications*, 93:273-289, 1983.

[38] H.P. Benson. On a domination property for vector maximization with respect to cones, *Journal of Optimization Theory and Applications*, 39(1):125-132, 1983.

[39] H.P. Benson and T.L. Morin. The vector maximization problem: Proper efficiency and stability, *SIAM Journal on Applied Mathematics*, 32(l):64-72, 1977.

[40] H.P. Benson and S. Sayin. Optimization over the efficient set: Four special cases, *Journal of Optimization Theory and Applications*, 80(1):3-18, 1994.

[41] H.P. Benson and E. Sun. New closedness results for efficient sets in multiple objective mathematical programming, *Journal of Mathematical Analysis and Applications*, 238:277-296, 1999.

[42] K. Bergstresser, A. Charnes and P.L. Yu. Generation of domination structures and nondominated solutions in multicriteria decision making, *Journal of Optimization Theory and Applications*, 18(1):3-13, 1976.

[43] D. Bhatia and S. Aggarwal. Optimality and duality for multiobjective nonsmooth programming, *European Journal of Operational Research*, 57: 360-367, 1992.

[44] D. Bhatia and P. Gupta. Generalized linear complementarity problem and multiobjective programming problem, *Optimization*, 46(2):199-214, 1999.

[45] D. Bhatia and P. Jain. Nondifferentiable multiobjective fractional programming with Hanson-Mond classes of functions, *Journal of Information & Optimization Sciences*, 12(l):35-47, 1991.

[46] D. Bhatia and P. Jain. On multiobjective fractional duality for Hanson-Mond classes of functions, *Journal of Information & Optimization Sciences*, 14:1-9, 1993.

[47] D. Bhatia and P. Jain. Generalized (F, ρ)-convexity and duality for nonsmooth multiobjective programs, *Optimization*, 31:153-164, 1994.

[48] D. Bhatia and P. Kumar. Pseudolinear vector optimization problems containing n-set functions, *Indian Journal of Pure and Applied Mathematics*, 28(4):439-453, 1997.

[49] D. Bhatia and A. Mehra. Optimality conditions and duality for multiobjective variational problems with generalized B-invexity, *Journal of Mathematical Analysis and Applications,* 234(2):341-360, 1999.

[50] M. Bianchi, N. Hadjisavvas and S. Schaible. Vector equilibrium problems with generalized monotone bifunctions, *Journal of Optimization Theory and Applications*, 92(2):527-542, 1997.

[51] G. Bigi and M. Pappalardo. Regularity conditions in vector optimization, *Journal of Optimization Theory and Applications*, 102(l):83-96, 1999.

[52] G.R. Bitran. Duality for nonlinear multiple-criteria optimization problems, *Journal of Optimization Theory and Applications*, 35(3):367-401, 1981.

[53] G.R. Bitran and T.L. Magnanti. The structure of admissible points with respect to cone dominance, *Journal of Optimization Theory and Applications*, 29(4):573-614, 1979.

[54] W. Gomez Bofill. Vector optimization: singularities, regularizations, *Mathematical Methods of Operations Research,* 47(3):473-497, 1998.

[55] N. Boissard. Super efficient solutions and super infima in vector optimization, *Journal of Mathematical Analysis and Applications,* 228(1):37-50, 1998.

[56] S. Bolintineanu. Necessary conditions for nonlinear suboptimization over the weakly-efficient set, *Journal of Optimization Theory and Applications,* 78(3):579-598, 1993.

[57] S. Bolintineanu and M. El Maghri. Second-order efficiency conditions and sensitivity of efficient points, *Journal of Optimization Theory and Applications,* 98(3):569-592, 1998.

[58] J.M. Borwein. Proper efficient points for maximizations with respect to cones, *SIAM Journal on Control and Optimization,* 15(l):57-63, 1977.

[59] J.M. Borwein. The geometry of Pareto efficiency over cones, *Optimization,* 11(2):235-248, 1980.

[60] J.M. Borwein. On the existence of Pareto efficient points, *Mathematics of Operations Research,* 8(l):64-73, 1983.

[61] J.M. Borwein and J.W. Nieuwenhuis. Two kinds of normality in vector optimization, *Mathematical Programming,* 28:181-185, 1984.

[62] J.M. Borwein and D. Zhuang. Super efficiency in vector optimization, *Transactions of the American Mathematical Society,* 338(1):105-122, 1993.

[63] W.W. Breckner. Derived sets for weak multiobjective optimization problems with state and control variables, *Journal of Optimization Theory and Applications,* 93(1):73-102, 1997.

[64] S. Brumelle. Duality for multiple objective convex programs, *Mathematics of Operations Research,* 6 (2):159-172, 1981.

[65] R. Cambini. Second order optimality conditions in multiobjective programming, *Optimization,* 44(2):139-160, 1998.

[66] R. Cambini and S. Komlósi. On polar generalized monotonicity in vector optimization, *Optimization,* 47(1-2):111-121, 2000.

[67] A. Cambini, A. Marchi and L. Martein. On nonlinear scalarization in vector optimization, *Journal of Information & Optimization Sciences,* 17(2):373-385, 1996.

[68] A. Cambini and L. Martein. Some optimality conditions in vector optimization, *Journal of Information & Optimization Sciences,* 10 (1):141-151, 1989.

[69] A. Cambini and L. Martein. Generalized concavity in multiobjective programming, in: J.-P. Crouzeix, J.-E. Martinez-Legaz and M. Volle (eds.), *Generalized Convexity, Generalized Monotonicity: Recent Results,* 453-467, Kluwer Academic Publishers, Dordrecht, 1998.

[70] A. Cambini, L. Martein and R. Cambini. Some optimality conditions in multiobjective programming, in: J. Climaco (ed.), *Multicriteria Analysis,* 168-178, Springer-Verlag, Berlin, 1997.

[71] A. Cambini, L. Martein and R. Cambini. A new approach to second order optimality conditions in vector optimization, in: R. Caballero, F. Ruiz and R. Steuer (eds.), *Advances in Multiple Objective and Goal Programming,* 219-227, Springer-Verlag, Berlin, 1997.

[72] E. Carrizosa and F. Plastria. Dominators for multiple-objective quasiconvex maximization problems, *Journal of Global Optimization,* 18(l):35-58, 2000.

[73] Y. Censor. Pareto optimality in multiobjective problems, *Applied Mathematics and Optimization,* 4(l):41-59, 1977.

[74] O. Chadli and H. Riahi. On generalized vector equilibrium problems, *Journal of Global Optimization,* 16(1):33-41, 2000.

[75] S. Chandra, B.D. Craven and B. Mond. Vector-valued Lagrangian and multiobjective fractional programming duality, *Numerical Functional Analysis and Optimization,* 11:239-254, 1990.

[76] S. Chandra, B.D. Craven and B. Mond. Multiobjective fractional programming duality: a Lagrangean approach, *Optimization,* 22(4):549-556, 1991.

[77] S.S. Chang, H.B. Thompson, and G.X.Z. Yuan. The existence theorems of solutions for generalized vector-valued variational-like inequalities, *Computers & Mathematics with Applications,* 37(7):1-9, 1999.

[78] V. Chankong and Y.Y. Haimes. Optimization-based methods for multiobjective decision-making - an overview, *Large Scale Systems - Theory and Applications,* 5(1), 1983.

[79] V. Chankong and Y.Y. Haimes. On the characterization of noninferior solutions of the vector optimization problem, *Automatica,* 18(6):697-707, 1983.

[80] V. Chankong and Y.Y. Haimes. *Multiobjective Decision Making: Theory and Methodology,* North-Holland, Amsterdam, 1983.

[81] G.Y. Chen. Necessary conditions of nondominated solutions in multicriteria decision making, *Journal of Mathematical Analysis and Applications,* 104(1):38-46, 1984.

[82] G.Y. Chen. Existence of solutions for a vector variational inequality: an extension of the Hartmann-Stampacchia theorem, *Journal of Optimization Theory and Applications,* 74(3):445-456, 1992.

[83] G.Y. Chen and G.M. Cheng. Vector variational inequality and vector optimization, in: Y. Sawaragi, K. Inoue and H. Nakayama (eds.), *Toward Interactive and Intelligent Decision Support Systems, Volume I,* 408-416, Springer-Verlag, Berlin, 1987.

[84] G.Y. Chen and B.D. Craven. Approximate dual and approximate vector variational inequality for multiobjective optimization, *Journal of the Australian Mathematical Society Series A,* 47(3):418-423, 1989.

[85] G.Y. Chen and B.D. Craven. A vector variational inequality and optimization over an efficient set, *Zeitschrift für Operations Research,* 34(1):1-12, 1990.

[86] G.Y. Chen and B.D. Craven. Existence and continuity of solutions for vector optimization, *Journal of Optimization Theory and Applications,* 81(3):459-468, 1994.

[87] G.Y. Chen, C.J. Goh and X.Q. Yang. The gap function of a convex multicriteria optimization problem, *European Journal of Operational Research,* 111(2):142-151, 1998.

[88] G.Y. Chen and J. Jahn. Optimality conditions for set-valued optimization problems, *Mathematical Methods of Operations Research,* 48:187-200, 1998.

[89] G.Y. Chen and S.J. Li. Existence of solutions for generalized vector quasivariational inequality, *Journal of Optimization Theory and Applications,* 90(2):321-334, 1996.

[90] G.Y.Chen and W.D. Rong. Characterizations of the Benson proper efficiency for nonconvex vector optimization, *Journal of Optimization Theory and Applications,* 98(2):365-384, 1998.

[91] X. Chen. Duality for multiobjective variational problems with invexity, *Journal of Mathematical Analysis and Applications,* 203(1):236-253, 1996.

[92] X. Chen. Symmetric duality for the multiobjective fractional variational problem with partial invexity, *Journal of Mathematical Analysis and Applications,* 245(1):105-123, 2000.

[93] K.L. Chew and E.U. Choo. Pseudolinearity and efficiency, *Mathematical Programming,* 28:226-239, 1984.

[94] E.U. Choo and D.R. Atkins. Proper efficiency in nonconvex multicriteria programming, *Mathematics of Operations Research*, 8(3):467-470, 1983.

[95] F.H. Clarke. *Optimization and Nonsmooth Analysis*, John Wiley & Sons, New York, 1983.

[96] L. Coladas, Z. Li and S. Wang. Two types of duality in multiobjective fractional programming, *Bulletin of the Australian Mathematical Society*, 54 (1):99-114, 1996.

[97] H.W. Corley. An existence result for maximizations with respect to cones, *Journal of Optimization Theory and Applications*, 31(2):277-281, 1980.

[98] H.W. Corley. Duality theory for maximizations with respect to cones, *Journal of Mathematical Analysis and Applications*, 84:560-568, 1981.

[99] H.W. Corley. Optimality conditions for maximizations of set-valued functions, *Journal of Optimization Theory and Applications*, 58(1):1-10, 1988.

[100] B.D. Craven. Invex functions and constrained local minima, *Bulletin of the Australian Mathematical Society*, 24:357-366, 1981.

[101] B.D. Craven. On sensitivity analysis for multicriteria optimization, *Optimization*, 19(4):513-523, 1988.

[102] B.D. Craven. Nonsmooth multiobjective programming, *Numerical Functional Analysis and Optimization*, 10 (1-2) :49-64, 1989.

[103] B.D. Craven. Quasimin and quasisaddlepoint for vector optimization, *Numerical Functional Analysis and Optimization*, 11(1-2):45-54, 1990.

[104] B.D. Craven and X.Q. Yang. A nonsmooth version of alternative theorem and nonsmooth multiobjective programming, *Utilitas Mathematica,* 40:117-128, 1991.

[105] J.-P. Crouzeix, J.-E. Martinez-Legaz and M. Volle (eds.). *Generalized Convexity, Generalized Monotonicity: Recent Results*, Kluwer Academic Publishers, Dordrecht, 1998.

[106] N.O. Da Cunha and E. Polak. Constrained minimization under vector-valued criteria in finite dimensional spaces, *Journal of Mathematical Analysis and Applications*, 19(1):103-124, 1967.

[107] A. Daniilidis and N. Hadjisavvas. Existence theorems for vector variational inequalities, *Bulletin of the Australian Mathematical Society*, 54(3):473-481, 1996.

[108] I. Das and J.E. Dennis. A closer look at drawbacks of minimizing weighted sums of objectives for Pareto set generation in multicriteria optimization problems, *Structural Optimization,* 14(l):63-69, 1997.

[109] I. Das and J.E. Dennis. Normal-boundary intersection: A new method for generating the Pareto surface in nonlinear multicriteria optimization problems, *SIAM Journal on Optimization,* 8(3):631-657, 1998.

[110] L.N. Das and S. Nanda. Proper efficiency conditions and duality for multiobjective programming problems involving semilocally invex functions, *Optimization,* 34(1):43-51, 1995.

[111] L.N. Das and S. Nanda. Symmetric dual multiobjective programming, *European Journal of Operational Research,* 97(1):167-171, 1997.

[112] J.P. Dauer and T.A. Fosnaugh. Optimization over the efficient set, *Journal of Global Optimization,* 7(3):261-277, 1995.

[113] J.P. Dauer and R.J. Gallagher. Positive proper efficient points and related cone results in vector optimization theory, *SIAM Journal on Control and Optimization,* 28(1):158-172, 1990.

[114] J. Dauer and Y.-H. Liu. Multi-criteria and goal programming, in: T. Gal and H.J. Greenberg (eds.), *Advances in Sensitivity Analysis and Parametric Programming,* Chapter 11, Kluwer Academic Publishers, Boston, 1997.

[115] J.P. Dauer and M.S.A. Osman. Decomposition of the parametric space in multiobjective convex programs using the generalized Tchebycheff norm, *Journal of Mathematical Analysis and Applications,* 107(1):156-166, 1985

[116] J.P. Dauer and W. Stadler. A survey of vector optimization in infinite-dimensional spaces. II, *Journal of Optimization Theory and Applications,* 51(2): 205-241, 1986.

[117] S. Deng. On approximate solutions in convex vector optimization, *SIAM Journal on Control and Optimization,* 35(6):2128-2136, 1997.

[118] S. Deng. Characterization of the nonemptiness and compactness of solution sets in convex vector optimization, *Journal of Optimization Theory and Applications,* 96(1):123-131, 1998.

[119] S. Deng. On efficient solutions in vector optimization, *Journal of Optimization Theory and Applications,* 96(1):201-209, 1998.

[120] D. Dentcheva and S. Helbig. On variational principles, level sets, well-posedness, and ε-solutions in vector optimization, *Journal of Optimization Theory and Applications,* 89(2):325-349, 1996.

[121] F. Di Guglielmo. Nonconvex duality in multiobjective optimization, *Mathematics of Operations Research,* 2(3):285-291, 1977.

[122] X.P. Ding. The generalized vector quasi-variational-like inequalities, *Computers & Mathematics with Applications,* 37(6):57-67, 1999.

[123] W. Dinkelbach and H. Isermann. On decision making under multiple criteria and under incomplete information, in: J.L. Cochrane and M. Zeleney (eds.), *Multiple Criteria Decision Making,* University of South California Press, 302-312, 1973.

[124] S. Dolecki and C. Malivert. Stability of efficient sets - continuity of mobile polarities, *Nonlinear Analysis - Theory Methods & Applications,* 12(12):1461-1486, 1988.

[125] S. Dolecki and C. Malivert. General duality in vector optimization, *Optimization,* 27(1-2): 97-119, 1993.

[126] J. Doležal. Necessary conditions for Pareto optimality in nondifferentiable problems, *Problems of Control and Information Theory,* 14(2):131-141, 1985.

[127] R. Durier. Weighting factor results in vector optimization, *Journal of Optimization Theory and Applications,* 58(3):411-430, 1988.

[128] R. Durier. On Pareto optima, the Fermat-Weber problem, and polyhedral gauges, *Mathematical Programming,* 47:65-79, 1990.

[129] R.R. Egudo. Proper efficiency and multiobjective duality in nonlinear programming, *Journal of Information & Optimization Sciences,* 8(2):155-166, 1987.

[130] R.R. Egudo. Multiobjective fractional duality, *Bulletin of the Australian Mathematical Society,* 37:367-378, 1988.

[131] R.R. Egudo. Efficiency and generalized convex duality for multiobjective programming, *Journal of Mathematical Analysis and Applications,* 138:84-94, 1989.

[132] R.R. Egudo and M.A. Hanson. Multiobjective duality with invexity, *Journal of Mathematical Analysis and Applications,* 126(2):467-477, 1987.

[133] R.R. Egudo, T. Weir and B. Mond. Duality without constraint qualification for multiobjective programming, *Journal of the Australian Mathematical Society Series B,* 33(4):531-544, 1992.

[134] M. Ehrgott. *Multicriteria Optimization,* Springer-Verlag, Berlin, 2000.

[135] M. Ehrgott, H.W. Hamacher, K. Klamroth, S. Nickel, A. Schöbel and M.M. Wiecek. A note on the equivalence of balance points and Pareto solutions in multiple-objective programming, *Journal of Optimization Theory and Applications,* 92(1):209-212, 1997.

[136] A.-Z.H. El-Banna. A study on parametric multiobjective programming problems without differentiability, *Computers & Mathematics with Applications,* 26(12):87-92, 1993.

[137] K.-H. Elster and R. Elster. Vector approximation problems in geometric vector optimization, *Optimization,* 27(4):321-342, 1993.

[138] K.-H. Elster and R. Elster. Vector variational inequality and geometric vector optimization, in: F. Giannessi and A. Maugeri (eds.), *Variational Inequalities and Network Equilibrium Problems,* 55-67, Plenum Press, New York, 1995.

[139] R. Elster, C. Gerth and A. Göpfert. Duality in geometric vector optimization, *Optimization,* 20(4):457-476, 1989.

[140] P. Favati and M. Pappalardo. On the reciprocal vector optimization problems, *Journal of Optimization Theory and Applications,* 47(2):181-193, 1985.

[141] F. Ferro. A minimax theorem for vector-valued functions, *Journal of Optimization Theory and Applications,* 60 (1):19-31, 1989.

[142] F. Ferro. A minimax theorem for vector-valued functions, Part 2, *Journal of Optimization Theory and Applications,* 68(l):35-48, 1991.

[143] F. Ferro. An optimization result for set-valued mappings and a stability property in vector problems with constraints, *Journal of Optimization Theory and Applications,* 90(l):63-77, 1996.

[144] J. Fu. Simultaneous vector variational inequalities and vector implicit complementarity problem, *Journal of Optimization Theory and Applications,* 93(1):141-151, 1997.

[145] L. Gajek and D. Zagrodny. Countably ordered sets and their applications in optimization, *Optimization,* 26:287-301, 1992.

[146] T. Gal and T. Hanne. On the development and future aspects of vector optimization and MCDM. A. tutorial, in: J. Climaco (ed.), *Multicriteria Analysis,* 130-145, Springer-Verlag, Berlin, 1997.

[147] T. Gal and K. Wolf. Stability in vector optimization - A survey, *European Journal of Operational Research,* 25:169-182, 1986.

[148] E.A. Galperin. Nonscalarized multiobjective global optimization, *Journal of Optimization Theory and Applications,* 75(l):69-85, 1992.

[149] W.B. Gearhart. Compromise solutions and estimation of the noninferior set, *Journal of Optimization Theory and Applications,* 28(l):29-47, 1979.

[150] W.B. Gearhart. Characterization of properly efficient solutions by generalized scalarization methods, *Journal of Optimization Theory and Applications,* 41(3):491-502, 1983.

[151] W.B. Gearhart. Families of differentiable scalarization functions, *Journal of Optimization Theory and Applications,* 62(3):321-332, 1989.

[152] A.M. Geoffrion. Proper efficiency and the theory of vector maximization, *Journal of Mathematical Analysis and Applications,* 22:618-630, 1968.

[153] J.C. Geromel and P.A.V. Ferreira. An upper bound on properly efficient solutions in multiobjective optimization, *Operations Research Letters,* 10 (2):83-86, 1991.

[154] C. Gerth and P. Weidner. Nonconvex separation theorems and some applications in vector optimization, *Journal of Optimization Theory and Applications,* 67(2):297-320, 1990.

[155] F. Giannessi. Theorem of alternative, quadratic programs and complementarity problems, in: R.W. Cottle, F. Giannessi and J.-L. Lions (eds.), *Variational Inequalities and Complementarity Problems: Theory and Applications,* 151-186, John Wiley and Sons, Chichester, 1980.

[156] F. Giannessi. On Minty variational principle, in: F. Giannessi, S. Komlosi and T. Rapcsak (eds.), *New Trends in Mathematical Programming,* 93-99, Kluwer Academic Publishers, Dordrecht, 1998.

[157] F. Giannessi (ed.). *Vector Variational Inequalities and Vector Equilibria: Mathematical Theories,* Kluwer Academic Publishers, Dordrecht, 2000.

[158] G. Giorgi and A. Guerraggio. The notion of invexity in vector optimization: smooth and nonsmooth case, in: J.-P. Crouzeix, J.-E. Martinez-Legas and M. Volle (eds.), *Generalized Convexity, Generalized Monotonocity: Recent Results,* 389-405, Kluwer Academic Publishers, Dordrecht, 1998.

[159] C.J. Goh and X.Q. Yang. On Minkowski metric and weighted Tchebyshev norm in vector optimization, *Optimization,* 43(4):353-365, 1998.

[160] A. Göpfert, E.-C. Henkel and C. Tammer. A smooth variational principle for vector optimization problems, in: R. Durier and C. Michelot (eds.), *Recent Developments in Optimization,* 142-154, Springer-Verlag, Berlin, 1995.

[161] C. Gros. Generalization of Fenchel's duality theorem for convex vector optimization, *European Journal of Operational Research,* 2:368-376, 1978.

[162] J. Guddat, F.G. Vasquez, K. Tammer and K. Wendler. *Multiobjective and Stochastic Optimization based on Parametric Optimization,* Akademie-Verlag, Berlin, 1985.

[163] A. Guerraggio, E. Molho and A. Zaffaroni. On the notion of proper efficiency in vector optimization, *Journal of Optimization Theory and Applications,* 82(1):1-21, 1994.

[164] T.R. Gulati and M.A. Islam. Efficiency and proper efficiency in nonlinear vector maximum problems, *European Journal of Operational Research,* 44(3):373-382, 1990.

[165] T.R. Gulati and M.A. Islam. Sufficiency and duality in multiobjective programming involving generalized *F*-convex functions, *Journal of Mathematical Analysis and Applications,* 183(1):181-195, 1994.

[166] T.R. Gulati, I. Husain and A. Ahmed. Multiobjective symmetric duality with invexity, *Bulletin of the Australian Mathematical Society,* 56(l):25-36, 1997.

[167] T.R. Gulati and N. Talaat. Duality in nonconvex multiobjective programming, *Asia-Pacific Journal of Operational Research,* 8(l):62-69, 1991.

[168] N. Hadjisavvas and S. Schaible. From scalar to vector equilibrium problems in the quasimonotone case, *Journal of Optimization Theory and Applications,* 96(2):297-309, 1998.

[169] Y.Y. Haimes and V. Chankong. Kuhn-Tucker multipliers as trade-offs in multiobjective decision-making analysis, *Automatica,* 15(l):59-72, 1979.

[170] M.A. Hanson. On sufficiency of the Kuhn-Tucker conditions, *Journal of Mathematical Analysis and Applications,* 80(2):544-550, 1981.

[171] M.A. Hanson and B. Mond. Further generalizations of convexity in mathematical programming, *Journal of Information & Optimization Sciences,* 3:22-35, 1982.

[172] M.A. Hanson and B. Mond. Necessary and sufficient conditions in constrained optimization, *Mathematical Programming,* 37:51-58, 1987.

[173] R. Hartley. On cone-efficiency, cone-convexity and cone-compactness, *SIAM Journal on Applied Mathematics*, 34(2) :211-222, 1978.

[174] G.B. Hazen. Differential characterizations of nonconical dominance in multiple objective decision making, *Mathematics of Operations Research*, 13(1), 174-189, 1988.

[175] G.B. Hazen and T.L. Morin. Optimality conditions in nonconical multiple objective programming, *Journal of Optimization Theory and Applications*, 40(1):25-60, 1983.

[176] G.B. Hazen and T.L. Morin. Nonconical optimality conditions: some additional results, *Journal of Optimization Theory and Applications,* 41(4):619-623, 1983.

[177] S. Helbig. Duality in disjunctive programming via vector optimization, *Mathematical Programming*. *Series A*, 65(1):21-41, 1994.

[178] S. Helbig. On a constructive approximation of the efficient outcomes in bicriterion vector optimization, *Journal of Global Optimization*, 5(l):35-48, 1994.

[179] M.I. Henig. Proper efficiency with respect to cones, *Journal of Optimization Theory and Applications*, 36(3):387-407, 1982.

[180] M.I. Henig. Existence and characterization of efficient decisions with respect to cones, *Mathematical Programming*, 23:111-116, 1982.

[181] M.I. Henig. The domination property in multicriteria optimization, *Journal of Mathematical Analysis and Applications*, 114:7-16, 1986.

[182] M.I. Henig. Value functions, domination cones and proper efficiency in multicriteria optimization, *Mathematical Programming*, 46:205-217, 1990

[183] M.I. Henig and J.T. Buchanan. Tradeoff directions in multiobjective optimization problems, *Mathematical Programming*, 78(3):357-374, 1997.

[184] W.S. Hsia and T.Y. Lee. Proper *D*-solutions of multiobjective programming problems with set functions, *Journal of Optimization Theory and Applications*, 53(2):247-258, 1987.

[185] W.S. Hsia and T.Y. Lee. Lagrangian function and duality theory in multiobjective programming with set-functions, *Journal of Optimization Theory and Applications*, 57(2):239-251, 1988.

[186] W.S. Hsia, T.Y. Lee and J.Y. Lee. Lagrange multiplier theorem of multiobjective programming problems with set functions, *Journal of Optimization Theory and Applications,* 70(1):137-155, 1991.

[187] Y.D. Hu. Theory and application of multiobjective optimization in China - a survey, *Annals of Operations Research,* 24(1-4):45-51, 1990.

[188] Y.D. Hu and E.J. Sun. Connectedness of the efficient set in strictly quasiconcave vector maximization, *Journal of Optimization Theory and Applications,* 78(3):613-622, 1993.

[189] X.X. Huang. Extended well-posedness properties of vector optimization problems, *Journal of Optimization Theory and Applications,* 106(1):165-182, 2000.

[190] C.L. Hwang, S.R. Paidy, K. Yoon and A.S.M. Masud. Mathematical programming with multiple objectives: A tutorial, *Computers and Operations Research,* 7:5-31, 1980.

[191] G. Isac, M.M. Kostreva and M.M. Wiecek. Multiple-objective approximation of feasible but unsolvable linear complementarity problems, *Journal of Optimization Theory and Applications,* 86(2):389-405, 1995.

[192] H. Isermann. On some relations between a dual pair of multiple objective linear programs, *Zeitschrift für Operations Research,* 22:33-41, 1978.

[193] Y. Ishizuka. Optimality conditions for directionally differentiable multi-objective programming problems, *Journal of Optimization Theory and Applications,* 72(1):91-111, 1992.

[194] Y. Ishizuka and K. Shimizu. Necessary and sufficient conditions for the efficient solutions of nondifferentiable multiobjective problems, *IEEE Transactions on Systems, Man, and Cybernetics,* SMC-14: 625-629, 1984.

[195] Y. Ishizuka and H.D. Tuan. Directionally differentiable multiobjective optimization involving discrete inclusions, *Journal of Optimization Theory and Applications,* 88(3):585-616, 1996.

[196] M.A. Islam. Sufficiency and duality in nondifferentiable multiobjective programming, *Pure and Applied Mathematical Sciences,* 39(l-2):31-39, 1994.

[197] E.H. Ivanov and R. Nehse. Some results on dual vector optimization problems, *Optimization,* 16(4):505-517, 1985.

[198] J. Jahn. Duality in vector optimization, *Mathematical Programming,* 25(3):343-353, 1983.

[199] J. Jahn. Scalarization in vector optimization, *Mathematical Programming,* 29:203-218, 1984.

[200] J. Jahn. Some characterizations of the optimal solutions of a vector optimization problem, *OR Spektrum,* 7:7-17, 1985.

[201] J. Jahn. A characterization of properly minimal elements of a set, *SIAM Journal on Control and Optimization,* 23(5):649-656, 1985.

[202] J. Jahn. *Mathematical Vector Optimization in Partially Ordered Linear Spaces,* Peter Lang, Frankfurt, 1986.

[203] J. Jahn. Existence theorems in vector optimization, *Journal of Optimization Theory and Applications,* 50(3):397-406, 1986.

[204] J. Jahn. Parametric approximation problems arising in vector optimization, *Journal of Optimization Theory and Applications,* 54(3):503-516, 1987.

[205] J. Jahn. Optimality conditions in set-valued vector optimization, in: G. Fandel and T. Gal (eds.), *Multiple Criteria Decision Making,* 22-30, Springer-Verlag, Berlin, 1997.

[206] J. Jahn. Theory of vector optimization: concepts of solutions, in: T. Gal, T.J. Stewart and T. Hanne (eds.), *Multicriteria Decision Making: Advances in MCDM Models, Algorithms, Theory, and Applications,* Chapter 2, Kluwer Academic Publishers, Dordrecht, 1999.

[207] J. Jahn and A. Merkel. Reference point approximation method for the solution of bicriterial nonlinear optimization problems, *Journal of Optimization Theory and Applications,* 74(1):87-103, 1992.

[208] J. Jahn and R. Rauh. Contingent epiderivatives and set-valued optimization, *Mathematical Methods of Operations Research,* 46:193-211, 1997.

[209] J. Jahn and E. Sachs. Generalized quasiconvex mappings and vector optimization, *SIAM Journal on Control and Optimization,* 24(2):306-322, 1986.

[210] V. Jeyakumar. Strong and weak invexity in mathematical programming, *Methods of Operations Research,* 55:109-125, 1985.

[211] V. Jeyakumar. Equivalence of saddle-points and optima, and duality for a class of nonsmooth convex problems, *Journal of Mathematical Analysis and Applications,* 130:334-343, 1988.

[212] V. Jeyakumar and B. Mond. On generalized convex mathematical programming, *Journal of the Australian Mathematical Society Series B,* 34:43-53, 1992.

[213] V. Jeyakumar and X.Q. Yang. Convex composite multi-objective nonsmooth programming, *Mathematical Programming,* 59:325-343, 1993.

[214] A. Jourani. Necessary conditions for extremality and separation theorems with applications to multiobjective optimization, *Optimization,* 44(4):327-350, 1998.

[215] E. Jurkiewicz. Stability of compromise solution in multicriteria decision making problems, *Journal of Optimization Theory and Applications,* 40(l):77-83, 1983.

[216] I. Kaliszewski. A characterization of properly efficient solutions by an augmented Tchebycheff norm, *Bulletin of the Polish Academy of Sciences. Technical Sciences,* 33(7-8):415-420, 1985.

[217] I. Kaliszewski. Norm scalarization and proper efficiency in vector optimization, *Foundations of Control Engineering,* 11(3):117-131, 1986.

[218] I. Kaliszewski. A modified weighted Tchebycheff metric for multiple objective programming, *Computers & Operations Research,* 14(4):315-323, 1987.

[219] I. Kaliszewski. *Quantitative Pareto Analysis by Cone Separation Technique,* Kluwer Academic Publishers, Boston, 1994.

[220] I. Kaliszewski and W. Michalowski. Efficient solutions and bounds on tradeoffs, *Journal of Optimization Theory and Applications,* 94(2), 381-394, 1997.

[221] P. Kanniappan. Necessary conditions for optimality of nondifferentiable convex multiobjective programming, *Journal of Optimization Theory and Applications,* 40(2):167-174, 1983.

[222] R.N. Kaul, D. Bhatia and S. Aggarwal. Continuous time inverse vector optimization problems, *Optimization,* 26(3-4), 303-314, 1992.

[223] R.N. Kaul, S.K. Suneja and C.S. Lalitha. Duality in pseudolinear multiobjective fractional programming, *Indian Journal of Pure and Applied Mathematics,* 24(5):279-290, 1993.

[224] R.N. Kaul, S.K. Suneja and M.K. Srivastrava. Optimality criteria and duality in multiple-objective optimization involving generalized invexity, *Journal of Optimization Theory and Applications,* 80(3):465-482, 1994.

[225] H. Kawasaki. Conjugate relations and weak subdifferentials, *Mathematics of Operations Research,* 6(4):593-607, 1981.

[226] H. Kawasaki. A duality theorem in nonlinear multiobjective programming, *Mathematics of Operations Research,* 7(1):95-110, 1982.

[227] H. Kawasaki. Second-order necessary conditions of the Kuhn-Tucker type under new constraint qualification, *Journal of Optimization Theory and Applications,* 57(2):253-264, 1994.

[228] K.R. Kazmi. Existence of solutions for vector optimization, *Applied Mathematics Letters,* 9(6):19-22, 1996.

[229] K.R. Kazmi. Some remarks on vector optimization problems, *Journal of Optimization Theory and Applications,* 96(1):133-138, 1998.

[230] P.Q. Khanh. Proper solutions of vector optimization problems, *Journal of Optimization Theory and Applications,* 74(1):105-130, 1992.

[231] P.Q. Khanh. Optimality conditions via norm scalarization in vector optimization, *SIAM Journal on Control and Optimization,* 31(3): 646-658, 1993.

[232] P.Q. Khanh. Sufficient optimality conditions and duality in vector optimization with invex-convexlike functions, *Journal of Optimization Theory and Applications,* 87(2):359-378, 1995.

[233] P.Q. Khanh and T.H. Nuong. On necessary optimality conditions in vector optimization problems, *Journal of Optimization Theory and Applications,* 58(1):63-81, 1988.

[234] P.Q. Khanh and T.H. Nuong. On necessary and sufficient conditions in vector optimization, *Journal of Optimization Theory and Applications,* 63(3):391-413, 1989.

[235] D.S. Kim. Generalized convexity and duality for multiobjective optimization problems, *Journal of Information & Optimization Sciences,* 13(3):383-390, 1992.

[236] D.S. Kim and G.M. Lee. Optimality conditions and duality theorems for multiobjective invex programs, *Journal of Information & Optimization Sciences,* 12(2):235-242, 1991.

[237] D.S. Kim and W.J. Lee. Symmetric duality for multiobjective variational problems with invexity, *Journal of Mathematical Analysis and Applications,* 218(1):34-48, 1998.

[238] D.S. Kim and W.J. Lee. Generalized symmetric duality for multiobjective variational problems with invexity, *Journal of Mathematical Analysis and Applications,* 234(1):147-162, 1999.

[239] D.S. Kim, G.L. Lee and H. Kuk. Duality for multiobjective fractional variational problems with generalized invexity, *Nihonkai Mathematical Journal,* 9(l):17-25, 1998.

[240] D.S. Kim, Y.B. Yun and H. Kuk. Second-order symmetric and self-duality in multiobjective programming, *Applied Mathematics Letters,* 10(2):17-22, 1997.

[241] A. Klinger. Improper solutions of the vector maximum problems, *Operations Research,* 15:570-572, 1967.

[242] S. Komlosi, T. Rapcsak and S. Schaible (eds.). *Generalized Convexity,* Springer-Verlag, Berlin, 1994.

[243] I.V. Konnov and S. Schaible. Duality for equilibrium problems under generalized monotonicity, *Journal of Optimization Theory and Applications,* 104(2):395-408, 2000.

[244] I.V. Konnov and J.C. Yao. On the generalized vector variational inequality problem, *Journal of Mathematical Analysis and Applications,* 206(1):42-58, 1997.

[245] I.V. Konnov and J.C. Yao. Existence of solutions for generalized vector equilibrium problems, *Journal of Mathematical Analysis and Applications,* 233(1):328-335, 1999.

[246] J.S.H. Kornbluth. Duality, indifference and sensitivity analysis in multiple objective linear programming, *Operations Research Quarterly,* 25(4):599-614, 1974.

[247] G.C. Kospidas. A new Pareto optimal solution in a Lagrange decomposable multiobjective optimization problem, *Journal of the Operational Research Society,* 42(5):401-411, 1991.

[248] M.M. Kostreva and M.M. Wiecek. Linear complementarity problems and multiple objective programming, *Mathematical Programming,* 60:349-359, 1993.

[249] I. Kouada. Upper-semi-continuity and cone-concavity of multivalued vector functions in a duality theory for vector optimization, *Mathematical Methods of Operations Research,* 46(2):169-192, 1997.

[250] H.W. Kuhn and A.W. Tucker. Nonlinear programming, in: J. Neyman (ed.) *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability,* 481-492. University of California Press, Berkeley, CA, 1951.

[251] H. Kuk, G.M. Lee and D.S. Kim. Nonsmooth multiobjective programs with V-ρ-invexity, *Indian Journal of Pure and Applied Mathematics,* 29(4):405-412, 1998.

[252] H. Kuk and T. Tanino. On nonsmooth minimax programming problems containing V-ρ-invexity, *Journal of Information & Optimization Sciences,* 21(3):437-444, 2000.

[253] H. Kuk, T. Tanino and M. Tanaka. Sensitivity analysis in vector optimization, *Journal of Optimization Theory and Applications,* 89(3):713-730, 1996.

[254] H. Kuk, T. Tanino and M. Tanaka. Sensitivity analysis in parametrized convex vector optimization, *Journal of Mathematical Analysis and Applications,* 202(2):511-522, 1996.

[255] H. Kuk, T. Tanino and M. Tanaka. Trade-off analysis for vector optimization problems via scalarization, *Journal of Information & Optimization Sciences,* 18(l):75-87, 1997.

[256] H.C. Lai and C.P. Ho. Duality theorem of nondifferentiable convex multiobjective programming, *Journal of Optimization Theory and Applications,* 50(3):407-420, 1986.

[257] S.N. Lal and A. Kumar. Some results on optimality and duality in multiobjective programming, *Bulletin of the Allahabad Mathematical Society,* 7:31-41, 1992.

[258] S.N. Lal, B. Nath and A. Kumar. Duality for some nondifferentiable static multiobjective programming problems, *Journal of Mathematical Analysis and Applications,* 186(3):862-867, 1994.

[259] G.M. Lee. Optimality conditions in multiobjective optimization problems, *Journal of Information & Optimization Sciences,* 13(1): 107-111, 1992.

[260] G.M. Lee. On efficiency in nonlinear fractional vector maximization problem, *Optimization,* 25:47-52, 1992.

[261] G.M. Lee. Nonsmooth invexity in multiobjective programming, *Journal of Information & Optimization Sciences,* 15(1):127-136, 1994.

[262] G.M. Lee. On relations between vector variational inequality and vector optimization problem, in: X.Q. Yang, A.I. Mees, M. Fisher and L. Jennings (eds.), *Progress in Optimization,* 167-179, Kluwer Academic Publishers, Dordrecht, 2000.

[263] G.M. Lee, D.S. Kim and C.L. Jo. Duality for multiobjective variational problems with generalized invexity, *Journal of Information & Optimization Sciences,* 19(l):12-23, 1998.

[264] G.M. Lee, D.S. Kim and H. Kuk. Existence of solutions for vector optimization problems, *Journal of Mathematical Analysis and Applications,* 220:90-98, 1998.

[265] G.M. Lee, D.S. Kim and B.S. Lee. Some existence theorems for generalized vector variational inequalities, *Bulletin of the Korean Mathematical Society,* 32(2):343-348, 1995.

[266] G.M. Lee, D.S. Kim and B.S. Lee. On noncooperative vector equilibrium, *Indian Journal of Pure and Applied Mathematics,* 27(8):735-739, 1996.

[267] G.M. Lee, D.S. Kim and B.S. Lee. On vector quasivariational-like inequality, *Bulletin of the Korean Mathematical Society,* 33:45-55, 1996.

[268] G.M. Lee, D.S. Kim and B.S. Lee. Generalized vector variational inequality, *Applied Mathematics Letters,* 9(1):39-42, 1996.

[269] G.M. Lee, D.S. Kim, B.S. Lee and G.Y. Chen. Generalized vector variational inequality and its duality for set-valued maps, *Applied Mathematics Letters,* 11(4):21-26, 1998.

[270] G.M. Lee, D.S. Kim, B.S. Lee and S.J. Cho. Generalized vector variational inequality and fuzzy extension, *Applied Mathematics Letters,* 6(6):47-51, 1993.

[271] G.M. Lee, D.S. Kim, B.S. Lee and S.J. Cho. On vector variational inequality, *Bulletin of the Korean Mathematical Society,* 33(4):553-564, 1996.

[272] G.M. Lee, D.S. Kim, B.S. Lee and N.D. Yen. Vector variational inequality as a tool for studying vector optimization problems, *Nonlinear Analysis: Theory, Methods and Applications,* 34:745-765, 1998.

[273] G.M. Lee and S.H. Kum. On implicit vector variational inequalities, *Journal of Optimization Theory and Applications,* 104(2):409-425, 2000.

[274] G.M. Lee, B.S. Lee and S.S. Chang. On vector quasivariational inequalities, *Journal of Mathematical Analysis and Applications,* 203:626-638,1996.

[275] G.M. Lee, B.S. Lee, D.S. Kim and G.Y. Chen. On vector variational inequalities for multifunctions, *Indian Journal of Pure and Applied Mathematics,* 28(5):633-639, 1997.

[276] G.M. Lee and H. Nakayama. Generalized trade-off directions in multiobjective optimization problems, *Applied Mathematics Letters,* 10(1):119-122, 1997.

[277] B. Lemaire. Approximation in multiobjective optimization, *Journal of Global Optimization,* 2(2):117-132, 1992.

[278] D. Li. Convexification of a noninferior frontier, *Journal of Optimization Theory and Applications,* 88(1):177-196, 1996.

[279] D. Li, J.-B. Yang and M.P. Biswal. Quantitative parametric connections between methods for generating noninferior solutions in

multiobjective optimization, *European Journal of Operational Research,* 117(1):84-99, 1999.

[280] S.X. Li. Quasiconcavity and nondominated solutions in multiobjective programming, *Journal of Optimization Theory and Applications,* 88(1):197-208, 1996.

[281] Z.F. Li. Benson proper efficiency in the vector optimization of set-valued maps, *Journal of Optimization Theory and Applications,* 98(3):623-649, 1998.

[282] Z.F. Li and G.Y. Chen. Lagrangian multipliers, saddle points, and duality in vector optimization of set-valued maps, *Journal of Mathematical Analysis and Applications,* 215(2):297-316, 1997.

[283] Z.F. Li and S.Y. Wang. Lagrange multipliers and saddle points in multiobjective programming, *Journal of Optimization Theory and Applications,* 83(1):63-81, 1994.

[284] Z.F. Li and S.Y. Wang. ε-approximate solutions in multiobjective optimization, *Optimization,* 44(2):161-174, 1998.

[285] Z.F. Li and S.Y. Wang. Connectedness of super efficient sets in vector optimization of set-valued maps, *Mathematical Methods of Operations Research,* 48(2):207-217, 1998.

[286] Z.M. Li. The optimality conditions of differentiable vector optimization problems, *Journal of Mathematical Analysis and Applications,* 201(l):35-43, 1996.

[287] Z.M. Li. The optimality conditions for vector optimization of set-valued maps, *Journal of Mathematical Analysis and Applications,* 237(2):413-424, 1999.

[288] J.G. Lin. Maximal vectors and multiobjective optimization, *Journal of Optimization Theory and Applications,* 18(l):41-64, 1976.

[289] J.G. Lin. Multiple-objective problems: Pareto-optimal solutions by method of proper equality constraints, *IEEE Transactions on Automatic Control,* AC-21(5):641-650, 1976

[290] J.G. Lin. Multiple-objective optimization by a multiplier method of proper equality constraints - part I: Theory, *IEEE Transactions on Automatic Control,* AC-24(4):567-573, 1979

[291] K.L. Lin, D.P. Yang and J.C. Yao. Generalized vector variational inequalities, *Journal of Optimization Theory and Applications,* 92(1):117-125, 1997.

[292] L.-J. Lin. Optimality of differentiable vector-valued n-set functions, *Journal of Mathematical Analysis and Applications,* 149:255-270, 1990.

[293] L.-J. Lin. On the optimality conditions of vector-valued n-set functions, *Journal of Mathematical Analysis and Applications*, 161:367-387, 1991.

[294] L.J. Lin. Pre-vector variational inequalities, *Bulletin of the Australian Mathematical Society*, 53(1):63-70, 1996.

[295] J.C. Liu. ε-duality theorem of nondifferentiable nonconvex multiobjective programming, *Journal of Optimization Theory and Applications*, 69(1):153-167, 1991.

[296] J.C. Liu. ε-Pareto optimality for nondifferentiable multiobjective programming via penalty function, *Journal of Mathematical Analysis and Applications*, 198(1):248-261, 1996.

[297] J.C. Liu. Optimality and duality for multiobjective fractional programming involving nonsmooth (F, ρ)-convex functions, *Optimization*, 36(4):333-346, 1996.

[298] J.C. Liu. Optimality and duality for multiobjective fractional programming involving nonsmooth pseudoconvex functions, *Optimization*, 37(1):27-40, 1996.

[299] J.C. Liu. Optimality and duality for multiobjective programming involving subdifferentiable set functions, *Optimization*, 39(3):239-252, 1997.

[300] J.C. Liu. ε-properly efficient solutions to nondifferentiable multiobjective programming problems, *Applied Mathematics Letters*, 12(6):109-113, 1999.

[301] J.C. Liu and K. Yokoyama. ε-optimality and duality for multiobjective fractional programming, *Computers & Mathematics with Applications*, 37(8):119-128, 1999.

[302] W. Liu and X. Gong. Proper efficiency for set-valued vector optimization problems and vector variational inequalities, *Mathematical Methods of Operations Research*, 51(3):443-457, 2000.

[303] P. Loridan. ε-solutions in vector minimization problems, *Journal of Optimization Theory and Applications*, 43(2):265-276, 1984.

[304] P. Loridan. Technical comment: "ε-duality theorem of nondifferentiable nonconvex multiobjective programming" [Journal of Optimization Theory and Applications, 69(1):153-167, 1991] by J.C. Liu. With a reply by Liu, *Journal of Optimization Theory and Applications*, 74(3):565-568, 1992.

[305] P. Loridan. Well-posedness in vector optimization, in: R. Lucchetti and J. Revalski (eds.), *Recent Developments in Well-Posed Variational Problems*, 171-192, Kluwer Academic Publishers, Dordrecht, 1995.

[306] D.T. Luc. On the domination property in vector optimization, *Journal of Optimization Theory and Applications,* 43(2):327-330, 1984.

[307] D.T. Luc. On duality theory in multiobjective programming, *Journal of Optimization Theory and Applications,* 43(4):557-582, 1984.

[308] D.T. Luc. Structure of the efficient point set, *Proceedings of the American Mathematical Society,* 95(3):433-440, 1985.

[309] D.T. Luc. On scalarizing method in vector optimization, in: G. Fandel, M. Grauer, A. Kurzhanski and A.P. Wierzbicki (eds.), *Large-Scale Modelling and Interactive Decision Analysis,* 46-51, Springer-Verlag, Berlin, 1986.

[310] D.T. Luc. About duality and alternative in multiobjective optimization, *Journal of Optimization Theory and Applications,* 53(2):303-307, 1987.

[311] D.T. Luc. Scalarization of vector optimization problems, *Journal of Optimization Theory and Applications,* 55(1):85-102, 1987.

[312] D.T. Luc. Connectedness of the efficient point sets in quasiconcave vector maximization, *Journal of Mathematical Analysis and Applications,* 122:346-354, 1987.

[313] D.T. Luc. *Theory of Vector Optimization,* Springer-Verlag, Berlin, 1989.

[314] D.T. Luc. An existence theorem in vector optimization, *Mathematics of Operations Research,* 14(4):693-699, 1989.

[315] D.T. Luc. Recession cones and the domination property in vector optimization, *Mathematical Programming,* 49(1):113-122, 1990.

[316] D.T. Luc. Contingent derivatives of set-valued maps and applications to vector optimization. *Mathematical Programming, Series A,* 50(1):99-111, 1991.

[317] D.T. Luc and J. Jahn. Axiomatic approach to duality in optimization, *Numerical Functional Analysis and Optimization,* 13:305-356, 1992.

[318] D.T. Luc and S. Schaible. Efficiency and generalized concavity, *Journal of Optimization Theory and Applications,* 94(1):147-153, 1997.

[319] D.T. Luc and C. Vargas. A saddlepoint theorem for set-valued maps, *Nonlinear Analysis - Theory Methods & Applications,* 18(1):1-7, 1992.

[320] R. Lucchetti. Some existence results and stability in multiobjective optimization, in: P. Serafini (ed.), *Mathematics of Multiobjective Optimization,* 179-187, Springer-Verlag, 1985.

[321] T. Maeda. Constraint qualifications in multiobjective optimization problems: differentiable case, *Journal of Optimization Theory and Applications,* 80(3):483-500, 1994.

[322] A.A.K. Majumdar. Optimality conditions in differentiable multiobjective programming, *Journal of Optimization Theory and Applications,* 92(2):419-427, 1997.

[323] C. Malivert. Fenchel duality in vector optimization, in: W. Oettli and D. Pallaschke (eds.), *Advances in Optimization,* 420-438, Springer-Verlag, Berlin, 1992.

[324] C. Malivert and N. Boissard. Structure of efficient sets for strictly quasi convex objectives, *Journal of Convex Analysis,* 1(2):143-150, 1994.

[325] O. Mangasarian. *Nonlinear Programming,* McGraw-Hill Book Company, New York, 1969.

[326] L. Martein. Lagrange multipliers and generalized differentiable functions in vector extremum problems, *Journal of Optimization Theory and Applications,* 63(2):281-297, 1989.

[327] L. Martein. Some results on regularity in vector optimization, *Optimization,* 20(6):787-798, 1989.

[328] L. Martein. An approach to Lagrangian duality in vector optimization, in: A. Cambini, E. Castagnoli, L. Martein and S. Schaible (eds.), *Generalized Convexity and Fractional Programming with Economic Applications,* 242-252, Springer-Verlag, Berlin, 1990.

[329] J.E. Martinez-Legaz. Lexicographical order and duality in multiobjective programming, *European Journal of Operational Research,* 33:342-348, 1988.

[330] J.E. Martinez-Legaz and I. Singer. Surrogate duality for vector optimization, *Numerical Functional Analysis and Optimization,* 9(5-6):547-568, 1987.

[331] P. Mazzoleni. Duality and reciprocity for vector programming, *European Journal of Operational Research,* 10:42-50, 1982.

[332] K.M. Miettinen. *Nonlinear Multiobjective Optimization,* Kluwer Academic Publishers, Dordrecht, 1999.

[333] K. Miettinen and M.M. Mäkelä. Tangent and normal cones in nonconvex multiobjective optimization, in: Y.Y. Haimes and R.E. Steuer (eds.), *Research and Practice in Multiple Criteria Decision Making,* 114-124, Springer-Verlag, Berlin, 2000.

[334] S.K. Mishra. Lagrange multipliers saddle point and scalarization in composite multiobjective nonsmooth programming, *Optimization,* 38(2):93-105, 1996.

[335] S.K. Mishra. Generalized proper efficiency and duality for a class of nondifferentiable multiobjective variational problems with *V*-invexity, *Journal of Mathematical Analysis and Applications,* 202(1):53-71, 1996.

[336] S.K. Mishra. Multiobjective second order symmetric duality with cone constraints, *European Journal of Operational Research,* 126:675-682, 2000.

[337] S.K. Mishra and R.N. Mukherjee. Duality for multiobjective fractional variational problems, *Journal of Mathematical Analysis and Applications,* 186:711-725, 1994.

[338] S.K. Mishra and R.N. Mukherjee. Generalized convex composite multi-objective nonsmooth programming and conditional proper efficiency, *Optimization,* 34(1):53-66, 1995.

[339] S.K. Mishra and R.N. Mukherjee. On generalized convex multiobjective nonsmooth programming, *Journal of the Australian Mathematical Society Series B,* 38:140-148, 1996.

[340] B. Mond and T. Weir. Generalized concavity and duality, in: S. Schaible and W.T. Ziemba (eds.), *Generalized Concavity in Optimization and Economics,* 263-279, Academic Press, New York, 1981.

[341] R.N. Mukherjee. Generalized convex duality for multiobjective fractional programs, *Journal of Mathematical Analysis and Applications,* 162:309-316, 1991.

[342] R.N. Mukherjee. Generalized pseudoconvex functions and multiobjective programming, *Journal of Mathematical Analysis and Applications,* 208(1):49-57, 1997.

[343] R.N. Mukherjee. Arcwise connected functions and applications in multiobjective optimization, *Optimization,* 39(2):151-163, 1997.

[344] R.N. Mukherjee and S.K. Mishra. Sufficient optimality criteria and duality for multiobjective variational problems with *V*-invexity, *Indian Journal of Pure and Applied Mathematics,* 25(89):801-813, 1994.

[345] R.N. Mukherjee and S.K. Mishra. Multiobjective programming with semilocally convex functions, *Journal of Mathematical Analysis and Applications,* 199(2):409-424, 1996.

[346] R.N. Mukherjee and S.K. Mishra. Generalized invexity and duality in multiobjective variational programs, *Journal of Mathematical Analysis and Applications,* 195(2):307-322, 1995.

[347] R.N. Mukherjee and Ch.P. Rao. Multiobjective fractional programming under generalized invexity, *Indian Journal of Pure and Applied Mathematics,* 27(12):1175-1183, 1996.

[348] P.H. Naccache. Connectedness of the set of nondominated outcomes in multicriteria optimization, *Journal of Optimization Theory and Applications,* 25(3):459-467, 1978.

[349] P.H. Naccache. Stability in multicriteria optimization, *Journal of Mathematical Analysis and Applications,* 68:441-453, 1979.

[350] C. Nahak and S. Nanda. Duality for multiobjective variational problems with invexity, *Optimization,* 36(3):235-248, 1996.

[351] C. Nahak and S. Nanda. Duality for multiobjective variational problems with pseudo-invexity, *Optimization,* 41(4):361-382, 1997.

[352] H. Nakayama. Geometric consideration of duality in vector optimization, *Journal of Optimization Theory and Applications,* 44(4):625-655, 1984.

[353] H. Nakayama. Lagrange duality and its geometric interpretation, in: P. Serafini (ed.), *Mathematics of Multiobjective Optimization,* 105-128, Springer-Verlag, 1985.

[354] H. Nakayama. Duality theory in vector optimization: an overview, in: Y.Y. Haimes and V. Chankong (eds.), *Decision Making with Multiple Objectives,* 109-125, Springer-Verlag, Berlin, 1985.

[355] H. Nakayama. Trade-off analysis using parametric optimization techniques, *European Journal of Operational Research,* 60:87-98, 1992.

[356] H. Nakayama. Duality in multi-objective optimization, in: T. Gal, T.J. Stewart and T. Hanne (eds.), *Multicriteria Decision Making: Advances in MCDM Models, Algorithms, Theory, and Applications,* Chapter 3, Kluwer Academic Publishers, Dordrecht, 1999.

[357] J.W. Nieuwenhuis. Supremal points and generalized duality, *Mathematische Operationsforschung und Statistik, Series Optimization,* 11(1):41-59, 1980.

[358] J.W. Nieuwenhuis. Properly efficient and efficient solutions for vector maximization problems in Euclidean space, *Journal of Mathematical Analysis and Applications,* 84:311-317, 1981.

[359] J.W. Nieuwenhuis. Some minimax theorem in vector-valued functions, *Journal of Optimization Theory and Applications,* 40(3):463-475, 1983.

[360] R. Osuna-Gomez, A. Beato-Moreno, P. Luque-Calvo and A. Rufian-Lizana. Invex and pseudoinvex functions in multiobjective programming, in: R. Caballero, F. Ruiz, R. E. Steuer (eds.),

Advances in Multiple Objective and Goal Programming, 228-234, Springer-Verlag, Berlin, 1997.

[361] R. Osuna-Gomez, A. Beato-Moreno and A. Rufian-Lizana. Generalized convexity in multiobjective programming, *Journal of Mathematical Analysis and Applications,* 233(1), 205-220, 1999.

[362] R. Osuna-Gomez, A. Rufian-Lizana and P. Ruiz-Canales. Invex functions and generalized convexity in multiobjective programming, *Journal of Optimization Theory and Applications,* 98(3):651-661, 1998.

[363] D. Pallaschke and S. Rolewicz. *Foundations of Mathematical Optimization: Convex Analysis without Linearity,* Kluwer Academic Publishers, Dordrecht, 1997.

[364] A. Pascoletti and P. Serafini. Scalarizing vector optimization problems, *Journal of Optimization Theory and Applications,* 42(4):499-524, 1984.

[365] R.B. Patel. Second order duality in multiobjective fractional programming, *Indian Journal of Mathematics,* 38(1):39-46, 1997.

[366] R.B. Patel. Duality in multiobjective programming with generalized convex functions, *Indian Journal of Mathematics,* 40(3):331-345, 1998.

[367] F. Patrone and S.H. Tijs. Unified approach to approximate solutions in games and multiobjective programming, *Journal of Optimization Theory and Applications,* 52(2):273-278, 1987.

[368] J.P. Penot and A. Sterna-Karwat. Parametrized multicriteria optimization - continuity and closedness of optimal multifunctions, *Journal of Mathematical Analysis and Applications,* 120(1):150-168, 1986.

[369] J.P. Penot and A. Sterna-Karwat. Parametrized multicriteria optimization - order continuity of the marginal multifunctions, *Journal of Mathematical Analysis and Applications,* 144(1):1-15, 1989.

[370] R. Pini and C. Singh. A survey of recent [1985-1995] advances in generalized convexity with applications to duality theory and optimality conditions, *Optimization,* 39:311-360, 1997.

[371] F. Plastria and E. Carrizosa. Geometrical characterization of weakly efficient points, *Journal of Optimization Theory and Applications,* 90(1):217-223, 1996.

[372] K. Pöhler. A theorem of Tucker's type for vector optimization problems with polyhedral cones, *Optimization,* 29(3):215-223, 1994.

[373] V. Preda. On efficiency and duality for multiobjective programs, *Journal of Mathematical Analysis and Applications,* 166:365-377, 1992.

[374] V. Preda. Generalized invex duality for vector programs, *Revue Roumaine de Mathmatiques Pures et Appliques,* 38(1):41-54, 1993.

[375] V. Preda. On proper efficiency and duality for multiobjective programming problems, *Revue Roumaine de Mathmatiques Pures et Appliques,* 38(6):545-554, 1993.

[376] V. Preda. Some optimality conditions for multiobjective programming problems with set functions, *Revue Roumaine de Mathmatiques Pures et Appliques,* 39(3):233-247, 1994.

[377] V. Preda. Optimality conditions and duality in multiple objective programming involving semilocally convex and related functions, *Optimization,* 36(3):219-230, 1996.

[378] V. Preda and I. Chitescu. On constraint qualification in multiobjective optimization problems: Semidifferentiable case, *Journal of Optimization Theory and Application,* 100(2):417-433, 1999.

[379] V. Preda and I.M. Stancu-Minasian. On optimality and duality in multiobjective nonsmooth programming, in: R. Caballero, F. Ruiz and R.E. Steuer (eds.), *Advances in Multiple Objective and Goal Programming,* 178-187, Springer-Verlag, Berlin, 1997.

[380] L.V. Reddy and R.N. Mukherjee. Composite nonsmooth multiobjective programs with V-ρ-invexity, *Journal of Mathematical Analysis and Applications,* 235(2):567-577, 1999.

[381] R.W. Reid and S.J. Citron. On noninferior performance index vectors, *Journal of Optimization Theory and Applications,* 7(1):11-28, 1971.

[382] T.W. Reiland. Generalized invexity for nonsmooth vector-valued mapping, *Numerical Functional Analysis and Optimization,* 10:1191-1202, 1989.

[383] T.W. Reiland. Nonsmooth invexity, *Bulletin of the Australian Mathematical Society,* 42:437-446, 1990.

[384] H. Reuter. An approximation method for the efficiency set of multiobjective programming problems, *Optimization,* 21(6):905-911, 1990.

[385] A.W. Robert and D.E. Varberg. *Convex Functions,* Academic Press, New York, 1973.

[386] W. Rödder. A generalized saddle point theory, *European Journal of Operational Research,* 1(1):55-59, 1977.

[387] S. Rolewicz. Remarks on sufficient conditions of optimality of vector optimization, *Mathematische Operationsforschung und Statistik, Series Optimization,* 15(1):37-40, 1984.

[388] S. Rolewicz. On sufficient conditions for optimality of Lipschitz functions and their applications to vector optimization, in: V.F. Demyanov and D. Pallaschke (eds.), *Nondifferentiable Optimization: Motivations and Applications,* 129-138, Springer-Verlag, Berlin, 1985.

[389] W.D. Rong and Y.N. Wu. Characterizations of super efficiency in cone-convexlike vector optimization with set-valued maps, *Mathematical Methods of Operations Research,* 48(2):247-258, 1998.

[390] M. van Rooyen, X. Zhou and S. Zlobec. A saddle-point characterization of Pareto optima, *Mathematical Programming,* 67:77-88, 1994.

[391] M.G. Rueda, M.A. Hanson and C. Singh. Optimality and duality with generalized convexity, *Journal of Optimization Theory and Applications,* 86(2):491-500, 1995.

[392] P. Ruiz-Canales and A. Rufian-Lizana. A characterization of weakly efficient points, *Mathematical Programming,* 68:205-212, 1995.

[393] M. Sakawa and H. Yano. Trade-off rates in the hyperplane method for multiobjective optimization problems, *European Journal of Operational Research,* 44(1):105-118, 1990.

[394] M. Sakawa and H. Yano. Generalized hyperplane methods for characterizing Λ-extreme points and trade-off rates for multiobjective optimization problems, *European Journal of Operational Research,* 57(3):368-380, 1992.

[395] M.E. Salkuvadze. *Vector-Valued Optimization Problems in Control Theory,* Academic Press, New York, 1979.

[396] M.E. Salukvadze and A.L. Topchishvili. Weakly-efficient solutions of limiting multicriteria optimization problems, *Journal of Optimization Theory and Applications,* 77(2):373-386, 1993.

[397] Y. Sawaragi, H. Nakayama and T. Tanino. *Theory of Multiobjective Optimization,* Academic Press, Orlando, 1985.

[398] S. Schaible and W.T. Ziemba. *Generalized Concavity in Optimization and Economics,* Academic Press, New York, 1981.

[399] M. Sekatzek. Contributions to a multiplier rule for multiobjective optimization problems, in: V.H. Nguyen, J.J. Strodiot and P. Tossings (eds.), *Optimization (Namur 1998),* 419-436. Springer, Berlin, 2000.

[400] P. Serafini. A unified approach for scalar and vector optimization, in: P. Serafini (ed.), *Mathematics of Multiobjective Optimization,* 89-104, Springer-Verlag, 1985.

[401] H.D. Sherali and A.L. Soyster. Preemptive and nonpreemptive multiobjective programming: relations and counterexamples, *Journal of Optimization Theory and Applications,* 39(2):173-186, 1983.

[402] D.S. Shi. Contingent derivatives of the perturbation map in multiobjective optimization, *Journal of Optimization Theory and Applications,* 70(2):385-396, 1991.

[403] D.S. Shi. Sensitivity analysis in convex vector optimization, *Journal of Optimization Theory and Applications,* 77(1):145-159, 1993.

[404] K. Shimizu, Y. Ishizuka and J.F. Bard. *Nondifferentiable and Two-Level Mathematical Programming,* Kluwer Academic Publishers, Boston, 1997.

[405] A.H. Siddiqi, R. Ahmad and M.F. Khan. Generalized multivalued vector variational inequalities, *Bulletin of the Calcutta Mathematical Society,* 88(6):471-476, 1996.

[406] A.H. Siddiqi, R. Ahmad and M.F. Khan. Existence of solution for generalized multivalued vector variational inequalities without convexity, *Indian Journal of Pure and Applied Mathematics,* 28(8):1057-1060, 1997.

[407] A.H. Siddiqi, Q.H. Ansari and R. Ahmad. On vector variational-like inequalities, *Indian Journal of Pure and Applied Mathematics,* 28(8):1009-1016, 1997.

[408] A.H. Siddiqi, Q.H. Ansari and A. Khaliq. On vector variational inequalities, *Journal of Optimization Theory and Applications,* 84(1):171-180, 1995.

[409] A.H. Siddiqi, M.F. Khan and R. Ahmad. An existence theorem for generalized multivalued vector variational inequality, *Bulletin of the Calcutta Mathematical Society,* 89(5):393-396, 1997.

[410] C. Singh. Optimality conditions in multiobjective differentiable programming, *Journal of Optimization Theory and Applications,* 53(1):115-123, 1987.

[411] C. Singh. Duality theory in multiobjective differentiable programming, *Journal of Information & Optimization Sciences,* 9:231-240, 1988.

[412] C. Singh. Multiobjective programming strong vector duality, *Journal of Information & Optimization Sciences,* 11(2):319-326, 1990.

[413] C. Singh, D. Bhatia and N. Rueda. Duality in nonlinear multiobjective programming using augmented Lagrangian functions, *Journal of Optimization Theory and Applications,* 88(3):659-670, 1996.

[414] C. Singh and M.A. Hanson. Saddlepoint theory for nondifferentiable multiobjective fractional programming, *Journal of Information & Optimization Sciences,* 7(1):41-48, 1986.

[415] C. Singh and M.A. Hanson. Multiobjective fractional programming duality theory, *Naval Research Logistics,* 38(6):925-933, 1991.

[416] C. Singh and M.A. Hanson. Generalized proper efficiency in multiobjective programming, *Journal of Information & Optimization Sciences,* 12(1):139-144, 1991.

[417] C. Singh and M.A. Hanson. Erratum: "Generalized proper efficiency in multiobjective programming" [Journal of Information & Optimization Sciences, 12(1):139-144, 1991], *Journal of Information & Optimization Sciences,* 13(1):173, 1992.

[418] C. Singh, S.K. Suneja and N.G. Rueda. Preinvexity in multiobjective fractional programming, *Journal of Information & Optimization Sciences,* 13(2):293-302, 1992.

[419] B. Skarpness and V.A. Sposito. A modified Fritz John optimality criterion, *Journal of Optimization Theory and Applications,* 31:113-115, 1980.

[420] R.M. Soland. Multicriteria optimization: A general characterization of efficient solutions, *Decision Sciences,* 10(1):26-38, 1979.

[421] W. Song. Duality for vector optimization of set-valued functions, *Journal of Mathematical Analysis and Applications,* 201(1):212-225, 1996

[422] W. Song. Optimality conditions and Lagrangian duality for vector optimization of invex set-valued functions, *Control and Cybernetics,* 26(1):69-85, 1997.

[423] W. Song. Connectivity of efficient solution sets in vector optimization of set-valued mappings, *Optimization,* 39(1):1-11, 1997.

[424] W. Song. Conjugate duality in set-valued vector optimization, *Journal of Mathematical Analysis and Applications,* 216(1):265-283, 1997

[425] W. Song. A generalization of Fenchel duality in set-valued vector optimization. Set-valued optimization, *Mathematical Methods of Operations Research,* 48(2):259-272, 1998.

[426] Y. Sonntag and C. Zalinescu. Comparison of existence results for efficient points, *Journal of Optimization Theory and Applications,* 105(1):161-188, 2000.

[427] W. Stadler. A survey of multicriteria optimization or the vector maximum problem, part I: 1776-1960, *Journal of Optimization Theory and Applications,* 29(1):1-52, 1979.

[428] T. Staib. On necessary and sufficient optimality conditions for multicriterial optimization problems, *ZOR – Methods and Models of Operations Research,* 35(3):231-248, 1991.

[429] A. Sterna-Karwat. Semicontinuity of multifunctions connected with optimization with respect to cones, *Journal of the Australian Mathematical Society Series A,* 40:183-193, 1986.

[430] A. Sterna-Karwat. Continuous dependence of solutions on a parameter in a scalarization method, *Journal of Optimization Theory and Applications,* 55(3):417-434, 1987.

[431] A. Sterna-Karwat. Approximating families of cones and proper efficiency in vector optimization, *Optimization,* 20(6):809-817, 1989.

[432] A. Sterna-Karwat. Convexity of the optimal multifunctions and its consequences in vector optimization, *Optimization,* 20(6):799-808, 1989.

[433] R. Steuer. *Multiple Criteria Optimization: Theory, Computation and Application,* John Wiley & Sons, New York, 1986.

[434] S.K. Suneja, C.S. Lalitha and S. Khurana. Optimality and duality theorems for non-smooth multiobjective fractional programs, *Indian Journal of Pure and Applied Mathematics,* 30(3):243-257, 1999.

[435] S.K. Suneja and M.K. Srivastava. Optimality and duality in nondifferentiable multiobjective optimization involving *d*-type I and related functions, *Journal of Mathematical Analysis and Applications,* 206(2):465-479, 1997.

[436] A. Taa. Necessary and sufficient conditions for multiobjective optimization problems, *Optimization,* 36(2):97-104, 1996.

[437] Ch. Tammer and K. Tammer. Generalization and sharpening of some duality relations for a class of vector optimization problems, *Zeitschrift für Operations Research,* 35(3):249-265, 1991.

[438] Ch. Tammer and K. Tammer. Duality results for convex vector optimization problems with linear restrictions, in: P. Kall (ed.), *System Modelling and Optimization,* 55-64, Springer-Verlag, Berlin, 1992.

[439] K. Tamura and S. Arai. On proper and improper efficient solutions of optimal problems with multicriteria, *Journal of Optimization Theory and Applications,* 38(2):191-205, 1982.

[440] K. Tamura and S. Miura. Necessary and sufficient conditions for local and global nondominated solutions in decision problems with multi-objectives, *Journal of Optimization Theory and Applications,* 28(4):501-523, 1979.

[441] T. Tanaka. Some minimax problems of vector-valued functions, *Journal of Optimization Theory and Applications,* 59(3):505-524, 1988.

[442] T. Tanaka. Two types of minimax theorems for vector-valued functions, *Journal of Optimization Theory and Applications,* 68(2):321-334, 1991.

[443] T. Tanaka. Generalized quasiconvexities, cone saddle-points, and minimax theorem for vector-valued functions, *Journal of Optimization Theory and Applications,* 81(2):355-377, 1994.

[444] T. Tanino. Saddle points and duality in multi-objective programming, *International Journal of Systems Science,* 13(3):323-335, 1982.

[445] T. Tanino. Sensitivity analysis in multiobjective optimization, *Journal of Optimization Theory and Applications,* 56(3):479-499, 1988.

[446] T. Tanino. Stability and sensitivity analysis in convex vector optimization, *SIAM Journal on Control and Optimization,* 26(3):521-536, 1988.

[447] T. Tanino. On supremum of a set in a multi-dimensional space, *Journal of Mathematical Analysis and Applications,* 130(2):386-397, 1988.

[448] T. Tanino. Stability and sensitivity analysis in nonlinear multiobjective programming, *Annals of Operations Research,* 27:97-114, 1990.

[449] T. Tanino. Conjugate duality in vector optimization, *Journal of Mathematical Analysis and Applications,* 167(1):84-97, 1992.

[450] T. Tanino. Supremum of a set in the multi-dimensional extended real space, in: W. Takahashi and T. Tanaka (eds.), *Nonlinear Analysis and Convex Analysis,* World Scientific, River Edge, 352-359, 1999.

[451] T. Tanino. Sensitivity analysis in MCDM, in: T. Gal, T.J. Stewart and T. Hanne (eds.), *Multicriteria Decision Making: Advances in MCDM Models, Algorithms, Theory, and Applications,* Chapter 7, Kluwer Academic Publishers, Dordrecht, 1999.

[452] T. Tanino and Y. Sawaragi. Duality theory in multiobjective programming, *Journal of Optimization Theory and Applications,* 27(4):509-529, 1979.

[453] T. Tanino and Y. Sawaragi. Stability of nondominated solutions in multicriteria decision making, *Journal of Optimization Theory and Applications,* 30(2):229-253, 1980.

[454] T. Tanino and Y. Sawaragi. Conjugate maps and duality in multiobjective optimization, *Journal of Optimization Theory and Applications,* 31(4):473-499, 1980.

[455] M.L. TenHuisen and M.M. Wiecek. Vector optimization and generalized Lagrangian duality, *Annals of Operations Research,* 51:15-32, 1994.

[456] M.L. TenHuisen and M.M. Wiecek. On the structure of the nondominated set for bicriteria programmes, *Journal of Multi-criteria Decision Analysis,* 5(3):232-243, 1996.

[457] P.T. Thach, H. Konno and D. Yokota. Dual approach to minimization on the set of Pareto-optimal solutions, *Journal of Optimization Theory and Applications,* 88(3):698-707, 1996.

[458] I. Valyi. Approximate solutions of vector optimization problems, in: A. Sydow, M. Thoma and R. Vichnevstsky (eds.), *Systems Analysis and Simulation,* 246-250, Mathematical Research 27, Akademie-Verlag, Berlin, 1985.

[459] I. Valyi. Approximate saddle-point theorems in vector optimization, *Journal of Optimization Theory and Applications,* 55(3):435-448, 1987.

[460] M.N. Vartak and I. Gupta. Generalized convexity in multiobjective programming, *Indian Journal of Pure and Applied Mathematics,* 20(l):10-39, 1989.

[461] R. Vetschera. A note on scalarizing functions under changing sets of criteria, *European Journal of Operational Research,* 52(1):113-118, 1991.

[462] J.-P. Vial. Strong convexity of sets and functions, *Journal of Mathematical Economics,* 9:187-205, 1982.

[463] J.-P. Vial. Strong and weak convexity of sets and functions, *Mathematics of Operations Research,* 8(2):231-259, 1983.

[464] H.F. Wang. Resolution of variational inequality problems by multiobjective programming, in: Y.Y. Haimes and R.E. Steuer (eds.), *Research and Practice in Multiple Criteria Decision Making,* 185-195, Springer-Verlag, Berlin, 2000.

[465] S.Y. Wang. Second-order necessary and sufficient conditions in multiobjective programming, *Numerical Functional Analysis and Applications,* 12:237-252, 1991.

[466] S.Y. Wang and Z. Li. Scalarization and Lagrange duality in multiobjective optimization, *Optimization,* 26(3-4):315-324, 1992.

[467] S.Y. Wang and F.M. Yang. A gap between multiobjective optimization and scalar optimization, *Journal of Optimization Theory and Applications,* 68(2):389-391, 1991.

[468] A.R. Warburton. Quasiconcave vector maximization: connectedness of the sets of Pareto-optimal and weak Pareto-optimal alternatives, *Journal of Optimization Theory and Applications,* 40:537-557, 1983.

[469] P. Weidner. On the characterization of efficient points by means of monotone functionals, *Optimization,* 19(1):53-69, 1988.

[470] P. Weidner. Complete efficiency and interdependencies between objective functions in vector optimization, *Zeitschrift für Operations Research,* 34(2):91-115, 1990.

[471] T. Weir. A dual for multiobjective fractional programming problem, *Journal of Information & Optimization Sciences,* 7:261-269, 1986.

[472] T. Weir. A duality theorem for a multiple objective fractional optimization problem, *Bulletin of the Australian Mathematical Society,* 34:415-425, 1986.

[473] T. Weir. Proper efficiency and duality for vector valued optimization problems, *Journal of the Australian Mathematical Society Series A,* 43:21-34, 1987.

[474] T. Weir. On efficiency, proper efficiency and duality in multiobjective programming, *Asia-Pacific Journal of Operational Research,* 7(1):46-54, 1990.

[475] T. Weir. Symmetric dual multiobjective fractional programming, *Journal of the Australian Mathematical Society Series A,* 50(1):67-74, 1991.

[476] T. Weir and B. Mond. Pre-invex functions in multiple objective optimization, *Journal of Mathematical Analysis and Applications,* 136:29-38, 1988.

[477] T. Weir and M. Mond. Generalized convexity and duality in multiple objective programming, *Bulletin of the Australian Mathematical Society,* 39:287-299, 1989.

[478] T. Weir, B. Mond and B.D. Craven. Weak minimization and duality, *Numerical Functional Analysis and Optimization,* 9:181-192, 1987.

[479] T. Weir, B. Mond and R.R. Egudo. Duality without constraint qualification for multiobjective fractional programming, *Asia-Pacific Journal of Operational Research,* 9(2):195-206, 1992.

[480] R.E. Wendell and D.N. Lee. Efficiency in multiple objective optimization problems, *Mathematical Programming,* 12:406-414, 1977.

[481] D.J. White. Optimality and efficiency I, *European Journal of Operational Research,* 4:346-356, 1980.

[482] D.J. White. Rational solution axioms and efficient sets, *European Journal of Operational Research,* 8:66-75, 1981.

[483] D.J. White. *Optimality and Efficiency,* Wiley, New York, 1982.

[484] D.J. White. Multiobjective programming and penalty functions, *Journal of Optimization Theory and Applications,* 43(4):583-599, 1984.

[485] D.J. White. Vector maximisation and Lagrange multipliers, *Mathematical Programming,* 31:192-205, 1985.

[486] D.J. White. Epsilon efficiency, *Journal of Optimization Theory and Applications,* 49(2):319-337, 1986.

[487] D.J. White. Weighting factor extensions for finite multiple-objective vector minimization problems, *European Journal of Operational Research,* 36:256-265, 1988.

[488] A.P. Wierzbicki. Basic properties of scalarization functionals for multiobjective optimization, *Mathematische Operationsforschung und Statistik, Series Optimization,* 8(1):55-60, 1977.

[489] A.P. Wierzbicki. On the completeness and constructiveness of parametric characterizations to vector optimization problems, *OR Spektrum,* 8:73-87, 1986.

[490] A.P. Wierzbicki. Reference point approaches, in: T. Gal, T.J. Stewart and T. Hanne (eds.), *Multicriteria Decision Making: Advances in MCDM Models, Algorithms, Theory, and Applications,* Kluwer Academic Publishers, Chapter 9, 1999.

[491] P. Wolfe. A duality theorem for nonlinear programming, *Quarterly of Applied Mathematics,* 19:239-244, 1961.

[492] Z. Xu. Mixed type duality in multiobjective programming problems, *Journal of Mathematical Analysis and Applications,* 198(3):621-635, 1996.

[493] Z. Xu. Necessary conditions for suboptimization over the weakly efficient set associated to generalized invex multiobjective programming, *Journal of Mathematical Analysis and Applications,* 201(2):502-515, 1996.

[494] S. Yamada, T. Tanino and M. Inuiguchi. An inner approximation method for optimization over the weakly efficient set, *Journal of Global Optimization,* 16(3):197-217, 2000.

[495] J.-B. Yang. Minimax reference point approach and its application for multiobjective optimization, *European Journal of Operational Research,* 126(3):541-556, 2000.

[496] X.M. Yang, X.Q. Yang and G.Y. Chen. Theorems of the alternatives and optimization with set-valued maps, *Journal of Optimization Theory and Applications,* 107(3):627-640, 2000.

[497] X.Q. Yang. Generalized convex functions and vector variational inequalities, *Journal of Optimization Theory and Applications,* 79(3):563-580, 1993.

[498] X.Q. Yang. Vector variational inequality and its duality, *Nonlinear Analysis,* 21(11):869-877, 1993.

[499] X.Q. Yang. Vector variational inequality and vector pseudolinear optimization, *Journal of Optimization Theory and Applications,* 95(3):729-734, 1997.

[500] X.Q. Yang. A generalized upper Dini-directional derivative in vector optimization, *Optimization,* 43(4):339-351, 1998.

[501] X.Q. Yang and C.J. Goh. On vector variational inequalities: application to vector equilibria, *Journal of Optimization Theory and Applications,* 95(2):431-443, 1997.

[502] X.Q. Yang and V. Jeyakumar. First and second-order optimality conditions for convex composite multiobjective optimization, *Journal of Optimization Theory and Applications,* 95(1):209-224, 1997.

[503] H. Yano and M. Sakawa. Trade-off rates in the weighted Tchebycheff norm method, *Large Scale Systems,* 13(2):167-177, 1987.

[504] H. Yano and M. Sakawa. A unified approach for characterizing Pareto optimal solutions of multiobjective optimization problems: the hyperplane method, *European Journal of Operational Research,* 39(1):61-70, 1989.

[505] K. Yokoyama. Epsilon approximate solutions for multiobjective programming problems, *Journal of Mathematical Analysis and Applications,* 203(1):142-149, 1996.

[506] J. Yu. Essential weak efficient solution in multiobjective optimization problems, *Journal of Mathematical Analysis and Applications,* 166(1):230-235, 1992.

[507] P.L. Yu. Cone convexity, cone extreme points, and nondominated solutions in decision problems with multiple objectives, *Journal of Optimization Theory and Applications,* 14(3):319-377, 1974.

[508] P.L. Yu. *Multiple-Criteria Decision Making: Concepts, Techniques and Extensions,* Plenum Press, New York, 1985.

[509] P.L. Yu and G. Leitmann. Compromise solutions, domination structures, and Salukvadze's solution, *Journal of Optimization Theory and Applications,* 13(3):362-378, 1974.

[510] P.L. Yu and G. Leitmann. Nondominated decisions and cone convexity in dynamic multicriteria decision problems, *Journal of Optimization Theory and Applications,* 14(5):573-584, 1974.

[511] S.J. Yu and J.C. Yao. On vector variational inequalities, *Journal of Optimization Theory and Applications,* 89(3):749-769, 1996.

[512] L.A. Zadeh. Optimality and non-scalar-valued performance criteria, *IEEE Transactions on Automatic Control,* AC-8:59-60, 1963.

[513] C. Zalinescu. A result on sets, with applications to vector optimization, *Zeitschrift für Operations Research,* 35(4):291-298, 1991.

[514] G.J. Zalmai. Proper efficiency conditions and duality models for a class of nonsmooth continuous-time multiobjective programming problems, *Journal of Statistics & Management Systems,* 1(2-3):175-197, 1998.

[515] J. Zhang. Higher order convexity and duality in multiobjective programming problems, in: A. Eberhard, R. Hill, D. Ralph and B.M. Glover (eds.), 101-117, *Progress in Optimization,* Kluwer Academic Publishers, 1999.

[516] J. Zhu, C.K. Zhong and Y.J. Cho. Generalized variational principle and vector optimization, *Journal of Optimization Theory and Applications,* 106(1):201-217, 2000.

[517] D.M. Zhuang. Bases of convex cones and Borwein's proper efficiency, *Journal of Optimization Theory and Applications,* 71(3):613-620, 1991.

[518] V.I. Zhukovskiy and M.E. Salukvadze. Sufficient conditions in vector-valued maximin problems, *Journal of Optimization Theory and Applications,* 90(3):523-534, 1996.

Chapter 3

GOAL PROGRAMMING IN THE PERIOD 1990-2000

Dylan F. Jones, Mehrdad Tamiz
School of Computer Science and Mathematics
University of Portsmouth
Mercantile House, Hampshire Terrace, Portsmouth PO1 2EG
United Kingdom
dylan.jones.mehrdad.tamiz@port.ac.uk

Abstract This chapter presents a bibliography of goal programming for the period 1990-2000. Goal programming is introduced and the main variants are defined. An analysis of applications by field is given. A survey of advances in various goal programming extension areas is conducted. The integration and combination of goal programming with other solution, analysis, and modelling techniques is examined. Conclusions are drawn and suggestions for future research directions are made. A list of over 280 references is presented.

Keywords: Goal programming, Bibliography, Review.

1. Introduction

Goal Programming (GP) is a multi-criteria decision making technique. It is traditionally seen as an extension of linear programming to include multiple objectives, expressed by means of the attempted achievement of goal target values for each objective. Another way of viewing the relationship is that linear programming can be considered to be a special case of goal programming with a single objective. All of these considerations place goal programming within the paradigm of multiple objective programming. Connections can be shown between goal programming models and compromise programming and reference point models under certain conditions [163, 191].

The ethos of GP lies in Simon's [284] concept of the satisficing of objectives. Simon conjectures that in today's complex organisations the decision makers (DM's) do not try to maximise a well defined utility function. In fact the conflicts of interest and the incompleteness of available information make it almost impossible to build a reliable mathematical representation of the DMs' preferences. On the contrary, within this kind of decision environment the DMs try and achieve a set of goals (or targets) as closely as possible. Although GP was not originally conceived within a satisficing philosophy it still provides a good framework in which to implement this kind of philosophy.

The roots of GP lie in a paper by Charnes, Cooper, and Ferguson in 1955 [268] in which they deal with executive compensation methods. A more explicit definition is given by Charnes and Cooper [267] in 1961 in which the term 'goal programming' is first used. Until the middle of the 1970's, GP applications reported in literature were rather scarce. Since that time, and chiefly due to seminal works by Lee [278] and Ignizio [274], an impressive boom of GP applications and technical improvements has arisen. It can be said that GP has been, and still is, the most widely used multi-criteria decision making technique [281]. Although Schniederjans [203] has detected a decline in the life cycle of GP with regard to theoretical developments, the number of cases along with the range of fields to which GP has been, and still is, applied is impressive, as shown in surveys by Romero [281], Schniederjans [283], and Tamiz, Jones, and El-Darzi [237].

GP models can be classified into a number of variants, each of which is characterised by a different underlying distance metric or utility function. Three of the major goal programming variants are classified in the remainder of Section 1.

1.1. Weighted Goal Programming

In the first variant the unwanted deviations from the target values are assigned weights according to their relative importance to the decision maker and minimised as an Archimedean sum. The underlying distance metric here is the L_1 'Manhattan' distance. This variant is known as weighted or non-preemptive GP(WGP). The algebraic formulation of a WGP is given as

$$\min\ z = \sum_{i=1}^{Q} \frac{1}{k_i}(u_i n_i + v_i p_i)$$

subject to,

$$f_i(\mathbf{x}) + n_i - p_i = b_i \quad i = 1 \ldots Q$$

$$\mathbf{x} \in C_s$$

where $f_i(\mathbf{x})$ is a linear function(objective) of $\mathbf{x}$, and b_i is the target value or goal for that objective. n_i and p_i represent the negative and positive deviations from this target value. u_i and v_i are the respective non-negative weights attached to these deviations in the achievement function z. These weights take the value zero if the minimisation of the corresponding deviational variable is unimportant to the decision maker. C_s is an optional set of hard constraints as found in linear programming. The function z of weighted deviational variables to be penalised is known as the achievement function.

Each deviational variable included in the achievement function is divided through by a normalisation constant k_i. This is needed to overcome the problem of incommensurability. That is, deviations from objectives measured in differing units being summed directly. A number of different types of normalisation constants are available for use dependent on the situation. These are detailed in Romero [281] and Tamiz and Jones [286].

A weighted goal programming model is used when all the objectives can be compared directly and the decision maker is willing and able to assign weights that reflect the relative importance of the objectives in the situation. Also weighted goal programming should be used when the decision maker is interested in a solution that gives a pure overall lowest sum of weighted deviations from the goals rather than an overall balance between the achievement of those goals. Under these circumstances weighted goal programming is a powerful tool that gives not only solutions, but a good deal of trade-off information between the objectives.

1.2. Lexicographic Goal Programming

Another major variant of GP is formed when the deviational variables are assigned into a number of priority levels and minimised in a lexicographic sense. A lexicographic minimisation being defined as a sequential minimisation of each priority whilst maintaining the minimal values reached by all higher priority level minimisations. This is known as lexicographic or pre-emptive GP(LGP), as introduced and chiefly developed by Ijiri [275], Lee [278], and Ignizio [274].

The algebraic representation of a LGP is given as:

$$Lex \ \min \ \mathbf{a} = (g_1(\mathbf{n}, \mathbf{p}), g_2(\mathbf{n}, \mathbf{p}), \ldots\ldots, g_L(\mathbf{n}, \mathbf{p}))$$

subject to,

$$f_i(\mathbf{x}) + n_i - p_i = b_i \quad i = 1, \ldots, Q$$

This model has L priority levels, and Q objectives. The achievement function **a** is an ordered vector of these L priority levels. n_i and p_i are deviational variables which represent the under and over achievement of the i'th goal respectively, $\mathbf{x}$ is the vector of decision variables to be determined. Any 'LP' style hard constraints are placed, by convention, in the first priority level. A standard 'g' (within priority level) function is given by:

$$g_l(\mathbf{n}, \mathbf{p}) = u_{l_1} n_1 + ... + u_{l_q} n_q + v_{l_1} p_1 + ... + v_{l_Q} p_Q$$

where u and v represent inter-priority level weights, as in weighted GP, a zero weight is given to any deviational variable whose minimisation is unimportant. If the deviational variables summed inside a priority level are measured in different units then a normalisation technique can be applied to overcome incommensurability, as described in the section on weighted goal programming above.

Lexicographic goal programming models have proved the most contentious in terms of the definition of their underlying utility function. Romero [281] provides a discussion on this topic as well as good and poor modelling practices when using lexicographic goal programming. This variant should be used when the decision maker has a natural ordering of the objectives in mind rather than a relativistic comparison. It is also used when the decision maker is unable or unwilling to provide the relevant relative importance of the objectives by means of weights. Lexicographic goal programming is historically the most used goal programming variant [237].

1.3. Chebyshev Goal Programming

Another less widely used but theoretically significant variant is that of Chebyshev Goal Programming, also known as Minmax goal programming. In this variant the maximum deviation from amongst the weighted set of deviations is minimised rather than the sum of the deviations themselves. The algebraic formulation of the Chebyshev GP model is given as:

$$\min \quad z = D$$

subject to,

$$\frac{1}{k_i}[u_i n_i + v_i p_i] \leq D \quad i = 1, \ldots, Q$$

$$f_i(\mathbf{x}) + n_i - p_i = b_i \quad i = 1, \ldots, Q$$

$$X \in C_s$$

This model has Q objectives. $f_i(\mathbf{x})$ is a linear function(objective) of $\mathbf{x}$, and b_i is the target value or goal for that objective. n_i and p_i represent the negative and positive deviations from this target value. u_i and v_i are the respective non-negative weights attached to these deviations in the achievement function z. These weights take the value zero if the minimisation of the corresponding deviational variable is unimportant to the decision maker. C_s is an optional set of hard constraints as found in linear programming. D is the maximum deviation to be minimised and is the sole component of the achievement function z. k_i is the normalisation constant for the i'th objective as defined in Section 1.1.

Chebyshev goal programming in the form detailed here requires the use of weights to reflect the importance of the objective to the decision maker and hence the same comments regarding the use and appropriateness of weights as made for weighted goal programming in Section 1.1 apply. The principle difference is that in Chebyshev goal programming the maximum deviation is being penalised. This leads to a L_∞ distance metric minimisation and implies an underlying utility function of a Rawlsian nature [191, 280]. The practical effect of this is to provide, as far as is possible, a balance between the levels of the objectives rather than a strict minimisation of their sum. This is felt to be appropriate to some extent to many real-life applications. Decision makers should look to utilize Chebyshev goal programming if their requirements are defined in terms of balance and fairness.

To conclude, Section 1 has detailed three major variants of goal programming. Further sub-variants are considered in the analysis given by the following Sections.

2. Details of Literature Review

The articles detailed in this chapter are drawn from a variety of sources, lists, and investigations in the topic of goal programming. The criterion for inclusion is an article that appears in a refereed journal and uses, describes, modifies, or advances goal programming in some way. Articles are drawn from the period 1990-2000. Readers interested in goal

programming articles before this date are referred to the bibliography in Romero [281]. Additionally, the 14 articles from the year 2000 probably do not represent the entirety of the goal programming literature during this year as there may be some time lag before articles appear in lists and databases.

The above definition produces some exclusions that must be noted. Books concerning goal programming are not included. Some books about goal programming, or with substantial goal programming content, published in the relevant time period include the works of Romero [281], Ignizio and Cavalier [274], and Schniederjans [283]. Articles published as part of conference proceedings are not included either. The interested reader is refereed to the proceedings of the MOPGP (multiple objective programming and goal programming) conference series [266, 285] for details of some of the developmental work on goal programming in the past decade.

The principle breakdown of goal programming articles in the analysis of this bibliography is conducted by application area. This is deemed appropriate given the applied nature and wealth of real-world applications of the goal programming technique. The following sub-section presents this analysis.

2.1. Analysis of Fields of Application

The articles are sub-divided into 16 fields of application. Each article is classified into one of these fields. Some articles span more than one field and could have been classified in either. In this case, the article is put into what was judged to be its major field (in terms of novelty and contributions). The fields chosen and the associated articles are detailed as follows:

- Academic Management [12, 14, 77, 82]
- Agricultural Management [8, 9, 15, 22, 29, 30, 56, 66, 75, 80, 86, 90, 95, 97, 113, 114, 172, 220, 242, 261]
- Classical OR Application [20, 31, 50, 63, 132, 169, 174, 259, 265]
- Energy Planning and Production [1, 25, 26, 64, 67, 68, 88, 101, 102, 133, 154, 176, 177, 178, 179, 180, 193, 211]
- Engineering (all types) [41, 47, 70, 73, 94, 118, 126, 141, 159, 183, 207, 212, 214, 227, 228, 229, 230, 253, 254]
- Environmental and Waste Management [4, 5, 42, 46, 43, 44, 45, 83, 147, 185, 188, 224, 244, 257]

- Finance [16, 52, 107, 173, 209]
- Health Planning [34, 74, 106, 116, 125]
- Information Technology and Computing [98, 110, 112, 117, 165, 168, 198, 199, 208, 247]
- Management and Strategic Planning [57, 60, 78, 89, 103, 104, 127, 162, 205, 204, 206, 218, 219, 223, 260]
- Military [71, 105, 160, 250]
- Natural Resource Management [6, 21, 33, 48, 53, 54, 62, 85, 115, 122, 131, 135, 136, 140, 150, 155, 156, 161, 170, 171, 187, 225, 226, 252, 255, 256]
- Production Planning [3, 7, 13, 17, 19, 27, 28, 69, 79, 91, 108, 119, 134, 142, 146, 157, 158, 166, 200, 210, 221, 222, 231, 248, 249, 258, 264]
- Socio-Economic Planning [2, 10, 11, 58, 84, 99, 100]
- Theoretical [18, 32, 35, 36, 37, 39, 49, 51, 55, 59, 61, 65, 72, 81, 92, 93, 96, 109, 111, 120, 121, 123, 124, 128, 129, 130, 137, 138, 139, 144, 145, 148, 149, 151, 153, 163, 164, 167, 175, 181, 182, 184, 186, 189, 190, 191, 192, 195, 196, 197, 201, 202, 203, 213, 216, 217, 232, 233, 234, 235, 236, 237, 238, 239, 240, 241, 243, 245, 246, 251, 262]
- Transportation and Distribution [23, 24, 38, 40, 76, 87, 143, 152, 194, 215, 263]

Where the 'classical OR Application' field includes goal programming as a solution and analysis tool for models in scheduling, queuing, and forecasting. Transportation and distribution models are given their own category. The area of finance is distinguished from the more general area of management and strategic planning in so much as it concerns models dedicated to portfolio or financial product composition and control. Water resource planning is included under the area of natural resource management.

An overview of the division of articles by subject is given by the Table 3.1.

Table 3.1 shows that goal programming is still advancing and being used on both the theoretical and practical levels. There continues to be a healthy range of applications represented. The level of 26.5% theoretical advance seems appropriate for a discipline that is now fairly well established but still being adapted and refined to new technologies and techniques as they emerge. This theme is expanded upon in Section 4.

Table 3.1. Division of articles by field of application.

Field of Application	Number of Articles	Percentage of Total
Academic Management	4	1.5
Agricultural Management	20	7.5
Classical OR Application	9	3.4
Energy Planning and Production	18	6.8
Engineering (all types)	19	7.2
Environmental and Waste Management	14	5.3
Finance	5	1.9
Health Planning	5	1.9
Information Technology and Computing	10	3.8
Management and Strategic Planning	15	5.7
Military	4	1.5
Natural Resource Management	26	9.8
Production Planning	27	10.2
Socio-Economic Planning	7	2.6
Theoretical	71	26.8
Transportation and Distribution	11	4.2

3. Classification of GP Extension Articles

Analysis of the main goal programming variants shows that of the articles surveyed that fell into the categories of lexicographic or weighted goal programming, 59% fell into the lexicographic category and 41% into the category of weighted goal programming. This is a change from the findings of Tamiz, Jones, and El-Darzi [237] who, based on a pre-1990 survey, record a split of 75% for the lexicographic category and 25% for the weighted category. This result excludes models from other variants and extensions as listed below, which may fall into either category.

Chebyshev goal programming still remains a minor topic in terms of articles published, with just [59, 252] explicitly using it as the major variant. However, some Chebyshev GP applications and theory may be hidden within the categories of fuzzy goal programming and multiple goal programming detailed below.

Various models and extensions of goal programming exist beyond or in conjunction with the main variants. These are detailed by extension and broken down by application area in the remainder of this Section.

3.1. Non-Linear Goal Programming

Under the traditional goal programming formulation all the goals, hard constraints, and the achievement function are assumed to be linear in

nature. If any one of these are not then the goal programme is classified as non-linear. The growth in computing power and sophistication of solution methods (for example by meta-heuristic techniques as defined in Section 4.1) have led to greater opportunities for the application, modelling, and solution of non-linear goal programming in a variety of disciplines that are intrinsically non-linear. The following list provides a breakdown of these by application area

- Computing and Information Technology [198]
- Engineering (all disciplines) $[19]^{(A)}$, [94, 108, 228] $[232]^{(B)}$
- Environmental and Waste Management [44]
- Finance [173]
- Natural Resource Management [135]
- Production Planning [166]
- Theoretical [36, 39, 65, 72, 193, 195, 196, 251] $[262]^{(B)}$

Where (A) denotes the use of simulated annealing as a solution tool and (B) denotes the use of a genetic algorithm as a solution tool.

3.2. Quadratic Goal Programming

A special case of non-linear goal programming is that of quadratic goal programming. Here it is assumed that the objectives, hard constraints, and achievement function are polynomial functions of order at most two. The survey found quadratic goal programming used in the following disciplines:

- Engineering (all disciplines) [126, 118]
- Health Planning [34]

3.3. Fractional Goal Progamming

A further special case of non-linear goal programming is fractional goal programming. A general case lexicographic fractional goal programme is defined as:

$$Lex \ \min \ \mathbf{a} = (g_1(\mathbf{n}, \mathbf{p}), g_2(\mathbf{n}, \mathbf{p}), ..., g_L(\mathbf{n}, \mathbf{p}))$$

subject to,

$$\frac{f_i(\mathbf{x})}{g_i(\mathbf{x})} + n_i - p_i = b_i \quad i = 1, \ldots, Q$$

with all notation following the definitions given for lexicographic goal programming in Section 1.2. The chief advantages of fractional goal programmes appear to be ease of analysis and separation of the objective measures $f_i(\mathbf{x})$ and $g_i(\mathbf{x})$, although Romero [189] cites pitfalls associated with the analysis of fractional goal programmes that can be avoided. The past decade has seen modest use of fractional goal programming, with three articles quoting its use [59, 164, 167].

3.4. Integer Goal Programming Models

Standard goal programming models have the same divisibility assumption as linear programming models. That is, all decision variables are assumed to be able to take any value within their feasible range. If this is not the case then the model is classified as an integer goal programming model. The following articles, classified by application area, use integer goal programming

- Academic Management [77]
- Engineering (all disciplines) [41, 70]
- Environmental and Waste Management [4]
- Health Planning [34]
- Management and Strategic Planning [223]
- Natural Resource Management [122, 225],
- Production Planning [69, 142, 223, 248]
- Theoretical [39]

3.5. Zero-one Goal Programming Models

A further subset of integer goal programming is that of zero-one goal programming. Here a subset of the integer decision variables are constrained to take either the value zero or one. This condition frequently represents a decision with two outcomes that needs to be made. The following articles, classified by application area, use zero-one goal programming:

- Academic Management [12]
- Agricultural Management [66]
- Classical OR Application [50]

- Engineering (all types) [73]
- Information Technology and Computing [117, 198]
- Management and Strategic Planning [162, 206, 260]
- Production Planning [79, 158]

3.6. Stochastic Goal Programming Models

The increase in computing power and technology in the decade under review has led to new possibilities in the area of stochastic goal programming. The classical goal programming model assumes that all coefficients and relationships are known with certainty. Stochastic goal programming relaxes these assumptions for a subset of coefficients, objectives, or goal target values. The following articles make use of stochastic goal programming.

- Classical OR Application [63]
- Natural Resource Management [48, 171]
- Production Planning [258]
- Theoretical [92, 93, 128, 129, 130, 148]
- Transportation and Distribution [143]

The dominance of theoretical papers in stochastic goal programming is an indication of the its newness as an accessible goal programming extension.

3.7. Fuzzy Goal Programming Models

An area of increased popularity is the variant of fuzzy goal programming. This combines the area of fuzzy set theory as defined by Zimmermann [287] with goal programming techniques in order to add modelling flexibility and accuracy to the goal programming model. Twenty three articles dealing with or using fuzzy goal programming are found in this survey. The breakdown by application area is given as:

- Engineering (all types) [73, 227]
- Environmental and Waste Management [42, 43, 44, 45]
- Health Planning [106]
- Natural Resource Management [115, 188]

- Theoretical [49, 51, 109, 138, 148, 149, 151, 181, 182, 201, 202, 246]

- Transportation and Distribution [22, 23, 40]

Again, the high percentage of theoretical papers is an indication of the newness of this extension.

3.8. Interactive Goal Programming Models

Goal programming interactive methodology was developed throughout the two decades prior to the material surveyed in this bibliography. A listing of these advances can be found in Tamiz and Jones [235]. The number of papers explicitly using a formal interactive methodology is quite small, although some level of interaction, whether it is on a formal or informal level, is implicit in the solution of any real-life goal programming model. The articles that add to the development of interactive goal programming in the survey are given by Reeves and Hedin [184], Roy and Wallenius [192], Sasaki, Gen, and Ida [202], and Tamiz and Jones [235].

3.9. Multiple Goal Programming Variants

A number of articles use more than one goal programming variant and compare solutions, or are dedicated into the exploration of the differences and connections between the goal programming variants. Articles in this category are divided into application areas as follows

- Agricultural Management [86]

- Theoretical [144, 145, 189, 191, 203, 235, 237, 238, 241]

4. Integration and Combination of Goal Programming with Other Techniques

An important trend in the past decade has been the integration of goal programming with various MCDM, Operational Research, Statistical, and other techniques. This signifies that each technique is no longer seen as a separate entity that is disconnected from other techniques and is therefore no longer used and theoretically developed in isolation. A healthy cross-utilization and fertilization has taken place. Many applications and theoretical developments have taken place that use a conjunction of techniques in some way. These are divided into categories and examined below.

4.1. Using Other Techniques for Solution of Goal Programming Models

Traditional linear goal programming methods use simplex-based optimization techniques to find solutions. Efficient algorithms are available for solution of both weighted and lexicographic cases. Examples of these include the dual algorithm of Ignizio [273] and the specialized solution mechanisms of the GPSYS system [276]. Non-linear models traditionally use non-linear programming methods adapted to a multiple objective framework.

Recent developments in the field of meta-heuristic methods allow for the solution of models that are difficult to solve by conventional methods due to complicating factors. Such complicating factors may include non-linear or non-convex objectives, many integer or binary decision variables, stochasticity, or non-standard constraints and/or underlying utility functions. Meta-heuristic techniques can be effectively used in goal programming in order to expand the range of models able to be solved and hence the range of applications. The following articles, classified by meta-heuristic method, present algorithms or applications that use meta-heuristics for goal programming solution.

- Evolutionary Programming [262]
- Genetic Algorithms [72, 130, 128, 232]
- Simulated Annealing [19]
- Tabu Search [17, 18]

Simulation Techniques are available for models which are unable to be expressed and solved analytically. Articles that use goal programming within a simulation environment are [47] and [125].

Network analysis models allow for a further specialised type of solution algorithm. Articles that use goal programming with network analysis of a solution tool are given by [169, 213, 214, 215, 243]. In addition to these, a recent article by Lee and Kim combines goal programming with the analytical network process [117].

4.2. Goal Programming and the Analytical Hierarchy Process

The analytical hierarchy process [282] is a well-known technique for the determination of weights of factors by a series of pairwise comparisons. This concept leads to a natural combination with goal programming whereby the AHP is used to determine the weights used in a weighted

goal programming model. This integration was pioneered by Gass [272] in the context of a military planning model. Other articles concentrate on the use of goal programming to make technical improvements to the method of the AHP. Articles that use a combination of goal programming and the AHP are divided by application as follows:

- Computing and Information Technology [208]
- Energy Planning and Production [26, 178, 179, 180]
- Environmental and Waste Management [5, 257]
- Health Planning [116]
- Management and Strategic Planning [205, 223]
- Production Planning [13, 263]
- Theoretical [32, 59, 96, 175]

4.3. Goal Programming and Data Envelopment Analysis

Another area of development within the Operational Research framework within the past quarter of a century is that of data envelopment analysis(DEA) [269]. A number of articles in this survey combine goal programming methods and DEA in various ways. A breakdown of these articles by application area is given as follows:

- Management and Strategic Planning [78, 218, 219]
- Socio-Economic Planning [10, 11]
- Theoretical [51, 92, 186, 216]

The dominance of articles in the socio-economic, management, and strategic planning areas is a reflection on the nature and application of DEA.

4.4. Goal Programming and Pareto Efficiency Considerations

A topic that is a cause of considerable concern within the MCDM community is the ability of goal programming, in its basic form, to give solutions that are Pareto inefficient. One of the underlying assumptions of multiple-objective theory is that no rational decision maker will choose a Pareto inefficient solution if a Pareto efficient solution that dominates it

is available. The essence of this conflict is the fact that goal programming is based on the Simonan concept of 'satisficing' [284], whereas multiple objective methods in general are based upon the concept of optimising.

However, the past decade has seen advances in goal programming that allow an inefficient solution to be transformed into an efficient one in accordance with the decision makers preferences as expressed in the original model. Details of these modifications are given by Tamiz and Jones [234]. Integer case extensions are given by Tamiz, Jones, and Mirrazavi [240]. These approaches effectively integrate the philosophies of satisficing and optimizing without diminishing the value of either.

The complete list of articles using Pareto analysis in combination with goal programming, broken down by application area, is given as:

- Engineering (all types) [126]
- Production Planning and Logistics [3]
- Theoretical [234, 240, 251].

With growing awareness of the linkages between goal programming and the more general field of multiple objective programming, the percentage of articles using this type of combination will hopefully increase, as discussed in Section 5.

4.5. Goal Programming and Statistical Techniques

It should be noted that goal programming has connections with various statistical techniques, especially with the technique of regression analysis. The original formulation of goal programming [268] was introduced in the context of 'constrained regression'. In fact weighted and Chebyshev goal programming models of certain forms can be used to form regression models for the metrics L_1 and L_∞ respectively. The following articles use goal programming in a statistical context:

- Finance [52]
- Theoretical [39, 123, 217]

This list does not include topics which are borderline between Statistics and Operational Research (e.g. Forecasting). Such models can be found in the 'Classical OR' field in the applications breakdown given in Section 2.

5. Conclusion and Comment

The last recorded similar analyses of goal programming are given by Schniederjans [203] and by Tamiz, Jones and Romero [238], although neither of which in the context of the analysis of a full goal programming bibliography such as is presented in this chapter. Schniederjans uses life cycle analysis to conclude that the number of goal programming articles is diminishing. Tamiz, Jones, and Romero are more optimistic and claim that the actual rate of publication and breadth of new theory and application in goal programming is still impressive. This bibliography shows an average of 24 articles a year published over the eleven year period in question. Although it is clear that many of the fundamental questions regarding goal programming have been answered, there are nevertheless many avenues for future research in theoretical development and continual stream of new application areas arising which can benefit from goal programming analysis. Predicting the future always involves a good deal of hypothesis and conjecture. However, an analysis of what, in the opinion of the authors, are some of the new challenges for goal programming is given in this Section.

5.1. Possible Future Research Directions

- **Further Integration of Goal Programming In the Multiple Objective Programming and Multi-Criteria Decision Making Paradigms.**

  An analysis of the articles shows a number of applications that use goal programming to produce information for another MCDM technique. Articles found in this category include the following:

  - Despotis DK [59] the use of goal programming in obtaining priority and preference information for MCDM
  - Diaby M and Martel JM [61] preference structure modelling by goal programming
  - Martel JM and Aouni B [137] modelling decision maker preferences by goal programming
  - Gonzalez-Pachon J and Romero C [81]: distance-based consensus methods using goal programming
  - Islam R, Biswal MP and Alam SS [96] Preference programming and inconsistent interval judgments by goal programming
  - Lam KF and Choo EU [111] Using goal programming for preference decomposition

– Moy JW, Lam KF and Choo EU [153] Deriving the partial values in MCDM by goal programming

This list excludes AHP applications which are listed separately in Section 4.2

In addition to this, there exist theoretical advances by Romero, Tamiz and Jones [191] and Ogryczak [163] linking goal programming with other MOP techniques such as compromise programming and the reference point method. The obstacle of Pareto efficiency considerations can be overcome by methods such as those of Tamiz and Jones [234] which allow conversion from the satisficing to the optimizing frameworks.

It is hoped that future work in goal programming will be conducted with a growing awareness of such linkages and a new generation of hybrid algorithms and formulations can be developed with symbiotic advantages for both goal programming and the wider field of MCDM.

- **Goal Programming Applications in the Computing and IT Field and Strategic Management Fields.**

  It is clear that a major growth area worldwide has arisen with the communications revolution and the advent of the World Wide Web. The issues involved give rise to many issues relating to the field of Operational Research [279]. It is inevitable that many of these models are going to involve multiple objectives and that a subset of these will be suitable for analysis and solution using goal programming methodology. The goal programming community should therefore look forward to the growth of the 'Computing and IT' application field as goal programming methods are applied to these new technologies.

  It is also likely that goal programming will continue to have a continuing presence as a strategic management tool to answer questions posed by the changes mentioned in the previous paragraph. New businesses and old businesses looking for new strategies in the rapidly changing markets present good opportunity for goal programming to be used, especially in conjunction with other decision analysis tools and techniques.

- **Goal Programming in Combination with Other Operational Research and Statistical Techniques.**

  Recent years have seen a strong trend towards the use of goal programming as part of a decision making or analysis framework

comprising of multiple Operational Research and Statistical tools. This is the opposite of the traditional 'stand alone' image of goal programming as a solution tool. This trend can be seen in this chapter by the number of articles combining goal programming with various methods, as detailed in Section 4. This is a welcome trend as goal programming can only benefit from such combinations. Combinations with techniques such as the AHP are providing the solution to questions regarding the setting of weights for certain models that have long been an issue in goal programming. It is also noted that a large number of the articles providing such combinations are from the latter part of the time period analysed, i.e. they are very recent.

These considerations lead to the conclusion that this trend is set to continue. The enhanced compatibility of the input and output of various computerised solution and analysis systems both in general and in the area of Operational Research should fuel this trend. It is likely that as new developments are seen in the field of Operational Research then more and more integration and combination of goal programming will take place. The area of combining goal programming with many existing, established Operational Research and Statistical techniques also still has many open questions and potential for future research.

- **Meta Heuristic Methods for Goal Programming Solution.**

As detailed in Section 4.1, there is a growing interest in the use of various meta-heuristic methods as a solution tool for goal programming. This is particularly relevant for 'hard-to-solve' goal programming models as defined in Section 4.1. Whilst the impact on goal programming is likely to be less than some more computationally expensive multi-objective techniques such as Pareto set enumeration, there are nevertheless good opportunities for goal programming to exploit such advances in solution technology.

A closer examination shows that this topic is in its infancy, with most of the articles concerning theoretical developments. There are many open questions regarding the internal structure of the algorithms in the presence of lexicographic achievement functions and multiple goals. Most of the development to date is found in genetic algorithm application to goal programming. There are newer meta-heuristic methods, such as ant colony optimization [270], whose application to goal programming has yet to be investigated.

The type of model that benefits from meta-heuristic solution (i.e. hard-to-solve) tends to arise particularly in the fields of engineering and of finance. It is probable the recently seen goal programming meta-heuristic theoretical advances will lead to an expansion of the scope and number of goal programming models in these fields.

- **Stability and Sensitivity of Goal Programmes.**

  Issues surrounding the stability and sensitivity of the solutions of goal programming models is another area for future development. There now exist many goal programming variants and extensions to which meta-heuristic methods provide alternative means of solution. There is a need to determine whether these variants and methods produce solutions that are robust in terms of parameter changes and imprecise data. The concept of the goal programming dual is developed by Ignizio [273] but has not received major attention since its introduction. The robustness of various methods and more ways of exploiting the goal programming dual remain open for further research.

