Francesca Poggiolesi

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Gentzen Calculi for Modal Propositional Logic

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Preface

Modal logic, the logic created to formalise the concepts of possibility and necessity, has a long and important history that begins with Aristotle, who claims that

> Since there is a difference according as something belongs, necessarily belongs, or may belong to something else (for many things belong indeed, but not necessarily, others neither necessarily nor indeed at all, but it is possible for them to belong), it is clear that there will be different syllogisms to prove each of these relations, and syllogisms with differently related terms, one syllogism concluding from what is necessary, another from what is, a third from what is possible. [2, Book I, part 8]

For many years modal logic was exclusively considered a (Hilbertian) syntactic tool. It was not until Kripke's discoveries of the early 1960s that modal logic was opened up to semantic research, and it is in this context that its reputation and development have blossomed. Indeed, today modal logic is mainly seen as a way of talking about frames and models, and as such it represents one of the most stimulating and widely studied fields of contemporary logic. But the development of modal logic from a proof-theoretical point of view has been lacking in comparison with its progress in semantics. There seems to be a feeling of general dissatisfaction and disagreement amongst practitioners. As Blackburn, Rijke and Venema note,

> modal proof theory and automated reasoning are still relatively youthful enterprises; they are exciting and active fields, but as yet there is little consensus about methods and few general results. [11, p. xvi]

In this book we would like to remedy this situation and to restore prestige to proof theory for modal logic.

Prerequisites

We assume that the reader has attended an undergraduate course in logic and has a good mastery of the rudiments of classical propositional logic (Hilbert-systems, truth tables). Some prior acquaintance with modal logic and classical Gentzen calculus is useful, but not essential (these topics are introduced in Sections 1.1 and 2.1, respectively). The rest of the book is self-contained and accompanies the reader up to the most recent and specialised research on this area.

What This Book Is About

Chapter 1 serves as an introduction to the Gentzen calculus from both a formal and a conceptual perspective, and contains a description of the properties that a good Gentzen calculus should satisfy. Some of these properties are linked to proof-theoretic semantics. We have thus tried to summarise the different views that characterise this topic. Nevertheless our presentation is far from comprehensive.

Chapter 2 is concerned with normal modal logic and ordinary sequent calculi for this logic. In the last section of the chapter we reach the idea of generalising the Gentzen calculus.

Chapter 3 deals with the syntactic generalisations of the Gentzen calculus, while *Chapter 4* deals with the semantic ones. The relationships between the extensions of the sequent calculus for modal logic are the subject of *Chapter 5*.

Chapter 6 introduces tree-hypersequent calculi from both an intuitive and a formal point of view. Tree-hypersequent calculi are proved to be adequate with respect to their corresponding Hilbert systems.

Cut-admissibility and decidability of the tree-hypersequent calculi, both proved in a purely syntactic fashion, will be in the foreground in *Chapter 7*, while *Chapter 8* will deal with a purely semantic proof of the adequacy of the tree-hypersequent calculi.

Chapter 9 contains a presentation of a hypersequent calculus for the system **S5**, while *Chapter 10* contains a tree-hypersequent calculus for the modal system **GL**, or the logic of provability. In the first case, proofs of adequacy, cut-admissibility and decidability are given; in the second case, proofs of adequacy and cut-admissibility are provided.

Chapter 11 is concerned with (i) the relationships between tree-hypersequent calculi and display calculi, (ii) the logics, other than modal logic, that the tree-hypersequent method has been applied to, and (iii) the future directions of work.

What This Book Is Not About

There exists a wide variety of research in proof theory for modal logic. Due to obvious limitations of size, it was not possible to take account of all this work. We have thus made a selection that we explain in detail.

As concerns modal logic we have only dealt with those systems that are nowadays considered the main systems of modal propositional logic, e.g. **K**, **KT**, **S4**, **S5**. All the others, as well as the entire field of first-order modal logic, have been completely omitted. As concerns proof theory, we have concentrated only on the sequent calculi; there is no mention of Gentzen's other creation, namely natural deduction, nor of more recent proof tools.

These omissions can be justified. The first is justifiable on the grounds that it is sufficiently difficult to find a sequent calculus for those modal systems that are considered the principal ones, and that also happen to be the simplest ones.

Finding sequent calculi for more complicated modal logics may be an interesting field for future research. On the other hand, the restriction to sequent calculi is justified since, among all attempts to adapt syntactic instruments to the case of modal logic, progress has been notoriously slow for the case of sequent calculi. We take this to be a reason in itself to concentrate on this subject.

Earlier Works

Parts or Chapters of the book borrow from various earlier papers, though in a substantially revised form. In particular, the argument proposed in Section 1.10 is a more precise version of the one presented in [101]. Moreover, the notation used in the tree-hypersequent calculi, as well as their presentation, has been improved in the course of time, and harmonised with the rest of the book. To be precise, the earlier work has been distributed as follows:

[101] Section 1.10	[104] Sections 5.1, 5.2, 11.1
[100] Chapters 6 and 7	[105] Chapters 6, 7, and 9
[99] Chapter 9	[103] Chapter 10

This book is the direct continuation of the work began during the author's Ph.D. thesis at the Department of Philosophy, University of Florence and IHPST, University of Paris 1.

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Brussels, Belgium Francesca Poggiolesi
September 2010

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An Overview of the Sequent Calculus

- D'ora in avanti sarò io a descrivere le città, - aveva detto il Kan. - Tu nei tuoi viaggi verificherai se esistono. [...] io ho costruito nella mia mente un modello di città da cui dedurre tutte le città possibili, - disse Kublai - Esso racchiude tutto quello che risponde alla norma.

[I. Calvino, Le città invisibili, Oscar Mondadori, 2004]

Chapter 1
What Is a Good Sequent Calculus?

In his doctoral thesis of 1935, the young and brilliant student Gerhard Gentzen introduced what is today known as the *sequent calculus*. Over the last eighty years the sequent calculus has been the central interest of several illustrious proof theorists. This has given rise to a broad literature and numerous results. Nevertheless, there still are problems and issues concerning the sequent calculus that need to be further developed and tackled. Amongst these, our attention has been attracted by one question that can be expressed as follows: what is a good sequent calculus? What, in other words, are the properties that a sequent calculus needs to satisfy to be considered good? The aim of this chapter is to attempt to find an answer to this question.

Before beginning our work on this task, let us underline two things. The first is that, although the literature on this topic is not very extensive, in [6, 27, 63, 145, 149], lists of properties that characterise a good sequent calculus can be found. The second is the importance of the question itself. Avron's words [6, pp. 1, 2] give an indication of why this is:

> it is clear that there is no limit to the number of logics that logicians (and non-logicians) can produce [...]. But what is a good logic? One simple answer might be: a logic which has applications. This answer is not satisfactory, though. First, systems of logic are frequently introduced before they find actual applications [...]. Second: Logic is an autonomous mathematical discipline, and as such should have its own independent criteria. One such criterion is the existence of a simple, illuminating semantic. This indeed is always a very good sign. A more important criterion (in my opinion, and since logics deal above all with proofs) *is the existence of a good proof system*. (Our emphasis.)

1.1 The Sequent Calculus Gcl

We begin our attempt to identify a good sequent calculus by introducing some basic logical notions which will prove useful later. We exploit the framework of classical propositional logic.

Definition 1.1 The *classical propositional language* $\mathcal{L}^c$ is composed of a denumerable stock of propositional letters (p_0, p_1, ...) and the logical operators $\bot$, $\wedge$, $\vee$ and $\rightarrow$. We shall use $p, q, \ldots$ as metavariables for propositional letters. Formulas are constructed as usual; α, β, ... will be used as metavariables for generic formulas. *PL* will denote the set of propositional letters while *WF* will denote the set of

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well-formed formulas of classical propositional logic. We finally introduce the following abbreviations:

$\alpha \leftrightarrow \beta := (\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$
$\neg\alpha := \alpha \rightarrow \bot$
$\top := \bot \rightarrow \bot$

Definition 1.2 We understand a *formal system* S, based on a formal language $\mathcal{L}$, to include:

- an effective set of well-formed formulas called *axioms*,
- an effective set of rules, called *inference rules*, for deriving theorems from the axioms.

The axioms and the inference rules that determine a formal system S are often called *postulates*.

There are various types of formal systems, the three most common are: Hilbert systems (**H**), Gentzen systems (**G**), natural deduction systems (**N**). In this book we are mainly concerned with Gentzen systems, also called Gentzen calculi or sequent calculi, but we will also work with Hilbert systems.

Definition 1.3 A *derivation (or deduction) in a Hilbert system* ***H*** is a finite (upward growing) tree with a single root; the nodes of the tree are labelled by formulas and the top nodes are labelled by axioms; for each non-terminal node, its label is connected with the labels of the immediate predecessor nodes according with one of the inference rules. The root of the tree is the conclusion of the whole derivation and its label is a theorem of the Hilbert system **H**, in symbols: $\vdash_{\mathbf{H}} \alpha$ (or, equivalently, $\vdash \alpha$ in **H**).

In order to introduce the notion of derivation in a Gentzen system, we first have to introduce the notion of sequent, which is the following:

Definition 1.4 Let $M, N, ...$ vary on finite or empty multisets of well-formed formulas; a *sequent* is an object of the form: $M \Rightarrow N$. M and N are called, respectively, the *antecedent* and the *consequent* of the sequent.

Definition 1.5 The *interpretation* τ of a sequent $M \Rightarrow N$ is: $(M \Rightarrow N)^\tau := \bigwedge M \rightarrow \bigvee N$. This means that the comma (that separates the formulas belonging to the multisets M and N) must be read as a conjunction in the antecedent and as a disjunction in the consequent, while the arrow corresponds to the implication.

Definition 1.6 A *derivation (or deduction) in a Gentzen system* ***G*** has the same structure of a derivation in a Hilbert system **H** except for the fact that the nodes are labelled by sequents rather than by formulas. A theorem in a Gentzen system **G** is the label of the root of a tree, in symbols: $\vdash_{\mathbf{G}} M \Rightarrow N$ (or, equivalently, $\vdash M \Rightarrow N$ in **G**).

We shall denote derivations in Gentzen or Hilbert systems by means of metavariables $d, d', ...$ For the definition of derivation in a natural deduction system we refer to [139, pp. 29, 30]. Let us now consider Gentzen systems in more detail.

Definition 1.7 The *inference rules* of the sequent calculus are ordered pairs or triples of sequents, arranged in either of these two forms:

$$\frac{s'}{s} \qquad \frac{s' \quad s''}{s}$$

The sequents above the horizontal line are called the *upper* sequents or the *premises* of the rule; the sequent below the line is called the *lower* sequent or the *conclusion* of the rule. The inference rules are divided into *structural* and *logical* rules: while the structural rules operate on the structure of the sequents regardless of the formulas that constitute them, the logical rules concern the introduction of a logical operator on the left or on the right side of the sequent.

The postulates of the sequent calculus **Gcl** for classical propositional logic are the following:

Axioms

Ax: $p \Rightarrow p$ $\qquad$ $A\bot$: $\bot \Rightarrow$

Structural Rules

Weakening and Contraction

$$\frac{M \Rightarrow N}{\alpha, M \Rightarrow N}\, WA \qquad \frac{M \Rightarrow N}{M \Rightarrow N, \alpha}\, WK$$

$$\frac{\alpha, \alpha, M \Rightarrow N}{\alpha, M \Rightarrow N}\, CA \qquad \frac{M \Rightarrow N, \alpha, \alpha}{M \Rightarrow N, \alpha}\, CK$$

Cut-Rule

$$\frac{M \Rightarrow N, \alpha \qquad \alpha, P \Rightarrow Q}{M, P \Rightarrow N, Q}\, cut_\alpha$$

Logical Rules

Propositional Rules

$$\frac{\alpha_i, M \Rightarrow N}{\alpha_0 \wedge \alpha_1, M \Rightarrow N}\, {\wedge A} \qquad \frac{M \Rightarrow N, \alpha \qquad M \Rightarrow N, \beta}{M \Rightarrow N, \alpha \wedge \beta}\, {\wedge K}$$

$$\frac{\alpha, M \Rightarrow N \quad \beta, M \Rightarrow N}{\alpha \vee \beta, M \Rightarrow N}\, {\vee A} \qquad \frac{M \Rightarrow N, \alpha_i}{M \Rightarrow N, \alpha_0 \vee \alpha_1}\, {\vee K}$$

$$\frac{M \Rightarrow N, \alpha \quad \beta, M \Rightarrow N}{\alpha \rightarrow \beta, M \Rightarrow N}\, {\rightarrow A} \qquad \begin{cases} \dfrac{\alpha, M \Rightarrow N}{M \Rightarrow N, \alpha \rightarrow \beta}\, {\rightarrow K_1} \\[2ex] \dfrac{M \Rightarrow N, \beta}{M \Rightarrow N, \alpha \rightarrow \beta}\, {\rightarrow K_2} \end{cases}$$

We call the A-rules, *left (introduction) rules*, and the K-rules, *right (introduction) rules*. Note that A stands for *Antezedens*, while K for *Konsequent*. A more common notation is L (left) for A and R (right) for K.

Throughout the book, we will use the following notation: in the rules $\wedge A$ and $\vee K$, we will write α_i for $i = 0, 1$.

Definition 1.8 In the rules of the sequent calculus, the formula occurrences in M, N, ... are called *side formulas* or *contexts*; the formula occurrence of the conclusion that is not a side formula is called the *principal* or *main* formula; the formula occurrences in the premises that are not side formulas are called *auxiliary*.

Definition 1.9 We associate to each derivation d in **G** a natural number $h(d)$, which stands for the *height* of the derivation d. Intuitively, the height corresponds to the length, minus one, of the longest branch in a tree-derivation d. We define $h(d)$ by induction on the construction of d.

- $d \equiv M \Rightarrow N$ $\qquad h(d) = 0$

- $d \equiv \dfrac{\begin{matrix} \vdots\, d_1 \\ M' \Rightarrow N' \end{matrix}}{M \Rightarrow N}\,\mathcal{R}$ $\qquad h(d) = h(d_1) + 1$

- $d \equiv \dfrac{\begin{matrix} \vdots\, d_1 \\ M' \Rightarrow N' \end{matrix} \quad \begin{matrix} \vdots\, d_2 \\ M'' \Rightarrow N'' \end{matrix}}{M \Rightarrow N}\,\mathcal{R}$ $\qquad h(d) = \max\,(h(d_1) + 1, h(d_2) + 1)$

Let $d \vdash^n_{\mathbf{G}} M \Rightarrow N$ denote: d is a derivation of $M \Rightarrow N$ in **G**, with $h(d) \leq n$.

Definition 1.10 A rule $\mathcal{R}$, belonging to a calculus **G**, is said to be (*height-preserving*) *eliminable* in **G** if, whenever there exists a derivation of height n of the premise of $\mathcal{R}$, then there also exists a derivation of the conclusion of $\mathcal{R}$, that does not contain any application of $\mathcal{R}$ (and with the height at most n).

If the rule $\mathcal{R}$ does not belong to the calculus **G**, but the condition above still holds, then $\mathcal{R}$ is said to be (*height-preserving*) *admissible* in **G**.

An important feature of **Gcl** consists in the fact that the cut-rule is eliminable in it, or, stated otherwise, the sequent calculi **Gcl** and **Gcl** minus the cut-rule are equivalent (a sequent is provable in **Gcl** if, and only if, it is provable in **Gcl** minus the cut-rule). The theorem that states this fact is usually called *cut-elimination theorem* and its proof consists in providing an algorithm that transforms derivations containing applications of the cut-rule to derivations not containing any application. Note that another way of proving that the cut-rule is eliminable in a calculus is obtained by showing that that calculus without the cut-rule is sound and complete with respect to the corresponding semantics. Nevertheless this second proof loses any constructive appeal. We will further analyse and discuss these issues in the following sections (in particular, Sections 1.3 and 1.4).

Definition 1.11 Let us consider a sequent calculus **G** and a logical rule $\mathcal{R} \in$ **G** such that, given $M' \Rightarrow N'$, $\mathcal{R}$ allows us to infer $M \Rightarrow N$. We say that $\mathcal{R}$ is a (*height-preserving*) *invertible* rule when its *inverse*, i.e. the rule that allows us to infer $M' \Rightarrow N'$ from $M \Rightarrow N$, is (height-preserving) admissible in the calculus **G**.

Theorem 1.12 *For all formulas α, and for all sequents $M \Rightarrow N$,*

*if $\vdash \alpha$ in **Hcl**, then $\vdash \Rightarrow \alpha$ in **Gcl**.*
*If $\vdash M \Rightarrow N$ in **Gcl**, then $\vdash \bigwedge M \rightarrow \bigvee N$ in **Hcl**.*

Proof The proof is by induction on the height of derivations in **Hcl** and **Gcl**, respectively (e.g. see [18], pp. 228–30). □

1.2 Formal Remarks

The sequent calculus for classical propositional logic, like the sequent calculi for other logics, can be presented in several ways and no particular way is to be preferred over the others. We divide the possible reformulations of the sequent calculus into *variants* and *alternatives*.[1] There are three standard alternatives, and they are obtained by modifying the very concept of sequent. Each alternative may have several variants, each variant being obtained by varying the postulates that compose a sequent calculus without varying the set of provable sequents. **Gcl** is then a certain variant of a certain alternative of the classical sequent calculus (which variant and which alternative will be specified below). We will present other variants and other alternatives of the Gentzen calculus, starting with the alternatives.

In Definition 1.4, p. 4, in order to define a sequent, we assumed M and N to be multisets of formulas, i.e. aggregates of formulas such that the order of the formulas does not count, but the number of times the same formula occurs does. It is possible to strengthen, as well as to weaken, this concept. On the one hand, we can assume M and N to be *sequences* of formulas, i.e. aggregates of formulas where both the order of the formulas and the number of times the same formula occurs count. On the other hand, we can assume M and N to be *sets* of formulas, i.e. aggregates of formulas where neither the order of the formulas nor the number of times the same formula occurs count. In the first case we claim to be in the *sequence alternative* of the sequent calculus, while in the second case we are working in the *set alternative* of the sequent calculus. If, finally, we work following Definition 1.4, we are employing the *multiset alternative*.[2]

We now read the postulates of the calculus **Gcl** as if they referred to sequents where M and N are sets of formulas. It follows as an immediate consequence that the contraction rules become superfluous. We call **Gcl*** the sequent calculus obtained this way. But if, on the other hand, we read the postulates of the calculus **Gcl** as if they referred to sequents where M and N are sequences of formulas, another consequence follows, in that we have to adjoin the two structural rules

$$\frac{M, \alpha, \beta, N \Rightarrow P}{M, \beta, \alpha, N \Rightarrow P}\ {EA} \qquad\qquad \frac{M \Rightarrow N, \alpha, \beta, P}{M \Rightarrow N, \beta, \alpha, P}\ {EK}$$

that are normally called *exchange* rules. We call **Gcl**** the calculus obtained in this way. **Gcl*** and **Gcl**** are, as we will see later, the same kind of variant of **Gcl**, but in the set and sequence alternative respectively. It can be shown that **Gcl**, **Gcl*** and **Gcl**** are in a certain sense "equivalent," i.e. given adequate translation functions between sequences, sets and multisets, we can show that they prove the same theorems. For a clear and detailed exposition of these equivalencies see [18, p. 227] and [139, p. 77].

Let us now consider the variants of the sequent calculus. First of all we wish to reassure the reader who may feel bewildered in the face of all these possibilities with Troelstra and Schwichtenberg's words [139, p. 51],

> Gentzen systems for **M**, **I** and **C** (minimal logic, intuitionistic logic and classical logic, respectively) have many variants. There is no reason for the reader to get confused by this fact. Firstly, we wish to stress that in dealing with Gentzen systems, no particular variant is to be preferred over all the others; one should choose a variant suited for the purpose at hand. Secondly, there is some method in the apparent confusion.

Next it is imperative to underline the following important fact. In the calculus **Gcl**, for each of the three connectives $\wedge$, $\vee$, and $\rightarrow$, we can choose between two equivalent formulations of the left introduction rules and two equivalent formulations of the right introduction rules. The A-rules that we have seen above are one formulation of the left introduction rules; the other is the following:

$$\frac{\alpha, \beta, M \Rightarrow N}{\alpha \wedge \beta, M \Rightarrow N}\ {\wedge A'} \qquad\qquad \frac{\alpha, M \Rightarrow N \quad \beta, P \Rightarrow Q}{\alpha \vee \beta, M, P \Rightarrow N, Q}\ {\vee A'}$$

$$\frac{M \Rightarrow N, \alpha \quad \beta, P \Rightarrow Q}{\alpha \rightarrow \beta, M, P \Rightarrow N, Q}\ {\rightarrow A'}$$

The K-rules that we have seen above are one formulation of the right introduction rules; the other is the following:

$$\frac{M \Rightarrow N, \alpha \quad P \Rightarrow Q, \beta}{M, P \Rightarrow N, Q, \alpha \wedge \beta}\ {\wedge K'} \qquad \frac{M \Rightarrow N, \alpha, \beta}{M \Rightarrow N, \alpha \vee \beta}\ {\vee K'} \qquad \frac{\alpha, M \Rightarrow N, \beta}{M \Rightarrow N, \alpha \rightarrow \beta}\ {\rightarrow K'}$$

We normally call the A'-rules and the K'-rules *multiplicative* (or *context-free*), while the A-rules and K-rules *additive* (or *context-sharing*). The proof of the equivalence of these two groups relies on the structural rules of weakening and contraction to such an extent that if we dropped (the effects of) either weakening or contraction, or both, we would not be able to define the same connectives. For this reason, in Gentzen calculi that differ from **Gcl** in that they lack (the effects of) the structural rules of weakening and contraction, one usually introduces different symbols for distinguishing the two groups mentioned above, e.g. the original Gentzen symbol $\wedge$ is kept for the conjunction introduced by additive rules, the new symbol $\otimes$ is added for the conjunction introduced by multiplicative rules (for more detailed proofs and definitions, see [94], pp. 11–15).

Definition 1.13 We call a variant of the Gentzen calculus in which the structural and the logical rules are taken as primitives and their roles are kept separate a *general* variant of the calculus.

Gcl is a general variant of the (multiset alternative) Gentzen system for classical propositional logic, as are **Gcl*** and **Gcl**** (of the set and sequent alternatives, respectively). If in **Gcl** we substituted the additive version of the logical rules with their equivalent multiplicative version, we would obtain another general variant of the (multiset alternative) sequent calculus for classical propositional logic. We designate the general variant as the "standard" one because it allows us to understand the sequent calculus in its full generality, as its name indicates.

Definition 1.14 A second variant of the sequent calculus was introduced by Dragalin [34], and we will call it *logical* variant of the Gentzen calculus. We designate it as this because, in this variant, all the structural rules are dropped since (it can be proved that) their effects are completely absorbed by the axioms and by the logical rules.

We are going to introduce $\mathbf{Gcl}_L$, which is a logical variant of the multiset alternative of the sequent calculus for classical propositional logic. (We leave to the reader the task of finding logical variants in the sequence and set alternatives.) $\mathbf{Gcl}_L$ is composed of:

Axioms

$Ax\colon p, M \Rightarrow N, p$ $\qquad\qquad$ $\bot A\colon \bot, M \Rightarrow N$

Logical Rules

Propositional Rules

$$\frac{\alpha, \beta, M \Rightarrow N}{\alpha \wedge \beta, M \Rightarrow N}\ {\wedge A'} \qquad \frac{M \Rightarrow N, \alpha \qquad M \Rightarrow N, \beta}{M \Rightarrow N, \alpha \wedge \beta}\ {\wedge K}$$

$$\frac{\alpha, M \Rightarrow N \qquad \beta, M \Rightarrow N}{\alpha \vee \beta, M \Rightarrow N}\ {\vee A} \qquad \frac{M \Rightarrow N, \alpha, \beta}{M \Rightarrow N, \alpha \vee \beta}\ {\vee K'}$$

$$\frac{M \Rightarrow N, \alpha \qquad \beta, M \Rightarrow N}{\alpha \rightarrow \beta, M \Rightarrow N}\ {\rightarrow A} \qquad \frac{\alpha, M \Rightarrow N, \beta}{M \Rightarrow N, \alpha \rightarrow \beta}\ {\rightarrow K'}$$

The sequent calculus $\mathbf{Gcl}_L$ is then obtained from the sequent calculus **Gcl** with the following three modifications:

- the original axioms are substituted by the generalised ones;
- the rules $\wedge A$, $\vee K$ and $\rightarrow K$ are substituted by the rules $\wedge A'$, $\vee K'$ and $\rightarrow K'$, respectively, and
- the structural rules are omitted.

$\mathbf{Gcl}_L$ is equivalent to **Gcl**, or, in other words, each of the structural rules of **Gcl** is (height-preserving) admissible in $\mathbf{Gcl}_L$. In $\mathbf{Gcl}_L$ the logical rules are height-preserving invertible.

Note that there can be many ways of changing a set of axioms and logical rules of a sequent calculus in order to obtain logical variants, but these changes must yield logical variants which are equivalent to each other and of course equivalent to the original calculus. Those changes that do not respect this condition are not modifications of the original set of axioms and rules but additions (or subtractions) of logical rules or axioms to it.

Definition 1.15 A third variant of the Gentzen system, which was introduced by Došen [31], is the *structural* variant. In this variant certain logical rules are dropped since (it can be proved that) their effects are completely absorbed by the remaining rules.

We call $\mathbf{Gcl}_S$ the structural variant of the multiset alternative of the sequent calculus for classical propositional logic. (We again leave to the reader the task of finding the structural variants in the sequence and set alternatives.) $\mathbf{Gcl}_S$ is composed of:

Axioms

Ax: $p \Rightarrow p$ $\qquad\qquad$ $\bot A$: $\bot \Rightarrow$

Structural Rules

Weakening and Contraction

$$\frac{M \Rightarrow N}{\alpha, M \Rightarrow N}\ WA \qquad\qquad \frac{M \Rightarrow N}{M \Rightarrow N, \alpha}\ WK$$

$$\frac{\alpha, \alpha, M \Rightarrow N}{\alpha, M \Rightarrow N}\ CA \qquad\qquad \frac{M \Rightarrow N, \alpha, \alpha}{M \Rightarrow N, \alpha}\ CK$$

Cut-Rule

$$\frac{M \Rightarrow N, \alpha \qquad \alpha, P \Rightarrow Q}{M, P \Rightarrow N, Q}\ cut_\alpha$$

Logical Rules

Propositional Rules

$$\frac{M \Rightarrow N, \alpha \quad M \Rightarrow N, \beta}{\overline{M \Rightarrow N, \alpha \wedge \beta}}\ (\wedge) \qquad \frac{\alpha, M \Rightarrow N \quad \beta, M \Rightarrow N}{\overline{\alpha \vee \beta, M \Rightarrow N}}\ (\vee)$$

$$\frac{\alpha, M \Rightarrow N, \beta}{\overline{M \Rightarrow N, \alpha \to \beta}}\ (\to)$$

where the double line in the logical rules means not only that the conclusion is derivable from the premise(s), but also that the premise(s) is (are) derivable from the conclusion.

The sequent calculus $\mathbf{Gcl}_S$ is then obtained from the sequent calculus **Gcl** with the following two modifications:

- one logical rule for each logical constant is omitted, and
- the logical rules of the calculus acquire a double line form (for this we take the rule $\rightarrow K$ in its multiplicative form).

$\mathbf{Gcl}_S$ is equivalent to **Gcl**, or, in other words, the logical rules that have been dropped are all admissible, with the use of the cut-rule, in $\mathbf{Gcl}_S$.

We conclude this section by pointing out that many other sequent calculi for many other logics can be obtained by modifying the one for classical logic. We consider two examples. The first one concerns the sequent calculus **Gil** for intuitionistic propositional logic, which can be obtained from **Gcl** by:

`[structural part]` restricting all axioms and rules to sequents with at most one formula on the right;

`[logical part]` replacing the rule $\rightarrow K$ by its multiplicative counterpart, $\rightarrow K'$, and substituting the rule $\rightarrow A$ with the following one:

$$\frac{M \Rightarrow \alpha \quad \beta, M \Rightarrow N}{\alpha \rightarrow \beta, M \Rightarrow N}$$

Because of the structural restriction, the rule that introduces the symbol $\vee$ on the right side of the sequent has uniquely an additive form.

By applying analogous modifications, we can obtain the (logical variant of the) Gentzen system $\mathbf{Gil}_L$ from $\mathbf{Gcl}_L$, and the (structural variant of the) Gentzen system $\mathbf{Gil}_S$ from $\mathbf{Gcl}_S$.

The second example concerns the sequent calculus **Gll** for the linear logic **ll** without exponentials. **Gll** can be obtained from **Gcl** by:

`[structural part]` omitting the (effects of the) rules of weakening and contraction;

`[logical part]` as for the logical part, several operations are required. First of all, we must add left and right introduction rules for the three connectives $\otimes$ (the multiplicative version of $\wedge$), $\oplus$ (the multiplicative version of $\vee$), and $\multimap$ (the multiplicative version of $\rightarrow$). Then, we must adjoin a single and involutory negation $\sim$ trough the following rules:

$$\frac{M \Rightarrow N, \alpha}{\sim \alpha, M \Rightarrow N} \qquad \frac{\alpha, M \Rightarrow N}{M \Rightarrow N, \sim \alpha}$$

> Finally we must split the logical constants $\bot$ and $\top$ into two operators, namely, $\bot$ and $\mathbf{0}$, and $\top$ and $\mathbf{1}$, respectively. Appropriate rules and axioms for each of them are added (see [139], p. 236).

By applying analogous modifications, we can obtain the (structural variant of the) Gentzen system $\mathbf{Gll}_S$ from $\mathbf{Gcl}_S$. $\mathbf{Gll}_L$ in this case coincides with **Gll**.

1.3 Philosophical Remarks

The sequent calculus, besides undoubtedly being an useful formal instrument of computation, also has philosophical significance. We would like to summarise the three main philosophical ideas directly linked with the Gentzen calculus.

1.3.1 Analyticity

The central notion of the first philosophical idea is that of *analytic proof*. Therefore we start by reminding the reader what an analytic proof is by contrasting it with synthetic proofs.

In this section, we use the term *proof* to denote a rational procedure by means of which one may establish the truth of a sentence. Depending on how this procedure is developed, one usually distinguishes between synthetic proofs and analytic proofs. Synthetic proofs start from acquired truths and are developed from the *top-down*, with the aim of determining the propositions whose truth is assured by the previous ones. By contrast, analytic proofs start from propositions whose truth is to be established and aim, by proceeding *bottom-up*, to reduce them to propositions whose truth has already been established.

Presented this way, the contrast between analytic and synthetic proofs is certainly interesting from an epistemological point of view, but it may hardly seem revealing from a logical point of view: indeed it seems to amount to little more than a distinction between different directions in which the same logical object can be read. But in fact the contrast between analytic and synthetic proofs does not simply boil down to a distinction between different readings of one and the same object, on the contrary it can involve deep issues. For example, we can generally state that while synthetic proofs privilege the *conciseness* of the structural process by neglecting the complexity of the expressions involved, the main concern in analytic proofs is to reduce, in a logically significant way, more *complex formulas to simpler ones*, without any care for the length of the inferential procedure.

Support for the analytic method has a long and venerable history in philosophy. This history extends back to ancient Greece, with both Plato and Aristotle being advocates of the analytic approach. The predilection of Plato for the analytic method was noted by Proclus [56, p. 213], in the following passage:

> The finest is the method which, by analysis, carries the thing sought up to an acknowledged principle, a method which Plato, as they say, communicated to Leodamas, and by which the latter too, is said to have discovered many things.

For his part, Aristotle demanded mathematicians to respect a methodological requirement strictly connected with the one of analysis and which consisted in the so-called "purity of methods" (the famous conviction of μετάβασις εἰς ἄλλο γένος): mathematicians should not use in their proofs concepts belonging to a theoretical domain different from that employed initially.

> You cannot prove anything by crossing from another kind – e.g. something geometrical by arithmetic [...]. Arithmetical demonstrations always contain the kind with which the demonstrations are concerned, and so too do all other demonstrations. Hence the kind must be the same, either *simpliciter* or in some respect, if a demonstration is to cross. [1, p. 12]

The pythagorean Hippocrates of Chius and the third century mathematician Pappus were also supporters of the analytic method. Pappus, especially, was the first to give, at the end of the book VII of his *Mathematical Collections*, a complete exposition of the analytic method (ἀνάλυσις): he expressed a clear preference for it over the synthetic one (σύνθεσις). He described the analysis as a process that

> takes that which is sought as if it were admitted and passes from it through its successive consequences to something which is admitted as a result of synthesis: for in analysis we assume that which is sought as if it were (already) done and we inquire what it is from which this results, and again what is the antecedent cause of the latter, and so on, until by so retracing our steps we come upon something already known or belonging to the class of first principles, and such a method we call analysis as being solution backwards. [96, p. 82]

In the early modern era, Descartes, Arnauld and Pascal attributed to analytic proofs a great fertility and strength: for them, it was through these procedures that the mathematician succeeded in convincing himself of the certainty of the proposition(s) at issue. As Descartes wrote,

> Analysis shows the true way by which a thing was methodically discovered and derived, as it were effect from cause, so that, if the reader care to follow it and give sufficient attention to everything, he understands the matter no less perfectly and makes it as much his own as if he had himself discovered it. [30, p. 128]

Galileo and Newton also showed a preference for analysis over synthesis. As Galileo [41, p. 51] said,

> That is what is done for the most part in the demonstrative sciences; this comes about because when the conclusion is true, one may by making use of the analytic method hit upon some propositions which are already demonstrated, or arrive at some axiomatic principle.

In the first half of the nineteenth century, analytic proofs found support from the great Bohemian thinker, Bernard Bolzano. Bolzano, in his *Beyträge zu einer begründeteren Darstellung der Mathematik* (1810) and later in his *Wissenschaftslehre* (1837), introduced a broader perspective on the notion of analyticity. This perspective was supported by a new idea of what mathematical proofs should be. For Bolzano to prove a mathematical truth meant to provide a *foundation* (Begründungen), i.e. a process which goes up from that truth to its *objective grounds*. Proofs became this way analytic procedures that start from a truth and go up to its reasons.

Let us remark that this position might be seen as a departure from the Aristotelian idea of "purity of methods." As Paoli [93, p. 227] says,

> Aristotle stated that mathematicians should not prove arithmetically geometrical truths. Concerning this Bolzano observes that Aristotle was wrong, since quite often the properties of the spatial magnitudes have *their objective reasons* in the properties of the general magnitudes. (English translation ours.)

Let us now return to Gentzen and to the sequent calculus. Note that in a sequent calculus where

(i) the cut-rule is admissible, or eliminable (see Definition 1.10, p. 6), and
(ii) in every other rule all the formulas that occur in the premises are subformulas of the formulas that occur in the conclusion,

we are sure that we can construct only *analytic derivations*. This is because, if conditions (i) and (ii) are respected, we know that it is not possible to lose any formula during the derivation process, and that we can only pass from logically more complex formulas to logically simpler ones (reading derivation process bottom-up). A sequent calculus is said to have the *subformula property* if, and only if, every provable sequent possesses a derivation such that every formula which occurs in it is a subformula of the formulas which occur in the conclusion. Observe that, if conditions (i) and (ii) are satisfied, then a sequent calculus has the subformula property. Gentzen seems to follow the long tradition presented above, in considering such a property of crucial relevance:

> Perhaps we may express the essential properties of such a normal form by saying: it is not roundabout. No concepts enter into the proof other then those contained in its final result, and their use was therefore essential to the achievement of that result. [42, pp. 87, 88]

This concludes the exposition of the analytic method and the enumeration of illustrious thinkers who have preferred and supported it. Last but not least we have quoted Gerhard Gentzen who seems to attribute great importance to the fact that the sequent calculus enjoys the subformula property, which ensures the analyticity of the calculus. We want to conclude the subsection by summing it up in the following claim:

(I) *A Gentzen calculus satisfying the subformula property is a tool for generating analytic derivations, where the importance of dealing with analytic derivations is witnessed by a long and important philosophical tradition.*

1.3.2 Logicality

The second philosophical idea linked with the Gentzen calculus was recently highlighted by Doŝen [33] and consists in the proof-theoretical attempt to answer the question: what is a logical constant? Or, more specifically, what criteria are there for logicality of expressions?

In order to explain Došen's solution to this question, let us proceed from his initial assumption about logic: (**a**) (formal) logic is the science of formal deductions. This conception of logic is not the only one, nor the standard one, but it nevertheless represents the appropriate starting point for getting to grips with Došen's argument. However, one may at this point be wondering what precisely it is that constitutes a formal deduction. Došen provides us with an answer: (**b**) a formal deduction is a structural deduction. But the question now is what a structural deduction is, and we may say a structural deduction is a deduction that employs *only the structural rules* of the sequent calculus, that is to say a deduction which can be described by restricting the sequent-language to structural sequents, i.e. schematic sequents where no constant of the object language appears. Moreover, for Došen, all deductions can be derived from structural deductions.[3]

Hopefully this will have alleviated any uncertainty about formal logic and structural deductions. Combining (**a**) and (**b**) yields an important conclusion: logic is essentially about structural deductions, which is to say, logic is essentially articulated at the *structural level*. Let us reflect upon the consequences of this observation. Our goal is to discover a criterion for the logicality of expressions. In other words, we are seeking a criterion which can determine whether a certain expression, such as a constant, is logical. If it is true that logic concerns structural elements, then we must answer the above question with the claim that (**c**) an expression is logical if and only if it can be analysed in purely structural terms. This is exactly what Došen claims on page 368 of [33].

Recall the structural variant of the sequent calculus $\mathbf{Gcl}_S$ that we have introduced with Definition 1.15, p. 10 This variant is composed of the axioms, the four structural rules of weakening and contraction, the structural cut-rule, and one double line logical rule for each logical constant. Consider the double line rule for the constant $\wedge$:

$$\frac{M \Rightarrow N, \alpha \quad M \Rightarrow N, \beta}{M \Rightarrow N, \alpha \wedge \beta}\ (\wedge)$$

Reading top-down, the rule says that we can link two formulas derivable from the same multiset M; bottom-up, it indicates how to go back to the structural metalanguage, once the conjunction symbol has been introduced. What is important here is that every object belonging to the upper sequents is purely schematic, i.e. these sequents are structural, and that, therefore, the double line rule $(\wedge)$ establishes an equivalence between a sequent with a main connective $\wedge$, occurring in the formula $\alpha \wedge \beta$, and two purely structural sequents. It follows that the rule can be considered as providing an analysis of the logical constant $\wedge$ in purely structural terms. In other words, following the assumption (**c**), this rule can be taken to represent the formal logical criterion for the constant $\wedge$. Since we can generalise the above remarks to each of the double line logical rules of the calculus $\mathbf{Gcl}_S$, it then appears that we have found the answer to the questions asked at the beginning of this subsection:

(II) *double line logical rules represent a formal criterion for the logicality of the constant they introduce in the framework given by the structural variant of the sequent calculus.*[4]

1.3.3 From Logicality to Inferentialism

Closely related to our point (II) is the position frequently labelled as *inferentialism*. Roughly speaking, inferentialism can be characterised as the thesis that the meaning of logical constants is fully defined by the inferential rules that govern their use.

At first glance, the reader may be led to wonder whether there is a meaningful distinction to be made between the *analysis* of logical constants in structural terms, discussed in the previous section, and the inferentialist thesis that inference rules *define* logical constants. We will alleviate this doubt by illustrating the distinction between an analysis *tout court* and a definition. More precisely we will see that there are two properties that definitions satisfy while analyses do not. It will turn out that not even analyses in structural terms satisfy these two properties and hence that they do indeed differ from definitions.

Let us start by explaining what it means to define the meaning of a logical constant. In this regard, we can distinguish at least two different points of view.[5] According to a realist conception, we grasp the meaning of a sentence when we know what it is for that sentence to be true, where the truth is thought of as something that a sentence either possesses or lacks independently of our capacity to recognise it. According to an anti-realist conception, on the other hand, we grasp the meaning of a sentence when we know how to use it, where the use has to be understood as "correct use;" for otherwise, as Wansing [147, pp. 6, 7] says,

> meaning would depend on the factual linguistic behaviour of certain language users, and hence meaning would be pragmatic rather than a genuinely semantic and speaker-independent notion.

This anti-realist conception of the meaning is the conceptual basis of the semantic theory often called *proof-theoretic semantics* (the term was coined in the early 1990s by Schroeder-Heister, see [125]). Inferentialism can be seen as the semantic engine of proof-theoretic semantics.

The difference between a realist approach and an anti-realist approach to the meaning of a logical constant can be illustrated by the standard techniques for defining constants. Let us suppose that we have a logical constant $\star$ of a language $\mathcal{L}$ and we want to give a realistic definition of $\star$. This involves two steps: first of all we specify a language $\mathcal{M}$ which does not contain $\star$, and then we formulate, in $\mathcal{M}$ plus $\star$, an equivalence between sentence(s) containing $\star$ and the same sentence(s) containing other appropriate symbol(s) belonging to the language $\mathcal{M}$; by means of this equivalence we specify the truth values of sentences containing $\star$. We recall that the sentence containing the $\star$ is usually called *definiendum*, while the ones not containing the $\star$ are called *definiens*.

Let us now pass to anti-realistic definitions. We suppose that we want to give an anti-realistic definition of the constant $\star$. This (again) only involves two steps: first of all we specify a system $\mathtt{T}$ which does not contain $\star$, and then we add to $\mathtt{T}$ the constant $\star$ *via* a set of inference rules.

The difference between realistic and anti-realistic definitions of logical constants should now be clear. Realistic definitions take as central the notions of language and truth, and exploit equivalences, in order to give the meaning of logical constants. By contrast, anti-realistic definitions take as central the notions of system and derivation, and exploit sets of rules to give the meaning of logical constants.

Returning to the question of analysis, let us consider a constant $\star$ of a language $\mathcal{L}$, and let us suppose that we want to analyse $\star$. There are only two steps to make: first of all we specify a language $\mathcal{M}$ which does not contain $\star$, and then we establish that a sentence **A** in $\mathcal{M}$ plus $\star$, in which the constant $\star$ only occurs once, is equivalent to a sentence **B** in $\mathcal{M}$. In this way we analyse the constant $\star$. Given this explanation of analysis, it is not at all obvious that there is any real difference with respect to definitions. The situation is made even more ambiguous by the fact that both definitions (realistic and anti-realistic) and analyses should satisfy the two following well-known properties:

A_1 {adequacy} analyses, as well as definitions, must be sound and complete (see for details [33, p. 369]),

A_2 {uniqueness} the logical constants $\star$ and $\star'$ can receive the same analysis, as well as the same definition (realistic or anti-realistic) if, and only if, $\star$ and $\star'$ are the same constant (for further details see Section 1.8).

Nevertheless an equivalence can satisfy A_1 and A_2, and therefore qualify as an analysis, without thereby amounting to a definition. It is the satisfaction of at least two further conditions that an analysis need not satisfy that distinguishes a definition from an analysis. The formulation of these two conditions varies depending on the type of definition considered. For the moment we just present those set up for realistic definitions. They were introduced by the Polish logician Leśniewski [76] and they are as follows:

E_1 {conservativeness} there exists no formula α, not containing any occurrence of the symbol $\star$, that is valid in $\mathcal{L}$ without being already valid in $\mathcal{L}$ minus $\star$.

E_2 {eliminability} for any formula α of $\mathcal{L}$ containing the symbol $\star$, the definition should enable us to find a formula β not containing the symbol $\star$, such that β is semantically equivalent to α.

As we will see in Sections 1.4 and 1.5, these two criteria, if respected, ensure that a definition of a logical constant gives the whole meaning of the logical constant, and nothing more; an analysis, by contrast, need not satisfy these two criteria, since it just does not give the meaning of logical constants.

In the light of this account of analysis and definitions, let us now concentrate on the analysis of logical constants in structural terms. We shall only consider the analysis of conjunction given by the double line logical rule ($\wedge$) (see the previous Section 1.3.2). In this analysis, the language $\mathcal{L}$ is the language of propositional logic, and $\mathcal{M}$ is the deductive metalanguage in which structural deductions can be explicitly described; more precisely, $\mathcal{M}$ is the language of structural sequents. The sentence **A** is the lower sequent of ($\wedge$), while the sentence **B** is the upper sequent of ($\wedge$). Double line stands for an equivalence; finally, it can be shown that ($\wedge$)

serves to characterise conjunction soundly, completely and uniquely, i.e. conditions A_1 and A_2 are met (e.g. see [31]).We can therefore conclude that ($\wedge$) represents an analysis, in structural terms, of conjunction. On the other hand, the structural analysis of conjunction given by ($\wedge$) does not satisfy conditions E_2, and condition E_1 also fails (see [33, p. 374]). Therefore analyses in structural terms differ from definitions.

By way of a summary, note that we can informally explain the distinction between analysis in structural terms and definitions by thinking about the question of what logical constants are, as Hacking famously did in [54]. This question can be understood in two different ways. One way corresponds to the attempt to capture what identifies logical constants. Following claim (II), analyses in terms of double line rules *are* what is common to logical constants. But having identified logical constants, one can still ask for their essential nature: how can we grasp the essential nature, which is to say the meaning, of logical constants? One possible answer is provided by inferentialism: inferential rules give the meaning to the constants of which they govern the use. As Došen says,

> Our attempt to analyse logical constant [...] should not be confused with [...] (inferentialism). First, the main goal of this program is to show that the meaning of logical constants can be given syntactically, whereas our analyses are neutral with respect to this claim, and are equally compatible with the view that the meaning of logical constants is to be given in a more conventional semantical framework. Second, the search for a criterion for being a logical constant does not always have a very important place in this program. [33, p. 378]

1.3.4 Harmony

One of the main attacks against inferentialism was famously given by Prior [108] who introduced the two tonk-rules

$$\frac{\alpha}{\alpha\ tonk\ \beta} \qquad \frac{\alpha\ tonk\ \beta}{\beta}$$

in a natural deduction system with the aim of showing that inferentialism was ill-founded. Indeed, the calculus resulting from their addition can prove $\alpha \vdash \beta$ for any formula α and β whatsoever, which is of course to say that it is trivial.

The consequence that Prior drew is that not any set of rules is meaning conferring. Although many commentators have taken this as an argument against proof-theoretic semantics, defenders of this position replied that there are natural constraints on logical rules which guarantee them to confer meaning to the constant they introduce, and that tonk and tonkish connectives do not satisfy these constraints. The constraint that has received by far the most attention is proof-theoretic *harmony*.

Informally speaking, harmony is supposed to balance two features of a logical connectives $\star$: (i) the conditions under which one is entitled to assert a sentence containing the $\star$ connective; (ii) the consequences one is entitled to draw from a sentence containing the $\star$ connective. There have been many attempts to formally capture the harmony requirement. These are:

– harmony as conservativeness (Belnap [8] and Dummett [35]),
– harmony as deductive equilibrium (Tennant [136, 137]),
– harmony as reduction (Prawitz [107], Read [113], Schroeder-Heister [126]).

The most recent and most developed account is harmony as reduction; this is what we will focus on next.

At this point, a further distinction is necessary. On the one hand, Prawitz and Dummett identify harmony with normalisation (a cut-elimination-like theorem for natural deduction), which consists of a set of reduction-steps. On the other hand, Read [114] has shown that this position turns out to be problematic in several systems of modal logic, and has also proposed a new characterisation of the harmony requirement, which we call, following Hjortland [61], *General Elimination Harmony*. This characterisation is a local constraint on pairs of natural deduction rules which allows us to construct elimination rules whose form totally depends on the form of the corresponding introduction rules.

Francez and Dyckhoff [39], improving on Read [113], present the General Elimination Harmony criterion in the following way. In a natural deduction system the schematic introduction rule for a logical constant $\star$ occurring in α is as follows:

$$\dfrac{\begin{matrix}[\Sigma_i]_{j_1^i,\ldots,j_m^i}\\ \vdots\,\pi_i^*\\ \beta_i\end{matrix}}{\alpha}\;\star I^{j_1^i,\ldots,j_m^i}$$

where $i = 1, \ldots, n$, $[\Sigma_i]_{j_1^i,\ldots,j_m^i}$ are (possibly empty) sets of assumptions discharged by $\star I$, and β_i are formulas. The corresponding *general* elimination rule is then as follows:

$$\dfrac{\begin{matrix}\vdots\,\pi\\ \alpha\end{matrix}\quad \begin{matrix}\vdots\,\pi_1\\ \Sigma_1\end{matrix}\quad \ldots\quad \begin{matrix}\vdots\,\pi_n\\ \Sigma_n\end{matrix}\quad \begin{matrix}[\beta_1]1\\ \vdots\,\pi_1'\\ \gamma\end{matrix}\quad \ldots\quad \begin{matrix}[\beta_n]n\\ \vdots\,\pi_n'\\ \gamma\end{matrix}}{\gamma}\;\star GE^{1,\ldots,n}$$

where γ is a formula. Note that in the case of the connectives $\wedge$ and $\rightarrow$, the general elimination rule does not coincide with the elimination rule but represents, as is witnessed by the name, its generalisation (for further details see [113]).

It can be shown that Prior's tonk-rules do not meet the General Elimination Harmony criterion and hence it can be claimed that they do not constitute a counterexample to inferentialism. Nevertheless, the criterion does not in itself guarantee that the calculus is conservative and hence non-trivial. Indeed if we consider the connective $\odot$ which has the following two rules:

$$\dfrac{\begin{matrix} M \\ \vdots \\ \neg\odot \end{matrix}}{\odot}\ \odot I \qquad \dfrac{\begin{matrix} M \\ \vdots \\ \odot \end{matrix} \quad \begin{matrix} N, [\neg\odot] \\ \vdots \\ \gamma \end{matrix}}{\gamma}\ \odot E$$

we can claim that the two rules are harmonious according to the General Elimination Harmony principle, but they are not conservative. Read then draws the conclusions that there is a discrepancy between harmony and conservativeness and that, though rules should of course be harmonious, there is no particular reason for them to be conservative. We would like to argue that this conclusion fails to account for an important detail.

The rules $\odot I$ and $\odot E$ proposed by Read do not satisfy a condition on rules which we will discuss in Section 1.7, namely the separation condition. In brief, this condition demands that logical rules only contain the logical constant which they introduce or eliminate. However, Read's rules contain two logical constants: $\odot$ but also the connective $\neg$. It follows that, in order to introduce the rules $\odot I$ and $\odot E$ to a system, one must already have introduced the rules for $\neg$. But in such systems, Read's claim that the rules $\odot I$ and $\odot E$ are in harmony is false: for the rule $\odot I$ is not only a rule that introduces $\odot$, but is also a rule that eliminates $\neg$, and likewise the rule $\odot E$ not only eliminates $\odot$, but also eliminates $\neg$. And, considered as elimination rules for the connective $\neg$, these rules do not satisfy the General Elimination Harmony criterion, because they are not derivable from the standard introduction rule for $\neg$. So, contrary to what Read claims, $\odot I$ and $\odot E$ do not show that the General Elimination Harmony criterion differs from conservativeness, for it is a case where both of these conditions are violated. One might nevertheless ask whether Read's criterion does represent a local, and hence desired, constraint for assuring conservativeness.

Generally speaking, Read criticises conservativeness as formalisation of harmony. If harmony is taken to be a sort of balance between introduction and elimination rules, then it is true that the criterion he proposes, with its local flavour, looks much more like the appropriate formalisation of harmony. On the other hand, if inferential rules are taken to be definitions of the symbol they introduce, then, following what we have said in the previous section, these rules must be conservative to ensure that they do not give anything more than the meaning of the symbol they define. In other words, here we do not defend conservativeness as the appropriate formalisation of the notion of harmony, but we do defend it as a a constraint that definitions must respect in general.

1.3.5 Inferentialism

In the course of our long discussion on inferentialism, we did not touch upon two important questions: which are the rules that are supposed to give meaning to logical constants, and what system do they belong to? Traditionally the preferred tools of proof-theoretic semantics are the natural deduction calculi. There are two reasons for this choice: on the one hand, natural deduction systems have a greater intuitive appeal than sequent calculi; on the other hand, they are best suited to handle intuitionistic logic, which is often the reference logic for supporters of proof theoretic semantics. Recently, however, things seem to have changed and sequent calculus has been rediscovered. According to Paoli, this has occurred for two reasons:

> First of all, intuitionistic logic is no longer the sole constructive logic on the market. Now we know that, if we embrace linear logic, we can retain the most pleasing aspects of intuitionism – such as the possibility of assigning a procedural or even a computational content to its deductions – without being committed to some of its less agreeable features, e.g., its unwieldy asymmetric sequent calculus. Secondly, cut-free sequent calculi are even more apt than natural deduction systems for a molecularistic semantics of logical constants: not only do we have separate rules for each connective, but we are also guaranteed that larger fragments conservatively extend smaller fragments containing fewer connectives. [95, p. 536]

In line with this new trend, we choose the sequent calculus as the reference system for inferentialism. Moreover, in line with what it is commonly accepted in this framework (see Hacking [54], Paoli [95] and Wansing [148]), we consider the left and the right introduction rules of the sequent calculus to provide the meaning of the constant they introduce. Finally, following a reflection of Sambin:

> one has to abandon the traditional scheme which says that the rule introducing a connective is always the rule operating on the right and the rule on the left is always the elimination rule. [120, p. 980]

and a remark by Read:

> For ‘A⇒ B’ is really no more than the introduction of a conditional into the assumption set, and such an introduction is governed by (what might be called) Gentzen’s left rule. [113, p. 135]

we think that both A-rules and K-rules provide the grounds for inferring a sentence containing the connective they define, in the antecedent and in the consequent of a sequent respectively.

We sum up these several positions in the following claim:

(III) *the left and the right introduction rules of the sequent calculus together can be considered as a definition of the symbol they introduce since they both give the grounds for asserting a sentence containing the connective they define.*

1.3.6 Concluding Remarks

The following general remark concludes this section. Currently there is a flourish of numerous different logics. The classical sequent calculus is often extended and modified in order to handle them. However, it seems reasonable and useful to request that these modifications are judged, at least in part, according to the extent to which they respect (i) the philosophical significance of the Gentzen calculus, and (ii) the mathematical strength of the Gentzen calculus. Each of the properties that will be introduced in Sections 1.4, 1.5, 1.6, 1.7, 1.8, 1.9, and 1.10 is nothing other than a condition for (I) or (II) or (III) to hold; in fact, these properties ensure that a new sequent calculus has the philosophical significance of the original one. By contrast, in order to verify the mathematical strength of the new calculus, we will have to "get our hands dirty" and work with it.

That said, we conjecture the existence of a link between the philosophical and the mathematical aspects of a Gentzen calculus: a sequent calculus that respects each of the properties that we will list below and which are tenable mainly for philosophical reasons will most likely also be a very good mathematical tool.

1.4 Subformula Property

The subformula property states that every provable sequent of a Gentzen calculus should possess a derivation such that every formula that occurs in it is a subformula of the formulas that occur in the conclusion. As we have seen in the Section 1.3.1, a sequent calculus has the subformula property provided that it satisfies the following two conditions:

(i) the cut-rule is admissible (or eliminable), and
(ii) in each of its rules all the formulas that occur in the premises are subformulas of the formulas that occur in the conclusion.

For reasons related to analyticity, it is essential for a Gentzen system to possess the subformula property. The relevant question is whether there are any other reasons for requiring that a sequent calculus has the subformula property. The answer is affirmative, and we turn to it in more detail.

The first reason is technical: the subformula property yields several results. To mention just one, it often allows us to prove the decidability of a given calculus.

The second reason is linked with our claim (III), which, recall, says that the left and right introduction rules of the sequent calculus constitute the definition of the symbol they introduce. In Section 1.3.3, we have explained that definitions, in order to be such, must be conservative and eliminable. We will now see what anti-realistic conservativeness consists in. In the next section we will deal with anti-realistic eliminability.

Let us first of all remind the reader that a realistic definition, whose central notions are those of language and truth, is conservative (see point E_1 above) when *it does not modify the truth value* of the sentences not containing the symbol to be defined, i.e. it is conservative when it does not give anything more than the meaning (in terms of truth values) of the symbol to be defined. By naturally adapting this explanation to the anti-realistic definitions, whose central notions are those of system and derivation, we can claim that an anti-realistic definition is conservative when *it does not allow one to prove* any new sentence formulated in the old vocabulary, i.e. it is conservative when it does not give anything more than the meaning (in terms of derivation) of the symbol to be defined.

This explanation of "anti-realistic conservativeness," though correct, may still be modified to better fit our requirements. Indeed, the above definition of anti-realistic conservativeness is intended to cover a wide range of systems, and we are not interested in systems in general, but only in Gentzen's systems. We therefore introduce a more detailed definition of anti-realistic conservativeness: a calculus $\mathbf{G}^{'}$ obtained by adding to the calculus **G** one or more connectives and rules concerning these connectives, is said to *conservatively extend* the calculus **G**, when $\mathbf{G}^{'}$ proves no sequent containing just the old connectives which was not already provable in **G**.

With the definition of anti-realistic conservativeness in hand, the question is now what property can guarantee that the logical rules of a sequent calculus, as definitions for the symbol they introduce, are conservative. According to many (e.g. see [71, 94, 147]) the answer is precisely the subformula property. Indeed if a sequent calculus $\mathbf{G}^{'}$ enjoys the subformula property, any formula which occurs in a derivation of **G** must occur as a subformula of the end-sequent itself and this is enough to ensure that no new rule of $\mathbf{G}^{'}$ needs to be used for establishing this end-sequent. Hence the subformula property enables us to show that the introduction rules of a sequent calculus are conservative, which is to say that they do not give anything more than the meaning of the symbol they introduce. This is the third good reason for considering this property as desirable for a sequent calculus.

We conclude this section with an important observation. Sometimes a new sequent calculus where the cut-rule is left out is proposed, e.g. the multiple sequent calculi, Section 3.1, or the semantic modal sequent calculi, Section 4.1. Quite often the cut-rule is not formulated since it is difficult to establish how it would be formulated. If one then wants to show that the sequent calculus has the subformula property, one must prove that it is complete with respect to the corresponding semantic class of frames or models (and then check that condition (ii) is respected). In such a situation the reader may think that the inability to state a cut-rule, and the inability to provide a syntactic cut-elimination proof are not serious problems: a syntactic proof seems superfluous if we already have a semantic proof; as for the cut-rule, we have argued that in any case we must be able show its redundancy, so why should we worry whether it can be formulated?

Let us start by underlying the importance of having a syntactic proof of cut-elimination. This proof is important for its deeply constructive character, and because it represents the *conditio sine qua non* for a constructive proof of the equivalence between a Gentzen system and a Hilbert system.[6]

By contrast, as regards the formulation of a cut-rule, Boolos [13] and D'agostino and Mondadori [29] have emphasised that the total absence of a cut-rule from a sequent calculus is quite undesirable. Let us briefly explain why. Let us start with a quote from Descartes:

> But it [the analytic method] contains nothing to incite belief in an inattentive and hostile reader; for if the very least thing brought forward escapes his notice, the necessity of the conclusion is lost [...] Synthesis contrariwise employs an opposite procedure, one in which

> the search goes as it were from effect to cause (...). It does indeed clearly demonstrates its conclusions, and it employs a long series of definitions, postulates, axioms, theorems and problems [...] Thus the reader, however hostile and obstinate, is compelled to render his assent. Yet this method is not as satisfactory as the other and does not equally well content the eager learner, because it does not show the way in which the matter taught was discovered. [30, p. 128]

Descartes, though a supporter of the analytic method, recognises that this method is not commonly adopted. Indeed, in our reasoning, we often employ synthetic proofs in which we use subsidiary conclusions that help us to shorten the process of demonstration. The cut-rule is nothing but the formal equivalent of this exploitation of subsidiary conclusions, and this is the reason why it is desirable for a sequent calculus to possess it: the cut-rule allows us to continue using auxiliary lemmas in a formal way.

Note that this argument also implies that an unrestricted cut-rule is to be preferred over an *analytic* cut-rule, where an application of cut

$$\frac{M \Rightarrow N, \alpha \quad \alpha, P \Rightarrow Q}{M, P \Rightarrow N, Q}$$

is said to be analytic if the cut-formula α is a subformula of some formula in the conclusion.

To conclude, we must distinguish between:

- Subformula property when the cut-rule is formulated and there exists a syntactic proof of cut-elimination.
- Subformula property when the cut-rule is formulated but there does not exist a syntactic proof of cut-elimination.
- Subformula property when the cut-rule is not formulated and there does not exist a syntactic proof of cut-elimination.
- Subformula property when the calculus has an analytic cut-rule.

Obviously a good sequent calculus should satisfy the first condition.

1.5 Admissibility of the Structural Rules

As we have said in Section 1.2, a sequent calculus can have several variants, each of which is suited for a different purpose. In addition to the general variant (see Definition 1.13, p. 9), which should be used to introduce any sequent calculus since it allows one to understand it in its full generality, we have also focussed on two other variants: the logical variant (see Definition 1.14, p. 9) and the structural variant (see Definition 1.15, p. 10). Not all sequent calculi have these two variants. In this section we explain the reasons why any good Gentzen system should have a logical variant, while in the next section we will explain why it should have the structural one too.

There are two reasons: a technical one and a philosophical one. The technical reason is the same as for the subformula property. In the framework offered by the logical variant of the Gentzen calculus, several results can be proved in an easy and elegant way, e.g. the invertibility of the logical rules, or the decidability of the calculus.

The philosophical reason, on the other hand, is linked with claim (III), and hence with the intuition that the logical rules define the constants they introduce. As we have said at the end of Section 1.3.3, (realistic or anti-realistic) definitions should satisfy not only the characteristics relating to analyses (see conditions A_1 and A_2), but also the criteria of conservativeness and eliminability. While we already know what realistic and anti-realistic conservativeness (see condition E_1 and the previous section, respectively) and realistic eliminability (see condition E_2) consist in, we must still determine what anti-realistic eliminability is. Let us try to plug that gap by starting to examine what exactly the eliminability criterion is, or better, what it is a criterion for. For this, consider a language $\mathcal{L}$ plus a constant $\star$, and the same language $\mathcal{L}$ minus the constant $\star$. A realistic definition of the constant $\star$ is proposed, i.e. an equivalence between a definiens and a definiendum. Is this a good definition? One could answer: a definition is a good definition if it gives the whole meaning of the expression (in this case a logical constant) that it is supposed to define. According to the realist conception, to give the meaning of an expression means to give the conditions under which the expression is true. Therefore, for a realist, to ask whether an equivalence gives the whole meaning of the logical constant that it defines amounts to asking the equivalence to enable us to find, for any sentence containing the constant $\star$, a sentence not containing the symbol $\star$ which has the same truth value. But this is exactly the eliminability criterion (compare with E_2) for realistic definitions. Therefore we can conclude that the eliminability criterion is a criterion for establishing whether a definition gives the whole meaning of the expression that it defines.

Let us now try to repeat the same reasoning for anti-realistic definitions in order to get an anti-realistic eliminability criterion. Consider a calculus **G** that does not contain the constant $\star$ nor a set of rules that introduce it, and a calculus $\mathbf{G}'$ that contains the constant $\star$ and a set of rules that introduce it. Now the question is: can these rules be considered as good definitions of the constant $\star$? Well, as above, one might answer that these rules are good definitions of the constant $\star$ if they give us the whole meaning of the constant $\star$. Following the anti-realist conception, to give the meaning of an expression means to give the conditions under which the expression can be asserted. Therefore, for an anti-realist, the question of whether a definition gives the whole meaning of the logical constant that it defines amounts to whether the logical rules that introduce $\star$ determine exactly which sentences containing the constant $\star$ can be asserted. But this is the anti-realistic eliminability criterion that we are looking for.

Nevertheless, our investigation has not reached its term. Given that we have found the eliminability criterion for anti-realistic definitions, we can now ask which property of the sequent calculus ensures that the logical rules satisfy this criterion. The answer is simple: the proof of the admissibility of the structural rules.[7] Indeed, if this requirement is not met, then there will be sequents involving the new constant which cannot be established *solely* on the basis of the logical rules for that symbol together with the old rules, restricted to the old vocabulary. If this requirement *is* met, on the other hand, then the logical rules for the new symbol enable us to infer all the sequents involving the new vocabulary, without having to apply the old rules except for formulas involving the old vocabulary.

Recall that a logical variant of the sequent calculus is composed of only axioms and logical rules. The cut-rule, as well as the other structural rules, are all admissible, and hence the two criteria of anti-realistic conservativeness and eliminability are satisfied, if we accept the argument proposed in the last two sections. We can thus draw the conclusion that, thanks to the introduction rules, it is a logical variant of the sequent calculus that allows us to get a handle on the meanings of the logical constants.

This conclusion is correct but incomplete. We should verify another fact: we not only need that the structural rules and the cut-rule are admissible, but also that the sequents of the form $\alpha \Rightarrow \alpha$ are. According to Belnap the admissibility of $\alpha \Rightarrow \alpha$ constitutes

> half of what is required to show that the meaning of formulas . . . is not context-sensitive, but that instead formulas mean the same in both antecedent and consequent position. (The [Cut] Elimination Theorem . . . is the other half of what is required for this purpose). [8, p. 383]

A similar remark can be found in [44, p. 31].

We can therefore claim that in a logical variant of the sequent calculus in which sequents of the form $\alpha \Rightarrow \alpha$ are admissible, the logical rules give the whole meaning of the symbol they introduce and nothing more, and that this meaning is not context-sensitive. This is one of the reasons for demanding from a good sequent calculus that it has a logical variant, if we find the idea expressed by claim (III) compelling.

1.5.1 Operational vs Global Meaning

In a recent article [95], Paoli, in order to defuse Quine's meaning-variance argument against the existence of deviant logics and genuine rivalry between logics (see [109–111]), introduces the distinction between the *operational* and the *global* meaning of logical constants:

- the operational meaning of a logical constant $\star$ is fully specified by the right and left introduction rules for $\star$;
- the global meaning of a logical constant $\star$ is fully specified by the class of provable sequents containing $\star$.

While the set of provable sequents varies from system to system, the logical rules are (or at least they might be selected in a way such that they are) always the same, therefore the two notions come apart. However, Paoli suggests identifying the meaning of logical constants with the operational meaning in order to block Quine's meaning-variance argument. This contradicts what we have been arguing here – namely that the logical rules give the whole meaning of the constant they introduce when they are considered in a logical variant of the sequent calculus, i.e. when they prove by themselves all the sequents containing the symbol they introduce. Let us thus examine Paoli's position.

There are at least two arguments against Paoli's notion of operational meaning. The first one was proposed by Hjortland [61, p. 16] who claims that

> Inferentialism leaves it open whether all inferential rules are meaning-conferring or only some (and does not even consider structural assumptions not in rule-form), but meaning-theoretically the choice makes considerable difference. [...] Inferentialism is based on the idea that, at least for logical constants, the entrenched use of the expressions fully determines their meaning. But, if some aspects of the inferential role of these expressions come short of being semantically significant, then we need a corresponding use-theoretic distinction to explain how meaning supervenes on some (systematic) use of an expression but not all (systematic) use of an expression.

The second argument is the following. If the rules of inference *tout court* determine the meaning of the constants they introduce, i.e. if the meaning of the constants is their operational meaning, then we must accept that in the classical sequent calculus the constants $\wedge$, $\vee$ and $\rightarrow$ may have (at least) two different meanings: an additive and a multiplicative one. This is clearly an unacceptable conclusion. To counter it, one could reply that the above conclusion is not entirely correct since additive and multiplicative rules can be shown to be equivalent, and so that the different meanings they provide are in reality the same. This is certainly true, but crucially the equivalence between the additive and multiplicative rules can be shown to hold *only* from a global point of view since we require structural rules to prove it (see Section 1.2). Hence the reply is not decisive, and also the second argument would appear to go against the operational meaning.

The problem is that, if we accept the view according to which the meaning of logical constants is global, and therefore may vary from system to system, we no longer seem to have a defence against Quine's attack. Shall we then accept the conclusion that logical constants of different calculi are incomparable? Not necessarily. If we start comparing constants for their logicality and not for their meaning, then we still have something to say against Quine's attack. Indeed, while the left and right introduction rules (of the logical variants of the sequent calculus) vary from calculus to calculus, the double line logical rules (of the structural variant of the sequent calculus) are always the same. To put it differently, the meaning of logical constants varies from calculus to calculus, but their logicality is invariant. This is precisely the point that we can exploit against Quine's argument: we cannot compare the meaning of, say, the constant $\wedge$ in two different calculi, but we can recognise that in both calculi $\wedge$ is a logical constant by means of the same double line logical rule.

The moral that we draw from this discussion is that the distinction between operational and global meaning made by Paoli encounters some problems, but if we substitute the logicality criterion for the operational meaning, we have a defensible position against Quine's objection.

1.6 Admissibility of the Logical Rules

We will use this section to argue that it is important for a good sequent calculus to have the structural variant. In fact, much of the job has already been done. Recall that in Section 1.3.2 we have fully explained the close link between the structural variant of the sequent calculus and the idea of analysing the logical constants. The argument was as follows.

We started out looking for a formal criterion for logicality of expressions that captures the following condition: a constant is logical if and only if it can be analysed in purely structural terms. We considered double line logical rules. These rules do no more than translate the logical constants that they introduce into structural expressions. We then concluded that double line logical rules are the desired formal criterion for logicality of expressions.

This conclusion, though correct, is incomplete. Indeed it only holds if the translation is made in a purely structural framework which ensures that the analysis has really been done in structural terms. Such a framework is provided by the structural variant of the sequent calculus.

Therefore, thanks to the double line logical rules, the structural variant of the sequent calculus provides us with the ideal means of getting a grip on what characterises logical constants. This is one of the reasons for demanding from a good sequent calculus that it has the structural variant, if we find the idea expressed by claim (II) compelling.

1.7 Explicitness, Separation and Symmetry

Explicitness, separation and symmetry describe natural conditions on the format that the logical rules should have in a logical variant of the sequent calculus, if we take them to define the symbol they introduce. Let us explain these three conditions one by one.[8]

The introduction rules for a constant $\star$ will be called *weakly explicit* if they exhibit $\star$ in their lower sequent only, and they will be called *explicit* if, in addition to being weakly explicit, they exhibit only one occurrence of $\star$ on the right or on the left side of the sequent arrow.

The introduction rules for a constant $\star$ will be called *separated*, if they do not exhibit any connective other than $\star$.

The introduction rules for a constant $\star$ will be called *weakly symmetric*, if every rule either belongs to the A-rules set (the set of left introduction rules), or to the K-rules set (the set of right introduction rules). The introduction rules for a constant $\star$ will be called *symmetric*, if they are weakly symmetric and both $\star A$ and $\star K$ are non-empty.

The separation property has been introduced by Zucker and Tragesser [151], while the other two by Wansing [145], who rightly says that

> separation, symmetry and explicitness of the rules imply that in a sequent calculus for a given logic $\mathbb{L}$, every connective that is explicitly definable in $\mathbb{L}$ also has separated, symmetric and explicit introduction rules. [145, p. 127]

To sum up, the three properties of explicitness, separation and symmetry are characterised by two important features: first of all, all three are local properties, and secondly they are all related to claim (III). (One may note that the explicitness and separation properties are linked with claim (II) as well.) This explains why we have presented them in a single section.

1.8 Uniqueness

The uniqueness property, proposed by Došen [31], is a property that on the one hand concerns the logical rules, and on the other hand it is related to claims (II) and (III). (As we have already seen in Section 1.3.3, point A_2, analyses and definitions should satisfy the uniqueness property).

We can describe this property in the following way. Suppose that $\mathtt{S}$ is a formal system containing the connective $\star$. Let $\mathtt{S}^{'}$ be the result of rewriting $\star$ everywhere in $\mathtt{S}$ as $\star'$, and let $\mathtt{S}^{''}$ be the result of the union of the two systems $\mathtt{S}$ and $\mathtt{S}^{'}$ in the language with both $\star$ and $\star'$. Let $\alpha_\star$ denote a formula (in this language) that contains a certain occurrence of $\star$, and let $\alpha_{\star'}$ denote the result of replacing this occurrence of $\star$ in α by $\star'$. The connectives $\star$ and $\star'$ are said to be uniquely characterised in $\mathtt{S}^{''}$ if, and only if, for every formula $\alpha_\star$ in the language of $\mathtt{S}^{''}$, $\alpha_\star$ is provable in $\mathtt{S}^{''}$ if, and only if, $\alpha_{\star'}$ is provable in $\mathtt{S}^{''}$. In [31] it is shown that the uniqueness property is a non-trivial property.

1.9 Syntactic Purity

One can distinguish two different notions of syntactic purity: a strong notion, and a weak notion. The first one claims that

> [a sequent calculus] should be independent of any particular semantics. One should not be able to guess, just from the form of the structures which are used, the intended semantic of a given proof system. [6, p. 2]

The weak notion claims that

> a sequent calculus should not make any use of explicit semantic elements.

In this section our aim is twofold: we would, first of all, aim to explain why the strong syntactic purity condition is too strong, and, secondly, to provide a defense of weak syntactic purity.

Our first aim is indeed quite simple in the light of an undoubtable correspondence that holds between the definitions, in terms of truth values, of the constants of classical propositional logic, and the logical rules of the sequent calculus $\mathbf{Gcl}_L$. Let us illustrate this correspondence within an example. Consider the realistic definition and the logical rules of the symbol $\wedge$, which, recall, are respectively the following:

$$\alpha \wedge \beta \text{ is true if, and only if, } \alpha \text{ is true and } \beta \text{ is true} \qquad \begin{cases} \dfrac{\alpha, \beta, M \Rightarrow N}{\alpha \wedge \beta, M \Rightarrow N} \wedge A' \\[2ex] \dfrac{M \Rightarrow N, \alpha \quad M \Rightarrow N, \beta}{M \Rightarrow N, \alpha \wedge \beta} \wedge K' \end{cases}$$

If in the logical rules we ignore the contexts and read the formulas on the left side of the sequent as false formulas, we have that the right introduction rule corresponds to the left-right direction of the equivalence:

$$\dfrac{M \Rightarrow N, \alpha \quad M \Rightarrow N, \beta}{M \Rightarrow N, \alpha \wedge \beta} \wedge K' \qquad \text{if } \alpha \text{ is true and } \beta \text{ is true, then } \alpha \wedge \beta \text{ is true}$$

while the left introduction rule corresponds to the right-left direction of the equivalence[9]:

$$\dfrac{\alpha, \beta, M \Rightarrow N}{\alpha \wedge \beta, M \Rightarrow N} \wedge A' \qquad \text{if } \alpha \text{ is false or } \beta \text{ is false, then } \alpha \wedge \beta \text{ is false}$$

Thus we can say that the logical rules of $\mathbf{Gcl}_L$ *reflect* at the syntactic level (or may be read in terms of) the semantic definitions of each constant: the elements of the structure of the sequent calculus (i.e. the sequent arrow and the comma) remind us of the metalinguistic elements of the definitions (i.e. *if .. then* and *and* and *or*); the positions of the formulas in the sequent (i.e. the left or the right sides of the sequent) remind us of the truth values in the equivalencies (i.e. false or true).

Given these remarks, there are two possible solutions: either we reject the strong syntactic purity property as too strong a property, or we are forced to admit that even the original Gentzen system **Gcl** would not be a good sequent calculus. Since the second solution is not acceptable, we draw the conclusion that the strong syntactic purity requirement should be abandoned. To those who could contest that classical logic is a questionable example since it is too special a case, we reply that although it is certainly true that classical logic represents a peculiar case – though not a unique one – it is nevertheless a case that must be taken into account, given its enormous importance.

Having achieved the first aim, let us now pass to the second aim: we would like to require from a Gentzen system that it does not use any explicit semantic element. This requirement seems to be widely accepted in proof theory. Nevertheless, nobody, as far as we know, has ever attempted to establish in a clear and detailed manner precisely what a semantic element is. We propose the following criterion which has been inspired by the remarks of Hein, Stewart and Stouppa [57, 134].

We say that a sequent (or a set of sequents) does not contain a semantic element if every element that serves to define the sequent (or set of sequents) may be translated in such a way that it forms, together with the translation of the other elements, a formula equivalent to the sequent. Classical sequents – objects of the form $M \Rightarrow N$ – do not contain any semantic element; indeed, as we can see in Definition 1.5, p. 4, each metalinguistic element finds its own translation and forms together with the other elements the formula $\bigwedge M \rightarrow \bigvee N$. More precisely, the comma must be read as a conjunction in the antecedent and as a disjunction in the consequent, while the sequent arrow corresponds to the implication. Semantic sequents (see Section 4.1), tableaux (see Section 4.2) and internalised forcing sequents (see Section 4.3) are, by contrast, examples of sequents that contain semantic elements. More precisely: in semantic sequents and tableaux, the symbol R cannot be translated (see Definitions 4.2, p. 78 and 4.9, p. 86, respectively); in internalised forcing sequents, neither the variables i, j, ..., nor relational atoms can be translated.

Having clarified this point, we are in a position to address the question of why we should want the sequent calculus to be free of semantic elements. There are at least three reasons. The first one relates to the methodological form of Ockham's razor: adding semantic parameters means adding redundant objects (since they cannot even be translated), but why should we complicate matters to obtain the same results? If it is possible for us to stay within the domain of syntactic objects, there is no rationale for burdening the derivations and the calculus with superfluous semantic elements.

The second and the third reasons are linked with claims (II) and (III) respectively. What these claims have in common is the more or less tacit assumption that in the sequent calculus we operate only with formulas and inferences, and not with truth values or variables ranging over possible worlds. This way we can state that logical rules offer the meaning of the logical constants in terms of their use, and that they provide us with a criterion of logicality. Therefore it is clear that if we betray this assumption by introducing explicit semantic elements, the Gentzen calculus can no longer defend the philosophical claims expressed by (II) and (III). We hence conclude that the use of semantic objects is a kind of modification that cannot be allowed since it violates two of the main philosophical constraints related with the Gentzen system.

1.10 Došen's Principle Redefined

Let us bring this chapter to a close with a discussion of the well known principle called "Došen's principle."

Došen's principle states that one sequent calculus can be obtained from another by systematically varying the structural rules, whilst leaving the logical rules intact. As Došen puts it, "The rules for the logical operations are never changed: all changes are made in the structural rules" [32, p. 353]. As opposed to the other properties, Došen's principle does not involve a characteristic that should be satisfied by any

Gentzen system if it is to be good, but instead it is taken to indicate a supposedly "good" way to obtain new Gentzen systems from old ones. Our aim in this section is twofold: we firstly want to put forward some arguments that Došen's principle is incorrect; secondly we want to propose a new property which captures what, in our opinion, Došen was aiming at. In order to achieve these goals, we start by briefly summarising the argument given by Došen in support of his principle.

Let us start by pointing out that – unlike many other logicians – Došen does not work with a general variant of the sequent calculus but with the structural one. The structural variant of the Gentzen calculus is indeed the variant which best reflects his own philosophical conception. To briefly sum up this conception, we can say that Došen believes that between the structural part and the logical part of the sequent calculus *there is a difference of level*: the first one has a higher and determinant level, while the second one has a lower and determinate level. This conception follows from the idea, already explained in Section 1.3.2, that logic is the science of *structural* deductions, and that logical form can be expressed in *structural* terms.

By considering three different sequent calculi (for three different logics) in their structural variants, Došen remarks that they have the same double line logical rules. This is the case, if, for example, we compare $\mathbf{Gcl}_S$ and $\mathbf{Gil}_S$. The conclusion he draws is that, if two good sequent calculi differ, they do so in their structural rules and not in their logical rules.

There are at least three arguments against this conclusion.

(1) First of all, Došen's principle has a direct and undesirable consequence: sequent calculi which respect it *may not necessarily have logical variants* (and this contradicts what has been established in Section 1.5). Consider, for example, the Hilbert systems of modal logics **K** and **KT**. **K** results from **Hcl** by adding the distribution axiom and the rule of necessitation; while **KT** results from **K** by adding the axiom T: $\Box\alpha \rightarrow \alpha$ (for further detail see Section 2.1). Consider also two corresponding Gentzen calculi (in one of their general variants) which satisfy Došen's Principle:

Gk	**Gkt**
	↓
	Gk + the structural rule **t**

At this point, the interesting question is: do **Gk** and **Gkt** have logical variants? Suppose that **Gk** and **Gkt** both have a logical variant. Let us call them $\mathbf{Gk}_L$ and $\mathbf{Gkt}_L$, respectively. $\mathbf{Gk}_L$ is equivalent to **Gk** and $\mathbf{Gkt}_L$ is equivalent to $\mathbf{Gkt}_L$, but of course $\mathbf{Gk}_L$ and $\mathbf{Gkt}_L$ are not equivalent. Nevertheless they have been obtained by the same set of axioms and logical rules. Therefore, following what we have said in Section 1.2, after Definition 1.14, one of the two has been obtained by the addition (or the elimination) of a logical rule or an axiom. In both cases, this contradicts our assumption that they satisfy Došen's Principle. Thus the conclusion follows that either **Gk** or **Gkt** does not have a logical variant.

(2) Secondly, if we do not specify that the sequent calculi must be considered in their *structural* variants, we can quite easily find counterexamples to Došen's principle. For example **Gcl** and **Gil** (see Section 1.2), which are general variants of classical and intuitionistic logics, respectively, differ in their logical and structural rules. Even $\mathbf{Gcl}_L$ and $\mathbf{Gil}_L$, which are logical variants of classical and intuitionistic logics, respectively, do not differ in their structural rules, but in their logical rules.

Note that one could still reply to this objection that it is *possible* to give structural or general variants of the classical and intuitionistic sequent calculi that differ only in their structural parts, so that Došen is in fact right. While this objection is certainly correct, it is based on too narrow a view of the Gentzen calculus. Indeed, we firmly believe that in order to fully understand a sequent calculus, one should consider it in its fullest generality, i.e. by taking into account the several variants and alternatives it might have. Only in this way can one appreciate its characteristics and properties. The same of course holds when two different Gentzen systems are to be compared. It would be misleading to confront them by taking into account only one of their variants. It would moreover generate the natural question of why we are choosing this variant for comparing them and not another one.

Hence in the case of the Gentzen calculi for classical and intuitionistic logic, if we take a broader view, we find out that they differ not only in structural aspects but also in logical ones.

(3) Finally, it seems that even Došen was reluctant to claim that sequent calculi considered in their general variants are distinguished by means of their structural rules. On the contrary, it seems that what he really wanted to assert is that sequent calculi *considered in their structural variant* are distinguished exclusively by means of their structural rules.

In light of this, one easy solution would be to slightly change Došen's principle by specifying that it only holds when the sequent calculi are considered in their structural variant. As a result the principle is no longer incorrect: $\mathbf{Gcl}_S$ and $\mathbf{Gil}_S$ respect the principle, and no undesirable consequence follows from it. Nevertheless we are not completely satisfied with this arrangement for the following reasons. If we observe *concrete Gentzen systems* (see Section 1.2), we notice that their logical parts and structural parts may well differ in importance but they are never unrelated. Every change at the structural level is reflected at the logical level, and every logical change comes from (if we think that the structural rules are somehow deeper than logical ones) or produces (if we do not think so) a structural change. Logical and structural parts go hand in hand, and Došen's principle, even when it is slightly modified, cannot account of such a relationship.

Therefore, a new property should (i) account for the equilibrium between logical rules and structural rules, and (ii) try to capture what Došen was aiming at. Our proposal is the following.

Došen's Principle Redefined. Given a general variant of a sequent calculus **G** which satisfies all the properties of a good sequent calculus, a general variant of a sequent calculus $\mathbf{G}^{'}$ can be obtained from it by both varying its logical and structural parts in such a way that even $\mathbf{G}^{'}$ can have a logical and a structural variant.

Let us consider this definition more carefully. We can easily observe that it respects the two conditions that we set ourselves: on the one hand, like the old Došen's principle, it indicates the way to obtain a Gentzen calculus from another one ("$\mathbf{G}^{'}$ can be obtained from it by"); on the other hand, it takes into account both the logical and the structural levels ("by both varying its logical and structural parts"). We therefore think that it is a good substitute for the original Došen's principle.

1.10.1 Modularity

In the literature, Došen's principle is sometimes also referred to as the "modularity property." We find this second name a possible source of misunderstanding. Došen's principle describes the relationships between different sequent calculi, while the modularity property requires the link between Hilbert systems and Gentzen systems to be straightforward. Therefore the two properties are related but not the same. In order to clarify the situation, consider the following example. Let us take a Hilbert system $\mathbf{H}^{'}$ obtained from the Hilbert system **H** by the addition of one new axiom. The modularity property demands the Gentzen system to systematically reflect the addition of this new axiom in its formalism. Došen's principle redefined tells us that this new axiom should correspond to a logical change and a structural change. Having stated this distinction, we claim that a good sequent calculus should enjoy both these properties.

Notes

1. This terminology comes from Casari [18].
2. There exists a fourth and less well-known alternative of the sequent calculus introduced by Gabbay [40] and obtained by dropping the (hidden) structural rule of associativity. In this alternative of the sequent calculus one deals with the so-called *structural databases* in place of sets, multisets or sequences.
3. By attaching this importance to structural deductions, Došen seems to endorse P. Hertz's position (e.g. see [58, 59]).
4. We underline, as Došen does, that the lack of double line logical rules cannot show us that a constant is *not* logical. Double line logical rules just serve to show what is common to all those constants which are assumed to be logical.

5. In [148] Wansing distinguishes many more.
6. For the reasons why we cannot prove constructively the equivalence between Gentzen system and Hilbert system without a syntactic proof of cut-elimination, see [18, pp. 228, 229].
7. Kremer in [71] reaches an analogous conclusion.
8. Each of the following definitions of explicitness, separation and symmetry, comes from [145, p. 127].
9. We point out that the left introduction rule does not properly reflect the right-left direction of the realistic equation, but its contraposition. However, as is well-known, the two expressions $\alpha \rightarrow (\beta \wedge \gamma)$ and $(\neg\beta \vee \neg\gamma) \rightarrow \neg\alpha$ are logically equivalent and therefore no problem arises for our argument.

Part II
Sequent Calculi for Modal Logic

La solution de ce terrible problème ne se trouve que dans un travail constant, soutenu, car les difficultés matérielles doivent être tellement vaincues, la main doit être si châtiée, si prête et si obéissante, que le sculpteur puisse lutter âme à âme avec cette insaisissable nature idéale qu'il faut transfigurer en la matérialisant.

[H. de Balzac, La cousine Bette, Folio Classique, Gallimard, 1972]

Chapter 2
Modal Logic and Ordinary Sequent Calculi

In the first chapter we introduced the Gentzen calculus for classical logic considered not only from a formal point of view, but also with respect to its philosophical importance. In this second chapter we will see how this calculus has been adapted to the case of modal logic, by examining the research carried out between the 1950s and the 1990s.

The first part of the chapter will be dedicated to a brief summary of the main notions and results of what is usually called *normal modal logic*. In the second part of the chapter we will present the ordinary sequent calculi that have been developed for modal logic. It will turn out that these calculi do not satisfy many of the properties of a good sequent calculus. In the last section we will begin to consider how one might generalise the classical sequent calculus.

2.1 Normal Modal Logic

Modal logic is the logic that results from classical logic by adding the two operators $\Box$ and $\Diamond$. The standard interpretation of $\Box$ and $\Diamond$ is in terms of, respectively, necessity and possibility: necessity and possibility are said to be alethic modalities from the Greek word ἀλήθεια, which means truth. Modalities qualify the truth of a sentence: a sentence is said to be possible if it can hold, and necessary if it must hold. However the two symbols $\Box$ and $\Diamond$ can be interpreted in many other ways: e.g. epistemically ($\Box$ being interpreted as "it is known that"), deontically ($\Box$ being interpreted as "it is compulsory that"), and mathematically ($\Box$ being interpreted as "it is provable that"). The term modal logic usually covers all interpretations.

Definition 2.1 The *propositional modal language* $\mathcal{L}^{\Box}$ is composed of the classical language $\mathcal{L}^{c}$ (see Definition 1.1, p. 3) plus the symbols $\Box$ and $\Diamond$. The well-formed formulas α of the modal language $\mathcal{L}^{\Box}$ are given by the rule:

$$\alpha ::= p \mid \bot \mid \alpha \wedge \beta \mid \alpha \vee \beta \mid \alpha \rightarrow \beta \mid \Box\alpha \mid \Diamond \alpha$$

WMF will denote the set of well-formed modal formulas

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Remark 2.2 From now on, we will take the two connectives $\neg$ and $\wedge$ and the sentential operator $\Box$ to be primitive; we will indicate this assumption by writing $\mathcal{L}^{\Box}_{\{\neg,\wedge,\Box\}}$. All the other connectives can be defined as usual, while the sentential operator $\Diamond$ can be defined in the following way: $\Diamond\alpha := \neg\Box\neg\alpha$.

Definition 2.3 We define the *complexity* of a formula α, $cmp(\alpha)$, in the following inductive way:

- $cmp(p) = 0$
- $cmp(\neg\alpha) = cmp(\Box\alpha) = cmp(\alpha) + 1$
- $cmp(\alpha \wedge \beta) = max(cmp(\alpha), cmp(\beta)) + 1$

Syntactically we will deal with normal modal Hilbert systems that can be defined as follows.

Definition 2.4 A *normal modal* system `NMS` is a set $\subseteq WMF$ such that:

- it contains all the classical tautologies and the distribution axiom

$$\Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)$$

- it also contains:
  - modus ponens: given α and $\alpha \to \beta$, prove β,
  - uniform substitution: given α, prove γ, where γ is obtained from α by uniformly replacing propositional letters in α by arbitrary formulas,
  - the necessitation rule: given α, prove $\Box\alpha$.

The notions of derivation and of theorem in a normal modal system can be easily obtained from the ones introduced in Definition 1.3, p. 4.

The system **Hk**, that from now on, following the tradition, we will simply call **K**, is the weakest normal system of modal logic: it contains the axioms and the rules that we have listed above. The other normal modal systems extend **K** with different axioms (we will come to this at p. 44).

Semantically we will work with Kripke semantics whose tools can be defined as follows.

Definition 2.5 A *frame* $\mathcal{F}$ is a pair (W, R) such that:

- W is a non empty set (of possible worlds), and
- $R \subseteq W \times W$ is a binary relation on W. R is usually called *accessibility relation*.

For reasons that will become clear later (see the beginning of Chapter 6), we want to stress a particular "form" of frame normally called *tree*.

Definition 2.6 A *tree-frame* $\mathcal{T}$, or simpler a *tree*, is a frame (W, R) that forms a finite (upward growing) tree with a single root (compare with Definiton 1.3, p. 4);

the nodes of the tree are labelled by the variables $i, j, \ldots$ of the set W, and the connection between nodes is established by the relation R.

Definition 2.7 Let $\mathcal{F} = (W, R)$. We define the *transitive closure* R^+ of R as the smallest transitive relation on W that contains R, that is

$$R^+ = \bigcap \{R^{'} | R^{'} \text{ is a transitive binary relation on } W \text{ and } R \subseteq R^{'}\}$$

Furthermore R^*, the *reflexive transitive closure* of R, is the smallest reflexive and transitive relation on W which contains R, that is

$$R^* = \bigcap \{R^{'} | R^{'} \text{ is a reflexive and transitive binary relation on } W \text{ and } R \subseteq R^{'}\}$$

Note that from Definitions 2.6 and 2.7, it follows that in a tree-frame every world j is reachable from the root i thanks to the transitive closure of the relation R.

Definition 2.8 A *model* $\mathfrak{M}$ is a pair $(\mathcal{F}, v)$, where $\mathcal{F}$ is a frame and v is the following valuation on $\mathcal{F}$:

$$v := W \otimes PL \rightarrow \{0, 1\}$$

We say that a model $\mathfrak{M} = (\mathcal{F}, v)$ is based on the frame $\mathcal{F}$.

Note that informally speaking the function v specifies which propositional letters are true in which worlds, i.e.

$$v(i, p) = \begin{cases} 1 : p \; is \; true \; at \; the \; world \; i, \\ 0 : p \; false \; at \; the \; world \; i. \end{cases}$$

Definition 2.9 Given a model $\mathfrak{M} = \langle (W, R), v \rangle$, $i \in W$, $\alpha \in WMF$, the relation

$$i \models_{\mathfrak{M}} \alpha \qquad [or \; v(i, \alpha) = 1, \; or \; \alpha \text{ is true at the world } i \text{ of the model } \mathfrak{M}]$$

usually called *satisfability relation*, is inductively defined in the following way:

- $i \models_{\mathfrak{M}} p$ iff $v(i, p) = 1$
- $i \models_{\mathfrak{M}} \neg\beta$ iff $i \not\models_{\mathfrak{M}} \beta$
- $i \models_{\mathfrak{M}} \beta \wedge \gamma$ iff $i \models_{\mathfrak{M}} \beta$ and $i \models_{\mathfrak{M}} \gamma$
- $i \models_{\mathfrak{M}} \Box\beta$ iff $(\forall j \in W)\,(iRj \rightarrow j \models_{\mathfrak{M}} \beta)$

Definition 2.10 We say that a formula α is *true in a model*, in symbols: $\models_{\mathfrak{M}} \alpha$, if it is true at every world in the model. We say that a formula α is *valid in a frame*, in symbols: $\models_{\mathcal{F}} \alpha$, if α is true in every model based on that frame. We say that a formula α is *valid in a class of frames*, in symbols: $\models_{\mathfrak{C}} \alpha$, if α is valid in every frame which belongs to that class.

There exists an interesting property called the *tree-model* property. In order to explain this property, we first need some preliminary definitions.

Definition 2.11 Let $\mathfrak{M}$ and $\mathfrak{M}^{'}$ be two models. A mapping $f : \mathfrak{M} = \langle (W, R), v\rangle \rightarrow \mathfrak{M}^{'} = \langle (W^{'}, R^{'}), v^{'}\rangle$ is a *bounded morphism* (also p-morphism), if it satisfies the following conditions:

- w and $f(w)$ satisfy the same propositional letters,
- f is a homomorphism with respect to the relation R (that is f is a homomorphism from (W, R) to $(W^{'}, R^{'})$ as first-order structures), and
- if $f(i)\ R^{'}\ j^{'}$, then there exists a j such that iRj and $f(j) = j^{'}$ (this is called the back condition).

If there is a surjective bounded morphism from $\mathfrak{M}$ to $\mathfrak{M}^{'}$, then we say that $\mathfrak{M}^{'}$ is a *bounded morphic image*, or a *bisimulation*, of $\mathfrak{M}$, and we write $\mathfrak{M} \twoheadrightarrow \mathfrak{M}^{'}$.

Lemma 2.12 Let $\mathfrak{M}$ and $\mathfrak{M}^{'}$ be two models such that f is a bounded morphism from $\mathfrak{M}$ to $\mathfrak{M}^{'}$. Then, for all modal formulas α, and all elements i of W, we have $i \models_{\mathfrak{M}} \alpha$ if, and only if, $f(i) \models_{\mathfrak{M}'} \alpha$. In words, modal satisfaction is invariant under bounded morphism.

Proof See [11], p. 62. □

A simple application of Lemma 2.12 is the above mentioned tree-model property, which says that any satisfiable formula can be satisfied in a *tree-like* model, where a tree-like model is a model based on a tree-frame.

Theorem 2.13 For all rooted-models $\mathfrak{M}$, there exists a tree-like model $\mathfrak{M}^{'}$ such that $\mathfrak{M} \twoheadrightarrow \mathfrak{M}^{'}$. Hence any $\mathfrak{M}$-satisfiable formula is satisfiable in a tree-like model.

Proof See [11], p. 63. We emphasise that the method used to construct the tree-like model $\mathfrak{M}^{'}$ from $\mathfrak{M}$ is well know in modal logic and computer science under the name of *unraveling*. □

Definition 2.14 A modal system S is said to be *sound* with respect to a class of frames $\mathfrak{C}$, if each theorem of S is valid in $\mathfrak{C}$.

Definition 2.15 A modal system S is said to be *complete* with respect to a class of frames $\mathfrak{C}$ if, and only if, every formula α valid in $\mathfrak{C}$ is a theorem of S.

As Fine [37] and Thomason [138] have demonstrated, there exist some incomplete modal systems.

Definition 2.16 A modal formula α *defines* (or *characterises*) a class of frames $\mathfrak{C}$ if, for all frames $\mathcal{F}$, $\mathcal{F}$ is in $\mathfrak{C}$ if, and only if, $\models_{\mathcal{F}} \alpha$. In short, a modal formula defines a class of frames if the formula pins down precisely the frames that are in that class via the concept of validity.

Following Definition 2.16, a modal formula can characterise a class of frames. Since a class of frames may enjoy a certain property, we also say that a modal formula defines a property if it defines the class of frames which enjoys that property.

On the other hand, since frames are nothing but relational structures, the properties that they enjoy can also be defined by non-modal formulas, e.g. by first-order and second-order formulas. It is natural to ask what sort of link there is between modal and non-modal formulas that describe the same frame property. To answer this question, let us consider the following definitions.

Definition 2.17 (Frame Languages) The first-order correspondent language of the modal language $\mathcal{L}^{\Box}{}_{\{\neg,\wedge,\Box\}}$ is the first-order language whose only descriptive symbols are the identity symbol together with an $(n+1)$-ary relation symbol R (that corresponds to the modal operator). We denote this language with $\mathcal{L}^1_{\Box}$.

Let PL^* be any set of propositional letters. The second-order correspondent language of the modal language $\mathcal{L}^{\Box}{}_{\{\neg,\wedge,\Box\}}$ over PL^* is the monadic second-order language obtained by augmenting $\mathcal{L}^1_{\Box}$ with a PL^*-indexed collection of monadic predicate variables, and by allowing only monadic second-order quantification. We denote this language with $\mathcal{L}^2_{\Box}$.

Definition 2.18 If a class of frames, or a property, can be defined by a modal formula α and by a formula β from one of the frame languages $\mathcal{L}^1_{\Box}$ and $\mathcal{L}^2_{\Box}$, then we say that α and β are *frame correspondents* of each other.

Frame definability is an inherently second-order notion, and the second-order correspondent of any modal formula can be straightforwardly computed using a simple second-order translation (see [11], p. 135). There are, however, many modal formulas with first-order correspondents. These formulas are of particular interest here. Although modal language(s) do(es) not define all the classes that can be defined by first-order formulas (the simple first-order formula $\forall x(\neg xRx)$, which defines the irreflexivity property, does not correspond to any modally expressible axiom), the first-order definable frames class that is modally definable can be isolated (see [46]).

We will not consider all the modal axioms which have first-order correspondents, but we will focus on a subset of those that have the form

$$G : \Diamond^h \Box^i \alpha \rightarrow \Box^j \Diamond^k \alpha \text{ where } h, i, j, k \geq 0$$

The set of this kind of axioms is called *Scott-Lemmon* set. An example of an axiom that does not belong to this set, since it does not match the G-form, even if its frame correspondent is a first-order formula (namely the formula $(\forall xy)(xRy \rightarrow yRy))$, is the axiom $\Box(\Box\alpha \rightarrow \alpha)$. Scott and Lemmon [75] have shown that each of the axioms belonging to the Scott-Lemmon set enjoys the following important feature.

Theorem 2.19 The condition on frames which corresponds exactly to any axiom of the G-form is the following*:*

$$(\textit{hjik-Convergence}) \quad (\forall xyz)(xR^h y \wedge xR^j z) \rightarrow (\exists w)(yR^i w \wedge zR^k w)$$

where R^n is the result of the composition of R with itself n-times*,* R^0 being the identity.

Name	Axiom	Semantic Frame Property
D	$\Box\alpha \rightarrow \Diamond\alpha$	$(\forall x)(\exists y)(xRy)$: seriality
T	$\Box\alpha \rightarrow \alpha$	$(\forall x)(xRx)$: reflexivity
4	$\Box\alpha \rightarrow \Box\Box\alpha$	$(\forall xyz)(xRy \wedge yRz \rightarrow xRz)$: transitivity
B	$\alpha \rightarrow \Box\Diamond\alpha$	$(\forall xy)(xRy \rightarrow yRx)$: symmetry
5	$\Diamond\alpha \rightarrow \Box\Diamond\alpha$	$(\forall xyz)(xRy \wedge xRz \rightarrow yRz)$: euclideanness

Fig. 2.1 Modal axioms and corresponding properties

Scott and Lemmon have also shown the adequacy (soundness and completeness) of any system that extends **K** with a selection of G-form axioms. This result has subsequently been generalised by Sahlqvist [119].

In what follows we will focus on the Scott-Lemmon axioms that are listed in Fig. 2.1, and that give rise to the principal Hilbert systems of normal modal logic, when they are added to the system **K**. Philosophically, these systems have an undoubted interest (e.g. see [62]); formally, they have the properties of the Scott-Lemmon axioms. We will call these systems SLH-systems. Here are some examples (following traditional notation, we write the names of the Hilbert systems in capital letters and without the initial **H**):

KD: **K** + the axiom *D*
KT: **K** + the axiom *T*
K4: **K** + the axiom 4
KB: **K** + the axiom *B*
S4: **KT** + the axiom 4
S5: **KT** + the axiom 5, or equivalently: **S4** + the axiom *B*

The remaining Hilbert systems are named according to the concatenation of the names of their axioms. We stress that each SLH-system (except the weakest one **K**, and the strongest one **S5**) extends and is extended by another system, so that they can be set out in a cube known as the cube of normal modal logics.

Finally we are also interested in the Hilbert system **GL**, from the initials of Gödel and Löb, or the logic of provability. Syntactically **GL** is equivalent to **K4** plus the Löb's axiom: $\Box(\Box\alpha \rightarrow \alpha) \rightarrow \Box\alpha$, while semantically it defines the class of transitive frames without infinite ascending R-chains. Even if this last property can only be defined by a second-order logic formula, and therefore is distinguished from the frame properties listed in Fig. 2.1, this system merits attention given its deep importance, both mathematical and philosophical.

2.2 Ordinary Sequent Calculi for Modal Logic

The first attempts at finding (good) sequent calculi for the SLH-systems plus **GL** exploit the standard sequent calculus: several logical rules for the symbol $\Box$ (and $\Diamond$, if taken as primitive) have been added to **Gcl**. This section presents the main results obtained this way.

Amongst others, Leivant [74], Mints [80] and Sambin and Valentini [121] agree on adding the rule

$$\frac{M \Rightarrow \alpha}{\Box M \Rightarrow \Box \alpha}\, k$$

to **Gcl** to obtain the sequent calculus **Gk** for the system **K**, where $\Box M = \{\Box\alpha \mid \alpha \in M\}$. As Sambin and Valentini stress, in the rule k the consequent must contain exactly one formula. If, for example, it contains two, we can prove the following formula:

$$\cfrac{\cfrac{\cfrac{\alpha \Rightarrow \alpha}{\Rightarrow \alpha, \neg\alpha}}{\Rightarrow \Box\alpha, \Box\neg\alpha}}{\Rightarrow \Box\alpha \vee \Box\neg\alpha}$$

which is not a theorem of **K**. On the other hand, if the consequent of the rule k is allowed to be empty, then we can get the sequent calculus for the system **KD**. This fact was also noted by Goble [45] who introduced the calculus **Gkd**. **Gkd** results from **Gk** by the addition of the rule

$$\frac{\alpha, M \Rightarrow}{\Box\alpha, \Box M \Rightarrow}\, d$$

Moreover Goble showed how to obtain a sequent calculus for the Hilbert system **KD4**: it suffices to substitute the rule k in **Gkd** with the rule

$$\frac{M' \Rightarrow \alpha}{\Box M \Rightarrow \Box \alpha}\, d4$$

where M' results from M by prefixing zero or more formulas in M by the symbol $\Box$. Following Ohnishi and Matsumoto [89], the sequent calculus **Gkt** for the system **KT** results from **Gk** by adjoining the rule

$$\frac{\alpha, M \Rightarrow N}{\Box\alpha, M \Rightarrow N}\, t$$

Given what we have seen so far, one could expect to add the following rule

$$\frac{M \Rightarrow N, \Box\alpha}{M \Rightarrow N, \Box\Box\alpha}\ 4^{+}$$

to **Gcl** in order to obtain the calculus for the system **K4**. It is easy to prove that **Gcl** plus the rule 4^{+} is sound and complete with respect to the system **K4**, but is not cut-free, a counterexample being the sequent $\Box(\Box\alpha \to \beta), \Box\alpha \Rightarrow \Box\beta$. Sambin and Valentini [121] solved the problem by straightening the rule k in following way:

$$\frac{M, \Box M \Rightarrow \alpha}{\Box M \Rightarrow \Box\alpha}\ 4$$

and by adding the new rule 4 to the calculus **Gcl**. This way they obtained the calculus **Gk4** for the system **K4**.

Following Takano [135], the sequent calculi **Gkb** and **Gkb4** result from **Gcl** by including, respectively, the rules

$$\frac{M \Rightarrow \Box N, \alpha}{\Box M \Rightarrow N, \Box\alpha}\ b \qquad \frac{M, \Box M \Rightarrow \Box N, \Box T, \alpha}{\Box M \Rightarrow \Box N, T, \Box\alpha}\ b4$$

Takano also presents the calculi **Gktb** and **Gkdb** which are obtained from **Gkb** by adjoining, respectively, the rule t and the rule

$$\frac{M \Rightarrow \Box N}{\Box M \Rightarrow N}\ db$$

The calculus for the system **S4** was first given by Curry [27] and Feys [36], and then analysed again by Ohinshi and Matsumoto [89]. **Gs4** results from **Gcl** by including the rule t and the rule

$$\frac{\Box M \Rightarrow \alpha}{\Box M \Rightarrow \Box\alpha}\ s4$$

Let us summarise our presentation with the following figure.[1] As Fig. 2.2 clearly shows, we have deemed it interesting to specify (i) the type of sequent used by the author(s), and (ii) the set of calculi explicitly mentioned by the author(s) in the article listed in the left column.

Articles	Type of sequent	**K**	**KD**	**KD4**	**KT**	**K4**	**KB**	**KB4**	**S4**	**GL**
[27]	sequences					X				
[45]	sequences		X	X	X				X	
[74]	sets	X								X
[89], [90]	sequences				X				X	
[135]	sequences						X	X		
[121]	sets	X				X				X

Fig. 2.2 Ordinary sequent calculi for modal logic

Concerning the rules for the symbol $\diamond$, when taken as primitive, the following rules represent the "possibility" counterparts of the rules k, t, and $s4$, respectively:

$$\frac{\alpha \Rightarrow N}{\diamond\alpha \Rightarrow \diamond N}\, k' \qquad \frac{M \Rightarrow N, \alpha}{M \Rightarrow N, \diamond\alpha}\, t' \qquad \frac{\alpha \Rightarrow \diamond N}{\diamond\alpha \Rightarrow \diamond N}\, s4'$$

The rules $d, d4, 4, b, b4, db$ do not have an analogue counterpart. Kripke [72] remarked that we cannot show the equivalence between $\Box$ and $\neg \diamond \neg$ and between $\diamond$ and $\neg\Box\neg$ by means of the rules k', t' and $s4'$. As a solution (only for the system **S4**) he proposed to reformulate the calculus **Gs4** by adding to the calculus **Gk** the rules t, t' and the following two:

$$\frac{\Box M \Rightarrow \diamond N, \alpha}{\Box M \Rightarrow \diamond N, \Box\alpha} \qquad \frac{\alpha, \Box M \Rightarrow \diamond N}{\diamond\alpha, \Box M \Rightarrow \diamond N}$$

As Wansing [149, p. 64] claims: "Such rules fail to give a separate account of the inferential behavior of $\Box$ and $\diamond$, since only the combined use of these operations is specified."

Let us now move to the calculus for the system **S5**, which turns out to be the most difficult one. We can cite many attempts: Braüner's [14] and Mints's[2] [81], Ohnishi and Matsumoto's [90], Rautenberg's [112], and finally Sato's [124]. We will briefly present each of them.

`Braüner and Mints's Calculi for S5.` Braüner's and Mints's calculi can be presented in the same paragraph since they rely on the same idea. They both make the most of the fact that **S5** can be embedded into monadic predicate logic, the first-order logic of unary predicates. The main difference between Braüner's calculus and Mints's calculus consists in their use of semantic elements: while Mints exploits this resource, Braüner does not. Let us present both calculi in detail.

Braüner's calculus, **Gs5b**, is a strict imitation of the Gentzen calculus for monadic predicate logic. More precisely, **Gs5b** is obtained by adding to **Gcl** the rule t (that seems to be the modal correspondent of the left introduction rule for the universal quantifier in the sequent calculus for monadic predicate logic), and a rule that introduces the $\Box$ on the right side of a sequent, furnished with a simple and precise side condition, just as for the right introduction rule for the universal quantifier in the sequent calculus for monadic predicate logic. Given the relevance of this side condition, let us try to explain it in detail. Before doing so, it is necessary to make further terminological specifications.

Definition 2.20 If in a derivation d a side formula α in the premise(s) is mapped to a side formula β in the conclusion by the obvious bijection between side formulas in the premise(s) and side formulas in the conclusion, then β is said to be *inherited* from α.

Definition 2.21 Two formula occurrences α and β in a derivation d are *immediately connected* in d if, and only if, there exists a rule instance $\mathcal{R}$ in d such that one of the following four conditions is satisfied:

– α is a principal formula of $\mathcal{R}$ and β is an auxiliary formula of $\mathcal{R}$ or vice versa.
– $\mathcal{R}$ is an axiom and α, as well as β, are principal formulas of $\mathcal{R}$.
– $\mathcal{R}$ is a cut and α, as well as β, are auxiliary formulas of $\mathcal{R}$.
– α and β are side formulas of $\mathcal{R}$ and α is inherited from β or vice versa.

A list of formula occurrences $\gamma_1, ..., \gamma_{\text{n}}$ in a derivation d is a *connection* between γ_1 and γ_{n} in d if, and only if, for each $i \in \{1, ..., \text{n}-1\}$, the formula occurrences γ_i and γ_{i+1} are immediately connected in d.

Intuitively, connections start in axioms and go through a derivation as expected. Thanks to the notion of connection, we can define the notion of dependency between formula occurrences in a derivation d. A formula in which each occurrence of a propositional letter is within the scope of a $\Box$ and a $\Diamond$ is called *modally closed*.

Definition 2.22 Let α and β be formula occurrences in a derivation d. We say that α and β are *dependent* in d if, and only if, there exists a connection between α and β in d which does not contain any occurrence of a modally closed formula.

We can now state the rule which introduces the symbol $\Box$ on the right side of the sequent in the calculus **Gs5b**,

$$\frac{M \Rightarrow N, \alpha}{M \Rightarrow N, \Box\alpha}\ s5b$$

where applications of this rule in a derivation d must be such that in d none of the formula occurrences in M and N *depend* on the displayed occurrence of α.

Rules for the symbol $\Diamond$, if taken as primitive, are the mirror image of the rules for the symbol $\Box$. This means that the side condition of $s5b$ also holds for the rule that introduces the $\Diamond$ on the left side of the sequent.

Let us now turn to Mints's calculus. Mints's calculus **Gs5m** employs indexed sequents that are sequents of the form

$$(\alpha_1, i_1), ..., (\alpha_n, i_n) \Rightarrow (\beta_1, j_1), ...(\beta_m, j_m)$$

where $\alpha_1, ..., \alpha_n$, $\beta_1, ..., \beta_m$ are well-formed modal formulas, and $i_1, ..., i_n$, $j_1, ..., j_m$ are natural numbers. Informally, an indexed sequent is a sequent where each formula is indexed by a natural number. Rules are supplemented with appropriate indeces; more precisely, the rules for the symbol $\Box$ are the following:

$$\frac{(\alpha, i), M \Rightarrow N}{(\Box\alpha, k), M \Rightarrow N}\ s5m_1 \qquad \frac{M \Rightarrow N, (\alpha, i)}{M \Rightarrow N, (\Box\alpha, k)}\ s5m_2$$

where in $s5m_2$ it holds that the index i should be different from each index occurring in M or in N. The rules for the symbol $\Diamond$ can be easily obtained from the ones for the symbol $\Box$. The other rules do not change indeces.

It is interesting to notice that every derivation in **Gs5m** can become a derivation in **Gs5b** by removing indeces, while the contrary, i.e. decorating every derivation in

Gs5b with indeces in such a way that it becomes a derivation in **Gs5m**, does not hold unless we assume that the axioms in **Gs5b** do not contain $\Box$ or $\Diamond$.

`Ohnishi and Matsmoto's Calculus for S5.` Ohnishi and Matsmoto's calculus for **S5** can be obtained from the calculus **Gs4** by modifying the rule $s4$ in the following way:

$$\dfrac{\Box M \Rightarrow \Box N, \alpha}{\Box M \Rightarrow \Box N, \Box\alpha}\, s5om$$

Unfortunately this calculus is not cut-free as the following proof of the axiom b shows.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha \Rightarrow \alpha}{\alpha, \neg\alpha \Rightarrow}\,\neg A}{\alpha, \Box\neg\alpha \Rightarrow}\,t}{\alpha \Rightarrow \neg\Box\neg\alpha}\,\neg K \qquad \dfrac{\dfrac{\dfrac{\Box\neg\alpha \Rightarrow \Box\neg\alpha}{\Rightarrow \Box\neg\alpha, \neg\Box\neg\alpha}\,\neg K}{\Rightarrow \Box\neg\alpha, \Box\neg\Box\neg\alpha}\,s5om}{\neg\Box\neg\alpha \Rightarrow \Box\neg\Box\neg\alpha}\,\neg A}{\alpha \Rightarrow \Box\neg\Box\neg\alpha}\,cut_{\neg\Box\neg\alpha}}{\Rightarrow \alpha \rightarrow \Box\neg\Box\neg\alpha}\,\rightarrow K$$

Reading the derivation bottom-up, it appears clearly that we come to a halt after the first inference: on the left side, the formula α is not preceded by a connective or modal operator, on the right side, we cannot apply the rule $s5om$, since, precisely, the only formula belonging to the antecedent is not boxed. Therefore, in order to reach the axioms, we need to use the cut-rule.

Although Ohnishi and Matsmoto's calculus is not cut-free, it nevertheless has the following property: any derivable sequent $M \Rightarrow N$, where each formula occurrence in M and in N is modally closed, has a cut-free derivation.

`Rautenberg's Calculus for S5.` This calculus[3] is obtained by adding to the calculus **Gs4** the two rules

$$\dfrac{M \Rightarrow N, \Diamond\Box\alpha, \Box\alpha}{M \Rightarrow N, \Box\alpha}\, s5r \qquad\qquad \dfrac{M \Rightarrow N, \alpha, \Box\alpha \quad \alpha, M \Rightarrow N, \Box\alpha}{M \Rightarrow N, \Box\alpha}\, sc$$

It must be emphasised that the right side rule is an example of what is normally called *analytic cut* (see Section 1.4). It is worth recalling that an analytic cut is a cut-rule with the additional condition that the cut-formula must be a subformula of one of the formulas of the conclusion.

`Sato's Calculus for S5.` The basic idea of this calculus consists in labelling each well-formed formula α, as positive α^+ or as negative α^-. The mapping is defined in the following way:

- $\bot^+ = \bot^- = \bot$
- $\top^+ = \top^- = \top$
- $p^+ = \top$

- $p^- = \bot$
- $(\alpha \to \beta)^+ = \alpha^- \to \beta^+$
- $(\alpha \to \beta)^- = \alpha^+ \to \beta^-$
- $(\Box\alpha)^+ = (\Box\alpha)^- = (\Box\alpha)$

Sato's calculus is then obtained by making two changes to Ohnishi and Matsumoto's calculus:

(i) the rule $\to A$ is substituted with the following one:

$$\frac{M \Rightarrow N, \alpha, \beta \quad \beta, M \Rightarrow N, \alpha \quad \alpha, \beta, M \Rightarrow N}{\alpha \to \beta, M \Rightarrow N} \to A_s$$

(ii) the rule $S5_s$ is added,

$$\frac{M \Rightarrow N, \alpha^-}{M \Rightarrow N, \Box\alpha} S5_s$$

Two observations are in order. The first one concerns the analyticity of the calculus: even if Sato's system is cut-free, it does not enjoy the subformula property. The second one concerns signed formulas: it is not clear how, during a derivation, an unsigned formula can become signed. Suppose we want to prove the formula $\alpha \to \Box(\Box(\alpha \to \bot) \to \bot)$. We have

$$\cfrac{\cfrac{\cfrac{\alpha, (\Box(\alpha \to \bot))^- \Rightarrow \bot^+}{\alpha \Rightarrow (\Box(\alpha \to \bot) \to \bot)^-} \to K}{\alpha \Rightarrow \Box(\Box(\alpha \to \bot) \to \bot)} S5_s}{\Rightarrow \alpha \to \Box(\Box(\alpha \to \bot) \to \bot)} \to K$$

In order to reach the axioms, we need a rule that unsigns the formula $\bot$. Such a rule is unavailable; on the other hand, if we understand the definitions of signed formulas as two-ways rules, we can complete the derivation thus:

$$\cfrac{\cfrac{\alpha \Rightarrow \bot, \alpha, \bot \qquad \alpha, \bot \Rightarrow \bot, \alpha \qquad \alpha, \bot, \alpha \Rightarrow \bot}{\alpha, \alpha \to \bot \Rightarrow \bot} \to A_s}{\alpha, \Box(\alpha \to \bot) \Rightarrow \bot} t$$

In conclusion, let us recall the only attempt made at finding an ordinary sequent calculus for the Hilbert system **GL**. This attempt is quite simple to present. Indeed the sequent calculus **Ggl** results from **Gcl** by the addition of the rule

$$\frac{M, \Box M, \Box\alpha \Rightarrow \alpha}{\Box M \Rightarrow \Box\alpha} gl$$

Several authors have tried to prove the cut-elimination theorem for this sequent calculus. The first proof was proposed by Leivant [74], but unfortunately it contains a gap, as Valentini [140] pointed out. A second attempt was made by Valentini himself (see again [140]) and his proof has given rise to an interesting discussion. Firstly, Moen [83] has pointed out that in Valentini's proof one makes an essential use of the notion of set in a sequent, i.e. if one dealt with multisets, instead of sets, the proof would not work; subsequently, Goré and Ramanayake [50] have shown that these difficulties do not subsist. Finally, there even exists a third proof offered by Sasaki [122]. This proof, thought it is quite complicated, has not given rise to discussion.

As a conclusion to this long enumeration of results, we would like to stress that in Goré [47], Shvarts [129] and Zeman [150] the reader can find ordinary sequent calculi for others systems of modal logic.

Now that the ordinary sequent calculi for the main systems of modal logic have been presented, we turn to their assessment. On the one hand, we can emphasise their qualities:

> Systems of this sort have many virtues; rules are simple and self-evident and some of the most popular logics, including **K**, **T**, **S4**, **GL**, obtained, practically simple proof procedures. [63, p. 17]

On the other hand, we cannot ignore their shortcomings: they lack almost all the properties of a good sequent calculus. To borrow Sambin and Valentini's words [121, p. 316], the problem does not seem to be that of choosing suitable rules for each modal logic, but that of finding rules (and calculi) that satisfy certain special properties. In order to solve this problem, provided we are convinced by Došen's redefined principle (see Section 1.10), the best strategy to adopt seems to consist in enhancing not only the logical rules of **Gcl**, but also the features of its deducibility relation. A similar conclusion is reached by Blamey and Humberstone [12, p. 776], as well as by a number of other proof theorists:

> This strongly suggests that the move from truth-functional to modal logic is not one best made simply by adding a new primitive connective with new rules governing it, but rather by extending one's conception of the objects to be manipulated by such rules.

We will deal with this topic in the next section and in the following chapters.

2.3 The Idea of Generalising the Gentzen Calculus

In the early 1980s, the failures of the search for a sequent calculus for modal logic gave rise to the idea that the standard Gentzen calculus could only account for classical and intuitionistic logics and should therefore be enriched. Logicians thus started creating methods capable of generating extensions of the sequent calculus and hence suitable for providing modal logic, as well as other logics, with computational tools. These methods can be divided in two groups: the first group consists of methods that generate purely syntactic sequent calculi, while the second group includes methods that extend the standard sequent calculus by adding explicit semantic elements, such as possible worlds or truth values. In the next chapter we will deal with calculi belonging to the first group, while in the fourth chapter we will deal with calculi belonging to the second group. The following pages will serve to clarify two important points.

Firstly, we may reasonably expect a proof theory for modal logic to have a unique framework in which it is possible to formulate good sequent calculi for at least all the SLH-systems. Only in this way is it possible to observe the relationships and the differences amongst several normal modal logics at the proof-theoretical level. The ordinary sequent calculus fails to provide such a framework; hence the decision to extend it. Now we will obviously require that these extensions of the sequent calculus be flexible enough to generate calculi for at least the SLH-systems, since there would be no benefit in formulating them otherwise. Given these considerations, let us apply the following condition.

Condition 2.1 *In order to be presented and analysed in this book, a generalisation of the standard sequent calculus should have been used for obtaining a wide set of calculi for the main systems of modal propositional logic.*

Those extensions that do not respect the above condition will be merely cited but not further examined.

The second point concerns the symbols taken as primitive. Here we take $\neg$, $\wedge$, $\Box$ as primitive, indicating this with $\mathcal{L}^{\Box}_{\{\neg,\wedge,\Box\}}$ (see Remark 2.2, p. 40). Consequently, when presenting the calculi for modal logic, we will only show the logical rules for these symbols without mentioning the rules for the connectives $\vee$, $\rightarrow$, and $\Diamond$. On the other hand, a feature of a good sequent calculus for modal logic should consist in (i) being able to prove the equivalences $\Box\alpha \leftrightarrow \neg\Diamond\neg\alpha$ and $\Diamond\alpha \leftrightarrow \neg\Box\neg\alpha$, and in (ii) having parallel rules for the two modal operators $\Box$ and $\Diamond$. In many cases these conditions are fulfilled and in such cases we will leave it to the reader to (i) prove the interdefinability of these two connectives, and (ii) deduce the rules for the symbol $\Diamond$ from the ones for the symbol $\Box$. By contrast, in those cases in which the rules for the symbols $\Box$ and $\Diamond$ are not the image of each other, we will analyse the situation in detail.

In conclusion, let us sum up how we will present the several generalisations that extend the sequent calculus: (i) we will first introduce the syntactic notation necessary to show the generalisation that we are interested in, (ii) we will define the notion of sequent used in the calculi and its intended interpretation, (iii) we will introduce the calculus for the system **K** in one of its *general variants* (see Definition 1.13, p. 9), and then we will illustrate the rules by which one obtains the calculi for the other systems of modal logic, (iv) we will give the main results that can be obtained within the generalisation and an example of derivation in the calculi, (v) we will make some general remarks, (vi) finally we will indicate open (interesting) problems when they arise.

Notes

1. Note that the attempt at finding a calculus for the system **GL** is presented at the end of the current section.
2. Mints [79] has also proposed a sequent calculus for a quantified version of **S5**. This calculus, even if cut-free, does not enjoy the subformula property.
3. We emphasise that in [112] Rautenberg proposes alternative formalisations of the calculi **Gkd** and **Gktb**. As it has been stressed by Goré [47], the cut-elimination theorem for these alternative formalisations seems difficult to prove.

Chapter 3
Purely Syntactic Methods

When thinking about classical sequents, a question naturally arises: can a more abstract version of them be found? There are at least six ways of answering this question in the affirmative:

– we can deal with more than just one sequent arrow (multiple sequent calculi);
– we can deal with more than just one antecedent and one succedent (higher-arity sequent calculi);
– we can deal with n different sequents at the same time (hypersequent calculi);
– we can deal with different ways of bunching formulas together (display calculi);
– we can deal with sequents as "vertical" objects (higher-dimensional sequent calculi);
– we can deal with sequents which have on both sides of the sequent arrow finite sets of sequents (higher-level sequent calculi).

Each of the methods that generate purely syntactic calculi arise from one of the above ideas. The methods inspired by the last two ideas are, respectively, Masini's [77], and Došen's [31]. We will not present these two generalisations in detail since none of them respects Condition 2.1. On the one hand, Masini's method has been applied to the system **KD** only. As Wansing [149, p. 75] remarks,

> This sequent system for **KD** admits cut-elimination, $\Box$ and $\Diamond$ are interdefinable, and the introduction rules are separate, symmetrical, and explicit, but no indication is given of how to present axiomatic extensions of KD as higher-dimensional sequent systems.

On the other hand, Došen's method has only been applied to the systems **S4** and **S5**. As Wansing [149, p. 74] has observed,

> In Došen's higher-level framework it is not clear how restrictions similar to the one used to obtain S4p/D from S5p/D would allow to capture further axiomatic systems of normal modal propositional logic.

Moreover Došen's calculi do not satisfy the cut-elimination theorem at the upper levels.

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The method of hypersequents does not respect Condition 2.1 either since it has only been applied to the system **S5** independently by Avron [6] and Restall [117]. However, this method is strictly related to the tree-hypersequent method, and therefore we will often refer to it in the third and last part of the book (see in particular, Chapters 6 and 9).

A new and recent generalisation of the sequent calculs, called *calculus of structures* (e.g. see [53, 132, 133]), also exists. It is a generalization rather different from the ones that were introduced in the foregoing: is not obtained by introducing a more abstract version of the notion of sequent, but by changing the structure of the sequent calculus itself. Since it breaks away from the original Gentzen calculus, we will not address it in what follows.

3.1 Multiple Sequent Calculi

Let us start our presentation of the syntactic sequent calculi for modal logic with the multiple sequent calculi that were introduced by Indrzejczak [63, 64]. These calculi are based on the idea, which seems to go back to Curry [27] and Zeman [150], of dealing with two different types of sequent arrow: the normal one ($\Rightarrow$) and the modal one ($\Box\!\!\Rightarrow$). Intuitively the difference between these two types of sequent arrow can be explained by considering that a classical sequent is said to be unsatisfiable if, simply, the antecedent is true and the consequent false. The same holds for modal sequents even if, in this case, we must make reference to two different worlds of Kripke semantics: the antecedent is true in one, while the consequent false in the other.

Syntactic Notation

- The structural connectives of the multiple sequent calculi are the sequent arrow, the comma and the unary connective "$-$."
- For any $\alpha \in WMF$, $-\alpha$ is a well-formed multiple structure. The set WMS of well-formed multiple structures is defined in the following way: $WMS := \{-\alpha \mid \alpha \in WMF\}$. $WMF_{-} := WMF \bigcup WMS$.
- For every $\alpha \in WMF_{-}$, $\alpha^* := \begin{cases} \beta, & if \;\; \alpha \equiv -\beta, \\ -\alpha, & otherwise. \end{cases}$
- $M^* := \{\alpha \mid -\alpha \in M\} \cup \{-\alpha \mid \alpha \in M\}$.
- We call B-formula any formula of the form $\Box\alpha$ or $\Box\neg\alpha$ that occurs in the antecedent of a sequent. Accordingly, $B[M]$ stands for: the multiset M is composed of B-formulas.

Definition 3.1 Given two WMF_{-} multisets[1] M and N, we define a *modal sequent* in the following inductive way:

- $M \;\Box\!\!\underset{0}{\Rightarrow} N := M \Rightarrow N$
- $M \;\Box\!\!\underset{n+1}{\Rightarrow} N := M \Box\Box\!\!\underset{n}{\Rightarrow} N$

In the calculi for the B-systems $n = 0, 1, ...$, in the rest of the calculi $n = 0, 1$.

We remark that the relation

$\Gamma \vdash \alpha$ if, and only if, for some $\gamma_1, ..., \gamma_n \in \Gamma$, $\vdash \gamma_1, ..., \gamma_n \Box\!\underset{n}{\Rightarrow} \alpha$

is not a consequence relation for many of the calculi considered below, because in general $\alpha \Box\!\underset{n}{\Rightarrow} \alpha$ does not hold.

Definition 3.2 Given a translation δ from the well-formed multiple structures to the well-formed formulas of the language $\mathcal{L}^{\Box}{}_{\{\neg,\wedge,\Box\}}$ such that $(-\alpha)^{\delta} := \neg\alpha$, the *translation* τ is defined in the following way:

$$(M \Box\!\underset{n}{\Rightarrow} N)^{\tau} := \bigwedge(M)^{\delta} \Rightarrow \overbrace{\Box ... \Box}^{n} \bigvee(N)^{\delta}$$

where $(M)^{\delta} := \{\alpha \mid \alpha \in M\} \cup \{\neg\alpha \mid -\alpha \in M\}$.

The calculus **Msk** for the system **K** is composed of:

Initial Sequents

$\alpha \Rightarrow \alpha$

Structural Rules

Weakening and Contraction

$$\frac{M \Box\!\underset{n}{\Rightarrow} N}{\alpha, M \Box\!\underset{n}{\Rightarrow} N}\,WA \qquad \frac{M \Box\!\underset{n}{\Rightarrow} N}{M \Box\!\underset{n}{\Rightarrow} N, \alpha}\,WK$$

$$\frac{\alpha, \alpha, M \Box\!\underset{n}{\Rightarrow} N}{\alpha, M \Box\!\underset{n}{\Rightarrow} N}\,CA \qquad \frac{M \Box\!\underset{n}{\Rightarrow} N, \alpha, \alpha}{M \Box\!\underset{n}{\Rightarrow} N, \alpha}\,CK$$

Shifting Rules

$$\frac{M \Rightarrow N, \alpha}{\alpha*, M \Rightarrow N}\,SA \qquad \frac{\alpha, M \Rightarrow N}{M \Rightarrow N, \alpha*}\,SK$$

Necessitation Rule

$$\frac{\Rightarrow N}{\Box\!\underset{1}{\Rightarrow} N}\,rn$$

Logical Rules

Propositional Rules

$$\frac{-\alpha, M \Box\!\underset{n}{\Rightarrow} N}{\neg\alpha, M \Box\!\underset{n}{\Rightarrow} N}\,\neg A \qquad \frac{M \Box\!\underset{n}{\Rightarrow} N, -\alpha}{M \Box\!\underset{n}{\Rightarrow} N, \neg\alpha}\,\neg K$$

$$\frac{\alpha, \beta, M \Box\!\underset{n}{\Rightarrow} N}{\alpha \wedge \beta, M \Box\!\underset{n}{\Rightarrow} N}\,\wedge A' \qquad \frac{M \Box\!\underset{n}{\Rightarrow} N, \alpha \quad P \Box\!\underset{n}{\Rightarrow} Q, \beta}{M, P \Box\!\underset{n}{\Rightarrow} N, Q, \alpha \wedge \beta}\,\wedge K'$$

Modal Rules

$$\left\{\begin{array}{l} \dfrac{M \underset{1}{\Box\!\!\Rrightarrow} N, -\alpha}{\Box\alpha, M \underset{1}{\Box\!\!\Rrightarrow} N}\ \Box A_1 \\[2em] \dfrac{\alpha \underset{n}{\Box\!\!\Rrightarrow} N}{\Box\alpha \underset{n+1}{\Box\!\!\Rrightarrow} N}\ \Box A_2 \end{array}\right. \qquad \dfrac{M \underset{n+1}{\Box\!\!\Rrightarrow} \alpha}{M \underset{n}{\Box\!\!\Rrightarrow} \Box\alpha}\ \Box K$$

There are several observations to make about the calculus **Msk**. The first and most evident is that there is no cut-rule. The second is the presence of shifting rules that make use of the symbol "$-$" and that shift a formula α from one side of the sequent to the other, i.e. they do what in **Gcl** is part of the "job" of the $\neg A$ and $\neg K$ rules (and also of the $\rightarrow A$ and $\rightarrow K$ rules, if the symbol $\rightarrow$ is taken as primitive). The distinction between shifting rules and logical rules was firstly introduced by Fitting [38] in order to prove the interpolation theorem; here it is used to guarantee that the logical rules might be applied to any type of sequent, classical ($n = 0$) and modal ($n > 0$).

A third and final remark: the calculus that results from **Msk** by dropping (the effects of) the necessitation rule is a calculus for the system **C**, where the axiomatization of **C** can be obtained by replacing the necessitation rule in **K** by the weaker rule: from $(\alpha \wedge \beta) \rightarrow \gamma$, prove $(\Box\alpha \wedge \Box\beta) \rightarrow \Box\gamma$. See Chellas [24].

In order to obtain the calculi for the remaining normal modal systems, we add combinations of the rules below to the calculus **Msk**. Each rule corresponds to one of the axioms (or frame properties) listed in Section 2.1, p. 44.

Special Structural Rules

$$\dfrac{M \underset{n+1}{\Box\!\!\Rrightarrow}}{M \underset{n}{\Box\!\!\Rrightarrow}}\ d \qquad \dfrac{M \underset{n+1}{\Box\!\!\Rrightarrow} N}{M \underset{n}{\Box\!\!\Rrightarrow} N}\ t$$

$$\dfrac{B(M) \underset{n}{\Box\!\!\Rrightarrow} N}{B(M) \underset{n+1}{\Box\!\!\Rrightarrow} N}\ 4 \qquad \dfrac{M \underset{n}{\Box\!\!\Rrightarrow} N}{(N)^* \underset{n}{\Box\!\!\Rrightarrow} (M)^*}\ b$$

We have thus introduced all the necessary rules for obtaining calculi for the SLH-systems. Let us dwell for a moment on an important point. As we have already noted at the beginning of this section, there is a precise difference between the calculi for the B-systems and the calculi for the systems that do not contain the B axiom. This difference consists in the way one can vary the n of the modal sequents: in the first case $n = 0, 1, ...$, while in the second case $n = 0, 1$. A curious reader could rightly wonder about the reason of this distinction. To see why, consider the rule $\Box A_2$ in the calculi for the systems without the B axiom. Intuitively, this rule, if we read it top-down, take us from an unique world i to two possible worlds i and j related by the accessibility relation R. In the systems without the B axiom, the new world j and its relationship with the old one is irrelevant, since we must never look backwards. Syntactically, this means that using the rule $\Box K$ after the rule $\Box A_2$ is not limitative, even if $\Box K$ erases the new world j and takes us back to a classical sequent. In the calculi for the B-systems matters are different. In these symmetric calculi the possibility of looking backwards must be left open: thus it is restrictive to demand that the rule $\Box K$ takes us only from modal sequents to classical sequents. This is why we must modify the structure of the sequents, and allow the reiteration of the symbol $\Box$ in front of the sequent n times.

Theorem 3.3 *Each of the calculi* **Msk**^{*2} *is sound and complete with respect to the corresponding class of frames.*

Proof The soundness proof is by induction on the height of derivations. The completeness proof, on the other hand, changes depending on whether it is applied to the calculi that do not contain the b rule (first group), or to the ones that contain such a rule (second group). The completeness proof for the first group is inspired by a method introduced by Smullyan [131], and then used by Fitting [38]. This method, in a nut-shell, consists in defining a set of consistency properties (which are sufficient to build a model) for the formal system under consideration. If the deducibility relation of the formal system satisfies these properties, then the formal system is shown to be complete. The completeness proof for the second group simply consists in the definition of an automatic procedure of proofs search. Finally, in both of these proofs Indrzejczak deals with sets of formulas, and does not use the rules of contraction. □

Example 3.4 Here is an example of a derivation in the calculi **Msk**^{*3}:

$$\begin{array}{cl}
\alpha \Rightarrow \alpha \qquad\qquad \beta \Rightarrow \beta & \\
\hline
\alpha \to \beta, \alpha \Rightarrow \beta & \to A' \\
\hline
\alpha \Rightarrow \beta, -(\alpha \to \beta) & SK \\
\hline
\Box\alpha \; \Box\!\Rightarrow_1 \beta, -(\alpha \to \beta) & \Box A_2 \\
\hline
\Box\alpha, \Box(\alpha \to \beta) \; \Box\!\Rightarrow_1 \beta & \Box A_1 \\
\hline
\Box\alpha, \Box(\alpha \to \beta) \Rightarrow \Box\beta & \Box K \\
\hline
\Rightarrow -\Box\alpha, -\Box(\alpha \to \beta), \Box\beta & SK^* \\
\hline
\Rightarrow -\Box(\alpha \to \beta), \Box\alpha \to \Box\beta & \to K \\
\hline
\Rightarrow \Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta) & \to K
\end{array}$$

At the end of his article [63] Indrzejczak remarks that in the case of the system **S5** the method of multiple sequents allows for considerable simplifications. For the sake of completeness, we will attempt to present this simplified multiple sequent calculus for the system **S5** (for a detailed analysis see [64]).

First of all, let the n of modal sequents be such that $n = 0, 1$. Therefore, we will simply write $\Rightarrow$, when $n = 0$, $\Box\!\Rightarrow$, when $n = 1$, and $(\Rightarrow)$, when $n = 0, 1$. The calculus **Mss5**$_s$ is composed of:

Initial Sequents

$\alpha \Rightarrow \alpha$

Structural Rules

Weakening and Contraction

$$\frac{M(\Rightarrow)N}{\alpha, M(\Rightarrow)N}\ WA \qquad \frac{M(\Rightarrow)N}{M(\Rightarrow)N, \alpha}\ WK$$

$$\frac{\alpha, \alpha, M(\Rightarrow)N}{\alpha, M(\Rightarrow)N}\ CA \qquad \frac{M(\Rightarrow)N, \alpha, \alpha}{M(\Rightarrow)N, \alpha}\ CK$$

Shifting Rules

$$\frac{M \Rightarrow N, \alpha}{\alpha^*, M \Rightarrow N}\ SA \qquad \frac{\alpha, M \Rightarrow N}{M \Rightarrow N, \alpha^*}\ SK$$

$$\frac{M \boxRightarrow N}{(N)^* \boxRightarrow (M)^*}\ S_s$$

Necessitation Rule

$$\frac{M \Rightarrow N}{M \boxRightarrow N}\ rn_s$$

where $M(N) = \emptyset$ or $M(N)$ contains only boxed formulas or their negations.

Logical Rules

Propositional Rules

$$\frac{-\alpha, M(\Rightarrow)N}{\neg\alpha, M(\Rightarrow)N}\ \neg A \qquad \frac{M(\Rightarrow)N, -\alpha}{M(\Rightarrow)N, \neg\alpha}\ \neg K$$

$$\frac{\alpha, \beta, M(\Rightarrow)N}{\alpha \wedge \beta, M(\Rightarrow)N}\ \wedge A' \qquad \frac{M(\Rightarrow)N, \alpha \quad P(\Rightarrow)Q, \beta}{M, P(\Rightarrow)N, Q, \alpha \wedge \beta}\ \wedge K'$$

Modal Rules

$$\frac{\alpha, M(\Rightarrow)N}{\Box\alpha, M(\Rightarrow)N}\ \Box A_s \qquad \frac{M \boxRightarrow \Box N, \alpha}{M \Rightarrow \Box N, \Box\alpha}\ \Box K_s$$

The calculus $\mathbf{Mss5}_s$ is sound and complete with respect to the system **S5** and it is also cut-free.

A final question pertaining to the multiple sequent calculi **Msk*** concerns what happens if the symbol $\diamond$ is taken as primitive. To answer this, we must once again distinguish between the *b*-calculi and the calculi that do not contain the *b* rule. In the first case, there is no particular change except for the addition of suitable rules for the symbol $\diamond$. In the second case, by contrast, the changes are more substantial and can be explained in detail as follows.

In the case of the symbol $\Box$, we introduced the sequent arrow $\Box\!\!\Rightarrow$. Semantically, the passage from a classical sequent arrow to a modal sequent arrow can be conveyed thus:

$$\begin{array}{ccc} \Rightarrow & \rhd & \Box\!\!\Rightarrow \\ \circ^i & \rhd & \circ^i \to \circ^j \end{array}$$

What happens in the case of the symbol $\diamond$? The answer appears clearly: we introduce another sequent arrow $\diamond\!\!\Rightarrow$, and we semantically interpret the passage from a classical sequent arrow to this new sequent arrow in the following way:

$$\begin{array}{ccc} \Rightarrow & \rhd & \diamond\!\!\Rightarrow \\ \circ^i & \rhd & \circ^j \to \circ^i \end{array}$$

Given this explanation, the reason why in the b-calculi the second modal sequent arrow becomes superfluous, should be clear. Indeed the b-calculi it holds that

$$\circ^j \rightleftarrows \circ^i$$

Let us now consider the calculi **Msk*** that do not contain the b rule and in which the symbol $\diamond$ is taken as primitive. The following modifications become necessary: first of all, as we have already seen, we must assume a second modal sequent arrow, $\diamond\!\!\Rightarrow$, such that $(M \underset{n}{\diamond\!\!\Rightarrow} N)^\tau := \diamond \bigwedge(M)^\delta \Rightarrow \bigvee(N)^\delta$. As a result we must add the rules

$$\frac{\alpha \diamond\!\!\Rightarrow N}{\diamond\alpha \Rightarrow N}\,\diamond A \qquad \frac{M \Rightarrow \alpha}{M \diamond\!\!\Rightarrow \diamond\alpha}\,\diamond K_1 \qquad \frac{-\alpha, M \diamond\!\!\Rightarrow N}{M \diamond\!\!\Rightarrow N, \diamond\alpha}\,\diamond K_2 \qquad \frac{M \Rightarrow}{M \diamond\!\!\Rightarrow}\,rp$$

$$\frac{M \Box\!\!\Rightarrow N}{(N)^* \diamond\!\!\Rightarrow (M)^*}\,tr \qquad \frac{M \diamond\!\!\Rightarrow N}{(N)^* \Box\!\!\Rightarrow (M)^*}\,tr$$

We also have the counterparts of the special structural rules

$$\frac{\diamond\!\!\Rightarrow N}{\Rightarrow N}\,\diamond d \qquad \frac{M \diamond\!\!\Rightarrow N}{M \Rightarrow N}\,\diamond t \qquad \frac{M \Rightarrow B(N)}{M \diamond\!\!\Rightarrow B(N)}\,\diamond 4$$

Remark 3.5 On the one hand, the calculi **Msk*** respect the main purpose of the author, which is the simplicity of derivations. On the other hand, the calculi **Msk*** fail to satisfy many relevant properties. First of all, as we have pointed out above, the calculi do not have an explicit cut-rule. Secondly, the calculi do not have a logical variant because of the special and shifting structural rules and, in the cases of certain calculi at least, because of the contraction rules. Thirdly, a structural variant is not available either since the cut-rule is absent. Finally, the explicitness and modularity properties, as well as the new definition of Došen's principle, are clearly not satisfied, and no new rule corresponds to the 5 axiom.

The notation we have used to present the multiple sequent calculi is slightly different from the one employed by the author. Moreover, we would like to bring the reader's attention to the fact that a recent work of Indrzejczak [65] discusses sequent calculi for modal hybrid logics.

Problem 3.6 (i) What would a cut-rule for the multiple-sequent calculi look like? (ii) Is there a syntactic way to prove the cut-elimination theorem? (iii) Is there a way to formalise the rule for the axiom 5?

3.2 Higher-Arity Sequent Calculi

The idea of increasing the arity of a sequent was first introduced by Schröter [52], and then further explored by Rousseau [118] and Gottwald [127]. The idea originated as a natural solution to the problem of the lack of a sequent calculus for the Lukasiewicz n-valued logics. Indeed, if two-place sequents were adequate to formalise two truth-values logics, then n-place sequents would have been suitable for formalising n-valued logics. More precisely, a classical sequent $M \Rightarrow N$ holds if, and only if, at least one of the M's is false or at least one of the N's is true. In other words, it holds if, and only if, at least one of the M's assume the value 0, or at least one of the N's assume the value 1. The n-valued sequent $M_0, M_1, ..., M_{n-1}$ for a n-valued logic holds if, and only if, there is a $j \leq n$ such that at least one of the M_j's assumes the value j.

This intuition has been taken up recently and adapted to modal logic. In this case one considers 4-place sequents in which the "two new truth values" are the necessarily true and the possibly false. Sato [123] was the first to apply the higher-arity sequent method to modal logic: unfortunately he only obtained a calculus for the system **S5**. Blamey and Humberstone [12] constructed calculi with 3-ary sequents, and 4-ary sequents, for all the SLH-systems. Let us turn to their work.

Definition 3.7 Given four *WMF* multisets[4] M, N, S and T, a *higher-arity sequent* is an object of the form

$$M \Rightarrow_S^T N := \begin{cases} M \Rightarrow N, \quad if \quad S, T = \emptyset, \\ \\ M \Rightarrow_S^T N, \text{ otherwise.} \end{cases}$$

Definition 3.8 The *interpretation* τ of a higher-arity sequent is the following:

- if the higher-arity sequent is a classical sequent $M \Rightarrow N$, then $(M \Rightarrow N)^\tau := \bigwedge M \to \bigvee N$,
- if the higher-arity sequent has the form $M \Rightarrow_S^T N$, then $(M \Rightarrow_S^T N)^\tau :=$

$$(\bigwedge M \wedge \bigwedge \Box S) \to (\bigvee N \vee \bigvee \Box T)$$

The calculus **H-ask** for the system **K** is composed of:

Initial Higher-Arity Sequents

$Ax\colon \alpha \Rightarrow^{\emptyset}_{\emptyset} \alpha$ $\quad$ $Vertical\ Ax\colon \emptyset \Rightarrow^{\alpha}_{\alpha} \emptyset$

Structural Rules

Weakening and Contraction

$$\frac{M \Rightarrow^{T}_{S} N}{\alpha, M \Rightarrow^{T}_{S} N}\ WA \qquad \frac{M \Rightarrow^{T}_{S} N}{M \Rightarrow^{T}_{S} N, \alpha}\ WK$$

$$\frac{\alpha, \alpha, M \Rightarrow^{T}_{S} N}{\alpha, M \Rightarrow^{T}_{S} N}\ CA \qquad \frac{M \Rightarrow^{T}_{S} N, \alpha, \alpha}{M \Rightarrow^{T}_{S} N, \alpha}\ CK$$

Higher-arity Weakening and Contraction

$$\frac{M \Rightarrow^{T}_{S} N}{M \Rightarrow^{T}_{\alpha,S} N}\ W_n A \qquad \frac{M \Rightarrow^{T}_{S} N}{M \Rightarrow^{T,\alpha}_{S} N}\ W_n K$$

$$\frac{M \Rightarrow^{T}_{\alpha,\alpha,S} N}{M \Rightarrow^{T}_{\alpha,S} N}\ C_n A \qquad \frac{M \Rightarrow^{T,\alpha,\alpha}_{S} N}{M \Rightarrow^{T,\alpha}_{S} N}\ C_n K$$

Cut-Rules

$$\frac{\alpha, M \Rightarrow^{T}_{S} N \qquad M \Rightarrow^{T}_{S} N, \alpha}{M \Rightarrow^{T}_{S} N}\ cut^{1}_{\alpha}$$

$$\frac{M \Rightarrow^{T}_{\alpha,S} N \qquad M \Rightarrow^{T,\alpha}_{S} N}{M \Rightarrow^{T}_{S} N}\ cut^{2}_{\alpha}$$

$$\frac{M \Rightarrow \alpha \qquad P \Rightarrow^{W}_{Z,\ \alpha} Q}{P \Rightarrow^{W}_{Z,\ M} Q}\ Ucut_{\alpha}$$

Logical Rules

Propositional Rules

$$\frac{M \Rightarrow^{T}_{S} N, \alpha}{\neg\alpha, M \Rightarrow^{T}_{S} N}\ \neg A \qquad \frac{\alpha, M \Rightarrow^{T}_{S} N}{M \Rightarrow^{T}_{S} N, \neg\alpha}\ \neg K$$

$$\frac{\alpha, \beta, M \Rightarrow^{T}_{S} N}{\alpha \wedge \beta, M \Rightarrow^{T}_{S} N}\ \wedge A' \qquad \frac{M \Rightarrow^{T}_{S} N, \alpha \qquad P \Rightarrow^{W}_{Z} Q, \beta}{M, P \Rightarrow^{T,W}_{S,Z} N, Q, \alpha \wedge \beta}\ \wedge K'$$

Modal Axioms

$$\Box\alpha \Rightarrow^{\alpha}_{\emptyset} \emptyset \quad \Box A \qquad\qquad \emptyset \Rightarrow^{\emptyset}_{\alpha} \Box\alpha \quad \Box K$$

Note that $Ucut_\alpha$ stands for $Undercut$ of α. Note also that in this calculus it is possible to derive the two rules

$$\dfrac{M \Rightarrow^{T}_{\alpha,\beta,S} N}{M \Rightarrow^{T}_{\alpha\wedge\beta,S} N}\,{\wedge_n A'} \qquad \dfrac{M \Rightarrow^{T,\,\alpha}_{S} N \qquad P \Rightarrow^{W,\,\beta}_{Z} Q}{M, P \Rightarrow^{T,W,\,\alpha\wedge\beta}_{S,Z} N, Q}\,{\wedge_n K'}$$

that are nothing but the logical rules for the connective $\wedge$ applied on a higher-level. As Blamey and Humberstone say [12, p. 774],

> This can be thought of as a way of saying that necessity distributes over conjunction without actually mentioning necessity.

Below are the proofs of their derivability. The symbol W^* denotes the repeated applications of the four rules of weakening (classical and higher-arity ones).

$\wedge_n A'$:

$$\dfrac{\dfrac{\dfrac{M \Rightarrow^{T}_{\alpha,\beta,S} N}{M \Rightarrow^{T}_{\alpha,\beta,\alpha\wedge\beta,S} N}\,W_nA \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha \Rightarrow \alpha}{\alpha,\beta \Rightarrow \alpha}\,WA}{\alpha\wedge\beta \Rightarrow \alpha}\,\wedge A' \qquad \emptyset \Rightarrow^{\alpha}_{\alpha} \emptyset}{\emptyset \Rightarrow^{\alpha}_{\alpha\wedge\beta} \emptyset}\,Ucut_\alpha}{M \Rightarrow^{T,\alpha}_{\beta,\alpha\wedge\beta,S} N}\,W^*}{M \Rightarrow^{T}_{\alpha\wedge\beta,\beta,S} N}\,cut^2_\alpha \qquad \dfrac{\dfrac{\dfrac{\dfrac{\beta \Rightarrow \beta}{\alpha,\beta \Rightarrow \beta}\,WA}{\alpha\wedge\beta \Rightarrow \beta}\,\wedge A' \qquad \emptyset \Rightarrow^{\beta}_{\beta} \emptyset}{\emptyset \Rightarrow^{\beta}_{\alpha\wedge\beta} \emptyset}\,Ucut_\beta}{M \Rightarrow^{T,\beta}_{\alpha\wedge\beta,S} N}\,W^*}{M \Rightarrow^{T}_{\alpha\wedge\beta,S} N}\,cut^2_\beta$$

$\wedge_n K$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha \Rightarrow \alpha \qquad \beta \Rightarrow \beta}{\alpha,\beta \Rightarrow \alpha\wedge\beta}\,\wedge K' \qquad \emptyset \Rightarrow^{\alpha\wedge\beta}_{\alpha\wedge\beta} \emptyset}{\emptyset \Rightarrow^{\alpha\wedge\beta}_{\alpha,\beta} \emptyset}\,Ucut_{\alpha\wedge\beta}}{M \Rightarrow^{T,\alpha\wedge\beta}_{\alpha,\beta,S} N}\,W^* \qquad \dfrac{M \Rightarrow^{T,\alpha}_{\beta,S} N}{M \Rightarrow^{T,\alpha,\alpha\wedge\beta}_{\beta,S} N}\,W_nA}{M \Rightarrow^{T,\alpha\wedge\beta}_{\beta,S} N}\,cut^2_\alpha}{M, P \Rightarrow^{T,W,\alpha\wedge\beta}_{\beta,S,Z} N, Q}\,W^* \qquad \dfrac{P \Rightarrow^{W,\beta}_{Z} Q}{M, P \Rightarrow^{T,W,\beta,\alpha\wedge\beta}_{S,Z} N, Q}}{M, P \Rightarrow^{T,W,\alpha\wedge\beta}_{S,Z} N, Q}\,cut^2_\beta$$

On the contrary, the rules

$$\dfrac{M \Rightarrow^{T,\alpha}_{S} N}{M \Rightarrow^{T}_{S,\,\neg\alpha} N}\,\neg_n A \qquad \dfrac{M \Rightarrow^{T}_{S,\,\alpha} N}{M \Rightarrow^{T,\,\neg\alpha}_{S} N}\,\neg_n K$$

are not derivable. This fact comes as no surprise but rather as a reassurance since these inferences are clearly not K-valid. The rule $\neg_n A$ becomes valid and derivable in the calculi for the D-systems.

Finally, it is worth stressing the particular form of the rules cut^1_α and cut^2_α in which the order of the premises is inverted with respect to the usual one.

In order to obtain the calculi for the remaining normal modal systems, we add combinations of the rules below to the calculus **H-ask**. Each rule corresponds to one of the axioms (or frame properties) listed in Section 2.1, p. 44.