- **Goal Programming Distance Metric and Variant Awareness**

  In order to build effective goal programming models, it is essential that those using goal programming are aware of the various variants and extensions that exist and their underlying utility function representations. This knowledge allows for the correct choice of goal programming variant, extension, or mix of variants and extensions. Articles that develop effective means of teaching goal programming such as those by Lee and Kim [120, 121] are particularly welcome in this context as they raise decision maker awareness of the goal programming discipline. Furthermore, Web-based tutorials that teach the major goal programming variants are now freely available [277].

  These developments are expected to lead to a greater diversity and more discerning use of goal programming variants and extensions. It is expected that the number of articles using variants such as Chebyshev goal programming or a mix of Chebyshev and weighted or lexicographic goal programming in various fields of application is set to increase.

Acknowledgments

The authors would like to thank the following colleagues whose comments and suggestions on the topic of goal programming over a period

of time have helped shape this bibliography: Rafael Caballero, Jim Ignizio, Simon Mardle, and Carlos Romero.

References

[1] Abouelela AA. (1993) A complete strategy for optimal reactive load flow, *International Journal of Electrical Power and Energy Systems,* **15**, 2, 71-77.

[2] Aderoba A. (1994) A goal programming model for national revenue allocation, *Industrial Engineering,* **23**, 5, 3-8.

[3] Alexander B, Barton G, Petrie J, Romagnoli J. (2000) Process synthesis and optimisation tools for environmental design: Methodology and structure, *Computers and Chemical Engineering,* **24**, 2-7, 1195-1200.

[4] Alidi AS. (1992) An integer goal programming model for hazardous waste treatment and disposal, *Applied Mathematical Modelling,* **16**, 12, 6 45-651.

[5] Alidi AS. (1996) A multiobjective optimization model for the waste management of the petrochemical industry, *Applied Mathematical Modelling,* **20**, 12, 925-933.

[6] Alidi AS, Alfaraj TN. (1994) Goal programming model for distributing and blending desalinized water, *Journal of Water Resources Planning and Management - ASCE,* **120**, 3, 267-275.

[7] Alp N, Murray SL. (1996) A goal programming model to evaluate the production decision through the productivity of sub-systems, *Computers and Industrial Engineering,* **31**, 1-2, 363-366.

[8] Amador F, Sumpsi JM, Romero C. (1998) A non-interactive methodology to assess farmers utility function: An application to large farms in Andalusia, Spain, *European Review of Agricultural Economics,* **25**, 4, 92-109.

[9] Aromolaran AB, Olayemi JK. (1999) Multiple objective farm planning for food crop farmers: A goal programming approach, *Discovery and Innovation,* **11**, 1-2, 91-103.

[10] Athanassopoulos AD. (1995) Goal programming and data envelopment analysis (GoDEA) for target-based multi-level planning: Allocating central grants to the Greek local authorities, *European Journal of Operational Research,* **87**, 3, 535-550.

[11] Athanassopoulos AD. (1996) Assessing the comparative spacial disadvantage (CSD) of regions in the European Union using non-radial data envelopment analysis methods, *European Journal of Operational Research,* **94**, 3, 439-452.

[12] Badri MA. (1996) A two-stage multiobjective scheduling model for faculty-course-time assignments, *European Journal of Operational Research,* **94**, 1, 16-28.

[13] Badri MA. (2000) Combining the analytical hierarchy process and goal programming for global facility location allocation problem, *International Journal of Production Economics,* **62**, 3, 237-248.

[14] Bafail AO, Moreb AA. (1993) Optimal allocation of students to different departments in an engineering college, *Computers and Industrial Engineering,* **25**, 1-4, 295-298.

[15] Baird IA, Cocks KD, Ive JR. (1992) Optimizing the allocation of plant nursery stocks to landholders in a participatory revegetation project, *Journal of Environmental Management,* **35**, 2, 113-130.

[16] Baki MF, Fraser NM. (1997) Error noted in a case study on fund allocation using goal programming, *European Journal of Operational Research,* **97**, 1, 213-214.

[17] Baykasoglu A, Gindy NNZ. (2000), MOCACEF 1.0: Multiple objective capability based approach to form part-machine groups for cellular manufacturing applications, *International Journal of Production Research,* **38**, 5 , 1133-1161.

[18] Baykasoglu A, Owen S, Gindy N. (2000) Solution of goal programming models using a basic Taboo Search algorithm, *Journal of the Operational Research Society,* **50**, 9, 960-973.

[19] BazarganLari M, Kaebernick H. (1997) An approach to the machine layout problem in a cellular manufacturing environment, *Production Planning and Control,* **8**, 1, 41-55.

[20] Bector CR, Gupta YP, Gupta MC. (1991) Optimal scheduling of jobs about a common due date on a single machine, *International Journal of Systems Science,* **22**, 12, 2541-2552.

[21] Berbel J, Zamora R. (1995) An application of MOP and GP to wildlife management (deer), *Journal of Environmental Management,* **44**, 1, 29-38.

[22] Bhattacharya A. (2000) A multiple criteria decision problem for optimal management of farm resources under uncertainty - a case study, *International Journal of Systems Science,* **31**, 6 , 699-703.

[23] Bhattacharya A, Rao JR, Tiwari RN. (1992) Fuzzy multicriteria facility location problem, *Fuzzy Sets and Systems,* **51**, 3, 277-287.

[24] Bhattacharya A, Rao JR, Tiwari RN. (1993) Bi-criteria multi-facility location problem in fuzzy environment, *Fuzzy Sets and Systems,* **56**, 2, 145-153.

[25] Bit AK, Alam SS. (1993) Optimal planning for allocation of coal energy by goal programming, *Industrial Engineering Journal,* **22**, 6, 8-12.

[26] Bose RK, Anandalingam G. (1996) Sustainable urban energy environment management with multiple objectives, *Energy,* **21**, 4, 305-318.

[27] Brauer DC, Naadimuthu G. (1990) A goal programming model for aggregate inventory and distribution planning, *Mathematical and Computer Modelling,* **14**, 1085-1090.

[28] Brauer DC, Naadimuthu G. (1992) A goal programming model for aggregate inventory and distribution planning, *Mathematical and Computer Modelling,* **16**, 3, 81-90.

[29] Broad LR. (1991) Spray drier optimization using goal programming, *Irish Journal of Food Science and Technology,* **15**, 1-2, 17-25.

[30] Brown LG, Mcclendon RW, Akbay KS. (1990) Goal programming for estimating initial influx of insect pests, *Agricultural Systems,* **34**, 4, 337-348.

[31] Brusco MJ, Johns TR. (1995) Improving the dispersion of surplus labour in personnel scheduling problems, *Computers and Industrial Engineering,* **28**, 4, 745-754.

[32] Bryson, N. (1995) A goal programming method for generating priority vectors, *Journal of the Operational Research Society,* **46**, 5, 641-648.

[33] Buongiorno J, Peyron JL, Houllier F, Bruciamacchie M. (1995) Growth and management of mixed-species uneven-aged forests in French Jura - Implications for economic returns and tree diversity, *Forest Science,* **41**, 3, 397-429.

[34] Butler TW, Karwan KR, Sweigart JR, Reeves GR. (1992) An integrative model based approach to hospital layout, *IIE Transactions,* **24**, 2, 144-152.

[35] Caballero R, Gomez T, Gonzalez M, Rey L, Ruiz F. (1998) Goal programming with dynamic goals, *Journal of Multi-Criteria Decision Analysis,* **7**, 217-229.

[36] Caballero R., Rey, L. and Ruiz, F. (1996) Determination of Satisfying and Efficient Solutions in Convex Multi-Objective Programming, *Optimization,* **37**, 125-137.

[37] Caballero R., Rey, L. and Ruiz, F. (1998) Lexicographic Improvement of the Target Values in Convex Goal Programming. *European Journal of Operational Research,* **107**, 644-655.

[38] Ceha R, Ohta H. (2000) Productivity change model in the airline industry: A parametric approach, *European Journal of Operational Research,* **121**, 3, 641-655 .

[39] Chakraborty TK. (1990) The determination of indifference quality level single sampling attribute plans with given relative slope, *Sankhya - The Indian Journal of Statistics Series B,* **52**, 238-245.

[40] Challam GA. (1994) Fuzzy goal programming (FGP) approach to a stochastic transportation problem under budgetary constraint, *Fuzzy Sets and Systems,* **66**, 3, 293-299.

[41] Chang CT, Hwang JR. (1996) A multiobjective approach to waste minimization in the utility systems of chemical processes, *Chemical Engineering Science,* **51**, 16, 3951-3965.

[42] Chang NB, Chen YL. (1997) Optimal balance between recycling and waste-to-energy prographs by grey fuzzy nonlinear goal programming. 1 Theory, *Journal of Environmental Science and Health Part A - Environmental Science and Engineering and Toxic and Hazardous Substance Control,* **32**, 3, 633-647.

[43] Chang NB, Chen YL, Chen HW. (1997) A fuzzy regression analysis for the construction cost estimation of wastewater treatment plants. 1 Theoretical development, *Journal of Environmental Science and Health Part A - Environmental Science and Engineering and Toxic and Hazardous Substance Control,* **32**, 4, 885-899.

[44] Chang NB, Wang SF. (1996) Managerial fuzzy optimal planning for solid-water waste management system, *Journal of Environmental Engineering,* **122**, 7, 649-658.

[45] Chang NB, Wang SF. (1997) A fuzzy goal programming approach for the optimal planning of metropolitan solid waste management systems, *European Journal of Operational Research,* **99**, 2, 303-321.

[46] Chang NB, Wang SF. (1997) Integrated analysis of recycling and incineration programs by goal programming, *Waste Management and Research,* **15**, 2, 121-136.

[47] Chang P, Bernold L. (1992) Multiobjective process optimization for construction, *Canadian Journal of Civil Engineering,* **19**, 1, 129-136.

[48] Changchit C, Terrell MP. (1993) A multiobjective reservoir operation model with stochastic inflows, *Computers and Industrial Engineering,* **24**, 2, 303-313.

[49] Chen HK. (1994) A note on a fuzzy goal programming algorithm by Tiwari, Dharmar, and Rao, *Fuzzy Sets and Systems,* **62**, 3, 287-290.

[50] Chen VYX. (1994) A 0-1 goal programming model for scheduling multiple maintenance projects at a copper mine, *European Journal of Operational Research,* **76**, 1, 176-191.

[51] Cooper WW, Huang ZM, Li SX. (1996) Satisficing DEA models under chance constraints, *Annals of Operations Research,* **66**, 279-295.

[52] Cooper WW, Lejas V, Sueyoshi T. (1997) Goal programming models and their duality relations for use in evaluating security portfolio and regression relations, *European Journal of Operational Research,* **98**, 2, 431-443.

[53] Cornett D, Williams WA. (1991) Goal programming for multiple land use planning at Mineral King California, *Journal of Soil and Water Conservation,* **46**, 5, 373-376.

[54] Crawley PD, Dandy GC. (1993) Optimal operation of multiple reservoir system, *Journal of Water Resources Planning and Management - ASCE,* **119**, 1, 1-17.

[55] Crowder LJ, Sposito VA. (1991) Sequential linear goal programming - implementation via MPSX/370E, *Computers and Operations Research,* **18**, 3, 291-296.

[56] Dekoeijer TJ, Renkama JA, Vanmensvoort JJM. (1995) Environmental economic analysis of mixed crop livestock farming, *Agricultural Systems,* **48**, 4, 515-530.

[57] Deporter EL, Ellis KP. (1990) Optimization of project networks with goal programming and fuzzy linear programming, *Computers and Industrial Engineering,* **19**, 1-4, 500-504.

[58] Dessent G, Hume B. (1990) Value for money and prison perimeters - goal programming for jails, *Journal of the Operational Research Society,* **41**, 7, 583-589.

[59] Despotis DK. (1996) Fractional minmax goal programming: A unified approach to priority estimation and preference analysis in MCDM *Journal of the Operational Research Society,* **47**, 8, 989-999.

[60] Dey PK, Tabucanon MT, Ogunlana SO. (1996) Hierarchical approach to project planning: The case of a petroleum pipeline construction, *Applied Mathematical Modelling,* **20**, 9, 683-698.

[61] Diaby M, Martel JM. (1997) Preference structure modelling for multi-objective decision making: A goal programming approach, *Journal of Multi-Criteria Decision Analysis,* **6**, 3, 150-154.

[62] Diaz-Balteiro L, Romero, C (1998) Modeling timber harvest scheduling problems with multiple criteria: An application in Spain, *Forest Science,* **44**, 1, 47-57

[63] Easton FF, Rossin DF. (1996) A stochastic goal program for employee scheduling, *Decision Sciences,* **27**, 3, 541-568.

[64] Elamin IM, Duffuaa SO, Yassein HA. (1990) Transmission line route selection by goal programming, *International Journal of Electrical Power and Energy Systems,* **12**, 2, 138-143.

[65] Elsayed MEM, Jang TS. (1994) Structural optimization using unconstrained nonlinear goal programming algorithm, *Computers and Structures,* **52**, 4, 723-727.

[66] Fiske WA, Dsouza GE, Fletcher JJ, Phipps TT, Bryan WB, Prigge EC. (1994) An economic and environmental assessment of alternative forage resource production systems - a goal programming approach, *Agricultural Systems,* **45**, 3, 259-270.

[67] Forouzbaksh F, Deiters RM, Kermanshahi BS. (1991) Optimal dynamic scheduling of a power generation system to satisfy multiple criteria, *Simulation,* **56**, 4, 241-249.

[68] Fouad AA, Tong JZ, Hill DJ. (1993) Stability constrained optimal rescheduling of generation, *IEEE Transactions on Power Systems,* **8**, 1, 105-112.

[69] Fukuwaka T, Hong SC. (1993) The determination of the optimal number of Kanbans in a just-in-time production system, *Computers and Industrial Engineering,* **24**, 4, 551-559.

[70] Gagnon RJ, Sheu C. (1997) A strategic MIGP model for acquiring advanced technologies, *Computers and Industrial Engineering,* **32**, 1, 145-168.

[71] Gallagher MA, Kelly EJ. (1991) A new methodology for military force structure analysis, *Operations Research,* **39**, 6, 877-885.

[72] Gen M, Ida K, Lee J, Kim J. (1997) Fuzzy non-linear goal prgramming using genetic algorithm, *Computers and Industrial Engineering,* **33**, 1-2, 39-42.

[73] Gen M, Ida K, Tsujimura Y, Kim CE. (1993) Large scale 0-1 fuzzy goal programming and its application to reliability optimization problem, *Computers and Industrial Engineering,* **24**, 4, 539-549.

[74] Ghandfaroush P. (1993) Optimal allocation of time in a hospital pharmacy using goal programming, *European Journal of Operational Research,* **70**, 2, 191-198.

[75] Ghosh D, Pal BB, Basu M. (1993) Determination of optimal land allocation in agricultural planning through goal programming with penalty functions, *Journal of the Operational Research Society of India,* **30**, 1, 15-34.

[76] Giannikos I. (1998) A multiobjective programming model for locating treatment sites and routing hazardous wastes, *European Journal of Operational Research,* **104**, 2 , 333-342.

[77] Giannikos I, ElDarzi E, Lees P. (1995) An integer goal programming approach to allocate offices to staff in an academic institution, *Journal of the Operational Research Society,* **46**, 6, 713-720.

[78] Giokos D. (1997) The use of goal programming and data envelopment analysis for estimating efficient marginal costs of outputs, *Journal of the Operational Research Society,* **48**, 3, 319-323.

[79] Gokcen H, Erel E. (1997) A goal programming approach to mixed-model assembly line balancing problem, *International Journal of Production Economics,* **48**, 2, 177-185.

[80] Gomez-Limon JA, Berbel, J. (2000) Multicriteria analysis of derived water demand functions: A Spanish case study, *Agricultural Systems,* **63**, 1 , 49-72.

[81] Gonzalez-Pachon J, Romero C. (1999) Distance-based consensus methods: A goal programming approach, *Omega,* **27**, 3, 341-347.

[82] Gosh D, Paul BB, Masu M. (1992) Implementation of goal programming in long-range resource planning in university management, *Optimization,* **24**, 3-4, 373-383.

[83] Gupta SM, Isaacs JA. (1997) Value analysis of disposal strategies for automobiles, *Computers and Industrial Engineering,* **33**, 1-2, 325-328.

[84] Habeeb YA. (1999) Adopting multicriteria planning to the Nigerian economy, *Journal of the Operational Research Society,* **42**, 10, 885-888.

[85] Harboe R. (1992) Multiobjective decision making techniques for reservoir operation, *Water Resources Bulletin,* **28**, 1, 103-110.

[86] Hayashi K. (2000) Multicriteria analysis for agricultural resource management: A critical survey and future perspectives, *European Journal of Operational Research ,* **122** , 2 , 486-500.

[87] Hemaida RS, Kwak NW. (1994) A linear goal programming model for transshipment problems with flexible supply and demand constraints, *Journal of the Operational Research Society,* **45**, 2, 215-224.

[88] Hida K, Yoshioka R. (1992) Optimization of axial enrichment and gadolina distributions for BWR fuel under control rod programming. Part 2: Optimization of reload fuel, *Journal of Nuclear Science and Technology,* **29**, 4, 313-324.

[89] Hobbs BF, Meier PM. (1994) Multicriteria methods for resource planning - an experimental comparison, *IEEE Transactions on Power Systems,* **9**, 4, 1811-1817.

[90] Holden ST. (1993) Peasant household modeling - farming systems evolution and sustainability in Northern Zambia, *Agricultural Economics,* **9**, 3, 241-267.

[91] Hoshino T, Yura K, Hitomi K. (1995) Optimization analysis for recycle-oriented manufacturing systems, *International Journal of Production Research,* **33**, 8, 2069-2078.

[92] Huang ZM, Li SX. (1996) Dominance stochastic models in data envelopment analysis, *European Journal of Operational Research,* **95**, 2, 390-403.

[93] Hussain ML. (1993) Qualitative analysis of basic notions in an iterative approach to possibilistic goal programming, *Fuzzy Sets and Systems,* **54**, 1, 39-46.

[94] Ishiyama A, Hirooka H. (1991) Magnetic shielding for MRI superconducting magnets, *IEEE Transactions on Magnetics,* **27**, 2, 1692-1695.

[95] Irwin SC, Scott JL, Jackson RG. (1996) A semen allocation system for livestock management: A case study in goal programming, *Computers and Electronics in Agriculture,* **16**, 1, 87-101.

[96] Islam R, Biswal MP, Alam SS. (1997) Preference programming and inconsistent interval judgments, *European Journal of Operational Research,* **97**, 1, 53-62.

[97] Johnson PJ, Oltenacu PA, Kaiser HM, Blake RW. (1991) Modelling parasite control programs for developing nations using goal programming, *Journal of Production Agriculture,* **4**, 1, 33-38.

[98] Jones DF, Tamiz M, Mirrazavi SK. (1998) Intelligent solution and analysis of goal programmes: The GPSYS system, *Decision Support Systems,* **23**, 329-332.

[99] Kalu TCU. (1994) Determining the impact of Nigeria Economic crisis on the multinational oil companies - a goal programming

approach, *Journal of the Operational Research Society,* **45**, 2, 165-177.

[100] Kalu, TCU. (1999) An algorithm for systems welfare interactive goal programming modelling, *European Journal of Operational Research,* **116**, 3, 508-529.

[101] Kambo NS, Handa BR, Bose RK. (1991) A linear goal programming model for urban energy economy environment protection, *Energy and Buildings,* **16**, 1-2, 537-551.

[102] Kanniappan P, Ramachandran T. (2000) Goal programming model for sustainable electricity production from Biomass, *International Journal of Energy Research,* **24** , 1, 1-18.

[103] Karpak B, Kumcu E, Kasuganti R. (1999) An application of visual interactive goal programming: A case in vendor selection decisions, *Journal of Multi-Criteria Decision Analysis,* **8**, 2, 93-105.

[104] Khorramshahgol R, Oworuwa AA. (1994) A goal programming approach to different investment decisions: A case study of fund allocation among different shopping malls, *European Journal of Operational Research,* **73**, 1, 17-22.

[105] Kim SH, Ahn BS, Choi SH. (1997) An efficient force planning system using multi-objective linear goal programming, *Computers and Operations Research,* **24**, 6, 569-580.

[106] Kono Y, Yamaguchi T. (1994) An interactive method for multigoal programming problems with fuzzy solution using fuzzy reasoning and its application to diet problem, *Journal of Japan Society for Fuzzy Theory and Systems,* **6**, 1, 187-198.

[107] Kornbluth JSH, Meade N, Salkin GR. (1993) Bond index funds - a duration approach, *Journal of the Operational Research Society,* **44**, 9, 945-953.

[108] Krishnamurty S, Turcic DA. (1992) Optimal synthesis of mechanisms using non-linear goal programming techniques, *Mechanism and Machine Theory,* **27**, 5, 599-612.

[109] Kuwano H. (1996) On the fuzzy multi-objective linear programming problem: Goal programming approach, *Fuzzy Sets and Systems,* **82**, 1, 57-64.

[110] Lam KF, Choo EU. (1993) A linear goal programming model for classification with nonmonotone attributes, *Computers and Operations Research,* **20**, 4, 403-408.

[111] Lam KF, Choo EU. (1995) Goal programming in preference decomposition, *Journal of the Operational Research Society,* **46**, 2, 205-213.

[112] Lam KF, Choo EU, Wedley WC. (1993) Linear goal programming in estimation of classification probabilities, *European Journal of Operational Research,* **67**, 1, 101-110.

[113] Lara P, Romero C. (1992) An interactive multigoal programming for determining livestock rations: An application to dairy cows in Andalusia (Spain). *Journal of the Operational Research Society,* **43**, 10, 945-953.

[114] Lara P, Romero C. (1994) Relaxation of nutrient requirements on livestock rations through interactive multigoal programming. *Agricultural Systems,* **45**, 3, 443-453.

[115] Lee CS, Wen GG. (1997) Fuzzy goal programming approach for water quality management in a river basin, *Fuzzy Sets and Systems,* **89**, 2, 181-192.

[116] Lee CW, Kwak NK. (1999) Information resource planning for a health-care system using an AHP-based goal programming approach, *Journal of the Operational Research Society,* **50**, 12 , 1191-1198.

[117] Lee JW, Kim SH. (2000) Using analytical network process and goal programming for interdependent information system project selection, *Computers and Operations Research,* **27**, 4 , 367-382.

[118] Lee KW, Park GJ. (1998) The modelling and optimization of an electrical vehicle using joint analysis, *International Journal of Vehicle Design,* **19**, 1, 14-28.

[119] Lee SM, Everett AM, Melkonian JH (1994) Optimizing lot size and setup time in a JIT environment - a multiple objective application, *Production Planning and Control,* **5**, 3, 308-319.

[120] Lee SM, Kim EB. (1992) An analysis of padagogical effects of alternative teaching approaches of goal programming, *Decision Sciences,* **23**, 4, 991-1002.

[121] Lee SM, Kim EB. (1995) SFG-GP: An effective tutoring system for goal programming, *Omega,* **23**, 3, 295-302.

[122] Lence BJ, Latheef MI, Burn DH. (1992) Reservoir management and thermal power generation, *Journal of Water Resources Planning and Management - ASCE,* **118**, 4, 388-405.

[123] Lewis RP Jr., Taha HA. (1995) An investigation of the use of goal programming to fit response surfaces, *European Journal of Operational Research,* **86**, 3, 537-548.

[124] Li HL. (1996) An efficient method for solving linear programming problems, *Journal of Optimization Theory and Applications,* **90**, 2, 465-469.

[125] Li NH, Li LX. (2000) Modelling staffing flexibility. A case of China, *European Journal of Operational Research,* **124**, 2 , 255-266.

[126] Lim YI, Floquet P, Joulia X, Kim SD. (1999) Multiobjective design in terms of economics and potential environment impact for process design and analysis in a chemical process simulator, *Industrial and Engineering Chemistry Research,* **38**, 12, 4729-4741.

[127] Lin TW, O'Leary DE. (1993) Goal programming applications in financial management, *Advances in Mathematical Programming and Financial Planning,* **3**, 211-230.

[128] Liu BD. (1996) Dependent chance goal programming and its genetic algorithm based approach, *Mathematical and Computer Modelling,* **24**, 7, 43-52.

[129] Liu BD. (1997) Dependent-chance programming: A class of stochastic optimization *Computers and Mathematics with Applications,* **34**, 12, 89-104.

[130] Liu BD, Iwamura K. (1997) Modelling stochastic decision systems using dependent chance programming, *European Journal of Operational Research,* **101**, 1, 193-203.

[131] Loganathan GV, Bhattacharya D. (1990) Goal programming techniques for optimal reservoir operations, *Journal of Water Resources Planning and Management - ASCE,* **116**, 6, 820-838.

[132] Love CE, Lam KF. (1994) Classifying and controlling errors in forecasting using multiple criteria goal programming, *Computers and Operations Research,* **21**, 9, 979-989.

[133] Lyu J, Gunasekaran A, Chen CY, Kao C. (1995) A goal programming model for the coal blending problem, *Computers and Industrial Engineering,* **28**, 4, 861-868.

[134] Malakooti B. (1991) A multiple criteria decision making approach for the assembly line balancing problem, *International Journal of Production Research,* **29**, 10, 1979-2001.

[135] Mao N, Mays LW. (1994) Goal programming models for determining freshwater inflows to estuaries, *Journal of Water Resources Planning and Management - ASCE,* **120** , 3, 316-329.

[136] Mardle S, Pascoe S, Tamiz M, Jones D. (2000) Resource allocation in North Sea demersal fisheries: A goal programming approach *Annals of Operations Research,* **94**, 321-342.

[137] Martel JM, Aouni B. (1990) Incorporating the decision makers preferences in the goal programming model, *Journal of the Operational Research Society,* **41**, 12, 1121-1132.

[138] Martel JM, Aouni B. (1998) Diverse imprecise goal programming model formulations, *Journal of Global Optimization,* **12**, 127-138.

[139] McGlone TA, Calantone RJ. (1992) A goal programming model for effective segment determination: A comment and application, *Decision Sciences,* **23**, 5, 1231-1239.

[140] McGregor MJ, Dent JB. (1993) An application of lexicographic goal programming to resolve the allocation of water from the Rakaia River (New Zealand), *Agricultural Systems,* **41**, 3, 349-367.

[141] Meadowcroft TA, Stephanopoulis G, Brosilow C. (1992) The modular multivariate controller. Part 1: Steady state properties, *AICHE Journal,* **38**, 8, 1254-1278.

[142] Mehrez A, Benarieh D. (204) All unit discounts multiitem inventory model with stochastic demand service level constraints and finite horizon, *International Journal of Production Research,* **29**, 8, 1615-1628.

[143] Min H. (1991) International intermodal choices via chance constrained goal programming, *Transportation Research Part A - Policy and Practice,* **25**, 6, 351-362.

[144] Min H. (1991) On misconceptions in goal programming - a response, *Journal of the Operational Research Society,* **42**, 10, 928-929.

[145] Min H. (1991) On the origin and persistence of misconceptions in goal programming, *Journal of the Operational Research Society,* **42**, 4, 301-312.

[146] Mitra A, Patankar JG. (1993) An integrated multicriteria model for warranty cost estimation and production, *IEEE Transactions on Engineering Management,* **40**, 3, 300-311.

[147] Mogharabi SN, Ravindran A. (1992) Liquid waste injection planning via goal programming, *Computers and Industial Engineering,* **22**, 4, 423-433.

[148] Mohamed RH. (1992) A chance constrained fuzzy goal program, *Fuzzy Sets and Systems,* **47**, 2, 183-186.

[149] Mohamed RH. (1997) The relationship between goal programming and fuzzy programming, *Fuzzy Sets and Systems,* **89**, 2, 215-222.

[150] Mohan S, Keskar JB. (1991) Optimization of multipurpose reservoir system operation, *Water Resources Bulletin,* **27**, 4, 621-629.

[151] Mohandas SU, Phelps TA, Ragsdell KM. (1990) Structural optimization using a fuzzy goal programming approach, *Computers and Structures,* **37**, 1, 1-8.

[152] Mohanty BK, Vijayaraghavan TAS. (1995) A multiobjective programming problem and its equivalent goal programming problem with appropriate priorities and aspiration levels - a fuzzy approach, *Computers and Operations Research*, **22**, 8, 771-778.

[153] Moy JW, Lam KF, Choo EU. (1997) Deriving the partial values in MCDM by goal programming, *Annals of Operational Research*, **74**, 277-288.

[154] Mukherjee K, Bera A. (1995) Application of goal programming in project selection decision - a case study from the Indian coal mining industry, *European Journal of Operational Research*, **82**, 1, 18-25.

[155] Muthukude P, Novak JL, Jolly C. (1990) A goal programming evaluation of the fisheries development plan for Sri Lankas costal fishing fleet 1998-1991, *American Journal of Agricultural Economics*, **72**, 5, 1367-1367.

[156] Muthukude P, Novak JL, Jolly C. (1991) A goal programming evaluation of fisheries development plans for Sri Lanka's costal fishing fleet, *Fisheries Research,* **12**, 4, 325-329.

[157] Myint S, Tabucanon MT. (1994) A multiple criteria approach to machine selection for flexible manufacturing systems, *International Journal of Production Economics,* **33**, 1-3, 121-131.

[158] Nagarur N, Vrat P, Duongsuwan W. (1997) Production planning and scheduling for injection moulding of pipe fittings - A case study, *International Journal of Production Economics,* **53**, 2, 157-170.

[159] Ng KYK. (1991) Goal programming solutions for heat and mass transfer problems - A feasibility study, *Computers and Chemical Engineering,* **15**, 8, 539-547.

[160] Niehaus RJ. (1995) Evolution of the strategy and structure of a human resource planning DSS application, *Decision Support Systems,* **14**, 3, 187-204.

[161] Njiti CF, Sharpe DM. (1994) A goal programming approach to the management of competition and conflict among land uses in the tropics - The Cameroon example, *Ambio*, **23**, 2, 112-119.

[162] Nudtassomboon N, Randhawa SU. (1997) Resource-constrained project scheduling with renewable and non-renewable resources and time-resource tradeoffs, *Computers and Industrial Engineering,* **32**, 1, 227-242.

[163] Ogryczak W. (1994) A goal programming model of the reference point method. *Annals of Operations Research*, **51**, 33-44.

[164] Ohta H, Yamaguchi T. (1996) Linear fractional goal programming in consideration of fuzzy solution, *European Journal of Operational Research,* **92**, 1, 157-165.

[165] OLeary DE, Schaffer J. (1996) The use of mathematical programming to verify rule-based knowledge, *Annals of Operations Research,* **65**, 181-193.

[166] Padmanabhan G, Vrat P. (1990) Analysis of multiitem inventory systems under resource constraints - a nonlinear goal programming approach, *Engineering Costs and Production Economics,* **20**, 2, 121-127.

[167] Pant JC, Shah S. (1992) Linear approach to linear fractional goal programming, *Journal of the Operational Research Society of India,* **29**, 4, 297-304.

[168] Pentzaropoulos GC, Giokas DI. (1993) Cost performance modelling and optimization of network flow balance via linear goal programming analysis, *Computer Communications,* **16**, 10, 645-652.

[169] Pentzaropoulos GC, Giokas DI. (1997) Near-optimal analysis of homogeneous central-server queuing networks, *Applied Stochastic Models and Data Analysis,* **13**, 1, 15-27.

[170] Peralta RC, Solaimanian J, Musharrafich GR. (1995) Optimal dispersed groundwater contaminant management - Modcon method, *Journal of Water Resources Planning and Management - American Society of Civil Engineers,* **121**, 6, 490-498.

[171] Pickens JB, Hof JG. (1991) Fuzzy goal programming in forestry - an application with special solution problems, *Fuzzy Sets and Systems,* **39**, 3, 239-246.

[172] Piech B, Rehman T. (1993) Application of multiple criteria decision making methods to farm planning - a case-study, *Agricultural Systems,* **41**, 3, 305-319.

[173] Powell JG, Premachandra IM. (1998) Accomodating diverse institutional investment objectives and constraints using non-linear goal programming, *European Journal of Operational Research,* **105**, 3, 447-456.

[174] Premachandra IM. (1993) A goal programming model for activity crashing in project networks, *International Journal of Operations and Production Management,* **13**, 6, 79-85.

[175] Ramanathan R. (1997) A note on the use of goal programming for the multiplicative AHP, *Journal of Multi-Criteria Decision Analysis,* **6**, 5, 296-307.

[176] Ramanathan R, Ganesh LS. (1993) A multiobjective programming approach to energy resource allocation problems, *International Journal of Energy Research,* **17**, 2, 105-119.

[177] Ramanathan R, Ganesh LS. (1994) A multiobjective evaluation approach of decentralized electricity generation options available to urban households, *Energy Conservation and Management,* **35**, 8, 661-670.

[178] Ramanathan R, Ganesh LS. (1995) A multiobjective evaluation of energy alternatives for water pumping in urban households, *Energy Sources,* **17**, 2, 223-239.

[179] Ramanathan R, Ganesh LS. (1995) Energy alternatives for lighting in households - an evaluation using an integrated goal programming - AHP model, *Energy,* **20**, 1, 63-72.

[180] Ramanathan R, Ganesh LS. (1996) Energy resource allocation incorporating qualitative and quantitative criteria: An integrated model using goal programming and AHP, *Socio-Economic Planning Sciences,* **29**, 3, 197-218.

[181] Ramik J. (2000) Fuzzy goals and fuzzy alternatives in goal programming problems, *Fuzzy Sets and Systems,* **111**, 1 , 81-86.

[182] Rao SS, Sundararaju K, Prakash BG, Balakrishna C. (1992) Fuzzy goal programming approach for structural optimization, *AIAA Journal,* **30**, 5, 1425-1432.

[183] Ravirala V, Grivas DA. (1995) Goal programming methodology for integrating pavement and bridge programs, *Journal of Transportation Engineering,* **121**, 4, 345-351.

[184] Reeves GR, Hedin SR. (1993) A generalized interactive goal programming procedure, *Computers and Operations Research,* **20**, 7, 747-753.

[185] Reid RA, Christensen DC, Harbur DR. (1994) An integrated modelling approach for identifying promising technologies to address waste management problems, *Radioactive Waste Management and Environmental Restoration,* **18**, 3, 197-208.

[186] Retzlaff-Roberts DL, Morey RC. (1993) A goal programming method of stochastic allocative data envelopment analysis, *European Journal of Operational Research,* **71**, 3, 379-397.

[187] Reznicek KK, Simonovic SP, Bector CR. (1991) Optimization of short term operation of a single multipurpose reservoir - a goal programming approach, *Canadian Journal of Civil Engineering,* **18**, 3, 397-406.

[188] Ridgley MA, Aerts JCJH. (1996) Using multicriterion optimization to classify ecosystems from multitemporal imagery, *Physical Geography,* **17**, 3, 282-293.

[189] Romero C. (1991) On misconceptions in goal programming, *Journal of the Operational Research Society,* **42**, 10, 927-928.

[190] Romero C. (1994) Carry on with redundancy in lexicographic goal programming, *European Journal of Operational Research,* **78**, 3, 441-442.

[191] Romero C, Tamiz M, Jones DF. (1998) Goal programming, compromise programming and reference point method formulations: Linkages and utility theorems, *Journal of the Operational Research Society,* **49**, 989-991.

[192] Roy A, Wallenius J. (1992) Nonlinear multiple objective optimization - an algorithm and some theory, *Mathematical Programming,* **55**, 2, 235-249.

[193] Roy S. (1993) A goal programming model approach to optimal price determination for inter-area energy trading, *International Journal of Energy Research,* **17**, 9, 847-862.

[194] Ryan MJ. (2000) The distribution problem, the more for less paradox and economies of scale and scope, *European Journal of Operational Research,* **121**, 1, 92-104.

[195] Saber HM, Ravindran A. (1992) A partitioning gradient based (PGB) algorithm for solving nonlinear goal programming problems, *Computers and Industrial Engineering,* **23**, 1-4, 291-294.

[196] Saber HM, Ravindran A. (1993) Nonlinear goal programming theory and practice - a survey, *Computers and Operations Research,* **20**, 3, 275-291.

[197] Sankaran S. (1990) Multiple objective decision making approach to cell formation - a goal programming model, *Mathematical and Computer Modelling,* **13**, 9, 71-82.

[198] Santhanam R, Kyparisis J. (1995) A multiple criteria decision model for information system project selection, *Computers and Operations Research,* **22**, 8, 807-818.

[199] Santhanam R, Schniederjans MJ. (1993) A model formulation system for information system project selection, *Computers and Operations Research,* **20**, 7, 755-767.

[200] Sarma GV, Sellami L, Houam KD. (1993) Application of lexicographic goal programming in production planning: Two case studies, *Journal of the Operational Research Society of India,* **30**, 2, 141-162.

[201] Sasaki M, Gen M, Ida K. (1990) Interactive sequential fuzzy goal programming, *Computers and Industrial Engineering,* **19**, 1-4, 567-571.

[202] Sastry VN, Tiwari RN, Sastri KS. (1992) Solution of optimal control problems with lumped parameters having single or multiple goal objectives in fuzzy environment, *Fuzzy Sets and Systems,* **48**, 2, 173-183.

[203] Schniederjans MJ. (1995) The life cycle of goal programming research as recorded in journal articles, *Operations Research,* **43**, 4, 551-557.

[204] Schniederjans MJ. (1995) Designing a quality control system in a service organization: A goal programming case study, *European Journal of Operational Research,* **81**, 2, 249-258.

[205] Schniederjans MJ, Gavin T. (1997) Using the analytic hierarchy process and multi-objective programming for selection of cost drivers in activity based costing, *European Journal of Operational Research,* **100**, 1, 72-80.

[206] Schniederjans MJ, Hoffman J. (1992) Multinational acquisition analysis: A zero-one goal programming model, *European Journal of Operational Research,* **62**, 1, 175-185.

[207] Schniederjans MJ, Hong SK. (1996) Multiobjective concurrent engineering: A goal programming approach, *IEEE Transactions on Engineering Management,* **43**, 2, 202-209.

[208] Schniederjans MJ, Wilson RL. (1991) Using the analytic hierarchy process and goal programming for information system project selection, *Information and Management,* **20**, 5, 333-342.

[209] Schniederjans, MJ, Zorn TS, Johnson RR. (1993) Allocating total wealth - a goal programming approach, *Computers and Operations Research,* **20**, 7, 679-685.

[210] Shafer SM, Rogers DF. (1991) A goal programming approach to the cell formation problem, *Journal of Operations Management,* **10**, 1, 28-43.

[211] Shiau TN, Chang JR. (1993) Multiobjective optimization of rotor bearing system with critical speed constraints, *Journal of Engineering for Gas Turbines and Power - Transactions of the ASME,* **115**, 2, 246-255.

[212] Shiau TN, Kuo CP, Hwang JR. (1997) Multi-objective optimization of a flexible rotor in magnetic bearings with critical speeds and control circuit conditions, *Journal of Engineering for Gas Turbines and Power - Transactions of the ASME,* **119**, 1, 186-195.

[213] Shim JP, Chun SG. (1991) Goal programming - the RPMS approach, *Journal of the Operational Research Society,* **42**, 1, 83-93.

[214] Shiraishi Y, Sakemi S, Fukuda K. (1993) A global routing algorithm based on the multicommodity network flow method, *IEICE Transactions on Fundamental of Electronics Communications and Computer Sciences,* **E76A**, 10, 1746-1754.

[215] Sinclair M, Van Oudheusden DL. (1997) 18-27 Bus trip scheduling in heavily congested cities, *European Journal of Operational Research,* **103**, 1, 18-27.

[216] Sueyoshi T. (1999) DEA discriminant analysis in the view of goal programming, *European Journal of Operational Research,* **115**, 3, 564-582.

[217] Sueyoshi T. (1991) Empirical regression quantile, *Journal of Operations Research Society of Japan,* **34**, 3, 250-262.

[218] Sueyoshi T. (1991) Estimation of stochastic frontier cost function using data envelopment analysis - an application to the AT and T divestiture, *Journal of the Operational Research Society,* **42**, 6, 463-477.

[219] Sueyoshi T. (1996) Divestiture of Nippon telegraph and telephone, *Management Science,* **42**, 9, 1326-1351.

[220] Sumpsi JM, Amador F, Romero C. (1997) On farmers' objectives: A multi-criteria approach, *European Journal of Operational Research,* **96**, 1, 64-71.

[221] Sun GQ, Ui T, Nakayasu H. (1995) Development of a practical goal programming system for solving semi-large size multicriterion planning problems, *International Journal of Production Economics,* **39**, 3, 227-242.

[222] Suresh NC. (1991) An extended multiobjective replacement model for flexible automation investments, *International Journal of Production Research,* **29**, 9, 1823-1844.

[223] Suresh NC, Kaparthi S. (1992) Flexible automation investments - a synthesis of 2 multiobjective modelling approaches, *Computers and Industrial Engineering,* **22**, 3, 257-272.

[224] Sushil. (1993) Application of physical system-theory and goal programming to modelling and analysis of waste management in national planning, *International Journal of Systems Science,* **24**, 5, 957-984.

[225] Sutardi, Bector CR, Goulter I, Cheng TCE. (1994) Multiobjective water resources investment under budgetary and sociotechnical un-

certainties, *IEEE Transactions on Engineering Management,* **41**, 1, 50-68.

[226] Suter WC, Calloway JA. (1994) Rough mill policies and practices examined by a multiple criteria goal program called ROMGOP, *Forest Products Journal,* **44**, 10, 19-28.

[227] Suzuki S. (1994) Multiobjective trajectory optimization by goal programming with fuzzy decisions, *Journal of Guidance Control and Dynamics,* **17**, 2, 297-303.

[228] Suzuki S. (1995) Multigoal programming for structure control design synthesis with fuzzy goals, *Japanese Materials Science and Engineering International Journal Series C - Dynamics Control Robotics Design and Manufacturing,* **38**, 3, 450-456.

[229] Suzuki S, Matsuda S. (1991) Structure control design synthesis of active flutter suppression system by goal programming, *Journal of Guidance Control and Dynamics,* **14**, 6, 1260-1266.

[230] Suzuki S, Yonezawa S. (1993) Simultaneous structure control design optimization of a wing structure with a gust load alleviation system, *Journal of Aircraft,* **30**, 2, 268-274.

[231] Tabucanon MT, Ong JL. (1993) Multiproduct multistage machine requirements planning models, *Applied Mathematical Modelling,* **17**, 3, 162-167.

[232] Taguchi T, Ida K, Gen M. (1997) Method for solving nonlinear goal programming with interval coefficients using genetic algorithm, *Computers and Industrial Engineering,* **33**, 3-4, 597-600.

[233] Tamiz M, Jones DF. (1995) Algorithmic improvements to the method of Martel and Aouni, *Journal of the Operational Research Society,* **46**, 2, 254-257.

[234] Tamiz M, Jones DF. (1996) Goal programming and Pareto efficiency, *Journal of Information and Optimization Sciences,* **17**, 2, 291-307.

[235] Tamiz M, Jones DF. (1997) Interactive frameworks for investigation of goal programming models: Theory and practice, *Journal of Multi-Criteria Decision Analysis,* **6**, 1, 52-60.

[236] Tamiz M, Jones DF. (1998) Goal programming: Recent developments in theory and practice, *International Journal of Management and Systems,* **14**, 1, 1-16.

[237] Tamiz M, Jones DF, ElDarzi E. (1995) A review of goal programming and its applications, *Annals of Operations Research,* **58**, 39-53.

[238] Tamiz M, Jones DF, Romero C. (1998) Goal programming for decision making: An overview of the current state-of-the-art, *European Journal of Operational Research,* **111**, 569-581.

[239] Tamiz M, Mardle SJ, Jones DF. (1996) Detecting IIS in infeasible linear programmes using techniques from goal programming, *Computers and Operations Research,* **23**, 2, 113-119.

[240] Tamiz M, Mirrazavi SK, Jones DF. (1999) Extensions of Pareto efficiency analysis to integer goal programming, *Omega,* **27**, 179-188.

[241] Thore S, Nagurney A, Pan J. (1992) Generalized goal programming and variational inequalities, *Operations Research Letters,* **12**, 4, 217-226.

[242] Vanberlo JM. (1993) A decision support tool for the vegetable processing industry - an integrative approach of market industry and agriculture, *Agricultural Systems,* **43**, 1, 91-109.

[243] Vanhulle MM. (1991) A goal programming network for linear programming, *Biological Cybernetics,* **65**, 4, 243-252.

[244] Vankooten GC. (1994) How to preserve wilderness areas without an endangered species act - examining trade-offs on Vancouver island with goal programming, *American Journal of Agricultural Economics,* **76**, 5, 1262-1262.

[245] Vitoriano B, Romero C. (1999) Extended interval goal programming, *Journal of the Operational Research Society,* **50**, 12 , 1280-1283.

[246] Wang HF, Fu CC. (1997) A generalization of fuzzy goal programming with preemptive structure, *Computers and Operations Research,* **24**, 9, 819-828.

[247] Wang YM, Lu WX. (1992) Multiobjective optimization approach to image reconstruction from projections, *Signal Processing,* **26**, 1, 17-26.

[248] Watanabe N, Hiraki S. (1996) An approximate solution to a JIT-based ordering system, *Computers and Industrial Engineering,* **31**, 3-4, 565-569.

[249] Wei JC, Olson DL, White EM. (1990) Simultaneous optimization in process quality control via prediction interval constrained programming, *Journal of the Operational Research Society,* **41**, 12, 1161-1167.

[250] Weigel HS, Wilcox SP. (1993) The army personnel decision support system, *Decision Support Systems,* **9**, 3, 281-306.

[251] Yang JB, Sen P. (1996) Preference modelling by estimating local utility functions for multiobjective optimization, *European Journal of Operational Research,* **95**, 1, 115-138.

[252] Yang YJ, Bhatt SK, Burn DH. (1993) Reservoir operating policies considering release change, *Civil Engineering Systems,* **10**, 1, 77-86.

[253] Yang YP, Chen YA. (1996) Multiobjective optimization of hard disk suspension assemblies. 2. Integrated structure and control design, *Computers and Structures,* **59**, 4, 771-782.

[254] Yang YP, Kuo CC. (1997) Passive and active design of hard disk suspension assemblies using multiobjective optimization techniques, *Computer Methods in Applied Mechanics and Engineering,* **145**, 1-2, 147-166.

[255] Yin YY. (1993) Integrated resource assessment and sustainable land use, *Environmental Management,* **17**, 3, 319-327.

[256] Yin YY. (1995) Designing a multisector model for land conversion management, *Journal of Environmental Management,* **44**, 3, 249-266.

[257] Yin YY, Cohen SJ. (1994) Identifying regional goals and policy concerns associated with global climate change, *Global Environmental Change - Human and Policy Dimensions,* **4**, 3, 245-260.

[258] Yu CS, Li HL. (2000) A robust optimization model for stochastic logistic problems, *International Journal of Production Economics,* **64**, 1-3, 385-397.

[259] Yura K. (1994) Production scheduling to satisfy workers preferences for days off and overtime under due date constraints, *International Journal of Production Economics,* **33**, 1-3, 265-270.

[260] Yura K, Ohashi K, Nakajima M, Yosimura M, Ota M, Hitomi K. (1994) Strategic planning for CIM to enhance the competitive ability, *Computers and Industrial Engineering,* **27**, 1-4, 127-130.

[261] Zekri S, Romero C. (1992) A methodology to assess the current situation in irrigated agriculture: An application to the village of Tauste (Spain). *Oxford Agrarian Studies,* **20**, 2, 75-88.

[262] Zheng DW, Gen M, Ida K. (1996) Evolution program for nonlinear goal programming, *Computers and Industrial Engineering,* **31**, 3-4, 907-911.

[263] Zhou ZY, Cheng SW, Hua, B. (2000) Supply chain optimization of continuous process industries with sustainability considerations *Computers and Chemical Engineering* , **24**, 2-7, 1151-1158

[264] Zhu ZW, Cernich WA, Meredith PH, Lanier PA. (1997) Scheduling customized bag production: An application of the group technology concept, *Industrial Management and Data Systems,* **97**, 1-2, 31.

[265] Zhu Z, McKnew MA. (1993) A goal programming workload balancing optimization model for ambulance allocation: An application to Shanghai, P.R. China, *Socio Economic Planning Sciences,* **27**, 2, 137-148.

Section B : General References

[266] Caballero R, Ruiz F, Steuer R. (eds.) (1997) *Advances in Multiobjective Programming and Goal Programming,* Lecture Notes in Economics and Mathematical Systems, **455**, Springer, Berlin.

[267] Charnes A, Cooper, WW. (1961) *Management Models and Industrial Applications of Linear Programming,* John Wiley and Sons, New York.

[268] Charnes A, Cooper WW, Ferguson R. (1955), Optimal estimation of executive compensation by linear programming, *Management Science,* **1**, 138-151.

[269] Charnes A, Cooper WW, Rhodes E. (1978) Measuring the efficiency of decision making units. *European Journal of Operations Research,* **2**, 429-444.

[270] Dorigo M, Di Caro G, Gambardella LM. (1999) Ant algorithms for discrete optimization, *Artificial Life,* **5**, 137-172.

[271] Flavell RB. (1976) A new goal programming formulation, *Omega,* **4**, 731-732.

[272] Gass SI. (1986) A process for determining priorities and weights for large-scale linear goal programmes, *Journal of the Operational Research Society,* **37**, 779-785.

[273] Ignizio, JP. (1982) *Linear Programming in Single and Multiple Objective Systems,* Prentice-Hall, Englewood Cliffs, New Jersey.

[274] Ignizio JP, Cavalier TM. (1994) *Linear Programming,* Prentice Hall, Englewood Cliffs, New Jersey.

[275] Ijiri Y. (1972) *Management Goals and Accounting for Control,* North Holland, Amsterdam.

[276] Jones DF. (1995) *The Design and Development of an Intelligent Goal Programming System,* PhD Thesis. University of Portsmouth, UK.

[277] Jones DF. (2001) Goal programming information site at: http://www.csm.port.ac.uk/mpg/GP.html

[278] Lee SM. (1972) *Goal programming for Decision Analysis,* Auerbach, Philadelphia.

[279] Menasce DA, Almeida VAF. (2000) *Scaling for e-Business: Technologies, Models, Performance, and Capacity Planning,* Prentice Hall, Englewood Cliffs, New Jersey.

[280] Rawls J. (1973) *A Theory of Justice,* Oxford University Press, Oxford.

[281] Romero C. (1991) *Handbook of Critical Issues in Goal Programming,* Pergamon Press, Oxford.

[282] Saaty TL. (1981) *The Analytical Hierarchy Process,* McGraw–Hill International, New York.

[283] Schniederjans MJ. (1995) *Goal Programming Methodology and Applications,* Kluwer Academic Publishers, Boston.

[284] Simon HA. (1955) *Models of Man,* J.Wiley & Sons, New York.

[285] Tamiz M. (1996) (ed.) *Multi-objective and Goal Programming: Theories and Applications,* Lecture Notes in Economics and Mathematical Systems, **432**, Springer, Berlin.

[286] Tamiz M, Jones DF. (1997) An example of good modelling practice in goal programming: Means overcoming incommensurability, in *Advances in Multi-objective Programming and Goal Programming,* Lecture Notes in Economics and Mathematical Systems, **455**, 29 - 37, Springer, Berlin.

[287] Zimmermann HJ. (1983) Using fuzzy sets in Operational Research, *European Journal of Operational Research,* **13**, 201-206.

Chapter 4

FUZZY MULTIOBJECTIVE AND MULTILEVEL OPTIMIZATION

Masatoshi Sakawa
Department of Artificial Complex Systems Engineering,
Graduate School of Engineering, Hiroshima University,
Kagamiyama 1-4-1, Higashi-Hiroshima 739-8527
Japan
sakawa@msl.sys.hiroshima-u.ac.jp

Abstract In this chapter, for handling and tackling the imprecise nature of human judgments, multiobjective optimization in a fuzzy environment is discussed. Starting with several basic definitions involving fuzzy sets, Bellman and Zadeh's approach to decision making in a fuzzy environment, called fuzzy decision, is outlined. Fundamental notions and methods of multiobjective, and interactive multiobjective programming are briefly reviewed. Then multiobjective linear programming and interactive multiobjective linear programming, both incorporating fuzzy goals of the decision maker (DM), are explained in detail by putting special emphasis on Pareto optimality. Multiobjective linear programming problems with fuzzy parameters, which reflect the experts' ambiguous or fuzzy understanding of the nature of the parameters in the problem-formulation process, are also formulated. By extending the usual Pareto optimality concepts, interactive decision-making methods, both without and with the fuzzy goals of the DM, for deriving a satisficing solution for the DM efficiently from an extended Pareto optimal solution set are presented. Finally, attention is focused on two-level linear programming problems and an interactive fuzzy programming method is introduced. In the interactive method, after determining the fuzzy goals of the DMs at both levels, a satisfactory solution is derived efficiently by updating the minimal satisfactory level of the upper level DM with considerations of overall satisfactory balance between both levels. Furthermore, the proposed method is extended to deal with two-level linear programming problems with fuzzy parameters.

Keywords: Fuzzy programming, Multiobjective programming, Multilevel programming, Interactive methods, Fuzzy goals, Fuzzy parameters.

1. Fuzzy Decision

1.1. Fuzzy Sets

In 1965, L.A. Zadeh [105] published his famous paper "Fuzzy sets" in Information and Control providing a new mathematical tool which enables us to describe and handle vague or ambiguous notions. In general, a fuzzy set initiated by Zadeh [105] is defined as follows:

Definition 5 (Fuzzy sets)
Let X denote a universal set. Then a fuzzy subset $\tilde{A}$ of X is defined by its membership function

$$\mu_{\tilde{A}} : X \to [0, 1] \tag{4.1}$$

which assigns to each element $x \in X$ a real number $\mu_{\tilde{A}}(x)$ in the interval [0,1], where the value, of $\mu_{\tilde{A}}(x)$ at x represents the grade of membership of x in $\tilde{A}$. Thus, the nearer the value of $\mu_{\tilde{A}}(x)$ is unity, the higher the grade of membership of x in $\tilde{A}$.

A fuzzy subset $\tilde{A}$ can be characterized as a set of ordered pairs of element x and grade $\mu_{\tilde{A}}(x)$ and is often written

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X\}. \tag{4.2}$$

When the membership function $\mu_{\tilde{A}}(x)$ contains only the two points 0 and 1, then $\mu_{\tilde{A}}(x)$ is identical to the characteristic function $c_A : X \to \{0, 1\}$, and hence, $\tilde{A}$ is no longer a fuzzy subset, but an ordinary set A.

As is well known, an ordinary set A is expressed as

$$A = \{x \in X \mid c_A(x) = 1\}, \tag{4.3}$$

through its characteristic function

$$c_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A. \end{cases} \tag{4.4}$$

Figure 4.1 illustrates the membership function $\mu_{\tilde{A}}(x)$ of a fuzzy subset $\tilde{A}$ together with the characteristic function $c_A(x)$ of an ordinary set A.

Observe that the membership function is an obvious extension of the idea of a characteristic function of an ordinary set because it takes on values between 0 and 1, not only 0 and 1.

As can be easily understood from the definition, a fuzzy subset is always defined as a subset of a universal set X. For the sake of convenience, a fuzzy subset is usually called a fuzzy set by omitting the term "sub." To distinguish an ordinary set from a fuzzy set, an ordinary set

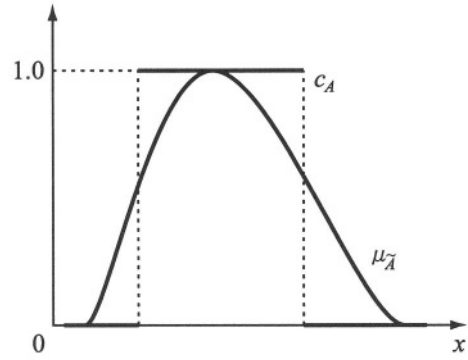


Figure 4.1. Membership function and characteristic function.

is called a nonfuzzy set or a crisp set. A fuzzy set is often denoted by $\tilde{A}, \tilde{B}, \tilde{C}, \ldots$, but it is sometimes written as *A, B, C,...*, for simplicity in the notation.

The concept of α-level sets serves as an important transfer between ordinary sets and fuzzy sets. It also plays an important role in the construction of a fuzzy set by a series of ordinary sets.

Definition 6 (α-level set)

The α-level set of a fuzzy set A is defined as an ordinary set A_α for which the degree of its membership function exceeds the level α:

$$A_\alpha = \{x \mid \mu_A(x) \geq \alpha\}, \quad \alpha \in [0, 1]. \tag{4.5}$$

Observe that the α-level set A_α can be defined by the characteristic function

$$c_{A_\alpha} = \begin{cases} 1, & \mu_A(x) \geq \alpha, \\ 0, & \mu_A(x) < \alpha, \end{cases} \tag{4.6}$$

since it is an ordinary set. Actually, an α-level set is an ordinary set whose elements belong to the corresponding fuzzy set to a certain degree α.

1.2. Fuzzy Numbers

Among fuzzy sets, numbers such as "approximately m" or "about n" can be defined as fuzzy sets of the real line R^1. Such fuzzy numbers are formally defined by Dubois and Prade [16, 17] as follows:

Definition 7 (Fuzzy numbers)

A fuzzy number $\tilde{m}$ is defined as any fuzzy set of the real line R^1, whose membership function $\mu_{\tilde{m}}(\cdot)$ is

(1) *A continuous mapping from* R^1 *to the closed interval* [0,1].

(2) $\mu_{\tilde{m}}(x) = 0$ *for all* $x \in (-\infty, a]$.

(3) *Strictly increasing and continuous on* $[a, m]$.

(4) $\mu_{\tilde{m}}(x) = 1$ *for* $x = m$.

(5) *Strictly decreasing and continuous on* $[m, b]$.

(6) $\mu_{\tilde{m}}(x) = 0$ *for all* $x \in [b, +\infty)$.

where a, b *and* m *are real numbers, and* $a < m < b$.

Figure 4.2 illustrates the graph of the possible shape of a fuzzy number $\tilde{m}$.

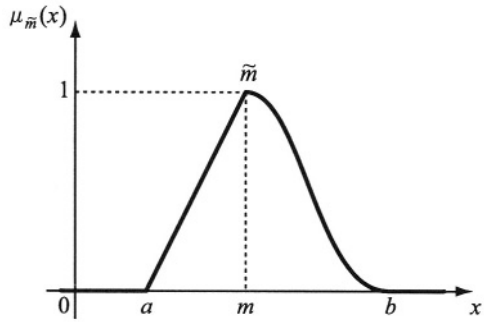


Figure 4.2. Fuzzy number.

Frequently, a fuzzy number m is called positive (negative), denoted by $\tilde{m} > 0$ ($\tilde{m} < 0$), if its membership function $\mu_{\tilde{m}}(x)$ satisfies $\mu_{\tilde{m}}(x) = 0$, $\forall x < 0$ ($\forall x > 0$).

From the definition of a fuzzy number $\tilde{m}$, it is significant to note that the α-level set $\tilde{m}_\alpha$ of a fuzzy number $\tilde{m}$ can be represented by the closed interval which depends on the value of α as is shown in Figure 4.3. Namely,

$$\tilde{m}_\alpha = \{x \in R^1 \mid \mu_{\tilde{m}}(x) \geq \alpha\} = [m_\alpha^L, m_\alpha^R] \tag{4.7}$$

where m_α^L or m_α^R represents the left or right extreme point of the α-level set $\tilde{m}_\alpha$, respectively.

1.3. Fuzzy Decision

In their 1970 paper "Decision making in a fuzzy environment," Bellman and Zadeh [2] introduced three basic concepts: fuzzy goal, fuzzy

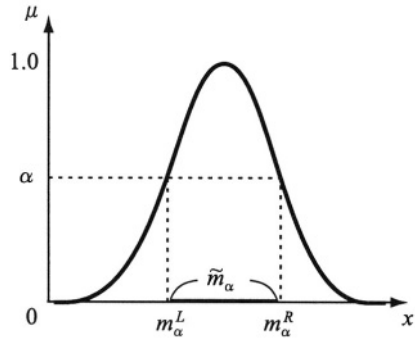


Figure 4.3. α-level set of fuzzy number $\tilde{m}$.

constraint, and fuzzy decision and explored the application of these concepts to decision making processes under fuzziness.

Let us now introduce the conceptual framework for decision making in a fuzzy environment.

Let X be a given set of possible alternatives which contains the solution of a decision making problem under consideration.

A fuzzy goal G is a fuzzy set on X characterized by its membership function

$$\mu_G : X \to [0, 1]. \tag{4.8}$$

A fuzzy constraint C is a fuzzy set on X characterized by its membership function

$$\mu_C : X \to [0, 1]. \tag{4.9}$$

Realizing that both the fuzzy goal and fuzzy constraint are desired to be satisfied simultaneously, Bellman and Zadeh [2] defined the fuzzy decision D resulting from the fuzzy goal G and fuzzy constraint C as the intersection of G and C.

To be more explicit, the fuzzy decision of Bellman and Zadeh is the fuzzy set D on X defined as

$$D = G \cap C \tag{4.10}$$

and is characterized by its membership function

$$\mu_D(\boldsymbol{x}) = \min(\mu_G(\boldsymbol{x}), \mu_C(\boldsymbol{x})). \tag{4.11}$$

The maximizing decision is then defined as

$$\underset{\boldsymbol{x} \in X}{\text{maximize}}\, \mu_D(\boldsymbol{x}) = \underset{\boldsymbol{x} \in X}{\text{maximize}}\ \min(\mu_G(\boldsymbol{x}), \mu_C(\boldsymbol{x})). \tag{4.12}$$

More generally, the fuzzy decision D resulting from k fuzzy goals $G_1, \ldots, G_k$ and m fuzzy constraints $C_1, \ldots, C_m$ is defined by

$$D = G_1 \cap \cdots \cap G_k \cap C_1 \cap \cdots \cap C_m \tag{4.13}$$

and the corresponding maximizing decision is defined as

$$\begin{aligned} &\underset{x \in X}{\text{maximize}}\, \mu_D(x) \\ &= \underset{x \in X}{\text{maximize}}\ \min\left(\mu_{G_1}(x), \ldots, \mu_{G_k}(x), \mu_{C_1}(x), \ldots, \mu_{C_m}(x)\right). \end{aligned} \tag{4.14}$$

It is significant to realize here that in the fuzzy decision defined by Bellman and Zadeh [2], the fuzzy goals and the fuzzy constraints enter into the expression for D in exactly the same way. In other words, in the definition of the fuzzy decision, there is no longer a difference between the fuzzy goals and the fuzzy constraints.

However, depending on the situations, other aggregation patterns for the fuzzy goal G and the fuzzy constraint C may be worth considering. When fuzzy goals and fuzzy constraints have unequal importance, Bellman and Zadeh [2] also suggested the convex fuzzy decision defined by

$$\mu_D^{co}(x) = \sum_{i=1}^{k} \alpha_i \mu_{G_i}(x) + \sum_{j=1}^{m} \beta_j \mu_{C_j}(x), \tag{4.15}$$

$$\sum_{i=1}^{k} \alpha_i + \sum_{j=1}^{m} \beta_j = 1, \quad \alpha_i, \beta_j \geq 0 \tag{4.16}$$

where the weighting coefficients reflect the relative importance among the fuzzy goals and constraints.

As an example of an alternative definition of a fuzzy decision, the product fuzzy decision defined by

$$\mu_D^{pr}(x) = \left(\prod_{i=1}^{k} \mu_{G_i}(x)\right) \cdot \left(\prod_{j=1}^{m} \mu_{C_j}(x)\right) \tag{4.17}$$

has been proposed.

For the convex fuzzy decision or the product fuzzy decision, similar to the maximizing decision for the fuzzy decision, the maximizing decision to select x^* such that

$$\mu_D^{co}(x^*) = \max_{x \in X} \left[\sum_{i=1}^{k} \alpha_i \mu_{G_i}(x)\right] \tag{4.18}$$

or

$$\mu_D^{pr}(\boldsymbol{x}^*) = \max_{\boldsymbol{x}\in X}\left[\left(\prod_{i=1}^{k}\mu_{G_i}(\boldsymbol{x})\right)\cdot\left(\prod_{j=1}^{m}\mu_{C_j}(\boldsymbol{x})\right)\right] \tag{4.19}$$

is also defined.

It should be noted here that among these three types of fuzzy decisions $\mu_D^{co}(\boldsymbol{x})$, $\mu_D^{pr}(\boldsymbol{x})$, and $\mu_D(\boldsymbol{x})$, the following relation holds:

$$\mu_D^{pr}(\boldsymbol{x}) \le \mu_D(\boldsymbol{x}) \le \mu_D^{co}(\boldsymbol{x}). \tag{4.20}$$

Example 9 Let $X = [0, \infty)$ be a set of alternatives. Suppose that we have a fuzzy goal G and a fuzzy constraint C expressed as "x should be much larger than 10" and "x should be substantially smaller than 30" where their membership functions are subjectively defined by

$$\mu_G(x) = \begin{cases} 0, & x \le 10 \\ 1 - (1 + (0.1(x-10))^2)^{-1}, & x > 10, \end{cases}$$

$$\mu_C(x) = \begin{cases} 0, & x \ge 30 \\ (1 + x(x-30)^{-2})^{-1}, & x < 30. \end{cases}$$

The fuzzy decision, the convex fuzzy decision, and the product fuzzy decision for this situation are depicted in Figure 4.4.

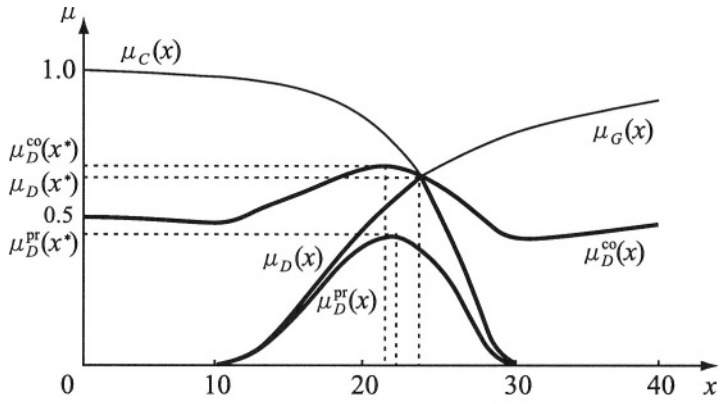


Figure 4.4. Fuzzy decision, convex fuzzy decision, and product fuzzy decision.

Further details of the theory and applications of fuzzy sets can be found in standard texts including Dubois and Prade [17], Kaufmann and Gupta [24], Klir and Yuan [24], Sakawa [47] and Zimmermann [110, 111].

2. Multiobjective Programming and Solution Concepts

The problem to optimize multiple conflicting objective functions simultaneously under given constraints is called the multiobjective programming problem and can be formulated as the following vector-minimization problem:

$$\left.\begin{array}{ll} \text{minimize} & \boldsymbol{f}(\boldsymbol{x}) \triangleq (f_1(\boldsymbol{x}), f_2(\boldsymbol{x}), \ldots, f_k(\boldsymbol{x}))^T \\ \text{subject to} & \boldsymbol{x} \in X \triangleq \{\boldsymbol{x} \in R^n \mid g_j(\boldsymbol{x}) \leq 0,\ j = 1, \ldots, m\} \end{array}\right\} \quad (4.21)$$

where $f_1(\boldsymbol{x}), \ldots, f_k(\boldsymbol{x})$ are k distinct objective functions of the decision vector $\boldsymbol{x}$, $g_1(\boldsymbol{x}), \ldots, g_m(\boldsymbol{x})$ are m inequality constraints and X is the feasible set of constrained decisions.

If we directly apply the notion of optimality for single-objective linear programming to this multiobjective programming, we arrive at the following notion of a complete optimal solution.

Definition 8 (Complete optimal solution)
$\boldsymbol{x}^* \in X$ is said to be a complete optimal solution if and only if there exists $\boldsymbol{x}^* \in X$ such that $f_i(\boldsymbol{x}^*) \leq f_i(\boldsymbol{x})$, $i = 1, \ldots, k$, for all $\boldsymbol{x} \in X$.

However, in general, such a complete optimal solution that simultaneously minimizes all of the multiple objective functions does not always exist when the objective functions conflict with each other. Thus, instead of a complete optimal solution, a new solution concept, called Pareto optimality, is introduced in multiobjective programming [8, 47, 96, 106].

Definition 9 (Pareto optimal solution)
$\boldsymbol{x}^* \in X$ is said to be a Pareto optimal solution if and only if there does not exist another $\boldsymbol{x} \in X$ such that $f_i(\boldsymbol{x}) \leq f_i(\boldsymbol{x}^*)$ for all $i = 1, \ldots, k$, and $f_j(\boldsymbol{x}) < f_j(\boldsymbol{x}^*)$ for at least one $j \in \{1, \ldots, k\}$.

As can be seen from the definition, a Pareto optimal solution consists of an infinite number of points. A Pareto optimal solution is sometimes called a noninferior solution since it is not inferior to other feasible solutions.

In addition to Pareto optimality, the following weak Pareto optimality is defined as a slightly weaker solution concept than Pareto optimality.

Definition 10 (Weak Pareto optimal solution)
$\boldsymbol{x}^* \in X$ is said to be a weak Pareto optimal solution if and only if there does not exist another $\boldsymbol{x} \in X$ such that $f_i(\boldsymbol{x}) < f_i(\boldsymbol{x}^*)$, $1, \ldots, k$.

For notational convenience, let X^{CO}, X^P, or X^{WP} denote complete optimal, Pareto optimal, or weak Pareto optimal solution sets, respectively. Then from their definitions, it can be easily understood that the

following relation holds:

$$X^{CO} \subseteq X^{P} \subseteq X^{WP}.$$

Several computational methods have been proposed for characterizing Pareto optimal solutions depending on the different methods to scalarize the multiobjective programming problems. Among the many possible ways of scalarizing the multiobjective programming problems, the weighting method, the constraint method, and the weighted minimax method have been studied as a means of characterizing Pareto optimal solutions of the multiobjective programming problems.

The details of multiobjective programming can be found in standard texts including Zeleny [106], Steuer [96], Chankong and Haimes [8], and Sakawa [47].

3. Interactive Multiobjective Programming

The STEP method (STEM) proposed by Benayoun et al. [3] seems to be known as one of the first interactive multiobjective linear programming techniques, but there have been some modifications and extensions (see, for example, Fichefet [20]; Choo and Atkins [10]). Essentially, the STEM algorithm consists of two major steps. Step 1 seeks a Pareto optimal solution that is near to the ideal point in the minimax sense. Step 2 requires the decision maker (DM) to compare the objective vector with the ideal vector and to indicate which objectives can be sacrificed, and by how much, in order to improve the current levels of unsatisfactory objectives. The STEM algorithm is quite simple to understand and implement, in the sense that the DM is required to give only the amounts to be sacrificed of some satisfactory objectives until all objectives become satisfactory. However, the DM will never arrive at the final solution if the DM is not willing to sacrifice any of the objectives. Moreover, in many practical situations, the DM will probably want to indicate directly the aspiration level for each objective rather than just specify the amount by which satisfactory objectives can be sacrificed.

Wierzbicki [103] developed a relatively practical interactive method called the reference point method (RPM) by introducing the concept of a reference point suggested by the DM which reflects in some sense the desired values of the objective functions. The basic idea behind the RPM is that the DM can specify reference values for the objective functions and change the reference objective levels interactively due to learning or improved understanding during the solution process. In this procedure, when the DM specifies a reference point, the corresponding scalarization problem is solved for generating the Pareto optimal solution which is, in a sense, close to the reference point or better than that

if the reference point is attainable. Then the DM either chooses the current Pareto optimal solution or modifies the reference point to find a satisficing solution.

Since then some similar interactive multiobjective programming methods have been developed along this line (see, for example, Steuer and Choo [97]). However, it is important to point out here that for dealing with the fuzzy goals of the DM for each of the objective functions of the multiobjective linear programming problem, Sakawa, Yano and Yumine [85] developed the extended fuzzy version of the RPM that supplies the DM with the trade-off information even if the fuzzy goals of the DM are not considered. Although the method will be outlined in the next section, it would certainly be appropriate to discuss here the RPM with trade-off information rather than the RPM proposed by Wierzbicki.

Consider the following multiobjective linear programming problem:

$$\left.\begin{array}{ll} \text{minimize} & z_1(\boldsymbol{x}) = \boldsymbol{c}_1\boldsymbol{x} \\ \text{minimize} & z_2(\boldsymbol{x}) = \boldsymbol{c}_2\boldsymbol{x} \\ & \vdots \\ \text{minimize} & z_k(\boldsymbol{x}) = \boldsymbol{c}_k\boldsymbol{x} \\ \text{subject to} & \boldsymbol{x} \in X \triangleq \{\boldsymbol{x} \in R^n \mid A\boldsymbol{x} \leq \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}\}, \end{array}\right\} \tag{4.22}$$

where

$$\begin{aligned} \boldsymbol{c}_i &= (c_{i1}, \ldots, c_{in}),\ i = 1, \ldots, k, \\ \boldsymbol{x} &= (\boldsymbol{x}_1^T, \ldots, \boldsymbol{x}_n^T)^T, \\ A &= \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}, \\ \boldsymbol{b} &= (b_1, \ldots, b_m)^T. \end{aligned} \tag{4.23}$$

For each of the conflicting objective functions $\boldsymbol{z}(\boldsymbol{x}) = (z_1(\boldsymbol{x}), \ldots, z_k(\boldsymbol{x}))^T$, assume that the DM can specify the so-called reference point $\bar{\boldsymbol{z}} = (\bar{z}_1, \ldots, \bar{z}_k)^T$ which reflects in some sense the desired values of the objective functions of the DM. Also assume that the DM can change the reference point interactively due to learning or improved understanding during the solution process. When the DM specifies the reference point $\bar{\boldsymbol{z}} = (\bar{z}_1, \ldots, \bar{z}_k)^T$, the corresponding Pareto optimal solution, which is, in the minimax sense, nearest to the reference point or better than that if the reference point is attainable, is obtained by solving the following

minimax problem:

$$\left.\begin{array}{ll} \text{minimize} & \max\limits_{i=1,\ldots,k}\{z_i(\boldsymbol{x}) - \bar{z}_i\} \\ \text{subject to} & \boldsymbol{x} \in X, \end{array}\right\} \tag{4.24}$$

or equivalently

$$\left.\begin{array}{ll} \text{minimize} & v \\ \text{subject to} & z_i(\boldsymbol{x}) - \bar{z}_i \leq v, \quad i = 1, \ldots, k \\ & \boldsymbol{x} \in X. \end{array}\right\} \tag{4.25}$$

The case of the two-objective functions in the z_1-z_2 plane is shown geometrically in Figure 4.5. For the two reference points $\bar{\boldsymbol{z}}^1 = (\bar{z}_1^1, \bar{z}_2^1)^T$ and $\bar{\boldsymbol{z}}^2 = (\bar{z}_1^2, \bar{z}_2^2)^T$ specified by the DM, solving the corresponding minimax problems yields the corresponding Pareto optimal solutions $\boldsymbol{z}^1(\boldsymbol{x}^1)$ and $\boldsymbol{z}^2(\boldsymbol{x}^2)$.

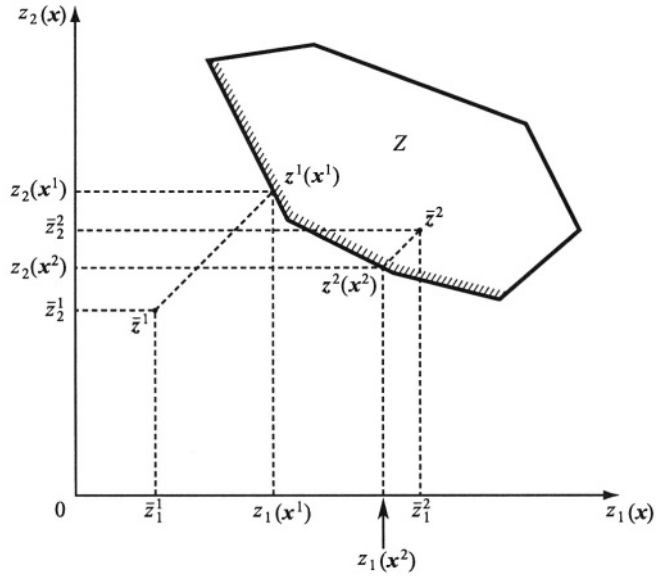


Figure 4.5. Graphical interpretation of minimax method.

The relationships between the optimal solutions of the minimax problem and the Pareto optimal concept of the multiobjective linear programming can be characterized by the following two theorems.

Theorem 4.1 (Minimax problem and Pareto optimality)

If $\boldsymbol{x}^* \in X$ is a unique optimal solution of the minimax problem for any reference point $\bar{\boldsymbol{z}}$, then $\boldsymbol{x}^*$ is a Pareto optimal solution of the multiobjective linear programming problem.

It should be noted that only weak Pareto optimality is guaranteed if the uniqueness of a solution is not guaranteed.

Theorem 4.2 (Pareto optimality and minimax problem)
If $\boldsymbol{x}^*$ is a Pareto optimal solution of the multiobjective linear programming problem, then $\boldsymbol{x}^*$ is an optimal solution of the minimax problem for some reference point $\bar{\boldsymbol{z}}$.

Now, given the Pareto optimal solution for the reference point specified by the DM by solving the corresponding minimax problem, the DM must either be satisfied with the current Pareto optimal solution or modify the reference point. To help the DM express a degree of preference, trade-off information between a standing objective function $z_1(\boldsymbol{x})$ and each of the other objective functions is very useful. Such a trade-off between $z_1(\boldsymbol{x})$ and $z_i(\boldsymbol{x})$ for each $i = 2, \ldots, k$ is easily obtainable since it is closely related to the strict positive simplex multipliers of the minimax problem. Let the simplex multipliers associated with the constraints of the minimax problem be denoted by π_i, $i = 1, \ldots, k$. If all $\pi_i > 0$ for each i, it can be proved that the following expression holds:

$$-\frac{\partial z_1(\boldsymbol{x})}{\partial z_i(\boldsymbol{x})} = \frac{\pi_1}{\pi_i}, \quad i = 2, \ldots, k. \tag{4.26}$$

We can now construct the interactive algorithm to derive the satisficing solution for the DM from the Pareto optimal solution set. The steps marked with an asterisk involve interaction with the DM. Observe that this interactive multiobjective linear programming method can be interpreted as the reference point method (RPM) with trade-off information.

Interactive multiobjective linear programming

Step 0: Calculate the individual minimum $z_i^{\min} = \min_{\boldsymbol{x} \in X} z_i(\boldsymbol{x})$ and maximum $z_i^{\max} = \max_{\boldsymbol{x} \in X} z_i(\boldsymbol{x})$ of each objective function under the given constraints.

Step 1*: Ask the DM to select the initial reference point by considering the individual minimum and maximum. If the DM finds it difficult or impossible to identify such a point, ideal point $z_i^{\min} = \min_{\boldsymbol{x} \in X} z_i(\boldsymbol{x})$ can be used for that purpose.

Step 2: For the reference point specified by the DM, solve the corresponding minimax problem to obtain the Pareto optimal solution together with the trade-off rate information between the objective functions.

Step 3*: If the DM is satisfied with the current levels of the Pareto optimal solution, stop. Then the current Pareto optimal solution is the satisficing solution for the DM. Otherwise, ask the DM to update the current reference point by considering the current values of the objective functions together with the trade-off rates between the objective functions and return to Step 2.

It should be stressed to the DM that any improvement of one objective function can be achieved only at the expense of at least one of the other objective functions.

Further details of the theory, methods and applications of interactive multiobjective programming can be found in Steuer [96], Chankong and Haimes [8], and Sakawa [47].

4. Fuzzy Multiobjective Linear Programming

In 1978, H.-J. Zimmermann [108] extended his fuzzy linear programming approach [107] to the following multiobjective linear programming problem with k linear objective functions $z_i(\boldsymbol{x}) = \boldsymbol{c}_i\boldsymbol{x}$, $i = 1, \ldots, k$:

$$\left.\begin{array}{ll} \text{minimize} & \boldsymbol{z}(\boldsymbol{x}) \triangleq (z_1(\boldsymbol{x}), z_2(\boldsymbol{x}), \ldots, z_k(\boldsymbol{x}))^T \\ \text{subject to} & A\boldsymbol{x} \leq \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0} \end{array}\right\} \tag{4.27}$$

where $\boldsymbol{c}_i = (c_{i1}, \ldots, c_{in})$, $i = 1, \ldots, k$, $\boldsymbol{x} = (x_1, \ldots, x_n)^T$, $\boldsymbol{b} = (b_1, \ldots, b_m)^T$ and $A = [a_{ij}]$ is an $m \times n$ matrix.

For each of the objective functions $z_i(\boldsymbol{x}) = \boldsymbol{c}_i\boldsymbol{x}$, $i = 1, \ldots, k$, of this problem, assume that the decision maker (DM) has a fuzzy goal such as " the objective function $z_i(\boldsymbol{x})$ should be substantially less than or equal to some value p_i." Then the corresponding linear membership function $\mu_i^L(z_i(\boldsymbol{x}))$ is defined as

$$\mu_i^L(z_i(\boldsymbol{x})) = \begin{cases} 0 & ;\ z_i(\boldsymbol{x}) \geq z_i^0 \\ \dfrac{z_i(\boldsymbol{x}) - z_i^0}{z_i^1 - z_i^0} & ;\ z_i^0 \geq z_i(\boldsymbol{x}) \geq z_i^1 \\ 1 & ;\ z_i(\boldsymbol{x}) \leq z_i^1 \end{cases} \tag{4.28}$$

where z_i^0 or z_i^1 denotes the value of the objective function $z_i(\boldsymbol{x})$ such that the degree of membership function is 0 or 1 respectively.

Figure 4.6 illustrates the graph of the possible shape of the linear membership function.

Using such linear membership functions $\mu_i^L(z_i(\boldsymbol{x}))$, $i = 1, \ldots, k$, and following the fuzzy decision of Bellman and Zadeh [2], the original mul-

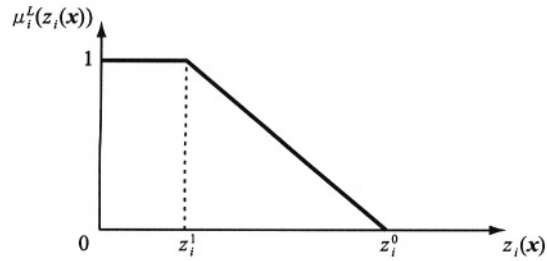


Figure 4.6. Linear membership function.

tiobjective linear programming problem can be interpreted as

$$\left.\begin{array}{ll} \text{maximize} & \min\limits_{i=1,\ldots,k} \{\mu_i^L(z_i(\boldsymbol{x}))\} \\ \text{subject to} & A\boldsymbol{x} \leq \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}. \end{array}\right\} \tag{4.29}$$

By introducing the auxiliary variable λ, it can be reduced to the following conventional linear programming problem:

$$\left.\begin{array}{ll} \text{maximize} & \lambda \\ \text{subject to} & \lambda \leq \mu_i^L(z_i(\boldsymbol{x})),\ i = 1, 2, \ldots, k \\ & A\boldsymbol{x} \leq \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}. \end{array}\right\} \tag{4.30}$$

By assuming the existence of the optimal solution $\boldsymbol{x}^{io}$ of the individual objective function minimization problem under the constraints defined by

$$\min_{\boldsymbol{x} \in X} z_i(\boldsymbol{x}),\ i = 1, 2, \ldots, k, \tag{4.31}$$

Zimmermann [108] suggested a way to determine the linear membership function $\mu_i^L(z_i(\boldsymbol{x}))$. To be more specific, using the individual minimum

$$z_i^{\min} = z_i(\boldsymbol{x}^{io}) = \min_{\boldsymbol{x} \in X} z_i(\boldsymbol{x}),\ i = 1, 2, \ldots, k, \tag{4.32}$$

together with

$$z_i^{\mathrm{m}} = \max(z_i(\boldsymbol{x}^{1o}), \ldots, z_i(\boldsymbol{x}^{i-1,o}), z_i(\boldsymbol{x}^{i+1,o}), \ldots, z_i(\boldsymbol{x}^{ko})), i = 1, 2, .., k, \tag{4.33}$$

he determined the linear membership function as in (4.28) by choosing $z_i^1 = z_i^{\min}$ and $z_i^0 = z_i^{\mathrm{m}}$. For this membership function, it can be easily shown that if the optimal solution of (4.29) or (4.30) is unique, it is also a Pareto optimal solution of the multiobjective linear programming problem.

In the case where not only fuzzy goals but also fuzzy constraints exist, using linear membership functions for fuzzy constraints, similar discussion can be made. Zimmermann [110] called the fuzzy decision the minimum operator, and for other aggregation patterns than the minimum operator, he considered the product fuzzy decision. He called the product fuzzy decision the product operator, and proposed using the product operator. In this case, the problem to be solved becomes

$$\left.\begin{array}{ll} \text{maximize} & \prod_{i=1}^{k} \mu_i^L(z_i(\boldsymbol{x})) \\ \text{subject to} & A\boldsymbol{x} \leq \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}. \end{array}\right\} \tag{4.34}$$

Unfortunately, with the product operator, even if we use the linear membership functions, the objective function of this problem becomes a nonlinear function, and hence, the linear programming method [14] cannot be applied.

In 1981, by considering the rate of increased membership satisfaction need not always be constant as in the case of the linear membership function proposed by Zimmermann, Leberling [29] introduced special nonlinear functions and showed that the resulting nonlinear programming problem can be equivalently converted to a conventional linear programming problem.

As another extension of the linear membership function of Zimmermann, in 1981, Hannan [22] proposed a different approach from Leberling. For each of the objective functions of the multiobjective linear programming problem, assuming that the DM could specify the degree of membership for several values of $z_i(\boldsymbol{x})$, he introduced the piecewise linear membership function. By adopting the piecewise linear membership function to represent the fuzzy goal of the DM for the multiobjective linear programming problem together with the fuzzy decision of Bellman and Zadeh [2], the problem to be solved can be converted to the ordinary linear programming problem.

However, suppose that the interaction with the DM establishes that the first membership function should be linear, the second hyperbolic, the third piecewise linear, and so forth. In such a situation, following the fuzzy decision of Bellman and Zadeh [2], the resulting problem becomes a nonlinear programming problem and cannot be solved by a linear programming method [14].

In 1983, to quantify the fuzzy goals of the DM by eliciting the corresponding membership functions, Sakawa [45] proposed using five types of membership functions: linear, exponential, hyperbolic, hyperbolic inverse, and piecewise linear functions. Through the use of these membership functions including nonlinear ones, the fuzzy goals of the DM are

quantified. Then following the fuzzy decision of Bellmann and Zadeh [2], the problem becomes a nonlinear programming problem. However, it can be reduced to a set of linear inequalities if some variable is fixed. Based on this idea, Sakawa [45] proposed a new method combining the use of the bisection method and the linear programming method [14].

5. Interactive Fuzzy Multiobjective Linear Programming

In the fuzzy approaches to multiobjective linear programming problems proposed by Zimmermann [108] and his successors [29, 22, 109], it has been implicitly assumed that the fuzzy decision of Bellman and Zadeh [2] is the proper representation of the fuzzy preferences of the decision maker (DM). Therefore, these approaches are preferable only when the DM feels that the fuzzy decision is appropriate when combining the fuzzy goals and/or constraints. However, such situations seem to occur rarely in practice and consequently it becomes evident that an interaction with the DM is necessary.

In this section, assuming that the DM has a fuzzy goal for each of the objective functions in multiobjective linear programming problems, we present an interactive fuzzy multiobjective linear programming method incorporating the desirable features of the interactive approaches into the fuzzy approaches.

Fundamental to the multiobjective linear programming is the concept of Pareto optimal solutions, also known as a noninferior solution.

However, considering the imprecise nature inherent in human judgments in multiobjective linear programming problems, the DM may have a fuzzy goal expressed as "$z_i(\boldsymbol{x})$ should be substantially less than or equal to some value p_i."

In a minimization problem, a fuzzy goal stated by the DM may be to achieve "substantially less than or equal to p_i." This type of statement can be quantified by eliciting a corresponding membership function.

To elicit a membership function $\mu_i(z_i(\boldsymbol{x}))$ from the DM for each of the objective functions $z_i(\boldsymbol{x})$, $i = 1, \ldots, k$, we first calculate the individual minimum $z_i^{\min} = \min_{\boldsymbol{x} \in X} z_i(\boldsymbol{x})$ and maximum $z_i^{\max} = \max_{\boldsymbol{x} \in X} z_i(\boldsymbol{x})$ of each objective function $z_i(\boldsymbol{x})$ under the given constraints.

Taking into account the calculated individual minimum and maximum of each objective function together with the rate of increase of membership of satisfaction, the DM must determine the subjective membership function $\mu_i(z_i(\boldsymbol{x}))$, which is a strictly monotone decreasing function with respect to $z_i(\boldsymbol{x})$. Here, it is assumed that $\mu_i(z_i(\boldsymbol{x})) = 0$ or $\to 0$ if $z_i(\boldsymbol{x}) \geq z_i^0$ and $\mu_i(z_i(\boldsymbol{x})) = 1$ or $\to 1$ if $z_i(\boldsymbol{x}) \leq z_i^1$.

So far, we have restricted ourselves to a minimization problem and consequently assumed that the DM has a fuzzy goal such as "$z_i(\boldsymbol{x})$ should be substantially less than or equal to p_i." In the fuzzy approaches, however, we can further treat a more general multiobjective linear programming problem in which the DM has two types of fuzzy goals expressed in words such as "$z_i(\boldsymbol{x})$ should be in the vicinity of r_i" (called fuzzy equal), "$z_i(\boldsymbol{x})$ should be substantially less than or equal to p_i" (called fuzzy min) or "$z_i(\boldsymbol{x})$ should be substantially greater than or equal to q_i" (called fuzzy max).

Such a generalized multiobjective linear programming problem may now be expressed as

$$\left.\begin{array}{rll} \text{fuzzy min} & z_i(\boldsymbol{x}) & i \in I_1 \\ \text{fuzzy max} & z_i(\boldsymbol{x}) & i \in I_2 \\ \text{fuzzy equal} & z_i(\boldsymbol{x}) & i \in I_3 \\ \text{subject to} & \boldsymbol{x} \in X & \end{array}\right\} \tag{4.35}$$

where $I_1 \cup I_2 \cup I_3 = \{1, \ldots, k\}$, $I_i \cap I_j = \emptyset$, $i, j = 1, 2, 3$, $i \neq j$.

Here "fuzzy min $z_i(\boldsymbol{x})$" or "fuzzy max $z_i(\boldsymbol{x})$" represents the fuzzy goal of the DM such as "$z_i(\boldsymbol{x})$ should be substantially less than or equal to p_i or greater than or equal to q_i," and "fuzzy equal $z_i(\boldsymbol{x})$" represents the fuzzy goal such as "$z_i(\boldsymbol{x})$ should be in the vicinity of r_i."

Concerning the membership function for the fuzzy goal of the DM such as "$z_i(\boldsymbol{x})$ should be in the vicinity of r_i," it is obvious that a strictly monotone increasing function $d_{iL}(z_i)$, $(i \in I_3)$ and a strictly monotone decreasing function $d_{iR}(z_i)$, $(i \in I_3)$, corresponding to the left and right sides of r_i must be determined through interaction with the DM.

Figures 4.7, 4.8 and 4.9 illustrate possible shapes of the fuzzy min, fuzzy max and fuzzy equal membership functions, respectively.

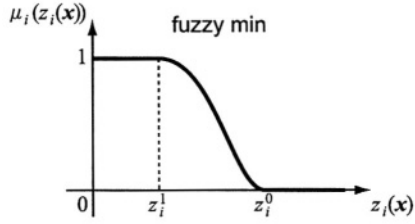


Figure 4.7. Fuzzy min membership function.

Having elicited the membership functions $\mu_i(z_i(\boldsymbol{x}))$, $i = 1, \ldots, k$, from the DM for each of the objective functions $z_i(\boldsymbol{x})$, $i = 1, \ldots, k$, the multiobjective linear programming problem and/or the generalized multiobjective linear programming problem can be converted into the fuzzy

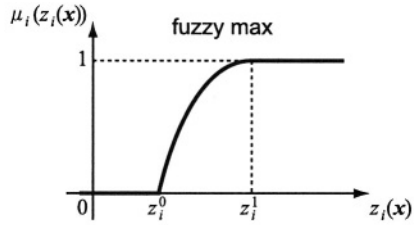


Figure 4.8. Fuzzy max membership function.

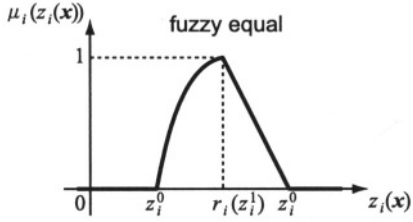


Figure 4.9. Fuzzy equal membership function.

multiobjective optimization problem defined by

$$\underset{x \in X}{\text{maximize}} \ (\mu_1(z_1(x)), \mu_2(z_2(x)), \ldots, \mu_k(z_k(x))). \tag{4.36}$$

When the fuzzy equal is included in the fuzzy goals of the DM, it is desirable that $z_i(x)$ should be as close to r_i as possible. Consequently, the notion of Pareto optimal solutions defined in terms of objective functions cannot be applied. For this reason, we introduce the concept of M-Pareto optimal solutions which is defined in terms of membership functions instead of objective functions. M refers to membership.

Definition 11 (M-Pareto optimal solution)

$x^* \in X$ is said to be an M-Pareto optimal solution to the generalized multiobjective linear programming problem if and only if there does not exist another $x \in X$ such that $\mu_i(z_i(x)) \geq \mu_i(z_i(x^*))$ for all $i = 1, \ldots, k$, and $\mu_j(z_j(x)) > \mu_j(z_j(x^*))$ for at least one $j \in \{1, \ldots, k\}$.

By introducing a general aggregation function

$$\mu_D(\boldsymbol{\mu}(\boldsymbol{z}(x))) = \mu_D(\mu_1(z_1(x)), \mu_2(z_2(x)), \ldots, \mu_k(z_k(x))), \tag{4.37}$$

a general fuzzy multiobjective decision making problem can be defined by

$$\underset{x \in X}{\text{maximize}} \ \mu_D(\boldsymbol{\mu}(\boldsymbol{z}(x))). \tag{4.38}$$

Observe that the value of $\mu_D(\boldsymbol{\mu}(\boldsymbol{z}(\boldsymbol{x})))$ can be interpreted as representing an overall degree of satisfaction with the DM's multiple fuzzy goals.

Probably the most crucial problem in the fuzzy multiobjective decision making problem is the identification of an appropriate aggregation function which well represents the DM's fuzzy preferences. If $\mu_D(\cdot)$ can be explicitly identified, then the fuzzy multiobjective decision making problem reduces to a standard mathematical programming problem. However, this rarely happens, and as an alternative, an interaction with the DM is necessary for finding the satisficing solution of the fuzzy multiobjective decision making problem.

In the interactive fuzzy multiobjective linear programming method proposed by Sakawa, Yano and Yumine [85], after determining the membership functions $\boldsymbol{\mu}(\boldsymbol{z}(\boldsymbol{x})) = (\mu_1(z_1(\boldsymbol{x})), \ldots, \mu_k(z_k(\boldsymbol{x})))^T$ for each of the objective functions $\boldsymbol{z}(\boldsymbol{x}) = (z_1(\boldsymbol{x}), \ldots, z_k(\boldsymbol{x}))^T$, for generating a candidate for the satisficing solution which is also M-Pareto optimal, the DM is then asked to specify the aspiration levels of achievement for the membership values of all membership functions, called the reference membership levels. The reference membership levels can be viewed as natural extensions of the reference point of Wierzbicki [103] in objective function spaces.

For the DM's reference membership levels $\bar{\boldsymbol{\mu}} = (\bar{\mu}_1, \ldots, \bar{\mu}_k)^T$, the corresponding M-Pareto optimal solution, which is nearest to the requirements in the minimax sense or better than that if the reference membership levels are attainable, is obtained by solving the following minimax problem

$$\underset{\boldsymbol{x} \in X}{\text{minimize}} \max_{i=1,\ldots,k} \{\bar{\mu}_i - \mu_i(z_i(\boldsymbol{x}))\}, \tag{4.39}$$

or equivalently

$$\left.\begin{array}{ll} \text{minimize} & v \\ \text{subject to} & \bar{\mu}_i - \mu_i(z_i(\boldsymbol{x})) \leq v, \ i = 1, \ldots, k \\ & \boldsymbol{x} \in X. \end{array}\right\} \tag{4.40}$$

The relationships between the optimal solutions of the minimax problem and the M-Pareto optimal concept of the multiobjective linear programming problem can be characterized by the following theorems.

Theorem 4.3 (Minimax problem and M-Pareto optimality)

If $\boldsymbol{x}^* \in X$ is a unique optimal solution to the minimax problem for some $\bar{\mu}_i$, $i = 1, \ldots, k$, then $\boldsymbol{x}^*$ is an M-Pareto optimal solution to the generalized multiobjective linear programming problem.

Theorem 4.4 (M-Pareto optimality and minimax problem)
If $\boldsymbol{x}^*$ is an M-Pareto optimal solution to the generalized multiobjective linear programming problem with $0 < \mu_i(z_i(\boldsymbol{x}^*)) < 1$ holding for all i, then there exists $\bar{\mu}_i$, $i = 1, \ldots, k$, such that $\boldsymbol{x}^*$ is an optimal solution to the minimax problem.

If all of the membership functions $\mu_i(z_i(\boldsymbol{x}))$, $i = 1, \ldots, k$, are linear, the minimax problem becomes a linear programming problem, and hence, we can obtain an optimal solution by directly applying the simplex method of linear programming [14].

However, with the strictly monotone decreasing or increasing membership functions, which may be nonlinear, the resulting minimax problem becomes a nonlinear programming problem. For notational convenience, denote the strictly monotone decreasing function for the fuzzy min and the right function of the fuzzy equal by $d_{iR}(z_i)$ $(i \in I_1 \cup I_3)$ and the strictly monotone increasing function for the fuzzy max and the left function of the fuzzy equal by $d_{iL}(z_i)$ $(i \in I_2 \cup I_3)$. Then in order to solve the formulated problem on the basis of the linear programming method, convert each constraint $\bar{\mu}_i - \mu_i(z_i(\boldsymbol{x})) \leq v$, $i = 1, \ldots, k$, of the minimax problem (4.40) into the following form using the strictly monotone property of $d_{iL}(\cdot)$ and $d_{iR}(\cdot)$:

$$\left.\begin{array}{ll} \text{minimize} & v \\ \text{subject to} & z_i(\boldsymbol{x}) \leq d_{iR}^{-1}(\bar{\mu}_i - v), \ i \in I_1 \cup I_3 \\ & z_i(\boldsymbol{x}) \geq d_{iL}^{-1}(\bar{\mu}_i - v), \ i \in I_2 \cup I_3 \\ & \boldsymbol{x} \in X. \end{array}\right\} \tag{4.41}$$

It is important to note here that, if the value of v is fixed, it can be reduced to a set of linear inequalities. Obtaining the optimal solution v^* to the above problem is equivalent to determining the minimum value of v so that there exists an admissible set satisfying the constraints of (4.41). Since v satisfies $\bar{\mu}_{\max} - 1 \leq v \leq \bar{\mu}_{\max}$, where $\bar{\mu}_{\max}$ denotes the maximum value of $\bar{\mu}_i$, $i = 1, \ldots, k$, we have the following method for solving this problem by combined use of the bisection method and the simplex method of linear programming [14]. Here, when $\bar{\mu}_i - v \leq 0$, set $\bar{\mu}_i - v = 0$ in view of the constraints $\bar{\mu}_i - v \leq \mu_i(z_i(\boldsymbol{x}))$ for $0 \leq \mu_i(z_i(\boldsymbol{x})) \leq 1$, $i = 1, \ldots, k$.

Step 1: Set $v = \bar{\mu}_{\max}$ and test whether an admissible set satisfying the constraints of (4.41) exists or not using phase one of the simplex method. If an admissible set exists, proceed. Otherwise, the DM must reassess the membership function.

Step 2: Set $v = \bar{\mu}_{\max} - 1$ and test whether an admissible set satisfying the constraints of (4.41) exists or not using phase one of the

simplex method. If an admissible set exists, set $v^* = \bar{\mu}_{\max} - 1$. Otherwise, go to the next step since the minimum v which satisfies the constraints of (4.41) exists between $\bar{\mu}_{\max} - 1$ and $\bar{\mu}_{\max}$.

Step 3: For the initial value of $v = \bar{\mu}_{\max} - 0.5$, update the value of v using the bisection method as follows:

$$\begin{cases} v_{n+1} = v_n - 1/2^{n+1} & \text{if an admissible set exists for } v_n, \\ v_{n+1} = v_n + 1/2^{n+1} & \text{if no admissible set exists for } v_n. \end{cases}$$

For each v_n, $n = 1, 2, \ldots$, test whether an admissible set of (4.41) exists or not using the sensitivity analysis technique for changes in the right-hand side of the simplex method and determine the minimum value of v satisfying the constraints of (4.41).

In this way, we can determine the optimal solution v^*. Then the DM selects an appropriate standing objective from among the objectives $z_i(\boldsymbol{x})$, $i = 1, \ldots, k$. For notational convenience in the following without loss of generality, let it be $z_1(\boldsymbol{x})$ and $1 \in I_1$. Then the following linear programming problem is solved for $v = v^*$:

$$\left.\begin{array}{ll} \text{minimize} & z_1(\boldsymbol{x}) \\ \text{subject to} & z_i(\boldsymbol{x}) \le d_{iR}^{-1}(\bar{\mu}_i - v^*), \ i(\neq 1) \in I_1 \cup I_3 \\ & z_i(\boldsymbol{x}) \ge d_{iL}^{-1}(\bar{\mu}_i - v^*), \ i(\neq 1) \in I_2 \cup I_3 \\ & \boldsymbol{x} \in X. \end{array}\right\} \tag{4.42}$$

The DM must either be satisfied with the current M-Pareto optimal solution or act on this solution by updating the reference membership levels. In order to help the DM express a degree of preference, trade-off information between a standing membership function $\mu_1(z_1(\boldsymbol{x}))$ and each of the other membership functions is very useful. Such trade-off information is easily obtainable since it is closely related to the simplex multipliers of the problem (4.42).

Let the simplex multipliers corresponding to the constraints $z_i(\boldsymbol{x})$, $i = 2, \ldots, k$, of the linear problem (4.42) be denoted by $\pi_i^* = \pi_i(\boldsymbol{x}^*)$, $i = 2, \ldots, k$, where $\boldsymbol{x}^*$ is an optimal solution of (4.42). If $\boldsymbol{x}^*$ is a non-degenerate solution of (4.42) and all the constraints of (4.42) are active, then by using the results in Haimes and Chankong [21], the trade-off information between the objective functions can be represented by

$$-\frac{\partial z_1(\boldsymbol{x})}{\partial z_i(\boldsymbol{x})} = \pi_i^*, \ i = 2, \ldots, k. \tag{4.43}$$

Hence, by the chain rule, the trade-off information between the membership functions is given by

$$-\frac{\partial \mu_1(z_1(\boldsymbol{x}))}{\partial \mu_i(z_i(\boldsymbol{x}))} = -\frac{\partial \mu_1(z_1(\boldsymbol{x}))}{\partial z_1(\boldsymbol{x})}\frac{\partial z_1(\boldsymbol{x})}{\partial z_i(\boldsymbol{x})}\left\{\frac{\partial \mu_i(z_i(\boldsymbol{x}))}{\partial z_i(\boldsymbol{x})}\right\}^{-1}, \quad i = 2, \ldots, k. \tag{4.44}$$

Therefore, for each $i = 2, \ldots, k$, we have the following expression:

$$-\frac{\partial \mu_1(z_1(\boldsymbol{x}))}{\partial \mu_i(z_i(\boldsymbol{x}))} = \pi_i^* \frac{\partial \mu_1(z_1(\boldsymbol{x}))/\partial z_1(\boldsymbol{x})}{\partial \mu_i(z_i(\boldsymbol{x}))/\partial z_i(\boldsymbol{x})}, \quad i = 2, \ldots, k. \tag{4.45}$$

It should be stressed here that in order to obtain the trade-off rate information from (4.45), all the constraints of the problem (4.42), must be active. Therefore, if there are inactive constraints, it is necessary to replace $\bar{\mu}_i$ for inactive constraints by $\bar{\mu}_i(z_i(\boldsymbol{x}^*))$ and solve the corresponding problem to obtain the simplex multipliers.

We can now construct the interactive algorithm in order to derive the satisficing solution for the DM from the M-Pareto optimal solution set where the steps marked with an asterisk involve interaction with the DM. This interactive fuzzy multiobjective programming method can also be interpreted as the fuzzy version of the reference point method (RPM) with trade-off information.

Interactive fuzzy multiobjective linear programming

Step 0: Calculate the individual minimum and maximum of each objective function under the given constraints.

Step 1*: Elicit a membership function from the DM for each of the objective functions.

Step 2: Set the initial reference membership levels to 1.

Step 3: For the reference membership levels, solve the corresponding minimax problem to obtain the M-Pareto optimal solution and the membership function value together with the trade-off rate information between the membership functions.

Step 4*: If the DM is satisfied with the current levels of the M-Pareto optimal solution, stop. Then the current M-Pareto optimal solution is the satisficing solution for the DM. Otherwise, ask the DM to update the current reference membership levels by considering the current values of the membership functions together with the trade-off rates between the membership functions and return to Step 3.

It should be stressed to the DM that any improvement of one membership function can be achieved only at the expense of at least one of the other membership functions.

In the next section, we will proceed to the multiobjective linear programming problems with fuzzy parameters as a generalized version of this section.

6. Interactive Fuzzy Multiobjective Linear Programming with Fuzzy Parameters

First, recall the multiobjective linear programming (MOLP) problems discussed thus far. For convenience in our subsequent discussion, consider the MOLP of the following form:

$$\left.\begin{array}{ll} \text{minimize} & (\boldsymbol{c}_1\boldsymbol{x}, \boldsymbol{c}_2\boldsymbol{x}, \ldots, \boldsymbol{c}_k\boldsymbol{x})^T \\ \text{subject to} & \boldsymbol{x} \in X = \{\boldsymbol{x} \in R^n \mid \boldsymbol{a}_j\boldsymbol{x} \leq b_j,\ j = 1, \ldots, m;\ \boldsymbol{x} \geq \boldsymbol{0}\} \end{array}\right\} \tag{4.46}$$

where x is an n-dimensional column vector of decision variables, $\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_k$ are n-dimensional cost factor row vectors, $a_1, a_2, \ldots, a_m$ are n-dimensional constraint row vectors, and $b_1, b_2, \ldots, b_m$ are constants.

In practice, however, it would certainly be more appropriate to consider that the possible values of the parameters in the description of the objective functions and the constraints usually involve the ambiguity of the experts' understanding of the real system. For this reason, in this chapter, we consider the following multiobjective linear programming problem involving fuzzy parameters (MOLP-FP):

$$\left.\begin{array}{ll} \text{minimize} & (\boldsymbol{C}_1\boldsymbol{x}, \boldsymbol{C}_2\boldsymbol{x}, \ldots, \boldsymbol{C}_k\boldsymbol{x})^T \\ \text{subject to} & \boldsymbol{x} \in X(\boldsymbol{A}, \boldsymbol{B}) = \{\boldsymbol{x} \in R^n \mid \boldsymbol{A}_j\boldsymbol{x} \leq B_j, \\ & \qquad\qquad\qquad\qquad j = 1, \ldots, m;\ \boldsymbol{x} \geq \boldsymbol{0}\} \end{array}\right\} \tag{4.47}$$

$\boldsymbol{C}_i = (C_{i1}, \ldots, C_{in})$, $\boldsymbol{A}_j = (A_{j1}, \ldots, A_{jn})$, B_j represent, respectively, fuzzy parameters involved in the objective function $\boldsymbol{C}_i\boldsymbol{x}$ and constraint $\boldsymbol{A}_j\boldsymbol{x} \leq B_j$.

These fuzzy parameters, reflecting the experts' ambiguous understanding of the nature of the parameters in the problem-formulation process, are assumed to be characterized as fuzzy numbers introduced by Dubois and Prade [16, 17].

We now assume that all of the fuzzy parameters $C_{i1}, \ldots, C_{in}, A_{j1}, \ldots, A_{jn}$, and B_j in the MOLP-FP are fuzzy numbers the membership functions of which are denoted by $\mu_{C_{i1}}(c_{i1}), \ldots, \mu_{C_{in}}(c_{in}), \mu_{A_{j1}}(a_{j1}), \ldots, \mu_{A_{jn}}(a_{jn})$, and $\mu_{B_j}(b_j)$, respectively. For simplicity in notation, define

the following vectors:

$$\boldsymbol{c} = (\boldsymbol{c}_1, \ldots, \boldsymbol{c}_k), \quad \boldsymbol{a} = (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m), \quad \boldsymbol{b} = (b_1, \ldots, b_m),$$

$$\boldsymbol{C} = (\boldsymbol{C}_1, \ldots, \boldsymbol{C}_k), \quad \boldsymbol{A} = (\boldsymbol{A}_1, \ldots, \boldsymbol{A}_m), \quad \boldsymbol{B} = (B_1, \ldots, B_m)$$

Observing that the MOLP-FP involves fuzzy numbers both in the objective functions and the constraints, it is evident that the notion of Pareto optimality defined for the MOLP cannot be applied directly. Thus, it seems essential to extend the notion of usual Pareto optimality in some sense. For that purpose, we first introduce the α-level set of the fuzzy numbers A_{jr}, B_j, and C_{ir}. To be more explicit, the α-level set of the fuzzy numbers A_{jr}, B_j, and C_{ir} is defined as the ordinary set $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})_\alpha$ for which the degree of their membership functions exceeds the level α:

$$(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})_\alpha = \{ (\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \mid \mu_{A_{jr}}(a_{jr}) \geq \alpha,\ \mu_{B_j}(b_j) \geq \alpha,\ \mu_{C_{ir}}(c_{ir}) \geq \alpha; \quad i = 1, \ldots, k,\ j = 1, \ldots, m,\ r = 1, \ldots, n\} \tag{4.48}$$

Now suppose that the decision maker (DM) decides that the degree of all of the membership functions of the fuzzy numbers involved in the MOLP-FP should be greater than or equal to some value α. Then for such a degree α, the MOLP-FP can be interpreted as the following nonfuzzy multiobjective linear programming (MOLP-FP($\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$)) problem which depends on the coefficient vector $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \in (\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})_\alpha$:

$$\left.\begin{array}{ll} \text{minimize} & (\boldsymbol{c}_1\boldsymbol{x}, \boldsymbol{c}_2\boldsymbol{x}, \ldots, \boldsymbol{c}_k\boldsymbol{x})^T \\ \text{subject to} & \boldsymbol{x} \in X(\boldsymbol{a}, \boldsymbol{b}) = \{\boldsymbol{x} \in R^n \mid \boldsymbol{a}_j\boldsymbol{x} \leq b_j,\ j = 1, \ldots, m\ ; \\ & \qquad\qquad\qquad\quad \boldsymbol{x} \geq \boldsymbol{0}\} \end{array}\right\} \tag{4.49}$$

Observe that there exists an infinite number of such MOLP-FP ($\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$) depending on the coefficient vector $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \in (\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})_\alpha$, and the values of $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ are arbitrary for any $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \in (\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})_\alpha$ in the sense that the degree of all of the membership functions for the fuzzy numbers in the MOLP-FP exceeds the level α. However, if possible, it would be desirable for the DM to choose $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \in (\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})_\alpha$ in the MOLP-FP($\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$) to minimize the objective functions under the constraints. From such a point of view, for a certain degree α, it seems to be quite natural to have the MOLP-FP as the following nonfuzzy α-multiobjective

linear programming (α-MOLP) problem:

$$\left.\begin{array}{ll} \text{minimize} & (\boldsymbol{c}_1\boldsymbol{x}, \boldsymbol{c}_2\boldsymbol{x}, \ldots, \boldsymbol{c}_k\boldsymbol{x})^T \\ \text{subject to} & \boldsymbol{x} \in X(\boldsymbol{a}, \boldsymbol{b}) = \{\boldsymbol{x} \in R^n \mid \boldsymbol{a}_j\boldsymbol{x} \leq b_j,\ j = 1, \ldots, m\ ; \\ & \qquad\qquad\qquad\qquad\quad \boldsymbol{x} \geq \boldsymbol{0}\} \\ & (\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \in (\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})_\alpha. \end{array}\right\} \tag{4.50}$$

It should be emphasized here that, in the α-MOLP, the parameters $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ are treated as decision variables rather than constants.

On the basis of the α-level sets of the fuzzy numbers, we can introduce the concept of an α-Pareto optimal solution to the α-MOLP as a natural extension of the Pareto optimality concept for the MOLP.

Definition 12 (α-Pareto optimal solution)
$\boldsymbol{x}^* \in X(\boldsymbol{a}^*, \boldsymbol{b}^*)$ is said to be an α-Pareto optimal solution to the α-MOLP if and only if there does not exist another $\boldsymbol{x} \in X(\boldsymbol{a}, \boldsymbol{b})$, $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \in (\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})_\alpha$ such that $\boldsymbol{c}_i\boldsymbol{x} \leq \boldsymbol{c}_i^*\boldsymbol{x}^*$, $i = 1, \ldots, k$ with strict inequality holding for at least one i, where the corresponding values of parameters $(\boldsymbol{a}^*, \boldsymbol{b}^*, \boldsymbol{c}^*)$ are called α-level optimal parameters.

Observe that α-Pareto optimal solutions and α-level optimal parameters can be obtained through a direct application of the usual scalarizing methods for generating Pareto optimal solutions by regarding the decision variables in the α-MOLP as $(\boldsymbol{x}, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$.

As can be seen from the definition of α-Pareto optimality, in general, α-Pareto optimal solutions to the α-MOLP consist of an infinite number of points.

In order to derive a satisficing solution for the DM efficiently from an α-Pareto optimal solution set, interactive programming methods have been presented by Sakawa et al. [77, 84, 47].

However, considering the imprecise nature of the DM's judgment, it is natural to assume that the DM may have imprecise or fuzzy goals for each of the objective functions in the α-MOLP. In a minimization problem, a goal stated by the DM may be to achieve "substantially less than or equal to some value p_i." This type of statement can be quantified by eliciting a corresponding membership function.

To elicit a membership function $\mu_i(\boldsymbol{c}_i\boldsymbol{x})$ from the DM for each of the objective functions $\boldsymbol{c}_i\boldsymbol{x}$, $i = 1, \ldots, k$, in the α-MOLP, we first calculate the individual minimum and maximum of each objective function under the given constraints for $\alpha = 0$ and $\alpha = 1$. By taking account of the calculated individual minimum and maximum of each objective function for $\alpha = 0$ and $\alpha = 1$ together with the rate of increase of membership

satisfaction, the DM may be able to determine a membership function $\mu_i(\boldsymbol{c}_i\boldsymbol{x})$ in a subjective manner which is a strictly monotone decreasing function with respect to $\boldsymbol{c}_i\boldsymbol{x}$. So far we have restricted ourselves to a minimization problem and consequently assumed that the DM has a fuzzy goal such as "$\boldsymbol{c}_i\boldsymbol{x}$ should be substantially less than or equal to p_i." In the fuzzy approaches, as discussed previously, we can further treat a more general case where the DM has two types of fuzzy goals, namely, fuzzy goals expressed in words such as "$\boldsymbol{c}_i\boldsymbol{x}$ should be in the vicinity of r_i" (called fuzzy equal) as well as "$\boldsymbol{c}_i\boldsymbol{x}$ should be substantially less than or equal to p_i or greater than or equal to q_i" (called fuzzy min or fuzzy max). Such a generalized α-MOLP (Gα-MOLP) problem may now be expressed as

$$\left.\begin{array}{rl} \text{fuzzy min} & \boldsymbol{c}_i\boldsymbol{x} \quad i \in I_1 \\ \text{fuzzy max} & \boldsymbol{c}_i\boldsymbol{x} \quad i \in I_2 \\ \text{fuzzy equal} & \boldsymbol{c}_i\boldsymbol{x} \quad i \in I_3 \\ \text{subject to} & \boldsymbol{x} \in X(\boldsymbol{a}, \boldsymbol{b}) \\ & (\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \in (\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})_\alpha \end{array}\right\} \tag{4.51}$$

where $I_1 \cup I_2 \cup I_3 = \{1, 2, \ldots, k\}$, $I_i \cap I_j = \emptyset$, $i, j = 1, 2, 3$, $i \neq j$.

To elicit a membership function $\mu_i(\boldsymbol{c}_i\boldsymbol{x})$ from the DM for a fuzzy goal like "$\boldsymbol{c}_i\boldsymbol{x}$ should be in the vicinity of r_i," it should be quite apparent that different functions can be utilized for both the left and right sides of r_i. Concerning the membership functions of the Gα-MOLP, it is reasonable to assume that $\mu_i(\boldsymbol{c}_i\boldsymbol{x})$, $i \in I_1$, and the right side functions of $\mu_i(\boldsymbol{c}_i\boldsymbol{x})$, $i \in I_3$, are strictly monotone increasing and continuous functions with respect to $\boldsymbol{c}_i\boldsymbol{x}$.

Here it is assumed that $d_{iR}(c_ix)$ is a strictly monotone decreasing continuous function with respect to c_ix and $d_{iL}(c_ix)$ is a strictly monotone increasing continuous function with respect to c_ix. Both may be linear or nonlinear. $(c_ix)_L^0$ and $(c_ix)_R^0$ are maximum values of unacceptable levels for c_ix, and $(c_ix)_L^1$ and $(c_ix)_R^1$ are minimum values of totally desirable levels for c_ix.

When a fuzzy equal is included in the fuzzy goals of the DM, it is desirable that c_ix should be as close to r_i as possible. Consequently, the notion of α-Pareto optimal solutions defined in terms of objective functions cannot be applied. For this reason, we introduce the concept of M-α-Pareto optimal solutions which is defined in terms of membership functions instead of objective functions, where M refers to membership.

Definition 13 (M-α-Pareto optimal solution)

$\boldsymbol{x}^* \in X(\boldsymbol{a}^*, \boldsymbol{b}^*)$ is said to be an M-α-Pareto optimal solution to the Gα-MOLP if and only if there does not exist another $\boldsymbol{x} \in X(\boldsymbol{a}, \boldsymbol{b})$,

$(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \in (\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})_\alpha$ *such that* $\mu_i(\boldsymbol{c}_i\boldsymbol{x}) \geq \mu_i(\boldsymbol{c}_i^*\boldsymbol{x}^*)$, $i = 1, \ldots, k$, *with strict inequality holding for at least one i, where the corresponding values of parameters* $(\boldsymbol{a}^*, \boldsymbol{b}^*, \boldsymbol{c}^*) \in (\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})_\alpha$ *are called* α*-level optimal parameters.*

Observe that the concept of M-α-Pareto optimal solutions defined in terms of membership functions is a natural extension to that of α-Pareto optimal solutions defined in terms of objective functions when fuzzy equal is included in the fuzzy goals of the DM.

Having elicited the membership functions $\mu_i(\boldsymbol{c}_i\boldsymbol{x})$, $i = 1, \ldots, k$, from the DM for each of the objective functions $\boldsymbol{c}_i\boldsymbol{x}$, $i = 1, \ldots, k$, if we introduce a general aggregation function $\mu_D(\cdot)$, a general fuzzy α-multiobjective decision making problem (Fα-MODMP) can be defined by

$$\left.\begin{array}{ll} \text{maximize} & \mu_D(\mu_1(\boldsymbol{c}_1\boldsymbol{x}), \mu_2(\boldsymbol{c}_2\boldsymbol{x}), \ldots, \mu_k(\boldsymbol{c}_k\boldsymbol{x}), \alpha) \\ \text{subject to} & (\boldsymbol{x}, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \in P(\alpha), \ \alpha \in [0, 1] \end{array}\right\} \tag{4.52}$$

where $P(\alpha)$ is the set of M-α-Pareto optimal solutions and corresponding α-level optimal parameters to the Gα-MOLP.

Probably the most crucial problem in the Fα-MODMP is the identification of an appropriate aggregation function which well represents the human decision makers' fuzzy preferences. If $\mu_D(\cdot)$ can be explicitly identified, then the Fα-MODMP reduces to a standard mathematical programming problem. However, this happens rarely, and as an alternative approach, it becomes evident that an interaction with the DM is necessary.

To generate a candidate for the satisficing solution, which is also M-α-Pareto optimal, in our decision making method, the DM is asked to specify the degree α of the α-level set and the reference membership values. Observe that the idea of the reference membership values, which first appeared in Sakawa, Yumine, and Yano [87], can be viewed as an obvious extension of the idea of the reference point in Wierzbicki [103].

Once the DM's degree α and reference membership values $\bar{\mu}_i$, $i = 1, \ldots, k$, are specified, the corresponding M-α-Pareto optimal solution, which is, in the minimax sense, nearest to the requirement or better than that if the reference levels are attainable, is obtained by solving the following minimax problem:

$$\min_{\substack{\boldsymbol{x} \in X(\boldsymbol{a}, \boldsymbol{b}) \\ (\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \in (\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})_\alpha}} \max_{i=1, \ldots, k} (\bar{\mu}_i - \mu_i(\boldsymbol{c}_i\boldsymbol{x})) \tag{4.53}$$

or equivalently

$$\left.\begin{array}{ll} \text{minimize} & v \\ \text{subject to} & \bar{\mu}_i - \mu_i(\boldsymbol{c}_i\boldsymbol{x}) \le v,\ i = 1, \dots, k, \\ & \boldsymbol{a}_j\boldsymbol{x} \le b_j,\ j = 1, \dots, m,\ \boldsymbol{x} \ge \boldsymbol{0}, \\ & (\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \in (\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})_\alpha. \end{array}\right\} \tag{4.54}$$

However, with the strictly monotone decreasing or increasing membership function, which may be nonlinear, the resulting problem becomes a nonlinear programming problem.

For notational convenience, denote the strictly monotone decreasing function for the fuzzy min and the right function of the fuzzy equal by $d_{iR}(z_i)$ $(i \in I_1 \cup I_3)$ and the strictly monotone increasing function for the fuzzy max and the left function of the fuzzy equal by $d_{iL}(z_i)$ $(i \in I_2 \cup I_3)$. Then in order to solve the formulated problem on the basis of the linear programming method, we first convert each constraint $\bar{\mu}_i - \mu_i(\boldsymbol{c}_i\boldsymbol{x}) \le v$, $i = 1, \dots, k$, of the minimax problem (4.54) into the following form using the strictly monotone property of $d_{iL}(\cdot)$ and $d_{iR}(\cdot)$:

$$\left.\begin{array}{l} \boldsymbol{c}_i\boldsymbol{x} \le d_{iR}^{-1}(\bar{\mu}_i - v),\ i \in I_1 \cup I_{3R} \\ \boldsymbol{c}_i\boldsymbol{x} \ge d_{iL}^{-1}(\bar{\mu}_i - v),\ i \in I_2 \cup I_{3L} \end{array}\right\} \tag{4.55}$$

Now we can introduce the following set-valued functions $S_{iR}(\cdot), S_{iL}(\cdot)$, and $T_j(\cdot, \cdot)$:

$$\left.\begin{array}{l} S_{iR}(\boldsymbol{c}_i) = \{(\boldsymbol{x}, v) \mid \boldsymbol{c}_i\boldsymbol{x} \le d_{iR}^{-1}(\bar{\mu}_i - v)\}, \quad i \in I_1 \cup I_{3R} \\ S_{iL}(\boldsymbol{c}_i) = \{(\boldsymbol{x}, v) \mid \boldsymbol{c}_i\boldsymbol{x} \ge d_{iL}^{-1}(\bar{\mu}_i - v)\}, \quad i \in I_2 \cup I_{3L} \\ T_j(\boldsymbol{a}_j, b_j) = \{\boldsymbol{x} \mid \boldsymbol{a}_j\boldsymbol{x} \le b_j\}, \quad j = 1, \dots, m \end{array}\right\} \tag{4.56}$$

It can be verified that the following relations hold for $S_{iR}(\cdot), S_{iL}(\cdot)$, and $T_j(\cdot, \cdot)$ when $\boldsymbol{x} \ge \boldsymbol{0}$.

Proposition 4 (Inclusion relations of set-valued functions)

(1) *If* $c_i^1 \le c_i^2$, *then* $S_{iR}(c_i^1) \supseteq S_{iR}(c_i^2)$ *and* $S_{iL}(c_i^1) \subseteq S_{iL}(c_i^2)$.
(2) *If* $a_j^1 \le a_j^2$, *then* $T_j(a_j^1, b_j) \supseteq T_j(a_j^2, b_j)$.
(3) *If* $b_j^1 \le b_j^2$, *then* $T_j(a_j, b_j^1) \subseteq T_j(a_j, b_j^2)$.

Using the properties of the α-level sets for the vectors of the fuzzy numbers C_i, A_j and the fuzzy numbers B_j, the feasible regions for c_i, a_j, and b_j can be denoted respectively by the closed intervals $[\boldsymbol{c}_{i\alpha}^L, \boldsymbol{c}_{i\alpha}^R]$, $[\boldsymbol{a}_{j\alpha}^L, \boldsymbol{a}_{j\alpha}^R]$, and $[b_{j\alpha}^L, b_{j\alpha}^R]$.

Consequently, using of the results in Proposition 4, we can obtain an optimal solution to (4.54) by solving the following problem:

$$\left.\begin{array}{ll} \text{minimize} & v \\ \text{subject to} & \boldsymbol{c}_{i\alpha}^{L}\boldsymbol{x} \le d_{iR}^{-1}(\bar{\mu}_i - v),\ i \in I_1 \cup I_{3R} \\ & \boldsymbol{c}_{i\alpha}^{R}\boldsymbol{x} \ge d_{iL}^{-1}(\bar{\mu}_i - v),\ i \in I_2 \cup I_{3L} \\ & \boldsymbol{a}_{j\alpha}^{L}\boldsymbol{x} \le b_{j\alpha}^{R},\ j = 1, \ldots, m,\ \boldsymbol{x} \ge \boldsymbol{0} \end{array}\right\} \tag{4.57}$$

It is important to note here that this formulation, if the value of v is fixed, can be reduced to a set of linear inequalities. Obtaining the optimal solution v^* to the above problem is equivalent to determining the minimum value of v so that there exists an admissible set satisfying the constraints of (4.57). Since v satisfies $\bar{\mu}_{\max} - 1 \le v \le \bar{\mu}_{\max}$, where $\bar{\mu}_{\max}$ denotes the maximum value of $\bar{\mu}_i$, $i = 1, \ldots, k$, we have the following method for solving this problem by combined use of the bisection method and the simplex method of linear programming [14].

Step 1: Set $v = \bar{\mu}_{\max}$ and test whether an admissible set satisfying the constraints of (4.57) exists or not by making use of phase one of the simplex method. If an admissible set exists, proceed. Otherwise, the DM must reassess the membership function.

Step 2: Set $v = \bar{\mu}_{\max} - 1$ and test whether an admissible set satisfying the constraints of (4.57) exists or not using phase one of the simplex method. If an admissible set exists, set $v^* = \bar{\mu}_{\max} - 1$. Otherwise, go to the next step since the minimum v which satisfies the constraints of (4.57) exists between $\bar{\mu}_{\max} - 1$ and $\bar{\mu}_{\max}$.

Step 3: For the initial value of $v = \bar{\mu}_{\max} - 0.5$, update the value of v using the bisection method as follows:

$$\begin{cases} v_{n+1} = v_n - 1/2^{n+1} & \text{if an admissible set exists for } v_n, \\ v_{n+1} = v_n + 1/2^{n+1} & \text{if no admissible set exists for } v_n. \end{cases} \tag{4.58}$$

For each v_n, $n = 1, 2, \ldots,$ test whether an admissible set of (4.57) exists or not using the sensitivity analysis technique for the changes in the right-hand side of the simplex method and determine the minimum value of v satisfying the constraints of (4.57).

In this way, we can determine the optimal solution v^*. Then the DM selects an appropriate standing objective from among the objectives $c_i x$, $i = 1, \ldots, k$. For notational convenience in the following without loss of generality, let it be $c_1 x$ and $1 \in I_1$. Then the following linear

programming problem is solved for $v = v^*$:

$$\left.\begin{array}{rl} \text{minimize} & \boldsymbol{c}_{1\alpha}^{L}\boldsymbol{x} \\ \text{subject to} & \boldsymbol{c}_{i\alpha}^{L}\boldsymbol{x} \le d_{iR}^{-1}(\bar{\mu}_i - v^*),\ i \in I_1 \cup I_{3R}, \\ & \boldsymbol{c}_{i\alpha}^{R}\boldsymbol{x} \ge d_{iL}^{-1}(\bar{\mu}_i - v^*),\ i \in I_2 \cup I_{3L}, \\ & \boldsymbol{a}_{j\alpha}^{L}\boldsymbol{x} \le b_{j\alpha}^{R},\ j = 1, \ldots, m,\ \boldsymbol{x} \ge \boldsymbol{0} \end{array}\right\} \tag{4.59}$$

For convenience in our subsequent discussion, we assume that the optimal solution $\boldsymbol{x}^*$ to (4.59) satisfies the following conditions:

$$\left.\begin{array}{l} \boldsymbol{c}_{i\alpha}^{L}\boldsymbol{x}^* = d_{iR}^{-1}(\bar{\mu}_i - v^*),\ i \in I_1 \cup I_{3R}, \\ \boldsymbol{c}_{i\alpha}^{R}\boldsymbol{x}^* = d_{iL}^{-1}(\bar{\mu}_i - v^*),\ i \in I_2 \cup I_{3L} \end{array}\right\} \tag{4.60}$$

where $I_3 = I_{3L} \cup I_{3R}$ and $I_{3L} \cap I_{3R} = \emptyset$.

It is interesting to note that $c_{i\alpha}^{L}, i \in I_1 \cup I_{3R}$, $c_{i\alpha}^{R}, i \in I_2 \cup I_{3L}$, and $a_{j\alpha}^{L}$, $b_{j\alpha}^{R}$, $j = 1, \ldots, m$, are α-level optimal parameters for any M-α-Pareto optimal solution.

The relationships between the optimal solutions to (4.57) and the M-α-Pareto optimal concept of the Gα-MOLP can be characterized by the following theorems.

Theorem 4.5 (Minimax problem and M-α-Pareto optimality)
If $\boldsymbol{x}^*$ is a unique optimal solution to (4.57), then $\boldsymbol{x}^*$ is an M-α-Pareto optimal solution to the Gα-MOLP.

Theorem 4.6 (M-α-Pareto optimality and minimax problem)
If $\boldsymbol{x}^*$ is an M-α-Pareto optimal solution to the Gα-MOLP, then $\boldsymbol{x}^*$ is an optimal solution to (4.57) for some $\bar{\mu} = (\bar{\mu}_1, \ldots, \bar{\mu}_k)$.

The proofs of these theorems follow directly from the definitions of optimality and M-α-Pareto optimality by making use of contradiction arguments.

It must be observed here that for generating M-α-Pareto optimal solutions using Theorem 4.5, uniqueness of solution must be verified. In the ad hoc numeral approach, however, to test the M-α-Pareto optimality of a current optimal solution $\boldsymbol{x}^*$, we formulate and solve the following linear programming problem:

$$\left.\begin{array}{rl} \text{maximize} & \displaystyle\sum_{i=1}^{k} \varepsilon_i \\ \text{subject to} & \boldsymbol{c}_{i\alpha}^{L}\boldsymbol{x} + \varepsilon_i = \boldsymbol{c}_{i\alpha}^{L}\boldsymbol{x}^*,\ \varepsilon_i \ge 0,\ i \in I_1 \cup I_{3R} \\ & \boldsymbol{c}_{i\alpha}^{R}\boldsymbol{x} - \varepsilon_i = \boldsymbol{c}_{i\alpha}^{R}\boldsymbol{x}^*,\ \varepsilon_i \ge 0,\ i \in I_2 \cup I_{3L} \\ & \boldsymbol{a}_{j\alpha}^{L}\boldsymbol{x} \le b_{j\alpha}^{R},\ j = 1, \ldots, m,\ \boldsymbol{x} \ge \boldsymbol{0} \end{array}\right\} \tag{4.61}$$

Let $\bar{\boldsymbol{x}}$ and $\bar{\varepsilon}_i$ be an optimal solution to this problem. If all $\bar{\varepsilon}_i = 0$, then $\boldsymbol{x}^*$ is an M-α-Pareto optimal solution. If at least one $\bar{\varepsilon}_i > 0$, as discussed previously, it can be easily shown that $\bar{\boldsymbol{x}}$ is an M-α-Pareto optimal solution.

Now given the M-α-Pareto optimal solution for the degree α and the reference membership values specified by the DM by solving the corresponding minimax problem, the DM must either be satisfied with the current M-α-Pareto optimal solution and α or update the reference membership values and/or the degree α. To help the DM express a degree of preference, trade-off information between a standing membership function and each of the other membership functions as well as between the degree α and the membership functions is very useful. Such trade-off information is easily obtainable since it is closely related to the simplex multipliers of the problem (4.59).

To derive the trade-off information, define the following Lagrangian function L corresponding to problem (4.59):

$$\begin{aligned} L = {} & \boldsymbol{c}_{1\alpha}^L \boldsymbol{x} + \sum_{i \in I_1 \cup I_{3R}} \pi_{iR} \{ \boldsymbol{c}_{i\alpha}^L \boldsymbol{x} - d_{iR}^{-1}(\bar{\mu}_i - v^*) \} \\ & + \sum_{i \in I_2 \cup I_{3L}} \pi_{iL} \{ d_{iL}^{-1}(\bar{\mu}_i - v^*) - \boldsymbol{c}_{i\alpha}^R \boldsymbol{x} \} + \sum_{j=1}^{m} \lambda_j (\boldsymbol{a}_{j\alpha}^L \boldsymbol{x} - b_{j\alpha}^R) \end{aligned} \tag{4.62}$$

where π_{iL}, π_{iR}, and λ_j are simplex multipliers corresponding to the constraints of (4.59).

Here we assume that problem (4.59) has a unique and nondegenerate optimal solution satisfying the following conditions:

(1) $\pi_{iR} > 0$, $i \in I_1 \cup I_{3R}$, $i \neq 1$
(2) $\pi_{iL} > 0$, $i \in I_2 \cup I_{3L}$.

Then by using the results in Haimes and Chankong [21], the following expression holds:

$$-\frac{\partial(\boldsymbol{c}_{1\alpha}^L \boldsymbol{x})}{\partial(\boldsymbol{c}_{i\alpha}^L \boldsymbol{x})} = \pi_{iR}, \quad i \in I_1 \cup I_{3R}, \quad i \neq 1 \tag{4.63}$$

$$-\frac{\partial(\boldsymbol{c}_{1\alpha}^L \boldsymbol{x})}{\partial(\boldsymbol{c}_{i\alpha}^R \boldsymbol{x})} = -\pi_{iL}, \quad i \in I_2 \cup I_{3L} \tag{4.64}$$

Furthermore, using the strictly monotone decreasing or increasing property of $d_{iR}(\cdot)$ or $d_{iL}(\cdot)$ together with the chain rule, if $d_{iR}(\cdot)$ and $d_{iL}(\cdot)$ are differentiable at the optimal solution to (4.59), it holds that

$$-\frac{\partial \mu_1(\boldsymbol{c}_{1\alpha}^L \boldsymbol{x})}{\partial \mu_i(\boldsymbol{c}_{i\alpha}^L \boldsymbol{x})} = \frac{d'_{1R}(\boldsymbol{c}_{1\alpha}^L \boldsymbol{x})}{d'_{iR}(\boldsymbol{c}_{i\alpha}^L \boldsymbol{x})} \pi_{iR}, \quad i \in I_1 \cup I_{3R}, \quad i \neq 1 \tag{4.65}$$

$$-\frac{\partial \mu_1(\boldsymbol{c}_{1\alpha}^L \boldsymbol{x})}{\partial \mu_i(\boldsymbol{c}_{i\alpha}^R \boldsymbol{x})} = \frac{d'_{1R}(\boldsymbol{c}_{1\alpha}^L \boldsymbol{x})}{d'_{iL}(\boldsymbol{c}_{i\alpha}^R \boldsymbol{x})} \pi_{iL}, \quad i \in I_2 \cup I_{3L} \tag{4.66}$$

where $d'_{iR}(\cdot)$ and $d'_{iL}(\cdot)$ denote the differential coefficients of $d_{iR}(\cdot)$ and $d_{iL}(\cdot)$, respectively.

Regarding a trade-off rate between $\mu_1(\boldsymbol{c}_{1\alpha}^L \boldsymbol{x})$ and α, the following relation holds based on the sensitivity theorem (for details, see, e.g., Luenberger [31] or Fiacco [19]):

$$\begin{aligned}\frac{\partial \mu_1(\boldsymbol{c}_{1\alpha}^L \boldsymbol{x})}{\partial \alpha} = d'_{1R}(\boldsymbol{c}_{1\alpha}^L \boldsymbol{x}) \Bigg\{ & \frac{\partial(\boldsymbol{c}_{1\alpha}^L)}{\partial \alpha} \boldsymbol{x} + \sum_{i \in I_1 \cup I_{3R}} \pi_{iR} \frac{\partial(\boldsymbol{c}_{i\alpha}^L)}{\partial \alpha} \boldsymbol{x} \\ & - \sum_{i \in I_2 \cup I_{3L}} \pi_{iL} \frac{\partial(\boldsymbol{c}_{i\alpha}^R)}{\partial \alpha} \boldsymbol{x} + \sum_{j=1}^{m} \lambda_j \left\{ \frac{\partial(\boldsymbol{a}_{j\alpha}^L)}{\partial \alpha} \boldsymbol{x} - \frac{\partial(b_{j\alpha}^R)}{\partial \alpha} \right\} \Bigg\}\end{aligned} \tag{4.67}$$

It should be noted that to obtain the trade-off rate information from (4.65) and (4.66), all the constraints of problem (4.59) must be active for the current optimal solution. Therefore, if there are inactive constraints, it is necessary to replace $\bar{\mu}_i$ for inactive constraints by $d_{iR}(\boldsymbol{c}_{i\alpha}^L \boldsymbol{x}^*) + v^*$ or $d_{iL}(\boldsymbol{c}_{i\alpha}^R \boldsymbol{x}^*) + v^*$ and solve the corresponding problem (4.59) for obtaining the simplex multipliers.

Now, following the above discussions, we can present the interactive algorithm to derive the satisficing solution for the DM from the M-α-Pareto optimal solution set. The steps marked with an asterisk involve interaction with the DM.

Interactive fuzzy multiobjective linear programming with fuzzy parameters

Step 0: (Individual minimum and maximum)

Calculate the individual minimum and maximum of each objective function under the given constraints for $\alpha = 0$ and $\alpha = 1$.

Step 1*: (Membership functions)

Elicit a membership function $\mu_i(\boldsymbol{c}_i \boldsymbol{x})$ from the DM for each of the objective functions.

Step 2*: (Initialization)

Ask the DM to select the initial value of α $(0 \leq \alpha \leq 1)$ and set the initial reference membership values $\bar{\mu}_i = 1$, $i = 1, \ldots, k$.

Step 3: (M-α-Pareto optimal solution)

For the degree α and the reference membership values specified by the DM, solve the minimax problem and perform the M-α-Pareto optimality test to obtain the M-α-Pareto optimal solution and the trade-off rates between the membership functions and the degree α.

Step 4*: (Termination or updating)

The DM is supplied with the corresponding M-α-Pareto optimal solution and the trade-off rates between the membership functions and the degree α. If the DM is satisfied with the current membership function values of the M-α-Pareto optimal solution and α, stop. Otherwise, the DM must update the reference membership values and/or the degree α by considering the current values of the membership functions and α together with the trade-off rates between the membership functions and the degree α and return to step 3.

Here it should be stressed to the DM that (1) any improvement of one membership function can be achieved only at the expense of at least one of the other membership functions for some fixed degree α and (2) the greater value of the degree α gives the worse values of the membership functions for some fixed reference membership values.

It is significant to point out here that all the results presented in this section have already been extended by the authors to deal with multiobjective linear fractional programming problems with fuzzy parameters. A successful generalization along this line can be found in Sakawa and Yano [76], and the interested readers might refer to them for details.

7. Related Works and Applications

So far multiobjective linear programming in a fuzzy environment is briefly discussed on the basis of the author's continuing research works. For further details of multiobjective linear and nonlinear programming in a fuzzy environment, including interactive computer programs and some applications, the readers might refer to Sakawa's 1993 book entitled "Fuzzy Sets and Interactive Multiobjective Optimization" [47].

In addition to this book, the book of Lai and Hwang [28], and Carlsson and Fullér [6] and the recently published two books of Sakawa [48, 49] together with the edited volumes of Kacprzyk and Orlovski [23], Verdegay and Delgado [101], Slowinski and Teghem [95], Delgado, Kacprzyk, Verdegay and Vila [15], and Slowinski [94] would be very useful for interested readers.

It is now appropriate to mention some application aspects of fuzzy multiobjective optimization. As we look at engineering, industrial, and management applications of fuzzy multiobjective optimization, we can see continuing advances. They can be found, for example, in the areas of an air pollution regulation problem [98], media selection in advertising [102], a transportation problem [100], environmental planning [75], water supply system development planning [93], operation of a packaging system in automated warehouses [85], pass scheduling for hot tandem mills [53], spatial planning problems [30], profit apportionment in concerns [37], a capital asset pricing model [38], a farm structure optimization problem [11], diet optimization problems [12], a forest management problem [40], quality control [7], wastewater management [18], fuzzy vehicle routing and scheduling [9], flexible scheduling in a machining center [50], a real size manpower allocation problem [1], multiobjective interval transportation problems [13], coal purchase planning in electric power plants [91], fuzzy job shop scheduling [52], and profit and cost allocation for a production and transportation problem [65].

8. Interactive Fuzzy Two-level Linear Programming

8.1. Two-level Programming Problems

Two-level programming problems, in which the upper level DM makes a decision first and the lower level DM makes a decision after understanding the decision of the upper level DM, admit of two interpretations. They depend on whether there is a cooperative relationship between the DMs or not.

Consider a decision problem in a decentralized firm as an example of a decision problem with cooperative DMs. Top management, an executive board, or headquarters interests itself in overall management policy such as long-term corporate growth or market share. In contrast, operation divisions of the firm are concerned with coordination of daily activities. After headquarters make a decision in accordance with the overall management policy, each division determines a goal to be achieved and tries to attain the goal, fully understanding the decision by the headquarters.

As an example of a decision problem without cooperative DMs, consider the Stackelberg duopoly: Firm 1 and Firm 2 supply homogeneous goods to a market. Suppose Firm 1 dominates Firm 2 in the market, and consequently Firm 1 first determines a level of supply and then Firm 2 decides its level of supply after it, realizes Firm 1's level of supply.

It seems that there exists cooperative relationship between the upper level DM and the lower level DM in the former problem while each DM does not have a motivation to cooperate each other in the latter problem.

As the former's mathematical programming problem, we can model such a problem as a single-objective large scale mathematical programming problems used the decomposition method or a multiobjective programming problem with objective functions of all levels. Bialas and Karwan remark that the two-level programming formulation is intend to supplement decomposition approach, not supplant it [5]. Naturally, the two-level formulation is noteworthy because a hierarchical structure of the decision problem is explicitly included in a mathematical model.

Studies on the latter have been seen in the literature on game theory. Such a situation is modelled as a Stackelberg game, in which there are two players, and one player determines a strategy and thereafter the other player decides a strategy [92]. Each player completely knows objective functions and constraints of an opponent and self, and the upper level DM first specifies a strategy and then the lower level DM specifies a strategy so as to optimize the objective with full knowledge of the decision of the upper level DM. According to the rule, the upper level DM also specifies the strategy so as to optimize the objective. Then a solution defined as the above mentioned procedure is called the Stackelberg (equilibrium) solution.

The Stackelberg solution has been employed as a solution concept when decision problems are modelled as two-level programming problems, whether there is a cooperative relationship between the DMs or not. Even if the objective functions of both DMs and the common constraint functions are linear, it is known that the two-level linear programming problem is a non-convex programming problem with special structure. Moreover, it should be noted that the Stackelberg solution does not always satisfy Pareto optimality because of its noncooperative nature. For obtaining the Stackelberg solution, a large number of computation methods have been developed [90].

In 1996, Lai [26] and Shih, Lai and Lee [89] have proposed a solution concept, which is different from the concept of a Stackelberg solution, for the two- or multi-level programming problems with cooperative DMs. Their method [27] is based on an idea that the lower level DM optimizes an objective function, taking a goal or preference of the upper level DM into consideration. The DMs elicit membership functions of fuzzy goals for their objective functions, and especially, the upper level DM also specifies those of fuzzy goals for the decision variables. The lower level DM solves a fuzzy programming problem with a constraint on a satisfactory degree of the upper level DM. Unfortunately, however, there

is a possibility that their method leads a final solution to an undesirable one because of inconsistency between the fuzzy goals of the objective function and those of the decision variables.

By eliminating the fuzzy goals for the decision variables to avoid such problems in the methods of Lai et al., Sakawa et al. [62] introduced interactive fuzzy programming for two-level linear programming problems. Moreover, from the viewpoint of experts' imprecise or fuzzy understanding of the nature of parameters in a problem-formulation process, they extend it to interactive fuzzy programming for two-level linear programming problems with fuzzy parameters [63]. These results are also extended to deal with two-level linear fractional programming problems [54, 64] and two-level linear and linear fractional programming problems in multiobjective environments [61, 57], considering diversity of evaluation by the DMs.

They also develop interactive fuzzy programming for two-level 0-1 programming problems through the genetic algorithms [59, 60]. and for two-level linear programming problems with multiple DMs at the lower level [55]. Moreover, these results are applied to real-world decision making problems [66].

Under these circumstances, in this section, interactive fuzzy programming for two-level linear programming problems [62] is introduced. In the interactive method, after determining the fuzzy goals of the DMs at both levels, a satisfactory solution is derived efficiently by updating the minimal satisfactory level of the upper level DM with considerations of overall satisfactory balance between both levels.

8.2. Interactive Fuzzy Two-level Linear Programming

Consider the following two-level linear programming problem:

$$\left.\begin{array}{ll} \underset{\text{Level 1}}{\text{minimize}} & z_1(\boldsymbol{x}_1, \boldsymbol{x}_2) = \boldsymbol{c}_{11}\boldsymbol{x}_1 + \boldsymbol{c}_{12}\boldsymbol{x}_2 \\ \underset{\text{Level 2}}{\text{minimize}} & z_2(\boldsymbol{x}_1, \boldsymbol{x}_2) = \boldsymbol{c}_{21}\boldsymbol{x}_1 + \boldsymbol{c}_{22}\boldsymbol{x}_2 \\ \text{subject to} & A_1\boldsymbol{x}_1 + A_2\boldsymbol{x}_2 \leq \boldsymbol{b} \\ & \boldsymbol{x}_1 \geq \boldsymbol{0},\ \boldsymbol{x}_2 \geq \boldsymbol{0}, \end{array}\right\} \tag{4.68}$$

where $\boldsymbol{x}_i$, $i = 1, 2$ is an n_i-dimensional decision variable column vector; $\boldsymbol{c}_{i1}$, $i = 1, 2$ is an n_1-dimensional constant row vector; $\boldsymbol{c}_{i2}$, $i = 1, 2$ is an n_2-dimensional constant row vector; $\boldsymbol{b}$ is an m-dimensional constant column vector; A_i, $i = 1, 2$ is an $m \times n_i$ constant matrix; $z_1(\boldsymbol{x}_1, \boldsymbol{x}_2)$ and $z_2(\boldsymbol{x}_1, \boldsymbol{x}_2)$, respectively, represent objective functions of the upper and the lower levels; and $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, respectively, represent decision variable vector of the upper and the lower levels. In the two-level linear

programming problem (4.68) with cooperative DMs, $\text{minimize}_{\text{Level 1}}$ and $\text{minimize}_{\text{Level 2}}$ mean that the DMs at the upper and the lower levels seek to minimize their objective functions under the given constraints.

For the sake of simplicity, we use the following notations: $\boldsymbol{x}^T = (\boldsymbol{x}_1^T, \boldsymbol{x}_2^T) \in R^{n_1+n_2}$, where T is transposition, $\boldsymbol{c}_1 = (\boldsymbol{c}_{11}, \boldsymbol{c}_{12})$, $\boldsymbol{c}_2 = (\boldsymbol{c}_{21}, \boldsymbol{c}_{22})$, and $A = [A_1\ A_2]$, and let DM1 denote the DM at the upper level and DM2 denote the DM at the lower level.

It is natural that DMs have fuzzy goals for their objective functions when they take fuzziness of human judgments into consideration. For each of the objective functions $z_i(\boldsymbol{x})$, $i = 1, 2$ of (4.68), assume that the DMs have fuzzy goals such as "the objective function $z_i(\boldsymbol{x})$ should be substantially less than or equal to some value p_i."

Let X denote the feasible region of Problem (4.68). The individual minimum

$$z_i^{\min} = \min_{\boldsymbol{x} \in X} z_i(\boldsymbol{x}),\ i = 1, 2 \tag{4.69}$$

and the individual maximum

$$z_i^{\max} = \max_{\boldsymbol{x} \in X} z_i(\boldsymbol{x}), i = 1, 2 \tag{4.70}$$

of the objective functions are referred to when the DMs elicit membership functions prescribing the fuzzy goals for the objective functions $z_i(\boldsymbol{x})$, $i = 1, 2$. The DMs determine the membership functions $\mu_i(z_i(\boldsymbol{x}))$, which are strictly monotone decreasing for $z_i(\boldsymbol{x})$, consulting the variation ratio of degree of satisfaction in the interval between the individual minimum (4.69) and the individual maximum (4.70). The domain of the membership function is the interval $[z_i^{\min}, z_i^{\max}]$, $i = 1, 2$, and the DM specifies the value z_i^0 of the objective function for which the degree of satisfaction is 0 and the value z_i^1 of the objective function for which the degree of satisfaction is 1. For the value undesired (larger) than z_i^0, it is defined that $\mu_i(z_i(\boldsymbol{x})) = 0$, and for the value desired (smaller) than z_i^1, it is defined that $\mu_i(z_i(\boldsymbol{x})) = 1$.

For example, we consider a linear membership function, which characterizes the fuzzy goal of the DM at each level. The corresponding linear membership function $\mu_i(z_i)$ is defined as:

$$\mu_i(z_i(\boldsymbol{x})) = \begin{cases} 0 & ;\ z_i(\boldsymbol{x}) > z_i^0 \\ \dfrac{z_i(\boldsymbol{x}) - z_i^0}{z_i^1 - z_i^0} & ;\ z_i^1 < z_i(\boldsymbol{x}) \le z_i^0 \\ 1 & ;\ z_i(\boldsymbol{x}) \le z_i^1, \end{cases} \tag{4.71}$$

where z_i^0 and z_i^1 denote the value of the objective function $z_i(x)$ such that the degree of membership function is 0 and 1, respectively, and it is assumed that the DMs subjectively assess z_i^0 and z_i^1.

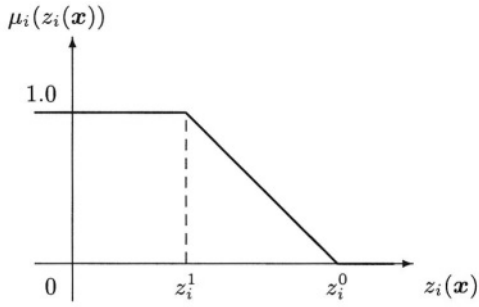


Figure 4.10. Linear membership function.

Zimmermann proposed a method for determining the parameters z_i^0 and z_i^1 of the linear membership function in the following way [108]. That is, using the individual minimum

$$z_i^{\min} = z_i(\boldsymbol{x}^{io}) = \min\{z_i(\boldsymbol{x}) \mid A_1\boldsymbol{x}_1 + A_2\boldsymbol{x}_2 \leq \boldsymbol{b}, \boldsymbol{x}_1 \geq \boldsymbol{0},\ \boldsymbol{x}_2 \geq \boldsymbol{0}\} \tag{4.72}$$

together with

$$z_i^{\mathrm{m}} = z_i(\boldsymbol{x}^{jo}),\ i = 1, 2,\ j = \begin{cases} 1 \text{ if } i = 2 \\ 2 \text{ if } i = 1, \end{cases} \tag{4.73}$$

the DMs determine the linear membership functions as in (4.71) by choosing $z_i^1 = z_i^{\min}$, $z_i^0 = z_i^{\mathrm{m}}$, $i = 1, 2$.

Having elicited the membership functions $\mu_1(z_1(\boldsymbol{x}))$ and $\mu_2(z_2(\boldsymbol{x}))$ from both DMs for the objective functions $z_1(\boldsymbol{x})$ and $z_2(\boldsymbol{x})$, the original two-level linear programming problem (4.68) can be interpreted as the membership function maximization problem defined by:

$$\left.\begin{array}{ll} \underset{\text{Level 1}}{\text{maximize}} & \mu_1(z_1(\boldsymbol{x})) \\ \underset{\text{Level 2}}{\text{maximize}} & \mu_2(z_2(\boldsymbol{x})) \\ \text{subject to} & A\boldsymbol{x} \leq \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0}. \end{array}\right\} \tag{4.74}$$

In Problem (4.74), $\boldsymbol{x} \in R^n$ is an n-dimensional decision variable vector and is divided into two vectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ which are n_1- and n_2-dimensional decision variable vectors of DM1 and DM2, respectively. However, because the two DMs make decisions cooperatively, the decision variable vector is represented simply by $\boldsymbol{x}$ without partition.

For deriving an overall satisfactory solution to the formulated problem (4.74), we first find the maximizing decision of the fuzzy decision

proposed by Bellman and Zadeh [2]. Namely, the following problem is solved for obtaining a solution which maximizes the smaller degree of satisfaction between the two DMs:

$$\left.\begin{array}{ll} \text{maximize} & \min\{\mu_1(z_1(\boldsymbol{x})), \mu_2(z_2(\boldsymbol{x}))\} \\ \text{subject to} & A\boldsymbol{x} \leq \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0}. \end{array}\right\} \tag{4.75}$$

By introducing the auxiliary variable λ, this problem can be transformed into the following equivalent maximization problem:

$$\left.\begin{array}{ll} \text{maximize} & \lambda \\ \text{subject to} & \mu_1(z_1(\boldsymbol{x})) \geq \lambda \\ & \mu_2(z_2(\boldsymbol{x})) \geq \lambda \\ & A\boldsymbol{x} \leq \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0}. \end{array}\right\} \tag{4.76}$$

Solving Problem (4.76), we can obtain a solution which maximizes the smaller satisfactory degree between those of both DMs. It should be noted that if the membership functions $\mu_i(z_i(\boldsymbol{x}))$, $i = 1, 2$ are linear membership functions such as (4.71), Problem (4.76) becomes linear programming problem. Let $\boldsymbol{x}^*$ denote an optimal solution to problem (4.76). Then, we define the satisfactory degree of both DMs under the constraints as

$$\lambda = \min\{\mu_1(z_1(\boldsymbol{x}^*)), \mu_2(z_2(\boldsymbol{x}^*))\}. \tag{4.77}$$

If DM1 is satisfied with the optimal solution $\boldsymbol{x}^*$, it follows that the optimal solution $\boldsymbol{x}^*$ becomes a satisfactory solution; however, DM1 is not always satisfied with the solution $\boldsymbol{x}^*$. It is quite natural to assume that DM1 would like to subjectively specify a minimal satisfactory level $\hat{\delta} \in [0, 1]$ for the membership function $\mu_1(z_1(\boldsymbol{x}))$.

Consequently, if DM1 is not satisfied with the solution $\boldsymbol{x}^*$ to Problem (4.76), the following problem is formulated:

$$\left.\begin{array}{ll} \text{maximize} & \mu_2(z_2(\boldsymbol{x})) \\ \text{subject to} & \mu_1(z_1(\boldsymbol{x})) \geq \hat{\delta} \\ & A\boldsymbol{x} \leq \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0}. \end{array}\right\} \tag{4.78}$$

where DM2's membership function is maximized under the condition that DM1's membership function $\mu_1(z_1(\boldsymbol{x}))$ is larger than or equal to the minimal satisfactory level $\hat{\delta}$ specified by DM1. It should be also noted that if the membership functions $\mu_i(z_i(\boldsymbol{x}))$, $i = 1, 2$ are linear

membership functions such as (4.71), Problem (4.78) becomes linear programming problem.

If there exists an optimal solution to Problem (4.78), it follows that DM1 obtains a satisfactory solution having a satisfactory degree larger than or equal to the minimal satisfactory level specified by DM1. However, the larger the minimal satisfactory level $\hat{\delta}$ is assessed, the smaller the DM2's satisfactory degree becomes when the objective functions of DM1 and DM2 conflict with each other. Consequently, a relative difference between the satisfactory degrees of DM1 and DM2 becomes larger and we cannot expect that the overall satisfactory balance between both levels is appropriate.

In order to take account of the overall satisfactory balance between both levels, DM1 needs to compromise with DM2 on DM1's own minimal satisfactory level. To do so, a ratio of satisfactory degrees between both DMs is defined as

$$\Delta = \frac{\mu_2(z_2(\boldsymbol{x}))}{\mu_1(z_1(\boldsymbol{x}))}, \tag{4.79}$$

which is originally introduced by Lai [26], is useful.

DM1 is guaranteed to have satisfactory degrees larger than or equal to the minimal satisfactory levels for all of the fuzzy goals because the corresponding constraints are involved in Problem (4.78). To take into account the overall satisfactory balance between both levels, we provide two methods for evaluating the ratio Δ of satisfactory degrees. In the first method, DM1 specifies the lower bound $\Delta_{\min}$ and the upper bound $\Delta_{\max}$ of the ratio and the ratio is evaluated by verifying that it is in the interval $[\Delta_{\min}, \Delta_{\max}]$. The condition that the overall satisfactory balance is appropriate is represented by

$$\Delta \in [\Delta_{\min}, \Delta_{\max}]. \tag{4.80}$$

In the second method, DM1 identify a fuzzy goal $\tilde{R}$ for the ratio Δ of satisfactory degrees, which is expressed in words such as "the ratio Δ should be in the vicinity of a certain value m" and gives the permissible level $\hat{\delta}_{\tilde{\Delta}}$ to the membership value of the fuzzy goal. The condition that the overall satisfactory balance is appropriate is represented by

$$\mu_{\tilde{R}}(\Delta) \geq \hat{\delta}_{\tilde{\Delta}}, \tag{4.81}$$

where $\mu_{\tilde{R}}$ denote a membership function of the fuzzy goal $\tilde{R}$.

The two methods have relevance to each other by interpreting the α-level set $\{\Delta \mid \mu_{\tilde{R}}(\Delta) \geq \hat{\delta}_{\tilde{\Delta}}\}$ as the interval $[\Delta_{\min}, \Delta_{\max}]$. Moreover, in multiobjective two-level programming problems, the ratio Δ is represented as a fuzzy number as we show in later sections and then we can

naturally extend the second method. From the relation between the two method, we explain only interactive procedure with the first method.

At an iteration l, let $\mu_1(z_1^l)$, $\mu_2(z_2^l)$, λ^l and $\Delta^l = \mu_2(z_2^l)/\mu_1(z_1^l)$ respectively denote DM1's and DM2's satisfactory degrees, a satisfactory degree of both levels and the ratio of satisfactory degrees between both DMs, and let the corresponding optimal solution be $\boldsymbol{x}^l$. The interactive process terminates if the following two conditions are satisfied and DM1 concludes the solution as an overall satisfactory solution.

Termination conditions of the interactive process

Condition 1: DM1's satisfactory degree is larger than or equal to the minimal satisfactory level $\hat{\delta}$ specified by DM1, i.e., $\mu_1(z_1^l) \geq \hat{\delta}$.

Condition 2: The ratio Δ^l of satisfactory degrees lies in the closed interval between the lower and the upper bounds specified by DM1, i.e., $\Delta^l \in [\Delta_{\min}, \Delta_{\max}]$.

Condition 1 is DM1's required condition for solutions, and Condition 2 is provided in order to keep overall satisfactory balance between both levels.

Unless these two conditions are satisfied simultaneously, DM1 needs to update the minimal satisfactory level $\hat{\delta}$.

Procedure for updating the minimal satisfactory level $\hat{\delta}$

Case 1: If Condition 1 is not satisfied, then DM1 decreases the minimal satisfactory level $\hat{\delta}$.

Case 2: If the ratio Δ^l exceeds its upper bound, then DM1 increases the minimal satisfactory level $\hat{\delta}$. Conversely, if the ratio Δ^l is below its lower bound, then DM1 decreases the minimal satisfactory level $\hat{\delta}$.

Case 3: Although Conditions 1 and 2 are satisfied, if DM1 is not satisfied with the obtained solution and judges that it is desirable to increase the satisfactory degree of DM1 at the expense of the satisfactory degree of DM2, then DM1 increases the minimal satisfactory level $\hat{\delta}$. Conversely, if DM1 judges that it is desirable to increase the satisfactory degree of DM2 at the expense of the satisfactory degree of DM1, then DM1 decreases the minimal satisfactory level $\hat{\delta}$.

If Condition 1 is not satisfied, because there does not exist any feasible solution, DM1 has to moderate the minimal satisfactory level. For Case 2, DM1 must adjust it so as to meet the bounds.

We are now ready to present an interactive algorithm for deriving an overall satisfactory solution to Problem (4.68), which is summarized in the following and is illustrated with a flowchart in Figure 4.11:

Interactive fuzzy two-level linear programming

Step 1: Ask DM1 to identify the membership function $\mu_1(z_1)$ of the fuzzy goal of DM1. Similarly, ask DM1 to identify the membership function $\mu_2(z_2)$ of the fuzzy goal of DM2.

Step 2: Set $l = 1$ and solve Problem (4.76), in which a smaller degree between the satisfactory degrees of DM1 and DM2 is maximized. If DM1 is satisfied with the obtained optimal solution, the solution becomes a satisfactory solution. Otherwise, ask DM1 to specify the minimal satisfactory level $\hat{\delta}$ together with the lower and the upper bounds $[\Delta_{\min}, \Delta_{\max}]$ of the ratio of satisfactory degrees Δ^l by considering the satisfactory degree λ of both DMs and the related information about the solution.

Step 3: Solve Problem (4.78), in which the satisfactory degree of DM2 is maximized under the condition that the satisfactory degree of DM1 is larger than or equal to the minimal satisfactory level, and then propose an optimal solution $\boldsymbol{x}^l$ to Problem (4.78) to DM1 together with λ^l, $\mu_1(z_1^l)$, $\mu_2(z_2^l)$ and Δ^l.

Step 4: If the solution proposed to DM1 satisfies the termination conditions and DM1 concludes the solution as a satisfactory solution, the algorithm stops.

Step 5: Ask DM1 to update the minimal satisfactory level $\hat{\delta}$ in accordance with the procedure of updating minimal satisfactory level.

Step 6: Solve Problem (4.78) and propose the obtained optimal solution to DM1 together with the related information. Return to Step 4.

Further details including an illustrative numerical example and an application can be found in Sakawa et al. [62, 66]. Extensions to linear fractional, nonlinear and 0-1 programming problems can be found in Sakawa et al. [54, 55, 56, 60].

9. Interactive Fuzzy Two-level Linear Programming with Fuzzy Parameters

When formulating a mathematical programming problem which closely describes and represents a real-world decision situation, various factors

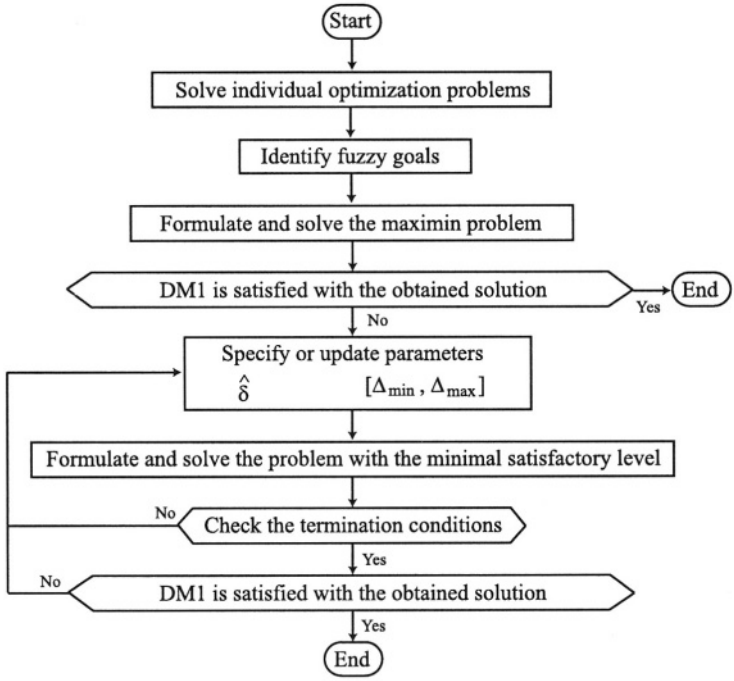


Figure 4.11. Flowchart of interactive fuzzy two-level programming.

of the real-world system should be reflected in the description of objective functions and constraints. Naturally, these objective functions and constraints involve many parameters whose possible values may be assigned by experts. In the conventional approaches, such parameters are required to be fixed at some values in an experimental and/or subjective manner through the experts' understanding of the nature of the parameters in the problem-formulation process.

It must be observed here that, in most real-world situations, the possible values of these parameters are often only imprecisely or ambiguously known to the experts. With this observation in mind, it would be certainly more appropriate to interpret the experts' understanding of the parameters as fuzzy numerical data which can be represented by means of fuzzy sets of the real line known as fuzzy numbers. The resulting mathematical programming problem involving fuzzy parameters would be viewed as a more realistic version than the conventional one [47, 84].

From this viewpoint, we assume that parameters involving in the objective functions and the constraints of the two-level linear programming problem are characterized by fuzzy numbers. As a result, a problem with

fuzzy parameters corresponding to Problem (4.68) is formulated as:

$$\left.\begin{array}{ll} \underset{\text{Level 1}}{\text{minimize}} & z_1(\boldsymbol{x}_1, \boldsymbol{x}_2) = \tilde{\boldsymbol{c}}_{11}\boldsymbol{x}_1 + \tilde{\boldsymbol{c}}_{12}\boldsymbol{x}_2 \\ \underset{\text{Level 2}}{\text{minimize}} & z_2(\boldsymbol{x}_1, \boldsymbol{x}_2) = \tilde{\boldsymbol{c}}_{21}\boldsymbol{x}_1 + \tilde{\boldsymbol{c}}_{22}\boldsymbol{x}_2 \\ \text{subject to} & \tilde{A}_1\boldsymbol{x}_1 + \tilde{A}_2\boldsymbol{x}_2 \leq \tilde{\boldsymbol{b}} \\ & \boldsymbol{x}_1 \geq \boldsymbol{0},\ \boldsymbol{x}_2 \geq \boldsymbol{0}, \end{array}\right\} \tag{4.82}$$

where $\tilde{\boldsymbol{c}}_{11}$, $\tilde{\boldsymbol{c}}_{12}$, $\tilde{\boldsymbol{c}}_{21}$, $\tilde{\boldsymbol{c}}_{22}$, $\tilde{A}_1$, $\tilde{A}_2$ and $\tilde{\boldsymbol{b}}$ are fuzzy parameters. For the sake of simplicity, we use the following notations: $\tilde{\boldsymbol{c}}_1 = (\tilde{\boldsymbol{c}}_{11}, \tilde{\boldsymbol{c}}_{12})$, $\tilde{\boldsymbol{c}}_2 = (\tilde{\boldsymbol{c}}_{21}, \tilde{\boldsymbol{c}}_{22})$, and $\tilde{A} = [\tilde{A}_1\ \tilde{A}_2]$.

Assuming that the fuzzy parameters $\tilde{\boldsymbol{c}}_{11}$, $\tilde{\boldsymbol{c}}_{12}$, $\tilde{\boldsymbol{c}}_{12}$, $\tilde{\boldsymbol{c}}_{22}$, $\tilde{A}_1$, $\tilde{A}_2$ and $\tilde{\boldsymbol{b}}$ are characterized by fuzzy numbers, let corresponding membership functions be denoted by:

$$(\mu_{\tilde{c}_{11,1}}(c_{11,1}), \ldots, \mu_{\tilde{c}_{11,n_1}}(c_{11,n_1})), \quad (\mu_{\tilde{c}_{12,1}}(c_{12,1}), \ldots, \mu_{\tilde{c}_{12,n_2}}(c_{12,n_2})),$$
$$(\mu_{\tilde{c}_{21,1}}(c_{21,1}), \ldots, \mu_{\tilde{c}_{21,n_1}}(c_{21,n_1})), \quad (\mu_{\tilde{c}_{22,1}}(c_{22,1}), \ldots, \mu_{\tilde{c}_{22,n_2}}(c_{22,n_2})),$$

$\mu_{\tilde{a}_{1,kj}}(a_{1,kj})$, $k = 1, \ldots, m$, $j = 1, \ldots, n_1$, $\mu_{\tilde{a}_{2,kj}}(a_{2,kj})$, $k = 1, \ldots, m$, $j = 1, \ldots, n_2$, and $(\mu_{\tilde{b}_1}(b_1), \ldots, \mu_{\tilde{b}_m}(b_m))$. We introduce the α-level set of the fuzzy numbers $\tilde{\boldsymbol{c}}$, $\tilde{\boldsymbol{b}}$ and $\tilde{A}$ defined as the ordinary set $(\tilde{\boldsymbol{c}}, \tilde{\boldsymbol{b}}, \tilde{A})_\alpha$ in which the degree of their membership functions exceeds the level α:

$$\begin{aligned} (\tilde{\boldsymbol{c}}, \tilde{\boldsymbol{b}}, \tilde{A})_\alpha = \{ (\boldsymbol{c}, \boldsymbol{b}, A) \mid\ & \mu_{\tilde{c}_{11,j}}(c_{11,j}) \geq \alpha,\ j = 1, \ldots, n_1, \\ & \mu_{\tilde{c}_{12,j}}(c_{12,j}) \geq \alpha,\ j = 1, \ldots, n_2, \\ & \mu_{\tilde{c}_{21,j}}(c_{21,j}) \geq \alpha,\ j = 1, \ldots, n_1, \\ & \mu_{\tilde{c}_{22,j}}(c_{22,j}) \geq \alpha,\ j = 1, \ldots, n_2, \\ & \mu_{\tilde{b}_k}(b_k) \geq \alpha,\ k = 1, \ldots, m, \\ & \mu_{\tilde{a}_{1,kj}}(a_{1,kj}) \geq \alpha,\ k = 1, \ldots, m,\ j = 1, \ldots, n_1, \\ & \mu_{\tilde{a}_{2,kj}}(a_{2,kj}) \geq \alpha,\ k = 1, \ldots, m,\ j = 1, \ldots, n_2\}. \end{aligned} \tag{4.83}$$

Now suppose that DM1 considers that the degree of all of the membership functions of the fuzzy numbers involved in the two-level linear programming problem should be greater than or equal to some value α. Then, for such a degree α, Problem (4.82) can be interpreted as the following nonfuzzy two-level linear programming problem which depends on a coefficient vector $(\boldsymbol{c}, \boldsymbol{b}, A) \in (\tilde{\boldsymbol{c}}, \tilde{\boldsymbol{b}}, \tilde{A})_\alpha$ [47, 84]:

$$\left.\begin{array}{ll} \underset{\text{Level 1}}{\text{minimize}} & z_1(\boldsymbol{x}_1, \boldsymbol{x}_2) = \boldsymbol{c}_{11}\boldsymbol{x}_1 + \boldsymbol{c}_{12}\boldsymbol{x}_2 \\ \underset{\text{Level 2}}{\text{minimize}} & z_2(\boldsymbol{x}_1, \boldsymbol{x}_2) = \boldsymbol{c}_{21}\boldsymbol{x}_1 + \boldsymbol{c}_{22}\boldsymbol{x}_2 \\ \text{subject to} & A_1\boldsymbol{x}_1 + A_2\boldsymbol{x}_2 \leq \boldsymbol{b} \\ & \boldsymbol{x}_1 \geq \boldsymbol{0},\ \boldsymbol{x}_2 \geq \boldsymbol{0}. \end{array}\right\} \tag{4.84}$$

Observe that there exist an infinite number of such problems (4.84) depending on the coefficient vector $(\boldsymbol{c}, \boldsymbol{b}, A) \in (\tilde{\boldsymbol{c}}, \tilde{\boldsymbol{b}}, \tilde{A})_\alpha$ and the values of $(\mathbf{c}, \mathbf{b}, A)$ are arbitrary for any $(\boldsymbol{c}, \boldsymbol{b}, A) \in (\tilde{\boldsymbol{c}}, \tilde{\boldsymbol{b}}, \tilde{A})_\alpha$ in the sense that the degree of all of the membership functions for the fuzzy numbers in Problem (4.84) exceeds the level α. However, if possible, it would be desirable for each DM to choose $(\boldsymbol{c}, \boldsymbol{b}, A) \in (\tilde{\boldsymbol{c}}, \tilde{\boldsymbol{b}}, \tilde{A})_\alpha$ in Problem (4.84) so as to minimize the objective function under the constraints. Assuming that DM1 chooses a degree of the α-level, from such a point of view, it seems to be quite natural to have understood the two-level linear programming problem with fuzzy parameters as the following nonfuzzy α-two-level linear programming problem [47, 84]:

$$\left.\begin{array}{ll} \underset{\text{Level 1}}{\text{minimize}} & z_1(\boldsymbol{x}_1, \boldsymbol{x}_2; \boldsymbol{c}_{11}, \boldsymbol{c}_{12}) = \boldsymbol{c}_{11}\boldsymbol{x}_1 + \boldsymbol{c}_{12}\boldsymbol{x}_2 \\ \underset{\text{Level 2}}{\text{minimize}} & z_2(\boldsymbol{x}_1, \boldsymbol{x}_2; \boldsymbol{c}_{21}, \boldsymbol{c}_{22}) = \boldsymbol{c}_{21}\boldsymbol{x}_1 + \boldsymbol{c}_{22}\boldsymbol{x}_2 \\ \text{subject to} & A_1\boldsymbol{x}_1 + A_2\boldsymbol{x}_2 \leq \boldsymbol{b} \\ & \boldsymbol{x}_1 \geq \mathbf{0},\ \boldsymbol{x}_2 \geq \mathbf{0} \\ & (\boldsymbol{c}, \boldsymbol{b}, A) \in (\tilde{\boldsymbol{c}}, \tilde{\boldsymbol{b}}, \tilde{A})_\alpha. \end{array}\right\} \tag{4.85}$$

It should be noted that the parameters $(\boldsymbol{c}, \boldsymbol{b}, A)$ are treated as decision variables rather than constants.

Similar to the two-level linear programming problems considered in the previous section, it is natural that DMs have fuzzy goals for their objective functions when they take fuzziness of human judgments into consideration.

After eliciting the membership functions $\mu_i(z_i(\boldsymbol{x}; \boldsymbol{c}_i))$, $i = 1, 2$, for deriving an overall satisfactory solution to the formulated problem (4.85), we first solve the following maximin problem for obtaining a solution which maximizes the smaller degree of satisfaction between the two DMs:

$$\left.\begin{array}{ll} \text{maximize} & \min\{\mu_1(z_1(\boldsymbol{x}; \boldsymbol{c}_1)), \mu_2(z_2(\boldsymbol{x}; \boldsymbol{c}_2))\} \\ \text{subject to} & A\boldsymbol{x} \leq \boldsymbol{b} \\ & \boldsymbol{x} \geq \mathbf{0} \\ & (\boldsymbol{c}, \boldsymbol{b}, A) \in (\tilde{\boldsymbol{c}}, \tilde{\boldsymbol{b}}, \tilde{A})_\alpha. \end{array}\right\} \tag{4.86}$$

By introducing the auxiliary variable λ, this problem can be transformed into the following equivalent maximization problem:

$$\left.\begin{array}{ll} \text{maximize} & \lambda \\ \text{subject to} & \mu_1(z_1(\boldsymbol{x}; \boldsymbol{c}_1)) \geq \lambda \\ & \mu_2(z_2(\boldsymbol{x}; \boldsymbol{c}_2)) \geq \lambda \\ & A\boldsymbol{x} \leq \boldsymbol{b} \\ & \boldsymbol{x} \geq \mathbf{0} \\ & (\boldsymbol{c}, \boldsymbol{b}, A) \in (\tilde{\boldsymbol{c}}, \tilde{\boldsymbol{b}}, \tilde{A})_\alpha. \end{array}\right\} \tag{4.87}$$

Unfortunately, Problem (4.87) is not a linear programming problem even if all the membership functions $\mu_i(z_i(\boldsymbol{x}; \boldsymbol{c}_i))$, $i = 1, 2$ are linear. To solve Problem (4.87) by using the linear programming technique, we introduce the set-valued functions:

$$\left.\begin{aligned} S_i(\boldsymbol{c}_i) &= \{(\boldsymbol{x}, \lambda) \mid \mu_i(z_i(\boldsymbol{x}, \boldsymbol{c}_i)) \geq \lambda\}, \ i = 1, 2 \\ T_j(A_j, b_j) &= \{\boldsymbol{x} \mid A_j \boldsymbol{x} \leq b_j\}, \ j = 1, \ldots, m, \end{aligned}\right\} \tag{4.88}$$

where A_j is a row vector corresponding to the jth row of the $m \times (n_1 + n_2)$ matrix A. Then it can be easily verified that the following relations hold for $S_i(\boldsymbol{c}_i)$ and $T_j(A_j, b_j)$ when $\boldsymbol{x} \geq \boldsymbol{0}$ [84, 47].

Proposition 5 (Inclusion relations of set-valued functions)

(1) *If* $\boldsymbol{c}_i^1 \leq \boldsymbol{c}_i^2$, *then* $S_i(\boldsymbol{c}_i^1) \supseteq S_i(\boldsymbol{c}_i^2)$.
(2) *If* $A_j^1 \leq A_j^2$, *then* $T_j(A_j^1, \cdot) \supseteq T_j(A_j^2, \cdot)$.
(3) *If* $b_j^1 \leq b_j^2$, *then* $T_j(\cdot, b_j^1) \subseteq T_j(\cdot, b_j^2)$.

From the properties of the α-level set for the vectors of fuzzy numbers $\tilde{\boldsymbol{c}}_1$, $\tilde{\boldsymbol{c}}_2$, $\tilde{\boldsymbol{b}}$ and the matrix of fuzzy numbers $\tilde{A}$, it should be noted that the feasible regions for $\boldsymbol{c}_1$, $\boldsymbol{c}_2$, b_j and A_j can be denoted respectively by the closed intervals $[\boldsymbol{c}_1^L, \boldsymbol{c}_1^R]$, $[\boldsymbol{c}_2^L, \boldsymbol{c}_2^R]$, $[b_j^L, b_j^R]$ and $[A_j^L, A_j^R]$.

Therefore, through the use of Proposition 5, we can obtain an optimal solution to Problem (4.87) by solving the following linear programming problem:

$$\left.\begin{array}{ll} \text{maximize} & \lambda \\ \text{subject to} & \mu_1(z_1(\boldsymbol{x}; \boldsymbol{c}_1^L)) \geq \lambda \\ & \mu_2(z_2(\boldsymbol{x}; \boldsymbol{c}_2^L)) \geq \lambda \\ & A^L \boldsymbol{x} \leq \boldsymbol{b}^R \\ & \boldsymbol{x} \geq \boldsymbol{0}. \end{array}\right\} \tag{4.89}$$

For the problem which maximizes DM2's membership function under the condition that DM1's membership function $\mu_1(z_1(\boldsymbol{x}; \boldsymbol{c}_1))$ is larger than or equal to the minimal satisfactory level $\hat{\delta}$ specified by DM1, we can also formulate the following problem including fuzzy parameters:

$$\left.\begin{array}{ll} \text{maximize} & \mu_2(z_2(\boldsymbol{x}; \boldsymbol{c}_2)) \\ \text{subject to} & \mu_1(z_1(\boldsymbol{x}; \boldsymbol{c}_1)) \geq \hat{\delta} \\ & A\boldsymbol{x} \leq \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \\ & (\boldsymbol{c}, \boldsymbol{b}, A) \in (\tilde{\boldsymbol{c}}, \tilde{\boldsymbol{b}}, \tilde{A})_\alpha. \end{array}\right\} \tag{4.90}$$

From the properties of the α-level set for the vectors of fuzzy numbers, Problem (4.90) can be also transformed into the following equivalent

problem:

$$\left.\begin{array}{ll} \text{maximize} & \mu_2(z_2(\boldsymbol{x}; \boldsymbol{c}_2^L)) \\ \text{subject to} & \mu_1(z_1(\boldsymbol{x}; \boldsymbol{c}_1^L)) \geq \hat{\delta} \\ & A^L \boldsymbol{x} \leq \boldsymbol{b}^R \\ & \boldsymbol{x} \geq \boldsymbol{0}. \end{array}\right\} \tag{4.91}$$

For the two-level linear programming problem with fuzzy parameters (4.82), we can provide the termination conditions of the interactive process and the procedure for updating the minimal satisfactory level $\hat{\delta}$ which are the same with the interactive fuzzy programming for the two-level linear programming problem (4.68), and give a similar algorithm with Problems (4.89) and (4.91) for deriving satisfactory solutions.

Further details including an illustrative numerical example can be found in Sakawa et al. [63]. Extensions to linear fractional, nonlinear and 0-1 programming problems with fuzzy parameters as a generalized version of this section can be found in Sakawa et al. [64, 58, 59].

The book "Nondifferentiable and Two-Level Mathematical Programming" of Shimizu, Ishizuka and Bard [90] and the recently published book by Lee and Shih [27] entitled "Fuzzy and Multi-Level Decision Making: An Interactive Computational Approach" which covers all of the major theoretical and practical advances in the wide range of fuzzy and multilevel decision making would be very useful for interested readers.

Acknowledgments

The author wishes to thank H. Yano and I. Nishizaki for their helpful comments on the first draft of this chapter.

References

[1] N. Abboud, M. Inuiguchi, M. Sakawa, and Y. Uemura, Manpower allocation using genetic annealing, *European Journal of Operational Research*, Vol. 111, pp. 405–420, 1998.

[2] R.E. Bellman and L.A. Zadeh, Decision making in a fuzzy environment, *Management Science*, Vol. 17, pp. 141–164, 1970.

[3] R. Benayoun, J. de Montgofier, J. Tergny and O. Larichev, Linear programming with multiple objective functions, Step method (STEM), *Mathematical Programming*, Vol. 1, pp. 366–375, 1971.

[4] W.F. Bialas and M.H. Karwan, On two-level optimization, *IEEE Transactions on Automatic Control*, Vol. AC-27, pp. 211–214, 1982.

[5] W.F. Bialas and M.H. Karwan, Two-level linear programming, *Management Science,* Vol. 30, pp. 1004–1020, 1984.

[6] C. Carlsson and R. Fullér, *Fuzzy Reasoning in Decision Making and Optimization,* Physica-Verlag, Heidelberg, 2002.

[7] T.K. Chakraborty, A class of single sampling inspection plans based on possibilistic programming problems, *Fuzzy Sets and Systems,* Vol. 63, pp. 35–43, 1994.

[8] V. Chankong and Y.Y. Haimes, *Multiobjective Decision Making: Theory and Methodology,* North-Holland, Amsterdam, 1983.

[9] R. Cheng and M. Gen, Fuzzy vehicle routing and scheduling problem using genetic algorithms, in F. Herrera and J.L. Verdegay (eds.), *Genetic Algorithms and Soft Computing,* Physica-Verlag, Heidelberg, pp. 683–709, 1996.

[10] E.U. Choo and D.R. Atkins, An interactive algorithm for multicriteria programming, *Computers & Operations Research,* Vol. 7, pp. 81–87,1980.

[11] P. Czyzak, Application of the "FLIP" method to farm structure optimization under uncertainty, in R. Slowinski and J. Teghem (eds.), *Stochastic versus Fuzzy Approaches to Multiobjective Mathematical Programming Problems under Uncertainty,* Kluwer Academic Publishers, Dordrecht, pp. 263–278, 1990.

[12] P. Czyzak and R. Slowinski, Solving muitiobjective diet optimization problems under uncertainty, in P. korhonen, A. Lewandowski and J. Wallenius (eds.) *Multiple Criteria Decision Support,* Springer-Verlag, Berlin, pp. 272–281, 1990.

[13] S.K. Das, A. Goswami and S.S. Alam, Multiobjective transportation problem with interval cost, source and destination parameters, *European Journal of Operational Research,* Vol. 117, pp. 100–112, 1999.

[14] G.B. Dantzig, *Linear Programming and Extensions,* Princeton University Press, New Jersey, 1961.

[15] M. Delgado, J. Kacprzyk, J.-L. Verdegay and M.A. Vila (eds.), *Fuzzy Optimization: Recent Advances,* Physica-Verlag, Heidelberg, 1994.

[16] D. Dubois and H. Prade, Operations on fuzzy numbers, *International Journal of Systems Science,* Vol. 9, pp. 613–626, 1978.

[17] D. Dubois and H. Prade, *Fuzzy Sets and Systems*: *Theory and Applications,* Academic Press, New York, 1980.

[18] L. Duckstein, A. Bardossy, A. Tecle and I. Bogardi, Fuzzy composite programming with application to wastewater management

under changing physical conditions, in M. Delgado, J. Kacprzyk, J.-L. Verdegay and M.A. Vila (eds.), *Fuzzy Optimization: Recent Advances,* Physica-Verlag, Heidelberg, pp. 199–219, 1994.

[19] A.V. Fiacco, *Introduction to Sensitivity and Stability Analysis in Nonlinear Programming,* Academic Press, New York, 1983.

[20] J. Fichefet, GPSTEM: an interactive multiobjective optimization method, *Progress in Operations Research,* North-Holland, Vol. 1, pp. 317–332, 1976.

[21] Y.Y. Haimes and V. Chankong, Kuhn-Tucker multipliers as trade-offs in multiobjective decision-making analysis, *Automatica,* Vol. 15, pp. 59–72, 1979.

[22] E.L. Hannan, Linear programming with multiple fuzzy goals, *Fuzzy Sets and Systems,* Vol. 6, pp. 235–248, 1981.

[23] J. Kacprzyk and S.A. Orlovski (eds.), *Optimization Models Using Fuzzy Sets and Possibility Theory,* D. Reidel Publishing Company, Dordrecht, 1987.

[24] A. Kaufmann and M.M. Gupta, *Introduction to Fuzzy Arithmetic: Theory and Applications,* Van Nostrand Reinhold, New York, 1991.

[25] G.J. Klir and B. Yuan (eds.), *Fuzzy Sets, Fuzzy Logic, and Fuzzy Systems: Selected Papers by Lotfi A. Zadeh,* World Scientific, Singapore, 1996.

[26] Y.J. Lai, Hierarchical optimization: a satisfactory solution, *Fuzzy Sets and Systems,* Vol. 77, pp. 321–335, 1996.

[27] E.S. Lee and H.S. Shih, *Fuzzy and Multi-Level Decision Making: An Interactive Computational Approach,* Springer, London, 2001.

[28] Y.J. Lai and C.L. Hwang, *Fuzzy Multiple Objective Decision Making: Methods and Applications,* Springer-Verlag, Berlin, 1994.

[29] H. Leberling, On finding compromise solution in multicriteria problems using the fuzzy min-operator, *Fuzzy Sets and Systems,* Vol. 6, pp. 105–228, 1981.

[30] Y. Leung, *Spatial Analysis and Planning under Imprecision,* North-Holland, Amsterdam, 1988.

[31] D.G. Luenberger, *Linear and Nonlinear Programming,* Second Edition, Addison-Wesley, California, 1984.

[32] M.K. Luhandjula, Compensatory operators in fuzzy linear programming with multiple objectives, *Fuzzy Sets and Systems,* Vol. 8, pp. 245–252, 1982.

[33] M.K. Luhandjula, Fuzzy approaches for multiple objective linear fractional optimization, *Fuzzy Sets and Systems,* Vol. 13, pp. 11-23, 1984.

[34] M.K. Luhandjula, Fuzzy optimization: an appraisal, *Fuzzy Sets and Systems,* Vol. 30, pp. 257–282, 1989.

[35] J.G. March and H.A. Simon, *Organizations,* Wiley, New York, 1958.

[36] S.A. Orlovski, Multiobjective programming problems with fuzzy parameters, *Control and Cybernetics,* Vol. 13, pp. 175–183, 1984.

[37] R. Ostermark, Profit apportionment in concerns with mutual owenership — an application of fuzzy inequalities, *Fuzzy Sets and Systems,* Vol. 26, pp. 283–297, 1988.

[38] R. Ostermark, Fuzzy linear constraints in the capital asset pricing model, *Fuzzy Sets and Systems,* Vol. 30, pp. 93–102, 1989.

[39] W. Pedrycz (ed.), *Fuzzy Evolutionary Computation,* Kluwer Academic Publishers, Boston, 1997.

[40] J.B. Pickens and J.G. Hof, Fuzzy goal programming in forestry: an application with special solution problems, *Fuzzy Sets and Systems,* Vol. 39, pp. 239–246, 1991.

[41] H. Rommelfanger, Interactive decision making in fuzzy linear optimization problems, *European Journal of Operational Research,* Vol. 41, pp. 210–217, 1989.

[42] H. Rommelfanger, Fuzzy linear programming and applications, *European Journal of Operational Research,* Vol. 92, pp. 512–527, 1989.

[43] M. Sakawa, An interactive computer program for multiobjective decision making by the sequential proxy optimization technique, *European Journal of Operational Research,* Vol. 9, No. 4, pp. 386–396, 1982.

[44] M. Sakawa, Interactive multiobjective decision making by the sequential proxy optimization technique: SPOT, *International Journal of Man-Machine Studies,* Vol. 14, pp. 193–213, 1981.

[45] M. Sakawa, Interactive computer programs for fuzzy linear programming with multiple objectives, *International Journal of Man-Machine Studies,* Vol. 18, pp. 489–503, 1983.

[46] M. Sakawa, Interactive fuzzy decision making for multiobjective nonlinear programming problems, in M. Grauer and A. P. Wierzbicki (eds.) *Interactive Decision Analysis,* Springer-Verlag, Berlin, pp. 105–112, 1984.

[47] M. Sakawa, *Fuzzy Sets and Interactive Multiobjective Optimization,* Plenum Press, New York, 1993.

[48] M. Sakawa, *Large Scale Interactive Fuzzy Multiobjective Optimization,* Physica-Verlag, Heidelberg, 2000.

[49] M. Sakawa, *Genetic Algorithms and Fuzzy Multiobjective Optimization,* Kluwer Academic Publishers, Boston, 2001.

[50] M. Sakawa, K. Kato, and T. Mori, Flexible scheduling in a machining center through genetic algorithms, *Computers & Industrial Engineering: An International Journal,* Vol. 30, No. 4, pp. 931–940, 1996.

[51] M. Sakawa, K. Kato, H. Sunada and T. Shibano, Fuzzy programming for multiobjective 0-1 programming problems through genetic algorithms, *European Journal of Operational Research,* Vol. 97, No. 1, pp. 149–158, 1997.

[52] M. Sakawa and R. Kubota, Fuzzy programming for multiobjective job shop scheduling with fuzzy processing time and fuzzy duedate through genetic algorithms, *European Journal of Operational Research,* Vol. 120, pp. 393–407, 2000.

[53] M. Sakawa, H. Narazaki, M. Konishi, K. Nose, and T. Morita, A fuzzy satisficing approach to multiobjective pass scheduling for hot tandem mills, in Y. Sawaragi, K. Inoue, and H. Nakayama (eds.), *Toward Interactive and Intelligent Decision Support Systems,* Vol. 1, Proceedings, Kyoto, Japan Springer-Verlag, pp. 363–373, 1987.

[54] M. Sakawa and I. Nishizaki, Interactive fuzzy programming for two-level linear fractional programming problems, *Fuzzy Sets and Systems,* Vol. 119, pp. 31–40, 2001.

[55] M. Sakawa and I. Nishizaki, Interactive fuzzy programming for cooperate two-level linear fractional programming problems with multiple decision makers, *International Journal of Fuzzy Systems,* Vol. 1, pp. 48–59, 1999.

[56] M. Sakawa and I. Nishizaki, Interactive fuzzy programming for multi-level nonconvex nonlinear problems through genetic algorithms, in Y. Yoshida (ed.) *Dynamical Aspects in Fuzzy Decision Making,* Physica-Verlag, Heidelberg, pp. 99–116, 2001.

[57] M. Sakawa and I. Nishizaki, Interactive fuzzy programming for decentralized two-level linear programming problems, *Fuzzy Sets and Systems,* Vol. 125, pp. 301–315, 2002.

[58] M. Sakawa and I. Nishizaki, Interactive fuzzy programming for two-level nonconvex programming problems with fuzzy parameters through genetic algorithms, *Fuzzy Sets and Systems* (in press).

[59] M. Sakawa, I. Nishizaki and M. Hitaka, Interactive fuzzy programming for multi-level 0-1 programming problems with fuzzy param-

eters through genetic algorithms, *Fuzzy Sets and Systems,* Vol. 117, pp. 95–112, 2001.

[60] M. Sakawa, I. Nishizaki and M. Hitaka, Interactive fuzzy programming for multi-level 0-1 programming problems through genetic algorithms, *European Journal of Operational Research,* Vol. 114, pp. 580–588, 1999.

[61] M. Sakawa, I. Nishizaki and Y. Oka, Interactive fuzzy programming for multiobjective two-level linear programming problems with partial information of preference, *International Journal of Fuzzy Systems,* Vol. 2, pp. 79–86, 2000.

[62] M. Sakawa, I. Nishizaki and Y. Uemura, Interactive fuzzy programming for multi-level linear programming problems, *Computers & Mathematics with Applications,* Vol. 36, pp. 71–86, 1998.

[63] M. Sakawa, I. Nishizaki and Y. Uemura, Interactive fuzzy programming for multi-level linear programming problems with fuzzy parameters, *Fuzzy Sets and Systems,* Vol. 109, pp. 3–19, 2000.

[64] M. Sakawa, I. Nishizaki and Y. Uemura, Interactive fuzzy programming for multi-level linear fractional programming problems with fuzzy parameters, *Fuzzy Sets and Systems,* Vol. 115, pp. 93–103, 2000.

[65] M. Sakawa, I. Nishizaki and Y. Uemura, Fuzzy programming and profit and cost allocation for a production and transportation problem, *European Journal of Operational Research,* Vol. 131, pp. 1–15, 2001.

[66] M. Sakawa, I. Nishizaki and Y. Uemura, A decentralized two-level transportation problem in a housing material manufacturer –Interactive fuzzy programming approach–, *European Journal of Operational Research* (in press).

[67] M. Sakawa and K. Sawada, An interactive fuzzy satisficing method for large-scale multiobjective linear programming problems with block angular structure, *European Journal of Operational Research,* Vol. 67, pp. 5–17, 1994.

[68] M. Sakawa and F. Seo, Interactive multiobjective decisionmaking for large-scale systems and its application to environmental systems, *IEEE Transactions Systems, Man and Cybernetics,* Vol. SMC-10, pp. 796–806, 1980.

[69] M. Sakawa and F. Seo, Interactive multiobjective decision making in environmental systems using sequential proxy optimization techniques (SPOT), *Automatica,* Vol. 18, pp. 155–165, 1982.

[70] M. Sakawa and T. Shibano, Interactive fuzzy programming for multiobjective 0-1 programming problems through genetic algorithms with double strings, in Da Ruan (ed.) *Fuzzy Logic Foundations and Industrial Applications,* Kluwer Academic Publishers, Boston, pp. 111–128, 1996.

[71] M. Sakawa and T. Shibano, Multiobjective fuzzy satisficing methods for 0-1 knapsack problems through genetic algorithms, in W. Pedrycz (ed.) *Fuzzy Evolutionary Computation,* Kluwer Academic Publishers, Boston, pp. 155-177, 1997.

[72] M. Sakawa and T. Shibano, An interactive fuzzy satisficing method for multiobjective 0-1 programming problems with fuzzy numbers through genetic algorithms with double strings, *European Journal of Operational Research,* Vol. 107, pp. 564–574, 1998.

[73] M. Sakawa and T. Shibano, An interactive approach to fuzzy multiobjective 0-1 programming problems using genetic algorithms, in M. Gen and Y. Tsujimura (eds.) *Evolutionary Computations and Intelligent Systems,* Gordon & Breach, Inc. (to appear).

[74] M. Sakawa and M. Tanaka, *Genetic Algorithms,* Asakura Publishing, 1995 (in Japanese).

[75] M. Sakawa and H. Yano, An interactive fuzzy satisficing method using augmented minimax problems and its application to environmental systems, *IEEE Transactions Systems, Man and Cybernetics,* Vol. SMC-15, No. 6, pp. 720–729, 1985.

[76] M. Sakawa and H. Yano, Interactive decision making for multiobjective linear fractional programming problems with fuzzy parameters, *Cybernetics and Systems*: *An International Journal,* Vol. 16, pp. 377–394, 1985.

[77] M. Sakawa and H. Yano, Interactive decision making for multiobjective linear problems with fuzzy parameters, in G. Fandel, M. Grauer, A. Kurzhanski and A. P. Wierzbicki (eds.) *Large-Scale Modeling and Interactive Decision Analysis,* pp. 88–96, 1986.

[78] M. Sakawa and H. Yano, An interactive fuzzy satisficing method for multiobjective linear programming problems with fuzzy parameters, *Large Scale Systems*: *Theory and Applications, Proceedings of the IFAC/IFORS Symposium,* 1986.

[79] M. Sakawa and H. Yano, An interactive fuzzy satisficing method for multiobjective nonlinear programming problems with fuzzy parameters, in R. Trappl (ed.) *Cybernetics and Systems '86,* D. Reidel Publishing Company, pp. 607–614, 1986.

[80] M.Sakawa and H. Yano, An interactive fuzzy satisfying method for multiobjective linear fractional programming problems, *Fuzzy Sets and Systems,* Vol. 28, pp. 129–144, 1988.

[81] M. Sakawa and H. Yano, Interactive decision making for multiobjective nonlinear programming problems with fuzzy parameters, *Fuzzy Sets and Systems,* Vol. 29, pp. 129–144, 1989.

[82] M. Sakawa and H. Yano, An interactive fuzzy satisficing method for multiobjective nonlinear programming problems with fuzzy parameters, *Fuzzy Sets and Systems,* Vol. 30, pp. 221–238, 1989.

[83] M. Sakawa and H. Yano, An interactive fuzzy satisficing method for generalized multiobjective linear programming problems with fuzzy parameters, *Fuzzy Sets and Systems,* Vol. 35, pp. 125–142, 1990.

[84] M. Sakawa and H. Yano, Trade-off rates in the hyperplane method for multiobjective optimization problems, *European Journal of Operational Research,* Vol. 44, pp. 105–118, 1990.

[85] M. Sakawa, H. Yano and T. Yumine, An interactive fuzzy satisficing method for multiobjective linear-programming problems and its application, *IEEE Transactions Systems, Man and Cybernetics,* Vol. SMC-17, No. 4, pp. 654–661, 1987.

[86] M. Sakawa and T. Yumine, Interactive fuzzy decision-making for multiobjective linear fractional programming problems, *Large Scale Systems,* Vol. 5, No. 2, pp. 105–114, 1983.

[87] M. Sakawa, T. Yumine and H. Yano, An interactive fuzzy satisficing method for multiobjective nonlinear programming problems, CP-84-18, *International Institute for Applied Systems Analysis,* Laxenburg, Austria, 1984.

[88] F. Seo and M. Sakawa, *Multiple Criteria Decision Analysis in Regional Planning*: *Concepts, Methods and Applications,* D. Reidel Publishing Company, Dordrecht, 1988.

[89] H.S. Shih, Y.J. Lai and E.S. Lee, Fuzzy approach for multi-level programming problems, *Computers and Operations Research,* Vol. 23, 73–91, 1996.

[90] K. Shimizu, Y. Ishizuka and J.F. Bard, *Nondifferentiable and Two-Level Mathematical Programming,* Kluwer Academic Publishers, Boston, 1997.

[91] I. Shiroumaru, M. Inuiguchi, and M. Sakawa, A fuzzy satisficing method for electric power plant coal purchase using genetic algorithms, *European Journal of Operational Research,* Vol. 126, pp. 218–230, 2000.

[92] M. Simaarn and J.B. Cruz, JR., On the Stackelberg strategy in nonzero-sum games, *Journal of Optimization Theory and Applications,* Vol. 11, pp. 533–555, 1973.

[93] R. Slowinski, A multicriteria fuzzy linear programming method for water supply system development planning, *Fuzzy Sets and Systems,* Vol. 19, pp. 217–237, 1986.

[94] R. Slowinski (ed.), *Fuzzy Sets in Decision Analysis, Operations Research and Statistics,* Kluwer Academic Publishers, Boston, 1998.

[95] R. Slowinski and J. Teghem (eds.), *Stochastic versus Fuzzy Approaches to Multiobjective Mathematical Programming Problems under Uncertainty,* Kluwer Academic Publishers, Dordrecht, 1990.

[96] R.E. Steuer, *Multiple Criteria Optimization: Theory, Computation, and Application,* John Wiley & Sons, New York, 1986.

[97] R.E. Steuer and E.U. Choo, An interactive weighted Tchebycheff procedure for multiple objective programming, *Mathematical Programming,* Vol. 26, pp. 326–344, 1983.

[98] G. Sommer and M.A. Pollatschek, A fuzzy programming approach to an air pollution regulation problem, in R. Trappl and G.J. Klir and L. Ricciardi (eds.), *Progress in Cybernetics and Systems Research,* Hemisphere, pp. 303–323, 1978.

[99] E.L. Ulungu and J. Teghem, Multi-objective combinatorial optimization problems: a survey, *Journal of Multicriteria Decision Analysis,* Vol. 3, pp. 83–104, 1994.

[100] J.-L. Verdegay, Application of fuzzy optimization in operational research, *Control and Cybernetics,* Vol. 13, pp. 229–239, 1984.

[101] J.-L. Verdegay and M. Delgado (eds.), *The Interface between Artificial Intelligence and Operations Research in Fuzzy Environment,* Verlag TÜV Rheinland, Köln, 1989.

[102] G. Wiedey and H.-J. Zimmermann, Media selection and fuzzy linear programming, *Journal of Operational Research Society,* Vol. 29, pp. 1071-1084, 1978.

[103] A.P. Wierzbicki, The use of reference objectives in multiobjective optimization, in G. Fandel and T. Gal (eds.) *Multiple Criteria Decision Making*: *Theory and Application,* Springer-Verlag, Berlin, pp. 468–486, 1980.

[104] A.P. Wierzbicki, A mathematical basis for satisficing decision making, *Mathematical Modeling,* Vol. 3, pp. 391–405, 1982.

[105] L.A. Zadeh, Fuzzy sets, *Information and Control,* Vol. 8, pp. 338–353, 1974.

[106] M. Zeleny, *Multiple Criteria Decision Making,* McGraw-Hill, New York, 1982.

[107] H.-J. Zimmermann, Description and optimization of fuzzy systems, *International Journal of General Systems,* Vol. 2, pp. 209–215, 1976.

[108] H.-J. Zimmermann, Fuzzy programming and linear programming with several objective functions, *Fuzzy Sets and Systems,* Vol. 1, pp. 45–55, 1978.

[109] H.-J. Zimmermann, Fuzzy mathematical programming, *Computers & Operations Research,* Vol. 10, pp. 291–298, 1983.

[110] H.-J. Zimmermann, *Fuzzy Sets, Decision-Making and Expert Systems,* Kluwer Academic Publishers, Boston, 1987.

[111] H.-J. Zimmermann, *Fuzzy Set Theory and Its Applications,* Kluwer Academic Publishers, Boston, 1985, Second Edition, 1991, Third edition, 1996.

Chapter 5

INTERACTIVE NONLINEAR MULTIOBJECTIVE PROCEDURES

Kaisa Miettinen
Department of Mathematical Information Technology
University of Jyväskylä
P.O. Box 35 (Agora), FIN-40351 Jyväskylä
Finland
miettine@mit.jyu.fi

Abstract An overview of the interactive methods for solving nonlinear multiple criteria decision making problems is given. In interactive methods, the decision maker progressively provides preference information so that the most satisfactory compromise can be found. The basic features of several methods are introduced and some theoretical results are provided. In addition, references to modifications and applications as well as to other methods are indicated.

Keywords: Multiple criteria decision making, Multiobjective optimization, Nonlinear optimization, Interactive methods.

1. Introduction

Nonlinear multiobjective optimization means multiple criteria decision making involving nonlinear functions of continuous decision variables. In these problems, the best possible compromise is to be found from an infinite number of alternatives represented by decision variables restricted by constraint functions.

Solving multiobjective optimization problems usually requires the participation of a human decision maker who is supposed to have better insight into the problem and to express preference relations between alternative solutions. The methods can be divided into four classes according to the role of the decision maker in the solution process. If the decision maker is not involved, we use methods where no articulation of

preference information is used, in other words, *no-preference methods.* If the decision maker expresses preference information after the solution process, we speak about *a posteriori methods* whereas *a priori methods* require articulation of preference information before the solution process. The most extensive method class is *interactive methods* where the decision maker specifies preference information progressively during the solution process. Here we concentrate on this last-mentioned class and introduce several examples of interactive methods.

Many real-world phenomena behave in a nonlinear way. Besides, linear problems can always be solved using methods created for nonlinear problems but not vice versa. For these reasons, we here devote ourselves to nonlinear problems. We assume that all the information involved is deterministic and that we have a single decision maker. Further information about the topics treated here can be found in [105].

2. Concepts

Let us begin by introducing several concepts and definitions. We study *multiobjective optimization problems* of the form

$$\begin{array}{ll} \text{minimize} & \{f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_k(\mathbf{x})\} \\ \text{subject to} & \mathbf{x} \in S \end{array} \tag{5.1}$$

involving k (≥ 2) *objective functions* $f_i : \mathbf{R}^n \to \mathbf{R}$ that we want to minimize simultaneously. The *decision (variable) vectors* $\mathbf{x}$ belong to the (nonempty) *feasible region* $S \subset \mathbf{R}^n$. The feasible region is formed by *constraint functions* but we do not fix them here.

We denote the image of the feasible region by $Z \subset \mathbf{R}^k$ and call it a *feasible objective region. Objective (function) values* form *objective vectors* $\mathbf{z} = \mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_k(\mathbf{x}))^T$. Note that if f_i is to be maximized, it is equivalent to minimize $-f_i$.

We call a multiobjective optimization problem *convex* if all the objective functions and the feasible region are convex. On the other hand, the problem is *nondifferentiable* if at least one of the objective or the constraint functions is nondifferentiable. (Here nondifferentiability means that the function is not necessarily continuously differentiable but that it is locally Lipschitz continuous.)

We assume that the objective functions are at least partly conflicting and possibly incommensurable. This means that it is not possible to find a single solution that would optimize all the objectives simultaneously. As the definition of optimality we employ Pareto optimality. A vector is Pareto optimal (or noninferior or efficient or nondominated) if none of its components can be improved without deterioration to at least one of the other components.

Definition 14 *A decision vector* $\mathbf{x}^* \in S$ *is (globally)* Pareto optimal *if there does not exist another vector* $\mathbf{x} \in S$ *such that* $f_i(\mathbf{x}) \leq f_i(\mathbf{x}^*)$ *for all* $i = 1, \ldots, k$ *and* $f_j(\mathbf{x}) < f_j(\mathbf{x}^*)$ *for at least one index* j.

An objective vector $\mathbf{z}^* \in Z$ *is Pareto optimal if there does not exist another vector* $\mathbf{z} \in Z$ *such that* $z_i \leq z_i^*$ *for all* $i = 1, \ldots, k$ *and* $z_j < z_j^*$ *for at least one index* j*; or equivalently,* $\mathbf{z}^*$ *is Pareto optimal if the decision vector corresponding to it is Pareto optimal.*

Local Pareto optimality is defined in a small environment of $\mathbf{x}^* \in S$. Naturally, any globally Pareto optimal solution is locally Pareto optimal. The converse is valid, for example, for convex multiobjective optimization problems; see [20, 105], among others.

For the sake of brevity, we usually speak about Pareto optimality in the sequel. In practice, however, we only have locally Pareto optimal solutions computationally available, unless some additional requirement, such as convexity, is fulfilled or unless we have global solvers available.

A *Pareto optimal set* consists of (an infinite number of) Pareto optimal solutions. In interactive methods, we usually move around the Pareto optimal set and forget the other solutions. However, one should remember that this limitation may have weaknesses. Namely, the real Pareto optimal set may remain unknown. This may be the case if an objective function is only an approximation of an unknown function or if not all the objective functions involved are explicitly expressed.

Moving from one Pareto optimal solution to another necessitates trading off. To be more specific, a *trade-off* reflects the ratio of change in the values of the objective functions concerning the increment of one objective function that occurs when the value of some other objective function decreases (see, for example, [22, 105]).

For any two solutions equally preferable to the decision maker there is a trade-off involving a certain increment in the value of one objective function that the decision maker is willing to tolerate in exchange for a certain amount of decrement in some other objective function while the preferences of the two solutions remain the same. This is called the *marginal rate of substitution.*

Usually, one of the objective functions is selected as a *reference function* when trade-offs and marginal rates of substitution are treated. The trade-offs and the marginal rates of substitution are generated with respect to it.

Sometimes Pareto optimal sets are not enough but we need wider or smaller sets: weakly and properly Pareto optimal sets, respectively. A vector is *weakly Pareto optimal* if there does not exist any other vector for which all the components are better. Weakly Pareto optimal solutions

are sometimes computationally easier to generate than Pareto optimal solutions. Thus, they have relevance from a technical point of view. On the other hand, a vector is *properly Pareto optimal* if unbounded trade-offs are not allowed. For a collection of different definitions of proper Pareto optimality, see, for example, [105].

Multiobjective optimization problems are usually solved by *scalarization.* It means that the problem is converted into one or a family of single (scalar) objective optimization problems. This produces a new problem with a real-valued objective function, possibly depending on some parameters.

Interactive methods differ from each other by the form how the problem is transformed into a single objective optimization problem, by the form in which information is provided by the decision maker and by the form in which information is given to the decision maker.

One way of inquiring the decision maker's opinions is to ask for satisfactory or desirable objective function values. They are called *aspiration levels* and denoted by $\bar{z}_i$, $i = 1, \ldots, k$. They form a vector $\bar{\mathbf{z}} \in \mathbf{R}^k$ to be called a *reference point.*

The ranges of the set of Pareto optimal solutions give valuable information to the decision maker about the possibilities and restrictions of the problem (assuming the objective functions are bounded over S). The components of the *ideal objective vector* $\mathbf{z}^\star \in \mathbf{R}^k$ are the individual optima of the objective functions. This vector represents the lower bounds of the Pareto optimal set. (In nonconvex problems, we need a global solver for minimizing the k functions.) Note that we sometimes need a vector that its strictly better than the ideal objective vector. This vector is called a *utopian objective vector* and denoted by $\mathbf{z}^{\star\star}$.

The upper bounds of the Pareto optimal set, that is, the components of a *nadir objective vector* $\mathbf{z}^{\text{nad}}$, are much more difficult to obtain. Actually, there is no constructive method for calculating the nadir objective vector for nonlinear problems. However, a rough estimate can be obtained by keeping in mind the points where each objective function attains its lowest value and calculating the values of the other objectives. The highest value obtained for each objective can be selected as the estimated component of $\mathbf{z}^{\text{nad}}$.

It is often assumed that the decision maker makes decisions on the basis of an underlying *value function* $U : \mathbf{R}^k \to \mathbf{R}$ representing her or his preferences among the objective vectors [74]. Even though value functions are seldom explicitly known, they are important in the development of solution methods and as a theoretical background. Thus, the value function is often presumed to be known implicitly.

The value function is usually assumed to be *strongly decreasing.* In other words, the preferences of the decision maker are assumed to increase if the value of one objective function decreases while all the other objective values remain unchanged. In brief, we can say that *less is preferred to more.* In that case, the maximal solution of U is assured to be Pareto optimal. Note that regardless of the existence of a value function, in what follows, we shall assume that lower objective function values are preferred to higher, that is, less is preferred to more by the decision maker.

An alternative to the idea of maximizing some value function is *satisficing decision making.* In this approach, the decision maker tries to achieve certain aspirations. If the aspirations are achieved, the solution is called a *satisficing solution.*

3. Methods

A large variety of methods has been developed for solving multiobjective optimization problems. We can say that none of them is generally superior to all the others. As mentioned earlier, we apply here the classification of the methods into four classes according to the participation of the decision maker in the solution process. This classification has originally been suggested in [64].

Here we discuss interactive methods. We divide these methods into ad hoc and non ad hoc methods (based on value functions) as suggested in [180]. Even if one knew the decision maker's value function, one would not exactly know how to respond to the questions posed by an *ad hoc* algorithm. On the other hand, in *non ad hoc* methods, the responses can be determined or at least confidently simulated based on a value function.

Before describing the methods, we mention several references for further information. This presentation is mainly based on [105]. Concepts and methods for multiobjective optimization are also treated in [15, 22, 39, 40, 64, 157, 165, 175, 179, 184, 196, 200, 225].

Interactive multiobjective optimization methods, in particular, are collected in [140, 166, 197, 210]. Furthermore, methods with applications to large-scale systems and industry are presented in [58, 173, 188].

We shall not discuss non-interactive methods here. However, we mention some of such methods by name and give references for further information. Examples of no-preference methods are the method of the global criterion [224, 227] and the multiobjective proximal bundle method [108]. From among a posteriori methods we mention the weighting method [49, 226], the ε-constraint method [57] and the hybrid method [29, 209]

as well as the method of weighted metrics [227] and the achievement scalarizing function approach [212, 213, 214, 216]. Multiobjective evolutionary algorithms are also a posteriori in nature, see, for example, [34] and references therein. A priori methods include the value function method [74], the lexicographic ordering [45] and the goal programming [23, 24, 66, 155, 156].

Let us next concentrate on interactive methods. In interactive methods, a solution pattern is formed and repeated several times. After every iteration, some information is given to the decision maker and (s)he is asked to answer some questions or to provide some other type of information. In this way, only part of the Pareto optimal points has to be generated and evaluated, and the decision maker can specify and correct her or his preferences and selections during the solution process when (s)he gets to know the problem better. Thus, the decision maker does not have to know any global preference structure.

There are three main stopping criteria in interactive methods. In the best situation, the decision maker finds a desirable solution and wants to stop. Alternatively, the decision maker gets tired and stops or some algorithmic stopping rule is fulfilled. In the last-mentioned case, one must check that the decision maker agrees to stop.

In what follows, we present several interactive methods. The idea is to describe a collection of methods based on different approaches. In addition, plenty of references are included. Note that although all the calculations take place in the decision variable space, we mostly speak about the corresponding objective vectors.

3.1. Interactive Surrogate Worth Trade-Off Method

The interactive surrogate worth trade-off (ISWT) method is introduced in [21] and [22], pp. 371–379. The ISWT method utilizes the *ε-constraint problem* where one of the objective functions is minimized subject to upper bounds on all the other objectives:

$$\begin{array}{ll} \text{minimize} & f_\ell(\mathbf{x}) \\ \text{subject to} & f_j(\mathbf{x}) \leq \varepsilon_j \quad \text{for all } j = 1, \ldots, k, \ j \neq \ell, \\ & \mathbf{x} \in S, \end{array} \tag{5.2}$$

where $\ell \in \{1, \ldots, k\}$ and ε_j are upper bounds for the other objectives.

Theorem 1 *The solution of (5.2) is weakly Pareto optimal. The point $\mathbf{x}^* \in S$ is Pareto optimal if and only if it solves (5.2) for every $\ell = 1, \ldots, k$, where $\varepsilon_j = f_j(\mathbf{x}^*)$ for $j = 1, \ldots, k$, $j \neq \ell$. A unique solution is Pareto optimal for any upper bounds.*

The idea of the ISWT method is to maximize an approximation of an underlying value function. A search direction is determined based on the opinions of the decision maker concerning trade-off rates at the current solution point. The step-size to be taken in the search direction is determined by solving several ε-constraint problems and asking the decision maker to select the most satisfactory solution.

It is assumed that the underlying value function exists and is implicitly known to the decision maker. In addition, it must be continuously differentiable and strongly decreasing. Furthermore, the objective and the constraint functions must be twice continuously differentiable and the feasible region has to be compact. Finally, it is assumed that the Pareto optimality of the solutions of the ε-constraint problem is guaranteed and that trade-off rate information is available in the Karush-Kuhn-Tucker (KKT) multipliers related to the ε-constraint problem.

Changes in objective function values between a reference function f_ℓ and all the other objectives are compared. For each $i = 1, \ldots, k$, $i \neq \ell$, the decision maker must answer the following question: Let an objective vector $\mathbf{z}^h$ be given. If the value of f_ℓ is decreased by λ_i^h units, then the value of f_i is increased by one unit (or vice versa) and the other objective values remain unaltered. How desirable do you find this trade-off?

The response of the decision maker indicating the degree of preference is called a *surrogate worth* value. According to [21, 22] the response must be an integer between 10 and –10 whereas it is suggested in [192] to use integers from 2 to –2.

The gradient of the underlying value function is then estimated with the help of the surrogate worth values. This gives a search direction with a steepest ascent for the value function. Several different steps are taken in the search direction and the decision maker must select the most satisfactory of them. In practice, the upper bounds of the ε-constraint problem are revised based on surrogate worth values with different step-sizes.

The main features of the ISWT method can be presented with four steps.

1 Select f_ℓ to be minimized and give upper bounds to the other objective functions. Set $h = 1$.

2 Solve (5.2) to get a solution $\mathbf{z}^h$. Trade-off rate information is obtained from the KKT multipliers.

3 Ask the decision maker for the surrogate worth values at $\mathbf{z}^h$.

4 If some stopping criterion is satisfied, stop. Otherwise, update the upper bounds with the help of the answers obtained in step 3 and

solve several ε-constraint problems. Let the decision maker choose the most preferred alternative $\mathbf{z}^{h+1}$ and set $h = h + 1$. Go to step 3.

As far as stopping criteria are concerned, one can always stop when the decision maker wants to do so. A common stopping criterion is the situation where all the surrogate worth values equal zero. One more criterion is the case when the decision maker wants to proceed only in an infeasible direction.

In the ISWT method, the decision maker is asked to specify surrogate worth values and compare Pareto optimal alternatives. It may be difficult for the decision maker to provide consistent surrogate worth values throughout the decision process. In addition, if there is a large number of objective functions, the decision maker has to specify a lot of surrogate worth values at each iteration. On the other hand, the easiness of the comparison of alternatives depends on the number of objectives and on the personal abilities of the decision maker.

The ISWT method can be regarded as a non ad hoc method. The sign of the surrogate worth values can be judged by comparing trade-off rates with marginal rates of substitution (obtainable from the value function). Furthermore, when comparing alternatives, it is easy to select the one with the highest value function value.

Modifications of the ISWT method are presented in [22, 25, 56, 59].

3.2. Geoffrion-Dyer-Feinberg Method

In the Geoffrion-Dyer-Feinberg (GDF) method, proposed in [50], the basic idea is related to that of the ISWT method. In both the methods, the underlying (implicitly known) value function is approximated and maximized. In the GDF method, the approximation is based on marginal rates of substitution.

It is assumed that an underlying value function exists, is implicitly known to the decision maker and is strongly decreasing with respect to the reference function f_ℓ. In addition, the corresponding value function with decision variables as variables must be continuously differentiable and concave on S. Furthermore, the objective functions have to be continuously differentiable and the feasible region S must be compact and convex.

Let $\mathbf{x}^h$ be the current solution. We can obtain a local linear approximation for the gradient of the value function with the help of marginal rates of substitution m_i^h involving a reference function f_ℓ and the other

functions f_i. Based on this information we solve the problem

$$\begin{aligned} &\text{maximize} && \left(\sum_{i=1}^{k} -m_i^h \nabla_x f_i(\mathbf{x}^h)\right)^T \mathbf{y} \\ &\text{subject to} && \mathbf{y} \in S, \end{aligned} \tag{5.3}$$

where $\mathbf{y} \in \mathbf{R}^n$ is the variable. Let us denote the solution by $\mathbf{y}^h$. Then, the search direction is $\mathbf{d}^h = \mathbf{y}^h - \mathbf{x}^h$.

The following problem is to find a step-size. The decision maker can be offered objective vectors where steps of different sizes are taken in the search direction starting from the current solution. Unfortunately, these alternatives are not necessarily Pareto optimal.

Now we can present the GDF algorithm.

1 Ask the decision maker to select f_ℓ. Set $h = 1$.

2 Ask the decision maker to specify marginal rates of substitution between f_ℓ and the other objectives at the current solution $\mathbf{z}^h$.

3 Solve (5.3). Set the search direction $\mathbf{d}^h$. If $\mathbf{d}^h = \mathbf{0}$, stop.

4 Determine with the help of the decision maker the appropriate step-size t^h to be taken in direction $\mathbf{d}^h$. Denote the corresponding solution by $\mathbf{z}^{h+1} = \mathbf{f}(\mathbf{x}^h + t^h \mathbf{d}^h)$.

5 Set $h = h + 1$. If the decision maker wants to continue, go to step 2. Otherwise, stop.

In the GDF method, the decision maker has to specify marginal rates of substitution and select the most preferred solution from a set of alternatives. The theoretical foundation of the method is convincing but the practical side is not as promising. At each iteration the decision maker has to determine $k - 1$ marginal rates of substitution in a consistent and correct way. On the other hand, it is obvious that in practice the task of selection becomes more difficult for the decision maker as the number of objective functions increases. Another drawback is that not all the solutions presented to the decision maker are necessarily Pareto optimal. They can naturally be projected onto the Pareto optimal set but this necessitates extra effort.

The GDF method is a non ad hoc method. The marginal rates of substitution and selections can be done with the help of value function information. Note that if the underlying value function is linear, the marginal rates of substitution are constant and only one iteration is needed.

Applications and modifications of the GDF method are described in [3, 36, 38, 44, 46, 61, 64, 69, 104, 106, 107, 124, 147, 158, 162, 175, 219].

3.3. Tchebycheff Method

The Tchebycheff method, proposed in [175, pp. 419–450] and [178] and refined in [176], is also known by the name *interactive weighted Tchebycheff procedure.* The idea in this weighting space reduction method is to develop a sequence of progressively smaller subsets of the Pareto optimal set until a final solution is located.

This method does not have too many assumptions. All that is assumed is that the objective functions are bounded (from below) over S. To start with, a (global) utopian objective vector $\mathbf{z}^{\star\star}$ is established. Then the distance from the utopian objective vector to the feasible objective region is minimized by solving the problem

$$\begin{aligned} &\text{lex minimize} && \max_{i=1,\ldots,k} \left[w_i^h (f_i(\mathbf{x}) - z_i^{\star\star}) \right], \sum_{i=1}^{k} (f_i(\mathbf{x}) - z_i^{\star\star}) \\ &\text{subject to} && \mathbf{x} \in S. \end{aligned} \tag{5.4}$$

The notation above means that if the min-max problem does not have a unique solution, the sum term is minimized subject to the obtained points.

Theorem 2 *The solution of (5.4) is Pareto optimal and any Pareto optimal solution can be found.*

In the Tchebycheff method, different Pareto optimal solutions are generated by altering the weighting vector $\mathbf{w}^h$. At each iteration h, the weighting vector space $W^h = \{\mathbf{w}^h \in \mathbf{R}^k \mid l_i^h < w_i^h < u_i^h, \ \sum_{i=1}^k w_i^h = 1\}$ is reduced to W^{h+1}, where $W^{h+1} \subset W^h$. At the first iteration, a sample of the whole Pareto optimal set is generated by solving (5.4) with well dispersed weighting vectors from $W = W^1$ (with $l_i^1 = 0$ and $u_i^1 = 1$). The space W^h is reduced by tightening the upper and the lower bounds for the weights.

Let $\mathbf{z}^h$ be the objective vector that the decision maker chooses from the sample at the iteration h and let $\mathbf{w}^h$ be the corresponding weighting vector in the problem. Now a concentrated group of weighting vectors centred around $\mathbf{w}^h$ is formed. In this way, a sample of Pareto optimal solutions centred about $\mathbf{z}^h$ is obtained.

We can now present the main features of the Tchebycheff algorithm.