Special Structural Rules

$$\frac{M \Rightarrow^{\emptyset}_{\emptyset} \emptyset}{\emptyset \Rightarrow^{\emptyset}_{M} \emptyset}\, d \qquad\qquad \emptyset \Rightarrow^{\emptyset}_{\alpha} \alpha \quad t$$

$$\frac{S \Rightarrow^{\emptyset}_{S} \alpha \qquad M \Rightarrow^{\emptyset}_{\alpha,\, S'} N}{M \Rightarrow^{\emptyset}_{S,S'} N}\, 4 \qquad\qquad \frac{S \Rightarrow^{N}_{\emptyset} \alpha \qquad M \Rightarrow^{T}_{\alpha,\, S} N}{M \Rightarrow^{T}_{S} N}\, b$$

Theorem 3.9 *Each of the calculi **H-ask*** is sound and complete with respect to the corresponding class of frames.*

Proof The soundness proof is by induction on the height of derivations. The completeness proof, instead, is developed following the Scott-Makinson adaptation of Henkin completeness proof (e.g. see [128]), which basically consists in constructing canonical structures. Finally, in both of these proofs, Blamey and Humberstone deal with sets of formulas and do not use the rules of contraction. □

Example 3.10 Below is an example of a derivation in the calculi **H-ask***:

$$\cfrac{\cfrac{\cfrac{\cfrac{\emptyset \Rightarrow^{\alpha}_{\alpha} \emptyset \qquad \Rightarrow_{\beta} \Box\beta}{\emptyset \Rightarrow^{\emptyset}_{\alpha\to\beta,\ \alpha} \Box\beta}\, \to A'_2 \qquad \Box(\alpha \to \beta) \Rightarrow^{\alpha\to\beta}}{\Box(\alpha \to \beta) \Rightarrow_{\alpha} \Box\beta}\, cut^2_{\alpha\to\beta} \qquad \Box\alpha \Rightarrow^{\alpha}}{\Box(\alpha \to \beta), \Box\alpha \Rightarrow \Box\beta}\, cut^2_{\alpha}}{\cfrac{\Box(\alpha \to \beta) \Rightarrow \Box\alpha \to \Box\beta}{\Rightarrow \Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)}\, \to K}\, \to K$$

At the formal level, at least three comments concerning the higher-arity sequent calculi are in order. The first remark is about two alternatives for obtaining the calculus **H-ask**. Indeed,

– the modal axioms $(\Box A)$ and $(\Box K)$ are interreplaceable with the following rules, respectively:

$$\frac{M \Rightarrow^{T}_{\alpha,S} N}{\Box\alpha, M \Rightarrow^{T}_{S} N}\, \Box A' \qquad\qquad \frac{\Box\alpha, M \Rightarrow^{T}_{S} N}{M \Rightarrow^{T}_{\alpha,S} N}\, \Box K'$$

Notice that even with these rules the calculi are not cut-free, as the derivation of the axiom $\neg\Box\neg\alpha \wedge \Box\beta \rightarrow \neg\Box\neg(\alpha \wedge \beta)$ shows.

– the *Vertical Ax.* and the Undercut-rule delivers the rule

$$\frac{M \Rightarrow^{\emptyset}_{\emptyset} \alpha}{\emptyset \Rightarrow^{\alpha}_{M} \emptyset}\, rn$$

and vice versa. In the rule rn the consequent should contain exactly one formula; if, for example, it does not contain any formula, that is to say if it is empty, then the rule rn becomes the rule d.

The second comment concerns a second way to obtain the calculus **H-askt**. The axiom t can be substituted by the rule

$$\frac{\alpha, M \Rightarrow^{\emptyset}_{S} N}{M \Rightarrow^{\emptyset}_{\alpha,S} N}\, t'$$

The third and final comment concerns the symbol $\diamond$. It is worth underlining that if the symbol $\diamond$ is taken as primitive, the rules that mirror the rules $\Box A'$ and $\Box K'$ fail to introduce the symbol $\diamond$ (they are not valid). On the other hand, Blamey and Humberstone do not offer any alternative solution, such as the one we have presented in the previous section for the multiple-sequent calculi.

Remark 3.11 The main purpose of the authors, as they themselves claim in the introduction of their paper, is to find a notion of sequent which reflects, at the purely syntactic level, several properties enjoyed by the classes of frames characterised by the SLH-axioms. In this respect, they certainly succeed, if we do not consider the fact that no new rule corresponds to the 5 axiom. On the other hand, the higher-arity sequent calculi have significant flaws: they are not cut-free, they do not have a logical variant, and the new definition of Došen's principle is not satisfied.

Problem 3.12 (i) Is there a way to modify and enrich the higher-arity sequent calculi in order to render them cut-free? (ii) Is there a way to formulate the rules for the symbol $\diamond$? (iii) Is there a way to formalise the rule for the axiom 5?

3.3 Display Sequent Calculi

The term *display logic* is normally used to refer to a general proof theoretic framework introduced by Belnap [8–10]. This framework has been fully exploited not only in the field of modal logic, where the work of Wansing [145, 147, 149] stands out, but also in other fields such as substructural logics [48, 115] and in particular subintuitionistic logic [146].

The basic idea behind display logic is to consider the sequent arrow as representing a deducibility relation between finite possible complex data. In line with this interpretation, one no longer works with finite multisets of formulas, but starts to deal with the so called *Gentzen terms* or *structures*, and adds new structural symbols that are applicable to such Gentzen terms or structures. Thanks to these innovations, it becomes possible to simulate, in the framework of the sequent calculus, the most natural and desirable data-operations that we can think of, e.g. combining or transferring the data, or moving the data around.

Syntactic Notation

- The structural connectives of display logic are the sequent arrow and the following four[5]:

  I: nullary operation,
  $\bullet$, $*$: unary operations,
  $\circ$: binary operation,
- M, N, ... vary on structures.

Let us explain how the four new structural connectives should be understood.

I is the empty structure,
$\circ$ is the structure composition,
$*$ shifts structures from one side to the other,
$\bullet$ marks the structure in its scope as intensional.

Definition 3.13 A *display structure* is given by the rule:

$$M ::= I \mid \alpha \mid \bullet\ M \mid M^* \mid M \circ N$$

Therefore every formula is considered to be as a structure, and the structural connectives are used to build up more complex structures in the obvious way. A sequent is now a relation between structures, as the following definition shows.

Definition 3.14 A *display sequent* is an object of the form $M \Rightarrow N$, where M and N are structures.

The structure $M(N)$ is the *antecedent* (*succedent*) of $M \Rightarrow N$. An *antecedent* (*succedent*) *part* of a sequent $M \Rightarrow N$ is a positive occurrence of a substructure of M or a negative occurrence of a substructure of N (a positive occurrence of a substructure of N or a negative occurrence of a substructure of M).

In order to translate display sequents, the language of modal logic $\mathcal{L}^{\Box}_{\{\neg,\wedge,\Box\}}$ is not enough. Not only do we need the two constants $\top$ and $\bot$, but, more peculiarly, we need the tense operator P for "sometimes in the past." Indeed, one of most meaningful characteristics of display logic consists in exploiting the well-known notion of residuated pair of unary operations (for a clear exposition of this and other related notions see [34, pp. 30–33]). This is done by means of the symbol $\bullet$ that can be either translated by the constant $\Box$, or by the tense operator P. $\Box$ – alias $[F]$ for "always in the future" – and P precisely form a residuated pair.

Definition 3.15 Let $\mathcal{LP}^{\Box}_{\{\neg,\wedge,\Box\}}$ be a (modal) tense language that extends the language $\mathcal{L}^{\Box}_{\{\neg,\wedge,\Box\}}$ by adding the falsity and the truth constants $\bot$ and $\top$, respectively, and the tense operator P. We give the following *translation* τ of sequents into formulas of the language $\mathcal{LP}^{\Box}_{\{\neg,\wedge,\Box\}}$:

$$(M \Rightarrow N)^{\tau} := (M)^{\tau_1} \rightarrow (N)^{\tau_2}$$

where τ_i $(i = 1, 2)$ is defined as follows:

$$(\alpha)^{\tau_i} = \alpha$$

$$(I)^{\tau_1} = \top$$
$$(I)^{\tau_2} = \bot$$

$$(M^*)^{\tau_1} = \neg(M)^{\tau_2}$$
$$(M^*)^{\tau_2} = \neg(M)^{\tau_1}$$

$$(M \circ N)^{\tau_1} = (M)^{\tau_1} \wedge (N)^{\tau_1}$$
$$(M \circ N)^{\tau_2} = (M)^{\tau_2} \vee (N)^{\tau_2}$$

$$(\bullet M)^{\tau_1} = P(M)^{\tau_1}$$
$$(\bullet M)^{\tau_2} = \Box(M)^{\tau_2}$$

Note that the display structural connectives I, $\circ$ and $\bullet$ do not differ from the classical structural connectives in being context-sensitive at the interpretational level.

The calculus **Dsk** for the system **K** is composed of:

Initial Sequents

$$p \Rightarrow p$$

Structural Rules

Weakening and Contraction

$$\frac{M_1 \Rightarrow N}{M_1 \circ M_2 \Rightarrow N}\, W \qquad \frac{M_1 \Rightarrow N}{M_2 \circ M_1 \Rightarrow N}\, W$$

$$\frac{M \circ M \Rightarrow N}{M \Rightarrow N}\, C$$

Associativity, Commutativity and Identity

$$\frac{M_1 \circ (M_2 \circ M_3) \Rightarrow N}{(M_1 \circ M_2) \circ M_3 \Rightarrow N}\, A \text{ (double line)} \qquad \frac{M_1 \circ M_2 \Rightarrow N}{M_2 \circ M_1 \Rightarrow N}\, C$$

$$\frac{M \Rightarrow N}{I \circ M \Rightarrow N}\, I+ \qquad \frac{M \Rightarrow N}{M \circ I \Rightarrow N}\, I+$$

$$\frac{I \circ M \Rightarrow N}{M \Rightarrow N}\, I- \qquad \frac{M \circ I \Rightarrow N}{M \Rightarrow N}\, I-$$

$$\frac{I \Rightarrow N}{M \Rightarrow N}\, IA \qquad \frac{M \Rightarrow I}{M \Rightarrow N}\, IK$$

Basic Structural Rules

$$\begin{array}{c} M \circ S \Rightarrow N \\ \hline\hline M \Rightarrow N \circ S^* \\ \hline\hline S \Rightarrow M^* \circ N \end{array} \qquad \begin{array}{c} M \Rightarrow N \circ T \\ \hline\hline M \circ T^* \Rightarrow N \\ \hline\hline N^* \circ M \Rightarrow T \end{array}$$

$$\begin{array}{c} M \Rightarrow N \\ \hline\hline N^* \Rightarrow M^* \\ \hline\hline M \Rightarrow N^{**} \end{array} \qquad \begin{array}{c} M \Rightarrow \bullet N \\ \hline\hline \bullet M \Rightarrow N \end{array}$$

Necessitation Rule

$$\frac{I \Rightarrow N}{\bullet I \Rightarrow N}\, rn$$

Cut-Rule

$$\frac{M \Rightarrow \alpha \qquad \alpha \Rightarrow Q}{M \Rightarrow Q}\, cut_\alpha$$

Logical Rules
Propositional Rules

$$\frac{\alpha^* \Rightarrow N}{\neg\alpha \Rightarrow N}\, \neg A \qquad \frac{M \Rightarrow \alpha^*}{M \Rightarrow \neg\alpha}\, \neg K$$

$$\frac{\alpha \circ \beta \Rightarrow N}{\alpha \wedge \beta \Rightarrow N}\, \wedge A' \qquad \frac{M \Rightarrow \alpha \qquad P \Rightarrow \beta}{M \circ P \Rightarrow \alpha \wedge \beta}\, \wedge K'$$

Modal Rules

$$\frac{\alpha \Rightarrow N}{\Box\alpha \Rightarrow \bullet N}\, \Box A \qquad \frac{\bullet M \Rightarrow \alpha}{M \Rightarrow \Box\alpha}\, \Box K$$

There are at least two remarks to make about this calculus. The first one concerns the basic structural rules, i.e. the rules that determine the simple and clear inferential behaviour of the four structural connectives. If two sequents are interderivable by means of the basic structural rules, then these sequents are said to be *structurally* *equivalent*. Note that the basic structural rules are not named; therefore if we need to use them in a derivation, we will indicate each of their applications with a generic sr.

The second remark concerns certain structural rules that are easily derivable from the structural rules assumed as primitive, and that, at the same time, are descriptive of the display calculus. Below are a few examples.

$$\frac{I \Rightarrow N}{I^* \Rightarrow N} \qquad \frac{M \Rightarrow N_1}{M \Rightarrow N_1 \circ N_2} \qquad \frac{M \Rightarrow N \circ N}{M \Rightarrow N}$$

So let us suppose that we want to derive the rule

$$\frac{I^* \Rightarrow N}{I \Rightarrow N}$$

we have

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{I^* \Rightarrow N}{I \circ I^* \Rightarrow N}}{N^* \circ I \Rightarrow I}}{N^* \circ I \Rightarrow N}}{I \circ N^* \Rightarrow N}}{I \Rightarrow N \circ N}}{I \Rightarrow N}$$

In order to obtain the calculi for the remaining normal modal systems, we add combinations of the rules below to the calculus **Dsk**. Each rule corresponds to one of the axioms (or frame properties) listed in Section 2.1, p. 44.

Special Structural Rules

$$\frac{\bullet M \circ \bullet N \Rightarrow I^*}{M \Rightarrow N^*}\,d \qquad \frac{M \Rightarrow \bullet N}{M \Rightarrow N}\,t$$

$$\frac{M \Rightarrow \bullet N}{M \Rightarrow \bullet \bullet N}\,4 \qquad \frac{(\bullet(M^*))^* \Rightarrow N}{\bullet M \Rightarrow N}\,b$$

$$\frac{(\bullet(M^*))^* \Rightarrow N}{\bullet((\bullet(M^*))^*) \Rightarrow N}\,5$$

Though we have already emphasised many important aspects of display logic, we still have not clarified the origin of its name. The reason consists in the fact that any substructure of a given display sequent s may be displayed as the entire antecedent or succedent, respectively, of a structurally equivalent sequent s'. More precisely, the Display Theorem states that:

Theorem 3.16 *For all sequents s of **Dsk***, and all antecedent (succedent) parts M of s, there exists a sequent s' structurally equivalent with s, such that M is the antecedent (succedent) of s'.*

Proof There are (at least) two proofs of this theorem: Belnap's [8] and Restall's [115]. □

If a logic satisfies the Display Theorem, it is said to satisfy the *display property*. As Wansing remarks [147, p. 36], the basic structural rules that we have presented above suffice to prove the Display Theorem. Nevertheless there are other combinations of basic structural rules that guarantee the display property.

The Display Theorem has both a technical and a philosophical significance. On the technical level, it allows an elegant and general proof of the cut-elimination theorem (see the proof of Theorem 3.18). On the philosophical level, the Display Theorem involves a property that is usually called *segregation*, and that can be seen as a straightening of the separation property (see Section 1.7). Indeed, the separation property requires a logical rule not to exhibit any other connective except the one it introduces (so that we can have a non holistic definition of the meaning of the introduced symbol). The segregation property, instead, requires a logical rule not to display any other formula (or structure) in its antecedent (or succedent) than the one(s) it is going to operate on. This way a logical rule is able to impart information about the meaning of the symbol it introduces without involving any reference to context. As Belnap [10, p. 81] explains,

> [t]he nub is this. If a rule for $\to$ only shows how $\alpha \to \beta$ behaves in context, then that rule is not merely explaining the meaning of $\to$. It is also and inextricably explaining the meaning of the context. Suppose we give sufficient conditions for
>
> $$\alpha \to \beta, M \Rightarrow N$$
>
> in part by the rule
>
> $$\frac{M \Rightarrow N, \alpha \quad \beta, M \Rightarrow N}{\alpha \to \beta, M, P \Rightarrow N, Q}$$
>
> Then we are not explaining $\alpha \to \beta$ alone. We are simultaneously involving the comma not just in our explicans (that would surely be all right), but in our explicandum. We are explaining two things at once. There is no way around this. You do not have to take it as a defect, but it is a fact. (Notation adjusted.)

Paoli [94] calls this view on the meaning-giving status of the logical rules the *undeterministic view*. He associates it with Sambin's position on the meaning of connectives and with his *visibility* requirement (see for further details [120]).

Theorem 3.17 *Each of the calculi **Dsk*** is sound and complete with respect to the corresponding Hilbert system.*

Proof By induction on the height of derivations in the appropriate Hilbert system and calculus, respectively. □

Theorem 3.18 *Each of the calculi **Dsk*** is cut-free.*

Proof The proof is developed following the technique first proposed by Curry [28] and then developed by Belnap [8]. This technique consists in enumerating eight conditions (see for further details [8], pp. 387–390) which, if satisfied by the calculus under consideration, ensure the eliminability of the cut-rule. Each of these conditions can be verified by sight except the last one, C8, which is the *Eliminability of principal constituents*. In this instance both premises of cut introduce the cut-formula. If the cut-formula has the form $\Box\alpha$, we have

$$\dfrac{\dfrac{\bullet M \Rightarrow \alpha}{M \Rightarrow \Box\alpha}\ \Box A \qquad \dfrac{\alpha \Rightarrow N}{\Box\alpha \Rightarrow \bullet N}\ \Box K}{M \Rightarrow \bullet N}\ cut_{\Box\alpha}$$

which we reduce to

$$\dfrac{\dfrac{\bullet M \Rightarrow \alpha \quad \alpha \Rightarrow N}{\bullet M \Rightarrow N}\ cut_{\alpha}}{M \Rightarrow \bullet N}\ sr$$

In the other cases, we proceed in the standard way. □

A calculus is said to be a *proper display calculus* if its rules satisfy the eight conditions mentioned in the proof of cut-elimination. If a system S can be presented as a proper display calculus, then S is said to be *properly displayable*. Belnap [8] proves that in every properly displayable system, a derivation of a sequent s can be converted into a derivation of s not containing any application of cut. Wansing [147] shows that for certain (modal) logics, it is possible to prove a strong cut-elimination theorem: every (sufficiently long) sequence of steps in the process of cut-elimination terminates. Despite these results, we cannot establish whether the calculi **Dsk*** are decidable.

Theorem 3.19 *The decidability of a display calculus is undecidable.*

Proof The proof involves a simulation of a Thue-process. The details can be found in [69]. A counterexample can be produced, based on a result found by Grefe and Kracht [70]. □

We would like to conclude this long list of results obtainable in display logic with the next one, which characterises the properly displayable extensions of **K**, and which has been established by Kracht.

Definition 3.20 A first-order sentence over two binary relations symbols R and $\breve{R}$ (that is the converse relation of R) is said to be *primitive* if it has the form $(\forall)(\exists)\alpha$, where every quantifier is restricted with respect to R and $\breve{R}$, and α is built up from $\wedge$, $\vee$ and the atomic formulas: $x = y$, xRy, $x\breve{R}y$, where at least one of x, y is not in the scope of an existential quantifier.

Theorem 3.21 *A class of Kripke frames is describable by a set of primitive first-order sentences if, and only if, the modal (and tense) logic of this class can be properly displayed.*

Proof See [70]. □

On the one hand, this theorem sheds light on the great expressive power of display logic at the first-order level. On the other hand, it also clarifies the limit of this method with respect to the system **GL** and other modal systems. Notice that there is also a syntactic variant of this theorem.

Definition 3.22 A modal axiom schema is said to be *primitive* if it has the form $\alpha \to \beta$, and α contains each propositional atom once, and α and β are built up from the symbols: $\top$, $\wedge$, $\vee$ and $\diamond$.

Theorem 3.23 *An axiomatic extension of* ***K*** *is properly displayable if, and only if, it can be axiomatized by a set of modal primitive axiom schema.*

Proof See [70]. □

Example 3.24 Below is an example of a derivation in the calculi **Dsk***:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha \Rightarrow \alpha \qquad \beta \Rightarrow \beta}{\alpha \to \beta \Rightarrow \alpha^* \circ \beta}\ {\to}A'}{\Box(\alpha \to \beta) \Rightarrow \bullet(\alpha^* \circ \beta)}\ \Box A}{\Box(\alpha \to \beta) \circ \Box\alpha \Rightarrow \bullet(\alpha^* \circ \beta)}\ W}{\bullet(\Box(\alpha \to \beta) \circ \Box\alpha) \Rightarrow \alpha^* \circ \beta}\ sr}{\alpha \Rightarrow \beta \circ (\bullet(\Box(\alpha \to \beta) \circ \Box\alpha))^*}\ sr}{\Box\alpha \Rightarrow \bullet(\beta \circ (\bullet(\Box(\alpha \to \beta) \circ \Box\alpha))^*)}\ \Box A}{\Box(\alpha \to \beta) \circ \Box\alpha \Rightarrow \bullet(\beta \circ (\bullet(\Box(\alpha \to \beta) \circ \Box\alpha))^*)}\ W}{\bullet(\Box(\alpha \to \beta) \circ \Box\alpha) \Rightarrow \beta \circ (\bullet(\Box(\alpha \to \beta) \circ \Box\alpha))^*}\ sr}{\bullet(\Box(\alpha \to \beta) \circ \Box\alpha) \circ \bullet(\Box(\alpha \to \beta) \circ \Box\alpha) \Rightarrow \beta}\ sr}{\bullet(\Box(\alpha \to \beta) \circ \Box\alpha) \Rightarrow \beta}\ C}{\Box(\alpha \to \beta) \circ \Box\alpha \Rightarrow \Box\beta}\ \Box K}{\Box(\alpha \to \beta) \Rightarrow (\Box\alpha \to \Box\beta)}\ {\to}K}{I \circ \Box(\alpha \to \beta) \Rightarrow (\Box\alpha \to \Box\beta)}\ I+}{I \Rightarrow \Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)}\ {\to}K$$

Remark 3.25 The display method is undoubtedly a quite powerful method, not only because of its applicability to a wide range of different logics, but also for the interesting results that can be obtained with it. Beyond all the results that we have already mentioned, Wansing [147] also proves (i) the admissibility of axioms of the form $\alpha \Rightarrow \alpha$, (ii) the interdefinability of the symbols $\Box$ and $\diamond$, (iii) the invertibility of the logical rules and of the modal rule $\Box K$ (by using the cut-rule), and (iv) the fact that the display rules satisfy the uniqueness property.

The display method is not, however, faultless: display calculi do not satisfy the redefined Došen's principle, and they do not have the logical variant (at least) because of the shifting rules.

The satisfiability of the subformula property by the display calculi has been at the source of an interesting discussion. According to Avron [6, p. 2] display calculi fail to meet this *desideratum*.

> A use of "structural connectives" that can arbitrarily be nested, usually violates this principle. It seems to me that this is the weak point of Belnap's framework of Display Logic.

While, according to Wansing [149, p. 71], what the display calculi do not satisfy is the *substructure property*, not the subformula property:

> In a sequent calculus with an enriched structural language, the subformula property need not be accompanied by a substructure property.

Without contributing to this debate directly, we can nevertheless emphasise a fact that is often overlooked. As opposed to the other methods, in the display calculi the new introduced symbols do not operate on sequents, but on formulas, turning them into structures. This gives them great expressive power, but also poses an interesting problem. What we prove with the display calculi are structures, not formulas and, in particular, not modal formulas. Therefore the question seems to be: can we really assert that these calculi are computational instruments for modal logic, that is to say a logic composed by formulas? In Section 4.3, we shall see how this question comes up again.

Let us recall that there is another method for generalising the Gentzen sequent calculus which was introduced by Cerrato [21] and which, like the display method, is characterised by a rise of the number of meta-linguistic connectives. The two meta-linguistic symbols added in this generalisation are $\langle\ \rangle$ and $[\ ]$. They can sign any formula – $\langle\alpha\rangle$, $[\alpha]$ – and are supposed to "stress on the modal nature of the formula". The classical structural rules operate on signed and unsigned formulas; the logical rules only affect unsigned formulas. The modal rules are

$$\frac{\alpha, M \Rightarrow N}{\langle\alpha\rangle, [M] \Rightarrow \langle N\rangle}\ \Box A \qquad \frac{M \Rightarrow N, \alpha}{[M] \Rightarrow \langle N\rangle, [\alpha]}\ \Box K$$

Finally, four extra rules must be added; these rules are called *duality rules* and they convert the two new meta-linguistic signs into modal operators, together with transforming the "possibly" into "necessarily," and vice versa. There are also structural or logical rules for obtaining calculi for all the SLH-systems.

Cerrato proves the cut-elimination theorem for the calculus for the system **K**. As Goré [22] points out, the other sequent calculi do not seem to be cut-free: the **S4**-theorem $\Diamond\Box(\Diamond\alpha \rightarrow \Box\Diamond\alpha)$, for example, cannot be proved without the use of the cut-rule.

Notes

1. In [63] M and N are *sets* of formulas.
2. With the notation: name of the calculus for the system **K** (in this case **Msk**) + *, we mean all the extensions of the calculus for the system **K** by combinations of special (structural or/and logical) rules. From now on we will take this assumption for granted.
3. Notice that we use the derived rules for the connective $\rightarrow$. Moreover in the case where repeated running applications of a same rule $\mathcal{R}$ take place, we write the rule $\mathcal{R}$ with the symbol * as index. From now on we will take these assumptions for granted.
4. In [12] M, N, S and T are *sets* of formulas.
5. Note that the comma is not used.

Chapter 4
Semantic Methods

As noted in Section 2.3, there are two types of methods for extending the ordinary sequent calculus: the first type modifies the structure of the classical sequent in a purely syntactic fashion (see the previous chapter); the second type enriches a classical sequent by adjoining semantic elements. This chapter will be entirely dedicated to the analysis of the calculi generated by means of this latter method.

At least two kinds of semantic elements can be introduced in a sequent: algebraic elements and Kripke semantics elements. We will only address those generalisations that exploit this second type of elements. Therefore, those extensions of the Gentzen calculus that use algebraic elements, and in particular those of Viganò [143], and Orlowska [91, 92], will not be considered further.

How is it possible to internalise semantic elements in the language or in the meta-language of the sequent calculus? In order to answer this question, we need to recall, first, that semantically modal logic can be seen as a tool for talking about frames (see Definition 2.5, p. 40), and, secondly, that modal logic possesses the so-called *tree-model* property which, roughly speaking, says that it is not limitative to work only with tree-like frames (see Proposition 2.13, p. 42). One important fact that we have not pointed out up to this point, and that happens to be quite useful here, is that there are at least three different albeit equivalent ways to present tree-frames. This is what we turn to next.

1. We can describe a tree-frame in a 'graphic' way. Given a non-empty set W of points and a binary relation $\rightarrow$, the following figures are examples of tree-frames:

   (A) (B)

   

   We then assume a binary relation R on W that extends $\rightarrow$ to hold. The properties of R vary depending on the properties enjoyed by the tree-frame.
2. We can describe a tree-frame by enumerating its worlds. Given a non-empty set W containing distinct finite sequences σ of natural numbers, $\sigma := \langle i_0, i_1, i_2, ...\rangle$, we can form a tree with a root 0 by concatenating the distinct finite sequences by means of the relation "being a proper initial segment of," where σ is a proper

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initial segment of τ when, if $\sigma = < a_1, ..., a_n >$ and $\tau = < b_1, ..., b_m >$, then $n < m$ and $a_i = b_i$ $(1 \leq i \leq n)$. Therefore, another way to describe the tree-frame figures (A) and (B) is

$$(A) \quad 0;\ 0,0;\ 0,1;\ 0,2 \qquad\qquad (B) \quad 0;\ 0,0;\ 0,0,0;\ 0,0,1$$

We then assume that a binary relation R holds between the finite sequences of natural numbers. R varies depending on the properties enjoyed by the tree-frame.

3. We can finally describe a tree-like frame by simply assuming a non-empty set of variables $W := \langle i, j, z, ...\rangle$, equipped with a binary relation R. Hence, another way to describe the tree-frame figures (A) and (B) is:

$\exists i, \exists j, \exists z, \exists w$, such that

$$(A) \quad iRj \text{ and } iRz \text{ and } iRw \qquad (B) \quad iRj \text{ and } jRz \text{ and } jRw$$

If the tree-frame enjoys certain properties as seriality or reflexivity, we will express them with the following first-order logic formulas: $\forall x \exists j(xRj)$ and $\forall x\ (xRx)$, respectively.

Each of the methods presented in this chapter makes use of one of these three notations in order to introduce Kripke semantics elements in the (meta-)language of the sequent calculus. While the methods which use the first two notations keep the tree-structure of the frames in their internalisation of Kripke semantics, the one which uses the third notation does not. Other methods use Kripke semantics, but we will not address them since they do not satisfy Condition 2.1, p. 52. These are: Kanger's [67], which generates calculi only for the systems **T**, **S4** and **S5**, and Kushida and Okada's [73], which only concerns a quantified version of **S4**. There is also Pliuškevičiene's [97, 98], which is for predicate modal logics, including those containing the Barcan axiom.

The idea of internalising Kripke semantics in the syntax, in order to provide modal propositional logic with computational instruments, has also been adopted in:[1] tableaux systems [19, 38, 49, 87], natural deduction [130, 142] and labelled deductive systems [40, 141].

4.1 Semantic Modal Sequent Calculi

Cerrato introduced three different methods for generalising the Gentzen calculus. The first, [21], which is purely syntactic, has already been presented at the end of Section 3.3: its basic idea consists in adding meta-linguistic symbols to separate modal behaviours from propositional ones. This method can be applied to several systems of modal logic, but it does not work very well for cut-elimination.

The second and third methods are similar: in one of them [20], Cerrato introduces *semantic modal sequents*, which are trees of sequents together with the accessibility relation R of Krikpe semantics; in the other [23], Cerrato uses trees of single sequences of formulas, called *modal tree-sequents*, and he operates without the aforementioned accessibility relation R. In this section we shall only deal with the former of these two methods, since it yields better results:

> Semantic modal sequents directly introduce Kripke accessibility relation into the structure of the calculus, leading to an uniform treatment that is cut-free for all those SLH-systems. The semantic proof exhibited [...] bypasses the problem of syntactic cut-elimination, that is afforded by modal tree-sequents. (Notation adjusted.) [22, p. 1]

To start with Cerrato's calculi, it is necessary to understand, at the intuitive level, how a semantic modal sequent is constructed. Let us recall the 'graphic' way of presenting tree-frames introduced at point 1. of the previous section. We had objects of the form[2]

$$\begin{array}{ccccc} & \circ & & \longrightarrow & \circ \\ \swarrow & & \searrow & & \\ \circ & & & \circ & \end{array} \qquad\qquad \begin{array}{ccc} & \circ & \\ & \downarrow & \\ & \circ & \\ \swarrow & & \searrow \\ \circ & & \circ \end{array}$$

By simply substituting classical sequents for points, according to determined rules that we will present below, we obtain semantic modal sequents

$$\begin{array}{ccccc} & \Gamma & & \longrightarrow & \Gamma_1 \\ \swarrow & & \searrow & & \\ \Gamma_2 & & & \Gamma_3 & \end{array} \qquad\qquad \begin{array}{ccc} & \Gamma & \\ & \downarrow & \\ & \Gamma_1 & \\ \swarrow & & \searrow \\ \Gamma_2 & & \Gamma_3 \end{array}$$

where Γ_i, $0 \leq i \leq 3$ is a classical sequent. A binary minimal relation R, that extends $\rightarrow$ and is called accessibility relation, is assumed to hold between sequents. Thus both, the relations $\rightarrow$ and the relation R, become part of the meta-linguistic apparatus of semantic modal sequents.

Note that in semantic modal sequents every formula is determined by two 'co-ordinates:' the position that the formula has in a sequent, i.e. on the left or on the right side of the sequent arrow, and the position that it has in the network made by sequents and arrows.

Syntactic Notation

- $\rightarrow$ and R are two new meta-linguistic symbols.
- G, H, ... denote trees of sequents.

Definition 4.1 A sequent in the semantic modal sequent calculi, or shortly a *semantic modal sequent*, is a triple $\langle W, \rightarrow, R\rangle$ where

W is a non-empty set of classical sequents,[3]

$\rightarrow$ is a strict tree-ordering on W; W and $\rightarrow$ give rise to trees of sequents that we can inductively define in the following way:

- if $M \Rightarrow N$ is a classical sequent, then $M \Rightarrow N$ is a tree of sequents;
- if $M \Rightarrow N$ is a classical sequent and G_1, ..., G_n are trees of sequents, then:

$$\begin{array}{ccc} & M \Rightarrow N & \\ \swarrow & .... & \searrow \\ G_1 & ... & G_n \end{array}$$ is a tree of sequents.

R is a binary relation on W that extends $\rightarrow$, called accessibility relation. R puts the corresponding semantic accessibility relation into sequents; namely, if Ax_1, ..., Ax_n are modal axioms, R is the minimal relation containing $\rightarrow$, $R(Ax_1)$, ... $R(Ax_n)$, where the correspondence between axioms and relations is given by the following table:

Axiom	$R(Ax)$
K	$\rightarrow$
T	reflexive closure of $\rightarrow$
4	transitive closure of $\rightarrow$
B	symmetric closure of $\rightarrow$
5	euclidean closure of $\rightarrow$

Definition 4.2 The *interpretation* τ of a semantic modal sequent is definable in the following inductive way:

- $(M \Rightarrow N)^\tau := \bigwedge M \rightarrow \bigvee N$
- $\begin{array}{c} M \Rightarrow N \\ \swarrow \ ... \ \searrow \\ G_1 ... G_n \end{array} := \quad \bigwedge M \rightarrow \left(\bigvee N \vee \Box G_1^\tau, ..., \vee \Box G_n^\tau\right)$

Note that only the relation $\rightarrow$ influences the translation τ.

If we assume the relation R to be $\rightarrow$, then we have that the calculus **Ssk** for the system **K** is composed of:

Initial Sequents

$$\alpha \Rightarrow \alpha$$

Structural Rules

Internal Weakening and Contraction

$$\frac{\begin{array}{c} H \\ \downarrow \\ M \Rightarrow N \\ \swarrow \ .... \ \searrow \\ G_1 \ ..... \ G_n \end{array}}{\begin{array}{c} H \\ \downarrow \\ \alpha, M \Rightarrow N \\ \swarrow \ .... \ \searrow \\ G_1 \ ..... \ G_n \end{array}} WA \qquad \frac{\begin{array}{c} H \\ \downarrow \\ M \Rightarrow N \\ \swarrow \ .... \ \searrow \\ G_1 \ ..... \ G_n \end{array}}{\begin{array}{c} H \\ \downarrow \\ M \Rightarrow N, \alpha \\ \swarrow \ .... \ \searrow \\ G_1 \ ..... \ G_n \end{array}} WK$$

$$\dfrac{\begin{array}{c} H \\ \downarrow \\ \alpha, \alpha, M \Rightarrow N \\ \swarrow \ldots \searrow \\ G_1 \ldots\ldots G_n \end{array}}{\begin{array}{c} H \\ \downarrow \\ \alpha, M \Rightarrow N \\ \swarrow \ldots \searrow \\ G_1 \ldots\ldots G_n \end{array}} CA \qquad \dfrac{\begin{array}{c} H \\ \downarrow \\ M \Rightarrow N, \alpha, \alpha \\ \swarrow \ldots \searrow \\ G_1 \ldots\ldots G_n \end{array}}{\begin{array}{c} H \\ \downarrow \\ M \Rightarrow N, \alpha \\ \swarrow \ldots \searrow \\ G_1 \ldots\ldots G_n \end{array}} CK$$

External Weakening and Merge

$$\dfrac{\begin{array}{c} H \\ \downarrow \\ M \Rightarrow N \\ \swarrow \ldots \searrow \\ G_1 \ldots\ldots G_n \end{array}}{\begin{array}{c} H \\ \swarrow \ldots \searrow \\ M \Rightarrow N \quad \ldots \quad S \Rightarrow T \\ \swarrow \ldots \searrow \\ G_1 \ldots\ldots G_n \end{array}} EW$$

Rule of merge:

from a semantic modal sequent s having two (points-)sequents $D \Rightarrow F$ and $M \Rightarrow N$ with $(D \Rightarrow F)\ R\ (M \Rightarrow N)$, and one (point-)sequent $S \Rightarrow T$ such that $(D \Rightarrow F) \rightarrow (S \Rightarrow T)$, infer the modal semantic sequent s' obtained from s by collapsing the sequent $S \Rightarrow T$ into the sequent $M \Rightarrow N$, in such a way that instead of $M \Rightarrow N$, we have $M, S \Rightarrow N, T$. Note that the rest of the structure in the semantic modal sequent rests unchanged.

Necessitation Rule

$$\dfrac{\begin{array}{c} G \\ \downarrow \end{array}}{\begin{array}{c} \Rightarrow \\ \downarrow \\ G \end{array}} rn$$

Logical Rules

Propositional Rules

$$\dfrac{\begin{array}{l} H \\ \downarrow \\ M \Rightarrow N, \alpha \\ \swarrow \ldots \searrow \\ G_1 \ldots\ldots G_n \end{array}}{\begin{array}{l} H \\ \downarrow \\ \neg\alpha, M \Rightarrow N \\ \swarrow \ldots \searrow \\ G_1 \ldots\ldots G_n \end{array}} \neg A \qquad \dfrac{\begin{array}{l} H \\ \downarrow \\ \alpha, M \Rightarrow N \\ \swarrow \ldots \searrow \\ G_1 \ldots\ldots G_n \end{array}}{\begin{array}{l} H \\ \downarrow \\ M \Rightarrow N, \neg\alpha \\ \swarrow \ldots \searrow \\ G_1 \ldots\ldots G_n \end{array}} \neg K$$

$$\dfrac{\begin{array}{c} H \\ \downarrow \\ \alpha_i, M \Rightarrow N \\ \swarrow \ldots \searrow \\ G_1 \ldots\ldots G_n \end{array}}{\begin{array}{c} H \\ \downarrow \\ \alpha_0 \wedge \alpha_1, M \Rightarrow N \\ \swarrow \ldots \searrow \\ G_1 \ldots\ldots G_n \end{array}} \wedge A \qquad \dfrac{\begin{array}{c} H \\ \downarrow \\ M \Rightarrow N, \alpha \\ \swarrow \ldots \searrow \\ G_1 \ldots\ldots G_n \end{array} \qquad \begin{array}{c} H \\ \downarrow \\ M \Rightarrow N, \beta \\ \swarrow \ldots \searrow \\ G_1 \ldots\ldots G_n \end{array}}{\begin{array}{c} H \\ \downarrow \\ M \Rightarrow N, \alpha \wedge \beta \\ \swarrow \ldots \searrow \\ G_1 \ldots\ldots G_n \end{array}} \wedge K$$

Modal Rules

Rule $\Box A$

from a semantic modal sequent s having two (points-)sequents $M \Rightarrow N$ and $\alpha, S \Rightarrow T$ with $(M \Rightarrow N)\ R\ (\alpha, S \Rightarrow T)$, infer the semantic modal sequent s' obtained from s by substituting (both in the domain and in the relations $\rightarrow$ and R) the sequents $M \Rightarrow N$ and $\alpha, S \Rightarrow T$ with $\Box\alpha, M \Rightarrow N$ and $S \Rightarrow T$, respectively.

$$\dfrac{\begin{array}{c} H \\ \downarrow \\ M \Rightarrow N \\ \swarrow \ldots \downarrow \ldots \searrow \\ G_1 \quad \Rightarrow \alpha \quad G_n \end{array}}{\begin{array}{c} H \\ \downarrow \\ M \Rightarrow N, \Box\alpha \\ \swarrow \ldots \searrow \\ G_1 \ldots\ldots G_n \end{array}} \Box K$$

Semantic Modal Sequent Calculi for other Normal Modal Systems

– In order to obtain semantic modal calculi for the systems containing the axiom D, we use the rule

$$\dfrac{\begin{array}{c} G \\ \downarrow \\ \Rightarrow \end{array}}{G}\, d$$

– In order to obtain semantic modal calculi for the systems containing the T axiom or the B axiom, we vary the rule of *merge* and the rule $\Box A$ in accordance with the fact that the accessibility relation R is the reflexive or the symmetric closure, respectively, of the relation $\rightarrow$.
– In order to obtain semantic modal calculi for the systems containing the 4 axiom or the 5 axiom, we vary the rule $\Box A$ in accordance with the fact that the

accessibility relation R is the transitive closure or the euclidean closure of the relation $\rightarrow$, respectively. Moreover we add the following two rules, respectively:

Rule $\tilde{4}$

from a semantic modal sequent s having a sequent $M \Rightarrow N$ that occurs at a node n different from the root of the tree, infer the semantic modal sequent s' obtained from s by moving the sequent $M \Rightarrow N$ to a node n' related to n by the relation R.

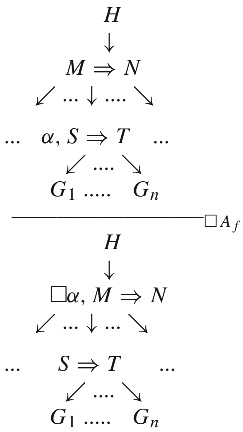
Rule $\tilde{5}$

from a semantic modal sequent s having a sequent $M \Rightarrow N$ that occurs at a node different from the root of the tree, infer the semantic modal sequent s' obtained from s by moving the sequent $M \Rightarrow N$ to an arbitrary node, except the root.

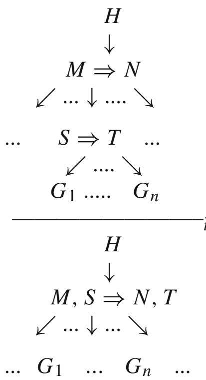
In the calculi for the systems containing the 4 and the 5 axiom, the rule of *merge* is applied *only* in those cases in which R is neither the transitive closure nor the euclidean closure of $\rightarrow$.

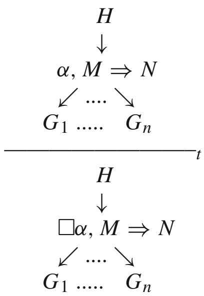
We can make several observations concerning the semantic sequent calculi. The first and most evident is that there is no cut-rule. The second observation concerns the rule of *merge* and the rule $\Box A^4$: indeed, it would be legitimate to ask why these rules have only been 'described,' but not graphically represented. There are two reasons for this. The first is that these rules can be described in general terms; so, for example, in order to get calculi for all the SLH-systems, it suffices to say that *merge* and $\Box A$ change in accordance with the relation R. By contrast, their graphical representation would have never allowed this level of simplicity and suppleness. The second reason why *merge* and $\Box A$ have only been described is related to the fact that it is difficult to draw certain variants of these rules. Let us, for example, consider the case of the semantic modal calculus **Sskt**. This calculus can be introduced in the following two equivalent ways: (i) the calculus **Ssk** is extended by assuming R to be the reflexive closure of $\rightarrow$ a; (ii) in the calculus **Ssk** the descriptions of the rules *merge* and $\Box A$ is substituted by their graphical representations, which are

$$\frac{\begin{array}{c} H \\ \swarrow \ \cdots\cdots \ \searrow \\ \begin{array}{cc} M \Rightarrow N & S \Rightarrow T \\ \swarrow \ \cdots\cdots \ \searrow & \swarrow \ \cdots\cdots \ \searrow \\ G_1 \ldots\ldots G_n & G'_1 \ldots\ldots G'_n \end{array} \end{array}}{\begin{array}{c} H \\ \downarrow \\ M, S \Rightarrow N, T \\ \swarrow \ \cdots\cdots \ \searrow \\ G_1 \ldots\ldots G_n \qquad G'_1 \ldots\ldots G'_n \end{array}} \, merge_f$$



and then the following two rules, that are just the variants of the rule of $merge_f$ and of the rule $\Box A_f$, respectively, if the relation R is the reflexive closure of $\rightarrow$, are added:





The foregoing shows that the graphical representations given in (ii) follow the instructions given in (i) *verbatim*. Thus, for the calculus for the system **KT**, we have a double option: we can either describe the rule of $merge$ and the rule $\Box A$ or graphical represent them. This option does not hold for all the semantic modal calculi, e.g. it seems rather difficult to represent in a graphic and correct way the variants of the rule of $merge$ and the rule $\Box A$ for the calculus **Sskt5**.

Let us now move to the third observation relating to the semantic sequent calculi. This observation concerns the difference between internal structural rules and propositional rules, on the one hand, and external structural rules and modal rules, on the other hand. While internal structural rules and propositional rules leave the structure of the tree unaltered, and affect only one sequent (for tree) at a time, the rule $\Box A$ affects more than one sequent at a time, while the rule $\Box K$, as well as the external structural rules, visibly modify the configuration of the tree.

Let us finally note that a derivation in this type of calculi, considered bottom-up, is nothing but an out-and-out construction of a tree-network, following the given rules; e.g. we start from

$$\Gamma$$

thanks to the rule $\Box K$, we can have

$$\begin{array}{c} \Gamma \\ \downarrow \\ \Gamma_1 \end{array}$$

and then with the rule of $merge$ we obtain

$$\begin{array}{ccc} & \Gamma & \\ \swarrow & & \searrow \\ \Gamma_1 & & \Gamma_2 \end{array}$$

and then with one of the propositional rules, we have

$$\begin{array}{ccc} & \Gamma' & \\ \swarrow & & \searrow \\ \Gamma_1 & & \Gamma_2 \end{array}$$

and so on ...

Theorem 4.3 *Each of the calculi* ***Ssk**** *is sound and complete with respect to the corresponding class of frames.*

Proof The soundness theorem is established thanks to the soundness result holding between the SLH-systems and the corresponding classes of frames, and the following three steps: (i) we extend the translation τ to a translation from sequent-style inferences to Hilbert-style inferences; (ii) we isolate the sequent(s) affected by the rule, and prove the corresponding implication; (iii) we transport that implication up all along the tree, so that, by modus ponens, we immediately have the thesis. The completeness theorem is established by means of a simplified version of the proof used in [43] for classical logic, adapted to modal systems. We underline that Cerrato proves the adequacy of his calculi by using a variant of the calculi where we only have generalised axioms, internal structural rules, propositional and modal rules, rules obtained from $\Box A$ by varying the relation R. $\square$

Example 4.4 Here is an example of a derivation in the calculi **Ssk***:

$$
\dfrac{\dfrac{\begin{array}{cc} \Rightarrow & \Rightarrow \\ \downarrow & \downarrow \\ \alpha \to \alpha & \beta \to \beta \end{array}}{\begin{array}{c} \Rightarrow \\ \downarrow \\ \alpha \to \beta, \alpha \Rightarrow \beta \end{array}}\,{\scriptstyle \to A'}}{\begin{array}{c} \Box(\alpha \to \beta), \Box\alpha \Rightarrow \\ \downarrow \\ \dfrac{\dfrac{\dfrac{\Rightarrow \beta}{\Box(\alpha \to \beta), \Box\alpha \Rightarrow \Box\beta}\,{\scriptstyle \Box K}}{\Box(\alpha \to \beta) \Rightarrow \Box\alpha \to \Box\beta}\,{\scriptstyle \to K}}{\Rightarrow \Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)}\,{\scriptstyle \to K} \end{array}}\,{\scriptstyle \Box A^*}
$$

Remark 4.5 As we can easily see from the rules of the calculi **Ssk*** or from the example of derivation in the calculi **Ssk***, the notation of the semantic calculi is rather awkward. Moreover, the cut-rule is not formulated, and therefore the structural variants are not available. In addition, the external structural rules, as well as the necessitation rule, are not mentioned in [20], nor in [22]. Finally, in a recent article by Restall [116], the reader can find an idea similar to the one of the semantic sequent calculi, which is not however fully developed.

4.2 Indexed Sequent Calculi

In the previous section we saw how it is possible to create semantic modal sequents simply by considering tree-frames formalised in a 'graphic way,' and by substituting points with classical sequents (according to determinate rules). In the current section we will complete a similar operation: we will consider finite sequences of natural numbers (which, as we have seen at point 2. at the beginning of the chapter, can form tree-frames, if conveniently concatenated), and we will 'attach' a classical sequent to each of them; in short we will deal with objects of the form

$$\sigma_1, s_1, \sigma_2, s_2, ..., \sigma_n, s_n$$

where σ_i and s_i $(1 \leq i \leq n)$ are an index and a classical sequent, respectively. These objects are called *tableaux* and can be interpreted thus:

$$s_1 \text{ is true at the world } \sigma_1, ..., s_n \text{ is true at a world } \sigma_n$$

Kripke [72] was the first to use tableaux in order to establish a completeness result with Kripke semantics; indeed, he was the one who posed the problem of the cut-elimination theorem for this type of calculi. Mints [82] solved the problem, and we shall present his results. However, before turning to this topic, let us mention some works in which Kripke-style systems are studied by modal-theoretical means; these are: Hughes and Cresswell's [62], Ohlbach's [88], and Wallen's [144]. As Mints underlines, in (most of) these works the sequences of natural numbers are applied to particular formulas, and not to a sequent as a whole, i.e. we have objects of the form: $(\sigma\alpha)$, $(\sigma\beta)\Rightarrow(\sigma\gamma),(\sigma\delta)$, instead of objects of the form: $\sigma\alpha, \beta \Rightarrow \gamma, \delta$.

Syntactic Notation

- $\sigma_1, \sigma_2, ...$ denote finite sequences of natural numbers, $\langle i_0, i_1, ..., i_p\rangle$.
- $\star$ will denote concatenation: the expression $\sigma_1 \star \sigma_2$ stands for the concatenation of the two sequences σ_1 and σ_2. The immediate successor of a sequence σ is any sequence of the form $\sigma \star \langle i\rangle$, which, for simplicity, is written $\sigma \star i$.
- $\Gamma, \Delta, ...$ denote classical sequents.
- $G, H, ...$ denote tableaux.

Definition 4.6 Given $n + 1$ classical sequents $\Gamma, \Gamma_1, ..., \Gamma_n$, and n distinct, finite and non-empty sequences of natural numbers $\sigma_1, ..., \sigma_n$, a sequent in the indexed sequent calculi, or simply, a *tableau*, is an object of the form

$$\Gamma; \sigma_1\Gamma_1; \sigma_2\Gamma_2; ...; \sigma_n\Gamma_n$$

where the σ_i $(1 \le i \le n)$, that we can call from now on the *indices* of the classical sequents, form a tree with a root Ø under inclusion of sequences:

$$\sigma < \tau \; if \; \tau = \sigma \star \sigma', \text{ for some } \sigma'$$

Moreover, it is assumed that a binary relation R of immediate accessibility between finite sequences is given with the following properties:

(i) $\sigma R \sigma \star i$,
(ii) $\sigma R \sigma'$ implies ($\sigma = \sigma'$, or $\sigma' = \sigma \star i$, or $\sigma = \sigma' \star i$).

The relation R is reflexive if $\sigma R\sigma$ holds for every sequence σ; R is symmetric if $\sigma R\sigma'$ implies $\sigma' R\sigma$. Notice that (ii) prevents R from being transitive.

We now have a precise definition of tableau; let us consider the following example of tableau:

$$(\circledcirc) \qquad M \Rightarrow N; \langle 0\rangle S \Rightarrow T; \langle 0, 0\rangle P \Rightarrow Q; \langle 0, 1\rangle Z \Rightarrow W$$

As the reader can observe, there are sequences of numbers ($\langle 0\rangle$, $\langle 0, 0\rangle$, $\langle 0, 1\rangle$), and to each sequence of numbers, or, simpler, to each index, a classical sequent is associated. Indices and sequents form a tree with root Ø under inclusion of sequents.

One cannot help but notice the strong resemblance between a tableau and a semantic modal sequent: e.g. the tableaux ◎ is nothing other than the following semantic modal sequent and vice versa,

$$\begin{array}{ccc} & M \Rightarrow N & \\ & \downarrow & \\ & S \Rightarrow T & \\ \swarrow & & \searrow \\ P \Rightarrow Q & & Z \Rightarrow W \end{array}$$

They are just two ways, a 'graphic' way and an 'indexed' way, of expressing the same kind of object. This resemblance will be fully analysed in Section 5.4.

In order to give the intended translation of tableaux, we first need the following two definitions.

Definition 4.7 Let us assume the notation

$$i \star (\Gamma; \sigma_1 \Gamma_1; ...; \sigma_n \Gamma_n)$$

which stands for the result of concatenating i in front of all indices of sequents, which is to say

$$\{i\}\ \Gamma;\ \ (\{i\} \star \sigma_1)\ \Gamma_1; ...;\ \ (\{i\} \star \sigma_n)\ \Gamma_n$$

Definition 4.8 If a tableau G contains non-empty indices, it can be written as

$$G = M \Rightarrow N; i_1 \star H_1; ...; i_n \star H_n$$

where $i_1, ..., i_n$ are all unit indices in G.

Definition 4.9 The *interpretation* τ of a tableaux with non-empty indices is definable in the following inductive way:

- $(M \Rightarrow N)^\tau := \bigwedge M \rightarrow \bigvee N$
- $(M \Rightarrow N\ ; i_1 \star G_1; ...; i_n \star G_n)^\tau := \bigwedge M \rightarrow \left(\bigvee N \vee \Box G_1^\tau \vee ... \vee \Box G_n^\tau\right)$

If we assume the relation R to be an accessibility relation that does not enjoy any particular property, then the calculus **Isk** for the system **K** is composed of:

Initial Sequents

$\alpha \Rightarrow \alpha$

Structural Rules

Internal Weakening and Contraction

$$\frac{G; \sigma\ M \Rightarrow N}{G; \sigma\ \alpha, M \Rightarrow N}\ WA \qquad \frac{G; \sigma\ M \Rightarrow N}{G; \sigma\ M \Rightarrow N, \alpha}\ WK$$

$$\frac{G; \sigma\ \alpha, \alpha, M \Rightarrow N}{G; \sigma\ \alpha, M \Rightarrow N}\ CA \qquad \frac{G; \sigma\ M \Rightarrow N, \alpha, \alpha}{G; \sigma\ M \Rightarrow N, \alpha}\ CK$$

External Weakening and Merge

$$\frac{G}{G; G'}\ EW \qquad \frac{G; \sigma \star i\ M \Rightarrow N; \sigma'\ P \Rightarrow Q}{G; \sigma'\ M, P \Rightarrow N, Q}\ merge$$

where in the rule of $merge$ we have $\sigma R \sigma'$ and i does not occur in G, σ'.

Necessitation Rule

$$\frac{G}{\emptyset \Rightarrow; i \star G}\ rn$$

Logical Rules

Propositional Rules

$$\frac{G; \sigma\ M \Rightarrow N, \alpha}{G; \sigma\ \neg\alpha, M \Rightarrow N}\ \neg A \qquad \frac{G; \sigma\ \alpha, M \Rightarrow N}{G; \sigma\ M \Rightarrow N, \neg\alpha}\ \neg K$$

$$\frac{G; \sigma\ \alpha_i, M \Rightarrow N}{G; \sigma\ \alpha_0 \wedge \alpha_1, M \Rightarrow N}\ \wedge A \qquad \frac{G; \sigma\ M \Rightarrow N, \alpha \quad G; \sigma\ M \Rightarrow N, \beta}{G; \sigma\ M \Rightarrow N, \alpha \wedge \beta}\ \wedge K$$

Modal Rules

$$\frac{G; \sigma\ M \Rightarrow N; \sigma'\ \alpha, S \Rightarrow T}{G; \sigma\ \Box\alpha, M \Rightarrow N; \sigma'\ S \Rightarrow T}\ \Box A \qquad \frac{G; \sigma\ M \Rightarrow N; \sigma \star i\ \Rightarrow \alpha}{G; \sigma\ M \Rightarrow N, \Box\alpha}\ \Box K$$

where in the rule $\Box A$ we have $\sigma R \sigma'$, while in the rule $\Box K$ the index $\sigma \star i$ is new, reading the rule bottom-up.

In order to define the cut-rule, we need to introduce some definitions and lemmas.

Definition 4.10 A *tableau is pure* if components of the indices σ_i are unique. A component is *unique* when it occurs in a given index at most once; if a component i occurs in σ_j and σ_k, $j \neq k$, then $\sigma_j < \sigma_k$ or $\sigma_k < \sigma_j$.

A *derivation is pure* if all tableaux in it are pure and a component i introduced by the rule $\Box K$ is encountered only above the rule which introduces it.

Lemma 4.11 *Every derivation can be made pure by renaming indices.*

Proof Go up the derivation and rename components i in the rule $\Box K$, so that the purity condition is satisfied. □

From now on we assume all tableaux and all derivations to be pure.

Definition 4.12 A *product* $\Gamma \centerdot \Delta$ of two classical sequents $\Gamma = M \Rightarrow N$ and $\Delta = P \Rightarrow Q$ is the sequent

$$\Gamma \centerdot \Delta = M, P \Rightarrow N, Q$$

The product of two tableaux with the same set of indices

$$U = \sigma_1 \Gamma_1; ...; \sigma_n \Gamma_n \qquad V = \sigma_1 \Delta_1; ...; \sigma_n \Delta_n$$

is, by definition, the tableau

$$U \centerdot V = \sigma_1 \Gamma_1 \centerdot \Delta_1; ...; \sigma_n \Gamma_n \centerdot \Delta_n$$

Finally, take two tableaux

$$U' = U; U_1 \qquad V' = V; V_1$$

with common indices in U and V, and different indices in U_1 and V_1, then the product is a tableau of the form

$$U' \centerdot V' = U \centerdot V; U_1; V_1$$

or, more informally, the common part is multiplied as before, and the remaining parts are added at the end.

We are now in a position to present the cut-rule, which is:

Cut-Rule

$$\frac{G; \sigma\ M \Rightarrow N, \alpha \qquad G'; \sigma\ \alpha, P \Rightarrow Q}{G \centerdot G'; \sigma\ M, P^* \Rightarrow N^*, Q} Mixcut_\alpha$$

where N and P might contain α, and N^* and P^* are the same multisets as N and P, respectively, except that they do not contain an occurrence of the formula α.

Indexed sequent calculi for other normal modal systems

In order to obtain the indexed sequent calculi for the systems containing the T axiom or the B axiom, we vary the rule of $merge$ and the rule $\Box A$ in accordance with the fact that the relation R is reflexive or symmetric, respectively. The calculi for the systems containing the 4 axiom are obtained by adjoining the logical and structural rules

$$\frac{G; \sigma M \Rightarrow N; \sigma^{'} \Box\alpha, S \Rightarrow T}{G; \sigma \Box\alpha, M \Rightarrow N; \sigma^{'} S \Rightarrow T} 4 \qquad \frac{G; \sigma \star i M \Rightarrow N}{G; \sigma^{'} \star i M \Rightarrow N} \tilde{4}$$

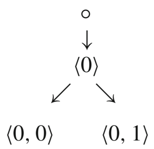
where in the rule 4 we have $\sigma R \sigma^{'}$, while in the rule $\tilde{4}$ we have $\sigma R \sigma^{'}$ and i does not occur in $G, \sigma^{'}$.

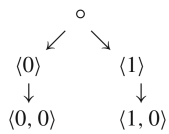
No rule is given for the axiom D, nor for the axiom 5.

As usual, some remarks are in order. Particularly, we would like to focus on the cut-rule, as well as on the rule of $merge$ and on the rule $\Box A$. Let us start with the cut-rule, which, at the first glance, might appear quite complex, although it is based on a simple and natural idea. Let us consider the tableau ◎ and the following:

$$M^{'} \Rightarrow N^{'}; \langle 0 \rangle S^{'} \Rightarrow T'; \langle 1 \rangle P^{'} \Rightarrow Q^{'}; \langle 0, 0 \rangle Z^{'} \Rightarrow W^{'}; \langle 1, 0 \rangle E \Rightarrow F$$

For the sake of clarity, let us draw the two trees that the indices of these tableaux describe, respectively,





Suppose that we want to apply a cut on the two sequents with the index $\langle 0 \rangle$, i.e. the sequents $S \Rightarrow T$ and $S^{'} \Rightarrow T'$. If, following the pattern of the classic cut, we are sure that, after the cut, we must collapse the sequents $S \Rightarrow T$ and $S^{'} \Rightarrow T'$ together, obtaining the sequent $S, S^{'} \Rightarrow T, T'$, we might wonder what to do with the rest of the structure. Following the definition of the $Mixcut$ rule presented above, we know that we must collapse the sequents that have the same index together (i.e. operating as in the case of the sequents where we cut), and leave the others separated. So, if we cut whichever formula belonging to both sequents $S \Rightarrow T$ and $S^{'} \Rightarrow T'$ of the two tableaux considered above, the resulting tableau is

$$M, M^{'} \Rightarrow N, N^{'}; \langle 0 \rangle S, (S^{'})^{*} \Rightarrow (T)^{*}, T'; \langle 1 \rangle P^{'} \Rightarrow Q^{'}; \langle 0, 0 \rangle P, Z^{'} \Rightarrow Q, W^{'};$$
$$\langle 0, 1 \rangle Z \Rightarrow W; \langle 1, 0 \rangle E \Rightarrow F$$

The indices of this tableau describe the tree

$$\begin{array}{ccccc} & & \circ & & \\ & \swarrow & & \searrow & \\ & \langle 0\rangle & & & \langle 1\rangle \\ \swarrow & & \searrow & & \downarrow \\ \langle 0,0\rangle & & \langle 0,1\rangle & & \langle 1,0\rangle \end{array}$$

As we shall see in Section 6.1, a similar idea will be used for tree-hypersequents.

The role of the cut-rule in indexed sequent calculi is now clear, and we can turn to the rule of *merge* and the rule $\Box A$. First of all, notice that in both these rules the relation R has such a central role that they vary depending on the property satisfied by R. This situation is similar to that of the rule of *merge* and of the rule $\Box A$ of the semantic sequent calculi (see the previous section). Nevertheless, there exists a deep difference between the two cases: in semantic sequent calculi we could only "describe" these rules. We could not formulate them, since this would have jeopardized correctness and generality. By contrast, in indexed sequent calculi, we can formulate both the rule of *merge* and the rule $\Box A$ without incurring a loss in generality and correctness, thanks to a more flexible notation.

Let us see in detail how the $\Box A$ and *merge* rules change depending on the property enjoyed by the relation R. First, we consider the rule $\Box A$ in which $\sigma R \sigma'$ holds. Let us suppose that R is an accessibility relation that does not enjoy any particular property, then we have $\sigma' = \sigma \star i$, and $\Box A$ becomes

$$\frac{G; \sigma M \Rightarrow N; \sigma \star i\ \alpha, S \Rightarrow T}{G; \sigma\ \Box\alpha, M \Rightarrow N; \sigma \star i\ \ S \Rightarrow T}\ \Box Af$$

Let us instead suppose that the relation R is reflexive, then we have

$$\frac{G; \sigma\ \alpha, M \Rightarrow N}{G; \sigma\Box\alpha, M \Rightarrow N}\ t$$

The same happens with the rule of *merge*; let us suppose that the relation R does not enjoy any particular property, we have

$$\frac{G; \sigma \star i M \Rightarrow N; \sigma \star j P \Rightarrow Q}{G; \sigma \star j\ M, P \Rightarrow N, Q}\ mergef$$

On the contrary, if R is reflexive, we have

$$\frac{G; \sigma \star i M \Rightarrow N; \sigma P \Rightarrow Q}{G; \sigma\ M, P \Rightarrow N, Q}\ \tilde{t}$$

Let us finally note that both the rules 4 and $\tilde{4}$ are subject to the same kind of variation.

We shall now present the results obtained in the indexed sequent calculi. Before doing so, it is important to note that Mints does not use the calculi **Isk*** in order to prove these results; rather, he uses a variant obtained by dropping the weakening rules (WA, WK and EW), the rule of merge and the necessitation rule, and substituting the axioms by the generalised ones. As for the calculi containing the rule 4, the rule $\tilde{4}$ is also dropped. Let us denote this variant of the calculi with $\mathbf{Isk}^*_+$.

Lemma 4.13 *The rules of external and internal weakening, necessitation, merge (in all its variants) and* $\tilde{4}$ *(whenever the rule* 4 *is present) are (height-preserving) admissible in the calculi* $\mathbf{Isk}^*_+$.

Proof By induction on the height of derivations. □

Theorem 4.14 *Each of the calculi* $\mathbf{Isk}^*_+$ *is sound and complete with respect to the corresponding Hilbert system.*

Proof By induction on the height of derivations in the appropriate Hilbert system and calculus, respectively. □

Lemma 4.15 *Every derivation of a pure sequent in any of the calculi* $\mathbf{Isk}^*_+$ *ending in a* $Mixcut$*, and containing no other application of* $Mixcut$*, can be transformed into a cut-free derivation of the same sequent by a finite number of standard reductions.*

Proof The proof is by induction on the complexity of the cut-formula (see Definition 2.3, p. 40), and by subinduction on the sum of the heights of the derivations of the premises of $Mixcut$. Note the use of a *mix*cut rule in order to solve the case of the contraction rules; for the rest, the proof is developed in the standard syntactic way, i.e. by distinguishing and analysing several cases depending on the last applied rule to the premises of the cut. The only difficult case is the one in which the cut-formula has the form $\Box\beta$, and has been introduced in the left premise by the rule $\Box K$, while in the right premise by the rule 4. This case is solved with the help of an auxiliary lemma where one makes essential use of the rule $\tilde{4}$ (see [82], pp. 686–688). □

Theorem 4.16 *Each of the calculi* $\mathbf{Isk}^*_+$ *is cut-free.*

Proof It follows from the previous lemma by induction on the number of cuts. □

Example 4.17 Below is an example of a derivation in the calculi **Isk***:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow\ ;\ 0\ \alpha \Rightarrow \alpha \qquad \Rightarrow\ ;\ 0\ \beta \Rightarrow \beta}{\Rightarrow\ ;\ 0\ \alpha, \alpha \to \beta \Rightarrow \beta}\ {\to}A'}{\Box(\alpha \to \beta), \Box\alpha \Rightarrow\ ;\ 0 \Rightarrow \beta}\ \Box A^*}{\Box(\alpha \to \beta), \Box\alpha \Rightarrow \Box\beta}\ \Box K}{\Box(\alpha \to \beta) \Rightarrow \Box\alpha \to \Box\beta}\ {\to}K}{\Rightarrow \Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)}\ {\to}K$$

Remark 4.18 As we have already stressed, the indexed sequent calculi have a more handy and flexible notation than the semantic modal sequent calculi. In addition, in this case the cut-rule is formulated. On the other hand, no rule corresponds to the axiom D nor to the axiom 5, and some delicate operations with the indices are here required.

Finally, let us point out that the calculi for systems containing the 4 axiom can be obtained in an alternative way: instead of postulating the rule 4, it is possible to extend the rule $\Box A$ allowing $\sigma R^{+}\sigma'$, where R^{+} is the transitive closure of R (see Definition 2.7, p. 41). We call this variant of the rule $\Box A$, $4'$. As the next chapter will show, the rule $4'$ will happen to be a quite useful rule.

Problem 4.19 Is it possible to formulate a 5 rule for the index sequent calculi (notice that it is quite easy to obtain a rule for the axiom d)?

4.3 Internalised Forcing (Relation) Sequent Calculi

The first two semantic methods for extending the Gentzen calculus that we have presented, namely the semantic modal method and the indexed method, have much in common. The third and last semantic generalisation of the classical sequent calculus, which is the object of this section, and which was introduced by Negri [85], departs considerably from the other two.

The main idea of Negri's generalisation consists in the internalisation, in the language of the sequent calculus, of the satisfiability relation of Kripke semantics (see Definition 2.9, p. 41). In order to internalise such a relation, Negri operates in the following way: (i) she considers as well-formed formulas objects of the form $i : \alpha$, where the symbol ":" simply substitutes the symbol $\models_{\mathfrak{M}}$. This means that now each modal formula α carries with itself the idea of being satisfied at a certain world i of a model $\mathfrak{M}$. (ii) She adds to the language and to the new well-formed formulas of the form $i : \alpha$, objects of the form iRj that (exactly as in Kripke semantics) should be interpreted as: the world i is related by the relation R to the world j (this idea is also common to Orlowska's relational proof systems [91]). This way the sequent calculus has all the (semantic) elements for expressing the notion of a formula α being satisfied in a model $\mathfrak{M}$. Consider for example the paradigmatic case of the satisfiability relation:

$$(\circledcirc) \qquad i \models_{\mathfrak{M}} \Box\beta \text{ iff } \forall j(iRj \to j \models_{\mathfrak{M}} \beta)$$

And suppose that we want to formulate in "sequent-calculus-terms" the right-left implication of the equivalence $\circledcirc$. We can express it with

$$\frac{iRj, j : \alpha \Rightarrow}{iRj, i : \Box\alpha \Rightarrow}$$

The rule, read bottom-up, says that if $\Box\alpha$ is true at a world i such that iRj, than α is true at j. The left-right implication of $\circledcirc$ will be instead conveyed by

$$\frac{iRj \Rightarrow j : \alpha}{\Rightarrow i : \Box\alpha}$$

If $\Box\alpha$ is false at a world i, then we can construct a world j such that iRj and j does not satisfy α.

Syntactic Notation

- Let $\mathcal{L}^{\Box}_{:\ \{\neg,\wedge,\Box\}}$ be a language that extends the language $\mathcal{L}^{\Box}{}_{\{\neg,\wedge,\Box\}}$ by adding (i) the symbol R for the accessibility relation, (ii) the symbol ":" to express the forcing relation, and (iii) a bunch of variables, i, j, z, ... which range in a set W.
- The set of well-formed modal formulas $WMF_:$ of the language $\mathcal{L}^{\Box}_{:\ \{\neg,\wedge,\Box\}}$ consists of
  - labelled formulas: $i : \alpha$, for every $\alpha \in WMF$, and for every $i \in W$, and
  - relational atoms: iRj, for every $i, j \in W$.

Definition 4.20 Given two $WMF_:$ multisets M and N, *an internalised forcing sequent* is an object of the form $M \Rightarrow N$.

The calculus **Ifsk** for the system **K** is composed of:

Initial Internalised Forcing Sequents

$$i : p \Rightarrow i : p \qquad\qquad iRj \Rightarrow iRj$$

Structural Rules

Weakening and Contraction

$$\frac{M \Rightarrow N}{i : \alpha, M \Rightarrow N}\ W_1A \qquad \frac{M \Rightarrow N}{M \Rightarrow N, i : \alpha}\ W_1K$$

$$\frac{i : \alpha, i : \alpha, M \Rightarrow N}{i : \alpha, M \Rightarrow N}\ C_1A \qquad \frac{M \Rightarrow N, i : \alpha, i : \alpha}{M \Rightarrow N, i : \alpha}\ C_1K$$

Relational Atoms Weakening and Contraction

$$\frac{M \Rightarrow N}{iRj, M \Rightarrow N}\ W_2A \qquad \frac{M \Rightarrow N}{M \Rightarrow N, iRj}\ W_2K$$

$$\frac{iRj, iRj, M \Rightarrow N}{iRj, M \Rightarrow N}\ C_2A \qquad \frac{M \Rightarrow N, iRj, iRj}{M \Rightarrow N, iRj}\ C_2K$$

Necessitation Rule

$$\frac{\Rightarrow i : \alpha}{\Rightarrow i : \Box\alpha}\,rn$$

Cut-Rule

$$\frac{M \Rightarrow N, i : \alpha \quad i : \alpha, P \Rightarrow Q}{M, P \Rightarrow N, Q}\,cut_{i:\alpha}$$

Logical Rules
Propositional Rules

$$\frac{M \Rightarrow N, i : \alpha}{i : \neg\alpha, M \Rightarrow N}\,\neg A \qquad \frac{i : \alpha, M \Rightarrow N}{M \Rightarrow N, i : \neg\alpha}\,\neg K$$

$$\frac{i : \alpha_i, M \Rightarrow N}{i : \alpha_0 \wedge \alpha_1, M \Rightarrow N}\,\wedge A \qquad \frac{M \Rightarrow N, i : \alpha \qquad M \Rightarrow N, i : \beta}{M \Rightarrow N, i : \alpha \wedge \beta}\,\wedge K$$

Modal Rules

$$\frac{j : \alpha, iRj, M \Rightarrow N}{i : \Box\alpha, iRj, M \Rightarrow N}\,\Box A \qquad \frac{iRj, M \Rightarrow N, j : \alpha}{M \Rightarrow N, i : \Box\alpha}\,\Box K$$

where in the rule $\Box K$ the variable j is new, reading the rule bottom-up.

Although the internalised forcing sequent calculi differ from the semantic modal calculi and from the indexed calculi with respect to how they internalise Kripke semantics in the Gentzen system, the modal rules of these three types of calculi are quite similar. More particularly, in the rule $\Box K$ of the internalised forcing calculi, the variable j is required not to have been used before, reading the rule bottom-up, as in Mints the index $\sigma \star i$ of the rule $\Box K$ is required to be new.

Before showing the special logical rules by means of which the calculi for the other normal modal systems can be obtained, we have to introduce a condition, said *closure condition*, that these rules should satisfy.

Definition 4.21 Consider a rule $\mathcal{R}$ of the form

$$\frac{\alpha, \beta_1, ..., \beta_m, \beta_{m+1}, \beta_{m+1}, M \Rightarrow N}{\beta_1, ..., \beta_m, M \Rightarrow N}$$

The result of the application of the *closure condition on* $\mathcal{R}$ is the rule $\mathcal{R}^c$ that is obtained from $\mathcal{R}$ by substituting the β_{m+1}, β_{m+1}, with a single β_{m+1}. $\mathcal{R}^c$ has then the form

$$\frac{\alpha, \beta_1, ..., \beta_m, \beta_{m+1}, M \Rightarrow N}{\beta_1, ..., \beta_m, M \Rightarrow N}$$

In order to obtain calculi for the SLH-systems, we add to the calculus **Ifsk** (i) combinations of the following special logical rules, each of which corresponds to one of the axioms (or frame properties) listed in Section 2.1, p. 44, and (ii) the rules that result from the application of the closure condition on the special logical rules (unless these rules happen to be redundant).

Special Logical Rules

$$\frac{iRj, M \Rightarrow N}{M \Rightarrow N}\, d \qquad \frac{iRi, M \Rightarrow N}{M \Rightarrow N}\, t$$

$$\frac{iRz, iRj, jRz, M \Rightarrow N}{iRj, jRz, M \Rightarrow N}\, 4 \qquad \frac{jRi, iRj, M \Rightarrow N}{iRj, M \Rightarrow N}\, b$$

$$\frac{iRz, iRj, jRz, M \Rightarrow N}{iRj, iRz, M \Rightarrow N}\, 5$$

where in the rule d the variable j does not belong to M nor to N.

In the case of the internalised forcing method, the procedure to obtain calculi for the SLH-systems is slightly different from the one used by the other methods. Suppose indeed that we want to obtain the calculus for the system **K** + **5**. Then we must consider the calculus **Ifsk** and add

- the rule 5, as usual; plus
- the rule

$$\frac{iRj, jRj, M \Rightarrow N}{iRj, M \Rightarrow N}\, 5^c$$

which is the rule that results from the application of the closure condition to the rule 5.

The procedure for obtaining the rule 5^c from the rule 5 is the following. Consider the rule 5 in the form

$$\frac{iRj, iRj, jRj, M \Rightarrow N}{iRj, iRj, M \Rightarrow N}$$

The premise of this rule contains two occurrences of the formula iRj. We thus apply the closure condition, i.e. we contract the two occurrences of the formula iRj in just one occurrence, and we obtain the rule 5^c.

Suppose that we want to obtain the internalised forcing calculus for the system **K+T+5**. According to the foregoing instructions, it would seem necessary to add to the calculus **Ifsk**

– the rule t (note that no new rule results from the application of the closure condition to the rule t),
– and the rules 5 and 5^c.

On the contrary, in this case, the rule 5^c is superfluous because of the rule t. Hence, in order to obtain the calculus for the system **K+T+5**, it suffices to add to the calculus **Ifsk** the rules t and 5. The conclusion that we can draw from this fact is that the internalised forcing sequent calculi do not satisfy the modularity property.

Thanks to her method, Negri is able to provide a calculus for the system **GL**. In this case (since it is a quite peculiar case, see Section 2.1), instead of adding a new logical rule, we must change our modal rules in the following way:

$$\frac{iRj, M \Rightarrow N, j : \Box\alpha \quad j : \alpha, iRj, M \Rightarrow N}{i : \Box\alpha, iRj, M \Rightarrow N}\ \Box A_{gl}$$

$$\frac{j : \Box\alpha, iRj, M \Rightarrow N, j : \alpha}{M \Rightarrow i : \Box\alpha}\ \Box K_{gl}$$

where in the rule $\Box K_{gl}$ the variable j is new, reading the rule bottom-up.