1 Set the set size P and an approximation for the number of iterations H. Set $l_i^1 = 0$ and $u_i^1 = 1$ for all $i = 1, \ldots, k$. Construct $\mathbf{z}^{\star\star}$. Set $h = 1$.

2 Form the weighting vector space W^h and generate $2P$ dispersed weighting vectors $\mathbf{w}^h \in W^h$.

3 Solve (5.4) for each of the $2P$ weighting vectors.

4 Present the P most different of the resulting objective vectors to the decision maker and let her or him choose the most preferred among them.

5 If $h = H$, stop.

6 Reduce W^h to get W^{h+1}, set $h = h + 1$ and go to step 2.

The problem (5.4) is solved more that P times so that solutions very close to each other do not have to be presented to the decision maker. On the other hand, the predetermined number of iterations is not necessarily conclusive. The decision maker can stop iterating when (s)he obtains a satisfactory solution or continue the solution process longer if necessary.

In this method, the decision maker is only asked to compare Pareto optimal objective vectors. The number of alternatives and criteria affects the easiness of the comparison. The personal capabilities of the decision maker are also important. Note that some consistency is required from the decision maker because the discarded parts of the weighting vector space cannot be restored.

It must be mentioned that a great deal of calculation is needed in the method. That is why it may not be applicable for large and complex problems. However, parallel computing can be utilized when generating the alternatives.

The Tchebycheff method is a non ad hoc method. It is easy to compare the alternative solutions with the help of a value function.

Applications and modifications of the Tchebycheff method are given in [1, 70, 146, 153, 167, 177, 186, 220].

3.4. Step Method

The step method (STEM) [9] is one of the first interactive methods developed for multiobjective optimization problems. Here we describe an extension for nonlinear problems according to [41] and [165, pp. 268–269].

STEM is based on the classification of the objective functions at the current iteration point $\mathbf{z}^h = \mathbf{f}(\mathbf{x}^h)$. In other words, it is assumed that the decision maker can indicate both functions that have acceptable values and those whose values are too high, that is, functions that are unacceptable. Then the decision maker is supposed to give up a little in the value(s) of some acceptable objective function(s) f_i $(i \in I^>)$ in order to improve the values of some unacceptable objective functions f_i $(i \in I^<)$ $(I^> \cup I^< = \{1, \ldots, k\})$. In other words, the decision maker is asked to specify upper bounds $\varepsilon_i^h > f_i(\mathbf{x}^h)$ for the functions in $I^>$.

The only requirement in the method is that the objective functions are bounded over S because distances are measured to the (global) ideal objective vector. The first problem to be solved is

$$\begin{aligned} &\text{minimize} && \max_{i=1,\ldots,k} \left[\frac{e_i}{\sum_{j=1}^{k} e_j} (f_i(\mathbf{x}) - z_i^\star) \right] \\ &\text{subject to} && \mathbf{x} \in S, \end{aligned} \tag{5.5}$$

where $e_i = \frac{1}{z_i^\star} \frac{z_i^{\text{nad}} - z_i^\star}{z_i^{\text{nad}}}$ as suggested in [41], or $e_i = \frac{z_i^{\text{nad}} - z_i^\star}{\max \left[|z_i^{\text{nad}}|, |z_i^\star| \right]}$ as suggested in [197].

Theorem 3 *The solution of (5.5) is weakly Pareto optimal. The problem has at least one Pareto optimal solution.*

After the decision maker has classified the objective functions, the feasible region is restricted according to the information of the decision maker. The weights of the relaxed objective functions are set equal to zero, that is $e_i = 0$ for $i \in I^>$. Then a new distance minimization problem

$$\begin{aligned} &\text{minimize} && \max_{i=1,\ldots,k} \left[\frac{e_i}{\sum_{j=1}^{k} e_j} (f_i(\mathbf{x}) - z_i^\star) \right] \\ &\text{subject to} && f_i(\mathbf{x}) \leq \varepsilon_i^h \quad \text{for all} \quad i \in I^>, \\ & && f_i(\mathbf{x}) \leq f_i(\mathbf{x}^h) \quad \text{for all} \quad i \in I^<, \\ & && \mathbf{x} \in S \end{aligned} \tag{5.6}$$

is solved.

The basic phases of the STEM algorithm are the following:

1 Calculate $\mathbf{z}^\star$ and $\mathbf{z}^{\text{nad}}$ and the weighting coefficients. Set $h = 1$. Solve (5.5). Denote the solution by $\mathbf{z}^h \in Z$.

2 Ask the decision maker to classify the objective functions at $\mathbf{z}^h$ into $I^>$ and $I^<$. If the latter class is empty, stop. Otherwise, ask the decision maker to specify relaxed upper bounds ε_i^h for $i \in I^>$.

3 Solve (5.6) and denote the solution by $\mathbf{z}^{h+1} \in Z$. Set $h = h + 1$ and go to step 2.

The procedure continues until the decision maker does not want to change any component of the current objective vector. If the decision maker is not satisfied with any of the components, then the procedure must also be stopped.

In STEM, we are moving from one weakly Pareto optimal solution to another. The idea of classification is quite simple for the decision maker. However, it may be difficult to estimate appropriate amounts of increment that would allow the desired amount of improvement in those functions whose values should be decreased.

STEM is an ad hoc method because the existence of a value function would not help in the classification process.

Applications and modifications of STEM are given in [6, 22, 33, 64].

3.5. Reference Point Method

The reference point method [211, 212, 214] is based on vectors formed of reasonable or desirable aspiration levels. These reference points are used to derive scalarizing functions having minimal solutions at weakly, properly or Pareto optimal points.

No specific assumptions are set in this method. The idea is to direct the search by changing the reference point $\bar{\mathbf{z}}^h$ in the spirit of satisficing decision making rather than optimizing any value function. It is important that reference points are intuitive and easy for the decision maker to specify and their consistency is not an essential requirement.

Note that specifying a reference point can be considered a way of classifying the objective functions. If the aspiration level is lower than the current objective value, that objective function is currently unacceptable, and if the aspiration level is equal to or higher than the current objective value, that function is acceptable. The difference here is that the reference point can be infeasible in every component. Naturally, trading off is unavoidable in moving from one Pareto optimal solution to another but different solutions can be obtained with different approaches.

Scalarizing functions used in the reference point method are so-called achievement (scalarizing) functions and the method relies on their properties. We can define so-called order-representing and order-approximating achievement functions (see [211, 212, 214] for definitions). An example of a problem with an order-representing achievement function is

$$\begin{aligned} &\text{minimize} && \max_{i=1,\ldots,k} [\, w_i (f_i(\mathbf{x}) - \bar{z}_i^h) \,] \\ &\text{subject to} && \mathbf{x} \in S, \end{aligned} \tag{5.7}$$

where $\mathbf{w}$ is some fixed positive weighting vector. An example of a problem with an order-approximating achievement function is

$$\begin{aligned} &\text{minimize} && \max_{i=1,\ldots,k} [\, w_i(f_i(\mathbf{x}) - \bar{z}_i^h)\,] + \rho \sum_{i=1}^{k} w_i(f_i(\mathbf{x}) - \bar{z}_i^h) \\ &\text{subject to} && \mathbf{x} \in S, \end{aligned} \tag{5.8}$$

where $\mathbf{w}$ is as above and $\rho > 0$.

Theorem 4 *If the achievement function is order-representing, then its solution is weakly Pareto optimal. If the function is order-approximating, then its solution is Pareto optimal and the solution is properly Pareto optimal if the function is also strongly increasing. Any (weakly) Pareto optimal solution can be found if the achievement function is order-representing. Finally, any properly Pareto optimal solution can be found if the function is order-approximating.*

The reference point technique of Wierzbicki is very simple. Before the solution process starts, some information is given to the decision maker about the problem. If possible, the ideal objective vector and the (approximated) nadir objective vector are presented. Another possibility is to minimize and maximize the objective functions individually in the feasible region (if it is bounded).

The basic steps are the following:

1 Select the achievement function. Present information about the problem to the decision maker. Set $h = 1$.

2 Ask the decision maker to specify a reference point $\bar{\mathbf{z}}^h \in \mathbf{R}^k$.

3 Minimize the achievement function and obtain a (weakly, properly or) Pareto optimal solution $\mathbf{z}^h$. Present it to the decision maker.

4 Calculate a number of k other (weakly, properly or) Pareto optimal solutions with perturbed reference points $\bar{\mathbf{z}}(i) = \bar{\mathbf{z}}^h + d^h\mathbf{e}^i$, where $d^h = \|\bar{\mathbf{z}}^h - \mathbf{z}^h\|$ and $\mathbf{e}^i$ is the ith unit vector for $i = 1, \ldots, k$.

5 Present the alternatives to the decision maker. If (s)he finds one of the $k + 1$ solutions satisfactory, stop. Otherwise, ask the decision maker to specify a new reference point $\bar{\mathbf{z}}^{h+1}$. Set $h = h + 1$and go to step 3.

The idea in perturbing the reference point in step 4 is that the decision maker gets a better conception of the possible solutions around the current solution. If the reference point is far from the Pareto optimal set, the decision maker gets a wider description of the Pareto optimal

set and if the reference point is near the Pareto optimal set, then a finer description of the Pareto optimal set is given.

In this method, the decision maker has to specify aspiration levels and compare objective vectors. The decision maker is free to change her or his mind during the process and can direct the solution process without being forced to understand complicated concepts and their meaning. On the other hand, the method does not necessarily help the decision maker to find improved solutions. What has been said about the comparison of alternatives in connection with the previous methods is naturally valid here.

The reference point method is an ad hoc method or a method having both non ad hoc and ad hoc features. On the one hand, a reference point cannot directly be defined based on a value function. On the other hand, alternatives are easy to compare when a value function is known.

Let us mention that a software family called DIDAS (Dynamic Interactive Decision Analysis and Support) has been developed on the basis of the reference point ideas of Wierzbicki. It is described, for example, in [218].

Applications and modifications of the reference point method are provided in [11, 55, 116, 143, 144, 168, 170, 172, 181, 195, 198, 199, 215, 217].

3.6. GUESS Method

The GUESS method is also called a *naïve method* [17]. The method is related to the reference point method.

It is assumed that the global vectors $\mathbf{z}^\star$ and $\mathbf{z}^{\text{nad}}$ are available. The structure of the method is very simple: the decision maker specifies a reference point (or a guess) $\bar{\mathbf{z}}^h$ and a solution is generated. Then the decision maker specifies a new reference point and so on.

The general idea is to maximize the minimum weighted deviation from the nadir objective vector. The problem to be solved is

$$\begin{array}{ll} \text{maximize} & \min\limits_{i=1,\ldots,k} \left[\dfrac{z_i^{\text{nad}} - f_i(\mathbf{x})}{z_i^{\text{nad}} - \bar{z}_i^h} \right] \\ \text{subject to} & \mathbf{x} \in S. \end{array} \tag{5.9}$$

Notice that the aspiration levels have to be strictly lower than the components of the nadir objective vector.

Theorem 5 *The solution of (5.9) is weakly Pareto optimal and any Pareto optimal solution can be found.*

The GUESS method has five basic steps.

1 Calculate $\mathbf{z}^\star$ and $\mathbf{z}^{\text{nad}}$ and present them to the decision maker. Set $h = 1$.

2 Let the decision maker specify upper or lower bounds to the objective functions if (s)he so desires. Update the problem, if necessary.

3 Ask the decision maker to specify a reference point $\bar{\mathbf{z}}^h$ between $\mathbf{z}^\star$ and $\mathbf{z}^{\text{nad}}$.

4 Solve (5.9) and present the solution to the decision maker.

5 If the decision maker is satisfied, stop. Otherwise, set $h = h + 1$ and go to step 2.

In step 2, upper or lower bounds mean adding constraints to the problem (5.9) but the ideal or the nadir objective vectors are not affected. The only stopping rule is the satisfaction of the decision maker. No guidance is given to the decision maker in setting new aspiration levels. This is typical of many reference point-based methods.

The GUESS method is simple to use and no consistency is required. The only information required from the decision maker is a reference point and possible upper and lower bounds. Note that inappropriate lower bounds may lead into solutions that are not weakly Pareto optimal. Unfortunately, the GUESS method relies heavily on the availability of the nadir objective vector, which is usually only an estimation.

The GUESS method is an ad hoc method. The existence of a value function would not help with reference points or bounds for the objective functions. The method has been compared to several other interactive methods in [16, 19, 31] and it has performed surprisingly well. The reasons may be its simplicity and flexibility. One can say that decision makers seem to prefer solution methods where they can feel that they are in control.

3.7. Satisficing Trade-Off Method

The satisficing trade-off method (STOM) [131, 136] utilizes classification and reference points. As its name suggests, STOM is based on satisficing decision making. The decision maker is asked to classify the objective functions at the current solution $\mathbf{z}^h = \mathbf{f}(\mathbf{x}^h)$ into three classes: the unacceptable objective functions whose values should be improved ($I^<$), the acceptable objective functions whose values may increase ($I^>$) and the acceptable objective functions whose values are acceptable as they are ($I^=$) (such that $I^< \cup I^> \cup I^= = \{1, \ldots, k\}$).

The decision maker only has to specify aspiration levels for the functions in $I^<$. The aspiration levels (that is, upper bounds) for the functions in $I^>$ can be derived using so-called automatic trade-off. In addition, the aspiration levels for the functions in $I^=$ are set equal to $f_i(\mathbf{x}^h)$. All the three kinds of aspiration levels form a reference point $\bar{\mathbf{z}}^h$.

Different scalarizing functions can be used in STOM. One alternative is to solve the problem

$$\begin{array}{ll} \text{minimize} & \max\limits_{i=1,\ldots,k} \left[\dfrac{f_i(\mathbf{x}) - z_i^{\star\star}}{\bar{z}_i^h - z_i^{\star\star}} \right] \\ \text{subject to} & \mathbf{x} \in S, \end{array} \tag{5.10}$$

where the reference point must be strictly worse than the utopian objective vector.

Theorem 6 *The solution of (5.10) is weakly Pareto optimal and any Pareto optimal solution can be found.*

If weakly Pareto optimal solutions are to be avoided, the problem to be solved is

$$\begin{array}{ll} \text{minimize} & \max\limits_{i=1,\ldots,k} \left[\dfrac{f_i(\mathbf{x}) - z_i^{\star\star}}{\bar{z}_i^h - z_i^{\star\star}} \right] + \rho \displaystyle\sum_{i=1}^{k} \dfrac{f_i(\mathbf{x})}{\bar{z}_i^h - z_i^{\star\star}} \\ \text{subject to} & \mathbf{x} \in S, \end{array} \tag{5.11}$$

where $\rho > 0$ is some sufficiently small scalar.

Theorem 7 *The solution of (5.11) is properly Pareto optimal and any properly Pareto optimal solution can be found.*

Here the utopian objective vector must be known globally. However, if some objective function f_j is not bounded from below on S, then some small scalar value can be used as $z_j^{\star\star}$.

Assuming all the functions involved are differentiable the scalarizing functions can be written in a differentiable form by introducing a scalar variable α to be optimized and setting it as an upper bound for each function in the max-term. Under certain assumptions, trade-off rate information can be obtained from the KKT multipliers connected to the solution of this formulation. In *automatic trade-off,* upper bounds for the functions in $I^>$ are derived with the help of this trade-off information.

Let us now describe the algorithm.

1 Select the scalarizing function. Calculate $\mathbf{z}^{\star\star}$. Set $h = 1$.

2 Ask the decision maker to specify a reference point $\bar{\mathbf{z}}^h \in \mathbf{R}^k$ such that $\bar{z}_i^h > z_i^{\star\star}$ for every $i = 1, \ldots, k$.

3 Minimize the scalarizing function used. Denote the solution by $\mathbf{z}^h$. Present it to the decision maker.

4 Ask the decision maker to classify the objective functions. If $I^{<} = \emptyset$, stop. Otherwise, ask the decision maker to specify new aspiration levels $\bar{z}_i^{h+1}$ for $I \in I^{<}$. Set $\bar{z}_i^{h+1} = z_i^h$ for $i \in I^{=}$.

5 Use automatic trade-off to obtain new levels (upper bounds) $\bar{z}_i^{h+1}$ for the functions in $I^{>}$. Set $h = h + 1$. and go to step 3.

The decision maker can modify the levels calculated based on trade-off rate information if they are not agreeable. On the other hand, the decision maker can specify those upper bounds herself or himself, if so desired. If trade-off rate information is not available, STOM is almost the same as the GUESS method. The only difference is the scalarizing function used.

There is no need to repeat comments mentioned in connection with STEM, the reference point method and the GUESS method. In all of them, the role of the decision maker is easy to understand. STOM requires even less input from the decision maker if automatic trade-off is used.

As said before, in practice, classifying the objective functions into three classes and specifying the amounts of increment and decrement for their values is a subset of specifying a new reference point. A new reference point is implicitly formed.

STOM is an ad hoc method as all the other classification-based methods. However, one must remember that the aim of the method is particularly in satisficing rather than optimizing some value function.

Modifications and applications of STOM are described in [115, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 146, 204].

3.8. Light Beam Search

The light beam search [67, 68] employs tools of multiattribute decision analysis (see, for example, [200]) together with reference point ideas. The basic setting is identical to the reference point method. The problem to be solved is

$$\begin{aligned} &\text{minimize} && \max_{i=1,\ldots,k} [\, w_i(f_i(\mathbf{x}) - \bar{z}_i^h)\,] + \rho \sum_{i=1}^{k} (f_i(\mathbf{x}) - \bar{z}_i^h) \\ &\text{subject to} && \mathbf{x} \in S, \end{aligned} \tag{5.12}$$

where $\mathbf{w}$ is a weighting vector, $\bar{\mathbf{z}}^h$ is the current reference point and $\rho > 0$.

Theorem 8 *The solution of (5.12) is properly Pareto optimal and any properly Pareto optimal solution can be found.*

The reference point is here assumed to be infeasible. It is also assumed that the objective and the constraint functions are continuously differentiable and that the objective functions are bounded over S. Furthermore, none of the objective functions is allowed to be more important than all the others together.

In the light beam search, the decision maker directs the search by specifying reference points. In addition, other solutions in the neighbourhood of the current solution are displayed. Thus, the idea is identical to that of the reference point method. The main difference is in the way the alternatives are generated. The motivation is to avoid comparing too similar alternatives or alternatives that are indifferent to the decision maker. To achieve this goal concepts of ELECTRE methods (in multiattribute decision analysis) are utilized (see, for example, [161]).

It is not always possible for the decision maker to distinguish between different alternatives. This means that there is an interval where indifference prevails. For this reason the decision maker is asked to provide *indifference thresholds* for each objective function. The line between indifference and preference does not have to be sharp, either. The hesitation between indifference and preference can be expressed by *preference thresholds.* Finally, a *veto threshold* prevents a good performance in some components from compensating for poor values on some other components.

In the light beam search, *outranking relations* are established between alternatives. A vector $\mathbf{z}^1$ is said to outrank $\mathbf{z}^2$ if $\mathbf{z}^1$ is at least as good as $\mathbf{z}^2$. The idea is to generate k new alternative objective vectors such that they outrank the current solution. In particular, incomparable or indifferent alternatives are not shown to the decision maker. The alternatives to be shown are called *characteristic neighbours.* The neighbours are determined by projecting the gradient of one objective function at a time onto the linear approximation of those constraints that are active in the current solution.

We can now outline the light beam algorithm.

1 If the decision maker can specify the best and the worst values for each objective function, denote them by $\mathbf{z}^\star$ and $\mathbf{z}^{\mathrm{nad}}$, respectively. Alternatively calculate $\mathbf{z}^\star$ and $\mathbf{z}^{\mathrm{nad}}$. Set $h = 1$ and $\bar{\mathbf{z}}^h = \mathbf{z}^\star$. Initialize the set of saved solutions as $B = \emptyset$. Ask the decision maker to specify an indifference threshold for each objective. If desired, (s)he can also specify preference and veto thresholds.

2 Calculate $\mathbf{z}^h$ by solving (5.12).

3 Present $\mathbf{z}^h$ to the decision maker. Calculate k Pareto optimal characteristic neighbours of $\mathbf{z}^h$ and present them as well to the

decision maker. If the decision maker wants to see alternatives between any two of the $k + 1$ alternatives displayed, set their difference as a search direction, take different steps in this direction and project them onto the Pareto optimal set before showing them to the decision maker. If the decision maker wants to save $\mathbf{z}^h$, set $B = B \cup \{\mathbf{z}^h\}$.

4 If the decision maker wants to revise the thresholds, save them, set $\mathbf{z}^h = \mathbf{z}^{h+1}$, $h = h + 1$ and go then to step 3. If the decision maker wants to give another reference point, denote it by $\bar{\mathbf{z}}^{h+1}$, set $h = h+1$ and go to step 2. If the decision maker wants to select one of the alternatives or one solution in B as a current solution, set it as $\mathbf{z}^{h+1}$, set $h = h + 1$ and go to step 3. If one of the alternatives is satisfactory, stop.

The option of saving desirable solutions in the set B increases the flexibility of the method. A similar option could be added to many other methods as well.

The name of the method comes from the idea of projecting a focused beam of light from the reference point onto the Pareto optimal set. The lighted part of the Pareto optimal set changes if the location of the spotlight, that is, the reference point or the point of interest in the Pareto optimal set are changed.

In the light beam search, the decision maker specifies reference points, compares alternatives and affects the set of alternatives in different ways. Specifying different thresholds may be demanding for the decision maker. Note, however, that the thresholds are not constant but can be altered at any time. The authors of the method point out that it may be computationally rather demanding to find the exact characteristic neighbours in a general case. It is noteworthy that the neighbours can be generated in parallel.

The light beam search is an ad hoc method because a value function could not directly determine new reference points. It could, however, be used in comparing alternatives. Remember that the thresholds are important here and they must come from the decision maker.

A modification of the method is described in [215].

3.9. Reference Direction Approach

The reference direction approach [81, 86] is also known by the name *visual interactive approach.* It contains ideas from, for example, the GDF method and the reference point method of Wierzbicki. However, more information is provided to the decision maker.

In reference point-based methods, a reference point is projected onto the Pareto optimal set by an achievement function. Here a whole so-called *reference direction* is projected onto the Pareto optimal set. It is a vector from the current iteration point $\mathbf{z}^h$ to the reference point $\bar{\mathbf{z}}^h$. In practice, steps of different sizes are taken along the reference direction and projected. The idea is to plot the objective function values on a computer screen as value paths. The decision maker can move the cursor back and forth and see the corresponding numerical values at each point.

The points along the reference direction are generated by solving the problem

$$\begin{aligned} &\text{minimize} && \max_{i \in I} \left[\frac{f_i(\mathbf{x}) - \bar{z}_i^h}{w_i} \right] \\ &\text{subject to} && \bar{\mathbf{z}}^h = \mathbf{z}^h + t\mathbf{d}^{h+1}, \\ & && \mathbf{x} \in S, \end{aligned} \tag{5.13}$$

where $I = \{i \mid w_i > 0\} \subset \{1, \ldots, k\}$ and t has different discrete nonnegative values. The weighting vector can be, for example, the reference point specified by the decision maker.

Theorem 9 *The solution of (5.13) is weakly Pareto optimal.*

The algorithm is as follows.

1 Find an arbitrary objective vector $\mathbf{z}^1$. Set $h = 1$.

2 Ask the decision maker to specify a reference point $\bar{\mathbf{z}}^h \in \mathbf{R}^k$ and set $\mathbf{d}^{h+1} = \bar{\mathbf{z}}^h - \mathbf{z}^h$.

3 Find the set Z^{h+1} of weakly Pareto optimal solutions with different values of t in (5.13).

4 Ask the decision maker to select the most preferred solution $\mathbf{z}^{h+1}$ in Z^{h+1}.

5 If $\mathbf{z}^h \neq \mathbf{z}^{h+1}$, set $h = h + 1$ and go to step 2. Otherwise, check the optimality conditions. If the conditions are satisfied, stop. Otherwise, set $h = h + 1$ and set $\mathbf{d}^{h+1}$ to be a search direction identified by the optimality checking procedure. Go to step 3.

Checking the optimality conditions in step 5 is the most complicated part of the algorithm. Thus far, no specific assumptions have been set on the value function. However, we can check the optimality of $\mathbf{z}^{h+1}$ if the cone containing all the feasible directions has a finite number of generators. We must then assume that an underlying value function

exists and is pseudoconcave on Z. In addition, S must be convex and compact and the constraint functions must be differentiable.

The role of the decision maker is similar in the reference point method and in the reference direction approach: specifying reference points and selecting the most preferred alternative. But by providing similar reference point information, in the reference direction approach, the decision maker can explore a wider part of the weakly Pareto optimal set. This possibility brings the task of comparing the alternatives.

The performance of the method depends greatly on how well the decision maker manages to specify the reference directions that lead to improved solutions. The consistency of the decision maker's answers is not important and it is not checked in the algorithm.

The reference direction approach can be characterized as an ad hoc method as the other reference point-based methods. The aim is to support the decision maker in getting to know the problem better.

A dynamic user interface to the reference direction approach and its adaptation to generalized goal programming is introduced in [88]. This method for linear multiobjective optimization problems is called the *Pareto race* and the software system implementing the Pareto race is called VIG (Visual Interactive Goal programming) [90, 91].

Applications and modifications of the reference direction approach are described in [10, 81, 82, 83, 84, 85, 87].

3.10. Reference Direction Method

The classification-based reference direction (RD) method [138, 139] is related to the reference direction approach. In the RD method, a current objective vector $\mathbf{z}^h$ is presented to the decision maker and (s)he is asked to specify a reference point $\bar{\mathbf{z}}^h$ consisting of desired levels for the objective functions. The idea is to move around the weakly Pareto optimal set, which is why some objective functions must be allowed to increase in order to attain lower values for some other objectives.

As mentioned earlier, specifying a reference point is equivalent to an implicit classification indicating those objective functions whose values should be decreased till they reach some acceptable aspiration level, those whose values are satisfactory at the moment, and those whose values are allowed to increase to some upper bound. We denote the three classes by $I^<$, $I^=$ and $I^>$, respectively. Furthermore, we denote the components of the reference point corresponding to $I^>$ by ε_i^h because we have upper bounds in question.

Here, as well as in the reference direction approach, steps are taken in the reference direction $\bar{\mathbf{z}}^h - \mathbf{z}^h$. However, now, the decision maker

specifies a priori the number of steps to be taken. The idea is to move step by step as long as the decision maker wants to. In this way, extra computation is avoided when only those alternatives are calculated that the decision maker wants to see.

Alternatives are produced by solving the problem

$$\begin{array}{ll} \text{minimize} & \max\limits_{i \in I^<} \left[\dfrac{f_i(\mathbf{x}) - z_i^h}{z_i^h - \bar{z}_i^h} \right] \\ \text{subject to} & f_i(\mathbf{x}) \leq \varepsilon_i^h + \alpha(z_i^h - \varepsilon_i^h) \quad \text{for all} \quad i \in I^>, \\ & f_i(\mathbf{x}) \leq z_i^h \quad \text{for all} \quad i \in I^=, \\ & \mathbf{x} \in S, \end{array} \tag{5.14}$$

where $0 \leq \alpha < 1$ is the step-size in the reference direction, $\bar{z}_i^h < z_i^h$ for $i \in I^<$ and $\varepsilon_i^h > z_i^h$ for $i \in I^>$.

Theorem 10 *The solution of (5.14) is weakly Pareto optimal for every* $0 \leq \alpha < 1$ *and any Pareto optimal solution can be found.*

The steps of the RD algorithm are the following:

1 Find a starting solution $\mathbf{z}^1$ and show it to the decision maker. Set $h = 1$.

2 If the decision maker does not want to decrease any component of $\mathbf{z}^h$, stop. Otherwise, ask the decision maker to specify $\bar{\mathbf{z}}^h$, where some of the components are lower and some higher or equal when compared to those of $\mathbf{z}^h$. If there are no higher values, set $P = r = 1$ and go to step 3. Otherwise, ask the decision maker to specify the maximum number of alternatives P (s)he wants to see. Set $r = 1$.

3 Set $\alpha = 1 - r/P$. Solve (5.14) and get $\mathbf{z}^h(r)$. Set $r = r + 1$.

4 Show $\mathbf{z}^h(r)$ to the decision maker. If (s)he is satisfied, stop. If $r \leq P$ and the decision maker wants to see another solution, go to step 3. Otherwise, if $r > P$ or the decision maker wants to change the reference point, set $\mathbf{z}^{h+1} = \mathbf{z}^h(r)$, $h = h + 1$ and go to step 2.

The RD method does not require artificial or complicated information from the decision maker; only reference points and the number of intermediate solutions are used. Some decision makers may appreciate the fact that they are not asked to compare several alternatives but only to decide whether another alternative is to be generated or not.

The decision maker must a priori determine the number of steps to be taken, and then intermediate solutions are calculated one by one as

long as the decision maker wants to. This has both positive and negative sides. On the one hand, it is computationally efficient since it may be unnecessary to calculate all the intermediate solutions. On the other hand, the number of steps to be taken cannot be changed.

The RD method is an ad hoc method because a value function would not help in specifying reference points or the numbers of steps to be taken. It could not even help in selecting the most preferred alternative. Here one must decide for one point at a time whether to calculate new alternatives or not. If the new alternative happens to be less preferred than its predecessor, one cannot return to the previous solution.

Applications and modifications of the RD method are described in [53, 111].

3.11. NIMBUS Method

The NIMBUS method is presented in [105, 108, 111]. The name NIMBUS comes from the words Nondifferentiable Interactive Multiobjective BUndle-based optimization System. As its name suggests, NIMBUS can handle even nondifferentiable problems.

In NIMBUS, it is assumed that the objective functions are bounded (from below) in S. In other words, we need a (global) ideal objective vector. If a nondifferentiable single objective solver is used, then we must also assume that the objective and the constraint functions are locally Lipschitz continuous.

NIMBUS offers flexible ways of performing interactive evaluation of the problem and determining the preferences of the decision maker during the solution process. Aspiration levels and classification are used as the means of interaction between the decision maker and the algorithm.

In the classification, the decision maker can easily indicate what kind of improvements are desirable and what kind of impairments are tolerable. The decision maker examines at every iteration h the current objective vector $\mathbf{z}^h$ and divides the objective functions into up to five classes. The classes are functions f_i whose values

- should be decreased $(i \in I^<)$,
- should be decreased to an aspiration level $\bar{z}_i^h < z_i^h$ $(i \in I^\leq)$,
- are satisfactory at the moment $(i \in I^=)$,
- are allowed to increase to a certain upper bound $\varepsilon_i^h > z_i^h$ $(i \in I^>)$, and
- are allowed to change freely $(i \in I^\diamond)$,

where $I^< \cup I^\leq \neq \emptyset$ and $I^= \cup I^> \cup I^\diamond \neq \emptyset$.

In addition to the classification, the decision maker is asked to specify the aspiration levels and the upper bounds. The difference between the classes $I^<$ and $I^\leq$ is that the functions in $I^<$ are to be minimized as far as possible but the functions in $I^\leq$ only as far as the aspiration level.

The decision maker can tune the order of importance inside the classes $I^<$ and $I^\leq$ with optional positive weighting coefficients w_i^h summing up to one. If the decision maker does not want to specify any weighting coefficients, they are set equal to one.

NIMBUS has more classes than STEM, STOM or the RD method. This means that the decision maker has more freedom in specifying the desired changes in the objective values. Note that not all of the classes have to be used. The existence of the class $I^\diamond$ means that some functions can be left unclassified.

After the classification, a problem

$$\begin{array}{ll} \text{minimize} & \max\limits_{\substack{i\in I^< \\ j\in I^\leq}} \left[w_i^h (f_i(\mathbf{x}) - z_i^\star), w_j^h \max\left[f_j(\mathbf{x}) - \bar{z}_j^h,\ 0\right]\right] \\ \text{subject to} & f_i(\mathbf{x}) \leq z_i^h \quad \text{for all} \quad i \in I^< \cup I^\leq \cup I^=, \\ & f_i(\mathbf{x}) \leq \varepsilon_i^h \quad \text{for all} \quad i \in I^>, \\ & \mathbf{x} \in S \end{array} \tag{5.15}$$

is solved, where $z_i^\star$ for $i \in I^<$ are components of the ideal objective vector.

Theorem 11 *The solution of (5.15) is weakly Pareto optimal if the set $I^<$ is nonempty and any Pareto optimal solution can be found.*

The solution of the problem is denoted by $\hat{\mathbf{x}}^h$. If the decision maker does not like the corresponding objective vector $\hat{\mathbf{z}}^h$, (s)he can explore intermediate solutions between $\mathbf{x}^h$ and $\hat{\mathbf{x}}^h$. This means that we calculate a search direction $\mathbf{d}^h = \hat{\mathbf{x}}^h - \mathbf{x}^h$ and provide more solutions by taking steps of different sizes in this direction. In other words, we generate $P-1$ new vectors $\mathbf{f}(\mathbf{x}^h + t_j \mathbf{d}^h)$, $j = 2, \ldots, P-1$, where $t_j = \frac{j-1}{P-1}$. Their Pareto optimal counterparts are presented to the decision maker, who then selects the most satisfying solution among the alternatives.

The NIMBUS algorithm is given below. The search procedure stops if the decision maker does not want to improve any objective function value.

1 Calculate $\mathbf{z}^\star$. Choose a Pareto optimal starting point $\mathbf{z}^1$. Set $h = 1$.

2 Ask the decision maker to classify the objective functions at $\mathbf{z}^h$ such that $I^= \cup I^> \cup I^\diamond \neq \emptyset$ and $I^< \cup I^\leq \neq \emptyset$. If either of the unions

is empty, stop. Otherwise, ask the decision maker for the aspiration levels and the upper bounds as well as the optional weights.

3 Calculate $\hat{\mathbf{z}}^h$ by solving (5.15). If $\hat{\mathbf{z}}^h = \mathbf{z}^h$, ask the decision maker whether (s)he wants to try another classification. If yes, set $\mathbf{z}^{h+1} = \mathbf{z}^h$, $h = h + 1$, and go to step 2; if no, stop.

4 Present $\mathbf{z}^h$ and $\hat{\mathbf{z}}^h$ to the decision maker. If the decision maker wants to see alternatives between them, calculate $\mathbf{d}^h$ and go to step 5. If the decision maker prefers $\mathbf{z}^h$, set $\mathbf{z}^{h+1} = \mathbf{z}^h$ and $h = h + 1$, and go to step 2. Otherwise, set $\mathbf{z}^{h+1} = \hat{\mathbf{z}}^h$, $h = h + 1$, and go to step 2.

5 Ask the decision maker to specify the desired number of intermediate alternatives P and calculate the vectors. Project them to be Pareto optimal.

6 Present the P alternatives to the decision maker and let her or him choose the most preferred one among them. Denote it by $\mathbf{z}^{h+1}$ and set $h = h + 1$. If the decision maker wants to continue, go to step 2. Otherwise, stop.

Since the Pareto optimality of the solutions produced is not guaranteed, we can check the final solution in the end by solving an additional problem [105]. Naturally, the decision maker may check Pareto optimality at any time during the solution process.

In NIMBUS, the decision maker is free to explore the (weakly) Pareto optimal set and also to change her or his mind if necessary. The selection of the most preferred alternative from a given set is also possible but not necessary. The decision maker can also extract undesirable solutions from further consideration. Unlike some other classification-based methods, NIMBUS does not depend entirely on how well the decision maker manages in the classification. It is important that the classification is not irreversible. If the solution obtained is not satisfactory, the decision maker can go back or explore intermediate points. The method aims at being flexible and the decision maker can select to what extent (s)he exploits the versatile possibilities available. The calculations are not too massive, either.

An implementation of NIMBUS is available on the Internet. This WWW-NIMBUS system is at the disposal of every Internet user at http://nimbus.mit.jyu.fi/. Positive sides of a WWW implementation are that the latest version of the system is always available and the user saves the trouble of installing the software. The operating system used or compilers available set no restrictions because all that is needed is a

WWW browser. Furthermore, WWW provides a graphical user interface with possibilities for visualizing the classification phase, alternative solutions etc. The system contains both a nondifferentiable local solver (proximal bundle method) (see [99], pp. 112–143) and a global solver (genetic algorithm). For details, see [110, 112]. (The first version of WWW-NIMBUS was implemented in 1995. Then, it was a pioneering interactive optimization system on the Internet.)

Being a classification-based method, NIMBUS is ad hoc in nature. A value function could only be used to compare different alternatives.

Applications and modifications of the NIMBUS method can be found in [104, 108, 109, 111, 113, 114].

3.12. Other Interactive Methods

The number of interactive methods developed for multiobjective optimization is large. So far, we have given several examples of them. Let us next mention references to some more methods. Methods based on goal programming are introduced in [73, 100, 101, 119, 120, 142, 171, 189, 206].

Methods based on weighted metrics are suggested in [35, 72, 79, 80, 96, 118, 122, 187, 227, 228, 229] whereas methods based on reference points can be found in [13, 32, 54, 60, 76, 97, 98, 103, 117, 141, 164, 190, 205, 207, 208]. Finally, methods based on miscellaneous ideas are described in [4, 5, 7, 8, 26, 27, 28, 42, 43, 47, 48, 63, 71, 75, 79, 93, 94, 95, 123, 137, 148, 149, 150, 151, 159, 160, 163, 169, 173, 174, 179, 182, 183, 185, 191, 202, 203, 221, 222, 223, 230, 231].

4. Comparing the Methods

None of the many multiobjective optimization methods can be claimed to be superior to the others in every aspect. One can say that selecting a multiobjective optimization method is a problem with multiple objectives itself. The properties of the problem and the capabilities and the desires of the decision maker have to be charted before a solution method can be chosen. Some methods may suit some problems and some decision makers better than some others.

Let us mention one property of the problem to be considered when selecting a method, that is, differentiability. Of the interactive methods described, the Tchebycheff method, STEM, the reference point method, the GUESS method, the RD method and NIMBUS can all be used to solve nondifferentiable problems assuming that a nondifferentiable single objective solver is available.

A decision tree is provided in [105] for easing the method selection. The tree is based on theoretical facts concerning the assumptions on the problem to be solved and the preferences of the decision maker. Further aspects to be taken into account when evaluating and selecting methods are collected, for example, in [12, 51, 62, 65, 105, 184, 193, 194].

In addition to theoretical properties, practical applicability also plays an important role in the selection of an appropriate method. The difficulty is that practical applicability is hard to determine without experience.

Some comparisons of the methods have been reported in the literature. They have been carried out with respect to a variety of criteria and under varied circumstances. Instead of a human decision maker one can sometimes employ value functions in the comparisons. Unfortunately, replacing the decision maker with a value function does not fully reflect the real usefulness of the methods. One of the problems is that value functions cannot really help in testing ad hoc methods.

Tests with human decision makers are described in [14, 16, 18, 19, 30, 31, 37, 89, 102, 145, 201] while tests with value functions are reported in [2, 52, 121, 152]. Finally, comparisons based on intuition are provided in [41, 77, 78, 92, 101, 103, 146, 154, 166, 197, 200].

5. Conclusions

We have outlined several interactive methods for solving nonlinear multiobjective optimization problems and indicated references to many more. One of the challenges in this area is spreading the word about the existing methods to those who solve real-world problems. Another challenge is to develop methods that support the decision maker even better. User-friendliness cannot be overestimated because interactive methods must be able to correspond to the characteristics of the decision maker. Specific methods for different areas of application that take into account the characteristics of the problems are also important.

An alternative to creating new methods is to use different methods in different phases of the solution process. This hybridization means that the positive features of various methods can be exploited to their best advantage in appropriate phases. In this way, it may also be possible to overcome some of the weaknesses of the methods.

The decision maker can be supported by using visual illustrations and further development of such tools is essential. For instance, one may visualize (parts of) the Pareto optimal set and, for example, use 3D slices of the feasible objective region (see [98], among others) and other tools. On the other hand, one can illustrate sets of alternatives by means of

bar charts, value paths, spider-web charts and petal diagrams etc. (see, for example, [105] and references therein), An example of illustrating alternatives is given in Figure 5.1 where five alternatives are depicted with the help of value paths, bar charts and petal diagrams. The figure has been generated with the WWW-NIMBUS system described in Section 3.11.

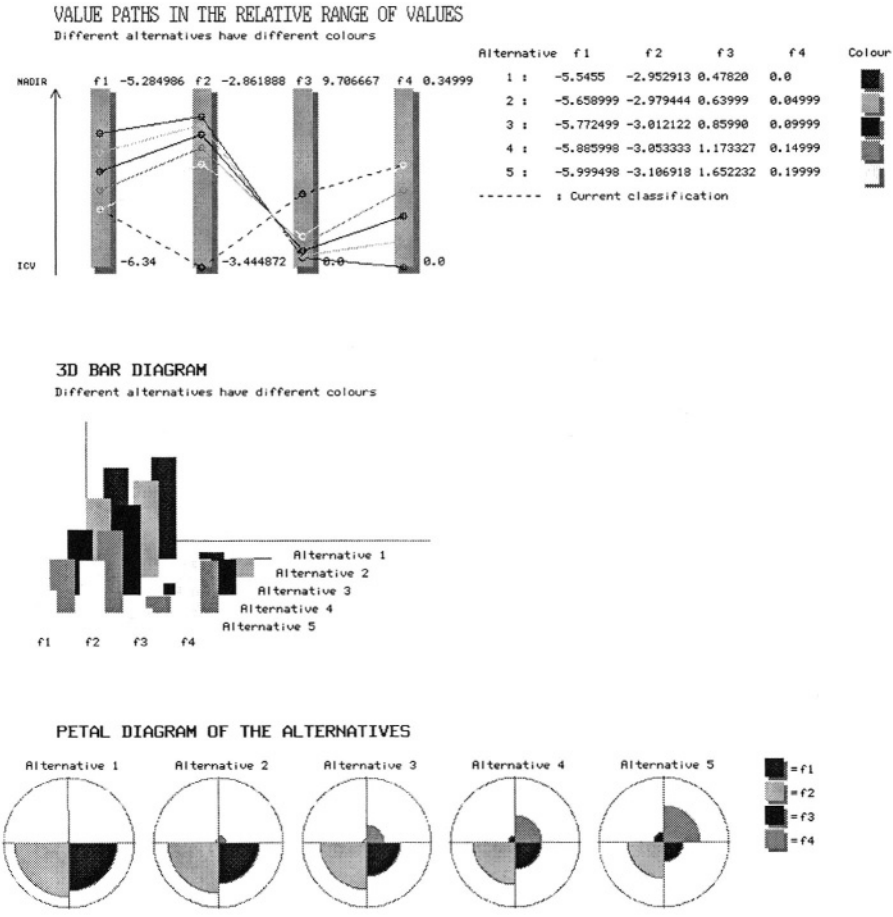


Figure 5.1. Visual illustrations in WWW-NIMBUS.

Acknowledgments

This research was supported by the Academy of Finland, grant #65760.

References

[1] P. J. Agrell, B. J. Lence, and A. Stam. An interactive multicriteria decision model for multipurpose reservoir management: the Shellmouth reservoir. *Journal of Multi-Criteria Decision Analysis,* 7:61–86, 1998.

[2] Y. Aksoy, T. W. Butler, and E. D. Minor III. Comparative studies in interactive multiple objective mathematical programming. *European Journal of Operational Research,* 89:408–422, 1996.

[3] J. E. Al-alvani, B. F. Hobbs, and B. Malakooti. An interactive integrated multiobjective optimization approach for quasiconcave/quasiconvex utility functions. In A. Goicoechea, L. Duckstein, and S. Zionts, editors, *Multiple Criteria Decision Making: Proceedings of the Ninth International Conference: Theory and Applications in Business, Industry, and Government,* pages 45–60. Springer-Verlag, New York, 1992.

[4] C. H. Antunes, M. J. Alves, A. L. Silva, and J. N. Clímaco. An integrated MOLP method base package – A guided tour of TOMMIX. *Computers & Operations Research,* 19:609–625, 1992.

[5] C. H. Antunes, M. P. Melo, and J. N. Clímaco. On the integration of an interactive MOLP procedure base and expert system technique. *European Journal of Operational Research,* 61:135–144, 1992.

[6] J.-P. Aubin and B. Näslund. An exterior branching algorithm. Working Paper 72–42, European Institute for Advanced Studies in Management, Brussels, 1972.

[7] N. Baba, H. Takeda, and T. Miyake. Interactive multi-objective programming technique using random optimization method. *International Journal of Systems Science,* 19:151–159, 1988.

[8] J. F. Bard. A multiobjective methodology for selecting subsystem automation options. *Management Science,* 32:1628–1641, 1986.

[9] R. Benayoun, J. de Montgolfier, J. Tergny, and O. Laritchev. Linear programming with multiple objective functions: Step method (STEM). *Mathematical Programming,* 1:366–375, 1971.

[10] H. P. Benson and Y. Aksoy. Using efficient feasible directions in interactive multiple objective linear programming. *Operations Research Letters,* 10:203–209, 1991.

[11] E. Bischoff. Two empirical tests with approaches to multiple-criteria decision making. In M. Grauer, M. Thompson, and

A. P. Wierzbicki, editors, *Plural Rationality and Interactive Decision Processes,* pages 344–347. Springer-Verlag, Berlin, Heidelberg, 1985.

[12] E. Bischoff. Multi-objective decision analysis – The right objectives? In G. Fandel, M. Grauer, A. Kurzhanski, and A. P. Wierzbicki, editors, *Large-Scale Modelling and Interactive Decision Analysis,* pages 155–160. Springer-Verlag, Berlin, 1986.

[13] P. Bogetoft, Å. Hallefjord, and M. Kok. On the convergence of reference point methods in multiobjective programming. *European Journal of Operational Research,* 34:56–68, 1988.

[14] K. Brockhoff. Experimental test of MCDM algorithms in a modular approach. *European Journal of Operational Research,* 22:159–166, 1985.

[15] J. T. Buchanan. Multiple objective mathematical programming: A review. *New Zealand Operational Research,* 14:1–27, 1986.

[16] J. T. Buchanan. An experimental evaluation of interactive MCDM methods and the decision making process. *Journal of the Operational Research Society,* 45:1050–1059, 1994.

[17] J. T. Buchanan. A naïve approach for solving MCDM problems: The GUESS method. *Journal of the Operational Research Society,* 48:202–206, 1997.

[18] J. T. Buchanan and J. L. Corner. The effects of anchoring in interactive MCDM solution methods. *Computers & Operations Research,* 24:907–918, 1997.

[19] J. T. Buchanan and H. G. Daellenbach. A comparative evaluation of interactive solution methods for multiple objective decision models. *European Journal of Operational Research,* 29:353–359, 1987.

[20] Y. Censor. Pareto optimality in multiobjective problems. *Applied Mathematics and Optimization,* 4:41–59, 1977.

[21] V. Chankong and Y. Y. Haimes. The interactive surrogate worth trade-off (ISWT) method for multiobjective decision-making. In S. Zionts, editor, *Multiple Criteria Problem Solving,* pages 42–67. Springer-Verlag, Berlin, New York, 1978.

[22] V. Chankong and Y. Y. Haimes. *Multiobjective Decision Making Theory and Methodology.* Elsevier Science Publishing Co., New York, 1983.

[23] A. Charnes and W. W. Cooper. *Management Models and Industrial Applications of Linear Programming,* volume 1. John Wiley & Sons, New York, 1961.

[24] A. Charnes and W. W. Cooper. Goal programming and multiple objective optimization; Part 1. *European Journal of Operational Research,* 1:39–54, 1977.

[25] T. Chen and B.-I. Wang. An interactive method for multiobjective decision making. In A. Straszak, editor, *Large Scale Systems: Theory and Applications 1983,* pages 277–282. Pergamon Press, Oxford, 1984.

[26] J. N. Clímaco and C. H. Antunes. Flexible method bases and man-machine interfaces as key features in interactive MOLP approaches. In P. Korhonen, A. Lewandowski, and J. Wallenius, editors, *Multiple Criteria Decision Support,* pages 207–216. Springer-Verlag, Berlin, 1991.

[27] J. N. Clímaco and C. H. Antunes. Man-machine interfacing in MCDA. In G. H. Tzeng, H. F. Wand, U. P. Wen, and P. L. Yu, editors, *Multiple Criteria Decision Making – Proceedings of the Tenth International Conference: Expand and Enrich the Domains of Thinking and Application,* pages 239–253. Springer-Verlag, New York, 1994.

[28] J. N. Clímaco, C. H. Antunes, and M. J. Alves. From TRIMAP to SOMMIX – building effective interactive MOLP computational tools. In G. Fandel and T. Gal, editors, *Multiple Criteria Decision Making: Proceedings of the Twelfth International Conference, Hagen (Germany),* pages 285–296. Springer-Verlag, Berlin, Heidelberg, 1997.

[29] H. W. Corley. A new scalar equivalence for Pareto optimization. *IEEE Transactions on Automatic Control,* 25:829–830, 1980.

[30] J. L. Corner and J. T. Buchanan. Experimental consideration of preference in decision making under certainty. *Journal of Multi-Criteria Decision Analysis,* 4:107–121, 1995.

[31] J. L. Corner and J. T. Buchanan. Capturing decision maker preference: Experimental comparison of decision analysis and MCDM techniques. *European Journal of Operational Research,* 98:85–97, 1997.

[32] J. P. Costa and J. N. Clímaco. A multiple reference point parallel approach in MCDM. In G. H. Tzeng, H. F. Wand, U. P. Wen, and P. L. Yu, editors, *Multiple Criteria Decision Making – Proceedings of the Tenth International Conference: Expand and Enrich the Domains of Thinking and Application,* pages 255–263, Springer-Verlag, New York, 1994.

[33] Y. Crama. Analysis of STEM-like solutions to multi-objective programming problems. In S. French, R. Hartley, L. C. Thomas, and D. J. White, editors, *Multi-Objective Decision Making,* pages 208–213. Academic Press, London, 1983.

[34] K. Deb. *Multi-Objective Optimization using Evolutionary Algorithms.* John Wiley & Sons, Chichester, 2001.

[35] A. Diaz. Interactive solution to multiobjective optimization problems. *International Journal for Numerical Methods in Engineering,* 24:1865–1877, 1987.

[36] J. S. Dyer. Interactive goal programming. *Management Science,* 19:62–70, 1972.

[37] J. S. Dyer. An empirical investigation of a man-machine interactive approach to the solution of the multiple criteria problem. In J. L. Cochrane and M. Zeleny, editors, *Multiple Criteria Decision Making,* pages 202–216. University of South Carolina Press. Columbia, South Carolina, 1973.

[38] J. S. Dyer. A time-sharing computer program for the solution of the multiple criteria problem. *Management Science,* 19:1379–1383, 1973.

[39] J. S. Dyer and R. K. Sarin. Multicriteria decision making. In A. G. Holzman, editor, *Mathematical Programming for Operations Researchers and Computer Scientists,* pages 123–148. Marcel Dekker, New York, 1981.

[40] M. Ehrgott. *Multicriteria Optimization.* Springer-Verlag, Berlin, Heidelberg, 2000.

[41] H. A. Eschenauer, A. Osyczka, and E. Schäfer. Interactive multicriteria optimization in design process. In H. Eschenauer, J. Koski, and A. Osyczka, editors, *Multicriteria Design Optimization Procedures and Applications,* pages 71–114. Springer-Verlag, Berlin, Heidelberg, 1990.

[42] H. A. Eschenauer, E. Schäfer, and H. Bernau. Application of interactive vector optimization methods with regard to problems in structural mechanics. In H. A. Eschenauer and G. Thierauf, editors, *Discretization Methods and Structural Optimization – Procedures and Applications,* pages 95–101. Springer-Verlag, Berlin, Heidelberg, 1989.

[43] P. A. V. Ferreira and J. C. Geromel. An interactive projection method for multicriteria optimization problems. *IEEE Transactions on Systems, Man, and Cybernetics,* 20:596–605, 1990.

[44] P. A. V. Ferreira and M. E. S. Machado. Solving multiple-objective problems in the objective space. *Journal of Optimization Theory and Applications,* 89:659–680, 1996.

[45] P. C. Fishburn. Lexicographic orders, utilities and decision rules: A survey. *Management Science,* 20:1442–1471, 1974.

[46] T. L. Friesz. Multiobjective optimization in transportation: The case of equilibrium network design. In N. N. Morse, editor, *Organizations: Multiple Agents with Multiple Criteria,* pages 116–127. Springer-Verlag, Berlin, Heidelberg, 1981.

[47] L. R. Gardiner and R. E. Steuer. Unified interactive multiple objective programming. *European Journal of Operational Research,* 74:391–406, 1994.

[48] L. R. Gardiner and R. E. Steuer. Unified interactive multiple objective programming: An open architecture for accommodating new procedures. *Journal of the Operational Research Society,* 45:1456–1466, 1994.

[49] S. Gass and T, Saaty. The computational algorithm for the parametric objective function. *Naval Research Logistics Quarterly,* 2:39–45, 1955.

[50] A. M. Geoffrion, J. S. Dyer, and A. Feinberg. An interactive approach for multi-criterion optimization, with an application to the operation of an academic department. *Management Science,* 19:357–368, 1972.

[51] M. Gershon and L. Duckstein. An algorithm for choosing of a multiobjective technique. In P. Hansen, editor, *Essays and Surveys on Multiple Criteria Decision Making,* pages 53–62. Springer-Verlag, Berlin, Heidelberg, 1983.

[52] M. Gibson, J. J. Bernardo, C. Chung, and R. Badinelli. A comparison of interactive multiple-objective decision making procedures. *Computers & Operations Research,* 14:97–105, 1987.

[53] V. G. Gouljashki, L. M. Kirilov, S. C. Narula, and V. S. Vassilev. A reference direction interactive algorithm of the multiple objective nonlinear integer programming. In G. Fandel and T. Gal, editors, *Multiple Criteria Decision Making: Proceedings of the Twelfth International Conference, Hagen (Germany),* pages 308–317. Springer-Verlag, Berlin, Heidelberg, 1997.

[54] J. Granat and M. Makowski. Interactive specification and analysis of aspiration-based preferences. *European Journal of Operational Research,* 122:469–485, 2000.

[55] M. Grauer, A. Lewandowski, and A. Wierzbicki. DIDASS – theory, implementation and experiences. In M. Grauer and A. P. Wierzbicki, editors, *Interactive Decision Analysis,* pages 22–30. Springer-Verlag, Berlin, 1984.

[56] Y. Y. Haimes. The surrogate worth trade-off (SWT) method and its extensions. In G. Fandel and T. Gal, editors, *Multiple Criteria Decision Making Theory and Application,* pages 85–108. Springer-Verlag, Berlin, Heidelberg, 1980.

[57] Y. Y. Haimes, L. S. Lasdon, and D. A. Wismer. On a bicriterion formulation of the problems of integrated system identification and system optimization. *IEEE Transactions on Systems, Man, and Cybernetics,* 1:296–297, 1971.

[58] Y. Y. Haimes, K. Tarvainen, T. Shima, and J. Thadathil. *Hierarchical Multiobjective Analysis of Large-Scale Systems.* Hemisphere Publishing Corporation, New York, 1990.

[59] W. A. Hall and Y. Y. Haimes. The surrogate worth trade-off method with multiple decision-makers. In M. Zeleny, editor, *Multiple Criteria Decision Making Kyoto 1975,* pages 207–233. Springer-Verlag, Berlin, Heidelberg, 1976.

[60] Å. Hallefjord and K. Jörnsten. An entropy target-point approach to multiobjective programming. *International Journal of Systems Science,* 17:639–653, 1986.

[61] T. Hemming. Some modifications of a large step gradient method for interactive multicriterion optimization. In N. N. Morse, editor, *Organizations: Multiple Agents with Multiple Criteria,* pages 128–139. Springer-Verlag, Berlin, Heidelberg, 1981.

[62] B. F. Hobbs. What can we learn from experiments in multiobjective decision analysis? *IEEE Transactions on Systems, Man, and Cybernetics,* 16:384–394, 1986.

[63] M. L. Hussein and F. S. A. El-Ghaffar. An interactive approach for vector optimization problems. *European Journal of Operational Research,* 89:185–192, 1996.

[64] C.-L. Hwang and A. S. M. Masud. *Multiple Objective Decision Making – Methods and Applications: A State-of-the-Art Survey.* Springer-Verlag, Berlin, Heidelberg, 1979.

[65] C.-L. Hwang and K. Yoon. *Multiple Attribute Decision Making Methods and Applications: A State-of-the-Art Survey.* Springer-Verlag, Berlin, Heidelberg, 1981.

[66] J. P. Ignizio. *Goal Programming and Extensions.* Lexington Books B.C. Heath and Company, Lexington, 1976.

[67] A. Jaszkiewicz and R. Słowiński. The light beam search – outranking based interactive procedure for multiple-objective mathematical programming. In P. M. Pardalos, Y. Siskos, and C. Zopounidis, editors, *Advances in Multicriteria Analysis,* pages 129–146. Kluwer Academic Publishers, Dordrecht, 1995.

[68] A. Jaszkiewicz and R. Słowiński. The 'light beam search' approach – an overview of methodology and applications. *European Journal of Operational Research,* 113:300–314, 1999.

[69] P. Jedrzejowicz and L. Rosicka. Multicriterial reliability optimization problem. *Foundations of Control Engineering,* 8:165–173, 1983.

[70] I. Kaliszewski. A modified weighted Tchebycheff metric for multiple objective programming. *Computers & Operations Research,* 14:315–323, 1987.

[71] I. Kaliszewski and W. Michalowski. Searching for psychologically stable solutions of multiple criteria decision problems. *European Journal of Operational Research,* 118:549–562, 1999.

[72] I. Kaliszewski, W. Michalowski, and G. Kersten. A hybrid interactive technique for the MCDM problems. In M. H. Karwan, J. Spronk, and J. Wallenius, editors, *Essays in Decision Making: A Volume in Honour of Stanley Zionts,* pages 48–59. Springer-Verlag, Berlin, Heidelberg, 1997.

[73] T. C. U. Kalu. An algorithm for systems welfare interactive goal programming modelling. *European Journal of Operational Research,* 116:508–529, 1999.

[74] R. L. Keeney and H. Raiffa. *Decisions with Multiple Objectives: Preferences and Value Tradeoffs.* John Wiley & Sons, Inc., New York, London, Sydney, 1976.

[75] S. H. Kim and T. Gal. A new interactive algorithm for multiobjective linear programming using maximally changeable dominance cone. *European Journal of Operational Research,* 64:126–137, 1993.

[76] L. M. Kirilov and V. S. Vassilev. A method for solving multiple objective linear programming problems. In J. Clímaco, editor, *Multicriteria Analysis,* pages 302–309. Springer-Verlag, Berlin, Heidelberg, 1997.

[77] M. Kok. Scalarization and the interface with decision makers in interactive multi objective linear programming. In P. Serafini, editor, *Mathematics of Multi Objective Optimization,* pages 433–438.Springer-Verlag, Wien, New York, 1985.

[78] M. Kok. The interface with decision makers and some experimental results in interactive multiple objective programming method. *European Journal of Operational Research*, 26:96–107, 1986.

[79] M. Kok and F. A. Lootsma. Pairwise-comparison methods in multiple objective programming, with applications in a long-term energy-planning model. *European Journal of Operational Research,* 22:44–55, 1985.

[80] M. M. Köksalan and H. Moskowitz. Solving the multiobjective decision making problem using a distance function. In G. H. Tzeng, H. F. Wand, U. P. Wen, and P. L. Yu, editors, *Multiple Criteria Decision Making – Proceedings of the Tenth International Conference: Expand and Enrich the Domains of Thinking and Application,* pages 101–107. Springer-Verlag, New York, 1994.

[81] P. Korhonen. Reference direction approach to multiple objective linear programming: Historical overview. In M. H. Karwan, J. Spronk, and J. Wallenius, editors, *Essays in Decision Making: A Volume in Honour of Stanley Zionts,* pages 74–92. Springer-Verlag, Berlin, Heidelberg, 1997.

[82] P. Korhonen and M. Halme. Using lexicographic parametric programming for searching a nondominated set in multiple objective linear programming. *Journal of Multi-Criteria Decision Analysis,* 5:291–300, 1996.

[83] P. Korhonen and J. Laakso. A visual interactive method for solving the multiple-criteria problem. In M. Grauer and A. P. Wierzbicki, editors, *Interactive Decision Analysis,* pages 146–153. Springer-Verlag, Berlin, 1984.

[84] P. Korhonen and J. Laakso. On developing a visual interactive multiple criteria method – an outline. In Y. Y. Haimes and V. Chankong, editors, *Decision Making with Multiple Objectives,* pages 272–281. Springer-Verlag, Berlin, Heidelberg, 1985.

[85] P. Korhonen and J. Laakso. Solving generalized goal programming problems using a visual interactive approach. *European Journal of Operational Research,* 26:355–363, 1986.

[86] P. Korhonen and J. Laakso. A visual interactive method for solving the multiple criteria problem. *European Journal of Operational Research,* 24:277–287, 1986.

[87] P. Korhonen and S. C. Narula. An evolutionary approach to support decision making with linear decision models. *Journal of Multi-Criteria Decision Analysis,* 2:111–119, 1993.

[88] P. Korhonen and J. Wallenius. A Pareto race. *Naval Research Logistics,* 35:615–623, 1988.

[89] P. Korhonen and J. Wallenius. Observations regarding choice behaviour in interactive multiple criteria decision-making environments: An experimental investigation. In A. Lewandowski and I. Stanchev, editors, *Methodology and Software for Interactive Decision Support,* pages 163–170. Springer-Verlag, Berlin, 1989.

[90] P. Korhonen and J. Wallenius. VIG – a visual and dynamic decision support system for multiple objective linear programming. In B. Karpak and S. Zionts, editors, *Multiple Criteria Decision Making and Risk Analysis Using Microcomputers,* pages 251–281. Springer-Verlag, Berlin, Heidelberg, 1989.

[91] P. Korhonen and J. Wallenius. A multiple objective linear programming decision support system. *Decision Support Systems,* 6:243–251, 1990.

[92] O. I. Larichev, O. A. Polyakov, and A. O. Nikiforov. Multicriterion linear programming problems. *Journal of Economic Psychology,* 8:389–407, 1987.

[93] R. Lazimy. Interactive relaxation method for a broad class of integer and continuous nonlinear multiple criteria problems. *Journal of Mathematical Analysis and Applications,* 116:553–573, 1986.

[94] R. Lazimy. Solving multiple criteria problems by interactive decomposition. *Mathematical Programming,* 35:334–361, 1986.

[95] E. R. Lieberman. Soviet multi-objective mathematical programming methods: An overview. *Management Science,* 37:1147–1165, 1991.

[96] G. V. Loganathan and H. D. Sherali. A convergent interactive cutting-plane algorithm for multiobjective optimization. *Operations Research,* 35:365–377, 1987.

[97] A. V. Lotov. Computer-based support for planning and negotiation on environmental rehabilitation of water resource systems. In D. P. Loucks, editor, *Restoration of Degraded Rivers: Challenges, Issues and Experiences,* pages 417–445. Kluwer Academic Publishers, Dordrecht, 1998.

[98] A. V. Lotov, V. A. Bushenkov, and O. L. Chernykh. Multicriteria DSS for river water-quality planning. *Microcomputers in Civil Engineering,* 12:57–67, 1997.

[99] M. M. Mäkelä and P. Neittaanmäki. *Nonsmooth Optimization: Analysis and Algorithms with Applications to Optimal Control.* World Scientific Publishing Co., Singapore, 1992.

[100] A. S. M. Masud and C. L. Hwang. Interactive sequential goal programming. *Journal of the Operational Research Society,* 32:391–400, 1981.

[101] A. S. M. Masud and X. Zheng. An algorithm for multiple-objective non-linear programming. *Journal of the Operational Research Society,* 40:895–906, 1989.

[102] W. Michalowski. Evaluation of a multiple criteria interactive programming approach: An experiment. *INFOR: Information Systems & Operational Research,* 25:165–173, 1987.

[103] W. Michalowski and T. Szapiro. A bi-reference procedure for interactive multiple criteria programming. *Operations Research,* 40:247–258, 1992.

[104] K. Miettinen. *On the Methodology of Multiobjective Optimization with Applications.* PhD thesis, University of Jyväskylä, Department of Mathematics, 1994. Report 60.

[105] K. Miettinen. *Nonlinear Multiobjective Optimization.* Kluwer Academic Publishers, Boston, 1999.

[106] K. Miettinen and M. M. Mäkelä. An interactive method for nonsmooth multiobjective optimization with an application to optimal control. *Optimization Methods and Software,* 2:31–44, 1993.

[107] K. Miettinen and M. M. Mäkelä. A nondifferentiable multiple criteria optimization method applied to continuous casting process. In A. Fasano and M. Primicerio, editors, *Proceedings of the Seventh European Conference on Mathematics in Industry,* pages 255–262. B.G. Teubner, Stuttgart, 1994.

[108] K. Miettinen and M. M. Mäkelä. Interactive bundle-based method for nondifferentiable multiobjective optimization: NIMBUS. *Optimization,* 34:231–246, 1995.

[109] K. Miettinen and M. M. Mäkelä. Interactive method NIMBUS for nondifferentiable multiobjective optimization problems. In J. Clímaco, editor, *Multicriteria Analysis,* pages 310–319. Springer-Verlag, Berlin, Heidelberg, 1997.

[110] K. Miettinen and M. M. Mäkelä. Interactive MCDM support system in the Internet. In T. Stewart and R. van den Honert, editors, *Trends in Multicriteria Decision Making: Proceedings of the 13th International Conference on Multiple Criteria Decision Making,* pages 419–428. Springer-Verlag, Berlin, 1998.

[111] K. Miettinen and M. M. Mäkelä. Comparative evaluation of some interactive reference point-based methods for multi-objective opti-

misation. *Journal of the Operational Research Society,* 50:949–959, 1999.

[112] K. Miettinen and M. M. Mäkelä. Interactive multiobjective optimization system WWW-NIMBUS on the Internet. *Computers & Operations Research,* 27:709–723, 2000.

[113] K. Miettinen, M. M. Mäkelä, and R. A. E. Mäkinen. Interactive multiobjective optimization system NIMBUS applied to nonsmooth structural design problems. In J. Doležal and J. Fidler, editors, *System Modelling and Optimization,* Proceedings of the 17th IFIP Conference on System Modelling and Optimization, Prague, Czech Republic, pages 379–385. Chapman &; Hall, London, 1996.

[114] K. Miettinen, M. M. Mäkelä, and T. Männikkö. Optimal control of continuous casting by nondifferentiable multiobjective optimization. *Computational Optimization and Applications,* 11:177–194, 1998.

[115] K. Mitani and H. Nakayama. A multiobjective diet planning support system using the satisficing trade-off method. *Journal of Multi-Criteria Decision Analysis,* 6:131–139, 1997.

[116] U. Mocci and L. Primicerio. Ring network design: An MCDM approach. In G. Fandel and T. Gal, editors, *Multiple Criteria Decision Making: Proceedings of the Twelfth International Conference, Hagen (Germany),* pages 491–500. Springer-Verlag, Berlin, Heidelberg, 1997.

[117] C. Mohan and H. T. Nguyen. Reference direction interactive method for solving multiobjective fuzzy programming problems. *European Journal of Operational Research,* 107:599–613, 1998.

[118] M. A. Moldavskiy. Singling out a set of undominated solutions in continuous vector optimization problems. *Soviet Automatic Control,* 14:47–53, 1981.

[119] D. E. Monarchi, C. C. Kisiel, and L. Duckstein. Interactive multiobjective programming in water resources: A case study. *Water Resources Research,* 9:837–850, 1973.

[120] D. E. Monarchi, J. E. Weber, and L. Duckstein. An interactive multiple objective decision-making aid using nonlinear goal programming. In M. Zeleny, editor, *Multiple Criteria Decision Making Kyoto 1975,* pages 235–253. Springer-Verlag, Berlin, Heidelberg, 1976.

[121] J. Mote, D. L. Olson, and M. A. Venkataramanan. A comparative multiobjective programming study. *Mathematical and Computer Modelling,* 10:719–729, 1988.

[122] A. M'silti and P. Tolla. An interactive multiobjective nonlinear programming procedure. *European Journal of Operational Research,* 64:115–125, 1993.

[123] H. Mukai. Algorithms for multicriterion optimization. *IEEE Transactions on Automatic Control,* 25:177–186, 1980.

[124] K. Musselman and J. Talavage. A tradeoff cut approach to multiple objective optimization. *Operations Research,* 28:1424–1435, 1980.

[125] H. Nakayama. On the components in interactive multiobjective programming methods. In M. Grauer, M. Thompson, and A. P. Wierzbicki, editors, *Plural Rationality and Interactive Decision Processes,* pages 234–247. Springer-Verlag, Berlin, Heidelberg, 1985.

[126] H. Nakayama. Sensitivity and trade-off analysis in multiobjective programming. In A. Lewandowski and I. Stanchev, editors, *Methodology and Software for Interactive Decision Support,* pages 86–93. Springer-Verlag, Berlin, 1989.

[127] H. Nakayama. Satisficing trade-off method for problems with multiple linear fractional objectives and its applications. In A. Lewandowski and V. Volkovich, editors, *Multiobjective Problems of Mathematical Programming,* pages 42–50. Springer-Verlag, Berlin, 1991.

[128] H. Nakayama. Trade-off analysis based upon parametric optimization. In P. Korhonen, A. Lewandowski, and J. Wallenius, editors, *Multiple Criteria Decision Support,* pages 42–52. Springer-Verlag, Berlin, 1991.

[129] H. Nakayama. Trade-off analysis using parametric optimization techniques. *European Journal of Operational Research,* 60:87–98, 1992.

[130] H. Nakayama. Engineering applications of multi-objective programming: Recent results. In G. H. Tzeng, H. F. Wand, U. P. Wen, and P. L. Yu, editors, *Multiple Criteria Decision Making – Proceedings of the Tenth International Conference: Expand and Enrich the Domains of Thinking and Application,* pages 369–378. Springer-Verlag, New York, 1994.

[131] H. Nakayama. Aspiration level approach to interactive multiobjective programming and its applications. In P. M. Pardalos, Y. Siskos, and C. Zopounidis, editors, *Advances in Multicriteria Analysis,* pages 147–174. Kluwer Academic Publishers, Dordrecht, 1995.

[132] H. Nakayama and K. Furukawa. Satisficing trade-off method with an application to multiobjective structural design. *Large Scale Systems,* 8:47–57, 1985.

[133] H. Nakayama, K. Kaneshige, S. Takemoto, and Y. Watada. An application of a multi-objective programming technique to construction accuracy control of cable-stayed bridges. *European Journal of Operational Research,* 87:731–738, 1995.

[134] H. Nakayama, K. Kaneshige, S. Takemoto, and Y. Watada. A construction accuracy control system of cable stayed bridge using a multi-objective programming technique. In G. Fandel and T. Gal, editors, *Multiple Criteria Decision Making: Proceedings of the Twelfth International Conference, Hagen (Germany),* pages 501–509. Springer-Verlag, Berlin, Heidelberg, 1997.

[135] H. Nakayama, J. Nomura, K. Sawada, and R. Nakajima. An application of satisficing trade-off method to a blending problem of industrial materials. In G. Fandel, M. Grauer, A. Kurzhanski, and A. P. Wierzbicki, editors, *Large-Scale Modelling and Interactive Decision Analysis,* pages 303–313. Springer-Verlag, Berlin, 1986.

[136] H. Nakayama and Y. Sawaragi. Satisficing trade-off method for multiobjective programming. In M. Grauer and A. P. Wierzbicki, editors, *Interactive Decision Analysis,* pages 113–122. Springer-Verlag, Berlin, 1984.

[137] H. Nakayama, T. Tanino, and Y. Sawaragi. An interactive optimization method in multicriteria decisionmaking. *IEEE Transactions on Systems, Man, and Cybernetics,* 10:163–169, 1980.

[138] S. C. Narula, L. Kirilov, and V. Vassilev. An interactive algorithm for solving multiple objective nonlinear programming problems. In G. H. Tzeng, H. F. Wand, U. P. Wen, and P. L. Yu, editors, *Multiple Criteria Decision Making – Proceedings of the Tenth International Conference: Expand and Enrich the Domains of Thinking and Application,* pages 119–127. Springer-Verlag, New York, 1994.

[139] S. C. Narula, L. Kirilov, and V. Vassilev. Reference direction approach for solving multiple objective nonlinear programming problems. *IEEE Transactions on Systems, Man, and Cybernetics,* 24:804–806, 1994.

[140] S. C. Narula and H. R. Weistroffer. Algorithms for multi-objective nonlinear programming problems: An overview. In A. G. Lockett and G. Islei, editors, *Improving Decision Making in Organisations,* pages 434–443. Springer-Verlag, Berlin, Heidelberg, 1989.

[141] S. C. Narula and H. R. Weistroffer. A flexible method for nonlinear multicriteria decisionmaking problems. *IEEE Transactions on Systems, Man, and Cybernetics,* 19:883–887, 1989.

[142] P. Nijkamp and J. Spronk. Interactive multiple goal programming: An evaluation and some results. In G. Fandel and T. Gal, editors, *Multiple Criteria Decision Making Theory and Applications,* pages 278–293. Springer-Verlag, Berlin, Heidelberg, 1980.

[143] W. Ogryczak. Preemptive reference point method. In J. Clímaco, editor, *Multicriteria Analysis,* pages 156–167. Springer-Verlag, Berlin, Heidelberg, 1997.

[144] M. Olbrisch. The interactive reference point approach as a solution concept for econometric decision models. In M. J. Beckmann, K.-W. Gaede, K. Ritter, and H. Schneeweiss, editors, *X. Symposium on Operations Research, Part 1, Sections 1-5,* pages 611–619. Verlag Anton Hain Meisenheim GmbH, Königstein/Ts., 1986.

[145] D. L. Olson. Review of empirical studies in multiobjective mathematical programming: Subject reflection of nonlinear utility and learning. *Decision Sciences,* 23:1–20, 1992.

[146] D. L. Olson. Tchebycheff norms in multi-objective linear programming. *Mathematical and Computer Modelling,* 17:113–124, 1993.

[147] K. R. Oppenheimer. A proxy approach to multi-attribute decision making. *Management Science,* 24:675–689, 1978.

[148] A. Osyczka and S. Kundu. A new method to solve generalized multicriteria optimization problems using the simple genetic algorithm. *Structural Optimization,* 10:94–99, 1995.

[149] R. Ramesh, M. H. Karwan, and S. Zionts. Theory of convex cones in multicriteria decision making. *Annals of Operations Research,* 16:131–147, 1988.

[150] R. Ramesh, M. H. Karwan, and S. Zionts. Interactive multicriteria linear programming: An extension of the method of Zionts and Wallenius. *Naval Research Logistics,* 36:321–335, 1989.

[151] R. Ramesh, M. H. Karwan, and S. Zionts. Preference structure representation using convex cones in multicriteria integer programming. *Management Science,* 35:1092–1105, 1989.

[152] G. R. Reeves and J. J. Gonzalez. A comparison of two interactive MCDM procedures. *European Journal of Operational Research,* 41:203–209, 1989.

[153] G. R. Reeves and K. R. MacLeod. Robustness of the interactive weighted Tchebycheff procedure to inaccurate preference information. *Journal of Multi-Criteria Decision Analysis,* 8:128–132, 1999.

[154] P. Rietveld. *Multiple Objective Decision Methods and Regional Planning.* North-Holland Publishing Company, Amsterdam, 1980.

[155] C. Romero. *Handbook of Critical Issues in Goal Programming.* Pergamon Press, Oxford, 1991.

[156] C. Romero. Extended lexicographic goal programming: a unifying approach. *Omega,* 29:63–71, 2001.

[157] R. E. Rosenthal. Principles of multiobjective optimization. *Decision Sciences,* 16:133–152, 1985.

[158] E. E. Rosinger. Interactive algorithm for multiobjective optimization. *Journal of Optimization Theory and Applications,* 35:339–365, 1981. Errata Corrige in Journal of Optimization Theory and Applications, 38:147–148, 1982.

[159] A. Roy and P. Mackin. Multicriteria optimization (linear and nonlinear) using proxy value functions. In P. Korhonen, A. Lewandowski, and J. Wallenius, editors, *Multiple Criteria Decision Support,* pages 128–134. Springer-Verlag, Berlin, 1991.

[160] A. Roy and J. Wallenius. Nonlinear multiobjective optimization: An algorithm and some theory. *Mathematical Programming,* 55:235–249, 1992.

[161] B. Roy. The outranking approach and the foundations of ELECTRE methods. In C. A. Bana e Costa, editor, *Readings in Multiple Criteria Decision Aid,* pages 155–183. Springer-Verlag, Berlin, Heidelberg, 1990.

[162] S. Sadagopan and A. Ravindran. Interactive algorithms for multiple criteria nonlinear programming problems. *European Journal of Operational Research,* 25:247–257, 1986.

[163] M. Sakawa. Interactive multiobjective decision making by the sequential proxy optimization technique: SPOT. *European Journal of Operational Research,* 9:386–396, 1982.

[164] M. Sakawa and K. Yauchi. An interactive fuzzy satisficing method for multiobjective nonconvex programming problems through floating point genetic algorithms. *European Journal of Operational Research,* 117:113–124, 1999.

[165] Y. Sawaragi, H. Nakayama, and T. Tanino. *Theory of Multiobjective Optimization.* Academic Press, Inc., Orlando, Florida, 1985.

[166] W. S. Shin and A. Ravindran. Interactive multiple objective optimization: Survey I – continuous case. *Computers & Operations Research,* 18:97–114, 1991.

[167] J. Silverman, R. E. Steuer, and A. W. Whisman. A multi-period, multiple criteria optimization system for manpower planning. *European Journal of Operational Research,* 34:160–170, 1988.

[168] A. M. J. Skulimowski. *Decision Support Systems Based on Reference Sets.* Wydawnictwa AGH, Kraków, 1996.

[169] R. Slowinski. Interactive multiobjective optimization based on ordinal regression. In A. Lewandowski and V. Volkovich, editors, *Multiobjective Problems of Mathematical Programming,* pages 93–100. Springer-Verlag, Berlin, 1991.

[170] A. Song and W.-M. Cheng. A method for multihuman and multicriteria decision making. In A. Sydow, S. G. Tzafestas, and R. Vichnevetsky, editors, *Systems Analysis and Simulation 1988 I: Theory and Foundations,* pages 213–216. Akademie-Verlag, Berlin, 1988.

[171] J. Spronk. Interactive multifactorial planning: State of the art. In C. A. Bana e Costa, editor, *Readings in Multiple Criteria Decision Aid,* pages 512–534. Springer-Verlag, Berlin, Heidelberg, 1990.

[172] A. Stam, M. Kuula, and H. Cesar. Transboundary air pollution in Europe: An interactive multicriteria tradeoff analysis. *European Journal of Operational Research,* 56:263–277, 1992.

[173] R. B. Statnikov. *Multicriteria Design: Optimization and Identification.* Kluwer Academic Publishers, Dordrecht, 1999.

[174] R. B. Statnikov and J. Matusov. Use of p_τ-nets for the approximation of the Edgeworth-Pareto set in multicriteria optimization. *Journal of Optimization Theory and Applications,* 91:543–560, 1996.

[175] R. E. Steuer. *Multiple Criteria Optimization: Theory, Computation, and Applications.* John Wiley & Sons, New York, 1986.

[176] R. E. Steuer. The Tchebycheff procedure of interactive multiple objective programming. In B. Karpak and S. Zionts, editors, *Multiple Criteria Decision Making and Risk Analysis Using Microcomputers,* pages 235–249. Springer-Verlag, Berlin, Heidelberg, 1989.

[177] R. E. Steuer. Implementing the Tchebycheff method in a spreadsheet. In M. H. Karwan, J. Spronk, and J. Wallenius, editors, *Essays in Decision Making: A Volume in Honour of Stanley Zionts,* pages 93–103. Springer-Verlag, Berlin, Heidelberg, 1997.

[178] R. E. Steuer and E.-U. Choo. An interactive weighted Tchebycheff procedure for multiple objective programming. *Mathematical Programming,* 26:326–344, 1983.

[179] R. E. Steuer and L. R. Gardiner. Interactive multiple objective programming: Concepts, current status, and future directions. In C. A. Bana e Costa, editor, *Readings in Multiple Criteria Decision Aid,* pages 413–444. Springer-Verlag, Berlin, Heidelberg, 1990.

[180] R. E. Steuer and L. R. Gardiner. On the computational testing of procedures for interactive multiple objective linear programming. In G. Fandel and H. Gehring, editors, *Operations Research,* pages 121–131. Springer-Verlag, Berlin, Heidelberg, 1991.

[181] R. E. Steuer, J. Silverman, and A. W. Whisman. A combined Tchebycheff/aspiration criterion vector interactive multiobjective programming procedure. *Management Science,* 39:1255–1260, 1993.

[182] R. E. Steuer and M. Sun. The parameter space investigation method of multiple objective nonlinear programming: A computational investigation. *Operations Research,* 43:641–648, 1995.

[183] R. E. Steuer and A. W. Whisman. Toward the consolidation of interactive multiple objective programming procedures. In G. Fandel, M. Grauer, A. Kurzhanski, and A. P. Wierzbicki, editors, *Large-Scale Modelling and Interactive Decision Analysis,* pages 232–241. Springer-Verlag, Berlin, 1986.

[184] T. J. Stewart. A critical survey on the status of multiple criteria decision making theory and practice. *Omega,* 20:569–586, 1992.

[185] M. Sun, A. Stam, and R. E. Steuer. Solving multiple objective programming problems using feed-forward artificial neural networks: The interactive FFANN procedure. *Management Science,* 42:835–849, 1996.

[186] M. Sun, A. Stam, and R. E. Steuer. Interactive multiple objective programming using Tchebycheff programs and artificial neural networks. *Computers & Operations Research,* 27:601–620, 2000.

[187] T. Sunaga, M. A. Mazeed, and E. Kondo. A penalty function formulation for interactive multiobjective programming problems. In M. Iri and K. Yajima, editors, *System Modelling and Optimization,* pages 221–230. Springer-Verlag, Berlin, 1988.

[188] M. T. Tabucanon. *Multiple Criteria Decision Making in Industry.* Elsevier Science Publishers B. V., Amsterdam, 1988.

[189] M. Tamiz and D. F. Jones. A general interactive goal programming algorithm. In G. Fandel and T. Gal, editors, *Multiple Criteria Decision Making: Proceedings of the Twelfth International Conference, Hagen (Germany),* pages 433–444. Springer-Verlag, Berlin, Heidelberg, 1997.

[190] C. G. Tapia and B. A. Murtagh. The use of preference criteria in interactive multiobjective mathematical programming. *Asia-Pacific Journal of Operational Research,* 6:131–147, 1989.

[191] C. G. Tapia and B. A. Murtagh. A Markovian process in interactive multiobjective decision-making. *European Journal of Operational Research,* 57:421–428, 1992.

[192] K. Tarvainen. On the implementation of the interactive surrogate worth trade-off (ISWT) method. In M. Grauer and A. P. Wierzbicki, editors, *Interactive Decision Analysis,* pages 154–161, Berlin, Heidelberg, 1984. Springer-Verlag.

[193] A. Tecle and L. Duckstein. A procedure for selecting MCDM techniques for forest resources management. In A. Goicoechea, L. Duckstein, and S. Zionts, editors, *Multiple Criteria Decision Making: Proceedings of the Ninth International Conference: Theory and Applications in Business, Industry, and Government,* pages 19–32. Springer-Verlag, New York, 1992.

[194] J. Teghem Jr., C. Delhaye, and P. L. Kunsch. An interactive decision support system (IDSS) for multicriteria decision aid. *Mathematical and Computer Modelling,* 12:1311–1320, 1989.

[195] A. Udink ten Cate. On the determination of the optimal temperature for the growth of an early cucumber crop in a greenhouse. In M. Grauer, M. Thompson, and A. P. Wierzbicki, editors, *Plural Rationality and Interactive Decision Processes,* pages 311–318. Springer-Verlag, Berlin, Heidelberg, 1985.

[196] D. Vanderpooten. Multiobjective programming: Basic concepts and approaches. In R. Slowinski and J. Teghem, editors, *Stochastic versus Fuzzy Approaches to Multiobjective Mathematical Programming under Uncertainty,* pages 7–22. Kluwer Academic Publishers, Dordrecht, 1990.

[197] D. Vanderpooten and P. Vincke. Description and analysis of some representative interactive multicriteria procedures. *Mathematical and Computer Modelling,* 12:1221–1238, 1989.

[198] R. Vetschera. Feedback-oriented group decision support in a reference point framework. In A. Lewandowski and V. Volkovich, editors, *Multiobjective Problems of Mathematical Programming,* pages 309–314. Springer-Verlag, 1991.

[199] R. Vetschera. A note on scalarizing functions under changing sets of criteria. *European Journal of Operational Research,* 52:113–118, 1991.

[200] P. Vincke. *Multicriteria Decision-Aid.* John Wiley & Sons, Inc., Chichester, 1992.

[201] J. Wallenius. Comparative evaluation of some interactive approaches to multicriterion optimization. *Management Science,* 21:1387–1396, 1975.

[202] J. Wallenius and S. Zionts. A research project on multicriteria decision making. In D. E. Bell, R. Keeney, and H. Raiffa, editors, *Conflicting Objectives in Decision,* pages 76–96. John Wiley & Sons, Inc., New York, NY, 1977.

[203] S. Wang. Algorithms for multiobjective and nonsmooth optimization. In P. Kleinschmidt, F. J. Radermacher, W. Schweitzer, and H. Wildermann, editors, *Methods of Operations Research 58,* pages 131–142. Athenäum Verlag, Frankfurt am Main, 1989.

[204] S. Wang. An interactive method for multicriteria decision making. In K. H. Phua, C. M. Wang, W. Y. Yeong, T. Y. Leong, H. T. Loh, K. C. Tan, and F. S. Chou, editors, *Optimization: Techniques and Applications,* Proceedings of the International Conference (ICOTA), volume 1, pages 307–316. World Scientific Publishing Co., River Edge, 1992.

[205] H. R. Weistroffer. Multiple criteria decision making with interactive over-achievement programming. *Operations Research Letters,* 1:241–245, 1982.

[206] H. R. Weistroffer. An interactive goal-programming method for non-linear multiple-criteria decision-making problems. *Computers & Operations Research,* 10:311–320, 1983.

[207] H. R. Weistroffer. A combined over- and under-achievement programming approach to multiple objective decision-making. *Large Scale Systems,* 7:47–58, 1984.

[208] H. R. Weistroffer. A flexible model for multi-objective optimization. In J. Jahn and W. Krabs, editors, *Recent Advances and Historical Developments of Vector Optimization,* pages 311–316. Springer-Verlag, Berlin, Heidelberg, 1987.

[209] R. E. Wendell and D. N. Lee. Efficiency in multiple objective optimization problems. *Mathematical Programming,* 12:406–414, 1977.

[210] D. J. White. A selection of multi-objective interactive programming methods. In S. French, R. Hartley, L. C. Thomas, and D. J. White, editors, *Multi-Objective Decision Making,* pages 99–126. Academic Press, London, 1983.

[211] A. P. Wierzbicki. The use of reference objectives in multiobjective optimization. In G. Fandel and T. Gal, editors, *Multiple Criteria Decision Making Theory and Applications,* pages 468–486. Springer-Verlag, Berlin, Heidelberg, 1980.

[212] A. P. Wierzbicki. A mathematical basis for satisficing decision making. *Mathematical Modelling,* 3:391–405, 1982.

[213] A. P. Wierzbicki. A methodological approach to comparing parametric characterizations of efficient solutions. In G. Fandel, M. Grauer, A. Kurzhanski, and A. P. Wierzbicki, editors, *Large-Scale Modelling and Interactive Decision Analysis,* pages 27–45. Springer-Verlag, Berlin, 1986.

[214] A. P. Wierzbicki. On the completeness and constructiveness of parametric characterizations to vector optimization problems. *OR Spektrum,* 8:73–87, 1986.

[215] A. P. Wierzbicki. Convergence of interactive procedures of multiobjective optimization and decision support. In M. H. Karwan, J. Spronk, and J. Wallenius, editors, *Essays in Decision Making: A Volume in Honour of Stanley Zionts,* pages 19–47. Springer-Verlag, Berlin, Heidelberg, 1997.

[216] A. P. Wierzbicki. Reference point approaches. In T. Gal, T. J. Stewart, and T. Hanne, editors, *Multicriteria Decision Making: Advances in MCDM Models, Algorithms, Theory, and Applications,* Chapter 9. Kluwer Academic Publishers, Boston, 1999.

[217] A. P. Wierzbicki and J. Granat. Multi-objective modeling for engineering application in decision support. In G. Fandel and T. Gal, editors, *Multiple Criteria Decision Making: Proceedings of the Twelfth International Conference, Hagen (Germany),* pages 529–540. Springer-Verlag, Berlin, Heidelberg, 1997.

[218] A. P. Wierzbicki and J. Granat. Multi-objective modeling for engineering applications: DIDASN++ system. *European Journal of Operational Research,* 113:372–389, 1999.

[219] H.-M. Winkels and M. Meika. An integration of efficiency projections into the Geoffrion approach for multiobjective linear programming. *European Journal of Operational Research,* 16:113–127, 1984.

[220] E. Wood, N. P. Greis, and R. E. Steuer. Linear and nonlinear applications of the Tchebycheff metric to the multi criteria water allocation problem. In S. Rinaldi, editor, *Environmental Systems Analysis and Management,* pages 363–376. North-Holland Publishing Company, Amsterdam, New York, 1982.

[221] J.-B. Yang. Gradient projection and local region search for multiobjective optimization. *European Journal of Operational Research,* 112:432–459, 1999.

[222] J.-B. Yang, C. Chen, and Z.-J. Zhang. The interactive step tradeoff method (ISTM) for multiobjective optimization. *IEEE Transactions on Systems, Man, and Cybernetics,* 20:688–695, 1990.

[223] J.-B. Yang and P. Sen. Preference modelling by estimating local utility functions for multiobjective optimization. *European Journal of Operational Research,* 95:115–138, 1996.

[224] P. L. Yu. A class of solutions for group decision problems. *Management Science,* 19:936–946, 1973.

[225] P. L. Yu. *Multiple-Criteria Decision Making Concepts, Techniques, and Extensions.* Plenum Press, New York, 1985.

[226] L. Zadeh. Optimality and non-scalar-valued performance criteria. *IEEE Transactions on Automatic Control,* 8:59–60, 1963.

[227] M. Zeleny. Compromise programming. In J. L. Cochrane and M. Zeleny, editors, *Multiple Criteria Decision Making,* pages 262–301. University of South Carolina Press, Columbia, South Carolina, 1973.

[228] M. Zeleny. *Linear Multiobjective Programming.* Springer-Verlag, Berlin, Heidelberg, 1974.

[229] M. Zeleny. The theory of displaced ideal. In M. Zeleny, editor, *Multiple Criteria Decision Making Kyoto 1975,* pages 153–206. Springer-Verlag, Berlin, Heidelberg, 1976.

[230] S. Zionts and J. Wallenius. An interactive programming method for solving the multiple criteria problem. *Management Science,* 22:652–663, 1976.

[231] S. Zionts and J. Wallenius. An interactive multiple objective linear programming method for a class of underlying nonlinear utility functions. *Management Science,* 29:519–529, 1983.

Chapter 6

EVOLUTIONARY ALGORITHMS AND MULTIPLE OBJECTIVE OPTIMIZATION

Carlos A. Coello Coello
CINVESTAV-IPN, Departamento de Ingeniería Eléctrica
Av. Instituto Politécnico Nacional No. 2508, Col. San Pedro Zacatenco
México, D.F. 07300
Mexico
ccoello@cs.cinvestav.mx

Carlos E. Mariano Romero
Instituto Mexicano de Tecnología del Agua
Paseo Cuauhnáhuac 8532, Col. Progreso
Jiutepec, Morelos 62550
Mexico
cmariano@tlaloc.imta.mx

Abstract This chapter presents a review of the most important evolutionary multiobjective optimization techniques developed to date. Using as a basis a simple taxonomy of approaches, we briefly describe and analyze the advantages and disadvantages of each of them, together with some of their applications reported in the literature. Other important issues such as diversity and some of the main techniques developed to preserve it, as well as the need of suitable test functions and metrics that can properly evaluate the performance of these multiobjective optimization techniques are also addressed. We conclude this chapter with a brief outline of some potential paths of future research in this area.

Keywords: Evolutionary algorithms, evolutionary multiobjective optimization, genetic algorithms, multiobjective optimization, vector optimization.

1. Introduction

The idea of using techniques based on the emulation of the mechanism of natural selection to solve problems can be traced as long back as the 1930s [12]. However, it was not until the 1960s that the three main techniques based on this notion were developed: genetic algorithms [75], evolution strategies [142] and evolutionary programming [50]. These approaches, which are now collectively denominated "evolutionary algorithms", have been very effective for single-objective optimization [58, 144, 51].

Evolutionary algorithms seem also particularly desirable for solving multiobjective optimization problems because they deal simultaneously with a set of possible solutions (the so-called population) which allows us to find several members of the Pareto optimal set in a single run of the algorithm, instead of having to perform a series of separate runs as in the case of the traditional mathematical programming techniques. Additionally, evolutionary algorithms are less susceptible to the shape or continuity of the Pareto front (e.g., they can easily deal with discontinuous and concave Pareto fronts), whereas these two issues are a real concern for mathematical programming techniques.

The potential of evolutionary algorithms in this field was indicated in the late 1960s by Rosenberg [132], but the first implementation was not produced until the mid-1980s [137, 138]. Since then, a considerable amount of research has been done in this area, now known as evolutionary multi-objective optimization (EMOO for short). The growing importance of this field is reflected by a significant increment (mainly during the last five years) of technical papers in international conferences and peer-reviewed journals, special sessions in international conferences and interest groups on the Internet[1].

The content of this chapter is organized as follows: first, we will define the terminology that we will adopt and we will describe the general multiobjective optimization problem. Then, we will give some basic notions of evolutionary algorithms. After that, we will analyze the main evolutionary multiobjective optimization techniques that have been proposed in the specialized literature. Each technique will be briefly described and criticized. We will also provide some sample applications of each. Then, we will describe some of the main approaches proposed to maintain diversity, emphasizing the importance that this process has in multiob-

[1]The first author maintains an EMOO repository with over 850 bibliographical entries at: http://delta.cs.cinvestav.mx/~ccoello/EM00, with mirrors at http://www.lania.mx/~ccoello/EM00/ and http://www.jeo.org/emo/

jective optimization. Test functions and metrics proposed for EMOO techniques are also discussed together with some representative applications reported in the literature. Finally, we will describe some of the potential research paths in this area.

2. Definitions

The emphasis of this chapter is the solution of multiobjective optimization problems (MOPs) of the form:

$$\text{minimize } [f_1(\vec{x}), f_2(\vec{x}), \ldots, f_k(\vec{x})] \tag{6.1}$$

subject to the m inequality constraints:

$$g_i(\vec{x}) \geq 0 \quad i = 1, 2, \ldots, m \tag{6.2}$$

and the p equality constraints:

$$h_i(\vec{x}) = 0 \quad i = 1, 2, \ldots, p \tag{6.3}$$

where k is the number of objective functions $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$. We call $\vec{x} = [x_1, x_2, \ldots, x_n]^T$ the vector of decision variables. We wish to determine from among the set $\mathcal{F}$ of all vectors which satisfy (6.2) and (6.3) the particular set of values $x_1^*, x_2^*, \ldots, x_n^*$ which yield the optimum values of all the objective functions.

2.1. Pareto Optimality

It is rarely the case that there is a single point that simultaneously optimizes all the objective functions. Therefore, we normally look for "trade-offs", rather than single solutions when dealing with multiobjective optimization problems. The notion of "optimality" is therefore, different. The most commonly adopted notion of optimality is that originally proposed by Francis Ysidro Edgeworth [44] and later generalized by Vilfredo Pareto [114]. Although some authors call this notion *Edgeworth-Pareto optimality* (see for example [152]), we will use the most commonly accepted term: *Pareto optimality*.

We say that a vector of decision variables $\vec{x}^* \in \mathcal{F}$ is *Pareto optimal* if there does not exist another $\vec{x} \in \mathcal{F}$ such that $f_i(\vec{x}) \leq f_i(\vec{x}^*)$ for all $i = 1, \ldots, k$ and $f_j(\vec{x}) < f_j(\vec{x}^*)$ for at least one j.

In words, this definition says that $\vec{x}^*$ is Pareto optimal if there exists no feasible vector of decision variables $\vec{x} \in \mathcal{F}$ which would decrease some criterion without causing a simultaneous increase in at least one other criterion. Unfortunately, this concept almost always gives not a single solution, but rather a set of solutions called the *Pareto optimal set.* The

vectors $\vec{x}^*$ corresponding to the solutions included in the Pareto optimal set are called *nondominated.* The image of the Pareto optimal set under the objective functions is called *Pareto front.*

3. Notions of Evolutionary Algorithms

The term evolutionary computing or evolutionary algorithms is generically applied to a set of biologically-inspired techniques (inspired by the Neo-Darwinian theory of natural evolution[2]. Although three main paradigms are normally considered (evolutionary programming [50, 51], evolution strategies [143, 144], and genetic algorithms [76, 58]), nowadays it becomes increasingly difficult to distinguish the differences among them, and researchers tend to use the broader term "evolutionary algorithms" to refer to any technique that is based in the principle of natural selection (or survival of the fittest) originally defined by Charles Darwin [29].

In nature, individuals have to adapt to their environment in order to survive in a process called "evolution", in which those features that make an individual more suited to compete are preserved when it reproduces, and those features that make it weaker are eliminated. Such features are controlled by units called genes which form sets called chromosomes. Over subsequent generations not only the fittest individuals survive, but also their fittest genes which are transmitted to their descendants during the sexual recombination process which is called crossover.

In general terms, to simulate an evolutionary process in a computer, we need the following [105]:

- A representation for potential solutions to the problem.
- A way to create an initial population of potential solutions (this is normally done randomly, but deterministic approaches can also be used).
- An evaluation function that plays the role of the environment, rating solutions in terms of their "fitness".
- Genetic operators that alter the composition of the offspring generated (normally, crossover and mutation).
- Values for various parameters that the evolutionary algorithm uses (population size, probabilities of applying genetic operators, etc.).

[2]The Neo-Darwinian theory of natural evolution combines the original evolutionary theory of Charles Darwin (based on the survival of the fittest), the selectionism of August Weismann and Mendel's inheritance laws. It is called "Neo-Darwinian", because it improves the original proposal of Charles Darwin.

These elements are important both for single- and for multi-objective optimization. However, in multi-objective optimization, two more issues must be kept in mind: how to select individuals so that they correspond to elements of the Pareto optimal set, and how to keep diversity to avoid convergence of all the population to a single solution.

4. Classifying Techniques

A considerable amount of EMOO techniques have been developed in recent years [19, 162]. In an attempt to discuss the most important approaches proposed, we decided to classify these techniques using the following scheme:

- Non-Pareto Techniques
  - Aggregating approaches
  - VEGA
  - Lexicographic ordering
  - The ε-constraint method
  - Target-vector approaches
  - Game theory
- Pareto-based Techniques
  - Pure Pareto ranking
  - MOGA
  - NSGA
  - NPGA
  - Non-generational approaches
- Recent Approaches
  - PAES
  - SPEA
  - Micro-Genetic Algorithm

5. Non-Pareto Techniques

Under this category, we will consider approaches that do not incorporate directly the concept of Pareto optimality (or Pareto dominance). The approaches discussed in this section are all very efficient (computationally speaking), but most of them are incapable of producing certain

portions of the Pareto front. Others could be appropriate to handle only a few objectives. However, their simplicity and efficiency has made them popular among a certain sector of researchers.

5.1. Aggregating Approaches

Perhaps the most straightforward approach to handle multiple objectives with any technique is to use a combination of all the objectives into a single one using either an addition, multiplication or any other combination of arithmetical operations that we could think of. These techniques are normally known as "aggregating functions", because they combine (or "aggregate") all the objectives of the problem into a single one. In fact, aggregating approaches are the oldest mathematical programming methods for multiobjective optimization, since they can be derived from the Kuhn-Tucker conditions for nondominated solutions [89].

An example of this approach is a sum of weights of the form:

$$\min \sum_{i=1}^{k} w_i f_i(\vec{x}) \tag{6.4}$$

where $w_i \geq 0$ are the weighting coefficients representing the relative importance of the k objective functions of our problem. It is usually assumed that

$$\sum_{i=1}^{k} w_i = 1 \tag{6.5}$$

Aggregating functions have been used with evolutionary algorithms in a number of occasions, with relative success in problems in which the behavior of the objective functions is more or less well-known.

It is normal practice in aggregating approaches to vary the weighting coefficients used, so that different portions of the Pareto front can be generated. However, it is important to realize that the weighting coefficients do not reflect proportionally the relative importance of the objectives (unless a proper scaling of the objectives takes place), but are only factors which, when varied, locate elements from the Pareto optimal set.

5.1.1 Advantages and Disadvantages. The main advantages of this method are its simplicity (it is easy to implement and use) and its efficiency (computationally speaking). Its main disadvantage is the difficulty to determine the appropriate weight coefficients to be used when we do not have enough information about the problem (this is

an important concern, particularly in real-world applications). Also, a proper scaling of the objectives requires a considerable amount of extra knowledge about the problem. To obtain this information could be a very expensive process (computationally speaking). A more serious drawback of this approach is that it cannot generate certain portions of the Pareto front when its shape is concave, regardless of the weights combination used [30]. Nevertheless, aggregating functions could be very useful to get a preliminary sketch of the Pareto front of a certain problem, or to provide prior information to be exploited by another approach.

5.1.2 Some Applications.

- Water quality control [15].
- Controller design [40].
- Design of optical filters for lamps [46].
- Improvement of wire-antenna geometries [166].

5.2. VEGA

This is the first actual implementation of an evolutionary multiobjective optimization technique, which was made by Schaffer [137, 138] in the mid-1980s. The approach was called the *Vector Evaluated Genetic Algorithm* (VEGA), and it basically consisted of a simple genetic algorithm (GA) with a modified selection mechanism. At each generation, a number of sub-populations were generated by performing proportional selection according to each objective function in turn. Thus, for a problem with k objectives, k sub-populations of size N/k each would be generated (assuming a total population size of N). These sub-populations would then be shuffled together to obtain a new population of size N, on which the GA would apply the crossover and mutation operators in the usual way. This process is illustrated in Figure 6.1.

Schaffer realized that the solutions generated by his system were nondominated in a local sense, because their nondominance was limited to the current population, and while a locally dominated individual is also globally dominated, the converse is not necessarily true [138]. An individual which is not dominated in one generation may become dominated by an individual who emerges in a later generation. Also, he noted a problem that in genetics is known as "speciation" (i.e., we could have the evolution of "species" within the population which excel on different aspects of performance). This problem arises because this technique selects individuals who excel in one dimension of performance, without

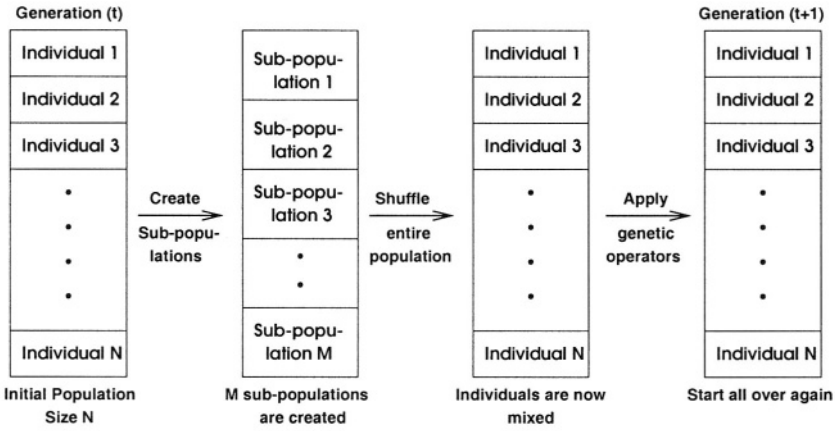


Figure 6.1. Scheme of VEGA's selection mechanism. It is assumed that the population size is N and that there are M objective functions.

looking at the other dimensions. The potential danger doing that is that we could have individuals with what Schaffer called "middling" performance[3] in all dimensions, which could be very useful for compromise solutions, but which will not survive under this selection scheme, since they are not in the extreme for any dimension of performance (i.e., they do not produce the best value for any objective function, but only moderately good values for all of them). Speciation is undesirable because it is opposed to our goal of finding a compromise solution. Schaffer suggested some heuristics to deal with this problem. For example, to use a heuristic selection preference approach for nondominated individuals in each generation, to protect our "middling" chromosomes. Also, crossbreeding among the "species" could be encouraged by adding some mate selection heuristics instead of using the random mate selection of the traditional GA.

5.2.1 Advantages and Disadvantages. VEGA is very simple and easy to implement, since only the selection mechanism of a traditional GA has to be modified. However, the shuffling and merging of all the sub-populations that VEGA does corresponds to averaging the fitness components associated with each of the objectives [60]. Since Schaffer used proportional fitness assignment [58], these fitness

[3]By "middling", Schaffer meant an individual with acceptable performance, perhaps above average, but not outstanding for any of the objective functions.

components were in turn proportional to the objectives themselves [53]. Therefore, the resulting expected fitness corresponded to a linear combination of the objectives where the weights depended on the distribution of the population at each generation as shown by Richardson et al. [128]. This means that VEGA has the same problems as the aggregating approaches (i.e., it is not able to generate concave portions of the Pareto front). Nevertheless, VEGA has been found useful in other domains such as constraint-handling, where its biased behavior can be of great help [154, 22].

5.2.2 Some Applications.

- Groundwater pollution containment [129].
- Optimum placement of aerodynamic actuators for aircraft control [131, 130].
- Design of combinational circuits at the gate-level [22].
- Constraint-handling in evolutionary algorithms used for single-objective optimization [21, 154, 153].

5.3. Lexicographic Ordering

In this method, the user is asked to rank the objectives in order of importance. The optimum solution $\vec{x}^*$ is then obtained by minimizing the objective functions, starting with the most important one and proceeding according to the assigned order of importance of the objectives.

Let the subscripts of the objectives indicate not only the objective function number, but also the priority of the objective. Thus, $f_1(\vec{x})$ and $f_k(\vec{x})$ denote the most and least important objective functions, respectively. Then the first problem is formulated as

$$\text{Minimize} \quad f_1(\vec{x}) \tag{6.6}$$

subject to

$$g_j(\vec{x}) \leq 0; \quad j = 1, 2, \ldots, m \tag{6.7}$$

and its solution $\vec{x}_1^*$ and $f_1^* = (\vec{x}_1^*)$ is obtained. Then the second problem is formulated as

$$\text{Minimize} \quad f_2(\vec{x}) \tag{6.8}$$

subject to

$$g_j(\vec{x}) \leq 0; \quad j = 1, 2, \ldots, m \tag{6.10}$$

$$f_1(\vec{x}) = f_1^* \tag{6.11}$$

and the solution of this problem is obtained as x_2^* and $f_2^* = f_2(x_2^*)$. This procedure is repeated until all k objectives have been considered. The ith problem is given by

$$\text{Minimize} \quad f_i(\vec{x}) \tag{6.12}$$

subject to

$$g_j(\vec{x}) \leq 0; \quad j = 1, 2, \ldots, m \tag{6.14}$$

$$f_l(\vec{x}) = f_l^*, \quad l = 1, 2, \ldots, i-1 \tag{6.15}$$

The solution obtained at the end, i.e., x_k^* is taken as the desired solution x^* of the problem.

Fourman [55] suggested a selection scheme based on lexicographic ordering. In a first version of his algorithm, objectives are assigned different priorities by the user and each pair of individuals are compared according to the objective with the highest priority. If this resulted in a tie, the objective with the second highest priority was used, and so on. In another version of this algorithm (that apparently worked quite well), an objective is randomly selected at each run.

5.3.1 Advantages and Disadvantages. This technique explores objective space unequally, in the sense that priority is given to solutions performing well in one objective over another(s). Or, in other words, one objective is optimized at all costs. This approach appears most suitable only when the importance of each objective (in comparison to the others) is clearly known.

Selecting randomly an objective (as in the case of Fourman [55]) is equivalent to a weighted combination of objectives, in which each weight is defined in terms of the probability that each objective has of being selected. However, the use of tournament selection with this approach makes an important difference with respect to other approaches such as VEGA, because the pairwise comparisons of tournament selection will make scaling information negligible [53]. This means that this approach may be able to see as convex a concave trade-off surface, although that really depends on the distribution of the population and on the problem itself. Its main weakness is that this approach will tend to favor certain objectives when many are present in the problem, because of the randomness involved in the process, and this will have the undesirable consequence of making the population to converge to a particular part of the Pareto front rather than to delineate it completely [27]. The main advantage of this approach is its simplicity and computational efficiency. These two properties make it highly competitive with other non-Pareto approaches such as a weighted sum of objectives or VEGA.

5.3.2 Some Applications.

- Symbolic layout compaction [55].
- Tuning of a fuzzy controller for the guidance of an autonomous vehicle in an elliptic road [56].

5.4. The ε-Constraint Method

This method is based on minimization of one (the most preferred or primary) objective function, and considering the other objectives as constraints bound by some allowable levels ε_i. Hence, a single objective minimization is carried out for the most relevant objective function f_1 subject to additional constraints on the other objective functions. The levels ε_i are then altered to generate the entire Pareto optimal set. The method may be formulated as follows:

1 Find the minimum of the rth objective function, i.e. find $\vec{x}^*$ such that

$$f_r(\vec{x}^*) = \min_{x \in \mathcal{F}} f_r(\vec{x}) \tag{6.16}$$

subject to additional constraints of the form

$$f_i(\vec{x}) \leq \varepsilon_i \quad \text{for } i = 1, 2, \ldots, k \quad \text{and } i \neq r \tag{6.17}$$

where ε_i are assumed values of the objective functions which we do not wish to exceed.

2 Repeat 1 for different values of ε_i. The information derived from a well chosen set of ε_i can be useful in making the decision. The search ends when the user finds a satisfactory solution.

It may be necessary to repeat the above procedure for different indices r.

To get adequate ε_i values, single-objective optimizations are normally carried out for each objective function in turn by using mathematical programming techniques (or independent EAs). For each objective function f_i $(i = 1, 2, \ldots, k)$, there is an optimal solution vector $\vec{x}_i^*$ for which $f_i(\vec{x}_i^*)$ is a minimum. Let $f_i(\vec{x}_i^*)$ be the lower bound on ε_i, i.e.

$$\varepsilon_i \geq f_i(\vec{x}_i^*) \quad i = 1, 2, \ldots, r-1, r+1, \ldots, k \tag{6.18}$$

and $f_i(\vec{x}_r^*)$ be the upper bound on ε_i, i.e.

$$\varepsilon_i \leq f_i(\vec{x}_r^*) \quad i = 1, 2, \ldots, r-1, r+1, \ldots, k \tag{6.19}$$

When the bounds ε_i are too low, there is no solution and at least one of these bounds must be relaxed.

This technique has been hybridized with EAs on several occasions. The idea is to use only one objective function at a time as the fitness function of the EA, and keep the others constant (constrained to a single value). Then, the EA is run several times varying the constrained values, so that the Pareto front of the problem can be generated.

5.4.1 Advantages and Disadvantages. The main disadvantage of this approach is its (potentially high) computational cost. Also, the encoding of the objective functions may be extremely difficult or even impossible for certain applications, particularly if there are too many objectives. Nevertheless, the relative simplicity of the technique (its main advantage) has made it popular among some researchers (particularly in engineering).

5.4.2 Some Applications.

- Preliminary design of a marine vehicle [94].
- Groundwater pollution containment problems [149].
- Fault tolerant system design [139].

5.5. Target-Vector Approaches

This category encompasses methods in which we have to define a set of goals (or targets) that we wish to achieve for each objective function under consideration. The EA in this case will then try to minimize the difference between the current solution generated and the vector of desirable goals (different metrics can be used for this purpose). Although target vector approaches can be considered as another aggregating approach, we decided to discuss them separately because these techniques can generate (under certain conditions) concave portions of the Pareto front, whereas approaches based on simple weighted sums cannot.

The most popular techniques included here are hybrids of EAs with: Goal Programming [32, 170, 135], Goal Attainment [171, 177], and the min-max algorithm [67, 23].