The calculus **Ifsgl** is obtained from the calculus **Ifsk** by (i) allowing initial sequent of the form

$$i : \Box\alpha, M \Rightarrow N, i : \Box\alpha$$

(ii) replacing the rules $\Box A$ and $\Box K$ by the rules $\Box A_{gl}$ and $\Box K_{gl}$, respectively, and (iii) adding the rule 4 and the following one:

$$\frac{}{iRi, M \Rightarrow N}\ irr.$$

We have thus seen that the internalised forcing method can be applied to all the SLH-systems plus **GL**; let us now present the results obtained in these calculi. Let us note, before doing so, that Negri presents her calculi in a logical variant, that we will denote with $\mathbf{Ifsk}^*_L$ and $\mathbf{Ifsgl}_L$, respectively.

Lemma 4.22 *In $\mathbf{Ifsk}^*_L$ and $\mathbf{Ifsgl}_L$ we have that: (i) the axioms of the form $\alpha \Rightarrow \alpha$ are admissible, (ii) the logical and modal rules are height-preserving invertible,*[5] *and (iii) the classical and relational atoms structural rules are (height-preserving) admissible.*

Proof By induction on the height of derivations. □

In order to be able to prove that the rule of necessitation is admissible in $\mathbf{Ifsk}^*_L$ and $\mathbf{Ifsgl}_L$, Negri uses a substitution lemma that she introduces thus [85, p. 516]:

> Although we are considering a propositional system, the use of explicit elements of the syntax creates a strong analogy to first-order logic. The *substitution lemma* is similar, both in the statement and in the proof, to the substitution lemma of the classical predicate calculus. (Our emphasis.)

Let us see how the substitution lemma works. First, we must define substitution in the following way:

$iRj\ (z/w) \equiv iRj$ if $w \neq i$ and $w \neq j$
$iRj\ (z/i) \equiv zRj$ if $i \neq j$
$iRj\ (z/j) \equiv iRz$ if $i \neq j$
$iRi\ (z/i) \equiv zRz$
$i : \alpha\ (z/j) \equiv i : \alpha$ if $j \neq i$
$i : \alpha\ (z/i) \equiv z : \alpha$

and extend the definition to multisets componentwise.

Lemma 4.23 *If $M \Rightarrow N$ is derivable in **Ifsk**$^*_L$, then $M(j/i) \Rightarrow N(j/i)$ is also derivable, with the same derivation height.*

Proof By induction on the height of the derivation of $M \Rightarrow N$. □

As we will see in a while, the substitution lemma is used by Negri in the proof of cut-elimination. We will make use of the substitution lemma in Sections 5.3 and 5.5.

Theorem 4.24 *Each of the calculi **Ifsk**$^*_L$ + **Ifsgl**$_L$ is sound and complete with respect to the corresponding class of frames.*

Proof The soundness proof is trivial, while the completeness one is by induction on the height of derivations. □

Theorem 4.25 *Each of the calculi **Ifsk**$^*_L$ is cut-free.*

Proof The proof is by induction on the complexity of the cut-formula (see Definition 2.3, p. 40) with subinduction on the sum of the heights of the derivations of the premises of cut, and it has the same structure of the proof of admissibility of cut presented in [86], Theorem 6.2.3. In case the rule d is considered, the proof follows the pattern of [84]. In all the cases of permutation of cuts that may result in a clash with the variable conditions in the rules $\Box K$ and d, an appropriate substitution (Lemma 4.23) prior to the permutation will be used. □

We have excluded the calculus **Ifsgl**$_L$ from Theorem 4.25 not because it is not cut-free, but because the cut-elimination proof for **Ifsgl**$_L$ requires the introduction of some important preliminary notions.

Definition 4.26 Consider the rule $\Box K_{gl}$

$$\frac{j : \Box\alpha, iRj, M \Rightarrow N, j : \alpha}{M \Rightarrow i : \Box\alpha}\ \Box K_{gl}$$

The variables j occurring in the premise of the rule are said to be *proper variables* (Negri call them *eigenvariables*).

Definition 4.27 Derivations are said to be *pure* when the proper variables used at step $\Box K_{gl}$ appear only in the subtree above the rule introducing them. (Clearly, by Lemma 4.23, such a condition can always be met.)

Definition 4.28 The *range* of a variable i in a derivation d is the (finite) set of worlds j such that either iRj or, for some $n \geqslant 1$, and for some $i_1, ..., i_n$, the atoms iRi_1, i_1Ri_2, ..., i_nRi_j appear in the antecedent of sequents of d. Ranges of variables are ordered by set inclusion.

The range of a variable i is, roughly, the set of all the variables to which i is directly (e.g. iRj) or more indirectly ($iRi_1, i_1Ri_2, ..., i_nRi_j$), connected by relational atoms. In the calculus **Ifsgl**$_L$ the contraction rules are *range-preserving admissible*, where a rule is said to be range-preserving admissible if the elimination of the rule does not increase the ranges of variables in the derivation.

Theorem 4.29 *The calculus* ***Ifsgl***$_L$ *is cut-free.*

Proof The proof follows the pattern of the proof of Theorem 4.25, but with a modified induction parameter. Indeed the proof is developed by induction on

1. the complexity of the cut-formula α,
2. the range of i,
3. the sum of the heights of the derivations of the premises of cut.

Following Negri, let us comment on how to treat cases of loops that may occur in a proof. If a loop $xRx_1, x_1Rx_2, ..., x_nRx$ occurs in the conclusion of cut, then the conclusion can be obtained without cut from the rule irr. and several applications of the rule 4. Otherwise, if there is no loop in the conclusion of cut, there is no loop in the premises either. The only way then to produce a loop would be by introducing proper variables at steps of $\Box K_{gl}$ that violate either the variable or the pureness condition. □

Negri proves that (certain) internalised forcing sequent calculi are decidable.

Theorem 4.30 *The calculi* ***Ifsk***, ***Ifskt***, ***Ifskb***, ***Ifsktb***, ***Ifss4*** *and* ***Ifss5*** *allow terminating proofs search.*

Proof The proof is developed in a purely syntactic way. We do not explain it in detail since we will use the same kind of proof for the method of tree-hypersequents, and therefore the interested reader can see Section 7.2. □

Example 4.31 Here is an example of a derivation in the calculi **Ifsk***:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{iRj, j:\alpha \Rightarrow j:\alpha \qquad j:\beta \Rightarrow j:\beta}{iRj, j:\alpha \to \beta, j:\alpha \Rightarrow j:\beta}\ {\to}A'}{iRj, i:\Box(\alpha \to \beta), i:\Box\alpha \Rightarrow j:\beta}\ \Box A^*}{i:\Box(\alpha \to \beta), i:\Box\alpha \Rightarrow i:\Box\beta}\ \Box K}{i:\Box(\alpha \to \beta) \Rightarrow i:\Box\alpha \to \Box\beta}\ {\to}K}{\Rightarrow i:\Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)}\ {\to}K$$

Let us finally note that the internalised forcing method can be used to give a syntactic account of those first-order frame properties that do not correspond to any modally expressible axioms (see Definition 2.18, p. 43 and what follows). For example, the irreflexivity property $\forall x(\neg x R x)$ corresponds to the zero-premise rule (that we have already seen before)

$$\overline{i R i, M \Rightarrow N}\ ^{irr.}$$

In the internalised sequent calculi obtained by adding to the calculus **Ifsk** this kind of rules, it is possible to prove, within a conservativity theorem, that no modal formula corresponds to such first-order definable frame properties.

Remark 4.32 The method introduced by Negri can be applied to several systems, even to **GL**, and it offers interesting results, such as decidability or admissibility of structural rules. On the other hand, it is not modular, it does not satisfy the Došen's principle redefined, and the calculi do not have a structural variant because of the lack of the special structural rules. Moreover the subformula property is not satisfied, as Negri herself remarks:

> Our calculi, although not satisfying a full subformula property, enjoy a subterm property: all terms in minimal derivations are terms found in the endsequent. [85, p. 508]

Restall [116, pp. 6, 7] proposes two solutions for repairing the lack of the subformula property in Negri's calculi,

> the rules here do not satisfy the subformula property [...]. We could repair this in two ways. One is to take relational facts to not be formulas properly so-called, or to take R the predicate to be present as a part of the operator □.

Neither of Restall's solutions is satisfactory. The first because, as Negri herself makes clear, relational atoms are part of the language of the internalised forcing calculi. The second because it is limited to the case of the rule $\Box K$, and does not consider the case of the special logical rules. In addition, even Restall [116, p. 7] mentions a third important reason for being sceptical about the possibility of avoiding the lack of subformula property,

> However, satisfying the subformula property in this way (the second way) seems unsatisfactory [...] we still have no genuine subformula property since we still have to deal with renaming variables.[...] Reworking our proof-theoretical analysis to deal with variables and quantification seems like a high price to pay to deal with modal inference which does not explicitly mention such things. (Brackets ours.)

By way of conclusion, let us bring up a problematic point that was also raised in the case of the display calculi (see Section 3.3). Note that in the internalised forcing sequent calculi, we do not deal with out-and-out modal formulas, but with modal formulas enriched with labels or with relational atoms. This means that we can never prove a modal formula of the form $\Box\alpha$, but only labelled objects of the form $i : \Box\alpha$. As a result, a question naturally arises: can we really assert that these calculi are computational instruments for modal logic, a logic where labels or relational atoms do not usually appear?

Problem 4.33 Is there a way to make the internalised forcing calculi modular?

Notes

1. The following list is by no means complete.
2. For the sake of uniformity with what follows, we let the trees grow top-down, and not bottom-up as in the figure of point 1. of the previous section; this change, of course, is not of any deep importance.
3. In [20] M, N are *sequences* of formulas.
4. What we are going to say on the rules $merge$ and $\Box A$ also holds for the rules $\tilde{4}$ and $\tilde{5}$.
5. Let us specify that the rule $\Box K_{gl}$ is just invertible.

Chapter 5
Comparing the Different Generalisations of the Sequent Calculus

We have thus introduced the main generalisations of the sequent calculus for modal propositional logic. Their analysis may be further developed, in particular from the perspective of deepening our understanding of the links between these generalisations. Wansing [147, p. 171] stresses the importance of this issue:

> In view of the diversity of these types of proof systems it becomes increasingly important to investigate their interrelations and their advantages and disadvantages.

In the following sections we will address this problem by comparing the sequent calculi introduced in Chapters 3 and 4. In the course of this discussion, other peculiarities and features of each calculus will emerge.

5.1 From Multiple Sequent Calculi to Display Sequent Calculi

We start the comparison between the numerous types of Gentzen systems for modal logic by showing how to embed the calculi **Msk*** in the calculi **Dsk***.

We firstly define a translation δ from the well-formed formulas and the well-formed multiple structures to display structures in the following way:

- $(\emptyset)^\delta := \begin{cases} I, & if\ it\ is\ the\ antecedent, \\ I^*, & if\ it\ is\ the\ consequent. \end{cases}$
- $(\alpha)^\delta := \alpha$
- $(-\alpha)^\delta := \alpha^*$
- $(M^*)^\delta := (M^\delta)^*$

Given the translation δ, we can define the translation τ in the following way:

- $(\alpha, M \underset{0}{\Box\!\!\Rightarrow} N, \beta)^\tau := (\alpha)^\delta \circ (M)^\delta \Rightarrow (N)^\delta \circ (\beta)^\delta$
- $(\alpha, M \underset{n}{\Box\!\!\Rightarrow} N, \beta)^\tau := \bullet^n((\alpha)^\delta \circ (M)^\delta) \Rightarrow (N)^\delta \circ (\beta)^\delta$

where $\bullet^n M$ stands for: $\overbrace{\bullet, \ldots, \bullet}^{n} M$.

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We are now in a position to prove the following theorem.

Theorem 5.1 *Let $M \underset{n}{\Box\!\!\Rightarrow} N$ be any modal sequent of the calculi **Msk***. Then every derivation of $M \underset{n}{\Box\!\!\Rightarrow} N$ in **Msk*** can be translated into a derivation of $(M \underset{n}{\Box\!\!\Rightarrow} N)^\tau$ in **Dsk***.*

Proof The proof is by induction on the derivation of the modal sequent $M \underset{n}{\Box\!\!\Rightarrow} N$ in **Msk***.[1]

If $M \underset{n}{\Box\!\!\Rightarrow} N$ is an initial sequent, then its translation τ is an admissible initial sequent of the form $\alpha \Rightarrow \alpha$ in display calculi. If $M \underset{n}{\Box\!\!\Rightarrow} N$ has been inferred by one of the structural rules or one of the shifting rules or one of the propositional rules for the connective $\neg$, then the procedure is straightforward. However, let us consider the example of the rule $\neg K$,

$$\dfrac{M \underset{n}{\Box\!\!\Rightarrow} N, -\alpha}{M \underset{n}{\Box\!\!\Rightarrow} N, \neg\alpha}\,\neg K \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\dfrac{\bullet^n (M)^\delta \Rightarrow (N)^\delta \circ \alpha^*}{\bullet^n (M)^\delta \circ ((N)^\delta)^* \Rightarrow \alpha^*}}{\bullet^n (M)^\delta \circ ((N)^\delta)^* \Rightarrow \neg\alpha}\,\neg K}{\bullet^n (M)^\delta \Rightarrow (N)^\delta \circ \neg\alpha}$$

If $M \underset{n}{\Box\!\!\Rightarrow} N$ has been inferred by the rule $\wedge K'$ (for the rule that $\wedge A'$ the procedure is analogous but easier), then we have

$$\dfrac{M \underset{n}{\Box\!\!\Rightarrow} N, \alpha \qquad P \underset{n}{\Box\!\!\Rightarrow} Q, \beta}{M, P \underset{n}{\Box\!\!\Rightarrow} N, Q, \alpha \wedge \beta}\,\wedge K' \qquad \rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\bullet^n (M)^\delta \Rightarrow (N)^\delta \circ \alpha}{(M)^\delta \Rightarrow \bullet^n ((N)^\delta \circ \alpha)}}{(M)^\delta \circ (P)^\delta \Rightarrow \bullet^n ((N)^\delta \circ \alpha)}}{\bullet^n ((M)^\delta \circ (P)^\delta) \Rightarrow (N)^\delta \circ \alpha}}{\bullet^n ((M)^\delta \circ (P)^\delta) \circ ((N)^\delta)^* \Rightarrow \alpha} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\bullet^n (P)^\delta \Rightarrow (Q)^\delta \circ \beta}{(P)^\delta \Rightarrow \bullet^n ((Q)^\delta \circ \beta)}}{(M)^\delta \circ (P)^\delta \Rightarrow \bullet^n ((Q)^\delta \circ \beta)}}{\bullet^n ((M)^\delta \circ (P)^\delta) \Rightarrow (Q)^\delta \circ \beta}}{\bullet^n ((M)^\delta \circ (P)^\delta) \circ ((Q)^\delta)^* \Rightarrow \beta}}{\bullet^n ((M)^\delta \circ (P)^\delta) \circ \bullet^n ((M)^\delta \circ (P)^\delta) \circ ((N)^\delta)^* \circ ((Q)^\delta)^* \Rightarrow \alpha \wedge \beta}\,\wedge K'}{\bullet^n ((M)^\delta \circ (P)^\delta) \circ \bullet^n ((M)^\delta \circ (P)^\delta) \Rightarrow (N)^\delta \circ (Q)^\delta \circ \alpha \wedge \beta}}{\bullet^n ((M)^\delta \circ (P)^\delta) \Rightarrow (N)^\delta \circ (Q)^\delta \circ \alpha \wedge \beta}\,C$$

If $M \underset{n}{\Box\!\!\Rightarrow} N$ is of the form $\Box\alpha, M \underset{1}{\Box\!\!\Rightarrow} N$ and has been inferred by the rule $\Box A_1$, then we have

$$\dfrac{M \underset{1}{\Box\!\!\Rightarrow} N, -\alpha}{\Box\alpha, M \underset{1}{\Box\!\!\Rightarrow} N}\,\Box A_1 \qquad \rightsquigarrow$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha \Rightarrow \alpha}{\Box\alpha \Rightarrow \bullet\alpha}\,\Box A}{\Box\alpha \circ (M)^\delta \Rightarrow \bullet\alpha}}{\bullet(\Box\alpha \circ (M)^\delta) \Rightarrow \alpha} \qquad \dfrac{\bullet(M)^\delta \Rightarrow (N)^\delta \circ (\alpha)^*}{\alpha \Rightarrow (N)^\delta \circ (\bullet(M)^\delta)^*}}{\bullet(\Box\alpha \circ (M)^\delta) \Rightarrow (N)^\delta \circ (\bullet(M)^\delta)^*}\,cut_\alpha}{\bullet(M)^\delta \Rightarrow (N)^\delta \circ (\bullet(\Box\alpha \circ (M)^\delta))^*}}{(M)^\delta \Rightarrow \bullet((N)^\delta \circ (\bullet(\Box\alpha \circ (M)^\delta))^*)}}{\Box\alpha \circ (M)^\delta \Rightarrow \bullet((N)^\delta \circ (\bullet(\Box\alpha \circ (M)^\delta))^*)}}{\bullet(\Box\alpha \circ (M)^\delta) \Rightarrow (N)^\delta \circ (\bullet(\Box\alpha \circ (M)^\delta))^*}}{\bullet(\Box\alpha \circ (M)^\delta) \circ \bullet(\Box\alpha \circ (M)^\delta) \Rightarrow (N)^\delta}}{\bullet(\Box\alpha \circ (M)^\delta) \Rightarrow (N)^\delta}\,C
$$

If $M \underset{n}{\Box\!\!\Rightarrow} N$ is of the form $\Box\alpha \underset{n}{\Box\!\!\Rightarrow} N$ and has been inferred by the rule $\Box A_2$, then we have

$$
\dfrac{\alpha \underset{n}{\Box\!\!\Rightarrow} N}{\Box\alpha \underset{n+1}{\Box\!\!\Rightarrow} N}\,\Box A_2 \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\dfrac{\bullet^n\alpha \Rightarrow (N)^\delta}{\alpha \Rightarrow \bullet^n(N)^\delta}}{\Box\alpha \Rightarrow \bullet^{n+1}(N)^\delta}\,\Box A}{\bullet^{n+1}\Box\alpha \Rightarrow (N)^\delta}
$$

If $M \underset{n}{\Box\!\!\Rightarrow} N$ is of the form $M \underset{n}{\Box\!\!\Rightarrow} \Box\alpha$ and has been inferred by the rule $\Box K$, then we have

$$
\dfrac{M \underset{n}{\Box\!\!\Rightarrow} \alpha}{M \underset{n-1}{\Box\!\!\Rightarrow} \Box\alpha}\,\Box K \qquad \rightsquigarrow \qquad \dfrac{\bullet^n(M)^\delta \Rightarrow \alpha}{\bullet^{n-1}(M)^\delta \Rightarrow \Box\alpha}\,\Box K
$$

We finally analyse each of the cases in which $M \underset{n}{\Box\!\!\Rightarrow} N$ has been inferred by one of the special structural rules d, t, b, 4.

$$
\dfrac{M \underset{n}{\Box\!\!\Rightarrow}}{M \underset{n-1}{\Box\!\!\Rightarrow}}\,d \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\bullet^n(M)^\delta \Rightarrow I^*}{\bullet^n(M)^\delta \circ \bullet I \Rightarrow I^*}}{\bullet^{n-1}(M)^\delta \Rightarrow I^*}\,d
$$

$$
\dfrac{M \underset{n}{\Box\!\!\Rightarrow} N}{M \underset{n-1}{\Box\!\!\Rightarrow} N}\,t \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\dfrac{\bullet^n(M)^\delta \Rightarrow (N)^\delta}{(M)^\delta \Rightarrow \bullet^n(N)^\delta}}{(M)^\delta \Rightarrow \bullet^{n-1}(N)^\delta}\,t}{\bullet^{n-1}(M)^\delta \Rightarrow (N)^\delta}
$$

$$
\dfrac{B(M) \underset{n}{\Box\!\!\Rightarrow} N}{B(M) \underset{n+1}{\Box\!\!\Rightarrow} N}\,4 \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\dfrac{\bullet^n(B(M))^\delta \Rightarrow (N)^\delta}{(B(M))^\delta \Rightarrow \bullet^n(N)^\delta}}{(B(M))^\delta \Rightarrow \bullet^{n+1}(N)^\delta}\,4}{\bullet^{n+1}(B(M))^\delta \Rightarrow (N)^\delta}
$$

$$\dfrac{M \ \Box\!\!\Rightarrow_n \ N}{N^* \Box\!\!\Rightarrow_n \ M^*}\, b \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\dfrac{\dfrac{\bullet^n (M)^\delta \Rightarrow (N)^\delta}{\bullet^n (M)^\delta \Rightarrow ((N)^\delta)^{**}}}{(M)^\delta \Rightarrow \bullet^n (((N)^\delta)^{**})}}{(\bullet^n (((N)^\delta)^{**}))^* \Rightarrow ((M)^\delta)^*}}{\bullet^n (((N)^\delta)^*) \Rightarrow ((M)^\delta)^*}\, b$$

□

5.2 From Higher-Arity Sequent Calculi to Display Sequent Calculi

We will now draw another comparison between modal sequent calculi. Specifically, we will consider higher-arity sequent calculi and display calculi. Analogously to the previous section, we will show that each derivation in the higher-arity sequent calculi can be transformed in a derivation in display calculi.

We firstly define the translation τ in the following way:

$$\left(M \Rightarrow^T_S N\right)^\tau := \bigwedge \Box S \circ M \Rightarrow \bigcirc \bullet T \circ N$$

where $\bigcirc \bullet T$ should be seen as analogous to $\bigwedge \Box S$: each formula belonging to T is preceded by the symbol $\bullet$, and linked to the others by the symbol $\circ$. Informally the translation τ can be explained in the following way: all the formulas of the higher-arity sequent are linked together by the symbol $\circ$, except the ones belonging to the multiset S that become a conjunction of boxed formulas. Moreover, each of the formulas belonging to the multiset T is preceded by the symbol $\bullet$. If $T \equiv N \equiv \emptyset$, we have

$$\left(M \Rightarrow^\emptyset_S \emptyset\right)^\tau := \bigwedge \Box S \circ M \Rightarrow I^*$$

If $S \equiv M \equiv \emptyset$, we have

$$\left(\emptyset \Rightarrow^T_\emptyset N\right)^\tau := I \Rightarrow \bigcirc \bullet T \circ N$$

We are now in a position to prove the following theorem.

Theorem 5.2 *Let $\mathcal{R} \subseteq \{d, t\}$ and $M \Rightarrow^S_T N$ be any higher-arity sequent of the calculi **H-ask** + $\mathcal{R}$. Then, every derivation of $M \Rightarrow^S_T N$ in **H-ask** + $\mathcal{R}$ can be translated into a derivation of $\left(M \Rightarrow^S_T N\right)^\tau$ in **Dsk** + $\mathcal{R}$.*

Proof The proof is by induction on the derivation of the higher-arity sequent $M \Rightarrow^T_S N$ in **H-ask** +$\mathcal{R}$.

If $M \Rightarrow^T_S N$ is an initial higher-arity sequent, then its translation τ is an admissible initial sequent of the form $\alpha \Rightarrow \alpha$ in display calculi. If $M \Rightarrow^T_S N$ has been inferred by one of the classical or higher-arity rules of weakening, or by the classical rules of contraction, or by one of the propositional rules, then the procedure is straightforward. If $M \Rightarrow^T_S N$ has been inferred by the rule $C_n A$ (for the rule $C_n K$ the procedure is analogous but easier), then we have

$$\dfrac{M \Rightarrow^T_{\alpha,\alpha,S} N}{M \Rightarrow^T_{\alpha,S} N}\, C_n A \qquad \rightsquigarrow$$

$$\begin{array}{cl}
\Box\alpha \wedge \Box\alpha \wedge \bigwedge \Box S \circ M \Rightarrow \bigcirc \bullet T \circ N & \\ \hline
\Box\alpha \wedge \Box\alpha \wedge \bigwedge \Box S \Rightarrow \bigcirc \bullet T \circ N \circ M^* & \\ \hline
\Box\alpha \wedge \Box\alpha \circ \bigwedge \Box S \Rightarrow \bigcirc \bullet T \circ N \circ M^* & \underline{\wedge A'} \\ \hline
\Box\alpha \wedge \Box\alpha \Rightarrow \bigcirc \bullet T \circ N \circ M^* \circ (\bigwedge \Box S)^* & \\ \hline
\Box\alpha \circ \Box\alpha \Rightarrow \bigcirc \bullet T \circ N \circ M^* \circ (\bigwedge \Box S)^* & \underline{\wedge A'} \\ \hline
\Box\alpha \Rightarrow \bigcirc \bullet T \circ N \circ M^* \circ (\bigwedge \Box S)^* & C \\ \hline
\Box\alpha \circ \bigwedge \Box S \Rightarrow \bigcirc \bullet T \circ N \circ M^* & \\ \hline
\Box\alpha \wedge \bigwedge \Box S \Rightarrow \bigcirc \bullet T \circ N \circ M^* & \wedge A' \\ \hline
\Box\alpha \wedge \bigwedge \Box S \circ M \Rightarrow \bigcirc \bullet T \circ N &
\end{array}$$

Note that in both the second and the fourth inferences counting from the top, we are exploiting the fact that in display calculi propositional rules are invertible (see Section 3.3); more precisely we are using the invertibility of the rule $\wedge A'$, that we have indicated with the notation $\underline{\wedge A'}$.

If $M \Rightarrow^T_S N$ is of the form $\Box\alpha \Rightarrow^\alpha_\emptyset \emptyset$, i.e. it is the modal axiom $\Box A$, then we have

$$\dfrac{\alpha \Rightarrow \alpha}{\Box\alpha \Rightarrow \bullet\alpha}\, \Box A$$

If $M \Rightarrow^T_S N$ is of the form $\emptyset \Rightarrow^\emptyset_\alpha \Box\alpha$, i.e. it is the modal axiom $\Box K$, then its translation τ is an admissible axiom of the form $\Box\alpha \Rightarrow \Box\alpha$ in display calculi. If $M \Rightarrow^T_S N$ is of the form $\emptyset \Rightarrow^\alpha_M \emptyset$, and has been derived by the necessitation rule

$$\dfrac{M \Rightarrow^\emptyset_\emptyset \alpha}{\emptyset \Rightarrow^\alpha_M \emptyset}\, rn$$

then we have a quite long derivation to deal with, and we develop it in the following several steps. Suppose that $M \equiv \gamma_1, ..., \gamma_n$. If we apply the translation τ on the premise of the necessitation rule, we obtain $\gamma_1 \circ ... \circ \gamma_n \Rightarrow \alpha$, and then we can derive

$$\begin{array}{cl}
\gamma_1 \circ ... \circ \gamma_n \Rightarrow \alpha & \\ \hline
\gamma_1 \wedge ... \wedge \gamma_n \Rightarrow \alpha & \\ \hline
\Box(\gamma_1 \wedge ... \wedge \gamma_n) \Rightarrow \bullet\alpha & \Box A
\end{array}$$

Let us then observe these n analogous proofs

$$\dfrac{\dfrac{\dfrac{\dfrac{\gamma_1 \Rightarrow \gamma_1}{\Box\gamma_1 \Rightarrow \bullet\gamma_1}\,\Box A}{\Box\gamma_1 \circ \ldots \circ \Box\gamma_n \Rightarrow \bullet\gamma_1}}{\Box\gamma_1 \wedge \ldots \wedge \Box\gamma_n \Rightarrow \bullet\gamma_1}}{\bullet(\Box\gamma_1 \wedge \ldots \wedge \Box\gamma_n) \Rightarrow \gamma_1} \quad \cdots \quad \dfrac{\dfrac{\dfrac{\dfrac{\gamma_n \Rightarrow \gamma_n}{\Box\gamma_n \Rightarrow \bullet\gamma_n}\,\Box A}{\Box\gamma_1 \circ \ldots \circ \Box\gamma_n \Rightarrow \bullet\gamma_n}}{\Box\gamma_1 \wedge \ldots \wedge \Box\gamma_n \Rightarrow \bullet\gamma_n}}{\bullet(\Box\gamma_1 \wedge \ldots \wedge \Box\gamma_n) \Rightarrow \gamma_n}$$

By applying the rule $\wedge K'$ on the n premises $\bullet(\Box\gamma_1 \wedge \ldots \wedge \Box\gamma_n) \Rightarrow \gamma_1 \ldots \bullet(\Box\gamma_1 \wedge \ldots \wedge \Box\gamma_n) \Rightarrow \gamma_n$, n-times, we obtain $\bullet(\Box\gamma_1 \wedge \ldots \wedge \Box\gamma_n) \circ \ldots \circ \bullet(\Box\gamma_1 \wedge \ldots \wedge \Box\gamma_n) \Rightarrow \gamma_1 \wedge \ldots \wedge \gamma_n$. Then we can proceed in the following way:

$$\dfrac{\dfrac{\bullet(\Box\gamma_1 \wedge \ldots \wedge \Box\gamma_n) \circ \ldots \circ \bullet(\Box\gamma_1 \wedge \ldots \wedge \Box\gamma_n) \Rightarrow \gamma_1 \wedge \ldots \wedge \gamma_n}{\bullet(\Box\gamma_1 \wedge \ldots \wedge \Box\gamma_n) \Rightarrow \gamma_1 \wedge \ldots \wedge \gamma_n}}{\Box\gamma_1 \wedge \ldots \wedge \Box\gamma_n \Rightarrow \Box(\gamma_1 \wedge \ldots \wedge \gamma_n)}\,\Box K$$

Note that what we have just proven is that the sequent $\bigwedge(\Box M) \Rightarrow \Box(\bigwedge M)$ is derivable in display calculi. We can now apply a cut and reach our conclusion

$$\dfrac{\Box\gamma_1 \wedge \ldots \wedge \Box\gamma_n \Rightarrow \Box(\gamma_1 \wedge \ldots \wedge \gamma_n) \qquad \Box(\gamma_1 \wedge \ldots \wedge \gamma_n) \Rightarrow \bullet\alpha}{\Box\gamma_1 \wedge \ldots \wedge \Box\gamma_n \Rightarrow \bullet\alpha}\,cut_{\Box(\gamma_1,\ldots,\gamma_n)}$$

If $M \Rightarrow^T_S N$ has been inferred by the rule cut^2_α (for the rule cut^1_α the procedure is analogous but easier), then we have

$$\dfrac{M \Rightarrow^T_{\alpha,S} N \quad M \Rightarrow^{T,\alpha}_S N}{M \Rightarrow^T_S N}\,cut^2_\alpha \quad \rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Box\alpha \wedge \bigwedge\Box S \circ M \Rightarrow \bigcirc\bullet T \circ N}{\Box\alpha \circ \bigwedge\Box S \circ M \Rightarrow \bigcirc\bullet T \circ N}\,\wedge A'}{(\bigcirc\bullet T \circ N)^* \circ \bigwedge\Box S \circ M \Rightarrow \Box\alpha^*}}{(\bigcirc\bullet T \circ N)^* \circ \bigwedge\Box S \circ M \Rightarrow \neg\Box\alpha} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\bigwedge\Box S \circ M \Rightarrow \bullet\alpha \circ \bigcirc\bullet T \circ N}{(\bigcirc\bullet T \circ N)^* \circ \bigwedge\Box S \circ M \Rightarrow \bullet\alpha}}{\bullet((\bigcirc\bullet T \circ N)^* \circ \bigwedge\Box S \circ M) \Rightarrow \alpha}}{(\bigcirc\bullet T \circ N)^* \circ \bigwedge\Box S \circ M \Rightarrow \Box\alpha}\,\Box K}{\bigwedge\Box S \circ M \Rightarrow \bigcirc\bullet T \circ N \circ \Box\alpha}}{\Box\alpha^* \Rightarrow \bigcirc\bullet T \circ N \circ (\bigwedge\Box S \circ M)^*}}{\neg\Box\alpha \Rightarrow \bigcirc\bullet T \circ N \circ (\bigwedge\Box S \circ M)^*}}{(\bigcirc\bullet T \circ N)^* \circ \bigwedge\Box S \circ M \Rightarrow \bigcirc\bullet T \circ N \circ (\bigwedge\Box S \circ M)^*}\,cut_{\neg\Box\alpha}}{\bigwedge\Box S \circ M \circ \bigwedge\Box S \circ M \Rightarrow \bigcirc\bullet T \circ N \circ \bigcirc\bullet T \circ N}}{\bigwedge\Box S \circ M \Rightarrow \bigcirc\bullet T \circ N}$$

We finally analyse the cases in which $M \Rightarrow^T_S N$ has been inferred by one of the special structural rules d, t.

$$\dfrac{M \Rightarrow^{\emptyset}_{\emptyset} \emptyset}{\emptyset \Rightarrow^{\emptyset}_{M} \emptyset}\, d \quad \rightsquigarrow \quad \dfrac{\bigwedge(\Box M) \Rightarrow \Box(\bigwedge M) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{M \Rightarrow I^*}{\bigwedge M \Rightarrow I^*}}{\Box(\bigwedge M) \Rightarrow \bullet(I^*)}}{\bullet\Box(\bigwedge M) \Rightarrow I^*}}{\bullet\Box(\bigwedge M) \circ \bullet I^* \Rightarrow I^*}}{\Box(\bigwedge M) \Rightarrow I^*}\, d}{\bigwedge(\Box M) \Rightarrow I^*}\, cut_{\Box(\bigwedge M)}$$

Note that we could use the sequent $\bigwedge(\Box M) \Rightarrow \Box(\bigwedge M)$, since we have already shown that it is derivable in display sequent calculi.

$$\emptyset \Rightarrow^{\emptyset}_{\alpha} \alpha \quad t \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\alpha \Rightarrow \alpha}{\Box\alpha \Rightarrow \bullet\alpha}}{\Box\alpha \Rightarrow \alpha}\, t$$

□

To conclude, there exists a result of Wansing [149], obtained by embedding multiple sequents in higher-arity sequents. It is important to emphasise that: (i) this result only holds for those multiple sequent calculi that do not contain the b rule; (ii) Wansing only shows that a sequent is provable in **Msk** + $\mathcal{R}$, where $\mathcal{R} \subseteq \{d, t, 4\}$, if, and only if, its translation in higher-arity terms is valid in the correspondent Hilbert system.

Problem 5.3 Is there a way to prove that, given a multiple sequent $M \; \Box\!\!\underset{n}{\Rightarrow} N$, every derivation of $M \; \Box\!\!\underset{n}{\Rightarrow} N$ in **Msk*** can be translated into a derivation of that sequent appropriately translated in higher-arity terms in the calculi **H-ask***?

5.3 From Indexed Sequent Calculi to Internalised Forcing Sequent Calculi

In the last two sections we compared different syntactic methods; the following two sections will be concerned with the relationships between semantic methods.

Let us start by showing how to embed indexed sequent calculi into internalised forcing sequent calculi. It is important to note that both these calculi have a logical variant (in the case of Negri's calculi, see Section 4.3). Following the notation introduced in the first chapter, we will indicate these logical variants with $\mathbf{Isk}^*_L$ and $\mathbf{Ifsk}_L + \mathcal{R}$, where $\mathcal{R} \subseteq \{t, 4, b\}$, respectively.

In the calculi $\mathbf{Isk}^*_L$, see Fig. 5.1 of the next page, instead of presenting the rule $\Box A$ in the form

$$\dfrac{G; \sigma\ \Box\alpha, M \Rightarrow N; \sigma'\ \alpha, S \Rightarrow T}{G; \sigma\ \Box\alpha, M \Rightarrow N; \sigma'\ S \Rightarrow T}\, \Box A$$

Initial Indexed Sequents

$$G; \sigma\ \alpha, M \Rightarrow N, \alpha$$

Logical Rules

Propositional Rules

$$\frac{G; \sigma\ M \Rightarrow N, \alpha}{G; \sigma\ \neg\alpha, M \Rightarrow N}\ \neg A \qquad \frac{G; \sigma\ \alpha, M \Rightarrow N}{G; \sigma\ M \Rightarrow N, \neg\alpha}\ \neg K$$

$$\frac{G; \sigma\ \alpha, \beta, M \Rightarrow N}{G; \sigma\ \alpha \wedge \beta, M \Rightarrow N}\ \wedge A' \qquad \frac{G; \sigma\ M \Rightarrow N, \alpha \quad G; \sigma\ M \Rightarrow N, \beta}{G; \sigma\ M \Rightarrow N, \alpha \wedge \beta}\ \wedge K$$

Modal Rules

$$\frac{G; \sigma\ \Box\alpha, M \Rightarrow N; \sigma \star i\ \alpha, S \Rightarrow T}{G; \sigma\ \Box\alpha, M \Rightarrow N; \sigma \star i\ S \Rightarrow T}\ \Box A \qquad \frac{G; \sigma\ M \Rightarrow N; \sigma \star i\ \Rightarrow \alpha}{G; \sigma\ M \Rightarrow N, \Box\alpha}\ \Box K$$

where in the rule $\Box K$ the index $\sigma \star i$ is new, reading the rule bottom-up

Special Logical Rules

$$\frac{G; \sigma\ \Box\alpha, \alpha, M \Rightarrow N}{G; \sigma\ \Box\alpha, M \Rightarrow N}\ t \qquad \frac{G; \sigma\ \alpha, M \Rightarrow N; \sigma \star i\ \Box\alpha, S \Rightarrow T}{G; \sigma\ M \Rightarrow N; \sigma \star i\ \Box\alpha, S \Rightarrow T}\ b$$

$$\frac{G; \sigma\ \Box\alpha, M \Rightarrow N; \sigma \star i\ \Box\alpha, S \Rightarrow T}{G; \sigma\ \Box\alpha, M \Rightarrow N; \sigma \star i\ S \Rightarrow T}\ 4_1 \qquad \frac{G; \sigma\ \Box\alpha, M \Rightarrow N; \sigma \star i\ \Box\alpha, S \Rightarrow T}{G; \sigma\ M \Rightarrow N; \sigma \star i\ \Box\alpha, S \Rightarrow T}\ 4_2$$

Fig. 5.1 Logical variant of the indexed sequent calculi

with the side condition $\sigma R \sigma'$ (as it is done in Section 4.2), and obtaining the rules t and b by simply requiring the relation R to be reflexive and symmetric, respectively, we have preferred to change the notation, and specify the three rules $\Box A$, t and b.[2] An analogous operation has been applied on the rule 4: if we want to obtain calculi for the systems **K4** or **S4**, we use the rule 4_1. If we want to obtain calculi for the systems **K4B** or **S5**, we use both the rules 4_1 and 4_2. These changes will prove useful in what follows.

We will show that the calculi $\mathbf{Isk}_L^*$ can be embedded into the calculi $\mathbf{Ifsk}_L + \mathcal{R}$ (which are shown in Fig. 5.2). The translation will proceed in two steps: firstly, the multisets of indexed sequents used by Mints, i.e. the tableaux, will be translated into multisets of internalised forcing sequents; then, these multisets will be reduced to single internalised forcing sequents. Let G be a tableaux and $\sigma M \Rightarrow N$ be a sequent that occurs in G. To complete the first stage, we define a function δ. We begin by defining δ on single indexed sequents $\sigma M \Rightarrow N$.

$(\sigma M \Rightarrow N)^\delta :=$

if $\sigma = \emptyset$ then the translation is simply $(M)^i \Rightarrow (N)^i$, where the notation $(M)^i$ ($(N)^i$) stands for: each formula α of M (or of N) is labelled by a variable i ($i : \alpha$).

Initial Internalised Forcing Sequents

$$i : p, M \Rightarrow N, i : p \qquad\qquad iRj, M \Rightarrow N, iRj$$

Logical Rules

Propositional Rules

$$\frac{M \Rightarrow N, i : \alpha}{i : \neg\alpha, M \Rightarrow N}\ \neg A \qquad\qquad \frac{i : \alpha, M \Rightarrow N}{M \Rightarrow N, i : \neg\alpha}\ \neg K$$

$$\frac{i : \alpha, i : \beta, M \Rightarrow N}{i : \alpha \wedge \beta, M \Rightarrow N}\ \wedge A' \qquad\qquad \frac{M \Rightarrow N,\ i : \alpha \qquad M \Rightarrow N, i : \beta}{M \Rightarrow N, i : \alpha \wedge \beta}\ \wedge K$$

Modal Rules

$$\frac{i : \Box\alpha, j{:}\alpha, iRj, M \Rightarrow N}{i : \Box\alpha, iRj, M \Rightarrow N}\ \Box A \qquad\qquad \frac{iRj, M \Rightarrow N, j : \alpha}{M \Rightarrow N, i : \Box\alpha}\ \Box K$$

where in the rule $\Box K$ the variable j is new, reading the rule bottom-up.

Special Logical Rules

$$\frac{iRi, M \Rightarrow N}{M \Rightarrow N}\ t \qquad\qquad \frac{jRi, iRj, M \Rightarrow N}{iRj, M \Rightarrow N}\ b$$

$$\frac{iRj, jRz, iRzM \Rightarrow N}{iRj, jRz, M \Rightarrow N}\ 4$$

Fig. 5.2 Logical variant of the internalised forcing sequent calculi

if $\sigma = \sigma' \star z$ and all the formulas of the sequent indexed by σ' have already been labelled by the variable i, then the translation is iRj, $(M)^j \Rightarrow (N)^j$, where each formula α of M and of N is labelled by the variable j $(j : \alpha)$ and the variable j has not been already used in the translation δ.

Then, we extend the definition of δ to tableaux as follows:

$$if\ G = M \Rightarrow N; z_1 \star S_1 \Rightarrow T_1; ...; z_n \star S_n \Rightarrow T_n$$

then $(G)^\delta$ is

$$(M)^i \Rightarrow (N)^i;\ iRj_1,\ (S_1)^{j_1} \Rightarrow (T_1)^{j_1};\ \ ...;\ iRj_n,\ (S_n)^{j_n} \Rightarrow (T_n)^{j_n}$$

The translation τ from Mints's tableaux into Negri's internalised forcing sequents can now be defined, using δ, as follows:

$$(G)^\tau := (G^\delta)^A \Rightarrow (G^\delta)^C$$

where

$(G^\delta)^A$ is the multiset composed by those elements (of the form $i : \alpha$, or iRj) which are in *antecedent* position in each sequent that comes out from the translation δ of the tableaux G.
$(G^\delta)^C$ is the multiset composed by those elements (of the form $i : \alpha$) which are in *consequent* position in each sequent that comes out from the translation δ of the tableaux G.

Therefore the translation τ of the tableaux G,

$$G = M \Rightarrow N; z_1 \star S_1 \Rightarrow T_1; ...; z_n \star S_n \Rightarrow T_n$$

is

$$\underbrace{iRj_1, ..., iRj_n, (M)^i, (S_1)^{j_1}, ..., (S_n)^{j_n}}_{(G^\delta)^A} \Rightarrow \underbrace{(N)^i, (T_1)^{j_1}, ..., (T_n)^{j_n}}_{(G^\delta)^C}$$

We are now ready to prove the following theorem.

Theorem 5.4 *Let G be any tableaux of the calculi* $\textbf{Isk}^*_L$*. Then every derivation of G in* $\textbf{Isk}^*_L$ *can be translated into a derivation of* $(G)^\tau$ *in* $\textbf{Ifsk}_L + \mathcal{R}$*, where* $\mathcal{R} \subseteq \{t, 4, b\}$.

Proof The proof is by induction on the derivation of the tableaux G in $\textbf{Isk}^*_L$.

If G is an initial sequent, then its translation τ is an initial internalised forcing sequent in $\textbf{Ifsk}_\textbf{L} + \mathcal{R}$. If G has been inferred by one of the propositional rules, then the procedure is straightforward. If G has been inferred by the modal rule $\Box A$, then we have

$$\frac{G; \sigma\ \Box\alpha, M \Rightarrow N; \sigma \star i\ \alpha, S \Rightarrow T}{G; \sigma\ \Box\alpha, M \Rightarrow N; \sigma \star i\ S \Rightarrow T}\Box A \quad \rightsquigarrow$$

$$\frac{(G^\delta)^A, iRj, i : \Box\alpha, j : \alpha, (M)^i, (S)^j \Rightarrow (N)^i, (T)^j, (G^\delta)^C}{(G^\delta)^A, iRj, i : \Box\alpha, (M)^i, (S)^j \Rightarrow (N)^i, (T)^j, (G^\delta)^C}\Box A$$

If G has been inferred by the modal rule $\Box K$, then we have

$$\frac{G; \sigma M \Rightarrow N; \sigma \star i \Rightarrow \alpha}{G; \sigma M \Rightarrow N, \Box\alpha}\Box K \quad \rightsquigarrow$$

$$\frac{(G^\delta)^A, iRj, (M)^i \Rightarrow (N)^i, j : \alpha, (G^\delta)^C}{(G^\delta)^A, (M)^i \Rightarrow (N)^i, i : \Box\alpha, (G^\delta)^C}\Box K$$

If G has been inferred by the rule t, then we have

$$\dfrac{G; \sigma\Box\alpha, \alpha, M \Rightarrow N}{G; \sigma\Box\alpha, M \Rightarrow N}\,t \quad \rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{(G^{\delta})^{A}, i : \Box\alpha, i : \alpha, (M)^{i} \Rightarrow (N)^{i}, (G^{\delta})^{C}}{(G^{\delta})^{A}, iRi, i : \Box\alpha, i : \alpha, (M)^{i} \Rightarrow (N)^{i}, (G^{\delta})^{C}}\,W_2A}{(G^{\delta})^{A}, iRi, i : \Box\alpha, (M)^{i} \Rightarrow (N)^{i}, (G^{\delta})^{C}}\,\Box A}{(G^{\delta})^{A}, i : \Box\alpha, (M)^{i} \Rightarrow (N)^{i}, (G^{\delta})^{C}}\,t$$

If G has been inferred by the rule 4_1, then we have

$$\dfrac{G; \sigma\,\Box\alpha, M \Rightarrow N ; \sigma \star i\,\Box\alpha, S \Rightarrow T}{G; \sigma\,\Box\alpha, M \Rightarrow N ; \sigma \star i\,S \Rightarrow T}\,4_1 \quad \rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{\Rightarrow i : \Box\alpha \rightarrow \Box\Box\alpha}{i : \Box\alpha \Rightarrow i : \Box\Box\alpha}\,\underline{\rightarrow K'} \qquad \dfrac{\dfrac{(G^{\delta})^{A}, iRj, j : \Box\alpha, i : \Box\alpha, (M)^{i}, (S)^{j} \Rightarrow (N)^{i}, (T)^{j}, (G^{\delta})^{C}}{(G^{\delta})^{A}, iRj, i : \Box\Box\alpha, j : \Box\alpha, i : \Box\alpha, (M)^{i}, (S)^{j} \Rightarrow (N)^{i}, (T)^{j}, (G^{\delta})^{C}}\,WA}{(G^{\delta})^{A}, iRj, i : \Box\Box\alpha, i : \Box\alpha, (M)^{i}, (S)^{j} \Rightarrow (N)^{i}, (T)^{j}, (G^{\delta})^{C}}\,\Box A}{(G^{\delta})^{A}, iRj, i : \Box\alpha, i : \Box\alpha, (M)^{i}, (S)^{j} \Rightarrow (N)^{i}, (T)^{j}, (G^{\delta})^{C}}\,cut\ \Box\Box\alpha}{(G^{\delta})^{A}, iRj, i : \Box\alpha, (M)^{i}, (S)^{j} \Rightarrow (N)^{i}, (T)^{j}, (G^{\delta})^{C}}\,CA$$

Note that in the first inference of the left-side derivation of the second deduction, we have used the height-preserving invertibility of the rule $\rightarrow K'$ (see Section 4.3), indicated with the notation $\underline{\rightarrow K'}$.

If G has been inferred by the rule 4_2, then we have

$$\dfrac{G; \sigma\,\Box\alpha, M \Rightarrow N ; \sigma \star i\,\Box\alpha, S \Rightarrow T}{G; \sigma\,M \Rightarrow N ; \sigma \star i\,\Box\alpha, S \Rightarrow T}\,4_2 \quad \rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{\Rightarrow j : \Box\alpha \rightarrow \Box\Box\alpha}{j : \Box\alpha \Rightarrow j : \Box\Box\alpha}\,\underline{\rightarrow K'} \qquad \dfrac{\dfrac{\dfrac{\dfrac{(G^{\delta})^{A}, iRj, j : \Box\alpha, i : \Box\alpha, (M)^{i}, (S)^{j} \Rightarrow (N)^{i}, (T)^{j}, (G^{\delta})^{C}}{(G^{\delta})^{A}, iRj, jRi, j : \Box\alpha, i : \Box\alpha, (M)^{i}, (S)^{j} \Rightarrow (N)^{i}, (T)^{j}, (G^{\delta})^{C}}\,W_2A}{(G^{\delta})^{A}, iRj, jRi, j : \Box\Box\alpha, j : \Box\alpha, i : \Box\alpha, (M)^{i}, (S)^{j} \Rightarrow (N)^{i}, (T)^{j}, (G^{\delta})^{C}}\,WA}{(G^{\delta})^{A}, iRj, jRi, j : \Box\alpha, j : \Box\Box\alpha, (M)^{i}, (S)^{j} \Rightarrow (N)^{i}, (T)^{j}, (G^{\delta})^{C}}\,\Box A}{(G^{\delta})^{A}, iRj, j : \Box\alpha, j : \Box\Box\alpha, (M)^{i}, (S)^{j} \Rightarrow (N)^{i}, (T)^{j}, (G^{\delta})^{C}}\,b}{(G^{\delta})^{A}, iRj, j : \Box\alpha, j : \Box\alpha, (M)^{i}, (S)^{j} \Rightarrow (N)^{i}, (T)^{j}, (G^{\delta})^{C}}\,cut\ \Box\Box\alpha}{(G^{\delta})^{A}, iRj, j : \Box\alpha, (M)^{i}, (S)^{j} \Rightarrow (N)^{i}, (T)^{j}, (G^{\delta})^{C}}\,CA$$

Note that we could use the rule b, since we are treating the case of the rule 4_2. Moreover, even in this case we have exploited the invertibility of the rule $\rightarrow K'$. Finally, we underline that the rule $4'$ too (see Remark 4.18, p. 91) can be translated (without the use of the cut-rule) in internalised forcing terms. On the other hand, the proof of this translation is quite long and tedious.

If G has been inferred by the rule b, then we have

$$\dfrac{G; \sigma\alpha, M \Rightarrow N; \sigma \star i\ \Box\alpha, S \Rightarrow T}{G; \sigma M \Rightarrow N; \sigma \star i\ \Box\alpha, S \Rightarrow T}\, b \quad \rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{(G^{\delta})^{A}, iRj, j : \Box\alpha, i : \alpha, (M)^{i}, (S)^{j} \Rightarrow (N)^{i}, (T)^{j}, (G^{\delta})^{C}}{(G^{\delta})^{A}, iRj, jRi, j : \Box\alpha, i : \alpha, (M)^{i}, (S)^{j} \Rightarrow (N)^{i}, (T)^{j}, (G^{\delta})^{C}}\, W_2A}{(G^{\delta})^{A}, iRj, jRi, j : \Box\alpha, (M)^{i}, (S)^{j} \Rightarrow (N)^{i}, (T)^{j}, (G^{\delta})^{C}}\, \Box A}{(G^{\delta})^{A}, iRj, j : \Box\alpha, (M)^{i}, (S)^{j} \Rightarrow (N)^{i}, (T)^{j}, (G^{\delta})^{C}}\, b$$

□

5.4 From Indexed Sequent Calculi to Semantic Modal Sequent Calculi and Vice Versa

Following what was announced previously, this section will deal with another relationship between semantic methods. More precisely, we will show how to embed the indexed sequent calculi **Isk*** in the semantic modal sequent calculi **Ssk***, *and vice versa*. These operations reveal a straightforward link between the two different semantic methods, as it was pointed out in Section 4.2.

Let us start by defining the translation τ from tableaux to trees of sequents in the following way. Let G be the tableau

$$M \Rightarrow N;\ i_1 \star H_1;\ \ldots;\ i_n \star H_n$$

which under the translation τ becomes

$$\begin{array}{ccc} & M \Rightarrow N & \\ & \swarrow \ \ldots . \ \searrow & \\ H_1 & \ldots . & H_n \end{array}$$

In reverse, let us define the translation ϕ from the trees of sequents to the tableaux in the following way. Let G be the tree of sequents

$$\begin{array}{ccc} & M \Rightarrow N & \\ & \swarrow \ \ldots . \ \searrow & \\ H_1 & \ldots . & H_n \end{array}$$

which under the translation ϕ becomes

$$M \Rightarrow N; i_1 \star H_1; \ldots; i_n \star H_n$$

Moreover, both authors assume a relation R (that does not affect the two translations τ and ϕ, and may enjoy, depending on the system that one wants to treat, several properties) to hold between the indexed sequents and the semantic sequents. This fact increases the similarity between the two methods.

In what follows the indexed sequent calculi for the modal systems containing the axiom 4 are obtained by means of the rule $4^{'}$ (see Remark 4.18, p. 91). and not by the usual rule 4. We will indicate this fact by writing $\mathbf{Isk}^{\circ}$ instead of $\mathbf{Isk}^{*}$.

Theorem 5.5 *Let G be any tableaux of the calculi* $\mathbf{Isk}^{\circ}$*. Then every derivation of G in* $\mathbf{Isk}^{\circ}$ *can be translated into a derivation of* $(G)^{\tau}$ *in* $\mathbf{Ssk}^{*}$*.*

Proof The proof is by straightforward induction on the derivation of the tableaux G in $\mathbf{Isk}^{\circ}$. □

Theorem 5.6 *Let* $\mathcal{R} \subseteq \{t, b, 4\}$ *and G be any tree of sequents of the calculi* $\mathbf{Ssk} + \mathcal{R}$*. Then every derivation of G in* $\mathbf{Ssk} + \mathcal{R}$ *can be translated into a derivation of* $(G)^{\phi}$ *in* $\mathbf{Isk}^{\circ}$*.*

Proof The proof is by straightforward induction on the derivation of the tree of sequents G in $\mathbf{Ssk} + \mathcal{R}$. □

5.5 From Display Sequent Calculi to Internalised Forcing Sequent Calculi

Let us finally turn to the relationship between a syntactic method, display logic, and a semantic method, namely, the one proposed by Negri.

Let us first of all consider the reasons why we chose display calculi and internalised forcing sequent calculi as the best candidates for building a bridge between syntactic and semantic methods. Even the first glance reveals that display calculi and internalised forcing calculi have several common points. They have similar disadvantages (see Remarks 3.25, p. 73, and 4.32, p. 99), but they also share a great expressive power (witnessed by at least two facts: (i) the other generalisations of the Gentzen calculus can be plugged into them, and (ii) they can be applied to a wide range of modal systems). The more interesting question is then whether the commonalities between display calculi and internalised forcing sequents are explicable by a deeper formal reason. The affirmative answer can be spelled out as follows.

In other generalisations of Gentzen calculi, the new proof tools – metalinguistic or linguistic, syntactic or semantic – are applied to the entire sequent or to the entire context. This is not the case for display calculi and internatised forcing sequents. Rather, for these types of calculi, the new proof tools are applied to single formulas. Our formal translation will rest on this common characteristic.

Given a display sequent $M \Rightarrow N$, its translation is the internalised forcing sequent

$$n_i^r(M), p_i^r(N) \Rightarrow p_i^l(N), n_i^l(M)$$

where n_i^r, n_i^l, p_i^r, p_i^l are translation functions that we shall define recursively.

Let us first provide an informal interpretation: the letters n for negative, and p for positive, indicate the position of the display structure M in the display sequent. The

	I	α	M^*	$M \circ N$	$\bullet M$
n_i^l	–	$i : \alpha$	$p_i^l(M)$	$n_i^l(M),\ n_i^l(N)$	$jRi\ ,\ n_j^l(M)$
n_i^r	–	–	$p_i^r(M)$	$n_i^r(M),\ n_i^r(N)$	$n_j^r(M)$
p_i^l	–	–	$n_i^l(M)$	$p_i^l(M),\ p_i^l(N)$	$iRj, p_j^l(M)$
p_i^r	–	$i : \alpha$	$n_i^r(M)$	$p_i^r(M),\ p_i^r(N)$	$p_j^r(M)$

Fig. 5.3 Translation functions

letters r for right, and l for left, indicate the position of the translated display structure M in the internalised forcing sequent. Finally, the letter i stands for the Kripke semantics world at which the translated display structure M is true. Functions are defined as in Fig. 5.3.

For the sake of clarity, let us make an example. Let us consider the display sequent $\bullet(\alpha^* \circ \beta) \circ \delta \Rightarrow \gamma^*$. It is translated as follows:

$$n_i^l(\bullet(\alpha^* \circ \beta)), n_i^l(\delta), p_i^l(\gamma^*) \Rightarrow n_i^r(\bullet(\alpha^* \circ \beta)), n_i^r(\delta), p_i^r(\gamma^*)$$

that becomes

$$jRi, j : \beta, i : \gamma, i : \delta \Rightarrow j : \alpha$$

This translation is based on a result by Restall [116], that we have managed to improve. We are now in a position to prove the following theorem.

Theorem 5.7 *Let $M \Rightarrow N$ be any display sequent of the calculi* **Dsk***. *Then every derivation of $M \Rightarrow N$ in* **Dsk*** *can be translated into a derivation of $n_i^r(M), p_i^r(N) \Rightarrow p_i^l(N), n_i^l(M)$ in* **Ifsk***.

Proof The proof is by induction on the derivation of the sequent $M \Rightarrow N$ in the calculi **Dsk***.

If $M \Rightarrow N$ is an initial sequent, then its translation τ is an initial internalised forcing sequent in $\mathbf{Ifsk_L} + \mathcal{R}$. If $M \Rightarrow N$ has been inferred by one of the structural rules, or one of the basic structural rules or one of the propositional rules, then the procedure is straightforward. If $M \Rightarrow N$ has been inferred by the modal rule $\Box A$, then we have

$$\dfrac{\alpha \Rightarrow N}{\Box\alpha \Rightarrow \bullet N}\,\Box A \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{i : \alpha, p_i^l(N) \Rightarrow p_i^r(N)}{j : \alpha, p_j^l(N) \Rightarrow p_j^r(N)}\,SL}{iRj, j : \alpha, p_j^l(N) \Rightarrow p_j^r(N)}\,W_2A}{iRj, i : \Box\alpha, p_j^l(N) \Rightarrow p_j^r(N)}\,\Box A$$

Note that in the first inference of the right-side derivation, we have used the substitution lemma that from now on we are going to indicate with SL.

If $M \Rightarrow N$ has been inferred by the modal rule $\Box K$, then we have

$$\dfrac{\bullet M \Rightarrow \alpha}{M \Rightarrow \Box\alpha}\,\Box K \quad\rightsquigarrow\quad \dfrac{\dfrac{jRi, n^l_j(M) \Rightarrow n^r_j(M), i : \alpha}{n^l_j(M) \Rightarrow n^r_j(M), j : \Box\alpha}\,\Box K}{n^l_i(M) \Rightarrow n^r_i(M), i : \Box\alpha}\,SL$$

If $M \Rightarrow N$ has been inferred by the rule d, then we have

$$\dfrac{\bullet M \circ \bullet N \Rightarrow I^*}{M \Rightarrow N^*}\,d \quad\rightsquigarrow\quad \dfrac{\dfrac{\dfrac{jRi,\, jRi, n^l_j(M),\, p^l_j(N) \Rightarrow n^r_j(M),\, p^r_j(N)}{jRi, n^l_j(M),\, p^l_j(N) \Rightarrow n^r_j(M),\, p^r_j(N)}\,CA}{n^l_j(M),\, p^l_j(N) \Rightarrow n^r_j(M),\, p^r_j(N)}\,d}{n^l_i(M),\, p^l_i(N) \Rightarrow n^r_i(M),\, p^r_i(N)}\,SL$$

If $M \Rightarrow N$ has been inferred by the rule t, then we have

$$\dfrac{M \Rightarrow \bullet N}{M \Rightarrow N}\,t \quad\rightsquigarrow\quad \dfrac{\dfrac{iRj, n^l_i(M),\, p^l_j(N) \Rightarrow n^r_i(M),\, p^r_j(N)}{iRi, n^l_i(M),\, p^l_i(N) \Rightarrow n^r_i(M),\, p^r_i(N)}\,SL}{n^l_i(M),\, p^l_i(N) \Rightarrow n^r_i(M),\, p^r_i(N)}\,t$$

If G has been inferred by the rule 4, then we have

$$\dfrac{M \Rightarrow \bullet N}{M \Rightarrow \bullet\bullet N}\,4 \quad\rightsquigarrow\quad \dfrac{\dfrac{\dfrac{iRj, n^l_i(M),\, p^l_j(N) \Rightarrow n^r_i(M),\, p^r_j(N)}{iRz, n^l_i(M),\, p^l_z(N) \Rightarrow n^r_i(M),\, p^r_z(N)}\,SL}{iRz, iRj,\, jRz, n^l_i(M),\, p^l_z(N) \Rightarrow n^r_i(M),\, p^r_z(N)}\,W_2A^*}{iRj,\, jRz, n^l_i(M),\, p^l_z(N) \Rightarrow n^r_i(M),\, p^r_z(N)}\,4$$

If G has been inferred by the rule b, then we have

$$\dfrac{(\bullet(M^*))^* \Rightarrow N}{\bullet M \Rightarrow N}\,b \quad\rightsquigarrow\quad \dfrac{\dfrac{iRj, n^l_j(M),\, p^l_i(N) \Rightarrow n^r_j(M),\, p^r_i(N)}{iRj,\, jRi, n^l_j(M),\, p^l_i(N) \Rightarrow n^r_j(M),\, p^r_i(N)}\,W_2A}{jRi, n^l_j(M),\, p^l_i(N) \Rightarrow n^r_j(M),\, p^r_i(N)}\,b$$

If G has been inferred by the rule 5, then we have

$$\dfrac{(\bullet(M^*))^* \Rightarrow N}{\bullet((\bullet(M^*))^*) \Rightarrow N}\,5 \quad\rightsquigarrow\quad \dfrac{\dfrac{\dfrac{iRj, n^l_j(M),\, p^l_i(N) \Rightarrow n^r_j(M),\, p^r_i(N)}{iRz, n^l_z(M),\, p^l_i(N) \Rightarrow n^r_z(M),\, p^r_i(N)}\,SL}{jRz,\, jRi, iRz, n^l_z(M),\, p^l_i(N) \Rightarrow n^r_z(M),\, p^r_i(N)}\,W_2A^*}{jRz,\, jRi, iRz, n^l_z(M),\, p^l_i(N) \Rightarrow n^r_z(M),\, p^r_i(N)}\,5$$

□

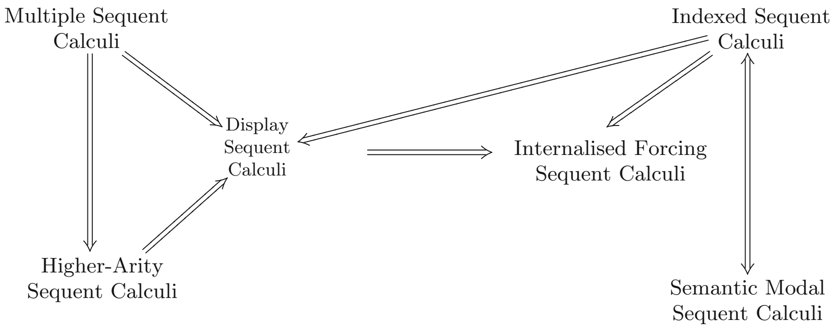


Fig. 5.4 Relationships between sequent calculi for modal logic

We have thus finished presenting the relationships that unite the several Gentzen systems for modal logic. They are summed up in Fig. 5.4: on the left side, are the syntactic methods, while on the right side are the semantic ones. There are two "bridges" connecting them: the first one is represented by the link between indexed sequent calculi and display calculi, and has been proved by Mints [82]; the second one is represented by the link between internalised forcing calculi and display calculi, which we have just proved.

Finally there are two other translations between sequent calculi for modal logic: the first one holds between the hypersequent calculus **Hss5** of Avron (see [6]) and the corresponding display calculus. This link has been established by Wansing [149]. The second one holds between display calculi and the calculi of structures (see the beginning of Chapter 3), and was established by Goré and Tiu [51].

Notes

1. Notice that, in order to shorten the derivations in the calculi **Dsk***, we may use several rules in a row, and indicate them with just one inference. For this reason and also not to overload the derivations, we will not specify the names of the rules used in the derivations, except for the most significant ones.
2. Mints himself suggests this alternative presentation.

Part III
Tree-Hypersequent Calculi

$\overset{\,\prime}{\alpha}\varrho\tau\iota\mu\alpha\nu\vartheta\acute{\alpha}\nu\omega$

[Euripide, Alcesti-Ciclope, Garzanti, 2006]

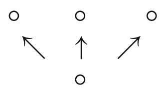
Chapter 6
On the Tree-Hypersequent Calculi

In the first part of the book we defined what it means for a sequent calculus to be good and we explained the reasons why it is important for a logic to have a good Gentzen calculus. In the second part of the book we set out the numerous attempts made at providing the main systems of modal logic with a proof calculus while showing their limits and their benefits. Our aim in this last part of the book is to present and deeply analyse a new method for generating good extensions of the sequent calculus for the SLH-systems plus **GL**.

This new method is called *tree-hypersequent* method, and before introducing it formally, let us briefly present the basic idea behind it. The basic idea of the tree-hypersequent method consists in reproducing, at the proof-theoretical level, the structure of the tree-frames of Kripke semantics. As we saw in Sections 4.1 and 4.2, this idea is not original. The novelty of this attempt lies in reflecting the structure of the tree-frames without the support of explicit semantic parameters; this represents a remarkable improvement at least on the conceptual level.

How do we internalise the structure of the tree-frames of Kripke semantics without the aid of explicit semantic parameters? We can use the following simple tree-frame of Kripke semantics to offer a good explanation:

(A)



Let us consider the worlds of this tree-frame: we have a root, i.e. a world at distance zero, and linked to this, three worlds at distance one. The worlds of a tree-frame will be represented, at the proof-theoretical level, by sequents, i.e. we will have

(B)

$$\Gamma_2 \qquad \Gamma_3 \qquad \Gamma_4$$

$$\Gamma_1$$

F. Poggiolesi, *Gentzen Calculi for Modal Propositional Logic*, Trends in Logic 32,
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where Γ_i, $1 \leqslant i \leqslant 4$ is a classical sequent. The sequent Γ_1 stands for the root, while the three sequents Γ_2, Γ_3, Γ_4 stand for the three worlds at distance one linked to the root.

Let us now come back to the figure (A), and restrict our attention on the three worlds at distance one. These three worlds, though all linked to the root and all at distance one, are separated. We want something in our meta-linguistic language that indicates this separation. We will use the semicolon to represent this, i.e. (B) becomes

(C)

$$\begin{array}{ccc} \Gamma_2; & \Gamma_3; & \Gamma_4 \\ & \Gamma_1 & \end{array}$$

The task is almost complete: the last element to internalise is the accessibility relation between worlds. A simple move would be the following:

(D)

$$\begin{array}{ccc} \Gamma_2; & \Gamma_3; & \Gamma_4 \\ \nwarrow & \uparrow & \nearrow \\ & \Gamma_1 & \end{array}$$

This move not only brings us back to semantic modal calculi (see Section 4.1), the disadvantages of which have already been shown, but, more importantly, it also happens to be redundant. Indeed, we can express the fact that the world-sequent Γ_1 is linked with the world-sequents Γ_2, Γ_3, Γ_4, simply by changing the order of the sequents, and by using the symbol "/" as follows:

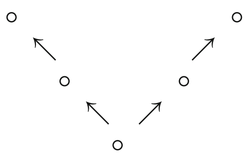
$$\Gamma_1 / \ \Gamma_2; \ \ \Gamma_3; \ \ \Gamma_4$$

This is an example of tree-hypersequent. We can intuitively interpret the object $\Gamma_1 / \ \Gamma_2; \ \Gamma_3; \ \Gamma_4$ as the world-sequent Γ_1 being linked to three other world-sequents Γ_2, Γ_3, Γ_4.

For the sake of clarity, let us give another couple of examples of tree-hypersequents. Below is the first one.

$$\Gamma_1/(\Gamma_2/\Gamma_3); \ (\Gamma_4/\Gamma_5)$$

which corresponds to the tree-frame

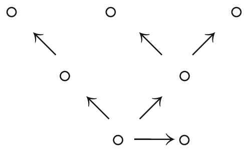


More precisely, the sequent Γ_1 corresponds to the root of the tree, the sequents Γ_2 and Γ_4 to the two worlds at distance one, each of which is linked to one of the two worlds at distance three, which are represented by the sequents Γ_3 and Γ_5, respectively.

Below is the second example. We consider the tree-hypersequent

$$\Gamma_1/\Gamma_2; (\Gamma_3/\Gamma_4); (\Gamma_5/\Gamma_6; \Gamma_7)$$

which intuitively corresponds to this tree-frame



More precisely the sequent Γ_1 corresponds to the root of the tree, the sequents Γ_2, Γ_3 and Γ_5 to the three worlds at distance one, and the sequents Γ_4, Γ_6, Γ_7 to the three worlds at distance three.

Thanks to these two examples, it should now be clear what a tree-hypersequents is, and what its interpretation is. However, the reason why tree-hypersequents are called thus remains to be elucidated. More specifically, while the term "tree" is evidently appropriated, we may still ask ourselves why they are also termed "hypersequents."

We introduced the term "hypersequent" at the beginning of Chapter 3: hypersequents are a syntactic generalisation of the Gentzen calculus, obtained by dealing with n sequents a time. More precisely, a hypersequent is an object of the form

$$\Gamma_1/\Gamma_2/.../\Gamma_n$$

which is to say n sequents separated by $n-1$ slashes. Note that the order of the sequents is irrelevant and that therefore a hypersequent is just a multiset of sequents. Hypersequents were first introduced by Pottinger [106] and then studied by Avron [3–6]. They are widely used in proof theory, e.g. [7, 25, 78].

Another way of looking at tree-hypersequents is by considering hypersequents. Tree-hypersequents are hypersequents where the order of the sequents is taken into account, and where the semicolon supplements the metalinguistic symbol slash.

The tree-hypersequent method was studied by Brünnler [15], Kashima [55, 66, 68], and Poggiolesi [100, 105]. While Kashima calls the tree-hypersequents, *nested-sequents*, Brünnler calls them *deep-sequents*. Kashima and Brünnler use the same notation, which is different from the one used here (first introduced in [100]). Moreover, Kashima and Brünnler only deal with semantic proofs (which will be presented in Chapter 8), while this book is mainly devoted to syntactic proofs, and builds on and improves the results presented in [60, 100, 102, 103, 105].

6.1 The Calculi Thsk*

Syntactic Notation

- ";" and "/" are two new meta-linguistic symbols.
- Γ, Δ, ... denote sequents (SEQ).
- G, H, ... denote tree-hypersequents (THS).
- $\underline{X}$, $\underline{Y}$, ... denote multisets of tree-hypersequents (MTHS).

For the sake of brevity, we will adopt the following convention.

Convention 6.1 *Given $\Gamma \equiv M \Rightarrow N$ and $\Pi \equiv P \Rightarrow Q$, instead of writing $\alpha, M \Rightarrow N, \beta$, we substitute $M \Rightarrow N$ with Γ, and we write α, Γ, β. Moreover, we write $\Gamma \centerdot \Pi$ instead of $M, P \Rightarrow N, Q$. Consequently, instead of writing $\alpha, M, P \Rightarrow N, Q, \beta$, we write $\alpha, \Gamma \centerdot \Pi, \beta$.*

Definition 6.1 The notion of *tree-hypersequent* is inductively defined in the following way:

- *if $\Gamma \in SEQ$, then $\Gamma \in THS$,*
- *if $\Gamma \in SEQ$ and $G_1, ..., G_n \in THS$, then $\Gamma / G_1; ...; G_n \in THS$.*

Definition 6.2 The *intended interpretation* τ of a tree-hypersequent is inductively defined in the following way:

- $(M \Rightarrow N)^\tau := \bigwedge M \rightarrow \bigvee N$
- $(\Gamma / G_1; ...; G_n)^\tau := \Gamma^\tau \vee \Box G_1^\tau \vee ... \vee \Box G_n^\tau$

In order to display the rules of the calculi, we will use the notation $G[*]$ defined as follows.

Definition 6.3 The notion of *zoom tree-hypersequent* (*ZTHS*) is inductively defined in the following way:

- $[*] \in ZTHS$,
- *if $G_1, ..., G_n \in THS$, then $[*]/G_1; ...; G_n \in ZTHS$,*
- *if $G_1[*] \in ZTHS$, $G_2, ..., G_n \in THS$, then $[*]/G_1[*]; ...; G_n \in ZTHS$,*
- *if $\Gamma \in SEQ$, $G_1[*] \in ZTHS$ and $G_2, ..., G_n \in THS$, then $\Gamma / G_1[*]; ...; G_n \in ZTHS$.*
- *if $\Gamma \in SEQ$, $G_1[*][*] \in ZTHS$, $G_2, ..., G_n \in THS$, then $\Gamma / G_1[*][*]; ...; G_n \in ZTHS$.*

Definition 6.4 For all zoom tree-hypersequents $G[*]$, or $G[*][*]$, and tree-hypersequents H and I, we define $G[H]$ and $G[H][I]$, the result of substituting H into $G[*]$, and the result of substituting H and I in $G[*][*]$, respectively, as follows:

- *if* $G[*] = [*]$, *then* $G[H] = H$
- *if* $G[*] = [*]/G_1; ...; G_n$ *and* $H = \Delta/J_1; ...; J_m$, *then* $G[H] = \Delta/G_1; ...; G_n; J_1; ...; J_m$
- *if* $G[*][*] = [*]/G_1[*]; ...; G_n$ *and* $H = \Delta/J_1; ...; J_m$, *then* $G[H][I] = \Delta/G_1[I]; ...; G_n; J_1; ...; J_m$
- *if* $G[*] = \Gamma/G_1[*], ..., G_n$, *then* $G[H] = \Gamma/G_1[H], ..., G_n$
- *if* $G[*][*] = \Gamma/G_1[*][*], ..., G_n$, *then* $G[H][I] = \Gamma/G_1[H][I], ..., G_n$

Note that a sequent is a tree-hypersequent so that Definition 6.4 also applies to the case of substituting a sequent into a zoom tree-hypersequent.

The last two definitions can be approached from the perspective of their intuitive meaning. $G[*]$ can be thought of as a tree-hypersequent G together with one hole $[*]$, where the hole should be understood, metaphorically, as a zoom by means of which we can focus on a particular part, $*$, of G. The substitution fills the hole with a sequent or a tree-hypersequent, and therefore allows us to make explicit the particular part of the tree-hypersequent that we want to concentrate on. Similarly for $G[*][*]$.

Let us consider an arbitrary tree-hypersequent G. Let us suppose that we want to focus on a particular sequent Γ of this tree-hypersequent G. Then, following Definitions 6.3 and 6.4, we write $G[\Gamma]$. By contrast, let us suppose that we want to focus, not on the sequent Γ, but on the tree-hypersequent $\Gamma/\underline{X}$. According to Definitions 6.3 and 6.4, we write $G^{'}[\Gamma/\underline{X}]$. From one case to another, we change notation: we go from $G[*]$ to $G^{'}[*]$. Indeed, even if the tree-hypersequent G is the same in both cases, while in the first case the multiset of tree-hypersequents $\underline{X}$ is included in the notation $G[*]$, in the second case it is not, and we must somehow indicate this change. In order to avoid any confusion in what follows (see in particular Sections 7.1 and 10.4), we will indicate changes of this type, not with a generic index as we have done above, but with the aid of (n occurrences of) the symbol *dot* over the tree-hypersequent that we want to work with. Consider the above example anew; if we denote with $G[\Gamma]$ the fact that we focus on the sequent Γ of the tree-hypersequent G, then the fact that we focus on the tree-hypersequent $\Gamma/\underline{X}$ of the same tree-hypersequent G, is signalled with $\dot{G}[\Gamma/\underline{X}]$.

According to the order (see Section 2.3, p. 52) adopted for presenting the other syntactic and semantic methods, we start by showing the calculus **Thsk** in one of its general variants. The calculus **Thsk** is composed of:

Initial Tree-hypersequents

$G[p \Rightarrow p]$

Structural Rules

Internal Weakening and Contraction

$$\frac{G[\Gamma]}{G[\alpha, \Gamma]}\,WA \qquad \frac{G[\Gamma]}{G[\Gamma, \alpha]}\,WK$$

$$\frac{G[\alpha, \alpha, \Gamma]}{G[\alpha, \Gamma]}\,CA \qquad \frac{G[\Gamma, \alpha, \alpha]}{G[\Gamma, \alpha]}\,CK$$

External Weakening and Merge

$$\frac{G[\Gamma]}{G[\Gamma/\Sigma]}\,EW \qquad \frac{G[\Delta/(\Gamma/\underline{X}); (\Pi/\underline{X}^{'})]}{G[\Delta/(\Gamma \centerdot \Pi/\underline{X}; \underline{X}^{'})]}\,merge$$

Necessitation Rule

$$\frac{G}{\Rightarrow /G}\,rn$$

Logical Rules

Propositional Rules

$$\frac{G[\Gamma, \alpha]}{G[\neg\alpha, \Gamma]}\,\neg A \qquad \frac{G[\alpha, \Gamma]}{G[\Gamma, \neg\alpha]}\,\neg K$$

$$\frac{G[\alpha_i, \Gamma]}{G[\alpha_0 \wedge \alpha_1, \Gamma]}\,\wedge A \qquad \frac{G[\Gamma, \alpha] \qquad G[\Gamma, \beta]}{G[\Gamma, \alpha \wedge \beta]}\,\wedge K$$

Modal Rules

$$\frac{G[\Gamma/(\alpha, \Sigma/\underline{X})]}{G[\Box\alpha, \Gamma/(\Sigma/\underline{X})]}\,\Box A \qquad \frac{G[\Gamma/ \Rightarrow \alpha]}{G[\Gamma, \Box\alpha]}\,\Box K$$

In order to introduce the cut-rule, we first need the following notions.

Definition 6.5 Given two tree-hypersequents $G[\Gamma]$ and $G^{'}[\Gamma^{'}]$, the relation of *equivalent position* between two of their sequents, in this case Γ and $\Gamma^{'}$, $G[\Gamma] \sim G^{'}[\Gamma^{'}]$, is inductively defined in the following way:

- $\Gamma \sim \Gamma^{'}$
- $\Gamma/\underline{X} \sim \Gamma^{'}/\underline{X}^{'}$
- *If* $H[\Gamma] \sim H^{'}[\Gamma^{'}]$, *then* $\Delta/H[\Gamma]; \underline{X} \sim \Delta^{'}/H^{'}[\Gamma^{'}]; \underline{X}^{'}$

Definition 6.6 Given two tree-hypersequents $G[\Gamma]$ and $G^{'}[\Gamma^{'}]$ such that $G[\Gamma] \sim G^{'}[\Gamma^{'}]$, the operation of *product*, $G[\Gamma] \otimes G^{'}[\Gamma^{'}]$, is inductively defined in the following way:

- $\Gamma \otimes \Gamma' := \Gamma \centerdot \Gamma'$
- $(\Gamma/\underline{X}) \otimes (\Gamma'/\underline{X}') := \Gamma \centerdot \Gamma'/\underline{X}; \underline{X}'$
- $(\Delta/H[\Gamma]; \underline{X}) \otimes (\Delta'/H'[\Gamma']; \underline{X}') := \Delta \centerdot \Delta'/(H[\Gamma] \otimes H'[\Gamma']); \underline{X}; \underline{X}'$

Cut-Rule

Given two tree-hypersequents $G[\Gamma, \alpha]$ and $G'[\alpha, \Pi]$ such that $G[\Gamma, \alpha] \sim G'[\alpha, \Pi]$, the cut-rule is

$$\frac{G[\Gamma, \alpha] \quad G'[\alpha, \Pi]}{G \otimes G'[\Gamma \centerdot \Pi]}\, cut_\alpha$$

The above definition clearly suggests that the cut-rule should respect two important conditions. The first one says that given two tree-hypersequents, we can cut on any two sequents belonging to them provided that they are in equivalent position. The second one says that after a cut the two *tree*-hypersequents on which we have applied the cut, should not be randomly mixed, but fused according to the inductive definition of product. These two conditions are fundamental because they serve to ensure that the result of a cut between two *tree*-hypersequents is still a *tree*-hypersequent, which is to say that the tree-shape is kept.

In order to obtain the calculi for the remaining systems, we add combinations of the pairs of rules (one logical and one structural) listed below to the calculus **Thsk**. Each pair corresponds to one of the axioms (or frame properties) listed in Section 2.1, p. 44.

Special Structural and Logical Rules

$$\frac{G[\Gamma/\alpha \Rightarrow]}{G[\Box\alpha, \Gamma]}\, d \qquad \frac{G[\Gamma/\Rightarrow]}{G[\Gamma]}\, \tilde{d}$$

$$\frac{G[\alpha, \Gamma]}{G[\Box\alpha, \Gamma]}\, t \qquad \frac{G[\Gamma/(\Sigma/\underline{X})]}{G[\Gamma \centerdot \Sigma/\underline{X}]}\, \tilde{t}$$

$$\frac{G[\Gamma/(\Box\alpha, \Sigma/\underline{X})]}{G[\Box\alpha, \Gamma/(\Sigma/\underline{X})]}\, 4 \qquad \frac{G[\Gamma/(\Sigma/\underline{X})]}{G[\Gamma/(\Rightarrow /\Sigma/\underline{X})]}\, \tilde{4}$$

$$\frac{G[\alpha, \Gamma/(\Sigma/\underline{X})]}{G[\Gamma/(\Box\alpha, \Sigma/\underline{X})]}\, b \qquad \frac{G[\Gamma/(\Sigma/(\Delta/\underline{X}); \underline{X}')]}{G[\Gamma \centerdot \Delta/(\Sigma/\underline{X}'); \underline{X}]}\, \tilde{b}$$

In the calculi where both the pair of rules for the axiom 4 and the pair of rules for the axiom b are present, we should add the pair of rules

$$\frac{G[\Box\alpha, \Gamma/(\Sigma/\underline{X})]}{G[\Gamma/(\Box\alpha, \Sigma/\underline{X})]}\, 5 \qquad \frac{G[\Gamma/(\Sigma/(\Delta/\underline{X}); \underline{X}')]}{G[\Gamma/(\Delta/\underline{X}); (\Sigma/\underline{X}')]}\, \tilde{5}$$

Contrary to all the other pairs of rules, 5 and $\tilde{5}$ do not reflect the strength and the power of the axiom 5: indeed the calculus **Thsk** plus 5 and $\tilde{5}$ is not cut-free, while the calculus **Thsk** plus the pair 5 and $\tilde{5}$, and the pair t and $\tilde{t}$ is not complete. Therefore, the pair of rules 5 and $\tilde{5}$ just serve to complete the calculi obtained by adding the rules 4, $\tilde{4}$, b and $\tilde{b}$ to the calculus **Thsk**. Brünnler [15] has proposed a partial solution to these problems, consisting in two other rules for the axiom 5 that, added to the calculus **Thsk**, form a tree-hypersequent calculus **Thsk5** which is sound and complete with respect to the system **K5**. On the other hand, **Thsk5** $+t+\tilde{t}$ is still incomplete (we cannot prove the axiom 4), and the cut-elimination proof for **Thsk5** is purely semantic.

In conclusion, the two disadvantages of the tree-hypersequent calculi come to the fore: they are not fully modular, and the axiom 5 is difficult to capture in this framework. Nevertheless their precious benefits will be the objects of the next sections.

6.2 Logical Variant of the Tree-Hypersequent Calculi

In this section we will start by introducing the tree-hypersequent calculi in one of their logical variants, and then we will show which structural rules are (height-preserving) admissible in this variant (for the definition of (height-preserving) admissibility, see Definition 1.10, p. 6). Moreover, we will prove that the propositional and modal rules are height-preserving invertible (for the definition of (height-preserving) invertibility, see Definition 1.11, p. 7). In the next chapter the admissibility of the cut-rule will be proved.

Thsk$_L$, a logical variant of the calculus **Thsk**, is composed of:

Initial Tree-hypersequents

$$G\,[p,\ \Gamma,\ p]$$

Logical Rules

Propositional Rules

$$\frac{G[\Gamma, \alpha]}{G[\neg\alpha, \Gamma]}\ \neg A \qquad \frac{G[\alpha, \Gamma]}{G[\Gamma, \neg\alpha]}\ \neg K$$

$$\frac{G[\alpha, \beta, \Gamma]}{G[\alpha \wedge \beta, \Gamma]}\ \wedge A' \qquad \frac{G[\Gamma, \alpha] \qquad G[\Gamma, \beta]}{G[\Gamma, \alpha \wedge \beta]}\ \wedge K$$

Modal Rules

$$\frac{G[\Box\alpha, \Gamma/(\alpha, \Sigma/\underline{X})]}{G[\Box\alpha, \Gamma/(\Sigma/\underline{X})]}\ \Box A \quad \frac{G[\Gamma/ \Rightarrow \alpha]}{G[\Gamma, \Box\alpha]}\ \Box K$$

$\mathbf{Thsk}_L$ is then obtained from **Thsk** by means of the following four changes:

- the initial tree-hypersequents are substituted with the generalised ones,
- the rule $\wedge A$ is substituted with its multiplicative counterpart $\wedge A'$,
- in the rule $\Box A$, the formula $\Box\alpha$ is added to the premise, and
- all the structural rules are dropped.

The first two and the last changes are the same as the ones that we used for obtaining the sequent calculus $\mathbf{Gcl}_L$ from the calculus **Gcl** (see Definition 1.14, p. 9). The third change is, on the other hand, similar to the one usually adopted with the rule that introduces the universal quantifier on the left side of the sequent, in the sequent calculus for first-order classical logic. Note that, even if the rule $\Box A$ now contains the formula $\Box\alpha$ in its premise, we do not consider the explicitness property (see Section 1.7) to be lost. Indeed the addition of the formula $\Box\alpha$ to the premise of the rule $\Box A$ is just the simplest way to obtain some results in the calculus $\mathbf{Thsk}_L$.

The tree-hypersequent calculi $\mathbf{Thsk}^*_L$ are obtained thanks to the

Special Logical Rules

$$\frac{G[\Box\alpha, \Gamma/\alpha \Rightarrow]}{G[\Box\alpha, \Gamma]}\,d \qquad \frac{G[\Box\alpha, \alpha, \Gamma]}{G[\Box\alpha, \Gamma]}\,t$$

$$\frac{G[\Box\alpha, \Gamma/(\Box\alpha, \Sigma/\underline{X})]}{G[\Box\alpha, \Gamma/(\Sigma/\underline{X})]}\,4 \qquad \frac{G[\alpha, \Gamma/(\Box\alpha, \Sigma/\underline{X})]}{G[\Gamma/(\Box\alpha, \Sigma/\underline{X})]}\,b$$

The following three remarks concerning the special logical rules are in order.

- The special logical rules have been modified in a way similar to that of the rule $\Box A$: the formula $\Box\alpha$ has been added to each of their premises.
- In the calculi where both the rules 4 and b are present, we should add the rule

$$\frac{G[\Box\alpha, \Gamma/(\Box\alpha, \Sigma/\underline{X})]}{G[\Gamma/(\Box\alpha, \Sigma/\underline{X})]}\,5$$

- Another way of obtaining the calculi for the systems containing the axiom 4 consists in substituting the rule 4 with the rule

$$\frac{G[\Box\alpha, \Gamma][\alpha, \Sigma]}{G[\Box\alpha, \Gamma][\Sigma]}\,4'$$

Recall that a special structural rule corresponds to each special logical rule (see the previous section). We will show that in the case where a special logical rule $\mathcal{R}$ is added to the calculus $\mathbf{Thsk}_L$, i.e. we have $\mathbf{Thsk}_L + \mathcal{R}$, the correspondent structural rule $\tilde{\mathcal{R}}$ is proved to be (height-preserving) admissible. As we will see in the next chapter, special structural rules have a fundamental role in the cut-elimination proof.

Definition 6.7 By analogy with Definition 1.8, p. 6, we will call *auxiliary* those sequents that are explicitly displayed in the premise(s) of the rules of the tree-hypersequent calculi.

Remark 6.8 In the following proofs of (height-preserving) admissibility of the structural rules and height-preserving invertibility of the propositional and modal rules, we will only take into account those cases in which the last applied rule operates on the auxiliary sequent(s) of the rule that we want to show to be admissible or invertible. All the other cases are dealt with easily, as Lemmas 6.20 and 6.21 prove at the end of the current section.

We will write (recall Definition 1.9, p. 6) $d \vdash^{n}_{\mathbf{Thsk}^*_L} G$, or shortly ${}^{n}G$, for: there exists a derivation d of G in $\mathbf{Thsk}^*_L$, with $h(d) \leq n$.

Lemma 6.9 *Tree-hypersequents of the form* $G[\alpha, \Gamma, \alpha]$*, with* α *an arbitrary modal formula, are derivable in* $\mathbf{Thsk}^*_L$.

Proof By straightforward induction on α. □

Lemma 6.10 *The rule of necessitation*

$$\frac{G}{\Rightarrow / G}\, rn$$

is height-preserving admissible in $\mathbf{Thsk}^*_L$.

Proof By induction on the derivation of the premise.

If G is an initial tree-hypersequent, then so is the conclusion. If G is inferred by a propositional rule, then the inference is clearly preserved. Let us consider the example of the rule $\neg K$,

$$\frac{{}^{\langle n-1\rangle} G[\alpha, \Gamma]}{{}^{\langle n\rangle} G[\Gamma, \neg\alpha]}\,\neg K \qquad \rightsquigarrow^{25} \qquad \frac{{}^{\langle n-1\rangle} \Rightarrow / G[\alpha, \Gamma]}{{}^{\langle n\rangle} \Rightarrow / G[\Gamma, \neg\alpha]}\,\neg K$$

If G is inferred by a modal rule, then the inference is preserved. Let us consider the example of the rule $\Box K$,

$$\frac{{}^{\langle n-1\rangle} G[\Gamma / \Rightarrow \alpha]}{{}^{\langle n\rangle} G[\Gamma, \Box\alpha]}\,\Box K \qquad \rightsquigarrow \qquad \frac{{}^{\langle n-1\rangle} \Rightarrow / G[\Gamma / \Rightarrow \alpha]}{{}^{\langle n\rangle} \Rightarrow / G[\Gamma, \Box\alpha]}\,\Box K$$

Finally, if G is inferred by one of the special logical rules, then the inference is preserved. Let us consider the example of the rule t,

$$\frac{{}^{\langle n-1\rangle} G[\Box\alpha, \alpha, \Gamma]}{{}^{\langle n\rangle} G[\Box\alpha, \Gamma]}\, t \qquad \rightsquigarrow \qquad \frac{{}^{\langle n-1\rangle} \Rightarrow / G[\Box\alpha, \alpha, \Gamma]}{{}^{\langle n\rangle} \Rightarrow / G[\Box\alpha, \Gamma]}\, t$$

□

Lemma 6.11 *The rules of internal and external weakening:*

$$\frac{G[\Gamma]}{G[\alpha, \Gamma]}\,WA \qquad \frac{G[\Gamma]}{G[\Gamma, \alpha]}\,WK \qquad \frac{G[\Gamma]}{G[\Gamma/\Sigma]}\,EW$$

are height-preserving admissible in $\mathbf{Thsk}^*_L$.

Proof By straightforward induction on the derivation of the premise. □

Lemma 6.12 *The rule of merge*

$$\frac{G[\Delta/(\Gamma/\underline{X}); (\Pi/\underline{X}')]}{G[\Delta/(\Gamma \centerdot \Pi/\underline{X}; \underline{X}')]}\,merge$$

is height-preserving admissible in $\mathbf{Thsk}^*_L$.

Proof By induction on the derivation of the premise. As the rule of merge has three auxiliary sequents, Δ, Γ and Π, we should, for each rule $\mathcal{R}$ applied to the premise, distinguish three subcases: one in which the rule $\mathcal{R}$ has been applied to the sequent Δ, one in which the rule $\mathcal{R}$ has been applied to the sequent Γ, and one in which the rule $\mathcal{R}$ has been applied to the sequent Π. On the other hand, these three subcases are similar, and therefore we do not need to analyse all of them; on the contrary, we will develop the proof by choosing the most significant one each time.

If the premise is an initial tree-hypersequent, then so is the conclusion. If the premise is inferred by a propositional rule, then the inference is preserved. Let us consider the example of the rule $\neg K$,

$$\frac{{}^{\langle n-1\rangle}G[\Delta/(\alpha, \Gamma/\underline{X}); (\Pi/\underline{X}')]}{{}^{\langle n\rangle}G[\Delta/(\Gamma, \neg\alpha/\underline{X}); (\Pi/\underline{X}')]}\,\neg K \qquad \rightsquigarrow$$

$$\frac{{}^{\langle n-1\rangle}G[\Delta/(\alpha, \Gamma \centerdot \Pi/\underline{X}; \underline{X}')]}{{}^{\langle n\rangle}G[\Delta/(\Gamma \centerdot \Pi, \neg\alpha/\underline{X}; \underline{X}')]}\,\neg K$$

If the premise is inferred by the modal rule $\Box K$, then the inference is preserved.

$$\frac{{}^{\langle n-1\rangle}G[\Delta/(\Gamma/\Rightarrow \alpha; \underline{X}); (\Pi/\underline{X}')]}{{}^{\langle n\rangle}G[\Delta/(\Gamma, \Box\alpha/\underline{X}); (\Pi/\underline{X}')]}\,\Box K \qquad \rightsquigarrow$$

$$\frac{{}^{\langle n-1\rangle}G[\Delta/(\Gamma \centerdot \Pi/\Rightarrow \alpha; \underline{X}; \underline{X}')]}{{}^{\langle n\rangle}G[\Delta/(\Gamma \centerdot \Pi, \Box\alpha/\underline{X}; \underline{X}')]}\,\Box K$$

If the premise is inferred by the rule $\Box A$, then the inference is preserved.

$$\frac{{}^{\langle n-1\rangle}G[\Box\alpha, \Delta/(\alpha, \Gamma/\underline{X}); (\Pi/\underline{X}')]}{{}^{\langle n\rangle}G[\Box\alpha, \Delta/(\Gamma/\underline{X}); (\Pi/\underline{X}')]}\,\Box A \qquad \rightsquigarrow$$

$$\frac{{}^{\langle n-1\rangle}G[\Box\alpha, \Delta/(\alpha, \Gamma \centerdot \Pi/\underline{X}; \underline{X}')]}{{}^{\langle n\rangle}G[\Box\alpha, \Delta/(\Gamma \centerdot \Pi/\underline{X}; \underline{X}')]}\,\Box A$$

If the premise is inferred by the rule d, then the case can be dealt with analogously to the case of the rule $\Box K$. If the premise is inferred by the rule t, then the case can be dealt with analogously to the case of the logical rules. If the premise is inferred by the rule 4 or by the rule b or by the rule 5, then the case can be dealt with analogously to the case of the rule $\Box A$. □

Lemma 6.13 *The rule* $\tilde{d}$

$$\frac{G[\Gamma / \Rightarrow]}{G[\Gamma]}\ \tilde{d}$$

is admissible in those tree-hypersequent calculi that contain the rule d.

Proof By induction on the derivation of the premise.

If the premise is an initial tree-hypersequent, then so is the conclusion. If the premise is inferred by a logical rule, then the inference is preserved. Let us consider the example of the rule $\neg K$,

$$\frac{{}^{\langle n-1\rangle}G[\alpha, \Gamma / \Rightarrow]}{{}^{\langle n\rangle}G[\Gamma, \neg\alpha / \Rightarrow]}\ \neg K \qquad \rightsquigarrow \qquad \frac{G[\alpha, \Gamma]}{G[\Gamma, \neg\alpha]}\ \neg K$$

If the premise is inferred by the modal rule $\Box K$, then the inference is preserved.

$$\frac{{}^{\langle n-1\rangle}G[\Gamma / \Rightarrow ; \Rightarrow \alpha]}{{}^{\langle n\rangle}G[\Gamma, \Box\alpha / \Rightarrow]}\ \Box K \qquad \rightsquigarrow \qquad \frac{G[\Gamma / \Rightarrow \alpha]}{G[\Gamma, \Box\alpha]}\ \Box K$$

If the premise is inferred by the rule $\Box A$, then we have (we analyse the following special case)

$$\frac{{}^{\langle n-1\rangle}G[\Box\alpha, \Gamma / \alpha \Rightarrow]}{{}^{\langle n\rangle}G[\Box\alpha, \Gamma / \Rightarrow]}\ \Box A \qquad \rightsquigarrow \qquad \frac{G[\Box\alpha, \Gamma / \alpha \Rightarrow]}{G[\Box\alpha, \Gamma]}\ d$$

If the premise is inferred by the rule d, then the case can be dealt with analogously to the case of the rule $\Box K$. If the premise is inferred by the rule t, then the case can be dealt with analogously to the case of the propositional rules. If the premise is inferred by the rule 4, then we have (we analyse the following special case)

$$\frac{{}^{\langle n-1\rangle}G[\Box\alpha, \Gamma / \Box\alpha \Rightarrow]}{{}^{\langle n\rangle}G[\Box\alpha, \Gamma / \Rightarrow]}\ 4 \qquad \rightsquigarrow \qquad \cfrac{\cfrac{\cfrac{G[\Box\alpha, \Gamma / \Box\alpha \Rightarrow]}{G[\Box\alpha, \Gamma / \Box\alpha, \alpha \Rightarrow]}\ WA}{G[\Box\alpha, \Gamma / \alpha \Rightarrow]}\ 4}{G[\Box\alpha, \Gamma]}\ d$$

If the premise is inferred by the rule b, then the case can be dealt with analogously to the case of the rule $\Box A$; if the premise is inferred by the rule 5, then the case can be dealt with analogously to the case of the rule 4. However, neither with the rule b nor with the rule 5, there is a special case to treat. □

Lemma 6.14 *The rule* $\tilde{t}$

$$\frac{G[\Gamma/(\Sigma/\underline{X})]}{G[\Gamma \centerdot \Sigma/\underline{X}]}\ \tilde{t}$$

is height-preserving admissible in those tree-hypersequent calculi that contain the rule t.

Proof By induction on the derivation of the premise. As the rule $\tilde{t}$ has two auxiliary sequents, Γ and Σ, we should, for each rule $\mathcal{R}$ applied to the premise, distinguish two subcases: one in which the rule $\mathcal{R}$ has been applied to the sequent Γ, and one in which the rule $\mathcal{R}$ has been applied to the sequent Σ. On the other hand, since the two subcases are similar, we will only sketch the proof for one of them.

If the premise is an initial tree-hypersequent, then so is the conclusion. If the premise is inferred by a propositional rule, then the inference is preserved. Let us consider the example of the rule $\neg K$,

$$\frac{^{\langle n-1\rangle}G[\alpha, \Gamma/(\Sigma/\underline{X})]}{^{\langle n\rangle}G[\Gamma, \neg\alpha/(\Sigma/\underline{X})]}\ \neg K \qquad \rightsquigarrow \qquad \frac{^{\langle n-1\rangle}G[\alpha, \Gamma \centerdot \Sigma/\underline{X}]}{^{\langle n\rangle}G[\Gamma \centerdot \Sigma, \neg\alpha/\underline{X}]}\ \neg K$$

If the premise is inferred by the modal rule $\Box K$, then the inference is preserved.

$$\frac{^{\langle n-1\rangle}G[\Gamma/\Rightarrow \alpha; (\Sigma/\underline{X})]}{^{\langle n\rangle}G[\Gamma, \Box\alpha/(\Sigma/\underline{X})]}\ \Box K \qquad \rightsquigarrow \qquad \frac{^{\langle n-1\rangle}G[\Gamma \centerdot \Sigma/\Rightarrow \alpha; \underline{X}]}{^{\langle n\rangle}G[\Gamma \centerdot \Sigma, \Box\alpha/\underline{X}]}\ \Box K$$

If the premise is inferred by the rule $\Box A$, then we have

$$\frac{^{\langle n-1\rangle}G[\Box\alpha, \Gamma/(\alpha, \Sigma/\underline{X})]}{^{\langle n\rangle}G[\Box\alpha, \Gamma/(\Sigma/\underline{X})]}\ \Box A \qquad \rightsquigarrow \qquad \frac{^{\langle n-1\rangle}G[\Box\alpha, \alpha, \Gamma \centerdot \Sigma/\underline{X}]}{^{\langle n\rangle}G[\Box\alpha, \Gamma \centerdot \Sigma/\underline{X}]}\ t$$

If the premise is inferred by the rule d, then the case can be dealt with analogously to the case of the rule $\Box K$. If the premise is inferred by the rule t, then the case can be dealt with analogously to the case of the propositional rules. If the premise is inferred by the rule 4, then we have

$$\frac{^{\langle n-1\rangle}G[\Box\alpha, \Gamma/(\Box\alpha, \Sigma/\underline{X})]}{^{\langle n\rangle}G[\Box\alpha, \Gamma/(\Sigma/\underline{X})]}\ 4 \qquad \rightsquigarrow \qquad \frac{^{\langle n-1\rangle}G[\Box\alpha, \Box\alpha, \Gamma \centerdot \Sigma/\underline{X}]}{^{\langle n\rangle}G[\Box\alpha, \Gamma \centerdot \Sigma/\underline{X}]}\ CA$$

If the premise is inferred by the rule b, then the case can be dealt with analogously to the case of the rule $\Box A$. If the premise is inferred by the rule 5, then the case can be dealt with analogously to the case of the rule 4. □

Lemma 6.15 *The rule* $\tilde{4}$

$$\frac{G[\Gamma/(\Sigma/\underline{X})]}{G[\Gamma/(\Rightarrow /(\Sigma/\underline{X}))]}\ \tilde{4}$$

is admissible in those tree-hypersequent calculi that contain the rule 4.

Proof By induction on the derivation of the premise. As the rule $\tilde{4}$ has two auxiliary sequents, Γ and Σ, we should, for each rule $\mathcal{R}$ applied to the premise, distinguish two subcases: one in which the rule $\mathcal{R}$ has been applied to the sequent Γ, and one in which the rule $\mathcal{R}$ has been applied to the sequent Σ. On the other hand, since these two subcases are similar, we will only sketch the proof for one of them.

If the premise is an initial tree-hypersequent, then so is the conclusion. If the premise is inferred by a logical rule, this inference is preserved. Let us consider the example of the rule $\neg K$,

$$\dfrac{{}^{\langle n-1\rangle}G[\alpha, \Gamma/(\Sigma/\underline{X})]}{{}^{\langle n\rangle}G[\Gamma, \neg\alpha/(\Sigma/\underline{X})]}\,\neg K \qquad \rightsquigarrow \qquad \dfrac{G[\alpha, \Gamma/(\Rightarrow /(\Sigma/\underline{X}))]}{G[\Gamma, \neg\alpha/(\Rightarrow /(\Sigma/\underline{X}))]}\,\neg K$$

If the premise is inferred by the modal rule $\Box K$, this inference is preserved.

$$\dfrac{{}^{\langle n-1\rangle}G[\Gamma/ \Rightarrow \alpha; (\Sigma/\underline{X})]}{{}^{\langle n\rangle}G[\Gamma, \Box\alpha/(\Sigma/\underline{X})]}\,\Box K \qquad \rightsquigarrow \qquad \dfrac{G[\Gamma/ \Rightarrow \alpha; (\Rightarrow /(\Sigma/\underline{X}))]}{G[\Gamma, \Box\alpha/(\Rightarrow /(\Sigma/\underline{X}))]}\,\Box K$$

If the premise is inferred by the rule $\Box A$, then we have

$$\dfrac{{}^{\langle n-1\rangle}G[\Box\alpha, \Gamma/(\alpha, \Sigma/\underline{X})]}{{}^{\langle n\rangle}G[\Box\alpha, \Gamma/(\Sigma/\underline{X})]}\,\Box A \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\dfrac{G[\Box\alpha, \Gamma/(\Rightarrow /(\alpha, \Sigma/\underline{X}))]}{G[\Box\alpha, \Gamma/(\Box\alpha \Rightarrow /(\alpha, \Sigma/\underline{X}))]}\,WA}{G[\Box\alpha, \Gamma/(\Box\alpha \Rightarrow /(\Sigma/\underline{X}))]}\,\Box A}{G[\Box\alpha, \Gamma/(\Rightarrow /(\Sigma/\underline{X}))]}\,4$$

If the premise is inferred by the rule d, then the case can be dealt with analogously to the case of the rule $\Box K$. If the premise is inferred by the rule t, then the case can be dealt with analogously to the case of the propositional rules. If the premise is inferred by the rule 4, then we have

$$\dfrac{{}^{\langle n-1\rangle}G[\Box\alpha, \Gamma/(\Box\alpha, \Sigma/\underline{X})]}{{}^{\langle n\rangle}G[\Box\alpha, \Gamma/(\Sigma/\underline{X})]}\,4 \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\dfrac{G[\Box\alpha, \Gamma/(\Rightarrow /(\Box\alpha, \Sigma/\underline{X}))]}{G[\Box\alpha, \Gamma/(\Box\alpha \Rightarrow /(\Box\alpha, \Sigma/\underline{X}))]}\,WA}{G[\Box\alpha, \Gamma/(\Box\alpha \Rightarrow /(\Sigma/\underline{X}))]}\,4}{G[\Box\alpha, \Gamma/(\Rightarrow /(\Sigma/\underline{X}))]}\,4$$

If the premise is inferred by the rule b, then we have[2]

$$\dfrac{{}^{\langle n-1\rangle}G[\alpha, \Gamma/(\Box\alpha, \Sigma/\underline{X})]}{{}^{\langle n\rangle}G[\Gamma/(\Box\alpha, \Sigma/\underline{X})]}\,b \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\dfrac{G[\alpha, \Gamma/(\Rightarrow /(\Box\alpha, \Sigma/\underline{X}))]}{G[\alpha, \Gamma/(\Box\alpha \Rightarrow /(\Box\alpha, \Sigma/\underline{X}))]}\,WA}{G[\Gamma/(\Box\alpha \Rightarrow /(\Box\alpha, \Sigma/\underline{X}))]}\,b}{G[\Gamma/(\Rightarrow /(\Box\alpha, \Sigma/\underline{X}))]}\,5$$

If the premise is inferred by the rule 5, then the case can be dealt with analogously to the case of the rule 4. $\Box$

Lemma 6.16 *The rule* $\tilde{b}$

$$\frac{G[\Gamma/(\Sigma/(\Delta/\underline{X}); \underline{X}')]}{G[\Gamma \centerdot \Delta/(\Sigma/\underline{X}'); \underline{X}]}\ \tilde{b}$$

is height-preserving admissible in those tree-hypersequent calculi that contain the rule b.

Proof By induction on the derivation of the premise. As the rule $\tilde{b}$ has three auxiliary sequents, Γ, Σ and Δ, we should, for each rule $\mathcal{R}$ applied to the premise, distinguish three subcases: one in which the rule $\mathcal{R}$ has been applied to the sequent Γ, one in which the rule $\mathcal{R}$ has been applied to the sequent Σ, and one in which the rule $\mathcal{R}$ has been applied to the sequent Δ. On the other hand, these three subcases are similar, and therefore we do not need to analyse all of them; on the contrary, we will develop the proof by choosing the most significant one each time.

If the premise is an initial tree-hypersequent, then so is the conclusion. If the premise is inferred by a propositional rule, this inference is preserved. Let us consider the example of the rule $\neg K$,

$$\frac{{}^{\langle n-1\rangle}G[\Gamma/(\Sigma/(\alpha, \Delta/\underline{X}); \underline{X}')]}{{}^{\langle n\rangle}G[\Gamma/(\Sigma/(\Delta, \neg\alpha/\underline{X}); \underline{X}')]}\ \neg K \qquad \rightsquigarrow$$

$$\frac{{}^{\langle n-1\rangle}G[\alpha, \Gamma \centerdot \Delta/(\Sigma/\underline{X}'); \underline{X}]}{{}^{\langle n\rangle}G[\Gamma \centerdot \Delta, \neg\alpha/(\Sigma/\underline{X}'); \underline{X}]}\ \neg K$$

If the premise is inferred by the modal rule $\Box K$, this inference is preserved.

$$\frac{{}^{\langle n-1\rangle}G[\Gamma/(\Sigma/(\Delta/ \Rightarrow \alpha; \underline{X}); \underline{X}')]}{{}^{\langle n\rangle}G[\Gamma/(\Sigma/(\Delta, \Box\alpha/\underline{X}); \underline{X}')]}\ \Box K \qquad \rightsquigarrow$$

$$\frac{{}^{\langle n-1\rangle}G[\Gamma \centerdot \Delta/(\Sigma/\underline{X}'); \Rightarrow \alpha; \underline{X}]}{{}^{\langle n\rangle}G[\Gamma \centerdot \Delta, \Box\alpha/(\Sigma/\underline{X}'); \underline{X}]}\ \Box K$$

If the premise is inferred by the rule $\Box A$, then we have

$$\frac{{}^{\langle n-1\rangle}G[\Gamma/(\Box\alpha, \Sigma/(\alpha, \Delta/\underline{X}); \underline{X}')]}{{}^{\langle n\rangle}G[\Gamma/(\Box\alpha, \Sigma/(\Delta/\underline{X}); \underline{X}')]}\ \Box A \qquad \rightsquigarrow$$

$$\frac{{}^{\langle n-1\rangle}G[\alpha, \Gamma \centerdot \Delta/(\Box\alpha, \Sigma/\underline{X}'); \underline{X}]}{{}^{\langle n\rangle}G[\Gamma \centerdot \Delta/(\Box\alpha, \Sigma/\underline{X}'); \underline{X}]}\ b$$

If the premise is inferred by the rule d, then the case can be dealt with analogously to the case of the rule $\Box K$. If the premise is inferred by the rule t, then the case can be dealt with analogously to the case of the propositional rules. If the premise is inferred by the rule 4, then we have[3]

$$\frac{^{\langle n-1\rangle}G[\Gamma/(\Box\alpha,\Sigma/(\Box\alpha,\Delta/\underline{X});\underline{X}^{'})]}{^{\langle n\rangle}G[\Gamma/(\Box\alpha,\Sigma/(\Delta/\underline{X});\underline{X}^{'})]}\,4 \qquad \rightsquigarrow$$

$$\frac{^{\langle n-1\rangle}G[\Box\alpha,\Gamma\,\centerdot\,\Delta/(\Box\alpha,\Sigma/\underline{X}^{'});\underline{X}]}{^{\langle n\rangle}G[\Gamma\,\centerdot\,\Delta/(\Box\alpha,\Sigma/\underline{X}^{'});\underline{X}]}\,5$$

If the premise is inferred by the rule b, then we have

$$\frac{^{\langle n-1\rangle}G[\Gamma/(\alpha,\Sigma/(\Box\alpha,\Delta/\underline{X});\underline{X}^{'})]}{^{\langle n\rangle}G[\Gamma/(\Sigma/(\Box\alpha,\Delta/\underline{X});\underline{X}^{'})]}\,b \qquad \rightsquigarrow$$

$$\frac{^{\langle n-1\rangle}G[\Box\alpha,\Gamma\,\centerdot\,\Delta/(\alpha,\Sigma/\underline{X}^{'});\underline{X}]}{^{\langle n\rangle}G[\Box\alpha,\Gamma\,\centerdot\,\Delta/(\Sigma/\underline{X}^{'});\underline{X}]}\,\Box A$$

If the premise is inferred by the rule 5, then we have

$$\frac{^{\langle n-1\rangle}G[\Gamma/(\Box\alpha,\Sigma/(\Box\alpha,\Delta/\underline{X});\underline{X}^{'})]}{^{\langle n\rangle}G[\Gamma/(\Sigma/(\Box\alpha,\Delta/\underline{X});\underline{X}^{'})]}\,5 \qquad \rightsquigarrow$$

$$\frac{^{\langle n-1\rangle}G[\Box\alpha,\Gamma\,\centerdot\,\Delta/(\Box\alpha,\Sigma/\underline{X}^{'});\underline{X}]}{^{\langle n\rangle}G[\Box\alpha,\Gamma\,\centerdot\,\Delta/(\Sigma/\underline{X}^{'});\underline{X}]}\,4$$

□

Lemma 6.17 *The rule* $\tilde{5}$

$$\frac{G[\Gamma/(\Sigma/(\Delta/\underline{X});\underline{X}^{'})]}{G[\Gamma/(\Delta/\underline{X});(\Sigma/\underline{X}^{'})]}\,\tilde{5}$$

is admissible in those tree-hypersequent calculi that contain the rule 5 *(and therefore also the rules* 4 *and b).*

Proof By induction on the derivation of the premise. As the rule $\tilde{5}$ has three auxiliary sequents Γ, Σ and Δ, we should, for each rule $\mathcal{R}$ applied to the premise, distinguish three subcases: one in which the rule $\mathcal{R}$ has been applied to the sequent Γ, one in which the rule $\mathcal{R}$ has been applied to the sequent Σ, and one in which the rule $\mathcal{R}$ has been applied to the sequent Δ. On the other hand, these three subcases are similar, and therefore we do not need to analyse all of them; on the contrary, we will develop the proof by choosing the most significant one each time.

If the premise is an initial tree-hypersequent, then so is the conclusion. If the premise is inferred by a propositional rule, then the inference is preserved. Let us consider the example of the rule $\neg K$,

$$\frac{^{\langle n-1\rangle}G[\Gamma/(\Sigma/(\alpha,\Delta/\underline{X});\underline{X}^{'})]}{^{\langle n\rangle}G[\Gamma/(\Sigma/(\Delta,\neg\alpha/\underline{X});\underline{X}^{'})]}\,\neg K \qquad \rightsquigarrow \qquad \frac{G[\Gamma/(\alpha,\Delta/\underline{X});(\Sigma/\underline{X}^{'})]}{G[\Gamma/(\Delta,\neg\alpha/\underline{X});(\Sigma/\underline{X}^{'})]}\,\neg K$$

If the premise is inferred by the modal rule $\Box K$, then the inference is preserved.

$$\frac{{}^{\langle n-1\rangle}G[\Gamma/(\Sigma/(\Delta/\Rightarrow\alpha;\underline{X});\underline{X}')]}{{}^{\langle n\rangle}G[\Gamma/(\Sigma/(\Delta,\Box\alpha/\underline{X});\underline{X}')]}\,\Box K \qquad\rightsquigarrow$$

$$\frac{G[\Gamma/(\Delta/\Rightarrow\alpha;\underline{X});(\Sigma/\underline{X}')]}{G[\Gamma/(\Delta,\Box\alpha/\underline{X});(\Sigma/\underline{X}');\underline{X}'']}\,\Box K$$

If the premise is inferred by the rule $\Box A$, then we have

$$\frac{{}^{\langle n-1\rangle}G[\Gamma/(\Box\alpha,\Sigma/(\alpha,\Delta/\underline{X});\underline{X}')]}{{}^{\langle n\rangle}G[\Gamma/(\Box\alpha,\Sigma/(\Delta/\underline{X});\underline{X}')]}\,\Box A \qquad\rightsquigarrow$$

$$\cfrac{\cfrac{\cfrac{G[\Gamma/(\alpha,\Delta/\underline{X});(\Box\alpha,\Sigma/\underline{X}')]}{G[\Box\alpha,\Gamma/(\alpha,\Delta/\underline{X});(\Box\alpha,\Sigma/\underline{X}')]}\,WA}{G[\Box\alpha,\Gamma/(\Delta/\underline{X});(\Box\alpha,\Sigma/\underline{X}')]}\,\Box A}{G[\Gamma/(\Delta/\underline{X});(\Box\alpha,\Sigma/\underline{X}')]}\,5$$

If the premise is inferred by the rule d, then the case can be dealt with analogously to the case of the rule $\Box K$. If the premise is inferred by the rule t, then the case can be dealt with analogously to the case of the propositional rules. If the premise is inferred by the rule 4, then we have

$$\frac{{}^{\langle n-1\rangle}G[\Gamma/(\Box\alpha,\Sigma/(\Box\alpha,\Delta/\underline{X});\underline{X}')]}{{}^{\langle n\rangle}G[\Gamma/(\Box\alpha,\Sigma/(\Delta/\underline{X});\underline{X}')]}\,4 \qquad\rightsquigarrow$$

$$\cfrac{\cfrac{\cfrac{G[\Gamma/(\Box\alpha,\Delta/\underline{X});(\Box\alpha,\Sigma/\underline{X}')]}{G[\Box\alpha,\Gamma/(\Box\alpha,\Delta/\underline{X});(\Box\alpha,\Sigma/\underline{X}')]}\,WA}{G[\Box\alpha,\Gamma/(\Delta/\underline{X});(\Box\alpha,\Sigma/\underline{X}')]}\,4}{G[\Gamma/(\Delta/\underline{X});(\Box\alpha,\Sigma/\underline{X}')]}\,5$$

If the premise is inferred by the rule b, then the case can be dealt with analogously to the case of the rule $\Box A$. Finally, if the premise is inferred by the rule 5, then the case can be dealt with analogously to the case of the rule 4. □

Lemma 6.18 *The propositional rules, the modal rules and the special logical rules of* $\mathbf{Thsk}^*_L$ *are height-preserving invertible.*

Proof The proof is by induction on the derivation of the premise of the rule considered. The cases of the propositional rules are dealt with in the classical way. The only differences – the fact that we are dealing with tree-hypersequents, and the cases where the last applied rule is one of the modal rules or one of the special logical rules – do not pose a major obstacle.

The rules $\Box A$, d, t, 4, b and 5 are trivially height-preserving invertible since each of their premises is obtained by weakening from the conclusion, and weakening is height-preserving admissible, as we have shown in Lemma 9.5, p. 178.

We now turn to showing the invertibility of the rule $\Box K$ in detail. If $G[\Gamma, \Box\alpha]$ is an initial tree-hypersequent, then so is $G[\Gamma/\Rightarrow\alpha]$. If $G[\Gamma, \Box\alpha]$ is obtained by a propositional rule $\mathcal{R}$, we apply the inductive hypothesis to the premise(s) $G[\Gamma^{'}, \Box\alpha]$ ($G[\Gamma^{''}, \Box\alpha]$), and we obtain derivation(s), of height $n-1$, of $G[\Gamma^{'}/\Rightarrow\alpha]$ ($G[\Gamma''/\Rightarrow\alpha]$). By applying the rule $\mathcal{R}$, we obtain a derivation of height n of $G[\Gamma/\Rightarrow\alpha]$. If $G[\Gamma, \Box\alpha]$ is of the form $\dot{G}[\Box\beta, \Gamma^{'}, \Box\alpha/(\Sigma/\underline{X})]$ and is obtained by the modal rule $\Box A$, we apply the inductive hypothesis to $\dot{G}[\Box\beta, \Gamma^{'}, \Box\alpha/(\beta, \Sigma/\underline{X})]$, and we obtain a derivation of height $n-1$ of $\dot{G}[\Box\beta, \Gamma^{'}/\Rightarrow\alpha; (\beta, \Sigma/\underline{X})]$. By applying the rule $\Box A$, we obtain a derivation of height n of $\dot{G}[\Box\beta, \Gamma^{'}/\Rightarrow\alpha; (\Sigma/\underline{X})]$.

If $G[\Gamma, \Box\alpha]$ is obtained by one of the special logical rules, or by the modal rule $\Box K$ in which $\Box\alpha$ is not the principal formula, then these cases can be dealt with analogously to the case of the rule $\Box A$. Finally, if $G[\Gamma, \Box\alpha]$ is preceded by the modal rule $\Box K$ and $\Box\alpha$ is the principal formula, the premise of the last step gives the conclusion. □

Lemma 6.19 *The rules of contraction*

$$\frac{G[\alpha, \alpha, \Gamma]}{G[\alpha, \Gamma]}\,CA \qquad \frac{G[\Gamma, \alpha, \alpha]}{G[\Gamma, \alpha]}\,CK$$

are height-preserving admissible in $\mathbf{Thsk}^{*}_{L}$.

Proof By induction on the derivation of the premises $G[\alpha, \alpha, \Gamma]$ and $G[\Gamma, \alpha, \alpha]$. We only analyse the case of the rule CK. The case of the rule CA is similar.

If $G[\Gamma, \alpha, \alpha]$ is an initial tree-hypersequent, so is $G[\Gamma, \alpha]$. If $G[\Gamma, \alpha, \alpha]$ is obtained by a rule $\mathcal{R}$ that does not have any of the two occurrences of the formula α as principal, we apply the inductive hypothesis to the premise(s) $G[\Gamma^{'}, \alpha, \alpha]$ ($G[\Gamma^{''}, \alpha, \alpha]$), obtaining derivation(s) of height $n-1$ of $G[\Gamma^{'}, \alpha]$ ($G[\Gamma^{''}, \alpha]$). By applying the rule $\mathcal{R}$ we obtain a derivation of height n of $G[\Gamma, \alpha]$.

If $G[\Gamma, \alpha, \alpha]$ is obtained by a propositional or modal rule and one of the two occurrences of the formula α is principal, then the rule that concludes $G[\Gamma, \alpha, \alpha]$ is a K-rule, and we must analyse the following three cases: $\neg K$, $\wedge K$, $\Box K$.

$\neg K$:

$$\frac{{}^{\langle n-1\rangle}G[\beta, \Gamma, \neg\beta]}{{}^{\langle n\rangle}G[\Gamma, \neg\beta, \neg\beta]}\,\neg K \quad \dashrightarrow^{4} \quad \cfrac{\cfrac{{}^{\langle n-1\rangle}G[\beta, \beta, \Gamma]}{{}^{\langle n-1\rangle}G[\beta, \Gamma]}\,i.h.}{{}^{\langle n\rangle}G[\Gamma, \neg\beta]}\,\neg K$$

$\wedge K$:

$$\frac{{}^{\langle n-1\rangle}G[\Gamma, \beta, \beta\wedge\gamma] \quad {}^{\langle n-1\rangle}G[\Gamma, \gamma, \beta\wedge\gamma]}{{}^{\langle n\rangle}G[\Gamma, \beta\wedge\gamma, \beta\wedge\gamma]}\,\wedge K \quad \dashrightarrow$$

$$\frac{\dfrac{{}^{\langle n-1\rangle}G[\Gamma, \beta, \beta]}{{}^{\langle n-1\rangle}G[\Gamma, \beta]}\,i.h. \quad \dfrac{{}^{\langle n-1\rangle}G[\Gamma, \gamma, \gamma]}{{}^{\langle n-1\rangle}G[\Gamma, \gamma]}\,i.h.}{{}^{\langle n\rangle}G[\Gamma, \beta\wedge\gamma]}\,\wedge K$$

$\Box K$:

$$\frac{{}^{\langle n-1\rangle}G[\Gamma, \Box\beta/\Rightarrow\beta]}{{}^{\langle n\rangle}G[\Gamma, \Box\beta, \Box\beta]}\,\Box K \quad \dashrightarrow \quad \dfrac{\dfrac{\dfrac{{}^{\langle n-1\rangle}G[\Gamma/\Rightarrow\beta;\Rightarrow\beta]}{{}^{\langle n-1\rangle}G[\Gamma/\Rightarrow\beta, \beta]}\,merge}{{}^{\langle n-1\rangle}G[\Gamma/\Rightarrow\beta]}\,i.h.}{{}^{\langle n\rangle}G[\Gamma, \Box\beta]}\,\Box K$$

□

$$\frac{G[\Gamma]}{G[\alpha, \Gamma]}\,WA \qquad \frac{G[\Gamma]}{G[\Gamma, \alpha]}\,WK$$

$$\frac{G[\alpha, \alpha, \Gamma]}{G[\alpha, \Gamma]}\,CA \qquad \frac{G[\Gamma, \alpha, \alpha]}{G[\Gamma, \alpha]}\,CK$$

$$\frac{G[\Gamma]}{G[\Gamma/\Sigma]}\,EW \qquad \frac{G[\Delta/(\Gamma); (\Pi/\underline{X}')]}{G[\Delta/(\Gamma \centerdot \Pi/\underline{X}; \underline{X}')]}\,merge$$

$$\frac{G}{\Rightarrow /G}\,rn$$

$$\frac{G[\Gamma/(\Sigma/\underline{X})]}{G[\Gamma \centerdot \Sigma/\underline{X}]}\,\tilde{t} \qquad \frac{G[\Gamma/(\Sigma/(\Delta/\underline{X}); \underline{X}')]}{G[\Gamma \centerdot \Delta/(\Sigma/\underline{X}'); \underline{X}]}\,\tilde{b}$$

Fig. 6.1 Height-preserving admissible rules

$$\frac{G[\Gamma/\Rightarrow]}{G[\Gamma]}\,\tilde{d}$$

$$\frac{G[\Gamma/(\Sigma/\underline{X})]}{G[\Gamma/(\Rightarrow/\Sigma/\underline{X})]}\,\tilde{4} \qquad \frac{G[\Gamma/(\Sigma/(\Delta/\underline{X}); \underline{X}')]}{G[\Gamma/(\Delta/\underline{X}); (\Sigma/\underline{X}')]}\,\tilde{5}$$

Fig. 6.2 Admissible rules

Lemma 6.20 *Let $G[H]$ be any tree-hypersequent of the calculi* $\mathbf{Thsk}_L^*$, *and $G^*[H]$ the result of the application of one of the (height-preserving admissible) rules - rn, WA, WK, EW, merge, $\tilde{t}$, $\tilde{b}$, CA and CK - or of one of the (admissible) rules - $\tilde{d}$, $\tilde{4}$, $\tilde{5}$ - on $G[H]$. If, for a rule $\mathcal{R}$, we have*

$$\frac{G[H^{'}]}{G[H]}\ \mathcal{R}$$

then it holds that

$$\frac{G^*[H^{'}]}{G^*[H]}\ \mathcal{R}$$

Proof By induction on the form of the tree-hypersequent $G[H]$. □

Lemma 6.21 *Let $G[H]$ be any tree-hypersequent of the calculi* $\mathbf{Thsk}_L^*$, *and $G[H^{'}]$ the result of the application of one of the propositional rules or of the rule $\Box K$ on $G[H]$. If, for a rule $\mathcal{R}$, we have*

$$\frac{G^*[H^{'}]}{G[H^{'}]}\ \mathcal{R}$$

then it holds that

$$\frac{G^*[H]}{G[H]}\ \mathcal{R}$$

Proof By induction on the form of the tree-hypersequent $G[H^{'}]$. □

Note that the results obtained in this section (except for Lemma 6.18) are all summed up in Figures 6.1 and 6.2.

6.3 Adequacy of the Tree-Hypersequent Calculi

Let us show that the calculi $\mathbf{Thsk}_L^*$ prove exactly the same formulas as their corresponding systems, which will be referred to as $\mathbf{K}^*$.

Theorem 6.22 *For all tree-hypersequents G, and all formulas α,*

[i] if $\vdash \alpha$ in $\mathbf{K}^*$, then $\vdash\Rightarrow \alpha$ in $\mathbf{Thsk}_L^*$.
[ii] If $\vdash G$ in $\mathbf{Thsk}_L^*$, then $\vdash (G)^\tau$ in $\mathbf{K}^*$.

Proof By induction on the height of derivations in $\mathbf{K}^*$ and $\mathbf{Thsk}_L^*$, respectively. Concerning [ii], we omit the proof which is easy but quite tedious.[29] However we sketch the technique to develop this proof. We must prove that [iia] the translations of the initial tree-hypersequents are $\mathbf{K}^*$-theorems, and that [iib] the translations of the rules of the $\mathbf{Thsk}^*_L$ calculi hold in $\mathbf{K}^*$. As for [iia] this is trivial; as for [iib] the procedure is the following. We firstly isolate the sequent(s) affected by the rule, and we prove the corresponding implication. Secondly, we transport the implication up to the tree so that, by modus ponens, the desired result appears immediately.

In order to better be acquainted with the calculi $\mathbf{Thsk}^*_L$, let us verify [i]. The classical axioms and the modus ponens rule are derived as usual. We restrict the current presentation to a derivation of the distribution axiom, and of the axioms: D, T, 4, B and 5.

$\mathbf{Thsk}^*_L \vdash \Rightarrow \Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Box(\alpha\to\beta),\Box\alpha\Rightarrow/\alpha\Rightarrow\alpha,\beta \qquad \Box(\alpha\to\beta),\Box\alpha\Rightarrow/\alpha,\beta\Rightarrow\beta}{\Box(\alpha\to\beta),\Box\alpha\Rightarrow/\alpha\to\beta,\alpha\Rightarrow\beta}\to A}{\Box(\alpha\to\beta),\Box\alpha\Rightarrow/\alpha\Rightarrow\beta}\Box A}{\Box(\alpha\to\beta),\Box\alpha\Rightarrow/\Rightarrow\beta}\Box A}{\Box(\alpha\to\beta),\Box\alpha\Rightarrow\Box\beta}\Box K}{\Box(\alpha\to\beta)\Rightarrow\Box\alpha\to\Box\beta}\to K}{\Rightarrow\Box(\alpha\to\beta)\to\Box\alpha\to\Box\beta}\to K
$$

$\mathbf{Thskd}^*_L \vdash \Rightarrow \Box\alpha \to \neg\Box\neg\alpha$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Box\alpha,\Box\neg\alpha\Rightarrow/\alpha\Rightarrow\alpha}{\Box\alpha,\Box\neg\alpha\Rightarrow/\neg\alpha,\alpha\Rightarrow}\neg A}{\Box\alpha,\Box\neg\alpha\Rightarrow/\alpha\Rightarrow}\Box A}{\Box\alpha,\Box\neg\alpha\Rightarrow}d}{\Box\alpha\Rightarrow\neg\Box\neg\alpha}\neg K}{\Rightarrow\Box\alpha\to\neg\Box\neg\alpha}\to K
$$

$\mathbf{Thskt}^*_L \vdash \Rightarrow \Box\alpha \to \alpha$

$$
\dfrac{\dfrac{\Box\alpha,\alpha\Rightarrow\alpha}{\Box\alpha\Rightarrow\alpha}t}{\Rightarrow\Box\alpha\to\alpha}\to K
$$

$\mathbf{Thsk4}^*_L \vdash \Rightarrow \Box\alpha \to \Box\Box\alpha$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Box\alpha\Rightarrow/\Box\alpha\Rightarrow/\alpha\Rightarrow\alpha}{\Box\alpha\Rightarrow/\Box\alpha\Rightarrow/\Rightarrow\alpha}\Box A}{\Box\alpha\Rightarrow/\Box\alpha\Rightarrow\Box\alpha}\Box K}{\Box\alpha\Rightarrow/\Rightarrow\Box\alpha}4}{\Box\alpha\Rightarrow\Box\Box\alpha}\Box K}{\Rightarrow\Box\alpha\to\Box\Box\alpha}\to K
$$

Thskb$^*_L$ $\vdash \Rightarrow \alpha \rightarrow \Box\neg\Box\neg\alpha$

$$
\begin{array}{cl}
\alpha \Rightarrow \alpha / \Box\neg\alpha \Rightarrow & \\ \hline
\alpha, \neg\alpha \Rightarrow / \Box\neg\alpha \Rightarrow & \neg A \\ \hline
\alpha \Rightarrow / \Box\neg\alpha \Rightarrow & b \\ \hline
\alpha \Rightarrow / \Rightarrow \neg\Box\neg\alpha & \neg K \\ \hline
\alpha \Rightarrow \Box\neg\Box\neg\alpha & \Box K \\ \hline
\Rightarrow \alpha \rightarrow \Box\neg\Box\neg\alpha & \rightarrow K
\end{array}
$$

Thskb45$^*_L$ $\vdash \Rightarrow \neg\Box\neg\alpha \rightarrow \Box\neg\Box\neg\alpha$

$$
\begin{array}{cl}
\Box\neg\alpha \Rightarrow / \alpha \Rightarrow \alpha ; \Box\neg\alpha \Rightarrow & \\ \hline
\Box\neg\alpha \Rightarrow / \neg\alpha, \alpha \Rightarrow ; \Box\neg\alpha \Rightarrow & \neg A \\ \hline
\Box\neg\alpha \Rightarrow / \neg\alpha \Rightarrow \neg\alpha ; \Box\neg\alpha \Rightarrow & \neg K \\ \hline
\Box\neg\alpha \Rightarrow / \Rightarrow \neg\alpha ; \Box\neg\alpha \Rightarrow & \Box A \\ \hline
\Rightarrow / \Rightarrow \neg\alpha ; \Box\neg\alpha \Rightarrow & 5 \\ \hline
\Rightarrow / \Rightarrow \neg\alpha ; \Rightarrow \neg\Box\neg\alpha & \neg K \\ \hline
\Rightarrow \Box\neg\alpha / \Rightarrow \neg\Box\neg\alpha & \Box K \\ \hline
\Rightarrow \Box\neg\alpha, \Box\neg\Box\neg\alpha & \Box K \\ \hline
\neg\Box\neg\alpha \Rightarrow \Box\neg\Box\neg\alpha & \neg A \\ \hline
\Rightarrow \neg\Box\neg\alpha \rightarrow \Box\neg\Box\neg\alpha & \rightarrow K
\end{array}
$$

□

The characteristic axioms D, T, 4, B and 5 can also be proved by means of their corresponding *special structural rules*. Let us consider the example of the axioms 4 and B established by means of the rules $\tilde{4}$ and $\tilde{b}$, respectively.

$$
\begin{array}{cl}
\Box\alpha \Rightarrow / \alpha \Rightarrow \alpha & \\ \hline
\Box\alpha \Rightarrow / \Rightarrow \alpha & \Box A \\ \hline
\Box\alpha \Rightarrow / \Rightarrow / \Rightarrow \alpha & \tilde{4} \\ \hline
\Box\alpha \Rightarrow / \Rightarrow \Box\alpha & \Box K \\ \hline
\Box\alpha \Rightarrow \Box\Box\alpha & \Box K \\ \hline
\Rightarrow \Box\alpha \rightarrow \Box\Box\alpha & \rightarrow K
\end{array}
\qquad
\begin{array}{cl}
\Rightarrow / \Box\neg\alpha \Rightarrow / \alpha \Rightarrow \alpha & \\ \hline
\Rightarrow / \Box\neg\alpha \Rightarrow / \neg\alpha, \alpha \Rightarrow & \neg A \\ \hline
\Rightarrow / \Box\neg\alpha \Rightarrow / \alpha \Rightarrow & \Box A \\ \hline
\Rightarrow / \Rightarrow \neg\Box\neg\alpha / \alpha \Rightarrow & \neg K \\ \hline
\alpha \Rightarrow / \Rightarrow \neg\Box\neg\alpha & \tilde{b} \\ \hline
\alpha \Rightarrow \Box\neg\Box\neg\alpha & \Box K \\ \hline
\Rightarrow \alpha \rightarrow \Box\neg\Box\neg\alpha & \rightarrow K
\end{array}
$$

More precisely, in [16] it is shown that tree-hypersequent calculi composed by generalised initial tree-hypersequents, propositional rules, modal rules, special structural rules and contraction rules are sound and complete with respect to their corresponding Hilbert systems. Moreover they are cut-free and modular.

Notes

1. The symbol $\rightsquigarrow$ means: the premise of the right side is obtained by applying the inductive hypothesis to the premise of the left side.
2. Note that we can use the rule 5, since we are in a calculus where both the rule 4 and the rule b are present.
3. Ditto.
4. The symbol $\dashrightarrow$ means: the premise of the right side is obtained by applying Lemma 6.18 to the premise of the left side.
5. In Chapter 8, the reader can find a detailed semantic proof of soundness.

Chapter 7
Syntactic Cut-Admissibility and Decidability

In Section 6.2 we proved that the weakening (internal and external) rules, the contraction rules, the rule of merge, the rule of necessitation and the special structural rules are all (height-preserving) admissible in the calculi $\mathbf{Thsk}_L^*$. These results are satisfactory both from a technical and from a conceptual point of view. The central question remains open: is the cut-rule admissible in the tree-hypersequent calculi? The answer is affirmative and it will be presented in the next section. More precisely, we will provide an algorithm for transforming derivations involving the cut-rule to derivations which are cut-free.

The second result, that will take up the remainder of this chapter, is the decidability of (certain) tree-hypersequent calculi. Once again, we shall develop proofs in a purely syntactic and constructive way.

7.1 Cut-Admissibility in the Tree-Hypersequent Calculi

This section will offer a proof that the cut-rule is admissible in the $\mathbf{Thsk}_L^*$ calculi. Consequently, we must first show the following lemma.

Lemma 7.1 *Given three zoom tree-hypersequents* $I[*]$, $J[*]$ *and* $H[*]$ *such that* $I[*] \sim J[*] \sim H[*]$, *if there is a rule* $\mathcal{R}$ *of* $\mathbf{Thsk}_L^*$ *and a sequent* Γ *such that*

$$\frac{J[\Gamma]}{I[\Gamma]}\ \mathcal{R}$$

then, for any Δ, *we have that*

$$\frac{J \otimes H[\Delta]}{I \otimes H[\Delta]}\ \mathcal{R}$$

Proof By induction on the form of the tree-hypersequents $I[*]$, $J[*]$ and $H[*]$.

(A) $I[*]$, $J[*]$ and $H[*] \equiv *$. Nothing to prove.
(B) $I[*]$, $J[*]$ and $H[*] \equiv$ resp. $*/\underline{X}$, $*/\underline{Y}$ and $*/\underline{Z}$. In this case, if we have

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$$\frac{\Gamma/\underline{X}}{\Gamma/\underline{Y}}\,\mathcal{R}$$

we want, for any Δ,

$$\frac{\Delta/\underline{X};\,\underline{Z}}{\Delta/\underline{Y};\,\underline{Z}}\,\mathcal{R}$$

This case is easily solvable since $\underline{X}$ and $\underline{Z}$, and $\underline{Y}$ and $\underline{Z}$, are kept completely separate.
(C)

$I[*]\colon \Phi/I'[*];\,\underline{X}$
$J[*]\colon \Phi'/J'[*];\,\underline{X}'$
$H[*]\colon \Phi''/H'[*];\,\underline{X}''$

Let

$$\frac{\Phi/I'[\Gamma];\,\underline{X}}{\Phi'/J'[\Gamma];\,\underline{X}'}\,\mathcal{R}$$

We want, for any Δ, that

$$\frac{\Phi\,\centerdot\,\Phi''/I'\otimes H'[\Delta];\,\underline{X};\,\underline{X}''}{\Phi'\,\centerdot\,\Phi''/J'\otimes H'[\Delta];\,\underline{X}';\,\underline{X}''}\,\mathcal{R}$$

We shall distinguish several subcases.
(C.1.) The rule $\mathcal{R}$ operates on Φ. We shall again distinguish between:

(C.1.1.) the rule $\mathcal{R}$ only operates on Φ,
(C.1.2.) the rule $\mathcal{R}$ operates between Φ and $I'[\Gamma];\,\underline{X}$.

(C.2.) The rule $\mathcal{R}$ operates on $\underline{X}$.
(C.3.) The rule $\mathcal{R}$ operates on I'.

We shall examine each of these subcases, starting with (C.1.1.).

(C.1.1.) Let us suppose that $\mathcal{R}$ is the one premise rule $\neg K$ (for the other rules the proceeding is analogous), we have

$$\frac{\beta,\,\Phi/I'[\Gamma];\,\underline{X}}{\Phi,\,\neg\beta/I'[\Gamma];\,\underline{X}}\,\neg K$$

Then, for any Δ, if we have as premise $\beta,\,\Phi\,\centerdot\,\Phi''/I'\otimes H'[\Delta];\,\underline{X};\,\underline{X}''$, we obtain the conclusion by applying the rule $\neg K$:

$$\frac{\beta,\,\Phi\,\centerdot\,\Phi''/I'\otimes H'[\Delta];\,\underline{X};\,\underline{X}''}{\Phi\,\centerdot\,\Phi'',\,\neg\beta/I'\otimes H'[\Delta];\,\underline{X};\,\underline{X}''}\,\neg K$$

(C.1.2.) Modal rules and special logical rules (except the rule t) are the only rules that can operate between Φ and $I^{'}[\Gamma]; \underline{X}$. We will analyse each of them in turn, starting with the modal rules.

[–] $\mathcal{R}$ is the rule $\Box K$. We must distinguish two subcases: (i) $I^{'}[\Gamma]; \underline{X}$ is of the form $\Rightarrow \beta; I^{'}[\Gamma]; \underline{X}^{'''}$ (i.e. $\Rightarrow \beta$ is one of the tree-hypersequents of the multiset $\underline{X}$); (ii) $I^{'}[\Gamma]; \underline{X}$ is of the form $\Rightarrow \beta; I^{'''}[\Gamma]; \underline{X}$ (i.e. $\Rightarrow \beta$ is included in the tree-hypersequent $I^{'}[\Gamma]$). Since these two subcases are similar, we will deal with just one of them, (i), and the other can be solved analogously. Suppose that we have

$$\frac{\Phi / \Rightarrow \beta; I^{'}[\Gamma]; \underline{X}^{'''}}{\Phi, \Box\beta / I^{'}[\Gamma]; \underline{X}^{'''}} \Box K$$

Then, for any Δ, if we have as premise $\Phi \centerdot \Phi^{''} / \Rightarrow \beta; I^{'} \otimes H^{'}[\Delta]; \underline{X}^{''}; \underline{X}^{'''}$, we obtain the conclusion by applying the rule $\Box K$:

$$\frac{\Phi \centerdot \Phi^{''} / \Rightarrow \beta; I^{'} \otimes H^{'}[\Delta]; \underline{X}^{''}; \underline{X}^{'''}}{\Phi \centerdot \Phi^{''}, \Box\beta / I^{'} \otimes H^{'}[\Delta]; \underline{X}^{''}; \underline{X}^{'''}} \Box K$$

[–] $\mathcal{R}$ is the rule $\Box A$. We must distinguish two subcases: (i) the case where $\Box A$ has been applied between Φ and $\underline{X}$, and (ii) the case where $\Box A$ has been applied between Φ and $I^{'}[\Gamma]$. Since these two subcases are similar, we will deal with just one of them, (i), and the other can be solved analogously. In order to show the resolution of (i), we assume that $\underline{X} \equiv (\beta, \Sigma / \underline{X}^{'}); \underline{X}^{'''}$ and we have

$$\frac{\Box\beta, \Phi / I^{'}[\Gamma]; (\beta, \Sigma / \underline{X}^{'}); \underline{X}^{'''}}{\Box\beta, \Phi / I^{'}[\Gamma]; (\Sigma / \underline{X}^{'}); \underline{X}^{'''}} \Box A$$

Then, for any Δ, if we have as premise, $\Box\beta, \Phi \centerdot \Phi^{''} / I^{'} \otimes H^{'}[\Delta]; (\beta, \Sigma / \underline{X}^{'}); \underline{X}^{'''}; \underline{X}^{''}$, we obtain the conclusion by applying the rule $\Box A$:

$$\frac{\Box\beta, \Phi \centerdot \Phi^{''} / I^{'} \otimes H^{'}[\Delta]; (\beta, \Sigma / \underline{X}^{'}); \underline{X}^{'''}; \underline{X}^{''}}{\Box\beta, \Phi \centerdot \Phi^{''} / I^{'} \otimes H^{'}[\Delta]; (\Sigma / \underline{X}^{'}); \underline{X}^{'''}; \underline{X}^{''}} \Box A$$

[–] If $\mathcal{R}$ is the rule d, the case can be dealt with similarly to the one where $\mathcal{R}$ is the rule $\Box K$.

[–] If $\mathcal{R}$ is the rule 4 or b or 5, the case can be dealt with similarly to the one where $\mathcal{R}$ is the rule $\Box A$.

(C.2.) This subcase can be dealt with similarly to the case (B).

(C.3.) In order to solve this subcase, we should analyse again $I^{'}[\Gamma]$, $J^{'}[\Gamma]$, $H^{'}[\Gamma]$. Let us suppose that $I^{'}[\Gamma]$, $J^{'}[\Gamma]$ and $H^{'}[\Gamma] \equiv \Gamma$. Then there is noting to prove. Let us suppose that $I^{'}[\Gamma]$, $J^{'}[\Gamma]$ and $H^{'}[\Gamma] \equiv \Gamma / \underline{W}$, $\Gamma / \underline{Y}$ and $\Gamma / \underline{Z}$, respectively. Then we have

$$\frac{\Phi / (\Gamma / \underline{W}); \underline{X}}{\Phi / (\Gamma / \underline{Y}); \underline{X}} \mathcal{R}$$

and we want, for any Δ, that

$$\dfrac{\Phi \centerdot \Phi''/(\Delta/\underline{W}; \underline{Z}); \underline{X}; \underline{X}''}{\Phi \centerdot \Phi''/(\Delta/\underline{Y}; \underline{Z}); \underline{X}; \underline{X}''}\ \mathcal{R}$$

By applying the inductive hypothesis to $I'[\Gamma]$, $J'[\Gamma]$ and $H'[\Gamma]$, we obtain that, for any Δ, $\Delta/\underline{Y}; \underline{Z}$, which is to say: $\Phi/(\Delta/\underline{Y}; \underline{Z}); \underline{X}$. Thanks to several applications of the rules of internal and external weakening, we have $\Phi \centerdot \Phi''/(\Delta/\underline{Y}; \underline{Z}); \underline{X}; \underline{X}''$.

Finally let us suppose that

$I[\Gamma]$: $\Sigma/I''[\Gamma]; \underline{Z}$
$J[\Gamma]$: $\Sigma'/J''[\Gamma]; \underline{Z}'$
$H[\Gamma]$: $\Sigma''/H''[\Gamma]; \underline{Z}''$

Then we have

$$\dfrac{\Phi/(\Sigma/I''[\Gamma]; \underline{Z}); \underline{X}}{\Phi/(\Sigma'/J''[\Gamma]; \underline{Z}'); \underline{X}}\ \mathcal{R}$$

and we want, for any Δ, that

$$\dfrac{\Phi \centerdot \Phi''/(\Sigma \centerdot \Sigma''/I'' \otimes H''[\Delta]; \underline{Z}; \underline{Z}''); \underline{X}; \underline{X}''}{\Phi \centerdot \Phi''/(\Sigma' \centerdot \Sigma''/J'' \otimes H''[\Delta]; \underline{Z}'; \underline{Z}''); \underline{X}; \underline{X}''}\ \mathcal{R}$$

By applying the inductive hypothesis to $I'[\Gamma]$, $J'[\Gamma]$ and $H'[\Gamma]$, we obtain that, for any Δ, $\Sigma' \centerdot \Sigma''/J'' \otimes H''[\Delta]; \underline{Z}'; \underline{Z}''$, which is to say: $\Phi/(\Sigma' \centerdot \Sigma''/J'' \otimes H''[\Delta]; \underline{Z}'; \underline{Z}''); \underline{X}; \underline{X}''$. Thanks to several applications of the rules of internal and external weakening, we have $\Phi \centerdot \Phi''/(\Sigma' \centerdot \Sigma''/J'' \otimes H''[\Delta]; \underline{Z}'; \underline{Z}''); \underline{X}; \underline{X}''$. □

Lemma 7.2 *Let* $G[\Gamma, \alpha]$ *and* $G'[\alpha, \Pi]$ *be such that* $G[\Gamma, \alpha] \sim G'[\alpha, \Pi]$. *If*

$$\dfrac{\overset{\vdots\, d_1}{G[\Gamma, \alpha]} \quad \overset{\vdots\, d_2}{G[\alpha, \Pi]}}{G \otimes G'[\Gamma \centerdot \Pi]}\ cut_\alpha$$

and d_1 *and* d_2 *do not contain any other application of the cut-rule, then we can construct a derivation of* $G \otimes G'[\Gamma \centerdot \Pi]$ *with no application of the cut-rule.*

Proof The proof is by induction on the complexity of the cut-formula $cmp(\alpha)$ (see Definition 2.3, p. 40), with subinduction on the sum of the heights of the derivations of the premises of the cut-rule. We will distinguish cases according to the last rule applied to the left premise.

Case 1. $G[\Gamma, \alpha]$ is an initial tree-hypersequent. Then either the conclusion is also a tree-hypersequent, or the cut can be replaced by various applications of the internal and external weakening rules to $G^{'}[\alpha, \Pi]$.

Case 2. $G[\Gamma, \alpha]$ is inferred by a rule $\mathcal{R}$ in which α is not principal. This case can be standardly solved by induction on the sum of the heights of the derivations of the premises of the cut-rule. As a matter of fact, no rule can change the position of the sequent where the cut occurs, and, on the other hand, the definition of product ensures that the structure of the tree-hypersequent stays unchanged, so that any problems are avoided. Some examples will help to clarify. More precisely, we will analyse those significant cases where the rule $\mathcal{R}$ has been applied on the sequent Γ, α. Note that the cases where $\mathcal{R}$ has been applied on a sequent different from Γ, α can be dealt with analogously, thanks to Lemma 7.1.

Let us suppose that the rule before $G[\Gamma, \alpha]$ is the rule $\Box K$ (the case where $\mathcal{R}$ is the rule d is analogous) applied to the sequent Γ, α and without α as principal formula.

$$\dfrac{\dfrac{G[\Gamma, \alpha / \Rightarrow \beta]}{G[\Gamma, \alpha, \Box\beta]}\,\Box K \qquad \begin{matrix}\vdots\\ G^{'}[\alpha, \Pi]\end{matrix}}{G \otimes G^{'}[\Gamma \centerdot \Pi, \Box\beta]}\,cut_{\alpha}$$

We reduce to

$$\dfrac{\dfrac{G[\Gamma, \alpha / \Rightarrow \beta] \qquad G^{'}[\alpha, \Pi]}{G \otimes G^{'}[\Gamma \centerdot \Pi / \Rightarrow \beta]}\,cut_{\alpha}}{G \otimes G^{'}[\Gamma \centerdot \Pi, \Box\beta]}\,\Box K$$

Let us suppose that the rule before $G[\Gamma, \alpha]$ is the rule $\Box A$ (the cases where $\mathcal{R}$ is the rule t, b, 4 or 5 are analogous) applied between the sequent Γ, α and the sequent immediately successive to it, and without α as principal formula.

$$\dfrac{\dfrac{\dot{G}[\Box\beta, \Gamma, \alpha / (\beta, \Sigma / \underline{X})]}{\dot{G}[\Box\beta, \Gamma, \alpha / (\Sigma / \underline{X})]}\,\Box A \qquad \begin{matrix}\vdots\\ G^{'}[\alpha, \Pi]\end{matrix}}{\dot{G} \otimes G^{'}[\Box\beta, \Gamma \centerdot \Pi / (\Sigma / \underline{X})]}\,cut_{\alpha}$$

We reduce to

$$\dfrac{\dfrac{\dot{G}[\Box\beta, \Gamma, \alpha / (\beta, \Sigma / \underline{X})] \qquad G^{'}[\alpha, \Pi]}{\dot{G} \otimes G^{'}[\Box\beta, \Gamma \centerdot \Pi / (\beta, \Sigma / \underline{X})]}\,cut_{\alpha}}{\dot{G} \otimes G^{'}[\Box\beta, \Gamma \centerdot \Pi / (\Sigma / \underline{X})]}\,\Box A$$

Case 3. $G[\Gamma, \alpha]$ is inferred by a rule $\mathcal{R}$ in which α is principal. We can distinguish two subcases: in one subcase $\mathcal{R}$ is a propositional rule, in the other $\mathcal{R}$ is a modal rule.
Case 3.1. Supposing, for illustration, that the rule that introduces $G[\Gamma, \alpha]$ is $\neg K$ and $\alpha \equiv \neg\beta$, we have

$$\dfrac{\dfrac{G[\beta, \Gamma]}{G[\Gamma, \neg\beta]}\ \neg K \qquad \begin{matrix}\vdots \\ G^{'}[\neg\beta, \Pi]\end{matrix}}{G \otimes G^{'}[\Gamma \centerdot \Pi]}\ cut_{\neg\beta}$$

By applying Lemma 6.18, p. 135 to $G^{'}[\neg\beta, \Pi]$, we obtain $G^{'}[\Pi, \beta]$. We replace the previous cut with the following, which is eliminable by induction on the complexity of the cut-formula:

$$\dfrac{G^{'}[\Pi, \beta] \quad G[\beta, \Gamma]}{G \otimes G^{'}[\Gamma \centerdot \Pi]}\ cut_{\beta}$$

Case 3.2. $\mathcal{R}$ is $\Box K$ and $\alpha \equiv \Box\beta$. We have the following situation:

$$\dfrac{\dfrac{G[\Gamma / \Rightarrow \beta]}{G[\Gamma, \Box\beta]}\ \Box K \qquad \begin{matrix}\vdots \\ G^{'}[\Box\beta, \Pi]\end{matrix}}{G \otimes G^{'}[\Gamma \centerdot \Pi]}\ cut_{\Box\beta}$$

We must consider the last rule $\mathcal{R}^{'}$ of d_2. If no rule $\mathcal{R}^{'}$ introduces $G^{'}[\Box\beta, \Pi]$ because $G^{'}[\Box\beta, \Pi]$ is an initial tree-hypersequent, then we can solve the case as in 1. If $\Box\beta$ is not principal in the rule $\mathcal{R}^{'}$, then we solve the case as in 2. Only those cases where $\mathcal{R}^{'}$ is one of the following rules: $\Box A$, d, t, 4, b, 5, are problematic. We will analyse each of them in turn.