5.5.1 Advantages and Disadvantages. The main advantage of these methods is their simplicity and their efficiency (computationally speaking) because they do not require a Pareto ranking procedure. However, their main disadvantage is the definition of the desired goals which requires some extra computational effort. Some target vector ap-

proaches have additional problems. For example, Wilson and MacLeod [171] found that goal attainment could generate, under certain circumstances, a misleading selection pressure. For example, if we have two candidate solutions which are the same in one objective function value but different in the other, they will still have the same goal-attainment value for their two objectives, which means that for the EA neither of them will be better than the other.

An additional problem with these techniques is that they will yield a nondominated solution only if the goals are chosen in the feasible domain, and such conditions may certainly limit their applicability.

5.5.2 Some Applications.

- Design of multiplierless IIR filters [171].
- Structural optimization [135, 67],
- Optimization of the counterweight balancing of a robot arm [25].

5.6. Game Theory

We can analyze this technique with reference to a simple optimization problem with two objectives and two decision variables whose graphical representation is shown in Figure 6.2. Let $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$ represent two scalar objectives and x_1 and x_2 two scalar variables. It is assumed that one player is associated with each objective. The first player wants to select a variable x_1 which will minimize his objective function f_1, and similarly the second player seeks a variable x_2 which will minimize his objective function f_2. If f_1 and f_2 are continuous, then the contours of constant values of f_1 and f_2 appear as shown in Figure 6.2. The dotted lines passing through O_1 and O_2 represent the loci of rational (minimizing) choices for the first and second player for a fixed value of x_2 and x_1, respectively. The intersection of these two lines, if it exists, is a candidate for the two objective minimization problem, assuming that the players do not cooperate with each other (*noncooperative game*). In Figure 6.2, the point $N(x_1^*, x_2^*)$ represents such an intersection point. This point, known as a *Nash equilibrium* solution, represents a stable equilibrium condition in the sense that no player can deviate unilaterally from this point for further improvement of his own criterion [110].

This point has the characteristic that

$$f_1(x_1^*, x_2^*) \leq f_1(x_1, x_2^*) \tag{6.20}$$

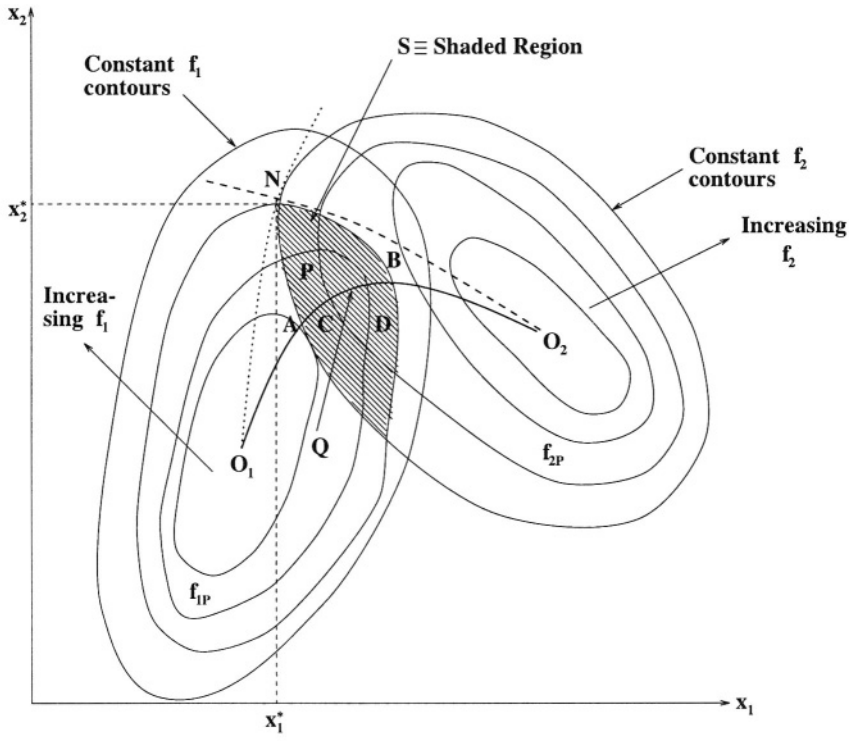


Figure 6.2. Example of cooperative and non-cooperative game solutions.

and

$$f_2(x_1^*, x_2^*) \leq f_2(x_1^*, x_2) \tag{6.21}$$

where x_1 may be to the left or right of x_1^* in (6.20) and x_2 may lie above or below x_2^* in (6.21).

5.6.1 Advantages and Disadvantages. The main advantage of this approach is that it is very efficient (computationally speaking). However, under certain circumstances, it could generate a single non-dominated vector instead of a set of them (as in [117]). Nevertheless, it is possible to extend this approach to k players (where k is the number of objectives of a problem), and to have several Nash equilibrium points, with which the entire Pareto front of a problem can actually be found, although a *cooperative game* may be preferred in that case over a *non-cooperative* approach [124, 123].

5.6.2 Some Applications.

- Truss optimization [37, 125].

- Minimization of the backscattering of aerodynamic reflectors [116, 117].

6. Pareto-Based Techniques

The idea of using Pareto-based fitness assignment was first proposed by Goldberg [58] to solve the problems of Schaffer's approach [138]. He suggested the use of nondominated ranking and selection to move a population toward the Pareto front in a multiobjective optimization problem. The basic idea is to find the set of strings in the population that are Pareto nondominated by the rest of the population. These strings are then assigned the highest rank and eliminated from further contention. Another set of Pareto nondominated strings are determined from the remaining population and are assigned the next highest rank. This process continues until the population is suitably ranked. Goldberg also suggested the use of some kind of niching technique to keep the GA from converging to a single point on the front [34]. A niching mechanism such as sharing [60] would allow the EA to maintain individuals all along the nondominated frontier.

6.1. Pure Pareto Ranking

Although several variations of Goldberg's proposal have been proposed in the literature (see the following subsections), several authors have used what we call "pure Pareto ranking". The idea in this case is to follow Goldberg's proposal as stated in his book [58].

6.1.1 Advantages and Disadvantages. The main weakness of Pareto ranking in general is that there is no efficient algorithm to check for nondominance in a set of feasible solutions (the conventional process is $O(kM^2)$, where k is the number of objectives and M is the population size). Therefore, any traditional algorithm to check for Pareto dominance exhibits a serious degradation in performance as we increase the size of the population and the number of objectives. Also, the use of sharing requires to estimate the value of the sharing factor, which is not easy, and the performance of the method relies a lot on this value. However, Pareto ranking is the most appropriate way to generate an entire Pareto front in a single run of an EA and its main advantage is that the approach is less susceptible to the shape or continuity

of the Pareto front, whereas these two issues are a serious concern for traditional mathematical programming techniques.

6.1.2 Applications.

- Optimal location of a network of groundwater monitoring wells [18].
- Pump scheduling [141, 136].
- Feasibility of full stern submarines [158].
- Optimal planning of an electrical power distribution system [121].

6.2. MOGA

Fonseca and Fleming [52] proposed a scheme called "Multi-Objective Genetic Algorithm" (MOGA), in which the rank of a certain individual corresponds to the number of chromosomes in the current population by which it is dominated. Consider, for example, an individual[4] x_i at generation t, which is dominated by $p_i^{(t)}$ individuals in the current generation. Its current position in the individuals' rank can be given by [52]:

$$\text{rank}(x_i, t) = 1 + p_i^{(t)} \tag{6.22}$$

All nondominated individuals are assigned rank 1, while dominated ones are penalized according to the population density of the corresponding region of the trade-off surface.

Fitness assignment is performed in the following way [52]:

1 Sort population according to rank.

2 Assign fitness to individuals by interpolating from the best (rank 1) to the worst (rank $n \leq N$) in the way proposed by Goldberg [58], according to some function, usually linear, but not necessarily.

3 Average the fitness of individuals with the same rank, so that all of them will be sampled at the same rate. This procedure keeps the global population fitness constant while maintaining appropriate selective pressure, as defined by the function used.

[4] An individual encodes the decision variables of the problem.

As Goldberg and Deb [59] point out, this type of blocked fitness assignment is likely to produce a large selection pressure that might produce premature convergence. To avoid that, Fonseca and Fleming [52] used a niche-formation method to distribute the population over the Pareto-optimal region, but instead of performing sharing on the parameter values, they have used sharing on the objective function values [150].

6.2.1 Advantages and Disadvantages. It has been indicated in the literature [150, 31] that the main drawback of MOGA is that it performs sharing on the objective value space, which implies that two different vectors with the same objective function values can not exist simultaneously in the population under this scheme. This is apparently undesirable, because these are precisely the kind of solutions that the user normally wants. However, nothing in the algorithm precludes it from performing sharing in decision variable space, and apparently this choice has been taken in some of the applications reported below.

The main advantage of MOGA is that it is efficient and relatively easy to implement [27, 162]. Its main weakness is that, as all the other Pareto ranking techniques, its performance is highly dependent on an appropriate selection of the sharing factor. However, it is important to add that Fonseca and Fleming [52] have developed a good methodology to compute this value for their approach.

6.2.2 Some Applications.

- Co-synthesis of hardware-software embedded systems [39].
- Design of active magnetic bearing controllers [140].
- Fault Diagnosis [100, 101, 99].
- Plane truss optimization [109, 3].

6.3. NSGA

Srinivas and Deb [150] proposed the "Nondominated Sorting Genetic Algorithm" (NSGA). This algorithm is based on several layers of classifications of the individuals as shown in Figure 6.3. Before selection is performed, the population is ranked on the basis of nondomination: all nondominated individuals are classified into one category (with a dummy fitness value, which is proportional to the population size, to provide an equal reproductive potential for these individuals). To maintain the diversity of the population, these classified individuals are shared with their dummy fitness values. Then this group of classified individuals is ignored and another layer of nondominated individuals is considered.

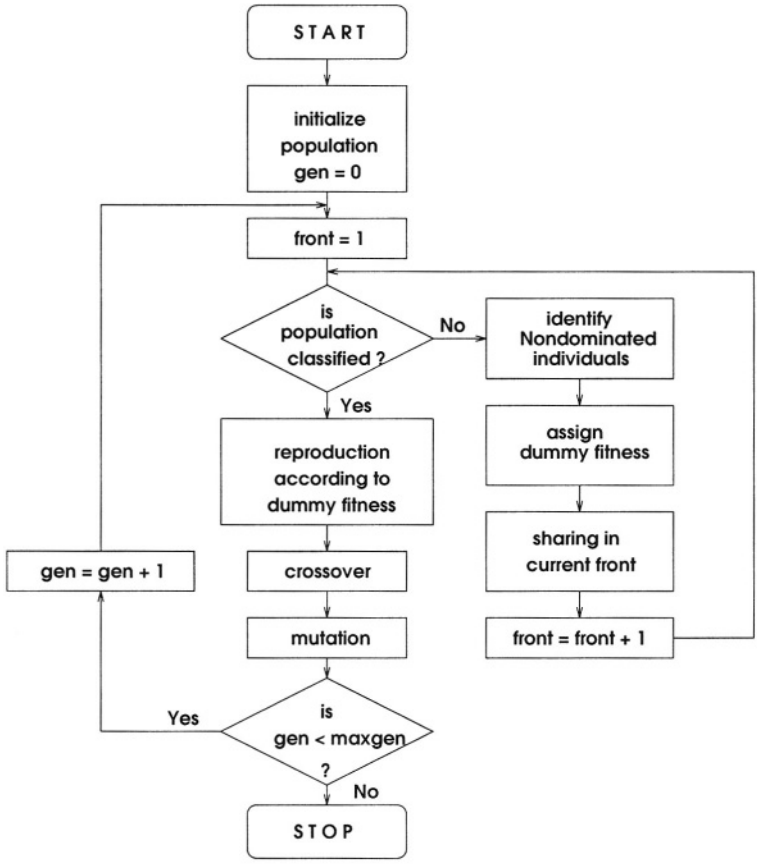


Figure 6.3. Flowchart of the Nondominated Sorting Genetic Algorithm (NSGA).

The process continues until all individuals in the population are classified. A stochastic remainder proportionate selection was adopted by the authors. Since individuals in the first front have the maximum fitness value, they always get more copies than the rest of the population. This allows to search for nondominated regions, and results in convergence of the population toward such regions. Sharing, by its part, helps to distribute the population over this region (i.e., the Pareto front of the problem).

6.3.1 Advantages and Disadvantages. Some researchers have reported that NSGA has a lower overall performance than MOGA, and it seems to be also more sensitive to the value of the sharing factor than MOGA [27, 162]. Other authors [180] report that the NSGA performed quite well in terms of "coverage" of the Pareto front (i.e., it

spreads in a more uniform way the population over the Pareto front) when applied to the 0/1 knapsack problem, but in these experiments no comparisons with MOGA were provided.

In any case, Deb et al. [33] have recently proposed a new version of this algorithm, called NSGA-II, which is more efficient (computationally speaking), uses elitism and a crowded comparison operator that keeps diversity without specifying any additional parameters. The new approach has not been extensively tested yet, but it certainly looks promising.

6.3.2 Some Applications.

- Computational fluid dynamics [98].
- Design of multilayer microwave absorbers [169], and thinned antenna arrays with digital phase shifters [168].
- Robust trajectory tracking problems [8].
- Design of optimal earth orbiting satellite constellations [103].

6.4. NPGA

Horn and Nafpliotis [77, 78] proposed a tournament selection scheme based on Pareto dominance. Two individuals randomly chosen are compared against a subset from the entire population (typically, around 10% of the population). When both competitors are either dominated or nondominated (i.e., there is a tie), the result of the tournament is decided through fitness sharing [60].

The pseudocode for Pareto domination tournaments assuming that all of the objectives are to be maximized is presented below [77]. S is an array of the N individuals in the current population, *random_pop_index* is an array holding the N indices of S, in a random order, and t_{dom} is the size of the comparison set.

```
function selection
        /* Returns an individual from the current population S */
begin
   shuffle(random_pop_index); /* Re-randomize random index array */
   candidate_1 = random_pop_index[1];
   candidate_2 = random_pop_index[2];
   candidate_1_dominated = false;
   candidate_2_dominated = false;
   for comparison_set_index = 3 to t_dom + 3 do
   /* Select t_dom individuals randomly from S */
      begin
```

```
            comparison_individual =
                random_pop_index[comparison_set_index];
            if S[comparison_individual] dominates S[candidate_1]
                then candidate_1_dominated = true;
            if S[comparison_individual] dominates S[candidate_2]
                then candidate_2_dominated = true;
        end /* end for loop */
    if ( candidate_1_dominated AND ¬ candidate_2_dominated )
        then return candidate_2;
    else if ( ¬ candidate_1_dominated AND candidate_2_dominated )
        then return candidate_1;
    else
        do sharing;

end
```

Horn and Nafpliotis [77, 78] also arrived at a form of fitness sharing in the objective domain, and suggested the use of a metric combining both the objective and the decision variable domains, leading to what they called *equivalent class sharing.*

6.4.1 Some Applications.

- Fault tolerant system design [139].
- Planning of a traffic route [64].
- Analysis of experimental spectra and monochromatic images [62].
- Partitioning and allocation of objects in heterogeneous distributed environments [17].

6.4.2 Advantages and Disadvantages. Since this approach does not apply Pareto selection to the entire population, but only to a segment of it at each run, its main advantage is that it is very fast and that it produces good nondominated fronts that can be kept for a large number of generations [27, 162]. However, its main disadvantage is that besides requiring a sharing factor, this approach also requires a good choice of the size of the set against which the two reference individuals will be compared (i.e., the tournament size), in order to perform well. This adds an extra parameter to the EA, which is also subject to certain fine tuning. Also, the NPGA has normally been used with population sizes considerably larger than usual with other approaches so that the

noise of the selection method can be tolerated by the emerging niches in the population [53].

6.5. Non-generational Approaches

Valenzuela-Rendón and Uresti-Charre [161] proposed a GA that uses non-generational selection and in which the fitness of an individual is calculated incrementally. The idea comes from Learning Classifier Systems (LCS),[5] in which it has been shown that a simple replacement of the worst individual in the population followed by an update of fitnesses of the rest of the population works better than a traditional (generational) GA. In the context of multiobjective optimization, what the authors did was to transform the problem with N objectives into another one with only two objectives: the minimization of domination count (weighted average of the number of individuals that have dominated this individual so far) and the minimization of the moving niche count (weighted average of the number of individuals that lie close according to a certain sharing function). Then, this biobjective optimization problem is transformed into a single objective optimization problem by performing a linear combination of these two objectives.

More recently, Borges & Barbosa [9] proposed another non-generational GA that reduces all the objectives of the problem to two measures related to dominance and population distribution. Such measures, however, are different in this case. The domination measure expresses the state of domination of a certain individual with respect to the current population. The neighbor density measure represents the size of the niche in which a certain individual is in. Fitness is then computed using a combination of these two measures. This approach presents several differences with respect to the previous one. For example, the dominance and neighborhood measures in this case consider the entire population instead of using a sampling of the population (as in the previous approach). Also, the several parameters required by the previous approach become unnecessary. This approach also compared well with respect to other EMOO techniques in several test functions.

6.5.1 Advantages and Disadvantages. The approach proposed by Valenzuela-Rendón and Uresti-Charre (1997) is really a more elaborate version of the weighted ranking techniques used by Bentley and Wakefield [6] (particularly the technique that they called weighted

[5]A classifier system is a machine learning system that learns syntactically simple string rules to guide its performance in an arbitrary environment [58].

average ranking—WAR). The main advantage of this approach is that it seems to provide good distributions in an efficient manner using well-known techniques taken from LCS. However, its main disadvantage is that it does not seem feasible to incorporate in this approach preferences of the objectives defined by the decision maker, which may be a drawback in real-world applications. Also, it does not seem clear how to define the six additional parameters (two more are fixed by the authors) required by this algorithm, which apparently require an empirical fine tuning as the other normal parameters of the GA (e.g., crossover and mutation rates).

The approach proposed by Borges & Barbosa [9] eliminates most of the drawbacks of Valenzuela-Rendón and Uresti-Charre's technique. However, the use of this approach has not been too widespread and we are not aware of its performance with a larger amount of objectives and in constrained search spaces.

6.5.2 Some Applications.

- Structural optimization [10].

7. Recent Approaches

Recently, several new EMOO approaches have been developed. We consider important to discuss briefly at least two of them: PAES and SPEA. Also, we will discuss some of our recent work regarding the use of a micro-genetic algorithm for multiobjective optimization.

7.1. PAES

The *Pareto Archived Evolution Strategy* (PAES) was introduced in [85] by Knowles & Corne. The idea of the approach is very simple. A (1+1) evolution strategy (i.e., a single parent that generates a single offspring) is used in combination with a historical archive that records all nondominated solutions previously found. This archive is used as a reference set against which each mutated individual will be compared. This is analogous to the tournament competitions held with the NPGA [78].

PAES also uses a novel approach to keep diversity, which consists of a crowding procedure that divides objective space in a recursive manner. Each solution is placed in a certain grid location based on the values of its objectives (which are used as its "coordinates" or "geographical location"). A map of this grid is maintained, indicating the number of solutions that reside in each grid location. Since the procedure is adaptive, no extra parameters are required (except for the number of

divisions of the objective space). Furthermore, the procedure has a lower computational complexity than traditional niching methods [85].

Since PAES is a very recent approach, only a few applications of it have been reported in the literature, all of them related to telecommunications problems [84, 85, 86].

7.2. SPEA

The *Strength Pareto Evolutionary Algorithm* (SPEA) was introduced by Zitzler & Thiele [181]. This approach was conceived as a way of integrating different EMOO techniques. SPEA uses an archive containing nondominated solutions previously found (the so-called external nondominated set). At each generation, nondominated individuals are copied to the external nondominated set. For each individual in this external set, a strength value is computed. This strength is similar to the ranking value of MOGA, since it is proportional to the number of solutions to which a certain individual dominates. The fitness of each member of the current population is computed according to the strengths of all external nondominated solutions that dominate it. Additionally, a clustering technique called "average linkage method" [107] is used to keep diversity.

SPEA has been used to explore trade-offs of software implementations for programmable digital signal processors (PDSP) [179] and to solve 0/1 knapsack problems [181].

7.3. A Micro-GA for Multiobjective Optimization

Currently, we have been experimenting with a a micro-GA (a GA with small population and a reinitialization mechanism [88]) for multiobjective optimization [26]. This approach uses two memories: the population memory, which is used as the source of diversity of the approach, and the external memory, which is used to archive members of the Pareto optimal set. Population memory is divided in two parts: a replaceable and a non-replaceable portion (the percentages of each can be regulated by the user).

The way in which this technique works is illustrated in Figure 6.4. First, an initial random population is generated. This population feeds the population memory, which is divided in two parts as indicated before. The non-replaceable portion of the population memory will never change during the entire run and is meant to provide the diversity required by the algorithm. The initial population of the micro-GA at the beginning of each of its cycles is taken (with a certain probability) from both portions of the population memory as to allow a greater diversity.

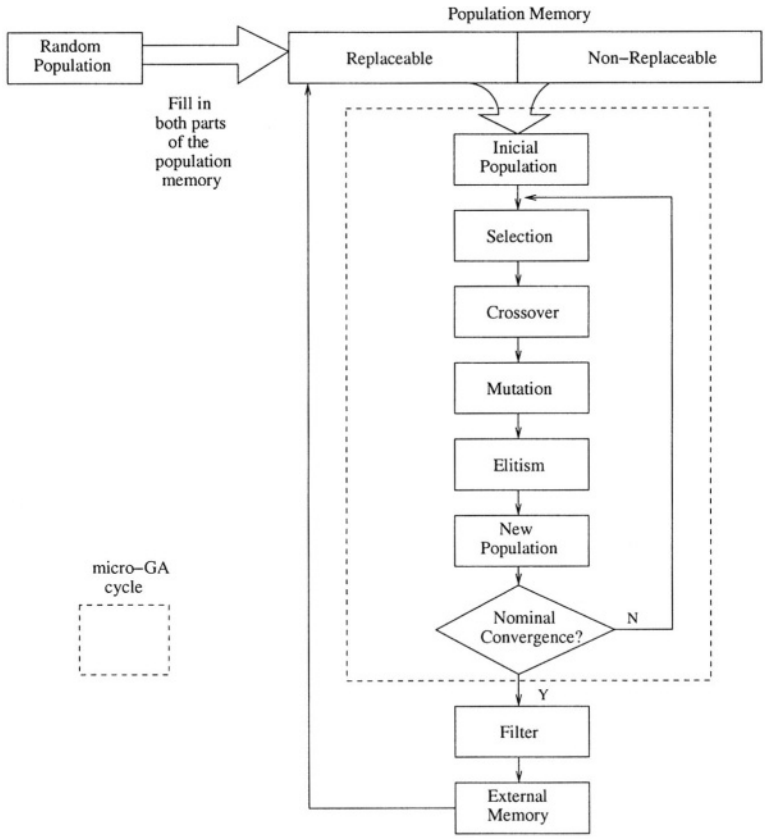


Figure 6.4. Diagram that illustrates the way in which the micro-GA for multiobjetive optimization works.

During each cycle, the micro-GA undergoes conventional genetic operators: tournament selection, two-point crossover, uniform mutation, and elitism (regardless of the amount of nondominated vectors in the population only one is arbitrarily selected at each generation and copied intact to the following one).

This approach uses three types of elitism. The first is based on the notion that if we store the nondominated vectors produced from each cycle of the micro-GA, we will not lose any valuable information obtained from the evolutionary process. The second is based on the idea that if we replace the population memory by the nominal solutions (i.e., the best solutions found when nominal convergence is reached), we will gradually converge, since crossover and mutation will have a higher probability of reaching the true Pareto front of the problem over time. This notion was hinted at by Goldberg [58]. Nominal convergence, in this case, is defined in terms of a certain (low) number of generations (typically, two to five in our case). However, similarities among the strings (either at the phenotypical or genotypical level) could also be used as a criterion for convergence. The third type of elitism is applied at certain intervals (defined by a parameter called "replacement cycle"). We take a certain amount of points from all the regions of the Pareto front generated so far and we use them to fill in the replaceable memory. Depending on the size of the replaceable memory, we choose as many points from the Pareto front as necessary to guarantee a uniform distribution. This process intends to use the best solutions generated so far as the starting point for the micro-GA, so that we can improve them (either by getting closer to the true Pareto front or by getting a better distribution). This also avoids that the content of the replaceable memory becomes homogeneous.

To keep diversity in the Pareto front, the micro-GA uses an approach similar to the adaptive grid proposed by Knowles & Corne [85]. The idea is that once the archive that stores nondominated solutions has reached its limit, the search space that this archive covers is divided, assigning a set of coordinates to each solution. Then, each newly generated nondominated solution will be accepted only if the geographical location to where the individual belongs is less populated than the most crowded location. Alternatively, the new nondominated solution could also be accepted if the individual belongs to a location outside the previously speficied boundaries. In other words, the less crowded regions are given preference so that the spread of the individuals on the Pareto front can be more uniform.

This approach allows the regulation of the amount of points from the Pareto front that the user wishes to find through the size of the external memory. Our preliminary results indicate that our micro-GA is able to generate the Pareto front of difficult test functions (i.e., disconnected and concave Pareto fronts) that have been previously adopted to evaluate EMOO techniques. Furthermore, the approach seems to exhibit a lower computational cost than the NSGA II and PAES while obtaining Pareto

fronts of similar quality. However, it also requires certain additional parameters and the sensitivity of the approach to them is still subject of ongoing research [26].

8. Diversity

Due to stochastic errors associated with its genetic operators, evolutionary algorithms tend to converge to a single solution when used with a finite population [34], As long as our goal is to find the global optimum (or at least a very good approximation of it), this behavior is acceptable. However, there are certain applications in which we are interested in finding not one, but several solutions. Multiobjective optimization is certainly one of those applications, because we want to find the entire Pareto front of a problem, and not only a single nondominated solution. The question is then how to keep the EA from converging to a single solution.

Early evolutionary computation researchers identified this convergence phenomenon of EAs, called *genetic drift* [36], and found that it happens in Nature as well. They correctly stated that the key to solve this problem is to find a way of preserving diversity in the population, and several proposals, modelled after natural systems were made. Holland [76] suggested the use of a "crowding" operator, which was intended to identify situations in which more and more individuals dominate an environmental niche, since in those cases the competition for limited resources increases rapidly, which will result in lower life expectancies and birth rate. DeJong [36] experimented with such a *crowding* operator, which was implemented by having a newly formed offspring to replace the existing individual more similar to itself. The similarity between two individuals was measured in the genotype by counting the number of bits along each chromosome that were equal in the two individuals being compared. DeJong used two parameters in his model: generation gap (G) and crowding factor (CF) [34]. The first parameter indicates the percentage of the population that is allowed to reproduce. The second parameter specifies the number of individuals initially selected as candidates to be replaced by a particular offspring [36]. Therefore, CF=1 means that no crowding will take place, and as we increase the value of CF, it becomes more likely that similar individuals replace one another [36].

Goldberg and Richardson [60] used a different approach in which the population was divided in different subpopulations according to the similarity of the individuals in two possible solution spaces: the decoded parameter space (phenotype) and the gene space (genotype). They defined

a sharing function $\phi(d_{ij})$ as follows [60]:

$$\phi(d_{ij}) = \begin{cases} 1 - \left(\frac{d_{ij}}{\sigma_{sh}}\right)^{\alpha}, & d_{ij} < \sigma_{share} \\ 0, & \text{otherwise} \end{cases} \tag{6.23}$$

where normally $\alpha = 1$, d_{ij} is a metric indicative of the distance between designs i and j, and σ_{share} is the sharing parameter which controls the extent of sharing allowed. The fitness of a design i is then modified as:

$$f_{s_i} = \frac{f_i}{\sum_{j=1}^{M} \phi(d_{ij})} \tag{6.24}$$

where M is the number of designs located in vicinity of the i-th design.

Deb and Goldberg [34] proposed a way of estimating the parameter σ_{share} in both phenotypical and genotypical space. In phenotypical sharing, the distance between 2 individuals is measured in decoded parameter space, and can be calculated with a simple Euclidean distance in a p-dimensional space, where p refers to the number of variables encoded in the GA; the value of d_{ij} can then be calculated as:

$$d_{ij} = \sqrt{\sum_{k=1}^{p} (x_{k,i} - x_{k,j})^2} \tag{6.25}$$

where $x_{1,i}, x_{2,i}, \ldots, x_{p,i}$ and $x_{1,j}, x_{2,j}, \ldots, x_{p,j}$ are the variables decoded from the EA.

To estimate the value of σ_{share}, Deb and Goldberg [34] proposed to use the expression:

$$\sigma_{share} = \frac{r}{\sqrt[p]{q}} = \frac{\sqrt{\sum_{k=1}^{p} (x_{k,max} - x_{k,min})^2}}{\sqrt[p]{2q}} \tag{6.26}$$

where r is the volume of a p-dimensional hypersphere of radius σ_{share} and q is the number of peaks that we want the EA to find.

In genotypical sharing, d_{ij} is defined as the Hamming distance between the strings and σ_{share} is the maximum number of different bits allowed between the strings to form separate niches in the population. The experiments performed by Deb and Goldberg [34] showed sharing as a better way of keeping diversity than crowding, and indicated that phenotypic sharing was better than genotypic sharing.

It should be added that much further work has been done regarding keeping the diversity in the population. Deb and Goldberg [34] suggested the use of restrictive mating with respect to the phenotypic

distance. The idea is to allow two individuals to reproduce only if they are very similar (i.e., if their phenotypic distance is less than a factor called σ_{share}). This is intended to produce distinct "species" (mating groups) in the population [106]. Other researchers such as Eshelman [47] and Eshelman & Schaffer [48] did exactly the opposite: they did not allow mating between individuals that were too similar (they said to be "preventing incest").

Smith et al. (1993) [148] proposed an approach, modelled after the immune system, that can maintain the diversity of the population without the use of an explicit sharing function. This approach has been actually used by Hajela et al. [66, 68] to handle constraints in structural optimization problems.

Poloni and Pediroda [119] proposed an interesting alternative to preserve diversity. They called their approach "local Pareto selection", and it basically consists of placing the population on a toroidal grid and choosing the members of the local tournament by means of a random walk in the neighborhoods of the given grid point.

Kita et al. [83] proposed the so called "Thermodynamical Genetic Algorithm" (TDGA) to maintain diversity when using a Pareto ranking technique for multiobjective optimization. The TDGA is inspired by the principle of minimal free energy used in simulated annealing [82]. The idea is to select the individuals for a new generation in such a way that the free energy F is minimized, and

$$F = \langle E \rangle - HT \tag{6.27}$$

where $\langle E \rangle$ is the mean energy of the system, H is the entropy and T is the temperature. The diversity of the population is controlled by adjusting T according to a certain schedule (as in simulated annealing). Presumably, T is less sensitive to the population size and to the size of the feasible region than traditional sharing functions [156].

Goldberg & Wang [61] proposed a coevolutionary adaptive niching scheme inspired on the economic model of monopolistic competition. The idea is to create two populations, one of businessmen and another one of customers. The population of customers is in fact the population of solutions to our problem (e.g., members of the Pareto optimal set) that will try to maximize a certain set of criteria, whereas the businessmen will try to locate themselves in such a way that their "profit" can be maximized. Customers will create niches according to their own criteria being optimized. Businessmen will then have to adapt to the current fitness landscape so that they can serve as many customers as possible. By enforcing a competition between these two populations, a uniform spread of the population of customers is expected to emerge.

Tan et al. [157] proposed the use of a dynamic sharing distance. The idea is to approximate the curvature of the trade-off curve formed by the nondominated solutions in objective space. The procedure then attempts to perform a uniform distribution of points along the Pareto front without requiring any prior parameters (the information required to bias the search is obtained from the evolutionary process itself).

Deb et al. [33] proposed the use of a crowding distance measure which represents the amount of solutions that lie within a certain neighborhood (in objective space). This approach is more efficient (computationally speaking) than traditional fitness sharing and does not require an extra parameter (i.e., σ_{share}).

Several other proposals exist (see [96] for a more detailed review of approaches to keep diversity). In fact, some researchers tend to develop their own variation of a certain technique or (in a few cases) to design an entirely new approach.

9. Test Functions

A very important aspect of this research area that has been generally disregarded in the technical literature is the use of appropriate test functions. In the early days of evolutionary multiobjective optimization, many researchers tested their approaches only with the two classic test functions provided by Schaffer in his seminal work on EMOO [138]. These functions are not only very simple (they have only two objectives), but are also unconstrained and do not show any of the most important aspects that would be interesting to analyze using an EMOO approach (e.g., ability of the algorithm to deal with concave or discontinuous Pareto fronts).

In recent years, several researchers have addressed the design of standard benchmarks against which any EMOO algorithm can be validated. Deb [31] has proposed ways to create controllable test problems for evolutionary multiobjective optimization techniques using single-objective optimization problems as a basis. Under this proposal, some problems that have been of great interest in evolutionary computation could be transformed into multiobjective optimization problems (e.g., deceptive and massively multimodal problems). Recently, this study has been extended to constrained multiobjective optimization problems [35].

Van Veldhuizen and Lamont [164, 165] have also proposed some guidelines to design a test function suite for evolutionary multiobjective optimization techniques (mainly combinatorial optimization problems). In more recent work, Van Veldhuizen [162] has also summarized most of

the test functions that have been previously suggested in the specialized literature.

Nevertheless, a more complete test suite is still required. Such a suite should contain problems of different degrees of difficulty (both constrained and unconstrained) and some real-world applications. If possible, good approximations of the true Pareto front of each problem should also be included. Furthermore, the test suite should be easily accessible (i.e., through the Internet), so that anyone interested in using it could use it. Such a test suite would become an important benchmark to validate any new EMOO technique developed.

10. Metrics

Closely related to the previous issue is the importance of defining good metrics to assess the effectiveness of an EMOO technique. The definition of such metrics is not an easy task since it is difficult to compare two vectors. Three are normally the issues to take into consideration to design a good metric in this domain [178]:

1 Minimize the distance of the Pareto front produced by our algorithm with respect to the true Pareto front (assuming we know its location).

2 Maximize the spread of solutions found, so that we can have a distribution of vectors as smooth and uniform as possible.

3 Maximize the amount of elements of the Pareto optimal set found.

There are several interesting proposals in the specialized literature that take into consideration at least one of these issues. The main proposals will briefly be described next:

1 **Enumeration**: Van Veldhuizen & Lament [164, 162] have proposed the use of parallel processing techniques to enumerate the entire intrinsic search space explored by an EA. This obviously allows to obtain the Pareto front that is global with respect to the granularity used. Knowing the global Pareto front of the problem, we can compare results against it, and devise different metrics for estimating how well our EA is performing.

   This approach might work with relatively short binary strings (Van Veldhuizen & Lamont [164] report success with strings ≤ 26 bits), but might not be suitable when using alphabets of higher cardinality (e.g., real-coded GAs) or longer binary strings.

2 **Spread**: Srinivas and Deb [150] proposed to measure the "spread" of points along the Pareto front using a statistical metric such as the chi-square distribution. This metric also assumes knowledge of the true Pareto front, and emphasizes the good distribution of points (determined through a set of niches) rather than a direct comparison between our Pareto front and the true Pareto front.

3 **Attainment Surfaces**: Fonseca and Fleming [54] proposed to draw a boundary in objective space separating those points which are dominated (by a certain set of points) from those which are nondominated. Such boundary was called "attainment surface". This attainment surface could then be used to determine the quality and the distribution of the nondominated points found by an EMOO approach. Multiple runs would then have to be performed and standard non-parametric statistical procedures would have to be applied to evaluate the quality of the nondominated vectors found. Several EMOO approaches can then be compared using this approach, but it is unclear how we can really assess how much better a certain approach is with respect to others [178].

4 **Generational Distance**: Van Veldhuizen & Lamont [163] proposed the use of a metric that estimates how far our current Pareto front is from the true Pareto front of a problem. This metric uses the Euclidean distance (measured in objective space) between each vector and the nearest member of the true Pareto front. Similar metrics have also been proposed by Schott [139], Rudolph [133], and Zitzler et al. [178]. The problem with this metric is that only distance to the true Pareto front is considered and not uniform spread along the Pareto front.

5 **Coverage**: Zitzler and Thiele [181] proposed two measures: the first concerns the size of the objective value space area which is covered by a set of nondominated solutions and the second compares directly two sets of nondominated solutions, using as a metric the fraction of the Pareto front covered by each of them. The first metric combines the three issues previously mentioned (distance, spread and amount of elements of the Pareto optimal set found) into a single value. Therefore, sets differing in more than one criterion could not be distinguished. The second metric is similar to the attainment surfaces of Fonseca & Fleming and it also has the same drawbacks.

In more recent work, Zitzler et al. [178] proposed several additional metrics for EMOO algorithms and also performed a detailed

comparative study using such metrics. More work in this area is, however, still needed.

11. Applications

An analysis of the evolution of the EMOO literature reveals some interesting facts. From the first EMOO approach published in 1985 [138] up to the first survey of the area published in 1995 [53], the number of published papers related to EMOO is relatively low. However, from 1995 to our days, the increase of EMOO-related papers is exponential. Today, the EMOO repository registers over 850 papers, from which a vast majority are applications. Given the large number of EMOO papers that currently exist, we will not attempt to produce a detailed review of applications in this section. Instead, we will delineate the most popular application fields, indicating some of the specific areas within them in which researchers have focused their efforts.

Current EMOO applications can be roughly classified in three large groups: engineering, industrial and scientific. Some specific areas within each of these groups are indicated next.

We will start with the engineering applications, which are, by far, the most popular in the literature. This should not be too surprising, since engineering disciplines normally have problems with better understood mathematical models which facilitates the use of evolutionary algorithms. A representative sample of engineering applications is the following (aeronautical engineering seems to be the most popular subdiscipline within this group):

- Electrical engineering [159, 108, 122]
- Hydraulic engineering [141, 136, 174]
- Structural engineering [95, 24, 173]
- Aeronautical engineering [72, 111, 167]
- Robotics [41, 57, 112]
- Control [40, 97, 42]
- Telecommunications [104, 86, 175]
- Civil engineering [49, 5, 81]
- Transport engineering [120, 2, 93]

Industrial applications occupy the second place in popularity in the EMOO literature. Within this group, scheduling is the most popular

subdiscipline. A representative sample of industrial applications is the following:

- Design and manufacture [63, 127, 113]
- Scheduling [155, 4, 14]
- Management [11, 87, 43]

Finally, we have a variety of scientific applications, from which the most popular are (for obvious reasons) those related to computer science:

- Chemistry [170, 74, 90]
- Physics [115, 117, 62]
- Medicine [176, 145, 92]
- Computer science [147, 13, 7]

The above distribution of applications indicates a strong interest for developing real-world applications of EMOO algorithms (something not surprising considering that most problems are of a multiobjective nature). Furthermore, the previous sample of EMOO applications should give a general idea of the application areas that have not been explored yet, some of which are mentioned in the following section.

12. Future Research Paths

Despite the noticeable increment in the amount of EMOO research in the last few years, there are still several open research areas. Some of them will be described next.

- **New Approaches:** Several new techniques have been proposed in the last few years. However, only a fistful of them have been adopted by a significant portion of the scientific community. In fact, some of these techniques widely used are already undergoing updates. MOGA [52], for example, has been recently hybridized with neural networks to improve its efficiency [42]. The NSGA [150] has undergone significant changes in its algorithmic structure and its diversity preservation approach, in order to make it more efficient [33]. But this may be only the beginning. We believe that the next few years will witness the development of many other new approaches (and updates of those currently in use). However, the focus of these developments will be different. Right now, for example, efficiency is the main issue. Researchers try to defeat

the inherent inefficiency associated with Pareto ranking and with traditional niching in order to produce new approaches whose computational cost is lower and therefore more suitable to be scaled to larger (real-world) problems. The use of local search with archival memories [85, 79, 26] and parallel selection strategies [104, 99, 73] are two of the alternatives currently explored, but several others are also possible. For example, little attention has been paid to the data structures used to store nondominated vectors in the current EMOO literature. In contrast, operational researchers have used efficient data structures for discrete multiobjective optimization (e.g., domination-free quad trees where a nondominated vector can be retrieved from the tree very efficiently). Checking if a new vector is dominated by the vectors in one of these trees can also be done very efficiently [65].

We also believe that multiobjective extensions of other heuristics will become popular in the next few years [102, 71, 28, 146, 126, 160, 16], as well as the hybridization of EAs with other heuristics (particularly to deal with multiobjective combinatorial optimization problems) [91, 38].

- **New Applications:** Despite the large amount of applications reported in the literature, many other domains remain practically unexplored. For example, the coordination of distributed agents is a problem that frequently involves globally conflicting solutions to multiple (local) objectives and it therefore lends itself naturally to a multiobjective optimization approach [118]. Other domain areas such as shape design [151] and constraint-handling [20] seem also very appropriate for testing new EMOO techniques. Additionally, EMOO researchers have not paid enough attention to multiobjective combinatorial optimization problems, which are not only challenging, but have also been studied in great depth [45]. Few EMOO researchers have actually used well-studied combinatorial optimization problems such as the 0/1 knapsack problem to validate EMOO approaches [181, 79, 80]. Finally, more real-world applications of EMOO techniques are also lacking in the current literature.

- **Theory:** There is a noticeable lack of research in theoretical issues related to EMOO. Most of the current studies available deal with convergence issues of EMOO algorithms [133, 134, 69, 70, 163], or with ways to compute niche sizes [52, 78]. However, many other important areas have not been studied. It would be very interesting to study, for example, the structure of fitness landscapes

in MOPs [172, 1]. Such study could provide some insights regarding the sort of problems that are particularly difficult for EAs and could also provide clues regarding the design of more powerful EMOO techniques. Furthermore, there is a need for detailed studies of the different aspects involved in the parallelization of EMOO techniques (e.g., load balancing, impact on Pareto convergence, performance issues, etc.), including new algorithms that are more suitable for parallelization than those currently in use.

- **Benchmarks:** We have mentioned some of the current work regarding the design of test functions that can be properly used to validate EMOO approaches. Despite these recent efforts, more work in this area is still necessary. Other domains such as constraint handling in the context of single-objective optimization could be used to validate in a more quantitative way the performance of EMOO approaches [20, 153]. A more systematic way of designing test functions is also required, focusing on the aspects that are more important to evaluate from an EMOO algorithm (e.g., its ability to deal with concave, discontinuous and highly-constrained search spaces). Closely related to this issue is the notorious lack of comparative studies in the current literature. Also, it is necessary to have more in-depth studies of metrics appropriate to evaluate the performance of EMOO techniques. Some of the efforts in that direction have also been discussed in this chapter, but more work is still required.

13. Summary

This chapter has reviewed some of the most important research done in evolutionary multiobjective optimization. We have discussed the main EMOO techniques currently in use, together with their advantages and disadvantages and some of their applications. Also, we have discussed the importance of diversity in the context of multiobjective optimization, reviewing some of the most important proposals found in the literature. Then, we have included a brief discussion of test functions and metrics used to validate EMOO techniques, addressing their importance to estimate (in a quantitative way) how good a certain technique is with respect to others. Finally, we have provided a representative sample of the types of applications of EMOO algorithms reported in the literature.

In the last section of this chapter, we have discussed some potential research areas that would be interesting to explore in more depth in the next few years. Some of them are already being studied, but others have not been addressed by any EMOO researchers. We expect that the general view of this relatively new field presented in this chapter can be of some use to the newcomers who want to become familiar with the research in this area in order to identify some possible research topic. Additionally, we also expect mature researchers and practitioners interested in evolutionary multiobjective optimization to find enough pointers as to allow them to initiate work in this area. As we mentioned before, this research discipline still has several open areas and possible application domains for those who may be interested.

Acknowledgments

The first author acknowledges support from the mexican Consejo Nacional de Ciencia y Tecnología (CONACyT) through project number 34201-A.

The second author acknowledges support from the mexican Consejo Nacional de Ciencia y Tecnología (CONACyT) through project number 33000-A, and from the Instituto Mexicano de Tecnología del Agua.

References

[1] L. Altenberg. NK fitness landscapes. In T. Bäck, D.B. Fogel, and Z. Michalewicz, editors, *Handbook of Evolutionary Computation,* Chapter B2.7.2. Oxford University Press, New York, NY, 1997.

[2] J.M. Anderson, T.M. Sayers, and M.G.H. Bell. Optimization of a fuzzy logic traffic signal controller by a multiobjective genetic algorithm. In *Proceedings of the Ninth International Conference on Road Transport Information and Control*, pages 186–190. IEE, London, UK, 1998.

[3] S. Azarm, B.J. Reynolds, and S. Narayanan. Comparison of two multiobjective optimization techniques with and within genetic algorithms. In *CD-ROM Proceedings of the 25th ASME Design Automation Conference,* Paper No. DETC99/DAC-8584. ASME Press, New York, NY, 1999.

[4] T.P. Bagchi. *Multiobjective Scheduling by Genetic Algorithms.* Kluwer Academic Publishers, Boston, MA, 1999.

[5] R. Balling and S. Wilson. The maximim fitness function for multiobjective evolutionary computation: Application to city planning.

In L. Spector, E.D. Goodman, A. Wu, W.B. Langdon, H.-M. Voigt, M. Gen, S. Sen, M. Dorigo, S. Pezeshk, M.H. Garzon, and E. Burke, editors, *Proceedings of the Genetic and Evolutionary Computation Conference (GECCO'2001)*, pages 1079–1084, Morgan Kaufmann Publishers, San Francisco, CA, 2001.

[6] P.J. Bentley and J.P. Wakefield. Finding acceptable solutions in the pareto-optimal range using multiobjective genetic algorithms. In P.K. Chawdhry, R.Roy, and R.K. Pant, editors, *Soft Computing in Engineering Design and Manufacturing,* Part 5, pages 231–240, Springer Verlag, London, UK, 1997.

[7] E. Bernadó i Mansilla and J.M. Garrell i Guiu. MOLeCS: Using multiobjective evolutionary algorithms for learning. In E. Zitzler, K. Deb, L. Thiele, C.A. Coello Coello, and D. Corne, editors, *First International Conference on Evolutionary Multi-Criterion Optimization,* pages 696–710. Lecture Notes in Computer Science No. 1993, Springer Verlag, Berlin, Germany, 2001.

[8] A.L. Blumel, E.J. Hughes, and B.A. White. Fuzzy autopilot design using a multiobjective evolutionary algorithm. In *2000 Congress on Evolutionary Computation,* volume 1, pages 54–61, IEEE Service Center, Piscataway, NJ, 2000.

[9] C.C.H. Borges and H.J.C. Barbosa. A non-generational genetic algorithm for multiobjective optimization. In *2000 Congress on Evolutionary Computation,* volume 1, pages 172–179, IEEE Service Center, Piscataway, NJ, 2000.

[10] C.C.H. Borges. *Algoritmos Genéticos para Otimização em Dinâmica de Estruturas* (In Portuguese). PhD thesis, Engheneria Civil, Universidade Federal do Rio de Janeiro, Brasil, 1999.

[11] R.A.C.M. Broekmeulen. Facility management of distribution centers for vegetables and fruits. In J. Biethahn and V. Nissen, editors, *Evolutionary Algorithms in Management Applications,* pages 199–210. Springer Verlag, Berlin, Germany, 1995.

[12] W. D. Cannon. *The Wisdom of the Body.* Norton and Company, New York, NY, 1932.

[13] A. Cardon, T. Galinho, and J.-P. Vacher. Genetic algorithms using multi-objectives in a multi-agent system. *Robotics and Autonomous Systems,* 33(2–3): 179–190, 2000.

[14] W.M. Carlyle, B. Kim, J.W. Fowler, and E.S. Gel. Comparison of multiple objective genetic algorithms for parallel machine scheduling problems. In E. Zitzler, K. Deb, L. Thiele, C.A. Coello Coello, and D. Corne, editors, *First International Conference on*

Evolutionary Multi-Criterion Optimization, pages 472–485. Lecture Notes in Computer Science No. 1993. Springer Verlag, Berlin, Germany, 2001.

[15] H.W. Chen and N.-B. Chang. Water pollution control in the river basin by fuzzy genetic algorithm-based multiobjective programming modeling. *Water Science and Technology,* 37(8):55–63, 1998.

[16] A.J. Chipperfield, J.F. Whidborne, and P.J. Fleming. Evolutionary algorithms and simulated annealing for MCDM. In T. Gal, T.J. Stewart, and T. Hanne, editors, *Multicriteria Decicion Making – Advances in MCDM Models, Algorithms, Theory, and Applications,* Chapter 16. Kluwer Academic Publishers, Boston, MA, 1999.

[17] S. Choi and C. Wu. Partitioning and allocation of objects in heterogeneous distributed environments using the niched pareto genetic-algorithm. In *Proceedings of the 5th Asia Pacific Software Engineering Conference (APSEC 98) Taipei, Taiwan, December 1998.* IEEE Service Center, Piscataway, NJ, 1998.

[18] S.E. Cieniawski, J.W. Eheart, and S. Ranjithan. Using genetic algorithms to solve a multiobjective groundwater monitoring problem. *Water Resources Research,* 31(2):399–409, 1995.

[19] C.A. Coello Coello. A comprehensive survey of evolutionary-based multiobjective optimization techniques. *Knowledge and Information Systems. An International Journal,* 1(3):269–308, 1999.

[20] C.A. Coello Coello. Constraint-handling using an evolutionary multiobjective optimization technique. *Civil Engineering Systems,* 17:319–346, 2000.

[21] C.A. Coello Coello. Treating constraints as objectives for single-objective evolutionary optimization. *Engineering Optimization,* 32(3):275–308, 2000.

[22] C.A. Coello Coello, A.H. Aguirre, and B.P. Buckles. Evolutionary multiobjective design of combinational logic circuits. In J. Lohn, A. Stoica, D. Keymeulen, and S. Colombano, editors, *Proceedings of the Second NASA/DoD Workshop on Evolvable Hardware,* pages 161–170, IEEE Computer Society, Los Alamitos, CA, 2000.

[23] C.A. Coello Coello and A.D. Christiansen. Two new GA-based methods for multiobjective optimization. *Civil Engineering Systems,* 15(3):207–243, 1998.

[24] C.A. Coello Coello and A.D. Christiansen. Multiobjective optimization of trusses using genetic algorithms. *Computers and Structures,* 75(6):647–660, 2000.

[25] C.A. Coello Coello, A.D. Christiansen, and A.H. Aguirre. Using a new GA-based multiobjective optimization technique for the design of robot arms. *Robotica,* 16(4):401–414, 1998.

[26] C.A. Coello Coello and G. Toscano. A micro-genetic algorithm for multiobjective optimization. In E. Zitzler, K. Deb, L. Thiele, C.A. Coello Coello, and D. Corne, editors, *First International Conference on Evolutionary Multi-Criterion Optimization,* pages 127–141. Lecture Notes in Computer Science No. 1993. Springer Verlag, Berlin, Germany, 2001.

[27] C.A. Coello Coello. *An Empirical Study of Evolutionary Techniques for Multiobjective Optimization in Engineering Design.* PhD thesis, Department of Computer Science, Tulane University, New Orleans, LA, 1996.

[28] P. Czyzak and A. Jaszkiewicz. Pareto simulated annealing – A metaheuristic technique for multiple-objective combinatorial optimization. *Journal of Multi-Criteria Decision Analysis,* 7:34–47, 1998.

[29] C. Darwin. *The Origin of Species by Means of Natural Selection or the Preservation of Favored Races in the Struggle for Life.* Random House, New York, NY, 1929.

[30] I. Das and J. Dennis. A closer look at drawbacks of minimizing weighted sums of objectives for pareto set generation in multicriteria optimization problems. *Structural Optimization,* 14(1):63–69, 1997.

[31] K. Deb. Evolutionary algorithms for multi-criterion optimization in engineering design. In K. Miettinen, M.M. Mäkelä, P. Neittaanmäki, and J. Periaux, editors, *Evolutionary Algorithms in Engineering and Computer Science,* pages 135–161. John Wiley & Sons, Chichester, UK, 1999.

[32] K. Deb. Solving goal programming problems using multi-objective genetic algorithms. In *1999 Congress on Evolutionary Computation,* pages 77–84, IEEE Service Center, Piscataway, NJ, 1999.

[33] K. Deb, S. Agrawal, A. Pratab, and T. Meyarivan. A fast elitist non-dominated sorting genetic algorithm for multi-objective optimization: NSGA-II. In M. Schoenauer, K. Deb, G. Rudolph, X. Yao, E. Lutton, J.J. Merelo, and H.-P. Schwefel, editors, *Proceedings of the Parallel Problem Solving from Nature VI Conference,* pages 849–858. Lecture Notes in Computer Science No. 1917. Springer Verlag, Berlin, Germany, 2000.

[34] K. Deb and D.E. Goldberg. An investigation of niche and species formation in genetic function optimization. In J.D. Schaffer, editor, *Proceedings of the Third International Conference on Genetic Algorithms,* pages 42–50. Morgan Kaufmann Publishers, San Mateo, CA, June 1989.

[35] K. Deb, A. Pratap, and T. Meyarivan. Constrained test problems for multi-objective evolutionary optimization. In E. Zitzler, K. Deb, L. Thiele, C.A. Coello Coello, and D. Corne, editors, *First International Conference on Evolutionary Multi-Criterion Optimization,* pages 284–298. Lecture Notes in Computer Science No. 1993. Springer Verlag, Berlin, Germany, 2001.

[36] A.K. DeJong. *An Analysis of the Behavior of a Class of Genetic Adaptive Systems.* PhD thesis, University of Michigan, Ann Arbor, MI, 1975.

[37] A.K. Dhingra and B. H. Lee. A genetic algorithm approach to single and multiobjective structural optimization with discrete-continuous variables. *International Journal for Numerical Methods in Engineering,* 37:4059–4080, 1994.

[38] R.P. Dick and N.K. Jha. CORDS: Hardware-software co-synthesis of reconfigurable real-time distributed embedded systems. In *Proceedings of the International Conference on Computer-Aided Design,* pages 62–68. IEEE Service Center, Piscataway, NJ, 1998.

[39] R.P. Dick and N.K. Jha. MOGAC: A multiobjective genetic algorithm for hardware-software co-synthesis of hierarchical heterogeneous distributed embedded systems. *IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems,* 17(10):920–935, 1998.

[40] D.C. Donha, D.S. Desanj, and M.R. Katebi. Genetic algorithm for weight selection in h_∞ control design. In T. Back, editor, *Proceedings of the Seventh International Conference on Genetic Algorithms,* pages 599–606. Morgan Kaufmann Publishers, San Mateo, CA, 1997.

[41] G.V. Dozier, S. McCullough, A. Homaifar, and L. Moore. Multiobjective evolutionary path planning via fuzzy tournament selection. In *IEEE International Conference on Evolutionary Computation (ICEC'98),* pages 684–689. IEEE Press, Piscataway, NJ, 1998.

[42] N.M. Duarte, A.E. Ruano, C.M. Fonseca, and P.J. Fleming. Accelerating multi-objective control system design using a neuro-genetic approach. In *2000 Congress on Evolutionary Computation,* volume 1, pages 392–397. IEEE Service Center, Piscataway, NJ, 2000.

[43] E.I. Ducheyne, R.R. De Wulf, and B. De Baets. Bi-objective genetic algorithm for forest management: A comparative study. In *Proceedings of the 2001 Genetic and Evolutionary Computation Conference. Late-Breaking Papers,* pages 63–66. Morgan Kaufman, San Francisco, CA, 2001.

[44] F.Y. Edgeworth. *Mathematical Physics.* P. Keagan, London, UK, 1881.

[45] M. Ehrgott and X. Gandibleux. A survey and annotated bibliography of multiobjective combinatorial optimization. *OR Spektrum,* 22:425–460, 2000.

[46] N.H. Eklund and M.J. Embrechts. GA-based multi-objective optimization of visible spectra for lamp design. In C.H. Dagli, A.L. Buczak, J. Ghosh, M.J. Embrechts, and O. Ersoy, editors, *Smart Engineering System Design: Neural Networks, Fuzzy Logic, Evolutionary Programming, Data Mining and Complex Systems,* pages 451–456. ASME Press, New York, NY, 1999.

[47] L.J. Eshelman. The CHC adaptive search algorithm: How to have safe search when engaging in nontraditional genetic recombination. In G.E. Rawlins, editor, *Foundations of Genetic Algorithms,* pages 265–283. Morgan Kaufmann Publishers, San Mateo, CA, 1991.

[48] L.J. Eshelman and J.D. Schaffer. Preventing premature convergence in genetic algorithms by preventing incest. In R.K. Belew and L.B. Booker, editors, *Proceedings of the Fourth International Conference on Genetic Algorithms,* pages 115–122. Morgan Kaufmann Publishers, San Mateo, CA, 1991.

[49] C.-W. Feng, L. Liu, and S.A. Burns. Using genetic algorithms to solve construction time-cost trade-off problems. *Journal of Computing in Civil Engineering,* 10(3):184–189, 1999.

[50] L.J. Fogel. *Artificial Intelligence through Simulated Evolution.* John Wiley, New York, NY, 1966.

[51] L.J. Fogel. *Artificial Intelligence through Simulated Evolution. Forty Years of Evolutionary Programming.* John Wiley & Sons, New York, NY, 1999.

[52] C.M. Fonseca and P.J. Fleming. Genetic algorithms for multiobjective optimization: Formulation, discussion and generalization. In S. Forrest, editor, *Proceedings of the Fifth International Conference on Genetic Algorithms,* pages 416–423. Morgan Kaufmann Publishers, San Mateo, CA, 1993.

[53] C.M. Fonseca and P.J. Fleming. An overview of evolutionary algorithms in multiobjective optimization. *Evolutionary Computation,* 3(1):1–16, 1995.

[54] C.M. Fonseca and P.J. Fleming. On the performance assessment and comparison of stochastic multiobjective optimizers. In H.-M. Voigt, W. Ebeling, I. Rechenberg, and H.-P. Schwefel, editors, *Parallel Problem Solving from Nature – PPSN IV,* pages 584–593. Lecture Notes in Computer Science No. 1141. Springer Verlag, Berlin, Germany, 1996.

[55] M.P. Fourman. Compaction of symbolic layout using genetic algorithms. In J.J. Grefenstette, editor, *Genetic Algorithms and their Applications: Proceedings of the First International Conference on Genetic Algorithms,* pages 141–153. Lawrence Erlbaum, Hillsdale, NJ, 1985.

[56] L. Gacôgne. Research of pareto set by genetic algorithm, application to multicriteria optimization of fuzzy controller. In *5th European Congress on Intelligent Techniques and Soft Computing EUFIT'97,* pages 837–845. Verlag Mainz, Aachen, Germany, 1997.

[57] L. Gacôgne. Multiple objective optimization of fuzzy rules for obstacles avoiding by an evolution algorithm with adaptative operators. In P. Ošmera, editor, *Proceedings of the Fifth International Mendel Conference on Soft Computing (Mendel'99),* pages 236–242. Brno University of Technology, Faculty of Mechanical Engineering, Institute of Automation and Computer Science. Brno, Czech Republic, 1999.

[58] D.E. Goldberg. *Genetic Algorithms in Search, Optimization and Machine Learning.* Addison-Wesley Publishing Company, Reading, MA, 1989.

[59] D.E. Goldberg and K. Deb. A comparison of selection schemes used in genetic algorithms. In G.J. E. Rawlins, editor, *Foundations of Genetic Algorithms,* pages 69–93. Morgan Kaufmann, San Mateo, CA, 1991.

[60] D.E. Goldberg and J. Richardson. Genetic algorithm with sharing for multimodal function optimization. In J.J. Grefenstette, editor, *Genetic Algorithms and Their Applications: Proceedings of the Second International Conference on Genetic Algorithms,* pages 41–49. Lawrence Erlbaum, Hillsdale, NJ, 1987.

[61] D.E. Goldberg and L. Wang. Adaptive niching via coevolutionary sharing. In D. Quagliarella, J. Périaux, C. Poloni, and G. Winter, editors, *Genetic Algorithms and Evolution Strategies in Engineering and Computer Science. Recent Advances and Industrial*

Applications, pages 22–38. John Wiley and Sons, Chichester, UK, 1998.

[62] I.E. Golovkin, R.C. Mancini, S.J. Louis, R.W. Lee, and L. Klein. Multi-criteria search and optimization: An application to x-ray plasma spectroscopy. In *2000 Congress on Evolutionary Computation,* volume 2, pages 1521–1527. IEEE Service Center, Piscataway, NJ, 2000.

[63] R. Groppetti and R. Muscia. On a genetic multiobjective approach for the integration and optimization of assembly product design and process planning. In P. Chedmail, J. C. Bocquet, and D. Dornfeld, editors, *Integrated Design and Manufacturing in Mechanical Engineering,* pages 61–70. Kluwer Academic Publishers, Dordrecht, The Netherlands, 1997.

[64] P. Haastrup and A. Guimaraes-Pereira. Exploring the use of multi-objective genetic algorithms for reducing traffic generated urban air and noise pollution. In *Proceedings of the 5th European Congress on Intelligent and Soft Computing,* pages 819–825. Verlag Mainz, Aachen, Germany, September 1997.

[65] W. Habenicht. Quad trees: A data structure for discrete vector optimization problems. In P. Hansen, editor, *Essays and Surveys on Multiple Criteria Decision Making,* pages 136–145. Lecture Notes in Economics and Mathematical Systems No. 209. Springer Verlag, Berlin, Germany, 1982.

[66] P. Hajela and J. Lee. Constrained genetic search via scheme adaptation: An immune network solution. *Structural Optimization,* 12(1):11–15, 1996.

[67] P. Hajela and C. Y. Lin. Genetic search strategies in multicriterion optimal design. *Structural Optimization,* 4:99–107, 1992.

[68] P. Hajela, J. Yoo, and J. Lee. GA based simulation of immune networks—applications in structural optimization. *Journal of Engineering Optimization,* 29:131–149, 1997.

[69] T. Hanne. On the convergence of multiobjective evolutionary algorithms. *European Journal of Operational Research,* 117(3):553–564, 2000.

[70] T. Hanne. Global multiobjective optimization using evolutionary algorithms. *Journal of Heuristics,* 6(3):347–360, 2000.

[71] M.P. Hansen. *Metaheuristics for multiple objective combinatorial optimization.* PhD thesis, Institute of Mathematical Modelling, Technical University of Denmark, Lyngby, Denmark, 1998.

[72] J.W. Hartmann. Low-thrust trajectory optimization using stochastic optimization methods. Master's thesis, Department of Aeronautical and Astronautical Engineering, University of Illinois at Urbana-Champaign, Urbana, IL, 1999.

[73] A. Herreros López. *Diseño de Controladores Robustos Multiobjetivo por Medio de Algoritmos Genéticos* (In Spanish). PhD thesis, Departamento de Ingeniería de Sistemas y Automática, Universidad de Valladolid, Valladolid, Spain, 2000.

[74] M. Hinchliffe, M. Willis, and M. Tham. Chemical process systems modelling using multi-objective genetic programming. In J.R. Koza, W. Banzhaf, K. Chellapilla, K. Deb, M. Dorigo, D.B. Fogel, M.H. Garzon, D.E. Goldberg, H. Iba, and R.L. Riolo, editors, *Proceedings of the Third Annual Conference on Genetic Programming,* pages 134–139. Morgan Kaufmann Publishers, San Mateo, CA, 1998.

[75] J.H. Holland. Outline for a logical theory of adaptive systems. *Journal of the Association for Computing Machinery,* 9:297–314, 1962.

[76] J.H. Holland. *Adaptation in Natural and Artificial Systems. An Introductory Analysis with Applications to Biology, Control and Artificial Intelligence.* University of Michigan Press, Ann Arbor, MI, 1975.

[77] J. Horn and N. Nafpliotis. Multiobjective optimization using the niched pareto genetic algorithm. Technical Report IlliGAl Report 93005, University of Illinois at Urbana-Champaign, Urbana, IL, 1993.

[78] J. Horn, N. Nafpliotis, and D.E. Goldberg. A niched pareto genetic algorithm for multiobjective optimization. In *Proceedings of the First IEEE Conference on Evolutionary Computation, IEEE World Congress on Computational Intelligence,* volume 1, pages 82–87. IEEE Service Center, Piscataway, NJ, 1994.

[79] A. Jaszkiewicz. On the performance of multiple objective genetic local search on the 0/1 knapsack problem. a comparative experiment. Technical Report RA-002/2000, Institute of Computing Science, Poznan University of Technology, Poznań, Poland, July 2000.

[80] K. Kato, M. Sakawa, and T. Ikegame. Interactive decision making for multiobjective block angular 0-1 programming problems with fuzzy parameters through genetic algorithms. *Japanese Journal of Fuzzy Theory and Systems,* 9(1):49–59, 1997.

[81] S. Khajehpour. *Optimal Conceptual Design of High-Rise Office Buldings.* PhD thesis, Civil Engineering Department, University of Waterloo, Ontario, Canada, 2001.

[82] S. Kirkpatrick, C.D. Gellatt, and M.P. Vecchi. Optimization by simulated annealing. *Science,* 220(4598):671–680, 1983.

[83] H. Kita, Y. Yabumoto, N. Mori, and Y. Nishikawa. Multi-objective optimization by means of the thermodynamical genetic algorithm. In H.-M. Voigt, W. Ebeling, I. Rechenberg, and H.-P. Schwefel, editors, *Parallel Problem Solving from Nature—PPSN IV,* pages 504–512. Lecture Notes in Computer Science No. 1141. Springer Verlag, Berlin, Germany, 1996.

[84] J.D. Knowles and D.W. Corne. The Pareto archived evolution strategy: A new baseline algorithm for multiobjective optimisation. In *1999 Congress on Evolutionary Computation,* pages 98–105. IEEE Service Center, Piscataway, NJ, 1999.

[85] J.D. Knowles and D.W. Corne. Approximating the nondominated front using the pareto archived evolution strategy. *Evolutionary Computation,* 8(2):149–172, 2000.

[86] J.D. Knowles, M.J. Oates, and D.W. Corne. Multiobjective evolutionary algorithms applied to two problems in telecommunications. *BT Technology Journal,* 18(4):51–64, 2000.

[87] M. Krause and V. Nissen. On using penalty functions and multicriteria optimisation techniques in facility layout. In J. Biethahn and V. Nissen, editors, *Evolutionary Algorithms in Management Applications.* Springer Verlag, Berlin, Germany, 1995.

[88] K. Krishnakumar. Micro-genetic algorithms for stationary and non-stationary function optimization. *SPIE Proceedings: Intelligent Control and Adaptive Systems,* 1196:289–296, 1989.

[89] H. W. Kuhn and A. W. Tucker. Nonlinear programming. In J. Neyman, editor, *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability,* pages 481–492. University of California Press, Berkeley, CA, 1951.

[90] A. Gaspar Kunha, P. Oliveira, and J.A. Covas. Genetic algorithms in multiobjective optimization problems: An application to polymer extrusion. In A.S. Wu, editor, *Proceedings of the 1999 Genetic and Evolutionary Computation Conference. Workshop Program,* pages 129–130, Orlando, FL, July 1999.

[91] S. Kurahashi and T. Terano. A genetic algorithm with tabu search for multimodal and multiobjective function optimization. In D. Whitley, D. Goldberg, E. Cantú-Paz, L. Spector, I. Parmee, and

H.-G. Beyer, editors, *Proceedings of the Genetic and Evolutionary Computation Conference (GECCO'2000),* pages 291–298. Morgan Kaufmann Publishers, San Francisco, CA, 2000.

[92] M. Lahanas, N. Milickovic, D. Baltas, and N. Zamboglou. Application of multiobjective evolutionary algorithms for dose optimization problems in brachytherapy. In E. Zitzler, K. Deb, L. Thiele, C.A. Coello Coello, and D. Corne, editors, *First International Conference on Evolutionary Multi-Criterion Optimization,* pages 574–587. Lecture Notes in Computer Science No. 1993. Springer Verlag, Berlin, Germany, 2001.

[93] N. Laumanns, M. Laumanns, and D. Neunzig. Multi-objective design space exploration of road trains with evolutionary algorithms. In E. Zitzler, K. Deb, L. Thiele, C.A. Coello Coello, and D. Corne, editors, *First International Conference on Evolutionary Multi-Criterion Optimization,* pages 612–623. Lecture Notes in Computer Science No. 1993. Springer Verlag, Berlin, Germany, 2001.

[94] D. Lee. Multiobjective design of a marine vehicle with aid of design knowledge. *International Journal for Numerical Methods in Engineering,* 40:2665–2677, 1997.

[95] X. Liu, D.W. Begg, and R.J. Fishwick. Genetic approach to optimal topology/controller design of adaptive structures. *International Journal for Numerical Methods in Engineering,* 41:815–830, 1998.

[96] S.W. Mahfoud. *Niching Methods for Genetic Algorithms.* PhD thesis, University of Illinois at Urbana-Champaign, Department of General Engineering, Urbana, IL, 1995.

[97] M. Mahfouf, M.F. Abbod, and D.A. Linkens. Multi-objective genetic optimization of the performance index of self-organizing fuzzy logic control algorithm using a fuzzy ranking approach. In H.J. Zimmerman, editor, *Proceedings of the Sixth European Congress on Intelligent Techniques and Soft Computing,* pages 1799–1808. Verlag Mainz, Aachen, Germany, 1998.

[98] N. Marco, S. Lanteri, J.-A. Desideri, and J. Périaux. A parallel genetic algorithm for multi-objective optimization in computational fluid dynamics. In K. Miettinen, M.M. Mäkelä, P. Neittaanmäki, and J. Périaux, editors, *Evolutionary Algorithms in Engineering and Computer Science,* pages 445–456. John Wiley & Sons, Chichester, UK, 1999.

[99] T. Marcu, L. Ferariu, and P. M. Frank. Genetic evolving of dynamic neural networks with application to process fault diagnosis.

In *Procedings of the EUCA/IFAC/IEEE European Control Conference ECC'99,* Karlsruhe, Germany, 1999. CD-ROM, F-1046,1.

[100] T. Marcu. A multiobjective evolutionary approach to pattern recognition for robust diagnosis of process faults. In R. J. Patton and J. Chen, editors, *IFAC Symposium on Fault Detection, Supervision and Safety for Technical Processes: SAFEPROCESS'97,* Kingston Upon Hull, UK, August 1997, pages 1183–1188. Elsevier Science, Amsterdam, The Netherlands, 1997.

[101] T. Marcu and P.M. Frank. Parallel evolutionary approach to system identification for process fault diagnosis. In P.S. Dhurjati and S. Cauvin, editors, *Procedings of the IFAC Workshop on 'On-line Fault Detection and Supervision in the Chemical Process Industries',* Solaize (Lyon), France, 1998, pages 113–118. Elsevier Science, Amsterdam, The Netherlands, 1998.

[102] C.E. Mariano Romero and E. Morales Manzanares. MOAQ an ant-Q algorithm for multiple objective optimization problems. In W. Banzhaf, J. Daida, A. E. Eiben, M. H. Garzon, V. Honavar, M. Jakiela, and R.E. Smith, editors, *Genetic and Evolutionary Computing Conference (GECCO 99),* volume 1, pages 894–901. Morgan Kaufmann Publishers, San Francisco, CA, July 1999.

[103] W. Mason, V. Coverstone-Carroll, and J. Hartmann. Optimal earth orbiting satellite constellations via a pareto genetic algorithm. In *1998 AIAA/AAS Astrodynamics Specialist Conference and Exhibit, Boston, MA, August 1998,* pages 169–177. Paper No. AIAA 98–4381, AIAA, Reston, VA.

[104] H. Meunier, E.-G. Talbi, and P. Reininger. A multiobjective genetic algorithm for radio network optimization. In *2000 Congress on Evolutionary Computation,* volume 1, pages 317–324. IEEE Service Center, Piscataway, NJ, July 2000.

[105] Z. Michalewicz. *Genetic Algorithms + Data Structures = Evolution Programs.* Third edition. Springer Verlag, Berlin, Germany, 1996.

[106] M. Mitchell. *An Introduction to Genetic Algorithms.* MIT Press, Cambridge, MA, 1996.

[107] J.N. Morse. Reducing the size of the nondominated set: Pruning by clustering. *Computers and Operations Research,* 7(1–2):55–66, 1980.

[108] D. Nam, Y.D. Seo, L.-J. Park, C.H. Park, and B. Kim. Parameter optimization of a voltage reference circuit using EP. In D.B. Fogel, editor, *Proceedings of the 1998 International Conference on*

Evolutionary Computation, pages 245–266. IEEE Service Center, Piscataway, NJ, 1998.

[109] S. Narayanan and S. Azarm. On improving multiobjective genetic algorithms for design optimization. *Structural Optimization,* 18:146–155, 1999.

[110] J. Nash. The bargaining problem. *Econometrica,* 18:155–162, 1950.

[111] S. Obayashi, T. Tsukahara, and T. Nakamura. Multiobjective evolutionary computation for supersonic wing-shape optimization. *IEEE Transactions on Evolutionary Computation,* 4(2): 182–187, 2000.

[112] M. Ortmann and W. Weber. Multi-criterion optimization of robot trajectories with evolutionary strategies. In L. Spector, E.B. Goodman, A. Wu et al., editors, *Proceedings of the 2001 Genetic and Evolutionary Computation Conference. Late-Breaking Papers,* pages 310–316. Morgan Kaufmann, San Francisco, CA, 2001.

[113] K.A. Osman, A.M. Higginson, and J. Moore. Improving the efficiency of vehicle water-pump designs using genetic algorithms. In C. Dagli, M. Akay, A. Buczak, O. Ersoy, and B. Fernandez, editors, *Smart Engineering Systems: Proceedings of the Artificial Neural Networks in Engineering Conference (ANNIE '98),* volume 8, pages 291–296. ASME Press, New York, NY, 1998.

[114] Vilfredo Pareto. *Cours D'Economic Politique,* volume I and II. F. Rouge, Lausanne, Switzerland, 1896.

[115] G.T. Parks. Multiobjective pressurised water reactor reload core design using a genetic algorithm. In G.D. Smith, N.C. Steele, and R.F. Albrecht, editors, *Artificial Neural Nets and Genetic Algorithms,* pages 53–57. Springer Verlag, Vienna, Austria, 1997.

[116] J. Périaux, M. Sefrioui, and B. Mantel. RCS multi-objective optimization of scattered waves by active control elements using GAs. In *Proceedings of the Fourth International Conference on Control, Automation, Robotics and Vision (ICARCV'96),* Singapore, 1996.

[117] J. Périaux, M. Sefrioui, and B. Mantel. GA multiple objective optimization strategies for electromagnetic backscattering. In D. Quagliarella, J. Périaux, C. Poloni, and G. Winter, editors, *Genetic Algorithms and Evolution Strategies in Engineering and Computer Science. Recent Advances and Industrial Applications,* pages 225–243. John Wiley and Sons, Chichester, UK, 1997.

[118] C.J. Petrie, T.A. Webster, and M.R. Cutkosky. Using pareto optimality to coordinate distributed agents. *Artificial Intelligence*

for Engineering Design, Analysis and Manufacturing, 9:269–281, 1995.

[119] C. Poloni and V. Pediroda. GA coupled with computationally expensive simulations: Tools to improve efficiency. In D. Quagliarella, J. Périaux, C. Poloni, and G. Winter, editors, *Genetic Algorithms and Evolution Strategies in Engineering and Computer Science. Recent Advances and Industrial Applications,* pages 267–288. John Wiley and Sons, Chichester, UK, 1997.

[120] M. Qiu. Prioritizing and scheduling road projects by genetic algorithm. *Mathematics and Computers in Simulation,* 43:569–574, 1997.

[121] I.J. Ramírez Rosado, J.L. Bernal Agustín, L.M. Barbosa Proença, and V. Miranda. Multiobjective planning of power distribution systems using evolutionary algorithms. In M.H. Hamza, editor, *8th IASTED International Conference on Modelling, Identification and Control – MIC'99, Innsbruck, Austria, February 1999,* pages 185–188, ACTA Press, Calgary, Canada, 1999.

[122] I.J. Ramírez Rosado and J.L. Bernal Agustín. Reliability and cost optimization for distribution networks expansion using an evolutionary algorithm. *IEEE Transactions on Power Systems,* 16(1):111–118, 2001.

[123] S.S. Rao. Multiobjective optimization in structural design with uncertain parameters and stochastic processes. *AIAA Journal,* 22(11):1670–1678, 1984.

[124] S.S. Rao. Game theory approach for multiobjective structural optimization. *Computers and Structures,* 25(1):119–127, 1987.

[125] S.S. Rao. Genetic algorithmic approach for multiobjective optimization of structures. In *ASME Annual Winter Meeting, Structures and Controls Optimization, New Orleans, LA, November 1993. ,* volume AD-Vol. 38, pages 29–38, ASME Press, New York, NY, 1993.

[126] T. Ray, R.P. Gokarn, and O.P. Sha. A global optimization model for ship design. *Computers in Industry,* 26:175–192, 1995.

[127] B. Rekiek. *Assembly Line Design (multiple objective grouping genetic algorithm and the balancing of mixed-model hybrid assembly line).* PhD thesis, Univerité Libre de Bruxelles, CAD/CAM Department, Brussels, Belgium, 2000.

[128] J.T. Richardson, M.R. Palmer, G. Liepins, and M. Hilliard. Some guidelines for genetic algorithms with penalty functions. In J.D.

Schaffer, editor, *Proceedings of the Third International Conference on Genetic Algorithms,* pages 191–197. Morgan Kaufmann Publishers, San Mateo, CA, 1989.

[129] B.J. Ritzel, J.W. Eheart, and S. Ranjithan. Using genetic algorithms to solve a multiple objective groundwater pollution containment problem. *Water Resources Research,* 30(5): 1589–1603, 1994.

[130] J.L. Rogers. Optimum actuator placement with a genetic algorithm for aircraft control. In C.H. Dagli, A.L. Buczak, J. Ghosh, M.J. Embrechts, and O. Ersoy, editors, *Smart Engineering System Design: Neural Networks, Fuzzy Logic, Evolutionary Programming, Data Mining, and Complex Systems (ANNIE'99),* pages 355–360. ASME Press, New York, NY, 1999.

[131] J.L. Rogers. A parallel approach to optimum actuator selection with a genetic algorithm. In *AIAA Guidance, Navigation, and Control Conference, Denver, CO, August 14–17 2000.* AIAA Paper No. 2000-4484. AIAA, Reston, VA.

[132] R.S. Rosenberg. *Simulation of genetic populations with biochemical properties.* PhD thesis, University of Michigan, Ann Arbor, MI, 1967.

[133] G. Rudolph. On a multi-objective evolutionary algorithm and its convergence to the pareto set. In *Proceedings of the 5th IEEE Conference on Evolutionary Computation,* pages 511–516. IEEE Press, Piscataway, NJ, 1998.

[134] G. Rudolph and A. Agapie. Convergence properties of some multi-objective evolutionary algorithms. In *Proceedings of the 2000 Conference on Evolutionary Computation,* volume 2, pages 1010–1016. IEEE Press, Piscataway, NJ, 2000.

[135] E. Sandgren. Multicriteria design optimization by goal programming. In H. Adeli, editor, *Advances in Design Optimization,* pages 225–265. Chapman & Hall, London, UK, 1994.

[136] D.A. Savic, G.A. Walters, and M. Schwab. Multiobjective genetic algorithms for pump scheduling in water supply. In *AISB International Workshop on Evolutionary Computing,* pages 227–236. Lecture Notes in Computer Science No. 1305. Springer Verlag, Berlin, Germany, April 1997.

[137] J.D. Schaffer. *Multiple Objective Optimization with Vector Evaluated Genetic Algorithms.* PhD thesis, Vanderbilt University, Nashville, TN, 1984.

[138] J.D. Schaffer. Multiple objective optimization with vector evaluated genetic algorithms. In J.J. Grefenstette, editor, *Genetic Algorithms and their Applications: Proceedings of the First International Conference on Genetic Algorithms,* pages 93–100. Lawrence Erlbaum, Hillsdale, NJ, 1985.

[139] J.R. Schott. Fault tolerant design using single and multicriteria genetic algorithm optimization. Master's thesis, Department of Aeronautics and Astronautics, Massachusetts Institute of Technology, Cambridge, MA, 1995.

[140] P. Schroder, A. J. Chipperfield, P. J. Fleming, and N. Grum. Multiobjective optimization of distributed active magnetic bearing controllers. In A.M.S. Zalzala and P.J. Fleming, editors, *Genetic Algorithms in Engineering Systems: Innovations and Applications,* pages 13–18. IEE, London, Uk, 1997.

[141] M. Schwab, D. A. Savic, and G. A. Walters. Multi-objective genetic algorithm for pump scheduling in water supply systems. In D. Torne and J.L. Shapiro, editors, *Evolutionary Computing AISBP Workshop 1997,* pages 227–236. Lecture Notes in Computer Science No. 1305. Springer Verlag, Berlin, Germany, 1997.

[142] H.-P. Schwefel. *Kybernetische Evolution als Strategie der experimentellen Forschung in der Strömungstechnik* (In German). Dipl.-Ing. thesis, Institute for Hydrodynamics, Technische Universität Berlin, Berlin, Germany, 1965.

[143] H.-P. Schwefel. *Numerische Optimierung von Computer-Modellen mittels der Evolutionsstrategie.* (In German). Birkhäuser, Basel, Switzerland, 1977.

[144] H.-P. Schwefel. *Numerical Optimization of Computer Models.* John Wiley & Sons, Chichester, UK, 1981.

[145] E. Science, N. Marvin, M. Bower, and R. J. Rowe. An evolutionary approach to constructing prognostic models. *Artificial Intelligence in Medicine,* 15(2):155–165, 1999.

[146] P. Serafini. Simulated annealing for multiple objective optimization problems. In G.H. Tzeng, H.F. Wang, U.P. Wen, and P.L. Yu, editors, *Proceedings of the Tenth International Conference on Multiple Criteria Decision Making: Expand and Enrich the Domains of Thinking and Application,* volume 1, pages 283–292. Springer Verlag, Berlin, Germany, 1994.

[147] M. Shibuya, H. Kita, and S. Kobayashi. Integration of multiobjective and interactive genetic algorithms and its application to animation design. In *Proceedings of IEEE Systems, Man, and*

Cybernetics, volume III, pages 646–651. IEEE Service Center, Piscataway, NJ, 1999.

[148] R.E. Smith, S. Forrest, and A.S. Perelson. Population diversity in an immune system model: Implications for genetic search. In L.D. Whitley, editor, *Foundations of Genetic Algorithms 2,* pages 153–165. Morgan Kaufmann Publishers, San Mateo, CA, 1993.

[149] K.C. Srigiriraju. *Noninferior Surface Tracing Evolutionary Algorithm (NSTEA) for Multi Objective Optimization.* Master's thesis, North Carolina State University, Raleigh, NC, 2000.

[150] N. Srinivas and K. Deb. Multiobjective optimization using nondominated sorting in genetic algorithms. *Evolutionary Computation,* 2(3):221–248, 1994.

[151] W. Stadler. Natural structural shapes (the static case). *The Quarterly Journal of Mechanics and Applied Mathematics,* XXXI(2):169–217, 1978.

[152] W. Stadler. Fundamentals of multicriteria optimization. In W. Stadler, editor, *Multicriteria Optimization in Engineering and the Sciences,* pages 1–25. Plenum Press, New York, NY, 1988.

[153] P.D. Surry and N.J. Radcliffe. The COMOGA method: Constrained optimisation by multiobjective genetic algorithms. *Control and Cybernetics,* 26(3):391–412, 1997.

[154] P.D. Surry, N.J. Radcliffe, and I.D. Boyd. A multi-objective approach to constrained optimisation of gas supply networks: The COMOGA method. In T.C. Fogarty, editor, *Evolutionary Computing. AISB Workshop. Selected Papers,* pages 166–180. Lecture Notes in Computer Science No. 993. Springer Verlag, Berlin, Germany, 1995.

[155] T. Tagami and T. Kawabe. Genetic algorithm based on a pareto neighborhood search for multiobjective optimization. In *Proceedings of the 1999 International Symposium of Nonlinear Theory and its Applications (NOLTA'99), Hawaii,* pages 331–334. Institute of Electronics, Information, and Commnication Engineers, Tokyo, Japan, 1999.

[156] H. Tamaki, H. Kita, and S. Kobayashi. Multi-objective optimization by genetic algorithms: A review. In T. Fukuda and T. Furuhashi, editors, *Proceedings of the 1996 International Conference on Evolutionary Computation (ICEC'96),* pages 517–522. IEEE Service Center, Piscataway, NJ, 1996.

[157] K. C. Tan, T. H. Lee, and E. F. Khor. Evolutionary algorithms with goal and priority information for multi-objective optimiza-

tion. In *1999 Congress on Evolutionary Computation,* pages 106–113. IEEE Service Center, Piscataway, NJ, 1999.

[158] M.W. Thomas. *A Pareto Frontier for Full Stern Submarines via Genetic Algorithm.* PhD thesis, Ocean Engineering Department, Massachusetts Institute of Technology, Cambridge, MA, 1998.

[159] E. Tsoi, K.P. Wong, and C. Che Fung. Hybrid GA/SA algorithms for evaluating trade-off between economic cost and environmental impact in generation dispatch. In D.B. Fogel, editor, *Proceedings of the Second IEEE Conference on Evolutionary Computation (ICEC'95),* pages 132–137. IEEE Press. Piscataway, NJ, 1995.

[160] E.L. Ulungu, J. Teghem, P. Fortemps, and D. Tuyttens. MOSA method: A tool for solving multiobjective combinatorial optimization problems. *Journal of Multi-Criteria Decision Analysis,* 8(4):221–236, 1999.

[161] M. Valenzuela-Rendón and E. Uresti-Charre. A non-generational genetic algorithm for multiobjective optimization. In T. Bäck, editor, *Proceedings of the Seventh International Conference on Genetic Algorithms,* pages 658–665. Morgan Kaufmann Publishers. San Mateo, CA, 1997.

[162] D.A. Van Veldhuizen. *Multiobjective Evolutionary Algorithms: Classifications, Analyses, and New Innovations.* PhD thesis, Department of Electrical and Computer Engineering. Air Force Institute of Technology, Wright-Patterson AFB, OH, 1999.

[163] D.A. Van Veldhuizen and G.B. Lamont. Evolutionary computation and convergence to a pareto front. In J.R. Koza, editor, *Late Breaking Papers at the Genetic Programming 1998 Conference,* pages 221–228. Stanford University Bookstore. Stanford, CA, 1998.

[164] D.A. Van Veldhuizen and G.B. Lamont. Multiobjective evolutionary algorithm research: A history and analysis. Technical Report TR-98-03, Department of Electrical and Computer Engineering, Graduate School of Engineering, Air Force Institute of Technology, Wright-Patterson AFB, OH, 1998.

[165] D.A. Van Veldhuizen and G.B. Lamont. Multiobjective evolutionary algorithm test suites. In J. Carroll, H. Haddad, D. Oppenheim, B. Bryant, and G.B. Lamont, editors, *Proceedings of the 1999 ACM Symposium on Applied Computing, San Antonioo 1999,* pages 351–357. ACM, New York, NY, 1999.

[166] D.A. Van Veldhuizen, B.S. Sandlin, R.M. Marmelstein, and G.B. Lamont. Finding improved wire-antenna geometries with genetic

algorithms. In D.B. Fogel, editor, *Proceedings of the 1998 International Conference on Evolutionary Computation,* pages 102–107. IEEE Service Center, Piscataway, NJ, 1998.

[167] J.F. Wang and J. Périaux. Multi-point optimization using GAs and Nash/Stackelberg games for high lift multi-airfoil design in aerodynamics. In *Proceedings of the Congress on Evolutionary Computation 2001 (CEC'2001),* volume 1, pages 552–559. IEEE Service Center, Piscataway, NJ, 2001.

[168] D.S. Weile and E. Michielssen. Integer coded pareto genetic algorithm design of constrained antenna arrays. *Electronics Letters,* 32(19):1744–1745, 1996.

[169] D.S. Weile, E. Michielssen, and D.E. Goldberg. Genetic algorithm design of pareto optimal broadband microwave absorbers. *IEEE Transactions on Electromagnetic Compatibility,* 38(3):518–525, 1996.

[170] P.B. Wienke, C. Lucasius, and G. Kateman. Multicriteria target optimization of analytical procedures using a genetic algorithm. *Analytical Chimica Acta,* 265(2):211–225, 1992.

[171] P.B. Wilson and M.D. Macleod. Low implementation cost IIR digital filter design using genetic algorithms. In *IEE/IEEE Workshop on Natural Algorithms in Signal Processing,* pages 4/1–4/8. IEE, London, UK, 1993.

[172] S. Wright. The roles of mutation, inbreeding, crossbreeding and selection in evolution. In D.F. Jones, editor, *Proceedings of the Sixth International Conference on Genetics,* volume 1, pages 356–366. Brooklyn Botanic Gardens, New York, NY, 1932.

[173] J. Wu and S. Azarm. On a new constraint handling technique for multi-objective genetic algorithms. In L. Spector, E.D. Goodman, A. Wu, W.B. Langdon, H.-M. Voigt, M. Gen, S. Sen, M. Dorigo, S. Pezeshk, M.H. Garzon, and E. Burke, editors, *Proceedings of the Genetic and Evolutionary Computation Conference (GECCO'2001),* pages 741–748. Morgan Kaufmann Publishers, San Francisco, CA, 2001.

[174] P.O. Yapo, H.V. Gupta, and S. Sorooshian. Multi-objective global optimization for hydrologic models. *Journal of Hydrology,* 204:83–97, 1998.

[175] H. Youssef, S.M. Sait, and S.A. Khan. Fuzzy evolutionary hybrid metaheuristic for network topology design. In E. Zitzler, K. Deb, L. Thiele, C.A. Coello Coello, and D. Corne, editors, *First*

International Conference on Evolutionary Multi-Criterion Optimization, pages 400–415. Lecture Notes in Computer Science No. 1993. Springer Verlag, Berlin, Germany, 2001.

[176] Y. Yu. Multi-objective decision theory for computational optimization in radiation therapy. *Medical Physics,* 24:1445–1454, 1997.

[177] R.S. Zebulum, M.A. Pacheco, and M. Vellasco. A multi-objective optimisation methodology applied to the synthesis of low-power operational amplifiers. In I.J. Cheuri and C.A. dos Reis Filho, editors, *Proceedings of the XIII International Conference in Microelectronics and Packaging Curitiba, Brazil, August 1998,* volume 1, pages 264–271. Brazilian Microelectronic Society, 1998.

[178] E. Zitzler, K. Deb, and L. Thiele. Comparison of multiobjective evolutionary algorithms: Empirical results. *Evolutionary Computation,* 8(2):173–195, 2000.

[179] E. Zitzler, J. Teich, and S.S. Bhattacharyya. Multidimensional exploration of software implementations for DSP algorithms. *Journal of VLSI Signal Processing,* 24(1):83–98, 2000.

[180] E. Zitzler and L. Thiele. Multiobjective optimization using evolutionary algorithms–A comparative study. In A.E. Eiben, editor, *Parallel Problem Solving from Nature V,* pages 292–301. Lecture Notes in Computer Science No. 1498. Springer Verlag, Berlin, Germany, 1998.

[181] E. Zitzler and L. Thiele. Multiobjective evolutionary algorithms: A comparative case study and the strength pareto approach. *IEEE Transactions on Evolutionary Computation,* 3(4):257–271, 1999.

This page intentionally left blank

Chapter 7

DATA ENVELOPMENT ANALYSIS IN MULTICRITERIA DECISION MAKING

Hirotaka Nakayama
Department of Applied Mathematics
Konan University
Kobe 658-8501
Japan
nakayama@konan-u.ac.jp

Masao Arakawa, Ye Boon Yun
Department of Reliability-based Information System Engineering
Kagawa University
Kagawa 760-0301
Japan
arakawa,yun@eng.kagawa-u.ac.jp

Abstract Multiple criteria decision analysis has been studied for helping decision makers to make their final decisions in MCDM (Multiple Criteria Decision Making) problems. One of the main tasks in this research is how to incorporate value judgments of decision makers in decision support systems. If decision makers can make their decisions by seeing efficiencies (or inefficiencies) of alternatives, the idea of DEA(Data Envelopment Analysis) can be applied to MCDM problems. In this event, it is important to know what value judgment the domination structure of each DEA model reflects. Moreover, a model which can treat a wide range of value judgments of decision makers is required. To this end, in this chapter, a generalized DEA model is proposed and discussed for practical use in MCDM problems.

Keywords: Data envelopment analysis, Multiple criteria decision making, DEA model, Generalized DEA model.

1. Introduction

Consider decision making problems with multiple criteria $f_1, \cdots, f_r$ which are to be maximized. Let S denote the set of alternatives. For this problem, $\boldsymbol{x}^o \in S$ or $\boldsymbol{f}^o(= \boldsymbol{f}(\boldsymbol{x}^o))$ is said to be *Pareto efficient* if and only if there does not exist $\boldsymbol{x} \in S$ such that $\boldsymbol{f}(\boldsymbol{x}) \geq \boldsymbol{f}(\boldsymbol{x}^o)$. Usually, a Pareto efficient solution is not necessarily uniquely determined, but there are several Pareto efficient solutions. In practical decision making, therefore, we have to determine a solution among the Pareto efficient solutions. To this end, value judgments of decision makers are introduced. The multi-attribute utility (value) analysis provides some mathematical form for these value judgments of decision makers. On the other hand, interactive multi-objective programming techniques search a decision making solution eliciting partial information (see Chapter 9 of this volume) on value judgments of decision makers. In any case, the final solution strongly depends on the value judgment.

The idea of data envelopment analysis(DEA) can be applied to multiple criteria decision making(MCDM) problems, if a final decision making solution is determined by seeing efficiencies (or inefficiencies) of alternatives. Let decision making units (DMUs) be identified with alternatives in MCDM problems. Then, it should be noted that efficiencies in DEA also depend on value judgments. It should be emphasized that the ratio of output to input is merely one of these value judgments. In many production activity analyses, the ratio of output to input is naturally adopted as such a value judgment. In applying DEA to a wide range of practical problems, however, there are some cases in which the ratio value judgement is not adequate. In other words, in some cases a DMU is not necessarily judged to be inefficient even though it is inefficient by the CCR model, which was named after Charnes, Cooper and Rhodes [8, 9].

The additive value may be represented by a linear weighted sum of each criterion. Under this circumstance, a value judgment is reflected by a set of weights to criteria. If a DMU maximizes a weighted sum of criteria, it can be regarded as efficient in terms of the value judgment. Therefore, a DMU can be said to be additive value efficient if it maximizes a weighted sum of criteria. The set of additive value efficient DMUs is identical to the set of efficient DMUs in the BCC model (or the additive model of DEA by Charnes et al. [8, 9]) which was named after Banker, Charnes and Cooper [3].

Depending on the situation, value judgments of decision makers can not necessarily be represented by a weighted sum of criteria. Nonlinear

value functions can be used for more general value judgments of decision makers (e.g., pseudo-concave value functions by Halme et al. [17]). The notion of efficiency without introducing any value judgment is the Pareto efficiency. We call this "the value free efficiency". The set of value free efficient DMUs is identical to that of the FDH (free disposable hull[1]) model [29]. In this chapter, we describe generalized DEA models which embed these value judgments in a unified model. The key idea of the model is to introduce a domination structure with one parameter varying from the value free structure to a ratio value structure.

2. Data Envelopment Analysis

DEA was suggested by Charnes, Cooper and Rhodes, and built on the idea of Farrell [13] which is concerned with the estimation of technical efficiency and efficient frontiers. The CCR model [8, 9] generalized the single output/single input ratio efficiency measure for each DMU to multiple outputs/multiple inputs situations by forming the ratio of a weighted sum of outputs to a weighted sum of inputs. DEA is a method for measuring the relative efficiency of DMUs performing similar tasks in a production system that consumes multiple inputs to produce multiple outputs.

The main characteristics of DEA are that (i) it can be applied to analyze multiple outputs and multiple inputs without preassigned weights, (ii) it can be used for measuring a relative efficiency based on the observed data without knowing information on the production function and (iii) decision makers' preferences can be incorporated in DEA models. Later, Banker, Charnes and Cooper suggested a model for distinguishing between technical efficiency and scale inefficiency in DEA. The BCC model [3] relaxed the constant returns to scale assumption of the CCR model and made it possible to investigate whether the performance of each DMU was conducted in regions of increasing, constant or decreasing returns to scale in multiple outputs and multiple inputs situations. In addition, Tulkens [29] introduced a relative efficiency on the non-convex free disposable hull of the observed data, and formulated a mixed integer programming problem to calculate the relative efficiency for each DMU. In addition to basic models as mentioned above, a number of extended models have been studied, for example, a cone ratio model [10], a polyhedral cone ratio model [7], Seiford and Thrall's model [24], Wei and Yu's model [30], and so on.

[1]The free disposable hull (FDH) by Deprins *et al.* [12] is a non-convex hull consisting of any points that perform less output with the same amount of input as the observed data, and/or those that perform more input with the same amount of output.

Relationships between DEA and multiple criteria decision analysis have been studied from several viewpoints by many authors. Belton [4] and Belton and Vickers [5] measured efficiency as a weighted sum of input and output. Stewart [25] showed the equivalence between the CCR model and some linear value function model for multiple outputs and multiple inputs. Joro et al. [18] proved structural correspondences between DEA models and multiple objective linear programming using an achievement scalarizing function proposed by Wierzbicki [31]. Especially, various ways of introducing preference information into DEA formulations have been developed. Golany [15] suggested a so-called target setting model, which allows decision makers to select the preferred set of output levels given the input levels of a DMU. Thanassoulis and Dyson [28] introduced models that can be used to estimate alternative output and input levels, in order to render relatively inefficient DMUs efficient. Zhu [32] proposed a model that calculates efficiency scores incorporating the decision makers' preference informations, whereas Korhonen [20] applied an interactive technique to progressively reveal preferences. Halme et al. [17] evaluated efficiency of a DMU in terms of pseudo-concave value functions, by considering a tangent cone of the feasible set at the most preferred solution of the decision maker. Agrell and Tind [1] showed correspondences among the CCR model [8], the BCC model [3] and the FDH model [29] and an MCDA model according to the property of a partial Lagrangean relaxation. Yun, Nakayama and Tanino [33] suggested a concept of "value free efficiency" in the observed data. They have proposed a generalized model for DEA, the so-called GDEA model, which can treat basic DEA models, specifically, the CCR model, the BCC model and the FDH model in a unified way. The GDEA model makes it possible to evaluate the efficiency of DMUs incorporating various preference structures of decision makers. Furthermore, a dual approach GDEA_D to GDEA can reveal domination relations among all DMUs.

The next section introduces notations used in this chapter and presents brief explanations on basic DEA models. In Section 3, the GDEA model based on a parametric domination is introduced. Section 4 presents a dual approach to GDEA, that is, the GDEA_D model based on a production possibility set. In Section 5, we compare the efficiency of GDEA and several DEA models for each DMU through illustrative examples. Finally, Section 6 applies GDEA models to multiple criteria decision making problems.

2. Basic DEA Models

In the following discussion, we assume that there exist n DMUs to be evaluated. Each DMU consumes varying amounts of m different inputs to produce p different outputs. Specifically, DMU j consumes amounts $\boldsymbol{x}_j := (x_{ij})$ of inputs $(i = 1, \cdots, m)$ and produces amounts $\boldsymbol{y}_j := (y_{kj})$ of outputs $(k = 1, \cdots, p)$. For these constants, which generally take the form of observed data, we assume $x_{ij} > 0$ for each $i = 1, \cdots, m$ and $y_{kj} > 0$ for each $k = 1, \cdots, p$. Further, we assume that there are no duplicated units in the observed data. The $p \times n$ output matrix for the n DMUs is denoted by $\boldsymbol{Y}$, and the $m \times n$ input matrix for the n DMUs is denoted by $\boldsymbol{X}$. $\boldsymbol{x}_o := (x_{1o}, \cdots, x_{mo})$ and $\boldsymbol{y}_o := (y_{1o}, \cdots, y_{po})$ are amounts of inputs and outputs of DMUo, which is evaluated. In addition, ε is a small positive number ("non-Archimedean") and $\boldsymbol{1}^T = (1, \cdots, 1)$ is a vector of all ones.

For convenience of explanation, the following notations for vectors $\boldsymbol{a}$ $(= (a_1, \cdots, a_m)^T)$ and $\boldsymbol{b}$ $(= (b_1, \cdots, b_m)^T)$ in $\mathbb{R}^m$ will be used.

$$\begin{aligned}
\boldsymbol{a} > \boldsymbol{b} &\iff a_i > b_i, \quad i = 1, \cdots, m, \\
\boldsymbol{a} \geqq \boldsymbol{b} &\iff a_i \geqq b_i, \quad i = 1, \cdots, m, \\
\boldsymbol{a} \geq \boldsymbol{b} &\iff a_i \geqq b_i, \quad i = 1, \cdots, m \quad \text{but} \quad \boldsymbol{a} \neq \boldsymbol{b}.
\end{aligned}$$

So far, a number of DEA models have been developed. Among them, the CCR model [8, 9], the BCC model [3] and the FDH model [29] are well known as basic DEA models. These models are based on the domination structure in the primal form, and moreover these are characterized by how to determine the production possibility set in the dual form: the convex cone, the convex hull and the free disposable hull for the observed data, respectively.

2.1. The CCR Model

The CCR model, which was suggested by Charnes, Cooper and Rhodes [8], is a fractional linear programming problem and can be solved by being transformed into an equivalent linear programming one. Therefore, the primal problem (CCR) with an input oriented model[2] can be

[2]The CCR model, the BCC model and the FDH model are dependent on the orientation. For instance, in an input orientation, one focuses on maximal movement toward the efficient frontier through proportional reduction of inputs, whereas in an output orientation one focuses on maximal movement via proportional augmentation of outputs. In this chapter, to condense the text, we deal with only the input oriented model for simplicity.

formulated as the following:

$$\begin{aligned} \underset{\mu_k,\,\nu_i}{\text{maximize}} \quad & \sum_{k=1}^{p} \mu_k y_{ko} \qquad \text{(CCR)} \\ \text{subject to} \quad & \sum_{i=1}^{m} \nu_i x_{io} = 1, \\ & \sum_{k=1}^{p} \mu_k y_{kj} - \sum_{i=1}^{m} \nu_i x_{ij} \leqq 0, \;\; j = 1, \cdots, n, \\ & \mu_k \geqq \varepsilon, \;\; \nu_i \geqq \varepsilon, \;\; k = 1, \cdots, p\,; \; i = 1, \cdots, m. \end{aligned}$$

The dual problem (CCR_D) to the problem (CCR) is given by

$$\begin{aligned} \underset{\theta,\,\boldsymbol{\lambda},\,\boldsymbol{s}_x,\,\boldsymbol{s}_y}{\text{minimize}} \quad & \theta - \varepsilon(\mathbf{1}^T \boldsymbol{s}_x + \mathbf{1}^T \boldsymbol{s}_y) \qquad (\mathrm{CCR}_D) \\ \text{subject to} \quad & \boldsymbol{X}\boldsymbol{\lambda} - \theta \boldsymbol{x}_o + \boldsymbol{s}_x = \mathbf{0}, \\ & \boldsymbol{Y}\boldsymbol{\lambda} - \boldsymbol{y}_o - \boldsymbol{s}_y = \mathbf{0}, \\ & \boldsymbol{\lambda} \geqq \mathbf{0}, \;\; \boldsymbol{s}_x \geqq \mathbf{0}, \;\; \boldsymbol{s}_y \geqq \mathbf{0}, \\ & \theta \in \mathbb{R}, \;\; \boldsymbol{\lambda} \in \mathbb{R}^n, \;\; \boldsymbol{s}_x \in \mathbb{R}^m, \boldsymbol{s}_y \in \mathbb{R}^p. \end{aligned}$$

The 'efficiency' in the CCR model is introduced as follows:

Definition 1 *(CCR-efficiency) A DMUo is CCR-efficient if and only if the optimal value* $\sum_{k=1}^{p} \mu_k^* y_{ko}$ *to the problem (CCR) equals one. Otherwise, the DMUo is said to be CCR-inefficient.*

Definition 2 *(*CCR_D*-efficiency)A DMUo is* CCR_D*-efficient if and only if for the optimal solution* $(\theta^*, \boldsymbol{\lambda}^*, \boldsymbol{s}_x^*, \boldsymbol{s}_y^*)$ *to the problem (*CCR_D*), the following two conditions are satisfied:*

(i) θ^* *is equal to one;*

(ii) the slack variables $\boldsymbol{s}_x^*$ *and* $\boldsymbol{s}_y^*$ *are all zero.*

Otherwise, the DMUo is CCR_D*-inefficient.*

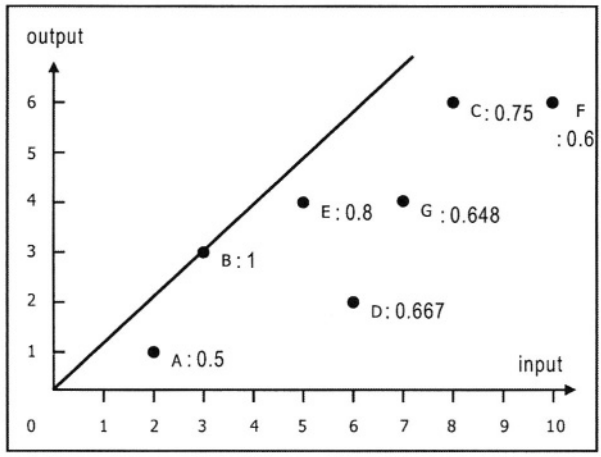
Note that the above two definitions are equivalent due to the well known duality of linear programming.

Additionally, the production possibility set P_1 in the dual form of the CCR model is the *convex cone* (or conical hull) generated by the observed data, which implies that the scale efficiency of a DMU is constant, that is to say, constant returns to scale. Namely, P_1 can be denoted by

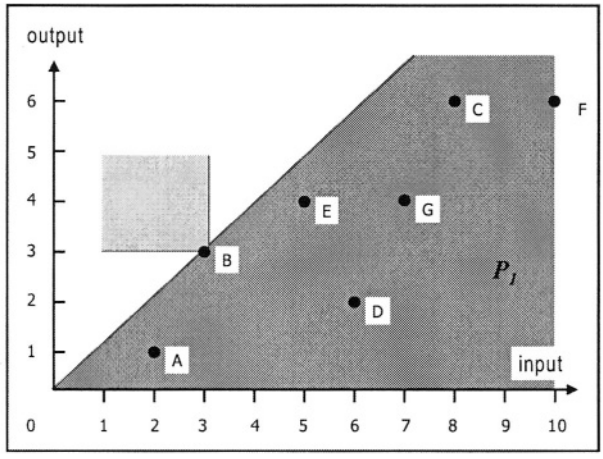
$$P_1 = \left\{ (\boldsymbol{y}, \boldsymbol{x}) \mid Y\boldsymbol{\lambda} \geqq \boldsymbol{y}, \; X\boldsymbol{\lambda} \leqq \boldsymbol{x}, \; \boldsymbol{\lambda} \geqq \mathbf{0} \right\}.$$

and the definition of CCR-efficiency (or CCR_D-efficiency) can be transformed into the following:

Definition 3 *DMUo is said to be Pareto efficient in* P_1 *if and only if there does not exist* $(\boldsymbol{y}, \boldsymbol{x}) \in P_1$ *such that* $(\boldsymbol{y}, -\boldsymbol{x}) \geq (\boldsymbol{y}_o, -\boldsymbol{x}_o)$.



(a) Primal form



(b) Dual form

Figure 7.1. CCR efficient frontier and production possibility set generated by the CCR model from the observed data.

It is readily seen that the Pareto efficiency in P_1 is equivalent to the CCR-efficiency. Fig. 7.1 shows a geometric interpretation on the relation between the primal form of the CCR model and the dual one.

2.2. The BCC Model

The BCC model of Banker et al. [3] is formulated similarly to that for the CCR model. The dual problem for the BCC model is obtained by adding the convexity constraint $\mathbf{1}^T\boldsymbol{\lambda} = 1$ to the dual problem (CCR_D) and thus, the variable u_o appears in the primal problem. The efficiency degree of a DMUo with respect to the BCC model can be measured by solving the problem.

$$
\begin{aligned}
\underset{\mu_k,\,\nu_i,\,u_o}{\text{maximize}} \quad & \sum_{k=1}^{p} \mu_k y_{ko} - u_o && \text{(BCC)}\\
\text{subject to} \quad & \sum_{i=1}^{m} \nu_i x_{io} = 1,\\
& \sum_{k=1}^{p} \mu_k y_{kj} - \sum_{i=1}^{m} \nu_i x_{ij} - u_o \leqq 0, \quad j = 1, \cdots, n,\\
& \mu_k \geqq \varepsilon, \quad \nu_i \geqq \varepsilon, \quad k = 1, \cdots, p\,;\ i = 1, \cdots, m.
\end{aligned}
$$

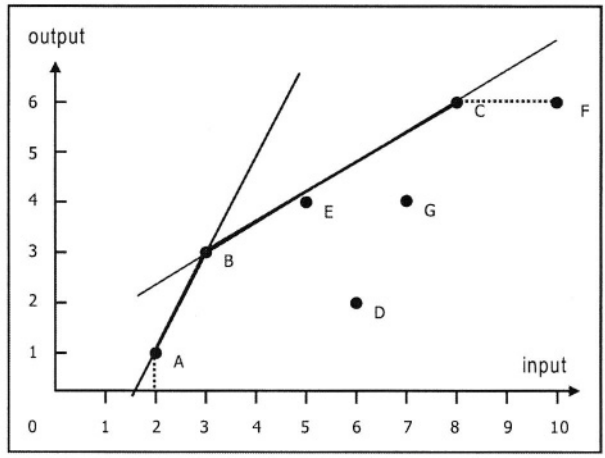
The dual problem (BCC_D) to the problem (BCC) is formulated as follows:

$$
\begin{aligned}
\underset{\theta,\,\boldsymbol{\lambda},\,\boldsymbol{s}_x,\,\boldsymbol{s}_y}{\text{minimize}} \quad & \theta - \varepsilon(\mathbf{1}^T\boldsymbol{s}_x + \mathbf{1}^T\boldsymbol{s}_y) && (\mathrm{BCC}_D)\\
\text{subject to} \quad & \boldsymbol{X}\boldsymbol{\lambda} - \theta\boldsymbol{x}_o + \boldsymbol{s}_x = \mathbf{0},\\
& \boldsymbol{Y}\boldsymbol{\lambda} - \boldsymbol{y}_o - \boldsymbol{s}_y = \mathbf{0},\\
& \mathbf{1}^T\boldsymbol{\lambda} = 1,\\
& \boldsymbol{\lambda} \geqq \mathbf{0}, \quad \boldsymbol{s}_x \geqq \mathbf{0}, \quad \boldsymbol{s}_y \geqq \mathbf{0},\\
& \theta \in \mathbb{R}, \quad \boldsymbol{\lambda} \in \mathbb{R}^n, \quad \boldsymbol{s}_x \in \mathbb{R}^m, \boldsymbol{s}_y \in \mathbb{R}^p.
\end{aligned}
$$

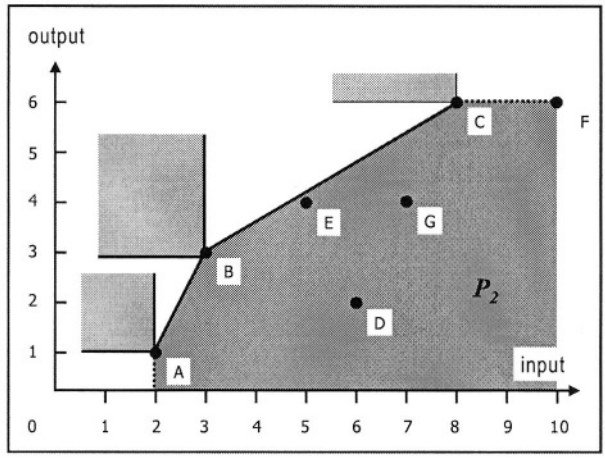
The 'efficiency' in the BCC model is given by the following two definitions which are equivalent to each other due to the duality of linear programming.

Definition 4 *(BCC-efficiency) A DMUo is BCC-efficient if and only if the optimal value* $\left(\sum_{k=1}^{p} \mu_k^* y_{ko} - u_o^*\right)$ *to the problem (BCC) equals one. Otherwise, the DMUo is said to be BCC-inefficient.*

Definition 5 *(BCC_D-efficiency) A DMUo is BCC_D-efficient if and only if for an optimal solution* $(\theta^*, \boldsymbol{\lambda}^*, \boldsymbol{s}_x^*, \boldsymbol{s}_y^*)$ *to the problem (BCC_D), the following two conditions are satisfied:*



(a) Primal form



(b) Dual form

Figure 7.2. BCC efficient frontier and production possibility set generated by the BCC model from the observed data.