$\Box A$:

$$\dfrac{\dfrac{G[\Gamma / \Rightarrow \beta]}{G[\Gamma, \Box\beta]}\ \Box K \qquad \dfrac{\dot{G}^{'}[\Box\beta, \Pi / (\beta, \Psi / \underline{Y})]}{\dot{G}^{'}[\Box\beta, \Pi / (\Psi / \underline{Y})]}\ \Box A}{G \otimes \dot{G}^{'}[\Gamma \centerdot \Pi / (\Psi / \underline{Y})]}\ cut_{\Box\beta}$$

We reduce to[1]

$$\dfrac{\dfrac{G[\Gamma / \Rightarrow \beta] \qquad \dfrac{G[\Gamma, \Box\beta] \quad \dot{G}^{'}[\Box\beta, \Pi / (\beta, \Psi / \underline{Y})]}{G \otimes \dot{G}^{'}[\Gamma \centerdot \Pi / (\beta, \Psi / \underline{Y})]}\ cut_{\Box\beta}}{G \otimes G \otimes \dot{G}^{'}[\Gamma \centerdot \Gamma \centerdot \Pi / (\Psi / \underline{Y})]}\ cut_{\beta}}{G \otimes \dot{G}^{'}[\Gamma \centerdot \Pi / (\Psi / \underline{Y})]}\ CA^{*}{+}CK^{*}{+}merge^{*}$$

where the first cut is eliminable by induction on the sum of the heights of the derivations of the premises of the cut-rule, and the second cut is eliminable by induction on the complexity of the cut-formula.

d:

$$\dfrac{\dfrac{G[\Gamma/\Rightarrow \beta]}{G[\Gamma,\Box\beta]}\,\Box K \qquad \dfrac{G'[\Box\beta,\Pi/\beta\Rightarrow]}{G'[\Box\beta,\Pi]}\,d}{G\otimes G'[\Gamma\centerdot\Pi]}\,cut_{\Box\beta}$$

We reduce to

$$\dfrac{\dfrac{\dfrac{G[\Gamma/\Rightarrow\beta] \qquad \dfrac{G[\Gamma,\Box\beta]\quad G'[\Box\beta,\Pi/\beta\Rightarrow]}{G\otimes G'[\Gamma\centerdot\Pi/\beta\Rightarrow]}\,cut_{\Box\beta}}{G\otimes G\otimes G'[\Gamma\centerdot\Gamma\centerdot\Pi/\Rightarrow]}\,cut_{\beta}}{G\otimes G\otimes G'[\Gamma\centerdot\Gamma\centerdot\Pi]}\,\tilde{d}}{G\otimes G'[\Gamma\centerdot\Pi]}\,CA^*{+}CK^*{+}merge^*$$

where the first cut is eliminable by induction on the sum of the heights of the derivations of the premises of the cut-rule, and the second cut is eliminable by induction on the complexity of the cut-formula. Note that, in order to obtain the conclusion of the cut-rule, we use the admissible rule $\tilde{d}$ (see Fig. 6.2, p. 137).

t:

$$\dfrac{\dfrac{G[\Gamma/\Rightarrow \beta]}{G[\Gamma,\Box\beta]}\,\Box K \qquad \dfrac{G'[\Box\beta,\beta,\Pi]}{G'[\Box\beta,\Pi]}\,t}{G\otimes G'[\Gamma\centerdot\Pi]}\,cut_{\Box\beta}$$

We reduce to

$$\dfrac{\dfrac{\dfrac{G[\Gamma/\Rightarrow\beta]}{G[\Gamma,\beta]}\,\tilde{t} \qquad \dfrac{G[\Gamma,\Box\beta]\quad G'[\Box\beta,\beta,\Pi]}{G\otimes G'[\beta,\Gamma\centerdot\Pi]}\,cut_{\Box\beta}}{G\otimes G\otimes G'[\Gamma\centerdot\Gamma\centerdot\Pi]}\,cut_{\beta}}{G\otimes G'[\Gamma\centerdot\Pi]}\,CA^*{+}CK^*{+}merge^*$$

where the first cut is eliminable by induction on the sum of the heights of the derivations of the premises, and the second cut is eliminable by induction on the complexity of the cut-formula. Note that, in order to cut the formula β, we must use the height-preserving admissible rule $\tilde{t}$ (see Fig. 6.1, p. 137).

b:

$$\dfrac{\dfrac{\ddot{G}[\Delta/(\Gamma/\Rightarrow\beta;\underline{X}')]}{\ddot{G}[\Delta/(\Gamma,\Box\beta/\underline{X}')]}\,\Box K \qquad \dfrac{\ddot{G}'[\beta,\Phi/(\Box\beta,\Pi/\underline{Y}')]}{\ddot{G}'[\Phi/(\Box\beta,\Pi/\underline{Y}')]}\,b}{\ddot{G}\otimes\ddot{G}'[\Delta\centerdot\Phi/(\Gamma\centerdot\Pi/\underline{X}';\underline{Y}')]}\,cut_{\Box\beta}$$

We reduce to

$$\dfrac{\dfrac{\dfrac{\ddot{G}[\Delta/(\Gamma/\Rightarrow\beta;\underline{X}^{'})]}{\ddot{G}[\Delta,\beta/(\Gamma/\underline{X}^{'})]}\ \tilde{b} \qquad \dfrac{\ddot{G}[\Delta/(\Gamma,\Box\beta/\underline{X}^{'})] \quad \ddot{G}^{'}[\beta,\Phi/(\Box\beta,\Pi/\underline{Y}^{'})]}{\ddot{G}\otimes\ddot{G}^{'}[\beta,\Delta\centerdot\Phi/(\Gamma\centerdot\Pi/\underline{X}^{'};\underline{Y}^{'})]}\ cut_{\Box\beta}}{\ddot{G}\otimes\ddot{G}\otimes\ddot{G}^{'}[\Delta\centerdot\Delta\centerdot\Phi/(\Gamma\centerdot\Pi/\underline{X}^{'};\underline{Y}^{'});(\Gamma/\underline{X}^{'})]}\ cut_{\beta}}{\ddot{G}\otimes\ddot{G}^{'}[\Delta\centerdot\Phi/(\Gamma\centerdot\Pi/\underline{X}^{'};\underline{Y}^{'})]}\ CA^{*}{+}CK^{*}{+}merge^{*}$$

where the first cut is eliminable by induction on the sum of the heights of the derivations of the premises of the cut-rule, and the second cut is eliminable by induction on the complexity of cut-formula. Note that, in order to cut the formula β, we must use the height-preserving admissible rule $\tilde{b}$ (see Fig. 6.1, p. 137).

We have left the cases where $\mathcal{R}$ is the rule 4 or the rule 5 for last, since they both require a similar and more complicated technique compared to the one that was used for the other rules. Below is the analysis of each of the cases.

4:

$$\dfrac{\dfrac{G[\Gamma/\Rightarrow\beta]}{G[\Gamma,\Box\beta]}\ \Box K \qquad \dfrac{\dot{G}^{'}[\Box\beta,\Pi/(\Box\beta,\Psi/\underline{Y})]}{\dot{G}^{'}[\Box\beta,\Pi/(\Psi/\underline{Y})]}\ 4}{G\otimes\dot{G}^{'}[\Gamma\centerdot\Pi/(\Psi/\underline{Y})]}\ cut_{\Box\beta}$$

In order to solve this case, we must analyse each of the rules that may have introduced the tree-hypersequent $\dot{G}^{'}[\Box\beta,\Pi/(\Box\beta,\Psi/\underline{Y})]$. We go up the derivation until either a rule $\mathcal{R}^{''}$ applies to a formula different from the $\Box\beta$'s, or a rule $\mathcal{R}^{''}$ different from 4 applies to some of the $\Box\beta$'s; this way we stop in front of the following situation (i.e. this way we stop in front of the tree-hypersequent that is the conclusion of the rule $\mathcal{R}^{''}$):

$$\circledcirc \quad \dot{G}^{'}[\Box\beta,\Pi/(\Box\beta,\Psi/(\Box\beta)\underline{Y})]$$

where with $(\Box\beta)\underline{Y}$ we indicate all the formulas $\Box\beta$ that can occur on the left side of the sequents that are on the same branch of the sequent Π, and belong to the multiset of tree-hypersequents $\underline{Y}$.

We analyse each of the rules that can have inferred this tree-hypersequent.

- $\circledcirc$ is an axiom. Then, as $\Box\beta$ cannot be principal, even the conclusion of the cut is an axiom, and the case is solved.
- $\circledcirc$ has been inferred by a rule $\mathcal{R}^{''}$ that does not have any $\Box\beta$ as principal formula. In this case the technique consists of (i) applying the rule 4, n-times, to the premise of the rule $\mathcal{R}^{''}$, and (ii) operating as in case 2.
- $\circledcirc$ has been inferred by a rule $\mathcal{R}^{''}$ that has one of the $\Box\beta$'s as principal formula. $\mathcal{R}^{''}$ can either be the rule $\Box A$ or one of the special logical rules. The technique for solving the cases where $\mathcal{R}^{''}$ is the rule $\Box A$ or the rule d or the rule t are analogous; therefore, we will only show in detail the case where $\mathcal{R}^{''}$ is the rule

$\Box A$ (the others can be dealt with similarly[2]). We will conclude by also analysing the cases where $\mathcal{R}''$ is the rule b or the rule 5.

- Let us suppose that $\mathcal{R}''$ is the rule $\Box A$. We shall first of all distinguish the following two subcases. (a) The rule $\Box A$ has been applied to two sequents belonging to $(\Box\beta), \underline{Y}$, let us suppose the sequents $\Box\beta, \Xi/\Box\beta, \beta, \Omega$. Hence we have the following situation:

$$\dfrac{\dfrac{G[\Gamma/\Rightarrow\beta]}{G[\Gamma,\Box\beta]}\,\Box K \qquad \dfrac{\dfrac{\vdots}{\dfrac{\dfrac{\overset{\dots}{G}'[\Box\beta,\Pi][\Box\beta,\Xi/(\Box\beta,\beta,\Omega/\underline{Y}'')]}{\overset{\dots}{G}'[\Box\beta,\Pi][\Box\beta,\Xi/(\Box\beta,\Omega/\underline{Y}'')]}\,\Box A}{}\,4}}{\overset{\dots}{G}'[\Box\beta,\Pi][\Xi/(\Omega/\underline{Y}'')]}\,4}{G\otimes\overset{\dots}{G}'[\Gamma\centerdot\Pi][\Xi/(\Omega/\underline{Y}'')]}\,cut_{\Box\beta}$$

We proceed in three steps.

(a1) We apply the rule 4, n-times, to the tree-hypersequent $\overset{\dots}{G}'[\Box\beta,\Pi][\Box\beta,\Xi/(\Box\beta,\beta,\Omega/\underline{Y}'')]$. As a result, we obtain the tree-hypersequent

$$\overset{\dots}{G}'[\Box\beta,\Pi][\Xi/(\beta,\Omega/\underline{Y}'')]$$

(a2) We apply the admissible rule $\tilde{4}$ (see Fig. 6.2, p. 137) to the tree-hypersequent $G[\Gamma/\Rightarrow\beta]$ a number of times sufficient to get $\Rightarrow\beta$ in an equivalent position with the sequent β, Ω of the tree-hypersequent $\overset{\dots}{G}'[\Box\beta,\Pi][\Xi/(\beta,\Omega/\underline{Y}'')]$. This way we obtain a tree-hypersequent where $\Rightarrow\beta$ is no longer after Γ, but n empty sequences after. Let us note this as $G[\Gamma][\Rightarrow\beta]$.

(a3) We are now in a position to apply two cuts: the first eliminable by induction on the sum of the heights, the second by induction on the complexity of the cut-formula.

$$\dfrac{\dfrac{G[\Gamma][\Rightarrow\beta] \qquad \dfrac{G[\Gamma,\Box\beta] \quad \overset{\dots}{G}'[\Box\beta,\Pi][\Xi/(\beta,\Omega/\underline{Y}'')]}{G\otimes\overset{\dots}{G}'[\Gamma\centerdot\Pi][\Xi/(\beta,\Omega/\underline{Y}'')]}\,cut_{\Box\beta}}{G\otimes G\otimes\overset{\dots}{G}'[\Gamma\centerdot\Gamma\centerdot\Pi][\Xi/(\Omega/\underline{Y}'')]}\,cut_{\beta}}{G\otimes\overset{\dots}{G}'[\Gamma\centerdot\Pi][\Xi/(\Omega/\underline{Y}'')]}\,CA^*{+}CK^*{+}merge^*$$

(b) The rule $\Box A$, with $\Box\beta$ as principal formula, has been applied to the sequents $\Box\beta, \Pi/\Box\beta, \beta, \Psi$. In this case we firstly proceed as in the subcase (a1) (i.e. we use n times the rule 4), and then as in the case $\Box A$ above.

- Let us now suppose that $\mathcal{R}''$ is the rule b. We shall first of all distinguish the following three subcases. (a) The rule b has been applied to a pair of sequents belonging to $(\Box\beta), \underline{Y}$. This case can be dealt with similarly to the subcase (a) above. (b) The rule b, with $\Box\beta$ as principal formula, has been applied to the sequents $\beta, \Phi/\Box\beta, \Pi$. This case can be dealt with similarly to the subcase (b) above. (c) The rule b has been applied to the sequents $\Box\beta, \beta, \Pi/\Box\beta, \Psi$. We have the following situation:

$$\dfrac{\dfrac{G[\Gamma/\Rightarrow\beta]}{G[\Gamma,\Box\beta]}\,\Box K \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dot{G}'[\Box\beta,\beta,\Pi/(\Box\beta,\Psi/(\Box\beta)\ \underline{Y})]}{\dot{G}'[\Box\beta,\Pi/(\Box\beta,\Psi/(\Box\beta)\ \underline{Y})]}\,b}{\vdots}\,4}{\dot{G}'[\Box\beta,\Pi/(\Psi/\underline{Y})]}\,4}{G\otimes\dot{G}'[\Gamma\centerdot\Pi/(\Psi/\underline{Y})]}\,cut_{\Box\beta}$$

We proceed in the following way. First of all, we apply the rule 4, n-times, to the tree-hypersequent $\dot{G}'[\Box\beta, \beta, \Pi/(\Box\beta, \Psi/(\Box\beta)\ \underline{Y})]$, obtaining this way the tree-hypersequent $\dot{G}'[\Box\beta, \beta, \Pi/(\Psi/\underline{Y})]$. Then we continue the derivation with two cuts: the first eliminable by induction on the sum of the heights, the second by induction on the complexity of the cut-formula.

$$\dfrac{\dfrac{\dfrac{\dfrac{G[\Gamma/\Rightarrow\beta]}{G[\Gamma/(\Rightarrow/\Rightarrow\beta)]}\,\tilde{4}}{G[\Gamma,\beta/\Rightarrow]}\,\tilde{b} \qquad \dfrac{G[\Gamma,\Box\beta] \qquad \dot{G}'[\Box\beta,\beta,\Pi/(\Psi/\underline{Y})]}{G\otimes\dot{G}'[\beta,\Gamma\centerdot\Pi/(\Psi/\underline{Y})]}\,cut_{\Box\beta}}{G\otimes G\otimes\dot{G}'[\Gamma\centerdot\Gamma\centerdot\Pi/(\Psi/\underline{Y});\Rightarrow]}\,cut_{\beta}}{G\otimes\dot{G}'[\Gamma\centerdot\Pi/(\Psi/\underline{Y})]}\,CA^*{+}CK^*{+}merge^*$$

- Let us finally suppose that $\mathcal{R}''$ is the rule 5. We shall distinguish the following two subcases: (a) the rule 5 has been applied to a pair of sequents belonging to $(\Box\beta), \underline{Y}$, let us suppose the sequents $\Box\beta, \Box\beta, \Xi/\Box\beta, \Omega$. Hence we have the following situation:

$$\dfrac{\dfrac{G[\Gamma/\Rightarrow\beta]}{G[\Gamma,\Box\beta]}\,\Box K \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dddot{G}'[\Box\beta,\Pi]\,[\Box\beta,\Box\beta,\Xi/(\Box\beta,\Omega/\underline{Y}'')]}{\dddot{G}'[\Box\beta,\Pi]\,[\Box\beta,\Xi/(\Box\beta,\Omega/\underline{Y}'')]}\,5}{\vdots}\,4}{\dddot{G}'[\Box\beta,\Pi]\,[\Xi/(\Omega/\underline{Y}'')]}\,4}{G\otimes\dddot{G}'[\Gamma\centerdot\Pi]\,[\Xi/(\Omega/\underline{Y}'')]}\,cut_{\Box\beta}$$

By applying the height-preserving admissible rule of contraction CA once, and n times the rule 4, we obtain a derivation of lower height of the tree-hypersequent

$\dddot{G}'[\Box\beta, \Pi]\,[\Xi/(\Omega/\underline{Y}'')]$. Therefore we are in a position to apply the following cut, which is eliminable by induction on the sum of the heights:

$$\dfrac{G[\Gamma, \Box\beta] \quad \dddot{G}'[\Box\beta, \Pi]\,[\Xi/(\Omega/\underline{Y}'')]}{G \otimes \dddot{G}'[\Gamma \,.\, \Pi]\,[\Xi/(\Omega/\underline{Y}'')]}\; cut_{\Box\beta}$$

(d) The rule 5, with $\Box\beta$ as principal formula, has been applied to the sequents $\Box\beta, \Phi/\Box\beta, \Pi$. In this case we apply the rule 4 n-time, and then we proceed as in the following case 5.

5:

$$\dfrac{\dfrac{\ddot{G}[\Delta/(\Gamma/\Rightarrow\beta;\underline{X}')]}{\ddot{G}[\Delta/(\Gamma, \Box\beta/\underline{X}')]}\,\Box K \qquad \dfrac{\ddot{G}'[\Box\beta, \Phi/(\Box\beta, \Pi/\underline{Y}')]}{\ddot{G}'[\Phi/(\Box\beta, \Pi/\underline{Y}')]}\,5}{\ddot{G} \otimes \ddot{G}'[\Delta \,.\, \Phi/(\Gamma \,.\, \Pi/\underline{X}';\underline{Y}')]}\; cut_{\Box\beta}$$

In order to solve this case, we must analyse each of the rules that may have introduced the tree-hypersequent $\ddot{G}'[\Box\beta, \Phi/(\Box\beta, \Pi/\underline{Y}')]$. We go up the derivation until either a rule $\mathcal{R}''$ applies to a formula different from the $\Box\beta$'s, or a rule $\mathcal{R}''$ different from 5 applies to some of the $\Box\beta$'s (the situation is analogous to the one encountered in the case of the rule 4). Let us indicate with the symbol ◎ the tree-hypersequent that is the conclusion of the rule $\mathcal{R}''$. We distinguish cases by the type of rule $\mathcal{R}''$.

- ◎ is an axiom. Then, as $\Box\beta$ cannot be principal, even the conclusion of the cut is an axiom, and the case is solved.
- ◎ has been inferred by a rule $\mathcal{R}''$ that does not have any $\Box\beta$ as principal formula. In this case the technique consists of (i) applying the rule 5, n-times, to the premise of the rule $\mathcal{R}''$, and (ii) operating as in case 2.
- ◎ has been inferred by a rule $\mathcal{R}''$ that has $\Box\beta$ as principal formula. $\mathcal{R}''$ can either be the rule $\Box A$ or one of the special logical rules. The technique for solving the cases where $\mathcal{R}''$ is the rule $\Box A$ or the rule d are analogous; therefore, we will only show in detail the case where $\mathcal{R}''$ is the rule $\Box A$ (the other can be dealt with similarly[3]). We will conclude by also analysing the case where $\mathcal{R}''$ is the rule t (in case $\mathcal{R}''$ is the rule b, the procedure is analogous) and the rule 4.

• Let us suppose that $\mathcal{R}''$ is the rule $\Box A$, we shall first of all distinguish the following two subcases. (a) The rule $\Box A$ has been applied to a pair of sequents that precede the sequent Π and are on its same branch, let us suppose the sequents $\Box\beta, \Xi/\Box\beta, \beta, \Omega$. Hence we have the following situation:

$$\dfrac{\dfrac{\ddot{G}[\Xi'/(\Omega'/\underline{Z}')][\Gamma/\Rightarrow\beta]}{\ddot{G}[\Xi'/(\Omega'/\underline{Z}')][\Gamma,\Box\beta]}\,\Box K \qquad \dfrac{\dfrac{\dfrac{\dfrac{\ddddot{G}'[\Box\beta,\Xi/(\Box\beta,\beta,\Omega/(\Box\beta)\underline{Z})][\Box\beta,\Pi]}{\ddddot{G}'[\Box\beta,\Xi/(\Box\beta,\Omega/(\Box\beta)\underline{Z})][\Box\beta,\Pi]}\,\Box A}{\vdots}\,5}{\ddddot{G}'[\Xi/(\Omega/\underline{Z})][\Box\beta,\Pi]}\,5}{}}{\ddot{G}\otimes\ddddot{G}'[\Xi\bullet\Xi'/(\Omega\bullet\Omega'/\underline{Z};\underline{Z}')][\Gamma\bullet\Pi]}\,cut_{\Box\beta}$$

where with $(\Box\beta)\underline{Z}$ we indicate all the formulas $\Box\beta$ that can occur on the left side of the sequents that are on the same branch of the sequent Π and belong to the multiset of tree-hypersequent $\underline{Z}$.
We proceed in three steps.

(a1) We apply the rule 5, n-times, to the tree-hypersequent $\ddddot{G}'[\Box\beta,\Xi/(\Box\beta,\beta,\Omega/(\Box\beta)\underline{Z})][\Box\beta,\Pi]$. As a result, we obtain the tree-hypersequent

$$\ddddot{G}'[\Xi/(\beta,\Omega/\underline{Z})][\Box\beta,\Pi]$$

(a2) We apply the admissible rule $\tilde{5}$ (see Fig. 6.2., p. 137) to the tree-hypersequent $\ddot{G}[\Xi'/(\Omega'/\underline{Z}')][\Gamma/\Rightarrow\beta]$ a number of times sufficient to get $\Rightarrow\beta$ in an equivalent position with the sequent β,Ω of the tree-hypersequent $\ddddot{G}'[\Xi/(\beta,\Omega/\underline{Z})][\Box\beta,\Pi]$. This way we obtain a tree-hypersequent where $\Rightarrow\beta$ is no longer after Γ, but n sequents before (and separated from the other (tree-hyper)sequents by the semicolon). Let us note this as $\ddot{G}[\Xi'/(\Omega'/\underline{Z}');\Rightarrow\beta]$.

(a3) We are now in a position to apply two cuts: the first eliminable by induction on the sum of the heights, the second by induction on the complexity of the cut formula.

$$\dfrac{\dfrac{\ddot{G}[\Xi'/(\Omega'/\underline{Z}');\Rightarrow\beta] \qquad \dfrac{\ddot{G}[\Xi'/(\Omega'/\underline{Z}')][\Gamma,\Box\beta] \quad \ddddot{G}'[\Xi/(\beta,\Omega/\underline{Z})][\Box\beta,\Pi]}{\ddot{G}\otimes\ddddot{G}'[\Xi\bullet\Xi'/(\beta,\Omega\bullet\Omega'/\underline{Z}';\underline{Z})][\Gamma\bullet\Pi]}\,cut_{\Box\beta}}{\ddot{G}\otimes\ddot{G}\otimes\ddddot{G}'[\Xi\bullet\Xi'\bullet\Xi'/(\Omega\bullet\Omega'/\underline{Z};\underline{Z}');(\Omega'/\underline{Z}')][\Gamma\bullet\Pi]}\,cut_{\beta}}{\ddot{G}\otimes\ddddot{G}'[\Xi\bullet\Xi'/(\Omega\bullet\Omega'/\underline{Z};\underline{Z}')][\Gamma\bullet\Pi]}\,CA^{*}{+}CK^{*}{+}merge^{*}$$

(b) The rule $\Box A$, with $\Box\beta$ as principal formula, has been applied to the sequents $\Box\beta,\Pi/\beta,\Psi$. In this case we firstly proceed as in the subcase (a1) (i.e. we use the rule 5 n-times), and then as in the case $\Box A$ above.

- Let us now suppose that $\mathcal{R}''$ is the rule t. We shall first of all distinguish the following three subcases. (a) The rule t has been applied to a pair of sequents that precede the sequent Π on its same branch. This case can be dealt with similarly to the subcase (a) above. (b) The rule t, with $\Box\beta$ principal formula, has been applied to the sequent $\Box\beta,\beta,\Pi$. This case can be dealt with similarly to the subcase (b) above. (c) The tree-hypersequent G has the form $\Delta/(\Gamma,\Box\beta/\underline{W});\underline{W}'$, while the tree-hypersequent $G^{'}$ has the form $\Phi/(\Box\beta, \Pi/\underline{V}); \underline{V}^{'}$, and we have the following situation:

$$\dfrac{\dfrac{\Delta/(\Gamma/\Rightarrow\beta; \underline{W}); \underline{W}^{'}}{\Delta/(\Gamma, \Box\beta/\underline{W}); \underline{W}^{'}}\;{\scriptstyle \Box K} \qquad \dfrac{\dfrac{\Box\beta, \beta, \Phi/(\Box\beta, \Pi/\underline{V}); \underline{V}^{'}}{\Box\beta, \Phi/(\Box\beta, \Pi/\underline{V}); \underline{V}^{'}}\;{\scriptstyle t}}{\Phi/(\Box\beta, \Pi/\underline{V}); \underline{V}^{'}}\;{\scriptstyle 5}}{\Delta \centerdot \Phi/(\Gamma \centerdot \Pi/\underline{W}; \underline{V}); \underline{W}^{'}; \underline{V}^{'}}\;{\scriptstyle cut_{\Box\beta}}$$

We firstly apply the rule 5 to the tree-hypersequent $\Box\beta, \beta, \Phi/(\Box\beta, \Pi/\underline{V}); \underline{V}^{'}$, and we obtain this way the tree-hypersequent $\beta, \Phi/(\Box\beta, \Pi/\underline{V}); \underline{V}^{'}$. Then we continue the derivation with two cuts: the first eliminable by induction on the sum of the heights, the second by induction on the complexity of the cut-formula.

$$\dfrac{\dfrac{\dfrac{\dfrac{\Delta/(\Gamma/\Rightarrow\beta; \underline{W}); \underline{W}^{'}}{\Delta/(\Gamma/\underline{W}); \Rightarrow\beta; \underline{W}^{'}}\;{\scriptstyle \tilde{5}}}{\Delta, \beta/(\Gamma/\underline{W}); \underline{W}^{'}}\;{\scriptstyle \tilde{t}} \qquad \dfrac{\Delta/(\Gamma, \Box\beta/\underline{W}); \underline{W}^{'} \qquad \beta, \Phi/(\Box\beta, \Pi/\underline{V}); \underline{V}^{'}}{\beta, \Delta \centerdot \Phi/(\Gamma \centerdot \Pi/\underline{W}; \underline{V}); \underline{W}^{'}; \underline{V}^{'}}\;{\scriptstyle cut_{\Box\beta}}}{\Delta \centerdot \Delta \centerdot \Phi/(\Gamma \centerdot \Pi/\underline{W}; \underline{V}); (\Gamma/\underline{W}); \underline{W}^{'}; \underline{W}^{'}; \underline{V}^{'}}\;{\scriptstyle cut_{\beta}}}{\Delta \centerdot \Phi/(\Gamma \centerdot \Pi/\underline{W}; \underline{V}); \underline{W}^{'}; \underline{V}^{'}}\;{\scriptstyle CA^{*}+CK^{*}+merge^{*}}$$

- Let us finally suppose that $\mathcal{R}^{''}$ is the rule 4. We shall distinguish the following two subcases. (a) The rule 4 has been applied to a pair of sequents that precede the sequent Φ on its same branch, let us suppose the sequents $\Box\beta, \Xi/\Box\beta, \Box\beta, \Omega$. Hence we have the following:

$$\dfrac{\dfrac{\dddot{G}[\Xi^{'}/(\Omega^{'}/\underline{Z}^{'})]\,[\Gamma/\Rightarrow\beta]}{\dddot{G}[\Xi^{'}/(\Omega^{'}/\underline{Z}^{'})]\,[\Gamma, \Box\beta]}\;{\scriptstyle \Box K} \qquad \dfrac{\begin{array}{c}\dfrac{\dfrac{\ddddot{G}^{'}[\Box\beta, \Xi/(\Box\beta, \Box\beta, \Omega/(\Box\beta)\underline{Z})]\,[\Box\beta, \Pi]}{\ddddot{G}^{'}[\Box\beta, \Xi/(\Box\beta, \Omega/(\Box\beta)\underline{Z})]\,[\Box\beta, \Pi]}\;{\scriptstyle 4}}{}\;{\scriptstyle 5}\\ \vdots\end{array}}{\ddddot{G}^{'}[\Xi/(\Omega/\underline{Z})]\,[\Box\beta, \Pi]}\;{\scriptstyle 5}}{\dddot{G} \otimes \ddddot{G}^{'}[\Xi \centerdot \Xi^{'}/(\Omega \centerdot \Omega^{'}/\underline{Z}^{'}; \underline{Z})]\,[\Gamma \centerdot \Pi]}\;{\scriptstyle cut_{\Box\beta}}$$

By applying the height-preserving admissible rule of contraction CA once, and n times the rule 5, we obtain a derivation of lower height of the tree-hypersequent $\ddddot{G}^{'}[\Xi/(\Omega/\underline{Z})]\,[\Box\beta, \Pi]$. Therefore we are in a position to apply the following cut, which is eliminable by induction on the sum of the heights:

$$\dfrac{\dddot{G}[\Xi^{'}/(\Omega^{'}/\underline{Z}^{'})]\,[\Gamma, \Box\beta] \qquad \ddddot{G}^{'}[\Xi/(\Omega/\underline{Z})]\,[\Box\beta, \Pi]}{\dddot{G} \otimes \ddddot{G}^{'}[\Xi \centerdot \Xi^{'}/(\Omega \centerdot \Omega^{'}/\underline{Z}^{'}; \underline{Z})]\,[\Gamma \centerdot \Pi]}\;{\scriptstyle cut_{\Box\beta}}$$

(b) The rule 4, with $\Box\beta$ as principal formula, has been applied to the sequents $\Box\beta, \Pi/\Box\beta, \Psi$. In this case we firstly proceed as in the subcase (a1) (i.e. we use n times the rule 5), and then as in the case 4 above. □

Theorem 7.3 *Every derivation d in* $\mathbf{Thsk}^*_L$ *can be effectively transformed into a derivation d' where there is no application of the cut-rule.*

Proof It follows from the previous Lemma 7.2 by induction on the number of cuts. □

7.2 Decidability of the Tree-Hypersequent Calculi

In this section, we would like to prove that (certain) tree-hypersequent calculi are decidable, i.e. that given any tree-hypersequent G belonging to these calculi, there is an algorithm that determines whether G is provable in them or not.

First of all, let us observe that our calculi satisfy the subformula property since (i) the cut-rule is admissible (see Theorem 7.3), and (ii) in each of their rules all the formulas that occur in the premise(s) are subformulas of the formulas that occur in the conclusion. Moreover, even the contraction rules are height-preserving admissible (see Lemma 6.19, p. 136). These facts would seem to eliminate any source of potentially non-terminating proof search; nevertheless, this is not the case because of the repetition of the principal formula in each of the special logical rules and in the rule $\Box A$. In order to avoid this problem, and prove that (certain) tree-hypersequent calculi are indeed decidable, we shall obtain a bound on the number of applications of the special logical rules and of the rule $\Box A$.

To do this, let us start by taking into account only *minimal derivations*, which is to say, derivations where shortenings are not possible. Then we prove, by means of the following lemmas and their corollaries, that in minimal derivations it is enough to apply the rules d, t, b and 5, only once on any given formula of the form $\Box\alpha$ occurring on the left side of the sequent, and the rules $\Box A$ and 4, only once on any given pair of sequents. This technique is mostly inspired by the one used in [85].

Lemma 7.4 *Each of the rules d, t, b and* 5 *permutes down with respect to the others, the propositional rules, the modal rules and the special logical rule* 4.

Proof We restrict our analysis to the case of the rule t, since the other three cases can be dealt with similarly. First of all, consider the permutation with one-premise propositional rules, which is straightforward. Consider the example of the rule $\neg K$,

$$\dfrac{\dfrac{G[\beta, \alpha, \Box\alpha, \Gamma]}{G[\beta, \Box\alpha, \Gamma]}\, t}{G[\Box\alpha, \Gamma, \neg\beta]}\, \neg K$$

$$\downarrow$$

$$\dfrac{\dfrac{G[\beta, \alpha, \Box\alpha, \Gamma]}{G[\alpha, \Box\alpha, \Gamma, \neg\beta]}\, \neg K}{G[\Box\alpha, \Gamma, \neg\beta]}\, t$$

Secondly consider the permutation with the two-premises rule $\wedge K$. We have the following derivation:

$$\dfrac{\dfrac{G[\alpha, \Box\alpha, \Gamma, \beta]}{G[\Box\alpha, \Gamma, \beta]}\,t \qquad \begin{matrix}\vdots\\ G[\Box\alpha, \Gamma, \gamma]\end{matrix}}{G[\Box\alpha, \Gamma, \beta \wedge \gamma]}\,\wedge K$$

$$\downarrow$$

$$\dfrac{\dfrac{G[\alpha, \Box\alpha, \Gamma, \beta] \qquad \dfrac{\begin{matrix}\vdots\\ G[\Box\alpha, \Gamma, \gamma]\end{matrix}}{G[\alpha, \Box\alpha, \Gamma, \gamma]}\,WA}{G[\alpha, \Box\alpha, \Gamma, \beta \wedge \gamma]}\,\wedge K}{G[\Box\alpha, \Gamma, \beta \wedge \gamma]}\,t$$

The transformation of the first derivation into the second one is achieved by means of an application of the height-preserving admissible rule of (internal) weakening WA.

Let us now consider the permutation in case of the modal rule $\Box K$,

$$\dfrac{\dfrac{G[\alpha, \Box\alpha, \Gamma/ \Rightarrow \beta]}{G[\Box\alpha, \Gamma/ \Rightarrow \beta]}\,t}{G[\Box\alpha, \Gamma, \Box\beta]}\,\Box K$$

$$\downarrow$$

$$\dfrac{\dfrac{G[\alpha, \Box\alpha, \Gamma/ \Rightarrow \beta]}{G[\alpha, \Box\alpha, \Gamma, \Box\beta]}\,\Box K}{G[\Box\alpha, \Gamma, \Box\beta]}\,t$$

Finally let us analyse the permutation in case of the special logical rule 4, we have

$$\dfrac{\dfrac{G[\alpha, \Box\alpha, \Gamma/(\Box\alpha, \Sigma/\underline{X})]}{G[\Box\alpha, \Gamma/(\Box\alpha, \Sigma/\underline{X})]}\,t}{G[\Box\alpha, \Gamma/(\Sigma/\underline{X})]}\,4$$

$$\downarrow$$

$$\dfrac{\dfrac{G[\alpha, \Box\alpha, \Gamma/(\Box\alpha, \Sigma/\underline{X})]}{G[\alpha, \Box\alpha, \Gamma/(\Sigma/\underline{X})]}\,4}{G[\Box\alpha, \Gamma/(\Sigma/\underline{X})]}\,t$$

□

Lemma 7.5 *The rules $\Box A$ and 4 permute down with respect to the other, the propositional rules and the special logical rules. They also permute with instances of the rule $\Box K$ in the case where their auxiliary formulas, α and $\Box\alpha$, respectively, are not active in the sequent where the auxiliary formula of $\Box K$ occurs.*

Proof The proof is analogous to the one of Lemma 7.4. □

Corollary 7.6 *In a minimal derivation in*[4]*, respectively,* $\mathbf{Thskd}^*_L$*,* $\mathbf{Thskt}^*_L$*,* $\mathbf{Thskb}^*_L$ *and* $\mathbf{Thskb45}^*_L$*, the rules d, t, b and* 5 *cannot be applied more than once on the same formula of the form* $\Box\alpha$ *occurring on the left side of the sequent.*

Proof Consider a minimal derivation where the rule d has been applied to the same formula of the form $\Box\alpha$ twice,

$$\dfrac{G'[\Box\alpha, \Gamma'/\alpha \Rightarrow]}{G'[\Box\alpha, \Gamma']}\, d$$

$$\vdots$$

$$\dfrac{G[\Box\alpha, \Gamma/\alpha \Rightarrow]}{G[\Box\alpha, \Gamma]}\, d$$

By permuting down d with respect to the steps in the dotted part of the derivation, we obtain a derivation of the same height ending with

$$\dfrac{\dfrac{G[\Box\alpha, \Gamma/\alpha \Rightarrow; \alpha \Rightarrow]}{G[\Box\alpha, \Gamma/\alpha \Rightarrow]}\, d}{G[\Box\alpha, \Gamma]}\, d$$

By applying the height-preserving admissible rules of *merge* and CA to the two occurrences of the formula α in place of the upper d, we obtain a shortened derivation, contrary to the assumption of minimality.

Let us suppose we have a minimal derivation where the rule t has been applied twice on the same formula $\Box\alpha$,

$$\dfrac{G'[\alpha, \Box\alpha, \Gamma']}{G'[\Box\alpha, \Gamma']}\, t$$

$$\vdots$$

$$\dfrac{G[\alpha, \Box\alpha, \Gamma]}{G[\Box\alpha, \Gamma]}\, t$$

By permuting down t with respect to the steps in the dotted part of the derivation, we obtain a derivation of the same height ending with

$$\dfrac{\dfrac{G[\alpha, \alpha, \Box\alpha, \Gamma]}{G[\alpha, \Box\alpha, \Gamma]}\, t}{G[\Box\alpha, \Gamma]}\, t$$

By applying the height-preserving admissible rule CA to the two occurrences of the formula α in place of the upper t, we obtain a shortened derivation, contrary to the assumption of minimality.

The cases of the rules b and 5 can be treated analogously to the previous ones. □

Corollary 7.7 *In a minimal derivation in, respectively,* $\mathbf{Thsk}^*_L$ *and* $\mathbf{Thsk4}^*_L$*, the rules* $\Box A$ *and* 4 *cannot be applied more than once on the same pair of sequents of any branch.*

Proof The proof is analogous to the one of Corollary 7.6. However, for the sake of clarity, we show the case of the rule $\Box A$. Let us suppose to have a minimal derivation where the rule $\Box A$ has been applied twice on the same pair of sequents,

$$\dfrac{G'[\Box\alpha, \Gamma'/(\alpha, \Sigma'/\underline{X})]}{G'[\Box\alpha, \Gamma'/(\Sigma'/\underline{X})]}\,\Box A$$

$$\vdots$$

$$\dfrac{G[\Box\alpha, \Gamma/(\alpha, \Sigma/\underline{X})]}{G[\Box\alpha, \Gamma/(\Sigma/\underline{X})]}\,\Box A$$

By permuting down $\Box A$ with respect to the steps in the dotted part of the derivation, we obtain a derivation of the same height ending with

$$\dfrac{\dfrac{G[\Box\alpha, \Gamma/(\alpha, \alpha, \Sigma/\underline{X})]}{G[\Box\alpha, \Gamma/(\alpha, \Sigma/\underline{X})]}\,\Box A}{G[\Box\alpha, \Gamma/(\Sigma/\underline{X})]}\,\Box A$$

By applying the height-preserving admissible rule CA to the two occurrences of the formula α in place of the upper $\Box A$, we obtain a shortened derivation, contrary to the assumption of minimality. □

We can finally prove that the modal logic **K** is decidable by showing effective bounds on proof search in the calculus $\mathbf{Thsk}_L$.

Theorem 7.8 *The calculus* $\mathbf{Thsk}_L$ *allows terminating proof search.*

Proof Place a tree-hypersequent G, for which we are looking for a proof search, at the root of the procedure. Apply first the propositional rules and then the modal rules. The propositional rules reduce the complexity of the tree-hypersequent. The rule $\Box K$ removes the modal constant $\Box$, and adds a new sequent, the rule $\Box A$ increases the complexity. However, by Corollary 7.7, the rule $\Box A$ cannot be applied more than once to the same pair of sequents. Therefore, the number of applications of the rule $\Box A$ is bounded by the number of sequents that may appear in the derivation. The latter, in its turn, is bounded by the number of sequents belonging to the tree-hypersequent to prove, and the sequents that can be introduced by applications of the rule $\Box K$.

Let us explain how to calculate explicit bounds. Let us first of all define the *negative* and *positive* parts of the tree-hypersequent G, as the union of the negative and positive parts of the translation into formulas of each of the sequents that compose G. Recall that any sequent $M \Rightarrow N$ belonging to a tree-hypersequent G, corresponds to

$$\bigwedge M \to \bigvee N$$

For any given tree-hypersequent G, let $n(\Box)$ be the number of $\Box$ in the negative part of the tree-hypersequent G, and $p(\Box)$ be the number of $\Box$ in the positive part of the tree-hypersequent G.

In case the root-tree-hypersequent is just a sequent, the number of applications of the rule $\Box A$ in a minimal derivation is bounded by

$$n(\Box) \cdot p\ (\Box)$$

In case the root-tree-hypersequent is a tree-hypersequent, and s is the number of sequents that occur in it, the number of applications of the rule $\Box A$ in a minimal derivation is bounded by

$$n(\Box) \cdot (p\ (\Box) + s)$$

□

By a similar argument we have:

Theorem 7.9 *The calculus* **Thskt**$_L$ *allows terminating proof search.*

Proof The order of the application of the rules rests unchanged: the rule t should indeed be applied after the others. On the other hand we should specify how to calculate explicit bounds for the rule t. Given the fact that this rule cannot be applied more than once on a same formula of the form $\Box\alpha$ occurring on the left side of the sequent, we have that the number of its applications in a minimal derivation is simply bounded by

$$n(\Box)$$

□

As for the decidability of the calculus **Thskd**$_L$, the situation is somewhat more difficult. Indeed the rule d, like the rule $\Box K$, creates a new sequent, and therefore the calculation of the bound on the applications of the rule $\Box A$ should take into account this fact. As a result, we have:

Theorem 7.10 *The calculus* **Thskd**$_L$ *allows terminating proof search.*

Proof The order of the applications of the rules slightly varies: we firstly apply the propositional rules, then the rule $\Box K$, then the rule d, and finally the rule $\Box A$. The number of applications of the rule d in a minimal derivation is bounded by

$$n(\Box)$$

In case the root-tree-hypersequent is just a sequent, the number of applications of the rule $\Box A$ in a minimal derivation is bounded by

$$n(\Box) \cdot (p\ (\Box) + n(\Box))$$

In case the root-tree-hypersequent is a tree-hypersequent, and s is the number of sequents which occurs in it, the number of applications of the rule $\Box A$ in a minimal derivation is bounded by

$$n(\Box) \cdot (p\ (\Box) + s + n(\Box))$$

□

Theorem 7.11 *The calculi* $\mathbf{Thskb}_L$ *and* $\mathbf{Thsktb}_L$ *allow terminating proof search.*

Proof The order of the application of the rules rests unchanged: the rule b is indeed applied after the others. Thanks to the above Theorems 7.8 and 7.9, we should only explain how to calculate explicit bounds for the rule b. The number of applications of the rule b in a minimal derivation is bounded by

$$n(\Box)$$

□

In $\mathbf{Thss4}_L$ the situation is more complicated. In order to illustrate and solve this situation, we use Negri's ideas (see [85, pp. 536, 537]) once more.

Let us start by explaining the problem. In the calculus $\mathbf{Thss4}_L$, the interaction of the rule 4 with the rule $\Box K$ means that one can construct chains of sequents on which the rule $\Box A$ can be applied *ad libitum*: this way the aim of finding a terminating proof-search seems to be impracticable. On the other hand, thanks to the height-preserving admissible rules of contraction and $\tilde{t}$, we can truncate an attempted proof search after a finite number of steps. Before showing how, let us illustrate this method with an example. Let us then try to find a derivation for the sequent $\Rightarrow \Box\neg\Box\alpha \to \Box\beta$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\Box\neg\Box\alpha \Rightarrow /\Box\neg\Box\alpha \Rightarrow \beta/ \Rightarrow \alpha/ \Rightarrow \alpha}}{\Box\neg\Box\alpha \Rightarrow /\Box\neg\Box\alpha \Rightarrow \beta/ \Rightarrow \alpha, \Box\alpha}\ \Box K}{\Box\neg\Box\alpha \Rightarrow /\Box\neg\Box\alpha \Rightarrow \beta/\neg\Box\alpha \Rightarrow \alpha}\ \neg A}{\Box\neg\Box\alpha \Rightarrow /\Box\neg\Box\alpha \Rightarrow \beta/ \Rightarrow \alpha}\ \Box A}{\Box\neg\Box\alpha \Rightarrow / \Rightarrow \beta/ \Rightarrow \alpha}\ 4}{\Box\neg\Box\alpha \Rightarrow / \Rightarrow \beta, \Box\alpha}\ \Box K}{\Box\neg\Box\alpha \Rightarrow /\neg\Box\alpha \Rightarrow \beta}\ \neg A}{\Box\neg\Box\alpha \Rightarrow / \Rightarrow \beta}\ \Box A}{\dfrac{\Box\neg\Box\alpha \Rightarrow \Box\beta}{\Rightarrow \Box\neg\Box\alpha \to \Box\beta}\ \to K}\ \Box K$$

Consider the top tree-hypersequent. By applying the height-preserving rule $\tilde{t}$ (see Fig. 6.1., p. 137) to it, we obtain a derivation of the same height of

$$\Box\neg\Box\alpha \Rightarrow /\Box\neg\Box\alpha \Rightarrow \beta / \Rightarrow \alpha, \alpha$$

By applying the rule of contraction CK, we obtain a derivation of the same height of

$$\Box\neg\Box\alpha \Rightarrow /\Box\neg\Box\alpha \Rightarrow \beta / \Rightarrow \alpha$$

with a resulting shortening by three steps of the original derivation. Since we can assume that the attempted proof search is for a minimal derivation, we have a contradiction, and thus we can conclude that the sequent is not derivable.

This argument can be formalised by providing a bound on the number of successive applications of the rule $\Box K$ with principal formula $\Box\alpha$, on sequents that occur one after another and that all belong to the same branch. Intuitively, only those applications that contribute to unfold all the boxed negative subformulas of the endsequent through steps of $\Box A$ are required. Additional steps are superfluous as they give rise to duplications as soon as the inner-most boxed formula in the negative part has been reached, as the above example shows.

Lemma 7.12 *In a minimal derivation of a tree-hypersequent in* **Thss4**$_L$*, for each formula* $\Box\alpha$ *in its positive part, there are at most* $n(\Box)$ *applications of the rule* $\Box K$ *iterated on a chain of sequents that occur one after another and all belong to the same branch, with principal formula* $\Box\alpha$*.*

Proof Let m be $n(\Box)$ and suppose that the antecedent of the derivable tree-hypersequent contains a formula of the form $\Box^m\mathcal{F}$, where $\Box^m$ denotes a block of m boxes. This assumption can be made without loss of generality. The modalities in the negative part of the sequent do not necessarily occur in a block, but may be interwoven with propositional connectives. However, these connectives can be unfolded by the application, root-first, of propositional rules without changing the number of applications of $\Box K$ that are necessary to reach the innermost non-modal formula. Suppose that we iterate $\Box K$ on a chain of successive sequents. After the first application of $\Box K$, we obtain a new sequent and application of $\Box A$ produces in this new sequent an antecedent containing $\Box^{m-1}\mathcal{F}$. After the second application of the rule $\Box K$, we obtain another new sequent, which succeeds to the previous one. By the rule 4 and the rule $\Box A$, we can produce an antecedent containing the formulas $\Box^{m-2}\mathcal{F}$, $\Box^{m-1}\mathcal{F}$ in this new sequent. After m applications, there will be a sequent at distance m, containing in the antecedent also the formulas $\mathcal{F}$, $\Box^{m-1}\mathcal{F}$ and in the succedent α. Let us apply the rule $\Box K$ once more. In this way, we create a $m + 1$ sequent, and the formulas $\mathcal{F}$, $\Box^{m-1}\mathcal{F}$ can be reproduced in the antecedent of this new sequent, thanks to the rules 4 and $\Box A$. These latter steps are superfluous. Indeed, we can apply the two admissible rules $\tilde{t}$ and CA, and eliminate the last $m + 1$ sequent, while maintaining the elimination height. This way we obtain a shorter derivation of the sequent reached after m steps of $\Box K$. □

Theorem 7.13 *The calculus* $\mathbf{Thss4}_L$ *allows terminating proof search.*

Proof The order of the application of the rules varies slightly: we firstly apply the propositional rules, then the rule $\Box K$, then the rule 4, and finally the rule $\Box A$ an t. The above Lemma 7.12 ensures that the interaction between the rule $\Box K$ and the rule 4 do not generate non terminating proof searches. Finally, we calculate explicit bound for the rule 4. In case the root-tree-hypersequent is just a sequent, the number of applications of the rule 4 in a minimal derivation is bounded by

$$n(\Box) \cdot p\ (\Box)$$

In case the root-tree-hypersequent is a tree-hypersequent and s is the number of sequents that occurs in it, the number of applications of the rule 4 in a minimal derivation is bounded by

$$n(\Box) \cdot (p\ (\Box) + s)$$

□

By Corollary 7.6 and Theorems 7.11 and 7.13, and the fact that the number of applications of the rule 5 in a minimal derivation is bounded by $n(\Box)$, we have that also the calculus $\mathbf{Thss5}_L$ allows terminating proof search.

Notes

1. Note that we use the notation $\mathcal{R}^* + \mathcal{R}'^* + \mathcal{R}''^*$ to stand for: repeated applications of the rules $\mathcal{R}$, $\mathcal{R}'$ and $\mathcal{R}''$ take place. The order of the several applications of these three rules is straightforward to deduce. From now on we will take this notation for granted.
2. The only difference consists in the use, illustrated in the cases d and t above, of the special structural rules $\tilde{d}$ and $\tilde{t}$, respectively.
3. The only difference consists in the use, illustrated in the case d, of the special structural rules $\tilde{d}$.
4. With the notation: name of the tree-hypersequent calculus + *, we mean all the extensions of that calculus by a combination of special logical rules. From now on we will take this assumption for granted.

Chapter 8
Semantic Adequacy

In Section 6.3 we proved that the tree-hypersequent calculi are sound and complete with respect to their corresponding Hilbert systems. All the proofs were purely syntactic. In this chapter, we tackle the same issues from a semantic point of view, giving alternative (though less constructive) proofs. This operation can shed further light on the tree-hypersequent calculi.

8.1 Semantic Validity of the Tree-Hypersequent Calculi

This section will provide a proof that each of the tree-hypersequent calculi $\mathbf{Thsk}^*_L$ is sound with respect to the corresponding class of Kripke frames. We must first introduce the following definition and lemma.

Definition 8.1 Let $\mathfrak{M} = \langle W, R, v \rangle$, $i \in W$, $G \in$ THS,

$$i \models_{\mathfrak{M}} G$$

is inductively defined in the following way:

- $i \models_{\mathfrak{M}} M \Rightarrow N \quad \text{iff} \quad \exists \beta \in M (i \not\models_{\mathfrak{M}} \beta) \text{ or } \exists \gamma \in N (i \models_{\mathfrak{M}} \gamma)$,
- $i \models_{\mathfrak{M}} \Gamma/\underline{X} \quad \text{iff} \quad i \models_{\mathfrak{M}} \Gamma \text{ or } \exists G \in \underline{X}\, \forall j (iRj \rightarrow j \models_{\mathfrak{M}} G)$.

By adopting the convention, for P multiset of formulas,

$$i \models_{\mathfrak{M}} P := \exists \alpha \in P (i \models_{\mathfrak{M}} \alpha)$$

we can more succinctly write

$$i \models_{\mathfrak{M}} M \Rightarrow N \quad \text{iff} \quad i \models_{\mathfrak{M}} \neg M, N$$

where $\neg M := \{\neg \beta \mid \beta \in M\}$.

By adopting the convention

$$i \models^*_{\mathfrak{M}} \ldots := \forall j (iRj \rightarrow j \models_{\mathfrak{M}} \ldots)$$

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we can more succinctly write

$$i \models_{\mathfrak{M}} \Gamma/\underline{X} \quad \text{iff} \quad i \models_{\mathfrak{M}} \Gamma \text{ or } \exists G \in \underline{X},\ i \models^{*}_{\mathfrak{M}} G$$

Given a class of frames $\mathfrak{C}$, we will write $i \models_{\mathfrak{C}} G$ to mean that for every model $\mathfrak{M}$ based on any frame that belongs to the class $\mathfrak{C}$, we have $i \models_{\mathfrak{M}} G$. We will use the notation $\mathfrak{C}f$ to indicate the class of all frames, while, to indicate the class of frames that enjoy certain properties, we will substitute the f in $\mathfrak{C}f$ with the name of the rule that corresponds to that property. For example we will write $\mathfrak{C}t$ to indicate the class of reflexive frames.

Lemma 8.2 *(a) For all sequents Γ, Δ, and all tree-hypersequents G,*

if $\forall i\ (i \models_{\mathfrak{C}f} \Gamma \to i \models_{\mathfrak{C}f} \Delta)$
then, $\forall i (i \models_{\mathfrak{C}f} G[\Gamma] \to i \models_{\mathfrak{C}f} G[\Gamma/\Delta])$

(b) For all sequents Γ_1, Γ_2, Δ, and all tree-hypersequents G,

if $\forall i\ (i \models_{\mathfrak{C}f} \Gamma_1$ *and* $i \models_{\mathfrak{C}f} \Gamma_2 \to i \models_{\mathfrak{C}f} \Delta)$
then, $\forall i (i \models_{\mathfrak{C}f} G[\Gamma_1]$ *and* $i \models_{\mathfrak{C}f} G[\Gamma_2] \to i \models_{\mathfrak{C}f} [\Gamma/\Delta])$

(c) For all tree-hypersequents (that are not sequents) J, H, and G,

if $\forall i\ (i \models_{\mathfrak{C}f} J \to i \models_{\mathfrak{C}f} H)$
then, $\forall i (i \models_{\mathfrak{C}f} G[J] \to i \models_{\mathfrak{C}f} G[J/H])$

Proof (a) By induction on G. (i) If $G \equiv \Gamma$, then the claim is obvious. (ii) If $G \equiv \Gamma/\underline{X}$, then the claim is obvious. (iii) If $G \equiv \Phi/G_1; ...; G_m; G_0[\Gamma]$, then $G[\Gamma/\Delta] \equiv \Phi/G_1; ...; G_m; G_0[\Gamma/\Delta]$. By inductive hypothesis we have that, for any i, $i \models_{\mathfrak{C}f} G_0[\Gamma]$, implies $i \models_{\mathfrak{C}f} G_0[\Gamma/\Delta]$. So we also have

$$\circledcirc \quad \text{for any } i,\ i \models^{*}_{\mathfrak{C}f} G_0[\Gamma],\ \text{implies } i \models^{*}_{\mathfrak{C}f} G_0[\Gamma/\Delta]$$

Now suppose $k \models_{\mathfrak{C}f} \Phi/G_1; ...; G_m; G_0[\Gamma]$, then $k \models_{\mathfrak{C}f} \Phi$ or $k \models^{*}_{\mathfrak{C}f} G_1$, or ..., or $k \models^{*}_{\mathfrak{C}f} G_m$, or $k \models^{*}_{\mathfrak{C}f} G_0[\Gamma]$. Therefore, by $\circledcirc$, we have $k \models_{\mathfrak{C}f} \Phi$, or $k \models^{*}_{\mathfrak{C}f} G_1$, or, ..., or $k \models^{*}_{\mathfrak{C}f} G_m$, or $k \models^{*}_{\mathfrak{C}f} G_0[\Delta]$, i.e. $k \models_{\mathfrak{C}f} \Phi/G_1; ...; G_m; G_0[\Delta]$.

The proofs of (b) and (c) can be developed in the same way as the one for (a). □

Theorem 8.3 *For all tree-hypersequents G, if $\vdash G$ in* **Thsk**$^{*}_{L}$*, then $\models_{\mathfrak{C}\mathfrak{f}^*} G$, where $\models_{\mathfrak{C}\mathfrak{f}^*} G$ stands for: G is valid in the corresponding class of frames.*

Proof By induction on the derivation of the premise. The validity of the initial tree-hypersequents and of the propositional rules is proved, exploiting Lemma 8.2, in the usual way. We just prove the validity of the modal rules and of the special logical rules.

– $\Box K$. Let us consider the rule in the form

$$\frac{\Gamma/\Rightarrow \alpha}{\Gamma, \Box\alpha}$$

By the inductive hypothesis, we have $\forall i$ ($i \models_{\mathfrak{Cf}} \Gamma$ or $i \models^*_{\mathfrak{Cf}} \alpha$). By definition of the forcing relation (see Definition 2.9, p. 41), we have $\forall i$ ($i \models_{\mathfrak{Cf}} \Gamma$ or $i \models_{\mathfrak{Cf}} \Box\alpha$), which is nothing other than the conclusion of the rule. By Lemma 8.2, we have that the rule $\Box K$ is valid in the class of all frames.

– $\Box A$. Let us consider the rule in the form

$$\frac{\Box\alpha, \Gamma/(\alpha, \Sigma/\underline{X})}{\Box\alpha, \Gamma/(\Sigma/\underline{X})}$$

By the inductive hypothesis, we have $\forall i$ ($i \models_{\mathfrak{Cf}} \neg\Box\alpha, \Gamma$ or $i \models^*_{\mathfrak{Cf}} \neg\alpha, \Sigma$ or $j \models^*_{\mathfrak{Cf}} \underline{X}$), i.e. $\forall i$ ($i \not\models_{\mathfrak{Cf}} \Box\alpha$ or $i \models_{\mathfrak{Cf}} \Gamma$ or $\forall j$ ($iRj \rightarrow j \not\models_{\mathfrak{Cf}} \alpha$ or $j \models_{\mathfrak{Cf}} \Sigma$ or $\forall z$ ($jRz \rightarrow z \models_{\mathfrak{Cf}} \underline{X}$))). From this, we obtain

1 $\forall i$ ($i \not\models_{\mathfrak{Cf}} \Box\alpha$ or $i \models_{\mathfrak{Cf}} \Gamma$ or $\forall j((iRj \rightarrow j \models_{\mathfrak{Cf}} \alpha) \rightarrow (iRj \rightarrow j \models_{\mathfrak{Cf}} \Sigma$ or $\forall z$ ($jRz \rightarrow z \models_{\mathfrak{Cf}} \underline{X}$))))
2 $\forall i$ ($i \not\models_{\mathfrak{Cf}} \Box\alpha$ or $i \models_{\mathfrak{Cf}} \Gamma$ or $\forall j(iRj \rightarrow j \models_{\mathfrak{Cf}} \alpha) \rightarrow \forall j$ ($iRj \rightarrow j \models_{\mathfrak{Cf}} \Sigma$ or $\forall z$ ($jRz \rightarrow z \models_{\mathfrak{Cf}} \underline{X}$)))
3 $\forall i$ ($i \not\models_{\mathfrak{Cf}} \Box\alpha$ or $i \models_{\mathfrak{Cf}} \Gamma$ or $\neg\forall j(iRj \rightarrow j \models_{\mathfrak{Cf}} \alpha)$ or $\forall j$ ($iRj \rightarrow j \models_{\mathfrak{Cf}} \Sigma$ or $\forall z$ ($jRz \rightarrow z \models_{\mathfrak{Cf}} \underline{X}$)))
4 $\forall i$ ($i \not\models_{\mathfrak{Cf}} \Box\alpha$ or $i \models_{\mathfrak{Cf}} \Gamma$ or $\exists j(iRj$ and $j \not\models_{\mathfrak{Cf}} \alpha)$ or $\forall j$ ($iRj \rightarrow j \models_{\mathfrak{Cf}} \Sigma$ or $\forall z$ ($jRz \rightarrow z \models_{\mathfrak{Cf}} \underline{X}$)))
5 $\forall i$ ($i \not\models_{\mathfrak{Cf}} \Box\alpha$ or $i \models_{\mathfrak{Cf}} \Gamma$ or $i \not\models_{\mathfrak{Cf}} \Box\alpha$ or $\forall j$ ($iRj \rightarrow j \models_{\mathfrak{Cf}} \Sigma$ or $\forall z$ ($jRz \rightarrow z \models_{\mathfrak{Cf}} \underline{X}$)))
6 $\forall i$ ($i \not\models_{\mathfrak{Cf}} \Box\alpha$ or $i \models_{\mathfrak{Cf}} \Gamma$ or $\forall j$ ($iRj \rightarrow j \models_{\mathfrak{Cf}} \Sigma$ or $\forall z$ ($jRz \rightarrow z \models_{\mathfrak{Cf}} \underline{X}$)))

The last line of the proof is the conclusion of the rule. From this argument and Lemma 8.2, we conclude that the rule $\Box A$ is valid in the class of all frames.

– d. Let us consider the rule in the form

$$\frac{\Box\alpha, \Gamma/\alpha \Rightarrow}{\Box\alpha, \Gamma}$$

By the inductive hypothesis, we have $\forall i$ ($i \models_{\mathfrak{Cd}} \neg\Box\alpha, \Gamma$ or $i \models^*_{\mathfrak{Cd}} \neg\alpha$), i.e. $\forall i$ ($i \not\models_{\mathfrak{Cd}} \Box\alpha$ or $i \models_{\mathfrak{Cd}} \Gamma$ or $\forall j$ ($iRj \rightarrow j \not\models_{\mathfrak{Cd}} \alpha$)). From this, we get

1 $\forall i$ ($i \not\models_{\mathfrak{Cd}} \Box\alpha$ or $i \models_{\mathfrak{Cd}} \Gamma$ or $\forall j$ ($iRj \rightarrow j \not\models_{\mathfrak{Cd}} \alpha$ or $j \models_{\mathfrak{Cd}} \bot$))
2 $\forall i$ ($i \not\models_{\mathfrak{Cd}} \Box\alpha$ or $i \models_{\mathfrak{Cd}} \Gamma$ or $\forall j$ (($iRj \rightarrow j \models_{\mathfrak{Cd}} \alpha) \rightarrow (iRj \rightarrow j \models_{\mathfrak{Cd}} \bot$)))
3 $\forall i$ ($i \not\models_{\mathfrak{Cd}} \Box\alpha$ or $i \models_{\mathfrak{Cd}} \Gamma$ or $\forall j$ ($iRj \rightarrow j \models_{\mathfrak{Cd}} \alpha) \rightarrow \forall j(iRj \rightarrow j \models_{\mathfrak{Cd}} \bot$))
4 $\forall i$ ($i \not\models_{\mathfrak{Cd}} \Box\alpha$ or $i \models_{\mathfrak{Cd}} \Gamma$ or $\neg\forall j$ ($iRj \rightarrow j \models_{\mathfrak{Cd}} \alpha)$ or $\forall j(iRj \rightarrow j \models_{\mathfrak{Cd}} \bot$))
5 $\forall i$ ($i \not\models_{\mathfrak{Cd}} \Box\alpha$ or $i \models_{\mathfrak{Cd}} \Gamma$ or $\exists j$ (iRj and $j \not\models_{\mathfrak{Cd}} \alpha)$ or $\forall j(iRj \rightarrow j \models_{\mathfrak{Cd}} \bot$))
6 $\forall i$ ($i \not\models_{\mathfrak{Cd}} \Box\alpha$ or $i \models_{\mathfrak{Cd}} \Gamma$ or $i \not\models_{\mathfrak{Cd}} \Box\alpha$ or $\forall j(iRj \rightarrow j \models_{\mathfrak{Cd}} \bot$))
7 $\forall i$ ($i \not\models_{\mathfrak{Cd}} \Box\alpha$ or $i \models_{\mathfrak{Cd}} \Gamma$ or $\neg\exists j$ (iRj))

Since the formula $\neg\exists j$ (iRj) contradicts the *seriality* property, we can drop it and this way obtain $\forall i$ ($i \models_{\mathfrak{Cd}} \neg\Box\alpha$ or $i \models_{\mathfrak{Cd}} \Gamma$), which is nothing other than the conclusion of the rule. From this argument and Lemma 8.2, we conclude that the rule d is valid in the class of serial frames.

– t. Let us consider the rule in the form

$$\frac{\Box\alpha, \alpha, \Gamma}{\Box\alpha, \Gamma}$$

By the inductive hypothesis, we have $\forall i\ (i \models_{\mathfrak{C}t} \neg\alpha, \neg\Box\alpha, \Gamma)$, i.e. $\forall i\ (i \not\models_{\mathfrak{C}t} \alpha$ or $i \not\models_{\mathfrak{C}t} \Box\alpha$ or $i \models_{\mathfrak{C}t} \Gamma)$. Since we are dealing with *reflexive* frames, we have

1 $\forall i\ (iRi$ and $i \not\models_{\mathfrak{C}t} \alpha$ or $i \not\models_{\mathfrak{C}t} \Box\alpha$ or $i \models_{\mathfrak{C}t} \Gamma)$
2 $\forall i\ (\exists j\ (iRj$ and $j \not\models_{\mathfrak{C}t} \alpha)$ or $i \not\models_{\mathfrak{C}t} \Box\alpha$ or $i \models_{\mathfrak{C}t} \Gamma)$
3 $\forall i\ (i \not\models_{\mathfrak{C}t} \Box\alpha$ or $i \not\models_{\mathfrak{C}t} \Box\alpha$ or $i \models_{\mathfrak{C}t} \Gamma)$
4 $\forall i\ (i \not\models_{\mathfrak{C}t} \Box\alpha$ or $i \models_{\mathfrak{C}t} \Gamma)$

The last line of the proof is the conclusion of the rule. From this argument and Lemma 8.2, we conclude that the rule t is valid in the class of reflexive frames.

– 4. Let us consider the rule in the form

$$\frac{\Box\alpha, \Gamma/(\Box\alpha, \Sigma/\underline{X})}{\Box\alpha, \Gamma/(\Sigma/\underline{X})}$$

By the inductive hypothesis, we have $\forall i\ (i \models_{\mathfrak{C}4} \neg\Box\alpha, \Gamma$ or $i \models^{*}_{\mathfrak{C}4} \neg\Box\alpha, \Sigma$ or $j \models^{*}_{\mathfrak{C}4} \underline{X})$, i.e. $\forall i\ (i \not\models_{\mathfrak{C}4} \Box\alpha$ or $i \models_{\mathfrak{C}4} \Gamma$ or $\forall j\ (iRj \to j \not\models_{\mathfrak{C}4} \Box\alpha$ or $j \models_{\mathfrak{C}4} \Sigma$ or $\forall z\ (jRz \to z \models_{\mathfrak{C}4} \underline{X})))$. From this, we have

1 $\forall i\ (i \not\models_{\mathfrak{C}4} \Box\alpha$ or $i \models_{\mathfrak{C}4} \Gamma$ or $\forall j\ (\neg iRj$ or $j \not\models_{\mathfrak{C}4} \Box\alpha$ or $j \models_{\mathfrak{C}4} \Sigma$ or $\forall z\ (jRz \to z \models_{\mathfrak{C}4} \underline{X})))$
2 $\forall i\ (i \not\models_{\mathfrak{C}4} \Box\alpha$ or $i \models_{\mathfrak{C}4} \Gamma$ or $\forall j\ (\neg iRj$ or $\exists w\ (jRw$ and $w \not\models_{\mathfrak{C}4} \alpha)$ or $j \models_{\mathfrak{C}4} \Sigma$ or $\forall z\ (jRz \to z \models_{\mathfrak{C}4} \underline{X})))$
3 $\forall i\ (i \not\models_{\mathfrak{C}4} \Box\alpha$ or $i \models_{\mathfrak{C}4} \Gamma$ or $\forall j\ \exists w\ (\neg iRj$ or $(jRw$ and $w \not\models_{\mathfrak{C}4} \alpha)$ or $j \models_{\mathfrak{C}4} \Sigma$ or $\forall z\ (jRz \to z \models_{\mathfrak{C}4} \underline{X})))$

From 3, since we are dealing with *transitive* frames, we have

4 $\forall i\ (i \not\models_{\mathfrak{C}4} \Box\alpha$ or $i \models_{\mathfrak{C}4} \Gamma$ or $\forall j\ \exists w\ (\neg iRj$ or $((\neg iRj$ or $iRw)$ and $w \not\models_{\mathfrak{C}4} \alpha)$ or $j \models_{\mathfrak{C}4} \Sigma$ or $\forall z\ (jRz \to z \models_{\mathfrak{C}4} \underline{X})))$
5 $\forall i\ (i \not\models_{\mathfrak{C}4} \Box\alpha$ or $i \models_{\mathfrak{C}4} \Gamma$ or $\forall j\ \exists w\ (\neg iRj$ or $(\neg iRj$ or $(iRw$ and $w \not\models_{\mathfrak{C}4} \alpha))$ or $j \models_{\mathfrak{C}4} \Sigma$ or $\forall z\ (jRz \to z \models_{\mathfrak{C}4} \underline{X})))$
6 $\forall i\ (i \not\models_{\mathfrak{C}4} \Box\alpha$ or $i \models_{\mathfrak{C}4} \Gamma$ or $\forall j\ \exists w\ (\neg iRj$ or $\neg iRj$ or $(iRw$ and $w \not\models_{\mathfrak{C}4} \alpha)$ or $j \models_{\mathfrak{C}4} \Sigma$ or $\forall z\ (jRz \to z \models_{\mathfrak{C}4} \underline{X})))$
7 $\forall i\ (i \not\models_{\mathfrak{C}4} \Box\alpha$ or $i \models_{\mathfrak{C}4} \Gamma$ or $\forall j\ (\neg iRj$ or $\exists w\ (iRw$ and $w \not\models_{\mathfrak{C}4} \alpha)$ or $j \models_{\mathfrak{C}4} \Sigma$ or $\forall z\ (jRz \to z \models_{\mathfrak{C}4} \underline{X})))$
8 $\forall i\ (i \not\models_{\mathfrak{C}4} \Box\alpha$ or $i \models_{\mathfrak{C}4} \Gamma$ or $\forall j\ (\neg iRj$ or $i \not\models_{\mathfrak{C}4} \Box\alpha$ or $j \models_{\mathfrak{C}4} \Sigma$ or $\forall z\ (jRz \to z \models_{\mathfrak{C}4} \underline{X})))$
9 $\forall i\ (i \not\models_{\mathfrak{C}4} \Box\alpha$ or $i \models_{\mathfrak{C}4} \Gamma$ or $i \not\models_{\mathfrak{C}4} \Box\alpha$ or $\forall j\ (\neg iRj$ or $j \models_{\mathfrak{C}4} \Sigma$ or $\forall z\ (jRz \to z \models_{\mathfrak{C}4} \underline{X})))$
10 $\forall i\ (i \not\models_{\mathfrak{C}4} \Box\alpha$ or $i \models_{\mathfrak{C}4} \Gamma$ or $\forall j\ (iRj \to j \models_{\mathfrak{C}4} \Sigma$ or $\forall z\ (jRz \to z \models_{\mathfrak{C}4} \underline{X})))$

The last line of the proof is the conclusion of the rule. From this argument and Lemma 8.2, we conclude that the rule 4 is valid in the class of transitive frames.

– b. Let us consider the rule in the form

$$\frac{\alpha, \Gamma/(\Box\alpha, \Sigma/\underline{X})}{\Gamma/(\Box\alpha, \Sigma/\underline{X})}$$

By the inductive hypothesis, we have $\forall i\ (i \models_{\mathfrak{C}b} \neg\alpha, \Gamma$ or $i \models^{*}_{\mathfrak{C}b} \neg\Box\alpha, \Sigma$ or $j \models^{*}_{\mathfrak{C}b} \underline{X})$, i.e. $\forall i\ (i \not\models_{\mathfrak{C}b} \alpha$ or $i \models_{\mathfrak{C}b} \Gamma$ or $\forall j\ (iRj \rightarrow j \not\models_{\mathfrak{C}b} \Box\alpha$ or $j \models_{\mathfrak{C}b} \Sigma$ or $\forall z\ (jRz \rightarrow z \models_{\mathfrak{C}b} \underline{X}))$. From this, we get

1 $\forall i\ (i \not\models_{\mathfrak{C}b} \alpha$ or $i \models_{\mathfrak{C}b} \Gamma$ or $\forall j\ (\neg iRj$ or $j \not\models_{\mathfrak{C}b} \Box\alpha$ or $j \models_{\mathfrak{C}b} \Sigma$ or $\forall z\ (jRz \rightarrow z \models_{\mathfrak{C}b} \underline{X})))$
2 $\forall i\ \forall j\ (i \not\models_{\mathfrak{C}b} \alpha$ or $\neg iRj$ or $i \models_{\mathfrak{C}b} \Gamma$ or $j \not\models_{\mathfrak{C}b} \Box\alpha$ or $j \models_{\mathfrak{C}b} \Sigma$ or $\forall z\ (jRz \rightarrow z \models_{\mathfrak{C}b} \underline{X}))$

From 2, since we are dealing with *symmetric* frames, we have

3 $\forall i\ \forall j\ ((jRi$ and $i \not\models_{\mathfrak{C}b} \alpha)$ or $\neg iRj$ or $i \models_{\mathfrak{C}b} \Gamma$ or $j \not\models_{\mathfrak{C}b} \Box\alpha$ or $j \models_{\mathfrak{C}b} \Sigma$ or $\forall z\ (jRz \rightarrow z \models_{\mathfrak{C}b} \underline{X}))$
4 $\forall i\ \forall j\ (\exists w(jRw$ and $w \not\models_{\mathfrak{C}b} \alpha)$ or $\neg iRj$ or $i \models_{\mathfrak{C}b} \Gamma$ or $j \not\models_{\mathfrak{C}b} \Box\alpha$ or $j \models_{\mathfrak{C}b} \Sigma$ or $\forall z\ (jRz \rightarrow z \models_{\mathfrak{C}b} \underline{X}))$
5 $\forall i\ \forall j\ (j \not\models_{\mathfrak{C}b} \Box\alpha$ or $\neg iRj$ or $i \models_{\mathfrak{C}b} \Gamma$ or $j \not\models_{\mathfrak{C}b} \Box\alpha$ or $j \models_{\mathfrak{C}b} \Sigma$ or $\forall z\ (jRz \rightarrow z \models_{\mathfrak{C}b} \underline{X}))$
6 $\forall i\ \forall j\ (\neg iRj$ or $i \models_{\mathfrak{C}b} \Gamma$ or $j \not\models_{\mathfrak{C}b} \Box\alpha$ or $j \models_{\mathfrak{C}b} \Sigma$ or $\forall z\ (jRz \rightarrow z \models_{\mathfrak{C}b} \underline{X}))$
7 $\forall i\ (i \models_{\mathfrak{C}b} \Gamma$ or $\forall j\ (iRj \rightarrow j \not\models_{\mathfrak{C}b} \Box\alpha$ or $j \models_{\mathfrak{C}b} \Sigma$ or $\forall z\ (jRz \rightarrow z \models_{\mathfrak{C}b} \underline{X})))$

The last line of the proof is the conclusion of the rule. From this argument and Lemma 8.2, we conclude that the rule b is valid in the class of symmetric frames.

– 5. Let us consider the rule in the form

$$\frac{\Box\alpha, \Gamma/(\Box\alpha, \Sigma/\underline{X})}{\Gamma/(\Box\alpha, \Sigma/\underline{X})}$$

By the inductive hypothesis, we have $\forall i\ (i \models_{\mathfrak{C}b4} \neg\Box\alpha, \Gamma$ or $i \models^{*}_{\mathfrak{C}b4} \neg\Box\alpha, \Sigma$ or $j \models^{*}_{\mathfrak{C}b4} \underline{X})$, i.e. $\forall i\ (i \not\models_{\mathfrak{C}b4} \Box\alpha$ or $i \models_{\mathfrak{C}b4} \Gamma$ or $\forall j\ (iRj \rightarrow j \not\models_{\mathfrak{C}b4} \Box\alpha$ or $j \models_{\mathfrak{C}b4} \Sigma$ or $\forall z\ (jRz \rightarrow z \models_{\mathfrak{C}b4} \underline{X})))$. From this, we get

1 $\forall i\ (i \not\models_{\mathfrak{C}b4} \Box\alpha$ or $i \models_{\mathfrak{C}b4} \Gamma$ or $\forall j\ (\neg iRj$ or $j \not\models_{\mathfrak{C}b4} \Box\alpha$ or $j \models_{\mathfrak{C}b4} \Sigma$ or $\forall z\ (jRz \rightarrow z \models_{\mathfrak{C}b4} \underline{X})))$
2 $\forall i\ \forall j\ (i \not\models_{\mathfrak{C}b4} \Box\alpha$ or $i \models_{\mathfrak{C}b4} \Gamma$ or $\neg iRj$ or $j \not\models_{\mathfrak{C}b4} \Box\alpha$ or $j \models_{\mathfrak{C}b4} \Sigma$ or $\forall z\ (jRz \rightarrow z \models_{\mathfrak{C}b4} \underline{X}))$
3 $\forall i\ \forall j\ (\exists w\ (iRw$ and $w \not\models_{\mathfrak{C}b4} \alpha)$ or $i \models_{\mathfrak{C}b4} \Gamma$ or $\neg iRj$ or $j \not\models_{\mathfrak{C}b4} \Box\alpha$ or $j \models_{\mathfrak{C}b4} \Sigma$ or $\forall z\ (jRz \rightarrow z \models_{\mathfrak{C}b4} \underline{X}))$
4 $\forall i\ \forall j\ \exists w((w \not\models_{\mathfrak{C}b4} \alpha$ and $iRw)$ or $i \models_{\mathfrak{C}b4} \Gamma$ or $\neg iRj$ or $j \not\models_{\mathfrak{C}b4} \Box\alpha$ or $j \models_{\mathfrak{C}b4} \Sigma$ or $\forall z\ (jRz \rightarrow z \models_{\mathfrak{C}b4} \underline{X}))$

From 4, since we are dealing with *transitive* and *symmetric* – and hence *euclidean* – frames, we have

5 $\forall i\ \forall j\ \exists w((w \nvDash_{\mathfrak{C}b4} \alpha$ and $(\neg iRj$ or $jRw))$ or $i \vDash_{\mathfrak{C}b4} \Gamma$ or $\neg iRj$ or $j \nvDash_{\mathfrak{C}b4} \Box\alpha$ or $j \vDash_{\mathfrak{C}b4} \Sigma$ or $\forall z\ (jRz \to z \vDash_{\mathfrak{C}b4} \underline{X}))$
6 $\forall i\ \forall j\ \exists w((w \nvDash_{\mathfrak{C}b4} \alpha$ and $jRw)$ or $\neg iRj$ or $i \vDash_{\mathfrak{C}b4} \Gamma$ or $\neg iRj$ or $j \nvDash_{\mathfrak{C}b4} \Box\alpha$ or $j \vDash_{\mathfrak{C}b4} \Sigma$ or $\forall z\ (jRz \to z \vDash_{\mathfrak{C}b4} \underline{X}))$
7 $\forall i\ \forall j\ (\exists w(w \nvDash_{\mathfrak{C}b4} \alpha$ and $jRw)$ or $i \vDash_{\mathfrak{C}b4} \Gamma$ or $\neg iRj$ or $j \nvDash_{\mathfrak{C}b4} \Box\alpha$ or $j \vDash_{\mathfrak{C}b4} \Sigma$ or $\forall z\ (jRz \to z \vDash_{\mathfrak{C}b4} \underline{X}))$
8 $\forall i\ \forall j\ (j \nvDash_{\mathfrak{C}b4} \Box\alpha$ or $i \vDash_{\mathfrak{C}b4} \Gamma$ or $\neg iRj$ or $j \nvDash_{\mathfrak{C}b4} \Box\alpha$ or $j \vDash_{\mathfrak{C}b4} \Sigma$ or $\forall z\ (jRz \to z \vDash_{\mathfrak{C}b4} \underline{X}))$
9 $\forall i\ \forall j\ (i \vDash_{\mathfrak{C}b4} \Gamma$ or $\neg iRj$ or $j \nvDash_{\mathfrak{C}b4} \Box\alpha$ or $j \vDash_{\mathfrak{C}b4} \Sigma$ or $\forall z\ (jRz \to z \vDash_{\mathfrak{C}b4} \underline{X}))$
10 $\forall i\ (i \vDash_{\mathfrak{C}b4} \Gamma$ or $\forall j\ (iRj \to j \nvDash_{\mathfrak{C}b4} \Box\alpha$ or $j \vDash_{\mathfrak{C}b4} \Sigma$ or $\forall z\ (jRz \to z \vDash_{\mathfrak{C}b4} \underline{X}))$

The last line of the proof is the conclusion of the rule. From this argument and Lemma 8.2, we conclude that the rule 5 is valid in the class of transitive and symmetric – and hence euclidean – frames. □

8.2 Semantic Completeness of the Tree-Hypersequent Calculi

This section is concerned with the proof that our calculi are complete with respect to the corresponding class of frames. In [15] Brünnler gives an elegant proof for this. He provides a terminating proof search procedure, which, when it fails, can serve as the basis for the construction of a countermodel. We turn to this below.

The first idea of Brünnler's proof is to prove completeness not directly of the $\mathbf{Thsk}_L^*$ calculi, but of different equivalent calculi that we are going to call $(\mathbf{Thsk}_L^*)^+$, and that are defined as follows.

For each rule $\mathcal{R}$, we define a rule $\mathcal{R}^+$ which keeps the main formula from the conclusion. For the rule $\Box A$ and the special logic rules we have $\mathcal{R} = \mathcal{R}^+$. For the other rules we have

$$\frac{G[\neg\alpha, \Gamma, \alpha]}{G[\neg\alpha, \Gamma]}\ (\neg A)^+ \qquad \frac{G[\alpha, \Gamma, \neg\alpha]}{G[\Gamma, \neg\alpha]}\ (\neg K)^+$$

$$\frac{G[\alpha, \beta, \alpha\wedge\beta, \Gamma]}{G[\alpha\wedge\beta, \Gamma]}\ (\wedge A')^+ \qquad \frac{G[\Gamma, \alpha, \alpha\wedge\beta] \qquad G[\Gamma, \beta, \alpha\wedge\beta]}{G[\Gamma, \alpha\wedge\beta]}\ (\wedge K)^+$$

$$\frac{G[\Gamma, \Box\alpha / \Rightarrow \alpha]}{G[\Gamma, \Box\alpha]}\ (\Box K)^+$$

where the sequent $\Gamma, \Box\alpha$ does not have any immediate successive sequent (or more succinctly, child-sequent) that contains the formula α on the left side.

$$\frac{G[\Box\alpha, \Gamma / \alpha \Rightarrow]}{G[\Box\alpha, \Gamma]}\ (d)^+$$

where the sequent $\Box\alpha, \Gamma$ does not have any child-sequent.

Definition 8.4 The *set tree-hypersequent* of the tree-hypersequent $\Gamma/G_1; ...; G_n$ is the underlying set of

$$\Delta/H_1; ...; H_n$$

where $H_1; ...; H_n$ are the set tree-hypersequent of $G_1; ...; G_n$. Clearly the set tree-hypersequent of a tree-hypersequent is still a tree-hypersequent since a set is a multiset.

Any rule $\mathcal{R}^+$ carries the proviso that for all of its premises the set tree-hypersequent is different from the set tree-hypersequent of the conclusion.

Given a calculus $\mathbf{Y} \in \mathbf{Thsk}^*_L$, the calculus $(\mathbf{Y})^+$ is obtained by replacing each rule $\mathcal{R}$ of $\mathbf{Y}$ by the corresponding rule $(\mathcal{R})^+$. The calculi $\mathbf{Thsk}^*_L$ and $\left(\mathbf{Thsk}^*_L\right)^+$ will happen to be equivalent. For now we prove the following.

Lemma 8.5 *For all tree-hypersequents G,*

$$if \ \vdash G \ in \ \left(\mathbf{Thsk}^*_L\right)^+, \ then \ \vdash G \ in \ \mathbf{Thsk}^*_L$$

Proof By straightforward induction on the height of derivations in $\left(\mathbf{Thsk}^*_L\right)^+$, using contraction and weakening. □

In order to prove completeness, let us introduce some closure relations. For the sake of brevity we will use the following names: se, re, tr, sy to denote the properties of: seriality, reflexivity, transitivity, symmetry, respectively.

Definition 8.6 Let $\rightarrow$ be a binary relation on a set W. Then $\leftarrow$ denotes its *inverse*, $\leftrightarrow$ its *symmetric closure*, $\rightarrow^+$ its *transitive closure,* and $\rightarrow^*$ its *reflexive-transitive closure.* More generally, for $X \subseteq \{re, tr, sy\}$, $\rightarrow^X$ denotes the smallest relation that includes $\rightarrow$ and has the properties in X.

Lemma 8.7 *Let $\rightarrow$ be a binary relation on a set W. Then for all $X \subseteq \{re, tr, sy\}$, the relation $\rightarrow^X$ is well-defined.*

Proof It is easy to check. □

Definition 8.8 Let $\rightarrow$ be a binary relation on a set W. Its serial closure, denoted $\rightarrow^{se}$, is obtained from $\rightarrow$ by adding loops on all elements of W that violate seriality. For $X \subseteq \{re, tr, sy\}$ the relation $\rightarrow^{X \cup se}$ is defined as $(\rightarrow^X)^{se}$.

Lemma 8.9 *Let $\rightarrow$ be a binary relation on a set W. If $\rightarrow$ satisfies a frame condition in $\{re, tr, sy\}$, then $\rightarrow^{se}$ also satisfies that frame condition.*

Proof Concerning reflexivity, this is clear since a reflexive relation is its own serial closure. Concerning symmetry, this is clear since only loops are added, which are their own inverses. Concerning transitivity, we see the contrapositive. Assume that $\rightarrow^{se}$ is not transitive. Then we have $i \rightarrow^{se} j$ and $j \rightarrow^{se} z$, but not $i \rightarrow^{se} z$. But then $i \neq j$ and $j \neq z$, and thus $i \rightarrow j$ and $j \rightarrow z$, but not $i \rightarrow z$.

Concerning euclideanness, we also see the contrapositive. Assume that $\to^{se}$ is not euclidean. Then we have $i \to^{se} j$ and $i \to^{se} z$, but not $j \to^{se} z$. But then $i \neq j$, and thus $i \to j$, but not $j \to z$. To show that $\to$ is not euclidean we need that $i \to z$. If $i \neq z$ then this is clear. Assume that $i = z$. Since $i \to^{se} z$, and since i does not violate seriality, we have $i \to z$. □

Definition 8.10 A leaf of a tree-hypersequent (thinking the tree-hypersequent as a tree-frame of Kripke semantics) is *cyclic* if in its branch there exists a sequent that contains the same set of formulas.

Definition 8.11 A sequent of a tree-hypersequent is *finished* for a tree-hypersequent calculus **Y** if no rule of that calculus applies to one of its formulas. A tree-hypersequent is finished for a tree-hypersequent calculus **Y** if all sequents that compose it are finished or cyclic.

Definition 8.12 We define a procedure *prove*$(G, (\mathbf{Y})^+)$, which takes a tree-hypersequent G and a calculus $(\mathbf{Y})^+ \in \left(\mathbf{Thsk}^*_L\right)^+$, and builds a derivation tree for G by applying rules from that calculus to non-initial and unfinished derivation leaves in a bottom-up fashion, as follows:

1. keep applying all the rules of $(\mathbf{Y})^+$ which are not the rules $(\Box K)^+$ and $(d)^+$ as long as possible;
2. wherever possible, apply the rules $(\Box K)^+$ and $(d)^+$ once.

Repeat this operation until each non-initial derivation leaf of the tree-hypersequent G is finished. If *prove* $(G, (\mathbf{Y})^+)$ terminates and all derivation leaves are initial then it succeeds; otherwise, i.e. if it terminates and there is a non initial derivation leaf, it fails.

Definition 8.13 The *size* of a tree-hypersequent G, $s(G)$, is the number of sequents that compose it. The *set of subformulas* of a tree-hypersequent G, denoted *sf(G)*, is the set of all subformulas of all formulas that compose all sequents that belong to the tree-hypersequent.

Lemma 8.14 *For all calculi* $\left(\mathbf{Thsk}^*_L\right)^+$ *and for all tree-hypersequents G, the procedure* prove $(G, (\mathbf{Y})^+)$ *terminates after at most* $2^{|sf(G)|}$ *iterations.*

Proof Consider a sequence of tree-hypersequents along a given branch of the derivation starting from the root. None of the rules that we can apply in accordance with step 1 creates new sequents in the tree-hypersequent, but each of them causes the set of formulas of some sequent belonging to the tree-hypersequent to strictly grow. By the subformula property, only finitely many formulas can occur in a sequent, so step 1 terminates. If after step 1 there is an unfinished leaf in a tree-hypersequent, then the size of the tree-hypersequent strictly grows in step 2. Since there are only $2^{|sf(G)|}$ different sets of formulas that can occur, each unfinished tree-hypersequent leaf has to be cyclic eventually. □

After the next definition, we will prove the completeness theorem for our $\left(\mathbf{Thsk}^*_L\right)^+$ calculi.

Definition 8.15 In this definition we exploit the strong analogy between tree-hypersequents and tree-frames of Kripke semantics. In particular, a tree-hypersequent H is an *immediate subtree* of a tree-hypersequent G if G is of the form $\Gamma/H; H_1; ...; H_n$. It is a *proper subtree* if it is an immediate subtree either of G or of a proper subtree of G, and it is a subtree if it is either a proper subtree of G or $G = H$. The *set of all subtrees* of G is denoted by $st(G)$.

Theorem 8.16 *For all $X \subseteq \{se, re, tr, sy\}$ and all tree-hypersequents G, we have that*

(i) if G is valid with respect to a X-frame, then the tree-hypersequent calculus $\mathbf{Y} \in \mathbf{Thsk}^*_L$ that corresponds to the X-frame is such that $\vdash G$ in $\mathbf{Y}$.
(ii) If $prove(G, (\mathbf{Y})^+)$ fails, then G is not valid in the frame that corresponds to the calculus $(\mathbf{Y})^+$.

Proof The contrapositive of (i) follows from (ii). If $\nvdash G$ in a calculus $\mathbf{Y} \in \mathbf{Thsk}^*_L$, then by Lemma 8.5, also $\nvdash G$ in $(\mathbf{Y})^+ \in \left(\mathbf{Thsk}^*_L\right)^+$, and thus, in particular, $prove(G, (\mathbf{Y})^+)$ cannot yield a derivation and by Lemma 8.14 has to fail. For (ii), we define a model $\mathfrak{M}$ on an X-frame, for $X \subseteq \{se, re, sy, tr\}$, for which we prove that it is a countermodel for G.

Let $G^\star$ be the set tree-hypersequent of the non initial tree-hypersequent obtained. Let Υ be the set of all cyclic leaves in $G^\star$. Let $W = st(G^\star)\backslash\Upsilon$. Let $f : \Upsilon \rightarrow W$ be some function which maps a cyclic leaf to a tree-hypersequent in W whose root carries the same set of formulas, and extend f to $st(G^*)$ by the identity on W. Define a binary relation $\rightarrow$ on W such that $I \rightarrow H$ if, and only if, either (i) H is an immediate subtree of I, or (ii) I has an immediate subtree $J \in \Upsilon$ and $f(J) = H$. Let $v(G, p)$ such that

$$v(G, p) = \begin{cases} 1 : p \;\; occurs \;\; on \;\; the \;\; left \;\; side \; of \; a \; sequent \; \Gamma \in G \\ 0 : otherwise \end{cases}$$

Let $\mathfrak{M} = (W, \rightarrow^X, v)$.

Claim 1. For all $I, H \in W$ such that $I \rightarrow^X H$, for all α occurring on the left side of a sequent that belongs to the tree-hypersequent G, we have: if $\Box\alpha \in I$, then $\alpha \in H$.
$X = \emptyset$. By the definition of $\rightarrow$ there is an immediate subtree of I whose root sequent carries the same set of formulas as the root node of H. By the $\Box A$ rule we have α in (the root sequent of) all immediate subtrees of I.
$X = \{re\}$. $I \rightarrow^{re} H$ if, and only if, $I \rightarrow H$ or $I = H$. In the second case $\alpha \in H$ follows from the t rule.
$X = \{tr\}$. $I \rightarrow^{tr} H$ if, and only if, there is a sequence

$$I = I_0 \rightarrow I_1 \rightarrow I_2 \rightarrow ... \rightarrow I_n = H$$

with $n \geqslant 1$. An induction on i gives us that $\Box\alpha \in I_i$, for $0 \leqslant i \leqslant n$, for the rule 4. That $\alpha \in I_n$ follows from that by the $\Box A$ rule.

$X = \{sy\}$. $I \rightarrow^{sy} H$ if, and only if, $I \rightarrow H$ or $H \rightarrow I$. In the second case $\alpha \in H$ follows from the b rule.
$X = \{re, sy\}$. $I \rightarrow^{re,sy} H$ if, and only if, $I \rightarrow H$ or $H \rightarrow I$ or $I = H$. In these cases $\alpha \in H$ follows from the $\Box A$, b and t rules, respectively.
$X = \{re, tr\}$. $I \rightarrow^{re,tr} H$ if, and only if, $I \rightarrow^{+} H$ or $I = H$. In the first case $\alpha \in H$ follows from the 4 rule, in the second case from the t rule.
$X = \{tr, sy\}$. $I \rightarrow^{tr,sy} H$ if, and only if, $I \leftrightarrow^{+} H$. Thus there is a tree-hypersequent J such that either $J \rightarrow H$ or $J \leftarrow H$. The rules 4 and 5 imply that $\Box\alpha$ is in every subtree of $G^{\star}$, and thus in particular in J. We have $\alpha \in H$ in the first case by the $\Box A$ rule, in the second case by the b rule.
$X = \{re, tr, sy\}$. $I \rightarrow^{re,tr,sy} H$ iff $I \leftrightarrow^{*} H$. We have $\Box A$ in all subtrees of $G^{\star}$ by the rules 4, 5, and thus also α by the t rule.
$X \subseteq \{re, tr, sy\}$ and $I \rightarrow^{X \cup se} H$. The argument for all these cases is similar to the arguments for the cases X without the serial closure. Indeed $I \rightarrow^{X \cup se} H$ if, and only if, $I \rightarrow^{X} H$ or ($I = H$ and there is no I' with $I \rightarrow^{X} I'$). In the second case, thanks to the rule $(d)^{+}$, there is no formula $\Box\alpha$ in I and thus our claim is trivially true.

Claim 2. For all $I \in W$, we have:

- for all $\alpha \in I$ such that they occur on the left side of the sequent, $I \models_{\mathfrak{M}} \alpha$
- for all $\alpha \in I$ such that they occur on the right side of the sequent, $I \not\models_{\mathfrak{M}} \alpha$

By induction on the complexity of the formula α. Concerning atoms, it follows from the definition of the valuation function, and the fact that $G^{\star}$ is not an initial tree-hypersequent. Concerning the propositional connectives, it is clear from the shape of the $(\neg)^{+}$ rules and the $(\wedge)^{+}$ rules. If $\alpha = \Box\beta$ and it occurs on the right side of the sequent, then by the $(\Box K)^{+}$ rule, we have at least one $H \in I$ with $\beta \in H$. By the inductive hypothesis, we have $H \not\models_{\mathfrak{M}} \beta$ and thus $I \not\models_{\mathfrak{M}} \Box\beta$. If $\alpha = \Box\beta$ and it occurs on the left side of the sequent, then, by Claim 1, we have $\beta \in H$ for all H with $I \rightarrow^{X} H$. Thus, by the inductive hypothesis, we have $H \models_{\mathfrak{M}} \beta$ and then $I \models_{\mathfrak{M}} \Box\beta$.

Claim 3. For all $I \in st(G^{\star})$, $f(I) \not\models_{\mathfrak{M}} I$.
By induction on the complexity of the tree-hypersequent I. If the tree-hypersequent I is just a classical sequent, then it follows from Claim 2 and the fact that a formula is in I if, and only if, it is in $f(I)$. So let

$$I = \Delta / I_1; ...; I_n \qquad for\ n > 0$$

Then $f(I) = I$. We have $f(I) \not\models_{\mathfrak{M}} \Delta$ by Claim 2, and $f(I) \not\models_{\mathfrak{M}} I_i$ because $I \rightarrow f(I_i)$ and by inductive hypothesis $f(I_i) \not\models_{\mathfrak{M}} I_i$.
Since all rules seen top-down preserve countermodels Claim 3 implies that $\not\models_{\mathfrak{M}} G$.
□

Chapter 9
A Hypersequent Calculus for the System S5

S5 is undoubtedly one of the most important and well-known of all SLH-systems. When considered from the point of view of Kripke semantics, **S5** is rather peculiar since it can be described in two different albeit equivalent ways. The first one (that we illustrated in Section 2.1) specifies the properties that the accessibility relation of a Kripke frame should satisfy: **S5** is sound and complete with respect to the class of reflexive, transitive and symmetric frames (or, equivalently, with respect to the class of reflexive and euclidean frames). A second and easier way to study **S5** semantically exploits Kripke frames where the accessibility relation is absent: **S5** is sound and complete with respect to the class of frames which are just non-empty sets of worlds. (From now on, we will call this kind of frame **S5** Kripke frames.) This second way is evidently simpler. It would then be useful and interesting to reflect this simplicity at the syntactic level, within a Gentzen system.

In the previous sections we introduced the calculus $\mathbf{Thss5}_L$ for the system **S5**. $\mathbf{Thss5}_L$ reflects, at the syntactic level, the more complex semantic description that can be given of **S5**: indeed it is composed of the rules t, 4, b and 5, that are meant to reflect the semantic properties of reflexivity, transitivity, symmetry and euclideaness, respectively. We have already proved (see Lemma 6.17, p. 134) that the rule

$$\frac{G[\Gamma/(\Sigma/(\Delta/\underline{X}); \underline{X}')]}{G[\Gamma/(\Delta/\underline{X}); (\Sigma/\underline{X}')]}\,\tilde{5}$$

is admissible in $\mathbf{Thss5}_L$.[1] Roughly speaking, this rule allows one to pass from the symbol / to the symbol ;, which is to say, in more intuitive terms, this rule allows one to pass from the presence of an accessibility relation to its absence. Given this result, an idea naturally arises: we could construct an alternative sequent calculus for **S5** where we still have n different sequents at a time, but where there is no longer an order on these sequents, i.e. there is no longer an accessibility relation over the set of worlds. Thus, in this sequent calculus, we no longer need to deal with the two symbols / and ;, but with one of them only.

This section will be dedicated to the realisation of such an idea: we will develop a new Gentzen system for the modal logic **S5**, which, by contrast with $\mathbf{Thss5}_L$,

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will reflect, at the syntactic level, the simplicity of the **S5** Kripke frames. In this new sequent calculus we will use hypersequents – where the only meta-linguistic symbol is the semi-colon – and not tree-hypersequents. Let us however emphasise that the return to hypersequents is motivated by the work with tree-hypersequents. In other words, hypersequents stand to tree-hypersequents, as **S5** Kripke frames stand to Kripke frames.

9.1 The Calculus ThS5$_L$

Syntactic Notation

- ; denote a meta-linguistic symbol.
- $\Gamma, \Delta, ...$: classical sequents.
- $G, H, ...$: hypersequents.

We will adopt the Convention 6.1, p. 122.

Definition 9.1 A *hypersequent* is a syntactic object of the form

$$\Gamma_1;\, \Gamma_2;\; ...;\; \Gamma_n$$

where Γ_i $(i = 1, ..., n)$ is a classical sequent.

Definition 9.2 The *interpretation* τ of a hypersequent is definable in the following inductive way:

- $(M \Rightarrow N)^\tau := \bigwedge M \to \bigvee N$
- $(\Gamma_1; \Gamma_2; ... ; \Gamma_n)^\tau := \Box\Gamma_1^\tau \vee \Box\Gamma_2^\tau \vee ... \vee \Box\Gamma_n^\tau$

A hypersequent is then a *multiset* of classical sequents, which is to say, the order of the sequents in a hypersequent does not count.

The postulates of the calculus **ThS5**$_L$ are[2]:

Initial Hypersequents

$G;\;\; p, \Gamma, p$

Logical Rules

Propositional Rules

$$\frac{G; \Gamma, \alpha}{G; \neg\alpha, \Gamma}\neg A \qquad\qquad \frac{G; \alpha, \Gamma}{G; \Gamma, \neg\alpha}\neg K$$

$$\frac{G; \alpha, \beta, \Gamma}{G; \alpha \wedge \beta, \Gamma}\wedge A' \qquad\qquad \frac{G; \Gamma, \alpha \qquad G; \Gamma, \beta}{G; \Gamma, \alpha \wedge \beta}\wedge K$$

Modal Rules

$$\frac{G; \alpha, \Box\alpha, \Gamma}{G; \Box\alpha, \Gamma}\ \Box A_1 \qquad\qquad \frac{G; \Gamma; \Rightarrow \alpha}{G; \Gamma, \Box\alpha}\ \Box K$$

$$\frac{G; \Box\alpha, \Gamma; \alpha, \Sigma}{G; \Box\alpha, \Gamma; \Sigma}\ \Box A_2$$

Two remarks on modal rules are in order. The first one concerns the rules $\Box A_i$ ($i = 1, 2$) only. The repetition of the principal formula $\Box\alpha$ in the premise of each of these rules only serves to prove in a short and simple manner the main results obtainable in the calculus, as was the case for the rule $\Box A$ and the special logical rules in the calculi $\mathbf{Thsk}^*_L$. The second remark concerns the three modal rules. It is easy to informally understand these rules if we compare the hypersequent to an **S5** Kripke frame, and the sequents that compose the hypersequent to different worlds of the **S5** Kripke frame. From this perspective, the rule $\Box K$ says, if read bottom-up, that, if the formula $\Box\alpha$ is false at a world i, then we can create a new world j where the formula α is false; on the other hand, the rules $\Box A_i$ tell us, if they are read bottom-up and considered together, that, if the formula $\Box\alpha$ is true at a world i, then the formula α is true in any world of the frame.

9.2 Admissibility of the Structural Rules in $\mathbf{ThS5}_L$

In this section we will show which structural rules are admissible in the calculus $\mathbf{ThS5}_L$. Moreover we will prove that all the propositional and modal rules are height-preserving invertible. The proof of the admissibility of the cut-rule will be shown in Section 9.4.

Lemma 9.3 *Hypersequents of the form* $G; \alpha, \Gamma, \alpha$*, with* α *an arbitrary modal formula, are derivable in* $\mathbf{ThS5}_L$*.*

Proof By straightforward induction on α. □

Lemma 9.4 *The rule of merge*

$$\frac{G; \Gamma; \Sigma}{G; \Gamma \centerdot \Sigma}\ merge$$

is height-preserving admissible in $\mathbf{ThS5}_L$*.*

Proof By induction on the derivation of the premise.

If the premise is an initial hypersequent, then so is the conclusion. If the premise is inferred by a propositional rule, then the inference is clearly preserved. Let us consider the example of the rule $\neg K$,

$$\frac{{}^{\langle n-1\rangle}G; \alpha, \Gamma; \Sigma}{{}^{\langle n\rangle}G; \Gamma, \neg\alpha; \Sigma}\ \neg K \quad \leadsto^3 \quad \frac{{}^{\langle n-1\rangle}G; \alpha, \Gamma \centerdot \Sigma}{{}^{\langle n\rangle}G; \Gamma \centerdot \Sigma, \neg\alpha}\ \neg K$$

If the premise is inferred by the modal rule $\Box K$, then the inference is preserved.

$$\frac{{}^{\langle n-1\rangle}G;\, \Gamma;\, \Sigma;\, \Rightarrow \alpha}{{}^{\langle n\rangle}G;\, \Gamma, \Box\alpha;\, \Sigma}\,\Box K \quad \rightsquigarrow \quad \frac{{}^{\langle n-1\rangle}G;\, \Gamma \centerdot \Sigma;\, \Rightarrow \alpha}{{}^{\langle n\rangle}G;\, \Gamma \centerdot \Sigma, \Box\alpha}\,\Box K$$

If the premise is inferred by the modal rule $\Box A_1$, then the inference is preserved.

$$\frac{{}^{\langle n-1\rangle}G;\, \alpha, \Box\alpha, \Gamma;\, \Sigma}{{}^{\langle n\rangle}G;\, \Box\alpha, \Gamma;\, \Sigma}\,\Box A_1 \quad \rightsquigarrow \quad \frac{{}^{\langle n-1\rangle}G;\, \Box\alpha, \alpha, \Gamma \centerdot \Sigma}{{}^{\langle n\rangle}G;\, \Box\alpha, \Gamma \centerdot \Sigma}\,\Box A_1$$

If the premise is inferred by the modal rule $\Box A_2$, there are two significant cases to analyse: the one where the rule $\Box A_2$ has been applied between the two sequents Γ and Σ; and the one where the rule $\Box A_2$ has been applied between a third sequent, let us call it Φ, and Γ (or, equivalently, Σ). The two situations are similar, therefore, we will limit a detailed analysis to the first case.

$$\frac{{}^{\langle n-1\rangle}G;\, \Box\alpha, \Gamma;\, \alpha, \Sigma}{{}^{\langle n\rangle}G;\, \Box\alpha, \Gamma;\, \Sigma}\,\Box A_2 \quad \rightsquigarrow \quad \frac{{}^{\langle n-1\rangle}G;\, \Box\alpha, \alpha, \Gamma \centerdot \Sigma}{{}^{\langle n\rangle}G;\, \Box\alpha, \Gamma \centerdot \Sigma}\,\Box A_1$$

□

Lemma 9.5 *The rule of external weakening*

$$\frac{G}{G;\, \Gamma}\, EW$$

is height-preserving admissible in **ThS5**$_L$.

Proof By straightforward induction on the derivation of the premise. □

Lemma 9.6 *The rule of internal weakening*

$$\frac{G;\, \Gamma}{G;\, \Gamma \centerdot \Sigma}\, IW$$

is height-preserving admissible in **ThS5**$_L$.

Proof It follows by the height-preserving admissibility of the two rules of merge and external weakening. □

Lemma 9.7 *The propositional and modal rules of* **ThS5**$_L$ *are height-preserving invertible.*

Proof The proof is by induction on the height of the derivation of the premise of the rule considered. The cases of the propositional rules are dealt with in the classical way. The only differences – the fact that we are dealing with hypersequents, and the cases where the last applied rule is one of the rules $\Box A_i$ or $\Box K$ – are dealt with easily.

The rules $\Box A_i$ are trivially height-preserving invertible since both their premises are obtained by weakening from their respective conclusions, and weakening is height-preserving admissible.

Let us show in detail the invertibility of the rule $\Box K$. If $G; \Gamma, \Box\alpha$ is an initial hypersequent, then so is $G; \Gamma; \Rightarrow \alpha$. If $G; \Gamma, \Box\alpha$ is obtained by a propositional rule $\mathcal{R}$, we apply the inductive hypothesis on the premise(s) $G^{'}; \Gamma^{'}, \Box\alpha$ ($G^{''}; \Gamma^{''}, \Box\alpha$), and we obtain derivation(s), of height $n-1$, of $G^{'}; \Gamma^{'}; \Rightarrow \alpha$ ($G^{''}; \Gamma^{''}; \Rightarrow \alpha$). By applying the rule $\mathcal{R}$, we obtain a derivation of height n of $G; \Gamma; \Rightarrow \alpha$. If $G; \Gamma, \Box\alpha$ is of the form $G; \Box\beta, \Gamma^{'}, \Box\alpha$, then it may have been obtained by the rule $\Box A_1$, as well as by the rule $\Box A_2$. Since the procedure is the same in both cases, analysing one of the rules is sufficient. We will consider the case of $\Box A_1$. We apply the inductive hypothesis on $G; \Box\beta, \beta, \Gamma^{'}, \Box\alpha$, and we obtain a derivation of height $n-1$ of $G; \Box\beta, \beta, \Gamma^{'}; \Rightarrow \alpha$. By applying the rule $\Box A_1$, we obtain a derivation of height n of $G; \Box\beta, \Gamma^{'}; \Rightarrow \alpha$.

If $G; \Gamma, \Box\alpha$ is obtained by the modal rule $\Box K$ and $\Box\alpha$ is not the principal formula, then this case can be treated analogously to the one of the rules $\Box A_i$. Finally, if $G; \Gamma, \Box\alpha$ is obtained by the modal rule $\Box K$ and $\Box\alpha$ is the principal formula, the premise of the last step gives the conclusion. □

Lemma 9.8 *The rules of contraction*

$$\frac{G; \alpha, \alpha, \Gamma}{G; \alpha, \Gamma}\, CA \qquad \frac{G; \Gamma, \alpha, \alpha}{G; \Gamma, \alpha}\, CK$$

are height-preserving admissible in $\mathbf{ThS5}_L$.

Proof By induction on the derivation of the premises $G; \alpha, \alpha, \Gamma$ and $G; \Gamma, \alpha, \alpha$. We only analyse the case of the rule CK. The case of the rule CA is similar.

If $G; \Gamma, \alpha, \alpha$ is an initial hypersequent, so is $G; \Gamma, \alpha$. If $G; \Gamma, \alpha, \alpha$ is obtained by a rule $\mathcal{R}$ that does not have any of the two occurrences of the formula α as principal, we apply the inductive hypothesis to the premise(s) $G^{'}; \Gamma^{'}, \alpha, \alpha$ ($G^{''}; \Gamma^{''}, \alpha, \alpha$), obtaining derivation(s) of height $n-1$ of $G^{'}; \Gamma^{'}, \alpha$ ($G^{''}; \Gamma^{''}, \alpha$). By applying the rule $\mathcal{R}$, we obtain a derivation of height n of $G; \Gamma, \alpha$.

If $G; \Gamma, \alpha, \alpha$ is obtained by a propositional or modal rule, and one of the two occurrences of the formula α is principal, then the rule that concludes $G; \Gamma, \alpha, \alpha$ is a K-rule, and we must analyse the following three cases: $\neg K$, $\wedge K$, $\Box K$.

$\neg K$:

$$\frac{{}^{\langle n-1\rangle}G; \beta, \Gamma, \neg\beta}{{}^{\langle n\rangle}G; \Gamma, \neg\beta, \neg\beta}\,\neg K \quad \dashrightarrow^{4} \quad \cfrac{\cfrac{{}^{\langle n-1\rangle}G; \beta, \beta, \Gamma}{{}^{\langle n-1\rangle}G; \beta, \Gamma}\, i.h.}{{}^{\langle n\rangle}G; \Gamma, \neg\beta}\,\neg K$$

$\wedge K$:

$$\frac{{}^{\langle n-1\rangle}G; \Gamma, \beta, \beta\wedge\gamma \quad {}^{\langle n-1\rangle}G; \Gamma, \gamma, \beta\wedge\gamma}{{}^{\langle n\rangle}G; \Gamma, \beta\wedge\gamma, \beta\wedge\gamma}\,\wedge K \quad \dashrightarrow$$

$$\dfrac{\dfrac{{}^{\langle n-1\rangle}G;\,\Gamma,\beta,\beta}{{}^{\langle n-1\rangle}G;\,\Gamma,\beta}\,i.h. \qquad \dfrac{{}^{\langle n-1\rangle}G;\,\Gamma,\gamma,\gamma}{{}^{\langle n-1\rangle}G;\,\Gamma,\gamma}\,i.h.}{{}^{\langle n\rangle}G;\,\Gamma,\beta\wedge\gamma}\,\wedge K$$

$\Box K$:

$$\dfrac{{}^{\langle n-1\rangle}G;\,\Gamma,\Box\beta;\Rightarrow\beta}{{}^{\langle n\rangle}G;\,\Gamma,\Box\beta,\Box\beta}\,\Box K \quad \dashrightarrow \quad \dfrac{\dfrac{\dfrac{{}^{\langle n-1\rangle}G;\,\Gamma;\Rightarrow\beta;\Rightarrow\beta}{{}^{\langle n-1\rangle}G;\,\Gamma;\Rightarrow\beta,\beta}\,merge}{{}^{\langle n-1\rangle}G;\,\Gamma;\Rightarrow\beta}\,i.h.}{{}^{\langle n\rangle}G;\,\Gamma,\Box\beta}\,\Box K$$

□

9.3 Adequacy of ThS5$_L$

This section will show that the sequent calculus **ThS5**$_L$ proves exactly the same formulas as the corresponding Hilbert system **S5**.

Theorem 9.9 *For all hypersequents G, and for all formulas α,*

(i) if $\vdash G$ *in* ***ThS5***$_L$*, then* $\vdash (G)^\tau$ *in* ***S5****.*
(ii) If $\vdash \alpha$ *in* ***S5****, then* $\vdash\Rightarrow \alpha$ *in* ***ThS5***$_L$*.*

Proof By induction on the height of derivations in **S5** and **ThS5**$_L$, respectively. (i) The case of the axioms is trivial, while, for the induction steps with the propositional rules, all we need is classical logic and the fact that if **S5** $\vdash \alpha_1 \to (\alpha_2 \to \ldots \to (\alpha_n \to \beta)\ldots)$, then **S5** $\vdash \Box\alpha_1 \to (\Box\alpha_2 \to \ldots \to (\Box\alpha_n \to \Box\beta)\ldots)$. For what concerns the induction steps for modal rules, we again exploit the fact that, if **S5** $\vdash \alpha_1 \to (\alpha_2 \to \ldots \to (\alpha_n \to \beta)\ldots)$, then **S5** $\vdash \Box\alpha_1 \to (\Box\alpha_2 \to \ldots \to (\Box\alpha_n \to \Box\beta)\ldots)$, and the axiom $\Box\alpha \to \alpha$.

(ii) The classical axioms and modus ponens are derived as usual. The derivations of the axiom T, the axiom 4, the axiom B are the same as the ones we have shown in the proof of Theorem 6.22, p. 138. The only difference consists in using one of the rules $\Box A_i$ instead of the special logical rules t, 4 and b. The derivation of the axiom 5 is different since now we deal with hypersequents.

ThS5$_L$ $\vdash\Rightarrow \neg\Box\neg\alpha \to \Box\neg\Box\neg\alpha$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow;\,\Box\neg\alpha\Rightarrow;\,\alpha\Rightarrow\alpha}{\Rightarrow;\,\Box\neg\alpha\Rightarrow;\,\Rightarrow\neg\alpha,\alpha}\,\neg K}{\Rightarrow;\,\Box\neg\alpha\Rightarrow;\,\neg\alpha\Rightarrow\neg\alpha}\,\neg A}{\Rightarrow;\,\Box\neg\alpha\Rightarrow;\,\Rightarrow\neg\alpha}\,\Box A_2}{\Rightarrow;\,\Rightarrow\neg\Box\neg\alpha;\,\Rightarrow\neg\alpha}\,\neg K}{\Rightarrow\Box\neg\alpha;\,\Rightarrow\neg\Box\neg\alpha}\,\Box K}{\Rightarrow\Box\neg\alpha,\Box\neg\Box\neg\alpha}\,\Box K}{\neg\Box\neg\alpha\Rightarrow\Box\neg\Box\neg\alpha}\,\neg A}{\Rightarrow\neg\Box\neg\alpha\to\Box\neg\Box\neg\alpha}\,\to K$$

□

9.4 Cut-Admissibility in ThS5$_L$

In this section we prove that the cut-rule is admissible in the calculus **ThS5**$_L$.

Lemma 9.10 *Let $G; \Gamma, \alpha$ and $G'; \alpha, \Pi$ be two hypersequents. If*

$$\dfrac{\begin{matrix}\vdots\, d_1 & & \vdots\, d_2 \\ G; \Gamma, \alpha & & G'; \alpha, \Pi\end{matrix}}{G; G'; \Gamma \centerdot \Pi}\, cut_\alpha$$

and d_1 and d_2 do not contain any other application of the cut-rule, then we can construct a derivation of $G; G'; \Gamma \centerdot \Pi$ with no application of the cut-rule.

Proof The proof is developed by induction on the complexity of the cut-formula (see Definition 2.3, p. 40), with subinduction on the sum of the heights of the derivations of the premises of the cut-rule. We will distinguish cases according to the last rule applied to the left premise.

Case 1. $G; \Gamma, \alpha$ is an initial hypersequent. Then either the conclusion is also a hypersequent, or the cut can be replaced by various applications of the internal and external weakening rules to $G'; \alpha, \Pi$.

Case 2. $G; \Gamma, \alpha$ is inferred by a rule $\mathcal{R}$ in which α is not principal. The reduction is done in the standard way by induction on the sum of the heights of the derivations of the premises of the cut-rule. However, for the sake of clarity, we consider the example of the rule $\Box K$,

$$\dfrac{\dfrac{G; \Gamma, \alpha \;; \Rightarrow \beta}{G; \Gamma, \alpha, \Box\beta}\,\Box K \qquad \begin{matrix}\vdots \\ G'; \alpha, \Pi\end{matrix}}{G; G'; \Gamma \centerdot \Pi, \Box\beta}\, cut_\alpha$$

We reduce to

$$\dfrac{\dfrac{G; \Gamma, \alpha; \Rightarrow \beta \qquad G'; \alpha, \Pi}{G; G'; \Gamma \centerdot \Pi \;; \Rightarrow \beta}\, cut_\alpha}{G; G'; \Gamma \centerdot \Pi, \Box\beta}\,\Box K$$

where this cut is eliminable by induction on the sum of the heights of the derivations of the premises of the cut-rule.

Case 3. $G; \Gamma, \alpha$ is inferred by a rule $\mathcal{R}$ in which α is principal. We can distinguish two subcases: in one subcase $\mathcal{R}$ is a propositional rule, in the other $\mathcal{R}$ is a modal rule.

Case 3.1. Supposing, for illustration, that the rule before $G; \Gamma, \alpha$ is $\neg K$ and $\alpha \equiv \neg\beta$, we have

$$\dfrac{\dfrac{G;\beta,\Gamma}{G;\Gamma,\neg\beta}\neg K \qquad \begin{matrix}\vdots\\ G^{'};\neg\beta,\Pi\end{matrix}}{G;G^{'};\Gamma\centerdot\Pi}\, cut_{\neg\beta}$$

By applying Lemma 9.7 to $G^{'};\neg\beta,\Pi$, we obtain $G^{'};\Pi,\beta$. Therefore, we can replace the previous cut with the following one, which is eliminable by induction on the complexity of the cut-formula:

$$\dfrac{G^{'};\Pi,\beta \quad G;\beta,\Gamma}{G;G^{'};\Gamma\centerdot\Pi}\, cut_{\beta}$$

Case 3.2. $\mathcal{R}$ is $\Box K$ and $\alpha\equiv\Box\beta$. We have the following situation:

$$\dfrac{\dfrac{G;\Gamma;\Rightarrow\beta}{G;\Gamma,\Box\beta}\Box K \qquad \begin{matrix}\vdots\\ G^{'};\Box\beta,\Pi\end{matrix}}{G;G^{'};\Gamma\centerdot\Pi}\, cut_{\Box\beta}$$

We must consider the last rule $\mathcal{R}^{'}$ of d_2. If no rule $\mathcal{R}^{'}$ introduces $G^{'};\Box\beta,\Pi$ because $G^{'};\Box\beta,\Pi$ is an initial hypersequent, then we can solve the case as in 1. If $\Box\beta$ is not principal in the rule $\mathcal{R}^{'}$, we solve the case as in 2. Only the case where $\mathcal{R}^{'}$ is one of the rules $\Box A_i$ is problematic. Since the procedure is the same in both cases, analysing one of the rule is sufficient. We will consider the rule $\Box A_2$ and the other can be dealt with in an analogous fashion.

$$\dfrac{\dfrac{G;\Gamma;\Rightarrow\beta}{G;\Gamma,\Box\beta}\Box K \qquad \dfrac{G^{'};\Box\beta,\Pi;\beta,\Psi}{G^{'};\Box\beta,\Pi;\Psi}\Box A_2}{G;G^{'};\Gamma\centerdot\Pi;\Psi}\, cut_{\Box\beta}$$

We reduce to

$$\dfrac{\dfrac{G;\Gamma;\Rightarrow\beta \qquad \dfrac{G;\Gamma,\Box\beta \quad G^{'};\Box\beta,\Pi;\beta,\Psi}{G;G^{'};\Gamma\centerdot\Pi;\beta,\Psi}\, cut_{\Box\beta}}{G;G;G^{'};\Gamma;\Gamma\centerdot\Pi;\Psi}\, cut_{\beta}}{G\ ;\ G^{'};\Gamma\centerdot\Pi;\Psi}\, CA^{*}+CK^{*}+merge^{*}$$

where the first cut is eliminable by induction on the sum of the heights of the derivations of the premises of the cut-rule, and the second cut is eliminable by induction on the complexity of cut-formula. □

Theorem 9.11 *Every derivation d in* **ThS5**$_L$ *can be effectively transformed into a derivation $d^{'}$ where there is no application of the cut-rule.*

Proof It follows from Lemma 9.10 by induction on the number of cuts. □

9.5 Decidability of ThS5$_L$

This section will prove that the calculus **ThS5**$_L$ is decidable. The situation is analogous to the case of the calculi **Thsk**$^*_L$. On the one hand, (i) the contraction rules are height-preserving admissible (see Lemma 9.8), (ii) the cut rule is admissible (see Theorem 9.11), and (iii) in each of the rules of the calculus **ThS5**$_L$, all the formulas that occur in the premise(s) are subformulas of the formulas that occur in the conclusion. Hence it seems that no difficulty arises. On the other hand, the repetition of the principal formula in each of the rules $\Box A_i$ is a source of potentially non-terminating proof search. In order to avoid this problem, and prove that our calculus is decidable, we shall apply the same technique of Section 7.2, and obtain a bound on the number of applications of the rules $\Box A_i$.

Let us take into account *minimal derivations*, which are, as we have already said, derivations where shortenings are not possible. Then we can prove that in minimal derivations applying the rule $\Box A_1$ once on any given pair of principal formulas, and the rule $\Box A_2$ once on any given pair of sequents is sufficient.

Lemma 9.12 *The rule $\Box A_1$ permutes down with respect to the rules $\neg A$, $\neg K$, $\wedge A'$, $\wedge K$, $\Box A_2$ and $\Box K$.*

Proof Let us first of all consider the permutation with one-premise propositional rules, which is straightforward. Let us consider the example of the rule $\neg K$,

$$\dfrac{\dfrac{G;\beta,\alpha,\Box\alpha,\Gamma}{G;\beta,\Box\alpha,\Gamma}\,\Box A_1}{G;\Box\alpha,\Gamma,\neg\beta}\,\neg K$$

$$\downarrow$$

$$\dfrac{\dfrac{G;\beta,\alpha,\Box\alpha,\Gamma}{G;\alpha,\Box\alpha,\Gamma,\neg\beta}\,\neg K}{G;\Box\alpha,\Gamma,\neg\beta}\,\Box A_1$$

Let us now consider the permutation with the two premises-rule $\wedge K$. We have the following derivation:

$$\dfrac{\dfrac{G;\alpha,\Box\alpha,\Gamma,\beta}{G;\Box\alpha,\Gamma,\beta}\,\Box A_1 \qquad \begin{array}{c}\vdots\\ G;\Box\alpha,\Gamma,\gamma\end{array}}{G;\Box\alpha,\Gamma,\beta\wedge\gamma}\,\wedge K$$

$$\downarrow$$

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ G;\alpha,\Box\alpha,\Gamma,\beta\end{array} \qquad \dfrac{G;\Box\alpha,\Gamma,\gamma}{G;\alpha,\Box\alpha,\Gamma,\gamma}\,IW}{G;\alpha,\Box\alpha,\Gamma,\beta\wedge\gamma}\,\wedge K}{G;\Box\alpha,\Gamma,\beta\wedge\gamma}\,\Box A_1$$

The transformation of the first derivation into the second one is achieved by means of an application of the height-preserving admissible rule of internal weakening.

Finally, this is the permutation in case of the rule $\Box K$:

$$\cfrac{\cfrac{G;\alpha,\Box\alpha,\Gamma;\Rightarrow\beta}{G;\Box\alpha,\Gamma;\Rightarrow\beta}\,\Box A_1}{G;\Box\alpha,\Gamma,\Box\beta}\,\Box K$$

$$\downarrow$$

$$\cfrac{\cfrac{G;\alpha,\Box\alpha,\Gamma;\Rightarrow\beta}{G;\alpha,\Box\alpha,\Gamma,\Box\beta}\,\Box K}{G;\Box\alpha,\Gamma,\Box\beta}\,\Box A_1$$

□

Lemma 9.13 *The rule $\Box A_2$ permutes down with respect to the rules $\neg A$, $\neg K$, $\wedge A'$, $\wedge K$, $\Box A_1$. It also permutes with instances of the rule $\Box K$ when the auxiliary formula α of its premise is not active in the sequent where the auxiliary formula of the premise of $\Box K$ occurs.*

Proof The proof is analogous to the one of Lemma 9.12. □

Corollary 9.14 *In a minimal derivation in* $\mathbf{ThS5}_L$, *the rule $\Box A_1$ cannot be applied more than once on the same pair of auxiliary formulas of any branch.*

Proof Let us suppose we have a minimal derivation where the rule $\Box A_1$ has been applied twice on the same pair of formulas,

$$\cfrac{G';\alpha,\Box\alpha,\Gamma'}{G';\Box\alpha,\Gamma'}\,\Box A_1$$

$$\vdots$$

$$\cfrac{G;\alpha,\Box\alpha,\Gamma}{G;\Box\alpha,\Gamma}\,\Box A_1$$

By permuting down $\Box A_1$ with respect to the steps in the dotted part of the derivation, we obtain a derivation of the same height ending with

$$\cfrac{\cfrac{G;\alpha,\alpha,\Box\alpha,\Gamma}{G;\alpha,\Box\alpha,\Gamma}\,\Box A_1}{G;\Box\alpha,\Gamma}\,\Box A_1$$

By applying the height-preserving admissible rule CA on the two occurrences of the formula α in place of the upper $\Box A_1$, we obtain a shorter derivation, contrary to the assumption of minimality. □

Corollary 9.15 *In a minimal derivation in* **ThS5**$_L$*, the rule* $\Box A_2$ *cannot be applied more than once on the same pair of sequents of any branch.*

Proof The proof is analogous to the one of Corollary 9.14. □

Now let us prove that the modal logic **S5** is decidable by showing effective bounds on proof search in the calculus **ThS5**$_L$.

Theorem 9.16 *The calculus* **ThS5**$_L$ *allows terminating proof search.*

Proof Place a hypersequent G, for which we are looking for a proof search, at the root of the procedure. Apply first the propositional rules and then the modal rules. The propositional rules reduce the complexity of the hypersequent. The rule $\Box K$ removes the modal constant $\Box$ and adds a new sequent, each of the rules $\Box A_i$ increases the complexity. However, by Corollary 9.14, the rule $\Box A_1$ cannot be applied more than once on the same pair of formulas, while, by Corollary 9.15, the rule $\Box A_2$ cannot be applied more than once on the same pair of sequents. Therefore, the number of applications of the two rules $\Box A_1$ and $\Box A_2$ is bounded by the number of $\Box$'s occurring in the negative part (see definition below) of the hypersequent to prove, and by the number of sequents that may appear in the derivation, respectively. The latter, in turn, is bounded by the number of sequents belonging to the hypersequent to prove, and the sequents which can be introduced by applications of the rule $\Box K$.

In order to calculate explicit bounds, we first define the *negative* and *positive* parts of the hypersequent $M_1 \Rightarrow N_1; ...; M_n \Rightarrow N_n$, as the negative and positive parts of each of the following conjuncts and disjuncts:

$$\bigwedge M_1 \to \bigvee N_1, ..., \bigwedge M_n \to \bigvee N_n$$

For any given hypersequent G, let $n(\Box)$ be the number of $\Box$'s in the negative part of the hypersequent G, and $p(\Box)$ be the number of $\Box$'s in the positive part of the hypersequent G. The number of applications of the rule $\Box A_1$ in a minimal derivation is bounded by

$$n(\Box)$$

In the case where the root-hypersequent is just a sequent, the number of applications of the rule $\Box A_2$ in a minimal derivation is bounded by

$$n(\Box) \cdot p\ (\Box)$$

In the case where the root-hypersequent is a hypersequent, and s is the number of sequents which occurs in it, the number of applications of the rule $\Box A_2$ in a minimal derivation is bounded by

$$n(\Box) \cdot (p\ (\Box) + s)$$

□

Notes

1. For the sake of completeness, we underline that it is an easy (but quite tedious) exercise to show that the rule $\tilde{5}$ is invertible.
2. For the sake of brevity, we present only a logical variant of the hypersequent calculus for the system **S5**. The reader interested in a general variant should be able to obtain it on his own.
3. Recall that the symbol $\rightsquigarrow$ means: the premise of the right side is obtained by applying the inductive hypothesis to the premise of the left side (see Section 6.2, p. 128).
4. The symbol $\dashrightarrow$ means: the premise of the right side is preceded by application of Lemma 9.7 to the premise of the left side.

Chapter 10
A Tree-Hypersequent Calculus for the Modal Logic of Provability

GL, the logic of provability, was elaborated at the end of the 1970s in order to "capture" the properties of the provability predicate of Peano arithmetic in the simple framework of modal logic.

As Section 2.1 highlighted, **GL** is sound and complete with respect to the class of transitive frames without infinite ascending $\mathcal{R}$-chains. The property of not having infinite ascending $\mathcal{R}$-chains is a property that cannot be described with a first-order logic formula. This is the first relevant difference between **GL** and the other SLH-systems. The second one, which is probably not unrelated to the first, consists in the fact that it is particularly difficult to find a sequent calculus for this system. Indeed we only know of two such attempts. The first attempt **Ggl** (that was introduced in Section 2.2) is Leivant's [74]. **Ggl** presents many disadvantages: the proof of cut-elimination is complicated, the structural rules are not eliminable, and the rules are neither explicit nor symmetric.

Some of these shortcomings have been overcome. The second attempt for a calculus for **GL**, which we owe to Negri (see Section 4.3), aims to do that. Nevertheless, this calculus is not free from imperfections either. In particular, we can single out three flaws: the violation of the explicitness and syntactic purity properties, and the requirement of axioms of the form $\Box\alpha \Rightarrow \Box\alpha$.

In the light of this state of affairs, it is worth proposing a new sequent calculus for **GL**, obtained by means of the tree-hypersequent method. As the next sections will make explicit, this sequent calculus has all the advantages of Negri's calculus (and also some of its disadvantages, e.g. axioms of the form $\Box\alpha \Rightarrow \Box\alpha$), since it is mainly inspired from it; moreover it is syntactic pure.

10.1 The Calculus $\mathbf{Thsgl}_L$

In order to get the tree-hypersequent calculus $\mathbf{Thsgl}_L$,[1] we will employ the same technique adopted by Negri. Therefore, contrary to what we have done for the several $\mathbf{Thsk}^*_L$ systems, we will not add a new special logical rule to the calculus $\mathbf{Thsk4}_L$, but we will modify it by (i) substituting the rules $\Box A$ and $\Box K$ with, respectively

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$$\frac{G[\Gamma/\Box\alpha \Rightarrow \alpha]}{G[\Gamma, \Box\alpha]}\ \Box K_{gl}$$

$$\frac{G[\Box\alpha, \Gamma/(\Sigma, \Box\alpha/\underline{X})] \qquad G[\Box\alpha, \Gamma/(\alpha, \Sigma/\underline{X})]}{G[\Box\alpha, \Gamma/(\Sigma/\underline{X})]}\ \Box A_{gl}$$

and (ii) allowing as initial tree-hypersequents, tree-hypersequents of the form

$$G[\Box\alpha, \Gamma, \Box\alpha]$$

In view of this second modification, we cannot properly consider **Thsgl**$_L$ to be a logical variant of the sequent calculus **Thsgl**. On the other hand, since the other structural rules are all admissible, this approximation is allowed.

The postulates of the calculus **Thsgl**$_L$ are:

Initial Tree-Hypersequents

$G\ [p, \Gamma, p]$ $\qquad$ $G\ [\Box\alpha, \Gamma, \Box\alpha]$

Logical Rules

Propositional Rules

$$\frac{G[\Gamma, \alpha]}{G[\neg\alpha, \Gamma]}\ \neg A \qquad \frac{G[\alpha, \Gamma]}{G[\Gamma, \neg\alpha]}\ \neg K$$

$$\frac{G[\alpha, \beta, \Gamma]}{G[\alpha \wedge \beta, \Gamma]}\ \wedge A' \qquad \frac{G[\Gamma, \alpha] \qquad G[\Gamma, \beta]}{G[\Gamma, \alpha \wedge \beta]}\ \wedge K$$

Modal Rules

$$\frac{G[\Box\alpha, \Gamma/(\Sigma, \Box\alpha/\underline{X})] \qquad G[\Box\alpha, \Gamma/(\alpha, \Sigma/\underline{X})]}{G[\Box\alpha, \Gamma/(\Sigma/\underline{X})]}\ \Box A_{gl}$$

$$\frac{G[\Gamma/\Box\alpha \Rightarrow \alpha]}{G[\Gamma, \Box\alpha]}\ \Box K_{gl}$$

Special Logical Rule

$$\frac{G[\Box\alpha, \Gamma/(\Box\alpha, \Sigma/\underline{X})]}{G[\Box\alpha, \Gamma/(\Sigma/\underline{X})]}\ 4$$

10.2 Admissibility of the Structural Rules in Thsgl$_L$

This section is dedicated to the proofs of the admissibility of the structural rules and the invertibility of the logical rules in **Thsgl**$_L$. In these proofs, thanks to Lemmas 6.9–6.12 and 6.15, we will merely check the cases where the last applied rule is one of the new modal rules, since all the others have already been verified. Note that in analysing these cases we will take into account the Remark 6.7, p. 128.

Lemma 10.1 *Tree-hypersequents of the form* $G[\alpha, \Gamma, \alpha]$*, with* α *arbitrary formula, are derivable in* $\mathbf{Thsgl}_L$*.*

Proof By straightforward induction on α. □

Lemma 10.2 *The rule of necessitation*

$$\frac{G}{\Rightarrow /G}\, rn$$

is height-preserving admissible in $\mathbf{Thsgl}_L$*.*

Proof By induction on the derivation of the premise.

If G is inferred by the modal rule $\Box K_{gl}$, then the inference is preserved.

$$\frac{{}^{\langle n-1\rangle}G[\Gamma/\Box\alpha \Rightarrow \alpha]}{{}^{\langle n\rangle}G[\Gamma, \Box\alpha]}\,\Box K_{gl} \quad \rightsquigarrow \quad \frac{{}^{\langle n-1\rangle} \Rightarrow /G[\Gamma/\Box\alpha \Rightarrow \alpha]}{{}^{\langle n\rangle} \Rightarrow /G[\Gamma, \Box\alpha]}\,\Box K_{gl}$$

If G is inferred by the modal rule $\Box A_{gl}$, then the inference is preserved.

$$\frac{{}^{\langle n-1\rangle}G[\Box\alpha, \Gamma/(\Sigma, \Box\alpha/\underline{X})] \qquad {}^{\langle n-1\rangle}G[\Box\alpha, \Gamma/(\alpha, \Sigma/\underline{X})]}{{}^{\langle n\rangle}G[\Box\alpha, \Gamma/(\Sigma/\underline{X})]}\,\Box A_{gl}$$

$\rightsquigarrow$

$$\frac{{}^{\langle n-1\rangle} \Rightarrow /G[\Box\alpha, \Gamma/(\Sigma, \Box\alpha/\underline{X})] \quad {}^{\langle n-1\rangle} \Rightarrow /G[\Box\alpha, \Gamma/(\alpha, \Sigma/\underline{X})]}{{}^{\langle n\rangle} \Rightarrow /G[\Box\alpha, \Gamma/(\Sigma/\underline{X})]}\,\Box A_{gl}$$

□

Lemma 10.3 *The rules of internal and external weakening*

$$\frac{G[\Gamma]}{G[\alpha, \Gamma]}\, WA \qquad \frac{G[\Gamma]}{G[\Gamma, \alpha]}\, WK \qquad \frac{G[\Gamma]}{G[\Gamma/\Sigma]}\, EA$$

are height-preserving admissible in $\mathbf{Thsgl}_L$*.*

Proof By straightforward induction on the derivation of the premise. □

Lemma 10.4 *The rule of merge*

$$\frac{G[\Delta/(\Gamma/\underline{X}); (\Pi/\underline{X}')]}{G[\Delta/(\Gamma \centerdot \Pi/\underline{X}; \underline{X}')]}\, merge$$

is height-preserving admissible in $\mathbf{Thsgl}_L$*.*

Proof By induction on the height of the derivation of the premise. As the rule of merge has three auxiliary sequents, Δ, Γ and Π, we should, for each rule $\mathcal{R}$ applied to the premise, distinguish three subcases: one in which the rule $\mathcal{R}$ has been applied to the sequent Δ, one in which the rule $\mathcal{R}$ has been applied to the sequent Γ, one in which the rule $\mathcal{R}$ has been applied to the sequent Π. On the other hand, since these subcases are similar, we will sketch the proof for one of them only.

If G is inferred by the modal rule $\Box K_{gl}$, then the inference is preserved.

$$\frac{{}^{\langle n-1\rangle}G[\Delta/(\Gamma/\Box\alpha \Rightarrow \alpha; \underline{X}); (\Pi/\underline{X}')]}{{}^{\langle n\rangle}G[\Delta/(\Gamma, \Box\alpha/\underline{X}); (\Pi/\underline{X}')]}\,\Box K_{gl} \quad \rightsquigarrow$$

$$\frac{{}^{\langle n-1\rangle}G[\Delta/(\Gamma \centerdot \Pi/\Box\alpha \Rightarrow \alpha; \underline{X}; \underline{X}')]}{{}^{\langle n\rangle}G[\Delta/(\Gamma \centerdot \Pi, \Box\alpha/\underline{X}; \underline{X}')]}\,\Box K_{gl}$$

If the premise is inferred by the rule $\Box A_{gl}$, then the inference is preserved.

$$\frac{{}^{\langle n-1\rangle}G[\Box\alpha, \Delta/(\Gamma, \Box\alpha/\underline{X}); (\Pi/\underline{X}')] \quad {}^{\langle n-1\rangle}G[\Box\alpha, \Delta/(\alpha, \Gamma/\underline{X}); (\Pi/\underline{X}')]}{{}^{\langle n\rangle}G[\Box\alpha, \Delta/(\Gamma/\underline{X}); (\Pi/\underline{X}')]}\,\Box A_{gl}$$

$$\rightsquigarrow$$

$$\frac{{}^{\langle n-1\rangle}G[\Box\alpha, \Delta/(\Gamma \centerdot \Pi, \Box\alpha/\underline{X}; \underline{X}')] \quad {}^{\langle n-1\rangle}G[\Box\alpha, \Delta/(\alpha, \Gamma \centerdot \Pi/\underline{X}; \underline{X}')]}{{}^{\langle n\rangle}G[\Box\alpha, \Delta/(\Gamma \centerdot \Pi/\underline{X}; \underline{X}')]}\,\Box A_{gl}$$

□

Lemma 10.5 *The rule* $\tilde{4}$

$$\frac{G[\Gamma/(\Sigma/\underline{X})]}{G[\Gamma/(\Rightarrow /\Sigma/\underline{X})]}\,\tilde{4}$$

is admissible in $\mathbf{Thsgl}_L$.

Proof By induction on the height of the derivation of the premise. As the rule $\tilde{4}$ has two auxiliary sequents, Γ and Σ, we should, for each rule $\mathcal{R}$ applied to the premise, distinguish two subcases: one in which the rule $\mathcal{R}$ has been applied to the sequent Γ, and one in which the rule $\mathcal{R}$ has been applied to the sequent Σ. On the other hand, since the two subcases are similar, we will sketch the proof for one of them only.

If G is inferred by the modal rule $\Box K_{gl}$, then the inference is preserved.

$$\frac{{}^{\langle n-1\rangle}G[\Gamma/\Box\alpha \Rightarrow \alpha; (\Sigma/\underline{X})]}{{}^{\langle n\rangle}G[\Gamma, \Box\alpha/(\Sigma/\underline{X})]}\,\Box K_{gl} \quad \rightsquigarrow \quad \frac{G[\Gamma/\Box\alpha \Rightarrow \alpha; (\Rightarrow /\Sigma/\underline{X})]}{G[\Gamma, \Box\alpha/(\Rightarrow /\Sigma/\underline{X})]}\,\Box K_{gl}$$

If G is inferred by the modal rule $\Box A_{gl}$, then the inference is preserved.

$$\dfrac{{}^{\langle n-1\rangle}G[\Box\alpha, \Gamma/(\Sigma, \Box\alpha/\underline{X})] \quad {}^{\langle n-1\rangle}G[\Box\alpha, \Gamma/(\alpha, \Sigma/\underline{X})]}{{}^{\langle n\rangle}G[\Box\alpha, \Gamma/(\Sigma/\underline{X})]}\ \Box A_{gl}$$

$\rightsquigarrow$

$$\dfrac{\dfrac{G[\Box\alpha, \Gamma/(\Rightarrow /\Sigma, \Box\alpha/\underline{X})]}{G[\Box\alpha, \Gamma/(\Box\alpha \Rightarrow /\Sigma, \Box\alpha/\underline{X})]}\ WA \quad \dfrac{G[\Box\alpha, \Gamma/(\Rightarrow /\alpha, \Sigma/\underline{X})]}{G[\Box\alpha, \Gamma/(\Box\alpha \Rightarrow /\alpha, \Sigma/\underline{X})]}\ WA}{\dfrac{G[\Box\alpha, \Gamma/(\Box\alpha \Rightarrow /\Sigma/\underline{X})]}{G[\Box\alpha, \Gamma/(\Rightarrow /\Sigma/\underline{X})]}\ 4}\ \Box A_{gl}$$

□

Lemma 10.6 *The propositional rules, the modal rules and the special logical rule of* $\mathbf{Thsgl}_L$ *are invertible.*

Proof The proof is by induction on the height of the derivation of the premise of the rule considered. The cases of the propositional rules are dealt with in the classical way. The only differences – the fact that we are dealing with tree-hypersequents, and the cases where the last applied rule is one of the modal rules or the special logical rule – are dealt with easily.

The rules $\Box A_{gl}$ and 4 are (height-preserving) invertible by the (height-preserving) admissibility of internal weakening.

Let us now consider the invertibility of the $\Box K_{gl}$ rule. If $G[\Gamma, \Box\alpha]$ is an initial tree-hypersequent and $\Box\alpha$ is not the principal formula, then $G[\Gamma/\Box\alpha \Rightarrow \alpha]$ is also an initial tree-hypersequent. On the other hand, if $G[\Gamma, \Box\alpha]$ is an initial tree-hypersequent and $\Box\alpha$ is the principal formula, then Γ will be of the form $\Box\alpha, M' \Rightarrow N$, and we need to prove that $G[\Box\alpha, M' \Rightarrow N/\Box\alpha \Rightarrow \alpha]$ is derivable. This follows by $\Box A_{gl}$ from the initial tree-hypersequent $G[\Box\alpha, M' \Rightarrow N/\Box\alpha \Rightarrow \Box\alpha, \alpha]$ and the derivable tree-hypersequent $G[\Box\alpha, M' \Rightarrow N/\Box\alpha, \alpha \Rightarrow \alpha]$. The rest of the proof continues in the standard way. □

Lemma 10.7 *The rules of contraction*

$$\dfrac{G[\alpha, \alpha, \Gamma]}{G[\alpha, \Gamma]}\ CA \qquad \dfrac{G[\Gamma, \alpha, \alpha]}{G[\Gamma, \alpha]}\ CK$$

are admissible in $\mathbf{Thsgl}_L$.

Proof By induction on the complexity of the formula α, $cmp(\alpha)$, with subinduction on the height of derivations of the premises. Let $CA_{<n}$ and CA_n mean that CA is admissible for $cmp(\alpha) < n$ and for $cmp(\alpha) = n$, respectively. Analogously for $CK_{<n}$ and CK_n. We prove successively that

(i) for every k: if $CA_{<k}$ and $CK_{<k}$, then CA_k,
(ii) for every k: if CA_k and $CK_{<k}$, then CK_k.

Thus, if $CA_{<k}$ and $CK_{<k}$, then CA_k and CK_k, and the conclusion follows by complete induction on k.

$k = 0$ is trivial. So suppose $k = n$, for $n > 0$. We treat this case in detail. As the reader will see, we are going to use the two (inductive) hypothesis $CA_{<n}$ and $\mathrm{CK}_{<n}$. We will indicate their use by $i.h.$

(i) We only analyse those cases in which $G[\alpha, \alpha, \Gamma]$ is obtained by a rule $\mathcal{R}$ that has one of the two occurrences of the formula α as principal. The others can be solved easily by subinduction on the height of derivations.

- $\alpha \equiv \neg\beta$ and has been obtained by the rule $\neg A$.

$$\dfrac{{}^{\langle n-1\rangle}G[\neg\beta, \Gamma, \beta]}{{}^{\langle n\rangle}G[\neg\beta, \neg\beta, \Gamma]}\,\neg A \quad \dashrightarrow^{2} \quad \dfrac{\dfrac{G[\Gamma, \beta, \beta]}{G[\Gamma, \beta]}\,i.h.}{G[\neg\beta, \Gamma]}\,\neg A$$

- $\alpha \equiv \beta \wedge \gamma$ and has been obtained by the rule $\wedge A'$.

$$\dfrac{{}^{\langle n-1\rangle}G[\beta\wedge\gamma, \beta, \gamma, \Gamma]}{{}^{\langle n\rangle}G[\beta\wedge\gamma, \beta\wedge\gamma, \Gamma]}\,\wedge A' \quad \dashrightarrow \quad \dfrac{\dfrac{\dfrac{G[\beta, \beta, \gamma, \gamma, \Gamma]}{G[\beta, \gamma, \gamma, \Gamma]}\,i.h.}{G[\beta, \gamma, \Gamma]}\,i.h.}{G[\beta\wedge\gamma, \Gamma]}\,\wedge A'$$

- $\alpha \equiv \Box\beta$ and has been obtained by the rule $\Box A_{gl}$.

$$\dfrac{{}^{\langle n-1\rangle}G[\Box\beta, \Box\beta, \Gamma/(\Sigma, \Box\beta/\underline{X})] \quad {}^{\langle n-1\rangle}G[\Box\beta, \Box\beta, \Gamma/(\beta, \Sigma/\underline{X})]}{{}^{\langle n\rangle}G[\Box\beta, \Box\beta, \Gamma/(\Sigma/\underline{X})]}\,\Box A_{gl} \quad \rightsquigarrow$$

$$\dfrac{G[\Box\beta, \Gamma/(\Sigma, \Box\beta/\underline{X})] \quad G[\Box\beta, \Gamma/(\beta, \Sigma/\underline{X})]}{G[\Box\beta, \Gamma/(\Sigma/\underline{X})]}\,\Box A_{gl}$$

- $\alpha \equiv \Box\beta$ and has been obtained by the rule 4.

$$\dfrac{{}^{\langle n-1\rangle}G[\Box\beta, \Box\beta, \Gamma/(\Box\beta, \Sigma/\underline{X})]}{{}^{\langle n\rangle}G[\Box\beta, \Box\beta, \Gamma/(\Sigma/\underline{X})]}\,4 \quad \rightsquigarrow$$

$$\dfrac{G[\Box\beta, \Gamma/(\Box\beta, \Sigma/\underline{X})]}{G[\Box\beta, \Gamma/(\Sigma/\underline{X})]}\,4$$

We have thus established (i) for $k = n$. We now turn to (ii) for $k = n$.

(ii) We will restrict our analysis to those cases in which $G[\Gamma, \alpha, \alpha]$ is obtained by a rule $\mathcal{R}$ that has one of the two occurrences of the formula α as principal. The others can be dealt with easily by subinduction on the height of derivations.

- $\alpha \equiv \neg\beta$ and has been obtained by the rule $\neg K$. This case can be dealt with analogously to the case (i) - $\neg A$ above.

– $\alpha \equiv \alpha \wedge \beta$ and has been obtained by the rule $\wedge K$. This case can be dealt with analogously to the case (i) - $\wedge A'$ above.
– $\alpha \equiv \Box\beta$ and has been obtained by the rule $\Box K_{gl}$

$$\dfrac{{}^{\langle n-1\rangle} G[\Gamma, \Box\beta / \Box\beta \Rightarrow \beta]}{{}^{\langle n\rangle} G[\Gamma, \Box\beta, \Box\beta]}\, \Box K_{gl} \quad \dashrightarrow \quad \dfrac{\dfrac{\dfrac{\dfrac{G[\Gamma / \Box\beta \Rightarrow \beta; \Box\beta \Rightarrow \beta]}{G[\Gamma / \Box\beta, \Box\beta, \Rightarrow \beta, \beta]}\, merge}{G[\Gamma / \Box\beta \Rightarrow \beta, \beta]}\, (i)}{G[\Gamma / \Box\beta \Rightarrow \beta]}\, i.h.}{G[\Gamma, \Box\beta]}\, \Box K_{gl}$$

□

Lemma 10.8 *Let $G[H]$ be any tree-hypersequent of the calculus* $\mathbf{Thsgl}_L$*, and $G^*[H]$ the result of the application of one of the (height-preserving admissible) rules - rn, WA, WK, EW, $merge$ - or of one of the (admissible) rules - CA, CK, $\tilde{4}$ - on $G[H]$. If, for a rule $\mathcal{R}$, we have*

$$\frac{G[H']}{G[H]}\, \mathcal{R}$$

then it holds that

$$\frac{G^*[H']}{G^*[H]}\, \mathcal{R}$$

Proof By induction on the form of the tree-hypersequent $G[H]$. □

Lemma 10.9 *Let $G[H]$ be any tree-hypersequent of the calculus* $\mathbf{Thsgl}_L$*, and $G[H']$ the result of the application of one of the propositional rules or of the rule $\Box K_{gl}$ on $G[H]$. If, for a rule $\mathcal{R}$, we have*

$$\frac{G^*[H']}{G[H']}\, \mathcal{R}$$

then it holds that

$$\frac{G^*[H]}{G[H]}\, \mathcal{R}$$

Proof By induction on the form of the tree-hypersequent $G[H']$. □

10.3 Adequacy of $\mathbf{Thsgl}_L$

This section will show that the sequent calculus $\mathbf{Thsgl}_L$ proves exactly the same formulas as the corresponding Hilbert system **GL**. In order to reach this result, the following lemma, established by Negri [85], is necessary.

Lemma 10.10 *For all interpretations in transitive frames without infinite ascending $\mathcal{R}$-chains, for all Kripke semantics words i, and for all formulas α,*

$$i \models \Box\alpha \quad \textit{if, and only if,} \quad \forall j\ (iRj \textit{ and } j \models \Box\alpha \ \rightarrow \ j \models \alpha)$$

Proof The direction from the right to the left is easily provable by taking the standard forcing relation and by reasoning *a fortiori*.

For the converse, assume the right-hand side and suppose $i \not\models \Box\alpha$. Then there exists a j_1 such that iRj_1 and $j_1 \not\models \alpha$. The assumption allows us to derive that $j_1 \not\models \Box\alpha$, and so that there exists a j_2 such that j_1Rj_2 and $j_2 \not\models \alpha$. By the transitivity of the relation $\mathcal{R}$, as we have iRj_1 and j_1Rj_2, we also have iRj_2 and, consequently, $j_2 \not\models \Box\alpha$. This way we can build an infinite ascending R-chain, iRj_1, j_1Rj_2, j_2Rj_3, ... against the hypothesis. □

Lemma 10.11 *For all the tree-hypersequents G, if $\vdash G$ in* ***Thsgl$_{\mathbf{L}}$***, *then $\models_{\mathfrak{Cfgl}} G$, where $\models_{\mathfrak{Cfgl}} G$ stands for: the translation τ of G is valid in the class of transitive frames without infinite ascending R-chains.*

Proof By induction on the derivation of the premise. The validity of the axiom, the propositional rules, and the special logical rule 4, is established as in the proof of Theorem 8.3, p. 166. We will prove the validity of the two new modal rules only.