(i) θ^* *is equal to one;*

(ii) the slack variables s_x^* *and* s_y^* *are all zero.*

Otherwise, the DMUo is said to be BCC_D*-inefficient.*

The presence of the constraint $\mathbf{1}^T\boldsymbol{\lambda} = 1$ in the dual problem (BCC_D) yields that the production possibility set P_2 in the BCC model is the *convex hull* generated by the observed data. Therefore, P_2 can be obtained

as

$$P_2 = \left\{ (\boldsymbol{y}, \boldsymbol{x}) \mid Y\boldsymbol{\lambda} \geqq \boldsymbol{y},\ X\boldsymbol{\lambda} \leqq \boldsymbol{x},\ \mathbf{1}^T\boldsymbol{\lambda} = 1,\ \boldsymbol{\lambda} \geqq \mathbf{0} \right\}$$

and the definition of BCC_D-efficiency can be transformed into the following:

Definition 6 *DMUo is said to be Pareto efficient in P_2 if and only if there does not exist $(\boldsymbol{y}, \boldsymbol{x}) \in P_2$ such that $(\boldsymbol{y}, -\boldsymbol{x}) \geq (\boldsymbol{y}_o, -\boldsymbol{x}_o)$.*

It is readily seen that the Pareto efficiency in P_2 is equivalent to the BCC-efficiency. Fig. 7.2 shows a geometric interpretation of the relation between the primal form of BCC model and the dual one.

2.3. The FDH Model

The FDH model by Tulkens [29] is formulated as follows:

$$\begin{aligned}
\underset{\theta, \boldsymbol{\lambda}, \boldsymbol{s}_x, \boldsymbol{s}_y}{\text{minimize}} \quad & \theta - \varepsilon(\mathbf{1}^T\boldsymbol{s}_x + \mathbf{1}^T\boldsymbol{s}_y) && (\mathrm{FDH}_D)\\
\text{subject to} \quad & X\boldsymbol{\lambda} - \theta\boldsymbol{x}_o + \boldsymbol{s}_x = \mathbf{0},\\
& Y\boldsymbol{\lambda} - \boldsymbol{y}_o - \boldsymbol{s}_y = \mathbf{0},\\
& \mathbf{1}^T\boldsymbol{\lambda} = 1;\ \lambda_j \in \{0,\ 1\} \text{ for each } j = 1, \cdots, n,\\
& \boldsymbol{\lambda} \geqq \mathbf{0},\ \boldsymbol{s}_x \geqq \mathbf{0},\ \boldsymbol{s}_y \geqq \mathbf{0},\\
& \theta \in \mathbb{R},\ \boldsymbol{\lambda} \in \mathbb{R}^n,\ \boldsymbol{s}_x \in \mathbb{R}^m, \boldsymbol{s}_y \in \mathbb{R}^p.
\end{aligned}$$

Here, it is seen that the problem (FDH_D) is a mixed integer programming problem, and hence the traditional linear optimization methods cannot apply to it. An optimal solution, however, can be obtained by means of a simple vector comparison procedure.

For a DMUo, the optimal solution θ^* to the problem (FDH_D) is equal to the value R_o^* defined by

$$R_o^* = \min_{j \in D(o)} \max_{i=1,\cdots m} \left\{ \frac{x_{ij}}{x_{io}} \right\},$$

where $D(o) = \left\{ j \mid \boldsymbol{x}_j \leq \boldsymbol{x}_o \text{ and } \boldsymbol{y}_j \geqq \boldsymbol{y}_o,\ j = 1, \cdots, n \right\}$. Therefore, R_o^* takes the place of θ^* showing the efficiency degree for DMUo in the FDH model. The 'efficiency' in the FDH model is given by the following.

Definition 7 *(FDH-efficiency) A DMUo is FDH-efficient if and only if $R_o^* = 1$. If $R_o^* < 1$, the DMUo is said to be FDH-inefficient.*

Definition 8 *(FDH_D-efficiency) A DMUo is FDH_D-efficient if and only if for an optimal solution $(\theta^*, \boldsymbol{\lambda}^*, \boldsymbol{s}_x^*, \boldsymbol{s}_y^*)$ to the problem (FDH_D), the following two conditions are satisfied:*

(i) θ^* *is equal to one;*

(ii) the slack variables $\boldsymbol{s}_x^*$ *and* $\boldsymbol{s}_y^*$ *are all zero.*

Otherwise, the DMUo is said to be FDH_D*-inefficient.*

It can be seen that the above two definitions are equivalent to each other, and the production possibility set P_3, which is a free disposable hull, is given by

$$P_3 = \left\{(\boldsymbol{y}, \boldsymbol{x}) \mid Y\boldsymbol{\lambda} \geqq \boldsymbol{y},\ X\boldsymbol{\lambda} \leqq \boldsymbol{x},\ \mathbf{1}^T\boldsymbol{\lambda} = 1,\ \lambda_j \in \{0,1\},\ j = 1, \cdots, n\right\}.$$

Besides, the definition of FDH-efficiency (or FDH_D-efficiency) can be transformed into the following:

Definition 9 *DMUo is said to be Pareto efficient in* P_3 *if and only if there does not exist* $(\boldsymbol{y}, \boldsymbol{x}) \in P_3$ *such that* $(\boldsymbol{y}, -\boldsymbol{x}) \geq (\boldsymbol{y}_o, -\boldsymbol{x}_o)$.

Fig. 7.3 shows a geometric interpretation on the relation between the primal form of the FDH model and the dual one.

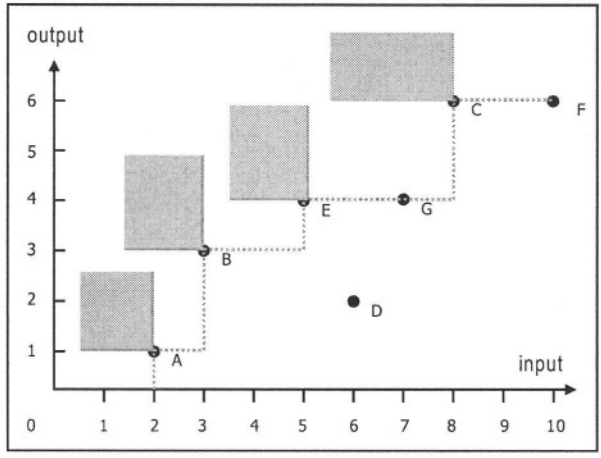
3. GDEA Based on Parametric Domination Structure

In this section, we formulate a GDEA model based on a domination structure and define a new 'efficiency' in the GDEA model. Next, we establish relationships between the GDEA model and basic DEA models mentioned in Section 2.

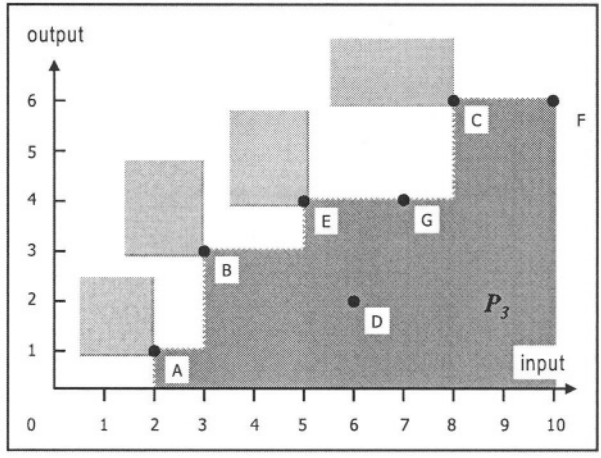
3.1. Relationships between GDEA and DEA

The generalized DEA model can be formulated by employing the augmented Tchebyshev scalarizing function [22]. Namely, the GDEA model, which can evaluate the efficiency in several basic models as special cases, is the following:

$$\begin{aligned}
&\underset{\Delta,\,\mu_k,\,\nu_i}{\text{maximize}} \quad \Delta && \text{(GDEA)}\\
&\text{subject to} \quad \Delta \leqq \tilde{d}_j + \alpha\left(\sum_{k=1}^{p}\mu_k(y_{ko} - y_{kj}) + \sum_{i=1}^{m}\nu_i(-x_{io} + x_{ij})\right),\\
&\qquad\qquad j = 1, \cdots, n,\\
&\qquad \sum_{k=1}^{p}\mu_k + \sum_{i=1}^{m}\nu_i = 1,\\
&\qquad \mu_k,\ \nu_i \geqq \varepsilon,\ k = 1, \cdots, p;\ i = 1, \cdots, m,
\end{aligned}$$



(a) Primal form



(b) Dual form

Figure 7.3. FDH efficient frontier and production possibility set generated by the FDH model from the observed data.

where $\tilde{d}_j = \max_{\substack{k=1,\cdots,p \\ i=1,\cdots,m}} \{\mu_k(y_{ko} - y_{kj}),\ \nu_i(-x_{io} + x_{ij})\}$ and α is a positive number.

Note that when $j = o$, the right-hand side of the inequality constraint in the problem (GDEA) is zero, and hence its optimal value is not greater than zero. We define 'efficiency' in the GDEA model as follows.

Definition 10 *(α-efficiency)For a given positive number α, DMUo is defined to be α-efficient if and only if the optimal value to the problem (GDEA) is equal to zero. Otherwise, DMUo is said to be α-inefficient.*

We here summarize theoretical properties on relationships among efficiencies in the basic DEA models and that in the GDEA model. For detailed proofs of the following theorems, see Yun et al. [34].

Theorem 1 *DMUo is FDH-efficient if and only if DMUo is α-efficient for some sufficiently small positive number α.*

Theorem 2 *DMUo is BCC-efficient if and only if DMUo is α-efficient for some sufficiently large positive number α.*

Theorem 3 *DMUo is CCR-efficient if and only if DMUo is α-efficient for some sufficiently large positive number α, when regarding the problem (GDEA) as the problem (GDEA′) in which the constraint $\sum_{k=1}^{p} \mu_k y_{ko} = \sum_{i=1}^{m} \nu_i x_{io}$ is added to the problem (GDEA).*

$$
\begin{aligned}
\underset{\Delta, \mu_k, \nu_i}{\text{maximize}} \quad & \Delta && \text{(GDEA′)}\\
\text{subject to} \quad & \Delta \leqq \tilde{d}_j + \alpha \left(\sum_{k=1}^{p} \mu_k (y_{ko} - y_{kj}) + \sum_{i=1}^{m} \nu_i (-x_{io} + x_{ij}) \right), \\
& \qquad j = 1, \cdots, n, \\
& \sum_{k=1}^{p} \mu_k y_{ko} - \sum_{i=1}^{m} \nu_i x_{io} = 0, \\
& \sum_{k=1}^{p} \mu_k + \sum_{i=1}^{m} \nu_i = 1, \\
& \mu_k, \ \nu_i \geqq \varepsilon, \ \ k = 1, \cdots, p; \ i = 1, \cdots, m,
\end{aligned}
$$

where $\tilde{d}_j = \max_{\substack{k=1,\cdots,p \\ i=1,\cdots,m}} \{\mu_k (y_{ko} - y_{kj}), \nu_i (-x_{io} + x_{ij})\}$ and α is a positive number.

From the stated theorems, it is seen that the CCR-efficiency, BCC-efficiency and FDH-efficiency for each DMU can be evaluated by varying the parameter α in the problem (GDEA).

3.2. An Illustrative Example

In this subsection, we explain the α-efficiency in the GDEA model with a simple illustrative example and reveal domination relations among all DMUs by GDEA. Assume that there are six DMUs which consume one input to produce one output, as seen in Table 7.1.

Table 7.1. An example of 1-input and 1-output.

DMU	*A*	*B*	*C*	*D*	*E*	*F*
input	2	3	4.5	4	6	5.5
output	1	3	3.5	2	5	4

Table 7.2 shows the results of efficiency in the basic DEA models and in the GDEA model. It can be seen in the upper half part of Table 7.2 that a DMU is efficient if the optimal value is equal to one in the CCR model, the BCC model and the FDH model, respectively. The lower half part of Table 7.2 shows the α-efficiency by changing parameter α. It can be seen that if $\alpha = 0.1$, the α-efficiency of each DMU is the same as the FDH-efficiency. If $\alpha = 10$, the α-efficiency of each DMU is the same as the BCC-efficiency, and moreover if $\alpha = 10$ in the problem (GDEA′), then the α-efficiency is equivalent to the CCR-efficiency. Furthermore, Figure 7.4 – Figure 7.6 represent the efficient frontier generated by varying α in the GDEA model.

Table 7.2. The optimal values in basic DEA models and GDEA model.

DMU	*A*	*B*	*C*	*D*	*E*	*F*
CCR model	0.50	**1.00**	0.78	0.50	0.83	0.73
BCC model	**1.00**	**1.00**	0.83	0.63	**1.00**	0.75
FDH model	**1.00**	**1.00**	**1.00**	0.75	**1.00**	**1.00**
(i) $\alpha = 10$ (GDEA′)	-9.33	**0.00**	-3.25	-11.33	-0.73	-3.74
(ii) $\alpha = 10$	**0.00**	**0.00**	-2.10	-11.00	**0.00**	-3.35
(iii) $\alpha = 3$	**0.00**	**0.00**	**0.00**	-4.00	**0.00**	-0.55
(iv) $\alpha = 1$	**0.00**	**0.00**	**0.00**	-2.00	**0.00**	**0.00**
(v) $\alpha = 0.1$	**0.00**	**0.00**	**0.00**	-1.10	**0.00**	**0.00**

Through this example, it was shown that by varying the value of parameter α, various efficiencies of the basic DEA models can be measured in a unified way on the basis of this GDEA model, and furthermore the relationships among efficiency for these models become transparent.

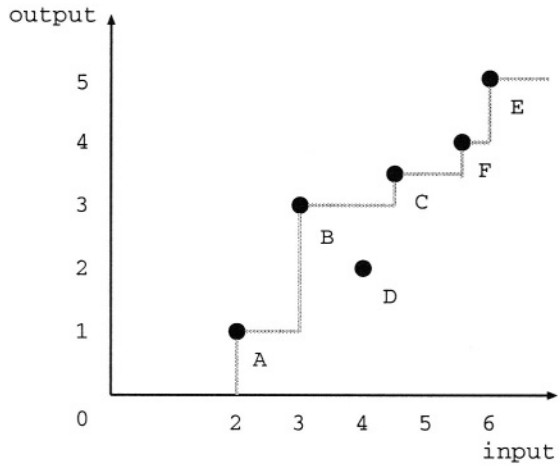


Figure 7.4. Efficient frontier generated by GDEA model with $\alpha = 10^{-6}$.

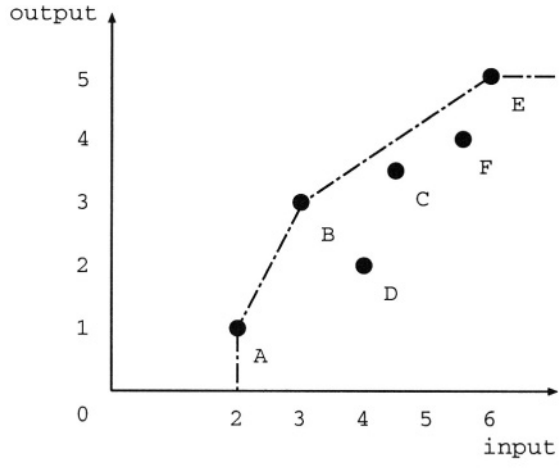


Figure 7.5. Efficient frontier generated by GDEA model with $\alpha = 10$.

4. GDEA Based on Production Possibility

In this section, we consider a dual approach to GDEA introduced in Section 3. We formulate a GDEA_D model based on the production possibility set and define 'efficiency' in the GDEA_D model. Next, we establish relationships between the GDEA_D model and dual models to basic DEA models mentioned in Section 2.

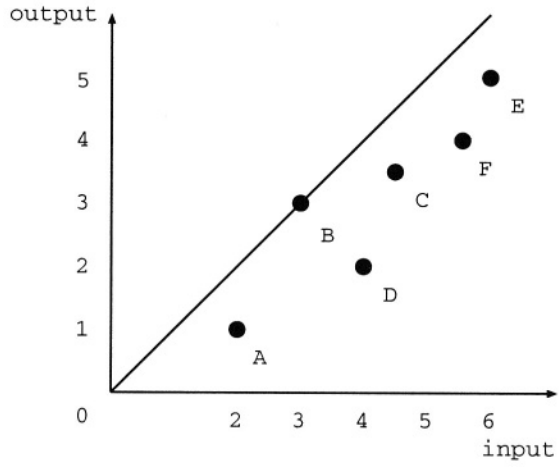


Figure 7.6. Efficient frontier generated by GDEA′ model with $\alpha = 10$.

4.1. Relationships between GDEA_D and DEA

To begin with, an output-input vector $\boldsymbol{z}_j$ of a DMUj, $j = 1, \cdots, n$ and output-input matrix Z of all DMUs, respectively, are defined by

$$\boldsymbol{z}_j := \begin{pmatrix} \boldsymbol{y}_j \\ -\boldsymbol{x}_j \end{pmatrix}, \ j = 1, \cdots, n \text{ and } Z := \begin{pmatrix} \boldsymbol{Y} \\ -\boldsymbol{X} \end{pmatrix}.$$

In addition, we define a $(p+m) \times n$ matrix Z_o by $Z_o := (\boldsymbol{z}_o, \cdots, \boldsymbol{z}_o)$, where o is index of the DMU to be evaluated.

The production possibility sets in the CCR model, the BCC model and the FDH model in Section 2 are reformulated as follows:

$$P_1' = \{\boldsymbol{z} \mid Z\boldsymbol{\lambda} \geqq \boldsymbol{z}, \ \boldsymbol{\lambda} \geqq \boldsymbol{0}\}$$

$$P_2' = \{\boldsymbol{z} \mid Z\boldsymbol{\lambda} \geqq \boldsymbol{z}, \ \boldsymbol{1}^T\boldsymbol{\lambda} = 1, \ \boldsymbol{\lambda} \geqq \boldsymbol{0}\}$$

$$P_3' = \{\boldsymbol{z} \mid Z\boldsymbol{\lambda} \geqq \boldsymbol{z}, \ \boldsymbol{1}^T\boldsymbol{\lambda} = 1, \ \lambda_j \in \{0,1\}, \ j = 1, \cdots, n\}$$

and the 'efficiencies' in these models are redefined:

Definition 11 *DMUo is said to be Pareto efficient in P_1' if and only if there does not exist $(\boldsymbol{y}, -\boldsymbol{x}) \in P_1'$ such that $(\boldsymbol{y}, -\boldsymbol{x}) \geq (\boldsymbol{y}_o, -\boldsymbol{x}_o)$.*

Definition 12 *DMUo is said to be Pareto efficient in P_2' if and only if there does not exist $(\boldsymbol{y}, -\boldsymbol{x}) \in P_2'$ such that $(\boldsymbol{y}, -\boldsymbol{x}) \geq (\boldsymbol{y}_o, -\boldsymbol{x}_o)$.*

Definition 13 *DMUo is said to be Pareto efficient in P_3' if and only if there does not exist $(\boldsymbol{y}, -\boldsymbol{x}) \in P_3'$ such that $(\boldsymbol{y}, -\boldsymbol{x}) \geq (\boldsymbol{y}_o, -\boldsymbol{x}_o)$.*

Remark 1 [18] *Here, Definitions 11-13 correspond to CCR-efficiency (or CCR_D-efficiency), BCC-efficiency (or BCC_D-efficiency) and FDH-efficiency (or FDH_D-efficiency), respectively.*

The dual problem to (GDEA′) introduced in Section 3 is formulated as follows:

$$\begin{aligned} \underset{\omega,\, \kappa,\, \boldsymbol{\lambda},\, \boldsymbol{s_z}}{\text{minimize}} \quad & \omega - \varepsilon \mathbf{1}^T \boldsymbol{s_z} \qquad (\text{GDEA}_D) \\ \text{subject to} \quad & \{\alpha(Z_o - Z) + D_{\boldsymbol{z}}\}\boldsymbol{\lambda} - \boldsymbol{\omega} + \boldsymbol{s_z} + \kappa \boldsymbol{z}_o = \mathbf{0} \\ & \mathbf{1}^T \boldsymbol{\lambda} = 1, \\ & \boldsymbol{\lambda} \geqq \mathbf{0}, \ \ \boldsymbol{s_z} \geqq \mathbf{0}, \end{aligned}$$

where $\boldsymbol{\omega} = (\omega, \cdots, \omega)$ and α is a given positive number. A $(p+m) \times n$ matrix $D_{\boldsymbol{z}} := (\boldsymbol{d}_1, \cdots, \boldsymbol{d}_n)$ is obtained by replacing the components of $(Z - Z_o)$ by 0 except for the maximal component in each row (if there exist plural maximal components, only one is chosen from among them). Especially, it is seen that when κ is fixed at 0, (GDEA_D) becomes the dual problem to (GDEA) since κ is a dual variable to the second constraint in (GDEA′).

We define an 'efficiency' for a DMUo in the GDEA_D model as follows:

Definition 14 *(α_D-efficiency) For a given positive α, DMUo is said to be α_D-efficient if and only if the optimal solution $(\omega^*, \kappa^*, \boldsymbol{\lambda}^*, \boldsymbol{s}_z^*)$ to the problem $(GDEA_D)$ satisfies the following two conditions:*

(i) ω^* is equal to zero;

(ii) the slack variable $\boldsymbol{s}_z^*$ is zero.

Otherwise, DMUo is said to be α_D-inefficient.

It should be noted particularly that for an optimal solution $(\omega^*, \kappa^*, \boldsymbol{\lambda}^*, \boldsymbol{s}_z^*)$ to problem (GDEA_D), ω^* is not greater than zero because of the strong duality of (GDEA) and (GDEA_D) (in linear programming terms), and the 'non-Archimedean' property of ε.

Here, we summarize theoretical properties on relationships among efficiencies in basic DEA models and the GDEA_D model. For detailed proofs of the following theorems, see Yun et al. [34].

Theorem 4 *Let κ be fixed at 0 in $(GDEA_D)$. DMUo is Pareto efficient in P_3' if and only if DMUo is α_D-efficient for some sufficiently small positive number α.*

Theorem 5 *Let κ be fixed at 0 in $(GDEA_D)$. DMUo is Pareto efficient in P_2' if and only if DMUo is α_D-efficient for some sufficiently large positive number α.*

Theorem 6 *DMUo is Pareto efficient in P_1' if and only if DMUo is α_D-efficient for some sufficiently large positive number α.*

From theorems stated above, it is also seen that the CCR_D-efficiency, BCC_D-efficiency and FDH_D-efficiency for each DMU can be evaluated by varying the parameter α in the problem (GDEA_D).

4.2. Optimal Solutions to (GDEA_D)

In this subsection, we explain the meaning of optimal solutions ω^*, $\boldsymbol{\lambda}^*$, $\boldsymbol{s}_{\boldsymbol{z}}^*$ to (GDEA_D). ω^* gives a measure of relative efficiency for DMUo. In other words, it represents the degree how inefficient DMUo is, that is, how far DMUo is from the efficient frontier generated with the given α. $\boldsymbol{\lambda}^* := (\lambda_1^*, \cdots, \lambda_n^*)$ represents a domination relation between DMUo and other DMUs. That is, it means that the DMUo is dominated by DMUj if λ_j for some $j \neq 0$ is positive. $\boldsymbol{s}_x^*$ represents the surplus of inputs and $\boldsymbol{s}_y^*$ the slack of outputs for performance of the DMUo.

Consider an illustrative example as shown in Table 7.3. The Table shows the results of the CCR-efficiency, BCC-efficiency and FDH-efficiency, respectively, in the example. Table 7.4 shows the optimal solution $(\omega^*, \kappa^*, \boldsymbol{\lambda}^*, \boldsymbol{s}_{\boldsymbol{z}}^*)$ to (GDEA_D) $(\varepsilon = 10^{-6})$ when α is given as 10^{-6} and κ is fixed at 0. Table 7.5 shows the optimal solution $(\omega^*, \kappa^*, \boldsymbol{\lambda}^*, \boldsymbol{s}_{\boldsymbol{z}}^*)$ to (GDEA_D) $(\varepsilon = 10^{-6})$ when α is given by 10 and κ is fixed at 0. Finally, Table 7.6 shows the optimal solution $(\omega^*, \kappa^*, \boldsymbol{\lambda}^*, \boldsymbol{s}_{\boldsymbol{z}}^*)$ to (GDEA_D) $(\varepsilon = 10^{-6})$ when α is given as 10.

Here, we can see that the FDH-efficiency, BCC-efficiency and CCR-efficiency are equivalent to the α-efficiency with $\alpha = 10^{-6}$ $(\kappa = 0)$, $\alpha = 10$ $(\kappa = 0)$ and $\alpha = 10$ (nonfixed κ) respectively, from the result of Table 7.4 – Table 7.6 and Figure 7.7 – Figure 7.9. In other words, the FDH-efficiency, BCC-efficiency and CCR-efficiency can be obtained by changing the parameter α in the GDEA_D model.

Now, we interpret a meaning of optimal solutions $(\omega^*, \kappa^*, \boldsymbol{\lambda}^*, \boldsymbol{s}_{\boldsymbol{z}}^*)$ to (GDEA_D). Note that ω^* gives a measure of relative efficiency for DMUo. In other words, it represents the degree how inefficient DMUo is, that is, how far DMUo is from the efficient frontier generated with the given α. $\boldsymbol{\lambda}^* := (\lambda_1^*, \cdots, \lambda_n^*)$ represents a domination relation between DMUo

Table 7.3. An Example of 1-input and 1-output and optimal value in the problems (CCR), (BCC) and (FDH).

DMU	input	output	CCR model	BCC model	FDH model
A	2	1	0.5	1	1
B	3	3	1	1	1
C	8	6	0.75	1	1
D	6	2	0.333	0.417	0.5
E	5	4	0.8	0.933	1
F	10	6	0.6	$1 - 2 \times 10^{-6}$	0.8
G	7	4	0.571	0.667	0.714

Table 7.4. Optimal solution to (GDEA_D) with $\alpha = 10^{-6}$ and $\kappa = 0$.

DMU	ω^*	$\boldsymbol{\lambda}^*$	$\boldsymbol{s}_z^* = (\boldsymbol{s}_x^*, \boldsymbol{s}_y^*)$
A	0	$\lambda_A^* = 1$	(0, 0)
B	0	$\lambda_B^* = 1$	(0, 0)
C	0	$\lambda_C^* = 1$	(0, 0)
D	−0.5	$\lambda_B^* = \lambda_E^* = 0.5$	(0, 0)
E	0	$\lambda_E^* = 1$	(0, 0)
F	0	$\lambda_C^* = 1$	(2, 0)
G	0	$\lambda_E^* = 1$	(2, 0)

and another DMUs. That is, it means that the DMUo is dominated by DMUj if λ_j for some $j \neq o$ is positive.

For example, as is seen in Table 7.4, the optimal solution for the DMU D is $\lambda_B^* = 0.5$ and $\lambda_E^* = 0.5$, and hence DMU D is dominated by DMU B and DMU E. (See Figure 7.7.) In addition, in Table 7.5, the optimal solution for the DMU E is $\lambda_B^* = 0.631$ and $\lambda_C^* = 0.369$, and hence DMU E is dominated by a linear combination of DMU B and DMU C. (See Figure 7.8.) As is seen in Table 7.6, the optimal solution for the DMU C is $\lambda_B^* = 1$, and hence DMU D is dominated by a point on the line through DMU B and the origin. (See Figure 7.9.) $\boldsymbol{s}_x^*$ represents the slack of inputs and $\boldsymbol{s}_y^*$ does the surplus of outputs for performance of the DMUo. For instance, DMU G has the optimal solution $\omega^* = 0$, $\lambda_E^* = 1$ and $(\boldsymbol{s}_x^*, \boldsymbol{s}_y^*) = (2, 0)$. DMU G is α-inefficient because $\boldsymbol{s}_x^*$ is not equal

Table 7.5. Optimal solution to (GDEA_D) with $\alpha = 10$ and $\kappa = 0$.

DMU	ω^*	$\boldsymbol{\lambda}^*$	$\boldsymbol{s}_{\boldsymbol{z}}^* = (\boldsymbol{s}_x^*,\ \boldsymbol{s}_y^*)$
A	0	$\lambda_A^* = 1$	(0, 0)
B	0	$\lambda_B^* = 1$	(0, 0)
C	0	$\lambda_C^* = 1$	(0, 0)
D	−7.803	$\lambda_B^* = 0.765,\ \lambda_C^* = 0.235$	(0, 0)
E	−0.441	$\lambda_B^* = 0.631,\ \lambda_C^* = 0.369$	(0, 0)
F	0	$\lambda_C^* = 1$	(20, 0)
G	−8.281	$\lambda_B^* = 0.378, \lambda_C^* = 0.622$	(0, 0)

Table 7.6. Optimal solution to (GDEA_D) with $\alpha = 10$ and non-fixed κ.

DMU	ω^*	$\boldsymbol{\lambda}^*$	$\boldsymbol{s}_{\boldsymbol{z}}^* = (\boldsymbol{s}_x^*,\ \boldsymbol{s}_y^*)$	κ^*
A	−11.333	$\lambda_C^* = 1$	(0, 0)	38.667
B	0	$\lambda_B^* = 1$	(0, 0)	0
C	−2.571	$\lambda_B^* = 1$	(0, 0)	−5.929
D	−24.500	$\lambda_C^* = 1$	(0, 0)	7.750
E	−2.778	$\lambda_B^* = 1$	(0, 0)	−3.444
F	−7.500	$\lambda_C^* = 1$	(0, 0)	−1.250
G	−8.727	$\lambda_C^* = 1$	(0, 0)	2.818

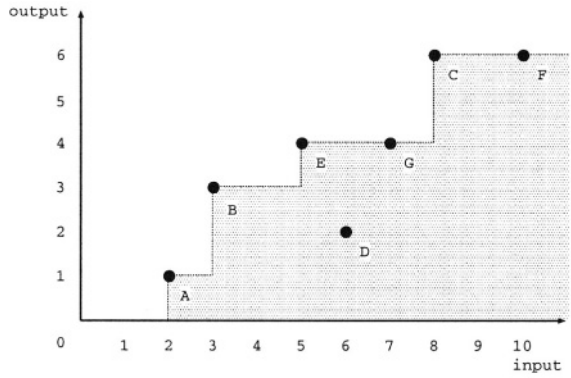


Figure 7.7. Efficient frontier generated by GDEA_D model with $\alpha = 10^{-6}$ and $\kappa = 0$.

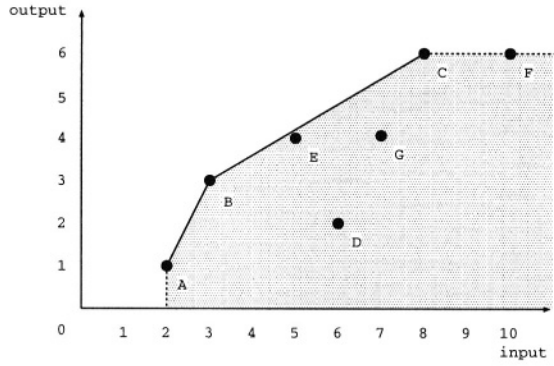


Figure 7.8. Efficient frontier generated by GDEA_D model with $\alpha = 10$ and $\kappa = 0$.

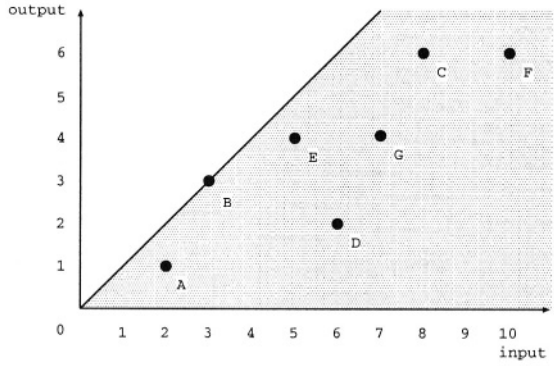


Figure 7.9. Efficient frontier generated by GDEA_D model with $\alpha = 10$ and non-fixed κ.

to zero although $\omega^* = 0$. It implies that DMU G has a larger surplus amount of input than DMU E with the same output.

5. Comparison between GDEA and DEA Models

Now, we compare the efficiency in basic DEA models and the GDEA model for data for thirteen Mexican commercial banks in two years (1990–1991) from Taylor et al. [27]. As is shown in Table 7.7, each bank has the total income as the single output. Total income is the sum of a bank's interest and non-interest income. Total deposits and total non-interest expense are the two inputs used to generate the output. Interest income includes interest earned from loan activities. To-

Table 7.7. Input and output values for 13 Mexican banks, 1990 – 1991 (billions of nominal pesos).

Bank	1990			1991		
	deposits	non-int. expense	int. income plus non-int. income	deposits	non-int. expense	int. income plus non-int. income
(1) Banamex	35313.90	2500.88	14247.10	57510.90	3670.33	15764.60
(2) Bancomer	34504.60	2994.70	12682.10	59965.00	3872.40	15877.00
(3) Serfin	30558.20	1746.50	11766.40	46987.20	2709.20	12694.10
(4) Intermac	7603.53	1011.40	3422.40	13458.00	1165.20	4212.20
(5) Cremi	1977.18	1628.80	2889.10	5108.97	760.60	2102.70
(6) Bancreser	2405.00	140.70	1050.50	3314.32	190.80	1681.10
(7) MercNort	2146.06	338.30	1320.10	3714.72	463.30	1377.40
(8) BCH	2944.00	260.8	1410.00	3728.00	402.90	1794.10
(9) Confia	1962.34	266.60	1568.00	3324.43	364.90	1944.40
(10) Bancen	1815.73	196.70	946.20	2544.96	242.70	848.80
(11) Promex	1908.23	251.30	1162.80	3080.00	320.40	1251.40
(12) Banoro	1372.78	169.60	598.20	2799.00	224.40	810.50
(13) Banorie	488.17	71.90	340.80	680.88	86.80	373.00

Table 7.8. DEA Mexican bank analysis, 13 banks, 1990. Output is total interest and non-interest income; inputs are total deposits and non-interest expense.

Bank	1990								
	CCR		BCC		GDEA				
	θ	class	θ	RTS	$\alpha = 10^3$ $(\boldsymbol{x}_o^T \nu = \boldsymbol{y}_o^T \mu)$	$\alpha = 10$	$\alpha = 1$	$\alpha = 0.5$	$\alpha = 0.1$
(1) Banamex	0.816	NE	1.000	D	-123.46	0.00	0.00	0.00	0.00
(2) Bancomer	0.646	NE	0.890	–	-744.67	-7282.88	-358.41	0.00	0.00
(3) Serfin	0.902	NE	1.000	D	-11.88	0.00	0.00	0.00	0.00
(4) Intermac	0.573	NE	0.809	–	-285.50	-1648.99	0.00	0.00	0.00
(5) Cremi	1.000	E	1.000	C	0.00	0.00	0.00	0.00	0.00
(6) Bancreser	1.000	E	1.000	C	0.00	0.00	0.00	0.00	0.00
(7) MercNort	0.750	NE	0.757	–	-126.73	-1078.91	-149.92	-102.55	-19.69
(8) BCH	0.829	NE	0.837	–	-70.89	-390.60	-11.27	-0.08	0.00
(9) Confia	1.000	E	1.000	C	0.00	0.00	0.00	0.00	0.00
(10) Bancen	0.778	NE	0.803	–	-94.29	-390.09	-8.06	0.00	0.00
(11) Promex	0.782	NE	0.797	–	-79.50	-506.79	-29.08	-6.76	0.00
(12) Banoro	0.588	NE	0.644	–	-299.20	-606.52	-12.81	0.00	0.00
(13) Banorie	0.862	NE	1.000	I	-58.55	0.00	0.00	0.00	0.00

E: Efficient D: Decreasing Returns to Scale(RTS) I: Increasing Returns to Scale

NE: Not Efficient C: Constant Returns to Scale

Table 7.9. DEA Mexican bank analysis, 13 banks, 1991. Output is total interest and non-interest income; inputs are total deposits and non-interest expense.

Bank	1991									
	CCR		BCC		GDEA					
	θ	class	θ	RTS	$\alpha = 10^3$ $(\boldsymbol{x}_o^T\nu = \boldsymbol{y}_o^T\mu)$	$\alpha = 15$	$\alpha = 10$	$\alpha = 1$	$\alpha = 0.5$	$\alpha = 0.1$
(1) Banamex	0.531	NE	1.000	D	-181.32	0.00	0.00	0.00	0.00	0.00
(2) Bancomer	0.511	NE	1.000	D	-281.95	0.00	0.00	0.00	0.00	0.00
(3) Serfin	0.532	NE	1.000	D	-136.52	0.00	0.00	0.00	0.00	0.00
(4) Intermac	0.569	NE	0.908	–	-257.11	-717.26	0.00	0.00	0.00	0.00
(5) Cremi	0.704	NE	0.772	–	-282.58	-3134.25	-1957.76	0.00	0.00	0.00
(6) Bancreser	1.000	E	1.000	C	0.00	0.00	0.00	0.00	0.00	0.00
(7) MercNort	0.634	NE	0.638	–	-284.80	-4371.50	-2999.54	-385.14	-212.60	-42.31
(8) BCH	0.826	NE	0.828	–	-112.88	-1481.79	-982.50	-99.34	-60.03	-15.61
(9) Confia	1.000	E	1.000	C	0.00	0.00	0.00	0.00	0.00	0.00
(10) Bancen	0.592	NE	0.612	–	-253.70	-1621.77	-1075.07	-50.54	0.00	0.00
(11) Promex	0.705	NE	0.715	–	-191.64	-2262.34	-1504.08	-74.49	0.00	0.00
(12) Banoro	0.535	NE	0.554	–	-295.19	-1410.08	-934.00	-80.67	-5.37	0.00
(13) Banorie	0.937	NE	1.000	I	-73.42	0.00	0.00	0.00	0.00	0.00

E: Efficient D: Decreasing Returns to Scale(RTS) I: Increasing Returns to Scale

NE: Not Efficient C: Constant Returns to Scale

tal non-interest income includes dividends, fees, and other non-interest revenue. The total deposits input variable includes the bank's interest paying deposit liabilities. Total non-interest expense includes personnel and administrative costs, commissions paid, banking support fund contributions and other non-interest operating costs. Thus, we evaluate the efficiency for each bank with the annual data, that is, consider α-efficiency corresponding to several values $\alpha = 0.1,\ 0.5,\ 1,\ 10,\ 15$ (only 1991) and 10^3. Therefore, Table 7.8 and Table 7.9 represent the results of analyses by the basic DEA models and the GDEA model.

As is shown in the tables, the GDEA model with $\alpha = 0.1$ provides FDH efficiency. It means that there is no change in α-efficient DMUs for smaller α than 0.1. In addition, the GDEA model with $\alpha = 10$ yields BCC efficiency in Table 7.8, while $\alpha = 15$ does in Table 7.9. Also, there is no change in α-efficiency of DMUs, even if taking greater α than 10 or 15. Moreover, CCR-efficiency can be conducted by taking α sufficiently large in the GDEA′ model. From this fact, we see that the number of efficient DMUs decreases as parameter α increases in general. Particularly, note the α-efficiency for $\alpha = 0.5$ and $\alpha = 1$: This represents an intermediate efficiency between FDH-efficiency and BCC-efficiency. In practice, there are decision making problems which cannot correspond to a special value judgment such as "ratio value efficiency" in the CCR model, "sum value efficiency" in the BCC model, and so on. In contrast to the existing DEA models, the GDEA model can incorporate various value judgments of decision makers by changing a parameter α, and then several kinds of efficiency of the basic DEA models can be measured in a unified way on the basis of the GDEA model. Furthermore, the relationships among efficiencies for these models become transparent by considering GDEA.

6. GDEA for Multiple Criteria Decision Making

6.1. Generation of Efficient Frontiers

In multi-objective optimization problems, there does not necessarily exist the solution that optimizes all objective functions, and then the concept which is called Pareto optimal solution (or efficient solution) is introduced [22]. Usually, there exist a number of Pareto optimal solutions, which are considered as candidates for final decision making solution [19]. It is an issue how decision makers choose one from the set of Pareto optimal solutions as the final solution. Consequently, interactive multi-objective optimization methods have been developed to this end. In many practical problems such as engineering design problems, however, criteria functions can not be given explicitly in terms of design variables. Under this circumstance, values of criteria functions for given

values of design variables are usually obtained by some analyses such as structural analysis, thermodynamical analysis or fluid mechanical analysis. These analyses require considerable computation time. Therefore, it is not unrealistic to apply existing interactive optimization methods to those problems.

Recently, multi-objective optimization methods using genetic algorithms (GA) have been studied actively by many authors [2, 11, 14, 16, 23, 26]. Genetic algorithms are useful for generating efficient frontiers with two or three objective functions. Decision making can be easily performed on the basis of visualized efficient frontiers. This is described in Chapter 5.1 of this book in more detail. We discuss here how we can utilize the GDEA effectively to generate efficient frontiers.

To begin with, we give a brief explanation on the ranking method [14]. Consider an individual x^o at a generation which is dominated by n individuals in the current population, then its rank is given by $(1+n)$. From this, we can see that all non-dominated individuals are assigned rank 1. In Fig. 7.10, each number in parentheses represents the rank of each individual and the curve represents the exact efficient frontier. The ranking method based on the relation of domination among individuals has a merit to be computationally simple. However, the ranking method has a shortcoming in the need to assess a large number of generations, since non-dominated individuals in the current generation such as C and G in Fig. 7.10 are often kept alive long, even though they are not Pareto optimal solutions in the final generation. Moreover, it is difficult to generate a smooth efficient frontier by the stated ranking method.

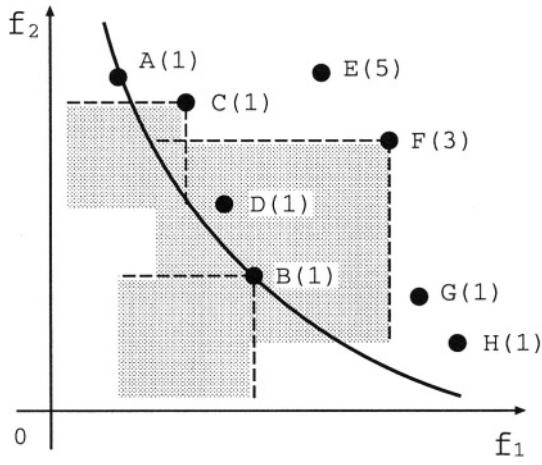


Figure 7.10. Ranking method.

Arakawa et al. [2] suggested a method using CCR-efficiency in order to overcome the shortcomings of the methods stated above. That is, the fitness θ of an individual $\boldsymbol{x}^o$ $(o = 1, \cdots, p)$ is given by solving the following linear programming problem:

$$\begin{aligned} &\underset{\theta, \boldsymbol{\lambda}}{\text{minimize}} && \theta \\ &\text{subject to} && \left[\boldsymbol{f}(\boldsymbol{x}^1), \cdots, \boldsymbol{f}(\boldsymbol{x}^p)\right] \boldsymbol{\lambda} - \theta \boldsymbol{f}(\boldsymbol{x}^o) \leqq 0, \\ & && \boldsymbol{\lambda} \geqq 0, \quad \boldsymbol{\lambda} \in \mathbb{R}^p. \end{aligned}$$

The optimal value θ^* to the above problem represents how far $\boldsymbol{f}(\boldsymbol{x}^o)$ is from the CCR-efficient frontier. We see that only when θ^* is equal to one, $\boldsymbol{f}(\boldsymbol{x}^o)$ is located on the CCR-efficient frontier. Selection is performed by θ^*.

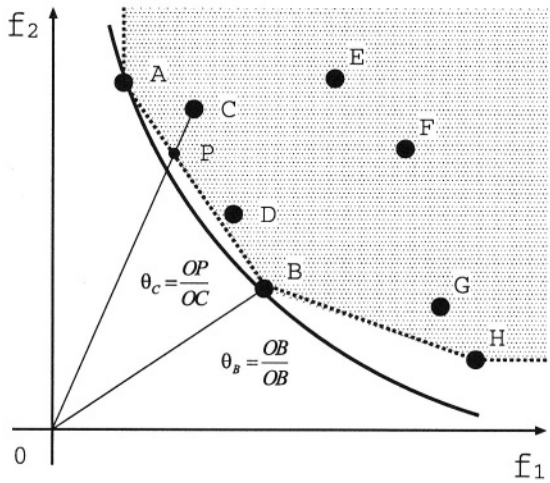


Figure 7.11. GA with DEA method.

In other words, this method investigates the relation of domination among individuals with respect to the shaded region (see Fig. 7.11). In Fig. 7.11, the solid curve represents the exact efficient frontier and the dotted line represents the CCR-efficient frontier at a generation. As the figure shows, individuals *C* and *G* are removed fast, and then a good approximation of the exact efficient frontier can be obtained efficiently. Therefore, when the efficient frontier is convex[3], non-Pareto solutions can be removed at a young generation. However, when the efficient

[3]Let $E \in \mathbb{R}^n$ be an efficient frontier in the objective space and let $\mathbb{R}^n_+$ be a non-negative orthant. Then we say *the efficient frontier is convex* if $(E + \mathbb{R}^n_+)$ is a convex set. Otherwise, the efficient frontier is said to be non-convex.

frontier is non-convex, the sunken part of it can not be generated by Arakawa et al.'s method [2].

6.2. Utilization of Generalized Data Envelopment Analysis

Utilizing the GDEA model instead of the traditional DEA models, we can overcome the shortcomings of both the ranking methods and Arakawa et al.'s method. In applying GA to problems with constraints, we introduce an augmented objective function using penalty functions imposed on constraints. Here, an augmented objective function of f_i $(i = 1, \cdots, m)$ in the problem (MOP) is given by

$$F_i(\boldsymbol{x}) = f_i(\boldsymbol{x}) + \sum_{j=1}^{l} p_j \left\{[g_j(\boldsymbol{x})]_+\right\}^{\alpha},$$

where p_j is a penalty coefficient, α is a penalty exponent and $[y]_+ = \max\{y,\ 0\}$.

As a result, the initial problem (MOP) can be converted into a problem to minimize the augmented objective function $(F_1(\boldsymbol{x}), \cdots, F_m(\boldsymbol{x}))$. Here, we need to prepare the data set in order to evaluate the degree of α-efficiency of an individual $\boldsymbol{x}^o$ in the current population. Let inputs and outputs in GDEA be substituted by the value of $F_i(\boldsymbol{x}^o)$. Then the problem (GDEA) reduces to the following problem (P).

$$\begin{aligned} \underset{\Delta,\nu}{\text{maximize}} \quad & \Delta \qquad\qquad (\text{P}) \\ \text{subject to} \quad & \Delta \leqq \tilde{d}_j - \alpha \sum_{i=1}^{m} \nu_i (F_i(\boldsymbol{x}^o) - F_i(\boldsymbol{x}^j)),\ \ j = 1, \cdots, p, \\ & \sum_{i=1}^{m} \nu_i = 1, \\ & \nu_i \geqq \varepsilon,\ \ i = 1, \cdots, m, \end{aligned}$$

where $\tilde{d}_j = \max_{i=1,\cdots,m} \left\{\nu_i \left(-F_i(\boldsymbol{x}^o) + F_i(\boldsymbol{x}^j)\right)\right\}$. ε is a sufficiently small number. α is the value of a monotonically decreasing function with respect to the number of generations.

Practically, α is given by

$$\alpha(t) := \omega \cdot \exp(-\beta \cdot t),\ \ t = 0, 1, \cdots, N,$$

where ω, β and N are positive fixed numbers. ω $(= \alpha(0))$ is determined to be sufficiently large as 10, 10^2 and 10^3. N (the number of generations

until the termination of computation) is given by the time limitation for decision making. For given ω and N, β is chosen by solving the equation $\alpha(N) = \omega \cdot \exp(-\beta \cdot N) \fallingdotseq 0$ (i.e., nearly equal to 0).

The degree of α-efficiency of an individual $\boldsymbol{x}^o$ in the current population is given by the optimal value Δ^* to the problem (P), and is considered as the fitness in GA. Therefore, the selection of an individual is determined by the degree of α-efficiency, i.e. if Δ^* equals to zero, the individual remains at the next generation. With making the best use of the stated properties of GDEA, it is possible to keep merits of ranking methods and the method using DEA, and at the same time, to overcome the shortcomings of existing methods. Namely, taking a large α can remove individuals which are located far from the efficient frontier, and taking a small α can generate non-convex efficient frontiers. (See Fig. 7.12.)

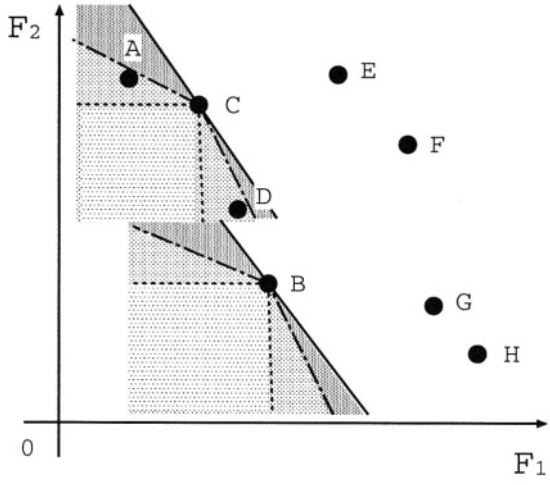


Figure 7.12. Geometric interpretation of α in (P).

Finally, the algorithm based on GDEA and GA is summarized as follows:

Step 1. (Initialization)
Generate p-individuals randomly. Here, the number of p is given prior.

Step 2. (Crossover and Mutation)
Make $p/2$ pairs randomly among the population. Making crossover each pair generates a new population. Mutate them according to the given probability of mutation.

Step 3. (Evaluation of Fitness by GDEA)
Evaluate the GDEA-efficiency by solving the problem (P)

Step 4. (Selection)
Select p individuals from the current population on the basis of the fitness given by GDEA-efficiency.

The process Step 2 – Step 4 is continued until the number of generations attains a given number.

6.3. Examples: Two-objective Optimization Problems

We consider the following example with two objective functions.

Example 1

$$\begin{aligned} \underset{\boldsymbol{x}}{\textit{minimize}} \quad & (f_1(\boldsymbol{x}),\ f_2(\boldsymbol{x})) = (x_1,\ x_2) \\ \textit{subject to} \quad & x_1^3 - 3x_1 - x_2 \leqq 0, \\ & x_1 \geqq -1,\ \ x_2 \leqq 2. \end{aligned}$$

The efficient frontier in Example 1 is non-concave and non-convex. In order to show the effectiveness of the GDEA method, we compare the results by the (a) ranking method, (b) DEA method and (c) GDEA method. Parameters in GA and the problem (P) are set as follows:

(i) the number of generations : 10, 20, 30

(ii) the size of population : 80
the representation of chromosome : 10 bits

(iii) the probability of crossover : 1
the probability of mutation : 0.05

(iv) $\alpha(t) = \begin{cases} 10 \times \exp(-0.6 \times t),\ t = 0, \cdots, 10 & \text{termination at 10 generations} \\ 10 \times \exp(-0.3 \times t),\ t = 0, \cdots, 20 & \text{termination at 20 generations} \\ 10 \times \exp(-0.2 \times t),\ t = 0, \cdots, 30 & \text{termination at 30 generations} \end{cases}$

(v) $\varepsilon = 10^{-6}$.

The elitist preserving selection [16] is adopted. The results are shown in Fig. 7.13. The horizontal axis and the vertical axis indicate the values of objective functions f_1 and f_2, respectively. The symbol • represents a Pareto optimal solution among all generations, and ○ represents a non-dominated individuals at some generation but not Pareto optimal among

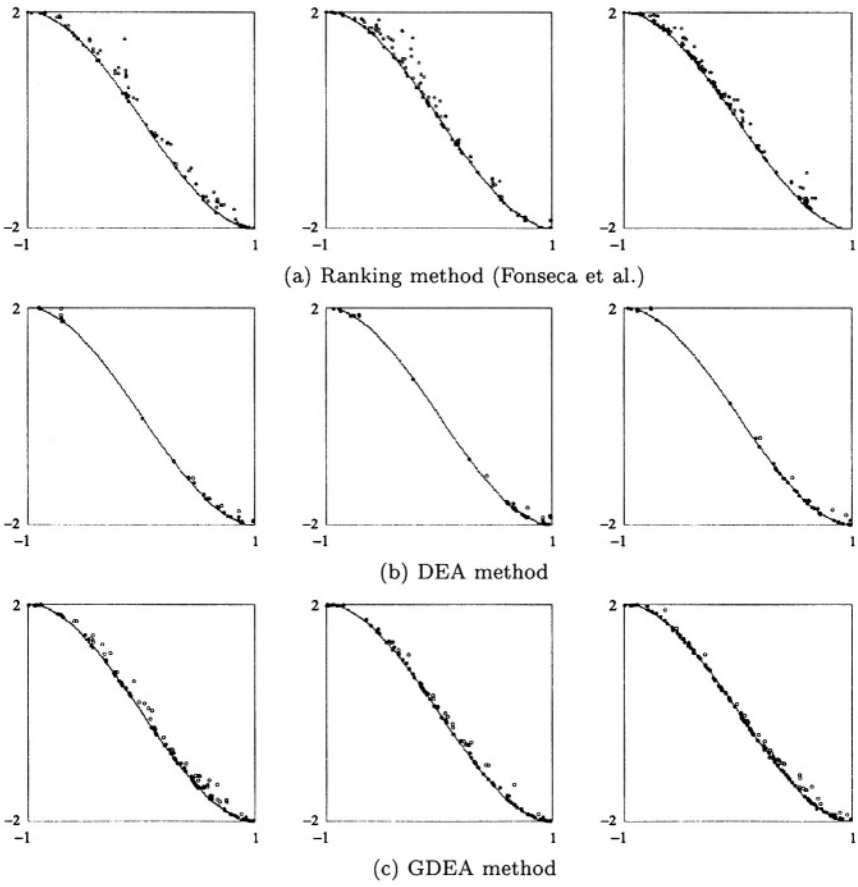


Figure 7.13. Results to the Example 1 (from left to right, 10, 20, 30 generations, respectively).

all generations. Note here that non-dominated individuals depend on the domination structure of each method: For example, individuals with rank 1 are non-dominated in the ranking method, the ones with $\theta^* = 1$ are non-dominated in the DEA method. In the GDEA method, non-dominated individuals are identical with α-efficient ones.

(a) Ranking method

The ranking method produced relatively many Pareto optimal solutions. However, there are also many non-Pareto optimal solutions among non-dominated individuals at each generation. Moreover, it is usually difficult to generate smooth efficient frontiers as shown in (a) of Fig. 7.13.

(b) DEA method

Many non-dominated individuals at each generation become finally Pareto optimal among the whole generation in (b) of Fig. 7.13, while the obtained Pareto optimal solutions are fewer than by the ranking method. However, the sunken parts of efficient frontier can not be generated by this method, and therefore, the DEA method cannot be applied to multi-objective optimization problems with non-convex functions.

(c) GDEA method

In (c) of Fig. 7.13, the largest number of Pareto optimal solutions are obtained among the stated methods. Moreover, efficient frontiers generated by the proposed method are smooth, even though they are non-convex. In addition, it is seen that almost all non-dominated individuals at each generation become the final Pareto optimal solutions.

In particular, it should be noted in the ranking method that non-dominated individuals obtained at intermediate generations are often not Pareto optimal solutions. In practical problems, we do not know when to stop the computation in advance. Usually, the computation is terminated at a relatively early generation due to the time limitation. It is an important requirement, therefore, that non-dominated individuals at intermediate generations are finally Pareto optimal solutions. The GDEA method has a desirable performance from this point of view.

7. Conclusions

In this chapter, we discussed several DEA models in multicriteria decision making. In particular, it has been observed that the GDEA model makes it possible to evaluate efficiencies of several DEA models in a unified way, and to incorporate various preference structures of decision makers. Through a numerical example, it has been shown that the mutual relations among all decision making units can be grasped by varying α in the GDEA model. Furthermore, interpreting the meaning of an optimal value to the GDEA_D model based on production possibility as a dual approach to the GDEA, it is possible to make a quantitative analysis of inefficiency on the basis of surplus of inputs and slack of outputs. Moreover, through an illustrative example, it has been shown that GDEA_D can reveal domination relations among all decision making units. Finally, it was shown that GDEA can be effectively applied to

drawing efficient frontiers in multi-objective (in particular, two objective) decision making problems. It is expected from the obtained results in this study that GDEA is useful for evaluating the efficiency of complex management systems in business, industry and social problems.

References

[1] Agrell, P.J. and J. Tind (2001). A Dual Approach to Nonconvex Frontier Models. *Journal of Productivity Analysis,* 16(2): 129–147.

[2] Arakawa, M., H. Nakayama, I. Hagiwara, and H. Yamakawa (1998). Multiobjective Optimization Using Adaptive Range Genetic Algorithms with Data Envelopment Analysis. In *A Collection of Technical Papers on the 7th Symposium on Multidisciplinary Analysis and Optimization,* Volume 3, 2074–2082. American Institute of Aeronautics and Astronautics, AIAA paper 98–4970.

[3] Banker, R.D., A. Charnes, and W.W. Cooper (1984). Some Models for Estimating Technical and Scale Inefficiencies in Data Envelopment Analysis. *Management Science,* 30:1078–1092.

[4] Belton, V. (1992). On Integrating Data Envelopment Analysis with Multiple Criteria Decision Analysis. In *Proceedings of the Ninth International Conference on Multiple Criteria Decision Making: Theory and Applications in Business, Industry and Commerce* (eds. A. Goicoechea, L. Duckstein and S. Zionts), 71–79. Springer-Verlag, Berlin.

[5] Belton, V., and S.P. Vickers (1993). Demystifying DBA – A Visual Interactive Approach Based on Multiple Criteria Analysis. *Journal of the Operational Research Society,* 44:883–896.

[6] Charnes, A., W.W. Cooper, B. Golany, L.M. Seiford, and J. Stutz (1985). Foundations of Data Envelopment Analysis for Pareto-Koopmans efficient empirical production functions. *Journal of Econometrics,* 30:91–107.

[7] Charnes, A., W.W. Cooper, Z.M. Huang, and D.B. Sun (1990). Polyhedral Cone Ratio DEA Models with an Illustrative Application to Large Commercial Banks. *Journal of Econometrics,* 46:73–91.

[8] Charnes, A., W.W. Cooper, and E. Rhodes (1978). Measuring the Efficiency of Decision Making Units. *European Journal of Operational Research,* 2:429–444.

[9] Charnes, A., W.W. Cooper, and E. Rhodes (1979). Measuring the Efficiency of Decision Making Units. *European Journal of Operational Research,* 3:339.

[10] Charnes, A., W.W. Cooper, Q.L. Wei, and Z.M. Huang (1989). Cone Ratio Data Envelopment Analysis and Multi-objective Programming. *International Journal of Systems Science,* 20:1099–1118.

[11] Coello, C.A.C. (1999). An Updated Survey of Evolutionary Multiobjective Optimization Techniques: State of Art and Future Trends. In *Proceedings of the 1999 Congress on Evolutionary Computation,* 3–13. IEEE Press, Piscataway, NJ.

[12] Deprins, D., L. Simar, and H. Tulkens (1984). Measuring Labor-Efficiency in Post Offices. In *The Performance of Public Enterprises: Concepts and Measurements* (eds. M. Marchand, P. Pestieu and H. Tulkens), 247–263. North Holland, Amsterdam.

[13] Farrell, M.J. (1957). The Measurement of Productive Efficiency. *Journal of the Royal Statistical Society A,* 120:253–281.

[14] Fonseca, C.M., and P.J. Fleming (1993). Genetic Algorithms for Multi-objective Optimization: Formulation, Discussion and Generalization. *Proceedings of the Fifth International Conference on Genetic Algorithms,* 416–426, Morgan Kaufmann Publishers, San Francisco, CA.

[15] Golany, B. (1988). An Interactive MOLP Procedure for the Extension of DEA to Effectiveness Analysis. *Journal of the Operational Research Society,* 39:725–734.

[16] D.E. Goldberg, D.E. (1989). *Genetic Algorithms in Search, Optimization and Machine Learning,* Addison-Wesley, Reading, MA.

[17] Halme, H., T. Joro, P. Korhonen, A. Salo, and J. Wallenius (1999). A Value Efficiency Approach to Incorporating Preference Information in Data Envelopment Analysis. *Management Science,* 45:103–115.

[18] Joro, T., P. Korhonen, and J. Wallenius (1998) Structural Comparison of Data Envelopment Analysis and Multiple Objective Linear Programming, *Management Science,* 44:962–970.

[19] Koopmans, T.C. (1951). Analysis of Production as an Efficient Combination of Activities. In *Analysis of Production and Allocation* (ed. T.C. Koopmans), 33–97. John Wiley, New York.

[20] Korhonen, P. (2001). Searching the Efficient Frontier in Data Envelopment Analysis. In *Aiding Decisions with Multiple Criteria (Essays in Honour of Bernard Roy)* (eds. Bouyssou, D., Jacquet-Lagreze, E., Perny, P., Slowinski, R., and Vanderpooten, D.), 543–558, Kluwer Academic Publishers, Dordrecht.

[21] Nakayama, H. (1995). Aspiration level approach to interactive multi-objective programming and its applications. In *Advances in Multicriteria Analysis* (eds. P.M. Pardalos, Y. Siskos, and C. Zopounidis), 147–174. Kluwer Academic Publishers, Dordrecht.

[22] Sawaragi, Y., H. Nakayama, and T. Tanino (1985). *Theory of Multiobjective Optimization.* Academic Press, Orlando.

[23] Schaffer, J.D. (1985). Multiple Objective Optimization with Vector Evaluated Genetic Algorithms. In *Genetic Algorithms and their Applications,* Proceedings of the First International Conference on Genetic Algorithms, 93–100. Lawrence Erlbaum Associates, Hillsdale, NJ.

[24] Seiford, S.M., and R.M. Thrall (1990). Recent Developments in DEA: The Mathematical Programming Approach to Frontier Analysis. *Journal of Econometrics,* 46:7–38.

[25] Stewart, T.J. (1996). Relationships Between Data Envelopment Analysis and Multiple Criteria Decision Analysis, *Journal of the Operational Research, Society*,47:654–665.

[26] Tamaki, H., H. Kita, and S. Kobayashi (1996). Multi-Objective Optimization by Genetic Algorithms: A Review. In *Proceedings of the 1996 International Conference on Evolutionary Computation ICEC'96* (eds. T. Fukuda and T. Furuhashi), 517–522. IEEE Press, Piscataway.

[27] Taylor, W.M., R.G. Thompson, R.M. Thrall, and P.S. Dharmapala (1997). DEA/AR Efficiency and Profitability of Mexican Banks: A Total Income Model. *European Journal of Operational* Research,98:346–363.

[28] Thanassoulis, E., and R.G. Dyson (1992). Estimating Preferred Target Input-Output Levels Using Data Envelopment Analysis. *European Journal of Operational Research* 56:80–97.

[29] Tulkens, H. (1993). On FDH Efficiency: Some Methodological Issues and Applications to Retail Banking, Courts, and Urban Transit. *Journal of Productivity Analysis,* 4:183–210.

[30] Wei, Q.L., and G. Yu(1997). Analyzing properties of K-Cones in the Generalized Data Envelopment Analysis Model. *Journal of Econometrics,* 80:63–84.

[31] Wierzbicki, A. (1980). The Use of Reference Objectives in Multiobjective Optimization. In *Multiple Objective Decision Making, Theory and Application* (eds. G. Fandel and T. Gal), 468–486. Springer-Verlag, Berlin.

[32] Zhu, J. (1996). Data Envelopment Analysis with Preference Structure. *Journal of the Operational Research Society,* 47:136–150.

[33] Yun, Y.B., H. Nakayama, and T. Tanino (2000). On Efficiency of Data Envelopment Analysis. In *Research and Practice in Multiple Criteria Decision Making* (eds. Y.Y. Haimes and R.E. Steuer), 208–217. Springer-Verlag, Berlin.

[34] Yun, Y.B., H. Nakayama, and T. Tanino (2000). A Generalized Model for Data Envelopment Analysis. Department of Applied Mathematics, Konan University, Research Paper 00-01.

[35] Yun, Y.B., H. Nakayama, T. Tanino, and M. Arakawa (2001). Generation of Efficient Frontiers in Multi-Objective Optimization Problems by Generalized Data Envelopment Analysis. *European Journal of Operational Research,* 129(3):586–595.

Chapter 8

MULTIOBJECTIVE COMBINATORIAL OPTIMIZATION – THEORY, METHODOLOGY, AND APPLICATIONS

Matthias Ehrgott
Department of Engineering Science
The University of Auckland
Private Bag 92019, Auckland
New Zealand
m.ehrgott@auckland.ac.nz

Xavier Gandibleux
LAMIH – UMR CNRS 8530
Université de Valenciennes
Campus "Le Mont Houy", F-59313 Valenciennes cedex 9
France
xavier.gandibleux@univ-valenciennes.fr

Abstract This chapter provides an annotated bibliography of multiple objective combinatorial optimization, MOCO. We present a general formulation of MOCO problems, describe their main characteristics, and review the main properties and theoretical results. One section is devoted to a brief description of the available solution methodology, both exact and heuristic. The main part of the chapter consists of an annotation of the existing literature in the field organized problem by problem. We conclude the chapter by stating open questions and areas of future research. The list of references comprises more than 400 entries.

Keywords: Combinatorial optimization, multiple objectives, metaheuristics, exact methods, bibliography

1. Introduction

Combinatorial Optimization is a field extensively studied by many researchers. Due to its potential for application in real world problems it has prospered over the last few decades. A good survey of the state of the art is provided by [74]. But as far as real world decision making is concerned, it is also well known, that decision makers have to deal with several – usually conflicting – objectives. The growth in the interest in theory and methodology of multicriteria decision making (MCDM) over the last thirty years as documented by the chapters in this volume, the survey of the activities in the field [366] and a bibliography of MCDM applications [421] is witness of this fact.

Thus it is somewhat surprising that a combination of both, i.e. multicriteria or multiobjective combinatorial optimization (MOCO) has not been studied widely. A few papers in the area have been published in the seventies, then the classical problems have been investigated in the eighties. Only in recent years – approximately since 1990 – a profound interest in the topic is evident. Since then several PhD theses have been written, specific methodologies have been developed, and the number of research papers in the field has grown considerably.

In this chapter we intend to give an overview over the literature in the field of multiobjective combinatorial optimization. In the following sections, we first present a brief introduction to the field, including a general problem formulation, description of several types of MOCO problems, and the most important theoretical properties of these problems (Sections 2 and 3). In Section 5 we explain the classification of literature that we used. This consists first of a classification of the problem treated and secondly of the methodology applied to solve it. Then we review existing methods to solve MOCO problems in Section 4. The main part of the chapter is devoted to the annotation of the literature (Section 6). The chapter is concluded by a brief discussion of open questions and areas of future research (Section 7).

Let us now describe the focus of this chapter. We compiled the literature on multiobjective combinatorial optimization accessible to us. We mainly consider papers that deal specifically with MOCO problems, thus our bibliography is certainly not complete on 0-1 programming with multiple objectives, and exclude most of the literature on general multiobjective integer programming. A similar statement can be made with respect to scheduling. Scheduling problems are specific problems with their own theory and methodology, which we will not describe in detail, but refer to Chapter 7.3. We should also mention, that there exist earlier survey papers related to MOCO, one general [398], and two specifically

devoted to multiobjective network design, [53, 54]. Our bibliography contains all the relevant literature listed there. However, it is more complete, e.g. we could include the new direction of using metaheuristics for MOCO problems. However, we are aware of the fact, that despite our best efforts the list will not be complete, so we apologize for any omissions.

The aim of the bibliography is twofold. First we want to provide a starting point for researchers and students interested in the field, giving a brief introduction and commenting on, thus guiding through, existing literature. For the experienced researcher the list is intended as structured overview of the field.

2. Multiple Objective Combinatorial Optimization Problems

The feasible set of a (multiobjective) combinatorial problem is defined as a subset $X \subseteq 2^A$ of the power set of a finite set $A = \{a_1, \ldots, a_n\}$. For example, consider the minimum spanning tree problem. $G = (V, A)$ is a graph with node set V and edge set A, the feasible set is the set of spanning trees of G and $X = \{S \subseteq A : S \text{ is a spanning tree of } G\}$.

Typically, in combinatorial optimization two types of objective functions are considered, namely the sum and the bottleneck objective:

$$z(S) = \sum_{a \in S} w(a), \text{ or}$$

$$z(S) = \max_{a \in S} w(a),$$

where $S \in X$ and $w : A \to \mathbb{Z}$ is some weight function.

A combinatorial problem can also be formulated in terms of binary variables. For this purpose we introduce a variable x_i for each element $a_i \in A$. Then, a feasible solution $S \in X$ can be represented by a binary vector $x \in \{0, 1\}^n$ if we define

$$x_i = \begin{cases} 1 & e_i \in S \\ 0 & \text{else.} \end{cases}$$

With this definition $S = \{a_i : x_i = 1\}$. It is therefore equivalent to speak about feasible solutions as subsets of A or about their representations by binary vectors. Accordingly S will be represented by a subset of $\{0, 1\}^n$.

In terms of the feasible set, this definition comprises (multiobjective versions of) the shortest path, minimum spanning tree, assignment, knapsack, travelling salesperson, or set covering problems, to mention only a few.

In a multicriteria combinatorial problem several weight functions $w_j : A \to \mathbb{Z}$ are given, yielding several objective functions z^j, $j = 1, \ldots, Q$ (usually of the sum or bottleneck type). The problem is then to solve

$$\text{“}\min_{S \in X}\text{”}(z^1(S), \ldots, z^Q(S)) \qquad \text{(MOCO)}$$

where the meaning of “min” has still to be defined.

Most often the minimization in (MOCO) is understood in the sense of efficiency (or Pareto optimality). A subset $S \in X$ is called efficient if there does not exist another feasible solution $S' \in X$ such that $z^j(S') \leq z^j(S)$ for all $j = 1, \ldots, Q$ with strict inequality for at least one of the objectives. The corresponding vector $z(S) = (z^1(S), \ldots, z^Q(S))$ is called nondominated. The set of Pareto optimal (efficient) solutions of (MOCO) will be denoted by E, the set of nondominated vectors by ND throughout the chapter. Sometimes we shall use the the term nondominated frontier for the set of all nondominated vectors.

However, besides efficiency, there are other definitions of the “min” term in the formulation of (MOCO). For example, one could consider lexicographic minimization, when objective vectors are compared lexicographically: $z(S_1) <_{lex} z(S_2)$ if $z^j(S_1) < z^j(S_2)$, where j is the smallest index such that $z^j(S_1) \neq z^j(S_2)$. This could be done with respect to one, or all permutations of the objective functions z^j.

Another possibility is to minimize the worst objective function, i.e.

$$\min_{S \in X} \max_{j=1,\ldots,Q} z^j(S).$$

We call this the max-ordering problem (following [94]) in order to distinguish it from the single objective bottleneck problem (note that both are often called min-max problems, which may create confusion).

A combination of the latter two is the lexicographic max-ordering problem, where the vector of objective values $z(S)$ is first resorted in a nonincreasing order of its components, and the resulting vectors are compared lexicographically, see [83, 85] for details.

In a real world decision context, finally a compromise has to be made among the many efficient solutions that (MOCO) may have. This is the reason why often the existence of a utility function is implicitly or explicitly assumed. A utility function assigns each criterion vector $z(S)$ a scalar overall utility. Then methods are developed to find a solution of maximum utility. This is a typical approach in interactive methods described later.

Closely related to combinatorial problems are multiobjective integer programming problems. These can be formulated as follows.

$$\begin{array}{ll} \text{“min”} & Cx \\ \text{subject to} & Ax = b \\ & x_i \geq 0, \text{ integer } i = 1, \ldots, n \end{array} \qquad \text{(MOIP)}$$

Here C is a $Q \times n$ objective matrix, A is an $m \times n$ constraint matrix, and $x \in \mathbb{R}^n$. There is a considerable amount of literature on these problems. We refer to some surveys that exist but will not consider the literature in detail. In this respect, [43, 381, 429] provide surveys of techniques to find efficient solutions for (MOIP), [380] gives an overview of interactive methods for (MOIP), and [313] surveys (MOIP) with binary variables.

In general, combinatorial optimization problems can be considered as special cases of integer (in particular binary) programming. A MOCO problem is distinguished by a specific set of constraints, that provides a structure to the problem. We focussed on such problems and do not intend to review literature on general multiobjective binary or integer programming.

To conclude this section, let us mention one particular case, namely, when the set of feasible solutions is an explicitly given finite set, e.g. $X = A$. In this case, all problems discussed above are efficiently solvable. Algorithms can be found in [86, 87] and [218]. For this reason, these problems are mathematically not particularly interesting and we omit them from further discussion.

To summarize, (MOCO) is a discrete optimization problem, with n variables x_i, $i = 1, \ldots, n$, Q objectives z^j, $j = 1, \ldots, Q$ and a specific constraint structure defining the feasible set X.

3. Properties of MOCO Problems

In this section we discuss some of the properties of MOCO problems. It is in order to mention here that there is a considerable number of erroneous statements, even in papers published in international standard refereed journals. We will point out the most important of these throughout the chapter, in the appropriate places.

By its nature, multiobjective combinatorial optimization deals with discrete, non continuous problems, although the objectives are usually linear functions. An essential consequence of this fact when trying to determine the set of all efficient solutions (or nondominated vectors in objective space) is, that it is not sufficient to aggregate the objectives through weighted sums.

It is long known that for multiobjective linear programming problems

$$\min\{Cx : Ax = b, x \geq 0\}$$

the set of efficient solutions is exactly the set of solutions that can be obtained by solving LP's

$$\min\left\{\sum_{j=1,\ldots,Q} \lambda_j c^j x : Ax = b, x \geq 0\right\},$$

where $\sum_{j=1}^{Q} \lambda_j = 1, \lambda_j > 0, j = 1, \ldots, n,$ see e.g. [183]. But the discrete structure of the MOCO problem makes this result invalid. Thus there usually exist efficient solutions, which are not optimal for any weighted sum of the objectives. This is true even in cases where the constraint matrix is totally unimodular, contrary to a proposition in [216] (see [399] for an example). These solutions are called nonsupported efficient solutions *NE,* whereas the remaining are called supported efficient solutions, *SE.* In early papers referring to MOCO, *NE* was usually not considered. Most authors focussed on scalarizing the objectives by means of weighting factors λ_j.

Nevertheless, the set *NE* is important. With more than one sum objective there are many more nonsupported than supported efficient solutions, see e.g. [413] for numerical results. But empirical results show that this is not necessarily the case when at most one objective is of the sum type and the others are bottleneck objectives [259, 261, 262].

Moreover, the nonsupported solutions contribute essentially to the difficulty of MOCO problems. Below, we shall briefly discuss the concepts of computational complexity of (MOCO). For introductions to the theory of $\mathcal{NP}$-completeness and $\#\mathcal{P}$-completeness we refer to [128] and [407, 408, 409], respectively. These notions deal with the difficulty of finding a, respectively counting the number of solutions of a (MOCO).

In order to transfer the notions of $\mathcal{P}, \mathcal{NP}$ and $\#\mathcal{P}$ to MOCO we first introduce a decision problem related to (MOCO) in a straightforward manner:

Given constants $k_1, \ldots, k_Q \in \mathbb{Z}$, *does there exist a feasible solution* $S \in X$ *such that* $z^j(S) \leq k_j$, $j = 1, \ldots, Q$? D(MOCO)

The corresponding counting problem is:

How many feasible solutions $S \in X$ *do satisfy* $z^j(S) \leq k_j$, $j = 1, \ldots, Q$? #(MOCO)

It turns out that the respective versions of (MOCO) in the sense of finding or counting efficient solutions are in general $\mathcal{NP}$- and $\#\mathcal{P}$-complete, respectively. This is true even for problems which have efficient algorithms in the single objective case. We refer to [102, 347] and [89] for results in this respect. Therefore the development of heuristics with guaranteed worst case performance (bounded error) is interesting. However, not much is known in this regard: [89] gives some general results on approximating the efficient set by a single solution, [303] uses a Tchebycheff metric to measure the error, and [326, 327] consider the existence of such algorithms. Some specific results about flow problems, shortest path, knapsack problems, and the TSP are discussed in Section 6.

Another aspect related to the difficulty of MOCO is the number of efficient solutions. It turns out that it may be exponential in the problem size, thus prohibiting any efficient method to determine all efficient solutions. Such results are known for the spanning tree, matroid base, shortest path, assignment, and travelling salesperson problems (see [103, 149, 349] for details). Consequently such problems are called intractable. Even the size of the set SE may be exponential, see [324]. However, numerical results available on the knapsack problem [413] show the number of supported solutions grows linearly with the problem size, but the number of nonsupported solution grows following an exponential function. However, other investigations indicate that for problems with bottleneck objectives (at most one sum objective), the ratio between the size of SE and NE is independent of the problem size in the asymmetric TSP [261], and that the number of efficient solutions decreases with increasing number of constraints for multi-constrained knapsack problems [262].

As far as the other definitions of optimality in (MOCO) are concerned, we note that the max-ordering problem with sum objectives is $\mathcal{NP}$-hard in general (see [41]), but can be reduced to a single objective problem in the case of bottleneck objectives [86]. Bounds and heuristic methods for the former problem have been investigated in [307]. At least one solution of the max-ordering problem is always efficient, but possibly nonsupported. Similarly, a lexicographic max-ordering solution, although always efficient and optimal for the max-ordering problem may be nonsupported, [86].

For lexicographic optimization it is known that a lexicographically optimal solution is always efficient, and even a supported efficient solution, see [149]. Lexicographic optimization can also be viewed as a special case of algebraic optimization, see [428].

In view of the new trend to apply metaheuristics and local search in MOCO problems (see Section 4 below), it is interesting to consider the issue of neighbourhoods of feasible solutions, and their relations to efficient solutions. Using a neighbourhood corresponding to Simplex basis pivots for the shortest path problem and exchanges of one edge for the spanning tree problem it was shown in [92, 93] that the set of efficient solutions can be an unconnected subset of X with respect to the neighbourhood. So it is possible that local search methods (in principle) cannot find all efficient solutions.

4. Solution Methods for MOCO Problems

In the context of multiobjective programming (MOP), it is usual to distinguish the methods following the role of the decision maker in the resolution process. Information provided by the decision maker often concerns his preferences. In "a priori mode", all the preferences are known at the beginning of the decision making process. The techniques used seek for a solution on the basis of these parameters. The best example is given by goal-programming methods. In "a posteriori mode" the set of all efficient solutions is generated for the considered problem. At the end, this set is analyzed according to the decision maker's preferences. Many approximation (heuristic) methods are conceived following this resolution mode. In the "interactive mode", the preferences are introduced by the decision maker during the resolution process. The methods involve a series of computing steps alternated with dialogue steps and can be viewed as the interactive determination of a satisfying compromise for the decision maker. Thus they require a high participation level on the part of the decision maker. Practical problems are often resolved according to the interactive mode.

The appropriate resolution mode is chosen considering the situation of the decision process. The method involved in the process could be exact or approximation methods.

4.1. Exact Methods

Here we discuss some of the methods used to solve MOCO problems. Many of these essentially combine the multiple objectives into one single objective. The most popular, and the one used first, is weighted sum scalarization. The problem solved is

$$\min \left\{ \sum_{j=1}^{Q} \lambda_j z^j(x) : x \in X \right\}, \qquad (P_\lambda)$$

where $0 \leq \lambda_j \leq 1$ and $\sum_{j=1}^{Q} \lambda_j = 1$. Varying the weights, it is known that all supported efficient solutions can be found, using results from [183] and linear programming [132]. The advantage of the method (especially for problems where the single objective version is solvable in polynomial time) is that for each $\lambda \in \mathbb{R}^Q$ the problem (with sum objectives) is only as difficult as the single objective counterpart of (MOCO). Parametric programming can be used to solve the problem for all λ.

The approach has been applied to many MOCO problems: see [166, 420] for shortest path, [7, 77, 78, 184, 363] for the transportation problem, [69] for assignment, [202, 227, 248] for network flow, [149, 341, 342] for spanning tree, [81, 321] for knapsack and [245] for location problems. In many of these papers, the existence of nonsupported efficient solutions was either not known, or ignored. When a sum and a bottleneck objective are present, the minimization of the sum of the objectives has been discussed in [266] and [306] for general combinatorial optimization problems.

Besides weighted sum scalarization the most important method for multiobjective programming involves constraints on some objective values. The ε-constraint problem, described in detail in [35], is the following

$$\begin{aligned} &\min_{x \in X} z^k(x) \\ &\text{subject to } z^j(x) \leq \varepsilon_j \quad j \neq k. \end{aligned}$$

Its usefulness for MOCO problems depends on the objective function type. With more than two sum objectives, the ε-constraint problem is often $\mathcal{NP}$-hard, see e.g. [128, problem ND30] for constrained shortest path. If, however, at most one sum objective is present, the constraints can be applied to the bottleneck objectives, and simply imply the exclusion of feasible solutions containing elements $a \in A$ with $z^j(a) > \varepsilon_j$ for some $j \neq k$. Moreover, assuming integer weights, the values of ε_j can be restricted to all integers between $\min_{a \in A} w_j(a)$ and $\max_{a \in A} w_j(a)$. Applications of the method to such cases are described for assignment, spanning tree, and 1-tree problems in [259, 260, 262], for asymmetric TSP in [261], and for knapsack problems in [264]. Recently a modification of

the method has been used to solve large scale bicriteria set-partitioning problems (with sum objectives) arising in airline crew scheduling [95].

Another well known approach in multicriteria optimization is the compromise solution method [424], where one tries to minimize the distance to an ideal point z^I or to a utopian point $z^U = z^I - \epsilon e)$, where $e = (1, \ldots, 1) \in \mathbb{R}^Q$ is the vector of all ones, and $\epsilon > 0$. The ideal point is defined according to the individual minima of each objective

$$z_j^I := \min_{x \in X} z^j(x).$$

Usually, the Tchebycheff norm is used as distance measure:

$$\min \left\{ \max_{j=1}^{Q} \{\lambda_j | z^j(x) - z_j^I|\} : x \in X \right\}.$$

Unfortunately, when we consider sum objectives, this type of problem is usually $\mathcal{NP}$-complete, see e.g. [277] for references on the shortest path problem. This explains why it is rarely used, even though, theoretically the whole of the efficient set can be found, see e.g. [332]. Using another norm, e.g. an l_p norm, $p \notin \{1, \infty\}$ leads to nonlinear objectives, and we found only one reference [405] using l_p norms for MOCO. Note that for $p = 1$, the compromise solution method coincides with the weighted sums approach.

A special approach to multiobjective optimization is goal programming, see e.g. [179, 229] and Chapter 7 for details. Here, for each of the objectives a target value (goal) is specified by the decision maker. The overall aim is to minimize the deviation from the specified goals. This approach is very popular and although it is sometimes considered a different field from multiobjective optimization we list the references here.

One approach that is popular for bicriteria problems is the use of ranking methods. First, define

$$z_j^I = \min_{x \in X} \{z^j(x)\}, \quad j = 1, 2 \tag{8.3}$$

and then

$$z_j^N := \min_{x \in X} \left\{ z^j(x) : z^i(x) = z_i^I \right\}, \quad j = 1, 2; \ \ i \neq j. \tag{8.4}$$

The ideal point $z^I = (z_1^I, z_2^I)$ and Nadir point $z^N = (z_1^N, z_2^N)$ define lower and upper bounds on the objective values of efficient solutions. Then

starting from a solution with $z^1(x) = z_1^I$, and finding second best, third best, ..., K-best solutions with respect to the first objective until z_1^N is reached, the efficient set can be determined. The approach has been used for the shortest path problem [44, 255] and the transportation problem [77]. Note that computation of the Nadir point z^N in the bicriteria case essentially means the solution of two lexicographic optimization problems.

A generalization of this approach to more than three objectives is not possible without knowledge of the Nadir point, which is difficult to obtain when $Q > 2$, see [214]. Note that a generalization of (8.4) (stated without proof in [255]) does not necessarily provide an upper bound on objective values of efficient solutions. Not even considering lexicographic optimization with respect to all permutations of objectives is guaranteed to produce upper bounds on objective values of efficient solutions, see [97, 98] for a recent discussion.

Moreover, the ranking approach can be effectively used to solve max-ordering problems with any number of criteria. First a weighting vector is chosen, then K-best solutions x^K are created according to the combined objective $\sum \lambda_j z^j$. When for the first time

$$\min_{k=1,\dots,K-1} \max_{j=1,\dots,Q} z^j(x^k) \leq \sum_{j=1}^{Q} \lambda_j z^j(x^K)$$

an optimal solution is among $\{x^1, \dots, x^K\}$. We refer to [84, 146], and [149] for applications to the uniform matroid, network flow problem, and spanning tree problem, respectively and [96] for a general procedure.

Let us now look at methods adapted from single objective combinatorial optimization. Among the very well established procedures is dynamic programming [21]. The method applies to sequential decision problems, which admit a recursion formula such as

$$\min \left(g_N(x_n) + \sum_{k=0}^{N-1} g_k(x_k, u_k) \right),$$

where g is a cost function depending on the state variable x_k and control variable u_k at stage k. Theoretically, this recursion can easily be adapted to the multiobjective case. Therefore dynamic programming algorithms appear most often for problems, where they have been established for the single objective versions earlier. These are the shortest path problem

[34, 165, 166, 215, 311, 319, 330, 358, 384], the knapsack problem [33, 38, 81, 199, 200, 201], the TSP [112, 391] and the transportation problem [123].

An implicit enumeration algorithm, which is widely used to solve hard combinatorial optimization problems is branch and bound. Its philosophy is to partition the problem into mutually disjoint and jointly exhaustive subproblems. Bounds are computed for subproblems and the process continues until an optimal solution is found. Much to our surprise, we could only find a few papers applying branch and bound for MOCO – to the knapsack problem, [396, 400, 413], the max-ordering shortest path problem, [311], and the cutting stock problem [210], The adaptation of branch and bound poses one difficult problem. Since we deal with nondominated vectors, bounds play the role of ideal/Nadir points for subproblems. Thus they may be difficult to compute, or bad, i.e. not discarding enough feasible, nonefficient solutions. Research on replacing single points as bounds by bound sets to better reflect the multicriteria nature (efficient frontier) has recently been initiated in [90].

Many authors used available single objective methods for a particular problem and adapted them to the multiobjective case. The more natural such a generalization is, the bigger the number of papers pursuing such an approach. We note the following.

- Shortest Path: [157, 251] for label setting and [26, 50, 52, 272, 356, 392, 393, 411] for label correcting methods
- Spanning Tree: [49, 149] for adaptations of Prim's algorithm and [341, 347] for the greedy algorithm
- Assignment: [247, 396, 399] for the Hungarian method
- Network Flow: [88, 226, 227, 228] for the out-of-kilter algorithm and [31, 109, 305, 345] for the network simplex method
- TSP: [89] for Christofides' algorithm

Finally, we explain a general framework for the exact solution of the problem of determining the efficient set for bicriteria (MOCO), the two phases method. The name goes back to [396] and [399] and is telling: In the first phase *SE* is found using the scalarization technique, and solving single objective problems. The necessary weights are easy to compute using information generated in the process. The second phase consists

of finding the nonsupported efficient solutions by problem specific methods, using bounds, reduced costs, etc. In fact, most of the algorithms known to the authors (with exception of the shortest path problem) that are capable of determining the whole of E are some modification of the two phases method, e.g. [88, 228, 346] (Network Flow), [396, 413], (Knapsack), [399] (Assignment) and [310](Spanning Tree).

4.2. Approximation Methods

The last two decades have been highlighted by the development and the improvement of approximative resolution methods, usually called "heuristics and metaheuristics". In the context of combinatorial optimization, the term heuristic is used as a contrast to methods that guarantee to find a global optimum, such as the "Hungarian method" for solving the assignment problem, or Johnson's method for 2-machine sequencing, or implicit enumeration schemes such as branch and bound or dynamic programming.

A heuristic is defined by [316] as a technique which seeks good (i.e. near-optimal) solutions at a reasonable computational cost without being able to guarantee optimality (or feasibility), or even in many cases to state how close to optimality a particular feasible solution is. Often heuristics are problem-specific, so that a method which works for one problem cannot be used to solve a different one.

In contrast, metaheuristics are powerful techniques applicable generally to a large number of problems. A metaheuristic refers to an iterative master strategy that guides and modifies the operations of subordinate heuristics by combining intelligently different concepts for exploring and exploiting the search space [134, 289]. A metaheuristic may manipulate a complete (or incomplete) single solution or a collection of solutions at each iteration. The family of metaheuristics includes, but is not limited to, constraint logic programming, genetic algorithms, evolutionary methods, neural networks, simulated annealing, tabu search, non-monotonic search strategies, greedy randomized adaptive search, ant colony systems, variable neighbourhood search, scatter search, and their hybrids. A comprehensive list of 1380 references on the theory and application of metaheuristics is presented in [289]. The success of these methods is due to the capacity of such techniques "to solve in practice" some hard combinatorial problems. As in the single objective case, a reasonable alternative to exact methods for solving large-scale instances of MOCO problem is to derive an approximation method. Such methods yield a good tradeoff between the quality of an approximation of the efficient solutions set denoted by $\widehat{E}$, and the time and memory requirements.

The introduction of metaheuristic techniques for the solution of MOP problems has mushroomed over the last ten years. This activity has given birth to multiobjective metaheuristics (MOMH), aiming to approximate the (sub)set of Pareto-optimal solutions. Chronologically, the literature reports methods based on genetic algorithms (GA, Schaffer 1984), artificial neural networks (ANN, Malakooti 1990), simulated annealing (SA, Serafini 1992), and tabu search (TS, Gandibleux 1997).

Two characteristics are found in the first methods: They are inspired exclusively either by evolutionary algorithms, or by neighborhood search algorithms. Future researchers may see methods which are inspired simultaneously by the two schools. Also, the pioneer methods were a direct derivation of single objective optimization metaheuristics. They have small adaptations in order to integrate the concept of efficient solution to optimize multiple objectives.

Evolutionary Algorithms. Evolutionary algorithms (EA) make use of a population of solutions. By maintaining a population of solutions such a method can search for many efficient solutions in parallel via self adaptation and cooperation mechanisms. Self adaptation means that the individuals evolve independently while cooperation implies an exchange of information among the individuals. Here the whole population contributes to the evolution process toward the efficient set. The generation mechanism is parallel along the frontier and we talk about "global convergence-based methods". This characteristic makes population-based methods very attractive for solving multiobjective problems with the advantage of being independent of the problem.

The first to introduce a multiobjective metaheuristic was Schaffer [337, 338]. He developed Vector Evaluated Genetic Algorithm (VEGA), which was an extension of Grefenstelle's GENESIS program [140] to include multiple objective functions. The vector extension concerns only the selection procedure. The main idea of VEGA is to divide the population into equal sized subpopulations. Each subpopulation is entrusted with optimization of a single objective. The selection procedure is performed independently for each objective while evolution (crossover and mutation operators) are performed on the union of subpopulations. As VEGA selects individuals who excel in one dimension of performance without looking at the other dimensions the speciation problem can arise from this approach. It implies that individuals with balanced performance on all objectives will not survive under this selection mechanism. Although some serious drawbacks are known, VEGA has had a strong influence up to now, and was at the origin of the Multiobjective Evolutionary Algorithms (MOEA) wave. Most MOMH are based on MOEA.

A list maintained on the web [48] counts several hundred papers only for multiobjective genetic algorithms, see also Chapter 5.1.

For a long time, the problems investigated were often unconstrained bi-objective problems with continuous variables and non-linear functions. EA are appreciated by the communities of engineers. This can explain the large number of applications of MOEA (in mechanical design, electronics, etc.) for solving real world problems. Surprisingly, few MOEA have been applied to solve MOCO problems. One only finds [130, 131] (Transportation Problem), [426] (Spanning Tree Problem), [188] (Travelling Salesperson Problem) [2, 127] (Knapsack Problem), [432] (Multi-constraint Knapsack Problem), [237] (Set Covering Problem), [390] (Containership Loading Design) and [271, 291, 379] (Scheduling Problems).

Neighborhood Search Algorithms. In neighborhood search algorithms (NSA) the generation relies upon one individual, a current solution x_n, and its neighbours $\{x\} \subseteq \mathcal{N}(x_n)$. Basically, starting from an initial solution and a weight vector $\lambda \in \Lambda$, the procedure approximates a part of the nondominated frontier corresponding to the search direction λ. A local aggregation mechanism of the objectives, often based on a weighted sum, produces the effect to focus the search on a part of the nondominated frontier. It defines a sequential generation implying a local convergence, i.e. a convergence located in an area of the efficient frontier. The principle is repeated for several search directions to approximate completely the nondominated frontier. NSA present an aggressive convergence due to less dispersion of the search. However, they need more effort in diversification to cover the efficient frontier completely.

The comparison of $x \in \mathcal{N}(x_n)$ with x_n according to Q objectives $z^j(x)$, $j = 1, \ldots, Q$ raises three possible situations. If $\Delta z^j = z^j(x) - z^j(x_n)$ is the difference between solution x and x_n in the objective j:

$\forall j\ \Delta z^j \leq 0$: All the objectives are improved for solution x. x dominates the current solution x_n and is always accepted.

$\exists j, j'\ \Delta z^j < 0$ and $\Delta z^{j'} > 0$: An improvement and a deterioration occur simultaneously for different criteria. Both solutions x and x_n are potentially efficient.

$\forall j\ \Delta z^j \geq 0$: All objectives are deteriorated with at least one strict inequality. Solution x is dominated by x_n.

For the two last cases a scalarizing function $s(z(x), \lambda)$ is often used to project the multidimensional objective space into a monodimensional one using λ. Such a function allows to produce a "local agregation"

of the objectives in order to compute the "weighted distance" $\Delta s = s(z(x), \lambda) - s(z(x_n), \lambda)$ between $z(x)$ and $z(x_n)$.

The majority of the methods developed on the principle of NSA were applied to bi-objective problems with discrete variables, linear functions and linear constraints (combinatorial optimization).

Recent methods are more and more hybridized. For example, a population of solutions comes into NSA based methods [62, 153]. This coupling aims at breaking the independent character of each search process, inherent to the sequential generation principle, by exploiting information available in the population.

4.2.1 Multiobjective Evolutionary Algorithms (MOEA). Since VEGA, significant progress concerns corrections of shortcomings observed in previous algorithms and propositions of new algorithmic primitives to generate a better approximation of E (like the use of non-domination ranking suggested by Goldberg [136]). Among the elements playing a significant role in a MOEA one finds the elite solutions. Zitzler underlines that recent studies suggest the use of elitism to improve MOEAs [223]. The results of numerical experiments show that the use of elite solutions must come with a strong rate of mutation in order to prevent a too fast specialization of the population. The contribution of elite solutions in the generation of the efficient frontier in the case of MOCO problems has been investigated by Gandibleux et al. [127, 271]. For the knapsack problem the use of greedy solutions or efficient supported solutions in the population has clearly shown a much better aptitude of the algorithm to generate the efficient solutions.

In the continuation of VEGA a lot of methods have contributed to the methodologic development of MOEAs. The most outstanding among them are briefly mentioned. Readers can find a complete survey on these methods in [46, 47, 114, 194].

[113]: **Multiple Objective Genetic Algorithm** (MOGA93) by Fonseca and Fleming, 1993. MOGA93 uses a ranking procedure where the rank of an individual is equal to the number of solutions which dominate this individual.

[361]: **Nondominated Sorting Genetic Algorithm** (NSGA) by Srinivas and Deb, 1994. NSGA implements Goldberg's ranking idea where the rank of an individual is equal to its domination layer computed by ranking the population on the basis of domination.

[172]: **Niched Pareto Genetic Algorithm** (NPGA) by Horn, Nafpliotis and Goldberg, 1994. NPGA combines the Pareto dominance princi-

ple and a Pareto tournament selection where two competing individuals and a set of individuals are compared to determine the winner of the tournament.

[275]: **Multiple Objective Genetic Algorithm** (MOGA95) by Murata and Ishibuchi, 1995. MOGA95 is not based on the Pareto ranking principle but on a weighted sum of objective functions to combine them into a scalar fitness function using weight values generated randomly in each iteration.

[431]: **Strength Pareto Evolutionary Algorithm** (SPEA) by Zitzler and Thiele, 1998. SPEA takes the best features of previous MOEAs and includes them in a single algorithm. Using the multiobjective multiconstraint knapsack problem as benchmark, SPEA shows advantages over the other MOEA under consideration in convergence to the efficient frontier [432].

[205]: **Pareto Archived Evolution Strategy** (PAES) by Knowles and Corne, 1999. PAES is an evolution strategy employing local search for the generation of new candidate solutions using a reference archive to compute the solution quality.

Other Methods Related to Evolutionary Algorithms: [20, 47, 48, 152, 151, 190].

4.2.2 Simulated Annealing. Serafini [348] was the first to use simulated annealing as a technique for multiobjective optimization problems. All multiobjective simulated annealing-based methods since then are still closely related to the original single objective method. They extend the single objective algorithm to cope with the notion of efficiency [396], The most recent methods include dynamic diversification mechanisms exploiting the set of potential efficient solutions to drive the approximation process [62, 295].

The methods presented subsequently are differentiated primarily by four points: (1) the rule for acceptance of a new solution with some probability depending on the temperature; (2) the scheme of decreasing the temperature; (3) the mechanism which guides the browsing of the efficient frontier; (4) the use of information drawn from a population of individuals. Often the authors tested various forms and definitions of acceptance rules. A lot of them have been suggested and discussed in [348].

[396]: **Multi-Objective Simulated Annealing** (MOSA) by Ulungu, 1993. The method uses a predefined set of weights. An independent SA process is then executed for each weight value. Each process generates a

set of potential solutions, which are then merged and filtered to provide the final approximation. Other references: [100, 241, 382, 394, 401, 402, 403].

[62]: **Pareto Simulated Annealing** (PSA) by Czyzak and Jaszkiewicz, 1996. PSA introduces the use of a sample of solutions which are simultaneously optimized toward the efficient frontier while they are dispersed over the whole frontier. Other references: [63, 64, 161, 159, 160, 161, 162, 163, 187, 191, 193].

[105, 295]: **The revised Engrand method.** This method is not based on a principle of search directions, and it does not need an aggregation mechanism for the objectives. Each objective is considered separately. Advanced strategies using the population of potential efficient solutions drive the approximation mechanism ensuring the detection of the whole efficient frontier. Other references: [106, 373, 374].

Other methods related to Simulated Annealing. The trip planning problem [135], interactive method for 0-1 multiobjective problems [5], bicriteria scheduling problems on a single machine [209].

4.2.3 Tabu Search. Metaheuristics dealing with multiple objective optimization problems are sometimes presented wrongly as MOMH. This occurs when multiobjective problems are solved with a single objective strategy, looking for a unique compromise solution. In this case the original multiobjective problem is transformed or managed as the optimization of one or several single objective problems. It is the case for the first papers describing the use of TS as technique for solving MOP [289]. In [66] a family of (P_λ) problems are solved to generate a subset of $\widehat{E}$. In [168] the method consists in solving a sequence of single objective problems considering in turn each objective z^j associated with a penalty term. Methods able to generated an approximation of the efficient solutions have been introduced later.

[126]: **Multiobjective Tabu Search** (MOTS) by Gandibleux et al., 1997. Its principle is based on a scalarizing function driven by a tabu search mechanism to browse, in an equilibrium way, the nondominated frontier. Intensification, diversification and tabu daemon are designed for the multiobjective case. Two tabu memories are used, one on the decision space, the second on the objective space. The former is an attribute-based tabu list preventing return to already visited solutions. The latter is connected with the objectives and based on an improvement measure of each objective. It is used for updating of weights using the pseudo criterion concept to diversify the search in the objective space. Other references: [124, 125].

[369]: **Interactive procedure using Tabu Search** by Sun, 1997. It is an interactive procedure for general multiple objective combinatorial optimization problems. The procedure works in a way similar to that of the Combined Tchebycheff/Aspiration Criterion Vector Method [365]. Tabu search is used to solve subproblems in order to find approximately efficient solutions. Other references: [3].

[153]: **Tabu Search approach inspired by PSA method** (MOTS*) by Hansen, 1998. A set of "generation solutions", each with its own tabu list is considered. These solutions are dispersed along the objective space in order to allow a search in different areas of the nondominated frontier. Weights are defined for each solution to force the search into a direction of the nondominated frontier and away from other current solutions that are efficient with respect to it. Diversification is ensured by the set of generation solutions and a drift criterion. Other references: [154, 410].

[18]: **TS algorithm for finding Pareto optimal solutions** by Baykasoglu et al., 1999. A candidate list is introduced as an opportunity to diversify the search. The components of the method are designed to handle any type of variables (integer, zero-one, continuous and mixed). Other references: [16, 17].

Other methods related to Tabu Search. A hybrid resolution process based on TS and GA [2], a hybrid and interactive resolution process based on SA and TS [5], the trip planning problem [135], scheduling problems [240].

4.2.4 Other Approaches and New Developments. At the beginning of the Nineties, the first works using Artificial Neural Networks (ANN) to solve MOP were published. However, the ANN approach remains marginal [244, 370, 371]. The first links of ant colony systems to MOMH are recent, meriting thus their presence in this section [182, 138, 351]. [207] presents a dedicated heuristic and [377] a stochastic search method. We mention also a paper concerning a comparison of neighbourhood search techniques for MOP [249].

Several aspects concerning MOMH and MOCO were discussed during recent international conferences. One finds new adaptations of metaheuristic such as GRASP, scatter search, ant systems, etc. Also, MOCO problems rarely tackled hitherto are now studied. One can find timetabling problems, space allocation problems, multi-period distribution management problems, vehicle routing problems, etc. Finally, aspects related to the computer are them also examined. Efficient data structures, such as the quad-tree have proven their efficiency to manage non-

dominated criterion vectors [142, 372]. Reusable software as object-oriented frameworks for multiobjective local search is in development. These new trends promise a lot of forthcoming papers.

5. Classification of the Literature

In this section, we describe the classification scheme we used below to annotate the references. We classify a paper according to four categories, namely combinatorial structure, objective function type, problem type, and method applied. The first three pertain to the description of the problem discussed in a given paper.

As indicated in Section 2, to classify a certain paper, we first have to identify the problem discussed. This consists of the combinatorial structure (i.e. shortest path, knapsack, etc.), the type and number of objectives (i.e. sum, bottleneck, or occasionally something else), and the type of problem (e.g. finding the efficient set, max-ordering, lexicographic).

In addition to the identification of the problem, we give the methodology used in the paper. We can distinguish between exact and approximation (or heuristic) methods, where exact means that the optimal solutions mentioned in the problem description are found, whereas approximation means that only some solutions representing this set, not necessarily optimal, are found.

So, we introduce a classification using positions

$$Pos1/Pos2/Pos3/Pos4:.$$

Below, we provide tables where the different entries for each position are listed.

Entries for $Pos2$ do not need a table, they simply define the number and type of objective functions considered. We could restrict ourselves to the sum and bottleneck objectives, with occasional exceptions explained where appropriate. Most of the papers that deal with other types of objectives are listed separately, because almost each of them would have required its own entry here. Note that Q stands for an arbitrary number of objectives.

We remark that sometimes two entries appear in one position. This means that one paper falls under two categories or that the approach applied in the paper is a combination of two methods. It may also happen that a single paper appears under several classifications if more than one problem was considered, or several methods proposed.

Table 8.1. Entries for $Pos1$: Combinatorial structure.

Entry	Explanation
P	Shortest Path Problem
AP	Assignment Problem
TP/TS	Transportation resp. Transshipment Problem
F	Network Flow Problem
ST	Spanning Tree Problem
MB/MI	Matroid Base resp. Matroid Intersection Problem
TSP	Travelling Salesperson Problem
KP	Knapsack Problem
SCH	Scheduling Problem
DL/NL	Discrete resp. Network Location Problem
SCP/SPA	Set Covering resp. Set Partitioning Problem
U	Unconstrained Problem
QAP	Quadratic Assignment Problem
VRP	Vehicle Routing Problem
FLP	Facility Layout Problem

Table 8.2. Entries for $Pos3$: Type of problem.

Entry	Explanation
E	Finding the efficient set
e	Finding a subset of the efficient set
SE, NE	Finding (non)supported efficient solutions
se, ne	Finding a subset of the (non)supported efficient solutions
$\hat{\bullet}$	Finding an approximation of $\bullet$
lex	Solving the lexicographic problem (preemptive priorities)
MO	Max-ordering problem
lexMO	Solving the lexicographic max-ordering problem
U	Optimizing a utility function
C/S	Finding a compromise respectively satisfying solution

6. Annotation of the Literature Problem by Problem

In this section we will give an annotated overview over the literature. We found it most convenient to organize the section according to the combinatorial structure of MOCO problems. Thus, we introduce eleven subsections, dealing with the most important combinatorial problems, in

Table 8.3. Entries for *Pos*4: Solution method applied.

Entry	Explanation
SP	Exact algorithm specifically designed for the problem
LS/LC	Label setting resp. label correcting method
DP	Algorithm based on dynamic programming
BB	Algorithm based on branch and bound
εC	Algorithm based on ε-constraint method
IA	Interactive method
H	Heuristic specifically designed for the problem
SA	Simulated annealing algorithm
TS	Tabu search algorithm
GA	Genetic or evolutionary algorithm
GP	Goal programming
2P	Two phases method
A	Approximation algorithm with worst case performance bound
LP	Method based on linear programming
DEA	Method using data envelopment analysis
Fuzzy	Method using fuzzy programming

terms of the number of papers available. In a last subsection we briefly mention other MOCO problems that have appeared in papers, but to a definitely smaller extent.

As an exception to this order, we briefly mention PhD theses in the subject, since they are also witness of the growing research efforts in the field. An increasing number of dissertations have been written on MOCO in recent years. Those that we found were not all dedicated to MOCO specifically, but use some MOCO problems in another context: [52] deals with the multiobjective shortest path problem for routing of hazardous material, [238] contains information about bicriteria spanning trees, [45, 204, 274, 430] are about evolutionary techniques in multiobjective optimization, and [86] presents some general results for certain general MOCO problems. Among those which are specifically dedicated to MOCO problems we mention [108] and [226] on the flow problem, [171] and [385] in scheduling. [153, 191] explores the use of metaheuristics for MOCO, and [396] introduces the two-phases method and develops it for the assignment and knapsack problem. [135] introduces the "Trip Planning Problem", as a preferences-based multiobjective travelling salesman problem with activity and lodging selection. [100] uses the MOSA method to solve a multiple objective Vehicle Routing Problem

with time windows. Finally fast approximation algorithms for MOCO problems are discussed in [326].

6.1. Shortest Path Problems

The multiobjective shortest path problem consists in finding in a network with vector weights on the edges "optimal" paths. The papers we found usually consider the problem with specified starting and ending node, or from a given starting node to all other nodes. The shortest path problem belongs to the most widely studied MOCO problems. There exists a survey on the topic [397], a bibliography on the Internet containing an abstract collection [254], and a classification of algorithms [355]. Our list contains all papers mentioned there, too.

Most problems in this category are $\mathcal{NP}$-complete: See [347] for the efficient paths problem with two sum objectives, [157] for intractability of the same problem. In [157] ten bicriteria shortest path problems are introduced and analyzed. In [92] an example shows that a result from [251] about the connectedness of efficient solutions is wrong, $\mathcal{NP}$-completeness of the max-ordering problem is mentioned in [277]. However, the multicriteria shortest path problem is an exceptional kind of problem, because a fully polynomial time approximation scheme is known, as presented in [415].

A variety of algorithms based on dynamic programming (e.g. [166, 215, 358]), label setting [157, 251] and label correcting methods (e.g. [26, 272, 356]) are available, with computational experiments [26, 174, 356] comparing different methods. In the biobjective case an algorithm based on ranking paths has also been proposed, [44, 255]. The general idea is also applicable to other MOCO problems with two objectives, as explained in Section 4.

Besides, several papers present formulations of specific problems in terms of multicriteria shortest paths, or consider other variations of the classical problem.

- P/2-$\sum$/E/LC: [26, 356, 392]
- P/2-$\sum$/E/LS: [157]
- P/2-$\sum$/E/2P,LC: [272]
- P/2-$\sum$/E/SP: [44, 174]
- P/2-$\sum$/E/DP: [65, 166]
- P/2-$\sum$/$\widehat{E}$/A: [157]
- P/1-$\sum$ 1-max/E/SP: [157, 252, 296]

- P/2-$\sum$/C/IA: [61, 110]
- P/2-$\sum$/U/SP: [166]
- P/2-$\sum$/U/IA: [278]
- P/2-$\sum$/ne/IA: [51]
- P/3-$\sum$/E/LC: [122]
- P/3-$\sum$/C/IA: [122]
- P/Q-$\sum$/SE/SP: [166, 420]
- P/Q-$\sum$/E/LS: [251]
- P/Q-$\sum$/E/LC: [50, 52, 393, 411]
- P/Q-$\sum$/E/DP: [165, 215, 312, 330, 358, 384]
- P/Q-$\sum$/E/SP: [11, 367]
- P/Q-$\sum$/$\hat{E}$/A: [415]
- P/Q-$\sum$/C/IA: [167]
- P/Q-$\sum$/C/SP: [418]
- P/Q-$\sum$/U/DP: [34]
- P/Q-$\sum$/U/SP: [10, 268]
- P/Q-$\sum$/MO/DP,BB: [311]
- P/Q-$\sum$/MO/LC: [277]
- P/Q-$\sum$/MO/SP: [290]
- P/Q-$\sum$/$\widehat{MO}$/A: [406, 415]
- P/Q-ND[1]/C/DEA: [70]
- P/Q-$\sum^2$/E/SP: [267]
- Other particular multiobjective path problems: [56, 57, 58, 59, 60, 75, 76, 143, 331, 422]
- Problems formulated as multiobjective shortest path problems, or where these appear as subproblems: [4, 250, 253, 256, 357]

[1] Any nondominated objective vectors.
[2] Weights are time-dependent nonnegative functions

6.2. The Assignment Problem

The multiobjective assignment problem is the following

$$\begin{aligned} &\text{“min”} \quad Cx \\ \text{subject to} \quad &\sum_{j=1}^{n} x_{ij} = 1 \quad i = 1, \ldots, n \\ &\sum_{i=1}^{n} x_{ij} = 1 \quad j = 1, \ldots, n \\ &x_{ij} \in \{0,1\} \end{aligned} \qquad \text{(MOAP).}$$

Total unimodularity of the constraint matrix guarantees that an optimal integer solution is found by linear programming methods, when only a single objective is considered. With the Hungarian method (see e.g. [279]), a very efficient algorithm is available.

The (MOAP) literature is again focussed on the determination of (supported) efficient solutions. In fact, (MOAP) belongs to the first MOCO problems studied. However, the first papers only deal with *SE*, using convex combinations of objectives [69], or goal programming [36]. However, nonsupported efficient solutions exist [399], and the problem is $\mathbb{NP}$-complete [347] and $\#\mathbb{P}$-complete [283] and an exponential number of efficient solutions may exist.

Exact algorithms to determine the whole set *E* [247, 399] have been developed. They make use of single objective methods and duality properties of the assignment problem. Recently we can also observe the application of metaheuristic techniques for the problem [394]. Quite a few papers deal with a special version of the problem: [19, 36, 419]. Other papers deal with variations of the problem or applications. These cannot really be classified according to the problem and methodology applied or discussed in detail. We list them separately.