– $\Box K_{gl}$. Let us consider the rule in the form

$$\frac{\Gamma/\Box\alpha \Rightarrow \alpha}{\Gamma, \Box\alpha}$$

By the inductive hypothesis, we have $\forall i$ ($i \models_{\mathfrak{Cfgl}} \Gamma$ or $i \models^*_{\mathfrak{Cfgl}} \neg\Box\alpha, \alpha$), i.e. $\forall i$ ($i \models_{\mathfrak{Cfgl}} \Gamma$ or $\forall j$ ($iRj \rightarrow j \not\models_{\mathfrak{Cfgl}} \Box\alpha$ or $j \models_{\mathfrak{Cfgl}} \alpha$)). From this, we get $\forall i$ ($i \models_{\mathfrak{Cfgl}} \Gamma$ or $\forall j$ ($iRj \rightarrow (j \models_{\mathfrak{Cfgl}} \Box\alpha \rightarrow j \models_{\mathfrak{Cfgl}} \alpha$))), and hence we have $\forall i$ ($i \models_{\mathfrak{Cfgl}} \Gamma$ or $\forall j$ (iRj and $j \models_{\mathfrak{Cfgl}} \Box\alpha \rightarrow j \models_{\mathfrak{Cfgl}} \alpha$)). By definition of the forcing relation in transitive frames without infinite ascending $\mathcal{R}$ - chains (see Lemma 10.10), we have $\forall i$ ($i \models_{\mathfrak{Cfgl}} \Gamma$ or $i \models_{\mathfrak{Cfgl}} \Box\alpha$), which is nothing other than the conclusion of the rule. Finally, by Lemma 8.2, p. 166, we have that the rule $\Box K_{gl}$ is valid in the class of transitive frames without infinite ascending $\mathcal{R}$-chains.

– $\Box A_{gl}$. Let us consider the rule in the form

$$\frac{\Box\alpha, \Gamma/(\Sigma, \Box\alpha/\underline{X}) \qquad \Box\alpha, \Gamma/(\alpha, \Sigma/\underline{X})}{\Box\alpha, \Gamma/(\Sigma/\underline{X})}$$

By the inductive hypothesis we have $\forall i$ ($i \models_{\mathfrak{Cfgl}} \neg\Box\alpha, \Gamma$ or $i \models^*_{\mathfrak{Cfgl}} \Box\alpha, \Sigma$ or $j \models^*_{\mathfrak{Cfgl}} \underline{X}$) and $\forall i$ ($i \models_{\mathfrak{Cfgl}} \neg\Box\alpha, \Gamma$ or $i \models^*_{\mathfrak{Cfgl}} \neg\alpha, \Sigma$ or $j \models^*_{\mathfrak{Cfgl}} \underline{X}$), i.e. $\forall i$ (($i \models_{\mathfrak{Cfgl}} \neg\Box\alpha, \Gamma$ or $i \models^*_{\mathfrak{Cfgl}} \Box\alpha, \Sigma$ or $j \models^*_{\mathfrak{Cfgl}} \underline{X}$) and ($i \models_{\mathfrak{Cfgl}} \neg\Box\alpha, \Gamma$ or $i \models^*_{\mathfrak{Cfgl}} \neg\alpha, \Sigma$ or $j \models^*_{\mathfrak{Cfgl}} \underline{X}$)). From this, we obtain

1 $\forall i\ (i \nvDash_{\mathfrak{cfgl}} \Box\alpha$ or $i \vDash_{\mathfrak{cfgl}} \Gamma$ or $((i \vDash^*_{\mathfrak{cfgl}} \Box\alpha, \Sigma$ or $j \vDash^*_{\mathfrak{cfgl}} \underline{X})$ and $(i \vDash^*_{\mathfrak{cfgl}} \neg\alpha, \Sigma$ or $j \vDash^*_{\mathfrak{cfgl}} \underline{X})))$
2 $\forall i\ (i \nvDash_{\mathfrak{cfgl}} \Box\alpha$ or $i \vDash_{\mathfrak{cfgl}} \Gamma$ or $(\forall j\ (iRj \to j \vDash_{\mathfrak{cfgl}} \Box\alpha$ or $j \vDash_{\mathfrak{cfgl}} \Sigma$ or $\forall z\ (jRz \to j \vDash_{\mathfrak{cfgl}} \underline{X}))$ and $\forall j\ (iRj \to j \nvDash_{\mathfrak{cfgl}} \alpha$ or $j \vDash_{\mathfrak{cfgl}} \Sigma$ or $\forall z\ (jRz \to z \vDash_{\mathfrak{cfgl}} \underline{X}))))$
3 $\forall i\ (i \nvDash_{\mathfrak{cfgl}} \Box\alpha$ or $i \vDash_{\mathfrak{cfgl}} \Gamma$ or $\forall j((iRj \to j \vDash_{\mathfrak{cfgl}} \Box\alpha$ or $j \vDash_{\mathfrak{cfgl}} \Sigma$ or $\forall z\ (jRz \to z \vDash_{\mathfrak{cfgl}} \underline{X}))$ and $(iRj \to j \nvDash_{\mathfrak{cfgl}} \alpha$ or $j \vDash_{\mathfrak{cfgl}} \Sigma$ or $\forall z\ (jRz \to z \vDash_{\mathfrak{cfgl}} \underline{X}))))$
4 $\forall i\ (i \nvDash_{\mathfrak{cfgl}} \Box\alpha$ or $i \vDash_{\mathfrak{cfgl}} \Gamma$ or $\forall j(iRj \to j \vDash_{\mathfrak{cfgl}} \Box\alpha$ or $j \vDash_{\mathfrak{cfgl}} \Sigma$ or $j \nvDash_{\mathfrak{cfgl}} \alpha$ or $j \vDash_{\mathfrak{cfgl}} \Sigma$ or $\forall z\ (jRz \to z \vDash_{\mathfrak{cfgl}} \underline{X})))$
5 $\forall i\ (i \nvDash_{\mathfrak{cfgl}} \Box\alpha$ or $i \vDash_{\mathfrak{cfgl}} \Gamma$ or $\forall j(iRj \to j \vDash_{\mathfrak{cfgl}} \Box\alpha$ or $j \vDash_{\mathfrak{cfgl}} \Sigma$ or $j \nvDash_{\mathfrak{cfgl}} \alpha$ or $\forall z\ (jRz \to z \vDash_{\mathfrak{cfgl}} \underline{X}))))$
6 $\forall i\ (i \nvDash_{\mathfrak{cfgl}} \Box\alpha$ or $i \vDash_{\mathfrak{cfgl}} \Gamma$ or $\forall j((iRj \to j \nvDash_{\mathfrak{cfgl}} \Box\alpha$ and $j \vDash_{\mathfrak{cfgl}} \alpha) \to (iRj \to j \vDash_{\mathfrak{cfgl}} \Sigma$ or $\forall z\ (jRz \to z \vDash_{\mathfrak{cfgl}} \underline{X}))))$
7 $\forall i\ (i \nvDash_{\mathfrak{cfgl}} \Box\alpha$ or $i \vDash_{\mathfrak{cfgl}} \Gamma$ or $\forall j(iRj \to j \nvDash_{\mathfrak{cfgl}} \Box\alpha$ and $j \vDash_{\mathfrak{cfgl}} \alpha) \to \forall j\ (iRj \to j \vDash_{\mathfrak{cfgl}} \Sigma$ or $\forall z\ (jRz \to z \vDash_{\mathfrak{cfgl}} \underline{X})))$
8 $\forall i\ (i \nvDash_{\mathfrak{cfgl}} \Box\alpha$ or $i \vDash_{\mathfrak{cfgl}} \Gamma$ or $\neg\forall j(iRj \to j \nvDash_{\mathfrak{cfgl}} \Box\alpha$ and $j \vDash_{\mathfrak{cfgl}} \alpha)$ or $\forall j(iRj \to j \vDash_{\mathfrak{cfgl}} \Sigma$ or $\forall z\ (jRz \to z \vDash_{\mathfrak{cfgl}} \underline{X})))$
9 $\forall i\ (i \nvDash_{\mathfrak{cfgl}} \Box\alpha$ or $i \vDash_{\mathfrak{cfgl}} \Gamma$ or $\exists j(iRj$ and $(j \vDash_{\mathfrak{cfgl}} \Box\alpha$ or $j \nvDash_{\mathfrak{cfgl}} \alpha))$ or $\forall j(iRj \to j \vDash_{\mathfrak{cfgl}} \Sigma$ or $\forall z\ (jRz \to z \vDash_{\mathfrak{cfgl}} \underline{X})))$
10 $\forall i\ (i \nvDash_{\mathfrak{cfgl}} \Box\alpha$ or $i \vDash_{\mathfrak{cfgl}} \Gamma$ or $i \nvDash_{\mathfrak{cfgl}} \Box\alpha$ or $\forall j(iRj \to j \vDash_{\mathfrak{cfgl}} \Sigma$ or $\forall z\ (jRz \to z \vDash_{\mathfrak{cfgl}} \underline{X})))$
11 $\forall i\ (i \nvDash_{\mathfrak{cfgl}} \Box\alpha$ or $i \vDash_{\mathfrak{cfgl}} \Gamma$ or $\forall j(iRj \to j \vDash_{\mathfrak{cfgl}} \Sigma$ or $\forall z\ (jRz \to z \vDash_{\mathfrak{cfgl}} \underline{X})))$

The last line of the proof is the conclusion of the rule. From this argument, by Lemma 8.2, we conclude that the rule $\Box A_{gl}$ is valid in the class of transitive frames without infinite ascending R-chains. □

Corollary 10.12 *For all the tree-hypersequents G, if $\vdash G$ in* $\mathbf{Thsgl}_L$*, then $\vdash (G)^\tau$ in* **GL**.

Proof By Lemma 10.11 and the completeness theorem between the class of transitive frames without infinite ascending R-chains and the Hilbert-style system **GL**. □

In order to prove the completeness of the calculus $\mathbf{Thsgl}_L$, we start by presenting the following lemma and its corollary.

Lemma 10.13 *All the tree-hypersequents of the form $G[\Box\alpha, \Gamma/(\Sigma, \Box\alpha/\underline{X})]$ are derivable in* $\mathbf{Thsgl}_L$.

Proof Root-first, by steps of $\Box K_{gl}$, 4 and $\Box A_{gl}$. □

Corollary 10.14 *The rule*

$$\frac{G[\Box\alpha, \Gamma/(\alpha, \Sigma/\underline{X})]}{G[\Box\alpha, \Gamma/(\Sigma/\underline{X})]}\,\Box A$$

is derivable in $\mathbf{Thsgl}_L$.

Proof By Lemma 10.13, the left premise of $\Box A_{gl}$ is derivable in $\mathbf{Thsgl}_L$. □

Although the two rules $\Box A$ and $\Box A_{gl}$ are interderivable, the use of $\Box A_{gl}$ is essential in the proof of cut-elimination, as we will see in a moment. However, it is possible to think of a tree-hypersequent calculus $\mathbf{Thsgl}^{\circ}_{L}$ obtained by substituting, in $\mathbf{Thsgl}_L$, the rule $\Box A_{gl}$ with the rule $\Box A$. The two tree-hypersequent calculi $\mathbf{Thsgl}_L$ and $\mathbf{Thsgl}^{\circ}_{L}$ are equivalent. $\mathbf{Thsgl}^{\circ}_{L}$ and Negri's calculus **G3KGL** have the same structural properties which are exposed in [85, p. 529].

Lemma 10.15 *For all formulas α, if $\vdash \alpha$ in* **GL**, *then $\vdash \Rightarrow \alpha$ in* $\mathbf{Thsgl}_L$.

Proof By induction on the height of derivations in **GL**. The derivation of the Löb's axiom will provide better acquaintance with the calculus $\mathbf{Thsgl}_L$.
$\mathbf{Thsgl}_L \vdash \Rightarrow \Box(\Box\alpha \to \alpha) \to \Box\alpha$

$$\dfrac{\dfrac{\dfrac{\dfrac{\Box(\Box\alpha \to \alpha) \Rightarrow /\Box\alpha \Rightarrow \alpha, \Box\alpha \qquad \Box(\Box\alpha \to \alpha) \Rightarrow /\alpha, \Box\alpha \Rightarrow \alpha}{\Box(\Box\alpha \to \alpha) \Rightarrow /\Box\alpha \to \alpha, \Box\alpha \Rightarrow \alpha}\to A}{\Box(\Box\alpha \to \alpha) \Rightarrow /\Box\alpha \Rightarrow \alpha}\Box A}{\Box(\Box\alpha \to \alpha) \Rightarrow \Box\alpha}\Box K_{gl}}{\Rightarrow \Box(\Box\alpha \to \alpha) \to \Box\alpha}\to K$$

□

Theorem 10.16 *The calculus* $\mathbf{Thsgl}_L$ *is sound and complete with respect to the system* **GL**.

Proof By Corollary 10.12 and Lemma 10.15. □

10.4 Cut-Admissibility in $\mathbf{Thsgl}_L$

This section shows that the cut-rule is admissible in the calculus $\mathbf{Thsgl}_L$. In order to prove this theorem, we must first introduce the following lemma and the following definition.

Lemma 10.17 *Given three zoom tree-hypersequents $I[*]$, $J[*]$ and $H[*]$, such that $I[*] \sim J[*] \sim H[*]$, if there is a rule $\mathcal{R}$ of* $\mathbf{Thsgl}_L$ *and a sequent Γ such that*

$$\dfrac{J[\Gamma]}{I[\Gamma]}\mathcal{R}$$

then, for any Δ, we have

$$\dfrac{J \otimes H[\Delta]}{I \otimes H[\Delta]}\mathcal{R}$$

Proof By induction on the form of the tree-hypersequents $I[*]$, $J[*]$ and $H[*]$. The proof is developed similarly to the proof of Lemma 7.1, p. 143. □

Definition 10.18 When we consider a finite tree-frame semantically, we can talk about its longest branch, i.e. a branch such that no other branch of the tree contains more worlds. Analogously, we can talk about the longest branch of a tree-hypersequent as the branch such that no other branch of the tree-hypersequent contains more sequents.

Let us call the *length* of a tree-hypersequent the number of sequents contained in its longest branch. The *position* of a sequent Γ in a tree-hypersequent G in a derivation d is defined as the difference between the length of the longest tree-hypersequent occurring in the derivation d and the number of sequents that precede Γ.

Lemma 10.19 *Let $G[\Gamma, \alpha]$ and $G'[\alpha, \Pi]$ be such that $G[\Gamma, \alpha] \sim G'[\alpha, \Pi]$. If*

$$\dfrac{\begin{array}{c}\vdots\, d_1\\ G[\Gamma, \alpha]\end{array} \quad \begin{array}{c}\vdots\, d_2\\ G'[\alpha, \Pi]\end{array}}{G \otimes G'[\Gamma \centerdot \Pi]}\ cut_\alpha$$

and d_1 and d_2 do not contain any other application of the cut-rule, then we can construct a derivation of $G \otimes G'[\Gamma \centerdot \Pi]$ with no application of the cut-rule.

Proof The proof is developed by induction on $cmp(\alpha)$, with subinduction on the position of the two sequents on which we apply the cut (see Definition 10.18), and with a third subinduction on the sum of the heights of the derivations of the premises of the cut-rule. We will distinguish cases according to the last rule applied on the left premise.

Case 1. $G[\Gamma, \alpha]$ is an initial tree-hypersequent. Then either the conclusion is also an initial tree-hypersequent, or the cut can be replaced by various applications of the internal and external weakening rules on $G'[\alpha, \Pi]$.

Case 2. $G[\Gamma, \alpha]$ is inferred by a rule $\mathcal{R}$ in which α is not principal. This case can be standardly solved by induction on the sum of the heights of the derivations of the premises of the cut-rule, with the help of Lemma 10.17. As a matter of fact, no rule can change the position of the sequent where the cut occurs, and, on the other hand, the definition of product ensures that the structure of the tree-hypersequent stays unchanged, so that any problems are avoided. However, for the sake of clarity, let us consider the example of the rule $\Box K_{gl}$.

$$\dfrac{\dfrac{G[\Gamma, \alpha / \Box\beta \Rightarrow \beta]}{G[\Gamma, \alpha, \Box\beta]}\ \Box K_{gl} \qquad \begin{array}{c}\vdots\\ G'[\alpha, \Pi]\end{array}}{G \otimes G'[\Gamma \centerdot \Pi, \Box\beta]}\ cut_\alpha$$

We reduce to

$$\dfrac{\dfrac{G[\Gamma, \alpha / \Box\beta \Rightarrow \beta] \quad G'[\alpha, \Pi]}{G \otimes G'[\Gamma \centerdot \Pi / \Box\beta \Rightarrow \beta]}\ cut_\alpha}{G \otimes G'[\Gamma \centerdot \Pi, \Box\beta]}\ \Box K_{gl}$$

Case 3. $G[\Gamma, \alpha]$ is inferred by a rule $\mathcal{R}$ in which α is principal. We distinguish two subcases: in one subcase $\mathcal{R}$ is a propositional rule, in the other $\mathcal{R}$ is a modal rule.

Case 3.1. Supposing, for the sake of illustration, that the rule that introduces $G[\Gamma, \alpha]$ is $\neg K$ and $\alpha \equiv \neg\beta$, we have

$$\dfrac{\dfrac{G[\beta, \Gamma]}{G[\Gamma, \neg\beta]}\,\neg K \qquad \begin{matrix}\vdots\\ G'[\neg\beta, \Pi]\end{matrix}}{G \otimes G'[\Gamma \centerdot \Pi]}\, cut_{\neg\beta}$$

By applying Lemma 10.6 on $G'[\neg\beta, \Pi]$, we obtain $G'[\Pi, \beta]$. We replace the previous cut with the following which is eliminable by induction on the complexity of the cut-formula[3]:

$$\dfrac{G'[\Pi, \beta] \quad G[\beta, \Gamma]}{G \otimes G'[\Gamma \centerdot \Pi]}\, cut_{\beta}$$

Case 3.2. $\mathcal{R}$ is $\Box K_{gl}$ and $\alpha \equiv \Box\beta$. We have the following situation:

$$\dfrac{\dfrac{G[\Gamma/\Box\beta \Rightarrow \beta]}{G[\Gamma, \Box\beta]}\,\Box K_{gl} \qquad \begin{matrix}\vdots\\ G'[\Box\beta, \Pi]\end{matrix}}{G \otimes G'[\Gamma \centerdot \Pi]}\, cut_{\Box\beta}$$

We must consider the last rule $\mathcal{R}'$ of d_2. If no rule $\mathcal{R}'$ introduces $G'[\Box\beta, \Pi]$ because $G'[\Box\beta, \Pi]$ is an initial tree-hypersequent, then we can solve the case as in 1. If $\Box\beta$ is not principal in the rule $\mathcal{R}'$, we solve the case as in 2. Only those cases where $\mathcal{R}'$ is the rule $\Box A_{gl}$ or the rule 4 are problematic. We will analyse each of them.
$\Box A_{gl}$:

$$\dfrac{\dfrac{G[\Gamma/\Box\beta \Rightarrow \beta]}{G[\Gamma, \Box\beta]}\,\Box K_{gl} \qquad \dfrac{\dot{G}'[\Box\beta, \Pi/(\Psi, \Box\beta/\underline{Y})] \quad \dot{G}'[\Box\beta, \Pi/(\beta, \Psi/\underline{Y})]}{\dot{G}'[\Box\beta, \Pi/(\Psi/\underline{Y})]}\,\Box A_{gl}}{G \otimes \dot{G}'[\Gamma \centerdot \Pi/(\Psi/\underline{Y})]}\, cut_{\Box\beta}$$

The reduction unfolds in several steps. We expose them one-by-one.

First step:

$$\dfrac{G[\Gamma, \Box\beta] \quad \dot{G}'[\Box\beta, \Pi/(\Psi, \Box\beta/\underline{Y})]}{G \otimes \dot{G}'[\Gamma \centerdot \Pi/(\Psi, \Box\beta/\underline{Y})]}\, cut_{\Box\beta}$$

this cut is eliminable by induction on the sum of the heights of the derivations of the premises of the cut-rule.

Second step:

$$\frac{G[\Gamma, \Box\beta] \quad \dot{G}'[\Box\beta, \Pi/(\beta, \Psi/\underline{Y})]}{G \otimes \dot{G}'[\Gamma \centerdot \Pi/(\beta, \Psi/\underline{Y})]}\ cut_{\Box\beta}$$

this cut is eliminable by induction on the sum of the heights of the derivations of the premises of the cut-rule.

Third step:

$$\frac{G \otimes \dot{G}'[\Gamma \centerdot \Pi/(\Psi, \Box\beta/\underline{Y})] \quad G[\Gamma/\Box\beta \Rightarrow \beta]}{G \otimes G \otimes \dot{G}'[\Gamma \centerdot \Gamma \centerdot \Pi/(\Psi, \beta/\underline{Y})]}\ cut_{\Box\beta}$$

this cut is eliminable by induction on the position of the sequents that contain the cut-formula.

Fourth step:

$$\dfrac{\dfrac{G \otimes G \otimes \dot{G}'[\Gamma \centerdot \Gamma \centerdot \Pi/(\Psi, \beta/\underline{Y})] \quad G \otimes \dot{G}'[\Gamma \centerdot \Pi/(\beta, \Psi/\underline{Y})]}{G \otimes G \otimes \dot{G}' \otimes G \otimes \dot{G}'[\Gamma \centerdot \Gamma \centerdot \Pi \centerdot \Gamma \centerdot \Pi/(\Psi \centerdot \Psi/\underline{Y}; \underline{Y})]}\ cut_{\beta}}{G \otimes \dot{G}'[\Gamma \centerdot \Pi/(\Psi/\underline{Y})]}\ CA^*{+}CK^*{+}merge^*$$

This cut is eliminable by induction on the complexity of the cut-formula.

4:

$$\dfrac{\dfrac{G[\Gamma/\Box\beta \Rightarrow \beta]}{G[\Gamma, \Box\beta]}\ \Box K_{gl} \quad \dfrac{\dot{G}'[\Box\beta, \Pi/(\Box\beta, \Psi/\underline{Y})]}{\dot{G}'[\Box\beta, \Pi/(\Psi/\underline{Y})]}\ 4}{G \otimes \dot{G}'[\Gamma \centerdot \Pi/(\Psi/\underline{Y})]}\ cut_{\Box\beta}$$

In order to solve this case, it is necessary to analyse each rule that may have introduced the tree-hypersequent $\dot{G}'[\Box\beta, \Pi/(\Box\beta, \Psi/\underline{Y})]$. We go up the derivation until either a rule $\mathcal{R}''$ applies to a formula different from the $\Box\beta$'s, or a rule $\mathcal{R}''$ different from 4 applies to some of the $\Box\beta$'s. Let us indicate with the symbol ◎ the tree-hypersequent that is the conclusion of this rule $\mathcal{R}''$. We make a distinction between cases according to the type of rule $\mathcal{R}''$ is.

- ◎ is an initial tree-hypersequent. If $\Box\beta$ is not the principal formula of the axiom, then even the conclusion of the cut is an initial tree-hypersequent and the case is solved. Otherwise, there is a sequence in ◎ that contains the formula $\Box\beta$ on both its right and left sides. This sequent can be the sequent $\Box\beta, \Pi$, and then we get the conclusion by several applications of the rules of external and internal weakening on the left premise of the cut. Otherwise, this sequent may be n steps after $\Box\beta, \Pi$. In this case we apply the (admissible) rule $\tilde{4}$, $n + 1$ times, to the left premise $G[\Gamma/\Box\beta \Rightarrow \beta]$, and we obtain a tree-hypersequent where $\Box\beta \Rightarrow \beta$

is no longer after Γ, but n empty sequences after it. By first applying the rule $\Box K_{gl}$, then the rules of external and internal weakening repetitively to this tree-hypersequent, we reach the conclusion.

– $\circledcirc$ has been inferred by a rule $\mathcal{R}''$ that does not have any $\Box\beta$ as principal formula. In this case the technique consists of (i) applying the rule 4, n-times, to the premise of the rule $\mathcal{R}''$, and (ii) operating as in case 2.
– $\circledcirc$ has been inferred by the modal rule $\Box A_{gl}$ that has $\Box\beta$ as principal formula. Let us first of all distinguish two subcases. (a) The rule $\Box A_{gl}$ has been applied to two sequents that are n steps after $\Box\beta, \Pi$, let us suppose the sequents $\Box\beta, \Xi / \Box\beta, \Omega$.

$$
\dfrac{\dfrac{G[\Gamma / \Box\beta \Rightarrow \beta]}{G[\Gamma, \Box\beta]}\ \Box K_{gl} \qquad \dfrac{\begin{array}{c}\dfrac{\dfrac{\ddot{G}'[\Box\beta, \Pi]\ [\Box\beta, \Xi/(\Box\beta, \Omega, \Box\beta/\underline{Y}''))] \quad \ddot{G}'[\Box\beta, \Pi]\ [\Box\beta, \Xi/(\Box\beta, \beta, \Omega/\underline{Y}''))]}{\ddot{G}'[\Box\beta, \Pi]\ [\Box\beta, \Xi/(\Box\beta, \Omega/\underline{Y}''))]}\ \Box A_{gl}}{}\ 4 \\ \vdots \end{array}}{\ddot{G}'[\Box\beta, \Pi]\ [\Xi/(\Omega/\underline{Y}''))]}\ 4}{G \otimes \ddot{G}'[\Gamma \centerdot \Pi]\ [\Xi/(\Omega/\underline{Y}''))]}\ cut_{\Box\beta}
$$

We proceed in three steps.

(a1) We apply the rule 4, n-times, to the premises of the rule $\Box A_{gl}$. We thus obtain the two tree-hypersequents

$$
\begin{array}{ll}
\text{Th1} & \ddot{G}'[\Box\beta, \Pi]\ [\Xi/(\Omega, \Box\beta/\underline{Y}''))] \\
\text{Th2} & \ddot{G}'[\Box\beta, \Pi]\ [\Xi/(\beta, \Omega/\underline{Y}''))]
\end{array}
$$

(a2) We apply the rule $\tilde{4}$ to the tree-hypersequent $G[\Gamma/\Box\beta \Rightarrow \beta]$ a number of times sufficient to get $\Box\beta \Rightarrow \beta$ in an equivalent position with the sequent β, Ω of the tree-hypersequent Th2 (and therefore also with the sequent $\Omega, \Box\beta$ of the tree-hypersequent Th1). We thus obtain a tree-hypersequent where $\Box\beta \Rightarrow \beta$ is no longer after Γ, but n empty sequences after. Let us note this as $G[\Gamma]\ [\Box\beta \Rightarrow \beta]$.

(a3) We are now in a position to apply the following reductions.

First step:

$$
\dfrac{G[\Gamma, \Box\beta] \quad \ddot{G}'[\Box\beta, \Pi]\ [\Xi/(\Omega, \Box\beta/\underline{Y}''))]}{G \otimes \ddot{G}'[\Gamma \centerdot \Pi]\ [\Xi/(\Omega, \Box\beta/\underline{Y}''))]}\ cut_{\Box\beta}
$$

this cut is eliminable by induction on the sum of the heights of the derivations of the premises of the cut-rule.

Second step:

$$
\dfrac{G[\Gamma, \Box\beta] \quad \ddot{G}'[\Box\beta, \Pi]\ [\Xi/(\beta, \Omega/\underline{Y}''))]}{G \otimes \ddot{G}'[\Gamma \centerdot \Pi]\ [\Xi/(\beta, \Omega/\underline{Y}''))]}\ cut_{\Box\beta}
$$

this cut is eliminable by induction on the sum of the heights of the derivations of the premises of the cut-rule.

Third step:

$$\frac{G \otimes \ddot{G}'[\Gamma \centerdot \Pi]\ [\Xi/(\Omega, \Box\beta/\underline{Y}'')] \quad G[\Gamma][\Box\beta \Rightarrow \beta]}{G \otimes G \otimes \ddot{G}'[\Gamma \centerdot \Gamma \centerdot \Pi][\Xi/(\Omega, \beta/\underline{Y}'')]}\ cut_{\Box\beta}$$

this cut is eliminable by induction on the position of the sequents that contain the cut-formula.

Fourth step:

$$\frac{\dfrac{G \otimes G \otimes \ddot{G}'[\Gamma \centerdot \Gamma \centerdot \Pi][\Xi/(\Omega, \beta/\underline{Y}'')] \quad G \otimes \ddot{G}'[\Gamma \centerdot \Pi]\ [\Xi/(\beta, \Omega/\underline{Y}'')]}{G \otimes G \otimes \ddot{G}' \otimes G \otimes \ddot{G}'[\Gamma \centerdot \Gamma \centerdot \Pi \centerdot \Gamma \centerdot \Pi]\ [\Xi \centerdot \Xi/(\Omega \centerdot \Omega//\underline{Y}''; \underline{Y}'')]}\ cut_{\beta}}{G \otimes \ddot{G}'[\Gamma \centerdot \Pi]\ [\Xi/(\Omega/\underline{Y}'')]}\ CA^{*}+CK^{*}+merge^{*}$$

this cut is eliminable by induction on the complexity of the cut-formula.

(b) The rule $\Box A_{gl}$, with $\Box\beta$ principal formula, has been applied to the sequents $\Box\beta, \Pi/\Box\beta, \Psi, \Box\beta$ and $\Box\beta, \Pi/\Box\beta, \beta, \Psi$. In this case we apply the rule 4, n-times (see (a1)), to the premises of the rule $\Box A_{gl}$, and then we simply proceed as in $\Box A_{gl}$. □

Theorem 10.20 *Every derivation d in* $\mathbf{Thsgl}_L$ *can be effectively transformed into a derivation d′ where there is no application of the cut-rule.*

Proof It follows from Lemma 10.19, by induction on the number of cuts. □

Notes

1. For the sake of brevity, we present only a logical variant of the tree-hypersequent calculus for the system **GL**. The reader interested in a general variant should be able to obtain it on his own.
2. This symbol means: the premise of the right side is concluded by application of Lemma 10.6 on the premise of the left side.
3. Since we eliminate the cut by primary induction on the complexity of the cut-formula, the fact that the logical rules are invertible, and not height-preserving invertible, does not affect the course of the proof.

Chapter 11
Further Results on Tree-Hypersequent Calculi

Tree-hypersequent calculi have several characteristics in addition to those presented above; the object of this chapter is to provide an overview of them. Specifically, in the first section, we will explore the link between tree-hypersequent calculi and display calculi. In this way we will supplement Table 5.4, p. 116. In the second section we will mention logics to which the tree-hypersequent method has been applied that are not modal logics. We will bring our analysis to a close by suggesting further developments employing the framework of tree-hypersequent calculi.

11.1 Tree-Hypersequent Calculi and Other Calculi

In Chapter 5, we showed the relationships between the several extensions of the classical sequent calculus for modal logic. In this section, we will deal with the proof that tree-hypersequents can be simulated by display sequents.

Before reaching this result, let us introduce some translations, which are similar to the ones Mints [82] introduced:

$$(\alpha_1, \ldots, \alpha_n \Rightarrow \beta_1, \ldots, \beta_m)^s = \alpha_1 \circ \ldots \circ \alpha_n \Rightarrow \beta_1 \circ \ldots \circ \beta_m$$
$$(M_1, \ldots, M_n \Rightarrow N_1, \ldots, N_m)^p = (M_1)^* \circ \ldots \circ (M_n)^* \circ N_1 \circ \ldots \circ N_m$$

In other words, the translation s allows one to substitute the comma with the symbol $\circ$; while the translation p allows one to transform, by moving the antecedent in the appropriate way, two-side sequents into one-side sequents.

For $G \equiv M \Rightarrow N / H_1; \ldots; H_n$, set

$$(G)^s = (M \Rightarrow N \circ \bullet(H_1)^{sp} \circ \ldots \circ \bullet(H_n)^{sp})^s$$

Remark 11.1 In what follows, if $\Gamma \equiv M \Rightarrow N$, then $\Gamma^\dagger$ will denote $M^* \circ N$. Moreover, we will use the notation $\bullet\underline{X}$ to mean that, if $\underline{X}$ is the multiset composed by $H_1, \ldots, H_n$, then each H_i, $1 \leq i \leq n$, is of the form $\bullet H_i$.

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Lemma 11.2 *Given a tree-hypersequent $G[H]$ in its translation $(G[H])^s$, we can always rewrite the tree-hypersequent H in the following way:*

$$R \Rightarrow (H)^{sp}$$

where R is some display structure that depends on G.

Proof By induction on the tree-hypersequent $G[H]$.

If $G[H] \equiv H$, then the procedure is straightforward. If $H \equiv M \Rightarrow N/I_1; ...; I_n$ and $G[H] \equiv M \Rightarrow N/I_1; ...; I_n; H_1; ...; H_m$, then we have

$$\cfrac{\cfrac{M \Rightarrow N \circ \bullet(I_1)^{sp} \circ ... \circ \bullet(I_n)^{sp} \circ \bullet(H_1)^{sp} \circ ... \circ \bullet(H_m)^{sp}}{(\bullet(H_1)^{sp} \circ ... \circ \bullet(H_m)^{sp})^* \circ M \Rightarrow N \circ \bullet(I_1)^{sp} \circ ... \circ \bullet(I_n)^{sp}}}{(\bullet(H_1)^{sp} \circ ... \circ \bullet(H_m)^{sp})^* \Rightarrow \Gamma^\dagger \circ \bullet(I_1)^{sp} \circ ... \circ \bullet(I_n)^{sp}}$$

If, finally, $G[H] \equiv M \Rightarrow N/G^{'}[H]; H_1; ...; H_m$, then we have $M \Rightarrow N \circ \bullet(G^{'}[H])^{sp} \circ \bullet(H_1)^{sp} \circ ... \circ \bullet(H_m)^{sp}$. By inductive hypothesis on $G^{'}[H]$, we obtain $(\bullet(G^{'})^{sp})^* \circ M \Rightarrow N \circ (H)^{sp} \circ \bullet(H_1)^{sp} \circ ... \circ \bullet(H_m)^{sp}$, and from this we easily obtain $(N \circ \bullet(H_1)^{sp} \circ ... \circ \bullet(H_m)^{sp})^* \circ (\bullet(G^{'})^{sp})^* \circ M \Rightarrow (H)^{sp}$. □

Theorem 11.3 *Let $G[H]$ be any tree-hypersequent of the calculi* $\textbf{Thsk}^*_L$. *Then every derivation of $G[H]$ in* $\textbf{Thsk}^*_L$ *can be translated into a derivation of $R \Rightarrow (H)^{sp}$, where R is some display structure that depends on G, in* $\textbf{Dsk}^*$.

Proof The proof is by induction on the derivation of the sequent $G[H]$.[1]

If $G[H]$ is an axiom, then $R \Rightarrow (H)^{sp}$ is derivable in $\textbf{Dsk}^*$ by several applications of the rules of weakening on admissible axioms of the form $\alpha \Rightarrow \alpha$. If $G[H]$ has been inferred by one of the propositional rules for the connectives $\neg$ and $\wedge$, then the procedure is straightforward. If $G[H]$ has been inferred by the modal rule $\Box K$, then we have

$$\frac{G[\Gamma/ \Rightarrow \alpha]}{G[\Gamma, \Box\alpha]}\,\Box K \qquad \rightsquigarrow$$

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{R \Rightarrow \Gamma^\dagger \circ \bullet(I^* \circ \alpha)}{(\Gamma^\dagger)^* \circ R \Rightarrow \bullet(I^* \circ \alpha)}}{\bullet((\Gamma^\dagger)^* \circ R) \Rightarrow I^* \circ \alpha}}{\bullet((\Gamma^\dagger)^* \circ R) \Rightarrow \alpha}}{(\Gamma^\dagger)^* \circ R \Rightarrow \Box\alpha}\,\Box K}{R \Rightarrow \Box\alpha \circ \Gamma^\dagger}$$

If $G[H]$ has been inferred by the modal rule $\Box A$, then we have

$$\frac{G[\Box\alpha, \Gamma/(\alpha, \Sigma/\underline{X})]}{G[\Box\alpha, \Gamma/(\Sigma/\underline{X})]}\,\Box A \qquad \rightsquigarrow$$

$$
\begin{array}{cl}
R \Rightarrow \Box\alpha^* \circ \varGamma^\dagger \circ \bullet(\alpha^* \circ \varSigma^\dagger \circ \bullet\underline{X}) & \\ \hline
(\Box\alpha^* \circ \varGamma^\dagger)^* \circ R \Rightarrow \bullet(\alpha^* \circ \varSigma^\dagger \circ \bullet\underline{X}) & \\ \hline
\bullet((\Box\alpha^* \circ \varGamma^\dagger)^* \circ R) \Rightarrow \alpha^* \circ \varSigma^\dagger \circ \bullet\underline{X} & \\ \hline
(\varSigma^\dagger \circ \bullet\underline{X})^* \circ \bullet((\Box\alpha^* \circ \varGamma^\dagger)^* \circ R) \Rightarrow \alpha^* & \\ \hline
\alpha \Rightarrow ((\varSigma^\dagger \circ \bullet\underline{X})^* \circ \bullet((\Box\alpha^* \circ \varGamma^\dagger)^* \circ R))^* & \\ \hline
\Box\alpha \Rightarrow \bullet((\varSigma^\dagger \circ \bullet\underline{X})^* \circ \bullet((\Box\alpha^* \circ \varGamma^\dagger)^* \circ R))^* & \Box A \\ \hline
\bullet\Box\alpha \Rightarrow ((\varSigma^\dagger \circ \bullet\underline{X})^* \circ \bullet((\Box\alpha^* \circ \varGamma^\dagger)^* \circ R))^* & \\ \hline
(\varSigma^\dagger \circ \bullet\underline{X})^* \circ \bullet((\Box\alpha^* \circ \varGamma^\dagger)^* \circ R) \Rightarrow (\bullet\Box\alpha)^* & \\ \hline
\bullet((\Box\alpha^* \circ \varGamma^\dagger)^* \circ R) \Rightarrow (\bullet\Box\alpha)^* \circ \varSigma^\dagger \circ \bullet\underline{X} & \\ \hline
(\Box\alpha^* \circ \varGamma^\dagger)^* \circ R \Rightarrow \bullet((\bullet\Box\alpha)^* \circ \varSigma^\dagger \circ \bullet\underline{X}) & \\ \hline
\Box\alpha \circ (\Box\alpha^* \circ \varGamma^\dagger)^* \circ R \Rightarrow \bullet((\bullet\Box\alpha)^* \circ \varSigma^\dagger \circ \bullet\underline{X}) & \\ \hline
\bullet(\Box\alpha \circ (\Box\alpha^* \circ \varGamma^\dagger)^* \circ R) \Rightarrow (\bullet\Box\alpha)^* \circ \varSigma^\dagger \circ \bullet\underline{X} & \\ \hline
\bullet\Box\alpha \Rightarrow (\bullet(\Box\alpha \circ (\Box\alpha^* \circ \varGamma^\dagger)^* \circ R))^* \circ \varSigma^\dagger \circ \bullet\underline{X} & \\ \hline
\Box\alpha \Rightarrow \bullet((\bullet(\Box\alpha \circ (\Box\alpha^* \circ \varGamma^\dagger)^* \circ R))^* \circ \varSigma^\dagger \circ \bullet\underline{X}) & \\ \hline
\Box\alpha \circ (\Box\alpha^* \circ \varGamma^\dagger)^* \circ R \Rightarrow \bullet((\bullet(\Box\alpha \circ (\Box\alpha^* \circ \varGamma^\dagger)^* \circ R))^* \circ \varSigma^\dagger \circ \bullet\underline{X}) & \\ \hline
\bullet(\Box\alpha \circ (\Box\alpha^* \circ \varGamma^\dagger)^* \circ R) \Rightarrow (\bullet(\Box\alpha \circ (\Box\alpha^* \circ A)^* \circ R))^* \circ \varSigma^\dagger \circ \bullet\underline{X} & \\ \hline
\bullet(\Box\alpha \circ (\Box\alpha^* \circ \varGamma^\dagger)^* \circ R) \circ \bullet(\Box\alpha \circ (\Box\alpha^* \circ A)^* \circ R) \Rightarrow \varSigma^\dagger \circ \bullet\underline{X} & \\ \hline
\bullet(\Box\alpha \circ (\Box\alpha^* \circ \varGamma^\dagger)^* \circ R) \Rightarrow \varSigma^\dagger \circ \bullet\underline{X} & CA \\ \hline
\Box\alpha \circ (\Box\alpha^* \circ \varGamma^\dagger)^* \circ R \Rightarrow \bullet(\varSigma^\dagger \circ \bullet\underline{X}) & \\ \hline
(\Box\alpha^* \circ \varGamma^\dagger)^* \circ R \Rightarrow \Box\alpha^* \circ \bullet(\varSigma^\dagger \circ \bullet\underline{X}) & \\ \hline
R \Rightarrow \Box\alpha^* \circ \Box\alpha^* \circ \varGamma^\dagger \circ \bullet(\varSigma^\dagger \circ \bullet\underline{X}) & \\ \hline
R \Rightarrow \Box\alpha^* \circ \varGamma^\dagger \circ \bullet(\varSigma^\dagger \circ \bullet\underline{X}) &
\end{array}
$$

If $G[H]$ has been inferred by the special logical rule t, then we have

$$
\dfrac{G[\Box\alpha, \alpha, \Gamma]}{G[\Box\alpha, \Gamma]}\, t \quad \rightsquigarrow
$$

$$
\begin{array}{cl}
R \Rightarrow \Box\alpha^* \circ \alpha^* \circ \Gamma^\dagger & \\ \hline
\alpha \Rightarrow R^* \circ \Box\alpha^* \circ \Gamma^\dagger & \\ \hline
\Box\alpha \Rightarrow \bullet(R^* \circ \Box\alpha^* \circ \Gamma^\dagger) & \Box A \\ \hline
\Box\alpha \Rightarrow R^* \circ \Box\alpha^* \circ \Gamma^\dagger & t \\ \hline
\Box\alpha \circ \Box\alpha \Rightarrow R^* \circ \Gamma^\dagger & \\ \hline
\Box\alpha \Rightarrow R^* \circ \Gamma^\dagger & \\ \hline
R \Rightarrow \Box\alpha^* \circ \Gamma^\dagger &
\end{array}
$$

If $G[H]$ has been inferred by the special logical rule b, then we have

$$\dfrac{G[\alpha, \Gamma/(\Box\alpha, \Sigma/\underline{X})]}{G[\Gamma/(\Box\alpha, \Sigma/\underline{X})]}\, b \qquad\qquad \rightsquigarrow$$

$$\begin{array}{cl}
R \Rightarrow \alpha^* \circ \Gamma^\dagger \circ \bullet(\Box\alpha^* \circ \Sigma^\dagger \circ \bullet C) & \\
\hline
(\Gamma^\dagger \circ \bullet(\Box\alpha^* \circ \Sigma^\dagger \circ \bullet C))^* \circ R \Rightarrow \alpha^* & \\
\hline
\alpha \Rightarrow ((\Gamma^\dagger \circ \bullet(\Box\alpha^* \circ \Sigma^\dagger \circ \bullet C))^* \circ R)^* & \\
\hline
\Box\alpha \Rightarrow \bullet((\Gamma^\dagger \circ \bullet(\Box\alpha^* \circ \Sigma^\dagger \circ \bullet C))^* \circ R)^* & \Box A \\
\hline
(\bullet((\Gamma^\dagger \circ \bullet(\Box\alpha^* \circ \Sigma^\dagger \circ \bullet C))^* \circ R)^*)^* \Rightarrow \Box\alpha^* & \\
\hline
\bullet((\Gamma^\dagger \circ \bullet(\Box\alpha^* \circ \Sigma^\dagger \circ \bullet C))^* \circ R) \Rightarrow \Box\alpha^* & b \\
\hline
\bullet((\Gamma^\dagger \circ \bullet(\Box\alpha^* \circ \Sigma^\dagger \circ \bullet C))^* \circ R) \Rightarrow \Box\alpha^* \circ \Sigma^\dagger \circ \bullet C & \\
\hline
(\Gamma^\dagger \circ \bullet(\Box\alpha^* \circ \Sigma^\dagger \circ \bullet C))^* \circ R \Rightarrow \bullet(\Box\alpha^* \circ \Sigma^\dagger \circ \bullet C) & \\
\hline
R \Rightarrow \bullet(\Box\alpha^* \circ \Sigma^\dagger \circ \bullet C) \circ \Gamma^\dagger \circ \bullet(\Box\alpha^* \circ \Sigma^\dagger \circ \bullet C) & \\
\hline
R \Rightarrow \Gamma^\dagger \circ \bullet(\Box\alpha^* \circ \Sigma^\dagger \circ \bullet C) & CK
\end{array}$$

If $G[H]$ has been inferred by the special logical rule 4, then we use the same procedure adopted for the rule $\Box A$ plus we exploit the special display structural rule 4. If $G[H]$ has been inferred by the special logical rule 5, then we use the same procedure adopted for the rule b plus we exploit the special display structural rule 4. □

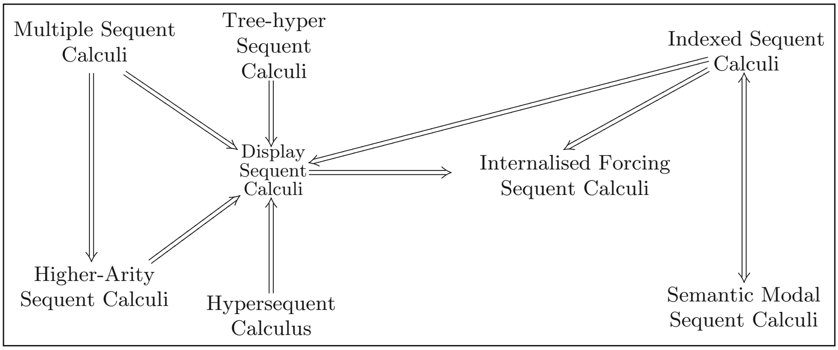

Fig. 11.1 Relationships between sequent calculi for modal logic

11.2 Tree-Hypersequent Calculi and Other Logics

In the present part of the book, we observed the applicability of the tree-hypersequent method to a broad number of modal logic systems, such as the systems **K**, **KT**, **KB**, **S4**, **S5**, **GL**. It is worth mentioning that the tree-hypersequent method has also been applied to other logics. A notable example is Kashima's result [55], which consists in the application of the tree-hypersequent method to first-order constructive logics with negation. Another significant instance of the application of the tree-hypersequent method to other logics can be found in the work of Ishigaki and Kikuchi [66], where tree-hypersequents are applied to subintuitionistic predicate logics. Also noteworthy is the work of Bünnler and Stüder, [17], on syntactic cut-elimination for common knowledge. Finally, Hill and Poggiolesi [60] have also put forward interesting results, where tree-hypersequents are used to provide propositional dynamic logic with a sequent calculus.

11.3 Tree-Hypersequent Calculi and Further Developments

There are at least three different routes for further developments of the tree-hypersequent framework. The first one concerns the results that are provable with it. It would be interesting to prove the interpolation theorem, or to find an algorithm that, given a modal axiom, it generates the corresponding tree-hypersequent rule (similarly to display calculi, see Section 3.3, and in line with the work developed in [26]).

Following the second route for future research would lead one to analyse the relationship between tree-hypersequent calculi and other proof-systems, in one of two ways. On the one hand, one could compare tree-hypersequent calculi to other types of calculi. This task would be fairly simple using tableaux systems, but it would be more arduous using natural deduction calculi. On the other hand, tree-hypersequent calculi could serve as inspiration to enhance our knowledge of more recent proof-tools. A particularly relevant and interesting example would be proof-nets. As Restall [117] also suggests, proof-nets for modal logic remain unexplored, and there is no doubt that tree-hypersequents could contribute to developing useful results.

A third and final option for future research would be to focus on the applicability of the tree-hypersequent method to other logics. We just saw that much has already been done in this respect, but there is still room for further enquiry. What about, for instance, the use of tree-hypersequents for temporal logics or for dynamic epistemic logics ?

Given what we have said in the book, in each of these potential developments, one may expect the tree-hypersequent method to yield valuable, and perhaps unexpected, results.

Note

1. In order to shorten the derivations in the calculi **Dsk***, we may use several rules in a row and indicate them with just one inference. For this reason and also not to burden the derivations, we will not specify the names of the rules used in the derivations except for the most significant ones.

References

1. Aristotle. *Posterior Analytics*. Clarendon Press, Oxford, 1993.
2. Aristotle. *Prior Analytics*. In http://classics.mit.edu/Aristotle/prior.html, 1994.
3. A. Avron. A constructive analysis of RM. *Journal of Symbolic Logic*, 52:939–951, 1987.
4. A. Avron. Natural 3-valued logics characterization and proof theory. *Journal of Symbolic Logic*, 56:276–294, 1991.
5. A. Avron. Using hypersequents in proof systems for non-classical logics. *Annals of Mathematics and Artificial Intelligence*, 4:225–248, 1991.
6. A. Avron. The method of hypersequents in the proof theory of propositional non-classical logic. In W. Hodges, M. Hyland, C. Steinhorn, and J. Strauss, editors, *Logic: from Foundations to Applications*, pp. 1–32. Oxford University Press, Oxford, 1996.
7. M. Baaz and R. Zach. Hypersequents and cut elimination for intuitionistic fuzzy logic. In G. Peter and H. Schwichtenberg, editors, *14th International Workshop, CSL 2000. Fischbachau, Germany, August 21–26, 2000. Proceedings*, pp. 187–201. Springer, Berlin, 2000.
8. N. D. Belnap. Display logic. *Journal of Philosophical Logic*, 11:375–417, 1982.
9. N. D. Belnap. Linear logic displayed. *Notre Dame Journal of Formal Logic*, 31:14–25, 1990.
10. N. D. Belnap. The display problem. In H. Wansing, editor, *Proof Theory of Modal Logic*, pp. 79–92. Kluwer Academic Publisher, Dordrecht, 1996.
11. P. Blackburn, M. de Rijke, and Y. Venema. *Modal Logic*. Cambridge University Press, Cambridge, 2001.
12. S. Blamey and L. Humberstone. A perspective on modal sequent logic. *Publications of the Research Institute for Mathematical Sciences, Kyoto University*, 27:763–782, 1991.
13. G. Boloos. Don't eliminate cut. *Journal of Philosophical Logic*, 13:373–378, 1984.
14. T. Braüner. A cut-free Gentzen formulation of the modal logic S5. *Logic Journal of the IGPL*, 8:629–643, 2000.
15. K. Brünnler. Deep sequent systems for modal logic. In G. Governatori, I. Hodkinson, and Y. Venema, editors, *Advances in Modal Logic, Vol 6*, pp. 107–119. College Publications, London, 2006.
16. K. Brünnler and L. Strassburger. Modular sequent systems for modal logic. In M. Giese and A Waaler, editors, *Proceeding of Tableaux 2009, LNAI 5607*, pp. 152–166. Springer, Berlin-Heidelberg, 2009.
17. K. Brünnler and T. Stüder. Syntactic cut-elimination for common knowledge. *Annals of Pure and Applied Logic*, 160:82–95, 2009.
18. E. Casari. *Introduzione alla Logica*. UTET, Torino, 1997.
19. L. Catach. Tableaux: A general theorem prover for modal logic. *Journal of Automated Reasoning*, 7:489–510, 1991.
20. C. Cerrato. Cut-free modal sequents for normal modal logics. *Notre Dame Journal of Formal Logic*, 34:564–582, 1993.
21. C. Cerrato. Modal sequents for normal modal logics. *Mathematical Logic Quarterly*, 39:231–240, 1993.

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DOI 10.1007/978-90-481-9670-8, © Springer Science+Business Media B.V. 2011

22. C. Cerrato. Modal sequents. In H. Wansing, editor, *Proof Theory of Modal Logic*, pp. 141–166. Kluwer Academic Publisher, Dordrecht, 1996.
23. C. Cerrato. Modal tree-sequents. *Mathematical Logic Quarterly*, 42:197–210, 1996.
24. B. F. Chellas. *Modal Logic*. Cambridge University Press, Cambridge, 1980.
25. A. Ciabattoni. Hypersequent calculi for some intermediate logics with bounded Kripke models. *Journal of Logic and Computation*, 11:283–294, 2001.
26. A. Ciabattoni, L. Strassburger, and K. Terui. Expanding the realm of systematic proof theory. In E. Grädel and R. Kahle, editors, *Lectures Notes in Computer Science*, pp. 163–178. Springer, Berlin-Heidelberg, 2009.
27. H. B. Curry. The elimination theorem when modality is present. *Journal of Symbolic Logic*, 17:249–265, 1952.
28. H. B. Curry. *Foundations of Mathematical Logic*. Dover, New York, 1977.
29. M. D'agostino and M. Mondadori. The taming of the cut. Classical refutations with analytic cut. *Journal of Logic and Computation*, 4:285–319, 1994.
30. R. Descartes. *Rules for the Direction of the Mind, Discourse on the Method, Meditations on First Philosophy, Objections against the Meditations and Replies, The Geometry*. William Benton, Chicago, London, Toronto, Geneva, Sydney, Tokyo, Manila, 1952.
31. K. Došen. Sequent-systems for modal logic. *Journal of Symbolic Logic*, 50:149–159, 1985.
32. K. Došen. Sequent systems and groupoid models I. *Studia Logica*, 47:353–389, 1988.
33. K. Došen. Logical constants as punctuation marks. *Notre Dame Journal of Formal Logic*, 30:362–381, 1989.
34. A. G. Dragalin. *Mathematical Intuitionism. Introduction to Proof Theory*. American Mathematical Society, Providence, 1988.
35. M. Dummett. *The logical basis of metaphysics*. Harvard University Press, Cambridge, 1991.
36. R. Feys. Les systèmes formalisés des modalités aristotéliciennes. *Revue Philosophique de Louvain*, 48:478–509, 1950.
37. K. Fine. An incomplete logic containing S4. *Theoria*, 40:23–29, 1974.
38. M. Fitting. *Proof Methods for Modal and Intuitionistic Logics*. Reidel, Dordrecht, 1983.
39. N. Francez and R. Dyckhoff. Proof-theoretic semantics for natural language. In *Proceedings of the 10th Meeting of the Association of Mathematics of Language (MOL), July 2007*, pp. 1–7. Los Angeles, forthcoming.
40. D. Gabbay. *Labelled Deductive Systems: Volume 1. Foundations*. Oxford University Press, Oxford, 1996.
41. Galileo. *Dialogue Concerning the Two Chief World Systems, Ptolemaic and Copernican*. University of California Press, Berkley, 1967.
42. G. Gentzen. Investigations into logical deductions. In M. E. Szabo, editor, *The Collected Papers of Gerhard Gentzen*, pp. 68–131. North-Holland, Amsterdam, 1969.
43. J.-Y. Girard. *Proof Theory and Logical Complexity, Volume 1*. Bibliopolis, Napoli, 1987.
44. J.-Y. Girard, Y. Lafont, and P. Taylor. *Proofs and Types*. Cambridge University Press, Cambridge, 1989.
45. L. Goble. Gentzen systems for modal logic. *Notre Dame Journal of Formal Logic*, 15:455–461, 1974.
46. R. I. Golblatt and S. K. Thomason. Axiomatic classes in propositional modal logic. In J. Crossley, editor, *Algebra and Logic*, pp. 163–173. Springer, Dordrecht, 1974.
47. R. Goré. *Cut-free Sequent and Tableau Systems for Propositional Normal Modal Logics*. Ph.D. Thesis, Technical Report No. 257, University of Cambridge, Cambridge Computer Laboratory, 1992.
48. R. Goré. Substructural logics on display. *Logic Journal of the IGPL*, 6:669–694, 1998.
49. R. Goré. Tableau methods for modal and temporal logics. In M. D'Agostino, D. M. Gabbay, R. Hähnle, and J. Posegga, editors, *Handbook of Tableau Methods*, pp. 297–396. Kluwer Academic Publisher, Dordrecht, 1999.
50. R. Goré and R. Ramanayake. Valentini's cut-elimination for provability logic resolved. In C. Areces and R. Goldblatt, editors, *Advances in Modal Logic, Vol 7*, pp. 67–86. College Publications, London, 2008.

51. R. Goré and A. Tiu. Classical modal display logic in the calculus of structures and minimal cut-free deep inference calculi for S5. *Journal of Logic and Computation*, 17:767–794, 2007.
52. S. Gottwald. *Mehrwertige Logik*. Akademie, Berlin, 1989.
53. A. Guglielmi. A system of interaction and structure. *ACM Transactions on computational Logic*, 8:1–64, 2007.
54. I. Hacking. What is logic? *Journal of Philosophy*, 76:285–319, 1979.
55. I. Hasuo and R. Kashima. Kripke completeness of first-order constructive logics with strong negation. *Logic Journal of the IGPL*, 11:615–646, 2003.
56. T. Heath. *A history of Greek Mathematics, Vol.1*. Courier Dover Publications, New York, 1981.
57. R. Hein and C. Stewart. Purity through unravelling. In P. Bruscoli, F. Lamarche, and C. Stewart, editors, *Structures and Deductions*, pp. 126–143. Technische Universit, Dresden, 2005.
58. P. Hertz. Uber axiomensysteme beliebiger satzsysteme. *Annalen der Philosophie und philosophischen Kritik*, 8:178–204, 1929.
59. P. Hertz. Uber axiomensysteme für beliebige satzsysteme. *Mathematische Annalen*, 101:457–514, 1929.
60. B. Hill and F. Poggiolesi. A contraction-free and cut-free sequent calculus for propositional dynamic logic. *Studia Logica*, 94:68–94, 2010.
61. O. Hjortland. Proof-theoretic harmony and structural assumptions. *http://olethhjortland.googlepages.com/papers*, pp. 1–20, 2007.
62. G. E. Hughes and M. J. Cresswell. *A New Introduction to Modal Logic*. Routledge, London-New York, 1996.
63. A. Indrezejczak. Generalised sequent calculus for propositional modal logics. *Logica Trianguli*, 1:15–31, 1997.
64. A. Indrezejczak. Cut-free double sequent calculus for S5. *Logic Journal of the IGPL*, 6:505–516, 1998.
65. A. Indrezejczak. Modal hybrid logic. *Logic and Logical Philosophy*, 16:147–257, 2007.
66. R. Ishigaki and K. Kikuchi. Tree-sequent method for subintuitionistic predicate logics. *Proceedings of the 16th International Conference on Automated Reasoning with Analytic Tableaux and Related Methods, LNAI*, 4548:149–164, 2007.
67. S. Kanger. *Provability in Logic*. Almqvist and Wiskell, Stockholm, 1957.
68. R. Kashima. Cut-free sequent calculi for some tense logics. *Studia Logica*, 53:119–135, 1994.
69. M. Kracht. Highway to the danger zone. *Journal of Logic and Computation*, 5:93–109, 1995.
70. M. Kracht. Power and weakness of the modal display calculus. In H. Wansing, editor, *Proof Theory of Modal Logic*, pp. 93–121. Kluwer Academic Publisher, Dordrecht, 1996.
71. M. Kremer. Logic and meaning: the philosophical significance of the sequent calculus. *Mind, New Series*, 97:50–72, 1988.
72. S. Kripke. Semantical analysis of modal logic I: Normal modal propositional calculi. *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, 11:3–16, 1968.
73. H. Kushida and M. Okada. A proof-theoretic study of the correspondence of classical logic and modal logic. *Journal of Symbolic Logic*, 68:1403–1414, 2003.
74. D. Leivant. On the proof theory of the modal logic for arithmetic provability. *Journal of Symbolic Logic*, 46:531–538, 1981.
75. E. Lemmon and D. Scott. *An Introduction to Modal Logic*. Blackwell, Oxford, 1977.
76. S. Leśniewski. Gründzuge eines neuen systems der grundlagen der mathematik. *Fundamenta Mathematicae*, 14:1–81, 1929.
77. A. Masini. 2-sequents calculus: A proof theory of modalities. *Annals of Pure and Applied Logic*, 58:229–246, 1992.
78. G. Metcalfe, N. Olivetti, and D. M. Gabbay. Sequent and hypersequent calculi for Abelian and Lukasiewicz logics. *ACM Transaction on Computational Logic (TOCL)*, 1:691–746, 1991.
79. G. Mints. Cut-free calculi of the S5 type. *Studies in constructive mathematics and mathematical logic. Part II. Seminars in Mathematics*, 8:79–82, 1970.

80. G. Mints. Gentzen-type systems and resolution rules. Part I. Propositional logic. In P. Martin-Löf and G. Mints, editors, *COLOG-88 Lecture Notes in Computer Science 417*, pp. 198–231. Springer, Berlin, 1990.
81. G. Mints. *A Short Introduction to Modal Logic*. CSLI Publications, Stanford, 1992.
82. G. Mints. Indexed systems of sequents and cut-elimination. *Journal of Philosophical Logic*, 26:671–696, 1997.
83. A. Moen. The proposed algorithms for eliminating cuts in the provability calculus GLS do not terminate. *NWPT 2001, Norwegian Computing Center, 2001-12-10*, 2001.
84. S. Negri. Contraction-free sequent calculi for geometric theories, with an application to Barr's theorem. *Archives of Mathematical Logic*, 42:389–401, 2003.
85. S. Negri. Proof analysis in modal logic. *Journal of Philosophical Logic*, 34:507–544, 2005.
86. S. Negri and J. von Plato. *Structural Proof Theory*. Cambridge University Press, Cambridge, 2004.
87. A. Nerode. Some lectures on modal logic. In F. L. Bauer, editor, *Logic, Algebra and Computation*, pp. 281–334. NATO Science Series, Springer, Amsterdam, Berlin, Oxford, Tokyo, Washington DC, 1991.
88. H. Ohlbach. Semantic-based translation methods for modal logics. *Journal of Logic and Computation*, 1:691–746, 1991.
89. M. Ohnishi and K. Matsumoto. Gentzen method in modal calculi. *Osaka Mathematical Journal*, 9:113–130, 1957.
90. M. Ohnishi and K. Matsumoto. Gentzen method in modal calculi II. *Osaka Mathematical Journal*, 11:115–120, 1959.
91. E. Orlowska. Relational proof systems for modal logics. In H. Wansing, editor, *Proof Theory of Modal Logic*, pp. 55–77. Kluwer Academic Publisher, Dordrecht, 1996.
92. E. Orlowska. Relational interpretations of modal logics. In H. Andreka, D. Monk, and I. Nemeti, editors, *Algebraic Logic. Colloquia Mathematica Societatis Janos Bolyai*, pp. 443–471. North Holland, Amsterdam, 1998.
93. F. Paoli. Bolzano e le dimostrazioni matematiche. *Rivista di Filosofia*, LXXVIII:221–242, 1991.
94. F. Paoli. *Substructural Logics: a Primer*. Kluwer Academic Publisher, Dordrecht-Boston-London, 2002.
95. F. Paoli. Quine and slater on paraconsistency and deviance. *Journal of Philosophical Logic*, 32:531–548, 2003.
96. Pappus. *Book 7 of the Collections*. Springer, New York-Berlin-Heidelberg-Tokyo, 1986.
97. A. Pliuškevičiene. Cut-free calculus for modal logics containing the Barcan axiom. In M. Kracht, M. de Rijke, and H. Wansing, editors, *Advances in Modal Logic '96*, pp. 157–172. CSLI Publications, Stanford, 1998.
98. A. Pliuškevičiene. Extended disjunction and existence properties for some predicate modal logics. *Journal of the IGPL*, 6:775–787, 1998.
99. F. Poggiolesi. A cut-free simple sequent calculus for modal logic S5. *Review of Symbolic Logic*, 1:3–15, 2008.
100. F. Poggiolesi. The method of tree-hypersequent for modal propositional logic. In D. Makinson, J. Malinowski, and H. Wansing, editors, *Towards Mathematical Philosophy*, pp. 31–51. Springer, Berlin, 2008.
101. F. Poggiolesi. A new definition of the Doŝen's principle. In M. Pelis, editor, *Logica Yearbook 2007*, pp. 133–142. Filosofia, Prague, 2008.
102. F. Poggiolesi. *Sequent Calculi for Modal Logic*. Ph.D. Thesis, Florence/ Paris, 2008.
103. F. Poggiolesi. A purely syntactic and cut-free sequent calculus for the modal logic of provability. *Review of Symbolic Logic*, 2:593–611, 2009.
104. F. Poggiolesi. Display calculi and other calculi: a comparison. *Synthese*, 173:259–279, 2010.
105. F. Poggiolesi. Reflecting the semantic features of S5 at the syntactic level. In C. Sinigaglia and M. D'Agostino, editors, *Proceedings of SILFS Conference*. College Publications, London, forthcoming.

106. G. Pottinger. Uniform cut-free formulations of T, S4 and S5 (abstract). *Journal of Symbolic Logic*, 48:900, 901, 1983.
107. D. Prawitz. Meaning and proofs: on the conflict between classical and intuitionistic logic. *Theoria*, 43:2–40, 1977.
108. A. N. Prior. A runabout inference ticket. *Analysis*, 21:38–39, 1960.
109. W. V. O. Quine. *Word and Object*. MIT Press, Cambridge, 1960.
110. W. V. O. Quine. *Philosophy of Logic*. Prentice-Hall, Englewood Cliffs, 1970.
111. W. V. O. Quine. *The Roots of Reference*. Open Court, La Salle, 1973.
112. W. Rautenberg. Modal tableau calculi and interpolation. *Journal of Philosophical Logic*, 12:403–423, 1983.
113. S. Read. Harmony and autonomy in classical logic. *Journal of Philosophical Logic*, 29:123–154, 2000.
114. S. Read. Harmony and modality. In C. Dégremont, L. Kieff, and H. Rückert, editors, *Dialogues, Logics and Other Strong Things: Essays in Honour of Shahid Rahman*, pp. 285–303. College Publications, London, 2008.
115. G. Restall. Displaying and deciding substructural logics. I: Logics with contraposition. *Journal of Philosophical Logic*, 27:179–216, 1998.
116. G. Restall. Comparing modal sequent systems. *Unpublished manuscript*, pp. 1–13, 2006.
117. G. Restall. Proofnets for S5: sequents and circuits for modal logic. In C. Dimitracopoulos, L. Newelski, and D. Normann, editors, *Logic Colloquium 2005*, pp. 151–172. Cambridge University Press, Cambridge, 2007.
118. G. Rousseau. Sequents in many-valued logic I. *Fundamenta Mathematicae*, 60:23–131, 1967.
119. H. Sahlqvist. Completeness and correspondence in first and second order semantics for modal logic. In S. Kanger, editor, *Proceedings of the Third Scandanavian Logic Symposium*, pp. 110–143. North Holland, Amsterdam, 1975.
120. G. Sambin, G. Battilotti, and C. Faggian. Basic logic: Reflection, symmetry, visibility. *Journal of Symbolic Logic*, 65:979–1013, 2000.
121. G. Sambin and S. Valentini. The modal logic of provability. The sequential approach. *Journal of Philosophical Logic*, 11:311–342, 1982.
122. K. Sasaki. Löb's axiom and cut-elimination theorem. *Journal of the Nanzan Academic Society Mathematical Sciences and Information Engineering*, 1:91–98, 2001.
123. M. Sato. A study of Kripke-type models for some modal logics by Gentzen's sequential method. *Publications of the Research Institute for Mathematical Sciences, Kyoto University*, 13:381–468, 1977.
124. M. Sato. A cut-free Gentzen-type system for the modal logic S5. *Journal of Symbolic Logic*, 45:67–84, 1980.
125. P. Schroeder-Heister. A natural extension of natural deduction (abstract). *Journal of Symbolic Logic*, 49:1284–1300, 1984.
126. P. Schroeder-Heister. Generalised definitional reflection and the inversion principle. *Logica Universalis*, 1:355–376, 2007.
127. M. Schroeter. Methoden zur axiomatisierung beliebiger aussagen und praedikatenkalkuele. *Zeitschrift fuer mathematische Logik und Grundlagen der Mathematik*, 1:214–251, 1955.
128. K. Sergerberg. *An Essay in Classical Modal Logic*. Filosofiska Studier (13), Uppsala, 1971.
129. G. Shvarts. Gentzen style systems for K45 and K45D. In A. Meyer and M. Taitslin, editors, *Logic at Botik '89, Lectures Notes in Computer Science 363*, pp. 245–256. Springer, Berlin, 1989.
130. A. Simpson. *Proof Theory and Semantics of Intuitionistic Modal Logic*. Ph.D. Thesis, School of Informatics, University of Edinburgh, Edinburgh, 1994.
131. R. M. Smullyan. *First-Order Logic*. Springer, Berlin, 1968.
132. C. Stewart and P. Stouppa. A systematic proof theory for several modal logics. In R. Schmidt, I. Pratt-Hartmann, M. Reynolds, and H. Wansing, editors, *Advances in Modal Logic, Vol 5*, pp. 309–333. College Publications, London, 2004.

133. P. Stouppa. *The Design of Modal Proof Theories: the Case of S5*. Master thesis, Technische Universitat Dresden, 2004.
134. P. Stouppa. A deep inference system for the modal logic S5. *Studia Logica*, 85:199–214, 2007.
135. M. Takano. Subformula property as a substitute for cut-elimination in modal propositional logics. *Mathematica Japonica*, 37:1129–1145, 1992.
136. N. Tennant. *The taming of the true*. Oxford University Press, Oxford, 1997.
137. N. Tennant. Inferentialism, logicism, harmony, and a counterpoint. In A. Miller, editor, *Essays for Crispin Wright: Logic, language, and mathematics*, Vol 2, pp. 105–132. Oxford University Press, London, 2007.
138. S.K. Thomason. An incompleteness theorem in modal logic. *Theoria*, 40:150–158, 1974.
139. A. S. Troelestra and H. Schwichtenberg. *Basic Proof Theory*. Cambridge University Press, Cambridge, 1996.
140. S. Valentini. The modal logic of provability: Cut-elimination. *Journal of Philosophical Logic*, 12:471–476, 1983.
141. L. Viganò. *Labelled Non-Classical Logics*. Kluwer, Boston-Dordrecht-London, 2000.
142. L. Viganò, D. Basin, and S. Matthews. Natural deduction for non-classical logics. *Studia Logica*, 60:119–160, 1998.
143. L. Viganò, P. Mateus, A. Sernadas, and C. Sernadas. Modal sequent calculi labelled with truth values: Completeness, duality and analyticity. *Logic Journal of IGPL*, 12:227–274, 2004.
144. L. Wallen. *Automated Deductions in Non-classical Logics*. MIT Press, Cambridge, Massachusetts, 1990.
145. H. Wansing. Sequent systems for normal modal propositional logics. *Journal of Logic and Computation*, 4:125–142, 1994.
146. H. Wansing. Displaying as temporalizing. Sequent systems for subintuitionistic logics. In S. Hakama, editor, *Logic, Language and Computation*, pp. 159–178. Kluwer Academic Publisher, Dordrecht, 1997.
147. H. Wansing. *Displaying Modal Logic*. Kluwer Academic Publisher, Dordrecht, 1998.
148. H. Wansing. The idea of a proof-theoretic semantics and the meaning of the logical operations. *Studia Logica*, 64:3–20, 2000.
149. H. Wansing. Sequent systems for modal logics. In *Handbook of Philosophical Logic, Vol 8*, pp. 61–145. Kluwer, Dordrecht, 2002.
150. J.J. Zeman. *Modal logic. The Lewis-Modal Systems*. Oxford University Press, Oxford, 1973.
151. J. Zucker and R. Tragesser. The adequacy problem for inferential logic. *Journal of Philosophical Logic*, 7:501–516, 1978.

Symbols and Notations

Below we list the symbol that either appear in the text more than just locally, or are important for other reasons. The more relevant conventions and notations in use throughout the book are found in Sections 1.1 and 2.1.

Logical Operators

$\neg, \wedge, \vee, \rightarrow, \leftrightarrow$	p. 3	(propositional operators)
$\bot$	p. 3	(falsity)
$\top$	p. 3	(truth)
$\Box, \Diamond$	p. 39	(modal operators)
$\Rightarrow$	p. 4	(sequent arrow)
$\bigwedge, \bigvee$	p. 4	(iterated conjunction and disjunction)

Languages and Well-Formed Formulas

$\mathcal{L}^c$	p. 3	(classical propositional language)
$\mathcal{L}^\Box$	p. 39	(modal propositional language)
$\mathcal{L}^\Box_{\neg,\wedge,\Box}$	p. 40	(modal propositional language restricted to the connectives $\neg$, $\wedge$, $\Box$)
WF	p. 3	(set of well-formed formulas)
PL	p. 3	(propositional letters)
WMF	p. 39	(set of well-formed modal formulas)

Measures on Formulas and Derivations

d, d'	p. 4	(formal derivations)
$cmp(\alpha)$	p. 40	(complexity of a formula)
$h(d)$	p. 6	(height of a derivation)
$s(G)$	p. 172	(size of a tree-hypersequent)
$sf(G)$	p. 172	(set of subformulas of a tree-hypersequent)

Turnstiles Symbols

$\vdash_{\mathbf{H}}$	p. 4	(deducibility in a Hilbert system)
$\vdash_{\mathbf{G}}$	p. 4	(deducibility in a Gentzen system)
$d \vdash^n_{\mathbf{G}}$	p. 6	(deducibility with height $\leq n$ in a Gentzen system)
$i \models_{\mathfrak{M}}$	p. 41	(satisfability)
$\models_{\mathfrak{C}}$	p. 41	(validity in a class of frames)

Formalisms

- general formalisms

`S`	p. 4	(formal system)
`NMS`	p. 40	(normal modal system)
`SLH`	p. 44	(Scott-Lemmon Hilbert system)
H	p. 4	(Hilbert system)
G	p. 4	(Gentzen system)
N	p. 4	(natural deduction system)

- non modal formalisms

Gcl	p. 5	(Gentzen system for classical logic)
$\mathbf{Gcl}_L$	p. 9	(logical variant of the Gentzen system for classical logic)
$\mathbf{Gcl}_S$	p. 10	(structural variant of the Gentzen system for classical logic)
Gil	p. 11	(Gentzen system for intuitionistic logic)
Gll	p. 11	(Gentzen system for linear logic without exponentials)
Hcl	p. 7	(Hilbert system for classical logic)

- Hilbert modal formalisms

K	p. 40	**KD**	p. 44
KT	p. 44	**K4**	p. 44
KB	p. 44	**S4**	p. 44
S5	p. 44	**GL**	p. 44

- Gentzen modal formalisms

Msk*	p. 59	(multiple sequent calculi)
H-ask*	p. 65	(higher-arity sequent calculi)
Dsk*	p. 71	(display sequent calculi)

Ssk*	p. 83	(semantic sequent calculi)
Isk*	p. 91	(indexed sequent calculi)
Ifsk*	p. 98	(internalised forcing sequent calculi)
Thsk*	p. 127	(tree-hypersequent calculi)

Syntactic Tools

Let $\star \in \{\wedge, \vee, \rightarrow, W, C\}$

$\mathcal{R}$	p. 6	(rule)
$\mathcal{R}^*$	p. 74	(repeated running application of the same rule)
$\mathcal{R}^* + \mathcal{R}'^* + \mathcal{R}''^*$	p. 163	(repeated running application of different rules)
$\star A$, $\star K$	p. 5	(left and right introduction rules)
Ax, $A\bot$	p. 5	(axioms)
cut_α	p. 5	(cut-rule)

Semantic Tools

$i, j, \ldots$	p. 40	(variables for possible worlds)
R	p. 40	(accessibility relation)
$\mathcal{F}$	p. 40	(frame)
$\mathfrak{M}$	p. 41	(model)
$\mathcal{T}$	p. 40	(tree-frame)

Other Notations

$\square$	p. 7	(end-of-proof symbol)
$^\delta, {}^\tau$	p. 4	(translations and embedding)
$\rightsquigarrow$	p. 141	(application of the inductive hypothesis)
$\dashrightarrow$	p. 141	(application of the invertibility of the logical rules)
$prove(G, (\mathbf{Y})^+)$	p. 172	(procedure for building derivation trees in tree-hypersequent calculi)

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