- AP/2-$\sum$/SE/SP: [69]
- AP/2-$\sum$/E/2P,SP: [247, 396, 399]
- AP/2-$\sum$/$\widehat{E}$/SA: [394]
- AP/2-$\sum$/lex/SP: [302]
- AP/2-$\sum$/C/IA: [300]
- AP/4-$\sum$/SE/SP: [265]
- AP/Q-$\sum$/E/SP: [343]

- AP/Q-$\sum$/$\widehat{E}$/SA: [382]
- AP/Q-$\sum$/S/GP: [36, 362]
- AP/Q-$\sum$/C/IA: [133]
- AP/1-$\sum$ 1-max, 2-max/E, SE/εC: [259, 260, 263]
- AP/1-$\sum$ 2-max, 3-max/E, SE/εC: [262, 263]
- Papers related to assignment models: [9, 13, 14, 19, 181, 225, 230, 235, 257, 258, 276, 298, 299, 419, 425]

6.3. Transportation and Transshipment Problems

Both are generalizations of the assignment problem, where the right hand side of the constraint may take positive integer values, and the variables any nonnegative integer. The transshipment problem has transshipment nodes in addition to demand and supply nodes. The transportation problem is given below.

$$\begin{aligned} &\text{“min”} \quad Cx \\ \text{subject to} \quad &\sum_{j=1}^{n} x_{ij} = a_i \quad i = 1, \ldots, m \\ &\sum_{i=1}^{m} x_{ij} = b_j \quad j = 1, \ldots, n \\ &x_{ij} \geq 0, \quad \text{integer.} \end{aligned} \tag{MOTP}$$

The transshipment problem has transshipment nodes in addition to supply and demand nodes. Again, in the single objective case total unimodularity and integer right hand sides imply that an optimal solution of the linear relaxation is also an optimal solution of the problem itself. Making use of this fact, most of the papers use a scalarization by means of weighted sums or goal programming approaches.

- TP/2-$\sum$/SE/LP: [7, 364]
- TP/1-$\sum$ 1-max/SE/LP: [7, 82, 309, 363]
- TP/1-$\sum$ 1-max/S/GP: [233]
- TP/Q-$\sum$/se, S/IA: [42, 319]
- TP/Q-$\sum$/SE/LP: [77, 78, 184, 352]

- TP/Q-$\sum$/SE/DP: [123]
- TP/Q-$\sum$/S/SP: [67]
- TP/Q-$\sum$/$\widehat{E}$/GA: [130, 131]
- TS/Q-$\sum$/S/GP: [219, 220, 270, 368]
- TP/Q-$\sum$/C/SP: [236]
- TP/Q-$\sum$/C/Fuzzy: [101]
- Other related problems and applications: [8, 203, 206, 224, 234, 287, 301, 314, 376, 378, 395]

6.4. Network Flow Problems

The network flow problem is a problem that actually is on the borderline between combinatorial and linear optimization. Its formulation is

$$\begin{array}{lll} \text{“min”} & Cx & \\ \text{subject to} & Ax & = \quad 0 \\ & l \leq x \leq u, & \end{array} \qquad \text{(MOFP)}$$

where A is the node-arc incidence matrix of a network. It is well known that with a single objective there always exist integer optimal solutions of the LP, due to the unimodularity of A, which is the reason for considering it a combinatorial problem.

In the multiobjective case we have to distinguish between the linear and the integer case. In the linear case, we know that $SE = E$. We deal with the papers in their relevance for the integer case. [324] demonstrated that an exponential number (in the number of node of the network) of extreme points among SE may occur. Most of the algorithms in the literature generalize methods for the single objective flow problem, e.g. the out-of-kilter method [227, 248] or elements from network simplex [31, 305, 345]. The algorithms for MO and lexMO problems [88, 146] are based on ranking approaches. For linear bicriteria network flow problems algorithms approximating the efficient set to any given precision ϵ are presented in [27, 121, 325] and generalized to bicriteria quadratic network flow problems in [423].

- F/2-$\sum$/SE/SP: [202, 227, 248, 305]
- F/2-$\sum$/SE/LP: [345]

- F/2-$\sum/\widehat{SE}$/A: [27, 121, 325, 328]
- F/2(3)-$\sum/E$/SP: [109, 173, 226, 228, 281, 282, 346]
- F/Q-$\sum/SE$/SP: [202]
- F/Q-$\sum/E$/SP: [88]
- F/Q-$\sum$/lex/SP: [30, 31]
- F/Q-$\sum$/MO/SP: [146]
- F/Q-$\sum$/lexMO/SP: [88]
- F/Q-$\sum$/C/IA: [108, 111]
- Other network flow problems: [8, 203, 253, 280, 315, 423]

6.5. The Spanning Tree Problem

The spanning tree problem is to find among all spanning trees of a given graph one that is "minimal" with respect to the edge weights. This problem appears in network design. It is known that the problem to find efficient solutions is $\mathcal{NP}$-complete [32] and intractable [149]. $\mathcal{NP}$-completeness also holds for the max-ordering problem [149]. The complexity status of a variety of multiobjective spanning tree problems, involving other than the typical sum and bottleneck objectives is studied in [32, 72, 73]. The algorithms that have been proposed to find efficient trees range from minimizing weighted sums [308, 341, 342] over generalizations of Prim's [49] and Kruskal's [342] method to approximation [149] and genetic algorithms [426], A counterexample to a sufficient condition for a spanning tree to be efficient [49] has been given in [149]. As far as local search methods are concerned, it is important to note that, defining trees to be adjacent if they have $n-2$ edges in common, can imply that the set of efficient spanning trees is not connected [92].

- ST/2-$\sum/SE$/SP: [149]
- ST/1-$\sum$ 1-max/SE/SP: [308]
- ST/2-$\sum/E$/2P,SP: [310]
- ST/2-$\sum/\widehat{E}$/H: [6, 149]
- ST/Q-$\sum/SE$/SP: [341, 342]
- ST/Q-$\sum/E$/SP: [49]

- ST/Q-$\sum$/$\widehat{E}$/GA: [426]
- ST/Q-$\sum$/MO/SP: [149]
- ST/1-$\sum$ 1-max, 2-max/E, SE/εC: [259, 260, 263]
- ST/1-$\sum$ 2-max, 3-max/E, SE/εC: [262,263]
- Other spanning tree problems with different objectives: [72, 73, 177, 178, 344]

6.6. Matroids and Matroid Intersections

The matroid base problem is a generalization of the spanning tree problem. With a single objective it can be solved by the greedy algorithm. A generalization of this result for finding efficient bases is given in [347]: For each efficient basis B, there exists a topological sorting of the elements (e.g. edges of a graph), such that the greedy algorithm finds B. A topological sorting is a total or linear order that respects the partial order given by the vector weights. The problem is $\mathcal{NP}$-complete, as was shown e.g. in [84, 347]. A matroid intersection problem is to find a set of minimal weight which is independent with respect to two matroids.

Few papers deal with these problems in the multiobjective case. We identified the following, mostly presenting exact algorithms, theoretical properties [137, 414], and complexity issues [84, 347].

- MB/2-$\sum$/SE, E/SP: [84, 347]
- MI/Q-$\sum$,1-max 1-$\sum$/Lex/SP: [427]
- MB/Q-$\sum$/MO/SP: [84, 137]
- MB/Q-$\sum$/$\widehat{\text{MO}}$/H: [414]

6.7. The Travelling Salesperson Problem

In combinatorial optimization, the TSP is widely studied. To find a shortest tour among n cities is $\mathcal{NP}$-complete even with one objective, for both the sum and bottleneck case. Moreover, the number of efficient solutions is expected to be exponential, see [103]. For approximation results, we refer to [89], where limits on the possibility of approximating efficient solution by one heuristic solution are derived and generalizations of the tree and Christofides heuristic are analyzed.

These might be reasons why investigation of the multiobjective version is not so common, and why research concentrates on exact algorithms

based on dynamic programming as well as heuristics. Some papers discuss special versions or generalizations of the TSP, such as various formulations of vehicle routing problems.

- TSP/1-$\sum$ 1-Π^3/E/DP: [112]
- TSP/1-$\sum$ 1-max/$\widehat{SE}$/H [353]
- TSP/1-$\sum$ 1-max/E, SE/εC/ [261]
- TSP/2,3-$\sum$/$\widehat{E}$/GA: [188]
- TSP/3-$\sum$/E/SP: [24]
- TSP/Q-$\sum$/E/DP: [391]
- TSP/Q-$\sum$/$\widehat{E}$/A: [89]
- TSP/Q-$\sum$/$\widehat{E}$/TS: [155]
- TSP/Q-$\sum$/$\widehat{MO}$/H: [141]
- TSP/3-$\sum$, 4-$\sum$/$\widehat{E}$/SA: [192]
- Other versions of the problem, e.g. vehicle routing (VRP): [56, 135, 170, 196, 197, 292, 293, 294, 375]

6.8. Knapsack Problems

The knapsack problem is one of the fundamental *NP*-complete combinatorial optimization problems. Its multiobjective formulation is

$$\begin{aligned} &\text{“min”} \quad Cx \\ \text{subject to} \quad &\sum_{i=1}^{n} a_i x_i \le b \\ &x_i \in \{0,1\}, \end{aligned} \tag{MOKP}$$

where all parameters are assumed to be positive integers. All papers that we found deal with the problem to identify or approximate *SE* or *E*. Finding *E* or *SE* are obviously *NP*-complete, too. Thus it is not surprising that the algorithms proposed are either based on implicit enumeration methods such as dynamic programming [81, 199, 200, 201], branch and bound [396, 400] or apply heuristic procedures, especially

[3] Π denotes an objective defined by the products of weights.

metaheuristics to approximate E [125, 153, 328, 329]. Recently a polynomial time approximation scheme was developed in [107]. Some papers also deal with an extension to time-dependent knapsack problems [200, 201]. An interactive decision support system for the capital budgeting problem is proposed in [383]. Metaheuristics have been used to solve multi-constraint knapsack problems [189, 432].

- KP/1-$\sum$ 1-max/SE/BB: [264]
- KP/2-$\sum$/SE/SP: [321]
- KP/2-$\sum$/SE/DP: [81]
- KP/2-$\sum$/E/DP: [33]
- KP/2-$\sum$/$\widehat{SE}$/H: [321]
- KP/2-$\sum$/E/2P,BB: [396, 400, 413]
- KP/2-$\sum$/$\widehat{E}$/TS: [125]
- KP/2-$\sum$/$\widehat{E}$/H: [329]
- KP/2-$\sum$/$\widehat{E}$/H: [328]
- KP/2-$\sum$/$\widehat{E}$/GA+TS: [2]
- KP/2-$\sum$/$\widehat{e}$/SA+TS: [5]
- KP/2-$\sum$/$\widehat{E}$/GA: [127]
- KP/Q-$\sum$/E/DP: [199, 200, 201]
- KP/Q-$\sum$/$\widehat{E}$/A: [107]
- KP/Q-$\sum$/$\widehat{E}$/TS: [153, 154]
- KP/Q-$\sum$/$\widehat{E}$/SA: [64, 382, 401, 402]
- KP/Q-$\sum$/SE/IA: [80]
- KP/Q-$\sum$/U/DP: [38]
- KP/Q-$\sum$/S/GP: [22, 71, 150, 195]
- KP/Q-$\sum$/$\widehat{E}$/IA+GA: [291]

6.9. Multiobjective Scheduling Problems

The scheduling problems constitute a particular category. Although these problems can often be formulated using 0-1 variables, they have generally no particular structure. Moreover, they have a usual classification defined according the shop organization which they refer to (single machine, parallel machines, flow shop, job shop, open shop, etc.). Also, the usual objective functions in scheduling have a specific sense (the makespan, the total flow time, the tardiness, etc.).

For example we look at [212]. Let us consider n jobs to be processed on a single machine at time zero. Let p_i and d_i denote the processing time and the due date of job i respectively. Let

$$\begin{aligned} J_i &: \text{job } i,\ i = 1, \ldots, n \\ C_i(\sigma) &: \text{completion time of job } i \text{ in schedule } \sigma \\ F(\sigma) &: \text{total flowtime of jobs in schedule } \sigma \\ T_{max}(\sigma) &: \text{maximum tardiness of schedule } \sigma \\ \Omega &: \text{set of all possible sequences.} \end{aligned}$$

Then the objective is to find a schedule σ^* such that

$$f(F(\sigma^*), T_{max}(\sigma^*)) = \min_{\sigma \in \Omega} f(F(\sigma), T_{max}(\sigma)),$$

where

$$F(\sigma) = \sum_{i=1}^{n} C_i(\sigma), \quad T_{max}(\sigma) = \max_i \{\max(C_i(\sigma) - d_i, 0)\}$$

and f is any arbitrary nondecreasing function of $F(\sigma)$ and $T_{max}(\sigma)$.

This problem is denoted by $1/d_i/f(\sum C_i, T_{max})$. A sequence σ is efficient with respect to total flowtime and maximum tardiness if there does not exist a sequence σ' with $F(\sigma') \leq F(\sigma)$ and $T_{max}(\sigma') \leq T_{max}(\sigma)$ with at least one of the above holding as a strict inequality.

We observe a constant interest on multiobjective scheduling problems during the last years, because the consideration of more than one objective is more in line with the real context of such practical problems. In a recent survey [388] and Chapter 7.3 more than one hundred papers are classified according to the usual notation introduced by Graham and extended by T'Kindt and Billaut to the multiobjective case. Also, the approximate resolution algorithms for scheduling problems and related problems (like [161, 209, 271, 379, 410]) often are inspired by multiobjective metaheuristic methods developed for MOCO problems. For these

reasons, we mention actual developments for this category of problems but for more details about multiobjective scheduling problems we refer to Chapter 7.3 and [37, 171, 385, 388].

- Single machine problems:
  - [12, 207, 208, 212, 213, 417], (SCH/2/$\hat{E}$/SA);
  - [209, 271, 328] (SCH/2/$\hat{E}$/GA);
  - [379] (SCH/Q/$\hat{E}$/GA);
- Multiple machine problems: [25, 175, 269, 336, 386, 389];
- Surveys: [387, 388];
- PhD theses: [171, 385];
- Papers on other scheduling or production management problems: cell formation problem (SCH/Q/CS/TS) [168] and resource constrained project scheduling (SCH/Q/$\hat{E}$/SA,TS) [410].

6.10. Location Problems

Location planning is an active area of research. The objective in a location problem is to find one (or more) locations, such that some objective, usually related to the distance to a set of existing facilities is minimized or maximized. These objectives usually are the weighted sum or maximum of individual distances. Moreover, location problems can be divided into three categories, namely planar, network and discrete problems. In planar location, the feasible set is (a subset of) the Euclidean plane. Network location problems deal with a network of nodes and arcs, new facilities can be built either on the nodes only, or also on arcs. Finally, for discrete location problems a set of potential sites is specified. Problems of the latter category are usually formulated as mixed integer programs. From the point of few of MOCO, we will consider only network and discrete location problems. For details about planar problems and single objective location problems, we refer to the specialized literature, e.g. [222, 221] for surveys. We refer also to two reviews on the topic in MOCO context, [55] and [317]. Most of the applications use a goal programming approach.

- NL/1-$\sum$, 1-max/E/SP: [412]
- NL/2-$\sum$/E/SP: [357]
- NL/Q-$\sum$/lex,E/SP: [147]

- NL/Q-$\sum$/E, SE/SP,IA: [322]
- NL/Q-$\sum$/E/SP: [158]
- NL/Q-$\sum$,Q-max/E/SP: [148]
- NL/Q-$\sum$/MO,lexMO/SP: [94]
- DL/Q-$\sum$, Q-max/E/SP: [286]
- DL/Q-$\sum$/E/DP: [304]
- DL/Q-$\sum$/SE/SP: [245]
- DL/Q-$\sum$/lexMO/SP: [285]
- DL/Q-$\sum$/U,S/IA,GP: [246]
- DL/Q-$\sum$/S/GP: [15]
- Warehouse location: [99, 139, 232]
- Others and applications: [169, 176, 211, 231, 288, 318, 323, 339, 340, 359]

6.11. Set Covering and Partitioning Problems

The set covering problem is concerned with selecting subsets of a set that cover all elements of the set at minimal cost. The variables are binary variables for each subset that may be chosen. Thus, the IP formulation of this *NP*-complete problem is given as follows:

$$\begin{aligned} &\text{“ min ”} \quad Cx \\ \text{subject to} \quad &\sum_{i=1}^{n} a_{ji} x_i \geq 1 \quad j = 1, \ldots, m \\ &x_i \in \{0, 1\}, \end{aligned} \tag{MOSCP}$$

where $a_{ji} = 1$ if element j is contained in subset i, and all coefficients of C are assumed positive.

The SCP has applications in the location of emergency facilities. Suppose there are m sites of potential emergency and n potential locations for emergency facilities, incurring cost c_i to build this site. Then the aim is to select – at minimal cost – enough sites to cover all risks.

The multiobjective set partitioning problem (MOSPA) requires equality constraints and the multiobjective set packing problem “$\leq$” constraints (MOSPP). Other applications of set covering and set partitioning problems arise in scheduling and rostering.

(MOSCP) and (MOSPA) have not gained much attention in the literature, and we found no references for (MOSPP). The main results in one of the references [333] are wrong. [164] deals with a particular problem. Note also that some of the problems discussed in the shortest path section 6.1 above and in the other MOCO problems section 6.12 below deal with aspects of "covering".

- SCP/Q-$\sum$/E/SP: [333]
- SCP/Q-$\sum$/SE/SP: [68]
- SCP/3-$\sum$/S/GP: [40]
- SPA/2-$\sum$/E/εC,BB: [95]
- Others: [144, 334]

6.12. Other MOCO Problems

In the previous sections we have discussed the most important multiobjective combinatorial optimization problems. Besides these there is some literature on other problems: Some classical problems have been discussed only in a few papers, others deal with problems that are so specific that they would require their own category. All of these are discussed summarily here.

In [145] a lexicographic flow problem is used to determine minimal cuts with a minimal number of arcs in a network. [354] deals with the one dimensional cutting stock problem with two objectives in a lexicographic context (priorities on the objectives). Both an exact and a heuristic algorithm are given. In [210] a branch and bound procedure to find all Pareto-optimal solutions is given, and [39] use a combination of objectives. In [1] an interactive approach is proposed to solve the multiobjective cutting stock problem.

We also found few references [185, 243] on the quadratic assignment problem (QAP) in a multicriteria context. This is closely related to the facility layout problem which is discussed in a number of papers. They actually propose approaches based on the quadratic assignment problem: [79, 115, 242, 320, 404]. Other references on the facility layout problem (FLP) are [185, 217, 350, 416]

Many of the papers listed in the surveys [53] and [54] about multiobjective transportation and routing problems also are among these specific problems. A variety of multiobjective routing problems is also discussed in [23], [100], and [135]. For network design problems we refer to [116, 117, 118, 119, 120, 129, 180, 186, 239, 284, 297]. Another problem which is combinatorial in nature has been discussed in [66]

(the channel minimization problem). In [273] a GA based heuristic is proposed to solve an extended formulation of the nominal airline crew rostering problem with two objectives.

The problem of minimizing several functions over the set of all permutations of a finite set is discussed in [104].

[28] presents a two step approach for a multicriteria timetabling problem. First, high quality timetables are generated with respect to each criterion separately. Second a compromise solution is searched. In [29], a hybrid heuristic using a population is described. Results are reported for a real instance of a biobjective space allocation problem.

[259] considers, among others 1-tree problems with one sum and one bottleneck objective or two bottleneck objectives. 1-trees are important as relaxations of the TSP. A 1-tree is a spanning tree on nodes $2,\ldots,n$ together with two edges incident on node 1.

7. Open Questions and Conclusions

Our survey of the state of the art in multiobjective combinatorial optimization clearly identifies potential areas of research and weak points in the existing literature. We briefly outline these below.

7.1. General Remarks

(a) Three is more than two plus one. Many of the existing methods concern the biobjective case (to various extents, depending on the problem). The multiobjective case is still hard to be solved, not only due to the computational complexity, but also due to the higher number of more efficient solutions of the MOCO problem.

(b) Theoretical results. Very few theoretical results are available about the properties of MOCO problems, like characterization of efficient solutions, the number of efficient solutions (supported and nonsupported) both in the worst case or on average, the topology of the nondominated frontier, the elicitation of lower and upper bounds, etc. Taking into account the fact that MOCO problems are almost always very hard in terms of computational complexity the need for a thorough theoretical understanding of MOCO problems is all the more evident. It is also clear that a better theoretical comprehension of these problems will contribute to the development of efficient solution methods.

(c) Adaptation of well known methods versus new methods. Many of the current extensions of methods useful for single objective optimization to the multiobjective situation have exhibited some diffi-

culties for finding E. One such example is the the VEGA method. MOCO problems have specific properties and need specific techniques to cope in an efficient way with these. Some adaptations such as MOSA, PSA, etc. could produce good results on a particular problem like the knapsack problem. The question is, whether such method show good performance when applied to other problems. From the evolution of these methods over the last years, one can have some doubts. No comparative studies on the performance of solution strategies like branch and bound or dynamic programming on a variety of problems are available.

(d) Applications of MOCO. Few papers refer to practical application of MOCO problems. Moreover, when the MOCO problem is extracted from a practical context, the resolution is often reduced to a single objective problem. For example, this is the case to the channel minimization problem of [66], but also for a lot of scheduling problems (see [385]). Thus there is a need to attract the attention of decision makers to the area of MOCO and solve the problems arising in practice in a real multicriteria context.

7.2. Remarks on Exact Methods

(a) Two versus many criteria. Especially for exact methods, i.e. those identifying the whole of E there is a huge gap between the bicriteria and the general case. Many procedures have been developed especially for bicriteria problems and cannot be modified to deal with the general case, a remark that is especially true for the two phases method. This gap is probably caused by the lack of theoretical understanding of MOCO problems with three or more objectives, as pointed out above.

(b) The two phases approach. As far as we know there are no procedures to compute supported efficient solutions in the multiobjective case. This would be of course the first step to an application of the two phases method in three or more criteria MOCO. Some difficulties are pointed out in [96] for max-ordering problems and in [360] for finding supported efficient solutions.

(c) Computation of bounds. For the effective adaptation of some bicriteria methods to the general case, knowledge of good lower and upper bounds on the efficient set is needed. The computation of the Nadir point (which is pretty easy in bicriteria problems) is an unsolved problem in general. Another research area would be to consider the computation of sets of solutions that constitute a set

of lower and upper bounds on E. The lack of such results makes it impossible to adapt certain procedures to general MOCO at this time. Some preliminary results are presented in [90].

(d) Problems not treated as MOCO. There is a wide variety of combinatorial problems that have not or hardly ever been investigated in a multicriteria context, as is evident from the problems list in Section 6.

(e) Level set approach. An important concept in MOP is that of level sets. It can be seen as a general framework for MOP, which allows a characterization of efficient solutions [91], as well as interactive procedures. Applications to MOCO could be promising but are not existing now.

7.3. Remarks on Heuristic Methods

(a) A real multiobjective metaheuristic for MOCO. Closely related to the remark about adaptation of single objective methods is the question of multiobjective metaheuristics to solve MOCO problems. We are not convinced of the efficiency of a real metaheuristic in the sense of a meta-method able to solve efficiently any MOCO. Each problem has its own specifics and a general MOMH cannot cope with all of these.

(b) Methods for obtaining quickly a first approximation of E. If a heuristic method defined according to the "a posteriori mode" is available, it is easy and alway possible to transform it to the "interactive mode". The main challenge for heuristic methods is then how to obtain very quickly a good approximation of the whole nondominated frontier. With such an approximation, the procedure could then be to continue either in increasing the approximation quality for the nondominated frontier or in focusing the approximation on a part of the nondominated frontier following the preference of a decision maker in the context of an interactive procedure.

(c) The quality of approximated solutions. This is an important question in the context of approximation methods: How to measure and compare approximations, and how to evaluate the quality of an approximation, especially for problems with multiple objectives? Ideas have been put forward in [156, 335, 394, 198]. Some attributes like coverage, uniformity and cardinality to judge the approximation to be satisfactory or not by a decision maker have

been defined. Such attributes are also useful when defining stopping rules in approximation methods, and again when the tuning of heuristic algorithms is examined. New attributes are then especially welcome.

(d) Using bounds and domination conditions to reduce the search space. In the continuation of the previous remark, all available information to bracket and reduce the decision space is welcome. Such information could be used for scanning the "core" of the problem, identifying and discarding irrelevant aspects of the problem investigated. Information could be derived from the decision space as well as from the objective space.

(e) Combination of exact and heuristic methods. For some MOCO problems, the resolution could be decomposed in several steps. For example, in a first step the procedure could try to identify the supported efficient solution using an exact method. Information could be extracted from the first results to reduce the search space and in a second step try to identify the nonsupported solutions by a heuristic method. Such a "semiexact" method is especially attractive for problems that can be efficiently solved as single objective combinatorial problems.

References

[1] R.M.S. Abd El-Aal. An interactive technique for the cutting stock problem with multiple objectives. *European Journal of Operational Research,* 78(3):304–317, 1994.

[2] F. Ben Abdelaziz, J. Chaouachi, and S. Krichen. A hybrid heuristic for multiobjective knapsack problems. In S. Voss, S. Martello, I. Osman, and C. Roucairol, editors, *Meta-Heuristics: Advances and Trends in Local Search Paradigms for Optimization,* pages 205–212. Kluwer Academic Publishers, Dordrecht, 1999.

[3] P. Agrell, M. Sun, and A. Stam. A tabu search multi-criteria decision model for facility location planning. In *Proceedings of the 1997 DSI Annual Meeting, San Diego, California,* volume 2, pages 908–910. Decision Sciences Institute, Atlanta, GA, 1997.

[4] R.K. Ahuja. Minimum cost-reliability ratio path problem. *Computers and Operations Research,* 15(l):83–89, 1988.

[5] M.J. Alves and J. Climaco. An interactive method for 0-1 multiobjective problems using simulated annealing and tabu search. *Journal of Heuristics,* 6(3):385–403, 2000.

[6] K.A. Andersen, K. Joernsten, and M. Lind. On bicriterion minimal spanning trees: An approximation. *Computers and Operations Research,* 23:1171–1182, 1996.

[7] Y.P. Aneja and K.P.K. Nair. Bicriteria transportation problem. *Management Science,* 25:73–78, 1979.

[8] J.L. Arthur and K.D. Lawrence. Multiple goal production and logistics planning in a chemical and pharmaceutical company. *Computers and Operations Research,* 9(2):127–137, 1982.

[9] J.L. Arthur and A. Ravindran. A multiple objective nurse scheduling problem. *AIIE Transactions,* 13:55–60, 1981.

[10] P. Aumüller and L. Bittner. Effiziente, optimale efficziente Wege in Graphen und diskret-differentielle Prozesse mit verktorwertigem Funktional. *Mathematische Operationsforschung und Statistik, Series Optimization,* 13(3):393–408, 1982.

[11] J.A. Azevedo and E.Q.V. Martins. An algorithm for the multiobjective shortest path problem on acyclic networks. *Investigação Operacional,* 11(l):52–69, 1991.

[12] M. Azizoglu, S.K. Kondakci, and M. Köksalan. Bicriteria scheduling: Minimizing flowtime and maximum earliness on a single machine. In J. Climaco, editor, *Multicriteria Analysis,* pages 279–288. Springer Verlag, Berlin, 1997.

[13] M.A. Badri. A two-stage multiobjective scheduling model for [faculty–course–time] assignments. *European Journal of Operational Research,* 94(l):16–28, 1997.

[14] M.A. Badri, D.L. Davis, D.F. Davis, and J. Hollingsworth. A multi-objective course scheduling model: combining faculty preferences for courses and times. *Computers and Operations Research,* 25(4):303–316, 1998.

[15] M.A. Badri, A.K. Mortagy, and C.A. Alsyed. A multi-objective model for locating fire stations. *European Journal of Operational Research,* 110(2):243–260, 1998.

[16] A. Baykasoglu. Goal programming using the multiple objective tabu search. *Journal of the Operational Research Society,* 52(12):1359–1369, 2001.

[17] A. Baykasoglu. MOAPPS 1.0: Aggregate production planning using the multiple objective tabu search. *International Journal of Production Research,* 39(16):3685–3702, 2001.

[18] A. Baykasoglu, S. Owen, and N. Gindy. A taboo search based approach to find the Pareto optimal set in multiple objective optimisation. *Journal of Engineering Optimization,* 31:731–748, 1999.

[19] E.M.L. Beale. Note on "A special multi-objective asignment problem" by D.J. White. *Journal of the Operational Research Society,* 35(8):769–770, 1984.

[20] P.J. Bentley and J.P. Wakefield. An analysis of multiobjective optimization within genetic algorithms. Technical Report ENG-PJB96, The University of Huddersfield, UK, 1996.

[21] D. Bertsekas. *Dynamic Programming.* Prentice Hall, Englewood Cliffs, NJ, 1987.

[22] K. Bhaskar. A multiple objective approach to capital budgeting. *Accounting and Business Research,* pages 25–46, winter 1979.

[23] B. Boffey. Multiobjective routing problems. *Top,* 3(2):167–220, 1995.

[24] P.C. Borges and M.P. Hansen. A basis for future successes in multiobjective combinatorial optimization. Technical Report IMM-REP-1998-8, Institute of Mathematical Modelling, Technical University of Denmark, Lyngby, 1998.

[25] P. Brandimarte and M. Calderini. A hierarchical bicriterion approach to integrated process plan selection and job shop scheduling. *International Journal of Production Research,* 33(1):161–181, 1995.

[26] J. Brumbaugh-Smith and D. Shier. An empirical investigation of some bicriterion shortest path algorithms. *European Journal of Operational Research,* 43(2):216–224, 1989.

[27] R.E. Burkard, G. Rote, G. Ruhe, and N. Sieber. Algorithmische Untersuchungen zu bikriteriellen kostenminimalen Flüssen in Netzwerken. *Wissenschaftliche Zeitung der technischen Hochschule Leipzig,* 13(6):333–341, 1989.

[28] E.K. Burke, Y. Bykov, and S. Petrovic. A multi-criteria approach to examination timetabling. In E.K. Burke and W. Erben, editors, *The Practice and Theory of Automated Timetabling III. Selected Papers from the 3rd International Conference on the Practice and Theory of Automated Timetabling (PATAT 2000), Konstanz, Germany, 16-18 August 2000,* volume 2079 of *Lecture Notes in Computer Science,* pages 118–131. Springer Verlag, Berlin, 2000.

[29] E.K. Burke, P. Cowling, J.D. Landa Silva, and S. Petrovic. Combining hybrid meta heuristics and populations for the multiobjective optimisation of space allocation problems. In L. Spector, D. Whitley, D. Goldberg, E. Cantu-Paz, I. Parmee, and H.-G. Beyer, editors, *Proceedings of the Genetic and Evolutionary Computation Conference 2001 (GECCO 2001). San Francisco, USA,*

7-11 July 2001, pages 1252–1259. Morgan Kaufmann, San Francisco, CA, 2001.

[30] H.I. Calvete and M. Mateo. An approach for the network flow problem with multiple objectives. *Computers and Operations Research,* 22(9):971–983, 1995.

[31] H.I. Calvete and M. Mateo. A sequential network-based approach for the multiobjective network flow problem with preemptive priorities. In M. Tamiz, editor, *Multi-Objective Programming and Goal Programming – Theory and Applications,* volume 432 of *Lecture Notes in Economics and Mathematical Systems,* pages 74–86. Springer Verlag, Berlin, 1996.

[32] P.M. Camerini, G. Galbiati, and F. Maffioli. The complexity of multi-constrained spanning tree problems. In L. Lovasz, editor, *Theory of Algorithms,* pages 53 – 101. North-Holland, Amsterdam, 1984.

[33] M.E. Captivo, J. Climaco, J. Figueira, E. Martins, and J.L. Santos. Solving multiple criteria {0, 1}-knapsack problems using a labeling algorithm. Technical Report 2, Faculdade de Economia, Unversidade de Coimbra, Portugal, 2000.

[34] R.L. Carraway, T.L. Morin, and H. Moskovitz. Generalized dynamic programming for multicriteria optimization. *European Journal of Operational Research,* 44:95–104, 1990.

[35] V. Chankong and Y.Y. Haimes. *Multiobjective Decision Making: Theory and Methodology.* Elsevier Science Publishing Co., New York, NY, 1983.

[36] A. Charnes, W.W. Cooper, R.J. Niehaus, and A. Stredry. Static and dynamic assignment models with multiple objectives and some remarks on organization design. *Management Science,* 15:365–375, 1969.

[37] C.L. Chen and R. Bulfin. Complexity of multiple machines, multicriteria scheduling problems. *European Journal of Operational Research,* 70:115–125, 1993.

[38] K.I. Cho and S.H. Kim. An improved interactive hybrid method for the linear multi-objective knapsack problem. *Computers and Operations Research,* 24(11):991–1003, 1997.

[39] C. Chu and J. Antonio. Approximation algorithms to solve real-life multicriteria cutting stock problems. *Operations Research,* 47(4):495–508, 1999.

[40] S.C.K. Chu. A goal programming model for crew duties generation. *Journal of Multi-Criteria Decision Analysis,* 10(3):143–151, 2001.

[41] S. Chung, H.W. Hamacher, F. Maffioli, and K.G. Murty. Note on combinatorial optimization with max-linear objective functions. *Discrete Applied Mathematics,* 42:139–145, 1993.

[42] J. Climaco, C. Henggeler Antunes, and M. Alves. Interactive decision support for multiobjective transportation problems. *European Journal of Operational Research,* 65:58–67, 1993.

[43] J. Climaco, C. Ferreira, and M.E. Captivo. Multicriteria integer programming: An overview of the different algorithmic approaches. In J. Climaco, editor, *Multicriteria Analysis,* pages 248–258. Springer Verlag, Berlin, 1997.

[44] J.C.M. Climaco and E.Q.V. Martins. A bicriterion shortest path algorithm. *European Journal of Operational Research,* 11:399–404, 1982.

[45] C.A. Coello. *An empirical study of evolutionary techniques for multiobjective optimization in engineering design.* PhD thesis, Department of Computer Science, Tulane University, New Orleans, LA, 1996.

[46] C.A. Coello. A comprehensive survey of evolutionary-based multiobjective optimization techniques. *Knowledge and Information Systems,* 1(3):269–308, 1999.

[47] C.A. Coello. An updated survey of GA-based multiobjective optimization techniques. *ACM Computing Surveys,* 32(2):109–143, 2000.

[48] C.A. Coello. List of references on evolutionary multiobjective optimization. http://www.lania.mx/~ccoello/EMOO/, 2001.

[49] H.W. Corley. Efficient spanning trees. *Journal of Optimization Theory and Applications,* 45(3):481–485, 1985.

[50] H.W. Corley and I.D. Moon. Shortest paths in networks with vector weights. *Journal of Optimization Theory and Applications,* 46(1):79–86, 1985.

[51] J.M. Coutinho-Rodrigues, J.C.N. Climaco, and J.R. Current. An interactive bi-objective shortest path approach: Searching for unsupported nondominated solutions. *Computers and Operations Research,* 26(8):789–798, 1999.

[52] R.G. Cox. *Routing of hazardous material.* PhD thesis, School of Civil and Environmental Engineering, Cornell University, Ithaca, NY, 1984.

[53] J. Current and M. Marsh. Multiobjective transportation network design: Taxonomy and annotation. *European Journal of Operational Research,* 65:4–19, 1993.

[54] J. Current and H. Min. Multiobjective design of transportation networks: Taxonomy and annotation. *European Journal of Operational Research,* 26:187–201, 1986.

[55] J. Current, H. Min, and D. Schilling. Multiobjective analysis of facility location decisions. *European Journal of Operational Research,* 49:295–307, 1990.

[56] J. Current, C. ReVelle, and J. Cohon. The application of location models to the multiobjective design of transportation networks. *Regional Science Review,* 14, 1985.

[57] J.R. Current, J.L. Cohon, and C.S. ReVelle. The shortest covering path problem: An application of locational constraints to network design. *Journal of Regional Science,* 24:161–183, 1984.

[58] J.R. Current, C.S. ReVelle, and J.L. Cohon. The maximum covering/shortest path problem: A multiobjective network design and routing formulation. *European Journal of Operational Research,* 21:189–199, 1985.

[59] J.R. Current, C.S. ReVelle, and J.L. Cohon. The median shortest path problem: A multiobjective approach to analyze cost vs. accesibility in the design of networks. *Transportation Science,* 21(3):188–197, 1987.

[60] J.R. Current, C.S. ReVelle, and J.L. Cohon. The minimum-covering/shortest path problem. *Decision Science,* 19:490–503, 1988.

[61] J.R. Current, C.S. ReVelle, and J.L. Cohon. An interactive approach to identify the best compromise solution for two objective shortest path problems. *Computers and Operations Research,* 17(2):187–198, 1990.

[62] P. Czyzak and A. Jaszkiewicz. A multiobjective metaheuristic approach to the localization of a chain of petrol stations by the capital budgeting model. *Control and Cybernetics,* 25(1):177–187, 1996.

[63] P. Czyzak and A. Jaszkiewicz. Pareto simulated annealing. In G. Fandel and T. Gal, editors, *Multiple Criteria Decision Making. Proceedings of the XIIth International Conference, Hagen (Germany),* volume 448 of *Lecture Notes in Economics and Mathematical Systems,* pages 297–307, 1997.

[64] P. Czyzak and A. Jaszkiewicz. Pareto simulated annealing – A metaheuristic technique for multiple objective combinatorial optimization. *Journal of Multi-Criteria Decision Analysis,* 7(l):34–47, 1998.

[65] H.G. Daellenbach and C.A. DeKluyver. Note on multiple objective dynamic programming. *Journal of the Operational Research Society,* 31:591–594, 1980.

[66] G. Dahl, K. Jörnsten, and A. Lokketangen. A tabu search approach to the channel minimization problem. In *Proceedings of the International Conference on Optimization Techniques and Applications (ICOTA'95), 5-8 July 1995, Chengdu, China,* pages 369–377. World Scientific, Singapore, 1995.

[67] S.K. Das, A. Goswami, and S.S. Alam. Multiobjective transportation problem with interval cost, source and destination parameters. *European Journal of Operational Research,* 117:100–112, 1999.

[68] M.S. Daskin and E.H. Stren. A hierarchical objective set covering model for emergency medical service vehicle deployment. *Transportation Science,* 15(2):137–152, 1981.

[69] H.M. Dathe. Zur Lösung des Zuordnungsproblems bei zwei Zielgrößen. *Zeitschrift für Operations Research,* 22:105–118, 1978.

[70] D. De Luca Cardillo and T. Fortuna. A DEA model for the efficiency evaluation of nondominated paths on a road network. *European Journal of Operational Research,* 121:549–558, 2000.

[71] R.F. Deckro, R.W. Spahr, and J.E. Hebert. Preference trade-offs in capital budgeting decisions. *IIE Transactions,* 17(4):332–337, 1985.

[72] M. Dell'Amico and F. Maffioli. On some multicriteria arborescence problems: Complexity and algorithms. *Discrete Applied Mathematics,* 65:191–206, 1996.

[73] M. Dell'Amico and F. Maffioli. Combining linear and non-linear objectives in spanning tree problems. *Journal of Combinatorial Optimization,* 4(2):253–269, 2000.

[74] M. Dell'Amico, F. Maffioli, and S. Martello, editors. *Annotated Bibliographies in Combinatorial Optimization.* J. Wiley & Sons, Chichester, 1997.

[75] R. Dial. A model and algorithm for multicriteria route-mode choice. *Transportation Research,* 13B:311–316, 1979.

[76] L.C. Dias and J.N. Climaco. Shortest path problems with partial information: Models and algorithms for detecting dominance. *European Journal of Operational Research,* 121:16–31, 2000.

[77] J.A. Diaz. Solving multiobjective transportation problems. *Ekonomicko Mathematicky Obzor,* 14:267–274, 1978.

[78] J.A. Diaz. Finding a complete description of all efficient solutions to a multiobjective transportation problem. *Ekonomicko Mathematicky Obzor,* 15:62–73, 1979.

[79] K.N. Dutta and S. Sahu. A multigoal heuristic for facilities design problems: MUGHAL. *International Journal of Production Research,* 20:147–154, 1982.

[80] R.F. Dyer, E.H. Foreman, and M.A. Mustafa. Decision support for media selection. *Journal of Advertising,* 21(1):59–72, 1992.

[81] M. Eben-Chaime. Parametric solution for linear bicriteria knapsack models. *Management Science,* 42(11):1565–1575, 1996.

[82] S.C. Egly and J.R. Wright. Microcomputer-based multiobjective personnel management model. *Journal of Computing in Civil Engineering,* 1:114–127, 1987.

[83] M. Ehrgott. Lexicographic max-ordering – A solution concept for multicriteria combinatorial optimization. In D. Schweigert, editor, *Methods of Multicriteria Decision Theory, Proceedings of the 5th Workshop of the DGOR-Working Group Multicriteria Optimization and Decision Theory,* pages 55–66. University of Kaiserslautern, 1995.

[84] M. Ehrgott. On matroids with multiple objectives. *Optimization,* 38(1):73–84, 1996.

[85] M. Ehrgott. A characterization of lexicographic max-ordering solutions. In A. Göpfert, J. Seeländer, and C. Tammer, editors, *Methods of Multicriteria Decision Theory, Proceedings of the 6th Workshop of the DGOR-Working Group Multicriteria Optimization and Decision Theory Alexisbad 1996,* volume 2389 of *Deutsche Hochschulschriften,* pages 193–202. Hänsel-Hohenhausen, Egelsbach, 1997.

[86] M. Ehrgott. *Multiple Criteria Optimization – Classification and Methodology.* Shaker Verlag, Aachen, 1997.

[87] M. Ehrgott. Discrete decision problems, multiple criteria optimization classes and lexicographic max-ordering. In T.J. Stewart and R.C. van den Honert, editors, *Trends in Multicriteria Decision Making,* volume 465 of *Lecture Notes in Economics and Mathematical Systems,* pages 31–44. Springer Verlag, Berlin, 1998.

[88] M. Ehrgott. Integer solutions of multicriteria network flow problems. *Investigação Operacional,* 19:229–243, 1999.

[89] M. Ehrgott. Approximation algorithms for combinatorial multicriteria optimization problems. *International Transcations in Operational Research,* 7:5–31, 2000.

[90] M. Ehrgott and X. Gandibleux. Bounds and bound sets for biobjective combinatorial optimization problems. In M. Köksalan and S. Zionts, editors, *Multiple Criteria Decision Making in the New Millenium,* volume 507 of *Lecture Notes in Economics and Mathematical Systems,* pages 242–253. Springer Verlag, Berlin, 2001.

[91] M. Ehrgott, H.W. Hamacher, K. Klamroth, S. Nickel, A. Schöbel, and M.M. Wiecek. A note on the equivalence of balance points and Pareto solutions in multiple-objective programming. *Journal of Optimization Theory and Applications,* 92(1):209–212, 1997.

[92] M. Ehrgott and K. Klamroth. Connectedness of efficient solutions in multiple criteria combinatorial optimization. *European Journal of Operational Research,* 97:159–166, 1997.

[93] M. Ehrgott and K. Klamroth. Nonconnected efficiency graphs in multiple criteria combinatorial optimization. In R. Caballero, F. Ruiz, and R.E. Steuer, editors, *Advances in Multiple Objective and Goal Programming,* volume 455 of *Lecture Notes in Economics and Mathematical Systems,* pages 140–150. Springer Verlag, Berlin, 1997.

[94] M. Ehrgott, S. Nickel, and H.W. Hamacher. Geometric methods to solve max-ordering location problems. *Discrete Applied Mathematics,* 93:3–20, 1999.

[95] M. Ehrgott and D.M. Ryan. Bicriteria robustness versus cost optimisation in tour of duty planning at Air New Zealand. In *Proceedings of the 35th Annual Conference of the Operational Research Society of New Zealand,* pages 31–39. ORSNZ, Auckland, 2000.

[96] M. Ehrgott and A.J.V. Skriver. Solving biobjective combinatorial max-ordering problems by ranking methods and a two-phases approach. Technical Report 2001-1, Department of Operations Research, University of Aarhus, Denmark, 2001.

[97] M. Ehrgott and D. Tenfelde. Nadir values: Computation and use in compromise programming. Report in Wirtschaftsmathematik 60/2000, University of Kaiserslautern, Department of Mathematics, 2000.

[98] M. Ehrgott and D. Tenfelde-Podehl. Computing nadir values in three objectives. In M. Köksalan and S. Zionts, editors, *Multiple Criteria Decision Making in the New Millenium,* volume 507 of *Lecture Notes in Economics and Mathematical Systems,* pages 219–228. Springer, Berlin, 2001.

[99] S. Eilon. Multicriteria warehouse location. *International Journal of Physical Distribution and Materials Management,* 12(1):42–45, 1982.

[100] N. El-Sherbeny. *Resolution of a vehicle routing problem with a multi-objective simulated annealing method.* PhD thesis, Faculté des Sciences, Université de Mons-Hainaut. Mons, Belgique, 2001.

[101] W.F.A. El-Wahed. A multi-objective transportation problem under fuzziness. *Fuzzy Sets and Systems,* 117:27–33, 2000.

[102] V.A. Emelichev and V.A. Perepelitsa. Complexity of vector optimization problems on graphs. *Optimization,* 22:903–918, 1991.

[103] V.A. Emelichev and V.A. Perepelitsa. On cardinality of the set of alternatives in discrete many-criterion problems. *Discrete Mathematics and Applications,* 2(5):461–471, 1992.

[104] V.A. Emelichev and V.G. Pokhilko. Sensitivity analysis of efficient solutions of the vector problem of minimisation of linear forms on a set of permutations. *Discrete Mathematics and Applications,* 10(4):367–378, 2000.

[105] P. Engrand. A multi-objective approach based on simulated annealing and its application to nuclear fuel management. In *Proceedings of the 5th ASME/SFEN/JSME International Conference on Nuclear Engineering. Icone 5, Nice, France 1997,* pages 416–423. American Society of Mechanical Engineers, New York, NY, 1997.

[106] P. Engrand and X. Mouney. Une méthode originate d'optimisation multi-objectif. Technical Report 98NJ00005, EDF-DER Clamart, France, 1998.

[107] T. Erlebach, H. Kellerer, and U. Pferschy. Approximating multiobjective knapsack problems. In F. Dehne, J.R. Sack, and R. Tamassia, editors, *Algorithms and Data Structures. 7th International Workshop, Providence, RI, August 8-10, 2001,* volume 2125 of *Lecture Notes in Computer Science,* pages 210–221. Springer Verlag, Berlin, 2001.

[108] J. Figueira. *L'approche interactive dans le cas des problèmes de flot multicritères.* PhD thesis, Université Paris-Dauphine, Paris, France, 1996.

[109] J. Figueira. On the integer bi-criteria network flow problem: A branch-and-bound approach. Technical Report 3, Faculdade de Economia, Universidade de Coimbra, Portugal, 2000.

[110] J. Figueira, M.E. Captivo, and J. Climaco. Sur le problème des chemins bi-critères: Une approche interactive a posteriori. Technical Report 149, LAMSADE, Université de Paris Dauphine, 1997.

[111] J. Figueira, H. M'Silti, and P. Tolla. Using mathematical programming heuristics in a multicriteria network flow context. *Journal of the Operational Research Society,* 49:878–885, 1998.

[112] R. Fischer and K. Richter. Solving a multiobjective traveling salesman problem by dynamic programming. *Mathematische Operationsforschung und Statistik, Series Optimization,* 13(2):247–252, 1982.

[113] C.M. Fonseca and P.J. Fleming. Genetic algorithms for multiobjective optimization: Formulation, discussion and generalization. In S. Forrest, editor, *Proceedings of the Fifth International Conference on Genetic Algorithms, San Mateo, California, 1993. University of Illinois at Urbana-Champaign,* pages 416–423. Morgan Kaufman, San Francisco, CA, 1993.

[114] C.M. Fonseca and P.J. Fleming. An overview of evolutionary algorithms in multiobjective optimization. *Evolutionary Computation,* 3(1):1–16, Spring 1995.

[115] J.C. Fortenberry and J.F. Cox. Multicriteria approach to the facilities layout problem. *International Journal of Production Research,* 23:773–782, 1985.

[116] T.L. Friesz. Multi-objective optimization in transportation: The case of equilibrium network design. In J.N. Morse, editor, *Organizations: Multiple Agents with Multiple Criteria,* volume 190 of *Lecture Notes in Economics and Mathematical Systems,* pages 116–127. Springer Verlag, Berlin, 1983.

[117] T.L. Friesz, G. Ananndalingam, N.J. Mehta, K. Nam, S.J. Shah, and R.L. Tobin. The multiobjective equilibrium network design problem revisited: A simulated annealing approach. *European Journal of Operational Research,* 65(1):44–57, 1993.

[118] T.L. Friesz and P.T. Harker. Multi-objective design of transportation networks: The case of spatial price equilibrium. In P. Hansen, editor, *Essays and Surveys on Multiple Criteria Decision Making,* volume 209 of *Lecture Notes in Economics and Mathematical Systems,* pages 86–93. Springer Verlag, Berlin, 1983.

[119] T.L. Friesz and P.T. Harker. Multicriteria spatial price equilibrium network design: Theory and computational results. *Transportation Research,* 17B:411–426, 1983.

[120] T.L. Friesz, F.A. Toureilles, and A.F.W. Han. Comparison of multicriteria optimization methods in transport project evaluation. *Transportation Research Record,* 751:38–41, 1980.

[121] B. Fruhwirth, R.E. Burkard, and G. Rote. Approximation of convex curves with application to the bicriterial minimum cost flow problem. *European Journal of Operational Research,* 42:326–388, 1989.

[122] V. Gabrel and D. Vanderpooten. Generation and selection of efficient paths in a multiple criteria graph: The case of daily planning the shots taken by a satellite with an interactive procedure. Technical Report 136, LAMSADE, Université Paris Dauphine, 1996.

[123] R.J. Gallagher and O.A. Saleh. Constructing the set of efficient objective values in linear multiple objective transportation problems. *European Journal of Operational Research,* 73:150–163, 1994.

[124] X. Gandibleux. Quick evaluation of the efficient solution set for the biojective knapsack problem. In Köksalan M. and Zionts S., editors, *Multiple Criteria Decision Making in the New Millennium,* volume 507 of *Lecture Notes in Economics and Mathematical Systems,* pages 254–264. Springer Verlag, Berlin, 2001.

[125] X. Gandibleux and A. Fréville. Tabu search based procedure for solving the 0/1 multiobjective knapsack problem: The two objective case. *Journal of Heuristics,* 6(3):361–383, 2000.

[126] X. Gandibleux, N. Mezdaoui, and A. Fréville. A tabu search procedure to solve multiobjective combinatorial optimization problems. In R. Caballero, F. Ruiz, and R. Steuer, editors, *Advances in Multiple Objective and Goal Programming,* volume 455 of *Lecture Notes in Economics and Mathematical Systems,* pages 291–300. Springer Verlag, Berlin, 1997.

[127] X. Gandibleux, H. Morita, and N. Katoh. The supported solutions used as a genetic information in a population heuristic. In E. Zitzler, K. Deb, L. Thiele, C.A. Coello Coello, and D. Corne, editors, *First International Conference on Evolutionary Multi-Criterion Optimization,* volume 1993 of *Lecture Notes in Computer Science,* pages 429–442. Springer Verlag, Berlin, 2001.

[128] M.R. Garey and D.S. Johnson. *Computers and Intractability – A Guide to the Theory of NP-Completeness.* Freeman, San Francisco, CA, 1979.

[129] M. Gen, K. Ida, and J. Kim. A spanning tree-based genetic algorithm for bicriteria topological network design. In *Proceedings*

of the 5th IEEE Conference on Evolutionary Computation, pages 15–20. IEEE Press, Piscataway, NJ, 1998.

[130] M. Gen and Y.Z. Li. Solving multi-objective transportation problems by spanning tree-based genetic algorithm. In I.C. Parmee, editor, *Adaptive Computing in Design and Manufacture: The Integration of Evolutionary and Adaptive Computing Technologies with Product/System Design and Realisation,* pages 95–108. Springer Verlag, Berlin, 1998.

[131] M. Gen and Y.Z. Li. Spanning tree based genetic algorithm for bicriteria transportation problem. *Computers and Industrial Engineering,* 35(3/4):531–534, 1998.

[132] A.M. Geoffrion. Proper efficiency and the theory of vector maximization. *Journal of Mathematical Analysis and Applications,* 22:618–630, 1968.

[133] K.C. Gilbert, D. Holmes, and R.E. Rosenthal. A multi-objective discrete optimisation model for land allocation. *Management Science,* 31(12):1509–1522, 1985.

[134] F. Glover and M. Laguna. *Tabu Search.* Kluwer Academic Publishers, Dordrecht, 1997.

[135] J.M. Godart. *Problèmes d'optimisation combinatoire à caractère économique dans le secteur du tourisme (organisation de voyages).* PhD thesis, Faculté Warocqué des sciences économiques, Université de Mons-Hainaut. Mons, Belgique, 2001.

[136] D.E. Goldberg. *Genetic Algorithms in Search, Optimization and Machine Learning.* Addison-Wesley Publishing Co., Reading, MA, 1989.

[137] D. Granot. A new exchange property for matroids and its application to max-min problems. *Zeitschrift für Operations Research,* 28:41–45, 1984.

[138] M. Gravel, W.L. Price, and C. Gagné. Scheduling continuous casting of aluminium using a multiple-objective ant colony optimization metaheuristic. *European Journal of Operational Research,* to appear, 2002.

[139] G.I. Green, C.S. Kim, and S.M. Lee. A multicriteria warehouse location model. *International Journal of Physical Distribution and Materials Management,* 11(1):5–13, 1981.

[140] J.J. Grefenstette. GENESIS: A system for using genetic search procedures. In *Proceedings of the 1984 Conference on Intelligent Systems and Machines,* pages 161–165, 1984.

[141] A. Gupta and A. Warburton. Approximation methods for multiple criteria travelling salesman problems. In Y. Sawaragi, editor, *Towards Interactive and Intelligent Decision Support Systems: Proceedings of the 7th International Conference on Multiple Criteria Decision Making, Kyoto,* volume 285 of *Lecture Notes in Economics and Mathematical Systems,* pages 211–217. Springer Verlag, Berlin, 1986.

[142] W. Habenicht. Quad trees – A datastructure for discrete vector optimization problems. In P. Hansen, editor, *Essays and Surveys on Multiple Criteria Decision Making,* volume 209 of *Lecture Notes Economics and Mathematical Systems,* pages 136–145. Springer Verlag, Berlin, 1983.

[143] D.K. Haider and A. Majumber. A method for selecting optimum number of stations for a rapid transit line: An application in Calcutta tube rail. In N.K. Jaiswal, editor, *Scientific Management of Transport Systems,* pages 97–108. North-Holland, Amsterdam, 1981.

[144] N.G. Hall and R.A. Vohra. Pareto optimality and a class of set covering problems. *Annals of Operations Research,* 43:279–284, 1993.

[145] H.W. Hamacher. Determining minimal cuts with a minimal number of arcs. *Networks,* 12:493–504, 1982.

[146] H.W. Hamacher. K best network flows. *Annals of Operations Research,* 57:65–72, 1995. Special Volume "Industrial Systems".

[147] H.W. Hamacher, M. Labbé, and S. Nickel. Multicriteria network location problems with sum objectives. *Networks,* 33:79–92, 1999.

[148] H.W. Hamacher, M. Labbé, S. Nickel, and A.J.V. Skriver. Multicriteria semi-obnoxious network location problems (MSNLP) with sum and center objectives. Working Paper 2000/6, Department of Operations Rsearch, University of Aarhus, Denmark, 2000.

[149] H.W. Hamacher and G. Ruhe. On spanning tree problems with multiple objectives. *Annals of Operations Research,* 52:209–230, 1994.

[150] E.L. Hannan. Allocation of library funds for books and standing orders – A multiple objective formulation. *Computers and Operations Research,* 5:109–114, 1978.

[151] T. Hanne. On the convergence of multiobjective evolutionary algorithms. *European Journal of Operational Research,* 117:553–564, 1999.

[152] T. Hanne. Global multiobjective optimization using evolutionary algorithms. *Journal of Heuristics,* 6(3):347–360, 2000.

[153] M.P. Hansen. *Metaheuristics for multiple objective combinatorial optimization.* PhD thesis, Institute of Mathematical Modelling, Technical University of Denmark, Lyngby, Denmark, 1998. Report IMM-PHD-1998-45.

[154] M.P. Hansen. Tabu search for multiobjective combinatorial optimization: TAMOCO. *Control and Cybernetics,* 29(3):799–818, 2000.

[155] M.P. Hansen. Use of substitute scalarizing functions to guide a local search based heuristics: The case of moTSP. *Journal of Heuristics,* 6(3):419–431, 2000.

[156] M.P. Hansen and A. Jaszkiewicz. Evaluating the quality of approximations to the non-dominated set. Technical Report IMM-REP-1998-7, Institute of Mathematical Modelling, Technical University of Denmark, Lyngby, 1998.

[157] P. Hansen. Bicriterion path problems. In G. Fandel and T. Gal, editors, *Multiple Criteria Decision Making Theory and Application,* volume 177 of *Lecture Notes in Economics and Mathematical Systems,* pages 109–127. Springer Verlag, Berlin, 1979.

[158] P. Hansen, J.F. Thisse, and R.E. Wendell. Efficient points on a network. *Networks,* 16:357–368, 1986.

[159] M. Hapke, A. Jaszkiewicz, and R. Slowinski. Fuzzy project scheduling with multiple criteria. In *Proceedings of Sixth IEEE International Conference on Fuzzy Systems, FUZZ-IEEE'97, July 1-5, Barcelona, Spain,* pages 1277–1282. IEEE Service Center, Piscataway, NJ, 1997.

[160] M. Hapke, A. Jaszkiewicz, and R. Slowinski. Fuzzy multi-mode resource-constrained project scheduling with multiple objectives,. In J. Weglarz, editor, *Recent Advances in Project Scheduling,* pages 355–382. Kluwer Academic Publishers, Dordrecht, 1998.

[161] M. Hapke, A. Jaszkiewicz, and R. Slowinski. Interactive analysis of multiple-criteria project scheduling problems. *European Journal of Operational Research,* 107(2):315–324, 1998.

[162] M. Hapke, A. Jaszkiewicz, and R. Slowinski. Pareto simulated annealing for fuzzy multi-objective combinatorial optimization. *Journal of Heuristics,* 6(3):329–354, 2000.

[163] M. Hapke, P. Kominek, A. Jaszkiewicz, and R. Slowinski. Integrated tools for software project scheduling under uncertainty. In

P. Brucker, S. Heitmann, J. Hurink, and S. Knust, editors, *Proceedings of the 7th International Workhop on Project Management and Scheduling PMS'2000, Osnabrück, Germany, April 17-19,* pages 149–151, 2000.

[164] R.M. Harnett and J.P. Ignizio. A heuristic program for the covering problem with multiple objectives. In J.L. Cochrane and M. Zeleny, editors, *Multiple Criteria Decision Making,* pages 738–740. University of South Carolina Press, Columbia, SC, 1973.

[165] R. Hartley. Vector optimal routing by dynamic programming. In P. Serafini, editor, *Mathematics of Multiobjective Optimization,* volume 289 of *CISM International Centre for Mechanical Sciences – Courses and Lectures,* pages 215–224. Springer Verlag, Wien, 1985.

[166] M.I. Henig. The shortest path problem with two objective functions. *European Journal of Operational Research,* 25:281–291, 1985.

[167] M.I. Henig. Efficient interactive methods for a class of multiattribute shortest path problems. *Management Science,* 40(7):891–897, 1994.

[168] A. Hertz, B. Jaumard, C. Ribeiro, and W. Formosinho Filho. A multi-criteria tabu search approach to cell formation problems in group technology with multiple objectives. *RAIRO – Recherche Opérationnelle/Operations Research,* 28(3):303–328, 1994.

[169] M.J. Hodgson, K.E. Rosing, and A.L.G. Storrier. Testing a bicriterion location-allocation model with real-world network traffic: The case of Edmonton, Canada. In J. Climaco, editor, *Multicriteria Analysis,* pages 484–495. Springer Verlag, Berlin, 1997.

[170] S.C. Hong and Y.B. Park. A heuristic for bi-objective vehicle routing with time window constraints. *International Journal of Production Economics,* 62:249–258, 1999.

[171] J.A. Hoogeveen. *Single-Machine Bicriteria Scheduling.* PhD thesis, Technische Universiteit Eindhoven, The Netherlands, 1992.

[172] J. Horn, N. Nafpliotis, and D.E. Goldberg. A niched Pareto genetic algorithm for multiobjective optimization. In *Proceedings of the First IEEE Conference on Evolutionary Computation, IEEE World Congress on Computational Intelligence, Orlando, FL, 29 June - 1 July 1994,* volume 1, pages 82–87. IEEE Service Center, Piscataway, NJ, 1994.

[173] F. Huarng, P. Pulat, and A. Ravindran. An algorithm for bicriteria integer network flow problem. In *Proceedings of the 10th Interna-*

tional Conference on Multiple Criteria Decision Making, Taipei, Taiwan, volume III, pages 305–318, 1992.

[174] F. Huarng, P.S. Pulat, and L.S. Shih. A computational comparison of some bicriterion shortest path algorithms. *Journal of the Chinese Institute of Industrial Engineers,* 13(2):121–125, 1996.

[175] K. Huckert, R. Rhode, O. Roglin, and R. Weber. On the interactive solution to a multicriteria scheduling problem. *Zeitschrift für Operations Research,* 24:47–60, 1980.

[176] J.W. Hultz, D. Klingman, T.G. Ross, and R.M. Soland. An interactive computer system for multicriteria faciliy location. *Computers and Operations Research,* 8(4):249–261, 1981.

[177] V.A. Hutson and C.S. ReVelle. Maximal direct covering tree problem. *Transportation Science,* 23:288–299, 1989.

[178] V.A. Hutson and C.S. ReVelle. Indirect covering tree problems on spanning tree networks. *European Journal of Operational Research,* 65:20–32, 1993.

[179] J.P. Ignizio. *Goal Programming and Extensions.* Lexington Books, Lexington, KY, 1976.

[180] J.P. Ignizio. An approach to the modelling and analysis of multiobjective generalized networks. *European Journal of Operational Research,* 12:357–361, 1983.

[181] J.P. Ignizio, D.F. Palmer, and C.M. Murphy. A multicriteria approach to supersystem architecture definition. *IEEE Transactions on Computers,* 31:410–418, 1982.

[182] S. Iredi, D. Merkle, and M. Middendorf. Bi-criterion optimization with multi colony ant algorithms. In E. Zitzler, K. Deb, L. Thiele, C.A. Coello Coello, and D. Corne, editors, *First International Conference on Evolutionary Multi-Criterion Optimization,* volume 1993 of *Lecture Notes in Computer Science,* pages 359–372. Springer Verlag, Berlin, 2001.

[183] H. Isermann. Proper efficiency and the linear vector maximum problem. *Operations Research,* 22:189–191, 1974.

[184] H. Isermann. The enumeration of all efficient solutions for a linear multiple-objective transportation problem. *Naval Research Logistics Quarterly,* 26:123–39, 1979.

[185] F.R. Jacobs. A layout planning system with multiple criteria and a variable domain representation. *Management Science,* 33(8):1020–1034, 1987.

[186] H.K. Jain and A. Dutta. Distributed computer system design: A multicriteria decision-making methodology. *Decision Sciences,* 17:437–453, 1986.

[187] A. Jaszkiewicz. A metaheuristic approach to multiple objective nurse scheduling. *Foundations of Computing and Decision Sciences Journal,* 22(3):169–184, 1997.

[188] A. Jaszkiewicz. Genetic local search for multiple objective combinatorial optimization. Working paper RA-014/98, Institute of Computing Science; Poznan University of Technology, Poland, 1998.

[189] A. Jaszkiewicz. Comparison of local search-based metaheuristics on the multiple objective knapsack problem. *Foundations of Computing and Decision Sciences Journal,* 26(1):99–120, 2001.

[190] A. Jaszkiewicz. Multiple objective genetic local search algorithm. In M. Köksalan and S. Zionts, editors, *Multiple Criteria Decision Making in the New Millennium,* volume 507 of *Lecture Notes in Economics and Mathematical Systems,* pages 231–240, 2001.

[191] A. Jaszkiewicz. *Multiple objective metaheuristic algorithms for combinatorial optimization.* Habilitation thesis, Poznan University of Technology, Poznan, Poland, 2001.

[192] A. Jaszkiewicz. On the computational effectiveness of multiple objective metaheuristics. In T. Trzaskalik and E. Michnik, editors, *Multiple Objective and Goal Programming – Recent Developments,* pages 86–100. Physica-Verlag, Heidelberg, 2002.

[193] A. Jaszkiewicz and A.B. Ferhat. Solving multiple criteria choice problems by interactive trichotomy segmentation. *European Journal of Operational Research,* 113(2):271–280, 1999.

[194] D. Jones, S.K. Mirrazavi, and M. Tamiz. Multi-objective metaheuristics: An overview of the current state-of-the-art. *European Journal of Operational Research,* 137(1):1–9, 2002.

[195] T.C.U. Kalu. Capital budgeting under uncertainty: An extended goal programming approach. *International Journal of Production Research,* 58(3):235–251, 1999.

[196] C.P. Keller. Algorithms to solve the orienteering problem. *European Journal of Operational Research,* 41:224–231, 1989.

[197] C.P. Keller and M.F. Goodchild. The multiobjective vending problem: A generalisation of the traveling salesman problem. *Environmantal Planning B: Planning Design,* 15:447–460, 1988.

[198] B. Kim, E.S. Gel, W.M. Carlyle, and J.W. Fowler. A new technique to compare algorithms for bi-criteria combinatorial optimization

problems. In M. Köksalan and S. Zionts, editors, *Multiple Criteria Decision Making in the New Millenium,* volume 507 of *Lecture Notes in Economics and Mathematical Systems,* pages 113–123. Springer Verlag, Berlin, 2001.

[199] K. Klamroth and M. Wiecek. Dynamic programming approaches to the multiple criteria knapsack problem. *Naval Research Logistics,* 47(1):57–76, 2000.

[200] K. Klamroth and M. Wiecek. Time-dependent capital budgeting with multiple criteria. In Y.Y. Haimes and R.E. Steuer, editors, *Research and Practice in Multiple Criteria Decision Making,* volume 487 of *Lecture Notes in Economics and Mathematical Systems,* pages 421–432. Springer Verlag, Berlin, 2000.

[201] K. Klamroth and M. Wiecek. A time-dependent single-machine scheduling knapsack problem. *European Journal of Operational Research,* 135:17–26, 2001.

[202] D. Klingman and J. Mote. Solution approaches for network flow problems with multiple criteria. *Advances in Management Studies,* 1(1):1–30, 1982.

[203] D. Klingman and N.V. Philips. Topological and computational aspects of preemptive multicriteria military personnel assignment problems. *Management Science,* 30(11):1362–1375, 1984.

[204] J.D. Knowles. *Local-Search and Hybrid Evolutionary Algorithms for Pareto Optimization.* PhD thesis, Department of Computer Science, University of Reading, UK, 2002.

[205] J.D. Knowles and D.W. Corne. The Pareto archived evolution strategy: A new baseline algorithm for multiobjective optimisation. In *Proceedings of the 1999 Congress on Evolutionary Computation. Washington, D.C.,* pages 98–105. IEEE Service Center, Piscataway, NJ, 1999.

[206] D.L. Knutson, L.M. Marquis, D.N. Ricchiute, and G.J. Saunders. A goal programming model for achieving racial balance in public schools. *Socio-Economic Planning Sciences,* 14:109–116, 1980.

[207] M. Köksalan. A heuristic approach to bicriteria scheduling. *Naval Research Logistics,* 46(7):777–789, 1999.

[208] M. Köksalan, M. Azizoglu, and S.K. Kondakci. Minimizing flowtime and maximum earliness on a single machine. *IIE Transactions,* 30:192–200, 1998.

[209] E. Koktener and M. Köksalan. A simulated annealing approach to bicriteria scheduling problems on a single machine. *Journal of Heuristics,* 6(3):311–327, 2000.

[210] A.W.J. Kolen and F.C.R. Spieksma. Solving a bi-criterion cutting stock problem with open-ended demand. *Journal of the Operational Research Society,* 51(11): 121–132, 2000.

[211] S. Kolli and G.W. Evans. A multiple objective integer programming approach for planning franchise expansion. *Computers and Industrial Engineering,* 37:543–561, 1999.

[212] S.K. Kondakci, M. Azizoglu, and M. Köksalan. Scheduling with multiple criteria. In *Proceedings of the 10th International Conference on Multiple Criteria Decision Making, Taipei, Taiwan,* volume III, pages 19–27, 1992.

[213] S.K. Kondakci, M. Azizoglu, and M. Köksalan. Bicriteria scheduling for minimizing flow time and maximum tardiness. *Naval Research Logistics Quarterly,* 43:929–936, 1996.

[214] P. Korhonen, S. Salo, and R.E. Steuer. A heuristic for estimating Nadir criterion values in multiple objective linear programming. *Operations Research,* 45(5):751–757, 1997.

[215] M.M. Kostreva and M.M. Wiecek. Time dependency in multiple objective dynamic programming. *Journal of Mathematical Analysis and Applications,* 173(1):289–307, 1993.

[216] P. Kouvelis and R.C. Carlson. Total unimodularity applications in bi-objective discrete optimization. *Operations Research Letters,* 11:61–65, 1992.

[217] M. Krause and V. Nissen. On using penalty functions and multicriteria optimisation techniques in facility layout. In J. Biethahn, editor, *Evolutionary Algorithms in Management Applications,* pages 153–166. Springer Verlag, Berlin, 1995.

[218] H.T. Kung, F. Luccio, and F. P. Preparata. On finding the maxima of a set of vectors. *Journal of the Association for Computing Machinery,* 22(4):469–476, 1975.

[219] N.K. Kwak and M.J. Schniederjans. A goal programming model for improved transportation problem solutions. *Omega: International Journal of Management Science,* 7:376–370, 1979.

[220] N.K. Kwak and M.J. Schniederjans. Goal programming solutions to transportation problems with variable supply and demand requirements. *Socio-Economic Planning Sciences,* 19:95–100, 1985.

[221] M. Labbé. Location problems. In M. Dell'Amico, F. Maffioli, and S. Martello, editors, *Annotated Bibliographies in Combinatorial Optimization,* pages 261–281. J. Wiley & Sons, Chichester, 1997.

[222] M. Labbé, D. Peeters, and J.F. Thisse. Location on networks. In *Network Routing,* volume 8 of *Handbooks in OR & MS,* pages 551–624. Elsevier Science B.V., Amsterdam, 1995.

[223] M. Laumanns, E. Zitzler, and L. Thiele. On the effect of archiving, elitism, and density based selection in evolutionary multi-objective optimization. In *Evolutionary Multi-Criteria Optimization. First International Conference, EMO 2001. Zürich, Switzerland, March 7–9, 2001. Proceedinsg,* volume 1993 of *Lecture Notes in Computer Science,* pages 181–196. Springer Verlag, Berlin, 2001.

[224] K.D. Lawrence and J.J. Burbridge. A multiple goal linear programming model for coordinated production and logistics planning. *International Journal of Production Research,* 14:215–222, 1976.

[225] K.D. Lawrence, G.R. Reeves, and S.M. Lawrence. A multiple objective shift allocation model. *IIE Transactions,* 16(4):323–328, 1984.

[226] H. Lee. *Integer solutions of bicriteria network flow problems.* PhD thesis, University of Oklahoma, Norman, OK, 1991.

[227] H. Lee and P.S. Pulat. Bicriteria network flow problems: Continuous case. *European Journal of Operational Research,* 51:119–126, 1991.

[228] H. Lee and P.S. Pulat. Bicriteria network flow problems: Integer case. *European Journal of Operational Research,* 66:148–157, 1993.

[229] S.M. Lee. *Goal Programming for Decision Analysis.* Auerbach Publications, Philadelphia, PA, 1972.

[230] S.M. Lee and E.R. Clayton. A goal programming model for academic resource allocation. *Management Science,* 18(8):395–408, 1972.

[231] S.M. Lee, G.I. Green, and C.S. Kim. Multiple criteria model for the location-allocation problem. *Computers and Operations Research,* 8(1):1–8, 1981.

[232] S.M. Lee and R. Luebbe. The multicriteria warehouse location problem revisited. *International Journal of Physical Distribution and Materials Management,* 17(3):56–59, 1987.

[233] S.M. Lee and L.J. Moore. Optimizing transportation problems with multiple objectives. *AIIE Transactions,* 5:333–338, 1973.

[234] S.M. Lee and L.J. Moore. Multicriteria school busing models. *Management Science,* 23(7):703–715, 1977.

[235] S.M. Lee and M.J. Schniederjans. A multicriteria assignment problem: A goal programming approach. *Interfaces,* 13(4):75–81, 1983.

[236] L. Li and K.K. Lai. A fuzzy approach to the multiobjective transportation problem. *Computers and Operations Research,* 27:43–57, 2000.

[237] G.E. Liepins, M.R. Hilliard, J. Richardson, and M. Palmer. Genetic algorithms application to set covering and travelling salesman problems. In D.E. Brown and C.C. White, editors, *Operations Research and Artificial Intelligence: The Integration of Problem-solving Strategies,* pages 29–57. Kluwer Academic Publishers, Norwell, MA, 1990.

[238] M. Lind. *Cooperative Game Theory and Multiple Criteria Decision Making.* PhD thesis, Department of Operations Research, University of Aarhus, Denmark, 1996.

[239] M. Los and C. Lardinois. Combinatorial programming, statistical optimization and the optimal transportation network problem. *Transportation Research,* 16B:89–124, 1982.

[240] T. Loukil Moalla, J. Teghem, and P. Fortemps. Solving multiobjective scheduling problems with tabu search. In *Workshop on Production Planning and Control, 2-4 October 2000. Mons, Belgium,* pages 18–26. Facultés Universitaires Catholiques de Mons, 2000.

[241] T. Loukil Moalla, J. Teghem, and D. Tuyttens. Solving multiobjective scheduling problems with the MOSA method. In *Workshop on Production Planning and Control, 2-4 October 2000. Mons, Belgium,* pages 12–17. Facultés Universitaires Catholiques de Mons, 2000.

[242] B. Malakooti. Multiple objective facility layout: A heuristic to generate efficient alternatives. *International Journal of Production Research,* 27:1225–1238, 1989.

[243] B. Malakooti and G.I. D'Souza. Multiple objective programmig for the quadratic assignment problem. *International Journal of Production Research,* 25:285–300, 1987.

[244] B. Malakooti, J. Wang, and E.C. Tandler. A sensor-based accelerated approach for multi-attribute machinability and tool life evaluation. *International Journal of Production Research,* 28:2373, 1990.

[245] J. Malczewski and W. Ogryczak. The multiple criteria location problem: 1. A generalized network model and the set of efficient solutions. *Environment and Planning A,* 27:1931–1960, 1995.

[246] J. Malczewski and W. Ogryczak. The multiple criteria location problem: 2. Preference-based techniques and interactive decision support. *Environment and Planning A,* 28:69–98, 1996.

[247] R. Malhotra, H.L. Bhatia, and M.C. Puri. Bi-criteria assignment problem. *Operations Research,* 19(2):84–96, 1982.

[248] R. Malhotra and M.C. Puri. Bi-criteria network problem. *Cahiers du Centre d'Etudes Recherche Operationelle,* 26(1/2):95–102, 1984.

[249] R. Marett and M. Wright. A comparison of neighborhood search techniques for multi-objective combinatorial problems. *Computers and Operations Research,* 23(5):465–483, 1996.

[250] E.Q.V. Martins. An algorithm to determine a path with minimal cost/capacity ratio. *Discrete Applied Mathematics,* 8:189–194, 1984.

[251] E.Q.V. Martins. On a multicriteria shortest path problem. *European Journal of Operational Research,* 16:236–245, 1984.

[252] E.Q.V. Martins. On a special class of bicriterion path problems. *European Journal of Operational Research,* 17:85–94, 1984.

[253] E.Q.V. Martins. On a particular quadratic network problem. *European Journal of Operational Research,* 29:317–327, 1987.

[254] E.Q.V. Martins. Bibliography of papers on multiobjective optimal path problems. www.mat.uc.pt/˜eqvm/Bibliografias.html, 1996.

[255] E.Q.V. Martins and J.C.N. Climaco. On the determination of the nondominated paths in a multiobjective network problem. *Methods of Operations Research,* 40:255–258, 1981.

[256] E.Q.V. Martins and J.L.E. Santos. An algorithm for the quickest path problem. *Operations Research Letters,* 20(4):195–198, 1997.

[257] A.J. Mehta and A.K. Rifai. Applications of linear programming vs. goal programming to assignment problems. *Akron Business and Economic Review,* 7:52–55, 1976.

[258] A.J. Mehta and A.K. Rifai. Goal programming application to assignment problem in marketing. *Journal of Marketing Science,* 7:108–116, 1979.

[259] I.I. Melamed and I.K. Sigal. An investigation of linear convolution of criteria in multicriteria discrete programming. *Computational Mathematics and Mathematical Physics,* 35(8):1009–1017, 1995.

[260] I.I. Melamed and I.K. Sigal. A computational investigation of linear parametrization of criteria in multicriteria discrete programming. *Computational Mathematics and Mathematical Physics,* 36(10):1341–1343, 1996.

[261] I.I. Melamed and I.K. Sigal. The linear convolution of criteria in the bicriteria traveling salesman problem. *Computational Mathematics and Mathematical Physics,* 37(8):902–905, 1997.

[262] I.I. Melamed and I.K. Sigal. Numerical analysis of tricriteria tree and assignment problems. *Computational Mathematics and Mathematical Physics,* 38(10):1707–1714, 1998.

[263] I.I. Melamed and I.K. Sigal. Combinatorial optimization problems with two and three criteria. *Doklady Mathematics,* 59(3):490–493, 1999.

[264] I.I. Melamed, I.K. Sigal, and N.Y. Vladimirova. Study of the linear parametrization of criteria in the bicriteria knapsack problem. *Computational Mathematics and Mathematical Physics,* 39(5):721–726, 1999.

[265] G. Mensch. Zur Berücksichtigung mehrerer Zielfunktionen bei der optimalen Personalanweisung. *Schmalenbach's Zeitschrift für betriebswirtschaftliche Forschung,* 23:200–207, 1971.

[266] M. Minoux. Solving combinatorial problems with combined min-max-min-sum objective and applications. *Mathematical Programming,* 45:361–372, 1989.

[267] Kostreva. M.M. and L. Lancaster. Multiple objective path optimization for time dependent objective functions. In T. Trzaskalik and E. Michnik, editors, *Multiple Objective and Goal Programming – Recent Developments,* pages 127–142. Physica-Verlag, Heidelberg, 2002.

[268] P. Modesti and A. Sciomachen. A utility measure for finding multiobjective shortest paths in urban multimodal transportation networks. *European Journal of Operational Research,* 111(3):495–508, 1998.

[269] S. Mohri, T. Masuda, and H. Ishii. Bi-criteria scheduling problem on three identical parallel machines. *International Journal of Production Economics,* 60-61:529–536, 1999.

[270] L.J. Moore, B.W. Taylor, and S.M. Lee. Analysis of a transshipment problem with multiple conflicting objectives. *Computers and Operations Research,* 5:39–46, 1978.

[271] H. Morita, X. Gandibleux, and N. Katoh. Experimental feedback on biobjective permutation scheduling problems solved with a population heuristic. *Foundations of Computing and Decision Sciences Journal,* 26(1):23–50, 2001.

[272] J. Mote, I. Murthy, and D.L. Olson. A parametric approach to solving bicriterion shortest path problems. *European Journal of Operational Research,* 53:81–92, 1991.

[273] W.E. Moudani, C.A.N. Cosenza, M. de Coligny, and F. Mora-Camino. A bi-criterion approach for the airlines crew rostering problem. In E. Zitzler, K. Deb, L. Thiele, C.A.C. Coello, and D. Corne, editors, *Evolutionary Multi-Criteria Optimization. First International Conference, EMO 2001, Zurich, Switzerland, March 7-9, 2001 Proceedings,* volume 1993 of *Lecture Notes in Computer Science,* pages 486–500. Springer Verlag, Berlin, 2001.

[274] T. Murata. *Genetic Algortithms for Multi-Objective Optimization.* PhD thesis, Osaka Prefecture University, Japan, 1997.

[275] T. Murata and H. Ishibuchi. MOGA: Multi-objective genetic algorithms. In *Proceedings of the 2nd IEEE International Conference on Evolutionary Computing, Perth, Australia,* pages 289–294. IEEE Service Center, Piscataway, NJ, 1995.

[276] C. Murphy and J.P. Ignizio. A methodology for multicriteria network partitioning. *Computers and Operations Research,* 11(2):1–11, 1984.

[277] I. Murthy and S.S. Her. Solving min-max shortest-path problems on a network. *Naval Research Logistics,* 39:669–683, 1992.

[278] I. Murthy and D. Olson. An interactive procedure using domination cones for bicriterion shortest path problems. *European Journal of Operational Research,* 72(2):418–432, 1994.

[279] K.G. Murty. *Network Programming.* Prentice Hall, Englewood Cliffs, NJ, 1992.

[280] A. Mustafa and M. Goh. Assigning faculty to courses: A multiple objective approach using DINAS. In H.G. Lee and K.Y. Tam, editors, *Proceedings of the First Asia-Pacific Decision Sciences Institute Conference,* pages 671–680, 1996.

[281] A. Mustafa and M. Goh. Characteristics of the efficient solutions of bicriteria and tricriteria network flow problems. In R. Caballero, F. Ruiz, and R.E. Steuer, editors, *Advances in Multiple Objective and Goal Programming,* volume 455 of *Lecture Notes in Economics and Mathematical Systems,* pages 131–139. Springer Verlag, Berlin, 1997.

[282] A. Mustafa and M. Goh. Finding integer efficient solutions for bicriteria and tricriteria network flow problems. *Computers and Operations Research,* 25(2):139–157, 1998.

[283] P. Neumayer. Complexity of optimization on vectorweighted graphs. In A. Bachem, U. Derigs, M. Jünger, and R. Schrader, editors, *Operations Research 93,* pages 359–361. Physica Verlag, Heidelberg, 1994.

[284] N. Nihan and E. Morlock. A planning model for multiple-mode transportation system operation. *Transportation Planning and Technology,* 3:59–73, 1976.

[285] W. Ogryczak. On the lexicographic minimax approach to location problems. *European Journal of Operational Research,* 100:566–585, 1997.

[286] W. Ogryczak. Location problems from the multiple criteria perspective: Efficient solutions. *Archives of Control Science,* 7(3-4), 1998.

[287] W. Ogryczak, K. Studzinski, and K. Zorychta. A solver for the multi-objective transshipment problem with facility location. *European Journal of Operational Research,* 43:53–64, 1989.

[288] J.P. Osleeb and S. Ratick. A mixed integer and multiple objective programming model to analyze coal handling in New England. *European Journal of Operational Research,* 12:302–313, 1983.

[289] I. Osman and G. Laporte. Metaheuristics: A bibliography. *Annals of Operations Research,* 63:513–623, 1996.

[290] A. Osyczka. The min-max approach to a multicriterion network optimization problem. In P. Dewilde, editor, *International Sympoium on Mathematical Theory of Networks and Systems,* pages 316–320. Western Periodicals Co., North Hollywood, CA, 1979.

[291] S. Pamuk and M. Köksalan. An interactive genetic algorithm applied to the multiobjective knapsack problem. In Köksalan M. and Zionts S., editors, *Multiple Criteria Decision Making in the New Millennium,* volume 507 of *Lecture Notes in Economics and Mathematical Systems,* pages 265–272. Springer Verlag, Berlin, 2001.

[292] Y.B. Park. A solution of the bicriteria vehicle scheduling problems with time and area-dependent travel speeds. *Computers and Industrial Engineering,* 38:173–178, 2000.

[293] Y.B. Park and C.P. Koelling. A solution of vehicle routing problems in a multiple objective context. *Engineering Costs and Production Economics,* 10:121–132, 1986.

[294] Y.B. Park and C.P. Koelling. An interactive computerized algorithm for multicriteria vehicle routing problems. *Computers and Industrial Engineering,* 16:477–490, 1989.

[295] G. Parks and A. Suppapitnarm. Multiobjective optimization of PWR reload core designs using simulated annealing. In *Proceedings of the International Conference on Mathematics and Computation, Reactor Physics and Environmental Analysis in Nuclear Applications. Madrid, Spain, September 1999,* volume 2, pages 1435–1444, 1999.

[296] B. Pelegrin and P. Fernandez. On the sum-max bicriterion path problem. *Computers and Operations Research,* 25(12):1043–1054, 1998.

[297] J. Perl. Goal-programming approach to multiobjective highway design model. *Transportation Research Record,* 751:41–51, 1980.

[298] K. Philipp and R. Staudte. Das *n*-dimensionale Zuweisungsproblem bei mehrfacher Zielsetzung. *Wissenschaftliche Zeitschrift der Technischen Hochschule Karl-Marx-Stadt,* 24:255–258, 1982.

[299] N.V. Phillips. A weighting function for pre-emptive multicriteria asignment problems. *Journal of the Operational Research Society,* 38(9):797–802, 1987.

[300] M.A. Pollatschek. Personnel assignment by multiobjective programming. *Zeitschrift für Operations Research,* 20:161–170, 1976.

[301] S. Prakash. A transportation problem with objectives to minimize total cost and duration of transportation. *Opsearch: Journal of the Operational Research Society of India,* 18(4):235–238, 1981.

[302] S. Praksh. A machine-assignment problem with two objectives. *Bulletin of the Technical University Istanbul,* 35:235–253, 1982.

[303] S.Y. Prasad. Approximation error analysis in bicriteria heuristics. *Journal of Multi-Criteria Decision Analysis,* 7(3):155–159, 1998.

[304] J. Puerto and E. Fernandéz. The multiobjective solution of the uncapacitated plant location problem. Report DR 2000/15, Department d'EIO, Universitat Politécnica de Catalunya, 2000.

[305] P.S. Pulat, F. Huarng, and H. Lee. Efficient solutions for the bicriteria network flow problem. *Computers and Operations Research,* 19(7):649–655, 1992.

[306] A.P. Punnen. On combined minmax-minsum optimization. *Computers and Operations Research,* 21(6):707–716, 1994.

[307] A.P. Punnen and Y.P. Aneja. Minmax combinatorial optimization. *European Journal of Operational Research,* 81:634–643, 1995.

[308] A.P. Punnen and K.P.K Nair. An $O(m \log n)$ algorithm for the max + sum spanning tree problem. *European Journal of Operational Research,* 89:423–426, 1996.

[309] V. Rajendra Prasad, N.P.K. Nair, and Y.P. Aneja. A generalized time-cost trade-off transportation problem. *Journal of the Operational Research Society,* 44:1243–1248, 1993.

[310] R.M. Ramos, S. Alonso, J. Sicilia, and C. González. The problem of the optimal biobjective spanning tree. *European Journal of Operational Research,* 111:617–628, 1998.

[311] K. Rana and R.G. Vickson. A model and solution algorithm for optimal routing of a time-chartered containership. *Transportation Science,* 22:83–96, 1988.

[312] P. Randolph and R. Ringeisen. Shortest paths through multicriteria networks. Technical Report ERI-76073, Engineering Research Institute, Iowa State University, Ames, IA, 1975.

[313] L.M. Rasmussen. Zero-one programming with multiple criteria. *European Journal of Operational Research,* 26:83–95, 1986.

[314] S.J. Ratick. Multiobjective programming with related bargaining games: An application to utility coal conversions. *Regional Science and Urban Economics,* 13:55–76, 1983.

[315] L.P. Rees, E.R. Clayton, and B.W. Taylor. A linear goal programming model of a multi-period, multi-commodity, network flow problem. *Journal of Business Logistics,* 8:117–138, 1987.

[316] C. Reeves. *Modern Heuristic Techniques for Combinatorial Problems.* McGrawHill, London, 1995.

[317] C. ReVelle, J.L. Cohon, and D. Shobys. Multiple objectives in facility location: A review. In J.N. Morse, editor, *Organizations: Multiple Agents with Multiple Criteria,* volume 190 of *Lecture Notes in Economics and Mathematical Systems,* pages 321–327. Springer Verlag, Berlin, 1981.

[318] C. ReVelle, D. Marks, and J.C. Liebman. An analysis of public and private sector location models. *Management Science,* 16:692–707, 1970.

[319] J. Ringuest and D. Rinks. Interactive solutions for the linear multi-objective transportation problem. *European Journal of Operational Research,* 32(1):96–106, 1987.

[320] M.J. Rosenblatt. The facilities layout problem: A multi-goal aproach. *International Journal of Production Research,* 17:323–332, 1979.

[321] M.J. Rosenblatt and Z. Sinuany-Stern. Generating the discrete efficient frontier to the capital budgeting problem. *Operations Research,* 37(3):384–394, 1989.

[322] G.T. Ross and R.M. Soland. A multicriteria approach to the location of public facilities. *European Journal of Operational Research,* 4:307–321, 1980.

[323] T. Ross and R. Soland. A multicriteria approach to the location of public facilities. *European Journal of Operational Research,* pages 307–321, 1980.

[324] G. Ruhe. Complexity results for multicriteria and parametric network flows using a pathological graph of Zadeh. *Zeitschrift für Operations Research,* 32:59–27, 1988.

[325] G. Ruhe and B. Fruhwirth. $\epsilon-$optimality for bicriteria programs and its application to minimum cost flows. *Computing,* 44(1):21–34, 1990.

[326] H.M. Safer. *Fast Approximation Schemes for Multi-Criteria Combinatorial Optimization.* PhD thesis, Sloan School of Management, MIT, Cambridge, MA, 1992.

[327] H.M. Safer and J.B. Orlin. Fast approximation schemes for multicriteria combinatorial optimization. Working paper 3756-95, Sloan School of Management, MIT, Cambridge, MA, 1995.

[328] H.M. Safer and J.B. Orlin. Fast approximation schemes for multicriteria flow, knapsack, and scheduling problems. Working paper 3757-95, Sloan School of Management, MIT, Cambridge, MA, 1995.

[329] F.S. Salman, J. Kalagnanam, and S. Murthy. Cooperative strategies for solving the bicriteria sparse multiple knapsack problem. In *1999 Congress on Evolutionary Computation,* pages 53–60. IEEE Service Center, Piscataway, NJ, 1999.

[330] N.G.F. Sancho. Multi-objective routing problem. *Engineering Optimization,* 10:71–76, 1986.

[331] N.G.F. Sancho. A new type of multiobjective routing problem. *Engineering Optimization,* 14:115–119, 1988.

[332] Y. Sawaragi, H. Nakayama, and T. Tanino. *Theory of Multiobjective Optimization.* Academic Press, Orlando, FL, 1985.

[333] R.R. Saxena and S.R. Arora. Enumeration technique for solving the multi-objective linear set covering problem. *Asia-Pacific Journal of Operational Research,* 12:87–97, 1995.

[334] R.R. Saxona and S.R. Arora. Linearization aproach to multiobjective quadratic set covering. *Optimization,* 43:145–156, 1998.

[335] S. Sayin. Measuring the quality of discrete representations of efficient sets in multiple objective mathematical programming. *Mathematical Programming,* 87(3):543–560, 2000.

[336] S. Sayin and S. Karabati. A bicriteria approach to the two-machine flow-shop scheduling problem. *European Journal of Operational Research,* 113(2):435–449, 1999.

[337] J.D. Schaffer. *Multiple Objective Optimization with Vector Evaluated Genetic Algorithms.* PhD thesis, Vanderbilt University, Nashville, TN, 1984.

[338] J.D. Schaffer. Multiple objective optimization with vector evaluated genetic algorithms. In J.J. Grefenstette, editor, *Genetic Algorithms and their Applications: Proceedings of the First International Conference on Genetic Algorithms,* pages 93–100. Lawrence Erlbaum, Pittsburgh, PA, 1985.

[339] M.J. Schniederjans, N.K. Kwak, and M.C. Helmer. An application of goal programming to resolve a site location problem. *Interfaces,* 12:65–72, 1982.

[340] P. Schoovaerts, L. Berghmans, and J. Teghem. Implementation of health facilities in a new city. *Journal of the Operational Research Society,* 35(12):1047–1054, 1984.

[341] D. Schweigert. Linear extensions and efficient trees. Preprint 172, University of Kaiserslautern, Department of Mathematics, 1990.

[342] D. Schweigert. Linear extensions and vector-valued spanning trees. *Methods of Operations Research,* 60:219–222, 1990.

[343] D. Schweigert. Vector-weighted matchings. In C.J. Colbourn and E.S. Mahmoodian, editors, *Combinatorics Advances,* volume 329 of *Mathematics and Its Applications,* pages 267–276. Kluwer Academic Publishers, Dordrecht, 1995.

[344] D. Schweigert. Ordered graphs and minimal spanning trees. *Foundation of Computing and Decision Sciences,* 24:219–229, 1999.

[345] A. Sedeño-Noda and C. González-Martín. The biobjective minimum cost flow problem. *European Journal of Operational Research,* 124:591–600, 2000.

[346] A. Sedeño-Noda and C. González-Martín. An algorithm for the biobjective integer minimum cost flow problem. *Computers and Operations Research,* 28(2):139–156, 2001.

[347] P. Serafini. Some considerations about computational complexity for multi objective combinatorial problems. In J. Jahn and W. Krabs, editors, *Recent advances and historical development of vector optimization,* volume 294 of *Lecture Notes in Economics and Mathematical Systems.* Springer Verlag, Berlin, 1986.

[348] P. Serafini. Simulated annealing for multiobjective optimization problems. In *Proceedings of the 10th International Conference on*

Multiple Criteria Decision Making, Taipei-Taiwan, volume I, pages 87–96, 1992.

[349] I.V. Sergienko and V.A. Perepelitsa. Finding the set of alternatives in discrete multicriterion problems. *Cybernetics,* 27(3):673–683, 1991.

[350] J.S. Shang. Multicriteria facility layout problem: An integrated approach. *European Journal of Operational Research,* 66(3) :291–304, 1993.

[351] P.S. Shelokar, S. Adhikari, R. Vakil, V.K. Jayaraman, and B.D. Kulkarni. Multiobjective ant algorithm for continuous function optimization: Combination of strength Pareto fitness assignment and thermo-dynamic clustering. *Foundations of Computing and Decision Sciences Journal,* 25(4):213–230, 2000.

[352] Y. Shi. A transportation model with multiple criteria and multiple constraint levels. *Mathematical Computer Modelling,* 21(4):13–28, 1995.

[353] I.K. Sigal. Algorithms for solving the two-criterion large-scale travelling salesman problem. *Computational Mathematics and Mathematical Physics,* 34(l):33–43, 1994.

[354] Z. Sinuany-Stern and I. Weiner. The one dimensional cutting stock problem using two objectives. *Journal of the Operational Research Society,* 45(2):231–236, 1994.

[355] A.J.V. Skriver. A classification of bicriteria shortest path (BSP) algorithms. *Asia-Pacific Journal of Operational Research,* 17(2):199–212, 2000.

[356] A.J.V. Skriver and K.A. Andersen. A label correcting approach for solving bicriterion shortest path problems. *Computers and Operations Research,* 27(6):507–524, 2000.

[357] A.J.V. Skriver, K.A. Andersen, and K. Holmberg. Bicriteria network location problems (BNLP) with criteria dependent lengths and minisum objectives. Working Paper 2001/4, Department of Operations Rsearch, University of Aarhus, Denmark, 2001.

[358] M. Sniedovich. A multi-objective routing problem revisited. *Engineering Optimization,* 13:99–108, 1988.

[359] R.M. Soland. The design of multiactivity multifacility systems. *European Journal of Operational Research,* 12:95–104, 1983.

[360] R.S. Solanki, P.A. Appino, and J.L. Cohon. Approximating the noninferior set in multiobjective linear programming problems. *European Journal of Operational Research,* 68:356–373, 1993.

[361] N. Srinivas and K. Deb. Multiobjective optimization using nondominated sorting in genetic algorithms. *Evolutionary Computation,* 2(3):221–248, 1994.

[362] V. Srinivasan and G.L. Thompson. Alternate formulations for static multiattribute assignment models. *Management Science,* 20:154–158, 1973.

[363] V. Srinivasan and G.L. Thompson. Algorithms for minimizing total cost, bottleneck time and bottleneck shipment in transportation problems. *Naval Research Logistics Quarterly,* 23:567–595, 1976.

[364] V. Srinivasan and G.L. Thompson. Determining cost vs. time Pareto-optimal frontiers in multi-modal transportation problems. *Transporation Science,* 11:1–19, 1977.

[365] R. Steuer, J. Silverman, and A. Whisman. A combined Tchebycheff/aspiration criterion vector interactive multiobjective programming procedure. *Management Science,* 39(10):1255–1260, 1993.

[366] R.E. Steuer, L.R. Gardiner, and J. Gray. A bibliographic survey of the activities and international nature of multiple criteria decision making. *Journal of Multi-Criteria Decision Analysis,* 5:195–217, 1996.

[367] B.S. Stewart and C.C. White. Multiobjective A^*. *Journal of the Association for Computing Machinery,* 38(4):775–814, 1991.

[368] T.J. Stewart and H.W. Ittmann. Two-stage optimisation in transportation problem. *Journal of the Operational Research Society,* 30:897–904, 1979.

[369] M. Sun. Applying tabu search to multiple objective combinatorial optimization problems. In *Proceedings of the 1997 DSI Annual Meeting, San Diego, California,* volume 2, pages 945–947. Decision Sciences Institute, Atlanta, GA, 1997.

[370] M. Sun, A. Stam, and R. Steuer. Solving multiple objective programming problems using feed-forward artificial neural networks: The interactive FFANN procedure. *Management Science,* 42(6):835–849, 1996.

[371] M. Sun, A. Stam, and R. Steuer. Interactive multiple objective programming using Tchebycheff programs and artificial neural networks. *Computers and Operations Research,* 27:601–620, 2000.

[372] M. Sun and R. Steuer. Quad-trees and linear lists for identifying nondominated criterion vectors. *INFORMS Journal on Computing,* 8(4):367–375, 1996.

[373] A. Suppapitnarm and G. Parks. Simulated annealing: An alternative approach to true multiobjective optimization. In A.S. Wu, editor, *Proceedings of the Genetic and Evolutionary Computation Conference (GECCO'99). Orlando, Florida,* 1999.

[374] A. Suppapitnarm, K. Seffen, G. Parks, and P. Clarkson. A simulated annealing algorithm for multiobjective optimization. *Engineering Optimization,* 33(1):59–85, 2000.

[375] C. Sutcliffe and J. Board. Optimal solution of a vehicle routing problem: Transporting mentally handicapped adults to an adult training center. *Journal of the Operational Research Society,* 41:61–67, 1990.

[376] C. Sutcliffe, J. Board, and P. Chesire. Goal programming and allocating children to secondary schools in Reading. *Journal of the Operational Research Society,* 35(8):719–730, 1984.

[377] V. Sysoev and A. Dolgui. A Pareto optimization approach for manufacturing system design. In *Proceedings of the International Conference on Industrial Engineering and Production Management (IEPM'99). July 1-15 1999, Glasgow, Scotland,* volume 2, pages 116–125. Facultés Universitaires Catholiques de Mons, 1999.

[378] F. Szidarovsky, L. Duckstein, and I. Bogardi. Multiobjective management of mining under water hazard by game theory. *European Journal of Operational Research,* 15:251–258, 1984.

[379] H. Tamaki, M. Mori, M. Araki, Y. Mishima, and H. Ogai. Multicriteria optimization by genetic algorithms: A case of scheduling in hot rolling process. In *Proceedings of the 3rd Conference of the Association of Asian-Pacific Operational Research Societies within IFORS (APORS94),* pages 374–381. World Scientific, Singapore, 1994.

[380] J. Teghem and P.L. Kunsch. Interactive methods for multiobjective integer linear programs. In G. Fandel, M. Grauer, A. Kurzhanski, and A.P. Wierzbicki, editors, *Large-scale modelling and interactive decision analysis,* volume 273 of *Lecture Notes in Economics and Mathematical Systems,* pages 75–87. Springer Verlag, Berlin, 1986.

[381] J. Teghem and P.L. Kunsch. A survey of techniques for finding efficient solutions to multi-objective integer linear programming. *Asia-Pacific Journal of Operational Research,* 3:95–108, 1986.

[382] J. Teghem, D. Tuyttens, and E. L. Ulungu. An interactive heuristic method for multi-objective combinatorial optimization. *Computers and Operations Research,* 27(7-8):621–634, 2000.

[383] J.M. Thizy, S. Pissarides, S. Rawat, and D.E. Lane. Interactive multiple criteria optimization for capital budgeting in a Canadian telecommunications company. In M. Tamiz, editor, *Multi-Objective Programming and Goal Programming – Theories and Applications,* volume 432 of *Lecture Notes in Economics and Mathematical Systems,* pages 128–147. Springer Verlag, Berlin, 1996.

[384] D. Thuente. Two algorithms for shortest paths through multiple criteria networks. In P.C.C. Wang, editor, *Information Linkages between Applied Mathematics and Industry. Proceedings of the First Annual Workshop, Naval Postgraduate School, Monterey, CA, 1978,* pages 567–573. Academic Press, London, 1979.

[385] V. T'Kindt. *Etude des problèmes d'ordonnancement multicritères.* PhD thesis, E3I, Université Francois Rabelais, Tours, France, 1999.

[386] V. T'Kindt, J.-C. Billaut, and H. Houngbossa. A multicriteria heuristic to solve a 2-stage hybrid flowshop scheduling problem. In *Proceedings of the International Conference on Industrial Engineering and Production Mangement (IEPM'99). July 1-15 1999, Glasgow, Scotland,* volume 2, pages 107–115. Facultés Universitaires Catholique de Mons, 1999.

[387] V. T'Kindt and J.C. Billaut. Some guidelines to solve multicriteria scheduling problems. In *1999 IEEE International Conference on Systems, Man, and Cybernetics Proceedings. October 10–12 1999, Tokyo, Japan,* volume 6, pages 463–468. IEEE Service Center, Piscataway, NJ, 1999.

[388] V. T'Kindt and J.C. Billaut. Multicriteria scheduling problems: A survey. *RAIRO Recherche Opérationnelle/Operations Research,* 35:143–163, 2001.

[389] V. T'Kindt, J.C. Billaut, and C. Proust. An interactive algorithm to solve a bicriteria scheduling problem on unrelated parallel machines. *European Journal of Operational Research,* 135(1):42–49, 2001.

[390] D.S. Todd and P. Sen. A multiple criteria genetic algorithm for containership loading. In T. Bäck, editor, *Proceedings of the Seventh International Conference on Genetic Algorithms (ICGA97).* Morgan Kaufmann, San Francisco, CA, 1997.

[391] C.T. Tung. A multicriteria Pareto-optimal algorithm for the traveling salesman problem. *Asia-Pacific Journal of Operational Research,* 11:103–115, 1994.

[392] C.T. Tung and K.L. Chew. A bicritrion Pareto-optimal path algorithm. *Asia-Pacific Journal of Operational Research,* 5:166–172, 1988.

[393] C.T. Tung and K.L. Chew. A multicriteria Pareto-optimal path algorithm. *European Journal of Operational Research,* 62:203–209, 1992.

[394] D. Tuyttens, J. Teghem, P. Fortemps, and K. Van Nieuwenhuyse. Performance of the MOSA method for the bicriteria assignment problem. *Journal of Heuristics,* 6(3):295–310, 2000.

[395] G.H. Tzeng, D. Teodorovic, and M.J. Hwang. Fuzzy bicriteria multi-index transportation problems for coal allocation planning of Taipower. *European Journal of Operational Research,* 95(1):62–72, 1996.

[396] E.L. Ulungu. *Optimisation combinatoire multicritère: Détermination de l'ensemble des solutions efficaces et méthodes interactives.* PhD thesis, Faculté des Sciences, Université de Mons-Hainaut. Mons, Belgium, 1993.

[397] E.L. Ulungu and J. Teghem. Multi-objective shortest path problem: A survey. In D. Glückaufova, D. Loula, and M. Cerný, editors, *Proceedings of the International Workshop on Multicriteria Decision Making: Methods – Algorithms – Applications at Liblice, Czechoslovakia,* pages 176–188. Institute of Economics, Czechoslovak Academy of Sciences, Prague, 1991.

[398] E.L. Ulungu and J. Teghem. Multi-objective combinatorial optimization problems: A survey. *Journal of Multi-Criteria Decision Analysis,* 3:83–104, 1994.

[399] E.L. Ulungu and J. Teghem. The two-phases method: An efficient procedure to solve bi-objective combinatorial optimization problems. *Foundations of Computing and Decision Sciences,* 20(2): 149–165, 1994.

[400] E.L. Ulungu and J. Teghem. Solving multi-objective knapsack problem by a branch-and-bound procedure. In J. Climaco, editor, *Multicriteria Analysis,* pages 269–278. Springer Verlag, Berlin, 1997.

[401] E.L. Ulungu, J. Teghem, and P. Fortemps. Heuristics for multi-objective combinatorial optimisation problem by simulated annealing. In Q. Wei J. Gu, G. Chen and S. Wang, editors, *MCDM: Theory and Applications,* pages 228–238. SCI-TECH Information Services, Windsor, UK, 1995.

[402] E.L. Ulungu, J. Teghem, P Fortemps, and D. Tuyttens. MOSA method: A tool for solving multi-objective combinatorial optimization problems. *Journal of Multi-Criteria Decision Analysis,* 8(4):221–236, 1999.

[403] E.L. Ulungu, J. Teghem, and C. Ost. Efficiency of interactive multi-objective simulated annealing through a case study. *Journal of the Operational Research Society,* 49:1044–1050, 1998.

[404] T.L. Urban. A multiple criteria model for the facilities layout problem. *International Journal of Production Research,* 25:1805–1812, 1987.

[405] A. Vainshtein. Vector shortest path problem in l_p norm. In *Simulation and Optimization of Complex Structure Systems,* pages 138–144. Omsk, 1987.

[406] A. Vainshtein. Vector shortest path problem and other vector optimization problems on graphs. In *Proceedings of the 17th Yugoslavian Symposium on Operations Research, Dubrovnik 1990.* University of Belgrad, 1990.

[407] L.G. Valiant. The complexity of computing the permanent. *Theoretical Computer Science,* 8:189–201, 1979.

[408] L.G. Valiant. The complexity of enumeration and reliability problems. *SIAM Journal on Computing,* 8(3):410 –421, 1979.

[409] L.G. Valiant and V.V. Vazirani. NP is as easy as detecting unique solutions. *Theoretical Computer Science,* 47:85–93, 1986.

[410] A. Viana and J. Pinho de Sousa. Using metaheuristics in multiobjective ressource constrained project scheduling. *European Journal of Operational Research,* 120(2):359–374, 2000.

[411] P. Vincke. Problèmes multicritères. *Cahiers du Centre d'Etudes de Recherche Operationelle,* 16:425–439, 1974.

[412] P. Vincke. Problèmes de localisation multicritères. *Cahiers du Centre d'Etudes de Recherche Operationelle,* 25:333–338, 1983.

[413] M. Visée, J. Teghem, M. Pirlot, and E.L. Ulungu. Two-phases method and branch and bound procedures to solve the bi-obective knapsack problem. *Journal of Global Optimization,* 12:139–155, 1998.

[414] A. Warburton. Worst case analysis of greedy and related heuristics for some min-max combinatorial optimization problems. *Mathematical Programming,* 33:234–241, 1985.

[415] A. Warburton. Aproximation of Pareto optima in multiple-objective shortest-path problems. *Operations Research,* 35(1):70 –79, 1987.

[416] G. Wäscher. Innerbetriebliche Standortplanung. *Schmalenbach's Zeitschrift für betriebswirtscahftliche Forschung,* 36:930–958, 1984.

[417] L. Van Wassenhove and L. Gelders. Solving a bicriterion scheduling problem. *European Journal of Operational Research,* 4:42–48, 1980.

[418] C.C. White, B.S. Stewart, and R.L. Carraway. Multiobjective, preference-based search in acyclic OR-graphs. *European Journal of Operational Research,* 56:357–363, 1992.

[419] D.J. White. A special multi-objective assignment problem. *Journal of the Operational Research Society,* 35(8):759–767, 1984.

[420] D.J. White. The set of efficient solutions for multiple objective shortest path problems. *Computers and Operations Research,* 9(2):101–107, 1987.

[421] D.J. White. A bibliography on the application of mathematical programming multiple-objective methods. *Journal of the Operational Research Society,* 41(8):669–691, 1990.

[422] A.B. Wijeratne, M.A. Turnquist, and P.B. Mirchandani. Multiobjective routing of hazardous materials in stochastic networks. *European Journal of Operational Research,* 65:33–43, 1993.

[423] X.Q. Yang and C.J. Goh. A method for convex curve approximation. *European Journal of Operational Research,* 97:205–212, 1997.

[424] P.L. Yu. *Multiple Criteria Decision Making: Concepts, Techniques and Extensions,* Plenum Press, New York, NY, 1985.

[425] S. Zanakis. A staff to job assignment (partitioning) problem with multiple objectives. *Computers and Operations Research,* 10(4):357–363, 1983.

[426] G. Zhou and M. Gen. Genetic algorithm approach on multi-criteria minimum spanning tree problem. *European Journal of Operational Research,* 114:141–152, 1999.

[427] U. Zimmermann. Matroid intersection problems with generalized objectives. In A. Prékopa, editor, *Survey of Mathematical Programming: Proceedings of the 9th International Mathematical Programming Symposium Budapest 1976,* volume 2, pages 383–392. North-Holland, Amsterdam, 1980.

[428] U. Zimmermann. *Linear and Combinatorial Optimization in Ordered Algebraic Structures,* volume 10 of *Annals of Discrete Mathematics.* North-Holland, Amsterdam, 1981.

[429] S. Zionts. A survey of multiple criteria integer programming methods. *Annals of Discrete Mathematics,* 5:389–398, 1979.

[430] E. Zitzler. *Evolutionary Algorithms for Multiobjective Optimization: Methods and Applications.* PhD thesis, Swiss Federal Institute of Technology (ETH), Zürich, Switzerland, 1999.

[431] E. Zitzler and L. Thiele. An evolutionary algorithm for multiobjective optimization: The strength Pareto approach. Technical Report 43, Computer Engineering and Communication Networks Lab (TIK), Swiss Federal Institute of Technology (ETH), Zürich, Switzerland, 1998.

[432] E. Zitzler and L. Thiele. Multiobjective evolutionary algorithms: A comparative case study and the strength Pareto approach. *IEEE Transactions on Evolutionary Computation,* 3(4):257–271, 1999.

Chapter 9

MULTICRITERIA SCHEDULING PROBLEMS

Vincent T'Kindt, Jean-Charles Billaut
Laboratoire d'Informatique
Ecole d'Ingénieurs en Informatique pour l'Industrie
64 avenue Jean Portalis, 37200 TOURS
France
{tkindt, billaut}@univ-tours.fr

Abstract We examine the problem of scheduling jobs on machines to minimize multiple conflictuous criteria. This problem is considered in the context of Multicriteria Optimization Theory. After having introduced the foundations, we define Multicriteria Scheduling Problems (MSP) and we present a framework for solving them. We review the literature on MSP, providing both complexity results and comments on recent advances.

Keywords: Scheduling, Multiple criteria, Complexity, Typology.

Introduction

Scheduling Theory appears in the mid 1950s. So far, problems increasingly become complex mainly due to underlying industrial applications. Considered workshop configurations are more and more tight to real ones since a lot of problems deal with pools of machines, with multipurpose machines, with multiprocessor tasks, etc. In the same way, the considered constraints become closer to real situations where orders have different release dates, due dates, and so on.

Unfortunately, most of the problems dealt with in the scheduling literature involve only one criterion to measure the quality of solutions. Nevertheless, along the planning levels different criteria may be considered. At a strategic level, at which the long term planning is determined, the goals are mainly to minimize costs related to material, financial and hu-

man investments. At a tactical level, a mid-term planning is established minimizing costs related to goods storage, supply chain and production system. At a scheduling level or short-term planning level, production costs, work-in-process costs as well as delivery delays have to be minimized. Even if other reasons can lead to considering a Multicriteria Scheduling Problem (MSP), it is clear enough that such a problem can have numerous practical issues.

Hence, taking multiple criteria into account allows to compute a more realistic solution for the decision maker [91]. This is emphasized while solving a practical scheduling problem. Multicriteria problems have been extensively studied in the literature, whatever the interest field. Syntheses on MSP already exist [30, 40, 56, 78] but they are mostly restricted to single machine problems. Moreover, none consider MSP within the context of multicriteria analysis although numerous works have been devoted to MultiCriteria Decision Aid/Making (MCDA/M). The purpose of this chapter is to fill that gap and to review recent advances in the last decade on MSP.

The remainder is organized as follows. Firstly, we introduce in Section 1 the basics of Scheduling Theory and in Section 2 that of Multicriteria Optimization Theory. Afterwards, a framework for solving MSP is presented in Section 3. Section 4 contains a synthesis of complexity results. The rest of the chapter is dedicated to the review of MSP according to the workshop configuration. Section 5 is devoted to single machine problems, Section 6 to parallel machines problems and Section 7 to shop scheduling problems.

1. Scheduling Theory

Numerous books present a synthesis of results concerning scheduling problems. We can cite for instance [11, 14, 86]. Several definitions of a scheduling problem are presented in the literature:

Scheduling is the allocation of limited resources to tasks over time. It is a decision-making process which involves the optimization of one or several objectives. [86]

We assume that jobs have to be processed on resources. Each job is composed of one or several operations, that can be processed in a given order. The resources are either renewable (like machines, men, files, etc.) or non-renewable (like money, raw materials, etc.). Renewable resources can be disjunctive (one operation is performed at a time) or cumulative (limited amounts of operations can be performed simultaneously). Sometimes, similar resources are gathered into stages which

leads to consider an assignment problem in addition to the scheduling problem.

1.1. Some Application Fields

Lee, Lei and Pinedo [69] present recent advances in scheduling theory and in scheduling practice.

Undoubtedly, numerous scheduling problems come from the production field and we generally consider that resources are machines. In Flexible Manufacturing Systems, machines have finite capacity input and output buffers, and transporters are needed to handle operations from one machine to another. Recent works related to this field are robotic cell scheduling and scheduling of Automated Guided Vehicles (AGV). Similarly, the *hoist scheduling problem,* that involves cyclic scheduling problems of hoists subject to time-window constraints, has its own characteristics. Besides, it has numerous practical issues as for instance in electroplating and chemical industries, printed circuit boards and industrial connectors production.

Jobs can also be considered as tasks and resources as processors. In this case, we meet different problems depending on the number of available processors. Tasks are often subject to deadlines and the problem of simply finding a feasible schedule is a hard one.

Many other applications like time-tabling problems, project scheduling under resource constraints, scheduling with batching, with lot-sizing, etc., can be dealt with.

Once we have a scheduling problem issued from a practical situation, it may be not easy to know how to solve it. First, we have to know if the problem - or a neighbourhood problem - has already been solved and if not, how to solve it. In order to refer to any scheduling problem, the notation introduced by Graham, Lawler, Lenstra and Rinnooy Kan [47] is commonly used. It consists in three fields $\alpha \mid \beta \mid \gamma$, where α refers to the configuration of the resources, β to the constraints, and γ to the optimized criterion.

1.2. Resource Environments (α)

The resource environment of a scheduling problem is described in the α-field of the problem notation. This field can be decomposed into $\alpha_1\alpha_2$ with α_1 the type of resource environment and α_2 the number of resources if it is known, $\alpha_2 = k$ if the number of resources is not known but is fixed, and $\alpha_2 = \emptyset$ if the number of resources is arbitrary. Several environments (noted in field α_1) are possible depending on whether there is an assignment problem or not.

1.2.1 Scheduling Problems.

- Single machine problems ($\alpha_1 = \emptyset$). There is only one resource and each job to perform is made up of only one operation. We say that jobs are mono-operation.
- Flowshop problems ($\alpha_1 = F$). The workshop is composed of m resources and each job is made up of m operations (multi-operation). The jobs have the same routing, i.e. they visit the resources of the workshop in the same order.
- Jobshop problems ($\alpha_1 = J$). The workshop is composed of m resources and each job is made up of m operations. The jobs have their own routing in the workshop.
- Openshop problems ($\alpha_1 = O$). The workshop is composed of m resources, each job is made up of m operations. The routing of the jobs is not fixed.
- Mixed shop problems ($\alpha_1 = X$). These problems are a combination of jobshop and openshop problems.

1.2.2 Scheduling and Assignment Problems. We assume that similar resources are gathered into stages. Accordingly, we assume that resources belonging to the same stage are able to process the same operations. For a given stage, different configurations are possible:

- resources are identical parallel machines (P): the operation processing time does not depend on the performing resource;
- resources are uniform parallel machines (Q): the operation processing time depends on the speed of the performing resource;
- resources are unrelated parallel machines (R): resources have different speeds that depend on the operations. Hence, the operation processing time depends on the performing resource.

Besides, different scheduling and assignment problems can be dealt with depending on the following resources configurations:

- Parallel machines ($\alpha_1 = P$ or Q or R). There is only one type of resources, that are gathered in one stage and jobs are mono-operation.
- Hybrid Flowshop ($\alpha_1 = HF$). The workshop is composed of k stages and each job is made up of k operations. The jobs have the same routing through the stages.

- Generalized Jobshop and Openshop ($\alpha_1 = GJ$ or GO). The workshop is composed of k stages. The jobs visit the stages according to their own routing or in a non specified order, respectively.

1.3. Constraints and Notations (β)

We note $J = \{J_1, J_2, ..., J_n\}$ the set of n jobs to schedule and m the number of resources. Operation j of job J_i is noted $O_{i,j}$ and the number of operations of job J_i is noted n_i (generally, $n_i = m$, $\forall i, i = 1, ..., n$).

The characteristics of the problem are specified in the β-field of the notation. However, some implicit constraints are not mentioned in this field. For example, we consider that resources are disjunctive, hence it is not possible to schedule more than one job on a resource at a time. When dealing with project scheduling problems, a particular notation is proposed by Herroelen, Demeulemeester and De Reyck in [55]. Furthermore, if job J_i is made up of several operations, operation $O_{i,j}$ can not be scheduled before the completion of operation $O_{i,j-1}$, $\forall j, j = 2, ..., n_i$.

Constraints related to the completion of jobs are of two types: due date or deadline. A due date d_i is a date before which job J_i has to be completed. However, the job can be late if needed. A deadline $\widetilde{d_i}$ is a due date for which tardiness is not allowed. Sometimes, such a problem may have no solution. Conversely, we define a release date r_i as a date before which it is not possible to start the processing of job J_i.

Numerous characteristics of jobs can be included in the field β. The most common are the following:

- *pmtn* indicates that preemption is allowed, e.g. it is possible to interrupt the processing of a job and to resume it later, even on another machine.

- *prec* indicates precedence relations between jobs. Usually, precedence relations are represented by a graph G. If job J_i precedes job J_k, it means that operation $O_{k,1}$ cannot start before the completion of operation O_{i,n_i}. If G is a particular graph (a tree, a chain, etc.), *prec* is replaced by *tree, chain,* etc.,

- *no − wait* is used for the multi-operation case. For each job J_i, $i = 1, ..., n$, it means that the start time of operation $O_{i,j}$ is equal to the completion time of operation $O_{i,j-1}$, for $j = 2, ..., n_i$.

- *nmit* stands for "no machine idle-time". This constraint imposes that when processing begins of the first job on a resource, processing continues uninterrupted until all jobs are completed on that resource.

- In some scheduling applications, setup times and removal times are not negligible and have to be considered. These durations depend on the job sequence (color transitions for example), or not. We refer to S_{nsd} and R_{nsd} as the non sequence dependent setup times and removal times, respectively. S_{sd} and R_{sd} refer to the sequence dependent setup times and removal times, respectively.

- Sometimes all the jobs share some characteristics as for instance a common due date, noted $d_i = d$, or a common job processing time noted $p_i = p$. If there is a common due date and if the due date has to be determined, the constraint is noted "$d_i = d$ *unknown*".

More details can be found in [11] for the constraints description.

1.4. Criteria (γ)

We note C_i the completion time of job J_i. For each job J_i, we define the following functions:

- $L_i = C_i - d_i$, the lateness of job J_i,
- $T_i = \max(0, C_i - d_i)$, the tardiness of job J_i,
- $E_i = \max(0, d_i - C_i)$, the earliness of job J_i,
- $U_i = 0$ if $C_i \leq d_i$, $U_i = 1$ otherwise, the unit penalty associated with a tardy job J_i.

Two types of objective function are used to evaluate a schedule: the "maximum" objective function and the "sum" objective function. The most common criteria considered in the scheduling literature are the following:

- $C_{max} = \max_{1 \leq i \leq n} C_i$ is the makespan, $L_{max} = \max_{1 \leq i \leq n} L_i$ is the maximum lateness, and $T_{max} = \max_{1 \leq i \leq n} T_i$ is the maximum tardiness,
- $\overline{C} = \sum_{i=1}^{n} C_i$ is the sum of job completion times or the total completion time, $\overline{T} = \sum_{i=1}^{n} T_i$ is the total tardiness, and $\overline{U} = \sum_{i=1}^{n} U_i$ is the total number of tardy jobs. These criteria can also be defined in a weighted form, if a weight w_i is associated with each job J_i. For example, we have $\overline{C}^w = \sum_{i=1}^{n} w_i C_i$, and $\overline{T}^w$ and $\overline{U}^w$ in the same way.

Each of these criteria is a function of the set of jobs completion times. In their general form, a criterion Z can be noted $Z(C_1, C_2, ..., C_n)$.

Definition 1 *We say that a criterion Z is a* regular criterion *if for any schedule S and S': $C_i(S) \leq C_i(S')$, $\forall i, i = 1, ..., n$, implies that $Z(C_1(S), C_2(S), ..., C_n(S)) \leq Z(C_1(S'), C_2(S'), ..., C_n(S'))$, with $C_i(S)$ and $C_i(S')$ the completion time of job J_i in S and in S', respectively.*

Definition 2 *We say that a schedule S is an* active schedule, *when it is not possible to find a schedule S' such that $C_i(S') \leq C_i(S)$, $\forall i, i = 1, ..., n$, with at least one strict inequality.*

For any regular criterion, the set of active schedules dominates the set of schedules, e.g. there always exists an optimal schedule which is an active schedule.

2. Overview of Multicriteria Optimization Theory

Multicriteria Optimization Theory (MOT) has been extensively studied in the literature. Generally, few hypotheses on the set of feasible solutions are assumed and the aim is to provide general results for optimizing criteria [32]. MSP are particular multicriteria optimization problems for which specific assumptions are made on that set, which emphasizes the need for considering MSP within the context of MOT.

Minimizing several conflicting criteria changes the way scheduling problems are dealt with. In most cases, one solution that optimally minimizes all the considered criteria does not exist. It implies that a new definition of optimality must be considered, namely Pareto optimality. In fact, two main definitions are encountered in multicriteria optimization literature: the definitions of Pareto optimality and of weak Pareto optimality.

Definition 3 *Let us consider K conflicting criteria Z_i to be minimized. $\mathcal{S}$ is the set of solutions and $\mathcal{Z}$ its image in the criteria space, $x \in \mathcal{S}$ is a Pareto optimum (or an efficient solution) iff $\nexists\, y \in \mathcal{S}$, $x \neq y$, such that $Z_i(y) \leq Z_i(x)$, $\forall i, i = 1, ..., K$, with at least one strict inequality. E is the set of all Pareto optima and Z_E its image in the criteria space (i.e. the set of non dominated criteria vectors).*

Definition 4 *Let us consider K conflicting criteria Z_i to be minimized. $\mathcal{S}$ is the set of solutions and $\mathcal{Z}$ its image in the criteria space. $x \in \mathcal{S}$ is a weak Pareto optimum (or a weak efficient solution) iff $\nexists\, y \in \mathcal{S}$, $x \neq y$, such that $Z_i(y) < Z_i(x)$, $\forall i, i = 1, ..., K$. WE is the set of all weak Pareto optima and Z_{WE} its image in the criteria space (i.e. the set of weakly non dominated criteria vectors). One has $E \subseteq WE$.*

A subset of set E is sometimes considered in the literature. It is called the set of proper Pareto optima and is noted *PRE* [46]. In the view of MSP, considering these solutions is useless since we mainly deal with problems with a finite number of linear constraints, for which we have $PRE = E$.

The decision maker is only interested in the Pareto optima, but unfortunately results available mostly allow computation of a subset (or the entire set) of WE. Roughly speaking, the computation of one Pareto optimum is done by aggregating the criteria into a new objective function, and hence by introducing new parameters like weights, goals or bounds. The optimal solutions of this new "single criterion" problem are Pareto optima. We briefly review, in the remainder of this section, methods to compute Pareto optima.

The method which is undoubtedly the most well known, involves the minimization of a convex combination of criteria. The impact in terms of computed Pareto optima depends on wether we have zero weights or convexity assumptions. The basic and more general result is due to Geoffrion [46]. It states a necessary and sufficient condition to compute proper Pareto optima, once we have a convex set $\mathcal{S}$, convex criteria Z_i and strictly positive weights. If the convexity hypothesis does not hold, some proper Pareto optima can not be computed by minimizing a convex combination of criteria, whatever the considered strictly positive weights. Such solutions are called *non supported proper Pareto optima,* whilst the ones that are minima for a convex combination of criteria are called *supported.* In the same way, conditions for computation of weak Pareto optima can be stated by considering that some, but not all, weights can be zero.

One of the most powerful methods is referred to as the parametric approach. A basic result [98] states a necessary and sufficient condition for computing Pareto optima, once a strictly increasing real valued function g over the K criteria has been defined. Consider the minimization problem of function g with the additional constraints $Z_i \leq b_i, \forall i, i = 1, ..., K$, where b_i are bounds. Then $x \in E$ if and only if it exists a bounds vector such that x is an optimal solution of the corresponding minimization problem.

Another classical method in MOT which is one of the most used for solving MSP is called the ϵ-constraint approach. We consider the minimization problem (P_ϵ^k) of a criterion, namely Z_k, whilst the others are subject to bound constraints, i.e. $Z_i \leq \epsilon_i, \forall i, i = 1, ..., K, i \neq k$. For a given criterion Z_k and a fixed bounds vector, the optimal solutions of the corresponding problem (P_ϵ^k) are weak Pareto optima. Unfortunately, some weak Pareto optima can not be computed by solving all the

possible problems (P_ϵ^k), Z_k being fixed or not. One result states that if the same criterion Z_k is always minimized, the set of optimal solutions of the possible problems (P_ϵ^k) is a subset of WE that contains E even if no convexity hypothesis is assumed.

Other classical methods are related to the introduction of goals. By considering a function (usually a metric) measuring the distance of a solution to the goals, we can derive results for the computation of Pareto optima. Moreover, goals are data that can be understood quite easily by a decision maker. The most used distance functions belong either to the Tchebycheff metric family, or to the L_p^λ metric family. We refer to [32] for more details on these methods.

In some situations no tradeoff is allowed between the criteria. It means that decreasing the value of a criterion is not allowed if it leads to increase a more important one. For these problems, no information is required from the decision maker once the order between the criteria is defined. We refer to this approach as the lexicographical, or hierarchical, approach. Let the criteria order be that of the indices, i.e. $Z_1 \to Z_2 \to \ldots \to Z_K$, $\mathcal{S}^0 = \mathcal{S}$ and $\mathcal{S}^i = \{x \in \mathcal{S}^{i-1} / Z_i(x) = \min_{y \in \mathcal{S}^{i-1}}(Z_i(y))\}$, $\forall i, i = 1, \ldots, K$. The lexicographical minimization problem is equivalent to find a solution $x^* \in \mathcal{S}^K$. Solution x^* is a Pareto optimum and all solutions belonging to a set $\mathcal{S}^i$ are weak Pareto optima.

The choice of a method for computing one Pareto optimum mainly depends on the information available from the decision maker. Nevertheless, it is not sufficient to solve the defined multicriteria optimization problem since it remains to choose when the information is given by the decision maker. As a matter of fact, he has to decide, among the set of Pareto optima, the one he prefers since none can be established as the ideal solution. The moments at which the decision maker can give this information lead to distinguish three classes of resolution contexts [36]. Consider that a method for computing Pareto optima has been obtained. If the values of the resulting parameters can be set with certainty by the decision maker, we fall in the class of a *priori* resolution contexts. If the decision maker has a partial idea of what are these values, he may want to interactively change them and test several Pareto optima before retaining one. In this case, we are in the class of *interactive* resolution contexts. The last case occurs when the decision maker has no idea of how to set the parameter values. It leads to the class of *a posteriori* resolution contexts, where the whole set of Pareto optima is computed and presented to the decision maker.

Besides the choice of a method for computing Pareto optima and of a resolution context, the algorithm designed to solve the multicriteria optimization problem can always be separated, from a practical point of

view, into two procedures. The first one, called the *taking criteria into consideration procedure,* aims to do the interaction with the decision maker following the chosen resolution context. Each time parameter values are fixed, a *solution procedure* is run with these values as input arguments. It returns to the former procedure a Pareto optimum. For MSP, the solving procedure is called *the scheduling procedure.*

For instance, consider the case where one Pareto optimum is computed by minimizing a convex combination of criteria in an interactive resolution context. First, initial weight values are set by the taking criteria into consideration procedure and the solution procedure is run. The returned Pareto optimum is shown to the decision maker who can change the weight values according to his preferences. With these new values, the execution of the solving procedure is iterated. This interactive process can be performed until the decision maker decides to stop.

3. Solving Multicriteria Scheduling Problems

3.1. A Definition of Multicriteria Scheduling Problems

Despite the fact that MSP are more and more studied in the last decade, no formal definition has been introduced. Thus, we may be a bit confused when dealing with some MSP. For instance, are problems involving a lexicographical order of criteria, MSP? Can we consider that the problem of minimizing a convex combination of criteria is an MSP, since only one objective function is involved? This kind of questions highlights the need for a definition of MSP.

Considering MCDA/M and MOT, it is clear that we are faced with an MSP when the problem definition exhibits multiple conflicting criteria. The notion of criterion is only related to a measure used by the decision maker to evaluate a solution and hence, to take a decision, without considering the relative importance of that criterion against the other ones. Together with the foundations seen in Section 2, it leads the following definition, illustrated in Figure 9.1.

Definition 5 *A Multicriteria Scheduling Problem consists in computing a Pareto optimal schedule for several conflicting criteria. This problem can be decomposed into three sub-problems:*

(a) ***A* modelling problem**, *whose resolution leads to define the MSP, i.e. the workshop configuration, the constraints and the optimized criteria.*

(b) ***A* taking criteria into consideration problem**, *whose resolution leads to choose a resolution context and a method to compute*

a Pareto optimum. We therefore provide a taking criteria into consideration procedure dedicated to interact with the decision maker.

(c) ***A* scheduling problem**, *whose resolution leads to find a schedule that optimizes the objective function defined at the previous step. We therefore provide a scheduling procedure dedicated to solve the problem once all additional parameters are fixed (see Section 2).*

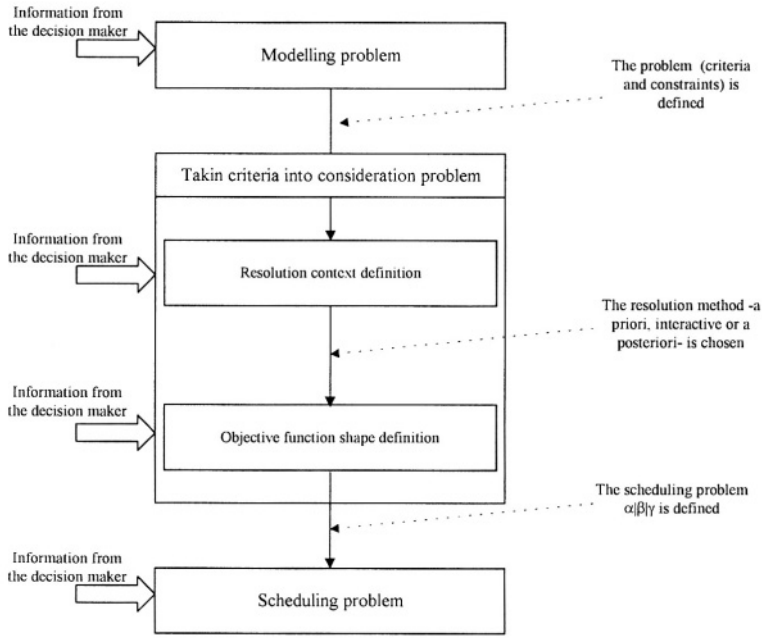


Figure 9.1. A framework for solving Multicriteria Scheduling Problems.

The **modelling problem** [91] is done according to the decision maker and firstly consists in choosing the relevant criteria. We assume they are conflicting, which means that the minimization of one criterion does not imply the minimization of others. Secondly, we exhibit the machine environment in which the MSP occurs, i.e. the resources (machines, men, etc.) available for processing of jobs and their organization. Thirdly, we identify the particular constraints related to the problem (see Section 1): job preemption, release dates, precedence constraints between jobs, etc.

Undoubtedly, the **taking criteria into consideration problem** does not exist in single criterion scheduling problems. For solving this

problem we need to get the decision maker's preferences, that are the resolution context and the method used to compute one Pareto optimum. Among all the possible choices, we usually choose the most relevant according to the information the decision maker can give us. If no tradeoff between criteria is allowed, he has to provide their optimization order. Otherwise, to express the tradeoffs that are allowed he may give some weights, goals to be reached or bounds not to be exceeded. These parameters are set according to the resolution context. If the decision maker is able to define the exact parameter values, an *a priori* resolution can be used. If he has an idea of these values but wants to try different ones, he may prefer an *interactive* resolution. At last, an *a posteriori* resolution is chosen if the decision maker has no idea of the value of the parameters. Therefore, he may want to have all the Pareto optima to choose the one he prefers. For instance, if he can give a weight for each criterion but he is not sure of their exact value, we may choose to compute a Pareto optimum by minimizing a convex combination of criteria in an interactive resolution context.

Once the resolution of the previous problem has been completed, we have to provide an algorithm that solves the scheduling problem defined when the parameter values are known. The **scheduling problem** can be described using the three-field notation presented in Section 1. The optimal solutions for the problem at hand are Pareto optima for the related MSP.

3.2. Extension of the Three-Field Notation for Multicriteria Scheduling Problems

The three-field notation mainly applies to single criterion problems, although some survey papers on MSP provide extensions [18, 30]. The framework presented in Section 3.1 highlights numerous possibilities for the criteria field of the notation, depending on the decisions taken while solving the taking criteria into consideration problem. Thus, the extensions quoted above may be quite restrictive and need to be revised.

We distinguish two levels and consequently, two possible notations for MSP. As mentioned in Definition 5, an MSP can be decomposed into three sub-problems. Once the modelling problem has been solved the MSP considered is well defined. Accordingly, it can be classified using the $\alpha \mid \beta \mid \gamma$ notation presented in Section 1 with the list of criteria in the γ-field separated by commas. For instance, a 2-machine flowshop with the total tardiness and the total earliness criteria, can be noted: $F2 \mid\mid \overline{E}, \overline{T}$.

Once the taking criteria into consideration problem has been solved, a scheduling problem, whose optimal solutions are Pareto optima, is defined. It can be useful to refer to this problem since the scheduling procedure designed solves it. Depending on the decisions previously taken and so on the method used to compute one Pareto optimum, different values for the γ field can be introduced in a second-level notation. We summarize them below:

- Z, if the aim is to minimize one single criterion. As usual, Z stands for C_{max}, T_{max}, etc.
- $F_\ell(Z_1, ..., Z_K)$ refers to the minimization of a convex combination of the K criteria.
- $P(Z_1, ..., Z_K)$ refers to the minimization of a non decreasing function over the K criteria, subject to upper bound constraints. This is the parametric analysis approach.
- $\epsilon(Z_u/Z_1, ..., Z_{u-1}, Z_{u+1}, Z_K)$ refers to an ϵ-constraint problem with one criterion Z_u being minimized subject to upper bound constraints for the others.
- $F_T(Z_1, ..., Z_K)$, $F_{Tp}(Z_1, ..., Z_K)$ and $F_{Tpa}(Z_1, ..., Z_K)$ refer to the problems of minimizing, respectively, a Tchebycheff metric, weighted Tchebycheff metric and weighted augmented Tchebycheff metric. For these problems the decision maker introduces goals.
- $F_s(Z_1, ..., Z_K)$ refers to the maximization of a particular distance function. It belongs to the goal-attainment approach [112].
- $Lex(Z_1, ..., Z_K)$ refers to the lexicographical minimization of the K criteria. The order considered is specified between the parenthesis. With this problem, no tradeoff between the criteria is allowed.
- $GP(Z_1, ..., Z_K)$ refers to the *Goal Programming* approach where the aim is not to minimize criteria but to find a solution satisfying defined goals.
- $\#(Z_1, ..., Z_K)$ refers to the problem of enumerating all the Pareto optima without using a special objective function. This notation is always related to an a posteriori resolution context where the provided algorithm proceeds by enumerating all the solutions in order to retain the Pareto optima.

For instance, it is possible to solve the $F2 \mid\mid \overline{E}, \overline{T}$ multicriteria scheduling problem, by iteratively solving either the $F2 \mid\mid F_\ell(\overline{E}, \overline{T})$ problem, or

the $F2 \mid\mid \epsilon(\overline{E}/\overline{T})$ problem, etc. In the rest of this chapter MSP will be mostly referred using the second level notation, i.e. the one of the scheduling problem involved.

4. Complexity Results

Complexity of MSP has been considered in surveys [18, 19, 56, 70, 78, 102], mostly related to single machine problems. In this section, new complexity results are investigated, depending on the kind of the objective function and depending on the criteria. Reduction trees are proposed to summarize the results and later used to deduce the complexity of some scheduling problems.

We note $A \propto B$ if problem A "Turing reduces" to problem B [44] and we note O_i the optimization problem associated to criterion Z_i.

4.1. Results Depending on the Objective Function

The MSP definition introduced in Section 3, highlights that for a given MSP scheduling problems with different objective functions can be solved. Hence, it might be interesting to tackle the links in terms of complexity between these problems. Both existing and new results are provided. Proofs are only given for new results. We denote O_{1s} as the problem of computing only one Pareto optimal solution.

Consider the problem of minimizing a lexicographical order of criteria, noted O_{Lex} and defined as follows.

O_{Lex}: Data: Let S be the set of feasible solutions.

Problem: Find a solution $x_0 \in \mathcal{S}$, such that $\forall i, i = 1, ..., K$, $Z_i(x_0) = \min_{x \in \mathcal{S}^{i-1}} Z_i(x)$ with $\mathcal{S}^i = \{x \in \mathcal{S}^{i-1} : Z_i(x) = \min_{y \in \mathcal{S}^{i-1}} Z_i(y)\}$ and $S^0 = S$.

Proposition 1 *We have (1.a)* $O_1 \propto O_{Lex}$ *[18] and (1.b)* $O_{1s} \propto O_{Lex}$.

Proof: We prove the second reduction *(1.b)*. The solution for problem O_{Lex} is a Pareto optimum, which is also a solution for problem O_{1s}. □

Consider the problem O_ϵ defined as follows.

O_ϵ: Data: Let $\mathcal{S}$ be the set of feasible solutions, Z_1 the first criterion and $K-1$ values ϵ_j, $\forall j, j = 2, ..., K$.

Problem: Find a solution x_0 in $\mathcal{S}$, such that $\forall j = 2, ..., K$, $Z_j(x_0) \leq \epsilon_j$ and $Z_1(x_0) = \min_{x' \in \mathcal{S}} Z_1(x')$.

Proposition 2 *We have (2.a)* $O_1 \propto O_\epsilon$ *and (2.b) if* O_1 *is polynomially solvable and* $K = 2$, *then* $O_{Lex} \propto O_\epsilon$.

Proof: *(2.a)* Consider an algorithm A which solves problem O_i by solving O_ϵ with $\epsilon_j = M$, where M is a sufficiently big number.

(2.b) Consider an algorithm A which solves problem O_{Lex} by solving $O_\epsilon(Z_2/Z_1)$ with $\epsilon_1 = Z_1^*$, the optimal value of criterion Z_1. □

Consider problem O_ℓ defined as follows.

O_ℓ: Data: Let $\mathcal{S}$ be the set of feasible solutions, $\alpha \in [0,1]^K$ and $\sum_{i=1}^K \alpha_i = 1$.

Problem: Find a solution x_0 in $\mathcal{S}$, such that:

$$\sum_{i=1}^{K} \alpha_i Z_i(x_0) = \min_{y \in \mathcal{S}} \sum_{i=1}^{K} \alpha_i Z_i(y).$$

Proposition 3 *We have (3.a)* $O_{Lex} \propto O_\ell$ *and (3,b)* $\forall i, i = 1, ..., K$, $O_i \propto O_\ell$.

Proof: *(3. a)* With the proper choice of weights, problem O_ℓ can be used to generate a solution to the lexicographical problem [32]. Therefore, a polynomial time algorithm for O_ℓ can give a solution to the O_{Lex} problem, which proves the result.

(3.b) Consider an algorithm A which solves problem O_i, by solving problem O_ℓ with $\alpha_i = 1$ and $\alpha_j = 0$ $\forall j, j = 1, ..., K$, $j \neq i$. □

4.2. Application to Scheduling Problems

4.2.1 Existing Results. In multicriteria scheduling literature, authors generally consider that criteria are either gathered in a linear combination (problem O_ℓ), considered in a lexicographical order (problem O_{Lex}), or subject to bounds (problem O_ϵ). Sometimes, the general multicriteria problem, related to the enumeration of the set of Pareto optima (problem $O_\#$) is tackled.

Chen and Bulfin [18, 19] provide complexity results for some bicriteria scheduling problems on a single machine and on traditional multiple machines scheduling problems including parallel machines, flowshop, jobshop or openshop. The criteria they consider belong to $\Gamma = \{C_{max}, T_{max}, \overline{C}, \overline{C}^w, \overline{U}, \overline{U}^w, \overline{T}, \overline{T}^w\}$.

Chen and Bulfin consider problems O_{Lex}, $O_\#$ and O_ℓ, and show that $O_1 \propto O_{Lex} \propto O_\#$. Besides, most of the scheduling problems are shown to be $\mathcal{NP}$-hard.

A simple reduction tree concerning scheduling criteria is well known (Figure 9.2). It is possible to extend this tree to bicriteria lexicographical problems.

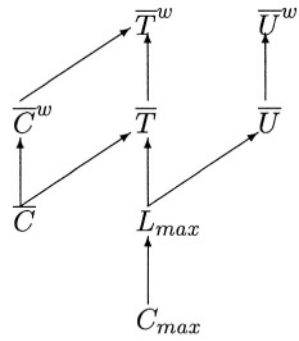


Figure 9.2. Reduction tree for criteria in scheduling [14, 86].

4.2.2 Bicriteria Lexicographical Scheduling Problems.

Proposition 4 *For all* $Z_1 \in \{C_{max}, L_{max}, T_{max}, \overline{C}, \overline{U}, \overline{T}, \overline{C}^{w'}, \overline{U}^{w'}, \overline{T}^{w'}\}$, *for all* $X \in \{C, U, T\}$, *it exists a polynomial Turing reduction such that* $\alpha \mid \beta \mid Lex(Z_1, \overline{X}) \propto \alpha \mid \beta \mid Lex(Z_1, \overline{X}^w)$.

Proof: The algorithm which solves the $\alpha \mid \beta \mid Lex(Z_1, \overline{X}^w)$ problem can be used to solve the corresponding $\alpha \mid \beta \mid Lex(Z_1, \overline{X})$ problem with all the weights w_i equal to 1. The returned solution is optimal for criterion Z_1 and minimizes $\overline{X}$. We obtain the reduction trees presented in Figure 9.3. □

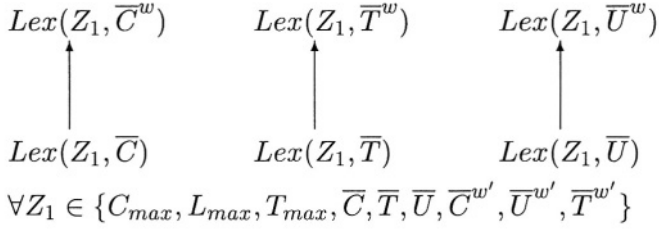


Figure 9.3. Reductions for lexicographical minimization (1).

Proposition 5 *For all* $Z_1 \in \{C_{max}, \overline{C}, \overline{C}^w\}$ *it exists a polynomial Turing reduction such that*

(5.a) $\alpha \mid \beta \mid Lex(Z_1, \overline{C}) \propto \alpha \mid \beta \mid Lex(Z_1, \overline{T})$,
(5.b) $\alpha \mid \beta \mid Lex(Z_1, \overline{C}^w) \propto \alpha \mid \beta \mid Lex(Z_1, \overline{T}^w)$ *and*
(5.c) $\alpha \mid \beta \mid Lex(Z_1, C_{max}) \propto \alpha \mid \beta \mid Lex(Z_1, L_{max})$

Proof: The reductions can be referred to as $\alpha \mid \beta \mid Lex(Z_1, Z_2) \propto \alpha \mid \beta \mid Lex(Z_1, Z_3)$. The proof we give holds for $Z_3 \in \{\overline{T}, \overline{T}^w, L_{max}\}$. The algorithm which solves the $\alpha \mid \beta \mid Lex(Z_1, Z_3)$ problem can be used to solve the $\alpha \mid \beta \mid Lex(Z_1, Z_2)$ problem with all the due dates equal to 0, because criterion Z_1 do not take the due dates into account. Hence, the returned solution is optimal with regards to criterion Z_1 and minimal for criterion Z_2. □

Proposition 6 *For all* $Z_1 \in \{C_{max}, \overline{C}, \overline{C}^w\}$ *it exists a polynomial Turing reduction such that*
(6.a) $\alpha \mid \beta \mid Lex(Z_1, L_{max}) \propto \alpha \mid \beta \mid Lex(Z_1, \overline{T})$ *and*
(6.b) $\alpha \mid \beta \mid Lex(Z_1, L_{max}) \propto \alpha \mid \beta \mid Lex(Z_1, \overline{U})$.

Proof: The proof is similar to the proof of the Turing reduction of the single criteria decision problems $L_{max} \propto \overline{T}$ and $L_{max} \propto \overline{U}$ [14]. □

The results of propositions 5 and 6, are summarized in the reduction tree presented in Figure 9.4.

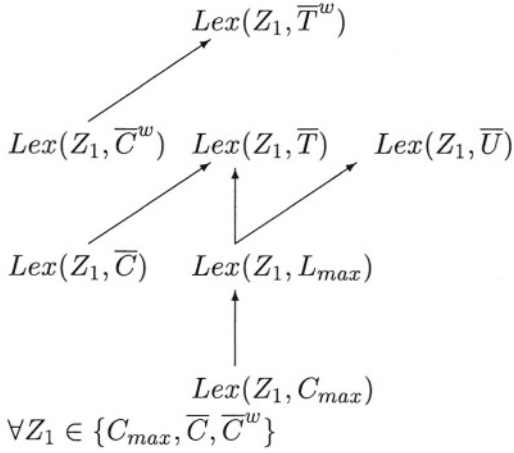


Figure 9.4. Reductions for lexicographical minimization (2).

4.3. Synthesis

We summarize in Table 9.1 the complexity of some well-known single criterion and single machine scheduling problems. We note "H" a $\mathcal{NP}$-hard problem, "H+" if it is $\mathcal{NP}$-hard in the strong sense, and "H–" if it

is $\mathcal{NP}$-hard in the ordinary sense. "P" refers to a polynomial problem and "O" to an open problem. Most of the scheduling problems $\alpha \mid\mid Z$ with $\alpha \in \{P, Q, R, F, J, O, HF,\ GJ,\ GO\}$ and $Z \in \Gamma$ are $\mathcal{NP}$-hard in the strong sense.

Table 9.1. Complexity results for problems of type $1 \mid\mid Z$.

Z	$\overline{C}^w$	$\overline{U}$	$\overline{U}^w$	$\overline{T}$	$\overline{T}^w$
	P	P	H−	H−	H+

We present in Table 9.2 some complexity results concerning single machine bicriteria scheduling problems of type O_{Lex} [18], that we extend to criteria belonging to $\Gamma \cup \{L_{max}\}$. The references given for each problem refer either to a paper ("[n]"), to a proposition in this chapter ("(n)") or to a trivial proof ("tr").

Table 9.2. Complexity results for problems of type $1 \mid\mid Lex(Z_1, Z_2)$.

Z_1	C_{max}	T_{max}	L_{max}	Z_2 $\overline{C}$	$\overline{C}^w$	$\overline{U}$
C_{max}	−	P [56]	P [56]	P [33, 56]	P tr	P tr
T_{max}	P [56],tr	−	−	P [97, 34, 56, 18](3.a)	H+ [18]	O [18]
L_{max}	P [56],tr	−	−	P [56](3.a)	H+ [16, 56]	O
$\overline{C}$	P [56],tr	P [56, 18]	P [56](6)	−	−	P [18](4)
$\overline{C}^w$	P [56],tr	P [18]	P [56](6.b)	−	−	P [18]
$\overline{U}$	P tr	O [18]	O	H [18, 34]	H+ [18]	−
$\overline{U}^w$	H− tr	H [18](1)	H (1)	H [18](1)	H+ [18](1)	−
$\overline{T}$	H− tr	H [18](1)	H (1)	H [18](1)	H+ [18](1)	H [18](1)
$\overline{T}^w$	H+ tr	H+ [18](1)	H+ (1)	H+ [18](1)	H+ [18](1)	H+ [18](1)

Notice that the $1 \mid d_i \mid Lex(\overline{C}, \overline{T}^w)$ problem is polynomially solvable, even if the $1 \mid\mid \overline{T}^w$ problem is $\mathcal{NP}$-hard in the strong sense. Remember that the problem of minimizing the first criterion only in the lexicographical order reduces to that of lexicographical minimization (Proposition 1.a).

From Table 9.2 and the literature, we produce Table 9.3, where results concerning single machine bicriteria scheduling problems of type O_ϵ are

Table 9.2 (continued)
Complexity results for problems 1 || $Lex(Z_1, Z_2)$.

Z_1 \ Z_2	$\overline{U}^w$	$\overline{T}$	$\overline{T}^w$
C_{max}	H− tr	H− tr	H+ tr
T_{max}	H [18]	H [18]	H [18]
L_{max}	O	O	O
$\overline{C}$	P [18]	P [18](4)	P [18]
$\overline{C}^w$	H [18]	H [18]	H+ [18]
$\overline{U}$	–	O [18]	H+ [18]
$\overline{U}^w$	–	H [18](1)	H+ [18](1)
$\overline{T}$	H [18](1)	–	–
$\overline{T}^w$	H+ [18](1)	–	–

presented. Similarly, Table 9.4 provides results for problems of type O_ℓ. We can note that the latter table is symmetric.

Notice that numerous bicriteria single machine scheduling problems involving criteria T_{max} and $\overline{U}$ are open.

Table 9.3. Complexity results for problems of type 1 || $\epsilon(Z_1/Z_2)$.

Z_1 \ Z_2	C_{max}	T_{max}	L_{max}	$\overline{C}$	$\overline{C}^w$	$\overline{U}$	$\overline{U}^w$	$\overline{T}$	$\overline{T}^w$
C_{max}	–	P [56]	P [56]	P [56]	O	O	O	O	O
T_{max}	P [56]	–	–	P [56]	O	O	O	O	O
L_{max}	P [56]	–	–	P [56]	H+ (6)	O	O	O	O
$\overline{C}$	O	P [56]	P [97]	–	–	H [80](2.b)	O	O	O
$\overline{C}^w$	O	H (2.b)	H+ [97](2.b)	–	–	H+ (2.b)	O	O	O
$\overline{U}$	O	O [80]	O	O	O	–	–	O	O
$\overline{U}^w$	H (2.a)	H (2.a)	H (2.a)	H (2.a)	H+ (2.a)	–	–	H (2.a)	H+ (2.a)
$\overline{T}$	H (2.a)	H (2.a)	H (2.a)	H (2.a)	H+ (2.a)	H (2.a)	H (2.a)	–	–
$\overline{T}^w$	H+ (2.a)	H+ (2.a)	H+ (2.a)	H+ (2.a)	H+ (2.a)	H+ (2.a)	H+ (2.a)	–	–

Table 9.4. Complexity results for problems of type $1 \mid\mid F_\ell(Z_1, Z_2)$.

Z_1	C_{max}	T_{max}	L_{max}	$\overline{C}$	$\overline{C}^w$	$\overline{U}$	$\overline{U}^w$	$\overline{T}$	$\overline{T}^w$
C_{max}	–	P [56]	P [56]	P [56]	O	O	H (3.b)	H (3.b)	H+ (3.b)
T_{max}		–	–	P [53, 56]	H [56, 18](3.a)	O [18]	H (3.b)	H (3.b)	H+ (3.b)
L_{max}		–	–	P [56]	H+ [56](3.a)	O	H (3.b)	H (3.b)	H+ (3.b)
$\overline{C}$				–	–	H [18]	H (3.b)	H (3.b)	H+ (3.b)
$\overline{C}^w$				–	–	H [18]	H+ (3.b)	H (3.b)	H+ (3.b)
$\overline{U}$						–	–	H (3.b)	H+ (3.b)
$\overline{U}^w$						–	–	H (3.b)	H+ (3.b)
$\overline{T}$								–	–
$\overline{T}^w$								–	–

All the multiple machines scheduling problems of type $\alpha \mid\mid Z$ with $\alpha \in \{F, J, O, HF, GJ, GO\}$ are $\mathcal{NP}$-hard in the strong sense, with any criterion Z of Γ, leading to the conclusion that they are $\mathcal{NP}$-hard in the strong sense with more than one criterion. Only problems of type $\alpha \mid\mid \overline{C}$ with $\alpha \in \{P, Q, R\}$ are polynomially solvable. We present in Table 9.5 complexity results for problems of type $P2 \mid\mid Lex(\overline{C}, Z_2)$. Since these problems are $\mathcal{NP}$-hard, the problems with a number of machines greater or equal to 2 are also $\mathcal{NP}$-hard, and the parallel machines scheduling problems with uniform or unrelated parallel machines are also $\mathcal{NP}$-hard.

Table 9.5. Complexity results for problems of type $P2 \mid\mid Lex(\overline{C}, Z_2)$.

Z_1	C_{max}	T_{max}	L_{max}	$\overline{C}$	$\overline{C}^w$	$\overline{U}$	$\overline{U}^w$	$\overline{T}$	$\overline{T}^w$
$\overline{C}$	H [19, 25]	H [19]	H (5.c)	–	–	H [19](6.b)	H [19](4)	H [19](6.a)	H [19](4)

Under some assumptions, an $\mathcal{NP}$-hard scheduling problem can be simplified and may become polynomially solvable. We present in Table 9.6 the complexity results of some lexicographical single machine problems with unit processing times (Section 5.1). For a given problem appearing in this table, *sr* stands for "*simple reduction*", since if the problem with arbitrary processing times is polynomially solvable, then the same problem with unit processing times is also polynomially solvable.

Table 9.6. Complexity results for problems of type $1 \mid p_j = 1 \mid Lex(Z_1, Z_2)$.

Z_1	Z_2: C_{max}	T_{max}	L_{max}	$\overline{C}$	$\overline{C}^w$	$\overline{U}$	$\overline{U}^w$	$\overline{T}$	$\overline{T}^w$
C_{max}	–	P *sr*	P *sr*	P *sr*	P *sr*	P *sr*	P tr	P tr	P tr
T_{max}	P *sr*	–	–	P [17]*sr*	P [17]	P [17]	P [17]	P [17]	P [17]
L_{max}	P *sr*	–	–	P *sr*	O	O	O	O	O
$\overline{C}$	P *sr*	P [17]*sr*	P *sr*	–	–	P [17]*sr*	P [17]*sr*	P [17]*sr*	P [17]*sr*
$\overline{C}^w$	P *sr*	P [17]*sr*	P *sr*	–	–	P [17]*sr*	P [17]	P [17]	P [17]
$\overline{U}$	P *sr*	P [17]	P	P [17]	P [17]	–	–	P [17]	P [17]
$\overline{U}^w$	P	P [17]	P	P [17]	P [17]	–	–	P [17]	P [17]
$\overline{T}$	P	P [17]	P	P [17]	P [17]	P [17]	P [17]	–	–
$\overline{T}^w$	P	P [17]	P	P [17]	P [17]	P [17]	P [17]	–	–

5. Single Machine Problems

Single machine scheduling problems have been extensively studied in the literature. They have practical issues, like scheduling in computers, and they provide basic models from which a lot of theoretical results and algorithms can be deduced for solving multiple machines problems. This is also valid for the multicriteria single machine problems, that are mainly encountered in literature on MSP. More accurately, works have been mostly dedicated to the bicriteria case and some surveys [30, 40, 56, 78] are devoted to this area.

We review the literature according to the complexity of the problems. We distinguish the ones that are polynomially solvable, the $\mathcal{NP}$-hard problems and those for which the complexity is unknown. We also consider problems according to the criteria. The first problem class contains those dealing with the *minimization of K ascending functions of the job completion times.* Problems related to the *minimization of the sum of job completion times* or to the *minimization of the weighted sum of job completion times,* with other criteria, are investigated. Problems where one criterion measures *setup time costs* as well as those with one *crashing time costs* criterion are also considered. The two last classes of problems dealt with concern problems with either only *due date based criteria* or *Just-in-Time criteria.* The latter are seldom considered as multicriteria problems in the literature even though they undoubtedly involve at least two criteria: one for measuring the earliness and the other for measuring the tardiness.

5.1. Polynomially Solvable Problems

5.1.1 Minimization of *K* Ascending Functions of Completion Times. Hoogeveen [57] deals with the problem of minimizing K functions of the completion times, noted f^i_{max} and assumed to be ascending. For a schedule S, we have $f^i_{max}(S) = \max_{1\leq j\leq n}\left(f^i_j(C_j(S))\right)$, $\forall i, i = 1, ..., K$, where f^i_j is an ascending function. For the problem, referred to as $1 \mid\mid \epsilon(f^1_{max}/f^2_{max}, ..., f^K_{max})$, two cases are investigated and a posteriori algorithms are proposed. The first case occurs for $K = 2$ and Hoogeveen shows that a modification of Lawler's algorithm [68], which solves the $1 \mid prec \mid f_{max}$ problem, leads to an optimal polynomial algorithm for the bicriteria problem once the ϵ value is fixed. This algorithm requires $O(n^2)$ time and the number of Pareto optima is at most $(n(n-1)/2 + 1$. Therefore, the enumeration of the whole set E requires $O(n^4)$ time.

The second considered case occurs for $K > 2$. The number of Pareto optima is at most $(n(n-1)/2 + 1)^{K-1}$ and Hoogeveen proposes an extension of the previously mentioned enumeration algorithm to the multicriteria case. The obtained algorithm requires $O(n^{K(K+1)-6})$ time, with $K > 2$.

5.1.2 Minimization of the Sum of Completion Times. The minimization of the sum of job completion times, with other criteria has been extensively studied in the literature. The $1 \mid d_i \mid \epsilon(\overline{C}/L_{max})$ problem is surely among the most studied MSP and various *a priori* algorithms [54, 97] as well as *a posteriori* algorithms [80, 107] have been provided.

Besides, if a general ascending function f_{max} of the job completion times is minimized instead of the L_{max} criterion, Emmons [34] produces an a priori algorithm for the $1 \mid\mid Lex(f_{max}, \overline{C})$ problem. This algorithm is based on Lawler's algorithm [68] for the $1 \mid prec \mid f_{max}$ problem. John [63] deals with the enumeration of the Pareto optima, namely the $1 \mid\mid \#(f_{max}, \overline{C})$ problem. The proposed algorithm starts with an optimal schedule for the $1 \mid\mid Lex(\overline{C}, f_{max})$ problem as a seed sequence and performs job permutations to get the whole set E. John shows that the number of Pareto optima is lower than $X = \frac{1}{4}(n^2 - 1)(p_{max} - p_{min})$ if n is even, where p_{max} and p_{min} stand for the maximum and minimum processing times, respectively and lower than $X = \frac{1}{4}n^2(p_{max} - p_{min})$ otherwise. The time complexity of the proposed algorithm is in $O(n^2X)$.

Hoogeveen and Van de Velde [59] consider the $1 \mid\mid \epsilon(f_{max}/\overline{C})$ problem for which they propose an a posteriori algorithm. They also provide a

tighter bound for the number of Pareto optima than John's one, since they show that it is at most equal to $(n(n-1)/2+1$.

5.1.3 Minimization of the Weighted Sum of Completion Times. Chen and Bulfin [17] deal with numerous problems with criterion $\overline{C}^w$, i.e. the weighted sum of job completion times, and when jobs have unit processing times. Such models may have potential applications in computer science or, as mentioned by Chen and Bulfin, in car production.

The $1 \mid p_i = 1, d_i \mid Lex(\overline{C}^w, Z_2)$ problems, where $Z_2 \in \{\overline{T}, \overline{T}^w, \overline{U}, \overline{U}^w, T_{max}\}$, can be solved in $O(n\log(n))$ time by greedy algorithms based on the WSPT rule: jobs are sorted in ascending order of the ratio $\frac{p_i}{w_i}$ and ties are broken according to the second criterion.

The $1 \mid p_i = 1, d_i \mid Lex(Z_1, \overline{C}^w)$ problems, where $Z_1 \in \{\overline{T}^w, \overline{U}, \overline{U}^w, T_{max}\}$, can be transformed into special assignment problems, solved in $O(n^3)$ time. A greedy algorithm requiring $O(n\log(n))$ time, can be produced for the $1 \mid p_i = 1, d_i \mid Lex(\overline{T}, \overline{C}^w)$ problem: jobs are sorted in ascending order of their due date *di* and ties are broken according to the WSPT rule.

Problems where a convex combination of criteria is minimized are also dealt with. Chen and Bulfin reduce the $1 \mid p_i = 1, d_i \mid F_\ell(\overline{C}^w, Z_2)$ problems, where $Z_2 \in \{\overline{T}, \overline{T}^w, \overline{U}, \overline{U}^w\}$, to particular assignment problems. These reductions can be used to enumerate the set of supported Pareto optima.

The $1 \mid p_i = 1, d_i \mid \epsilon(\overline{C}^w/T_{max})$ problem is also investigated and an a posteriori algorithm is provided.

5.1.4 Minimization of Setup Time Costs. The problem of minimizing costs of changing tools with job classes and orders has been tackled by Gupta, Ho and Van der Veen [49] and can be stated as follows. $\mathcal{M}$ orders $\mathcal{O}_j$, made up of n_j jobs, have to be scheduled on one machine. It is assumed that jobs can also be separated into k classes B_ℓ and we have $|B_\ell \cap \mathcal{O}_j| = 1$, $\forall \ell, \ell = 1, ..., k, \forall j, j = 1, ..., \mathcal{M}$. Moreover, if job $J_u \in B_p$ is processed immediately before job $J_v \in B_q$, with $p \neq q$, a setup time $C_{p,q}$ occurs between the processing of these two jobs. In order to minimize the makespan, the sum of the setup times has to be minimized. This setup time costs criterion is defined by:

$$\overline{SC} = \sum_{i=1}^{n} C_{[i],[i+1]} \tag{9.2}$$

where $n = \sum_{j=1}^{\mathcal{M}} n_j$ and $C_{[i],[i+1]}$ is the value of the setup time between jobs in position i and $i+1$ in the schedule under consideration. Besides,

we are also interested in minimizing the carrying cost, which is defined by:

$$\overline{AC} = \sum_{j=1}^{\mathcal{M}} \max_{J_\ell, J_i \in \mathcal{O}_j} (0; C_i - C_\ell) \tag{9.4}$$

This cost represents the storage of partially processed orders while waiting for their completion. Gupta, Ho and Van der Veen consider the two possible lexicographical problems.

The $1 \mid$ *classes, orders,* $S_{sd} \mid Lex(\overline{SC}, \overline{AC})$ problem can be solved by a $O(n \log(\mathcal{M}))$ time algorithm. An optimal schedule for the $\overline{SC}$ criterion can be obtained by successively scheduling all the jobs belonging to the same class. Hence, only the scheduling of classes has to be done. Concerning the second criterion, jobs have to be re-arranged inside each class in order to minimize the $\overline{AC}$ criterion.

The resolution of the $1 \mid$ *classes, orders,* $S_{sd} \mid Lex(\overline{AC}, \overline{SC})$ problem leads to a $O(n)$ time algorithm. The $\overline{AC}$ criterion is minimized by successively scheduling all the jobs belonging to the same order. For each order, the first scheduled job is the one with the greatest value of the processing time plus the setup time whatever the next scheduled job. The minimization of criterion $\overline{SC}$ is done by scheduling the orders and the remaining jobs inside the orders. It can be performed by solving a particular case of the traveling salesman problem, solvable in polynomial time.

5.1.5 Minimization of Crashing Time Costs. These problems are also known as problems with controllable processing times, since each job J_i has a processing time $p_i \in [\underline{p}_i; \overline{p}_i]$ which has to be determined, where $\underline{p}_i$ and $\overline{p}_i$ are fixed bounds. The crashing time costs criterion is therefore defined by $\overline{CC}^w = \sum_{i=1}^{n} w_i(\overline{p}_i - p_i)$. It has been considered in the literature together with the T_{max} criterion [106, 109] or the $\overline{C}$ criterion [109].

5.1.6 Minimization of Due Date Based Criteria. Among the problems with unit processing times tackled by Chen and Bulfin [17], several problems with only due date based criteria are dealt with.

The $1 \mid p_i = 1, d_i \mid Lex(\overline{T}^w, Z_2^1)$ problems, where $Z_2^1 \in \{\overline{U}, \overline{U}^w\}$, as well as the $1 \mid p_i = 1, d_i \mid Lex(\overline{U}, Z_2^2)$ and $1 \mid p_i = 1, d_i \mid Lex(\overline{U}^w, Z_2^2)$ problems, where $Z_2^2 \in \{\overline{T}, \overline{T}^w\}$, can be reduced to an assignment problem solvable in $O(n^3)$ time. This also holds for the $1 \mid p_i = 1, d_i \mid Lex(T_{max}, Z_2^3)$ and $1 \mid p_i = 1, d_i \mid Lex(Z_2^3, T_{max})$ problems, with $Z_2^3 \in \{\overline{U}, \overline{U}^w, \overline{T}^w\}$.

When the sum of job tardiness is involved, that is for the $1 \mid p_i = 1, d_i \mid Lex(\overline{T}^w, Z_2^1)$ problems, with $Z_2^1 \in \{\overline{U}, \overline{U}^w\}$, a greedy algorithm based on the EDD rule can be applied: the jobs are scheduled in ascending order of their due date d_i and ties are broken according to the second criterion.

For problems involving the minimization of a convex combination of criteria, that is the $1 \mid p_i = 1, d_i \mid F_\ell(\overline{U}, Z_2^2)$ and $1 \mid p_i = 1, d_i \mid F_\ell(\overline{U}^w, Z_2^2)$ problems, with $Z_2^2 \in \{\overline{T}, \overline{T}^w\}$, Chen and Bulfin propose reductions to particular assignment problems, that can be used to enumerate the set of supported Pareto optima.

5.1.7 Minimization of Just-in-Time Criteria. Just-in-Time (JiT) scheduling has been emphasized thanks to numerous practical issues. From a chronological point of view, it first appeared in the framework of car production. So far, numerous scheduling models with one machine have been considered with the purpose of minimizing a convex combination of the sum of earliness and the sum of tardiness.

Hoogeveen [58] considers two problems where each job has a target start time s_i and a due date d_i. The jobs' tardiness is measured in comparison with the due dates but not the jobs' earliness since a particular kind of earliness is defined, namely the job promptness. Let t_i be the start time of job J_i in a schedule S. The promptness of job J_i is defined by $P_i = s_i - t_i$. The purpose is to minimize the maximum promptness, that is criterion $P_{max} = \max_{1 \leq i \leq n}(P_i)$, and the maximum lateness noted L_{max}. We assume that $d_i - s_i \leq p_i, \forall i, i = 1, ..., n$ and the two considered problems differ depending on whether machine idle time is allowed or not. They can be referred to as $1 \mid s_i, d_i, d_i - s_i \leq p_i \mid \epsilon(L_{max}/P_{max})$ and $1 \mid s_i, d_i, d_i - s_i \leq p_i, nmit \mid \epsilon(L_{max}/P_{max})$ problems.

For a given value ϵ, constraint $P_{max} \leq \epsilon$ leads to consider only the minimization of criterion L_{max} together with job release dates $r_i = s_i - \epsilon, \forall i, i = 1, ..., n$. When the *nmit* constraint holds, a $O(n \log(n))$ time a priori algorithm is presented, whilst otherwise the time complexity becomes $O(n^2 \log(n))$.

For the $1 \mid s_i, d_i, d_i - s_i \leq p_i, nmit \mid \epsilon(L_{max}/P_{max})$ problem an a posteriori algorithm is provided and Hoogeveen shows that the number of Pareto optima is at most n.

Kondakci, Emre and Koksalan [65] study problems with unit processing times, for which the earliness is defined in comparison with due dates. They focus on the minimization of the number of late jobs as a measure of the tardiness. No machine idle time is allowed although it might allow to decrease the earliness of some jobs. Problems dealt with can be noted $1 \mid p_i = 1, d_i, nmit \mid \epsilon(\overline{E}/\overline{U})$, $1 \mid p_i = 1, d_i, nmit \mid \epsilon(E_{max}/\overline{U})$

and $1 \mid p_i = 1, d_i, \textit{nmit} \mid \epsilon(F_\ell(\overline{E}, \overline{T})/\overline{U})$ where $F_\ell(\overline{E}, \overline{T}) = \overline{E} + \overline{T}$. As jobs have unit processing times and no machine idle time is allowed, the aim becomes to determine the scheduled job for each position in the sequence, once the ϵ bound on criterion $\overline{U}$ is fixed. Hence, we can build for each scheduling problem an assignment problem with the additional upper bound constraint on criterion $\overline{U}$. A posteriori algorithms based on the enumeration of all relevant ϵ values, are proposed.

Ahmed and Sundararaghavan [2] tackle a problem with a *non restrictive* common due date and particular *symmetrical weights.* A common due date is said to be *non restrictive* if the optimal solution of the dummy problem obtained by increasing this common due date by one unit is no better than the optimal solution of the original problem. We have symmetrical weights if jobs have weights and if for each job the weight for its earliness is equal to the one for its tardiness. Ahmed and Sundararaghavan consider that the weights are equal to the job processing times and the tackled problem can be noted $1 \mid d_i = d \geq \sum p_i \mid F_\ell(\overline{E}^w, \overline{T}^w)$ where $F_\ell(\overline{E}^w, \overline{T}^w) = \sum_{i=1}^{n} p_i(E_i + T_i)$. As all the due dates are equal, inserting machine idle times, after the start time of the first job, is not interesting. For the optimal schedule of the problem for which all the jobs are processed continuously, idle time insertion does not lead to a decrease of the objective function value. A greedy algorithm, based on LPT rule is presented: jobs are sorted in descending order of their processing time. Starting from this sequence, n schedules S_i are built considering that the job in position i in S_i completes at time d. The schedule with the lowest value of the objective function is the optimal solution.

The $1 \mid d_i = d$ *unknown, nmit* $\mid F_\ell(\overline{E}, \overline{T}, d)$ problem, with the criteria function $F_\ell(\overline{E}, \overline{T}, d) = \alpha\overline{E} + \beta\overline{T} + \gamma nd$, is probably one of the basic problems with common due date and no machine idle time. As the value of d is unknown, the optimal solution of this problem is such that d is non restrictive. From a theoretical point of view, problems with non restrictive common due date and unknown common due date are equivalent. Panwalker, Smith and Seidmann [85] propose for the previously quoted problem an $O(n \log(n))$ time algorithm. Chen [20] considers an extension of this problem, where jobs can be grouped into classes for delivery. All the jobs completed early or on-time are set into the same class and are delivered to the customer at time d. Other classes are made up of tardy jobs and accordingly the aim is to minimize the number of these tardy classes, noted $\overline{B}$. This $1 \mid d_i = d$ *unknown, nmit, classes* $\mid F_\ell(\overline{E}, \overline{T}, \overline{B}, d)$ problem can be solved by a dynamic programming algorithm requiring $O(n^5)$ time.

When we deal with non regular criteria, such as criteria $\overline{E}$ or E_{max}, it might be interesting to insert machine idle time, in order to decrease

their value. Therefore, two problems have to be solved: the first one concerns the sequencing of jobs, whilst the second one is related to the computation of job start times once the job sequence is fixed. The latter is referred to as the optimal timing problem. Although it depends on the criteria and objective function, it is usually polynomially solvable since it can be modelled as a linear program. The basic work on optimal timing is due to Garey, Tarjan and Wilfong [45] and is related to a particular objective function. Szwarc and Mukhopadyay [101] consider the $1 \mid d_i \mid F_\ell(\overline{E}^w, \overline{T}^w)$ problem and propose a decomposition into blocks to solve the optimal timing problem. Assume that the job sequence is that of the job index. Two consecutive jobs J_i and J_{i+1} belonging to the same block are such that $d_{i+1} - d_i < p_{i+1}$. Accordingly, it is only necessary to insert machine idle time between blocks and an algorithm requiring $O(cn)$ time is presented, where c is the number of blocks. This algorithm is faster than that of Davis and Kanet [28]. Problems with up to 500 jobs are solved within 2 seconds.

5.2. $\mathcal{NP}$-hard Problems

5.2.1 Minimization of the Sum of the Completion Times.

Most of the $\mathcal{NP}$-hard problems involving the minimization of criterion $\overline{C}$ date back to the 1980s. Azizoglu, Kondakci and Koksalan [65] investigate the resolution of the $1 \mid d_i,\ nmit \mid \epsilon(\overline{C}/E_{max})$ problem which is strongly $\mathcal{NP}$-hard, like the $1 \mid d_i,\ nmit \mid Lex(E_{max}, \overline{C})$ problem. An a priori heuristic algorithm is presented and afterwards extended to an a posteriori heuristic algorithm to compute a subset of set *WE*. No computational experiments are reported.

5.2.2 Minimization of the Weighted Sum of the Completion Times.

An extensively studied MSP in the literature relates to the minimization of criteria $\overline{C}^w$ and L_{max}. The $1 \mid d_i \mid \epsilon(\overline{C}^w/L_{max})$ problem is strongly, $\mathcal{NP}$-hard since the $1 \mid d_i \mid Lex(L_{max}, \overline{C}^w)$ problem is, too [56]. For the former, special cases have been investigated [8, 16, 33, 97]. A priori heuristic algorithms have been provided both by Heck and Roberts [54], Burns [15] and Miyazaki [77].

5.2.3 Minimization of Setup Time Costs.

Bourgade, Aguilera, Penz and Binder [12] deal with a scheduling problem related to glass bottle production. Sequence dependent setup times occur when switching from the processing of a job to the processing of the following and the aim is to minimize the sum of the setup times, noted $\overline{SC}$. Two $1 \mid d_i, S_{sd} \mid F(\overline{SC}, T_{max})$ problems, with different objective functions F, are tackled. For each one, a mixed integer linear programming

formulation is proposed with one of two following objective functions:

$$F(\overline{SC}, T_{max}) = \frac{\overline{SC}}{\delta} + \alpha \sum_{i=1}^{n} \exp^{\max(0;T_i - T^*_{max})} \tag{9.6}$$

$$F(\overline{SC}, T_{max}) = \frac{\overline{SC}}{\delta} + \max(0; (T_{max} - T^*_{max}) \times (1 + \alpha)) \tag{9.7}$$

with T^*_{max} the optimal value of criterion T_{max} for the $1 \mid d_i \mid T_{max}$ problem, and δ and α given weights. As the $1 \mid S_{sd} \mid \overline{SC}$ problem is $\mathcal{NP}$-hard, the two investigated problems are also $\mathcal{NP}$-hard. Branch and bound algorithms to solve the two bicriteria problems, once δ and α are known, are also presented. Computational experiments exhibit that optimal solutions for the first objective function may be dominated solutions for the criteria T_{max} and $\overline{SC}$.

5.2.4 Minimization of Just-in-Time Criteria. One of the basic JiT scheduling problems is certainly the $1 \mid d_i \mid F_\ell(\overline{E}, \overline{T})$ problem which has been considered in a priori resolution contexts. A schedule for this problem can be decomposed into blocks of jobs such that machine idle time occurs only between these blocks, and jobs within a block are continuously processed. Szwarc [100] investigates the sequencing of two consecutive jobs without inserted machine idle time. He proposes sufficient conditions for scheduling jobs within the blocks, once the schedule decomposition is known. A branching scheme to be used in an enumerative algorithm like a branch and bound algorithm, deduced from that sufficient conditions is proposed.

The $1 \mid d_i \mid F_\ell(\overline{E}, \overline{T})$ problem, where $F_\ell(\overline{E}, \overline{T}) = \overline{E} + \overline{T}$, is studied by Kim and Yano [64] who provide heuristic and exact algorithms to solve it. The different heuristic algorithms first build jobs sequences, using priority rules and afterwards apply Garey, Tarjan and Wilfong's optimal timing algorithm [45] to deduce schedules. If only two conflicting jobs J_i and J_j have to be sequenced, i.e. jobs such that $d_j - d_i < p_j$, Kim and Yano show that J_i is sequenced before J_j either if $d_i + p_j - d_j < d_j + p_i - d_i$ or if they can be scheduled before d_i and $p_i > p_j$, or if they are scheduled after $\max(d_i - p_i; d_j - p_j)$ and $p_i < p_j$. Otherwise, job J_j is sequenced before job J_i. These results are particular cases of the results presented by Szwarc [100]. Kim and Yano afterwards present two lower bounds and a branch and bound algorithm in which the previous results are used as dominance conditions. Computational experiments report that the exact algorithm solves problems with up to 20 jobs.

The same problem is also tackled by Fry, Armstrong, Darby-Dowman and Philipoom [39] who provide a branch and bound algorithm based

on a decomposition into blocks schedule. This exact algorithm is able to solve problems with up to 25 jobs.

When the jobs earliness is no longer measured in comparison with the job due dates but with target job start times, Koulamas [66] proposes both exact and heuristic algorithms to solve the $1 \mid s_i, d_i \mid F_\ell(\overline{T}, \overline{P})$ problem. Criterion $\overline{P}$ refers to the sum of the job promptness and is defined by $\overline{P} = \sum_{i=1}^{n} P_i$, where $P_i = \max(0; s_i - t_i)$ and t_i is the real start time of job J_i. Computational experiments report that among the seven proposed heuristic algorithms, two give near optimal solutions.

When no machine idle time is allowed, Azizoglu, Kondakci and Kirca [5] propose an adaptation of Ow and Morton's heuristic [84], initially designed for the $1 \mid d_i, \textit{nmit} \mid F_\ell(\overline{E}^w, \overline{T}^w)$ problem, to solve the $1 \mid d_i \mid F_\ell(\overline{E}, \overline{T})$ problem. A branch-and-bound algorithm is also presented and computational experiments report that it can solve problems with up to 20 jobs, in the best case.

The generalized problem with weighted criteria, $1 \mid d_i \mid F_\ell(\overline{E}^w, \overline{T}^w)$, has been studied a lot. Fry, Armstrong and Blackstone [38, 41] propose heuristic algorithms as well as a mixed integer linear program of the problem. Two Tabu Search algorithms are provided by James and Buchanan [61, 62]. The first one works on jobs sequences and each time a schedule has to be evaluated, the optimal timing problem is solved using the linear program proposed in [38]. The second Tabu Search algorithm considers a particular jobs sequence encoding and a heuristic procedure is used to built a feasible schedule from a coded sequence.

The *no machine idle time* case of the generalized problem with weighted criteria, denoted $1 \mid d_i, \textit{nmit} \mid F_\ell(\overline{E}^w, \overline{T}^w)$, has been dealt with by Ow and Morton [83, 84]. For this problem, once the jobs sequence is known, a schedule is built by starting jobs as early as possible. Ow and Morton provide a sequencing rule, noted EXP-ET and different filtered beam search algorithms to compute near optimal schedules. A filtered beam search is a heuristic procedure based on a branch and bound scheme in which *a priori* bad nodes are cut, thus limiting the size of the search tree. Computational experiments report that the average deviation of the best heuristic from lower bounds is between 5% and 10%.

Li [73] proposes a neighborhood based heuristic algorithm where at each iteration the neighbors are obtained using n k-NAPI (Non Adjacent Pairwise Interchange), $\forall k, k = 0,..., n-1$, operators on a seed sequence. The initial seed sequence is obtained by sorting the jobs according to the EXP-ET rule. At a current iteration, the 0-NAPI operator is used to build n neighbors: each corresponding schedule is obtained by considering the swapping of two jobs with 0 job between them. Among the

neighbors that are better than the seed sequence, the best one is chosen as the new seed sequence and the 0-NAPI operator is used once again until no neighbor improves the seed sequence. At this time, the 1-NAPI operator is considered and so on. Once no improved sequence obtained by the $(n - 1)$-NAPI operator has been computed, the current iteration process is repeated. Li also provides a branch-and-bound algorithm where lower bounds, based on Lagrangean relaxation, are used to prune nodes. Computational experiments report that problems up to 25 jobs are optimally solved within 100 seconds.

Liaw [75] presents heuristic and exact algorithms close to Li's ones. The heuristic algorithm is also based on a neighborhood search where the seed sequence is obtained by the EXP-ET rule. No computational experiments comparing this heuristic with Li's one are reported. A branch and bound algorithm in which the lower bounds are based on Lagrangean relaxation, is provided. Nevertheless, computational experiments show that it does not strongly outperform Li's exact algorithm.

Almeida and Centeno [4] also produce heuristic algorithms for the $1 \mid d_i, nmit \mid F_\ell(\overline{E}^w, \overline{T}^w)$ problem. They consider different Tabu Search, Simulated Annealing and local search algorithms. Computational experiments only report comparisons between these heuristic algorithms.

When all the jobs share the same common due date, Van den Akker, Hoogeveen and Van de Velde [105] investigate the problem denoted $1 \mid d_i = d \geq \sum p_i, nmit \mid F_\ell(\overline{E}^w, \overline{T}^w)$. We refer to α_i as the unit earliness weight of job J_i in the objective function and β_i as its unit tardiness weight. They restrict their search for an optimal schedule to the dominant class of schedules such that one job completes at time d, all the early jobs are sorted in descending order of the ratio $\frac{p_i}{\alpha_i}$ and all the tardy jobs are sorted in ascending order of the ratio $\frac{p_i}{\beta_i}$. An exact algorithm combining column generation and Lagrangean relaxation is presented and computational experiments report that problems up to 125 jobs can be solved within few minutes.

Azizoglu and Webster [6] investigate that problem once jobs classes are defined and setup times occur between two jobs belonging to different classes processed successively. This problem can be noted $1 \mid d_i = d \geq \sum p_i, nmit, classes, S_{sd} \mid F_\ell(\overline{E}^w, \overline{T}^w)$. Azizoglu and Webster show that the schedule class which is dominant for the problem tackled by Van den Akker, Hoogeveen and Van de Velde ([105]) is also dominant for the problem with setup times. A branch-and-bound algorithm is presented as well as a heuristic procedure based on a filtered beam search.

Webster, Job and Gupta [111] consider the problem denoted $1 \mid d_i = d$ *unknown, nmit,* $S_{sd} \mid F_\ell(\overline{E}^w, \overline{T}^w)$, for which inserting machine idle times does not improve the best schedule. A genetic algorithm is pro-

vided and computational experiments report that it outperforms Azizoglu's and Webster's branch-and-bound limited to one hour of CPU time.

Dileepan and Sen [31] deal with the $1 \mid d_i, nmit \mid F(\overline{E}, \overline{T}, \overline{C})$ problem, where $F(\overline{E}, \overline{T}, \overline{C}) = (1-\alpha)\sum_{i=1}^{n}(L_i^2) + \alpha\overline{C}$. Criterion $\overline{C}$ usually reflects holding costs and is sometimes considered in JiT scheduling problems. Dileepan and Sen present a branch-and-bound algorithm and computational experiments.

5.3. Open Problems

Few open problems have been considered and mostly date back before the 1990s. Fry and Leong [42] have tackled the $1 \mid d_i \mid F_\ell(\overline{E}, \overline{C})$ problem for which they provide a mixed integer linear programming formulation. Vickson [108] deals with a scheduling problem with controllable job processing times, the $1 \mid p_i \in [\underline{p}_i; \overline{p}_i] \mid F_\ell(\overline{C}^w, \overline{CC}^w)$ problem and presents a branch-and-bound algorithm. The problem of minimizing criteria $\overline{U}$ and T_{max} has been investigated by Shantikumar [95] who proposes a branch-and-bound algorithm to solve the $1 \mid d_i \mid Lex(\overline{U}, T_{max})$ problem. Later on, Nelson, Sarin and Daniels [80] propose an a posteriori algorithm for the $1 \mid d_i \mid \epsilon(\overline{U}/T_{max})$ problem which computes a subset of the set of weak Pareto optima. A heuristic procedure is also described.

Adamopoulos and Pappis [1] consider an open JiT scheduling problem, noted $1 \mid d_i$ *unknown, nmit* $\mid F_\ell(\overline{E}^w, \overline{T}^w)$. Besides, they assume that the weights of jobs both for earliness and tardiness are functions of the processing times. Branch-and-bound algorithms are presented.

6. Parallel Machines Problems

When jobs are mono-operation and resources are of the same type, we have to solve a parallel machines scheduling problem. As in Section 5, we review the literature according to problem complexity.

6.1. Polynomially Solvable Problems

6.1.1 Minimization of the Maximum Completion Time.

Leung and Young [71] consider identical parallel machines with job preemption allowed and the minimization of the makespan subject to an optimal total completion time value. This problem is a lexicographical minimization problem that can be noted $P \mid pmtn \mid Lex(\overline{C}, C_{max})$. The authors show that it can be solved in polynomial time. Notice that both $P \mid pmtn \mid C_{max}$ and $P \mid pmtn \mid \overline{C}$ problems are polynomially solvable. Leung and Young suppose that the number of jobs n is equal to $r \times m$

and that jobs are numbered according to SPT rule. The $(r-1) \times m$ first jobs are scheduled and assigned according to SPT-FAM rule, that is "Shortest Processing Time on First Available Machine first" rule. Then, Sahni's algorithm [92] is used to schedule the m remaining jobs. The algorithm runs in $O(n \log(n))$ time.

Mac Cormick and Pinedo [76] consider uniform parallel machines and the problem is to find the set E of Pareto optima for the makespan and the total completion time criteria. The problem is noted $Q \mid pmtn \mid C_{max}, \overline{C}$. The preemption of jobs is allowed. An ϵ-constraint approach is used to compute one Pareto optimum. The enumeration of set E is conducted by solving all relevant problems (P_ϵ) defined as follows:

$$\begin{aligned} &\text{Min } \overline{C}(s) \\ &\text{s.t. } C_{max}(s) \leq \epsilon \text{ and } s \in \mathcal{S} \end{aligned}$$

This problem is noted $Q \mid pmtn \mid \epsilon(\overline{C}/C_{max})$. Mac Cormick and Pinedo propose an algorithm to solve problem (P_ϵ) by using rules to schedule jobs and to assign them to resources. This algorithm requires $O(mn)$ time.

T'Kindt, Billaut and Proust [103] consider an industrial problem that tackles the scheduling of glass bottle production on unrelated parallel machines. The authors suppose that job preemption is allowed and that the tools changing costs are negligible. However, an order cannot be processed on different machines at the same time, since this would induce the multiplication of expensive equipment. Hence, job splitting is not allowed. Processing one job on one machine allows to realize a margin and changing the color associated to one furnace implies to stop the production on all the associated machines. So, the objective is to maximize the margin and to minimize the difference of machine workloads. The aim is to propose to a decision maker an algorithm which allows to select the Pareto optimum he prefers. This selection is done in an interactive way. Using a linear programming model of the problem, the authors solve the problem noted $R \mid pmtn \mid \epsilon(F_\ell(I_{max}, -\overline{M})/C_{max})$. By modifying the weights associated to the criteria, either all the Pareto optima can be obtained (in an *a posteriori* method), or only the most interesting for the decision maker (in an *interactive* method).

6.1.2 Minimization of Just-in-Time Criteria. Sundararaghavan and Ahmed [99] consider identical parallel machines and jobs with a non restrictive common due date d. The authors suppose that it is not possible to introduce idle times on machines and the objective is to minimize the sum of the total earliness and the total tardi-

ness. The problem is noted $P \mid d_i = d$ *non restr, nmit* $\mid F_\ell(\overline{E}, \overline{T})$, with $F_\ell(\overline{E}, \overline{T}) = \overline{E} + \overline{T}$ and the authors propose an algorithm that generalizes the algorithm for the $1 \mid d_i = d$, *nmit* $\mid F_\ell(\overline{E}, \overline{T})$ problem (see Section 5). The authors build a V-shaped solution by scheduling and assigning iteratively jobs on machines. A V-shaped schedule is such that the early and on-time jobs are sequenced in descending order of their processing time whilst the tardy jobs are sequencing in the ascending order of their processing time.

A more general problem is the problem noted $P \mid d_i = d$ *unknown, nmit* $\mid F_\ell(\overline{E}, \overline{T})$, with $F_\ell(\overline{E}, \overline{T}) = \alpha\overline{E} + \beta\overline{T}$. This problem is solved by Emmons [35] who proposed an algorithm that builds a V-shaped schedule on each machine. The decision to put each job in the set of early jobs or in the set of tardy jobs of one machine depends on the value of α and β.

If processing times are equal to a common value p, it is possible to introduce the due date d in the objective function and the problem remains polynomial. Cheng and Chen [22] study the $P \mid d_i = d$ *unknown,* $p_i = p$, *nmit* $\mid F_\ell(\overline{E}, \overline{T}, d)$ problem with $F_\ell(\overline{E}, \overline{T}, d) = \alpha\overline{E} + \beta\overline{T} + \gamma nd$. The authors remark that if $\gamma \geq \beta$, then the problem is equivalent to the polynomial problem $P \mid p_i = p \mid \overline{C}$. **If** $\gamma < \beta$, then the authors define two classes of resources: class A which contains the $m-h$ first resources, each performing k jobs and class B which contains the remaining resources each performing $k+1$ jobs, with h defined by $h = n - km$ and $k = \lfloor \frac{n}{m} \rfloor$. Some coefficients allow to determine the common due date, the starting time of processing on class A machines and on class B machines.

Alidaee and Ahmadian [3] consider a problem with unrelated parallel machines, jobs with processing times as variables, with a common due date and a criterion defined by a linear combination of total tardiness, total earliness and total weighted compression costs. The problem can be noted $R \mid p_{i,k} \in [\underline{p}_{i,k}; \overline{p}_{i,k}], d_i = d$ *unknown* $\mid F_\ell(\overline{T}, \overline{E}, \overline{CC}^w)$ with $\overline{CC}^w$ denoting the weighted cost of reducing job processing times. This problem is equivalent to a transportation problem and can be solved in $O(n^3)$ time.

6.2. $\mathcal{NP}$-hard Problems

6.2.1 Minimization of the Maximum Completion Time.

Gupta and Ruiz-Torres [51] consider identical parallel machines and the problem to minimize the makespan given that the total completion time is minimum. If preemption is allowed, we have seen that the problem is polynomially solvable [71]. Gupta and Ruiz-Torres consider that preemption is not allowed and the problem becomes $\mathcal{NP}$-hard. The

problem can be noted $P \mid\mid Lex(\overline{C}, C_{max})$. The authors propose first a heuristic algorithm called U with complexity $O(n\log(n) + nm)$ and second a heuristic called M inspired by algorithm "Multifit" of Coffman, Garey and Johnson [26] for the $P \mid\mid C_{max}$ problem. Algorithm M runs in $O(Kn^3/m)$ time, with K a fixed number of iterations.

6.2.2 Minimization of Just-in-Time Criteria. In [10], Biskup and Cheng consider a problem with identical parallel machines and they are interested in two objectives: achieving small deviations from a common due date and short flowtimes. The problem considered is noted $P \mid d_i = d \mid F_\ell(\overline{E}, \overline{T}, \overline{C})$. They show that the problem is $\mathcal{NP}$-hard for $m = 2$. Then, they study some polynomially solvable cases with d given and $p_i = p$, $\forall i, i = 1, ..., n$. After that, the authors propose a heuristic algorithm, that calls an LP solver.

Li and Cheng [72] consider jobs with a common due date and no machine idle time allowed. The objective is to minimize a function defined by $f_{max}(\overline{E}^w, \overline{T}^w) = \max_{1 \le i \le n} w_i(E_i + T_i)$. The problem can be noted $P \mid d_i = d$ *non restr, nmit* $\mid f_{max}(\overline{E}^w, \overline{T}^w)$. The authors show that the problem is strongly $\mathcal{NP}$-hard and they propose a heuristic algorithm H with a worst case bound of the performance ratio $f^H_{max}/f^*_{max} \le 2m$.

Chen and Powell [21] consider jobs with a given common due date unrestrictively large and associated to each job an earliness penalty weight and a tardiness penalty weight. The problem is noted $P \mid d_i = d$ *non restr* $\mid F_\ell(\overline{E}^w, \overline{T}^w)$. The authors propose an integer programming formulation of the problem and a branch-and-bound algorithm, that can solve problems with up to 60 jobs within reasonable cpu time.

Balakrishnan, Kanet and Sridharan [7] consider a problem with uniform parallel machines, with sequence dependent setup times on machines, to switch from one job to another job and with ready times associated to jobs. The problem can be noted $Q \mid r_i, S_{sd} \mid F_\ell(\overline{E}, \overline{T})$. The authors propose a mixed-integer formulation for the problem and a relaxation of the model used as an approximate method.

6.3. Open Problems

6.3.1 Minimization of the Sum of Completion Times. Alidaee and Ahmadian [3] consider a problem with unrelated parallel machines and controllable job processing times. The problem is noted $R \mid p_{i,k} \in [\underline{p}_{i,k}; \overline{p}_{i,k}] \mid F_\ell(\overline{C}, \overline{CC}^w)$ with $\overline{CC}^w$ the total weighted compression cost. The authors show that there exists an optimal schedule and optimal processing times such that a job is either at the maximum

processing time value $\overline{p}_{i,k}$ or at the minimum processing time value $\underline{p}_{i,k}$ (fully compressed time).

6.3.2 Minimization of Just-in-Time Criteria. The $P \mid d_i = d$ *unknown, nmit* $\mid F_\ell(\overline{E}, \overline{T})$ problem has several optimal solutions, and Emmons [33] considers two extensions of this problem. The first one is the $P \mid d_i = d$*unknown, nmit* $\mid Lex(F_\ell(\overline{E}, \overline{T}), C_{max})$ problemand the second one is the $Q \mid d_i = d$ *unknown, nmit* $\mid F_\ell(\overline{E}, \overline{T})$ problem. The complexity of these problems is unknown and the author proposes polynomial algorithms to determine a solution.

7. Shop Problems

In this Section we discuss about shop scheduling problems. These problems are used for modeling production processes. The multicriteria scheduling literature concerning shop problems is composed essentially of flowshop scheduling problems. Hence, the section is divided in three subsections as follows: first a survey on two-machine flowshop problems, then on multiple machine flowshop problems and finally on the other shop scheduling problems.

7.1. Two-Machine Flowshop Problems

We consider that the shop is composed of two machines noted M_1 and M_2. Each job has to visit machine M_1 first and then machine M_2, and jobs are processed in the same order on each machine. The problem is to find a sequence of jobs that minimizes the criteria.

The most common criteria involved for this problem are criteria $\overline{C}$ and C_{max}.

Rajendran [87] considers the problem $F2 \mid pmtn \mid Lex(C_{max}, \overline{C})$. This problem is $\mathcal{NP}$-hard in the strong sense [19] and the author proposes two heuristics and a branch-and-bound algorithm. This algorithm allows to solve problems with up to 10 jobs. For the same problem, Nepalli, Chen and Gupta [81] propose a genetic algorithm. Gupta, Neppali and Werner [50] propose nine heuristic algorithms and study polynomially solvable cases of this problem. Gupta, Hennig and Werner [48] implement several neighborhood techniques like Tabu Search and Simulated Annealing. More recently, T'Kindt, Gupta and Billaut [104] propose one heuristic algorithm and different branch-and-bound algorithms. The most efficient branch-and-bound can solve problems with up to 27 jobs.

Nagar, Heragu and Haddock [79] consider a linear combination of these criteria. The problem is noted $F2 \mid pmtn \mid F_\ell(C_{max}, \overline{C})$. This problem is strongly $\mathcal{NP}$-hard too and the authors propose a heuris-

tic algorithm and a branch-and-bound algorithm. For this problem, Sivrikaya-Serifoglu and Ulusoy [96] propose another heuristic algorithm and three branch-and-bound procedures. The most efficient branch-and-bound can solve problems with up to 18 jobs. Yeh [114] propose a heuristic algorithm and a branch-and-bound procedure limited to problems with 14 jobs. If release dates are specified for each job, the problem is noted $F2 \mid pmtn, r_i \mid F_\ell(C_{max}, \overline{C})$. For this problem, Chou and Lee [24] present an integer linear programming formulation of the problem and a heuristic algorithm of Beam Search type.

Sayin and Karabati [93] consider the problem to obtain the Pareto optima. The problem is noted $F2 \mid pmtn \mid C_{max}, \overline{C}$. To obtain the Pareto set, the authors use an ϵ-constraint approach and the problems to be solved are noted $F2 \mid pmtn \mid \epsilon(\overline{C}/C_{max})$. These problems are proved to be strongly $\mathcal{NP}$-hard and the authors propose two branch-and-bound algorithms. Problems with less than 20 jobs can be solved.

Daniels and Chambers [27] consider criteria C_{max} and T_{max} and search the set of Pareto optima. This $\mathcal{NP}$-hard problem is noted $F2 \mid pmtn \mid C_{max}, T_{max}$. The authors propose a branch-and-bound algorithm with associated dominance conditions and a heuristic procedure.

Liao, Yu and Joe [74] consider criteria C_{max} and $\overline{U}$ and search the set of Pareto optima. It is not difficult to show that this problem is $\mathcal{NP}$-hard in the strong sense. The authors propose a branch-and-bound procedure with some dominance conditions. The experimentations show that the algorithm can solve problems with up to 20 jobs and that the average number of Pareto optima is less than 2. This leads to the conclusion that the criteria are not conflicting. Then, the authors consider criteria C_{max} and $\overline{T}$. The authors search the set of Pareto optima. This problem is strongly $\mathcal{NP}$-hard and the authors propose a branch-and-bound algorithm, that can solve problems with up to 30 jobs.

7.2. m-Machine Flowshop Problems

We consider now that the shop is composed by m machines. Each job has to visit the machines from machine M_1 to machine M_m.

Selen and Hott [94] and Wilson [113] consider criteria C_{max} and $\overline{C}$ in a lexicographical order. They suppose that jobs are processed in the same order on each machine. The problem is noted $F \mid pmtn \mid Lex(C_{max}, \overline{C})$. This problem is strongly $\mathcal{NP}$-hard and the authors present an integer programming formulation of problem $F \mid pmtn \mid C_{max}$ and $F \mid pmtn, C_{max} = C^*_{max} \mid \overline{C}$.

For the same criteria, Gangadharan and Rajendran [43] and Rajendran [88, 89] propose a method to find any Pareto optimum.

Daniels and Chambers consider criteria C_{max} and T_{max} and search the set of Pareto optima. The authors propose a heuristic procedure and show that 50% of the Pareto optima can be found by their algorithm.

Zegordi, Itoh and Enkawa [115] consider a just-in-time flowshop scheduling problem noted $F \mid pmtn, nmit \mid F_\ell(\overline{E}^w, \overline{T}^w)$. This problem is strongly $\mathcal{NP}$-hard and the authors propose a Simulated Annealing algorithm, that gives near optimal solutions.

Nowicki [82] considers the problem noted $F \mid p_{i,j} \in [\underline{p}_{i,j}, \overline{p}_{i,j}], pmtn \mid F_\ell(C_{max}, \overline{CC}^w)$. This problem is $\mathcal{NP}$-hard and the author propose a heuristic algorithm. Cheng and Shakhlevich [23] consider a similar problem where all the operations of one job have the same processing time: $p_{i,j} = p_i, \forall i = 1, ..., n$. Processing time are controllable and the authors search the set of Pareto optima. The problem is noted $F \mid p_i \in [\underline{p}_i, \overline{p}_i], pmtn \mid F_\ell(C_{max}, \overline{CC}^w)$. This problem is polynomially solvable and the authors propose an enumeration algorithm based on a linear programming model.

7.3. Other Shop Scheduling Problems

7.3.1 Scheduling Problems. The literature contains few papers on jobshop and openshop multicriteria scheduling problems.

Huckert, Rhode, Roglin and Weber [60] consider a jobshop problem with five criteria. They propose an interactive algorithm, like STEM method [9], that calls a greedy algorithm to find a solution to the jobshop problem.

Deckro, Herbert and Winkofsky [29] consider a jobshop scheduling problem with four criteria. The authors consider the goal programming approach to solve the problem.

Gupta and Werner [52] consider a two-machine openshop problem, with criteria C_{max} and $\overline{C}$ in a lexicographical order. The problem is noted $O2 \mid\mid Lex(C_{max}, \overline{C})$. The authors show that if the makespan is given by the largest job ($C^*_{max} = \max_{1 \leq i \leq n}(p_{i,1} + p_{i,2})$), then the problem can be solved in polynomial time. Otherwise, the authors propose a heuristic algorithm. For the same problem, Kyparisis and Koulamas [67] study several polynomially solvable cases and propose different heuristic algorithms. The authors consider the three-machine openshop problem with the same objective function. The problem is noted $O3 \mid\mid Lex(C_{max}, \overline{C})$ and the authors consider a polynomial solvable case.

7.3.2 Assignment and Scheduling Problems. When machines are duplicated and gathered into stages, the problem is to find a

starting date and a performing machine for each operation. Some papers in the literature deal with the multicriteria hybrid flowshop problem.

Fortemps, Ost, Pirlot, Teghem and Tuyttens [37] consider an industrial problem in a chemical firm. They define some equipments that are used for the production of jobs. The workshop can be considered like a three-stage hybrid flowshop, with a set of constraints and the following criteria: the makespan, a penalty proportional to the tardiness of the jobs and a penalty reflecting the violation of the availability period of some equipments. The authors propose two metaheuristics: a Tabu Search algorithm and a Simulated Annealing algorithm.

Riane, Meskens and Artiba [90] consider a k-stage hybrid flowshop, with criteria C_{max} and $\overline{C}$ gathered into a linear combination. The problem is noted $FHk,\ (Pm^{(\ell)})^{\ell=1,\ldots,k}\ ||\ F_\ell(C_{max}, \overline{C})$. This problem is strongly $\mathcal{NP}$-hard and the authors propose a mixed integer linear programming (MILP) formulation of the problem and a Tabu Search algorithm. The authors consider also the problem with the ϵ-constraint approach, i.e. the problem noted $FHk, (Pm^{(\ell)})^{\ell=1,\ldots,k}\ ||\ \epsilon(\overline{C}/C_{max})$. The authors propose a MILP formulation and a Tabu Search algorithm.

Vignier, Billaut and Proust [110] consider a k-stage hybrid flowshop, with the hypothesis that at each time and at each stage, at most one machine may not be available for processing a job. This unavailability period may correspond for instance to a maintenance activity. The authors suppose that jobs can not start before a ready date and the criteria they consider are C_{max} and T_{max}. The problem can be noted $FHk,\ (Pm^{(\ell)}(t))^{\ell=1,\ldots,k}\ ||\ \epsilon(C_{max}/T_{max})$ and the authors propose a branch-and-bound algorithm based on the branch-and-bound of Brah and Hunsucker [13]. Different lower bounds are proposed and compared.

References

[1] G.L. Adamopoulos and C.P. Pappis. Scheduling jobs with different job-dependent earliness and tardiness penalties using the SLK method. *European Journal of Operational Research,* 88:336–344, 1996.

[2] M.S. Ahmed and P.S. Sundararaghavan. Minimizing the weighted sum of late and early completion penalties in a single machine. *IIE Transactions,* 22(3):288–290, 1990.

[3] B. Alidaee and A. Ahmadian. Two parallel machine sequencing problems involving controllable job processing times. *European Journal of Operational Research,* 70:335–341, 1993.

[4] M.T. Almeida and M. Centeno. A composite heuristic for the single machine early/tardy job scheduling problem. *Computers*

and Operations Research, 25(7/8):625–635, 1998.

[5] M. Azizoglu, S. K. Kondakci, and O. Kirca. Bicriteria scheduling problem involving total tardiness and total earliness penalties. *International Journal of Production Economics,* 23:17–24, 1991.

[6] M. Azizoglu and S. Webster. Scheduling job families about an unrestricted common due date on a single machine. *International Journal of Production Research,* 35:1321–1330, 1997.

[7] N. Balakrishnan, J.J. Kanet, and S.V. Sridharan. Early/tardy scheduling with sequence dependent setups on uniform parallel machines. *Computers and Operations Research,* 26:127–141, 1999.

[8] S.P. Bansal. Single machine scheduling to minimize weighted sum of completion times with secondary criterion – A branch and bound approach. *European Journal of Operational Research*, 5(3):177–181, 1980.

[9] R. Benayoun, J. de Mongolfier, J. Tergny, and O. Laritchev. Linear programming with multiple objective funtions: Step method (STEM). *Mathematical Programming,* 1(3):366–375, 1971.

[10] D. Biskup and T.C.E. Cheng. Multiple-machine scheduling with earliness, tardiness and completion time penalties. *Computers and Operations Research,* 26:45–57, 1999.

[11] J. Blazewicz, K.H. Ecker, E. Pesch, G. Schmidt, and J. Weglarz. *Scheduling Computer and Manufacturing Processes.* Springer-Verlag, Berlin, 1996.

[12] V. Bourgade, L.M. Aguilera, B. Penz, and Z. Binder. Problème industriel d'ordonnancement bicritère sur machine unique: Modélisation et aide à la décision. *RAIRO-APII-JESA Journal Euopéen des Systèmes Automatisés,* 29(3):331–341, 1995.

[13] S.A. Brah and J.L. Hunsucker. Branch and bound algorithm for the flow shop with multiple processors. *European Journal of Operational Research,* 51:88–99, 1991.

[14] P. Brucker. *Scheduling Algorithms.* Springer Verlag, Berlin, 2nd edition, 1998.

[15] R.N. Burns. Scheduling to minimize the weighted sum of completion times with secondary criteria. *Naval Research Logistics Quarterly,* 23(1):125–129, 1976.

[16] S. Chand and H. Schneeberger. A note on the single machine scheduling problem with minimum weighted completion time and maximum allowable tardiness. *Naval Research Logistics Quarterly,* 33(3):551–557, 1986.

[17] C.-L. Chen and R.L. Bulfin. Scheduling unit processing time jobs on a single machine with multiple criteria. *Computers and Operations Research,* 17(1):1–7, 1990.

[18] C.-L. Chen and R.L. Bulfin. Complexity of single machine, multi-criteria scheduling problems. *European Journal of Operational Research,* 70:115–125, 1993.

[19] C.-L. Chen and R.L. Bulfin. Complexity of multiple machines, multi-criteria scheduling problems. *3rd Industrial Engineering Research Conference (IERC'94), Atlanta (USA),* pages 662–665, 1994.

[20] Z.-L. Chen. Scheduling and common due date assignment with earliness and tardiness penalties and batch delivery costs. *European Journal of Operational Research,* 93:49–60, 1996.

[21] Z.-L. Chen and W.B. Powell. A column generation based decomposition algorithm for a parallel machine just-in-time scheduling problem. *European Journal of Operational Research,* 116:220–232, 1999.

[22] T.C.E. Cheng and Z.-L. Chen. Parallel-machine scheduling problems with earliness and tardiness penalties. *Journal of the Operational Research Society,* 45(6):685–695, 1994.

[23] T.C.E. Cheng and N. Shakhlevich. Proportionate flow shop with controllable processing times. *Journal of Scheduling,* 2:253–265, 1999.

[24] F.-D. Chou and C.-E. Lee. Two-machine flowshop scheduling with bicriteria problem. *Computers and Industrial Engineering,* 36:549-564, 1999.

[25] E.G. Coffman. *Computer and Job-Shop Scheduling Theory.* John Wiley & Sons, New York, 1976.

[26] E.G. Coffman, M.R. Garey, and D.S. Johnson. An application of bin-packing to multi-processor scheduling. *SIAM Journal of Computing,* 7(1):1–17, 1978.

[27] R.L. Daniels and R.J. Chambers. Multiobjective flow-shop scheduling. *Naval Research Logistics,* 37:981–995, 1990.

[28] J.S. Davis and J.J. Kanet. Single machine scheduling with early and tardy completion costs. *Naval Research Logistics,* 40:85–101, 1993.

[29] R.F. Deckro, J.E. Herbert, and E.P. Winkofsky. Multiple criteria job-shop scheduling. *Computers and Operations Research,* 9(4):279–285, 1982.

[30] P. Dileepan and T. Sen. Bicriterion static scheduling research for a single machine. *Omega,* 16(1):53–59, 1988.

[31] P. Dileepan and T. Sen. Bicriterion jobshop scheduling with total flowtime and sum of squared lateness. *Engineering Costs and Production Economics,* 21:295–299, 1991.

[32] M. Ehrgott. *Multicriteria Optimization,* volume 491 of *Lecture Notes in Economics and Mathematical Systems.* Springer-Verlag, Berlin, 2000.

[33] H. Emmons. A note on a scheduling problem with dual criteria. *Naval Research Logistics Quarterly,* 22(4):615–616, 1975.

[34] H. Emmons. One machine sequencing to minimize mean flow time with minimum number tardy. *Naval Research Logistics Quarterly,* 22(4):585–592, 1975.

[35] H. Emmons. Scheduling to a common due date on parallel uniform processors. *Naval Research Logistics,* 34:803–810, 1987.

[36] G.W. Evans. An overview of techniques for solving multiobjective mathematical programs. *Management Science,* 30(11):1268–1282, 1984.

[37] P. Fortemps, C. Ost, M. Pirlot, J. Teghem, and D. Tuyttens. Using metaheuristics for solving a production scheduling problem in a chemical firm: A case study. *International Journal of Production Economics,* 46-47(1-3):13–26, 1996.

[38] T.D. Fry, R.D. Armstrong, and R.H. Blackstone. Minimizing weighted absolute deviation in single machine scheduling. *IIE Transactions,* 19(4):445–450, 1987.

[39] T.D. Fry, R.D. Armstrong, K. Darby-Dowman, and P.R. Philipoom. A branch and bound procedure to minimize mean absolute lateness on a single processor. *Computers and Operations Research,* 23(2):171–182, 1996.

[40] T.D. Fry, R.D. Armstrong, and H. Lewis. A framework for single machine multiple objective sequencing research. *Omega,* 17(6):595–607, 1989.

[41] T.D. Fry and R.H. Blackstone. Planning for idle time : A rationale for underutilization of capacity. *International Journal of Production Research,* 26(12):1853–1859, 1988.

[42] T.D. Fry and G.K. Leong. A bi-criterion approach to minimizing inventory costs on a single machine when early shipments are forbidden. *Computers and Operations Research,* 14(5):363–368, 1987.

[43] R. Gangadharan and C. Rajendran. A simulated annealing heuristic for scheduling in a flowshop with bicriteria. *Computers and Industrial Engineering,* 27(1-4):473–476, 1994.

[44] M.R. Garey and D.S. Johnson. *Computers and Intractability: A Guide to the Theory of $\mathcal{NP}$-Completeness.* W.H. Freeman and Company, San Francisco. 1979.

[45] M.R. Garey, R.E. Tarjan, and G.T. Wilfong. One-processor scheduling with symmetric earliness and tardiness penalties. *Mathematics of Operations Research,* 13(2):330–348, 1988.

[46] A.M. Geoffrion. Proper efficiency and the theory of vector maximization. *Journal of Mathematical Analysis and Applications,* 22:618–630, 1968.

[47] R.L. Graham, E.L. Lawler, J.K. Lenstra, and A.H.G. Rinnooy Kan. Optimization and approximation in deterministic sequencing and scheduling: A survey. *Annals of Discrete Mathematics,* 5:287–326, 1979.

[48] J.N.D. Gupta, K. Hennig, and F. Werner. Local search heuristic for the two-stage flowshop problems with secondary criterion. *Computers and Operations Research,* 29(2):113–149, 2002.

[49] J.N.D. Gupta, J.C. Ho, and A.A.A. Van der Veen. Single machine hierarchical scheduling with customer orders and multiple job classes. *Annals of Operations Research,* 70:127–143, 1997.

[50] J.N.D. Gupta, V.R. Neppalli, and F. Werner. Minimizing total flow time in a two-machine flowshop problem with minimum makespan. *International Journal of Production Economics,* 69(3):323–338, 2001.

[51] J.N.D. Gupta and A. J. Ruiz-Torres. Minimizing makespan subject to minimum total flow-time on identical parallel machines. *European Journal of Operational Research,* 125:370–380, 2000.

[52] J.N.D. Gupta and F. Werner. On the solution of 2-machine flow and open shop scheduling problems with secondary criteria. *15th ISPE/IEE International Conference on CAD/CAM, Robotics, and Factories of the Future, Aguas de Lindoia, Sao Paulo (Brazil),* pages MW4-1 – MW4-6, 1999.

[53] S.K. Gupta and T. Sen. Minimizing a quadratic function of job lateness on a single machine. *Engineering Costs and Production Economics,* 7(3):187–194, 1983.

[54] H. Heck and S. Roberts. A note on the extension of a result on scheduling with secondary criteria. *Naval Research Logistics Quarterly,* 19(3):4, 1972.

[55] W. Herroelen, E. Demeulemeester, and B. De Reyck. A classification scheme for project scheduling. In J. Weglarz, editor, *Project Scheduling*: *Recent Models, Algorithms and Applications,* pages 1–26. Kluwer Academic Publishers, Dordrecht, 1998.

[56] J.A. Hoogeveen. *Single-Machine Bicriteria Scheduling.* PhD thesis, Centrum voor Wiskunde en Informatica, Amsterrdam (The Netherlands), 1992.

[57] J.A. Hoogeveen. *Single machine scheduling to minimize a function of K maximum cost criteria.* In: *Single-Machine Bicriteria Scheduling, PhD thesis,* pages 67–77. Centrum voor Wiskunde en Informatica, Amsterdam (The Netherlands), 1992.

[58] J.A. Hoogeveen. Minimizing maximum promptness and maximum lateness on a single machine. *Mathematics of Operations Research,* 21(1):100–114, 1996.

[59] J.A. Hoogeveen and S.L. Van de Velde. Minimizing total completion time and maximum cost simultaneously is solvable in polynomial time. *Operations Research Letters,* 17:205–208, 1995.

[60] K. Huckert, R. Rhode, O. Roglin, and R. Weber. On the interactive solution to a multicriteria scheduling problem. *Zeitchrift fur Operations Research,* 24:47–60, 1980.

[61] R.J.W. James and J.T. Buchanan. A neighbourhood scheme with a compressed solution space for the early/tardy scheduling problem. *European Journal of Operational Research,* 102:513–527, 1997.

[62] R.J.W. James and J.T. Buchanan. Performance enhancements to tabu search for the early/tardy scheduling problem. *European Journal of Operational Research,* 106:254–265, 1998.

[63] T.C. John. Tradeoff solutions in single machine production scheduling for minimizing flow time and maximum penalty. *Computers and Operations Research,* 16(5):471–479, 1984.

[64] Y.-D. Kim and C.A. Yano. Minimizing mean tardiness and earliness in single-machine scheduling problems with unequal due dates. *Naval Research Logistics,* 41:913–933, 1994.

[65] S.K. Kondakci, E. Emre, and M. Koksalan. Scheduling of unit processing time jobs on a single machine. In G. Fandel and T. Gal, editors, *Multiple Criteria Decision Making,* volume 448 in *Lecture Notes in Economics and Mathematical Systems,* pages 654–660. Springer-Verlag, Berlin, 1997.

[66] C. Koulamas. Single-machine scheduling with time windows and earliness/tardiness penalties. *European Journal of Operational Research,* 91:190–202, 1996.

[67] G.J. Kyparisis and C. Koulamas. Open shop scheduling with makespan and total completion time criteria. *Computers and Operations Research,* 27:15–27, 2000.

[68] E.L. Lawler. Optimal sequencing of a single machine subject to precedence constraints. *Management Science,* 19:544–546, 1973.

[69] C-Y. Lee, L. Lei, and M. Pinedo. Current trends in deterministic scheduling. *Annals of Operations Research,* 70:1–41, 1997.

[70] C.-Y. Lee and G.L. Vairaktarakis. Complexity of single machine hierarchical scheduling: A survey. In P.M. Pardalos, editor, *Complexity in Numerical Optimization,* pages 269–298. World Scientific Publishing, Singapore, 1993.

[71] J.Y.-T. Leung and G.H. Young. Minimizing schedule length subject to minimum flow time. *SIAM Journal on Computing,* 18(2):314–326, 1989.

[72] C.-L. Li and T.C.E. Cheng. The parallel machine min-max weighted absolute lateness scheduling problem. *Naval Research Logistics,* 41:33–46, 1994.

[73] G. Li. Single machine earliness and tardiness scheduling. *European Journal of Operational Research,* 96:546–558, 1997.

[74] C.-J. Liao, W.-C. Yu, and C.-B. Joe. Bicriterion scheduling in the two-machine flowshop. *Journal of the Operational Research Society,* 48:929–935, 1997.

[75] C.-F. Liaw. A branch-and-bound algorithm for the single machine earliness and tardiness scheduling problem. *Computers and Operations Research,* 26:679–693, 1999.

[76] S.T. McCormick and M.L. Pinedo. Scheduling n independent jobs on m uniform machines with both flowtime and makespan objectives: A parametric analysis. *O.R.S.A. Journal on Computing,* 7(1):63–77, 1995.

[77] S. Miyazaki. One machine scheduling problem with dual criteria. *Journal of the Operational Research Society of Japan,* 24(1):37–50, 1981.

[78] A. Nagar, J. Haddock, and S.S. Heragu. Multiple and bicriteria scheduling: A literature survey. *European Journal of Operational Research,* 81:88–104, 1995.

[79] A. Nagar, S.S. Heragu, and J. Haddock. A branch-and-bound approach for a two-machine flowshop scheduling problem. *Journal of the Operational Research Society,* 46:721–734, 1995.

[80] R.T. Nelson, R.K. Sarin, and R.L. Daniels. Scheduling with multiple performance measures: The one-machine case. *Management Science,* 32(4):464-479, 1986.

[81] V.R. Neppalli, C.-L. Chen, and J.N.D. Gupta. Genetic algorithms for the two-stage bicriteria flowshop problem. *European Journal of Operational Research,* 95:356–373, 1996.

[82] E. Nowicki. An approximation algorithm for the m-machine permutation flow shop schedulingproblem with controllable processing times. *European Journal of Operational Research,* 70:342–349, 1993.

[83] P.S. Ow and T.E. Morton. Filtered beam search in scheduling. *International Journal of Production Research,* 26(l):35–62, 1988.

[84] P.S. Ow and T.E. Morton. The single machine early/tardy problem. *Management Science,* 35(2):177–190, 1989.

[85] S.S. Panwalker, M.L. Smith, and A. Seidmann, Common due date assignment to minimize total penalty for the one machine scheduling problem. *Operations Research,* 30(2):391–399, 1982.

[86] M. Pinedo. *Scheduling – Theory, Algorithms, and Systems.* Prentice Hall, Englewood Cliffs, New Jersey, 1995.

[87] C. Rajendran. Two-stage flowshop scheduling problem with bicriteria. *Journal of the Operational Research Society,* 43(9):871–884, 1992.

[88] C. Rajendran. A heuristic for scheduling in flowshop and flowline-based manufacturing cell with multi-criteria. *International Journal of Production Research,* 32(11):2541–2558, 1994.

[89] C. Rajendran. Heuristics for scheduling in flowshop with multiple objectives. *European Journal of Operational Research,* 82:540–555, 1995.

[90] F. Riane, N. Meskens, and A. Artiba. Bicriteria scheduling hybrid flowshop problems. *International Conference on Industrial Engineering and Production Management (IEPM'97), Lyon 1997,* pages 34–43. Facultés catholiques Universitaires de Mons, Belgium, 1997.

[91] B. Roy. *Méthodologie Multicritère d'Aide à la Décision.* Economica, Paris, 1985.

[92] S. Sahni. Preemptive scheduling with due dates. *Operations Research,* 27:925–934, 1979.

[93] S. Sayin and S. Karabati. A bicriteria approach to the two-machine flow shop scheduling problem. *European Journal of Operational Research,* 113:435–449, 1999.

[94] W.J. Selen and D.D. Hott. A mixed integer goal-programming formulation of a flowshop scheduling problem. *Journal of the Operational Research Society,* 37:1121–1128, 1986.

[95] J.G. Shantikumar. Scheduling n jobs on one machine to minimize the maxium tardiness with minimum number tardy. *Computers and Operations Research,* 10(3):255–266, 1983.

[96] F.S. Sivrikaya-Serifoglu and G. Ulusoy. A bicriteria two machine permutation flowshop problem. *European Journal of Operational Research,* 107:414–430, 1998.

[97] W.E. Smith. Various optimizers for single-stage production. *Naval Research Logistics Quarterly,* 3(l):59–66, 1956.

[98] R.M. Soland. Multicriteria optimization: A general characterization of efficient solutions. *Decision Sciences,* 10:27–38, 1979.

[99] P.S. Sundararaghavan and M.U. Ahmed. Minimizing the sum of absolute lateness in single-machine and multimachine scheduling. *Naval Research Logistics Quarterly,* 31 (2):325–333, 1984.

[100] W. Szwarc. Adjacent orderings in single machine scheduling with earliness and tardiness penalties. *Naval Research Logistics,* 40:229–243, 1993.

[101] W. Szwarc and S.K. Mukhopadhyay. Optimal timing schedules in earliness-tardiness single machine sequencing. *Naval Research Logistics,* 42:1109–1114, 1995.

[102] V. T'kindt and J.-C. Billaut. Some guidelines to solve multicriteria scheduling problems. *IEEE Conference on Systems, Man and Cybernetics (SMC'99), SMC/IEEE, Tokyo (Japon),* volume IV, pages 463–468. IEEE Service Center, Piscataway, NJ, 1999.

[103] V. T'kindt, J.-C. Billaut, and C. Proust. Solving a bicriteria scheduling problem on unrelated parallel machines occuring in the glass bottle industry. *European Journal of Operational Research,* 135:42–49, 2001.

[104] V. T'kindt, J.N.D. Gupta, and J.-C. Billaut. Exact and heuristic resolution of a two machine bicriteria flowshop scheduling problem. *Computers and Operations Research,* in press, 2002.

[105] M. Van den Akker, H. Hoogeveen, and S. Van de Velde. A combined column generation and lagrangian relaxation algorithm for common due date scheduling. *6th Workshop on Project Management and Scheduling (PMS'98), EURO, Istanbul (Turquie),* 1998.

[106] L. Van Wassenhove and K.R. Baker. A bicriterion approach to time/cost trade-offs in sequencing. *European Journal of Operational Research,* 11(1):48–54, 1982.

[107] L. Van Wassenhove and L.F. Gelders. Solving a bicriterion scheduling problem. *European Journal of Operational Research,* vol 4:42–48, 1980.

[108] R.G. Vickson. Choosing the job sequence and processing times to minimize total processing plus flow cost on a single machine. *Operations Research,* 28(5): 115–167, 1980.

[109] R.G. Vickson. Two single machine sequencing problems involving controllable job processing times. *IIE Transactions,* 12(3):158–162, 1980.

[110] A. Vignier, J.-C. Billaut, and C. Proust. Solving k-stage hybrid flowshop scheduling problems. *Multiconference on Computational Engineering in Systems Applications (CESA '96), IEEE-SMC/IMACS, Lille,* pages 250–258. IEEE Service Center, Piscataway, NJ, 1996.

[111] S. Webster, P.D. Job, and A. Gupta. A genetic algorithm for scheduling job families on a single machine with arbitrary earliness/tardiness penalties and an unrestricted common due date. *International Journal of Production Research,* 36(9):2543–2551, 1998.

[112] A. Wierzbicki. The use of reference objectives in multiobjective optimization. In G. Fandel and T. Gal, editors, *Multiple Criteria Decision Making, Theory and Application,* volume 177 in *Lecture Notes in Economics and Mathematical Systems,* pages 468–486. Springer-Verlag, Berlin, 1990.

[113] J.M. Wilson. Alternative formulations of a flow-shop scheduling problem. *Journal of the Operational Research Society,* 40(4):395–399, 1989.

[114] W.-C. Yeh. A new branch-and-bound approach for the n/2/flowshop/aF+bCmax flowshop scheduling problem. *Computers and Operations Research,* 26:1293–1310, 1999.

[115] S.H. Zegordi, K. Itoh, and T. Enkawa. A knowledgeable simulated annealing scheme for the early/tardy flow shop scheduling problem. *International Journal of Production Research,* 33(5): 1449–1466, 1995.

This page intentionally left blank

Index